MATH 114 Final Exam Question 5

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5. **Suppose I have a set of k nonzero vectors, $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$. These vectors are all orthogonal. That is, $\vec{v}_i \cdot \vec{v}_j = 0$ if $i \neq j$. What is the result of applying the Gram-Schmidt technique to this set of vectors?**

   If the vectors are all orthogonal, the projection operations used in the Gram-Schmidt technique will always return the zero vector.

$$\begin{aligned} \operatorname{proj}_{\mathbf{v}} \mathbf{u} &= \frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}} \mathbf{v} && \text{(The definition of proj)} \\ &= \frac{0}{\mathbf{v} \cdot \mathbf{v}} \mathbf{v} && (\mathbf{u} \cdot \mathbf{v} = 0) \\ &= \mathbf{0} \end{aligned}$$

   Let's define the Gram-Schmidt process for generating a set of orthogonal vectors $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_\mathbf{n}\}$ from a set of nonzero vectors $\{\mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_\mathbf{n}\}$ as follows:

$$\begin{aligned} \mathbf{u}_1 &= \mathbf{s}_1 \\ \mathbf{u}_2 &= \mathbf{s}_2 - \operatorname{proj}_{\mathbf{u}_1} \mathbf{s}_2 \\ \mathbf{u}_3 &= \mathbf{s}_3 - \operatorname{proj}_{\mathbf{u}_1} \mathbf{s}_3 - \operatorname{proj}_{\mathbf{u}_2} \mathbf{s}_3 \\ &\vdots \end{aligned}$$

   When used on our set of already orthogonal vectors, the projections will return $\mathbf{0}$:

$$\begin{aligned} \mathbf{u}_1 &= \mathbf{s}_1 \\ \mathbf{u}_2 &= \mathbf{s}_2 - \mathbf{0} \\ \mathbf{u}_3 &= \mathbf{s}_3 - \mathbf{0} - \mathbf{0} \\ &\vdots \end{aligned}$$

   Leaving us with an identical set of vectors.