PHYS 234 Assignment 5

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1. **Consider a spin-$\frac{1}{2}$ particle with a magnetic moment. (You should consider the following parts of the question to follow each other in time.)**

   (a) **At time $t = 0$, the observable S_x is measured, with the result $+\frac{\hbar}{2}$. What is the state vector $|\psi(t = 0)\rangle$ immediately after the measurement?**

   The vector is $|\psi(0)\rangle = \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle)$.

   (b) **Immediately after the measurement, a magnetic field $\vec{B} = B_0\hat{z}$ is applied and the particle is allowed to evolve for a time T. What is the state of the system at time $t = T$? *(What are the eigenstates of the Hamiltonian? Is the initial state (from (a)) an eigenstate of the Hamiltonian?)***

   The Hamiltonian is

$$H = \omega_0 \mathrm{S}_z,$$

   where $\omega_0 = \frac{gqB_0}{2m}$.

   Since H and S_z are proportional to each other, the eigenstates of H are the eigenstates of S_z, which we already know are $|+\rangle$ and $|-\rangle$.

   The initial state from part (a) is not an eigenstate of the Hamiltonian.

   The state of the system after a time T is

$$\begin{aligned}
|\psi(T)\rangle &= \frac{1}{\sqrt{2}}\left(e^{-iHt/\hbar}|+\rangle + e^{-iHt/\hbar}|-\rangle\right) \\
&= \frac{1}{\sqrt{2}}\left(e^{-i\omega_0 T/2}|+\rangle + e^{i\omega_0 T/2}|-\rangle\right) \\
&= \frac{1}{\sqrt{2}}e^{-i\omega_0 T/2}\left(|+\rangle + e^{i\omega_0 T}|-\rangle\right)
\end{aligned}$$

   We can ignore the overall phase factor, so

$$|\psi(T)\rangle = \frac{1}{\sqrt{2}}\left(|+\rangle + e^{i\omega_0 T}|-\rangle\right)$$

   (c) **At $t = T$, the magnetic field is very rapidly changed to $\vec{B} = B_0\hat{y}$. After another time interval T, a measurement of S_x is carried out once more. What is the probability that a value $+\frac{\hbar}{2}$ is found?**

   First, I'll transform $|\psi(T)\rangle$ to the new energy eigenbasis, which is $|+\rangle_y$ and $|-\rangle_y$, using the transformation matrix

$$U_{z\to y} = \begin{bmatrix} \langle +|+\rangle_y & \langle +|-\rangle_y \\ \langle -|+\rangle_y & \langle -|-\rangle_y \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}.$$

$$
\begin{aligned}
|\psi(T)\rangle_y &= U_{z\to y}\,|\psi(T)\rangle \\
&= \frac{1}{2}\begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}\begin{bmatrix} 1 \\ e^{i\omega_0 T} \end{bmatrix} \\
&= \frac{1}{2}\begin{bmatrix} 1 + e^{i\omega_0 T} \\ 1 - e^{i\omega_0 T} \end{bmatrix} \\
&= \frac{1}{2}\left(1 + e^{i\omega_0 T}\right)|+\rangle_y + \frac{i}{2}\left(1 - e^{i\omega_0 T}\right)|-\rangle_y
\end{aligned}
$$

After another time interval T, the state becomes

$$
\begin{aligned}
|\psi(2T)\rangle_y &= \frac{1}{2}\left(1 + e^{i\omega_0 T}\right)e^{-i\omega_0 T/2}\,|+\rangle_y + \frac{i}{2}\left(1 - e^{i\omega_0 T}\right)e^{i\omega_0 T/2}\,|-\rangle_y \\
&= e^{-i\omega_0 T/2}\left(\frac{1}{2}\left(1 + e^{i\omega_0 T}\right)|+\rangle_y + \frac{i}{2}\left(1 - e^{i\omega_0 T}\right)e^{i\omega_0 T}\,|-\rangle_y\right)
\end{aligned}
$$

Again, we can get rid of the overall phase factor:

$$
|\psi(2T)\rangle_y = \frac{1}{2}\left(1 + e^{i\omega_0 T}\right)|+\rangle_y + \frac{i}{2}\left(1 - e^{i\omega_0 T}\right)e^{i\omega_0 T}\,|-\rangle_y
$$

Now, to find the probability of measuring $|+\rangle_x$, we need to get $|+\rangle_x$ into the S_y basis using the transformation matrix

$$
U_{x\to y} = \begin{bmatrix} {}_x\langle +|+\rangle_y & {}_x\langle -|+\rangle_y \\ {}_x\langle +|-\rangle_y & {}_x\langle -|-\rangle_y \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1+i & 1-i \\ 1+i & 1-i \end{bmatrix}.
$$

$$
\begin{aligned}
|+_x\rangle_y &= U_{x\to y}\begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} 1+i & 1-i \\ 1+i & 1-i \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
&= \frac{1}{2}\begin{bmatrix} 1+i \\ 1-i \end{bmatrix}
\end{aligned}
$$

Then,

$$
\begin{aligned}
\left|{}_{y}\langle +_x | \psi(2T)\rangle_{y}\right|^2 &= \left|\frac{1}{4}\begin{bmatrix}1-i & 1+i\end{bmatrix}\begin{bmatrix}1+e^{i\omega_0 T}\\ i\left(1-e^{i\omega_0 T}\right)e^{i\omega_0 T}\end{bmatrix}\right|^2 \\
&= \frac{1}{16}\left|(1-i)\left(1+e^{i\omega_0 T}\right)+(1+i)\left(i\left(1-e^{i\omega_0 T}\right)e^{i\omega_0 T}\right)\right|^2 \\
&= \frac{1}{16}\left|1+e^{i\omega_0 T}-i-ie^{i\omega_0 T}+(1+i)\left(ie^{i\omega_0 T}-ie^{2i\omega_0 T}\right)\right|^2 \\
&= \frac{1}{16}\left|1+e^{i\omega_0 T}-i-ie^{i\omega_0 T}+ie^{i\omega_0 T}-ie^{2i\omega_0 T}-e^{i\omega_0 T}+e^{2i\omega_0 T}\right|^2 \\
&= \frac{1}{16}\left|1-i-ie^{2i\omega_0 T}+e^{2i\omega_0 T}\right|^2 \\
&= \frac{1}{16}\left|1-i+e^{2i\omega_0 T}(1-i)\right|^2 \\
&= \frac{1}{16}\left|(1-i)\left(1+e^{2i\omega_0 T}\right)\right|^2 \\
&= \frac{1}{16}(1-i)(1+i)\left(1+e^{2i\omega_0 T}\right)\left(1+e^{-2i\omega_0 T}\right) \\
&= \frac{1}{16}\cdot 2\cdot 2(1+\cos(2\omega_0 T)) \\
&= \frac{1}{4}(1+\cos(2\omega_0 T)) \\
&= \frac{\cos^2(\omega_0 T)}{2}
\end{aligned}
$$