PHYS 234 Assignment 2

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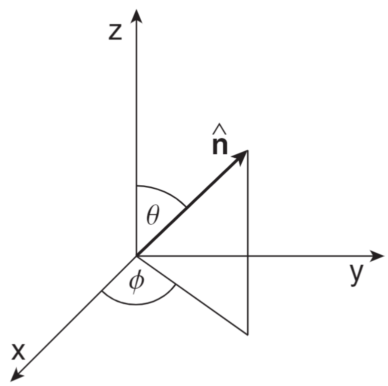
May 29, 2020

2. **<u>Spin Operator in an Arbitrary Direction</u>**
Find the representation of the spin operator $S_n := \vec{S} \cdot \hat{n}$ that measures the projection of the spin-$\frac{1}{2}$ particle along the $\hat{n}$ direction. Calculate its representation in the S_z-basis. Here, $\hat{n}$ is the unit vector in spherical-polar coordinates.



$$\hat{n} = \sin(\theta)\cos(\phi)\hat{x} + \sin(\theta)\sin(\phi)\hat{y} + \cos(\theta)\hat{z}$$

and

$$\vec{S} = S_x\hat{x} + S_y\hat{y} + S_z\hat{z}$$

where S_x, S_y, and S_z are the three components of the spin-$\frac{1}{2}$ operator.

(a) **Show that**

$$S_n = \frac{\hbar}{2}\begin{bmatrix} \cos\theta & \sin(\theta)e^{-i\phi} \\ \sin(\theta)e^{i\phi} & -\cos\theta \end{bmatrix}.$$

$$\begin{aligned} S_n &= \mathbf{S} \cdot \hat{\mathbf{n}} \\ &= S_x \sin(\theta)\cos(\phi) + S_y \sin(\theta)\sin(\phi) + S_z \cos(\theta) \end{aligned}$$

$$S_x = \frac{\hbar}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad S_y = \frac{\hbar}{2}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \qquad S_z = \frac{\hbar}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$
\begin{aligned}
S_n &= \frac{\hbar}{2}\left(\begin{bmatrix} 0 & \sin\theta\cos\phi \\ \sin\theta\cos\phi & 0 \end{bmatrix} + \begin{bmatrix} 0 & -i\sin\theta\sin\phi \\ i\sin\theta\sin\phi & 0 \end{bmatrix} + \begin{bmatrix} \cos\theta & 0 \\ 0 & -\cos\theta \end{bmatrix}\right) \\
&= \frac{\hbar}{2}\begin{bmatrix} \cos\theta & \sin\theta\cos\phi - i\sin\theta\sin\phi \\ \sin\theta\cos\phi + i\sin\theta\sin\phi & -\cos\theta \end{bmatrix} \\
&= \frac{\hbar}{2}\begin{bmatrix} \cos\theta & \sin\theta(\cos\phi - i\sin\phi) \\ \sin\theta(\cos\phi + i\sin\phi) & -\cos\theta \end{bmatrix} \\
&= \frac{\hbar}{2}\begin{bmatrix} \cos\theta & \sin(\theta)e^{-i\phi} \\ \sin(\theta)e^{i\phi} & -\cos\theta \end{bmatrix}
\end{aligned}
$$

(b) Show that the eigenvalues of S_n are $\pm\frac{\hbar}{2}$, as expected from the S-G experiment.

$$
\begin{aligned}
(S_n - \lambda I)\left|\psi\right\rangle &= \mathbf{0} \\
\left(\frac{\hbar}{2}\begin{bmatrix} \cos\theta & \sin(\theta)e^{-i\phi} \\ \sin(\theta)e^{i\phi} & -\cos\theta \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right)\left|\psi\right\rangle &= \mathbf{0} \\
\begin{vmatrix} \frac{\hbar}{2}\cos\theta - \lambda & \frac{\hbar}{2}\sin(\theta)e^{-i\phi} \\ \frac{\hbar}{2}\sin(\theta)e^{i\phi} & -\frac{\hbar}{2}\cos\theta - \lambda \end{vmatrix} &= 0 \\
-\left(\frac{\hbar}{2}\cos\theta - \lambda\right)\left(\frac{\hbar}{2}\cos\theta + \lambda\right) - \left(\frac{\hbar}{2}\sin\theta\right)^2\left(e^{i\phi - i\phi}\right) &= 0 \\
-\left(\frac{\hbar}{2}\cos\theta\right)^2 + \lambda^2 - \left(\frac{\hbar}{2}\sin\theta\right)^2 &= 0 \\
\lambda^2 &= \left(\frac{\hbar}{2}\cos\theta\right)^2 + \left(\frac{\hbar}{2}\sin\theta\right)^2 \\
&= \left(\frac{\hbar}{2}\right)^2 \\
\lambda &= \pm\frac{\hbar}{2}
\end{aligned}
$$

(c) Show that the eigenvectors of S_n can be represented as

$$
\begin{aligned}
\boldsymbol{\left|+\right\rangle_n = \cos\left(\frac{\theta}{2}\right)\left|+\right\rangle + \sin\left(\frac{\theta}{2}\right)e^{i\phi}\left|-\right\rangle} \\
\boldsymbol{\left|-\right\rangle_n = \sin\left(\frac{\theta}{2}\right)\left|+\right\rangle - \cos\left(\frac{\theta}{2}\right)e^{i\phi}\left|-\right\rangle}
\end{aligned}
$$

The eigenvector $\left|\psi\right\rangle$ corresponding to the eigenvalue $\lambda = +\frac{\hbar}{2}$ is $\left|+\right\rangle_n$, and the eigenvector $\left|\psi\right\rangle$ corresponding to the eigenvalue $\lambda = -\frac{\hbar}{2}$ is $\left|-\right\rangle_n$.

$$
\begin{aligned}
(S_n - \lambda I)\left|\psi\right\rangle &= \mathbf{0} \\
\left(\frac{\hbar}{2}\begin{bmatrix} \cos\theta & \sin(\theta)e^{-i\phi} \\ \sin(\theta)e^{i\phi} & -\cos\theta \end{bmatrix} - \frac{\hbar}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\left|+\right\rangle_n &= \mathbf{0} \\
\begin{bmatrix} \cos\theta - 1 & \sin(\theta)e^{-i\phi} \\ \sin(\theta)e^{i\phi} & -\cos\theta - 1 \end{bmatrix}\left|+\right\rangle_n &= \mathbf{0}
\end{aligned}
$$

$$\left[\begin{array}{cc|c} \cos\theta - 1 & \sin(\theta)e^{-i\phi} & 0 \\ \sin(\theta)e^{i\phi} & -\cos\theta - 1 & 0 \end{array}\right] \xrightarrow{R_2 - \frac{\sin(\theta)e^{i\phi}}{\cos\theta - 1}R_1} \left[\begin{array}{cc|c} \cos\theta - 1 & \sin(\theta)e^{-i\phi} & 0 \\ 0 & -\cos\theta - 1 - \frac{(\sin(\theta)e^{-i\phi})(\sin(\theta)e^{i\phi})}{\cos\theta - 1} & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|c} \cos\theta - 1 & \sin(\theta)e^{-i\phi} & 0 \\ 0 & -\frac{(\cos\theta+1)(\cos\theta-1)}{\cos\theta-1} - \frac{\sin^2(\theta)}{\cos\theta-1} & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|c} \cos\theta - 1 & \sin(\theta)e^{-i\phi} & 0 \\ 0 & -\frac{\cos^2(\theta)-1+\sin^2(\theta)}{\cos\theta-1} & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|c} \cos\theta - 1 & \sin(\theta)e^{-i\phi} & 0 \\ 0 & 0 & 0 \end{array}\right]$$

If $|+\rangle_n = a\,|+\rangle + b\,|-\rangle$, then the above matrix corresponds to the equation

$$\begin{aligned} (\cos\theta - 1)a + \sin(\theta)e^{-i\phi}b = 0 \\ (1 - \cos\theta)a = \sin(\theta)e^{-i\phi}b \\ \sin^2\left(\tfrac{\theta}{2}\right)a = \frac{1}{2}\sin(\theta)e^{-i\phi}b \\ \sin^2\left(\tfrac{\theta}{2}\right)e^{i\phi}a = \sin(\theta)\cos^2\left(\tfrac{\theta}{2}\right)b \end{aligned}$$

So $|+\rangle_n = \sin^2\left(\frac{\theta}{2}\right)e^{i\phi}\,|+\rangle + \sin(\theta)\cos^2\left(\frac{\theta}{2}\right)|-\rangle$.

(I couldn't get the equation to match the form above.)

(d) For which values of θ and ϕ does the state $|+\rangle_n$ reduce to $|+\rangle_x$ and $|+\rangle_y$?

$|+\rangle_x$ was $\frac{1}{\sqrt{2}}(|+\rangle + |-\rangle)$, so for $|+\rangle_n$ to equal $|+\rangle_x$, we must set

$$\cos\frac{\theta}{2} = \frac{1}{\sqrt{2}}$$

and

$$\sin\left(\frac{\theta}{2}\right)e^{i\phi} = \frac{1}{\sqrt{2}}.$$

Starting with the first equation:

$$\begin{aligned} \cos\frac{\theta}{2} &= \frac{1}{\sqrt{2}} \\ \frac{\theta}{2} &= \frac{\pi}{4} \\ \theta &= \frac{\pi}{2} \end{aligned}$$

Then the second equation:

$$\begin{aligned} \sin\left(\frac{\theta}{2}\right)e^{i\phi} &= \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}}e^{i\phi} &= \frac{1}{\sqrt{2}} \\ e^{i\phi} &= 1 \\ \phi &= 0 \end{aligned}$$

So for $|+\rangle_n$ to reduce to $|+\rangle_x$, θ must be $\frac{\pi}{2}$, and ϕ must be 0.
Next, $|+\rangle_y$ was $\frac{1}{\sqrt{2}}(|+\rangle + i\,|-\rangle)$, so for $|+\rangle_n$ to equal $|+\rangle_y$, we must set

$$\cos\frac{\theta}{2} = \frac{1}{\sqrt{2}}$$

and

$$\sin\left(\frac{\theta}{2}\right) e^{i\phi} = i\frac{1}{\sqrt{2}}.$$

The solution to the first equation we found earlier to be $\theta = \frac{\pi}{2}$. We then only have to solve the second equation:

$$\begin{aligned}
\sin\left(\frac{\theta}{2}\right) e^{i\phi} &= i\frac{1}{\sqrt{2}} \\
\frac{1}{\sqrt{2}} e^{i\phi} &= i\frac{1}{\sqrt{2}} \\
e^{i\phi} &= i \\
\phi &= \frac{\pi}{2}
\end{aligned}$$

So for $|+\rangle_n$ to reduce to $|+\rangle_y$, both θ and ϕ must be $\frac{\pi}{2}$.

(e) Suppose that a measurement of S_z is carried out on a particle in the $|-\rangle_n$ state. What is the probability that the measurement yields:

(i) $\frac{\hbar}{2}$?

The probability is:

$$\begin{aligned}
|\langle +|-\rangle_n|^2 &= \left|\langle +| \left(\sin\left(\tfrac{\theta}{2}\right)|+\rangle - \cos\left(\tfrac{\theta}{2}\right) e^{i\phi}\,|-\rangle\right)\right|^2 \\
&= \left|\sin\left(\tfrac{\theta}{2}\right)\right|^2 \\
&= \sin^2\left(\tfrac{\theta}{2}\right)
\end{aligned}$$

(ii) $-\frac{\hbar}{2}$?

The probability is:

$$\begin{aligned}
|\langle -|-\rangle_n|^2 &= \left|\langle -| \left(\sin\left(\tfrac{\theta}{2}\right)|+\rangle - \cos\left(\tfrac{\theta}{2}\right) e^{i\phi}\,|-\rangle\right)\right|^2 \\
&= \left|-\cos\left(\tfrac{\theta}{2}\right) e^{i\phi}\right|^2 \\
&= \cos^2\left(\tfrac{\theta}{2}\right)
\end{aligned}$$