# MATH 128 End-of-Term Assignment 1

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March 27, 2020

1. **Write out your student number and then determine the solution to each of the following initial value problems where $N_7$ and $N_8$ are the seventh and eighth digits of your student number:**

   My student number is 20845794.

   (a) $\frac{dy}{dx} = y\cos(x), \quad y(0) = e^{N_7}$

   This is a seperable differential equation. I'll solve it by separating the $y$'s from the $x$'s and integrating:

$$\begin{aligned}
\frac{dy}{dx} &= y\cos(x) \\
\frac{1}{y}\,dy &= \cos(x)\,dx \\
\int \frac{1}{y}\,dy &= \int \cos(x)\,dx \\
\ln|y| &= \sin(x) + C \\
|y| &= e^{\sin(x)+C} \\
y &= \pm e^{\sin(x)} e^{C}
\end{aligned}$$

   Then, substituting $A = \pm e^{C}$:

$$y = Ae^{\sin(x)} \tag{1}$$

   The seventh digit of my student number is $N_7 = 9$, so

$$\begin{aligned}
y(0) = e^{N_7} &= Ae^{\sin(0)} \\
e^9 &= Ae^0 \\
A &= e^9.
\end{aligned}$$

   Substituting this back into equation 1:

$$\begin{aligned}
y &= e^9 e^{\sin(x)} \\
&= e^{\sin(x)+9}
\end{aligned}$$

**(b)** $\frac{dy}{dx} + \frac{2}{x}y = x^{N_8}, \quad y(1) = 0$

This is a first-order linear differential equation in the form $y' + P(x)y = Q(x)$. First, I will rewrite the equation and find $P(x)$:

$$\begin{aligned} \frac{dy}{dx} + \frac{2}{x}y &= x^{N_8} \\ y' + \frac{2}{x}y &= x^{N_8} \\ P(x) &= \frac{2}{x} \end{aligned} \tag{2}$$

Then, the integrating factor is $I(x) = e^{\int P(x)\,dx}$:

$$\begin{aligned} I(x) &= e^{\int P(x)\,dx} \\ &= e^{\int \frac{2}{x}\,dx} \\ &= e^{2\ln|x|} \\ &= e^{\ln(x^2)} \\ &= x^2 \end{aligned}$$

Multiplying both sides of equation 2 by $I(x)$:

$$\begin{aligned} x^2y' + x^2\frac{2}{x}y &= x^2x^{N_8} \\ \frac{d}{dx}(x^2y) &= x^{2+N_8} \end{aligned}$$

The eighth digit of my student number is $N_8 = 4$:

$$\begin{aligned} \frac{d}{dx}(x^2y) &= x^{2+4} \\ &= x^6 \\ \int \frac{d}{dx}(x^2y)\,dx &= \int x^6\,dx \\ x^2y &= \frac{x^7}{7} + C \\ y &= \frac{x^5}{7} + C \end{aligned} \tag{3}$$

Next, I'm going to find $C$.

$$\begin{aligned} y(1) = 0 &= \frac{1^5}{7} + C \\ C &= -\frac{1}{7} \end{aligned}$$

Substituting $C$ back into equation 3:

$$\begin{aligned} y &= \frac{x^5}{7} + \left(-\frac{1}{7}\right) \\ &= \frac{x^5 - 1}{7} \end{aligned}$$

2. **A patient receives periodic intravenous injections of a drug. Let $y(t)$ denote the drug concentration (in $\mathrm{mg\,mL^{-1}}$) in the patient's bloodstream at time $t$ with initial concentration $y(0) = L$.**

   - **Every $T$ time units, an injection increases the concentration by a quantity $d$ $\mathrm{mg\,mL^{-1}}$—that is, $y(t)$ increases by $d$ (a jump discontinuity) at times $t = T, 2T, 3T, \ldots$.**
   - **In between doses, the drug concentration decreases exponentially, according to the differential equation $y'(t) = -ky(t)$ for some positive constant $k$.**

   **Determine $T$ (as a function of $k$, $d$, and $L$) so that immediately after each dose, the value of $y(t)$ is $L$—that is, immediately before the dose, the value is $L - d$. (This is the most frequent dosing strategy that ensures the concentration is never above $L$.)**

   First, I'll solve the differential equation which is separable:

$$\begin{aligned}
y'(t) &= -ky(t) \\
\frac{dy}{dt} &= -ky \\
\frac{1}{y}\,dy &= -k\,dt \\
\int \frac{1}{y}\,dy &= \int -k\,dt \\
\ln|y| &= -kt + C \\
|y| &= e^{-kt+C} \\
y &= \pm e^{-kt}e^{C}
\end{aligned}$$

   Substituting $A = \pm e^{C}$:

$$y = Ae^{-kt} \tag{4}$$

   Then, to find $A$, I'll use the fact that $y(0) = L$.

$$\begin{aligned}
y(0) = L &= Ae^{-k\cdot 0} \\
A &= L
\end{aligned}$$

   Substituting $A$ back into equation 4:

$$y = Le^{-kt}$$

   Now, I'll find the time $t$ at which $y = L - d$:

$$\begin{aligned}
L - d &= Le^{-kt} \\
1 - \frac{d}{L} &= e^{-kt} \\
-kt &= \ln\left(1 - \frac{d}{L}\right) \\
t &= -\frac{1}{k}\ln\left(1 - \frac{d}{L}\right)
\end{aligned}$$

   This is the amount of time after $t = 0$ at which the first injection is needed. Since the concentration of the drug decreases by the same curve each time, $T = t = -\frac{1}{k}\ln\left(1 - \frac{d}{L}\right)$.

3. **Glaciers are rivers of ice. The point at which a glacier ends is called its *terminus*. The thickness, $T$, of a glacier can be described as a function of the distance $x$ from the terminus: $T = T(x)$. That thickness function can be shown to satisfy the differential equation**

$$T\frac{dT}{dx} = \frac{\tau}{\rho g}$$

**where $\tau$ is the coefficient of friction at the bottom of the glacier, $\rho$ is the density of ice in the glacier, and $g$ is acceleration due to gravity.**

   (a) **What is the order of this differential equation?**

   The order is 1.

   (b) **Is this differential equation separable? Is it linear?**

   It is separable (in fact, it's already separated), but it is not linear as $T$ is in the same term as its derivative.

   (c) **Determine the general solution of the differential equation model.**

   The differential equation is solved as follows:

$$\begin{aligned} T\frac{dT}{dx} &= \frac{\tau}{\rho g} \\ T\,dT &= \frac{\tau}{\rho g}\,dx \\ \int T\,dT &= \int \frac{\tau}{\rho g}\,dx \\ \frac{1}{2}T^2 &= \frac{\tau}{\rho g}x + C \\ T^2 &= \frac{2\tau}{\rho g}x + C \\ T &= \sqrt{\frac{2\tau}{\rho g}x + C} \end{aligned}$$

   (d) **Given the initial condition $T(0) = 0$, determine the thickness of the glacier at a distance of 1 km from its terminus. Take $\rho = 917\,\mathrm{kg\,m^{-3}}$, $g = 9.8\,\mathrm{m\,s^{-2}}$, and $\tau = 75\,000\,\mathrm{N\,m^{-2}}$.**

   First, I'll find the value of $C$ using the inital value $T(0) = 0$:

$$\begin{aligned} T(0) = 0 &= \sqrt{\frac{2\tau}{\rho g}\cdot 0 + C} \\ C &= 0 \end{aligned}$$

   Therefore, $T = \sqrt{\frac{2\tau}{\rho g}x}$.

$$\begin{aligned} T(1000\,\mathrm{m}) &= \sqrt{\frac{2\tau}{\rho g}x} \\ &= \sqrt{\frac{2(75\,000\,\mathrm{N\,m^{-2}})}{(917\,\mathrm{kg\,m^{-3}})(9.8\,\mathrm{m\,s^{-2}})}\cdot 1000\,\mathrm{m}} \\ &= 129.196\,\mathrm{m} \end{aligned}$$

**4. Consider the initial value problem**

$$\frac{dy}{dt} = y + t, \quad y(0) = 1$$

**(a) Construct a direction (slope) field for the differential equation on a plot with $-2 \leq t \leq 2$ and $0 \leq y \leq 4$ showing slopes at all 25 lattice points. On your direction (slope) field, sketch the solution curve which satisfies the given initial value problem.**