PHYS 234 Assignment 2

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4. **State Tomography**
It is known that there is a 90% probability of obtaining $S_z = \frac{\hbar}{2}$ if a measurement of S_z is carried out on a spin-$\frac{1}{2}$ particle. In addition, it is known that there is a 20% probability of obtaining $S_y = \frac{\hbar}{2}$ if a measurement of S_y is carried out. Determine the spin state of a particle as completely as possible from this information. What is the probability of obtaining $S_x = -\frac{\hbar}{2}$ if a measurement of S_x is carried out?

From the first criterion:

$$|\langle +|\psi\rangle|^2 = \tfrac{9}{10}$$
$$|\langle +|\psi\rangle| = \tfrac{3}{\sqrt{10}}$$

If $|\psi\rangle = a\,|+\rangle + b\,|-\rangle$:

$$\begin{aligned}
|\langle +|\,(a\,|+\rangle + b\,|-\rangle)| &= \tfrac{3}{\sqrt{10}} \\
|a| &= \tfrac{3}{\sqrt{10}} \\
a &= \frac{3}{\sqrt{10}}e^{i\alpha}
\end{aligned}$$

And from the second criterion:

$$\begin{aligned}
\left|{}_y\langle +|\psi\rangle\right|^2 &= \frac{1}{5} \\
\left|{}_y\langle +|\psi\rangle\right|^2 &= \frac{1}{5} \\
\left|\tfrac{1}{\sqrt{2}}(\langle +| - i\,\langle -|)(a\,|+\rangle + b\,|-\rangle)\right|^2 &= \frac{1}{5} \\
\left|\tfrac{1}{\sqrt{2}}(a - bi)\right|^2 &= \frac{1}{5} \\
\frac{1}{2}(|a|^2 + |b|^2 + ab^*i - a^*bi) &= \frac{1}{5} \\
1 + ab^*i - a^*bi &= \frac{2}{5} \\
a|b|e^{-i\beta}i - a^*|b|e^{i\beta}i &= -\frac{3}{5} \\
|b|\left(ae^{-i\beta} - a^*e^{i\beta}\right) &= \frac{3i}{5} \\
|a||b|\left(e^{i\alpha}e^{-i\beta} - e^{-i\alpha}e^{i\beta}\right) &= \frac{3i}{5} \\
|a||b|\left(e^{i(\alpha-\beta)} - e^{i(\beta-\alpha)}\right) &= \frac{3i}{5}
\end{aligned}$$

Arbitrarily set $\beta = 0$:

$$\begin{aligned}
|a||b|\left(e^{i\alpha} - e^{-i\alpha}\right) &= \frac{3i}{5} \\
|b|\left(e^{i\alpha} - e^{-i\alpha}\right) &= \frac{3\sqrt{10}i}{15} \\
b &= \frac{\sqrt{10}i}{5\left(e^{i\alpha} - e^{-i\alpha}\right)}
\end{aligned}$$

So $|\psi\rangle = \frac{3}{\sqrt{10}}e^{i\alpha}\left|+\right\rangle + \frac{\sqrt{10}i}{5(e^{i\alpha}-e^{-i\alpha})}\left|-\right\rangle$.

Now, we find the probability of measuring $S_x = -\frac{\hbar}{2}$:

$$\begin{aligned}
\left|{}_x\langle -|\psi\rangle\right|^2 &= \left|\frac{1}{\sqrt{2}}(\langle +| - \langle -|)\left(\frac{3}{\sqrt{10}}e^{i\alpha}\left|+\right\rangle + \frac{\sqrt{10}i}{5\left(e^{i\alpha} - e^{-i\alpha}\right)}\left|-\right\rangle\right)\right|^2 \\
&= \left|\frac{1}{\sqrt{2}}\left(\frac{3}{\sqrt{10}}e^{i\alpha} - \frac{\sqrt{10}i}{5\left(e^{i\alpha} - e^{-i\alpha}\right)}\right)\right|^2
\end{aligned}$$