PHYS 234 Assignment 2

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May 29, 2020

1. **<u>Computing Probabilities</u>**
   A beam of spin-$\frac{1}{2}$ particles is prepared in the state

$$|\psi\rangle = \tfrac{3i}{5}\left|+\right\rangle_y + \tfrac{4}{5}\left|-\right\rangle_y .$$

The state is given in terms of the S_y eigenvectors. Since we derived relations for the S_z operator in class, I will express $|\psi\rangle$ in terms of S_z eigenvectors to ease the calculations.

First, I will confirm that $|\psi\rangle$ is normalized:

$$\begin{aligned}
\langle\psi|\psi\rangle &\stackrel{?}{=} 1 \\
\left(-\tfrac{3i}{5}\,{}_y\langle+| + \tfrac{4}{5}\,{}_y\langle-|\right)\left(\tfrac{3i}{5}\left|+\right\rangle_y + \tfrac{4}{5}\left|-\right\rangle_y\right) &\stackrel{?}{=} 1 \\
\frac{9}{25} + \frac{16}{25} &\stackrel{?}{=} 1 \\
1 &= 1
\end{aligned}$$

Then, I find the projection of $|\psi\rangle$ onto $|+\rangle$:

$$\begin{aligned}
\langle+|\psi\rangle &= \langle+|\left(\tfrac{3i}{5}\left|+\right\rangle_y + \tfrac{4}{5}\left|-\right\rangle_y\right) \\
&= \tfrac{3i}{5}\left\langle+|+\right\rangle_y + \tfrac{4}{5}\left\langle+|-\right\rangle_y
\end{aligned}$$

From class, $\left|+\right\rangle_y = \frac{1}{\sqrt{2}}(|+\rangle + i\,|-\rangle)$ and $\left|-\right\rangle_y = \frac{1}{\sqrt{2}}(|+\rangle - i\,|-\rangle)$:

$$\begin{aligned}
&= \tfrac{3i}{5}\langle+|\cdot\tfrac{1}{\sqrt{2}}(|+\rangle + i\,|-\rangle) + \tfrac{4}{5}\langle+|\cdot\tfrac{1}{\sqrt{2}}(|+\rangle - i\,|-\rangle) \\
&= \tfrac{3i}{5\sqrt{2}}(\langle+|+\rangle + i\,\langle+|-\rangle) + \tfrac{4}{5\sqrt{2}}(\langle+|+\rangle - i\,\langle+|-\rangle) \\
&= \frac{4+3i}{5\sqrt{2}}
\end{aligned}$$

And again with $|-\rangle$:

$$\begin{aligned}
\langle-|\psi\rangle &= \langle-|\left(\tfrac{3i}{5}\left|+\right\rangle_y + \tfrac{4}{5}\left|-\right\rangle_y\right) \\
&= \tfrac{3i}{5}\left\langle-|+\right\rangle_y + \tfrac{4}{5}\left\langle-|-\right\rangle_y \\
&= \tfrac{3i}{5}\langle-|\cdot\tfrac{1}{\sqrt{2}}(|+\rangle + i\,|-\rangle) + \tfrac{4}{5}\langle-|\cdot\tfrac{1}{\sqrt{2}}(|+\rangle - i\,|-\rangle) \\
&= \tfrac{3i}{5\sqrt{2}}(\langle-|+\rangle + i\,\langle-|-\rangle) + \tfrac{4}{5\sqrt{2}}(\langle-|+\rangle - i\,\langle-|-\rangle) \\
&= \frac{3i^2}{5\sqrt{2}} - \frac{4i}{5\sqrt{2}} \\
&= \frac{-3-4i}{5\sqrt{2}}
\end{aligned}$$

So $|\psi\rangle$ is then

$$|\psi\rangle = \tfrac{1}{5\sqrt{2}}((4+3i)\,|+\rangle - (3+4i)\,|-\rangle).$$

(a) What are the possible measurement results of the spin operator S_x and with what probability would they occur?

The two possible measurement results are $+\frac{\hbar}{2}$ and $-\frac{\hbar}{2}$.

The probability of $+\frac{\hbar}{2}$ is:

$$\begin{aligned}\left|{}_x\langle +|\psi\rangle\right|^2 &= \left|{}_x\langle +| \cdot \tfrac{1}{5\sqrt{2}}((4+3i)\,|+\rangle - (3+4i)\,|-\rangle)\right|^2 \\ &= \left|\tfrac{4+3i}{5\sqrt{2}}\,{}_x\langle +|+\rangle - \tfrac{3+4i}{5\sqrt{2}}\,{}_x\langle +|-\rangle\right|^2\end{aligned}$$

From class, $|+\rangle_x = \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle)$:

$$\begin{aligned}&= \left|\tfrac{4+3i}{5\sqrt{2}} \cdot \tfrac{1}{\sqrt{2}}(\langle +| + \langle -|)\,|+\rangle - \tfrac{3+4i}{5\sqrt{2}} \cdot \tfrac{1}{\sqrt{2}}(\langle +| + \langle -|)\,|-\rangle\right|^2 \\ &= \left|\frac{4+3i}{5\cdot 2} - \frac{3+4i}{5\cdot 2}\right|^2 \\ &= \left|\frac{1-i}{10}\right|^2 \\ &= \frac{1+1}{100} \\ &= \frac{1}{50}\end{aligned}$$

The probability of $-\frac{\hbar}{2}$ is:

$$\begin{aligned}\left|{}_x\langle -|\psi\rangle\right|^2 &= \left|{}_x\langle +| \cdot \tfrac{1}{5\sqrt{2}}((4+3i)\,|+\rangle - (3+4i)\,|-\rangle)\right|^2 \\ &= \left|\tfrac{4+3i}{5\sqrt{2}}\,{}_x\langle -|+\rangle - \tfrac{3+4i}{5\sqrt{2}}\,{}_x\langle -|-\rangle\right|^2\end{aligned}$$

From class, $|-\rangle_x = \frac{1}{\sqrt{2}}(|+\rangle - |-\rangle)$:

$$\begin{aligned}&= \left|\tfrac{4+3i}{5\sqrt{2}} \cdot \tfrac{1}{\sqrt{2}}(\langle +| - \langle -|)\,|+\rangle - \tfrac{3+4i}{5\sqrt{2}} \cdot \tfrac{1}{\sqrt{2}}(\langle +| - \langle -|)\,|-\rangle\right|^2 \\ &= \left|\frac{4+3i}{5\cdot 2} + \frac{3+4i}{5\cdot 2}\right|^2 \\ &= \left|\frac{7-7i}{10}\right|^2 \\ &= \frac{49+49}{100} \\ &= \frac{49}{50}\end{aligned}$$

(b) What are the possible measurement results of the spin operator S_y and with what probability would they occur?

The two possible measurement results are $+\frac{\hbar}{2}$ and $-\frac{\hbar}{2}$.

The probability of $+\frac{\hbar}{2}$ is:

$$\begin{aligned}
\left|{}_y\langle + | \psi \rangle\right|^2 &= \left|{}_y\langle +| \cdot \tfrac{1}{5\sqrt{2}}\left((4+3i)\,|+\rangle - (3+4i)\,|-\rangle\right)\right|^2 \\
&= \left|\tfrac{4+3i}{5\sqrt{2}}\ {}_y\langle + | + \rangle - \tfrac{3+4i}{5\sqrt{2}}\ {}_y\langle + | - \rangle\right|^2
\end{aligned}$$

From class, $|+\rangle_y = \frac{1}{\sqrt{2}}(|+\rangle + i\,|-\rangle)$:

$$\begin{aligned}
&= \left|\tfrac{4+3i}{5\sqrt{2}} \cdot \tfrac{1}{\sqrt{2}}\left(\left(\langle +| + i\,\langle -|\right)|+\rangle - \tfrac{3+4i}{5\sqrt{2}} \cdot \tfrac{1}{\sqrt{2}}\left(\left(\langle +| + i\,\langle -|\right)|-\rangle\right|^2 \\
&= \left|\frac{4+3i}{5\cdot 2} - \frac{3+4i}{5\cdot 2}i\right|^2 \\
&= \left|\frac{4+3i}{5\cdot 2} - \frac{-4+3i}{5\cdot 2}\right|^2 \\
&= \left|\frac{8}{10}\right|^2 \\
&= \frac{16}{25}
\end{aligned}$$

The probability of $-\frac{\hbar}{2}$ is:

$$\begin{aligned}
\left|{}_y\langle - | \psi \rangle\right|^2 &= \left|{}_y\langle +| \cdot \tfrac{1}{5\sqrt{2}}\left((4+3i)\,|+\rangle - (3+4i)\,|-\rangle\right)\right|^2 \\
&= \left|\tfrac{4+3i}{5\sqrt{2}}\ {}_y\langle - | + \rangle - \tfrac{3+4i}{5\sqrt{2}}\ {}_y\langle - | - \rangle\right|^2
\end{aligned}$$

From class, $|-\rangle_y = \frac{1}{\sqrt{2}}(|+\rangle - i\,|-\rangle)$:

$$\begin{aligned}
&= \left|\tfrac{4+3i}{5\sqrt{2}} \cdot \tfrac{1}{\sqrt{2}}\left(\left(\langle +| - i\,\langle -|\right)|+\rangle - \tfrac{3+4i}{5\sqrt{2}} \cdot \tfrac{1}{\sqrt{2}}\left(\left(\langle +| - i\,\langle -|\right)|-\rangle\right|^2 \\
&= \left|\frac{4+3i}{5\cdot 2} + \frac{3+4i}{5\cdot 2}i\right|^2 \\
&= \left|\frac{4+3i}{5\cdot 2} + \frac{-4+3i}{5\cdot 2}\right|^2 \\
&= \left|\frac{6i}{10}\right|^2 \\
&= \frac{36}{100} \\
&= \frac{9}{25}
\end{aligned}$$

(c) What are the possible measurement results of the spin operator S_z and with what probability would they occur?

The two possible measurement results are $+\frac{\hbar}{2}$ and $-\frac{\hbar}{2}$.

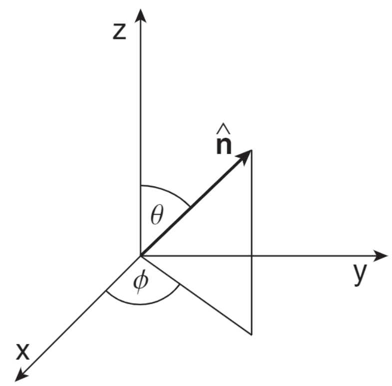
The probability of $+\frac{\hbar}{2}$ is:

$$\begin{aligned}
|\langle + | \psi \rangle|^2 &= \left| \langle + | \cdot \tfrac{1}{5\sqrt{2}} \left((4+3i) \left| + \right\rangle - (3+4i) \left| - \right\rangle \right) \right|^2 \\
&= \left| \frac{4+3i}{5\sqrt{2}} \right|^2 \\
&= \frac{16+9}{25 \cdot 2} \\
&= \frac{1}{2}
\end{aligned}$$

The probability of $-\frac{\hbar}{2}$ is:

$$\begin{aligned}
|\langle - | \psi \rangle|^2 &= \left| \langle - | \cdot \tfrac{1}{5\sqrt{2}} \left((4+3i) \left| + \right\rangle - (3+4i) \left| - \right\rangle \right) \right|^2 \\
&= \left| -\frac{3+4i}{5\sqrt{2}} \right|^2 \\
&= \frac{9+16}{25 \cdot 2} \\
&= \frac{1}{2}
\end{aligned}$$

2. **Spin Operator in an Arbitrary Direction**
Find the representation of the spin operator $S_n := \vec{S} \cdot \hat{n}$ that measures the projection of the spin-$\frac{1}{2}$ particle along the $\hat{n}$ direction. Calculate its representation in the S_z-basis. Here, $\hat{n}$ is the unit vector in spherical-polar coordinates.



$$\hat{n} = \sin(\theta)\cos(\phi)\hat{x} + \sin(\theta)\sin(\phi)\hat{y} + \cos(\theta)\hat{z}$$

and

$$\vec{S} = S_x\hat{x} + S_y\hat{y} + S_z\hat{z}$$

where S_x, S_y, and S_z are the three components of the spin-$\frac{1}{2}$ operator.

(a) Show that

$$S_n = \frac{\hbar}{2}\begin{bmatrix} \cos\theta & \sin(\theta)e^{-i\phi} \\ \sin(\theta)e^{i\phi} & -\cos\theta \end{bmatrix}.$$

$$\begin{aligned} S_n &= \mathrm{S}\cdot\hat{\mathbf{n}} \\ &= S_x\sin(\theta)\cos(\phi) + S_y\sin(\theta)\sin(\phi) + S_z\cos(\theta) \end{aligned}$$

$$S_x = \frac{\hbar}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad S_y = \frac{\hbar}{2}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \qquad S_z = \frac{\hbar}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\begin{aligned} S_n &= \frac{\hbar}{2}\left(\begin{bmatrix} 0 & \sin\theta\cos\phi \\ \sin\theta\cos\phi & 0 \end{bmatrix} + \begin{bmatrix} 0 & -i\sin\theta\sin\phi \\ i\sin\theta\sin\phi & 0 \end{bmatrix} + \begin{bmatrix} \cos\theta & 0 \\ 0 & -\cos\theta \end{bmatrix}\right) \\ &= \frac{\hbar}{2}\begin{bmatrix} \cos\theta & \sin\theta\cos\phi - i\sin\theta\sin\phi \\ \sin\theta\cos\phi + i\sin\theta\sin\phi & -\cos\theta \end{bmatrix} \\ &= \frac{\hbar}{2}\begin{bmatrix} \cos\theta & \sin\theta(\cos\phi - i\sin\phi) \\ \sin\theta(\cos\phi + i\sin\phi) & -\cos\theta \end{bmatrix} \\ &= \frac{\hbar}{2}\begin{bmatrix} \cos\theta & \sin(\theta)e^{-i\phi} \\ \sin(\theta)e^{i\phi} & -\cos\theta \end{bmatrix} \end{aligned}$$

(b) Show that the eigenvalues of S_n are $\pm\frac{\hbar}{2}$, as expected from the S-G experiment.

$$
\begin{aligned}
(S_n - \lambda I)\,|\psi\rangle &= \mathbf{0}\\
\left(\frac{\hbar}{2}\begin{bmatrix}\cos\theta & \sin(\theta)e^{-i\phi}\\ \sin(\theta)e^{i\phi} & -\cos\theta\end{bmatrix} - \begin{bmatrix}\lambda & 0\\ 0 & \lambda\end{bmatrix}\right)|\psi\rangle &= \mathbf{0}\\
\begin{vmatrix}\frac{\hbar}{2}\cos\theta - \lambda & \frac{\hbar}{2}\sin(\theta)e^{-i\phi}\\ \frac{\hbar}{2}\sin(\theta)e^{i\phi} & -\frac{\hbar}{2}\cos\theta - \lambda\end{vmatrix} &= 0\\
-\left(\tfrac{\hbar}{2}\cos\theta - \lambda\right)\left(\tfrac{\hbar}{2}\cos\theta + \lambda\right) - \left(\tfrac{\hbar}{2}\sin\theta\right)^2\left(e^{i\phi - i\phi}\right) &= 0\\
-\left(\tfrac{\hbar}{2}\cos\theta\right)^2 + \lambda^2 - \left(\tfrac{\hbar}{2}\sin\theta\right)^2 &= 0\\
\lambda^2 &= \left(\tfrac{\hbar}{2}\cos\theta\right)^2 + \left(\tfrac{\hbar}{2}\sin\theta\right)^2\\
&= \left(\frac{\hbar}{2}\right)^2\\
\lambda &= \pm\frac{\hbar}{2}
\end{aligned}
$$

(c) Show that the eigenvectors of S_n can be represented as

$$
\begin{aligned}
|+\rangle_n &= \cos\left(\tfrac{\theta}{2}\right)|+\rangle + \sin\left(\tfrac{\theta}{2}\right)e^{i\phi}\,|-\rangle\\
|-\rangle_n &= \sin\left(\tfrac{\theta}{2}\right)|+\rangle - \cos\left(\tfrac{\theta}{2}\right)e^{i\phi}\,|-\rangle
\end{aligned}
$$

The eigenvector $|\psi\rangle$ corresponding to the eigenvalue $\lambda = +\frac{\hbar}{2}$ is $|+\rangle_n$, and the eigenvector $|\psi\rangle$ corresponding to the eigenvalue $\lambda = -\frac{\hbar}{2}$ is $|-\rangle_n$.

$$
\begin{aligned}
(S_n - \lambda I)\,|\psi\rangle &= \mathbf{0}\\
\left(\frac{\hbar}{2}\begin{bmatrix}\cos\theta & \sin(\theta)e^{-i\phi}\\ \sin(\theta)e^{i\phi} & -\cos\theta\end{bmatrix} - \frac{\hbar}{2}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right)|+\rangle_n &= \mathbf{0}\\
\begin{bmatrix}\cos\theta - 1 & \sin(\theta)e^{-i\phi}\\ \sin(\theta)e^{i\phi} & -\cos\theta - 1\end{bmatrix}|+\rangle_n &= \mathbf{0}
\end{aligned}
$$

$$
\begin{aligned}
\left[\begin{array}{cc|c}\cos\theta - 1 & \sin(\theta)e^{-i\phi} & 0\\ \sin(\theta)e^{i\phi} & -\cos\theta - 1 & 0\end{array}\right]
&\xrightarrow{R_2 - \frac{\sin(\theta)e^{i\phi}}{\cos\theta - 1}R_1}
\left[\begin{array}{cc|c}\cos\theta - 1 & \sin(\theta)e^{-i\phi} & 0\\ 0 & -\cos\theta - 1 - \frac{(\sin(\theta)e^{-i\phi})(\sin(\theta)e^{i\phi})}{\cos\theta - 1} & 0\end{array}\right]\\
&\sim \left[\begin{array}{cc|c}\cos\theta - 1 & \sin(\theta)e^{-i\phi} & 0\\ 0 & -\frac{(\cos\theta + 1)(\cos\theta - 1)}{\cos\theta - 1} - \frac{\sin^2(\theta)}{\cos\theta - 1} & 0\end{array}\right]\\
&\sim \left[\begin{array}{cc|c}\cos\theta - 1 & \sin(\theta)e^{-i\phi} & 0\\ 0 & -\frac{\cos^2(\theta) - 1 + \sin^2(\theta)}{\cos\theta - 1} & 0\end{array}\right]\\
&\sim \left[\begin{array}{cc|c}\cos\theta - 1 & \sin(\theta)e^{-i\phi} & 0\\ 0 & 0 & 0\end{array}\right]
\end{aligned}
$$

If $|+\rangle_n = a|+\rangle + b|-\rangle$, then the above matrix corresponds to the equation

$$\begin{aligned}(\cos\theta - 1)a + \sin(\theta)e^{-i\phi}b &= 0\\ (1-\cos\theta)a &= \sin(\theta)e^{-i\phi}b\\ \sin^2\left(\frac{\theta}{2}\right)a &= \frac{1}{2}\sin(\theta)e^{-i\phi}b\\ \sin^2\left(\frac{\theta}{2}\right)e^{i\phi}a &= \sin(\theta)\cos^2\left(\frac{\theta}{2}\right)b\end{aligned}$$

So $|+\rangle_n = \sin^2\left(\frac{\theta}{2}\right)e^{i\phi}|+\rangle + \sin(\theta)\cos^2\left(\frac{\theta}{2}\right)|-\rangle$.
(I couldn't get the equation to match the form above.)

(d) For which values of θ and ϕ does the state $|+\rangle_n$ reduce to $|+\rangle_x$ and $|+\rangle_y$?

$|+\rangle_x$ was $\frac{1}{\sqrt{2}}(|+\rangle + |-\rangle)$, so for $|+\rangle_n$ to equal $|+\rangle_x$, we must set

$$\cos\frac{\theta}{2} = \frac{1}{\sqrt{2}}$$

and

$$\sin\left(\frac{\theta}{2}\right)e^{i\phi} = \frac{1}{\sqrt{2}}.$$

Starting with the first equation:

$$\begin{aligned}\cos\frac{\theta}{2} &= \frac{1}{\sqrt{2}}\\ \frac{\theta}{2} &= \frac{\pi}{4}\\ \theta &= \frac{\pi}{2}\end{aligned}$$

Then the second equation:

$$\begin{aligned}\sin\left(\frac{\theta}{2}\right)e^{i\phi} &= \frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}e^{i\phi} &= \frac{1}{\sqrt{2}}\\ e^{i\phi} &= 1\\ \phi &= 0\end{aligned}$$

So for $|+\rangle_n$ to reduce to $|+\rangle_x$, θ must be $\frac{\pi}{2}$, and ϕ must be 0.
Next, $|+\rangle_y$ was $\frac{1}{\sqrt{2}}(|+\rangle + i|-\rangle)$, so for $|+\rangle_n$ to equal $|+\rangle_y$, we must set

$$\cos\frac{\theta}{2} = \frac{1}{\sqrt{2}}$$

and

$$\sin\left(\frac{\theta}{2}\right)e^{i\phi} = i\frac{1}{\sqrt{2}}.$$

The solution to the first equation we found earlier to be $\theta = \frac{\pi}{2}$. We then only have to solve the second equation:

$$\begin{aligned}
\sin\left(\frac{\theta}{2}\right) e^{i\phi} &= i\frac{1}{\sqrt{2}} \\
\frac{1}{\sqrt{2}} e^{i\phi} &= i\frac{1}{\sqrt{2}} \\
e^{i\phi} &= i \\
\phi &= \frac{\pi}{2}
\end{aligned}$$

So for $|+\rangle_n$ to reduce to $|+\rangle_y$, both θ and ϕ must be $\frac{\pi}{2}$.

(e) Suppose that a measurement of S_z is carried out on a particle in the $|-\rangle_n$ state. What is the probability that the measurement yields:

(i) $\frac{\hbar}{2}$?

The probability is:

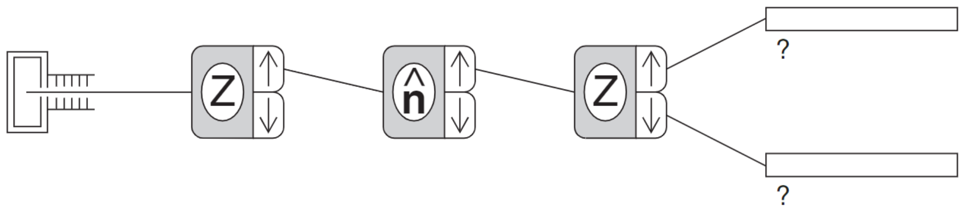
$$\begin{aligned}
|\langle +|-\rangle_n|^2 &= \left|\langle +| \left(\sin\left(\tfrac{\theta}{2}\right)|+\rangle - \cos\left(\tfrac{\theta}{2}\right) e^{i\phi}\,|-\rangle\right)\right|^2 \\
&= \left|\sin\left(\tfrac{\theta}{2}\right)\right|^2 \\
&= \sin^2\left(\tfrac{\theta}{2}\right)
\end{aligned}$$

(ii) $-\frac{\hbar}{2}$?

The probability is:

$$\begin{aligned}
|\langle -|-\rangle_n|^2 &= \left|\langle -| \left(\sin\left(\tfrac{\theta}{2}\right)|+\rangle - \cos\left(\tfrac{\theta}{2}\right) e^{i\phi}\,|-\rangle\right)\right|^2 \\
&= \left|-\cos\left(\tfrac{\theta}{2}\right) e^{i\phi}\right|^2 \\
&= \cos^2\left(\tfrac{\theta}{2}\right)
\end{aligned}$$

3. **<u>Three Stern-Gerlach Analyzers with Arbitrary Direction</u>**
A beam of spin-$\frac{1}{2}$ particles is sent through a series of three S-G analyzers, as shown in the figure. The second S-G analyzer is aligned along the $\hat{n}$-direction.



(a) **Find the probability that particles transmitted through the first S-G analyzer are measured to have spin down at the third S-G analyzer.**
This would be the probability of the particles going through the first two analyzers multiplied by the probability of those particles going through the third analyzer.

$$\begin{aligned}
|\langle+|+\rangle_n|^2 \left|{}_n\langle+|-\rangle\right|^2 &= \left|\langle+|\left(\cos\left(\frac{\theta}{2}\right)|+\rangle + \sin\left(\frac{\theta}{2}\right)e^{i\phi}|-\rangle\right)\right|^2 \left|\left(\cos\left(\frac{\theta}{2}\right)\langle+| + \sin\left(\frac{\theta}{2}\right)e^{-i\phi}\langle-|\right)|-\rangle\right|^2 \\
&= \left|\cos\left(\frac{\theta}{2}\right)\right|^2 \left|\sin\left(\frac{\theta}{2}\right)e^{-i\phi}\right|^2 \\
&= \cos^2\left(\frac{\theta}{2}\right)\sin^2\left(\frac{\theta}{2}\right) \\
&= \frac{\sin^2\theta}{4}
\end{aligned}$$

(b) **How must the angle θ of the second S-G analyzer be oriented so as to maximize the probability that particles are meaured to have spin down at the third S-G analyzer? What is this maximum fraction?**
Since the maximum of $\sin\theta$ is 1, we mush find a θ which will set the probability to $\frac{1}{4}$:

$$\begin{aligned}
\frac{\sin^2\theta}{4} &= \frac{1}{4} \\
\sin^2\theta &= 1 \\
\theta &= \frac{\pi}{2}
\end{aligned}$$

In other words, the second analyzer must be oriented along the xy-plane, and the maximum probability is $\frac{1}{4}$.

4. **State Tomography**
It is known that there is a 90% probability of obtaining $S_z = \frac{\hbar}{2}$ if a measurement of S_z is carried out on a spin-$\frac{1}{2}$ particle. In addition, it is known that there is a 20% probability of obtaining $S_y = \frac{\hbar}{2}$ if a measurement of S_y is carried out. Determine the spin state of a particle as completely as possible from this information. What is the probability of obtaining $S_x = -\frac{\hbar}{2}$ if a measurement of S_x is carried out?

From the first criterion:

$$|\langle + | \psi \rangle|^2 = \tfrac{9}{10}$$
$$|\langle + | \psi \rangle| = \tfrac{3}{\sqrt{10}}$$

If $|\psi\rangle = a\,|+\rangle + b\,|-\rangle$:

$$|\langle +| \left(a\,|+\rangle + b\,|-\rangle\right)| = \tfrac{3}{\sqrt{10}}$$
$$|a| = \tfrac{3}{\sqrt{10}}$$
$$a = \frac{3}{\sqrt{10}} e^{i\alpha}$$

And from the second criterion:

$$\left|{}_y\langle + | \psi \rangle\right|^2 = \frac{1}{5}$$
$$\left|{}_y\langle + | \psi \rangle\right|^2 = \frac{1}{5}$$
$$\left|\tfrac{1}{\sqrt{2}}(\langle +| - i\,\langle -|)(a\,|+\rangle + b\,|-\rangle)\right|^2 = \frac{1}{5}$$
$$\left|\tfrac{1}{\sqrt{2}}(a - bi)\right|^2 = \frac{1}{5}$$
$$\frac{1}{2}(|a|^2 + |b|^2 + ab^* i - a^* bi) = \frac{1}{5}$$
$$1 + ab^* i - a^* bi = \frac{2}{5}$$
$$a|b|e^{-i\beta} i - a^*|b|e^{i\beta} i = -\frac{3}{5}$$
$$|b|\left(ae^{-i\beta} - a^* e^{i\beta}\right) = \frac{3i}{5}$$
$$|a||b|\left(e^{i\alpha}e^{-i\beta} - e^{-i\alpha}e^{i\beta}\right) = \frac{3i}{5}$$
$$|a||b|\left(e^{i(\alpha-\beta)} - e^{i(\beta-\alpha)}\right) = \frac{3i}{5}$$

Arbitrarily set $\beta = 0$:

$$|a||b|\left(e^{i\alpha} - e^{-i\alpha}\right) = \frac{3i}{5}$$
$$|b|\left(e^{i\alpha} - e^{-i\alpha}\right) = \frac{3\sqrt{10}i}{15}$$
$$b = \frac{\sqrt{10}i}{5\left(e^{i\alpha} - e^{-i\alpha}\right)}$$

So $|\psi\rangle = \frac{3}{\sqrt{10}}e^{i\alpha}|+\rangle + \frac{\sqrt{10}i}{5(e^{i\alpha}-e^{-i\alpha})}|-\rangle$.

Now, we find the probability of measuring $S_x = -\frac{\hbar}{2}$:

$$\begin{aligned} \left|{}_x\langle -|\psi\rangle\right|^2 &= \left|\frac{1}{\sqrt{2}}(\langle +| - \langle -|)\left(\frac{3}{\sqrt{10}}e^{i\alpha}|+\rangle + \frac{\sqrt{10}i}{5\left(e^{i\alpha}-e^{-i\alpha}\right)}|-\rangle\right)\right|^2 \\ &= \left|\frac{1}{\sqrt{2}}\left(\frac{3}{\sqrt{10}}e^{i\alpha} - \frac{\sqrt{10}i}{5\left(e^{i\alpha}-e^{-i\alpha}\right)}\right)\right|^2 \end{aligned}$$