MATH 114 Final Exam Question 2

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2. Find the eigenvectors and corresponding eigenvalues of the rotation matrix $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$.

Starting with $A\mathbf{u} = \lambda\mathbf{u}$ (where λ is the eigenvalues):

$$\begin{aligned}
A\mathbf{u} &= \lambda\mathbf{u} \\
(A - \lambda I)\mathbf{u} &= \mathbf{0} \\
\begin{bmatrix} \cos\theta - \lambda & -\sin\theta \\ \sin\theta & \cos\theta - \lambda \end{bmatrix}\mathbf{u} &= \mathbf{0} \\
\begin{vmatrix} \cos\theta - \lambda & -\sin\theta \\ \sin\theta & \cos\theta - \lambda \end{vmatrix} &= \mathbf{0} \\
(\cos\theta - \lambda)(\cos\theta - \lambda) - (-\sin\theta)(\sin\theta) &= \mathbf{0} \\
\cos^2\theta - 2\lambda\cos\theta + \lambda^2 + \sin^2\theta &= \mathbf{0} \\
\lambda^2 - 2\lambda\cos\theta + 1 &= \mathbf{0} \\
\lambda &= \frac{2\cos\theta \pm \sqrt{(-2\cos\theta)^2 - 4(1)(1)}}{2(1)} \\
&= \frac{2\cos\theta \pm \sqrt{4\cos^2\theta - 4}}{2} \\
&= \cos\theta \pm \sqrt{\cos^2\theta - 1} \\
&= \cos\theta \pm i\sin\theta
\end{aligned}$$

Using these eigenvalues, we can find the eigenvectors. Starting with $\lambda = \cos\theta + i\sin\theta$:

$$\begin{aligned}
A\mathbf{u} &= \lambda\mathbf{u} \\
(A - \lambda I)\mathbf{u} &= \mathbf{0} \\
\begin{bmatrix} \cos\theta - (\cos\theta + i\sin\theta) & -\sin\theta \\ \sin\theta & \cos\theta - (\cos\theta + i\sin\theta) \end{bmatrix}\mathbf{u} &= \mathbf{0} \\
\begin{bmatrix} -i\sin\theta & -\sin\theta \\ \sin\theta & -i\sin\theta \end{bmatrix}\mathbf{u} &= \mathbf{0}
\end{aligned}$$

As an augmented matrix:

$$\left[\begin{array}{cc|c} -i\sin\theta & -\sin\theta & 0 \\ \sin\theta & -i\sin\theta & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} -i & -1 & 0 \\ 1 & -i & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -i & 0 \\ 1 & -i & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -i & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Using $u_2 = s$ as a free variable:

$$\begin{aligned} 1u_1 - iu_2 &= 0 \\ u_1 &= si \end{aligned}$$

Then, the eigenvector is $\begin{bmatrix} si \\ s \end{bmatrix} = s \begin{bmatrix} i \\ 1 \end{bmatrix}$.

Next, with $\lambda = \cos\theta - i\sin\theta$:

$$\begin{aligned} A\mathbf{u} &= \lambda\mathbf{u} \\ (A - \lambda I)\mathbf{u} &= \mathbf{0} \\ \begin{bmatrix} \cos\theta - (\cos\theta - i\sin\theta) & -\sin\theta \\ \sin\theta & \cos\theta - (\cos\theta - i\sin\theta) \end{bmatrix}\mathbf{u} &= \mathbf{0} \\ \begin{bmatrix} i\sin\theta & -\sin\theta \\ \sin\theta & i\sin\theta \end{bmatrix}\mathbf{u} &= \mathbf{0} \end{aligned}$$

As an augmented matrix:

$$\left[\begin{array}{cc|c} i\sin\theta & -\sin\theta & 0 \\ \sin\theta & i\sin\theta & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} i & -1 & 0 \\ 1 & i & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} -1 & -i & 0 \\ 1 & i & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} -1 & -i & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Using $u_2 = s$ as a free variable:

$$\begin{aligned} -1u_1 - iu_2 &= 0 \\ u_1 &= si \end{aligned}$$

and

$$\begin{aligned} 1u_1 - iu_2 &= 0 \\ u_1 &= -si \end{aligned}$$

So the eigenvector is $\begin{bmatrix} -si \\ s \end{bmatrix} = s \begin{bmatrix} -i \\ 1 \end{bmatrix}$.