MATH 114 Assignment 9, Q2

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2. **In class we have been working only with 2×2 matrices. Partly this is because we usually can't easily find the roots of polynomials of larger degree. (Octave and Matlab will be using approximation techniques to solve them.)**

   Consider the matrix

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 3 & 4 \\ 2 & 2 & 4 \end{bmatrix}.$$

   (a) **Find the eigenvalues. (You should find that $\lambda = 0$ is a solution; then what remains is a quadratic, and that gives you the other two.)**

$$\begin{aligned}
\begin{bmatrix} 1 & 0 & 1 \\ 1 & 3 & 4 \\ 2 & 2 & 4 \end{bmatrix} \mathbf{u} &= \lambda \mathbf{u} \\
\begin{bmatrix} 1 & 0 & 1 \\ 1 & 3 & 4 \\ 2 & 2 & 4 \end{bmatrix} \mathbf{u} - \lambda \mathbf{u} &= 0 \\
\left(\begin{bmatrix} 1 & 0 & 1 \\ 1 & 3 & 4 \\ 2 & 2 & 4 \end{bmatrix} - \lambda I_3 \right) \mathbf{u} &= 0 \\
\det \left(\begin{bmatrix} 1 & 0 & 1 \\ 1 & 3 & 4 \\ 2 & 2 & 4 \end{bmatrix} - \lambda I_3 \right) &= 0 \\
\begin{vmatrix} 1-\lambda & 0 & 1 \\ 1 & 3-\lambda & 4 \\ 2 & 2 & 4-\lambda \end{vmatrix} &= 0 \\
(1-\lambda) \begin{vmatrix} 3-\lambda & 4 \\ 2 & 4-\lambda \end{vmatrix} - 0 \begin{vmatrix} 1 & 4 \\ 2 & 4-\lambda \end{vmatrix} + 1 \begin{vmatrix} 1 & 3-\lambda \\ 2 & 2 \end{vmatrix} &= 0 \\
(1-\lambda)(3-\lambda)(4-\lambda) - (1-\lambda)(4)(2) - 0 + (1)(2) - (3-\lambda)(2) &= 0 \\
(\lambda^2 - 4\lambda + 3)(4-\lambda) - (8 - 8\lambda) + 2 - (6 - 2\lambda) &= 0 \\
-\lambda^3 + 8\lambda^2 - 19\lambda + 12 - 8 + 8\lambda + 2 - 6 + 2\lambda &= 0 \\
-\lambda^3 + 8\lambda^2 - 9\lambda &= 0 \\
\lambda(\lambda^2 - 8\lambda + 9) &= 0
\end{aligned}$$

One solution is $\lambda = 0$; the other two will need the quadratic formula:

$$\begin{aligned}\lambda &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{8 \pm \sqrt{(-8)^2 - 4(1)(9)}}{2(1)} \\ &= \frac{8 \pm \sqrt{28}}{2} \\ &= 4 \pm \sqrt{7}\end{aligned}$$

So the eigenvalues are $\lambda = 0$, $\lambda = 4 + \sqrt{7}$, and $\lambda = 4 - \sqrt{7}$.

(b) Find the eigenvector for $\boldsymbol{\lambda = 0}$.

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 3 & 4 \\ 2 & 2 & 4 \end{bmatrix} \mathbf{u} = 0\mathbf{u}$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 3 & 4 \\ 2 & 2 & 4 \end{bmatrix} \mathbf{u} = \mathbf{0}$$

Setting up an augmented matrix:

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 1 & 3 & 4 & 0 \\ 2 & 2 & 4 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 3 & 3 & 0 \\ 0 & 2 & 2 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 2 & 2 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

This represents the system

$$\begin{aligned} u_1 + u_3 &= 0 \\ u_2 + u_3 &= 0 \\ 0u_3 &= 0 \end{aligned}$$

where u_3 is a free variable, so it will be called s. This gives

$$\begin{aligned} u_1 &= -s \\ u_2 &= -s \\ u_3 &= s \end{aligned}$$

Finally, we get:

$$\mathbf{u} = s \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

Therefore, the eigenvector is $\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$.