PHYS 234 Assignment 1

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1. **Relationship between trigonometric functions and complex exponentials**

   (a) **Starting from the power series representation of the exponential function e^x, derive Euler's formula:**

$$e^{i\theta} = \cos\theta + i\sin\theta$$

where $i = \sqrt{-1}$.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad \cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}, \quad \sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

To start, it could be useful to write out the infinite sums in full:

$$\begin{aligned} e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} &= \frac{x^0}{0!} + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \frac{x^8}{8!} + \frac{x^9}{9!} + \dots \\ &= 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \frac{x^8}{8!} + \frac{x^9}{9!} + \dots \end{aligned}$$

$$\begin{aligned} \cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} &= \frac{(-1)^0 x^0}{0!} + \frac{(-1)^1 x^2}{2!} + \frac{(-1)^2 x^4}{4!} + \frac{(-1)^3 x^6}{6!} + \frac{(-1)^4 x^8}{8!} + \dots \\ &= 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \dots \end{aligned}$$

$$\begin{aligned} \sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} &= \frac{(-1)^0 x^1}{1!} + \frac{(-1)^1 x^3}{3!} + \frac{(-1)^2 x^5}{5!} + \frac{(-1)^3 x^7}{7!} + \frac{(-1)^4 x^9}{9!} + \dots \\ &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \dots \end{aligned}$$

When written out like this, it is easy to see that the expansion of $\cos x$ is extremely similar to that of $\sin x$, but $\cos x$ contains all the even terms, while $\sin x$ contains all the odd terms. Another observation to make is that the expansion of e^x is astoundingly similar to $\cos x + \sin x$:

$$\cos x + \sin x = 1 + x - \frac{x^2}{2} - \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} - \frac{x^6}{6!} - \frac{x^7}{7!} + \frac{x^8}{8!} + \frac{x^9}{9!} - \dots$$

The only difference is the signs on some of the terms. The pattern appears to be $+ + - - + + - - + + - - \dots$, which is suspiciously similar to the signs of the powers of i:

$$i^0 = 1,\ i^1 = i,\ i^2 = -1,\ i^3 = -i,\ i^4 = 1,\ i^5 = i,\ i^6 = -1,\ \dots$$

The thought comes to mind: what if instead of e^x, we wrote out e^{ix} instead? Then, the x^n portion of that infinite sum would lead to the same sign pattern as in $\cos x + \sin x$.

$$\begin{aligned} e^{ix} &= 1 + ix + \frac{(ix)^2}{2} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \frac{(ix)^5}{5!} + \frac{(ix)^6}{6!} + \frac{(ix)^7}{7!} + \frac{(ix)^8}{8!} + \frac{(ix)^9}{9!} - \dots \\ &= 1 + ix - \frac{x^2}{2} - \frac{ix^3}{3!} + \frac{x^4}{4!} + \frac{ix^5}{5!} - \frac{x^6}{6!} - \frac{ix^7}{7!} + \frac{x^8}{8!} + \frac{ix^9}{9!} - \dots \end{aligned}$$

Now the only difference between e^{ix} and $\cos x + \sin x$ is a factor of i on the odd terms. But, as we saw earlier, the odd terms come from $\sin x$! If we instead write the expansion of $\cos x + i \sin x$ (multiplying $\sin x$ by i), we get the two expressions to exactly match. This leads us to writing the formula

$$e^{ix} = \cos x + i \sin x.$$

(b) Use Euler's Formula to express the trigonometric functions $\cos\theta$ and $\sin\theta$ in terms of the complex exponential functions $e^{\pm i\theta}$.

What happens to Euler's formula when we use e^{-ix} instead of e^{ix}?

$$\begin{aligned} e^{-ix} &= \cos(-x) + i\sin(-x) \\ &= \cos x - i \sin x \end{aligned}$$

Notice that the sign of $\sin x$ changes but the sign of $\cos x$ does not. Naturally, this means that

$$e^{ix} + e^{-ix} = 2\cos x,$$

and

$$e^{ix} - e^{-ix} = 2i \sin x.$$

Solving each of these for $\sin x$ and $\cos x$, we get

$$\cos x = \frac{e^{ix} + e^{-ix}}{2}$$

and

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i}.$$