PHYS 234 Assignment 4

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3. The density matrix for an ensemble of spin-$\frac{1}{2}$ particles in the S_z basis is

$$\hat{\rho} \xrightarrow{S_z \text{ basis}} \begin{bmatrix} \frac{1}{4} & n \\ n^* & p \end{bmatrix}.$$

(a) What value must p have? Why?

Since the trace of $\hat{\rho}$ must be 1, p must be $\frac{3}{4}$.

(b) What value(s) must n have for the density matrix to represent a pure state?

For $\hat{\rho}$ to be a pure ensemble, $\hat{\rho}$ must equal $\hat{\rho}^2$.

$$\begin{aligned}\begin{bmatrix} \frac{1}{4} & n \\ n^* & \frac{3}{4} \end{bmatrix} &= \begin{bmatrix} \frac{1}{4} & n \\ n^* & \frac{3}{4} \end{bmatrix}\begin{bmatrix} \frac{1}{4} & n \\ n^* & \frac{3}{4} \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{8} + |n|^2 & n \\ n^* & \frac{3}{4} \end{bmatrix}\end{aligned}$$

The only constraint we have on the value of n is that $\frac{1}{8} + |n|^2 = \frac{1}{4}$:

$$\begin{aligned}\frac{1}{8} + |n|^2 &= \frac{1}{4} \\ |n|^2 &= \frac{1}{8} \\ n &= \frac{1}{\sqrt{8}} e^{i\theta}\end{aligned}$$

θ is some angle.

(c) What pure state is represented when n takes its maximum possible real value? Express your answer in terms of the state $|+\rangle_n$:

$$|+\rangle_n = \cos\left(\frac{\theta}{2}\right)|+\rangle + e^{i\phi}\sin\left(\frac{\theta}{2}\right)|-\rangle$$

The maximum possible real value for n is $\frac{1}{\sqrt{8}}$. ($\theta = 0$)

$$\begin{aligned}\rho &= |+\rangle_{n\,n}\langle+| \\ &= \begin{bmatrix} \cos\frac{\theta}{2} \\ e^{i\phi}\sin\frac{\theta}{2} \end{bmatrix}\begin{bmatrix} \cos\frac{\theta}{2} & e^{i\phi}\sin\frac{\theta}{2} \end{bmatrix} \\ &= \begin{bmatrix} \cos^2\frac{\theta}{2} & e^{i\phi}\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right) \\ e^{i\phi}\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right) & e^{2i\phi}\sin^2\frac{\theta}{2} \end{bmatrix}\end{aligned}$$

So we have

$$\begin{aligned}
\cos^2\frac{\theta}{2} &= \frac{1}{4}\\
\cos\frac{\theta}{2} &= \frac{1}{2}\\
\frac{\theta}{2} &= \frac{\pi}{3}\\
\theta &= \frac{2\pi}{3}
\end{aligned}$$

and

$$\begin{aligned}
e^{2i\phi}\sin^2\frac{\pi}{3} &= \frac{3}{4}\\
e^{2i\phi}\frac{3}{4} &= \frac{3}{4}\\
2i\phi &= 0\\
\phi &= 0.
\end{aligned}$$

Finally, we get that

$$|+\rangle_n = \frac{\sqrt{3}}{2}\,|+\rangle + \frac{1}{2}\,|-\rangle\,.$$