PHYS 234 Assignment 1

Brandon Tsang

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1. Relationship between trigonometric functions and complex exponentials

(a) Starting from the power series representation of the exponential function e^x, derive Euler's formula:

$$e^{i\theta} = \cos\theta + i\sin\theta$$

where $i = \sqrt{-1}$.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad \cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}, \quad \sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

To start, it could be useful to write out the infinite sums in full:

$$\begin{aligned} e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} &= \frac{x^0}{0!} + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \frac{x^8}{8!} + \frac{x^9}{9!} + \dots \\ &= 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \frac{x^8}{8!} + \frac{x^9}{9!} + \dots \end{aligned}$$

$$\begin{aligned} \cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} &= \frac{(-1)^0 x^0}{0!} + \frac{(-1)^1 x^2}{2!} + \frac{(-1)^2 x^4}{4!} + \frac{(-1)^3 x^6}{6!} + \frac{(-1)^4 x^8}{8!} + \dots \\ &= 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \dots \end{aligned}$$

$$\begin{aligned} \sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} &= \frac{(-1)^0 x^1}{1!} + \frac{(-1)^1 x^3}{3!} + \frac{(-1)^2 x^5}{5!} + \frac{(-1)^3 x^7}{7!} + \frac{(-1)^4 x^9}{9!} + \dots \\ &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \dots \end{aligned}$$

When written out like this, it is easy to see that the expansion of $\cos x$ is extremely similar to that of $\sin x$, but $\cos x$ contains all the even terms, while $\sin x$ contains all the odd terms. Another observation to make is that the expansion of e^x is astoundingly similar to $\cos x + \sin x$:

$$\cos x + \sin x = 1 + x - \frac{x^2}{2} - \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} - \frac{x^6}{6!} - \frac{x^7}{7!} + \frac{x^8}{8!} + \frac{x^9}{9!} - \dots$$

The only difference is the signs on some of the terms. The pattern appears to be $+ + - - + + - - + + - - \dots$, which is suspiciously similar to the signs of the powers of i:

$$i^0 = 1,\ i^1 = i,\ i^2 = -1,\ i^3 = -i,\ i^4 = 1,\ i^5 = i,\ i^6 = -1,\ \dots$$

The thought comes to mind: what if instead of e^x, we wrote out e^{ix} instead? Then, the x^n portion of that infinite sum would lead to the same sign pattern as in $\cos x + \sin x$.

$$\begin{aligned} e^{ix} &= 1 + ix + \frac{(ix)^2}{2} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \frac{(ix)^5}{5!} + \frac{(ix)^6}{6!} + \frac{(ix)^7}{7!} + \frac{(ix)^8}{8!} + \frac{(ix)^9}{9!} - \dots \\ &= 1 + ix - \frac{x^2}{2} - \frac{ix^3}{3!} + \frac{x^4}{4!} + \frac{ix^5}{5!} - \frac{x^6}{6!} - \frac{ix^7}{7!} + \frac{x^8}{8!} + \frac{ix^9}{9!} - \dots \end{aligned}$$

Now the only difference between e^{ix} and $\cos x + \sin x$ is a factor of i on the odd terms. But, as we saw earlier, the odd terms come from $\sin x$! If we instead write the expansion of $\cos x + i \sin x$ (multiplying $\sin x$ by i), we get the two expressions to exactly match. This leads us to writing the formula

$$e^{ix} = \cos x + i \sin x.$$

2. Calculations using quantum states

$$\begin{aligned} |\psi_1\rangle &= 3\,|+\rangle - i\,|-\rangle \\ |\psi_2\rangle &= e^{i\pi/3}\,|+\rangle + |-\rangle \\ |\psi_3\rangle &= 7i\,|+\rangle - 2\,|-\rangle \end{aligned}$$

(a) For each of the states $|\psi_j\rangle$ above ($j = 1, 2, 3$), find the corresponding normalized state $|\psi_j\rangle_N$.

(b) Using the bra-ket notation, calculate all 9 inner products ${}_N\langle\psi_i|\psi_j\rangle_N$ for $i = 1, 2, 3$ and $j = 1, 2, 3$ using the normalized states.

(c) For each state $|\psi_i\rangle$, find the state $|\phi_i\rangle$ with unit norm, $\langle\phi_i|\phi_i\rangle = 1$ that is orthogonal to it. Recall the orthogonality conditions for the basis states: $\langle+|+\rangle = \langle-|-\rangle = 1$ and $\langle+|-\rangle = \langle-|+\rangle = 0$.

(d) Postulate 4 of quantum mechanics tells us that the complex square of the inner product $|\langle a|b\rangle|^2$ is the probability of measuring a particular quantum state. For each of the normalized states $|\psi\rangle_N$, calculate the probability of measuring each of the six states indicated below.

$$\begin{aligned} |1\rangle &= |+\rangle \\ |2\rangle &= |-\rangle \\ |3\rangle &= \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle) \\ |4\rangle &= \frac{1}{\sqrt{2}}(|+\rangle - |-\rangle) \\ |5\rangle &= \frac{1}{\sqrt{2}}(|+\rangle + i\,|-\rangle) \\ |6\rangle &= \frac{1}{\sqrt{2}}(|+\rangle - i\,|-\rangle) \end{aligned}$$