MA5206 GRADUATE ANALYSIS II
OUTLINE

Notation and Review

A. Vector spaces

All vector spaces considered in this course will be over the real field $\mathbb{R}$ or the complex field $\mathbb{C}$. The symbol $\mathbb{K}$ is often used to refer to either of these fields.

Let A and B be subsets of a vector space E and let $\alpha \in \mathbb{K}$. The sets $A + B$ and αA are defined by

$$A + B = \{a + b : a \in A, b \in B\}, \qquad \alpha A = \{\alpha a : a \in A\}.$$

We also let $A - B = A + (-1)B$. If $x \in E$, then $\{x\} + B$ is also written as $x + B$.

A subset C of a vector space is **convex** if $\alpha x + (1-\alpha)y \in C$ for all $x, y \in C$ and all $\alpha \in [0, 1]$. Let A be a subset of a vector space. The **convex hull** of A is the set $\operatorname{co} A$ consisting of all points of the form $\sum_{k=1}^{n} a_k x_k$, where $n \in \mathbb{N}$, $x_k \in A$, $a_k \geq 0$ and $\sum_{k=1}^{n} a_k = 1$. The convex hull of A is the smallest convex set containing A.

Let X and Y be vector spaces over the same field ($\mathbb{R}$ or $\mathbb{C}$). The **direct sum** or **product vector space** of X and Y is the vector space

$$X \oplus Y = \{(x, y) : x \in X, y \in Y\}$$

with the operations

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2), \ \alpha(x, y) = (\alpha x, \alpha y).$$

Suppose that Y is a vector subspace of a vector space X. Define an equivalence relation $\sim$ on X by $x \sim y$ if and only if $x - y \in Y$. The equivalence classes are precisely the sets $x + Y$, where $x \in X$. Note that this representation is not unique, i.e., there may be distinct x and y such that $x + Y = y + Y$. The **quotient vector space** X/Y is defined as follows. The elements (points) of X/Y are the equivalence classes of $\sim$. If $x + Y, y + Y \in X/Y$ and $\alpha \in \mathbb{K}$, define

$$(x + Y) + (y + Y) = (x + y) + Y, \qquad \alpha(x + Y) = \alpha x + Y.$$

These operations are well defined (independent of the representation of the equivalence classes) and X/Y is a vector space under these operations.

B. Metric spaces

A **metric space** is a pair (X, d) where X is a nonempty set and $d : X \times X \to \mathbb{R}$ is a function such that

(1) $d(x, y) \geq 0$ for all $x, y \in X$;
(2) $d(x, y) = 0$ if and only if $x = y$;
(3) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(4) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

For example, $\mathbb{K}$ can be given the **standard metric** $d(x, y) = |x - y|$ for all $x, y \in \mathbb{K}$. A sequence $(x_k)_{k=1}^{\infty}$ in a metric space (X, d) is **convergent** if there exists $x \in X$ such that $\lim_{k\to\infty} d(x_k, x) = 0$. The sequence is **Cauchy** if for any $\varepsilon > 0$, there exists $k_0 \in \mathbb{N}$ such that $d(x_j, x_k) < \varepsilon$ for all $j, k \geq k_0$. (X, d) is **complete** if every Cauchy sequence in X is convergent.
A metric space is a special case of a *topological space*, which will be discussed below.

1. A crash course on general topology

A **topology** on a set X is a collection $\mathcal{T}$ of subsets of X such that

(1) $\emptyset, X \in \mathcal{T}$;
(2) $U \cap V \in \mathcal{T}$ if $U, V \in \mathcal{T}$;
(3) $\cup_\alpha U_\alpha \in \mathcal{T}$ if (U_α) is any collection of sets in $\mathcal{T}$.

Elements of the topology $\mathcal{T}$ are called **open sets** in X. A **closed set** in X is a set W such that its complement W^c is open. A **topological space** is a set X together with a topology $\mathcal{T}$ on X. A topological space $(X, \mathcal{T})$ is **Hausdorff** if for every pair of distinct points x and y in X, there are open sets U, V such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$.

Let A be a set in a topological space X. The **interior of** A, denoted by $\operatorname{int} A$, is the union of all open subsets of A. The **closure of** A, denoted by $\overline{A}$, is the intersection of all closed sets in X containing A. $\overline{A}$ is a closed set.

Proposition 1. *Let $(X, \mathcal{T})$ be a topological space.*

(1) *The sets $\emptyset$ and X are closed.*
(2) *If $F_1, \dots, F_n$ are closed sets in X, then $F_1 \cup \dots \cup F_n$ is a closed set,*
(3) *If F_α is a closed set for each α, then $\cap F_\alpha$ is a closed set.*
(4) *For any subset A of X, a point $x \in \overline{A}$ if and only if for any open set U containing x, $U \cap A \neq \emptyset$.*

A **basis** for a topology $\mathcal{T}$ on a set X is a subset $\mathcal{B}$ of $\mathcal{T}$ such that for all $x \in X$ and all $U \in \mathcal{T}$ with $x \in U$, there exists $B \in \mathcal{B}$ such that $x \in B \subseteq U$.

Proposition 2. *Suppose that $\mathcal{B}$ is a collection of subsets of X such that (i) every $x \in X$ is contained in some set $U \in \mathcal{B}$, (ii) for all $x \in X$ and all $U, V \in \mathcal{B}$ such that $x \in U \cap V$, there exists $W \in \mathcal{B}$ with $x \in W \subseteq U \cap V$. Let $\mathcal{T}$ be the collection of all arbitrary unions of members of $\mathcal{B}$. (Including the empty set, which is taken to be the union of an empty collection.) Then $\mathcal{T}$ is a topology on X and $\mathcal{B}$ is a basis for $\mathcal{T}$. Moreover, $\mathcal{T}$ is the only topology on X that has $\mathcal{B}$ as a basis.*

Remark. Proposition 2 says that any basis generates only one topology. However, a topology may have more than one basis.

As an example, let us specify a basis for the topology on a metric space. Let (X, d) be a metric space. If $x \in X$ and $r > 0$, the **ball** centered at x with radius r is the set

$$B(x, r) = \{y \in X : d(x, y) < r\}.$$

The set $\mathcal{B}$ of all balls in X is a basis for a topology on X (Check!), called the **metric topology**. The metric topology on a metric space is Hausdorff.

1.1. Subspace topology.

Let $(X, \mathcal{T})$ be a topological space and let Y be a subset of X. Then

$$\mathcal{T}_Y = \{U \cap Y : U \in \mathcal{T}\}$$

is a topology on Y (Check!), called the **subspace topology** on Y.

Proposition 3. *Let $(X, \mathcal{T})$ be a topological space and let Y be a subset of X. A subset F of Y is closed in the subspace topology if and only if there is a closed set H in X such that $F = H \cap Y$.*

1.2. **Product topology.** Let $(X, \mathcal{T}_1)$ and $(Y, \mathcal{T}_2)$ be two topological spaces. Define a basis for a topology on $X \times Y$ by $\mathcal{B} = \{U \times V : U \in \mathcal{T}_1, V \in \mathcal{T}_2\}$ (Check!). The topology generated by $\mathcal{B}$ is called the **product topology** on $X \times Y$.

Let $(X_\alpha, \mathcal{T}_\alpha)_{\alpha \in I}$ be a collection of topological spaces. Take X to be the Cartesian product $X = \prod_{\alpha \in I} X_\alpha$. Thus $X = \{(x_\alpha)_{\alpha \in I} : x_\alpha \in X_\alpha \text{ for all } \alpha\}$. If J is a finite subset of I and U_α is a set in $\mathcal{T}_\alpha$ for each $\alpha \in J$, let

$$\prod_{\alpha \in J} U_\alpha = \{(x_\alpha)_{\alpha \in I} \in X : x_\alpha \in U_\alpha \text{ for all } \alpha \in J\}.$$

The collection of all such sets is a basis for a topology on X (Check!), called the **product topology**. The product topology is Hausdorff if each $(X_\alpha, \mathcal{T}_\alpha)$ is Hausdorff.

1.3. **Quotient topology.** Let $(X, \mathcal{T})$ be a topological space and let $q : X \to Y$ be a surjective map from X onto a set Y. Define

$$\mathcal{T}_Y = \{U \subseteq Y : q^{-1}(U) \in \mathcal{T}\}.$$

Then $\mathcal{T}_Y$ is a topology on Y called the **quotient topology** (induced by q).

1.4. **Continuous functions.** Let $(X, \mathcal{T}_1)$ and $(Y, \mathcal{T}_2)$ be topological spaces. A function $f : X \to Y$ is said to be **continuous** at a point $x_0 \in X$ if for any open set V in Y containing $f(x_0)$, there exists an open set U in X containing x_0 such that $f(U) \subseteq V$. f is continuous on X if it is continuous at every point in X.

Proposition 4. *If $\mathcal{T}_1$ and $\mathcal{T}_2$ are metric topologies generated by metrics d_1 and d_2 respectively, then f is continuous at x_0 if and only if $(f(x_n))_{n=1}^\infty$ converges to $f(x_0)$ for every sequence $(x_n)_{n=1}^\infty$ in X that converges to x_0.*

Theorem 5. *Let $f : X \to Y$ be a function from a topological space $(X, \mathcal{T}_1)$ to a topological space $(Y, \mathcal{T}_2)$. The following are equivalent.*

1. *f is continuous on X.*
2. *$f^{-1}(V)$ is open for every open set V in Y.*
3. *$f^{-1}(W)$ is closed for every closed set W in Y.*

Proposition 6. *Let $(X, \mathcal{T})$ be a topological space and let $(Y_\alpha, \mathcal{T}_\alpha)$, $\alpha \in I$, be topological spaces. Set $Y = \prod_{\alpha \in I} Y_\alpha$ and let $\mathcal{T}'$ be the product topology on Y. For each $\beta \in I$, let $\pi_\beta : Y \to Y_\beta$ be the map $\pi_\beta((y_\alpha)) = y_\beta$. A function $f : X \to Y$ is continuous if and only if $\pi_\beta \circ f$ is continuous for each $\beta \in I$.*

Proposition 7. *Let $(X, \mathcal{T})$ be a topological space and let $q : X \to Y$ be a surjective map from X onto a set Y. Let $\mathcal{T}_Y$ be the quotient topology induced by q. For any topological space $(Z, \mathcal{T}_Z)$, a function $f : Y \to Z$ is continuous if and only if $f \circ q$ is continuous from X to Z.*

1.5. **Compact spaces.** Let $(X, \mathcal{T})$ be a topological space. A subset K of X is **compact** if given any collection of open sets $\mathcal{U}$ in X such that $K \subseteq \cup_{U \in \mathcal{U}} U$, there are finitely many sets $U_1, \ldots, U_k \in \mathcal{U}$ such that $K \subseteq U_1 \cup \cdots \cup U_k$.

Proposition 8. *Let $(X, \mathcal{T})$ be a Hausdorff topological space. If K is a compact subset of X, then K is closed.*

The following result is well known.

Theorem 9. *Let (X, d) be a metric space and let K be a subset of X. The following are equivalent.*

(1) *K is compact in the metric topology.*
(2) *Every sequence in K has a subsequence that converges to an element of K.*
(3) *K is complete and totally bounded.*

Recall that a subset K of a metric space X is **totally bounded** if for any $\varepsilon > 0$, there exist finitely many points $x_1, \ldots, x_n$ in X (or, equivalently, in K) such that $K \subseteq \cup_{i=1}^{n} B(x_i, \varepsilon)$.

Theorem 10. *Let $(X, \mathcal{T}_1)$ and $(Y, \mathcal{T}_2)$ be topological spaces. If K is a compact subset of X and $f : X \to Y$ is continuous, then the image $f(K)$ is compact.*

Corollary 11. *(Extreme value theorem) Let $(X, \mathcal{T}_1)$ be a topological space. If K is a compact subset of X and $f : X \to \mathbb{R}$ is continuous, then there exist $x_1, x_2 \in K$ such that $f(x_1) \leq f(x) \leq f(x_2)$ for all $x \in K$.*

The following theorem is an important result in general topology. We assume it without proof.

Theorem 12. *(Tychonoff's Theorem) Let $(X_\alpha, \mathcal{T}_\alpha)_{\alpha \in I}$ be a collection of compact topological spaces. Then the product space $\prod_{\alpha \in I} X_\alpha$ is compact in the product topology.*

2. Normed spaces and inner product spaces

It is well known that on $\mathbb{R}^n$ or $\mathbb{C}^n$, one can define an inner product and the length (or norm) of a vector, as follows:

$$\langle x, y\rangle = \sum_{k=1}^{n} x_k \overline{y_k} \quad \text{and} \quad \|x\| = \sqrt{\langle x, x\rangle} = \sqrt{\sum_{k=1}^{n} |x_k|^2},$$

where $x = (x_k)_{k=1}^n$ and $y = (y_k)_{k=1}^n$. We wish to generalize these notions to any vector space.

Definition 13. *Let X be a vector space over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. An* **inner product** *on X is a function $\langle \cdot, \cdot\rangle : X \times X \to \mathbb{K}$ that satisfies the following properties.*

1. *$\langle x, x\rangle \geq 0$ for all $x \in X$,*
2. *$\langle x, x\rangle = 0$ if and only if $x = 0$,*
3. *$\langle x, y\rangle = \overline{\langle y, x\rangle}$ for all $x, y \in X$,*
4. *$\langle \alpha x + y, z\rangle = \alpha\langle x, z\rangle + \langle y, z\rangle$ for all $\alpha \in \mathbb{K}$ and all $x, y, z \in X$.*

A vector space together with an inner product defined on it is called an **inner product space**.

Example 14.

1. *$\mathbb{R}^n$ or $\mathbb{C}^n$ are inner product spaces with the inner product defined above.*
2. *Let (Ω, Σ, μ) be a measure space. Define $\langle \cdot, \cdot\rangle : L^2(\Omega, \Sigma, \mu) \times L^2(\Omega, \Sigma, \mu) \to \mathbb{K}$ by*
$$\langle f, g\rangle = \int f\overline{g}\, d\mu \text{ for all } f, g \in L^2(\Omega, \Sigma, \mu).$$
$\langle \cdot, \cdot\rangle$ satisfies properties (1), (3) and (4) in Definition 13. However, $\langle f, f\rangle = 0$ only implies that $f = 0$ a.e. It is customary to adopt the convention that two functions in $L^2(\Omega, \Sigma, \mu)$ are treated as the same element if $f = g$ a.e. In particular, $f = 0$ a.e. means that f is the 0 element in $L^2(\Omega, \Sigma, \mu)$. Then condition (2) in Definition 13 is satisfied as well.
3. *Take (Ω, Σ, μ) be the the set $\{1, \ldots, n\}$ with the counting measure. Then $L^2(\Omega, \Sigma, \mu)$ with the inner product in (2) coincides with $\mathbb{R}^n$ or $\mathbb{C}^n$ with the inner product in (1).*
4. *Take (Ω, Σ, μ) to be $\mathbb{N}$ with the counting measure. $L^2(\Omega, \Sigma, \mu)$ is usually called ℓ^2 in this case. The inner product defined in (2), when applied to ℓ^2, is given by*
$$\langle x, y\rangle = \sum_{k=1}^{\infty} x_k \overline{y_k},$$
where $x = (x_k)_{k=1}^\infty, y = (y_k)_{k=1}^\infty$ are elements in ℓ^2.

All examples above are essentially of the same kind: L^2. We will see later on that this is not a coincidence.

Definition 15. *Let X be a vector space over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. A* **norm** *on X is a function $\|\cdot\| : X \times X \to \mathbb{R}$ that satisfies the following properties.*

1. *$\|x\| \geq 0$ for all $x \in X$,*
2. *$\|x\| = 0$ if and only if $x = 0$,*
3. *(Homogeneity) $\|\alpha x\| = |\alpha|\|x\|$ for all $\alpha \in \mathbb{K}$ and all $x \in X$,*
4. *(Triangle inequality) $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in X$.*

A vector space together with a norm defined on it is called a **normed space**.

Example 16. (1) *Let (Ω, Σ, μ) be a measure space and let $1 \leq p \leq \infty$. Consider the space $L^p(\Omega, \Sigma, \mu)$, where once again two functions that are equal a.e. are treated as the same element in $L^p(\Omega, \Sigma, \mu)$. Define*

$$\|\cdot\|_p : L^p(\Omega, \Sigma, \mu) \to \mathbb{R} \text{ by } \|f\|_p = (\int |f|^p \, d\mu)^{1/p}, 1 \leq p < \infty$$

or $\|f\|_\infty$ = essential supremum of $|f|$. Then $\|\cdot\|_p$ is a norm on $L^p(\Omega, \Sigma, \mu)$. (The triangle inequality is Minkowski's inequality).

(2) *If (Ω, Σ, μ) is the set $\{1, \ldots, n\}$ with the counting measure, then $L^p(\Omega, \Sigma, \mu)$ is also denoted as $\ell^p(n)$. Thus $\ell^p(n)$ is the vector space $\mathbb{R}^n$ or $\mathbb{C}^n$ endowed with the p-norm*

$$\|x\| = (\sum_{k=1}^{n} |x_k|^p)^{1/p} \text{ or } \max_k |x_k| \text{ for } p = \infty, \text{ if } x = (x_k)_{k=1}^m.$$

(3) *If (Ω, Σ, μ) is the set $\mathbb{N}$ with the counting measure, then $L^p(\Omega, \Sigma, \mu)$ is usually denoted by ℓ^p.*

(4) *Let K be a compact Hausdorff topological space and let $C(K)$ be the space of all scalar valued (i.e., real or complex valued) functions on K. Then $C(K)$ is a normed space with the norm*

$$\|f\| = \sup_{t \in K} |f(t)|.$$

Proposition 17. *(Cauchy-Schwarz inequality) Let X be an inner product space. For any $x, y \in X$,*

$$|\langle x, y \rangle|^2 \leq \langle x, x \rangle \, \langle y, y \rangle.$$

Example 18. *Let X be an inner product space with inner product $\langle \cdot, \cdot \rangle$. Then the equation $\|x\| = \sqrt{\langle x, x \rangle}$ defines a norm on X.*

Example 18 tells us that every inner product space is a normed space. In turn, every normed space is a metric space.

Proposition 19. *Let X be a normed space with norm $\|\cdot\|$. Define $d : X \times X \to \mathbb{R}$ by $d(x, y) = \|x - y\|$. Then d is a metric on X.*

Thus we have the inclusions

inner product spaces $\subseteq$ normed spaces $\subseteq$ metric spaces.

If a normed space is complete with respect to the metric generated by the norm, then it is called a **Banach space**. If an inner product space is complete with respect to the metric generated by the norm generated by the inner product, then it is called a **Hilbert space**. Similar to the inclusions above, we have

$$\text{Hilbert spaces} \subseteq \text{Banach spaces} \subseteq \text{complete metric spaces.}$$

Example 20.
(1) *If $1 \le p \le \infty$ and (Ω, Σ, μ) is a measure space, then $L^p(\Omega, \Sigma, \mu)$ is a Banach space. This follows from the Riesz-Fischer Theorem.*
(2) *In particular, $L^2(\Omega, \Sigma, \mu)$ is a Hilbert space.*
(3) *The space $C(K)$ from Example 16(4) is a Banach space. (Check!)*

2.1. **Subspaces, direct sums and quotient spaces.** It is possible to construct new inner product spaces/normed spaces from existing ones. We describe here two basic constructions.

Proposition 21. *(Subspaces) Let X be an inner product space (respectively, a normed space) and let Y be a vector subspace of X. Then Y is an inner product space (respectively, a normed space) with the inner product (respectively, norm) inherited from X.*
Let X be a Hilbert space (respectively, a Banach space) and let Y be a subspace of X. Then Y (as a subspace of X) is a Hilbert space (respectively, a Banach space) if and only if it is a closed set in X.

Example 22. *Let $X = \ell^\infty$, the space of all bounded scalar sequences with the sup-norm $\|x\| = \sup_k |x_k|$, where $x = (x_k)_{k=1}^\infty$. Then X is a Banach space*

(1) *Define c to be the subspace of X consisting of all $x = (x_k)_{k=1}^\infty$ such that $\lim_k x_k$ exists (in $\mathbb{K}$). Then c is a closed subspace of X and hence it is also a Banach space.*
(2) *Define c_0 to be the subspace of X consisting of all $x = (x_k)_{k=1}^\infty$ such that $\lim_k x_k = 0$. Then c_0 is a closed subspace of c and also of X. Hence it is also a Banach space.*
(3) *Define c_{00} to be the subspace of X consisting of all $x = (x_k)_{k=1}^\infty$ such that the set $\{k : x_k \neq 0\}$ is finite. Then c_{00} is <u>not</u> a closed subspace of X and hence it is <u>not</u> a Banach space.*

Proposition 23. *(Direct sums) Let X and Y be normed spaces and let $1 \le p \le \infty$. Then $X \oplus_p Y$ is the vector space direct sum*

$$X \oplus Y = \{(x, y) : x \in X, y \in Y\}$$

with the norm

$$\|(x, y)\| = (\|x\|^p + \|y\|^p)^{1/p} \text{ or } \max\{\|x\|, \|y\|\} \text{ if } p = \infty.$$

$X \oplus_p Y$ is a Banach space if and only if both X and Y are Banach spaces.

Proposition 24. *(Quotient spaces) Let X be a normed space and let Y be a <u>closed</u> subspace of X. Define $\| \cdot \|_{X/Y}$ on the quotient vector space X/Y*

(see part A of "Notation and Review") by

$$\|x + Y\|_{X/Y} = \inf\{\|x + y\| : y \in Y\}.$$

Then $\| \cdot \|_{X/Y}$ *defines a norm on* X/Y. *It is called the* **quotient norm**. *If* X *is a Banach space, then* X/Y *is also a Banach space under the quotient norm.*

3. Bounded linear operators

Let X and Y be normed spaces. We characterize the continuous linear operators $T : X \to Y$. We will also define a norm on the space of $L(X, Y)$ all continuous linear operators from X into Y.

Proposition 25. *Let X and Y be normed spaces and let $T : X \to Y$ be a linear operator. Then the following are equivalent.*

(1) *T is continuous on X,*
(2) *T is continuous at 0,*
(3) *There exists $C \in \mathbb{R}$ such that $\|Tx\| \leq C\|x\|$ for all $x \in X$.*

A linear operator T that satisfies condition (3) of Proposition 25 is called a **bounded linear operator**. Thus a bounded linear operator between normed spaces is the same as a continuous linear operator between normed spaces.

Definition 26. *Let X and Y be normed spaces. A linear operator $T : X \to Y$ is called an* **isomorphism** *if T is a bijection and both T and T^{-1} are continuous. If $T : X \to Y$ is an isomorphism from X onto a subspace Z of Y, we say that T is an* **into isomorphism** *from X into Y. X and Y are said to be* **isomorphic** *if there is an isomorphism T from X onto Y.*

Corollary 27. *Let X and Y be normed spaces and let $T : X \to Y$ be a linear operator. Then T is an into isomorphism if and only if there are constants $0 < c \leq C < \infty$ so that*

$$c\|x\| \leq \|Tx\| \leq C\|x\| \text{ for all } x \in X.$$

Definition 28. *Let X and Y be normed spaces. A linear operator $T : X \to Y$ is called an* **into isometry** *if $\|Tx\| = \|x\|$ for all $x \in X$. X and Y are said to be* **isometric** *if there is an isometry T from X* onto *Y.*

Example 29. *Let c_0 and c be the Banach spaces given in Example 22. Define a map $T : c_0 \to c$ by*

$$Tx = (x_1 + x_{k+1})_{k=1}^{\infty}, \text{ where } x = (x_k)_{k=1}^{\infty}.$$

Then T is an isomorphism from c_0 onto c. Thus c_0 and c are isomorphic. T is not an isometry. In fact, c_0 and c are not isometric, i.e., there is no isometry that maps c_0 onto c.

Proposition 30. *(Operator norm) Let X and Y be normed spaces. Denote by $L(X, Y)$ the space of all bounded linear operators from X to Y. For any $T \in L(X, Y)$, let*

$$\|T\| = \sup_{\|x\| \leq 1} \|Tx\|.$$

This defines a norm on $L(X, Y)$, called the **operator norm**. *If Y is a Banach space, then $L(X, Y)$ is complete with respect to the operator norm and thus is a Banach space.*

The operator norm can also be expressed in the following ways.

$$\|T\| = \sup_{\|x\|<1} \|Tx\| = \sup_{\|x\|=1} \|Tx\|.$$

For any $T \in L(X,Y)$ and any $x \in X$, we have $\|Tx\| \le \|T\|\,\|x\|$.

Example 31. *The supremum in the definition of the operator norm: $\|T\| = \sup_{\|x\|\le 1} \|Tx\|$ cannot be replaced by a maximum in general. That is, the supremum may not be attained. Consider the operator $T : c_0 \to \mathbb{K}$ defined by $Tx = \sum_{k=1}^{\infty} \frac{x_k}{2^k}$, where $x = (x_k)_{k=1}^{\infty}$. Then $T \in L(c_0, \mathbb{K})$, $\|T\| = 1$ and $\|Tx\| < 1$ for all $x \in c_0$ with $\|x\| \le 1$.*

A subset Z of a normed space X is **dense** if the closure of Z, $\overline{Z}$, is X.

Proposition 32. *Let X be a normed space, Y be a Banach space and let Z be a dense* subspace *of X. Suppose that $S : Z \to Y$ is a bounded linear operator. There is a unique bounded linear operator $T : X \to Y$ such that $S = T_{|Z}$. Furthermore, $\|T\| = \|S\|$. If S is an isometry, then so is T.*

Example 33. *Consider the complex Hilbert space $L^2(\mathbb{R})$, where $\mathbb{R}$ is endowed with Lebesgue measure. Recall that the Schwartz class $\mathcal{S}$ is the space of all C^∞ functions f on $\mathbb{R}$ such that* $\sup |x^k f^{(j)}(x)| < \infty$ *for all $k \in \mathbb{N}$ and all $j \in \mathbb{N} \cup \{0\}$. We may regard $\mathcal{S}$ as a subspace of $L^2(\mathbb{R})$. In particular $\mathcal{S}$ contains $C_c^\infty(\mathbb{R})$, the space of all C^∞ functions with compact support, which is dense in $L^2(\mathbb{R})$. Hence $\mathcal{S}$ is dense in $L^2(\mathbb{R})$.*
For $f \in \mathcal{S}$, define the **Fourier transform** *$\widehat{f}$ by*

$$\widehat{f}(x) = \int f(y) e^{-ixy}\, d\lambda(y).$$

Then $\widehat{f} \in \mathcal{S}$. The map $S : \mathcal{S} \to \mathcal{S} \subseteq L^2(\mathbb{R}, \frac{\lambda}{2\pi})$ given by $Sf = \widehat{f}$ is an isometry from $\mathcal{S}$ (with the L^2-norm) into $L^2(\mathbb{R}, \frac{\lambda}{2\pi})$. By Proposition 32, S extends uniquely to an isometry $T : L^2(\mathbb{R}) \to L^2(\mathbb{R}, \frac{\lambda}{2\pi})$. Furthermore, T maps $L^2(\mathbb{R})$ onto $L^2(\mathbb{R}, \frac{\lambda}{2\pi})$. For any $f \in L^2(\mathbb{R})$, we call Tf the Fourier transform of f.

Remark. It is clear that the Fourier transform defines a bounded linear operator of norm ≤ 1 from $L^1(\mathbb{R})$ to $L^\infty(\mathbb{R}, \frac{\lambda}{2\pi})$. Example 33 shows that it also defines a bounded linear operator of norm 1 from $L^2(\mathbb{R})$ to $L^2(\mathbb{R}, \frac{\lambda}{2\pi})$. Using an interpolation method, it is possible to show that the Fourier transform defines a bounded linear operator of norm 1 from $L^p(\mathbb{R})$ into $L^q(\mathbb{R}, \frac{\lambda}{2\pi})$, where $1 < p < 2$ and $q = p/(p-1)$. This result is called the Hausdorff-Young inequality. The Hausdorff-Young inequality cannot be extended to the range $2 < p < \infty$.

Every metric space X has a *completion*; that is, a complete metric space Y containing X as a dense subspace. The same is true for normed and inner product spaces.

Proposition 34. *(Completion) Let X be a normed space (respectively, an inner product space). There is a Banach space (respectively, Hilbert space) Y so that X is isometric to a dense subspace of Y. If Z is another completion of X, then Y and Z are isometric.*

4. Finite dimensional normed spaces

A **finite dimensional normed space** is a normed space that is finite dimensional as a vector space.

Theorem 35. (1) *Let X be an n-dimensional normed space, where $n \in \mathbb{N}$. Then X is isomorphic to $\ell^1(n)$.*
(2) *Any two n-dimensional normed spaces are isomorphic.*
(3) *Let X be a finite dimensional normed space and let Y be any normed space. Then any linear operator $T : X \to Y$ is bounded.*
(4) *Let Y be a finite dimensional subspace of a normed space X. Then Y is closed in X.*

Example 36. *Suppose that $1 \le p < q \le \infty$. The formal identity $I : \ell^p(n) \to \ell^q(n)$ is an isomorphism. For any $x \in \ell^p(n)$,*

$$n^{\frac{1}{q}-\frac{1}{p}} \|x\|_p \le \|Ix\|_q \le \|x\|_p.$$

Let X be a normed space. The set $B_X = \{x \in X : \|x\| \le 1\}$ is called the **(closed unit) ball** of X.

Theorem 37. *Let X be a normed space. Then X is finite dimensional if and only if B_X is compact (with respect to the metric generated by the norm).*

Thought question. From Theorem 37, if X is an infinite dimensional normed space, then there is a sequence $(x_k)_{k=1}^{\infty}$ in B_X with no convergent subsequence. Produce such sequences in ℓ^p, $L^p[0,1]$, $1 \le p \le \infty$, and $C[0,1]$.

5. Hilbert space

Proposition 38. *(Parallelogram Law) Let X be an inner product space. Then*

$$\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2) \text{ for all } x, y \in X.$$

Example 39. *Consider the space $L^p(\Omega, \Sigma, \mu)$, where $p \neq 2$ and $L^p(\Omega, \Sigma, \mu)$ is at least 2-dimensional. Then the p-norm $\|\cdot\|_p$ fails the Parallelogram Law and hence $L^p(\Omega, \Sigma, \mu)$ cannot be an inner product space with* any *inner product.*

Proposition 40. *(Polarization identity) Let X be an inner product space. For any $x, y \in X$,*

$$\langle x, y\rangle = \begin{cases} \frac{1}{4}(\|x+y\|^2 - \|x-y\|^2) & \text{for real scalars} \\ \frac{1}{4}\sum_{k=1}^{4} i^k \|x + i^k y\|^2 & \text{for complex scalars.} \end{cases}$$

Let X be an inner product space. Two vectors $x, y \in X$ are said to be **orthogonal**, written $x \perp y$, if $\langle x, y\rangle = 0$. A set of vectors A is orthogonal if $x \perp y$ whenever, $x, y \in A$ and $x \neq y$. A set of vectors A is **orthonormal** if it is orthogonal and $\|x\| = 1$ for all $x \in A$.

Proposition 41. *Let A be an orthogonal set in an inner product space X such that $x \neq 0$ for all $x \in A$. Then A is linearly independent in X.*

Proposition 42. *(Gram-Schmidt process) Let $\{x_1, \ldots, x_n\}$ be a linearly independent set of vectors in an inner product space X. There is an orthonormal set $\{e_1, \ldots, e_n\}$ in X such that* $\operatorname{span}\{x_1, \ldots, x_k\} = \operatorname{span}\{e_1, \ldots e_k\}$ *for all $1 \leq k \leq n$.*

Corollary 43. *Every finite dimensional inner product space has an orthonormal basis, i.e., a basis (in the vector space sense) that is an orthonormal set. Every finite dimensional inner product space X is isometric to $\ell^2(n)$, where n is the dimension of X.*

Lemma 44. *Let $\{e_1, \ldots, e_n\}$ be an orthonormal set in an inner product space X. For any $x \in X$ and any scalars $a_1 \ldots, a_n$,*

$$\|x - \sum_{k=1}^{n} a_k e_k\|^2 = \|x\|^2 - 2\operatorname{Re}(\sum_{k=1}^{n} \overline{a_k}\langle x, e_k\rangle) + \sum_{k=1}^{n} |a_k|^2.$$

Proposition 45. *(Bessel's inequality) Let $(e_\gamma)_{\gamma\in\Gamma}$ be an orthonormal set in an inner product space X. For any $x \in X$,*

$$\sum_{\gamma\in\Gamma} |\langle x, e_\gamma\rangle|^2 \leq \|x\|^2$$

Here the sum on the left is defined to be $\sup_F \sum_{\gamma\in F} |\langle x, e_\gamma\rangle|^2$*, with the supremum taken over all finite subsets F of Γ.*

If S is a subset of a metric space X and $x \in X$, then the distance from x to set S is

$$d(x,S) = \inf\{d(x,y) : y \in S\}.$$

Proposition 46. *(Nearest point) Let C be a closed convex set in a Hilbert space X and let $x \in X$. There is a unique point $u \in C$ such that $\|x-u\| = d(x,C)$.*

Proposition 47. *Let Y be a finite dimensional subspace of a Hilbert space X. Suppose that $\{e_1,\dots,e_k\}$ is an orthonormal basis for Y. For any $x \in X$,*

$$\|x - \sum_{k=1}^{n} \langle x, e_k\rangle e_k\| = d(x,Y).$$

Proposition 48. *(Orthogonal complement and orthogonal decomposition) Let Y be a closed subspace of a Hilbert space X. Define*

$$Y^{\perp} = \{x \in X : x \perp y \text{ for all } y \in Y\}.$$

Then $Y^{\perp}$ is a closed subspace of X, called the **orthogonal complement** *of Y. For any $x \in X$, there is a unique representation $x = y + z$ with $y \in Y$ and $z \in Y^{\perp}$.*

Theorem 49. *(Riesz Representation Theorem) Let X be a Hilbert space and let f be a bounded linear functional on X, i.e., f is a bounded linear operator from X to $\mathbb{K}$. There is a unique $y \in X$ such that*

$$f(x) = \langle x, y\rangle \text{ for all } x \in X.$$

Corollary 50. *(Hilbert adjoint operator) Let X be a Hilbert space and let T be a bounded linear operator from X to itself. There is a unique bounded linear operator $T^* : X \to X$ so that*

$$\langle T^* x, y\rangle = \langle x, Ty\rangle \text{ for all } x, y \in X.$$

Moreover, $\|T^*\| = \|T\|$.

Corollary 51. *(Lax-Milgram) Let X be a Hilbert space and let $B : X \times X \to \mathbb{K}$ be a function that satisfies the following conditions.*

(1) *(Sesquilinear form) For all $x, y, z \in X$ and any $\alpha \in \mathbb{K}$,*

$$B(\alpha x + y, z) = \alpha B(x,z) + B(y,z) \text{ and } B(x, \alpha y + z) = \overline{\alpha} B(x,y) + B(x,z).$$

(2) *There is a constant $C < \infty$ so that*

$$|B(x,y)| \le C\|x\|\|y\| \text{ for all } x, y \in X.$$

(3) *There is a constant $c > 0$ so that*

$$|B(x,x)| \ge c\|x\|^2 \text{ for all } x \in X.$$

Then there is an isomorphism from T from X onto X so that

$$B(x,y) = \langle Tx, y\rangle \text{ for all } x, y \in X.$$

For any bounded linear functional f on X, there is a unique $y \in X$ such that $f(x) = B(x,y)$ for all $x \in X$. Moreover, $\|y\| \le \frac{\|f\|}{c}$.

Theorem 52. *(Characterization of orthonormal basis in Hilbert space) Let X be a Hilbert space and let $(e_\gamma)_{\gamma\in\Gamma}$ be an orthonormal set in X. The following are equivalent.*

(1) *For any $x \in X$, $\sum_{\gamma\in\Gamma}\langle x, e_\gamma\rangle e_\gamma$ converges to x, in the sense that for any $\varepsilon > 0$, there exists a finite set $F_0 \subseteq \Gamma$ so that*

$$\|\sum_{\gamma\in F}\langle x, e_\gamma\rangle e_\gamma - x\| < \varepsilon \text{ for any finite set } F \text{ such that } F_0 \subseteq F \subseteq \Gamma,$$

(2) $\operatorname{span}\{e_\gamma : \gamma \in \Gamma\}$ *is dense in X,*

(3) *If $x \in X$ and $x \perp e_\gamma$ for all $\gamma \in \Gamma$, then $x = 0$,*

(4) *(Parseval's identity) For any $x \in X$, $\|x\|^2 = \sum_{\gamma\in\Gamma}|\langle x, e_\gamma\rangle|^2$.*

If an orthonormal set $(e_\gamma)_{\gamma\in\Gamma}$ satisfies any of the equivalent conditions of Theorem 52, it is said to be an **orthonormal basis** of X.

Caution. If X is an infinite dimensional Hilbert space, then an orthonormal basis of X is *not* a basis of X in the vector space sense. Specifically, an orthonormal basis of X does not span X if X is an infinite dimensional Hilbert space.

Proposition 53. *Every Hilbert space has an orthonormal basis.*

If Γ is an arbitrary set, let $\ell^2(\Gamma)$ be the L^2 space on Γ with respect to the counting measure.

Corollary 54. *Let X be a Hilbert space and let $(e_\gamma)_{\gamma\in\Gamma}$ be an orthonormal basis for X. Then the linear operator $T : X \to \ell^2(\Gamma)$ defined by $Tx = (\langle x, e_\gamma\rangle)_{\gamma\in\Gamma}$ is an isometry from X onto $\ell^2(\Gamma)$.*

6. Hahn-Banach Theorem

Definition 55. *Let X be a* real *vector space. A function $p : X \to \mathbb{R}$ is called a* **sublinear functional** *if $p(x+y) \le p(x) + p(y)$ for all $x, y \in X$, and $p(\alpha x) = \alpha p(x)$ for all $0 \le \alpha \in \mathbb{R}$ and all $x \in X$.*

Lemma 56. *Let X be a real vector space and let $p : X \to \mathbb{R}$ be a sublinear functional on X. Suppose that Y is a vector subspace of X and $g : Y \to \mathbb{R}$ is a linear functional such that $g(y) \le p(y)$ for all $y \in Y$. If $u \in X$, then there exists a linear functional $f : \operatorname{span}(Y \cup \{u\}) \to \mathbb{R}$ such that $f(y) = g(y)$ for all $y \in Y$ and that $f(x) \le p(x)$ for all $x \in \operatorname{span}(Y \cup \{u\})$.*

Proposition 57. *(General Hahn-Banach extension theorem) Let X be a real vector space and let $p : X \to \mathbb{R}$ be a sublinear functional on X. Suppose that Y is a vector subspace of X and $g : Y \to \mathbb{R}$ is a linear functional such that $g(y) \le p(y)$ for all $y \in Y$. Then there exists a linear functional $f : X \to \mathbb{R}$ such that $f(y) = g(y)$ for all $y \in Y$ and that $f(x) \le p(x)$ for all $x \in X$.*

Definition 58. *Let X be a normed space. The space $L(X, \mathbb{K})$ (the space of all* **bounded linear functionals***) is called the* **dual space** *of X and is denoted by X'. (In many books, the symbol X^* is also used.) It is a Banach space under the operator norm*

$$\|f\| = \sup_{\|x\| \le 1} |f(x)| \text{ for all } f \in X'.$$

Theorem 59. *(Hahn-Banach extension theorem for normed spaces) Let $(X, \|\cdot\|)$ be a normed space and let Y be a subspace of X. If $g \in Y'$, then there exists $f \in X'$ such that $f(y) = g(y)$ for all $y \in Y$ and that $\|f\| = \|g\|$.*

In the final equation of the theorem, $\|f\|$ refers to the norm of f in X' and $\|g\|$ refers to the norm of g in Y'.

Corollary 60. *Let Y be a closed subspace of a normed space X and let $x \in X \setminus Y$. There exists $f \in X'$ of norm 1 such that $f(x) = \|x\|$.*

Corollary 61. *Let Y be a subspace of a normed space X. Suppose that the only $f \in X'$ such that $f(y) = 0$ for all $y \in Y$ is the 0 functional. Then Y is dense in X.*

Proposition 62. *(Adjoint operator) Let X and Y be normed spaces and let $T : X \to Y$ be a bounded linear operator. For each $g \in Y'$, define $f_g : X \to \mathbb{K}$ by $f_g(x) = g(Tx)$. Then $f_g \in X'$. The map $T' : Y' \to X'$ defined by $T'g = f_g$ is a bounded linear operator from Y' to X'. It is called the* **adjoint** *of T. Thus*

$$(T'g)(x) = g(Tx) \text{ for all } x \in X \text{ and all } g \in Y'.$$

Furthermore, $\|T'\| = \|T\|$.

Proposition 63. *(Duality between subspaces and quotients) Let X be a normed space and let Y be a closed subspace.*

6. Hahn-Banach Theorem

Definition 55. *Let X be a real vector space. A function $p : X \to \mathbb{R}$ is called a* **sublinear functional** *if $p(x+y) \leq p(x) + p(y)$ for all $x, y \in X$, and $p(\alpha x) = \alpha p(x)$ for all $0 \leq \alpha \in \mathbb{R}$ and all $x \in X$.*

Lemma 56. *Let X be a real vector space and let $p : X \to \mathbb{R}$ be a sublinear functional on X. Suppose that Y is a vector subspace of X and $g : Y \to \mathbb{R}$ is a linear functional such that $g(y) \leq p(y)$ for all $y \in Y$. If $u \in X$, then there exists a linear functional $f : \operatorname{span}(Y \cup \{u\}) \to \mathbb{R}$ such that $f(y) = g(y)$ for all $y \in Y$ and that $f(x) \leq p(x)$ for all $x \in \operatorname{span}(Y \cup \{u\})$.*

Proposition 57. *(General Hahn-Banach extension theorem) Let X be a real vector space and let $p : X \to \mathbb{R}$ be a sublinear functional on X. Suppose that Y is a vector subspace of X and $g : Y \to \mathbb{R}$ is a linear functional such that $g(y) \leq p(y)$ for all $y \in Y$. Then there exists a linear functional $f : X \to \mathbb{R}$ such that $f(y) = g(y)$ for all $y \in Y$ and that $f(x) \leq p(x)$ for all $x \in X$.*

Definition 58. *Let X be a normed space. The space $L(X, \mathbb{K})$ (the space of all* **bounded linear functionals***) is called the* **dual space** *of X and is denoted by X'. (In many books, the symbol X^* is also used.) It is a Banach space under the operator norm*

$$\|f\| = \sup_{\|x\| \leq 1} |f(x)| \text{ for all } f \in X'.$$

Theorem 59. *(Hahn-Banach extension theorem for normed spaces) Let $(X, \|\cdot\|)$ be a normed space and let Y be a subspace of X. If $g \in Y'$, then there exists $f \in X'$ such that $f(y) = g(y)$ for all $y \in Y$ and that $\|f\| = \|g\|$.*

In the final equation of the theorem, $\|f\|$ refers to the norm of f in X' and $\|g\|$ refers to the norm of g in Y'.

Corollary 60. *Let Y be a closed subspace of a normed space X and let $x \in X \setminus Y$. There exists $f \in X'$ of norm 1 such that $f(y) = 0$ for all $y \in Y$ and that $f(x) = d(x, Y)$.*

Corollary 61. *Let Y be a subspace of a normed space X. Suppose that the only $f \in X'$ such that $f(y) = 0$ for all $y \in Y$ is the 0 functional. Then Y is dense in X.*

Proposition 62. *(Adjoint operator) Let X and Y be normed spaces and let $T : X \to Y$ be a bounded linear operator. For each $g \in Y'$, define $f_g : X \to \mathbb{K}$ by $f_g(x) = g(Tx)$. Then $f_g \in X'$. The map $T' : Y' \to X'$ defined by $T'g = f_g$ is a bounded linear operator from Y' to X'. It is called the* **adjoint** *of T. Thus*

$$(T'g)(x) = g(Tx) \text{ for all } x \in X \text{ and all } g \in Y'.$$

Furthermore, $\|T'\| = \|T\|$.

Proposition 63. *(Duality between subspaces and quotients) Let X be a normed space and let Y be a closed subspace.*

(1) *Let $q : X \to X/Y$ be the quotient map given by $q(x) = x + Y$. Then the adjoint q' is an isometry from $(X/Y)'$ into X'.*
(2) *Let $i : Y \to X$ be the inclusion map $i(y) = y$. Set*
$$Z = \ker i' = \{f \in X' : i'(f) = 0\}.$$
Then Z is a closed subspace of X'. Let $Q : X' \to X'/Z$ be the quotient map $Qf = f + Z$. There is an isometry j from X'/Z onto Y' such that $i' = j \circ Q$. In particular, Y' is isometric to a quotient space of X'.

Let X be a normed space. Since X' is also a normed space (in fact, a Banach space), it has a dual $X'' = (X')'$. X'' is the space of bounded linear functionals on X'.

Proposition 64. *(Canonical embedding of X in X'') Let X be a normed space. For each $x \in X$, define*
$$F_x : X' \to \mathbb{K} \text{ by } F_x(f) = f(x).$$
Then $F_x \in X''$. Moreover, the map $J : X \to X''$ defined by $Jx = F_x$ is an isometry from X into X''.

The map J_X is called the **canonical embedding of X into X''**. A normed space X is **reflexive** if J_X maps X onto X''.

Proposition 65. *Let X and Y be normed spaces and let $T : X \to Y$ be a bounded linear map. Denote the canonical embeddings from X into X'' and Y into Y'' by J_X and J_Y respectively. Then $J_Y T = T'' J_X$.*

Proposition 66. (1) *Any finite dimensional normed space is reflexive.*
(2) *If X is a reflexive normed space, then X is complete and hence a Banach space.*
(3) *A normed space is reflexive if and only if all of its closed subspaces are reflexive.*
(4) *Let X and Y be isomorphic normed spaces. Then either both X and Y are reflexive or neither one is reflexive.*
(5) *A Banach space X is reflexive if and only if X' is reflexive.*

Proposition 67. *The spaces c_0 and ℓ^1 are not reflexive.*

Proposition 68. *Let (Ω, Σ, μ) be a measure space. Both $L^1(\Omega, \Sigma, \mu)$ and $L^\infty(\Omega, \Sigma, \mu)$ are nonreflexive unless finite dimensional.*

Proposition 69. *Let K be an infinite compact Hausdorff space. Then $C(K)$ is not reflexive.*

Proposition 70. *Any Hilbert space is a reflexive Banach space.*

We will return to consider the reflexivity of L^p spaces for $1 < p < \infty$, $p \neq 2$.

Example 78. *It is known that if $f \in L^1(\mathbb{R})$ with respect to Lebesgue measure, then the Fourier transform*

$$\widehat{f}(x) = \int f(y)e^{-ixy}\, d\lambda(y)$$

is a continuous function on $\mathbb{R}$ such that $\lim_{|x|\to\infty} \widehat{f}(x) = 0$. However: There exists a continuous function g on $\mathbb{R}$ with $\lim_{|x|\to\infty} g(x) = 0$ so that g is not the Fourier transform of any function in $L^1(\mathbb{R})$.

8. Weak and weak* topologies. Locally convex spaces. Separation theorems.

Proposition 79. *(Weak and weak* topologies) Let X be a normed space with dual space X'.*

(1) *Let I be a finite subset of X', $x \in X$ and $r > 0$. Define*
$$B_I(x,r) = \{y \in X : |f(x) - f(y)| < r \text{ for all } f \in I\}.$$
Then the family
$$\mathcal{B} = \{B_I(x,r) : I \text{ is a finite subset of } X',\ x \in X,\ r > 0\}$$
is a basis for a topology on X, called the **weak topology**.

(2) *Let I be a finite subset of X, $f \in X'$ and $r > 0$. Define*
$$B_I(f,r) = \{g \in X' : |f(x) - g(x)| < r \text{ for all } x \in I\}.$$
Then the family
$$\mathcal{B} = \{B_I(f,r) : I \text{ is a finite subset of } X,\ f \in X',\ r > 0\}$$
is a basis for a topology on X', called the **weak* topology**.

It is easy to check that every weakly open set in X is norm open and every weak* open set in X' is norm open. I will denote the weak topology on X by w and the weak* topology on X' by w^*. These topologies are locally convex vector topologies in the following sense.

Definition 80. *Let X be a vector space and let $\mathcal{T}$ be a topology on X. $\mathcal{T}$ is a* **vector topology** *and $(X,\mathcal{T})$ is a* **topological vector space (TVS)** *if the maps*
$$+ : X \times X \to X,\ +(x,y) = x + y \text{ and} \cdot : \mathbb{K} \times X \to X,\ \cdot(\alpha, x) = \alpha x$$
are continuous. Here, X is given the topology $\mathcal{T}$, $\mathbb{K}$ is given the norm topology, and $X \times X$ and $\mathbb{K} \times X$ are given the respective product topologies. A vector topology $\mathcal{T}$ is **locally convex** *if for any $x \in X$ and any $U \in \mathcal{T}$ with $x \in U$, there exists a convex set $V \in \mathcal{T}$ such that $x \in V \subseteq U$. If $\mathcal{T}$ is a locally convex vector topology on X, we say that $(X,\mathcal{T})$ is a* **locally convex topological vector space (LCTVS)**.

Proposition 81. *Let X be a normed space. Then (X,w) and (X',w^*) are locally convex topological vector spaces. Both topologies are Hausdorff.*

Proposition 82. *Let X be a normed space.*

(1) *$(X,w)' = X'$, i.e., a linear functional f on X is norm continuous if and only if it is continuous with respect to the weak topology.*
(2) *$(X',w^*)' = X$, i.e., a linear functional F on X' is continuous with respect to the weak* topology if and only if there exists (unique) $x \in X$ such that $F(f) = f(x)$ for all $f \in X'$.*

Proposition 83. *Let X and Y be normed spaces. Suppose that $T : X \to Y$ is a bounded linear operator. Then T is continuous with respect to the weak topologies on X and Y respectively.*

The following geometric versions of the Hahn-Banach Theorem are extremely useful. We will see some applications shortly.

Theorem 84. *(First separation theorem) Let A and B be disjoint, nonempty, convex sets in a TVS X. Suppose that A is open. Then there exist $f \in X'$ and $\gamma \in \mathbb{R}$ such that*

$$\operatorname{Re} f(x) < \gamma \leq \operatorname{Re} f(y) \quad \text{for all } x \in A \text{ and all } y \in B.$$

Theorem 85. *(Second separation theorem) Let A and B be disjoint, nonempty, convex sets in an LCTVS X. Suppose that A is compact and B is closed. Then there exist $f \in X'$ and $\gamma_1, \gamma_2 \in \mathbb{R}$ such that*

$$\operatorname{Re} f(x) \leq \gamma_1 < \gamma_2 \leq \operatorname{Re} f(y) \quad \text{for all } x \in A \text{ and all } y \in B.$$

Theorem 86. *(Mazur) Let X be normed space and let C be a convex subset of X. Then the norm closure of C and the weak closure of C coincide.*

Corollary 87. *Let X be a normed space and let (x_k) be a sequence in X. Suppose that there exists $x_0 \in X$ so that $\lim_{k\to\infty} f(x_k) = f(x_0)$ for all $f \in X'$. Then there is a sequence (y_k) in $\operatorname{co}\{x_k : k \in \mathbb{N}\}$ so that $\lim_{k\to\infty} y_k = x_0$ in norm.*

Example 88. *Let $X = L^2[0, 2\pi]$. Fix a sequence (c_k) in $\mathbb{R}$ so that $\lim c_k = \infty$. Set $g_k(x) = e^{ic_k x}$, $k \in \mathbb{N}$. Then $g_k \in X$. For any $f \in L^2[0, 2\pi]$, extend it to a function on $\mathbb{R}$ be defining it to be 0 outside the interval $[0, 2\pi]$. Then we may regard f as a function in $L^2(\mathbb{R})$. The value of its Fourier transform at c_k is*

$$\widehat{f}(c_k) = \int_0^{2\pi} f(y) e^{-ic_k y}\, d\lambda(y) = \int f \overline{g_k}\, d\lambda = \langle f, g_k\rangle,$$

where the last inner product is the standard inner product on $L^2[0, \pi]$. Since $L^2[0, 2\pi]$ is a Hilbert space, by the Riesz Representation Theorem (Theorem 49), any $F \in X'$ is determined by some function $f \in L^2[0, 2\pi]$ so that

$$F(h) = \langle h, f\rangle \text{ for all } h \in L^2[0, 2\pi].$$

Then

$$F(g_k) = \langle g_k, f\rangle = \overline{\widehat{f}(c_k)} \to 0 \text{ by the Riemann-Lebesgue Lemma.}$$

By Corollary 87, there is a sequence (h_k) in $\operatorname{co}\{g_k : k \in \mathbb{N}\}$ such that $\|h_k\|_2 \to 0$.

Remark. *If c_k's are taken to be integers (diverging to ∞), then (g_k) is a bounded orthogonal sequence. So the averages $\frac{1}{n}\sum_{k=1}^{n} g_k$ converge to 0 in $L^2[0, 2\pi]$.*

It is an important observation that the weak and weak* topologies are manifestations of certain product topologies.

Proposition 89. *Let X be a normed space.*

(1) *The map $i_X : X \to \mathbb{K}^{X'}$ defined by $i_X(x) = (f(x))_{f \in X'}$ is injective. If X is given the weak topology and $\mathbb{K}^{X'}$ is given the product topology, then $i_X : X \to i_X(X)$ is a continuous function with a continuous inverse.*

(2) *The map $j_{X'} : X' \to \mathbb{K}^X$ defined by $j_{X'}(f) = (f(x))_{x \in X}$ is injective. If X' is given the weak* topology and $\mathbb{K}^X$ is given the product topology, then $j_{X'} : X' \to j_{X'}(X')$ is a continuous function with a continuous inverse.*

Theorem 90. *(Banach-Alaoglu) Let X be a normed space. Then $B_{X'}$ is compact in the weak* topology.*

Theorem 91. *(Gantmacher) Let X be a normed space and let $J : X \to X''$ be the canonical embedding. (See Proposition 64.) Then the closure of $J(B_X)$ in the weak* topology on X'' (generated by X') is equal to $B_{X''}$.*

The next result should be compared with Theorem 37.

Theorem 92. *Let X be a normed space. Then X is reflexive if and only if B_X is compact in the weak topology.*

9. Radon-Nikodym Theorem and the dual of L^p

Let (Ω, Σ, μ) be a finite measure space and let $1 \leq p < \infty$. Set $q = p/(p-1)$ ($q = \infty$ if $p = 1$). The main point of this section is to show that the dual space of $L^p(\Omega, \Sigma, \mu)$ can be represented as the space $L^q(\Omega, \Sigma, \mu)$. The main tool needed is the Radon-Nikodym Theorem. Let Σ be a σ-algebra of subsets of a set Ω and let μ, ν be measures defined on Σ. We say that ν is **absolutely continuous with respect to** μ if $\nu(E) = 0$ for all $E \in \Sigma$ with $\mu(E) = 0$.

Theorem 93. *(Radon-Nikodym Theorem) Let μ and ν be finite measures defined on a σ-algebra Σ of subsets of Ω. Assume that ν is absolutely continuous with respect to μ. Then there exists a nonnegative Σ-measurable function f such that $\nu(E) = \int_E f \, d\mu$ for all $E \in \Sigma$. Moreover, f is uniquely determined up to equality μ-a.e.*

Suppose that $\mathbb{K} = \mathbb{R}$. A functional $F \in (L^p(\Omega, \Sigma, \mu))'$ is said to be **positive** if $F(f) \geq 0$ for any $f \in L^p(\Omega, \Sigma, \mu)$ such that $f \geq 0$ μ-a.e.

Lemma 94. *Suppose that $\mathbb{K} = \mathbb{R}$ and that $F \in (L^p(\Omega, \Sigma, \mu))'$. There exists a positive $G \in (L^p(\Omega, \Sigma, \mu))'$ such that both $G - F$ and $G + F$ are positive. G can be chosen so that $\|G\| = \|F\|$.*

Theorem 95. *(The dual of L^p) Let (Ω, Σ, μ) be a finite measure space. Suppose that $1 \leq p < \infty$ and $q = p/(p-1)$ ($1/0 = \infty$). Any $g \in L^q(\Omega, \Sigma, \mu)$ determines a bounded linear functional F_g on $L^p(\Omega, \Sigma, \mu)$ by $F_g(f) = \int fg \, d\mu$. Moreover, $\|F_g\| = \|g\|_q$. Conversely, for any $F \in (L^p(\Omega, \Sigma, \mu))'$, there is a unique $g \in L^q(\Omega, \Sigma, \mu)$ such that $F = F_g$. The map $T : g \mapsto F_g$ is an isometry from $L^q(\Omega, \Sigma, \mu)$ onto $(L^p(\Omega, \Sigma, \mu))'$.*

Corollary 96. *(Reflexivity of L^p) Let (Ω, Σ, μ) be a finite measure space and let $1 < p < \infty$. Then $L^p(\Omega, \Sigma, \mu)$ is a reflexive Banach space.*

Remark. Theorem 93 continues to hold if μ and ν are only assumed to be σ-finite. Theorem 95 and Corollary 96 hold for all measure spaces if $1 < p < \infty$. If $p = 1$, Theorem 95 still holds if (Ω, Σ, μ) is σ-finite. Theorem 95 never holds if $p = \infty$ and $L^\infty(\Omega, \Sigma, \mu)$ is infinite dimensional.

10. The space $C(K)$

In this section, we study the space $C(K)$ as a Banach space, where K is a compact Hausdoff topological space. Recall that the norm on $C(K)$ is the sup-norm: $\|f\| = \sup_{t\in K} |f(t)|$. For notational convenience, I will assume that $\mathbb{K} = \mathbb{R}$ in this section.
Let $\mathcal{B}$ denote the Borel sets in K, i.e, the smallest σ-algebra generated by the open sets in K. A measure defined on the measurable space $(K, \mathcal{B})$ is called a **Borel measure**. A Borel measure μ is **regular** if for every $E \in \mathcal{B}$,

$$\mu(E) = \sup\{\mu(F) : F \subseteq E, F \text{ compact}\} = \inf\{\mu(O) : E \subseteq O, O \text{ open}\}.$$

Proposition 97. *Let μ_1 and μ_2 be regular Borel measures on $(K, \mathcal{B})$. Define*

$$F : C(K) \to \mathbb{R} \text{ by } F(f) = \int f\, d\mu_1 - \int f\, d\mu_2.$$

Then $F \in (C(K))'$ and $\|F\| \leq \mu_1(K) + \mu_2(K)$.

A functional $F \in C(K)'$ is said to be **positive** if $F(f) \geq 0$ for any $f \in C(K)$ with $f \geq 0$.

Lemma 98. *Let $F \in C(K)'$. There is a positive $G \in C(K)'$ such that $G - F$ and $G + F$ are both positive. G can be chosen so that $\|F\| = \|G\|$.*

Theorem 99. *Let $F \in C(K)'$ be a positive functional. There is a unique regular Borel measure on $(K, \mathcal{B})$ so that*

$$F(f) = \int f\, d\mu \text{ for all } f \in C(K).$$

Moreover, $\|F\| = \mu(K)$.
If H is a general element of $C(K)'$, there is a unique pair of regular Borel measures (μ_1, μ_2) on $(K, \mathcal{B})$ so that $\|H\| = \mu_1(K) + \mu_2(K)$ and that

$$H(f) = \int f\, d\mu_1 - \int f\, d\mu_2 \text{ for all } f \in C(K).$$

We give several applications of Theorem 99 below. Recall that two topological spaces V and W are **homeomorphic** if there is a continuous bijection $h : V \to W$ so that h^{-1} is also continuous.

Proposition 100. *For each $t \in K$, define $\delta_t : C(K) \to \mathbb{R}$ by $\delta_t(f) = f(t)$. Then $\delta_t \in C(K)'$ and $\|\delta_t\| = 1$.*
Give the set $S = \{\delta_t : t \in K\}$ the weak topology (as a subset of $C(K)'$). The function $h : K \to S$ defined by $h(t) = \delta_t$ is a homeomorphism.*
$B_{C(K)'}$ is the weak-closure of the convex hull $\mathrm{co}\{\pm\delta_t : t \in K\}$.*

A normed space is said to be **separable** if it contains a countable dense subset.

Theorem 101. *Let K be a compact Hausdorff topological space. The space $C(K)$ is separable if and only if the topology on K is given by a metric.*

Let Δ be the Cantor set. We will use the following result from the theory of metric spaces.

Theorem 102. *Let K be a compact metric space. There is a continuous function π from Δ onto K.*

Theorem 103. *($C(\Delta)$ as a universal separable Banach space) Let X be a separable Banach space. Then X is isometric to a subspace of $C(\Delta)$.*

Let X be a Banach space. An element $x \in B_X$ is an **extreme point** of B_X if $x = \frac{y+z}{2}$, $y, z \in B_X$ implies that $y = z = x$.

Theorem 104. *(Banach-Stone Theorem) Let K and L be compact Hausdorff spaces. Then $C(K)$ is isometric to $C(L)$ if and only if K and L are homeomorphic.*

11. Compact operators. Spectral theorem of compact self-adjoint operators on Hilbert space.

Let X and Y be normed spaces. A linear map $T : X \to Y$ is **compact** if $\overline{TB_X}$ is a compact set in Y. A compact map is always bounded. (Verify!) The set of compact linear maps from X to Y is denoted by $K(X, Y)$.
A bounded linear operator $T : X \to Y$, where X and Y are normed spaces, is a **finite rank** operator if $\operatorname{range} T$ is finite dimensional.

Proposition 105. *Let X and Y be normed spaces. Any finite rank operator $T \in L(X, Y)$ is compact.*

$K(X, Y)$ has the following properties which are similar to those of an "ideal" in the sense of abstract algebra.

Proposition 106. *Let W, X, Y and Z be normed spaces.*

(1) *If $T, S \in K(X, Y)$ and $\alpha \in \mathbb{K}$, then $\alpha T + S \in K(X, Y)$.*
(2) *If $T \in K(X, Y)$, $R \in L(Y, Z)$, $S \in L(W, X)$, then $RT \in K(X, Z)$ and $TS \in K(W, Y)$.*

Proposition 107. *Let X be a normed space and let Y be a Banach space. Let (T_n) be a sequence in $K(X, Y)$ and suppose that $\lim_{n\to\infty} \|T_n - T\| = 0$ for some bounded linear map $T \in L(X, Y)$. Then $T \in K(X, Y)$.*

Theorem 108. *(Schauder) Let $T : X \to Y$ be a bounded linear map between Banach spaces X and Y. Then T is compact if and only if T' is compact.*

Let X be a Banach space. Denote the identity operator on X by $I : X \to X$.

Proposition 109. *Let X be a Banach space and let $K : X \to X$ be a compact linear map. If $\ker(I - K) \neq \{0\}$, then $I - K$ cannot be surjective.*

Let Y be a closed subspace of a Banach space X. Y is said to have **finite codimension in** X if X/Y is finite dimensional. In this case, $\dim(X/Y)$ is called the **codimension** of Y in X and is denoted by $\operatorname{codim} Y$.

Theorem 110. *(Fredholm alternative) Let X be a Banach space and let $K : X \to X$ be a compact linear map. Define $T = I - K$, where I is the identity map on X.*

(1) *The kernel of T, $\ker T = \{x \in X : Tx = 0\}$, is a finite dimensional subspace of X.*
(2) *The range of T, $\operatorname{range} T = \{Tx : x \in X\}$, is a closed subspace of X with finite codimension.*
(3) *The kernel of T' is a finite dimensional subspace of X'.*
(4) *The range of T' is a closed subspace of X' with finite codimension.*
(5) *A vector x belongs to $\operatorname{range} T$ if and only if $f(x) = 0$ for all $f \in \ker T'$.*
(6) $\dim \ker T = \operatorname{codim} \operatorname{range} T = \dim \ker T' = \operatorname{codim} \operatorname{range} T'$.

Definition 111. *Let X be a Banach space and let $T : X \to X$ be a bounded linear map. A scalar λ is in the* **spectrum of** *T, $\sigma(T)$, if $\lambda I - T$ does* *not have an inverse in* $L(X)$*. An* **eigenvalue** *of* T *is a scalar* λ *such that* $\ker(\lambda I - T) \neq \{0\}$*. The subspace* $\ker(\lambda I - T)$ *is called the* **eigenspace** *of* T *corresponding to* λ*. We denote it by* X_λ*.*

Theorem 112. *Let* X *be a Banach space and* $K : X \to X$ *a compact linear map.*

(1) $\sigma(K) = \sigma(K')$.
(2) *If* λ *is a nonzero number in* $\sigma(K)$*, then* λ *is an eigenvalue of* K *and and eigenvalue of* K'*.*
(3) *The spectrum of* K *is either a finite set or of the form* $\{\lambda_n : n \in \mathbb{N}\} \cup \{0\}$ *for a sequence* (λ_n) *that converges to* 0*. If* X *is infinite dimensional,* $0 \in \sigma(K)$*.*

11.1. **Compact operators on Hilbert space. Spectral theorem.** Let X be a Hilbert space and let T be a bounded linear operator on X. Recall from Corollary 50 that the Hilbert adjoint operator T^* is the bounded linear operator on X determined by the equation

$$\langle Tx, y\rangle = \langle x, T^*y\rangle$$

for all $x, y \in X$. Moreover, $\|T^*\| = \|T\|$.

Caution. A Hilbert space X may be regarded as a Banach space. Then any bounded linear operator on X also has a Banach adjoint T'. (See Proposition 62.) The Banach and Hilbert adjoints act on different spaces, so they cannot be equal.

Proposition 113. *Let* S *and* T *be bounded linear operators on a Hilbert space* X *and let* α *be a scalar. Then*

(1) $\|T\| = \|T^*\|$,
(2) $(S+T)^* = S^* + T^*$,
(3) $(\alpha T)^* = \overline{\alpha} T^*$,
(4) $(ST)^* = T^* S^*$,
(5) $T^{**} = T$.

Definition 114. *Let* X *be a Hilbert space. An bounded linear operator* $T : X \to X$ *is said to be* **self-adjoint** *if* $T^* = T$*.*

Definition 115. *A linear map* $P : X \to X$ *on a Hilbert space* X *is an* **orthogonal projection** *if there is a closed subspace* Y *of* X *so that* $Py = y$ *for all* $y \in Y$ *and* $Pz = 0$ *for all* $z \in Y^\perp$*.*

Proposition 116. *Let* X *be a Hilbert space. Every closed subspace* Y *is the range of an orthogonal projection. A bounded linear map* $P : X \to X$ *is an orthogonal projection if and only if* $P^2 = P$ *and* $P^* = P$*.*

Proposition 117. *Let* T *be a bounded self-adjoint operator on a Hilbert space* X*. Then*

(1) *Any eigenvalue of* T *is a real number.*

(2) *If* λ *and* μ *are distinct eigenvalues of* T*, then* $X_\lambda \perp X_\mu$*, i.e.,* $\langle x, y\rangle = 0$ *if* $x \in X_\lambda$ *and* $y \in X_\mu$*.*
(3) *If* λ *is an eigenvalue of* T *and* P_λ *denotes the orthogonal projection onto* X_λ*, then* $TP_\lambda = P_\lambda T$*.*
(4) $\|T\| = \sup\{|\langle Tx, x\rangle| : \|x\| \leq 1\}$*.*
(5) *Either* $\|T\|$ *or* $-\|T\|$ *belongs to* $\sigma(T)$*.*

Theorem 118. *(Spectral theorem for compact self-adjoint operators) Let* T *be a compact self-adjoint operator on a Hilbert space* X*.*

(1) *The spectrum of* T *is either a finite set or of the form* $\{\lambda_n : n \in \mathbb{N}\} \cup \{0\}$ *for a real sequence* (λ_n) *that converges to* 0*. If* X *is infinite dimensional,* $0 \in \sigma(T)$*.*
(2) *Each* $\lambda \in \sigma(T) \setminus \{0\}$ *is an eigenvalue of* T*. Denote the orthogonal projection onto the eigenspace* $\ker(\lambda I - T)$ *by* P_λ*.*
(3) $T = \sum_{\lambda\in\sigma(T)\setminus\{0\}} \lambda P_\lambda$*, where the sum converges in operator norm.*

If $\sigma(T)$ is a finite set, then the sum in (3) has only finitely many terms and convergence does not come into play. If $\sigma(T)$ is an infinite set, then $\sigma(T) \setminus \{0\}$ is an infinite sequence (λ_n) that converges to 0. Then the sum in (3) is defined to be $\sum_{n=1}^{\infty} \lambda_n P_{\lambda_n}$.