PHYS 234 Assignment 1

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3. <u>Eigenvalues and Eigenvectors</u>
Find the eigenvalues and eigenvectors of the following matrices:

(a) $\begin{bmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{bmatrix}$

If the matrix is represented by A, then

$$(A - \lambda I)\mathbf{v} = 0$$

where I is the identity matrix with dimensions of A and λ represents the eigenvalues. Solving this equation:

$$\begin{aligned}
\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right)\mathbf{v} &= 0 \\
\begin{bmatrix} -\lambda & 1 \\ 1 & -\lambda \end{bmatrix}\mathbf{v} &= 0 \\
\begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} &= 0 \\
(-\lambda)(-\lambda) - (1)(1) &= 0 \\
\lambda^2 - 1 &= 0 \\
\lambda &= \pm 1
\end{aligned}$$

These are the eigenvalues. To find their associated eigenvectors, we substitute them into the original equation. For $\lambda = -1$:

$$\begin{aligned}
\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\right)\mathbf{v} &= 0 \\
\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\mathbf{v} &= 0
\end{aligned}$$

$$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & 1 & 0 \end{array}\right] \xrightarrow{R_2 - R_1} \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

If $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$, the above matrix corresponds to a solution of $v_1 + v_2 = 0$, or $v_1 = -v_2$. Therefore,

$$\mathbf{v} = \begin{bmatrix} -v_2 \\ v_2 \end{bmatrix} = v_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

and $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ is the eigenvector for $\lambda = -1$.

For $\lambda = 1$:

$$\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\mathbf{v} = 0$$
$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\mathbf{v} = 0$$

$$\left[\begin{array}{cc|c} -1 & 1 & 0 \\ 1 & -1 & 0 \end{array}\right] \xrightarrow{R_2+R_1} \left[\begin{array}{cc|c} -1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

If $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$, the above matrix corresponds to a solution of $-v_1 + v_2 = 0$, or $v_1 = v_2$. Therefore,

$$\mathbf{v} = \begin{bmatrix} v_2 \\ v_2 \end{bmatrix} = v_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is the eigenvector for $\lambda = 1$.

This same procedure will be followed for the rest of this question.

(b) $\begin{bmatrix} \mathbf{4} & \mathbf{1} \\ \mathbf{1} & \mathbf{-2} \end{bmatrix}$

To find the eigenvalues:

$$(A - \lambda I)\mathbf{v} = 0$$
$$\left(\begin{bmatrix} 4 & 1 \\ 1 & -2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right)\mathbf{v} = 0$$
$$\begin{bmatrix} 4-\lambda & 1 \\ 1 & -2-\lambda \end{bmatrix}\mathbf{v} = 0$$
$$\begin{vmatrix} 4-\lambda & 1 \\ 1 & -2-\lambda \end{vmatrix} = 0$$
$$(4-\lambda)(-2-\lambda) - (1)(1) = 0$$
$$-8 - 4\lambda + 2\lambda + \lambda^2 - 1 = 0$$
$$\lambda^2 - 2\lambda - 9 = 0$$
$$\lambda = 1 \pm \sqrt{10}$$

Then, finding the eigenvector for $\lambda = 1 - \sqrt{10}$:

$$(A - \lambda I)\mathbf{v} = 0$$
$$\left(\begin{bmatrix} 4 & 1 \\ 1 & -2 \end{bmatrix} - \begin{bmatrix} 1-\sqrt{10} & 0 \\ 0 & 1-\sqrt{10} \end{bmatrix}\right)\mathbf{v} = 0$$
$$\begin{bmatrix} 3+\sqrt{10} & 1 \\ 1 & \sqrt{10}-3 \end{bmatrix}\mathbf{v} = 0$$

$$\left[\begin{array}{cc|c} 3+\sqrt{10} & 1 & 0 \\ 1 & \sqrt{10}-3 & 0 \end{array}\right] \xrightarrow{R_1-\left(3+\sqrt{10}\right)R_2} \left[\begin{array}{cc|c} 0 & 0 & 0 \\ 1 & \sqrt{10}-3 & 0 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{cc|c} 1 & \sqrt{10}-3 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

If $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$, the above matrix corresponds to a solution of $v_1 + \left(\sqrt{10}-3\right) v_2 = 0$, or $v_1 = \left(3-\sqrt{10}\right) v_2$. Therefore,

$$\mathbf{v} = \begin{bmatrix} \left(3-\sqrt{10}\right) v_2 \\ v_2 \end{bmatrix} = v_2 \begin{bmatrix} 3-\sqrt{10} \\ 1 \end{bmatrix}$$

so the eigenvector for $\lambda = 1-\sqrt{10}$ is $\begin{bmatrix} 3-\sqrt{10} \\ 1 \end{bmatrix}$.

Finally, finding the eigenvector for $\lambda = 1+\sqrt{10}$:

$$(A-\lambda I)\mathrm{v} = 0$$
$$\left(\begin{bmatrix} 4 & 1 \\ 1 & -2 \end{bmatrix} - \begin{bmatrix} 1+\sqrt{10} & 0 \\ 0 & 1+\sqrt{10} \end{bmatrix}\right) \mathbf{v} = 0$$
$$\begin{bmatrix} 3-\sqrt{10} & 1 \\ 1 & -3-\sqrt{10} \end{bmatrix} \mathbf{v} = 0$$

$$\left[\begin{array}{cc|c} 3-\sqrt{10} & 1 & 0 \\ 1 & -3-\sqrt{10} & 0 \end{array}\right] \xrightarrow{R_1-\left(3-\sqrt{10}\right)R_2} \left[\begin{array}{cc|c} 0 & 0 & 0 \\ 1 & -3-\sqrt{10} & 0 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{cc|c} 1 & -3-\sqrt{10} & 0 \\ 0 & 0 & 0 \end{array}\right]$$

If $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$, the above matrix corresponds to a solution of $v_1 + \left(-3-\sqrt{10}\right) v_2 = 0$, or $v_1 = \left(3+\sqrt{10}\right) v_2$. Therefore,

$$\mathbf{v} = \begin{bmatrix} \left(3+\sqrt{10}\right) v_2 \\ v_2 \end{bmatrix} = v_2 \begin{bmatrix} 3+\sqrt{10} \\ 1 \end{bmatrix}$$

so the eigenvector for $\lambda = 1+\sqrt{10}$ is $\begin{bmatrix} 3+\sqrt{10} \\ 1 \end{bmatrix}$.

(c) $\begin{bmatrix} \mathbf{0} & -\boldsymbol{i} \\ \boldsymbol{i} & \mathbf{0} \end{bmatrix}$ *Sorry, incomplete.*