PHYS 234 Assignment 2

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1. **<u>Computing Probabilities</u>**
   A beam of spin-$\frac{1}{2}$ particles is prepared in the state

$$|\psi\rangle = \tfrac{3i}{5}\,|+\rangle_y + \tfrac{4}{5}\,|-\rangle_y\,.$$

The state is given in terms of the S_y eigenvectors. Since we derived relations for the S_z operator in class, I will express $|\psi\rangle$ in terms of S_z eigenvectors to ease the calculations.

First, I will confirm that $|\psi\rangle$ is normalized:

$$\begin{aligned}
\langle\psi|\psi\rangle &\overset{?}{=} 1 \\
\left(-\tfrac{3i}{5}\,{}_y\langle+| + \tfrac{4}{5}\,{}_y\langle-|\right)\left(\tfrac{3i}{5}\,|+\rangle_y + \tfrac{4}{5}\,|-\rangle_y\right) &\overset{?}{=} 1 \\
\frac{9}{25} + \frac{16}{25} &\overset{?}{=} 1 \\
1 &= 1
\end{aligned}$$

Then, I find the projection of $|\psi\rangle$ onto $|+\rangle$:

$$\begin{aligned}
\langle+|\psi\rangle &= \langle+|\left(\tfrac{3i}{5}\,|+\rangle_y + \tfrac{4}{5}\,|-\rangle_y\right) \\
&= \tfrac{3i}{5}\,\langle+|+\rangle_y + \tfrac{4}{5}\,\langle+|-\rangle_y
\end{aligned}$$

From class, $|+\rangle_y = \frac{1}{\sqrt{2}}(|+\rangle + i\,|-\rangle)$ and $|-\rangle_y = \frac{1}{\sqrt{2}}(|+\rangle - i\,|-\rangle)$:

$$\begin{aligned}
&= \tfrac{3i}{5}\,\langle+|\cdot\tfrac{1}{\sqrt{2}}(|+\rangle + i\,|-\rangle) + \tfrac{4}{5}\,\langle+|\cdot\tfrac{1}{\sqrt{2}}(|+\rangle - i\,|-\rangle) \\
&= \tfrac{3i}{5\sqrt{2}}(\langle+|+\rangle + i\,\langle+|-\rangle) + \tfrac{4}{5\sqrt{2}}(\langle+|+\rangle - i\,\langle+|-\rangle) \\
&= \frac{4+3i}{5\sqrt{2}}
\end{aligned}$$

And again with $|-\rangle$:

$$\begin{aligned}
\langle-|\psi\rangle &= \langle-|\left(\tfrac{3i}{5}\,|+\rangle_y + \tfrac{4}{5}\,|-\rangle_y\right) \\
&= \tfrac{3i}{5}\,\langle-|+\rangle_y + \tfrac{4}{5}\,\langle-|-\rangle_y \\
&= \tfrac{3i}{5}\,\langle-|\cdot\tfrac{1}{\sqrt{2}}(|+\rangle + i\,|-\rangle) + \tfrac{4}{5}\,\langle-|\cdot\tfrac{1}{\sqrt{2}}(|+\rangle - i\,|-\rangle) \\
&= \tfrac{3i}{5\sqrt{2}}(\langle-|+\rangle + i\,\langle-|-\rangle) + \tfrac{4}{5\sqrt{2}}(\langle-|+\rangle - i\,\langle-|-\rangle) \\
&= \frac{3i^2}{5\sqrt{2}} - \frac{4i}{5\sqrt{2}} \\
&= \frac{-3-4i}{5\sqrt{2}}
\end{aligned}$$

So $|\psi\rangle$ is then

$$|\psi\rangle = \tfrac{1}{5\sqrt{2}}((4+3i)\,|+\rangle - (3+4i)\,|-\rangle).$$

(a) What are the possible measurement results of the spin operator S_x and with what probability would they occur?

The two possible measurement results are $+\frac{\hbar}{2}$ and $-\frac{\hbar}{2}$.

The probability of $+\frac{\hbar}{2}$ is:

$$\begin{aligned}\left|{}_x\langle +|\psi\rangle\right|^2 &= \left|{}_x\langle +| \cdot \tfrac{1}{5\sqrt{2}}((4+3i)\,|+\rangle - (3+4i)\,|-\rangle)\right|^2 \\ &= \left|\tfrac{4+3i}{5\sqrt{2}}\,{}_x\langle +|+\rangle - \tfrac{3+4i}{5\sqrt{2}}\,{}_x\langle +|-\rangle\right|^2\end{aligned}$$

From class, $|+\rangle_x = \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle)$:

$$\begin{aligned}&= \left|\tfrac{4+3i}{5\sqrt{2}} \cdot \tfrac{1}{\sqrt{2}}(\langle +| + \langle -|)\,|+\rangle - \tfrac{3+4i}{5\sqrt{2}} \cdot \tfrac{1}{\sqrt{2}}(\langle +| + \langle -|)\,|-\rangle\right|^2 \\ &= \left|\frac{4+3i}{5\cdot 2} - \frac{3+4i}{5\cdot 2}\right|^2 \\ &= \left|\frac{1-i}{10}\right|^2 \\ &= \frac{1+1}{100} \\ &= \frac{1}{50}\end{aligned}$$

The probability of $-\frac{\hbar}{2}$ is:

$$\begin{aligned}\left|{}_x\langle -|\psi\rangle\right|^2 &= \left|{}_x\langle +| \cdot \tfrac{1}{5\sqrt{2}}((4+3i)\,|+\rangle - (3+4i)\,|-\rangle)\right|^2 \\ &= \left|\tfrac{4+3i}{5\sqrt{2}}\,{}_x\langle -|+\rangle - \tfrac{3+4i}{5\sqrt{2}}\,{}_x\langle -|-\rangle\right|^2\end{aligned}$$

From class, $|-\rangle_x = \frac{1}{\sqrt{2}}(|+\rangle - |-\rangle)$:

$$\begin{aligned}&= \left|\tfrac{4+3i}{5\sqrt{2}} \cdot \tfrac{1}{\sqrt{2}}(\langle +| - \langle -|)\,|+\rangle - \tfrac{3+4i}{5\sqrt{2}} \cdot \tfrac{1}{\sqrt{2}}(\langle +| - \langle -|)\,|-\rangle\right|^2 \\ &= \left|\frac{4+3i}{5\cdot 2} + \frac{3+4i}{5\cdot 2}\right|^2 \\ &= \left|\frac{7-7i}{10}\right|^2 \\ &= \frac{49+49}{100} \\ &= \frac{49}{50}\end{aligned}$$

(b) What are the possible measurement results of the spin operator S_y and with what probability would they occur?

The two possible measurement results are $+\frac{\hbar}{2}$ and $-\frac{\hbar}{2}$.

The probability of $+\frac{\hbar}{2}$ is:

$$\begin{aligned}\left|{}_y\langle +|\psi\rangle\right|^2 &= \left|{}_y\langle +| \cdot \tfrac{1}{5\sqrt{2}}\left((4+3i)\,|+\rangle - (3+4i)\,|-\rangle\right)\right|^2 \\ &= \left|\tfrac{4+3i}{5\sqrt{2}}\ {}_y\langle +|+\rangle - \tfrac{3+4i}{5\sqrt{2}}\ {}_y\langle +|-\rangle\right|^2\end{aligned}$$

From class, $|+\rangle_y = \frac{1}{\sqrt{2}}(|+\rangle + i\,|-\rangle)$:

$$\begin{aligned}&= \left|\tfrac{4+3i}{5\sqrt{2}} \cdot \tfrac{1}{\sqrt{2}}\left(\left(\langle +| + i\,\langle -|\right)|+\rangle - \tfrac{3+4i}{5\sqrt{2}} \cdot \tfrac{1}{\sqrt{2}}\left(\left(\langle +| + i\,\langle -|\right)|-\rangle\right.\right.\right|^2 \\ &= \left|\frac{4+3i}{5\cdot 2} - \frac{3+4i}{5\cdot 2}i\right|^2 \\ &= \left|\frac{4+3i}{5\cdot 2} - \frac{-4+3i}{5\cdot 2}\right|^2 \\ &= \left|\frac{8}{10}\right|^2 \\ &= \frac{16}{25}\end{aligned}$$

The probability of $-\frac{\hbar}{2}$ is:

$$\begin{aligned}\left|{}_y\langle -|\psi\rangle\right|^2 &= \left|{}_y\langle +| \cdot \tfrac{1}{5\sqrt{2}}\left((4+3i)\,|+\rangle - (3+4i)\,|-\rangle\right)\right|^2 \\ &= \left|\tfrac{4+3i}{5\sqrt{2}}\ {}_y\langle -|+\rangle - \tfrac{3+4i}{5\sqrt{2}}\ {}_y\langle -|-\rangle\right|^2\end{aligned}$$

From class, $|-\rangle_y = \frac{1}{\sqrt{2}}(|+\rangle - i\,|-\rangle)$:

$$\begin{aligned}&= \left|\tfrac{4+3i}{5\sqrt{2}} \cdot \tfrac{1}{\sqrt{2}}\left(\left(\langle +| - i\,\langle -|\right)|+\rangle - \tfrac{3+4i}{5\sqrt{2}} \cdot \tfrac{1}{\sqrt{2}}\left(\left(\langle +| - i\,\langle -|\right)|-\rangle\right.\right.\right|^2 \\ &= \left|\frac{4+3i}{5\cdot 2} + \frac{3+4i}{5\cdot 2}i\right|^2 \\ &= \left|\frac{4+3i}{5\cdot 2} + \frac{-4+3i}{5\cdot 2}\right|^2 \\ &= \left|\frac{6i}{10}\right|^2 \\ &= \frac{36}{100} \\ &= \frac{9}{25}\end{aligned}$$

(c) What are the possible measurement results of the spin operator S_z and with what probability would they occur?

The two possible measurement results are $+\frac{\hbar}{2}$ and $-\frac{\hbar}{2}$.

The probability of $+\frac{\hbar}{2}$ is:

$$\begin{aligned}
|\langle + | \psi \rangle|^2 &= \left| \langle + | \cdot \tfrac{1}{5\sqrt{2}} \left((4+3i) \left| + \right\rangle - (3+4i) \left| - \right\rangle \right) \right|^2 \\
&= \left| \frac{4+3i}{5\sqrt{2}} \right|^2 \\
&= \frac{16+9}{25 \cdot 2} \\
&= \frac{1}{2}
\end{aligned}$$

The probability of $-\frac{\hbar}{2}$ is:

$$\begin{aligned}
|\langle - | \psi \rangle|^2 &= \left| \langle - | \cdot \tfrac{1}{5\sqrt{2}} \left((4+3i) \left| + \right\rangle - (3+4i) \left| - \right\rangle \right) \right|^2 \\
&= \left| -\frac{3+4i}{5\sqrt{2}} \right|^2 \\
&= \frac{9+16}{25 \cdot 2} \\
&= \frac{1}{2}
\end{aligned}$$