PHYS 234 Assignment 4

Brandon Tsang

June 12, 2020

1. A spin-1 particle is in the state

$$|\psi\rangle \xrightarrow{S_z \text{ basis}} \frac{1}{\sqrt{14}}\begin{bmatrix}1\\2\\3i\end{bmatrix}.$$

(a) What are the probabilities that a measurement of S_z will yield the value $\hbar$, 0, or $-\hbar$ for this state? What is $\langle S_z\rangle$?

The probability of $\hbar$:

$$\begin{aligned}|\langle\psi|1\rangle|^2 &= \left|\frac{1}{\sqrt{14}}\begin{bmatrix}1 & 2 & -3i\end{bmatrix}\begin{bmatrix}1\\0\\0\end{bmatrix}\right|^2\\ &= \frac{1}{14}|1|^2\\ \boxed{|\langle\psi|1\rangle|^2 = \frac{1}{14}}\end{aligned}$$

The probability of 0:

$$\begin{aligned}|\langle\psi|0\rangle|^2 &= \left|\frac{1}{\sqrt{14}}\begin{bmatrix}1 & 2 & -3i\end{bmatrix}\begin{bmatrix}0\\1\\0\end{bmatrix}\right|^2\\ &= \frac{1}{14}|2|^2\\ \boxed{|\langle\psi|0\rangle|^2 = \frac{2}{7}}\end{aligned}$$

The probability of $-\hbar$:

$$\begin{aligned}|\langle\psi|1\rangle|^2 &= \left|\frac{1}{\sqrt{14}}\begin{bmatrix}1 & 2 & -3i\end{bmatrix}\begin{bmatrix}0\\0\\1\end{bmatrix}\right|^2\\ &= \frac{1}{14}|-3i|^2\\ \boxed{|\langle\psi|1\rangle|^2 = \frac{9}{14}}\end{aligned}$$

The expectation value can be calculated with

$$\langle S \rangle = \sum_i P_i a_i$$

where S is the operator, and P_i is the probability of measuring the ith eigenvalue (a_i) of that operator. For S_z, this is

$$\langle S_z \rangle = \frac{1}{14}\hbar + \frac{2}{7} \cdot 0 + \frac{9}{14} \cdot -\hbar$$
$$\boxed{\langle S_z \rangle = -\frac{5}{7}\hbar}.$$

(b) What is $\langle S_x \rangle$ for this state? *Suggestion: Use matrix mechanics to evaluate the expectation value.*

The expectation value can also be calculated using

$$\langle S \rangle = \langle \psi | S | \psi \rangle .$$

First, we need to get $|\psi\rangle$ in terms of S_x basis vectors. We need to find the transformation matrix which will do so:

$$U_{z\to x} = \begin{bmatrix} {}_x\langle 1|1\rangle & {}_x\langle 1|0\rangle & {}_x\langle 1|-1\rangle \\ {}_x\langle 0|1\rangle & {}_x\langle 0|0\rangle & {}_x\langle 0|-1\rangle \\ {}_x\langle -1|1\rangle & {}_x\langle -1|0\rangle & {}_x\langle -1|-1\rangle \end{bmatrix}$$

The S_x basis vectors are:

$$|1\rangle_x = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{\sqrt{2}} \\ \frac{1}{2} \end{bmatrix} \qquad |0\rangle_x = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{bmatrix} \qquad |-1\rangle_x = \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{\sqrt{2}} \\ \frac{1}{2} \end{bmatrix}$$

So

$$U_{z\to x} = \begin{bmatrix} \frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{1}{2} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{2} & -\frac{1}{\sqrt{2}} & \frac{1}{2} \end{bmatrix}.$$

Therefore, $|\psi\rangle$ in the S_x basis is

$$U_{z\to x}|\psi\rangle = \frac{1}{\sqrt{14}} \begin{bmatrix} \frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{1}{2} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{2} & -\frac{1}{\sqrt{2}} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3i \end{bmatrix}$$
$$= \frac{1}{\sqrt{14}} \begin{bmatrix} \frac{1}{2} + \frac{2\sqrt{2}}{2} + \frac{3i}{2} \\ \frac{\sqrt{2}}{2} + 0 - \frac{3i}{\sqrt{2}} \\ \frac{1}{2} - \frac{2\sqrt{2}}{2} + \frac{3i}{2} \end{bmatrix}$$
$$|\psi\rangle_x = \frac{1}{2\sqrt{14}} \begin{bmatrix} 1 + 2\sqrt{2} + 3i \\ \sqrt{2} - 3\sqrt{2}i \\ 1 - 2\sqrt{2} + 3i \end{bmatrix}$$

The S_x operator is given by

$$S_x = \frac{\hbar}{\sqrt{2}} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix},$$

so the expectation value is

$$\begin{aligned}
{}_x\langle\psi| S_x |\psi\rangle_x &= \frac{\hbar}{56\sqrt{2}} \begin{bmatrix} 1+2\sqrt{2}+3i & \sqrt{2}-3\sqrt{2}i & 1-2\sqrt{2}+3i \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1+2\sqrt{2}+3i \\ \sqrt{2}-3\sqrt{2}i \\ 1-2\sqrt{2}+3i \end{bmatrix} \\
&= \frac{\hbar}{56\sqrt{2}} \begin{bmatrix} 1+2\sqrt{2}+3i & \sqrt{2}-3\sqrt{2}i & 1-2\sqrt{2}+3i \end{bmatrix} \begin{bmatrix} \sqrt{2}-3\sqrt{2}i \\ 2+6i \\ \sqrt{2}-e\sqrt{2}i \end{bmatrix} \\
&= \frac{\hbar}{56\sqrt{2}} \left(\sqrt{2}-3\sqrt{2}i\right) \left(1+2\sqrt{2}+3i+2+6i+1-2\sqrt{2}+3i\right) \\
&= \frac{\hbar}{56\sqrt{2}} \left(\sqrt{2}-3\sqrt{2}i\right) (4+12i) \\
&= \frac{\hbar}{56\sqrt{2}} \cdot 40\sqrt{2} \\
\end{aligned}$$

$$\boxed{\langle S_z \rangle = \frac{5}{7}\hbar}$$

(c) What is the probability that a measurement of S_x will yield the value $\hbar$ for this state?

$$\begin{aligned}
|\langle\psi|1\rangle_x|^2 &= \left| \frac{1}{\sqrt{14}} \begin{bmatrix} 1 & 2 & -3i \end{bmatrix} \begin{bmatrix} \frac{1}{2} \\ \frac{1}{\sqrt{2}} \\ \frac{1}{2} \end{bmatrix} \right|^2 \\
&= \frac{1}{14} \left| \frac{1+2\sqrt{2}-3i}{2} \right|^2 \\
&= \frac{1}{56} \left(1+2\sqrt{2}-3i\right) \left(1+2\sqrt{2}+3i\right) \\
&= \frac{1}{56} \left(18+4\sqrt{2}\right)
\end{aligned}$$

$$\boxed{|\langle\psi|1\rangle_x|^2 = \frac{1}{28} \left(9+2\sqrt{2}\right)}$$