PHYS 234 Assignment 4

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2. Given the density operator

$$\hat{\rho} = \frac{3}{4} |+\rangle \langle +| + \frac{1}{4} |-\rangle \langle -|$$

(a) Construct the density matrix.

$|+\rangle \langle +|$ is $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $|-\rangle \langle -|$ is $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, so

$$\begin{aligned} \hat{\rho} &= \frac{3}{4} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \frac{1}{4} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} \frac{3}{4} & 0 \\ 0 & \frac{1}{4} \end{bmatrix}. \end{aligned}$$

(b) Show that this is the density operator for a mixed state.

If $\hat{\rho} \neq \hat{\rho}^2$, then the density operator describes a mixed state.

$$\begin{aligned} \begin{bmatrix} \frac{3}{4} & 0 \\ 0 & \frac{1}{4} \end{bmatrix} &\stackrel{?}{=} \begin{bmatrix} \frac{3}{4} & 0 \\ 0 & \frac{1}{4} \end{bmatrix} \begin{bmatrix} \frac{3}{4} & 0 \\ 0 & \frac{1}{4} \end{bmatrix} \\ &\neq \begin{bmatrix} \frac{9}{16} & 0 \\ 0 & \frac{1}{8} \end{bmatrix} \end{aligned}$$

$\hat{\rho} \neq \hat{\rho}^2$, so this density operator describes a mixed state.

(c) Determine $\langle S_x \rangle$, $\langle S_y \rangle$, and $\langle S_z \rangle$ for this state.

If $\hat{\rho}$ is the density operator for the state $|\psi\rangle$, then to find the expectation value of the S_x operator:

$$\begin{aligned} \langle \psi | S_x | \psi \rangle &= \text{tr}(S_x \hat{\rho}) \\ &= \text{tr}\left(\frac{\hbar}{2} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{3}{4} & 0 \\ 0 & \frac{1}{4} \end{bmatrix} \right) \\ &= \frac{\hbar}{2} \text{tr}\left(\begin{bmatrix} 0 & \frac{1}{4} \\ \frac{3}{4} & 0 \end{bmatrix} \right) \\ \boxed{\langle S_x \rangle = 0\hbar} \end{aligned}$$

And then for the S_y operator:

$$\begin{aligned}
\langle\psi| S_y |\psi\rangle &= \mathrm{tr}(S_y\hat{\rho}) \\
&= \mathrm{tr}\left(\frac{\hbar}{2}\begin{bmatrix}0 & -i\\ i & 0\end{bmatrix}\begin{bmatrix}\frac{3}{4} & 0\\ 0 & \frac{1}{4}\end{bmatrix}\right) \\
&= \frac{\hbar}{2}\,\mathrm{tr}\left(\begin{bmatrix}0 & -\frac{i}{4}\\ \frac{3i}{4} & 0\end{bmatrix}\right) \\
\boxed{\langle S_y\rangle = 0\hbar}
\end{aligned}$$

And finally for the S_z operator:

$$\begin{aligned}
\langle\psi| S_z |\psi\rangle &= \mathrm{tr}(S_z\hat{\rho}) \\
&= \mathrm{tr}\left(\frac{\hbar}{2}\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}\frac{3}{4} & 0\\ 0 & \frac{1}{4}\end{bmatrix}\right) \\
&= \frac{\hbar}{2}\,\mathrm{tr}\left(\begin{bmatrix}\frac{3}{4} & 0\\ 0 & -\frac{1}{4}\end{bmatrix}\right) \\
&= \frac{\hbar}{2}\frac{1}{2} \\
\boxed{\langle S_z\rangle = \frac{1}{4}\hbar}
\end{aligned}$$

(d) Find states $|\psi_1\rangle$ and $|\psi_2\rangle$ for which the density operator can be expressed in the form

$$\hat{\rho} = \frac{1}{2}|\psi_1\rangle\langle\psi_1| + \frac{1}{2}|\psi_2\rangle\langle\psi_2|\,.$$

Let $|\psi_1\rangle = \begin{bmatrix}a\\ b\end{bmatrix}$ and $|\psi_2\rangle = \begin{bmatrix}c\\ d\end{bmatrix}$. Then,

$$\begin{aligned}
\hat{\rho} = \begin{bmatrix}\frac{3}{4} & 0\\ 0 & \frac{1}{4}\end{bmatrix} &= \frac{1}{2}\begin{bmatrix}a\\ b\end{bmatrix}\begin{bmatrix}a & b\end{bmatrix} + \frac{1}{2}\begin{bmatrix}c\\ d\end{bmatrix}\begin{bmatrix}c & d\end{bmatrix} \\
&= \frac{1}{2}\begin{bmatrix}a^2 & ab\\ ab & b^2\end{bmatrix} + \frac{1}{2}\begin{bmatrix}c^2 & cd\\ cd & d^2\end{bmatrix} \\
&= \frac{1}{2}\begin{bmatrix}a^2+c^2 & ab+cd\\ ab+cd & b^2+d^2\end{bmatrix} \\
\begin{bmatrix}\frac{3}{2} & 0\\ 0 & \frac{1}{2}\end{bmatrix} &= \begin{bmatrix}a^2+c^2 & ab+cd\\ ab+cd & b^2+d^2\end{bmatrix}
\end{aligned}$$

This gives us three (linearly dependent) equations:

$$\begin{aligned}
a^2 + c^2 &= \frac{3}{2} \\
b^2 + d^2 &= \frac{1}{2} \\
ab + cd &= 0
\end{aligned}$$

And if we use the fact that the trace of $\frac{1}{2}\begin{bmatrix} a^2+c^2 & ab+cd \\ ab+cd & b^2+d^2 \end{bmatrix}$ is 1, then we have a fourth:

$$\begin{aligned} \mathrm{tr}\left(\frac{1}{2}\begin{bmatrix} a^2+c^2 & ab+cd \\ ab+cd & b^2+d^2 \end{bmatrix}\right) &= 1 \\ \frac{1}{2}(a^2+b^2+c^2+d^2) &= 1 \\ a^2+b^2+c^2+d^2 &= 2 \end{aligned}$$

It is easier to guess solutions than to actually solve them, so I will do that.
Since $ab+cd=0$, a trivial solution would be to have [a or b] and [c or d] equal zero. That way, we'll get $0+0=0$. I will arbitrarily choose b and c to equal zero.
Next,

$$\begin{aligned} b^2+d^2 &= \frac{1}{2} \\ d^2 &= \frac{1}{2} \\ d &= \sqrt{\frac{1}{2}} \end{aligned}$$

and

$$\begin{aligned} a^2+c^2 &= \frac{3}{2} \\ a^2 &= \frac{3}{2} \\ a &= \sqrt{\frac{3}{2}}. \end{aligned}$$

To check if the solutions work:

$$\begin{aligned} \begin{bmatrix} \frac{3}{4} & 0 \\ 0 & \frac{1}{4} \end{bmatrix} &\stackrel{?}{=} \frac{1}{2}\begin{bmatrix} a^2+c^2 & ab+cd \\ ab+cd & b^2+d^2 \end{bmatrix} \\ &\stackrel{?}{=} \frac{1}{2}\begin{bmatrix} \frac{3}{2}+0 & 0+0 \\ 0+0 & 0+\frac{1}{2} \end{bmatrix} \\ &= \begin{bmatrix} \frac{3}{4} & 0 \\ 0 & \frac{1}{4} \end{bmatrix} \end{aligned}$$

And we can confirm that this set of solutions is valid. These solutions give the states

$$\boxed{\begin{aligned} |\psi_1\rangle &= \sqrt{\frac{3}{2}}\,|+\rangle \\ |\psi_2\rangle &= \sqrt{\frac{1}{2}}\,|-\rangle\,. \end{aligned}}$$