PHYS 234 Assignment 1

Brandon Tsang

May 18, 2020

2. <u>Calculations using quantum states</u>

$$\begin{aligned}
|\psi_1\rangle &= 3\,|+\rangle - i\,|-\rangle \\
|\psi_2\rangle &= e^{i\pi/3}\,|+\rangle + |-\rangle \\
|\psi_3\rangle &= 7i\,|+\rangle - 2\,|-\rangle
\end{aligned}$$

(a) For each of the states $|\psi_j\rangle$ above ($j = 1, 2, 3$), find the corresponding normalized state $|\psi_j\rangle_{\mathrm{N}}$.

For $|\psi_1\rangle$:

$$\begin{aligned}
\langle C\psi_1|C\psi_1\rangle &= 1 \\
1 &= C^*(3\,\langle+| + i\,\langle-|) \cdot C(3\,|+\rangle - i\,|-\rangle) \\
&= CC^*(9\,\langle+|+\rangle - 3i\,\langle+|-\rangle + 3i\,\langle-|+\rangle - i^2\,\langle-|-\rangle) \\
&= CC^*(9+1) \\
|C|^2 &= \frac{1}{10} \\
C &= \frac{1}{\sqrt{10}}
\end{aligned}$$

Therefore, $|\psi_1\rangle_{\mathrm{N}} = \frac{1}{\sqrt{10}}\,|\psi_1\rangle = \frac{3}{\sqrt{10}}\,|+\rangle - \frac{i}{\sqrt{10}}\,|-\rangle$.

For $|\psi_2\rangle$:

$$\begin{aligned}
\langle C\psi_2|C\psi_2\rangle &= 1 \\
1 &= C^*\left(e^{-i\pi/3}\,\langle+| + \langle-|\right) \cdot C\left(e^{i\pi/3}\,|+\rangle + |-\rangle\right) \\
&= CC^*\left(\langle+|+\rangle + e^{-i\pi/3}\,\langle+|-\rangle + e^{i\pi/3}\,\langle-|+\rangle + \langle-|-\rangle\right) \\
&= CC^*(1+1) \\
|C|^2 &= \frac{1}{2} \\
C &= \frac{1}{\sqrt{2}}
\end{aligned}$$

Therefore, $|\psi_2\rangle_{\mathrm{N}} = \frac{1}{\sqrt{2}}\,|\psi_2\rangle = \frac{1}{\sqrt{2}}e^{i\pi/3}\,|+\rangle + \frac{1}{\sqrt{2}}\,|-\rangle$.

For $|\psi_3\rangle$:

$$\begin{aligned}
\langle C\psi_3|C\psi_3\rangle &= 1 \\
1 &= C^*(-7i\langle +| - 2\langle -|) \cdot C(7i|+\rangle - 2|-\rangle) \\
&= CC^*(-49i^2\langle +|+\rangle + 14i\langle +|-\rangle - 14i\langle -|+\rangle + 4\langle -|-\rangle) \\
&= CC^*(49+4) \\
|C|^2 &= \frac{1}{53} \\
C &= \frac{1}{\sqrt{53}}
\end{aligned}$$

Therefore, $|\psi_3\rangle_{\mathrm{N}} = \frac{1}{\sqrt{53}}|\psi_3\rangle = \frac{7i}{\sqrt{53}}|+\rangle - \frac{2}{\sqrt{53}}|-\rangle$.

(b) Using the bra-ket notation, calculate all 9 inner products ${}_{\mathrm{N}}\langle\psi_i|\psi_j\rangle_{\mathrm{N}}$ for $i = 1, 2, 3$ and $j = 1, 2, 3$ using the normalized states.

$i = 1,\ j = 1$:

$${}_{\mathrm{N}}\langle\psi_1|\psi_1\rangle_{\mathrm{N}} = 1 \quad \text{(by definition)}$$

$i = 1,\ j = 2$:

$$\begin{aligned}
{}_{\mathrm{N}}\langle\psi_1|\psi_2\rangle_{\mathrm{N}} &= \left(\tfrac{1}{\sqrt{10}}\langle\psi_1|\right)\left(\tfrac{1}{\sqrt{2}}|\psi_2\rangle\right) \\
&= \tfrac{1}{\sqrt{20}}\langle\psi_1|\psi_2\rangle \\
&= \tfrac{1}{\sqrt{20}}(3\langle +| + i\langle -|)\left(e^{i\pi/3}|+\rangle + |-\rangle\right) \\
&= \frac{1}{\sqrt{20}}\left(3e^{i\pi/3}\langle +|+\rangle + 3\langle +|-\rangle + ie^{i\pi/3}\langle -|+\rangle + i\langle -|-\rangle\right) \\
&= \frac{1}{\sqrt{20}}\left(3e^{i\pi/3} + i\right)
\end{aligned}$$

$i = 1,\ j = 3$:

$$\begin{aligned}
{}_{\mathrm{N}}\langle\psi_1|\psi_3\rangle_{\mathrm{N}} &= \left(\tfrac{1}{\sqrt{10}}\langle\psi_1|\right)\left(\tfrac{1}{\sqrt{53}}|\psi_3\rangle\right) \\
&= \tfrac{1}{\sqrt{530}}\langle\psi_1|\psi_3\rangle \\
&= \tfrac{1}{\sqrt{530}}(3\langle +| + i\langle -|)(7i|+\rangle - 2|-\rangle) \\
&= \tfrac{1}{\sqrt{530}}(21i\langle +|-\rangle - 6\langle +|-\rangle + 7i^2\langle -|+\rangle - 2i\langle -|-\rangle) \\
&= \tfrac{1}{\sqrt{530}}(21i - 2i) \\
&= \frac{19i}{\sqrt{530}}
\end{aligned}$$

$i = 2,\ j = 1$:

$$\begin{aligned}
{}_{\mathrm{N}}\langle\psi_2|\psi_1\rangle_{\mathrm{N}} &= {}^*_{\mathrm{N}}\langle\psi_1|\psi_2\rangle^*_{\mathrm{N}} \\
&= \mathrm{conj}\left(\frac{1}{\sqrt{20}}\left(3e^{i\pi/3} + i\right)\right) \\
&= \frac{1}{\sqrt{20}}\left(3e^{-i\pi/3} - i\right)
\end{aligned}$$

$i = 2,\ j = 2$:

$$_{\rm N}\langle\psi_2|\psi_2\rangle_{\rm N} = 1 \quad \text{(by definition)}$$

$i = 2,\ j = 3$:

$$\begin{aligned}
_{\rm N}\langle\psi_2|\psi_3\rangle_{\rm N} &= \left(\tfrac{1}{\sqrt{2}}\langle\psi_2|\right)\left(\tfrac{1}{\sqrt{53}}|\psi_3\rangle\right)\\
&= \tfrac{1}{\sqrt{106}}\langle\psi_2|\psi_3\rangle\\
&= \frac{1}{\sqrt{106}}\left(e^{-i\pi/3}\langle+| + \langle-|\right)(7i|+\rangle - 2|-\rangle)\\
&= \frac{1}{\sqrt{106}}\left(7ie^{-i\pi/3}\langle+|+\rangle - 2e^{-i\pi/3}\langle+|-\rangle + 7i\langle-|+\rangle - 2\langle-|-\rangle\right)\\
&= \frac{1}{\sqrt{106}}\left(7e^{i\pi/6} - 2\right)
\end{aligned}$$

$i = 3,\ j = 1$:

$$\begin{aligned}
_{\rm N}\langle\psi_3|\psi_1\rangle_{\rm N} &= {}^{*}_{\rm N}\langle\psi_1|\psi_3\rangle^{*}_{\rm N}\\
&= \operatorname{conj}\left(\tfrac{19i}{\sqrt{530}}\right)\\
&= -\frac{19i}{\sqrt{530}}
\end{aligned}$$

$i = 3,\ j = 2$:

$$\begin{aligned}
_{\rm N}\langle\psi_3|\psi_2\rangle_{\rm N} &= {}^{*}_{\rm N}\langle\psi_2|\psi_3\rangle^{*}_{\rm N}\\
&= \operatorname{conj}\left(\frac{1}{\sqrt{106}}\left(7e^{i\pi/6} - 2\right)\right)\\
&= \frac{1}{\sqrt{106}}\left(7e^{-i\pi/6} - 2\right)
\end{aligned}$$

$i = 3,\ j = 3$:

$$_{\rm N}\langle\psi_3|\psi_3\rangle_{\rm N} = 1 \quad \text{(by definition)}$$

(c) **For each state $|\psi_i\rangle$, find the state $|\phi_i\rangle$ with unit norm, $\langle\phi_i|\phi_i\rangle = 1$ that is orthogonal to it. Recall the orthogonality conditions for the basis states: $\langle+|+\rangle = \langle-|-\rangle = 1$ and $\langle+|-\rangle = \langle-|+\rangle = 0$.**

If $|\psi_i\rangle$ and $|\phi_i\rangle$ are to be orthogonal, they must satisfy the orthogonality condition:

$$\langle\phi_i|\psi_i\rangle = 0$$

Let's test this out with $|\psi_1\rangle$ and $|\phi_1\rangle$ to see if it works. First, we define $|\phi_1\rangle$ to be some linear combination of the basis states:

$$|\phi_1\rangle = a|+\rangle + b|-\rangle$$

Then, we apply the orthogonality condition.

$$\begin{aligned}
\langle\phi_1|\psi_1\rangle = 0 &= (a^*\langle+| + b^*\langle-|)(3|+\rangle - i|-\rangle)\\
&= 3a^*\langle+|+\rangle - b^*i\langle-|-\rangle\\
a^* &= \frac{1}{3}b^*i\\
a &= -\frac{1}{3}bi
\end{aligned}$$

If this is correct, I should be able to pick any pair of a and b which satisfy this equation, and the $|\phi_1\rangle$ they make should be orthogonal to $|\psi_1\rangle$. I will randomly pick $a = 1$ and $b = 3i$, so

$$|\phi_1\rangle = |+\rangle + 3i\,|-\rangle\,.$$

Now, we verify that this is orthogonal to $|\psi_1\rangle$:

$$\begin{aligned}
0 &\stackrel{?}{=} \langle\phi_1|\psi_1\rangle \\
&\stackrel{?}{=} (\langle+| - 3i\,\langle-|)(3\,|+\rangle - i\,|-\rangle) \\
&\stackrel{?}{=} 3\,\langle+|+\rangle + 3i^2\,\langle-|-\rangle \\
&\stackrel{?}{=} 3 - 3 \\
&= 0
\end{aligned}$$

Great! Now all that's left to do is normalize $|\phi_1\rangle$ and we're done.

$$\begin{aligned}
\langle C\phi_1|C\phi_1\rangle &= 1 \\
1 &= C^*(\langle+| - 3i\,\langle-|) \cdot C(|+\rangle + 3i\,|-\rangle) \\
&= CC^*(\langle+|+\rangle - 9i^2\,\langle-|-\rangle) \\
&= |C|^2(1+9) \\
|C|^2 &= \frac{1}{10} \\
C &= \frac{1}{\sqrt{10}} \\
|\phi_1\rangle_{\mathrm{N}} &= \tfrac{1}{\sqrt{10}}\,|+\rangle + \tfrac{3i}{\sqrt{10}}\,|-\rangle
\end{aligned}$$

Now I will repeat the process for finding $|\phi_2\rangle$ and $|\phi_3\rangle$. For $|\phi_2\rangle$:

$$|\phi_2\rangle = a_2\,|+\rangle + b_2\,|-\rangle$$

Applying the orthogonality condition:

$$\begin{aligned}
\langle\phi_2|\psi_2\rangle = 0 &= (a_2^*\,\langle+| + b_2^*\,\langle-|)\left(e^{i\pi/3}\,|+\rangle + |-\rangle\right) \\
&= a_2^* e^{i\pi/3}\,\langle+|+\rangle + b_2^*\,\langle-|-\rangle \\
a_2^* &= -e^{-i\pi/3} b_2^* \\
a_2 &= -e^{i\pi/3} b_2
\end{aligned}$$

Randomly picking $a_2 = -1$ and $b_2 = e^{-i\pi/3}$:

$$|\phi_2\rangle = -\,|+\rangle + e^{-i\pi/3}\,|-\rangle$$

Normalizing:

$$\begin{aligned}
\langle C_2\phi_2|C_2\phi_2\rangle &= 1 \\
1 &= C_2^*\left(-\,\langle+| + e^{i\pi/3}\,\langle-|\right) \cdot C_2\left(-\,|+\rangle + e^{-i\pi/3}\,|-\rangle\right) \\
&= C_2 C_2^*(\langle+|+\rangle + 0\,\langle-|-\rangle) \\
|C_2|^2 &= 1 \\
C_2 &= 1
\end{aligned}$$

$$|\phi_2\rangle_\text{N} = -|+\rangle + e^{-i\pi/3}|-\rangle$$

Finally, finding $|\phi_3\rangle$:

$$|\phi_3\rangle = a_3|+\rangle + b_3|-\rangle$$

Applying the orthogonality condition:

$$\begin{aligned}
\langle\phi_3|\psi_3\rangle = 0 &= (a_3^*\langle+| + b_3^*\langle-|)(7i|+\rangle - 2|-\rangle) \\
&= 7a_3^* i\langle+|+\rangle - 2b_3^*\langle-|-\rangle \\
a_3^* &= \frac{2b_3^*}{7i} \\
&= -\tfrac{2}{7}b_3^* i \\
a_3 &= \tfrac{2}{7}b_3 i
\end{aligned}$$

Randomly picking $a_3 = 2$ and $b_3 = -7i$:

$$|\phi_3\rangle = 2|+\rangle - 7i|-\rangle$$

Normalizing:

$$\begin{aligned}
\langle C_3\phi_3|C_3\phi_3\rangle &= 1 \\
1 &= C_3^*(2\langle+| + 7i\langle-|) \cdot C_3(2|+\rangle - 7i|-\rangle) \\
&= C_3C_3^*(4\langle+|+\rangle - 49i^2\langle-|-\rangle) \\
|C_3|^2 &= \frac{1}{53} \\
C_3 &= \frac{1}{\sqrt{53}}
\end{aligned}$$

$$|\phi_3\rangle_\text{N} = \tfrac{2}{\sqrt{53}}|+\rangle - \tfrac{7i}{\sqrt{53}}|-\rangle$$

(d) Postulate 4 of quantum mechanics tells us that the complex square of the inner product $|\langle a|b\rangle|^2$ is the probability of measuring a particular quantum state. For each of the normalized states $|\psi_i\rangle_\text{N}$, calculate the probability of measuring each of the six states indicated below.

$$\mathbf{|1\rangle = |+\rangle}$$

With $|\psi_1\rangle_\text{N}$:

$$\begin{aligned}
|\langle 1|\psi_1\rangle_\text{N}|^2 &= \left|\langle+| \cdot \tfrac{1}{\sqrt{10}}(3|+\rangle - i|-\rangle)\right|^2 \\
&= \left|\tfrac{3}{\sqrt{10}}\langle+|+\rangle - \tfrac{i}{\sqrt{10}}\langle+|-\rangle\right|^2 \\
&= \left|\tfrac{3}{\sqrt{10}}\right|^2 \\
&= \frac{9}{10}
\end{aligned}$$

With $|\psi_2\rangle_\mathrm{N}$:

$$\begin{aligned}
|\langle 1|\psi_2\rangle_\mathrm{N}|^2 &= \left|\langle +| \cdot \tfrac{1}{\sqrt{2}} \left(e^{i\pi/3} |+\rangle + |-\rangle\right)\right|^2 \\
&= \left|\tfrac{1}{\sqrt{2}} e^{i\pi/3} \langle +|+\rangle - \tfrac{1}{\sqrt{2}} \langle +|-\rangle\right|^2 \\
&= \left|\tfrac{1}{\sqrt{2}} e^{i\pi/3}\right|^2 \\
&= \frac{1}{2}
\end{aligned}$$

With $|\psi_3\rangle_\mathrm{N}$:

$$\begin{aligned}
|\langle 1|\psi_3\rangle_\mathrm{N}|^2 &= \left|\langle +| \cdot \tfrac{1}{\sqrt{53}} (7i |+\rangle - 2 |-\rangle)\right|^2 \\
&= \left|\tfrac{7i}{\sqrt{53}} \langle +|+\rangle - \tfrac{2}{\sqrt{53}} \langle +|-\rangle\right|^2 \\
&= \left|\tfrac{7i}{\sqrt{53}}\right|^2 \\
&= \frac{49}{53}
\end{aligned}$$

$|2\rangle = |-\rangle$

With $|\psi_1\rangle_\mathrm{N}$:

$$\begin{aligned}
|\langle 2|\psi_1\rangle_\mathrm{N}|^2 &= \left|\langle -| \cdot \tfrac{1}{\sqrt{10}} (3 |+\rangle - i |-\rangle)\right|^2 \\
&= \left|\tfrac{3}{\sqrt{10}} \langle -|+\rangle - \tfrac{i}{\sqrt{10}} \langle -|-\rangle\right|^2 \\
&= \left|-\tfrac{i}{\sqrt{10}}\right|^2 \\
&= \frac{1}{10}
\end{aligned}$$

With $|\psi_2\rangle_\mathrm{N}$:

$$\begin{aligned}
|\langle 2|\psi_2\rangle_\mathrm{N}|^2 &= \left|\langle -| \cdot \tfrac{1}{\sqrt{2}} \left(e^{i\pi/3} |+\rangle + |-\rangle\right)\right|^2 \\
&= \left|\tfrac{1}{\sqrt{2}} e^{i\pi/3} \langle -|+\rangle - \tfrac{1}{\sqrt{2}} \langle -|-\rangle\right|^2 \\
&= \left|-\tfrac{1}{\sqrt{2}}\right|^2 \\
&= \frac{1}{2}
\end{aligned}$$

With $|\psi_3\rangle_\mathrm{N}$:

$$\begin{aligned}
|\langle 2|\psi_3\rangle_\mathrm{N}|^2 &= \left|\langle -| \cdot \tfrac{1}{\sqrt{53}}(7i\,|+\rangle - 2\,|-\rangle)\right|^2 \\
&= \left|\tfrac{7i}{\sqrt{53}}\,\langle -|+\rangle - \tfrac{2}{\sqrt{53}}\,\langle -|-\rangle\right|^2 \\
&= \left|-\tfrac{2}{\sqrt{53}}\right|^2 \\
&= \frac{4}{53}
\end{aligned}$$

$$\mathbf{|3\rangle = \tfrac{1}{\sqrt{2}}(|+\rangle + |-\rangle)}$$

With $|\psi_1\rangle_\mathrm{N}$:

$$\begin{aligned}
|\langle 3|\psi_1\rangle_\mathrm{N}|^2 &= \left|\tfrac{1}{\sqrt{2}}(\langle +| + \langle -|) \cdot \tfrac{1}{\sqrt{10}}(3\,|+\rangle - i\,|-\rangle)\right|^2 \\
&= \left|\tfrac{1}{\sqrt{20}}(3\,\langle +|+\rangle - i\,\langle -|-\rangle)\right|^2 \\
&= \left|\tfrac{1}{\sqrt{20}}(3 - i)\right|^2 \\
&= \frac{10}{20} \\
&= \frac{1}{2}
\end{aligned}$$

With $|\psi_2\rangle_\mathrm{N}$:

$$\begin{aligned}
|\langle 3|\psi_2\rangle_\mathrm{N}|^2 &= \left|\frac{1}{\sqrt{2}}(\langle +| + \langle -|) \cdot \frac{1}{\sqrt{2}}\left(e^{i\pi/3}\,|+\rangle + |-\rangle\right)\right|^2 \\
&= \left|\frac{1}{4}\left(e^{i\pi/3}\,\langle +|+\rangle + \langle -|-\rangle\right)\right|^2 \\
&= \left|\frac{1}{4}\left(e^{i\pi/3} + 1\right)\right|^2 \\
&= \frac{1}{16}\left(e^{i\pi/3} + 1\right)\left(e^{-i\pi/3} + 1\right) \\
&= \frac{1}{16}\left(1 + e^{i\pi/3} + e^{-i\pi/3} + 1\right) \\
&= \frac{1}{16}\left(2 + 2\cos\tfrac{\pi}{3}\right) \\
&= \frac{3}{16}
\end{aligned}$$

With $|\psi_3\rangle_{\mathrm{N}}$:

$$\begin{aligned}
|\langle 3|\psi_3\rangle_{\mathrm{N}}|^2 &= \left|\tfrac{1}{\sqrt{2}}(\langle +| + \langle -|) \cdot \tfrac{1}{\sqrt{53}}(7i\,|+\rangle - 2\,|-\rangle)\right|^2 \\
&= \left|\tfrac{1}{\sqrt{106}}(7i\,\langle +|+\rangle - 2\,\langle -|-\rangle)\right|^2 \\
&= \left|\tfrac{1}{\sqrt{106}}(7i - 2)\right|^2 \\
&= \frac{53}{106} \\
&= \frac{1}{2}
\end{aligned}$$

$|4\rangle = \frac{1}{\sqrt{2}}(|+\rangle - |-\rangle)$

With $|\psi_1\rangle_{\mathrm{N}}$:

$$\begin{aligned}
|\langle 4|\psi_1\rangle_{\mathrm{N}}|^2 &= \left|\tfrac{1}{\sqrt{2}}(\langle +| - \langle -|) \cdot \tfrac{1}{\sqrt{10}}(3\,|+\rangle - i\,|-\rangle)\right|^2 \\
&= \left|\tfrac{1}{\sqrt{20}}(3\,\langle +|+\rangle + i\,\langle -|-\rangle)\right|^2 \\
&= \left|\tfrac{1}{\sqrt{20}}(3 + i)\right|^2 \\
&= \frac{10}{20} \\
&= \frac{1}{2}
\end{aligned}$$

With $|\psi_2\rangle_{\mathrm{N}}$:

$$\begin{aligned}
|\langle 4|\psi_2\rangle_{\mathrm{N}}|^2 &= \left|\frac{1}{\sqrt{2}}(\langle +| - \langle -|) \cdot \frac{1}{\sqrt{2}}\left(e^{i\pi/3}\,|+\rangle + |-\rangle\right)\right|^2 \\
&= \left|\frac{1}{4}\left(e^{i\pi/3}\,\langle +|+\rangle - \langle -|-\rangle\right)\right|^2 \\
&= \left|\frac{1}{4}\left(e^{i\pi/3} - 1\right)\right|^2 \\
&= \frac{1}{16}\left(e^{i\pi/3} - 1\right)\left(e^{-i\pi/3} - 1\right) \\
&= \frac{1}{16}\left(1 - e^{i\pi/3} - e^{-i\pi/3} + 1\right) \\
&= \frac{1}{16}\left(2 - 2\cos\tfrac{\pi}{3}\right) \\
&= \frac{1}{16}
\end{aligned}$$

With $|\psi_3\rangle_\mathrm{N}$:

$$
\begin{aligned}
|\langle 4|\psi_3\rangle_\mathrm{N}|^2 &= \left|\tfrac{1}{\sqrt{2}}(\langle +| - \langle -|) \cdot \tfrac{1}{\sqrt{53}}(7i\,|+\rangle - 2\,|-\rangle)\right|^2 \\
&= \left|\tfrac{1}{\sqrt{106}}(7i\,\langle +|+\rangle + 2\,\langle -|-\rangle)\right|^2 \\
&= \left|\tfrac{1}{\sqrt{106}}(7i + 2)\right|^2 \\
&= \frac{53}{106} \\
&= \frac{1}{2}
\end{aligned}
$$

$$
\boldsymbol{|5\rangle = \tfrac{1}{\sqrt{2}}(|+\rangle + i\,|-\rangle)}
$$

With $|\psi_1\rangle_\mathrm{N}$:

$$
\begin{aligned}
|\langle 5|\psi_1\rangle_\mathrm{N}|^2 &= \left|\tfrac{1}{\sqrt{2}}(\langle +| + i\,\langle -|) \cdot \tfrac{1}{\sqrt{10}}(3\,|+\rangle - i\,|-\rangle)\right|^2 \\
&= \left|\tfrac{1}{\sqrt{20}}(3\,\langle +|+\rangle - i^2\,\langle -|-\rangle)\right|^2 \\
&= \left|\tfrac{1}{\sqrt{20}}(3 + 1)\right|^2 \\
&= \frac{16}{20} \\
&= \frac{4}{5}
\end{aligned}
$$

With $|\psi_2\rangle_\mathrm{N}$:

$$
\begin{aligned}
|\langle 5|\psi_2\rangle_\mathrm{N}|^2 &= \left|\frac{1}{\sqrt{2}}(\langle +| + i\,\langle -|) \cdot \frac{1}{\sqrt{2}}\left(e^{i\pi/3}\,|+\rangle + |-\rangle\right)\right|^2 \\
&= \left|\frac{1}{4}\left(e^{i\pi/3}\,\langle +|+\rangle + i\,\langle -|-\rangle\right)\right|^2 \\
&= \left|\frac{1}{4}\left(e^{i\pi/3} + i\right)\right|^2 \\
&= \frac{1}{16}\left(e^{i\pi/3} + i\right)\left(e^{-i\pi/3} - i\right) \\
&= \frac{1}{16}\left(1 - ie^{i\pi/3} + ie^{-i\pi/3} - i^2\right) \\
&= \frac{1}{16}\left(2 - i(e^{i\pi/3} - ie^{-i\pi/3})\right) \\
&= \frac{1}{16}\left(2 - i\left(2i\sin\tfrac{\pi}{3}\right)\right) \\
&= \frac{1}{16}\left(2 + \sqrt{3}\right) \\
&= \frac{2 + \sqrt{3}}{16}
\end{aligned}
$$

With $|\psi_3\rangle_\mathrm{N}$:

$$
\begin{aligned}
|\langle 5|\psi_3\rangle_\mathrm{N}|^2 &= \left|\tfrac{1}{\sqrt{2}}(\langle +| + i\langle -|) \cdot \tfrac{1}{\sqrt{53}}(7i\,|+\rangle - 2\,|-\rangle)\right|^2 \\
&= \left|\tfrac{1}{\sqrt{106}}(7i\,\langle +|+\rangle - 2i\,\langle -|-\rangle)\right|^2 \\
&= \left|\tfrac{1}{\sqrt{106}}(5i)\right|^2 \\
&= \frac{25}{106} \\
&= \frac{1}{2}
\end{aligned}
$$

$$\boldsymbol{|6\rangle = \tfrac{1}{\sqrt{2}}(|+\rangle - i\,|-\rangle)}$$

With $|\psi_1\rangle_\mathrm{N}$:

$$
\begin{aligned}
|\langle 6|\psi_1\rangle_\mathrm{N}|^2 &= \left|\tfrac{1}{\sqrt{2}}(\langle +| - i\langle -|) \cdot \tfrac{1}{\sqrt{10}}(3\,|+\rangle - i\,|-\rangle)\right|^2 \\
&= \left|\tfrac{1}{\sqrt{20}}(3\,\langle +|+\rangle + i^2\,\langle -|-\rangle)\right|^2 \\
&= \left|\tfrac{1}{\sqrt{20}}(3 - 1)\right|^2 \\
&= \frac{4}{20} \\
&= \frac{1}{5}
\end{aligned}
$$

With $|\psi_2\rangle_\mathrm{N}$:

$$
\begin{aligned}
|\langle 6|\psi_2\rangle_\mathrm{N}|^2 &= \left|\frac{1}{\sqrt{2}}(\langle +| - i\langle -|) \cdot \frac{1}{\sqrt{2}}\left(e^{i\pi/3}\,|+\rangle + |-\rangle\right)\right|^2 \\
&= \left|\frac{1}{4}\left(e^{i\pi/3}\,\langle +|+\rangle - i\,\langle -|-\rangle\right)\right|^2 \\
&= \left|\frac{1}{4}\left(e^{i\pi/3} - i\right)\right|^2 \\
&= \frac{1}{16}\left(e^{i\pi/3} - i\right)\left(e^{-i\pi/3} + i\right) \\
&= \frac{1}{16}\left(1 + ie^{i\pi/3} - ie^{-i\pi/3} - i^2\right) \\
&= \frac{1}{16}\left(2 + i(e^{i\pi/3} - ie^{-i\pi/3})\right) \\
&= \frac{1}{16}\left(2 + i\left(2i\sin\tfrac{\pi}{3}\right)\right) \\
&= \frac{1}{16}\left(2 - \sqrt{3}\right) \\
&= \frac{2 - \sqrt{3}}{16}
\end{aligned}
$$

With $|\psi_3\rangle_{\mathrm{N}}$:

$$
\begin{aligned}
|\langle 6|\psi_3\rangle_{\mathrm{N}}|^2 &= \left|\tfrac{1}{\sqrt{2}}(\langle +| - i\,\langle -|)\cdot\tfrac{1}{\sqrt{53}}(7i\,|+\rangle - 2\,|-\rangle)\right|^2 \\
&= \left|\tfrac{1}{\sqrt{106}}(7i\,\langle +|+\rangle + 2i\,\langle -|-\rangle)\right|^2 \\
&= \left|\tfrac{1}{\sqrt{106}}(9i)\right|^2 \\
&= \frac{81}{106}
\end{aligned}
$$