PHYS 234 Assignment 1

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1. **Relationship between trigonometric functions and complex exponentials**

   (a) **Starting from the power series representation of the exponential function e^x, derive Euler's formula:**

$$\boldsymbol{e^{i\theta} = \cos\theta + i\sin\theta}$$

where $i = \sqrt{-1}$.

$$\boldsymbol{e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad \cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}, \quad \sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}}$$

To start, it could be useful to write out the infinite sums in full:

$$\begin{aligned} e^x &= \sum_{n=0}^{\infty} \frac{x^n}{n!} = \frac{x^0}{0!} + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \frac{x^8}{8!} + \frac{x^9}{9!} + \dots \\ &= 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \frac{x^8}{8!} + \frac{x^9}{9!} + \dots \end{aligned}$$

$$\begin{aligned} \cos x &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = \frac{(-1)^0 x^0}{0!} + \frac{(-1)^1 x^2}{2!} + \frac{(-1)^2 x^4}{4!} + \frac{(-1)^3 x^6}{6!} + \frac{(-1)^4 x^8}{8!} + \dots \\ &= 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \dots \end{aligned}$$

$$\begin{aligned} \sin x &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = \frac{(-1)^0 x^1}{1!} + \frac{(-1)^1 x^3}{3!} + \frac{(-1)^2 x^5}{5!} + \frac{(-1)^3 x^7}{7!} + \frac{(-1)^4 x^9}{9!} + \dots \\ &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \dots \end{aligned}$$

When written out like this, it is easy to see that the expansion of $\cos x$ is extremely similar to that of $\sin x$, but $\cos x$ contains all the even terms, while $\sin x$ contains all the odd terms. Another observation to make is that the expansion of e^x is astoundingly similar to $\cos x + \sin x$:

$$\cos x + \sin x = 1 + x - \frac{x^2}{2} - \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} - \frac{x^6}{6!} - \frac{x^7}{7!} + \frac{x^8}{8!} + \frac{x^9}{9!} - \dots$$

The only difference is the signs on some of the terms. The pattern appears to be $+ + - - + + - - + + - - \dots$, which is suspiciously similar to the signs of the powers of i:

$$i^0 = 1,\ i^1 = i,\ i^2 = -1,\ i^3 = -i,\ i^4 = 1,\ i^5 = i,\ i^6 = -1,\ \dots$$

The thought comes to mind: what if instead of e^x, we wrote out e^{ix} instead? Then, the x^n portion of that infinite sum would lead to the same sign pattern as in $\cos x + \sin x$.

$$\begin{aligned} e^{ix} &= 1 + ix + \frac{(ix)^2}{2} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \frac{(ix)^5}{5!} + \frac{(ix)^6}{6!} + \frac{(ix)^7}{7!} + \frac{(ix)^8}{8!} + \frac{(ix)^9}{9!} - \dots \\ &= 1 + ix - \frac{x^2}{2} - \frac{ix^3}{3!} + \frac{x^4}{4!} + \frac{ix^5}{5!} - \frac{x^6}{6!} - \frac{ix^7}{7!} + \frac{x^8}{8!} + \frac{ix^9}{9!} - \dots \end{aligned}$$

Now the only difference between e^{ix} and $\cos x + \sin x$ is a factor of i on the odd terms. But, as we saw earlier, the odd terms come from $\sin x$! If we instead write the expansion of $\cos x + i \sin x$ (multiplying $\sin x$ by i), we get the two expressions to exactly match. This leads us to writing the formula

$$e^{ix} = \cos x + i \sin x.$$

(b) Use Euler's Formula to express the trigonometric functions $\cos\theta$ and $\sin\theta$ in terms of the complex exponential functions $e^{\pm i\theta}$.

What happens to Euler's formula when we use e^{-ix} instead of e^{ix}?

$$\begin{aligned} e^{-ix} &= \cos(-x) + i \sin(-x) \\ &= \cos x - i \sin x \end{aligned}$$

Notice that the sign of $\sin x$ changes but the sign of $\cos x$ does not. Naturally, this means that

$$e^{ix} + e^{-ix} = 2\cos x,$$

and

$$e^{ix} - e^{-ix} = 2i \sin x.$$

Solving each of these for $\sin x$ and $\cos x$, we get

$$\cos x = \frac{e^{ix} + e^{-ix}}{2}$$

and

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i}.$$

2. **Calculations using quantum states**

$$\begin{aligned}
|\psi_1\rangle &= 3\,|+\rangle - i\,|-\rangle \\
|\psi_2\rangle &= e^{i\pi/3}\,|+\rangle + |-\rangle \\
|\psi_3\rangle &= 7i\,|+\rangle - 2\,|-\rangle
\end{aligned}$$

(a) **For each of the states $|\psi_j\rangle$ above ($j = 1, 2, 3$), find the corresponding normalized state $|\psi_j\rangle_\mathrm{N}$.**

For $|\psi_1\rangle$:

$$\begin{aligned}
\langle C\psi_1|C\psi_1\rangle &= 1 \\
1 &= C^*(3\,\langle +| + i\,\langle -|) \cdot C(3\,|+\rangle - i\,|-\rangle) \\
&= CC^*(9\,\langle +|+\rangle - 3i\,\langle +|-\rangle + 3i\,\langle -|+\rangle - i^2\,\langle -|-\rangle) \\
&= CC^*(9+1) \\
|C|^2 &= \frac{1}{10} \\
C &= \frac{1}{\sqrt{10}}
\end{aligned}$$

Therefore, $|\psi_1\rangle_\mathrm{N} = \frac{1}{\sqrt{10}}\,|\psi_1\rangle = \frac{3}{\sqrt{10}}\,|+\rangle - \frac{i}{\sqrt{10}}\,|-\rangle$.

For $|\psi_2\rangle$:

$$\begin{aligned}
\langle C\psi_2|C\psi_2\rangle &= 1 \\
1 &= C^*\left(e^{-i\pi/3}\,\langle +| + \langle -|\right) \cdot C\left(e^{i\pi/3}\,|+\rangle + |-\rangle\right) \\
&= CC^*\left(\langle +|+\rangle + e^{-i\pi/3}\,\langle +|-\rangle + e^{i\pi/3}\,\langle -|+\rangle + \langle -|-\rangle\right) \\
&= CC^*(1+1) \\
|C|^2 &= \frac{1}{2} \\
C &= \frac{1}{\sqrt{2}}
\end{aligned}$$

Therefore, $|\psi_2\rangle_\mathrm{N} = \frac{1}{\sqrt{2}}\,|\psi_2\rangle = \frac{1}{\sqrt{2}}e^{i\pi/3}\,|+\rangle + \frac{1}{\sqrt{2}}\,|-\rangle$.

For $|\psi_3\rangle$:

$$\begin{aligned}
\langle C\psi_3|C\psi_3\rangle &= 1 \\
1 &= C^*(-7i\,\langle +| - 2\,\langle -|) \cdot C(7i\,|+\rangle - 2\,|-\rangle) \\
&= CC^*(-49i^2\,\langle +|+\rangle + 14i\,\langle +|-\rangle - 14i\,\langle -|+\rangle + 4\,\langle -|-\rangle) \\
&= CC^*(49+4) \\
|C|^2 &= \frac{1}{53} \\
C &= \frac{1}{\sqrt{53}}
\end{aligned}$$

Therefore, $|\psi_3\rangle_\mathrm{N} = \frac{1}{\sqrt{53}}\,|\psi_3\rangle = \frac{7i}{\sqrt{53}}\,|+\rangle - \frac{2}{\sqrt{53}}\,|-\rangle$.

(b) Using the bra-ket notation, calculate all 9 inner products ${}_{\mathrm{N}}\langle\psi_i|\psi_j\rangle_{\mathrm{N}}$ for $i = 1, 2, 3$ and $j = 1, 2, 3$ using the normalized states.

$i = 1,\ j = 1$:

$$ {}_{\mathrm{N}}\langle\psi_1|\psi_1\rangle_{\mathrm{N}} = 1 \quad \text{(by definition)} $$

$i = 1,\ j = 2$:

$$
\begin{aligned}
{}_{\mathrm{N}}\langle\psi_1|\psi_2\rangle_{\mathrm{N}} &= \left(\tfrac{1}{\sqrt{10}}\langle\psi_1|\right)\left(\tfrac{1}{\sqrt{2}}|\psi_2\rangle\right) \\
&= \tfrac{1}{\sqrt{20}}\langle\psi_1|\psi_2\rangle \\
&= \tfrac{1}{\sqrt{20}}(3\langle+| + i\langle-|)\left(e^{i\pi/3}|+\rangle + |-\rangle\right) \\
&= \frac{1}{\sqrt{20}}\left(3e^{i\pi/3}\langle+|+\rangle + 3\langle+|-\rangle + ie^{i\pi/3}\langle-|+\rangle + i\langle-|-\rangle\right) \\
&= \frac{1}{\sqrt{20}}\left(3e^{i\pi/3} + i\right)
\end{aligned}
$$

$i = 1,\ j = 3$:

$$
\begin{aligned}
{}_{\mathrm{N}}\langle\psi_1|\psi_3\rangle_{\mathrm{N}} &= \left(\tfrac{1}{\sqrt{10}}\langle\psi_1|\right)\left(\tfrac{1}{\sqrt{53}}|\psi_3\rangle\right) \\
&= \tfrac{1}{\sqrt{530}}\langle\psi_1|\psi_3\rangle \\
&= \tfrac{1}{\sqrt{530}}(3\langle+| + i\langle-|)(7i|+\rangle - 2|-\rangle) \\
&= \tfrac{1}{\sqrt{530}}(21i\langle+|-\rangle - 6\langle+|-\rangle + 7i^2\langle-|+\rangle - 2i\langle-|-\rangle) \\
&= \tfrac{1}{\sqrt{530}}(21i - 2i) \\
&= \frac{19i}{\sqrt{530}}
\end{aligned}
$$

$i = 2,\ j = 1$:

$$
\begin{aligned}
{}_{\mathrm{N}}\langle\psi_2|\psi_1\rangle_{\mathrm{N}} &= {}^{*}_{\mathrm{N}}\langle\psi_1|\psi_2\rangle^{*}_{\mathrm{N}} \\
&= \mathrm{conj}\left(\frac{1}{\sqrt{20}}\left(3e^{i\pi/3} + i\right)\right) \\
&= \frac{1}{\sqrt{20}}\left(3e^{-i\pi/3} - i\right)
\end{aligned}
$$

$i = 2,\ j = 2$:

$$ {}_{\mathrm{N}}\langle\psi_2|\psi_2\rangle_{\mathrm{N}} = 1 \quad \text{(by definition)} $$

$i = 2,\ j = 3$:

$$
\begin{aligned}
{}_{\mathrm{N}}\langle\psi_2|\psi_3\rangle_{\mathrm{N}} &= \left(\tfrac{1}{\sqrt{2}}\langle\psi_2|\right)\left(\tfrac{1}{\sqrt{53}}|\psi_3\rangle\right) \\
&= \tfrac{1}{\sqrt{106}}\langle\psi_2|\psi_3\rangle \\
&= \frac{1}{\sqrt{106}}\left(e^{-i\pi/3}\langle+| + \langle-|\right)(7i|+\rangle - 2|-\rangle) \\
&= \frac{1}{\sqrt{106}}\left(7ie^{-i\pi/3}\langle+|+\rangle - 2e^{-i\pi/3}\langle+|-\rangle + 7i\langle-|+\rangle - 2\langle-|-\rangle\right) \\
&= \frac{1}{\sqrt{106}}\left(7e^{i\pi/6} - 2\right)
\end{aligned}
$$

$i = 3,\ j = 1$:

$$\begin{aligned}
{}_{\text{N}}\langle\psi_3|\psi_1\rangle_{\text{N}} &= {}^*_{\text{N}}\langle\psi_1|\psi_3\rangle^*_{\text{N}} \\
&= \text{conj}\left(\tfrac{19i}{\sqrt{530}}\right) \\
&= -\frac{19i}{\sqrt{530}}
\end{aligned}$$

$i = 3,\ j = 2$:

$$\begin{aligned}
{}_{\text{N}}\langle\psi_3|\psi_2\rangle_{\text{N}} &= {}^*_{\text{N}}\langle\psi_2|\psi_3\rangle^*_{\text{N}} \\
&= \text{conj}\left(\frac{1}{\sqrt{106}}\left(7e^{i\pi/6} - 2\right)\right) \\
&= \frac{1}{\sqrt{106}}\left(7e^{-i\pi/6} - 2\right)
\end{aligned}$$

$i = 3,\ j = 3$:

$$ {}_{\text{N}}\langle\psi_3|\psi_3\rangle_{\text{N}} = 1 \quad \text{(by definition)}$$

(c) For each state $|\psi_i\rangle$, find the state $|\phi_i\rangle$ with unit norm, $\langle\phi_i|\phi_i\rangle = 1$ that is orthogonal to it. Recall the orthogonality conditions for the basis states: $\langle +|+\rangle = \langle -|-\rangle = 1$ and $\langle +|-\rangle = \langle -|+\rangle = 0$.

If $|\psi_i\rangle$ and $|\phi_i\rangle$ are to be orthogonal, they must satisfy the orthogonality condition:

$$\langle\phi_i|\psi_i\rangle = 0$$

Let's test this out with $|\psi_1\rangle$ and $|\phi_1\rangle$ to see if it works. First, we define $|\phi_1\rangle$ to be some linear combination of the basis states:

$$|\phi_1\rangle = a\,|+\rangle + b\,|-\rangle$$

Then, we apply the orthogonality condition.

$$\begin{aligned}
\langle\phi_1|\psi_1\rangle = 0 &= (a^*\,\langle +| + b^*\,\langle -|)(3\,|+\rangle - i\,|-\rangle) \\
&= 3a^*\,\langle +|+\rangle - b^* i\,\langle -|-\rangle \\
a^* &= \frac{1}{3}b^* i \\
a &= -\frac{1}{3}bi
\end{aligned}$$

If this is correct, I should be able to pick any pair of a and b which satisfy this equation, and the $|\phi_1\rangle$ they make should be orthogonal to $|\psi_1\rangle$. I will randomly pick $a = 1$ and $b = 3i$, so

$$|\phi_1\rangle = |+\rangle + 3i\,|-\rangle\,.$$

Now, we verify that this is orthogonal to $|\psi_1\rangle$:

$$\begin{aligned}
0 &\stackrel{?}{=} \langle\phi_1|\psi_1\rangle \\
&\stackrel{?}{=} (\langle +| - 3i\,\langle -|)(3\,|+\rangle - i\,|-\rangle) \\
&\stackrel{?}{=} 3\,\langle +|+\rangle + 3i^2\,\langle -|-\rangle \\
&\stackrel{?}{=} 3 - 3 \\
&= 0
\end{aligned}$$

Great! Now all that's left to do is normalize $|\phi_1\rangle$ and we're done.

$$\begin{aligned}\langle C\phi_1|C\phi_1\rangle &= 1 \\ 1 &= C^*(\langle +| - 3i\langle -|) \cdot C(|+\rangle + 3i|-\rangle) \\ &= CC^*(\langle +|+\rangle - 9i^2\langle -|-\rangle) \\ &= |C|^2(1+9) \\ |C|^2 &= \frac{1}{10} \\ C &= \frac{1}{\sqrt{10}}\end{aligned}$$

$$|\phi_1\rangle_{\mathrm{N}} = \tfrac{1}{\sqrt{10}}|+\rangle + \tfrac{3i}{\sqrt{10}}|-\rangle$$

Now I will repeat the process for finding $|\phi_2\rangle$ and $|\phi_3\rangle$. For $|\phi_2\rangle$:

$$|\phi_2\rangle = a_2|+\rangle + b_2|-\rangle$$

Applying the orthogonality condition:

$$\begin{aligned}\langle\phi_2|\psi_2\rangle = 0 &= (a_2^*\langle +| + b_2^*\langle -|)\left(e^{i\pi/3}|+\rangle + |-\rangle\right) \\ &= a_2^* e^{i\pi/3}\langle +|+\rangle + b_2^*\langle -|-\rangle \\ a_2^* &= -e^{-i\pi/3}b_2^* \\ a_2 &= -e^{i\pi/3}b_2\end{aligned}$$

Randomly picking $a_2 = -1$ and $b_2 = e^{-i\pi/3}$:

$$|\phi_2\rangle = -|+\rangle + e^{-i\pi/3}|-\rangle$$

Normalizing:

$$\begin{aligned}\langle C_2\phi_2|C_2\phi_2\rangle &= 1 \\ 1 &= C_2^*\left(-\langle +| + e^{i\pi/3}\langle -|\right) \cdot C_2\left(-|+\rangle + e^{-i\pi/3}|-\rangle\right) \\ &= C_2C_2^*(\langle +|+\rangle + 0\langle -|-\rangle) \\ |C_2|^2 &= 1 \\ C_2 &= 1\end{aligned}$$

$$|\phi_2\rangle_{\mathrm{N}} = -|+\rangle + e^{-i\pi/3}|-\rangle$$

Finally, finding $|\phi_3\rangle$:

$$|\phi_3\rangle = a_3|+\rangle + b_3|-\rangle$$

Applying the orthogonality condition:

$$\begin{aligned}\langle\phi_3|\psi_3\rangle = 0 &= (a_3^*\langle +| + b_3^*\langle -|)(7i|+\rangle - 2|-\rangle) \\ &= 7a_3^* i\langle +|+\rangle - 2b_3^*\langle -|-\rangle \\ a_3^* &= \frac{2b_3^*}{7i} \\ &= -\tfrac{2}{7}b_3^* i \\ a_3 &= \tfrac{2}{7}b_3 i\end{aligned}$$

Randomly picking $a_3 = 2$ and $b_3 = -7i$:

$$|\phi_3\rangle = 2\,|+\rangle - 7i\,|-\rangle$$

Normalizing:

$$\begin{aligned}
\langle C_3\phi_3|C_3\phi_3\rangle &= 1 \\
1 &= C_3^*(2\,\langle +| + 7i\,\langle -|) \cdot C_3(2\,|+\rangle - 7i\,|-\rangle) \\
&= C_3C_3^*(4\,\langle +|+\rangle - 49i^2\,\langle -|-\rangle) \\
|C_3|^2 &= \frac{1}{53} \\
C_3 &= \frac{1}{\sqrt{53}} \\
|\phi_3\rangle_\mathrm{N} &= \tfrac{2}{\sqrt{53}}\,|+\rangle - \tfrac{7i}{\sqrt{53}}\,|-\rangle
\end{aligned}$$

(d) Postulate 4 of quantum mechanics tells us that the complex square of the inner product $|\langle a|b\rangle|^2$ is the probability of measuring a particular quantum state. For each of the normalized states $|\psi_i\rangle_\mathrm{N}$, calculate the probability of measuring each of the six states indicated below.

$$\boldsymbol{|1\rangle = |+\rangle}$$

With $|\psi_1\rangle_\mathrm{N}$:

$$\begin{aligned}
|\langle 1|\psi_1\rangle_\mathrm{N}|^2 &= \left|\langle +| \cdot \tfrac{1}{\sqrt{10}}(3\,|+\rangle - i\,|-\rangle)\right|^2 \\
&= \left|\tfrac{3}{\sqrt{10}}\,\langle +|+\rangle - \tfrac{i}{\sqrt{10}}\,\langle +|-\rangle\right|^2 \\
&= \left|\tfrac{3}{\sqrt{10}}\right|^2 \\
&= \frac{9}{10}
\end{aligned}$$

With $|\psi_2\rangle_\mathrm{N}$:

$$\begin{aligned}
|\langle 1|\psi_2\rangle_\mathrm{N}|^2 &= \left|\langle +| \cdot \tfrac{1}{\sqrt{2}}\left(e^{i\pi/3}\,|+\rangle + |-\rangle\right)\right|^2 \\
&= \left|\tfrac{1}{\sqrt{2}}e^{i\pi/3}\,\langle +|+\rangle - \tfrac{1}{\sqrt{2}}\,\langle +|-\rangle\right|^2 \\
&= \left|\tfrac{1}{\sqrt{2}}e^{i\pi/3}\right|^2 \\
&= \frac{1}{2}
\end{aligned}$$

With $|\psi_3\rangle_\mathrm{N}$:

$$\begin{aligned}
|\langle 1|\psi_3\rangle_\mathrm{N}|^2 &= \left|\langle +| \cdot \tfrac{1}{\sqrt{53}}(7i\,|+\rangle - 2\,|-\rangle)\right|^2 \\
&= \left|\tfrac{7i}{\sqrt{53}}\,\langle +|+\rangle - \tfrac{2}{\sqrt{53}}\,\langle +|-\rangle\right|^2 \\
&= \left|\tfrac{7i}{\sqrt{53}}\right|^2 \\
&= \frac{49}{53}
\end{aligned}$$

$|2\rangle = |-\rangle$

With $|\psi_1\rangle_{\mathrm{N}}$:

$$\begin{aligned}|\langle 2|\psi_1\rangle_{\mathrm{N}}|^2 &= \left|\langle -| \cdot \tfrac{1}{\sqrt{10}}(3\,|+\rangle - i\,|-\rangle)\right|^2 \\ &= \left|\tfrac{3}{\sqrt{10}}\,\langle -|+\rangle - \tfrac{i}{\sqrt{10}}\,\langle -|-\rangle\right|^2 \\ &= \left|-\tfrac{i}{\sqrt{10}}\right|^2 \\ &= \frac{1}{10}\end{aligned}$$

With $|\psi_2\rangle_{\mathrm{N}}$:

$$\begin{aligned}|\langle 2|\psi_2\rangle_{\mathrm{N}}|^2 &= \left|\langle -| \cdot \tfrac{1}{\sqrt{2}}\left(e^{i\pi/3}\,|+\rangle + |-\rangle\right)\right|^2 \\ &= \left|\tfrac{1}{\sqrt{2}}e^{i\pi/3}\,\langle -|+\rangle - \tfrac{1}{\sqrt{2}}\,\langle -|-\rangle\right|^2 \\ &= \left|-\tfrac{1}{\sqrt{2}}\right|^2 \\ &= \frac{1}{2}\end{aligned}$$

With $|\psi_3\rangle_{\mathrm{N}}$:

$$\begin{aligned}|\langle 2|\psi_3\rangle_{\mathrm{N}}|^2 &= \left|\langle -| \cdot \tfrac{1}{\sqrt{53}}(7i\,|+\rangle - 2\,|-\rangle)\right|^2 \\ &= \left|\tfrac{7i}{\sqrt{53}}\,\langle -|+\rangle - \tfrac{2}{\sqrt{53}}\,\langle -|-\rangle\right|^2 \\ &= \left|-\tfrac{2}{\sqrt{53}}\right|^2 \\ &= \frac{4}{53}\end{aligned}$$

$|3\rangle = \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle)$

With $|\psi_1\rangle_{\mathrm{N}}$:

$$\begin{aligned}|\langle 3|\psi_1\rangle_{\mathrm{N}}|^2 &= \left|\tfrac{1}{\sqrt{2}}(\langle +| + \langle -|) \cdot \tfrac{1}{\sqrt{10}}(3\,|+\rangle - i\,|-\rangle)\right|^2 \\ &= \left|\tfrac{1}{\sqrt{20}}(3\,\langle +|+\rangle - i\,\langle -|-\rangle)\right|^2 \\ &= \left|\tfrac{1}{\sqrt{20}}(3 - i)\right|^2 \\ &= \frac{10}{20} \\ &= \frac{1}{2}\end{aligned}$$

With $|\psi_2\rangle_\mathrm{N}$:

$$\begin{aligned}
|\langle 3|\psi_2\rangle_\mathrm{N}|^2 &= \left|\frac{1}{\sqrt{2}}(\langle +| + \langle -|) \cdot \frac{1}{\sqrt{2}}\left(e^{i\pi/3}\,|+\rangle + |-\rangle\right)\right|^2 \\
&= \left|\frac{1}{4}\left(e^{i\pi/3}\,\langle +|+\rangle + \langle -|-\rangle\right)\right|^2 \\
&= \left|\frac{1}{4}\left(e^{i\pi/3} + 1\right)\right|^2 \\
&= \frac{1}{16}\left(e^{i\pi/3} + 1\right)\left(e^{-i\pi/3} + 1\right) \\
&= \frac{1}{16}\left(1 + e^{i\pi/3} + e^{-i\pi/3} + 1\right) \\
&= \frac{1}{16}\left(2 + 2\cos\tfrac{\pi}{3}\right) \\
&= \frac{3}{16}
\end{aligned}$$

With $|\psi_3\rangle_\mathrm{N}$:

$$\begin{aligned}
|\langle 3|\psi_3\rangle_\mathrm{N}|^2 &= \left|\tfrac{1}{\sqrt{2}}(\langle +| + \langle -|) \cdot \tfrac{1}{\sqrt{53}}(7i\,|+\rangle - 2\,|-\rangle)\right|^2 \\
&= \left|\tfrac{1}{\sqrt{106}}(7i\,\langle +|+\rangle - 2\,\langle -|-\rangle)\right|^2 \\
&= \left|\tfrac{1}{\sqrt{106}}(7i - 2)\right|^2 \\
&= \frac{53}{106} \\
&= \frac{1}{2}
\end{aligned}$$

$|4\rangle = \frac{1}{\sqrt{2}}(|+\rangle - |-\rangle)$

With $|\psi_1\rangle_\mathrm{N}$:

$$\begin{aligned}
|\langle 4|\psi_1\rangle_\mathrm{N}|^2 &= \left|\tfrac{1}{\sqrt{2}}(\langle +| - \langle -|) \cdot \tfrac{1}{\sqrt{10}}(3\,|+\rangle - i\,|-\rangle)\right|^2 \\
&= \left|\tfrac{1}{\sqrt{20}}(3\,\langle +|+\rangle + i\,\langle -|-\rangle)\right|^2 \\
&= \left|\tfrac{1}{\sqrt{20}}(3 + i)\right|^2 \\
&= \frac{10}{20} \\
&= \frac{1}{2}
\end{aligned}$$

With $|\psi_2\rangle_{\mathrm{N}}$:

$$\begin{aligned}
|\langle 4|\psi_2\rangle_{\mathrm{N}}|^2 &= \left|\frac{1}{\sqrt{2}}(\langle +| - \langle -|) \cdot \frac{1}{\sqrt{2}}\left(e^{i\pi/3}|+\rangle + |-\rangle\right)\right|^2 \\
&= \left|\frac{1}{4}\left(e^{i\pi/3}\langle +|+\rangle - \langle -|-\rangle\right)\right|^2 \\
&= \left|\frac{1}{4}\left(e^{i\pi/3} - 1\right)\right|^2 \\
&= \frac{1}{16}\left(e^{i\pi/3} - 1\right)\left(e^{-i\pi/3} - 1\right) \\
&= \frac{1}{16}\left(1 - e^{i\pi/3} - e^{-i\pi/3} + 1\right) \\
&= \frac{1}{16}\left(2 - 2\cos\tfrac{\pi}{3}\right) \\
&= \frac{1}{16}
\end{aligned}$$

With $|\psi_3\rangle_{\mathrm{N}}$:

$$\begin{aligned}
|\langle 4|\psi_3\rangle_{\mathrm{N}}|^2 &= \left|\tfrac{1}{\sqrt{2}}(\langle +| - \langle -|) \cdot \tfrac{1}{\sqrt{53}}(7i|+\rangle - 2|-\rangle)\right|^2 \\
&= \left|\tfrac{1}{\sqrt{106}}(7i\langle +|+\rangle + 2\langle -|-\rangle)\right|^2 \\
&= \left|\tfrac{1}{\sqrt{106}}(7i + 2)\right|^2 \\
&= \frac{53}{106} \\
&= \frac{1}{2}
\end{aligned}$$

$|5\rangle = \frac{1}{\sqrt{2}}(|+\rangle + i|-\rangle)$

With $|\psi_1\rangle_{\mathrm{N}}$:

$$\begin{aligned}
|\langle 5|\psi_1\rangle_{\mathrm{N}}|^2 &= \left|\tfrac{1}{\sqrt{2}}(\langle +| + i\langle -|) \cdot \tfrac{1}{\sqrt{10}}(3|+\rangle - i|-\rangle)\right|^2 \\
&= \left|\tfrac{1}{\sqrt{20}}(3\langle +|+\rangle - i^2\langle -|-\rangle)\right|^2 \\
&= \left|\tfrac{1}{\sqrt{20}}(3 + 1)\right|^2 \\
&= \frac{16}{20} \\
&= \frac{4}{5}
\end{aligned}$$

With $|\psi_2\rangle_N$:

$$
\begin{aligned}
|\langle 5|\psi_2\rangle_N|^2 &= \left|\frac{1}{\sqrt{2}}(\langle +| + i\,\langle -|) \cdot \frac{1}{\sqrt{2}}\left(e^{i\pi/3}\,|+\rangle + |-\rangle\right)\right|^2 \\
&= \left|\frac{1}{4}\left(e^{i\pi/3}\,\langle +|+\rangle + i\,\langle -|-\rangle\right)\right|^2 \\
&= \left|\frac{1}{4}\left(e^{i\pi/3} + i\right)\right|^2 \\
&= \frac{1}{16}\left(e^{i\pi/3} + i\right)\left(e^{-i\pi/3} - i\right) \\
&= \frac{1}{16}\left(1 - ie^{i\pi/3} + ie^{-i\pi/3} - i^2\right) \\
&= \frac{1}{16}\left(2 - i(e^{i\pi/3} - ie^{-i\pi/3})\right) \\
&= \frac{1}{16}\left(2 - i\left(2i\sin\tfrac{\pi}{3}\right)\right) \\
&= \frac{1}{16}\left(2 + \sqrt{3}\right) \\
&= \frac{2+\sqrt{3}}{16}
\end{aligned}
$$

With $|\psi_3\rangle_N$:

$$
\begin{aligned}
|\langle 5|\psi_3\rangle_N|^2 &= \left|\tfrac{1}{\sqrt{2}}(\langle +| + i\,\langle -|) \cdot \tfrac{1}{\sqrt{53}}(7i\,|+\rangle - 2\,|-\rangle)\right|^2 \\
&= \left|\tfrac{1}{\sqrt{106}}(7i\,\langle +|+\rangle - 2i\,\langle -|-\rangle)\right|^2 \\
&= \left|\tfrac{1}{\sqrt{106}}(5i)\right|^2 \\
&= \frac{25}{106} \\
&= \frac{1}{2}
\end{aligned}
$$

$|6\rangle = \frac{1}{\sqrt{2}}(|+\rangle - i\,|-\rangle)$

With $|\psi_1\rangle_N$:

$$
\begin{aligned}
|\langle 6|\psi_1\rangle_N|^2 &= \left|\tfrac{1}{\sqrt{2}}(\langle +| - i\,\langle -|) \cdot \tfrac{1}{\sqrt{10}}(3\,|+\rangle - i\,|-\rangle)\right|^2 \\
&= \left|\tfrac{1}{\sqrt{20}}(3\,\langle +|+\rangle + i^2\,\langle -|-\rangle)\right|^2 \\
&= \left|\tfrac{1}{\sqrt{20}}(3 - 1)\right|^2 \\
&= \frac{4}{20} \\
&= \frac{1}{5}
\end{aligned}
$$

With $|\psi_2\rangle_{\mathrm{N}}$:

$$\begin{aligned}
|\langle 6|\psi_2\rangle_{\mathrm{N}}|^2 &= \left|\frac{1}{\sqrt{2}}(\langle +| - i\,\langle -|) \cdot \frac{1}{\sqrt{2}}\left(e^{i\pi/3}\,|+\rangle + |-\rangle\right)\right|^2 \\
&= \left|\frac{1}{4}\left(e^{i\pi/3}\,\langle +|+\rangle - i\,\langle -|-\rangle\right)\right|^2 \\
&= \left|\frac{1}{4}\left(e^{i\pi/3} - i\right)\right|^2 \\
&= \frac{1}{16}\left(e^{i\pi/3} - i\right)\left(e^{-i\pi/3} + i\right) \\
&= \frac{1}{16}\left(1 + ie^{i\pi/3} - ie^{-i\pi/3} - i^2\right) \\
&= \frac{1}{16}\left(2 + i(e^{i\pi/3} - ie^{-i\pi/3})\right) \\
&= \frac{1}{16}\left(2 + i\left(2i\sin\tfrac{\pi}{3}\right)\right) \\
&= \frac{1}{16}\left(2 - \sqrt{3}\right) \\
&= \frac{2 - \sqrt{3}}{16}
\end{aligned}$$

With $|\psi_3\rangle_{\mathrm{N}}$:

$$\begin{aligned}
|\langle 6|\psi_3\rangle_{\mathrm{N}}|^2 &= \left|\tfrac{1}{\sqrt{2}}(\langle +| - i\,\langle -|) \cdot \tfrac{1}{\sqrt{53}}(7i\,|+\rangle - 2\,|-\rangle)\right|^2 \\
&= \left|\tfrac{1}{\sqrt{106}}(7i\,\langle +|+\rangle + 2i\,\langle -|-\rangle)\right|^2 \\
&= \left|\tfrac{1}{\sqrt{106}}(9i)\right|^2 \\
&= \frac{81}{106}
\end{aligned}$$

3. **<u>Eigenvalues and Eigenvectors</u>**
Find the eigenvalues and eigenvectors of the following matrices:

(a) $\begin{bmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{bmatrix}$

If the matrix is represented by A, then

$$(A - \lambda I)\mathbf{v} = 0$$

where I is the identity matrix with dimensions of A and λ represents the eigenvalues. Solving this equation:

$$\begin{aligned}
\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right)\mathbf{v} &= 0 \\
\begin{bmatrix} -\lambda & 1 \\ 1 & -\lambda \end{bmatrix}\mathbf{v} &= 0 \\
\begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} &= 0 \\
(-\lambda)(-\lambda) - (1)(1) &= 0 \\
\lambda^2 - 1 &= 0 \\
\lambda &= \pm 1
\end{aligned}$$

These are the eigenvalues. To find their associated eigenvectors, we substitute them into the original equation. For $\lambda = -1$:

$$\begin{aligned}
\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\right)\mathbf{v} &= 0 \\
\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\mathbf{v} &= 0
\end{aligned}$$

$$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & 1 & 0 \end{array}\right] \xrightarrow{R_2 - R_1} \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

If $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$, the above matrix corresponds to a solution of $v_1 + v_2 = 0$, or $v_1 = -v_2$. Therefore,

$$\mathbf{v} = \begin{bmatrix} -v_2 \\ v_2 \end{bmatrix} = v_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

and $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ is the eigenvector for $\lambda = -1$.

For $\lambda = 1$:

$$\begin{aligned}
\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\mathbf{v} &= 0 \\
\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\mathbf{v} &= 0
\end{aligned}$$

$$\left[\begin{array}{cc|c} -1 & 1 & 0 \\ 1 & -1 & 0 \end{array}\right] \xrightarrow{R_2+R_1} \left[\begin{array}{cc|c} -1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

If $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$, the above matrix corresponds to a solution of $-v_1 + v_2 = 0$, or $v_1 = v_2$. Therefore,

$$\mathbf{v} = \begin{bmatrix} v_2 \\ v_2 \end{bmatrix} = v_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is the eigenvector for $\lambda = 1$.

This same procedure will be followed for the rest of this question.

(b) $\begin{bmatrix} \mathbf{4} & \mathbf{1} \\ \mathbf{1} & \mathbf{-2} \end{bmatrix}$

To find the eigenvalues:

$$\begin{aligned}
(A - \lambda I)\mathbf{v} &= 0 \\
\left(\begin{bmatrix} 4 & 1 \\ 1 & -2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right)\mathbf{v} &= 0 \\
\begin{bmatrix} 4-\lambda & 1 \\ 1 & -2-\lambda \end{bmatrix}\mathbf{v} &= 0 \\
\begin{vmatrix} 4-\lambda & 1 \\ 1 & -2-\lambda \end{vmatrix} &= 0 \\
(4-\lambda)(-2-\lambda) - (1)(1) &= 0 \\
-8 - 4\lambda + 2\lambda + \lambda^2 - 1 &= 0 \\
\lambda^2 - 2\lambda - 9 &= 0 \\
\lambda &= 1 \pm \sqrt{10}
\end{aligned}$$

Then, finding the eigenvector for $\lambda = 1 - \sqrt{10}$:

$$\begin{aligned}
(A - \lambda I)\mathbf{v} &= 0 \\
\left(\begin{bmatrix} 4 & 1 \\ 1 & -2 \end{bmatrix} - \begin{bmatrix} 1-\sqrt{10} & 0 \\ 0 & 1-\sqrt{10} \end{bmatrix}\right)\mathbf{v} &= 0 \\
\begin{bmatrix} 3+\sqrt{10} & 1 \\ 1 & \sqrt{10}-3 \end{bmatrix}\mathbf{v} &= 0
\end{aligned}$$

$$\left[\begin{array}{cc|c} 3+\sqrt{10} & 1 & 0 \\ 1 & \sqrt{10}-3 & 0 \end{array}\right] \xrightarrow{R_1-\left(3+\sqrt{10}\right)R_2} \left[\begin{array}{cc|c} 0 & 0 & 0 \\ 1 & \sqrt{10}-3 & 0 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{cc|c} 1 & \sqrt{10}-3 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

If $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$, the above matrix corresponds to a solution of $v_1 + \left(\sqrt{10} - 3\right) v_2 = 0$, or $v_1 = \left(3 - \sqrt{10}\right) v_2$. Therefore,

$$\mathbf{v} = \begin{bmatrix} \left(3-\sqrt{10}\right) v_2 \\ v_2 \end{bmatrix} = v_2 \begin{bmatrix} 3-\sqrt{10} \\ 1 \end{bmatrix}$$

so the eigenvector for $\lambda = 1 - \sqrt{10}$ is $\begin{bmatrix} 3 - \sqrt{10} \\ 1 \end{bmatrix}$.

Finally, finding the eigenvector for $\lambda = 1 + \sqrt{10}$:

$$\begin{aligned}(A - \lambda I)\mathrm{v} &= 0 \\ \left(\begin{bmatrix} 4 & 1 \\ 1 & -2 \end{bmatrix} - \begin{bmatrix} 1+\sqrt{10} & 0 \\ 0 & 1+\sqrt{10} \end{bmatrix}\right)\mathbf{v} &= 0 \\ \begin{bmatrix} 3-\sqrt{10} & 1 \\ 1 & -3-\sqrt{10} \end{bmatrix}\mathbf{v} &= 0\end{aligned}$$

$$\left[\begin{array}{cc|c} 3-\sqrt{10} & 1 & 0 \\ 1 & -3-\sqrt{10} & 0 \end{array}\right] \xrightarrow{R_1-(3-\sqrt{10})R_2} \left[\begin{array}{cc|c} 0 & 0 & 0 \\ 1 & -3-\sqrt{10} & 0 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{cc|c} 1 & -3-\sqrt{10} & 0 \\ 0 & 0 & 0 \end{array}\right]$$

If $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$, the above matrix corresponds to a solution of $v_1 + \left(-3 - \sqrt{10}\right) v_2 = 0$, or $v_1 = \left(3 + \sqrt{10}\right) v_2$. Therefore,

$$\mathbf{v} = \begin{bmatrix} \left(3+\sqrt{10}\right) v_2 \\ v_2 \end{bmatrix} = v_2 \begin{bmatrix} 3+\sqrt{10} \\ 1 \end{bmatrix}$$

so the eigenvector for $\lambda = 1 + \sqrt{10}$ is $\begin{bmatrix} 3 + \sqrt{10} \\ 1 \end{bmatrix}$.

(c) $\begin{bmatrix} \mathbf{0} & -\boldsymbol{i} \\ \boldsymbol{i} & \mathbf{0} \end{bmatrix}$ *Sorry, incomplete.*

4. Matrix Operations

Given the following matrices:

$$\boldsymbol{A} = \begin{bmatrix} -1 & 0 & i \\ 3 & 0 & 2 \\ i & -2i & 2 \end{bmatrix} \qquad \boldsymbol{B} = \begin{bmatrix} 2 & 0 & i \\ 0 & i & 0 \\ 1 & 2 & 2 \end{bmatrix}$$

compute the following:

(a) $\boldsymbol{A + B}$

$$\begin{aligned} A + B &= \begin{bmatrix} -1 & 0 & i \\ 3 & 0 & 2 \\ i & -2i & 2 \end{bmatrix} + \begin{bmatrix} 2 & 0 & i \\ 0 & i & 0 \\ 1 & 2 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 2i \\ 3 & i & 2 \\ 1+i & 2-2i & 4 \end{bmatrix} \end{aligned}$$

(b) $\boldsymbol{AB}$

$$\begin{aligned} AB &= \begin{bmatrix} -1 & 0 & i \\ 3 & 0 & 2 \\ i & -2i & 2 \end{bmatrix} \begin{bmatrix} 2 & 0 & i \\ 0 & i & 0 \\ 1 & 2 & 2 \end{bmatrix} \\ &= \begin{bmatrix} -2+i & 2i & i \\ 8 & 4 & 4+3i \\ 2+2i & 6 & 3 \end{bmatrix} \end{aligned}$$

(c) $[A, B]$ (commutator of A and B)

The commutator of two matrices is defined as

$$[A, B] = AB - BA.$$

We already know AB, so we must find BA.

$$BA = \begin{bmatrix} 2 & 0 & i \\ 0 & i & 0 \\ 1 & 2 & 2 \end{bmatrix} \begin{bmatrix} -1 & 0 & i \\ 3 & 0 & 2 \\ i & -2i & 2 \end{bmatrix}$$
$$= \begin{bmatrix} -3 & 2 & 4i \\ 3i & 0 & 2i \\ 5+2i & -4i & 8+i \end{bmatrix}$$

Then,

$$[A, B] = AB - BA = \begin{bmatrix} -2+i & 2i & i \\ 8 & 4 & 4+3i \\ 2+2i & 6 & 3 \end{bmatrix} - \begin{bmatrix} -3 & 2 & 4i \\ 3i & 0 & 2i \\ 5+2i & -4i & 8+i \end{bmatrix}$$
$$= \begin{bmatrix} -5+i & -2+2i & -3i \\ 8-3i & 4 & 4+i \\ -3 & 6+4i & -5-i \end{bmatrix}$$

(d) A^{T} (transpose)

$$A^{\mathrm{T}} = \begin{bmatrix} -1 & 3 & i \\ 0 & 0 & -2i \\ i & 2 & 2 \end{bmatrix}$$

(e) $A^{\dagger}$ (complex transpose)

$$A^{\mathrm{T}} = \begin{bmatrix} -1 & 3 & -i \\ 0 & 0 & 2i \\ -i & 2 & 2 \end{bmatrix}$$

(f) Verify by direct calculation that $(AB)^{\mathrm{T}} = B^{\mathrm{T}}A^{\mathrm{T}}$.

First, we calculate $(AB)^{\mathrm{T}}$:

$$(AB)^{\mathrm{T}} = \begin{bmatrix} -2+i & 8 & 2+2i \\ 2i & 4 & 6 \\ i & 4+3i & 3 \end{bmatrix}$$

Next, we calculate $B^{\mathrm{T}}A^{\mathrm{T}}$:

$$B^{\mathrm{T}}A^{\mathrm{T}} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & i & 2 \\ i & 0 & 2 \end{bmatrix} \begin{bmatrix} -1 & 3 & i \\ 0 & 0 & -2i \\ i & 2 & 2 \end{bmatrix}$$
$$= \begin{bmatrix} -2+i & 8 & 2+2i \\ 2i & 4 & 6 \\ i & 4+3i & 3 \end{bmatrix}$$

We can see that they are the same, so it is true that $(AB)^{\mathrm{T}} = B^{\mathrm{T}}A^{\mathrm{T}}$.