This week's Riddler Classic asks:

Suppose [there are] two distinct points are anywhere on the coordinate plane. If I tell you that a parabola with a vertical line of symmetry passes through those two points, where on the plane could that parabola's vertex be?

Take two points $(x_0, y_0), (x_1, y_1)$ where $x_0 \neq x_1$ and $y_0 \neq y_1$. The problem specifies that the parabola has a vertical axis of symmetry. Every parabola with a vertial axis of symmetry has vertex (d, k), where

$$d = -b/2c \tag{1}$$

$$k = -\frac{b^2 - 4ac}{4c} = a - cd^2 \tag{2}$$

and $y = a + bx + cx^2$.

Therefore, given our two points, for each vertical axis of symmetry, we have a system of three equations in three unknowns[1] :

$$y_0 = a + bx_0 + cx_0^2 \tag{3}$$

$$y_1 = a + bx_1 + cx_1^2 \tag{4}$$

$$0 = 2ad + b \tag{5}$$

Whenever $d \neq \frac{x_1 + x_0}{2}$, these equations uniquely pin down the parabola, and therefore the vertex. [2] These equations are linear in (a, b, c), so one can solve using elementary linear algebra techniques (direct substituion, or row reducing a matrix). We have

$$a = y_0 + \frac{2x_0 d(y_1 - y_0)}{-2d(x_1 - x_0) + x_1^2 - x_0^2} - \frac{x_0^2(y_1 - y_0)}{-2d(x1_x 0) + x_1^2 - x_0^2} \tag{6}$$

$$b = \frac{-2d(y_1 - y_0)}{-2d(x_1 - x_0) + x1^2 - x0^2} \tag{7}$$

$$c = \frac{(y_1 - y_0)}{-2d(x1_x 0) + x_1^2 - x_0^2} \tag{8}$$

So the set of vertices is $\{(d, a - cd^2) : d \in \mathbb{R}, d \neq \frac{x_1 + x_0}{2}, a \text{ given by } (6), c \text{ given by } (8)\}$.

[1] If $y_0 = y_1$, then $(3) + (5) \implies (4)$, so the system is linearly dependent. All (d, k) with $k \neq 0$ are possible vertices.

[2] If $d = \frac{x_1 + x_0}{2}$, then because the parabola has vertical symmetry, (x_1, y_0) and (x_0, y_0) must both lie on the parabola. But this is impossible since (x_1, y_1) lies on the parabola, and $y_0 \neq y_1$ by assumption.

Figure 1 displays an example parabola for selected values of d, and Figure 2 illustrates the example vertex for selected values.

Curiously, it appears as if there's an oblique limiting asymptote! We can confirm this.

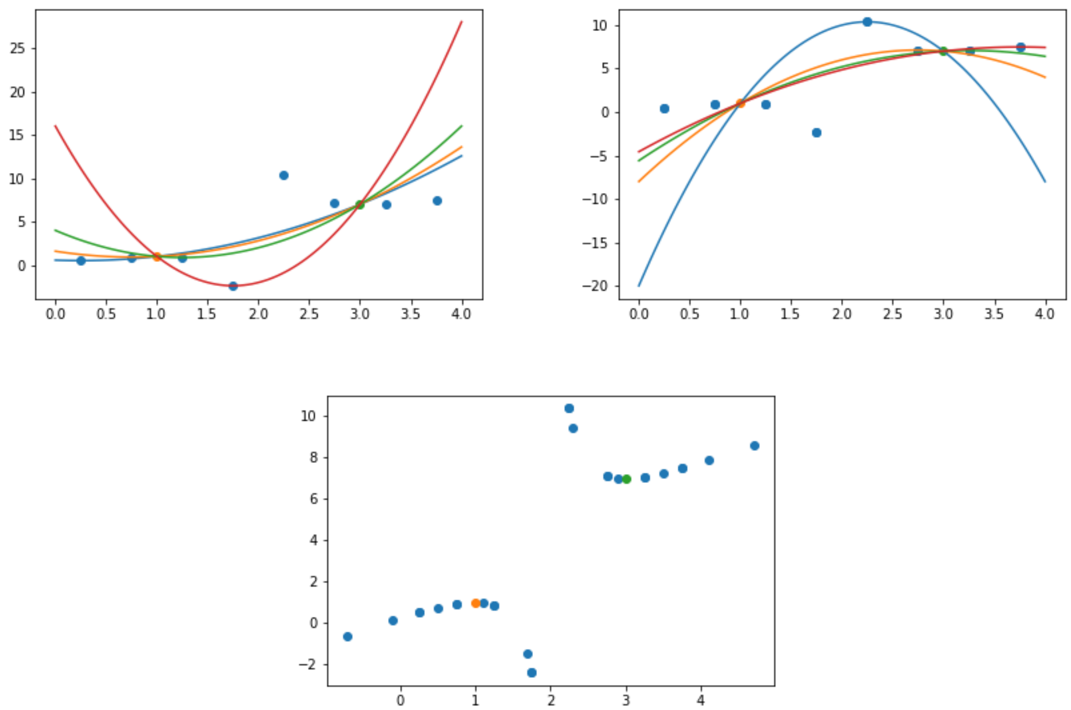
As $d \to \infty$,

$$\frac{k}{d} = \frac{y_0}{d} + \frac{2x_0(y_1 - y_0) - \frac{x_0^2}{d}(y_1 - y_0) - d^2(y_1 - y_0)}{x_1^2 - x_0^2 - 2d(x_1 - x_0)} \to \frac{y_1 - y_0}{2(x_1 - x_0)} \equiv m \tag{9}$$

$$\begin{aligned} k - md = \frac{y_0}{d} + \frac{2x_0(y_1 - y_0) - \frac{x_0^2}{d}(y_1 - y_0) - d^2(y_1 - y_0)}{x_1^2 - x_0^2 - 2d(x_1 - x_0)} - \frac{d(y_1 - y_0)}{2(x_1 - x_0)} \\ \to \frac{y_1 + y_0}{2} - \frac{(x_1 + x_0)(y_1 - y_0)}{4(x_1 - x_0)} \equiv e \end{aligned} \tag{10}$$

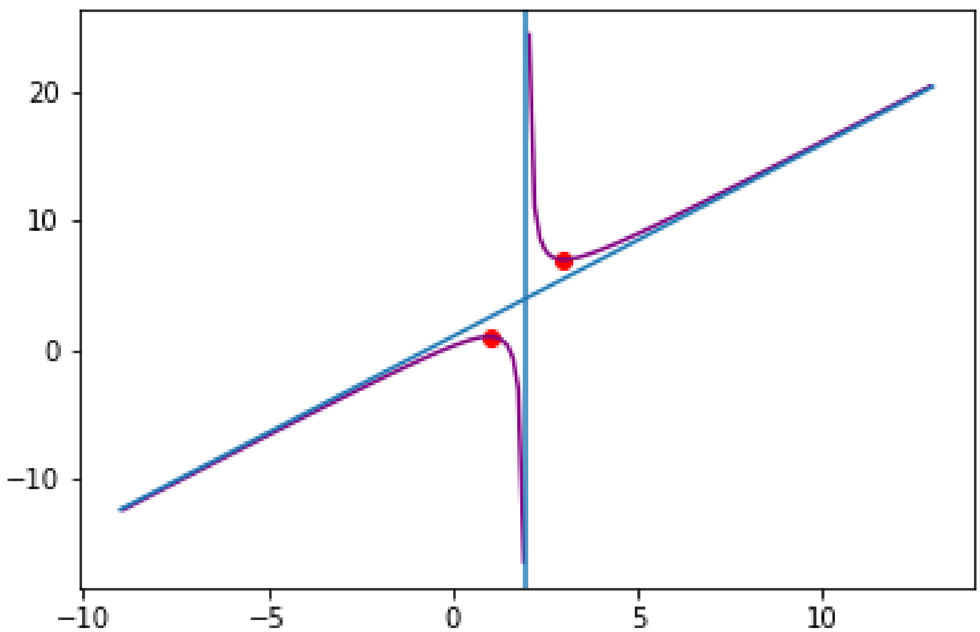
As shown in Figure 4, there are two asymptotes: one at $y = mx + e$, and the other at $x = \frac{x_1 + x_0}{2}$.

Figure 1: Parabola, vertex for selected values



$$(x_0, y_0) = (1, 0), (x_1, y_1) = (3, 7)$$

Figure 2: Vertices for selected values, with asymptotes



$$(x_0, y_0) = (1, 0), (x_1, y_1) = (3, 7)$$