MATH 128 End-of-Term Assignment 2

Brandon Tsang

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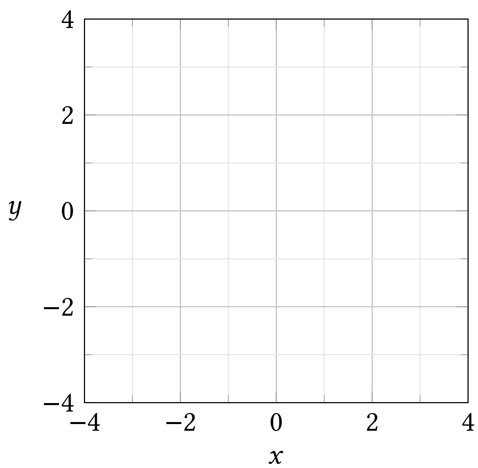
1. **Consider the function $f(x, y) = \sqrt{x^2 + y^2}$.**

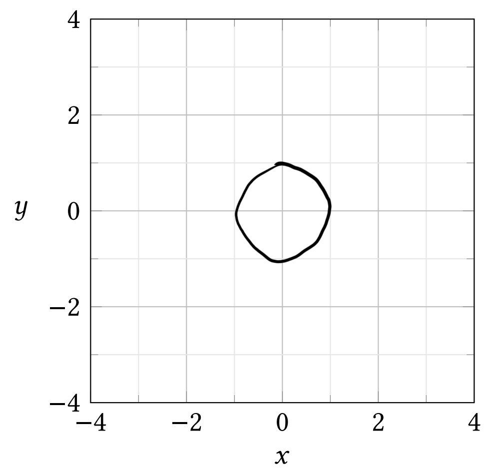
   (a) State the domain and range of f.
   The expression under the radical must be greater than or equal to zero (i.e., $x^2 + y^2 \geq 0$). However, x^2 and y^2 are always positive or zero, so the domain is $\{x, y \in \mathbb{R}\}$.
   The range is $\{z \in \mathbb{R} \mid z \geq 0\}$ since a square root is always positive or zero.

   (b) Sketch a contour plot of $f(x, y)$ illustrating the level curves defined by $f(x, y) = k$ for $k = 0, 1, 2, 3$.

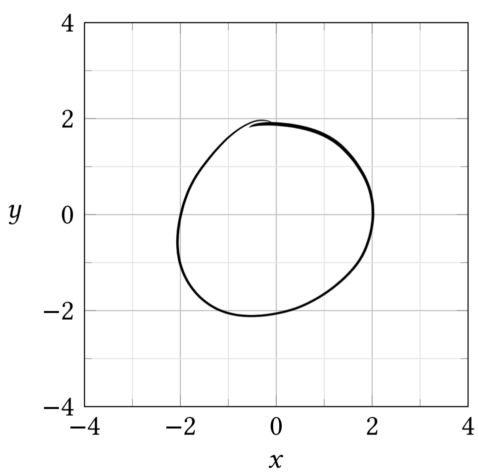
Contour plot for $k = 0$



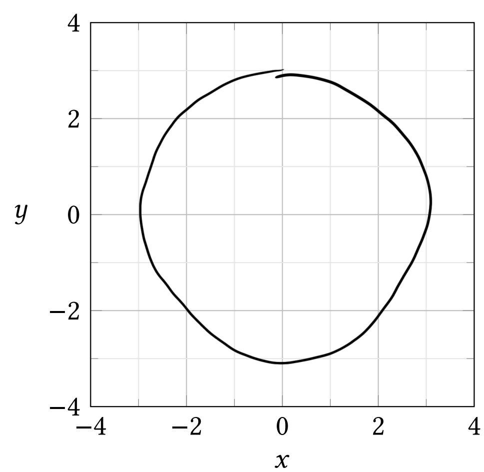
Contour plot for $k = 1$



Contour plot for $k = 2$



Contour plot for $k = 3$



(c) Consider the surface $z = f(x, y)$. Determine equations for the cross-sections $z = f(0, y)$ and $z = f(x, 0)$ (i.e., the curves of intersection between the surface and the yz and xz planes, respectively).

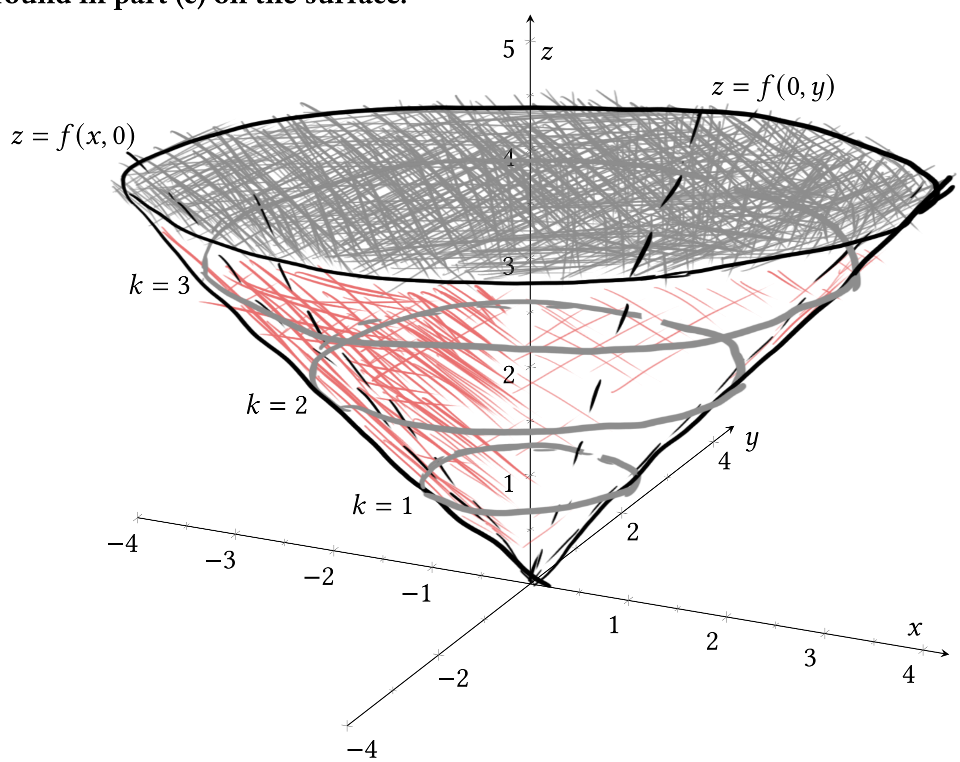
For $z = f(0, y)$:

$$\begin{aligned} z &= \sqrt{0^2 + y^2} \\ &= \sqrt{y^2} \\ &= |y| \end{aligned}$$

And for $z = f(x, 0)$:

$$\begin{aligned} z &= \sqrt{x^2 + 0^2} \\ &= \sqrt{x^2} \\ &= |x| \end{aligned}$$

(d) Sketch (by hand) a graph of the surface $z = f(x, y)$. Make the graph large enough so as to be able to draw and label the curves you found in part (b) and the cross-sections you found in part (c) on the surface.



2. (a) **Convert $(r, \theta) = \left(2, \frac{\pi}{6}(N_7 + 1)\right)$ to Cartesian coordinates where N_7 is the seventh digit of your student number.**

My student number is 20845794, so $N_7 = 9$. $\frac{\pi}{6}(N_7 + 1)$ then becomes $\frac{5\pi}{3}$. Finding x:

$$\begin{aligned} x &= r\cos\theta \\ &= 2\cos\frac{5\pi}{3} \\ &= 2 \cdot \frac{1}{2} \\ &= 1 \end{aligned}$$

Then, finding y:

$$\begin{aligned} y &= r\sin\theta \\ &= 2\sin\frac{5\pi}{3} \\ &= 2 \cdot -\frac{\sqrt{3}}{2} \\ &= -\sqrt{3} \end{aligned}$$

So $\left(2, \frac{\pi}{6}(N_7 + 1)\right)$ in Cartesian coordinates is $\left(1, -\sqrt{3}\right)$.

(b) **Convert $(x, y) = \left(-\sqrt{3}(N_8 + 1), N_8 + 1\right)$ to polar coordinates where N_8 is the eighth digit of your student number.**

$N_8 = 4$, so $x = -\sqrt{3}(N_8 + 1)$ becomes $x = 5\sqrt{3}$ and $y = N_8 + 1$ becomes $y = 5$. Finding r:

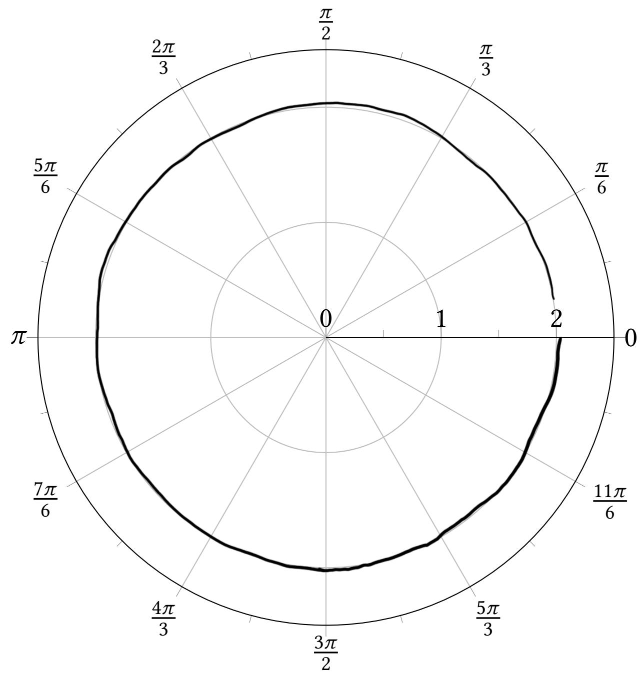
$$\begin{aligned} r &= \sqrt{x^2 + y^2} \\ &= \sqrt{\left(5\sqrt{3}\right)^2 + 5^2} \\ &= \sqrt{75 + 25} \\ &= 10 \end{aligned}$$

Then, finding θ:

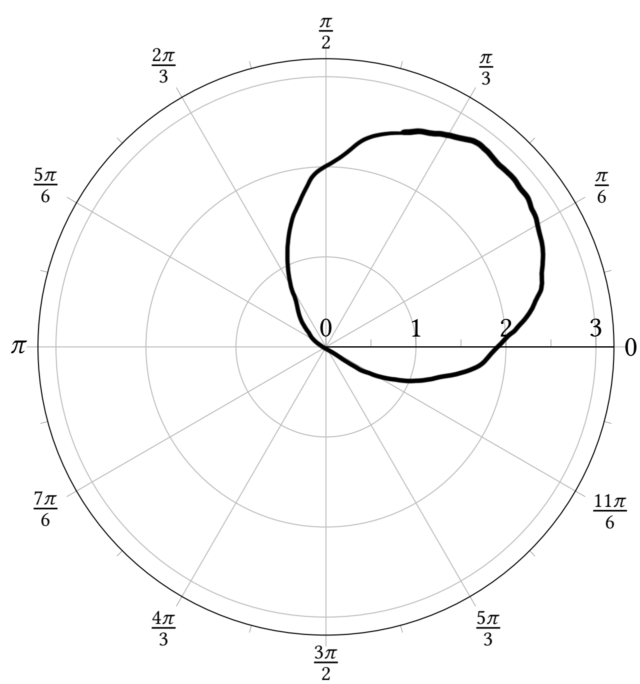
$$\begin{aligned} \theta &= \tan^{-1}\frac{y}{x} \\ &= \tan^{-1}\frac{5}{5\sqrt{3}} \\ &= \tan^{-1}\frac{1}{\sqrt{3}} \\ &= \frac{\pi}{6} \end{aligned}$$

So $\left(-\sqrt{3}(N_8 + 1), N_8 + 1\right)$ in polar coordinates is $\left(10, \frac{\pi}{6}\right)$.

(c) In the xy-plane, sketch the curve defined by $r(\theta) = 2$ for $0 \le \theta \le 2\pi$.



(d) In the xy-plane, sketch the curve defined by $r(\theta) = 2\cos\theta + 2\sin\theta$ for $0 \le \theta \le \pi$.



3. **Let $f(x, y) = \sqrt{xy}$.**

   (a) **Compute the first partial derivatives of $f(x, y)$.**

   With respect to x:

$$
\begin{aligned}
\frac{\partial f}{\partial x} &= \sqrt{y}\left(\frac{1}{2}x^{-\frac{1}{2}}\right) \\
&= \frac{\sqrt{y}}{2\sqrt{x}}
\end{aligned}
$$

   And with respect to y:

$$
\begin{aligned}
\frac{\partial f}{\partial y} &= \sqrt{x}\left(\frac{1}{2}y^{-\frac{1}{2}}\right) \\
&= \frac{\sqrt{x}}{2\sqrt{y}}
\end{aligned}
$$

   (b) **Compute the second partial derivatives of $f(x, y)$.**

   First, $f_{xx}(x, y)$:

$$
\begin{aligned}
\frac{\partial^2 f}{\partial x^2} &= \frac{\partial}{\partial x}\left(\frac{\sqrt{y}}{2\sqrt{x}}\right) \\
&= \frac{\sqrt{y}}{2}\left(-\frac{1}{2}x^{-\frac{3}{2}}\right) \\
&= -\frac{\sqrt{y}}{4x^{\frac{3}{2}}}
\end{aligned}
$$

   Next, $f_{yy}(x, y)$:

$$
\begin{aligned}
\frac{\partial^2 f}{\partial y^2} &= \frac{\partial}{\partial y}\left(\frac{\sqrt{x}}{2\sqrt{y}}\right) \\
&= \frac{\sqrt{x}}{2}\left(-\frac{1}{2}y^{-\frac{3}{2}}\right) \\
&= -\frac{\sqrt{x}}{4y^{\frac{3}{2}}}
\end{aligned}
$$

   Since $f_{xy}(x, y) = f_{yx}(x, y)$, I will only compute $f_{xy}(x, y)$.

$$
\begin{aligned}
f_{xy}(x, y) &= \frac{\partial}{\partial y}\left(\frac{\sqrt{y}}{2\sqrt{x}}\right) \\
&= \frac{1}{2\sqrt{x}}\left(\frac{1}{2}y^{-\frac{1}{2}}\right) \\
&= \frac{1}{4\sqrt{xy}}
\end{aligned}
$$

(c) Find an equation for the plane tangent to the surface $z = f(x, y)$ when $(x, y) = (1, 1)$.

The equation for a plane tangent to the surface of a function $f(x, y)$ at $x = x_0$ and $y = y_0$ is

$$z = f(x_0, y_0) + f_x(x_0, y_0) \cdot (x - x_0) + f_y(x_0, y_0) \cdot (y - y_0).$$

Finding the coefficients:

$$\begin{aligned} f(1, 1) &= \sqrt{1 \cdot 1} \\ &= 1 \end{aligned}$$

$$\begin{aligned} f_x(1, 1) &= \frac{\sqrt{1}}{2\sqrt{1}} \\ &= \frac{1}{2} \end{aligned}$$

$$\begin{aligned} f_y(1, 1) &= \frac{\sqrt{1}}{2\sqrt{1}} \\ &= \frac{1}{2} \end{aligned}$$

Then, putting it all together:

$$\begin{aligned} z &= 1 + \frac{1}{2}(x - 1) + \frac{1}{2}(y - 1) \\ &= 1 + \frac{x}{2} - \frac{1}{2} + \frac{y}{2} - \frac{1}{2} \\ &= \frac{x + y}{2} \end{aligned}$$

(d) Determine the linearization of $f(x, y)$ at $(x, y) = (1, 1)$ and use it to approximate $f(1.1, 0.8)$.

The linearization of $f(x, y)$ is the equation of the plane from part **(c)**.

$$\begin{aligned} f(x, y) &\approx \frac{x + y}{2} \\ f(1.1, 0.8) &\approx \frac{1.1 + 0.8}{2} \\ &\approx 0.95 \end{aligned}$$

4. **The wave equation is a partial differential equation which, for waves propagating in one spatial dimension, is given by**

$$f_{tt}(x,t) = a^2 f_{xx}(x,t)$$

where x is position, t is time, and a is a constant. Determine whether each of the following functions is a solution to the wave equation:

   (a) **$f(x,t) = \sin(x)\cos(at)$**

   Taking two partial derivatives with respect to t:

$$\begin{aligned}\frac{\partial f}{\partial t} &= \sin(x) \cdot -a\sin(at)\\ \frac{\partial^2 f}{\partial t^2} &= -a\sin(x) \cdot a\cos(at)\\ &= -a^2\sin(x)\cos(at)\end{aligned}$$

   Then with respect to x:

$$\begin{aligned}\frac{\partial f}{\partial x} &= \cos(at)\cos(x)\\ \frac{\partial^2 f}{\partial x^2} &= \cos(at) \cdot -\sin(x)\\ &= -\sin(x)\cos(at)\end{aligned}$$

   Then,

$$\begin{aligned}f_{tt}(x,t) &\stackrel{?}{=} a^2 f_{xx}(x,t)\\ -a^2\sin(x)\cos(at) &\stackrel{?}{=} a^2(-\sin(x)\cos(at))\\ -a^2\sin(x)\cos(at) &= -a^2\sin(x)\cos(at)\end{aligned}$$

   so $f(x,t) = \sin(x)\cos(at)$ is a solution.

   (b) **$f(x,t) = e^{-at}\sin(x)$**

   Taking two partial derivatives with respect to t:

$$\begin{aligned}\frac{\partial f}{\partial t} &= \sin(x) \cdot -ae^{-at}\\ \frac{\partial^2 f}{\partial t^2} &= -a\sin(x) \cdot -ae^{-at}\\ &= a^2 e^{-at}\sin(x)\end{aligned}$$

   Then with respect to x:

$$\begin{aligned}\frac{\partial f}{\partial x} &= e^{-at}\cos(x)\\ \frac{\partial^2 f}{\partial x} &= e^{-at} \cdot -\sin(x)\\ &= -e^{-at}\sin(x)\end{aligned}$$

Then,

$$\begin{aligned} f_{tt}(x,t) &\stackrel{?}{=} a^2 f_{xx}(x,t) \\ a^2e^{-at}\sin(x) &\stackrel{?}{=} a^2(-e^{-at}\sin(x)) \\ a^2e^{-at}\sin(x) &\neq -a^2e^{-at}\sin(x) \end{aligned}$$

so $f(x,t) = e^{-at}\sin(x)$ is not a solution.

(c) $\boldsymbol{f(x,t) = (x-at)^4}$

Taking two partial derivatives with respect to t:

$$\begin{aligned} \frac{\partial f}{\partial t} &= 4(x-at)^3 \cdot -a \\ \frac{\partial^2 f}{\partial t^2} &= -4a \cdot 3(x-at)^2 \cdot -a \\ &= 12a^2(x-at)^2 \end{aligned}$$

Then with respect to x:

$$\begin{aligned} \frac{\partial f}{\partial x} &= 4(x-at)^3 \\ \frac{\partial^2 f}{\partial x} &= 12(x-at)^2 \end{aligned}$$

Then,

$$\begin{aligned} f_{tt}(x,t) &\stackrel{?}{=} a^2 f_{xx}(x,t) \\ 12a^2(x-at)^2 &\stackrel{?}{=} a^2 \cdot 12(x-at)^2 \\ 12a^2(x-at)^2 &= 12a^2(x-at)^2 \end{aligned}$$

so $f(x,t) = (x-at)^4$ is a solution.

5. Evaluate the following iterated integrals:

(a) $\boldsymbol{\int_{x=0}^{2}\int_{y=0}^{2} x^2y^2\,dy\,dx}$

$$\begin{aligned} \int_{x=0}^{2}\int_{y=0}^{2} x^2y^2\,dy\,dx &= \int_0^2 x^2\left[\frac{1}{3}y^3\right]_0^2 dx \\ &= \frac{8}{3}\int_0^2 x^2\,dx \\ &= \frac{8}{3}\left[\frac{1}{3}x^3\right]_0^2 \\ &= \frac{8}{3}\left(\frac{8}{3}\right) \\ &= \frac{64}{9} \end{aligned}$$

(b) $\boldsymbol{\int_1^\infty \int_1^\infty xe^{-xy}\,dy\,dx}$

$$\begin{aligned}
\int_{x=1}^{\infty}\int_{y=1}^{\infty} xe^{-xy}\,dy\,dx &= -\int_1^\infty \left(\lim_{k\to\infty}\int_{-x}^{-kx} e^{u_1}\,du_1\right)\,dx && \begin{aligned} u_1 &= -xy \\ du_1 &= -x\,dy \end{aligned} \\
&= -\int_1^\infty \left(\lim_{k\to\infty}[e^u]_{-x}^{-kx}\right)\,dx \\
&= -\int_1^\infty \left(\lim_{k\to\infty}(e^{-kx} - e^{-x})\right)\,dx \\
&= -\int_1^\infty (-e^{-x})\,dx \\
&= -\int_{-1}^{-\infty} e^{u_2}\,du_2 && \begin{aligned} u_2 &= -x \\ du_2 &= -dx \end{aligned} \\
&= -\lim_{k\to\infty}[e^{u_2}]_{-1}^{-k} \\
&= -\lim_{k\to\infty}(e^{-k} - e^{-1}) \\
&= -(-e^{-1}) \\
&= \frac{1}{e}
\end{aligned}$$

6. **A thin, square sheet of metal measures L units by L units. The mass density (measured in terms of mass per unit area) is given by the function $\rho(x,y) = \rho_0\left(1 + \frac{xy}{L^2}\right)$ where ρ_0 is a constant. Determine the total mass of the sheet in terms of ρ_0 and L by evaluating**

$$\boldsymbol{M = \int_{x=0}^{L}\int_{y=0}^{L} \rho(x,y)\,dy\,dx.}$$

$$\begin{aligned}
M &= \int_0^L \int_0^L \rho_0\left(1 + \frac{xy}{L^2}\right)\,dy\,dx \\
&= \rho_0 \int_0^L \left[y + \frac{x}{L^2}\frac{1}{2}y^2\right]_0^L\,dx \\
&= \rho_0 \int_0^L \left(L + \frac{xL^2}{2L^2}\right)\,dx \\
&= \rho_0 \int_0^L \left(L + \frac{x}{2}\right)\,dx \\
&= \rho_0 \left[Lx + \frac{1}{2}\frac{1}{2}x^2\right]_0^L \\
&= \rho_0 \left(L^2 + \frac{1}{4}L^2\right) \\
&= \frac{5}{4}L^2\rho_0
\end{aligned}$$

The mass of the sheet is $\frac{5}{4}L^2\rho_0$.