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The Theory of $\mathcal{H}(b)$ Spaces

Volume 2

Emmanuel Fricain
and Javad Mashreghi

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The Theory of $\mathcal{H}(b)$ Spaces

Volume 2

An $\mathcal{H}(b)$ space is defined as a collection of analytic functions that are in the image of an operator. The theory of $\mathcal{H}(b)$ spaces bridges two classical subjects, complex analysis and operator theory, which makes it both appealing and demanding.

Volume 1 of this comprehensive treatment is devoted to the preliminary subjects required to understand the foundation of $\mathcal{H}(b)$ spaces, such as Hardy spaces, Fourier analysis, integral representation theorems, Carleson measures, Toeplitz and Hankel operators, various types of shift operators and Clark measures. Volume 2 focuses on the central theory. Both books are accessible to graduate students as well as researchers: each volume contains numerous exercises and hints, and figures are included throughout to illustrate the theory. Together, these two volumes provide everything the reader needs to understand and appreciate this beautiful branch of mathematics.

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Volume 2

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To our families:
Keiko & Shahzad
Hugo & Dorsa, Parisa, Golsa

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Preface

In 1915, Godfrey Harold Hardy, in a famous paper published in the *Proceedings of the London Mathematical Society*, first put forward the "theory of Hardy spaces" H^p. Having a Hilbert space structure, H^2 also benefits from the rich theory of Hilbert spaces and their operators. The mutual interaction of analytic function theory, on the one hand, and operator theory, on the other, has created one of the most beautiful branches of mathematical analysis. The Hardy–Hilbert space H^2 is the glorious king of this seemingly small, but profoundly deep, territory.

In 1948, in the context of dynamics of Hilbert space operators, A. Beurling classified the closed invariant subspaces of the forward shift operator on ℓ^2. The genuine idea of Beurling was to exploit the forward shift operator S on H^2. To that end, he used some analytical tools to show that the closed subspaces of H^2 that are invariant under S are precisely of the form ΘH^2, where Θ is an inner function. Therefore, the orthogonal complement of the Beurling subspace ΘH^2, the so-called *model subspaces* K_Θ, are the closed invariant subspaces of H^2 that are invariant under the backward shift operator S^*. The model subspaces have rich algebraic and analytic structures with applications in other branches of mathematics and science, for example, control engineering and optics.

The word "model" that was used above to describe K_Θ refers to their application in recognizing the Hilbert space contractions. The main idea is to identify (via a unitary operator) a contraction as the adjoint of multiplication by z on a certain space of analytic functions on the unit disk. As Beurling's theorem says, if we restrict ourselves to closed subspaces of H^2 that are invariant under S^*, we just obtain K_Θ spaces, where Θ runs through the family of inner functions. This point of view was exploited by B. Sz.-Nagy and C. Foiaş to construct a model for Hilbert space contractions. Another way is to consider submanifolds (not necessarily closed) of H^2 that are invariant under S^*. Above half a century ago, such a modeling theory was developed by L. de Branges and J. Rovnyak. In this context, they introduced the so-called $\mathcal{H}(b)$ spaces. The de Branges–Rovnyak model is, in a certain sense, more flexible, but it causes certain difficulties. For example, the inner product in $\mathcal{H}(b)$ is not given by an explicit integral formula, contrary to the case for K_Θ, which is actually the inner product of H^2, and this makes the treatment of $\mathcal{H}(b)$ functions more difficult.

The original definition of $\mathcal{H}(b)$ spaces uses the notion of *complementary space*, which is a generalization of the orthogonal complement in a Hilbert space. But $\mathcal{H}(b)$ spaces can also be viewed as the range of a certain operator involving Toeplitz operators. This point of view was a turning point in the theory of $\mathcal{H}(b)$ spaces. Adopting the new definition, D. Sarason and several others made essential contributions to the theory. In fact, they now play a key role in many other questions of function theory (solution of the Bieberbach conjecture by de Branges, rigid functions of the unit ball of H^1, Schwarz–Pick inequalities), operator theory (invariant subspaces problem, composition operators), system theory and control theory. An excellent but very concise account of the theory of $\mathcal{H}(b)$ spaces is available in Sarason's masterpiece [166]. However, there are many results, both new and old, that are not covered there. On the other hand, despite many efforts, the structure and properties of $\mathcal{H}(b)$ spaces still remain mysterious, and numerous natural questions still remain open. However, these spaces have a beautiful structure, with numerous applications, and we hope that this work attracts more people to this domain.

In this context, we have tried to provide a rather comprehensive *introduction* to the theory of $\mathcal{H}(b)$ spaces. That is why Volume 1 is devoted to the foundation of $\mathcal{H}(b)$ spaces. In Volume 2, we discuss $\mathcal{H}(b)$ spaces and their applications. However, two facts should be kept in mind: first, we just treat the scalar case of $\mathcal{H}(b)$ spaces; and second, we do not discuss in detail the theory of model operators, because there are already two excellent monographs on this topic [138]; [184]. Nevertheless, some of the tools of model theory are implicitly exploited in certain topics. For instance, to treat some natural questions such as the inclusion between two different $\mathcal{H}(b)$ spaces, we use a geometric representation of $\mathcal{H}(b)$ spaces that comes from the relation between Sz.-Nagy–Foiaş and de Branges–Rovnyak modeling theory. Also, even if the main point of view that has been adopted in this book is based on the definition of $\mathcal{H}(b)$ via Toeplitz operators, the historical definition of de Branges and Rovnyak is also discussed and used at some points.

In the past decade, both of us have made several transatlantic trips to meet each other and work together on this book project. For these visits, we have been financially supported by Université Claude Bernard Lyon I, Université Lille 1, Université Laval, McGill University, Centre Jacques-Cartier (France), CNRS (Centre National de la Recherche Scientifique, France), ANR (Agence Nationale de la Recherche), FQRNT (Fonds Québécois de la Recherche sur la Nature et les Technologies) and NSERC (Natural Sciences and Engineering Research Council of Canada). We also benefited from the warm hospitality of CIRM (Centre International des Rencontres Mathématiques, Luminy), CRM (Centre de Recherches Mathématiques, Montréal) and the Fields Institute (Toronto). We thank them all warmly for their support.

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16

The spaces $\mathcal{M}(A)$ and $\mathcal{H}(A)$

In this chapter, we introduce the notion of *complementary space*, which generalizes the classic geometric notion of orthogonal complement. This notion of complementary space is central in the theory of $\mathcal{H}(b)$ spaces. In Section 16.1, we study the bounded (contractively or isometrically) embeddings. This leads to the definition of $\mathcal{M}(A)$ spaces. Then, in Section 16.2, we characterize the relations between two $\mathcal{M}(A)$ spaces. In Section 16.3, we describe the linear functional on $\mathcal{M}(A)$. In Section 16.4, we give our first definition of complementary space based on an operatorial point of view. As we will see in the next chapter, this operatorial point of view seems particularly interesting in the context of $\mathcal{H}(b)$ spaces and Toeplitz operators. In Section 16.5, we describe the relation between $\mathcal{H}(A)$ and $\mathcal{H}(A^*)$. This relation, though very simple, is probably one of the most useful results in the theory of $\mathcal{H}(b)$ spaces. The overlapping space is introduced and described in Section 16.6. In Sections 16.7 and 16.8, we give useful results concerning some decomposition of $\mathcal{M}(A)$ and $\mathcal{H}(A)$ spaces. In Section 16.9, we introduce our second definition of complementary space and show that it coincides with the first one. Finally, in the last section, we show how the Julia operator can be used to connect this notion of complementary spaces to the more familiar geometric structure of orthogonal complements.

16.1 The space $\mathcal{M}(A)$

Suppose that $\mathcal{H}_1$ and $\mathcal{H}_2$ are Hilbert spaces and $\mathcal{H}_1 \subset \mathcal{H}_2$. We do not necessarily assume that $\mathcal{H}_1$ inherits the Hilbert structure of $\mathcal{H}_2$. They can have different Hilbert space structures. The assumption $\mathcal{H}_1 \subset \mathcal{H}_2$ ensures that the inclusion mapping

$$\begin{array}{rccc} i: & \mathcal{H}_1 & \longrightarrow & \mathcal{H}_2 \\ & x & \longmapsto & x \end{array}$$

is well defined. If this mapping is bounded, i.e. if there is a constant $c > 0$ such that

$$\|x\|_{\mathcal{H}_2} \le c\,\|x\|_{\mathcal{H}_1} \qquad (x \in \mathcal{H}_1), \tag{16.1}$$

we say that $\mathcal{H}_1$ is *boundedly* contained in $\mathcal{H}_2$ and write $\mathcal{H}_1 \Subset \mathcal{H}_2$. If the mapping i is a contraction, i.e. $c \le 1$, we say that $\mathcal{H}_1$ is *contractively* included in $\mathcal{H}_2$ and write $\mathcal{H}_1 \hookrightarrow \mathcal{H}_2$. Finally, if

$$\|x\|_{\mathcal{H}_2} = \|x\|_{\mathcal{H}_1} \qquad (x \in \mathcal{H}_1),$$

we say that $\mathcal{H}_1$ is *isometrically* contained in $\mathcal{H}_2$. If it happens that the set identity $\mathcal{H}_1 = \mathcal{H}_2$ holds and, moreover, $\mathcal{H}_1$ and $\mathcal{H}_2$ have the same Hilbert space structure, i.e. $\|x\|_{\mathcal{H}_2} = \|x\|_{\mathcal{H}_1}$ for all possible x, then we write $\mathcal{H}_1 \doteq \mathcal{H}_2$. It is important to distinguish between the set identity $\mathcal{H}_1 = \mathcal{H}_2$ and the Hilbert space identity $\mathcal{H}_1 \doteq \mathcal{H}_2$.

A very special case of the above phenomenon is when $\mathcal{H}_1$ is a closed subspace of $\mathcal{H}_2$ and inherits its Hilbert space structure. In this case, $\mathcal{H}_1$ is isometrically embedded inside $\mathcal{H}_2$. In the next section, we will look at this phenomenon from a slightly different angle.

The inequality (16.1) reveals some facts about the topologies of $\mathcal{H}_1$ and $\mathcal{H}_2$. If $\mathcal{E}$ is a closed (or open) subset of $\mathcal{H}_2$, then $\mathcal{E} \cap \mathcal{H}_1$ is closed (or open) in $\mathcal{H}_1$ with respect to the topology of $\mathcal{H}_1$. However, the topology of $\mathcal{H}_1$ is usually richer. In other words, the topology of $\mathcal{H}_1$ is finer than the topology it inherits from $\mathcal{H}_2$. That is why, if Λ is a continuous function on $\mathcal{H}_2$, then its restriction to $\mathcal{H}_1$ remains continuous. We will treat this fact in more detail in Section 16.3. As a special case, if $\mathcal{E} \subset \mathcal{H}_1 \subset \mathcal{H}_2$ is closed in $\mathcal{H}_2$, then $\mathcal{E}$ is also closed in $\mathcal{H}_1$. However, if $\mathcal{E}$ is closed in $\mathcal{H}_1$, we cannot conclude that it is also closed in $\mathcal{H}_2$. The following result reveals the relation between different closures of a set in $\mathcal{H}_1$.

Lemma 16.1 *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two Hilbert spaces, assume that $\mathcal{H}_1$ is boundedly embedded into $\mathcal{H}_2$, and let $\mathcal{E} \subset \mathcal{H}_1$. Then*

$$\mathrm{Clos}_{\mathcal{H}_2}(\mathrm{Clos}_{\mathcal{H}_1}\mathcal{E}) = \mathrm{Clos}_{\mathcal{H}_2}\mathcal{E}.$$

Proof For simplicity, put $\mathcal{F} = \mathrm{Clos}_{\mathcal{H}_1}\mathcal{E}$. Since $\mathcal{E} \subset \mathcal{F}$, we have

$$\mathrm{Clos}_{\mathcal{H}_2}\mathcal{E} \subset \mathrm{Clos}_{\mathcal{H}_2}\mathcal{F}.$$

To prove the converse, let $x \in \mathrm{Clos}_{\mathcal{H}_2}\mathcal{F}$ and fix any $\varepsilon > 0$. Then there exists $y \in \mathcal{F}$ such that $\|x - y\|_{\mathcal{H}_2} \le \varepsilon/2$. But, since $y \in \mathcal{F}$ and $\mathcal{F} = \mathrm{Clos}_{\mathcal{H}_1}\mathcal{E}$, there exists $z \in \mathcal{E}$ such that $\|y - z\|_{\mathcal{H}_1} \le \varepsilon/2C$, where C is the constant of embedding of $\mathcal{H}_1$ into $\mathcal{H}_2$, i.e.

$$\|x\|_{\mathcal{H}_2} \le C\,\|x\|_{\mathcal{H}_1} \qquad (x \in \mathcal{H}_1).$$

Therefore, we have $\|y-z\|_{\mathcal{H}_2} \le \varepsilon/2$ and then

$$\|x-z\|_{\mathcal{H}_2} \le \|x-y\|_{\mathcal{H}_2} + \|y-z\|_{\mathcal{H}_2} \le \varepsilon.$$

Therefore, $x \in \mathrm{Clos}_{\mathcal{H}_2}\mathcal{E}$. □

Suppose that $\mathcal{H}_1$ is a Hilbert space, $\mathcal{H}_2$ is a set and $A : \mathcal{H}_1 \longrightarrow \mathcal{H}_2$ is a set bijection between $\mathcal{H}_1$ and $\mathcal{H}_2$. Then the map A can be served to transfer the Hilbert space structure of $\mathcal{H}_1$ to $\mathcal{H}_2$. It is enough to define

$$\langle Ax, Ay\rangle_{\mathcal{H}_2} = \langle x, y\rangle_{\mathcal{H}_1} \tag{16.2}$$

for all $x, y \in \mathcal{H}_1$. The algebraic operations on $\mathcal{H}_2$ are defined similarly. If $\mathcal{H}_2$ is a linear space and A is an algebraic isomorphism between $\mathcal{H}_1$ and $\mathcal{H}_2$, the latter requirement is already fulfilled. In this case, (16.2) puts an inner product, maybe a new one, on $\mathcal{H}_2$.

The above construction sounds very elementary. Nevertheless, it has profound consequences. In fact, it is the main ingredient in the definition of $\mathcal{H}(b)$ spaces. To move in this direction, suppose that $\mathcal{H}_1$ and $\mathcal{H}_2$ are Hilbert spaces and that $A \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$. By the first homomorphism theorem, the operator A induces an isomorphism between the quotient space $\mathcal{H}_1/\ker A$ and $\mathcal{R}(A)$. Hence, by (16.2), the identity

$$\langle Ax, Ay\rangle_{\mathcal{R}(A)} = \langle x + \ker A,\ y + \ker A\rangle_{\mathcal{H}_1/\ker A} \qquad (x, y \in \mathcal{H}_1) \tag{16.3}$$

gives a Hilbert space structure on $\mathcal{R}(A)$. We denote this Hilbert space by $\mathcal{M}(A)$. The norm of $x + \ker A$ in $\mathcal{H}_1/\ker A$ is originally defined by

$$\|x + \ker A\|_{\mathcal{H}_1/\ker A} = \inf_{z \in \ker A} \|x + z\|_{\mathcal{H}_1}.$$

But, for each $z \in \ker A$,

$$\begin{aligned}\|x+z\|^2_{\mathcal{H}_1} &= \|P_{(\ker A)^\perp}x + (z + P_{\ker A}x)\|^2_{\mathcal{H}_1}\\ &= \|P_{(\ker A)^\perp}x\|^2_{\mathcal{H}_1} + \|z + P_{\ker A}x\|^2_{\mathcal{H}_1},\end{aligned}$$

and thus we easily see that

$$\|x + \ker A\|_{\mathcal{H}_1/\ker A} = \|P_{(\ker A)^\perp}x\|_{\mathcal{H}_1} \qquad (x \in \mathcal{H}_1).$$

Hence, by the polarization identity (1.16), we have

$$\langle x+\ker A,\ y+\ker A\rangle_{\mathcal{H}_1/\ker A} = \langle P_{(\ker A)^\perp}x,\ P_{(\ker A)^\perp}y\rangle_{\mathcal{H}_1} \qquad (x, y \in \mathcal{H}_1).$$

Moreover, by (1.27),

$$\langle P_{(\ker A)^\perp}x,\ P_{(\ker A)^\perp}y\rangle_{\mathcal{H}_1} = \langle x,\ P_{(\ker A)^\perp}y\rangle_{\mathcal{H}_1} = \langle P_{(\ker A)^\perp}x,\ y\rangle_{\mathcal{H}_1}.$$

Therefore, the definition (16.3) reduces to

$$\begin{aligned}\langle Ax, Ay\rangle_{\mathcal{M}(A)} &= \langle P_{(\ker A)^\perp}x,\ P_{(\ker A)^\perp}y\rangle_{\mathcal{H}_1}\\ &= \langle x,\ P_{(\ker A)^\perp}y\rangle_{\mathcal{H}_1}\\ &= \langle P_{(\ker A)^\perp}x,\ y\rangle_{\mathcal{H}_1}\end{aligned} \tag{16.4}$$

for each $x, y \in \mathcal{H}_1$. In particular, for each $x \in \mathcal{H}_1$,

$$\|Ax\|_{\mathcal{M}(A)} = \|P_{(\ker A)^\perp}x\|_{\mathcal{H}_1}. \tag{16.5}$$

Moreover, if at least one of x or y is orthogonal to $\ker A$, then, by (16.4),

$$\langle Ax, Ay\rangle_{\mathcal{M}(A)} = \langle x, y\rangle_{\mathcal{H}_1}. \tag{16.6}$$

The rather trivial inequality

$$\|Ax\|_{\mathcal{M}(A)} \le \|x\|_{\mathcal{H}_1} \qquad (x \in \mathcal{H}_1), \tag{16.7}$$

which is a direct consequence of (16.5), will also be frequently used. The preceding formulas should be kept in mind throughout the text.

On $\mathcal{R}(A)$ we now have two inner products. One is inherited from $\mathcal{H}_2$ and the new one imposed by A. In the following, when we write $\mathcal{M}(A)$ we mean that $\mathcal{R}(A)$ is endowed with the latter structure. If this is not the case, we will explicitly mention which structure is considered on $\mathcal{R}(A)$. Let us explore the relation between these two structures. Since A is a bounded operator, we have

$$\|Ax\|_{\mathcal{H}_2} = \|AP_{(\ker A)^\perp}x\|_{\mathcal{H}_2} \le \|A\|_{\mathcal{L}(\mathcal{H}_1,\mathcal{H}_2)}\,\|P_{(\ker A)^\perp}x\|_{\mathcal{H}_1} \qquad (x \in \mathcal{H}_1).$$

Therefore, by (16.5),

$$\|Ax\|_{\mathcal{H}_2} \le \|A\|_{\mathcal{L}(\mathcal{H}_1,\mathcal{H}_2)}\,\|Ax\|_{\mathcal{M}(A)} \qquad (x \in \mathcal{H}_1). \tag{16.8}$$

This inequality means that the inclusion map

$$\begin{array}{rccc} i: & \mathcal{M}(A) & \longrightarrow & \mathcal{H}_2\\ & w & \longmapsto & w \end{array}$$

is continuous and its norm is at most $\|A\|$. In fact, by (16.7),

$$\|Ax\|_{\mathcal{H}_2} \le \|i\|\,\|Ax\|_{\mathcal{M}(A)} \le \|i\|\,\|x\|_{\mathcal{H}_1} \qquad (x \in A).$$

Thus, considering (16.8), we deduce that

$$\|i\|_{\mathcal{L}(\mathcal{M}(A),\mathcal{H}_2)} = \|A\|_{\mathcal{L}(\mathcal{H}_1,\mathcal{H}_2)}. \tag{16.9}$$

Moreover,

$$i^* = AA^*. \tag{16.10}$$

Indeed, let $y \in \mathcal{H}_2$ and $Ax \in \mathcal{M}(A)$, with $x \in \mathcal{H}_1$ and $x \perp \ker A$. Then we have

$$\langle Ax, i^*y\rangle_{\mathcal{M}(A)} = \langle Ax, y\rangle_{\mathcal{H}_2} = \langle x, A^*y\rangle_{\mathcal{H}_1} = \langle Ax, AA^*y\rangle_{\mathcal{M}(A)},$$

which proves that $i^*y = AA^*y$. We will see in Section 16.8 that, in a sense, the operator i^* plays the role of an orthogonal projection of $\mathcal{H}_2$ onto $\mathcal{M}(A)$.

If A is invertible, then the relations (16.7), (16.8) and

$$\|x\|_{\mathcal{H}_1} = \|A^{-1}Ax\|_{\mathcal{H}_1} \leq \|A^{-1}\| \, \|Ax\|_{\mathcal{H}_2}$$

imply that the norms in $\mathcal{H}_1$, $\mathcal{H}_2$ and $\mathcal{M}(A)$ (which as a set is equal to $\mathcal{H}_2$) are equivalent, i.e.

$$\|x\|_{\mathcal{H}_1} \asymp \|Ax\|_{\mathcal{H}_2} \asymp \|Ax\|_{\mathcal{M}(A)}. \tag{16.11}$$

If A is a bounded operator, the above construction puts $\mathcal{M}(A)$ boundedly inside $\mathcal{H}_2$. If A is a contraction, i.e. $\|A\| \leq 1$, then $\mathcal{M}(A)$ is contractively contained in $\mathcal{H}_2$; and if $\|w\|_{\mathcal{M}(A)} = \|w\|_{\mathcal{H}_2}$, $w \in \mathcal{M}(A)$, then $\mathcal{M}(A)$ is isometrically contained in $\mathcal{H}_2$. Based on the conventions made in Section 16.1, we emphasize that, for $A, B \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$, the notation $\mathcal{M}(A) \doteq \mathcal{M}(B)$ means not only that the algebraic equality $\mathcal{M}(A) = \mathcal{M}(B)$ holds, but also that the Hilbert space structures coincide, i.e.

$$\langle w_1, w_2 \rangle_{\mathcal{M}(A)} = \langle w_1, w_2 \rangle_{\mathcal{M}(B)}$$

for all possible elements w_1 and w_2. Clearly, in the light of the polarization identity, the latter is equivalent to

$$\|w\|_{\mathcal{M}(A)} = \|w\|_{\mathcal{M}(B)}$$

for all possible elements w.

The relation (16.5) contains all the information regarding the definition of the structure of $\mathcal{M}(A)$. In short, the structure of $\mathcal{M}(A)$ is the same as that of $\mathcal{H}_1/\ker A$. This fact is explained in another language in the following result.

Theorem 16.2 *Let $A \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ and define*

$$\begin{array}{rccc} \mathbb{A}: & \mathcal{H}_1 & \longrightarrow & \mathcal{M}(A) \\ & x & \longmapsto & Ax. \end{array}$$

Then $\mathbb{A}$ is a bounded operator, i.e. $\mathbb{A} \in \mathcal{L}(\mathcal{H}_1, \mathcal{M}(A))$, and, moreover, $\mathbb{A}^*$ is an isometry on $\mathcal{M}(A)$.

Proof The inequality (16.7) can be rewritten as

$$\|\mathbb{A}x\|_{\mathcal{M}(A)} = \|Ax\|_{\mathcal{M}(A)} \leq \|x\|_{\mathcal{H}_1} \qquad (x \in \mathcal{H}_1).$$

This means that $\mathbb{A}$ is a bounded operator. In order to show that $\mathbb{A}^*$ is an isometry on $\mathcal{M}(A)$, by Corollary 7.23, it is enough to show that $\mathbb{A}$ is a surjective partial isometry. That $\mathbb{A}$ is surjective is a trivial consequence of the definition of $\mathcal{M}(A)$. Moreover, $\ker \mathbb{A} = \ker A$. Hence, by (16.5),

$$\|\mathbb{A}x\|_{\mathcal{M}(A)} = \|Ax\|_{\mathcal{M}(A)} = \|P_{(\ker A)^\perp}x\|_{\mathcal{H}_1} = \|P_{(\ker \mathbb{A})^\perp}x\|_{\mathcal{H}_1}$$

for each $x \in \mathcal{H}_1$. Thus, $\mathbb{A}$ is a partial isometry (see the original definition (7.14)). □

The definition of spaces $\mathcal{M}(A)$ is closely related to the notion of bounded embeddings introduced at the beginning of this section. Indeed, if $\mathcal{M}$ is a Hilbert space that is boundedly contained in another Hilbert space $\mathcal{H}$, then the inclusion map

$$\begin{array}{rccc} i: & \mathcal{M} & \longrightarrow & \mathcal{H} \\ & x & \longmapsto & x \end{array}$$

is bounded from $\mathcal{M}$ into $\mathcal{H}$. Now, since for any $x \in \mathcal{M} = \mathcal{M}(i)$, we have

$$\|x\|_{\mathcal{M}(i)} = \|i(x)\|_{\mathcal{M}(i)} = \|x\|_{\mathcal{M}},$$

the space $\mathcal{M}$ coincides with $\mathcal{M}(i)$, that is

$$\mathcal{M} \doteq \mathcal{M}(i).$$

Conversely, if $\mathcal{M} \doteq \mathcal{M}(A)$, where $A : \mathcal{H}_1 \longrightarrow \mathcal{H}$ is bounded, then $\mathcal{M}$ is boundedly contained in $\mathcal{H}$. Thus, we get the following result.

Theorem 16.3 *Let $\mathcal{M}$ and $\mathcal{H}$ be two Hilbert spaces. Then the following assertions are equivalent.*

(i) *The space $\mathcal{M}$ is boundedly contained in $\mathcal{H}$ (respectively contractively; respectively isometrically).*
(ii) *There exists a bounded operator $A \in \mathcal{L}(\mathcal{H}_1, \mathcal{H})$ (respectively a contraction; respectively an isometry) such that*

$$\mathcal{M} \doteq \mathcal{M}(A). \tag{16.12}$$

In the next section, we examine the problem of uniqueness in the representation of $\mathcal{M}$ given by (16.12). See also Exercise 16.2.2.

The following result shows that, if $A \in \mathcal{L}(\mathcal{H})$ is an orthogonal projection, then in fact we do not obtain a new structure on $\mathcal{M}(A)$. The Hilbert space structure of $\mathcal{M}(A)$ is precisely the one it has in the first place as a closed subspace of $\mathcal{H}$.

Lemma 16.4 *Let M be a closed subspace of $\mathcal{H}$, and let $P_M \in \mathcal{L}(\mathcal{H})$ denote the orthogonal projection on M. Then*

$$\mathcal{M}(P_M) \doteq M,$$

i.e. $\mathcal{M}(P_M) = M$ and $\|w\|_{\mathcal{M}(P_M)} = \|w\|_{\mathcal{H}}$ for all $w \in M$.

Proof The identity $\mathcal{M}(P_M) = M$ is an immediate consequence of the definition of an orthogonal projection. Remember that $\ker P_M = M^\perp$, and since M is closed, $(M^\perp)^\perp = M$. Hence, by (16.5),

$$\|P_M x\|_{\mathcal{M}(P_M)} = \|P_{(\ker P_M)^\perp} x\|_{\mathcal{H}} = \|P_M x\|_{\mathcal{H}} \qquad (x \in \mathcal{H}_1). \qquad \square$$

Lemma 16.5 *Let $A \in \mathcal{L}(\mathcal{H}_1, \mathcal{H})$ and $B \in \mathcal{L}(\mathcal{H})$. Then*

$$\|Bw\|_{\mathcal{M}(BA)} \le \|w\|_{\mathcal{M}(A)} \qquad (w \in \mathcal{M}(A)).$$

Proof It is clear that $B\,\mathcal{M}(A) \subset \mathcal{M}(BA)$. Put $w = Ax$, $x \in \mathcal{H}_1$. Hence, by (16.5),

$$\|Bw\|_{\mathcal{M}(BA)} = \|P_{(\ker BA)^\perp} x\|_{\mathcal{H}_1} \quad \text{and} \quad \|w\|_{\mathcal{M}(A)} = \|P_{(\ker A)^\perp} x\|_{\mathcal{H}_1}.$$

But, since $\ker BA \supset \ker A$, we have

$$\|P_{(\ker BA)^\perp} x\|_{\mathcal{H}_1} \le \|P_{(\ker A)^\perp} x\|_{\mathcal{H}_1}.$$

Therefore, we deduce that $\|Bw\|_{\mathcal{M}(BA)} \le \|w\|_{\mathcal{M}(A)}$. □

Exercises

Exercise 16.1.1 Let $\mathcal{H}$ be a set endowed with two inner products whose corresponding norms are complete and equivalent, i.e.

$$c\,\|x\|_1 \le \|x\|_2 \le C\,\|x\|_1 \qquad (x \in \mathcal{H}),$$

where c and C are positive constants. Show that $(\mathcal{H}, \langle\cdot,\cdot\rangle_1)$ is boundedly contained in $(\mathcal{H}, \langle\cdot,\cdot\rangle_2)$, and vice versa.

Exercise 16.1.2 Let $(X, \mathcal{A})$ be a measurable space, and let μ and ν be two positive measures on the σ-algebra $\mathcal{A}$. Suppose that

$$\mu(E) \le \nu(E) \tag{16.13}$$

for all $E \in \mathcal{A}$. Show that $L^2(\nu)$ is contractively contained in $L^2(\mu)$.

Hint: The assumption (16.13) can be rewritten as

$$\int_X \chi_E\, d\mu \le \int_X \chi_E\, d\nu,$$

where χ_E is the characteristic function of E. Take linear combinations with positive coefficients, and then apply the monotone convergence theorem to obtain

$$\int_X \varphi\, d\mu \le \int_X \varphi\, d\nu$$

for all positive measurable functions φ. Hence, deduce $\|f\|_{L^2(\mu)} \le \|f\|_{L^2(\nu)}$.

Exercise 16.1.3 Let $\varphi \in L^\infty(\mathbb{T})$, and consider the multiplication operator

$$\begin{array}{rccc} M_\varphi : & L^2(\mathbb{T}) & \longrightarrow & L^2(\mathbb{T}) \\ & f & \longmapsto & \varphi f, \end{array}$$

which was studied in Section 7.2. Show that

$$\|\varphi f\|_{\mathcal{M}(M_\varphi)} = \left(\frac{1}{2\pi}\int_{\mathbb{T}\setminus E} |f(e^{it})|^2\,dt\right)^{1/2} \qquad (f \in L^2(\mathbb{T}))$$

and that

$$\langle \varphi f, \varphi g\rangle_{\mathcal{M}(M_\varphi)} = \frac{1}{2\pi}\int_{\mathbb{T}\setminus E} f(e^{it})\,\overline{g(e^{it})}\,dt \qquad (f, g \in L^2(\mathbb{T})),$$

where $E = \{\zeta \in \mathbb{T} : \varphi(\zeta) = 0\}$. The first identity reveals that $\mathcal{M}(M_\varphi) = \varphi L^2(\mathbb{T})$ is contractively contained in $L^2(\mathbb{T})$. Under what condition is $\mathcal{M}(M_\varphi)$ isometrically contained in $L^2(\mathbb{T})$?

Exercise 16.1.4 Let Θ be an inner function for the open unit disk, and let

$$\begin{array}{rccc} M_\Theta: & H^2(\mathbb{D}) & \longrightarrow & H^2(\mathbb{D}) \\ & f & \longmapsto & \Theta f. \end{array}$$

Show that

$$\|\Theta f\|_{\mathcal{M}(M_\Theta)} = \|f\|_{H^2(\mathbb{D})} = \|\Theta f\|_{H^2(\mathbb{D})} \qquad (f \in H^2(\mathbb{D})).$$

Thus $\mathcal{M}(M_\Theta) = \Theta H^2$ is isometrically contained in $H^2(\mathbb{D})$.
Hint: M_Θ is injective and $|\Theta| = 1$ almost everywhere on $\mathbb{T}$.

Exercise 16.1.5 Let $A \in \mathcal{L}(H_1, H_2)$ and $\alpha \in \mathbb{C}$, $\alpha \neq 0$. Show that

$$\|w\|_{\mathcal{M}(\alpha A)} = \frac{\|w\|_{\mathcal{M}(A)}}{|\alpha|} \qquad (w \in \mathcal{M}(A)).$$

16.2 A characterization of $\mathcal{M}(A) \subset \mathcal{M}(B)$

If the operators $A \in \mathcal{L}(\mathcal{H}_1, \mathcal{H})$ and $B \in \mathcal{L}(\mathcal{H}_2, \mathcal{H})$ are such that $\mathcal{M}(A) \Subset \mathcal{M}(B)$, then we surely have $\mathcal{M}(A) \subset \mathcal{M}(B)$. Conversely, if the set inclusion $\mathcal{M}(A) \subset \mathcal{M}(B)$ holds, then the inclusion mapping

$$\begin{array}{rccc} i: & \mathcal{M}(A) & \longrightarrow & \mathcal{M}(B) \\ & w & \longmapsto & w \end{array}$$

is well defined. But, in fact, more is true. The way that the structures of $\mathcal{M}(A)$ and $\mathcal{M}(B)$ are defined forces i to be a bounded operator and thus $\mathcal{M}(A)$ is boundedly contained in $\mathcal{M}(B)$.

Lemma 16.6 *Let $A \in \mathcal{L}(\mathcal{H}_1, \mathcal{H})$ and $B \in \mathcal{L}(\mathcal{H}_2, \mathcal{H})$ be such that $\mathcal{M}(A) \subset \mathcal{M}(B)$. Then $\mathcal{M}(A) \Subset \mathcal{M}(B)$.*

Proof We need to show that the inclusion $i : \mathcal{M}(A) \longrightarrow \mathcal{M}(B)$ is a bounded operator. The justification is based on the closed graph theorem. Let $(w_n)_{n\geq 1}$ be a sequence in $\mathcal{R}(A)$ that converges to w in $\mathcal{M}(A)$ and to w' in $\mathcal{M}(B)$. Note that $iw_n = w_n$. Since $\mathcal{M}(A)$ and $\mathcal{M}(B)$ are both boundedly embedded into H, the sequence $(w_n)_{n\geq 1}$ also tends to w and to w' in the norm of H. Then, by uniqueness of the limit, we must have $w = w'$. Hence, the closed graph theorem implies that i is continuous. $\square$

Lemma 16.6 shows that the new notation $\Subset$ is not needed in the study of $\mathcal{M}(A)$ spaces. However, we emphasize that $\mathcal{M}(A) = \mathcal{M}(B)$ is not equivalent to $\mathcal{M}(A) \doteq \mathcal{M}(B)$. The identity $\mathcal{M}(A) = \mathcal{M}(B)$ implies that

$$c\,\|w\|_{\mathcal{M}(B)} \leq \|w\|_{\mathcal{M}(A)} \leq C\,\|w\|_{\mathcal{M}(B)},$$

while in the definition of $\mathcal{M}(A) \doteq \mathcal{M}(B)$ we assumed that

$$\|w\|_{\mathcal{M}(A)} = \|w\|_{\mathcal{M}(B)}.$$

To use Lemma 16.6, we naturally ask under what conditions the set inclusion $\mathcal{M}(A) \subset \mathcal{M}(B)$ holds. Let us treat a sufficient condition. Suppose that there is a bounded operator $C \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$, with $\|C\| \leq c$, such that $A = BC$. Since, for each $x \in \mathcal{H}_1$, $Ax = B(Cx)$, we have the set inclusion $\mathcal{M}(A) \subset \mathcal{M}(B)$. Thus, by Lemma 16.6, $\mathcal{M}(A) \Subset \mathcal{M}(B)$. Moreover, by (16.7) and the fact that $\|C\| \leq c$, we have

$$\|Ax\|_{\mathcal{M}(B)} = \|BCx\|_{\mathcal{M}(B)} \leq \|Cx\|_{\mathcal{H}_2} \leq c\,\|x\|_{\mathcal{H}_1}.$$

By (16.5), replacing x by $P_{(\ker A)^\perp} x$ gives us

$$\|Ax\|_{\mathcal{M}(B)} \leq c\,\|Ax\|_{\mathcal{M}(A)} \qquad (x \in \mathcal{H}_1).$$

Hence, the norm of i is less than or equal to c. This means that $\mathcal{M}(A)$ is boundedly contained in $\mathcal{M}(B)$ and, in particular, if $c = 1$, $\mathcal{M}(A)$ is contractively contained in $\mathcal{M}(B)$. What is surprising is that the existence of C is also necessary for the bounded inclusion of $\mathcal{M}(A)$ in $\mathcal{M}(B)$.

Theorem 16.7 *Let $A \in \mathcal{L}(\mathcal{H}_1, \mathcal{H})$ and $B \in \mathcal{L}(\mathcal{H}_2, \mathcal{H})$, and let $c > 0$. Then the following are equivalent.*

(i) $AA^* \leq c^2 BB^*$.
(ii) *There is an operator $C \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$, with $\|C\| \leq c$, such that $A = BC$.*
(iii) *We have $\mathcal{M}(A) \subset \mathcal{M}(B)$ with*

$$\|w\|_{\mathcal{M}(B)} \leq c\,\|w\|_{\mathcal{M}(A)} \qquad (w \in \mathcal{M}(A)),$$

i.e. the inclusion $i : \mathcal{M}(A) \longrightarrow \mathcal{M}(B)$ is a bounded operator of norm less than or equal to c.

In particular, $\mathcal{M}(A) \hookrightarrow \mathcal{M}(B)$ if and only if $AA^* \leq BB^*$.

Proof (i) $\Longleftrightarrow$ (ii) This is the content of Theorem 7.11.

(ii) $\Longrightarrow$ (iii) This was discussed above.

(iii) $\Longrightarrow$ (ii) Take an element $w = Ax \in \mathcal{M}(A)$, with some $x \in \mathcal{H}_1$. Hence, for each $x \in \mathcal{H}_1$, there is a $y \in \mathcal{H}_2$ such that

$$Ax = By. \tag{16.14}$$

The element y is not necessarily unique. However, if $By = By'$, with $y, y' \in \mathcal{H}_2$, then $B(y - y') = 0$ and thus $y - y' \in \ker B$. In other words, we have $P_{(\ker B)^\perp} y = P_{(\ker B)^\perp} y'$. Therefore, the mapping

$$\begin{array}{rccc} C: & \mathcal{H}_1 & \longrightarrow & \mathcal{H}_2 \\ & x & \longmapsto & P_{(\ker B)^\perp} y, \end{array}$$

with $y \in \mathcal{H}_2$ given by (16.14), is well defined and

$$BCx = BP_{(\ker B)^\perp} y = By = Ax \qquad (x \in \mathcal{H}_1).$$

This means that the definition of C is adjusted such that the identity $A = BC$ holds. Moreover, by (16.5) and (16.7) and our assumption,

$$\begin{aligned} \|Cx\|_{\mathcal{H}_2} &= \|P_{(\ker B)^\perp} y\|_{\mathcal{H}_2} \\ &= \|By\|_{\mathcal{M}(B)} \\ &= \|Ax\|_{\mathcal{M}(B)} \\ &\le c\,\|Ax\|_{\mathcal{M}(A)} \\ &\le c\,\|x\|_{\mathcal{H}_1} \qquad\qquad (x \in \mathcal{H}_1). \end{aligned}$$

Hence, C is a bounded operator of norm less than or equal to c. □

We gather some important corollaries below. The first one follows immediately from Theorem 16.7.

Corollary 16.8 *Let $A \in \mathcal{L}(\mathcal{H}_1, \mathcal{H})$ and $B \in \mathcal{L}(\mathcal{H}_2, \mathcal{H})$. Then the following statements hold.*

(i) $\mathcal{M}(A) \doteq \mathcal{M}(B)$ *if and only if* $AA^* = BB^*$.

(ii) $\mathcal{M}(A) \doteq \mathcal{M}(|A|)$, *where* $|A| = (AA^*)^{1/2}$.

If the linear manifold $\mathcal{R}(A)$ is closed in H, then it inherits the Hilbert space structure of H. One may wonder if this Hilbert space structure coincides with the one we put on $\mathcal{R}(A)$ and called it $\mathcal{M}(A)$. The following corollary answers this question.

Corollary 16.9 *Let $A \in \mathcal{L}(\mathcal{H}_1, \mathcal{H})$. Then $\mathcal{R}(A)$ is a closed subspace of $\mathcal{H}$ and $\|w\|_{\mathcal{M}(A)} = \|w\|_{\mathcal{H}}$, for each $w \in \mathcal{M}(A)$, if and only if A is a partial isometry. In this case, we have*

$$\mathcal{M}(A) \doteq \mathcal{M}(AA^*).$$

Proof If A is a partial isometry, then, by Theorem 7.22, $P = AA^*$ is an orthogonal projection and thus $|A| = P$. Hence, by Corollary 16.8(ii), $\mathcal{M}(A) \doteq \mathcal{M}(P)$. This means that $\mathcal{R}(A) = \mathcal{R}(P)$ and $\|w\|_{\mathcal{M}(A)} = \|w\|_{\mathcal{M}(P)}$, for each $w \in \mathcal{M}(A)$. But, by Lemma 16.4, $\mathcal{R}(P)$ is a closed subspace of H and $\|w\|_{\mathcal{M}(P)} = \|w\|_{\mathcal{H}}$ for each $w \in \mathcal{M}(P)$.

Now, suppose that $M = \mathcal{R}(A)$ is a closed subspace of H. Then the identity $\mathcal{M}(A) = \mathcal{M}(P_M)$ is trivial. Then, by Lemma 16.4 and our assumptions, we have $\|w\|_{\mathcal{M}(A)} = \|w\|_{\mathcal{H}} = \|w\|_{\mathcal{M}(P_M)}$, for each $w \in \mathcal{M}(A)$. In other words, we have the stronger relation $\mathcal{M}(A) \doteq \mathcal{M}(P_M)$. Hence, by Corollary 16.8(i),

$$AA^* = P_M P_M^* = P_M.$$

Therefore, again by Theorem 7.22, A is a partial isometry. In this case, the relations

$$\mathcal{M}(A) \doteq \mathcal{M}(P_M) \doteq \mathcal{M}(AA^*)$$

were implicitly established above. $\square$

In Theorem 16.7, the condition $AA^* \leq c^2 BB^*$ was studied. The following result is a slightly more generalized version of one part of this theorem. It answers the following natural question. If $\mathcal{M}(A)$ and $\mathcal{M}(B)$ are boundedly contained in H, and if $C : H \longrightarrow H$ is a bounded operator, under what conditions does C map continuously $\mathcal{M}(B)$ into $\mathcal{M}(A)$?

Corollary 16.10 *Let $A, B \in \mathcal{L}(\mathcal{H}_1, \mathcal{H})$, and let $C \in \mathcal{L}(\mathcal{H})$. Then:*

(i) $\mathcal{M}(B) \subset C\mathcal{M}(A)$ *if and only if there exists $c > 0$ such that*

$$BB^* \leq c^2 CAA^*C^*;$$

(ii) $C\mathcal{M}(A) \subset \mathcal{M}(B)$ *if and only if there exists $c > 0$ such that*

$$CAA^*C^* \leq c^2 BB^*. \tag{16.15}$$

Moreover, if the inequality (16.15) is satisfied, then the mapping

$$\begin{array}{rccc} \mathbf{C}: & \mathcal{M}(A) & \longrightarrow & \mathcal{M}(B) \\ & w & \longmapsto & Cw \end{array}$$

is a well-defined operator in $\mathcal{L}(\mathcal{M}(A), \mathcal{M}(B))$ and

$$\|\mathbf{C}\|_{\mathcal{L}(\mathcal{M}(A),\mathcal{M}(B))} \leq c.$$

Proof (i) By Theorem 16.7 and Lemma 16.6, the operator inequality $BB^* \leq c^2 CAA^*C^*$ is equivalent to the fact that $\mathcal{M}(B) \subset \mathcal{M}(CA)$. But $\mathcal{M}(CA) = C\mathcal{M}(A)$, which gives the first assertion.

(ii) The proof has the same flavor. Using once more Theorem 16.7 and Lemma 16.6, we see that the operator inequality (16.15) is equivalent to the set inclusion $\mathcal{M}(CA) \subset \mathcal{M}(B)$ and, since $\mathcal{M}(CA) = C\mathcal{M}(A)$, that gives the desired equivalence. It remains to check that $\mathbf{C}$ is a bounded operator of norm less than or equal to c. Using Theorem 16.7 once more, we see that the condition (16.15) implies that

$$\|w\|_{\mathcal{M}(B)} \le c\|w\|_{\mathcal{M}(CA)} \qquad (w \in \mathcal{M}(CA)).$$

Now put $w = Cx$, $x \in \mathcal{M}(A)$, and then apply Lemma 16.5 to get

$$\|Cx\|_{\mathcal{M}(B)} \le c\,\|Cx\|_{\mathcal{M}(CA)} \le c\,\|x\|_{\mathcal{M}(A)} \qquad (x \in \mathcal{M}(A)). \qquad \square$$

Without any serious difficulty, we will denote the operator $\mathbf{C}$ also by C. In particular, the relation

$$CAA^*C^* \le BB^*$$

ensures that C is a contraction from $\mathcal{M}(A)$ into $\mathcal{M}(B)$.

Corollary 16.11 *Let $A, B \in \mathcal{L}(\mathcal{H}_1, \mathcal{H})$, and let $C \in \mathcal{L}(\mathcal{H})$ be such that*

$$CAA^*C^* = BB^*. \tag{16.16}$$

Assume further that C is injective. Then, the mapping C is a unitary operator from $\mathcal{M}(A)$ onto $\mathcal{M}(B)$.

Proof According to Corollary 16.8, equation (16.16) implies that $\mathcal{M}(CA) = \mathcal{M}(B)$, that is $C\mathcal{M}(A) = \mathcal{M}(B)$ and

$$\|w\|_{\mathcal{M}(CA)} = \|w\|_{\mathcal{M}(B)}$$

for any $w \in \mathcal{M}(CA)$. Hence, C maps $\mathcal{M}(A)$ onto $\mathcal{M}(B)$ and for any $w \in \mathcal{M}(A)$ we have

$$\|Cw\|_{\mathcal{M}(CA)} = \|Cw\|_{\mathcal{M}(B)}. \tag{16.17}$$

If we write $w = Ax$, with $x \in \mathcal{H}_1 \ominus \ker A$, then

$$\|Cw\|_{\mathcal{M}(CA)} = \|CAx\|_{\mathcal{M}(CA)} = \|P_{(\ker CA)^\perp} x\|_{\mathcal{H}_1}.$$

But we always have $\ker A \subset \ker CA$ and, since C is assumed to be injective, the reverse inclusion is also true. Hence $\ker A = \ker CA$ and thus $P_{(\ker CA)^\perp} x = x$. Then we get

$$\|Cw\|_{\mathcal{M}(B)} = \|x\|_{\mathcal{H}_1} = \|Ax\|_{\mathcal{M}(A)} = \|w\|_{\mathcal{M}(A)}.$$

Hence C is unitary operator from $\mathcal{M}(A)$ onto $\mathcal{M}(B)$. $\square$

If $B \in \mathcal{L}(\mathcal{H}_2, \mathcal{H})$ is injective, then B puts an isometric isomorphic copy of $\mathcal{H}_2$ in $\mathcal{H}$, which we denote by $\mathcal{M}(B)$. This fact is an immediate consequence of the definition of $\mathcal{M}(B)$. This result is mentioned below in further detail.

Corollary 16.12 *Let $A \in \mathcal{L}(\mathcal{H}_1, \mathcal{H})$ and $B \in \mathcal{L}(\mathcal{H}_2, \mathcal{H})$. Then the following are equivalent.*

(i) $AA^* = BB^*$ *and* B *is injective.*
(ii) $\mathcal{M}(A) \doteq \mathcal{M}(B)$ *and* B *is injective.*
(iii) B *is an isometry from* $\mathcal{H}_2$ *onto* $\mathcal{M}(A)$.

Proof The equivalence (i) $\Longleftrightarrow$ (ii) follows from Corollary 16.8.

(ii) $\Longleftrightarrow$ (iii) By hypothesis, $\mathcal{M}(A) = \mathcal{M}(B)$ and

$$\|Bx\|_{\mathcal{M}(A)} = \|Bx\|_{\mathcal{M}(B)} \qquad (x \in \mathcal{H}_2).$$

But, since B is injective, we have

$$\|Bx\|_{\mathcal{M}(B)} = \|x\|_{\mathcal{H}_2} \qquad (x \in \mathcal{H}_2).$$

Thus,

$$\|Bx\|_{\mathcal{M}(A)} = \|x\|_{\mathcal{H}_2} \qquad (x \in \mathcal{H}_2).$$

This identity shows that B is an isometry from H_2 onto $\mathcal{M}(A)$.

(iii) $\Longrightarrow$ (ii) By assumption, we have $\mathcal{R}(A) = \mathcal{R}(B)$ and

$$\|Bx\|_{\mathcal{M}(A)} = \|x\|_{\mathcal{H}_2} \qquad (x \in \mathcal{H}_2).$$

That B is an isometry implies $\ker B = \{0\}$. Hence,

$$\|Bx\|_{\mathcal{M}(B)} = \|x\|_{\mathcal{H}_2}$$

for every $x \in (\ker B)^\perp = \mathcal{H}_2$. Thus,

$$\|Bx\|_{\mathcal{M}(A)} = \|Bx\|_{\mathcal{M}(B)} \qquad (x \in \mathcal{H}_2).$$

This means that $\mathcal{M}(A) \doteq \mathcal{M}(B)$. □

Exercises

Exercise 16.2.1 Let $A \in \mathcal{L}(\mathcal{H}_1, \mathcal{H})$ and $B \in \mathcal{L}(\mathcal{H}_2, \mathcal{H})$. Show that $\mathcal{M}(|A|) \doteq \mathcal{M}(|B|)$ if and only if $|A| = |B|$.
Hint: Use Corollary 16.8(i).

Exercise 16.2.2 Let $\mathcal{M}$ and $\mathcal{H}$ be two Hilbert spaces and assume that $\mathcal{M}$ is boundedly contained in $\mathcal{H}$. Show that there is a unique positive operator $T \in \mathcal{L}(\mathcal{H})$ such that $\mathcal{M} \doteq \mathcal{M}(T)$.
Hint: For the existence, consider $i = i_{\mathcal{M}} : \mathcal{M} \longrightarrow \mathcal{H}$, the inclusion mapping from $\mathcal{M}$ into $\mathcal{H}$. Then, use Corollary 16.8(ii) to show that $\mathcal{M} \doteq \mathcal{M}(|i|)$, where $|i| = (ii^*)^{1/2}$. For the uniqueness, assume that there exist two positive operators $T_1, T_2 \in \mathcal{L}(\mathcal{H})$ such that $\mathcal{M} \doteq \mathcal{M}(T_1) = \mathcal{M}(T_2)$. Use Corollary 16.8(i) to deduce that

$$T_1 T_1^* = T_2 T_2^*$$

and conclude using the positivity of T_1 and T_2 and the uniqueness of the positive square root.

Exercise 16.2.3 Let $\mathcal{E}$, $\mathcal{H}$ and $\mathcal{H}_*$ be Hilbert spaces such that $\mathcal{E} \subset \mathcal{H}_*$, and let $T : \mathcal{H} \longrightarrow \mathcal{H}_*$ be a bounded operator. Show that the following assertions are equivalent:

(i) $\mathcal{E} \overset{\Longleftarrow}{=} \mathcal{M}(T)$;
(ii) $TT^* = ii^*$, where $i : \mathcal{E} \longrightarrow \mathcal{H}_*$ is the embedding operator.

Hint: Use the fact that $\mathcal{E} \overset{\Longleftarrow}{=} \mathcal{M}(i)$ and apply Corollary 16.8.

16.3 Linear functionals on $\mathcal{M}(A)$

Let $A \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$. Suppose that

$$\Lambda : \mathcal{H}_2 \longrightarrow \mathbb{C}$$

is a bounded linear functional on $\mathcal{H}_2$. Then, by Riesz's theorem (Theorem 1.24), there is a unique $w \in \mathcal{H}_2$ such that

$$\Lambda z = \langle z, w \rangle_{\mathcal{H}_2} \qquad (z \in \mathcal{H}_2).$$

According to (16.8), the inclusion map

$$i : \mathcal{M}(A) \longrightarrow \mathcal{H}_2$$

is continuous. Hence,

$$\Lambda \circ i : \mathcal{M}(A) \longrightarrow \mathbb{C}$$

is a bounded linear functional on $\mathcal{M}(A)$. Thus, again by Riesz's theorem, there is a unique $w' \in \mathcal{M}(A)$ such that

$$(\Lambda \circ i)(z) = \langle z, w' \rangle_{\mathcal{M}(A)} \qquad (z \in \mathcal{M}(A)).$$

We naturally proceed to find the relation between w and w'. Note that $\Lambda \circ i$ is precisely the restriction of Λ to $\mathcal{M}(A)$, which, according to our general convention, we also denote by $\boldsymbol{\Lambda}$.

Theorem 16.13 *Let $A \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$. Let $w \in \mathcal{H}_2$, and let*

$$\Lambda z = \langle z, w \rangle_{\mathcal{H}_2} \qquad (z \in \mathcal{H}_2)$$

be the corresponding bounded linear functional on H_2. Then its restriction

$$\boldsymbol{\Lambda} : \mathcal{M}(A) \longrightarrow \mathbb{C}$$

is a bounded linear functional on $\mathcal{M}(A)$ and

$$\boldsymbol{\Lambda}(Ax) = \langle Ax, AA^* w \rangle_{\mathcal{M}(A)} \qquad (x \in \mathcal{H}_1).$$

Moreover,

$$\|\mathbf{\Lambda}\|_{\mathcal{M}(A)^*} = \|A^*w\|_{\mathcal{H}_1}.$$

Remark: We recall that, by Riesz's theorem,

$$\|\Lambda\|_{H_2^*} = \|w\|_{\mathcal{H}_2}.$$

Proof By the definition of the adjoint operator, we have

$$\Lambda(Ax) = \langle Ax, w\rangle_{\mathcal{H}_2} = \langle x, A^*w\rangle_{\mathcal{H}_1} \qquad (x \in \mathcal{H}_1).$$

But, by Theorem 1.30,

$$A^*w \in \mathcal{R}(A^*) \subset (\ker A)^\perp.$$

Hence, by (16.6),

$$\langle x, A^*w\rangle_{\mathcal{H}_1} = \langle Ax, AA^*w\rangle_{\mathcal{M}(A)} \qquad (x \in \mathcal{H}_1).$$

Therefore, we can write

$$\mathbf{\Lambda}(Ax) = \langle Ax, AA^*w\rangle_{\mathcal{M}(A)} \qquad (x \in \mathcal{H}_1).$$

This representation shows that

$$\|\mathbf{\Lambda}\|_{\mathcal{M}(A)^*} = \|AA^*w\|_{\mathcal{M}(A)}.$$

However, by (16.5) and the fact that $A^*w \in (\ker A)^\perp$, we have

$$\|AA^*w\|_{\mathcal{M}(A)} = \|A^*w\|_{\mathcal{H}_1}. \qquad \square$$

16.4 The complementary space $\mathcal{H}(A)$

If A is a Hilbert space contraction, then $AA^* \leq I$ and thus $(I - AA^*)^{1/2}$ is well defined (see Exercise 2.4.5). Therefore, we can consider the linear manifold $\mathcal{R}((I - AA^*)^{1/2})$ and put a Hilbert space structure on it, as explained in the previous section and denoted by $\mathcal{M}((I - AA^*)^{1/2})$. We call

$$\mathcal{H}(A) = \mathcal{M}((I - AA^*)^{1/2})$$

the *complementary space* of $\mathcal{M}(A)$, and the intersection $\mathcal{M}(A) \cap \mathcal{H}(A)$ is called the *overlapping space*. In the rest of this chapter we study $\mathcal{H}(A)$ and its relation to $\mathcal{M}(A)$.

Lemma 16.14 *Let $A \in \mathcal{L}(\mathcal{H}_1, \mathcal{H})$ be a contraction. Then $\mathcal{H}(A)$ is a closed subspace of $\mathcal{H}$ and $\|w\|_{\mathcal{H}(A)} = \|w\|_{\mathcal{H}}$, for each $w \in \mathcal{H}(A)$, if and only if A is a partial isometry. In this case, the set identity*

$$\mathcal{H}(A) = \mathcal{R}(I - AA^*)$$

holds.

Proof By Corollary 16.9, $\mathcal{H}(A)$ is a closed subspace of $\mathcal{H}$ and $\|w\|_{\mathcal{H}(A)} = \|w\|_{\mathcal{H}}$, for each $w \in \mathcal{H}(A)$, if and only if $(I - AA^*)^{1/2}$ is a partial isometry. But, by Theorem 7.22, this happens if and only if $I - AA^*$ is an orthogonal projection. Clearly, $I - AA^*$ is an orthogonal projection if and only if AA^* is an orthogonal projection. Finally, again by Theorem 7.22, AA^* is an orthogonal projection if and only if A is a partial isometry.

In this case, since $I - AA^*$ is an orthogonal projection, we have $(I - AA^*)^{1/2} = I - AA^*$, and thus the set identity $\mathcal{H}(A) = \mathcal{R}(I - AA^*)$ holds. □

For an operator $A \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$, we clearly have the set inclusions

$$\mathcal{R}(AA^*) \subset \mathcal{R}(A) \subset \mathcal{H}_2.$$

Therefore, $\mathcal{R}(AA^*)$ is a linear submanifold of $\mathcal{M}(A)$. We show that, with respect to the topology of $\mathcal{M}(A)$, in a sense $\mathcal{R}(AA^*)$ is a large set.

Lemma 16.15 *Let $A \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$. Then the linear manifold $\mathcal{R}(AA^*)$ is dense in $\mathcal{M}(A)$.*

Proof To show that $\mathcal{R}(AA^*)$ is dense in $\mathcal{M}(A)$, we use a standard Hilbert space technique. If 0 is the only vector in $\mathcal{M}(A)$ that is orthogonal to $\mathcal{R}(AA^*)$, then this linear manifold is dense in $\mathcal{M}(A)$. Thus let $w \in \mathcal{M}(A)$ be such that

$$\langle w, z\rangle_{\mathcal{M}(A)} = 0$$

for all $z \in \mathcal{R}(AA^*)$. We proceed to show that $w = 0$. By definition, $w = Ax$, for some $x \in \mathcal{H}_1$, and $z = AA^*y$, where y runs through $\mathcal{H}_2$. Remember that $A^*y \perp \ker A$. Hence, by (16.6),

$$\begin{aligned} 0 &= \langle w, z\rangle_{\mathcal{M}(A)} \\ &= \langle Ax, AA^*y\rangle_{\mathcal{M}(A)} \\ &= \langle x, A^*y\rangle_{\mathcal{H}_1} \\ &= \langle Ax, y\rangle_{\mathcal{H}_2} \\ &= \langle w, y\rangle_{\mathcal{H}_2} \end{aligned}$$

for all $y \in \mathcal{H}_2$. Therefore, $w = 0$. □

We now write Lemma 16.15 for $\mathcal{H}(A)$ spaces. This is the version that we mostly need.

Corollary 16.16 *Let $A \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ be a Hilbert space contraction. Then the linear manifold $\mathcal{R}(I - AA^*)$ is dense in $\mathcal{H}(A)$. Moreover, for each $z \in \mathcal{H}_2$ and $w \in \mathcal{H}(A)$,*

$$\|(I - AA^*)z\|^2_{\mathcal{H}(A)} = \|(I - AA^*)^{1/2}z\|^2_{\mathcal{H}_2} = \|z\|^2_{\mathcal{H}_2} - \|A^*z\|^2_{\mathcal{H}_1}$$

and

$$\langle w,\ (I - AA^*)z\rangle_{\mathcal{H}(A)} = \langle w, z\rangle_{\mathcal{H}_2}.$$

Proof For the first part, it is enough to apply Lemma 16.15 to the self-adjoint operator $(I - AA^*)^{1/2} \in \mathcal{L}(\mathcal{H}_2)$ and see that $\mathcal{R}(I - AA^*)$ is dense in $\mathcal{H}(A)$.

To prove the first identity, note that $(I - AA^*)^{1/2}z \perp \ker(I - AA^*)^{1/2}$. Thus, by (16.5),

$$\begin{aligned}
\|(I - AA^*)z\|^2_{\mathcal{H}(A)} &= \|(I - AA^*)^{1/2}z\|^2_{\mathcal{H}_2}\\
&= \langle (I - AA^*)^{1/2}z,\ (I - AA^*)^{1/2}z\rangle_{\mathcal{H}_2}\\
&= \langle (I - AA^*)z,\ z\rangle_{\mathcal{H}_2}\\
&= \|z\|^2_{H_2} - \|A^*z\|^2_{\mathcal{H}_1}.
\end{aligned}$$

For the second relation, we write $w = (I - AA^*)^{1/2}w'$, where $w' \perp \ker (I - AA^*)^{1/2}$. Hence, by (16.6),

$$\begin{aligned}
\langle w,\ (I - AA^*)z\rangle_{\mathcal{H}(A)} &= \langle (I - AA^*)^{1/2}w',\ (I - AA^*)z\rangle_{\mathcal{H}(A)}\\
&= \langle w',\ (I - AA^*)^{1/2}z\rangle_{\mathcal{H}_2}\\
&= \langle (I - AA^*)^{1/2}w',\ z\rangle_{\mathcal{H}_2}\\
&= \langle w, z\rangle_{\mathcal{H}_2}.
\end{aligned}$$

This completes the proof. □

Given an element $y \in \mathcal{H}$, we sometimes need to know if it belongs to a given complementary space $\mathcal{H}(A)$ or not. The following result is a characterization of this type.

Theorem 16.17 *Let A be a contraction on a Hilbert space H and let $y \in \mathcal{H}$. Then $y \in \mathcal{H}(A)$ if and only if*

$$\sup_{0 \le r < 1} \|(I - r^2AA^*)^{-1/2}y\|_{\mathcal{H}} < +\infty.$$

Moreover, if $y = (I - AA^*)^{1/2}x$ with $x \perp \ker(I - AA^*)^{1/2}$, then

$$\lim_{r\to 1} \|(I - r^2AA^*)^{-1/2}y - x\|_{\mathcal{H}} = 0,$$

and if $y_1, y_2 \in \mathcal{H}(A)$, then

$$\langle y_1, y_2\rangle_{\mathcal{H}(A)} = \lim_{r\to 1}\langle (I - r^2AA^*)^{-1/2}y_1,\ (I - r^2AA^*)^{-1/2}y_2\rangle_{\mathcal{H}}. \quad (16.18)$$

In particular, for each $y \in \mathcal{H}(A)$,

$$\|y\|_{\mathcal{H}(A)} = \lim_{r\to 1} \|(I - r^2AA^*)^{-1/2}y\|_{\mathcal{H}}.$$

Proof Put $B_r = I - r^2AA^*$, $0 \le r < 1$, and $B = I - AA^*$. Then the conditions of Theorem 7.10 are clearly satisfied and thus we deduce that y belongs to the range of $B^{1/2}$, which is $\mathcal{H}(A)$, if and only if

$$\sup_{0\le r<1} \|B_r^{-1/2}y\|_{\mathcal{H}} = \sup_{0\le r<1} \|(I - r^2AA^*)^{-1/2}y\|_{\mathcal{H}} < +\infty.$$

The first equality was also established in Theorem 7.10. Now, if $y_i \in \mathcal{H}(A)$, $i = 1, 2$, then $y_i = (I - AA^*)^{1/2}x_i$, with $x_i \perp \ker(I - AA^*)^{1/2}$, and, by the first equality, $(I - r^2AA^*)^{-1/2}y_i$ converges to x_i in H, as $r \longrightarrow 1$. Hence,

$$\langle y_1, y_2\rangle_{\mathcal{H}(A)} = \langle x_1, x_2\rangle_{\mathcal{H}} = \lim_{r\to 1}\langle (I-r^2AA^*)^{-1/2}y_1,\ (I-r^2AA^*)^{-1/2}y_2\rangle_{\mathcal{H}}.$$

□

Exercises

Exercise 16.4.1 Let $A \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$. Show that

$$\|w\|_{\mathcal{M}(A)} \le \|A\|_{\mathcal{L}(\mathcal{H}_1,\mathcal{H}_2)}\|w\|_{\mathcal{M}(AA^*)} \qquad (w \in \mathcal{M}(AA^*)).$$

Hint: Write $w = AA^*x$, where $x \perp \ker AA^*$.
Remark: This means that $\mathcal{M}(AA^*)$ is boundedly contained in $\mathcal{M}(A)$. This fact also follows from Lemma 16.6.

Exercise 16.4.2 Let $A \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ be a Hilbert space contraction. Show that

$$\|w\|_{\mathcal{H}(A)} \le \|w\|_{\mathcal{M}(I-AA^*)} \qquad (w \in \mathcal{M}(I - AA^*)).$$

Hint: Apply Exercise 16.4.1 to the operator $(I - AA^*)^{1/2}$.
Remark: This means that $\mathcal{M}(I - AA^*)$ is contractively contained in $\mathcal{H}(A)$.

16.5 The relation between $\mathcal{H}(A)$ and $\mathcal{H}(A^*)$

In this section we explore the relation between $\mathcal{H}(A)$ and $\mathcal{H}(A^*)$. In particular, we obtain a frequently used identity that exhibits the bridge between the inner products in $\mathcal{H}(A)$ and $\mathcal{H}(A^*)$.

Theorem 16.18 *Let $A \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ be a contraction, and let $w \in \mathcal{H}_2$. Then $w \in \mathcal{H}(A)$ if and only if $A^*w \in \mathcal{H}(A^*)$. Moreover, if $w_1, w_2 \in \mathcal{H}(A)$, then*

$$\langle w_1, w_2\rangle_{\mathcal{H}(A)} = \langle A^*w_1, A^*w_2\rangle_{\mathcal{H}(A^*)} + \langle w_1, w_2\rangle_{\mathcal{H}_2}.$$

In particular, for each $w \in \mathcal{H}(A)$,

$$\|w\|^2_{\mathcal{H}(A)} = \|A^*w\|^2_{\mathcal{H}(A^*)} + \|w\|^2_{\mathcal{H}_2}.$$

Proof We recall the intertwining relation (7.12):

$$A^*(I - AA^*)^{1/2} = (I - A^*A)^{1/2}A^*.$$

Hence, the set inclusion $A^*\mathcal{H}(A) \subset \mathcal{H}(A^*)$ follows immediately. This is equivalent to saying that

$$w \in \mathcal{H}(A) \quad \Longrightarrow \quad A^*w \in \mathcal{H}(A^*).$$

To prove the inverse, let $w \in \mathcal{H}_2$ be such that $A^*w \in \mathcal{H}(A^*)$. Thus, by definition, there is $x \in \mathcal{H}_1$ such that

$$A^*w = (I - A^*A)^{1/2}x.$$

By the intertwining relation, the trivial identity

$$w = (I - AA^*)w + AA^*w = (I - AA^*)w + A(I - A^*A)^{1/2}x$$

can be rewritten as

$$w = (I - AA^*)^{1/2}[(I - AA^*)^{1/2}w + Ax]. \tag{16.19}$$

Hence, $w \in \mathcal{H}(A)$. In other words, we also have

$$A^*w \in \mathcal{H}(A^*) \quad \Longrightarrow \quad w \in \mathcal{H}(A).$$

To prove the identity for the inner products, let $w_1, w_2 \in \mathcal{H}(A)$. Hence, there are $y_1, y_2 \in \mathcal{H}_2$ such that

$$w_k = (I - AA^*)^{1/2}y_k \qquad (k = 1, 2).$$

Without loss of generality, we assume that $y_k \perp \ker(I - AA^*)$. This assumption has two consequences: first,

$$\langle w_1, w_2 \rangle_{\mathcal{H}(A)} = \langle y_1, y_2 \rangle_{\mathcal{H}_2},$$

and second, $A^*y_k \perp \ker(I - A^*A)$ (see Exercise 1.8.3). But, by the intertwining relation, we have

$$A^*w_k = (I - A^*A)^{1/2}A^*y_k \qquad (k = 1, 2).$$

Therefore, we also have

$$\langle A^*w_1, A^*w_2 \rangle_{\mathcal{H}(A^*)} = \langle A^*y_1, A^*y_2 \rangle_{\mathcal{H}_1}.$$

Now, a direct calculation shows that

$$\begin{aligned}
\langle w_1, w_2 \rangle_{H_2} &= \langle (I - AA^*)^{1/2}y_1,\ (I - AA^*)^{1/2}y_2 \rangle_{\mathcal{H}_2} \\
&= \langle (I - AA^*)y_1,\ y_2 \rangle_{\mathcal{H}_2} \\
&= \langle y_1, y_2 \rangle_{\mathcal{H}_2} - \langle A^*y_1, A^*y_2 \rangle_{\mathcal{H}_1} \\
&= \langle w_1, w_2 \rangle_{\mathcal{H}(A)} - \langle A^*w_1, A^*w_2 \rangle_{\mathcal{H}(A^*)}.
\end{aligned}$$

This completes the proof. $\square$

Applying Theorem 16.18 to the operator A^* gives the following result.

Corollary 16.19 *Let $A \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ be a contraction. Then a vector $x \in \mathcal{H}_1$ belongs to $\mathcal{H}(A^*)$ if and only if $Ax \in \mathcal{H}(A)$. Moreover, if $x_1, x_2 \in \mathcal{H}(A^*)$, then*

$$\langle x_1, x_2 \rangle_{\mathcal{H}(A^*)} = \langle Ax_1, Ax_2 \rangle_{\mathcal{H}(A)} + \langle x_1, x_2 \rangle_{\mathcal{H}_1}.$$

In particular, for each $x \in \mathcal{H}(A^*)$,

$$\|x\|^2_{\mathcal{H}(A^*)} = \|Ax\|^2_{\mathcal{H}(A)} + \|x\|^2_{\mathcal{H}_1}.$$

16.6 The overlapping space $\mathcal{M}(A) \cap \mathcal{H}(A)$

As we mentioned in Section 16.4, the intersection $\mathcal{M}(A) \cap \mathcal{H}(A)$ is called the overlapping space. We first show that the overlapping space is precisely the image of $\mathcal{H}(A^*)$ under the operator A. Then we exploit this observation to characterize the trivial overlapping space.

Lemma 16.20 *Let $A \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ be a Hilbert space contraction. Then we have the set identity*

$$\mathcal{M}(A) \cap \mathcal{H}(A) = A\mathcal{H}(A^*).$$

Moreover, the operator A acts as a contraction from $\mathcal{H}(A^*)$ into $\mathcal{H}(A)$.

Proof By Corollary 16.19, $A\mathcal{H}(A^*) \subset \mathcal{H}(A)$. Moreover, by definition, we have $A\mathcal{H}(A^*) \subset \mathcal{M}(A)$. Hence, $A\mathcal{H}(A^*) \subset \mathcal{M}(A) \cap \mathcal{H}(A)$. To prove the other inclusion, let $w \in \mathcal{M}(A) \cap \mathcal{H}(A)$. Therefore, $w = Ax$, for some $x \in \mathcal{H}_1$, and $Ax \in \mathcal{H}(A)$. Thus, again by Corollary 16.19, we necessarily have $x \in \mathcal{H}(A^*)$, and this means $w = Ax \in A\mathcal{H}(A^*)$. If we apply Theorem 16.18 to A^*, then, for each $w \in \mathcal{H}(A^*)$, we have

$$\|w\|^2_{\mathcal{H}(A^*)} = \|Aw\|^2_{\mathcal{H}(A)} + \|w\|^2_{\mathcal{H}_1} \geq \|Aw\|^2_{\mathcal{H}(A)},$$

which exactly means that A acts as a contraction from $\mathcal{H}(A^*)$ into $\mathcal{H}(A)$. □

We naturally wonder when the overlapping space is trivial, i.e. $\mathcal{M}(A) \cap \mathcal{H}(A) = \{0\}$. We are now able to fully characterize this situation.

Theorem 16.21 *Let $A \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ be a Hilbert space contraction. Then the following are equivalent:*

(i) *A is a partial isometry;*
(ii) *$\mathcal{M}(A)$ is a closed subspace of $\mathcal{H}$ and inherits its Hilbert space structure;*
(iii) *$\mathcal{H}(A)$ is a closed subspace of $\mathcal{H}$ and inherits its Hilbert space structure;*
(iv) *$\mathcal{M}(A)$ and $\mathcal{H}(A)$ are orthogonal complements of each other;*

(v) $\mathcal{M}(A) \cap \mathcal{H}(A) = \{0\}$;
(vi) $\mathcal{H}(A^*) \subset \ker A$.

Moreover, under the preceding equivalent conditions, we have the orthogonal decomposition

$$\mathcal{H} = \mathcal{M}(A) + \mathcal{H}(A).$$

Proof The equivalence (i) $\Longleftrightarrow$ (ii) was proved in Corollary 16.9.

The equivalence (i) $\Longleftrightarrow$ (iii) was proved in Lemma 16.14.

(i) $\Longrightarrow$ (iv) If A is a partial isometry, then $\mathcal{M}(A)$ and $\mathcal{H}(A)$ are the range of complementary orthogonal projections AA^* and $I - AA^*$. Hence $\mathcal{M}(A)$ and $\mathcal{H}(A)$ are orthogonal complements of each other.

(iv) $\Longrightarrow$ (v) This is trivial.

(v) $\Longrightarrow$ (vi) This is an immediate consequence of Lemma 16.20.

(vi) $\Longrightarrow$ (i) By assumption $A(I - A^*A)^{1/2} = 0$. If so, then certainly we have $A(I - A^*A) = 0$. Hence, $A = AA^*A$, which implies $(AA^*)^2 = AA^*$. In other words, AA^* is an orthogonal projection. Therefore, by Theorem 7.22, A is a partial isometry.

The orthogonal decomposition $\mathcal{H} = \mathcal{M}(A) + \mathcal{H}(A)$ is a consequence of (iv). □

16.7 The algebraic sum of $\mathcal{M}(A_1)$ and $\mathcal{M}(A_2)$

Given two operators $A_1 \in \mathcal{L}(\mathcal{H}_1, \mathcal{H})$ and $A_2 \in \mathcal{L}(\mathcal{H}_2, \mathcal{H})$, we can form the Hilbert spaces $\mathcal{M}(A_1)$ and $\mathcal{M}(A_2)$ inside $\mathcal{H}$. Looking at them as linear submanifolds of $\mathcal{H}$, we can form the algebraic sum

$$\mathcal{M}(A_1) + \mathcal{M}(A_2) = \{w_1 + w_2 : w_1 \in \mathcal{M}(A_1) \text{ and } w_2 \in \mathcal{M}(A_2)\}.$$

We may naturally ask if this sum can be regarded as a new Hilbert space $\mathcal{M}(A)$, for a suitable operator A. The affirmative answer is explained in more detail in the following result.

Theorem 16.22 *Let $A_1 \in \mathcal{L}(\mathcal{H}_1, \mathcal{H})$ and $A_2 \in \mathcal{L}(\mathcal{H}_2, \mathcal{H})$, and let $A = [A_1 A_2] \in \mathcal{L}(\mathcal{H}_1 \oplus \mathcal{H}_2,\ \mathcal{H})$ be defined by*

$$A(x_1, x_2) = A_1 x_1 + A_2 x_2 \qquad (x_1 \in \mathcal{H}_1,\ x_2 \in \mathcal{H}_2).$$

Then the following hold.

(i) *$\mathcal{M}(A)$ decomposes as*

$$\mathcal{M}(A) = \mathcal{M}(A_1) + \mathcal{M}(A_2).$$

(ii) *For each representation*

$$w = w_1 + w_2,$$

where $w_i \in \mathcal{M}(A_i)$, $i = 1, 2$, and $w \in \mathcal{M}(A)$, we have

$$\|w\|^2_{\mathcal{M}(A)} \le \|w_1\|^2_{\mathcal{M}(A_1)} + \|w_2\|^2_{\mathcal{M}(A_2)}.$$

(iii) *For each $w \in \mathcal{M}(A)$, there is a unique pair of points $w_1 \in \mathcal{M}(A_1)$ and $w_2 \in \mathcal{M}(A_2)$ such that $w = w_1 + w_2$ and*

$$\|w\|^2_{\mathcal{M}(A)} = \|w_1\|^2_{\mathcal{M}(A_1)} + \|w_2\|^2_{\mathcal{M}(A_2)}.$$

Proof (i) By definition, for each $x_1 \in \mathcal{H}_1$ and $x_2 \in \mathcal{H}_2$, we have

$$A(x_1 \oplus x_2) = A_1x_1 + A_2x_2.$$

This immediately implies the set identity $\mathcal{M}(A) = \mathcal{M}(A_1) + \mathcal{M}(A_2)$.

(ii) If $w = w_1 + w_2$ with $w_i \in \mathcal{M}(A_i)$, $i = 1, 2$, then we can write $w_i = A_ix_i$ with $x_i \perp \ker A_i$. Note that a given $w \in \mathcal{H}(A)$ is not necessarily written in a unique way in the form $w = w_1 + w_2$, and in fact it may have infinitely many such representations. Then we can write

$$w = w_1 + w_2 = A_1x_1 + A_2x_2 = A(x_1 \oplus x_2).$$

Therefore, by Corollary 16.8(ii) and (16.5),

$$\begin{aligned}
\|w\|^2_{\mathcal{M}(A)} &= \|A(x_1 \oplus x_2)\|^2_{\mathcal{M}(A)} \\
&\le \|x_1 \oplus x_2\|^2_{\mathcal{H}_1 \oplus \mathcal{H}_2} \\
&= \|x_1\|^2_{\mathcal{H}_1} + \|x_2\|^2_{\mathcal{H}_2} \\
&= \|w_1\|^2_{\mathcal{M}(A_1)} + \|w_2\|^2_{\mathcal{M}(A_2)}.
\end{aligned}$$

(iii) Among all possible representations

$$w = w_1 + w_2 = A(x_1 \oplus x_2),$$

if we choose x_1 and x_2 such that $x_1 \oplus x_2 \perp \ker A$, then, in the light of (1.45), we certainly have $x_i \perp \ker A_i$. Hence, in the last paragraph of (ii) equality holds everywhere. Thus, this choice of x_1 and x_2 gives at least a suitable pair w_1 and w_2 for which $\|w\|^2_{\mathcal{M}(A)} = \|w_1\|^2_{\mathcal{M}(A_1)} + \|w_2\|^2_{\mathcal{M}(A_2)}$ holds. But, to have this equality, we need $x_1 \oplus x_2 \perp \ker A$ and this choice of $x_1 \oplus x_2$ is unique. Hence, in return, w_1 and w_2 are also unique. □

We now give an explicit example to reveal the contents of the above result. Let $A_1 \in \mathcal{L}(\mathbb{C}^3)$ be defined by

$$A_1(1,0,0) = (0,0,0), \qquad A_1(0,1,0) = (0,1,0), \qquad A_1(0,0,1) = (0,0,1),$$

and put $A_2 = -A_1$ and $A = A_1 \oplus A_2 \in \mathcal{L}(\mathbb{C}^3 \oplus \mathbb{C}^3 \longrightarrow \mathbb{C}^3)$. Then

$$w = A((\alpha, \beta, \gamma) \oplus (\alpha', \beta', \gamma')) = (0, \beta - \beta', \gamma - \gamma'). \tag{16.20}$$

There are infinitely many ways to write $w = w_1 + w_2$ with $w_i \in \mathcal{M}(A_i)$. For example, the above equality suggests that

$$w_1 = A_1((\alpha, \beta, \gamma)) = A_1((0, \beta, \gamma)) = (0, \beta, \gamma) \tag{16.21}$$

and

$$w_2 = A_2((\alpha', \beta', \gamma')) = A_2((0, \beta', \gamma')) = (0, -\beta', -\gamma'). \tag{16.22}$$

But, we may equally take $w_2 = 0$ and

$$w_1 = A_1((0, \beta - \beta', \gamma - \gamma')) = (0, \beta - \beta', \gamma - \gamma'). \tag{16.23}$$

We naturally seek the unique representation that is promised in Theorem 16.23. To do so, first note that

$$\ker A_1 = \ker A_2 = \{(\alpha, 0, 0) : \alpha \in \mathbb{C}\},$$

and

$$\ker A = \{(\alpha, \beta, \gamma) \oplus (\alpha', \beta, \gamma) : \alpha, \alpha', \beta, \gamma \in \mathbb{C}\},$$

which imply that

$$(\ker A)^{\perp} = \{(0, \beta, \gamma) \oplus (0, -\beta, -\gamma) : \beta, \gamma \in \mathbb{C}\}. \tag{16.24}$$

Observe that

$$\ker A_1 \oplus \ker A_2 \subsetneqq \ker A.$$

This proper inclusion has some important consequences.

According to (16.24), the *good* representation for $w = (0, \beta - \beta', \gamma - \gamma')$ is

$$w = A\left(\left(0, \frac{\beta - \beta'}{2}, \frac{\gamma - \gamma'}{2}\right) \oplus \left(0, -\frac{\beta - \beta'}{2}, -\frac{\gamma - \gamma'}{2}\right)\right).$$

Note that

$$\left(0, \frac{\beta - \beta'}{2}, \frac{\gamma - \gamma'}{2}\right) \oplus \left(0, -\frac{\beta - \beta'}{2}, -\frac{\gamma - \gamma'}{2}\right) \perp \ker A.$$

For this unique choice, we have the unique decomposition $w = w_1 + w_2$, where

$$w_1 = A_1\left(0, \frac{\beta - \beta'}{2}, \frac{\gamma - \gamma'}{2}\right) = \left(0, \frac{\beta - \beta'}{2}, \frac{\gamma - \gamma'}{2}\right)$$

and

$$w_2 = A_2\left(0, -\frac{\beta-\beta'}{2}, -\frac{\gamma-\gamma'}{2}\right) = \left(0, \frac{\beta-\beta'}{2}, \frac{\gamma-\gamma'}{2}\right).$$

Since, moreover,

$$\left(0, \frac{\beta-\beta'}{2}, \frac{\gamma-\gamma'}{2}\right) \perp \ker A_1$$

and

$$\left(0, -\frac{\beta-\beta'}{2}, -\frac{\gamma-\gamma'}{2}\right) \perp \ker A_2,$$

we deduce that

$$\begin{aligned}\|w\|^2_{\mathcal{M}(A)} &= \left\|\left(0, \frac{\beta-\beta'}{2}, \frac{\gamma-\gamma'}{2}\right) \oplus \left(0, -\frac{\beta-\beta'}{2}, -\frac{\gamma-\gamma'}{2}\right)\right\|^2 \\ &= \frac{|\beta-\beta'|^2 + |\gamma-\gamma'|^2}{2},\end{aligned}$$

$$\|w_1\|^2_{\mathcal{M}(A_1)} = \left\|\left(0, \frac{\beta-\beta'}{2}, \frac{\gamma-\gamma'}{2}\right)\right\|^2 = \frac{|\beta-\beta'|^2 + |\gamma-\gamma'|^2}{4}$$

and

$$\|w_2\|^2_{\mathcal{M}(A_2)} = \left\|\left(0, -\frac{\beta-\beta'}{2}, -\frac{\gamma-\gamma'}{2}\right)\right\|^2 = \frac{|\beta-\beta'|^2 + |\gamma-\gamma'|^2}{4}.$$

Hence, it is no wonder that, for the *good* representation, we have the norm identity

$$\|w\|^2_{\mathcal{M}(A)} = \|w_1\|^2_{\mathcal{M}(A_1)} + \|w_2\|^2_{\mathcal{M}(A_2)}.$$

For any other representation $w = w_1 + w_2$, with $w_i \in \mathcal{M}(A_i)$, we would certainly have

$$\|w\|^2_{\mathcal{M}(A)} < \|w_1\|^2_{\mathcal{M}(A_1)} + \|w_2\|^2_{\mathcal{M}(A_2)}.$$

For example, in the representation (16.21) and (16.22), we have

$$\|w_1\|^2_{\mathcal{M}(A_1)} = \|(0, \beta, \gamma)\|^2 = |\beta|^2 + |\gamma|^2$$

and

$$\|w_2\|^2_{\mathcal{M}(A_2)} = \|(0, \beta', \gamma')\|^2 = |\beta'|^2 + |\gamma'|^2.$$

In this case, the inequality

$$|\beta|^2 + |\gamma|^2 + |\beta'|^2 + |\gamma'|^2 > \frac{|\beta-\beta'|^2 + |\gamma-\gamma'|^2}{2}$$

is equivalent to

$$\|w_1\|^2_{\mathcal{M}(A_1)} + \|w_2\|^2_{\mathcal{M}(A_2)} > \|w\|^2_{\mathcal{M}(A)}.$$

In a similar manner, the representation (16.23) implies $\|w_2\|_{\mathcal{M}(A_2)} = 0$ and

$$\|w_1\|^2_{\mathcal{M}(A_1)} = \|(0, \beta - \beta', \gamma - \gamma')\|^2 = |\beta - \beta'|^2 + |\gamma - \gamma'|^2.$$

Hence, again, we are faced with the trivial inequality

$$\begin{aligned} \|w_1\|^2_{\mathcal{M}(A_1)} + \|w_2\|^2_{\mathcal{M}(A_2)} &= |\beta - \beta'|^2 + |\gamma - \gamma'|^2 \\ &> \frac{|\beta - \beta'|^2 + |\gamma - \gamma'|^2}{2} = \|w\|^2_{\mathcal{M}(A)}. \end{aligned}$$

A slightly different version of the above algebraic decomposition will be studied in Theorem 16.23.

16.8 A decomposition of $\mathcal{H}(A)$

If an operator decomposes as $A = A_2 A_1$, we naturally ask about the relation between $\mathcal{H}(A)$, on the one hand, and $\mathcal{H}(A_1)$ and $\mathcal{H}(A_2)$, on the other. In this section we address this important question.

Theorem 16.23 *Let $A_1 \in \mathcal{L}(\mathcal{H}_3, \mathcal{H}_1)$ and $A_2 \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ be contractions, and let $A = A_2 A_1$. Then the following hold.*

(i) *$\mathcal{H}(A)$ decomposes as*

$$\mathcal{H}(A) = A_2 \mathcal{H}(A_1) + \mathcal{H}(A_2).$$

(ii) *For any representation*

$$w = A_2 w_1 + w_2,$$

where $w_i \in \mathcal{H}(A_i)$, $i = 1, 2$, and $w \in \mathcal{H}(A)$, we have

$$\|w\|^2_{\mathcal{H}(A)} \le \|w_1\|^2_{\mathcal{H}(A_1)} + \|w_2\|^2_{\mathcal{H}(A_2)}.$$

(iii) *For each $w \in \mathcal{H}(A)$ there is a unique pair of points $w_1 \in \mathcal{H}(A_1)$ and $w_2 \in \mathcal{H}(A_2)$ such that $w = A_2 w_1 + w_2$ and*

$$\|w\|^2_{\mathcal{H}(A)} = \|w_1\|^2_{\mathcal{H}(A_1)} + \|w_2\|^2_{\mathcal{H}(A_2)}.$$

(iv) *$\mathcal{H}(A_2)$ is contractively contained in $\mathcal{H}(A)$.*

(v) *The operator A_2 acts as a contraction from $\mathcal{H}(A_1)$ into $\mathcal{H}(A)$.*

Proof The proof has the same spirit as the proof of Theorem 16.22. In fact, we can even appeal to this result, and we give a shorter proof below. However, here we provide a complete and independent proof.

(i) Consider the operators $B_1 = A_2(I - A_1A_1^*)^{1/2} \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ and $B_2 = (I - A_2A_2^*)^{1/2} \in \mathcal{L}(\mathcal{H}_2)$:

$$\begin{tikzcd}
\mathcal{H}_3 \arrow[d, "A_1"'] \arrow[dr, "A"] & \mathcal{H}_2 \arrow[d, "(I-AA^*)^{1/2}"] \\
\mathcal{H}_1 \arrow[r, "A_2"'] & \mathcal{H}_2 \\
\mathcal{H}_1 \arrow[u, "(I-A_1A_1^*)^{1/2}"] \arrow[ur, "B_1"'] & \mathcal{H}_2 \arrow[u, "(I-A_2A_2^*)^{1/2}"']
\end{tikzcd} \tag{16.25}$$

Then, by (1.43), we can write

$$\begin{aligned}
I - AA^* &= I - (A_2A_1)(A_2A_1)^* \\
&= A_2(I - A_1A_1^*)A_2^* + (I - A_2A_2^*) \\
&= B_1B_1^* + B_2B_2^* \\
&= BB^*,
\end{aligned} \tag{16.26}$$

where $B = [B_1 \; B_2] \in \mathcal{L}(\mathcal{H}_1 \oplus \mathcal{H}_2,\ \mathcal{H}_2)$. Therefore, by Corollary 16.8(ii), and that by definition $\mathcal{H}(A) = \mathcal{M}((I - AA^*)^{1/2})$, we have

$$\mathcal{H}(A) \eqcirc \mathcal{M}(B).$$

Hence, at least, we have the set identities

$$\begin{aligned}
\mathcal{H}(A) &= \mathcal{M}(B) \\
&= \mathcal{M}(B_1) + \mathcal{M}(B_2) \\
&= A_2\mathcal{M}((I - A_1A_1^*)^{1/2}) + \mathcal{M}((I - A_2A_2^*)^{1/2}) \\
&= A_2\mathcal{H}(A_1) + \mathcal{H}(A_2).
\end{aligned}$$

(ii) If $w = A_2w_1 + w_2$ with $w_i \in \mathcal{H}(A_i)$, $i = 1, 2$, then we can write $w_i = (I - A_iA_i^*)^{1/2}x_i$ with $x_i \perp \ker(I - A_iA_i^*)$. Then we have

$$\begin{aligned}
w &= A_2w_1 + w_2 \\
&= A_2(I - A_1A_1^*)^{1/2}x_1 + (I - A_2A_2^*)^{1/2}x_2 \\
&= B_1x_1 + B_2x_2 \\
&= B(x_1 \oplus x_2).
\end{aligned}$$

Therefore, by Corollary 16.8(ii) and (16.5),

$$\begin{aligned}
\|w\|^2_{\mathcal{H}(A)} &= \|w\|^2_{\mathcal{M}(B)} \\
&= \|B(x_1 \oplus x_2)\|^2_{\mathcal{M}(B)}
\end{aligned}$$

$$\begin{aligned}
&\leq \|x_1 \oplus x_2\|^2_{\mathcal{H}_1 \oplus \mathcal{H}_2} \\
&= \|x_1\|^2_{\mathcal{H}_1} + \|x_2\|^2_{\mathcal{H}_2} \\
&= \|w_1\|^2_{\mathcal{H}(A_1)} + \|w_2\|^2_{\mathcal{H}(A_2)}.
\end{aligned}$$

(iii) Among all possible representations

$$w = A_2 w_1 + w_2 = B(x_1 \oplus x_2),$$

if we choose x_1 and x_2 such that $x_1 \oplus x_2 \perp \ker B$, then, in the light of (1.45), we certainly have $x_i \perp \ker(I - A_i A_i^*)$. Hence, in the last paragraph of (ii) equality holds everywhere. Thus, this choice of x_1 and x_2 gives at least a suitable pair w_1 and w_2 for which $\|w\|^2_{\mathcal{H}(A)} = \|w_1\|^2_{\mathcal{H}(A_1)} + \|w_2\|^2_{\mathcal{H}(A_2)}$ holds. But, to have this equality, we need $x_1 \oplus x_2 \perp \ker B$, and this choice of $x_1 \oplus x_2$ is unique. Hence, in return, w_1 and w_2 are unique too.

(iv) By (i), $\mathcal{H}(A_2) \subset \mathcal{H}(A)$. For each $w_2 \in \mathcal{H}(A_2)$, consider the representation $w = A_2 0 + w_2$. Hence, by (ii),

$$\|w_2\|_{\mathcal{H}(A)} = \|w\|_{\mathcal{H}(A)} \leq \|w_2\|_{\mathcal{H}(A_2)}.$$

This means that $\mathcal{H}(A_2)$ is contractively contained in $\mathcal{H}(A)$.

(v) By (i), $A_2 \mathcal{H}(A_1) \subset \mathcal{H}(A)$. For each $w_1 \in \mathcal{H}(A_1)$, consider the representation $w = A_2 w_1 + 0$. Hence, by (ii),

$$\|A_2 w_1\|_{\mathcal{H}(A)} = \|w\|_{\mathcal{H}(A)} \leq \|w_1\|_{\mathcal{H}(A_1)}.$$

This means that A_2 acts as a contraction from $\mathcal{H}(A_1)$ into $\mathcal{H}(A)$. □

In part (iii) of the preceding theorem, the existence of a unique pair of w_1 and w_2 was established. However, we did not offer a procedure or formula to find them. We are able to do this in the following special case. In Corollary 16.16, we saw that $\mathcal{M}(I - AA^*)$ is a dense submanifold of $\mathcal{H}(A)$. Let $w \in \mathcal{M}(I - AA^*)$. Hence, there is $y \in H_2$ such that

$$w = (I - AA^*)y. \tag{16.27}$$

Let

$$x_1 = B_1^* y \quad \text{and} \quad x_2 = B_2^* y,$$

where B_1 and B_2 are as in the proof of Theorem 16.23. Then, by Theorem 1.30,

$$x_1 \oplus x_2 = B_1^* y \oplus B_2^* y = B^* y \in \mathcal{R}(B^*) \subset (\ker B)^\perp.$$

Moreover, by (1.43) and (16.26),

$$\begin{aligned}
B(x_1 \oplus x_2) = B_1 x_1 + B_2 x_2 &= (B_1 B_1^* + B_2 B_2^*)y \\
&= BB^* y = (I - AA^*)y = w.
\end{aligned}$$

Therefore, the unique pair for an element of the form $w = (I - AA^*)y$ is given by

$$w_1 = (I - A_1A_1^*)^{1/2}x_1 = (I - A_1A_1^*)A_2^*y \tag{16.28}$$

and

$$w_2 = (I - A_2A_2^*)^{1/2}x_2 = (I - A_2A_2^*)y. \tag{16.29}$$

The decomposition $\mathcal{H}(A) = A_2\mathcal{H}(A_1) + \mathcal{H}(A_2)$ is an algebraic direct sum of $A_2\mathcal{H}(A_1)$ and $\mathcal{H}(A_2)$ provided that

$$A_2\mathcal{H}(A_1) \cap \mathcal{H}(A_2) = \{0\}.$$

However, the relations

$$\mathcal{H}(A) = \mathcal{M}(B), \quad \mathcal{M}(B_1) = A_2\mathcal{H}(A_1), \quad \mathcal{M}(B_2) = \mathcal{H}(A_2)$$

and (1.47) show that the decomposition $\mathcal{H}(A) = A_2\mathcal{H}(A_1) + \mathcal{H}(A_2)$ is an algebraic direct sum of $A_2\mathcal{H}(A_1)$ and $\mathcal{H}(A_2)$ if and only if

$$\ker B = \ker B_1 \oplus \ker B_2.$$

Assuming the decomposition is an algebraic direct sum, if

$$w = A_2w_1 + w_2 = A_2w_1' + w_2'$$

then we must have $A_2w_1 = A_2w_1'$ and $w_2 = w_2'$. Hence, the choice of w_2 in the representation $w = A_2w_1 + w_2$ is unique. However, there is still some freedom for w_1.

Corollary 16.24 *Let $A_1 \in \mathcal{L}(\mathcal{H}_3, \mathcal{H}_1)$ and $A_2 \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ be contractions, and let $A = A_2A_1$. Suppose that*

$$A_2\mathcal{H}(A_1) \cap \mathcal{H}(A_2) = \{0\}.$$

Then the following hold.

(i) *$\mathcal{H}(A_2)$ is contained isometrically in $\mathcal{H}(A)$.*
(ii) *The operator A_2 acts as a partial isometry from $\mathcal{H}(A_1)$ into $\mathcal{H}(A)$.*
(iii) *Relative to the Hilbert space structure of $\mathcal{H}(A)$, the subspaces $A_2\mathcal{H}(A_1)$ and $\mathcal{H}(A_2)$ are complementary orthogonal subspaces of $\mathcal{H}(A)$. In other words, the decomposition $\mathcal{H}(A) = A_2\mathcal{H}(A_1) + \mathcal{H}(A_2)$ is in fact an orthogonal direct sum.*

Proof We use the notation in the proof of Theorem 16.23. The assumption $A_2\mathcal{H}(A_1) \cap \mathcal{H}(A_2) = \{0\}$ means that the decomposition

$$\mathcal{H}(A) = A_2\mathcal{H}(A_1) + \mathcal{H}(A_2)$$

is an algebraic direct sum. Based on this notation, this is equivalent to saying that

$$\mathcal{R}(B) = \mathcal{R}(B_1) + \mathcal{R}(B_2)$$

is an algebraic direct sum. Therefore, by (1.47),

$$P_{(\ker B)^\perp}(x_1 \oplus x_2) = P_{(\ker B_1)^\perp} x_1 \oplus P_{(\ker B_2)^\perp} x_2 \tag{16.30}$$

for all $x_1 \in \mathcal{H}_1$ and $x_2 \in \mathcal{H}_2$. The identity is the main ingredient in our proof.

(i) Let $w_2 \in \mathcal{H}(A_2)$. Hence $w_2 = B_2 x_2 = (I - A_2 A_2^*)^{1/2} x_2$ with $x_2 \in \mathcal{H}_2$. Therefore, by (16.30),

$$\begin{aligned}
\|w_2\|_{\mathcal{H}(A)} &= \|w_2\|_{\mathcal{M}(B)} \\
&= \|B_2 x_2\|_{\mathcal{M}(B)} \\
&= \|B(0 \oplus x_2)\|_{\mathcal{M}(B)} \\
&= \|P_{(\ker B)^\perp}(0 \oplus x_2)\|_{\mathcal{H}_1 \oplus \mathcal{H}_2} \\
&= \|0 \oplus P_{(\ker B_2)^\perp} x_2\|_{\mathcal{H}_1 \oplus \mathcal{H}_2} \\
&= \|P_{(\ker B_2)^\perp} x_2\|_{\mathcal{H}_2} \\
&= \|B_2 x_2\|_{\mathcal{M}(B_2)} \\
&= \|w_2\|_{\mathcal{H}(A_2)}.
\end{aligned}$$

Hence, $\mathcal{H}(A_2)$ is contained isometrically in $\mathcal{H}(A)$.

(ii) Let us temporarily use the notation

$$\begin{array}{rccc}
\mathbf{A}_2 : & \mathcal{H}(A_1) & \longrightarrow & \mathcal{H}(A) \\
 & w_1 & \longmapsto & A_2 w_1
\end{array}$$

for the restriction of A_2 to $\mathcal{H}(A_1)$. By Theorem 16.23(iv), this operator is well defined and $\ker \mathbf{A}_2 = (\ker A_2) \cap \mathcal{H}(A_1)$. To show that $\mathbf{A}_2$ is a partial isometry, we need to verify that, if $w_1 \in \mathcal{H}(A_1)$, with $w_1 \perp \ker \mathbf{A}_2$, with respect to the inner product of $\mathcal{H}(A_1)$, then $\|A_2 w_1\|_{\mathcal{H}(A)} = \|w_1\|_{\mathcal{H}(A_1)}$.

Fix $w_1 \in \mathcal{H}(A_1)$ with $w_1 \perp \ker \mathbf{A}_2$ in $\mathcal{H}(A_1)$. Then there exists $x_1 \in \mathcal{H}_1$, $x_1 \perp \ker(I - A_1 A_1^*)$, such that $w_1 = (I - A_1 A_1^*)^{1/2} x_1$. But, x_1 also satisfies $x_1 \perp \ker B_1$. To verify this fact, let $x \in \ker B_1$, which means that $A_2 (I - A_1 A_1^*)^{1/2} x = 0$. This identity implies that $(I - A_1 A_1^*)^{1/2} x \in \ker A_2 \cap \mathcal{H}(A_1) = \ker \mathbf{A}_2$. Hence, we have

$$\langle x_1, x \rangle_{\mathcal{H}_1} = \langle w_1,\ (I - A_1 A_1^*)^{1/2} x \rangle_{\mathcal{H}(A_1)} = 0.$$

Since this is true for every $x \in \ker B_1$, we obtain that $x_1 \perp \ker B_1$. Therefore, by (16.30),

$$\begin{aligned}
\|A_2 w_1\|_{\mathcal{H}(A)} &= \|A_2 (I - A_1 A_1^*)^{1/2} x_1\|_{\mathcal{H}(A)} \\
&= \|B_1 x_1\|_{\mathcal{M}(B)}
\end{aligned}$$

$$\begin{aligned}
&= \|B(x_1 \oplus 0)\|_{\mathcal{M}(B)} \\
&= \|P_{(\ker B_1)^\perp} x_1\|_{\mathcal{H}_1} \\
&= \|x_1\|_{\mathcal{H}_1} \\
&= \|w_1\|_{\mathcal{H}(A_1)},
\end{aligned}$$

which certifies that $\mathbf{A}_2$ is a partial isometry, or, equivalently, A_2 acts as a partial isometry from $\mathcal{H}(A_1)$ into $\mathcal{H}(A)$.

(iii) By parts (i) and (ii), with respect to the structure of $\mathcal{H}(A)$, the sets $\mathcal{H}(A_2)$ and $A_2\mathcal{H}(A_1)$ are closed subspaces of $\mathcal{H}(A)$. Now let $w_i \in \mathcal{H}(A_i)$, $i = 1, 2$. Hence, $w_i = (I - AA^*)^{1/2} x_i$ with $x_i \in \mathcal{H}_i$. Therefore, by (16.30),

$$\begin{aligned}
\langle A_2 w_1, w_2 \rangle_{\mathcal{H}(A)} &= \langle A_2 w_1, w_2 \rangle_{\mathcal{M}(B)} \\
&= \langle B_1 x_1, B_2 x_2 \rangle_{\mathcal{M}(B)} \\
&= \langle B(x_1 \oplus 0),\ B(0 \oplus x_2) \rangle_{\mathcal{M}(B)} \\
&= \langle P_{(\ker B)^\perp}(x_1 \oplus 0),\ P_{(\ker B)^\perp}(0 \oplus x_2) \rangle_{\mathcal{H}_1 \oplus \mathcal{H}_2} \\
&= \langle P_{(\ker B_1)^\perp} x_1 \oplus 0,\ 0 \oplus P_{(\ker B_2)^\perp} x_2 \rangle_{\mathcal{H}_1 \oplus \mathcal{H}_2} \\
&= \langle P_{(\ker B_1)^\perp} x_1,\ 0 \rangle_{\mathcal{H}_1} + \langle 0,\ P_{(\ker B_2)^\perp} x_2 \rangle_{\mathcal{H}_2} = 0.
\end{aligned}$$

Hence, $A_2\,\mathcal{H}(A_1)$ and $\mathcal{H}(A_2)$ are complementary orthogonal subspaces of $\mathcal{H}(A)$. □

To apply Corollary 16.24, we certainly need to verify the condition

$$A_2\mathcal{H}(A_1) \cap \mathcal{H}(A_2) = \{0\}.$$

This is not an easy task. However, in some special cases, it clearly holds. For example, by Lemma 16.20, we have

$$A_2\mathcal{H}(A_1) \cap \mathcal{H}(A_2) \subset \mathcal{M}(A_2) \cap \mathcal{H}(A_2) = A_2\mathcal{H}(A_2^*).$$

Hence, whenever A_2 satisfies $A_2^* A_2 = I$, then we have $\mathcal{H}(A_2^*) = \{0\}$ and thus we conclude that

$$A_2^* A_2 = I \implies A_2\mathcal{H}(A_1) \cap \mathcal{H}(A_2) = \{0\}. \tag{16.31}$$

If $\mathcal{M}$ is a closed subspace of the Hilbert space $\mathcal{H}$, then we have $\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^\perp$. We look at this identity from a different point of view. By Lemma 16.4, we have $\mathcal{M}(P_\mathcal{M}) = \mathcal{M}$ and $\mathcal{H}(P_\mathcal{M}) = \mathcal{M}^\perp$, and thus we can write

$$\mathcal{H} = \mathcal{M}(P_\mathcal{M}) \oplus \mathcal{H}(P_\mathcal{M}).$$

In the following, we generalize this observation.

Theorem 16.25 *Let $A \in \mathcal{L}(\mathcal{H}_1, \mathcal{H})$ be a Hilbert space contraction. Then*

$$\mathcal{H} = \mathcal{M}(A) + \mathcal{H}(A).$$

For each decomposition $w = w_1 + w_2$, with $w \in \mathcal{H}$, $w_1 \in \mathcal{M}(A)$ and $w_2 \in \mathcal{H}(A)$, we have

$$\|w\|^2_{\mathcal{H}} \leq \|w_1\|^2_{\mathcal{M}(A)} + \|w_2\|^2_{\mathcal{H}(A)}.$$

Moreover,

$$\|w\|^2_{\mathcal{H}} = \|w_1\|^2_{\mathcal{M}(A)} + \|w_2\|^2_{\mathcal{H}(A)}$$

if and only if

$$w_1 = AA^*w \quad \textit{and} \quad w_2 = (I - AA^*)w.$$

Proof In this proof we write T instead of A. This is because we want to apply Theorem 16.23 and use the notation there, but the operator A that appears in that theorem is not the same as the one introduced in the present theorem.

Consider the decomposition

$$0 = T0,$$

where on the left-hand side we have $0 \in \mathcal{L}(\mathcal{H}_1, \mathcal{H})$ and on the right-hand side 0 represents the zero operator in $\mathcal{L}(\mathcal{H}_1)$. Hence, we have the decomposition $A = A_2 A_1$ with $A = 0 \in \mathcal{L}(\mathcal{H}_1, \mathcal{H})$, $A_2 = T$ and $A_1 = 0 \in \mathcal{L}(\mathcal{H}_1)$. Hence, $\mathcal{H}(A) \circeq \mathcal{H}$ and $\mathcal{H}(A_1) \circeq \mathcal{H}_1$:

$$\begin{tikzcd}
\mathcal{H}_1 \arrow[d, "0"'] \arrow[dr, "0"] & \mathcal{H} \arrow[d, "I"] \\
\mathcal{H}_1 \arrow[r, "A"'] & \mathcal{H} \\
\mathcal{H}_1 \arrow[u, "I"] & \mathcal{H} \arrow[u, "(I-AA^*)^{1/2}"']
\end{tikzcd} \tag{16.32}$$

The diagram (16.32) is a simplified version of diagram (16.25).

Thus the decomposition $\mathcal{H}(A) = A_2\mathcal{H}(A_1) + \mathcal{H}(A_2)$ obtained in Theorem 16.23(i) can be written as

$$\mathcal{H} = T\mathcal{H}_1 + \mathcal{H}(T) = \mathcal{M}(T) + \mathcal{H}(T),$$

and if $z = Tz_1 + z_2$ with $z_1 \in \mathcal{H}_1$ and $z_2 \in \mathcal{H}(T)$, then, by Theorem 16.23(ii),

$$\|z\|^2_{\mathcal{H}} \leq \|z_1\|^2_{\mathcal{H}_1} + \|z_2\|^2_{\mathcal{H}(T)}.$$

In particular, if we take $z_1 \perp \ker T$, we obtain

$$\|z\|^2_{\mathcal{H}} \leq \|Tz_1\|^2_{\mathcal{M}(T)} + \|z_2\|^2_{\mathcal{H}(T)}.$$

Finally, by (16.27), (16.28) and (16.29), the unique pair z_1 and z_2 for which

$$\|z\|_{\mathcal{H}}^2 = \|z_1\|_{\mathcal{H}_1}^2 + \|z_2\|_{\mathcal{H}(T)}^2$$

holds is given by

$$z_1 = (I - A_1A_1^*)A_2^*z = T^*z$$

and

$$z_2 = (I - A_2A_2^*)z = (I - TT^*)z.$$

But

$$z_1 \in \mathcal{R}(T^*) \subset (\ker T)^\perp$$

implies that $\|z_1\|_{\mathcal{H}_1} = \|Tz_1\|_{\mathcal{M}(T)}$. To be consistent with the notation of the theorem, just take $w = z$, $w_1 = Tz_1$ and $w_2 = z_2$. $\square$

We are now able to better understand Theorem 16.21. Generally speaking, an algebraic direct sum is not necessarily an orthogonal direct sum. However, the decomposition $\mathcal{H} = \mathcal{M}(A) + \mathcal{H}(A)$ is a special case. If this is an algebraic direct sum, then it means that $\mathcal{M}(A) \cap \mathcal{H}(A) = \{0\}$. Therefore, Theorem 16.21 ensures that it is in fact an orthogonal direct sum.

16.9 The geometric definition of $\mathcal{H}(A)$

De Branges and Rovnyak had a geometric point of view and gave a different definition of the complementary space $\mathcal{H}(A)$. In this section we treat their definition and show that it is equivalent to that given in Section 16.4. The latter definition is due to Sarason and opened a new world for these spaces. As the motivation for their definition, let us make an observation. According to Theorem 16.25, for each $z \in \mathcal{M}(A)$ and $w \in \mathcal{H}(A)$, we have

$$\|w + z\|_{\mathcal{H}_2}^2 \le \|w\|_{\mathcal{H}(A)}^2 + \|z\|_{\mathcal{M}(A)}^2.$$

Writing this inequality as

$$\|w + z\|_{\mathcal{H}_2}^2 - \|z\|_{\mathcal{M}(A)}^2 \le \|w\|_{\mathcal{H}(A)}^2$$

immediately implies that

$$\sup_{z \in \mathcal{M}(A)} \left(\|w + z\|_{\mathcal{H}_2}^2 - \|z\|_{\mathcal{M}(A)}^2\right) \le \|w\|_{\mathcal{H}(A)}^2. \tag{16.33}$$

Moreover, the supremum is attained.

Suppose that $\mathcal{H}$ is a Hilbert space and let $\mathcal{M}$ be a Hilbert space contractively contained in $\mathcal{H}$. As we explained at the end of Section 16.1, we have $\mathcal{M} = \mathcal{M}(i)$, where i is the inclusion map $i = i_{\mathcal{M}} : \mathcal{M} \longrightarrow \mathcal{H}$. Then de Branges and

Rovnyak defined the complementary space of $\mathcal{M}$ as the set of all $w \in \mathcal{H}$ such that

$$\sup_{z\in\mathcal{M}} (\|w+z\|_{\mathcal{H}}^2 - \|z\|_{\mathcal{M}}^2) < \infty. \tag{16.34}$$

By (16.33), surely each element of $\mathcal{H}(i)$ satisfies this property. But, in fact, this is a characterization property.

Lemma 16.26 *Let $A \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ be a Hilbert space contraction, and let $w \in \mathcal{H}_2$ be such that*

$$\sup_{z\in\mathcal{M}(A)} (\|w+z\|_{\mathcal{H}_2}^2 - \|z\|_{\mathcal{M}(A)}^2) < \infty. \tag{16.35}$$

Then $w \in \mathcal{H}(A)$ and

$$\|w\|_{\mathcal{H}(A)}^2 \le \sup_{z\in\mathcal{M}(A)} (\|w+z\|_{\mathcal{H}_2}^2 - \|z\|_{\mathcal{M}(A)}^2).$$

Proof If we succeed in showing that $A^*w \in \mathcal{H}(A^*)$, then, by Theorem 16.18, w would be in $\mathcal{H}(A)$. Hence, we proceed to show that $A^*w \in \mathcal{H}(A^*)$. Let

$$c = \sup_{x\in\mathcal{H}_1} (\|w+Ax\|_{\mathcal{H}_2}^2 - \|x\|_{\mathcal{H}_1}^2). \tag{16.36}$$

Since $\|Ax\|_{\mathcal{M}(A)} \le \|x\|_{\mathcal{H}_1}$ and the equality holds whenever $x \perp \ker A$, then

$$c = \sup_{z\in\mathcal{M}(A)} (\|w+z\|_{\mathcal{H}_2}^2 - \|z\|_{\mathcal{M}(A)}^2) < \infty.$$

Let $\gamma \in \mathbb{T}$ be such that $\langle w, Ax\rangle_{\mathcal{H}_2} = \gamma|\langle w, Ax\rangle_{\mathcal{H}_2}|$. Replace x by $t\gamma x$, where $t \in \mathbb{R}$, in (16.36). Hence,

$$\|w+t\gamma Ax\|_{\mathcal{H}_2}^2 - \|t\gamma x\|_{\mathcal{H}_1}^2 \le c \qquad (x \in \mathcal{H}_1).$$

This is equivalent to

$$t^2\|(I-A^*A)^{1/2}x\|_{\mathcal{H}_1}^2 - 2t|\langle w, Ax\rangle_{\mathcal{H}_2}| + c - \|w\|_{\mathcal{H}_2}^2 \ge 0.$$

Thus

$$|\langle w, Ax\rangle_{\mathcal{H}_2}| \le C\,\|(I-A^*A)^{1/2}x\|_{\mathcal{H}_1} \qquad (x \in \mathcal{H}_1),$$

where $C = (c - \|w\|_{\mathcal{H}_2}^2)^{1/2}$. Since, by Corollary 16.16,

$$\|(I-A^*A)^{1/2}x\|_{\mathcal{H}_1} = \|(I-A^*A)x\|_{\mathcal{H}(A^*)},$$

we can write

$$|\langle A^*w, x\rangle_{\mathcal{H}_1}| \le C\,\|(I-A^*A)x\|_{\mathcal{H}(A^*)} \qquad (x \in \mathcal{H}_1).$$

By the same corollary, $\mathcal{R}(I-A^*A)$ is a dense submanifold of $\mathcal{H}(A^*)$. Hence, the last inequality says that the map

$$\begin{array}{ccc} \mathcal{R}(I-A^*A) & \longrightarrow & \mathbb{C} \\ (I-A^*A)x & \longmapsto & \langle x, A^*w\rangle_{\mathcal{H}_1} \end{array}$$

gives a bounded linear functional on $\mathcal{R}(I - A^*A)$, whose norm is less than or equal to C. We can extend it by continuity to a bounded linear functional on $\mathcal{H}(A^*)$. Hence, by Riesz's theorem, there is an element $y \in \mathcal{H}(A^*)$, $\|y\|_{\mathcal{H}(A^*)} \leq C$, such that

$$\langle x, A^*w\rangle_{\mathcal{H}_1} = \langle (I - A^*A)x,\ y\rangle_{\mathcal{H}(A^*)} \qquad (x \in \mathcal{H}_1).$$

Since, by Corollary 16.16, $\langle (I - A^*A)x,\ y\rangle_{\mathcal{H}(A^*)} = \langle x, y\rangle_{\mathcal{H}_1}$, we thus have

$$\langle x, A^*w\rangle_{\mathcal{H}_1} = \langle x, y\rangle_{\mathcal{H}_1} \qquad (x \in \mathcal{H}_1).$$

Therefore, $A^*w = y \in \mathcal{H}(A^*)$ and $\|A^*w\|_{\mathcal{H}(A^*)} \leq C$. Theorem 16.18 now implies that $w \in \mathcal{H}(A)$ and

$$\begin{aligned} \|w\|^2_{\mathcal{H}(A)} &= \|w\|^2_{\mathcal{H}} + \|A^*w\|^2_{\mathcal{H}(A^*)} \\ &\leq \|w\|^2_{\mathcal{H}} + C^2 = c \\ &= \sup_{z \in \mathcal{M}(A)} (\|w+z\|^2_{\mathcal{H}_2} - \|z\|^2_{\mathcal{M}(A)}). \end{aligned}$$

This completes the proof. □

Combining Lemma 16.26 and (16.33), we obtain the following result. It shows that the definition of de Branges and Rovnyak is equivalent to the definition of Sarason.

Corollary 16.27 *Let $A \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ be a Hilbert space contraction and let $w \in \mathcal{H}_2$. Then the following are equivalent:*

(i) $w \in \mathcal{H}(A)$;
(ii) $\sup_{z \in \mathcal{M}(A)}(\|w+z\|^2_{\mathcal{H}_2} - \|z\|^2_{\mathcal{M}(A)}) < +\infty$.

Moreover, for such an element, we have

$$\|w\|^2_{\mathcal{H}(A)} = \sup_{z \in \mathcal{M}(A)} (\|w+z\|^2_{\mathcal{H}_2} - \|z\|^2_{\mathcal{M}(A)}).$$

Given a Hilbert space $\mathcal{H}$ and $\mathcal{M}$, such that $\mathcal{M}$ is contractively contained in $\mathcal{H}$, we denote by $\mathcal{M}'$ the complementary space of $\mathcal{M}$ defined by (16.34) and we put

$$\|w\|^2_{\mathcal{M}'} = \sup_{z \in \mathcal{M}} (\|w+z\|^2_{\mathcal{H}} - \|z\|^2_{\mathcal{M}}) \qquad (w \in \mathcal{M}'). \tag{16.37}$$

If $i_{\mathcal{M}} : \mathcal{M} \longrightarrow \mathcal{H}$ is the inclusion map, then Corollary 16.27 says that

$$\mathcal{M}' = \mathcal{H}(i_{\mathcal{M}}).$$

In particular, we see that $\mathcal{M}'$ is a Hilbert space that is also contractively contained in $\mathcal{H}$. Remember also that, if $\mathcal{M}$ is a closed subspace of $\mathcal{H}$, then Theorem 16.21 gives that $\mathcal{M}'$ coincides with the orthogonal complement of $\mathcal{M}$ in $\mathcal{H}$.

We state the decomposition $\mathcal{H} = \mathcal{M}(A) + \mathcal{H}(A)$ in a slightly different language below. This version will be needed in our discussion of $\mathcal{H}(b)$ spaces.

Corollary 16.28 *Let $\mathcal{H}$ be a Hilbert space, let $\mathcal{M}$ be a Hilbert space that is contractively contained in $\mathcal{H}$ and let $\mathcal{N}$ be the complementary space of $\mathcal{M}$ in $\mathcal{H}$. Denote by $i_{\mathcal{M}}$ (respectively $i_{\mathcal{N}}$) the canonical injection of $\mathcal{M}$ (respectively $\mathcal{N}$) into $\mathcal{H}$. Then, for each $x \in \mathcal{H}$, we have*

$$x = i^*_{\mathcal{M}} x + i^*_{\mathcal{N}} x$$

and

$$\|x\|^2_{\mathcal{H}} = \|i^*_{\mathcal{M}} x\|^2_{\mathcal{M}} + \|i^*_{\mathcal{N}} x\|^2_{\mathcal{N}}.$$

Moreover, if $x = x_1 + x_2$, with $x_1 \in \mathcal{M}$ and $x_2 \in \mathcal{N}$, which satisfy

$$\|x\|^2_{\mathcal{H}} = \|x_1\|^2_{\mathcal{M}} + \|x_2\|^2_{\mathcal{N}},$$

then we necessarily have $x_1 = i^*_{\mathcal{M}} x$ and $x_2 = i^*_{\mathcal{N}} x$.

Proof We know that $\mathcal{M} \doteq \mathcal{M}(i_{\mathcal{M}})$ and $\mathcal{N} \doteq \mathcal{M}(i_{\mathcal{N}})$. Moreover, according to Corollary 16.27, we have $\mathcal{N} \doteq \mathcal{H}(i_{\mathcal{M}}) \doteq \mathcal{M}((I - i_{\mathcal{M}} i^*_{\mathcal{M}})^{1/2})$. Thus Corollary 16.12 implies that

$$i_{\mathcal{N}} i^*_{\mathcal{N}} = I - i_{\mathcal{M}} i^*_{\mathcal{M}}.$$

Therefore $I = i_{\mathcal{N}} i^*_{\mathcal{N}} + i_{\mathcal{M}} i^*_{\mathcal{M}}$, which gives

$$x = i^*_{\mathcal{M}} x + i^*_{\mathcal{N}} x \qquad (x \in H).$$

Furthermore, an application of Theorem 16.25 to $A = i_{\mathcal{M}}$ gives

$$\begin{aligned}\|x\|^2_{\mathcal{H}} &= \|i_{\mathcal{M}} i^*_{\mathcal{M}} x\|^2_{\mathcal{M}(i_{\mathcal{M}})} + \|(I - i_{\mathcal{M}} i^*_{\mathcal{M}})x\|^2_{\mathcal{H}(i_{\mathcal{M}})} \\ &= \|i^*_{\mathcal{M}} x\|^2_{\mathcal{M}} + \|i^*_{\mathcal{N}} x\|^2_{\mathcal{N}}.\end{aligned}$$

The second point of Corollary 16.28 follows also immediately from Theorem 16.25. □

This corollary explains why the notion of complementary space can be seen as a generalization of the orthogonal complement, and the map i^*_M (respectively i^*_N) can be seen as a generalization of the orthogonal projection onto M (respectively onto N). We end this section with a result about subspaces that are invariant under the shift operators.

Let $\mathcal{M}$ be a Hilbert space contractively contained in another Hilbert space $\mathcal{H}$, and let $i_{\mathcal{M}} : \mathcal{M} \longrightarrow \mathcal{H}$ be the inclusion map. Then, $\mathcal{M} \doteq \mathcal{M}(i_{\mathcal{M}})$ and the complementary space of $\mathcal{M}$ is $\mathcal{N} = \mathcal{H}(i_{\mathcal{M}})$. In particular, the space $\mathcal{N}$ is itself contractively contained in $\mathcal{H}$. It is natural to wonder what is the complementary space of $\mathcal{N}$. In other words, what is $\mathcal{M}''$?

Using the relation $I - i_{\mathcal{M}} i_{\mathcal{M}}^* = i_{\mathcal{N}} i_{\mathcal{N}}^*$ and Corollary 16.8, we get that

$$\mathcal{N} = \mathcal{H}(i_{\mathcal{M}}) = \mathcal{M}((I - i_{\mathcal{M}} i_{\mathcal{M}}^*)^{1/2}) = \mathcal{M}(i_{\mathcal{N}}).$$

Hence, the complementary space of $\mathcal{N}$ is $\mathcal{H}(i_{\mathcal{N}})$. Using once more the relation $I - i_{\mathcal{M}} i_{\mathcal{M}}^* = i_{\mathcal{N}} i_{\mathcal{N}}^*$ and Corollary 16.8, we get that

$$\mathcal{H}(i_{\mathcal{N}}) = \mathcal{M}(i_{\mathcal{N}}) = \mathcal{M},$$

which gives that the complementary space of $\mathcal{N}$ is $\mathcal{M}$. In other words, we have proved that, if we start with a space $\mathcal{M}$, contractively contained in $\mathcal{H}$, then

$$\mathcal{M}'' = \mathcal{M}.$$

Theorem 16.29 *Let $\mathcal{M}$ be a Hilbert space contained in H^2. Then the following assertions are equivalent.*

(i) *$\mathcal{M}$ is invariant under the backward shift S^*, i.e. $S^*\mathcal{M} \subset \mathcal{M}$, and*

$$\|S^* f\|_{\mathcal{M}}^2 \le \|f\|_{\mathcal{M}}^2 - |f(0)|^2 \qquad (f \in \mathcal{M}). \tag{16.38}$$

(ii) *The following hold:*

(a) *$\mathcal{M}$ is contractively contained in H^2;*

(b) *$S\mathcal{M}' \subset \mathcal{M}'$ and*

$$\|Sg\|_{\mathcal{M}'} \le \|g\|_{\mathcal{M}'} \qquad (g \in \mathcal{M}').$$

Proof (i) $\Longrightarrow$ (ii) Assume first that $\mathcal{M}$ is invariant under the backward shift S^* and satisfies (16.38). Let us prove first that condition (a) holds. Write $T = S^*|\mathcal{M}$. By induction, from (16.38) we get

$$\|T^n f\|_{\mathcal{M}}^2 \le \|f\|_{\mathcal{M}}^2 - \sum_{k=0}^{n-1} |T^k f(0)|^2 \qquad (n \ge 1,\ f \in \mathcal{M}).$$

Hence,

$$\sum_{k=0}^{n-1} |\hat{f}(k)|^2 \le \|f\|_{\mathcal{M}}^2 \qquad (n \ge 1,\ f \in \mathcal{M}). \tag{16.39}$$

Note that $T^k f(0) = (S^{*k} f)(0) = \hat{f}(k)$. But, for each $f \in H^2$, we have

$$\|f\|_2^2 = \sum_{k=0}^{\infty} |\hat{f}(k)|^2.$$

Therefore, letting $n \longrightarrow \infty$ in (16.39) gives

$$\|f\|_2 \le \|f\|_{\mathcal{M}} \qquad (f \in \mathcal{M}).$$

This inequality means precisely that $\mathcal{M}$ is contractively contained in H^2 and thus its complementary space $\mathcal{M}'$ is well defined.

Let us now prove that (b) holds. Let $f \in \mathcal{M}$, and let $g \in H^2$. Then using (16.38), we have

$$\begin{aligned}\|Sg+f\|_2^2 - \|f\|_{\mathcal{M}}^2 &\le \|Sg+f\|_2^2 - \|S^*f\|_{\mathcal{M}}^2 - |f(0)|^2 \\ &= \|Sg+SS^*f\|_2^2 - \|S^*f\|_{\mathcal{M}}^2,\end{aligned}$$

because

$$\|Sg+f\|_2^2 = \|Sg+SS^*f - f(0)\|_2^2 = \|Sg+SS^*f\|_2^2 + |f(0)|^2.$$

Therefore, we have

$$\|Sg+f\|_2^2 - \|f\|_{\mathcal{M}}^2 \le \|g+S^*f\|_2^2 - \|S^*f\|_{\mathcal{M}}^2, \tag{16.40}$$

for every $g \in H^2$ and every $f \in \mathcal{M}$. If furthermore we assume that $g \in \mathcal{M}'$, then we get

$$\|Sg+f\|_2^2 - \|f\|_{\mathcal{M}}^2 \le \sup_{f\in\mathcal{M}} (\|g+S^*f\|_2^2 - \|S^*f\|_{\mathcal{M}}^2) \le \|g\|_{\mathcal{M}'}^2.$$

By definition, this means that $Sg \in \mathcal{M}'$ and $\|Sg\|_{\mathcal{M}'} \le \|g\|_{\mathcal{M}'}$.

(ii) $\Longrightarrow$ (i) Assume that $\mathcal{M}$ is contractively contained in H^2 and the shift S acts as a contraction on $\mathcal{M}'$. Let $f \in \mathcal{M}$ and $g \in \mathcal{M}'$. We have

$$\|S^*f+g\|_2^2 - \|g\|_{\mathcal{M}'}^2 = \|SS^*f+Sg\|_2^2 - \|g\|_{\mathcal{M}'}^2,$$

and using the fact that $SS^*f+Sg \perp 1$, we also have

$$\|SS^*f+Sg\|_2^2 = \|SS^*f+Sg+f(0)\|_2^2 - |f(0)|^2 = \|f+Sg\|_2^2 - |f(0)|^2.$$

Hence,

$$\|S^*f+g\|_2^2 - \|g\|_{\mathcal{M}'}^2 = \|f+Sg\|_2^2 - |f(0)|^2 - \|g\|_{\mathcal{M}'}^2.$$

Since S acts as a contraction on $\mathcal{M}'$, we deduce that

$$\|S^*f+g\|_2^2 - \|g\|_{\mathcal{M}'}^2 \le \|f+Sg\|_2^2 - \|Sg\|_{\mathcal{M}'}^2 - |f(0)|^2.$$

Using the definition of the norm of the complementary space (see (16.37)), and the fact that $\mathcal{M}'' = \mathcal{M}$, we get

$$\|S^*f+g\|_2^2 - \|g\|_{\mathcal{M}'}^2 \le \|f\|_{\mathcal{M}}^2 - |f(0)|^2.$$

Now taking the supremum over all $g \in \mathcal{M}'$, we deduce that $S^*f \in \mathcal{M}$ and

$$\|S^*f\|_{\mathcal{M}}^2 \le \|f\|_{\mathcal{M}}^2 - |f(0)|^2. \qquad \square$$

Theorem 16.29 says that, if $\mathcal{H}$ is a Hilbert space contractively contained in H^2 and such that $S\mathcal{H} \hookrightarrow \mathcal{H}$, then $S^*\mathcal{H}' \hookrightarrow \mathcal{H}'$, but the converse is not true. Let us provide a simple example. Let $\mathcal{M} = \mathrm{Span}(1, z)$ endowed with a new scalar product such that

$$\|1\|_{\mathcal{M}} = \sqrt{2}, \quad \|z\|_{\mathcal{M}} = \sqrt{3} \quad \text{and} \quad 1 < \langle 1, z\rangle_{\mathcal{M}} < \sqrt{2}.$$

Then, we can easily check that $S^*\mathcal{M} \hookrightarrow \mathcal{M} \hookrightarrow H^2$. Putting $f(z) = 1 - z$, we see that $\|S^*f\|_{\mathcal{M}}^2 = \|1\|_{\mathcal{M}}^2 = 2$ and

$$\|f\|_{\mathcal{M}}^2 - |f(0)|^2 = 1 + \|z\|_{\mathcal{M}}^2 - 2\Re\langle 1, z\rangle_{\mathcal{M}} - 1 = 3 - 2\Re\langle 1, z\rangle_{\mathcal{M}}.$$

In particular,

$$\|f\|_{\mathcal{M}}^2 - |f(0)|^2 < \|S^*f\|_{\mathcal{M}}^2,$$

and Theorem 16.29 implies that $S\mathcal{M}'$ cannot be contained contractively in $\mathcal{M}'$.

The inequality (16.38) might look a bit strange. But note that

$$\|S^*f\|_2^2 = \|f\|_2^2 - |f(0)|^2 \qquad (f \in H^2),$$

and if $\mathcal{H}$ is a Hilbert space that is isometrically contained in H^2, then the inequality holds in the general case. In Section 18.8, we will show that the space $\mathcal{H}(b)$ also satisfies (16.38), and in Sections 23.5 and 25.4, we will study this inequality more precisely depending on whether b is an extreme or nonextreme point of the closed unit ball of H^∞.

Exercise

Exercise 16.9.1 Let $A \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$, $B \in \mathcal{L}(\mathcal{H}_3, \mathcal{H}_2)$, $C \in \mathcal{L}(\mathcal{H}_3, \mathcal{H}_1)$ be contractions such that $B = AC$. Show that, for any $f \in \mathcal{H}(A)$, we have

$$\|f\|_{\mathcal{H}(A)}^2 = \sup_{g \in \mathcal{H}(C)} (\|f + Ag\|_{\mathcal{H}(B)}^2 - \|g\|_{\mathcal{H}(C)}^2).$$

Hint: Apply Theorem 16.23 to get

$$\sup_{g \in \mathcal{H}(C)} (\|f + Ag\|_{\mathcal{H}(B)}^2 - \|g\|_{\mathcal{H}(C)}^2) \le \|f\|_{\mathcal{H}(A)}^2.$$

For the other direction, let $\varepsilon > 0$. Apply Corollary 16.27 to show that there exists $h \in \mathcal{H}_1$ such that

$$\|f + Ah\|_{\mathcal{H}_2}^2 - \|h\|_{\mathcal{H}_1}^2 \ge \|f\|_{\mathcal{H}(A)}^2 - \varepsilon.$$

Then, apply Theorem 16.25 with $h = Ch_1 + h_2$, $h_1 = C^*h$, $h_2 = (I - CC^*)h$, which gives that

$$\|h\|_{\mathcal{H}_1}^2 = \|h_1\|_{\mathcal{H}_3}^2 + \|h_2\|_{\mathcal{H}(C)}^2.$$

Hence

$$\begin{aligned}\|f + Ah_2\|_{\mathcal{H}(B)}^2 - \|h_2\|_{\mathcal{H}(C)}^2 &= \|f + Ah - Bh_1\|_{\mathcal{H}(B)}^2 - \|h_2\|_{\mathcal{H}(C)}^2 \\ &\ge \|f + Ah\|_{\mathcal{H}_2} - \|h_1\|_{\mathcal{H}_3}^2 - \|h_2\|_{\mathcal{H}(C)}^2 \\ &= \|f + Ah\|_{\mathcal{H}_2}^2 - \|h\|_{\mathcal{H}_1}^2 \\ &\ge \|f\|_{\mathcal{H}(A)}^2 - \varepsilon.\end{aligned}$$

16.10 The Julia operator $\mathfrak{J}(A)$ and $\mathcal{H}(A)$

As we have seen in Theorem 16.25, the complementary space $\mathcal{H}(A)$ generalizes in some sense the notion of orthogonality. There is a more direct way in which complementarity is related to orthogonality. In Section 7.3, we introduced the Julia operator $J(A)$ associated with a contraction $A \in \mathcal{L}(\mathcal{H}_1, \mathcal{H})$. Let us recall that $J(A) \in \mathcal{L}(\mathcal{H} \oplus \mathcal{H}_1)$ is defined by

$$J(A) = \begin{bmatrix} D_{A^*} & A \\ -A^* & D_A \end{bmatrix},$$

where $D_A = (I - A^*A)^{1/2}$ and $D_{A^*} = (I - AA^*)^{1/2}$. We also recall that $\mathcal{D}_A = \mathrm{Clos}_{\mathcal{H}}(D_A\mathcal{H}_1)$ and $\mathcal{D}_{A^*} = \mathrm{Clos}_{\mathcal{H}_1}(D_{A^*}\mathcal{H})$. According to Theorem 7.18, $J(A)$ is a unitary operator on $\mathcal{H} \oplus \mathcal{H}_1$.

We now define a related operator, which is also called the *Julia operator*. The operator

$$\mathfrak{J}(A) : \mathcal{H}_1 \oplus \mathcal{H} \longrightarrow \mathcal{H} \oplus \mathcal{H}_1$$

is defined by

$$\mathfrak{J}(A) = \begin{bmatrix} A & D_{A^*} \\ D_A & -A^* \end{bmatrix}.$$

There is a simple relation between these two operators. The connection is via the unitary operator $U : \mathcal{H} \oplus \mathcal{H}_1 \longrightarrow \mathcal{H}_1 \oplus \mathcal{H}$ defined by $U(x \oplus y) = y \oplus x$, where $x \oplus y \in \mathcal{H} \oplus \mathcal{H}_1$. If we identify this operator with its matrix, then we have

$$U = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} : \mathcal{H} \oplus \mathcal{H}_1 \longrightarrow \mathcal{H}_1 \oplus \mathcal{H}.$$

Hence

$$J(A)U = \begin{bmatrix} D_{A^*} & A \\ -A^* & D_A \end{bmatrix} \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} = \begin{bmatrix} A & D_{A^*} \\ D_A & -A^* \end{bmatrix},$$

which means that

$$\mathfrak{J}(A) = J(A)U.$$

In particular, we deduce from Theorem 7.18 that $\mathfrak{J}(A)$ is a unitary operator from $\mathcal{H}_1 \oplus \mathcal{H}$ onto $\mathcal{H} \oplus \mathcal{H}_1$.

Theorem 16.30 *Let $A \in \mathcal{L}(\mathcal{H}_1, \mathcal{H})$ be a Hilbert space contraction and assume that A is one-to-one. Write*

$$\mathcal{X}_1 = \mathfrak{J}(A)(\mathcal{H}_1 \oplus \{0\}), \qquad \mathcal{X}_2 = \mathfrak{J}(A)(\{0\} \oplus \mathcal{D}_{A^*}),$$

and let P_1 be the orthogonal projection of $\mathcal{H} \oplus \mathcal{D}_A$ onto its first coordinate $\mathcal{H}$. Then the following hold.

(i) $\mathfrak{J}(A)(\mathcal{H}_1 \oplus \mathcal{D}_{A^*}) = \mathcal{H} \oplus \mathcal{D}_A$.
(ii) $\mathcal{H} \oplus \mathcal{D}_A = \mathcal{X}_1 \oplus \mathcal{X}_2$.
(iii) $P_{1|\mathcal{X}_1}$ *is unitary from* $\mathcal{X}_1$ *onto* $\mathcal{M}(A)$.
(iv) $P_{1|\mathcal{X}_2}$ *is unitary from* $\mathcal{X}_2$ *onto* $\mathcal{H}(A)$.

Proof (i) Let $x \oplus y \in \mathcal{H}_1 \oplus \mathcal{D}_{A^*}$. Then

$$\mathfrak{J}(A)\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} Ax + D_{A^*}y \\ D_A x - A^* y \end{bmatrix}.$$

It is clear that $Ax + D_{A^*}y \in \mathcal{H}$ and $D_A x \in \mathcal{D}_A$. The fact that $A^*y \in \mathcal{D}_A$ for $y \in \mathcal{D}_{A^*}$ follows from (7.12). Hence

$$\mathfrak{J}(A)(\mathcal{H}_1 \oplus \mathcal{D}_{A^*}) \subset \mathcal{H} \oplus \mathcal{D}_A. \tag{16.41}$$

Note that

$$\mathfrak{J}(A^*) = \begin{bmatrix} A^* & D_A \\ D_{A^*} & -A \end{bmatrix} = \mathfrak{J}(A)^*, \tag{16.42}$$

and thus if we apply (16.41) to A^*, we get

$$\mathfrak{J}(A)^*(\mathcal{H} \oplus \mathcal{D}_A) \subset \mathcal{H}_1 \oplus \mathcal{D}_{A^*}.$$

Since $\mathfrak{J}(A)\mathfrak{J}(A)^* = I$, we obtain

$$\mathcal{H} \oplus \mathcal{D}_A \subset \mathfrak{J}(A)(\mathcal{H}_1 \oplus \mathcal{D}_{A^*}),$$

which gives (i).

(ii) This part follows immediately from the fact that $\mathfrak{J}(A)$ is a unitary operator from $\mathcal{H}_1 \oplus \mathcal{D}_{A^*}$ onto $\mathcal{H} \oplus \mathcal{D}_A$ and that

$$\mathcal{H}_1 \oplus \mathcal{D}_{A^*} = (\mathcal{H}_1 \oplus \{0\}) \oplus (\{0\} \oplus \mathcal{D}_{A^*}).$$

(iii) We have

$$P_1\mathcal{X}_1 = P_1(\{Ax \oplus D_A x : x \in \mathcal{H}_1\}) = \{Ax : x \in \mathcal{H}_1\} = \mathcal{M}(A).$$

Moreover, if $x_1 \in \mathcal{X}_1$, then $x_1 = Ax \oplus D_A x$ for some $x \in \mathcal{H}_1$. Then, since A is one-to-one, we have

$$\|P_1 x_1\|_{\mathcal{M}(A)} = \|Ax\|_{\mathcal{M}(A)} = \|x\|_{\mathcal{H}_1}.$$

On the other hand, with (7.33), we also have

$$\|x_1\|^2_{\mathcal{H}\oplus\mathcal{H}_1} = \|Ax\|^2_{\mathcal{H}} + \|D_A x\|^2_{\mathcal{H}_1} = \|x\|^2_{\mathcal{H}_1},$$

which gives the assertion (iii).

(iv) We have

$$P_1\mathcal{X}_2 = P_1(\{D_{A^*}y \oplus (-A^*y) : y \in \mathcal{D}_{A^*}\}) = \{D_{A^*}y : y \in \mathcal{D}_{A^*}\} = \mathcal{H}(A).$$

Now if $x_2 \in \mathcal{X}_2$, then $x_2 = D_{A^*}y \oplus (-A^*y)$ for some $y \in \mathcal{D}_{A^*}$, and we have

$$\|P_1 x_2\|_{\mathcal{H}(A)} = \|D_{A^*}y\|_{\mathcal{H}(A)} = \|y\|_{\mathcal{H}}.$$

On the other hand, once more using (7.33), we also have

$$\|x\|^2_{\mathcal{H}\oplus\mathcal{H}_1} = \|D_{A^*}y\|^2_{\mathcal{H}} + \|A^*y\|_{\mathcal{H}_1} = \|y\|^2_{\mathcal{H}},$$

which gives the result. □

Theorem 16.30 says that the orthogonal decomposition of $\mathcal{H} \oplus \mathcal{D}_A$ as $\mathcal{X}_1 \oplus \mathcal{X}_2$ is mapped by projecting onto the first coordinate into the complementary decomposition $\mathcal{H} = \mathcal{M}(A) + \mathcal{H}(A)$ (which is not in general, as we have already seen, a direct sum). So the rather exotic definition of complementary spaces is in fact the projection of a more familiar geometric structure.

Notes on Chapter 16

The main part of this chapter is taken from [166]. The notion of complementary space, which is the heart of our study, was introduced in the context of square-summable power series by de Branges and Rovnyak in their book [65] and their paper [64].

Section 16.1

The notion of a Hilbert space boundedly contained into another is crucial in the theory developed by de Branges and Rovnyak. Theorem 16.3 is taken from [139]. See also [19].

Section 16.2

Theorem 16.7 is known as Douglas's criterion and can be found in [67]. This theorem as well as Corollaries 16.8 and 16.9 can also be found in [166, chap. I]. Exercise 16.2.2 is taken from [19, corollary 3.3], but the method presented here is slightly different. Exercise 16.2.3 comes from [139, lemma 5.7].

Section 16.3

The description of linear functionals on $\mathcal{M}(A)$ is taken from [166, sec. I.3].

Section 16.4

The notions of contractive containment and complementary spaces were crucial in de Branges's proof of the Bieberbach conjecture. See [63, 141, 142].

The terminology and notion of complementary space are due to de Branges. In the context of square-summable power series, it was introduced by de Branges and Rovnyak in [65]. Nevertheless, the definition of complementary space $\mathcal{H}(A)$ used in this book is due to Sarason and appears in [160]. See Section 16.9 for the original definition of de Branges and Rovnyak. In [64] de Branges and Rovnyak used "overlapping space" in a different way. In this text, we use this term in the sense introduced by Lotto and Sarason [123].

The presentation of this section is taken from [166, chap. I]. A special version of Theorem 16.17 appears in [160] without proof.

Section 16.5

Theorem 16.18 on the relation between $\mathcal{H}(A)$ and $\mathcal{H}(A^*)$ is due to Lotto and Sarason [123, lemma 2.1].

Section 16.6

The description of the overlapping space $\mathcal{H}(A) \cap \mathcal{M}(A)$ is due to Lotto and Sarason [123, lemma 2.1].

Section 16.8

The decomposition of $\mathcal{H}(A)$ given by Theorem 16.23 is due to de Branges and Rovnyak [65, problem 52]. See also [64, appdx, theorem 4]. Theorem 16.25 is also due to de Branges and Rovnyak [65, theorem 8 and problem 36]. The presentation used in this text comes from [166].

Section 16.9

The geometric definition of the complementary space $\mathcal{H}(A)$ given in this section is due to de Branges and Rovnyak. The definition we choose to introduce the complementary space emphasizes the role of the contraction A and it will be successful (as we will see later) in the context of the Toeplitz operator on H^2.

A vector-valued version of Theorem 16.29 appears in Nikolskii and Vasyunin [139, theorem 7.4]. See also Ando [19, theorem 4.3]. The example at the end of the section showing that (16.38) is important in Theorem 16.29 comes from [139, theorem 7.4]. Exercise 16.9.1 is due to de Branges and

Rovnyak [65, theorem 11], who proved the formula in the context of square-summable power series.

Section 16.10

The connection between the Julia operator $\mathfrak{J}(A)$ and $\mathcal{H}(A)$ is taken from Timotin [187]. Nevertheless, in the particular case where $A = T_b$, it is implicitly present in the paper of Nikolskii and Vasyunin [139] when they studied the connection between the de Branges–Rovnyak and the Sz.-Nagy–Foiaş models.

17

Hilbert spaces inside H^2

In this chapter, our ambient Hilbert space is H^2, the Hardy space of the open unit disk $\mathbb{D}$. Using contractive Toeplitz operators on H^2, we apply the theory developed in Chapter 16 to obtain some new Hilbert spaces of analytic functions that live inside H^2. In particular, we introduce the $\mathcal{H}(b)$ spaces.

In Section 17.1, we introduce the $\mathcal{M}(u)$ spaces, where $u \in L^\infty(\mathbb{T})$. When u is analytic, this space is particularly simple and in this case we characterize some inclusions between two $\mathcal{M}(u)$ spaces. In Section 17.2, we study the case when u is antianalytic. Then, in Section 17.3, we introduce the space that is the main theme of our book, the $\mathcal{H}(b)$ space. The rest of the book is devoted to understanding this important (and as yet mysterious) space. The cousin of the $\mathcal{H}(b)$ space, the $\mathcal{H}(\bar{b})$ space, is introduced in Section 17.4. As we will see, it is impossible to separate the study of $\mathcal{H}(b)$ from that of $\mathcal{H}(\bar{b})$. The relation between two $\mathcal{H}(\bar{b})$ spaces is studied in Section 17.5. Since more tools are needed, the analogous result for $\mathcal{H}(b)$ spaces is stated in Chapter 27. In Section 17.6, we show that $\mathcal{M}(u)$ and $\mathcal{M}(\bar{u})$, $u \in H^\infty$, are invariant with respect to the backward shift operator S^*. In Section 17.7, we prove that $\mathcal{M}(u)$ is contractively included into $\mathcal{M}(\bar{u})$. The problem of the similarity of S and $S_{\mathcal{H}}$ is studied in Section 17.8. In Section 17.9, we describe the invariant subspaces with respect to $X_{\bar{u}} = S^*{}_{\mathcal{M}(\bar{u})}$. In Section 17.10, we extend the result of Beurling proved in Volume 1 (see Section 8.8). Recall that Beurling's theorem describes the closed subspaces of H^2 that are invariant with respect to S. Here, we describe the Hilbert spaces $\mathcal{H}$, which are contractively included in H^2, which are invariant with respect to S and such that S acts as an isometry.

17.1 The space $\mathcal{M}(u)$

Let $u \in L^\infty(\mathbb{T})$. Then the Hilbert spaces $\mathcal{M}(T_u)$ and $\mathcal{M}(T_{\bar{u}})$ are well defined. For simplicity, we denote them respectively by $\mathcal{M}(u)$ and $\mathcal{M}(\bar{u})$. We study some of their elementary results below. We will be concerned with the special, but important, case $u \in H^\infty(\mathbb{T})$, $u \not\equiv 0$. In this situation, we have the set identity

$$\mathcal{M}(u) = T_u H^2 = uH^2, \tag{17.1}$$

and, since T_u is injective,

$$\|T_u f\|_{\mathcal{M}(u)} = \|uf\|_{\mathcal{M}(u)} = \|f\|_2 \qquad (f \in H^2). \tag{17.2}$$

In particular, if u is inner, then $\|f\|_2 = \|uf\|_2$, and thus (17.2) becomes

$$\|T_u f\|_{\mathcal{M}(u)} = \|T_u f\|_2 \qquad (f \in H^2).$$

Hence, in the case when u is inner, the space $\mathcal{M}(u)$ is a closed subspace of H^2. According to Theorem 8.32, the class $\{\mathcal{M}(u) : u \text{ inner}\}$ describes exactly the class of closed subspaces of H^2 that are invariant under the forward shift operator S.

In the following result, we discuss some relations between two different $\mathcal{M}(u)$ spaces.

Theorem 17.1 *Let $u_1, u_2 \in H^\infty$. Then the following hold.*

(i) *$\mathcal{M}(u_1) \subset \mathcal{M}(u_2)$ if and only if $u_1/u_2 \in H^\infty$.*
(ii) *$\mathcal{M}(u_1) \hookrightarrow \mathcal{M}(u_2)$ if and only if $u_1 = u_2 b$, where b is in the closed unit ball of H^∞.*
(iii) *$\mathcal{M}(u_1) \overset{=}{=} \mathcal{M}(u_2)$ if and only if $u_1 = \gamma u_2$ for some constant $\gamma \in \mathbb{T}$.*

Proof (i) It follows from Lemma 16.6 and Theorem 16.7 that $\mathcal{M}(u_1) \subset \mathcal{M}(u_2)$ if and only if there is a constant $c > 0$ such that

$$T_{u_1} T_{\bar{u}_1} \le c^2 T_{u_2} T_{\bar{u}_2}. \tag{17.3}$$

Hence, applying (17.3) to the Cauchy kernel k_z, $z \in \mathbb{D}$, gives

$$\|T_{\bar{u}_1} k_z\|_2^2 \le c^2 \, \|T_{\bar{u}_2} k_z\|_2^2 \qquad (z \in \mathbb{D}).$$

But, by (12.7), $T_{\bar{u}_i} k_z = \overline{u_i(z)}\, k_z$, and thus we obtain

$$|u_1(z)|^2 \le c^2 \, |u_2(z)|^2 \qquad (z \in \mathbb{D}).$$

This implies that $u_1/u_2 \in H^\infty$. Conversely, if $u_1 = u_2 b$, where $b \in H^\infty$, then we have

$$\mathcal{M}(u_1) = u_1 H^2 = u_2 b H^2 \subset u_2 H^2 = \mathcal{M}(u_2).$$

(ii) It follows from Theorem 16.7 that $\mathcal{M}(u_1) \hookrightarrow \mathcal{M}(u_2)$ if and only if

$$T_{u_1} T_{\bar{u}_1} \le T_{u_2} T_{\bar{u}_2}.$$

Arguing as in (i), we get

$$|u_1(z)|^2 \le |u_2(z)|^2 \qquad (z \in \mathbb{D}).$$

This means that u_1/u_2 belongs to the closed unit ball of H^∞. Conversely, if $u_1 = u_2 b$, where b is in the closed unit ball of H^∞, then, since $T_b T_{\bar{b}} \leq I$, we have

$$T_{u_1} T_{\bar{u}_1} = T_{u_2} T_b T_{\bar{b}} T_{\bar{u}_2} \leq T_{u_2} T_{\bar{u}_2},$$

which, using Theorem 16.7 once more, reveals that $\mathcal{M}(u_1) \hookrightarrow \mathcal{M}(u_2)$.

(iii) It follows immediately from part (ii) that $\mathcal{M}(u_1) \coloneqq \mathcal{M}(u_2)$ if and only if $u_1 = u_2 b$, with b and b^{-1} both in the closed unit ball of H^∞. Therefore, by Corollary 4.24, b must be an outer function and $|b| = 1$ a.e. on $\mathbb{T}$. Hence, b is a constant of modulus one. Conversely, it is trivial that, if $u_1 = \gamma u_2$, where $\gamma \in \mathbb{T}$, then we have $\mathcal{M}(u_1) \coloneqq \mathcal{M}(u_2)$. $\square$

When $u \in H^\infty$, the space $\mathcal{M}(u)$ is trivially invariant under the shift operator S. It is natural to ask if it could also be invariant under the backward shift operator. The following result answers this question.

Lemma 17.2 *Let $u \in H^\infty$, $u \not\equiv 0$. The following assertions are equivalent:*

(i) *$S^*\mathcal{M}(u) \subset \mathcal{M}(u)$;*
(ii) *$S^* u \in \mathcal{M}(u)$;*
(iii) *the function $1/u$ belongs to H^2.*

Proof The implication (i) $\Longrightarrow$ (ii) is clear because $u \in \mathcal{M}(u)$.

Let us now prove the implication (ii) $\Longrightarrow$ (iii). If $S^* u \in \mathcal{M}(u)$, then there exists $g \in H^2$ such that $S^* u = ug$. Hence $u - u(0) = zug$, which gives

$$(1 - zg)u = u(0). \tag{17.4}$$

If $u(0) = 0$, then $(1 - zg)u \equiv 0$. Since $u \not\equiv 0$, we get that $1 - zg(z) = 0$, $z \in \mathbb{D}$. Taking $z = 0$ gives a contradiction. Hence $u(0) \neq 0$ and (17.4) implies that $u(z) \neq 0$ for any $z \in \mathbb{D}$. Thus we can write

$$\frac{1}{u} = \frac{1 - zg}{u(0)},$$

which gives $1/u \in H^2$.

It remains to establish the implication (iii) $\Longrightarrow$ (i). Assume that $1/u \in H^2$ and let $f \in H^2$. Write

$$(S^*(uf))(z) = \frac{u(z)f(z) - u(0)f(0)}{z} = u(z)g(z),$$

with

$$g(z) = \frac{f(z) - u(0)f(0)/u(z)}{z} \qquad (z \in \mathbb{D}).$$

Note that the condition $1/u \in H^2$ implies in particular that $u(z) \neq 0$, $z \in \mathbb{D}$. Thus, the function $f - u(0)f(0)/u$ belongs to H^2 and it vanishes at the origin. Hence, we can conclude that $g \in H^2$, which gives that $S^*(uf) \in \mathcal{M}(u)$. $\square$

Note that, if u is inner, then Lemma 17.2 reveals that $S^*\mathcal{M}(u) \subset \mathcal{M}(u)$ if and only if u is constant. We thus recover Theorem 8.36.

17.2 The space $\mathcal{M}(\bar{u})$

For $u \in H^\infty$, according to (17.1), T_u is an injective multiplication operator on H^2, and that is why $\mathcal{M}(u)$ has a simple structure. The structure of $\mathcal{M}(\bar{u})$ is more complex and needs extra attention. The first result shows that, to study the space $\mathcal{M}(\bar{u})$, we can always assume, if necessary, that u is outer.

Lemma 17.3 *Let $\varphi \in H^\infty$ and Θ be an inner function. Then,*

$$\mathcal{M}(\overline{\Theta\varphi}) \doteq \mathcal{M}(\bar{\varphi}).$$

In particular, if $u \in H^\infty$, let $u = u_i u_o$ be the canonical factorization of u into its inner part u_i and outer part u_o. Then we have $\mathcal{M}(\bar{u}) \doteq \mathcal{M}(\bar{u}_o)$.

Proof It follows from Corollary 16.8 that $\mathcal{M}(\overline{\Theta\varphi}) \doteq \mathcal{M}(\bar{\varphi})$ if and only if

$$T_{\overline{\Theta\varphi}} T_{\Theta\varphi} = T_{\bar{\varphi}} T_{\varphi}.$$

But, using Theorem 12.4 and that $|\Theta| = 1$ a.e. on $\mathbb{T}$, we have

$$T_{\overline{\Theta\varphi}} T_{\Theta\varphi} = T_{\bar{\varphi}} T_{|\Theta|^2} T_{\varphi} = T_{\bar{\varphi}} T_{\varphi}.$$

The second part of the theorem is just an application of the first part taking $\Theta = u_i$ and $\varphi = u_o$. □

The main advantage of using an outer function is that $T_{\bar{u}_o}$ is injective (see Theorem 12.19). Hence, for each $f \in \mathcal{M}(\bar{u})$, there is a *unique* $g \in H^2$ such that $f = T_{\bar{u}_o} g$ and, moreover,

$$\|f\|_{\mathcal{M}(\bar{u})} = \|g\|_2. \tag{17.5}$$

In general, $\mathcal{M}(u)$ is far from being dense in H^2. More precisely, by Theorem 8.16, we know that the closure of $\mathcal{M}(u)$ in H^2 is $u_i H^2$, where u_i is the inner part of u. Hence, $\mathcal{M}(u)$ is a dense subspace of H^2 if and only if u is outer.

The following result shows that this is not the case for $\mathcal{M}(\bar{u})$. In fact, for any choice of u, $\mathcal{M}(\bar{u})$ contains all the analytic polynomials.

Theorem 17.4 *Let $u \in H^\infty$, $u \not\equiv 0$. Then $\mathcal{M}(\bar{u})$ contains $\mathcal{P}$, the linear manifold of all analytic polynomials, as an infinite-dimensional dense subset.*

Proof By Lemma 17.3, we have $\mathcal{M}(\bar{u}) = \mathcal{M}(\bar{u}_o)$, where u_o is the outer part of u. Now, since $u_o(0) \neq 0$, it appears from the proof of Theorem 12.6 that, if p is an analytic polynomial of degree n, then $T_{\bar{u}_o}p$ is a polynomial of degree n. Hence, we certainly have

$$\mathcal{P} \subset \mathcal{M}(\bar{u}).$$

But, by (16.7), for each analytic polynomial p, we have

$$\|T_{\bar{u}}f - T_{\bar{u}}p\|_{\mathcal{M}(\bar{u})} \leq \|f - p\|_2.$$

Since the analytic polynomials are dense in H^2, and, again by Theorem 12.6, $T_{\bar{u}}p$ is an analytic polynomial, we conclude that $\mathcal{P}$ is dense in $\mathcal{M}(\bar{u})$. □

Note that Theorem 17.4 also reveals that $\mathcal{M}(\bar{u})$ is a dense submanifold of H^2, since it contains all the analytic polynomials.

The identity $\mathcal{M}(\bar{u}) \doteq \mathcal{M}(\bar{u}_o)$ suggests that we should confine ourselves to the outer function. That idea is also affirmatively supported by the following result.

Theorem 17.5 *Let $u_1, u_2 \in H^\infty$, and let $u_k = u_{i_k}u_{o_k}$, $k = 1, 2$, be the canonical factorization of u_k into its inner part u_{i_k} and outer part u_{o_k}. Then the following hold.*

(i) *$\mathcal{M}(\bar{u}_1) \subset \mathcal{M}(\bar{u}_2)$ if and only if $u_{o_1}/u_{o_2} \in H^\infty$.*
(ii) *$\mathcal{M}(\bar{u}_1) \hookrightarrow \mathcal{M}(\bar{u}_2)$ if and only if $u_{o_1} = u_{o_2}b$, where b is in the closed unit ball of H^∞.*
(iii) *$\mathcal{M}(\bar{u}_1) \doteq \mathcal{M}(\bar{u}_2)$ if and only if $u_{o_1} = \gamma u_{o_2}$, where γ is a constant in $\mathbb{T}$.*

Proof (i) According to Theorem 16.7 and Lemma 16.6, we have $\mathcal{M}(\bar{u}_1) \subset \mathcal{M}(\bar{u}_2)$ if and only if there is a constant $c > 0$ such that

$$T_{\bar{u}_1}T_{u_1} \leq c^2 T_{\bar{u}_2}T_{u_2}. \tag{17.6}$$

By Theorem 12.4, this is equivalent to

$$T_{|u_1|^2} \leq c^2 T_{|u_2|^2} = T_{|cu_2|^2},$$

and Theorem 12.3 says that this happens if and only if

$$|u_1|^2 \leq c^2 |u_2|^2 \qquad \text{(a.e. on } \mathbb{T}).$$

Since an inner function is unimodular on $\mathbb{T}$, we see that the preceding inequality is equivalent to

$$|u_{o_1}| \leq c\,|u_{o_2}| \qquad \text{(a.e. on } \mathbb{T}).$$

This means that $u_{o_1}/u_{o_2} \in L^\infty$. But since u_{o_2} is outer, Corollary 4.28 implies that u_{o_1}/u_{o_2} is in fact in H^∞.

Conversely, assume that $u_{o_1} = u_{o_2} b$, for some function $b \in H^\infty$. Then, by Theorem 12.4,

$$T_{\bar{u}_{o_1}} = T_{\bar{u}_{o_2}} T_{\bar{b}},$$

which implies that $\mathcal{M}(\bar{u}_{o_1}) \subset \mathcal{M}(\bar{u}_{o_2})$. Finally, Lemma 17.3 gives

$$\mathcal{M}(\bar{u}_1) = \mathcal{M}(\bar{u}_{o_1}) \subset \mathcal{M}(\bar{u}_{o_2}) = \mathcal{M}(\bar{u}_2).$$

(ii) This follows immediately from the proof of (i), since in this case we have $c = 1$.

(iii) This follows from (ii) and an argument similar to that used in the proof of Theorem 17.1(iii). $\square$

17.3 The space $\mathcal{H}(b)$

If $\varphi \in L^\infty(\mathbb{T})$ satisfies $\|\varphi\|_\infty \le 1$, then, by Theorem 12.2, the corresponding Toeplitz operator T_φ is a contraction on the Hilbert space H^2. Hence, the Hilbert space $\mathcal{H}(T_\varphi)$ is well defined. For simplicity, we denote the complementary space $\mathcal{H}(T_\varphi)$ by $\mathcal{H}(\varphi)$. By the same token, the norm and inner product in $\mathcal{H}(\varphi)$ will be denoted respectively by $\|\cdot\|_\varphi$ and $\langle \cdot, \cdot \rangle_\varphi$.

Our main concern is when φ is a nonconstant analytic function in the closed unit ball of H^∞. In this case, by tradition, we use b instead of φ. Therefore, from now on, we assume that

(i) $b \in H^\infty$,

(ii) b is not a constant, and

(iii) $\|b\|_\infty \le 1$,

and the corresponding Hilbert spaces created by T_b are denoted by $\mathcal{M}(b)$ and $\mathcal{H}(b)$, i.e.

$$\mathcal{M}(b) = \mathcal{M}(T_b) \tag{17.7}$$

and

$$\mathcal{H}(b) = \mathcal{M}((I - T_b T_{\bar{b}})^{1/2}). \tag{17.8}$$

We recall that the inner product in $\mathcal{H}(b)$ is defined by

$$\|(I - T_b T_{\bar{b}})^{1/2} f\|_b = \|f\|_2 \qquad (f \in H^2 \ominus \ker(I - T_b T_{\bar{b}})^{1/2}). \tag{17.9}$$

As we saw in Section 17.1, the structure of $\mathcal{M}(b)$ is simple. We study the structure of $\mathcal{H}(b)$ in detail. In fact, this is the main goal of this book.

As we discussed in Section 16.9, we know that $\mathcal{H}(b)$ and $\mathcal{M}(b)$ are complementary spaces of each other. In particular, a function $f \in H^2$ belongs to $\mathcal{H}(b)$ if and only

$$\sup_{g\in H^2} (\|f + bg\|_2^2 - \|g\|_2^2) < \infty,$$

and, in this situation, we have

$$\|f\|_b^2 = \sup_{g\in H^2} (\|f + bg\|_2^2 - \|g\|_2^2).$$

Similarly, a function $f \in H^2$ belongs to $\mathcal{M}(b)$ if and only if

$$\sup_{g\in \mathcal{H}(b)} (\|f + g\|_2^2 - \|g\|_b^2) < \infty,$$

and, for such functions, we have

$$\|f\|_{\mathcal{M}(b)}^2 = \sup_{g\in \mathcal{H}(b)} (\|f + g\|_2^2 - \|g\|_b^2).$$

In the following, we just rewrite Corollary 16.16 for the Toeplitz operator $T_b \in \mathcal{L}(H^2)$. More results will be discussed in the following sections.

Corollary 17.6 *With respect to the Hilbert space structure of $\mathcal{H}(b)$, the linear manifold $\mathcal{R}(I - T_b T_{\bar{b}})$ is dense in $\mathcal{H}(b)$. Moreover, for each $f \in H^2$ and $g \in \mathcal{H}(b)$, we have*

$$\|(I - T_b T_{\bar{b}})f\|_{\mathcal{H}(b)}^2 = \|(I - T_b T_{\bar{b}})^{1/2} f\|_{H^2}^2 = \|f\|_{H^2}^2 - \|T_{\bar{b}} f\|_{H^2}^2$$

and

$$\langle g,\ (I - T_b T_{\bar{b}})f\rangle_{\mathcal{H}(b)} = \langle g, f\rangle_{H^2}.$$

Exercises

Exercise 17.3.1 Show that the forward shift operator $S \in \mathcal{L}(H^2)$ is a contraction from $\mathcal{H}(b)$ into H^2.
Hint: We have $\|Sf\|_2 \leq \|f\|_2$ and $\|f\|_2 \leq \|f\|_b$ for each $f \in \mathcal{H}(b)$.

Exercise 17.3.2 Let $f = 1 - \overline{b(0)}b$. Show that $f \in \mathcal{H}(b)$ and

$$\|f\|_{\mathcal{H}(b)}^2 = 1 - |b(0)|^2.$$

Hint: Apply Corollary 17.6. Note that $f = (I - T_b T_{\bar{b}})1$.

17.4 The space $\mathcal{H}(\bar{b})$

To better understand the structure of $\mathcal{H}(b)$, we will also naturally be faced with its cousin

$$\mathcal{H}(\bar{b}) = \mathcal{M}((I - T_{\bar{b}}T_b)^{1/2}) = \mathcal{M}(T_{1-|b|^2}^{1/2}) \tag{17.10}$$

and

$$\mathcal{M}(\bar{b}) = \mathcal{M}(T_{\bar{b}}).$$

By Lemma 16.20, we recall that the overlapping space is described as

$$\mathcal{M}(b) \cap \mathcal{H}(b) = T_b\mathcal{H}(\bar{b}), \tag{17.11}$$

and the operator T_b acts as a contraction from $\mathcal{H}(\bar{b})$ into $\mathcal{H}(b)$.

Theorem 17.7 *The overlapping space $\mathcal{M}(b) \cap \mathcal{H}(b)$ is dense in $\mathcal{H}(b)$ if and only if b is an outer function.*

Proof The intertwining relation (7.12), i.e.

$$T_b(I - T_{\bar{b}}T_b)^{1/2} = (I - T_bT_{\bar{b}})^{1/2}T_b,$$

combined with (17.11) give

$$\mathcal{M}(b) \cap \mathcal{H}(b) \subset (I - T_bT_{\bar{b}})^{1/2}T_bH^2 \subset \mathcal{H}(b).$$

Since $(I - T_bT_{\bar{b}})^{1/2}$ is a partial isometry from H^2 onto $\mathcal{H}(b)$, we deduce that $\mathcal{M}(b) \cap \mathcal{H}(b)$ is dense in $\mathcal{H}(b)$ if and only if T_bH^2 is dense in H^2. By Theorem 8.16, this is equivalent to saying that b is an outer function. □

Note that, in the opposite case, when b is an inner function, then we know that $\mathcal{H}(b)$ and $\mathcal{M}(b)$ are orthogonal complements to each other.

The relations between the inner products of $\mathcal{H}(b)$ and $\mathcal{H}(\bar{b})$ are special cases of the Lotto–Sarason theorem (Theorem 16.18 and Corollary 16.19). For further reference, we restate this result below.

Theorem 17.8 *Let $f \in H^2$. Then $f \in \mathcal{H}(b)$ if and only if $T_{\bar{b}}f \in \mathcal{H}(\bar{b})$ and*

$$\langle f_1, f_2\rangle_b = \langle f_1, f_2\rangle_2 + \langle T_{\bar{b}}f_1, T_{\bar{b}}f_2\rangle_{\bar{b}} \qquad (f_1, f_2 \in \mathcal{H}(b)).$$

Similarly, if $g \in H^2$, then $g \in \mathcal{H}(\bar{b})$ if and only if $T_bg \in \mathcal{H}(b)$ and

$$\langle g_1, g_2\rangle_{\bar{b}} = \langle g_1, g_2\rangle_2 + \langle T_bg_1, T_bg_2\rangle_b \qquad (g_1, g_2 \in \mathcal{H}(\bar{b})).$$

As Theorem 17.8 suggests, the structure and properties of $\mathcal{H}(b)$ are closely related to those of $\mathcal{H}(\bar{b})$. In the following chapters, we will see that the properties of these two spaces depend profoundly on whether b is an extreme or nonextreme point of the closed unit ball of H^∞. Indeed, in the case when b is a nonextreme point of the closed unit ball of H^∞, as we will see, there exists an outer function $a \in H^\infty$ such that $\mathcal{H}(\bar{b}) = \mathcal{M}(\bar{a})$ and then $\mathcal{H}(\bar{b})$ will have an easier structure, which also implies special properties for $\mathcal{H}(b)$. Nevertheless, certain general properties of $\mathcal{H}(b)$ and $\mathcal{H}(\bar{b})$ do not depend on whether b is an extreme or nonextreme point of the closed unit ball of H^∞. These general properties will be studied, in particular, in Chapters 18 and 20.

Let us recall that, according to Theorem 17.4, we know that the space $\mathcal{M}(\bar{b})$ contains the linear manifold of all analytic polynomials $\mathcal{P}$, and $\mathcal{P}$ is dense in $\mathcal{M}(\bar{b})$.

Theorem 17.9 *The space $\mathcal{H}(\bar{b})$ is contractively contained in $\mathcal{H}(b)$, i.e.*

$$\mathcal{H}(\bar{b}) \hookrightarrow \mathcal{H}(b).$$

Proof By Theorem 12.10,

$$I - T_{\bar{b}}T_b \le I - T_b T_{\bar{b}}.$$

Hence, by Theorem 16.7, $\mathcal{M}((I - T_{\bar{b}}T_b)^{1/2}) = \mathcal{H}(\bar{b})$ is contractively contained in $\mathcal{M}((I - T_b T_{\bar{b}})^{1/2}) = \mathcal{H}(b)$. □

Corollary 16.16 can be applied to the operator $T_{\bar{b}} \in \mathcal{L}(H^2)$ to obtain the following result for $\mathcal{H}(\bar{b})$ spaces.

Corollary 17.10 *With respect to the Hilbert space structure of $\mathcal{H}(\bar{b})$, the linear manifold $\mathcal{M}(I - T_{\bar{b}}T_b)$ is dense in $\mathcal{H}(\bar{b})$. Moreover, for each $f \in H^2$ and $g \in \mathcal{H}(\bar{b})$, we have*

$$\|(I - T_{\bar{b}}T_b)f\|^2_{\mathcal{H}(\bar{b})} = \|f\|^2_{H^2} - \|T_b f\|^2_{H^2} = \int_{\mathbb{T}} (1 - |b|^2)|f|^2 \, dm$$

and

$$\langle g,\ (I - T_{\bar{b}}T_b)f\rangle_{\mathcal{H}(\bar{b})} = \langle g, f\rangle_{H^2}.$$

Exercises

Exercise 17.4.1 Fix b. Let Θ be any inner function, and put $f = \Theta - P_+(\Theta|b|^2)$. Show that $f \in \mathcal{H}(\bar{b})$ and

$$\|f\|^2_{\mathcal{H}(\bar{b})} = 1 - \|b\|^2_{H^2}.$$

Hint: Apply Corollary 17.10. Note that $f = (I - T_{\bar{b}}T_b)\Theta$.

Exercise 17.4.2 Let $f \in \mathcal{H}(\bar{b})$. Show that $(1 - b)f \in \mathcal{H}(b)$.
Hint: Use (17.11) and Theorem 17.9.

17.5 Relations between different $\mathcal{H}(\bar{b})$ spaces

In Lemma 17.3, we saw that $\mathcal{M}(\bar{u}) = \mathcal{M}(\bar{u}_o)$, where u_o is the outer part of $u \in H^\infty$. The following result has the same flavor. It shows that, in studying the space $\mathcal{H}(\bar{b})$, we could always assume that b is outer.

Lemma 17.11 *Let b be in the closed unit ball of H^∞ and let Θ be an inner function. Then*

$$\mathcal{H}(\overline{\Theta b}) = \mathcal{H}(\bar{b}).$$

In particular, if $b = b_i\, b_o$ is the canonical factorization of b into its inner part b_i and outer part b_o, then we have $\mathcal{H}(\bar{b}) = \mathcal{H}(\bar{b}_o)$.

Proof By Theorem 12.4 and the fact that $|\Theta| = 1$ a.e. on $\mathbb{T}$, we have

$$I - T_{\overline{\Theta b}} T_{\Theta b} = I - T_{|b\,\Theta|^2} = I - T_{|b|^2} = I - T_{\bar{b}} T_b.$$

Hence, Corollary 16.8 ensures that $\mathcal{H}(\overline{\Theta b}) = \mathcal{H}(\bar{b})$. This result can also be derived directly from (17.10). □

Theorem 17.12 *Let b_1 and b_2 be two functions in the closed unit ball of H^∞, and let $b_j = b_{i_j} b_{o_j}$ be the canonical factorization of b_j into its inner part b_{i_j} and outer part b_{o_j}. Then the following hold.*

(i) *$\mathcal{H}(\bar{b}_2) \subset \mathcal{H}(\bar{b}_1)$ if and only if*

$$\sup_{\zeta \in \mathbb{T}} \frac{1 - |b_2(\zeta)|^2}{1 - |b_1(\zeta)|^2} < \infty. \tag{17.12}$$

(ii) *$\mathcal{H}(\bar{b}_2) \hookrightarrow \mathcal{H}(\bar{b}_1)$ if and only if $b_{o_1} = b_{o_2} b$ with some b in the closed unit ball of H^∞.*

(iii) *$\mathcal{H}(\bar{b}_2) = \mathcal{H}(\bar{b}_1)$ if and only if $b_{o_1} = \gamma b_{o_2}$ for some constant $\gamma \in \mathbb{T}$.*

Proof (i) According to Lemma 16.6 and Theorem 16.7, the inclusion $\mathcal{H}(\bar{b}_2) \subset \mathcal{H}(\bar{b}_1)$ holds if and only if there exists a constant $c > 0$ such that

$$I - T_{\bar{b}_2} T_{b_2} \le c^2 (I - T_{\bar{b}_1} T_{b_1}).$$

By Theorem 12.4, this is equivalent to

$$T_{1-|b_2|^2} \le T_{c^2(1-|b_1|^2)},$$

and Theorem 12.3 reveals that the preceding inequality is equivalent to

$$1 - |b_2|^2 \le c^2 (1 - |b_1|^2) \qquad \text{(a.e. on } \mathbb{T}).$$

(ii) Arguing as in (i), $\mathcal{H}(\bar{b}_2) \hookrightarrow \mathcal{H}(\bar{b}_1)$ if and only if

$$1 - |b_2|^2 \le 1 - |b_1|^2 \qquad \text{(a.e. on } \mathbb{T}).$$

In other words, $|b_1(\zeta)| \le |b_2(\zeta)|$ a.e. on $\mathbb{T}$. Since inner parts are unimodular, we deduce that

$$|b_{o_1}(\zeta)| \le |b_{o_2}(\zeta)|$$

for almost all $\zeta \in \mathbb{T}$. Therefore, $b_{o_1}/b_{o_2} \in L^\infty(\mathbb{T})$. But, since b_{o_2} is outer, Corollary 4.28 ensures that $b_{o_1}/b_{o_2} \in H^\infty$.

Conversely, assume that $b_{o_1} = b_{o_2} b$ with some function b in the closed unit ball of H^∞. Then

$$I - T_{\overline{b_{o_1}}} T_{b_{o_1}} = I - T_{\overline{b_{o_2}}} T_{\bar{b}} T_b T_{b_{o_2}} \ge I - T_{\overline{b_{o_2}}} T_{b_{o_2}}.$$

Remember that $T_{\bar{b}} T_b \le I$. Then, according to Theorem 16.7, we have $\mathcal{H}(\overline{b_{o_2}}) \hookrightarrow \mathcal{H}(\overline{b_{o_1}})$. It remains to apply Lemma 17.11 to get the required result.

(iii) It follows immediately from (ii) that $\mathcal{H}(\bar{b}_2) \mathrel{\text{\d{=}}} \mathcal{H}(\bar{b}_1)$ if and only if $b_{o_1} = b_{o_2} b$, with b and b^{-1} both in the closed unit ball of H^∞. Hence, by Corollary 4.24, b must be an outer function and at the same time $|b| = 1$ a.e. on $\mathbb{T}$. Therefore, b is necessarily a constant of modulus one. Conversely, if $b_{o_1} = \gamma b_{o_2}$, with $\gamma \in \mathbb{T}$, then we obviously have $\mathcal{H}(\overline{b_{o_2}}) \mathrel{\text{\d{=}}} \mathcal{H}(\overline{b_{o_1}})$ and the result follows once more from Lemma 17.11. $\square$

Corollary 17.13 *Let b be a function in the closed unit ball of H^∞. Then*

$$\mathcal{H}(\bar{b}) = \mathcal{H}(\bar{b}^2).$$

Proof Note that

$$1 \le \frac{1 - |b|^4}{1 - |b|^2} \le 2, \quad \text{a.e. on } \mathbb{T}.$$

Then Theorem 17.12 implies the equality $\mathcal{H}(\bar{b}) = \mathcal{H}(\bar{b}^2)$. $\square$

Let b be an outer function in the closed unit ball of H^∞. Then it has no zeros on $\mathbb{D}$ and so we can define b^r by taking any logarithm of b. Note that b^r is also an outer function in the closed unit ball of H^∞. Since

$$\frac{1 - x^r}{1 - x} \asymp 1, \quad x \in [0, 1),$$

we see that $1 - |b|^r \asymp 1 - |b|$. Then Theorem 17.12 immediately implies the following generalization of Corollary 17.13.

Corollary 17.14 *Let b be an outer function in the closed unit ball of H^∞. Then*

$$\mathcal{H}(\bar{b}) = \mathcal{H}(\bar{b}^r)$$

for any $r > 0$.

We will see in Section 28.2 some situation where $\mathcal{H}(b^r)$ itself does not depend on r.

17.6 $\mathcal{M}(\bar{u})$ is invariant under S and S^*

Studying the invariant subspaces of an operator is the core of our text. The following lemma is a result of this type, which shows that $\mathcal{M}(\bar{u})$ is invariant under S and S^*.

Theorem 17.15 *Let $u, \varphi \in H^\infty$. Then $\mathcal{M}(\bar{u})$ is invariant under $T_{\bar{\varphi}}$. In particular, $\mathcal{M}(\bar{u})$ is invariant under the backward shift operator $S^* = T_{\bar{z}}$. Moreover, the norm of the operator*

$$\begin{array}{ccc} \mathcal{M}(\bar{u}) & \longrightarrow & \mathcal{M}(\bar{u}) \\ f & \longmapsto & T_{\bar{\varphi}} f \end{array}$$

does not exceed $\|\varphi\|_\infty$.

Proof Without loss of generality, assume that $\|\varphi\|_\infty = 1$. By Corollary 16.10, the inequality $RAA^*R^* \leq AA^*$ implies that $R\mathcal{M}(A) \subset \mathcal{M}(A)$ and that R acts contractively from $\mathcal{M}(A)$ into itself. In our setting, take $R = T_{\bar{\varphi}}$ and $A = T_{\bar{u}}$. Hence, we need to show that

$$T_{\bar{\varphi}} T_{\bar{u}} T_u T_\varphi \leq T_{\bar{u}} T_u.$$

But, by Theorem 12.4, this inequality is equivalent to

$$T_{|u|^2(1-|\varphi|^2)} \geq 0,$$

which, in the light of Theorem 12.3(ii), is in fact true.

This result can also be proven directly without appealing to Corollary 16.10. Fix any $f \in \mathcal{M}(\bar{u})$. Then, by (17.5), there is a unique $g \in H^2$ such that $f = T_{\bar{u}_o} g$ and $\|f\|_{\mathcal{M}(\bar{u})} = \|g\|_2$. Hence, by Theorem 12.4,

$$T_{\bar{\varphi}} f = T_{\bar{\varphi}} T_{\bar{u}_o} g = T_{\bar{u}_o} T_{\bar{\varphi}} g.$$

This identity shows that $T_{\bar{\varphi}} f \in \mathcal{M}(\bar{u})$ and $\|T_{\bar{\varphi}} f\|_{\mathcal{M}(\bar{u})} = \|T_{\bar{\varphi}} g\|_2$. Therefore,

$$\|T_{\bar{\varphi}} f\|_{\mathcal{M}(\bar{u})} \leq \|T_{\bar{\varphi}}\| \, \|g\|_2 = \|\varphi\|_\infty \, \|f\|_{\mathcal{M}(\bar{u})}.$$ □

The preceding result enables us to define the bounded operator

$$\begin{array}{cccc} X_{\bar{u}} : & \mathcal{M}(\bar{u}) & \longrightarrow & \mathcal{M}(\bar{u}) \\ & f & \longmapsto & S^* f. \end{array}$$

We may also denote this operator simply by $S^*_{|\mathcal{M}(\bar{u})}$. However, to avoid difficulties arising in the determination of the adjoint, spectrum, etc., of this operator, the notation $X_{\bar{u}}$ is preferable.

Theorem 17.16 *Let $u \in H^\infty$. Then the space $\mathcal{M}(\bar{u})$ is invariant under S and, moreover, the operator*

$$\begin{array}{rccc} Z_{\bar{u}}: & \mathcal{M}(\bar{u}) & \longrightarrow & \mathcal{M}(\bar{u}) \\ & f & \longmapsto & Sf \end{array}$$

is bounded.

Proof According to Lemma 17.3, without loss of generality, we assume that u is outer. For each $f \in H^2$, we have

$$(T_{\bar{u}}S - ST_{\bar{u}})f = P_+(\chi_1 \bar{u} f) - \chi_1 P_+(\bar{u} f) = \widehat{\bar{u}f}(-1) = c \in \mathbb{C},$$

and, by the Cauchy–Schwarz inequality,

$$|c| = |\widehat{\bar{u}f}(-1)| \leq \|u\|_2 \, \|f\|_2. \tag{17.13}$$

See Exercise 8.2.2. Also note that $T_{\bar{u}}1 = \overline{u(0)}$. Hence,

$$S(T_{\bar{u}}f) = T_{\bar{u}}Sf - c = T_{\bar{u}}(Sf - c/\overline{u(0)}). \tag{17.14}$$

Therefore, we conclude that

$$S(T_{\bar{u}}f) \in \mathcal{M}(\bar{u}) \qquad (f \in H^2).$$

This means that $S\mathcal{M}(\bar{u}) \subset \mathcal{M}(\bar{u})$. Moreover, it follows from (17.13) and (17.14) that

$$\|S(T_{\bar{u}}f)\|_{\mathcal{M}(\bar{u})} = \|Sf - c/\overline{u(0)}\|_2 \leq (1 + \|u\|_2/|u(0)|)\, \|f\|_2.$$

Hence, by (17.5),

$$\|S(T_{\bar{u}}f)\|_{\mathcal{M}(\bar{u})} \leq (1 + \|u\|_2/|u(0)|)\, \|T_{\bar{u}}f\|_{\mathcal{M}(\bar{u})}.$$

This means that S acts as a bounded operator from $\mathcal{M}(\bar{u})$ into itself. □

Incidentally, the last inequality shows that

$$\|Z_{\bar{u}}\|_{\mathcal{M}(\bar{u})} \leq 1 + \frac{\|u\|_2}{|u(0)|}.$$

We saw that $\mathcal{M}(\bar{u}) \subset H^2$ is a reproducing kernel Hilbert space that is invariant under S and S^*. Therefore, the shift operator $S_{\mathcal{M}(\bar{u})}$ is an example of the abstract operator, which was treated in Section 9.5.

17.7 Contractive inclusion of $\mathcal{M}(u)$ in $\mathcal{M}(\bar{u})$

In the following result, we characterize the equality of $\mathcal{M}(u)$ and $\mathcal{M}(\bar{u})$, when u is a function in H^∞. Note that, for the definition of $\mathcal{M}(u)$ and $\mathcal{M}(\bar{u})$, we do not need to assume that $\|u\|_\infty \leq 1$.

Theorem 17.17 *Let $u \in H^\infty$. Then we have*

$$\mathcal{M}(u) \hookrightarrow \mathcal{M}(\bar{u}),$$

i.e. $\mathcal{M}(u)$ is contractively included in $\mathcal{M}(\bar{u})$. Furthermore, if u is outer, then the following are equivalent:

(i) $\mathcal{M}(u) = \mathcal{M}(\bar{u})$;
(ii) $T_{u/\bar{u}} \in \mathcal{L}(H^2)$ *is invertible.*

Proof For all $f \in H^2$, we have

$$\|T_{\bar{u}} f\|_2^2 = \|P_+(\bar{u} f)\|_2^2 \le \|\bar{u} f\|_2^2 = \|u f\|_2^2 = \|T_u f\|_2^2,$$

which, using (1.32) and (12.2) and the fact that $T_{\bar{u}}^* = T_u$, implies

$$T_u T_u^* \le T_{\bar{u}} T_{\bar{u}}^*.$$

Hence, by Theorem 16.7, $\mathcal{M}(u)$ is contractively included in $\mathcal{M}(\bar{u})$.

To prove the equivalence of (i) and (ii), first note that, according to Theorem 12.4, we have $T_{\bar{u}} T_{u/\bar{u}} = T_u$, and thus

$$T_{\bar{u}} T_{u/\bar{u}} H^2 = \mathcal{M}(u). \tag{17.15}$$

(i) $\Longrightarrow$ (ii) By Theorem 12.19, the operator $T_{\bar{u}}$ is injective. In the light of (17.15) and the assumption (i), we now have

$$T_{\bar{u}} T_{u/\bar{u}} H^2 = \mathcal{M}(u) = \mathcal{M}(\bar{u}) = T_{\bar{u}} H^2,$$

which is equivalent to

$$T_{u/\bar{u}} H^2 = H^2.$$

In other words, $T_{u/\bar{u}}$ is onto. Therefore, by Theorem 12.24, $T_{u/\bar{u}}$ is invertible.

(ii) $\Longrightarrow$ (i) If $T_{u/\bar{u}}$ is invertible, then $T_{u/\bar{u}} H^2 = H^2$. Thus, by (17.15),

$$\mathcal{M}(u) = T_{\bar{u}} T_{u/\bar{u}} H^2 = T_{\bar{u}} H^2 = \mathcal{M}(\bar{u}). \qquad \square$$

Exercise

Exercise 17.7.1 Let $u \in H^\infty$ be an outer function, $u_1 \in H^\infty$ and assume that $u_1/u \in L^\infty(\mathbb{T})$. Show that $\mathcal{M}(\bar{u}) = \mathcal{M}(u_1)$ if and only if the Toeplitz operator $T_{u_1/\bar{u}}$ is invertible.
Hint: See the proof of Theorem 17.17.

17.8 Similarity of S and $S_{\mathcal{H}}$

In this section, we study further properties of the abstract forward shift operator $S_{\mathcal{H}}$, which was introduced in Section 9.5. We recall that an analytic reproducing kernel Hilbert space $\mathcal{H}$ on the open unit disk $\mathbb{D}$ satisfies $(H1)$ if the function

$$\begin{array}{rccc} \chi_1: & \mathbb{D} & \longrightarrow & \mathbb{C} \\ & z & \longmapsto & z \end{array}$$

is a multiplier of $\mathcal{H}$.

Theorem 17.18 *Suppose that $\mathcal{H}$ is a reproducing kernel Hilbert space of analytic functions on $\mathbb{D}$ that is contained in H^2 and satisfies $(H1)$. Then the following are equivalent.*

(i) *$S_{\mathcal{H}}$ is similar to S.*
(ii) *There is a function $u \in H^\infty$ such that the set identity $\mathcal{H} = \mathcal{M}(u)$ holds.*

Proof (i) $\Longrightarrow$ (ii) Let $A : H^2 \longrightarrow \mathcal{H}$ be an isomorphism such that

$$S_{\mathcal{H}} A = AS. \tag{17.16}$$

This is summarized in the following commutative diagram.

$$\begin{tikzcd} H^2 \arrow[r, "S"] \arrow[d, "A"'] & H^2 \arrow[d, "A"] \\ \mathcal{H} \arrow[r, "S_{\mathcal{H}}"'] & \mathcal{H} \end{tikzcd} \tag{17.17}$$

Since $\mathcal{H}$ is embedded continuously into H^2, the operator

$$\begin{array}{rccc} \mathbf{A}: & H^2 & \longrightarrow & H^2 \\ & f & \longmapsto & Af \end{array}$$

is well defined and bounded and (17.16) implies that

$$S\mathbf{A} = \mathbf{A}S.$$

Therefore, by Theorem 12.27, there is a function $u \in H^\infty$ such that $\mathbf{A} = T_u$. Hence,

$$\mathcal{M}(u) = \mathcal{R}(T_u) = T_u H^2 = \mathbf{A}H^2 = AH^2 = \mathcal{H},$$

which gives the required set identity $\mathcal{H} = \mathcal{M}(u)$.

(ii) $\Longrightarrow$ (i) Since $\mathcal{H} = \mathcal{M}(u)$, as we explained in deriving (9.8), the closed graph theorem implies that the norms $\|\cdot\|_{\mathcal{H}}$ and $\|\cdot\|_{\mathcal{M}(u)}$ are equivalent.

This means that the operator

$$\begin{array}{rccc} i: & \mathcal{M}(u) & \longrightarrow & \mathcal{H} \\ & f & \longmapsto & f \end{array}$$

is a Banach space isomorphism. Now, consider the mapping

$$\begin{array}{rccc} \mathbf{T}_u: & H^2 & \longrightarrow & \mathcal{M}(u) \\ & f & \longmapsto & T_u f. \end{array}$$

In other words, $\mathbf{T}_u$ is the Toeplitz operator T_u, whose range is restricted to $\mathcal{M}(u)$, and the elements of its range are measured by the norm of $\mathcal{M}(u)$. By Theorem 12.19, T_u is an injective operator. Hence, in the first place, $\mathbf{T}_u$ is bijective. Second, by the definition of norm,

$$\|\mathbf{T}_u f\|_{\mathcal{M}(u)} = \|T_u f\|_{\mathcal{M}(u)} = \|f\|_2 \qquad (f \in H^2).$$

Thus, $\mathbf{T}_u$ is a unitary operator from H^2 onto $\mathcal{M}(u)$. Therefore, the operator $V_u : H^2 \longrightarrow \mathcal{H}$, defined by $V_u = i\mathbf{T}_u$, is a Banach space isomorphism from H^2 onto $\mathcal{H}$. In other words, we have the following diagram.

$$\begin{tikzcd} H^2 \arrow[rr, "V_u"] \arrow[dr, "\mathbf{T}_u"'] & & \mathcal{H} \\ & \mathcal{M}(u) \arrow[ur, "i"'] & \end{tikzcd} \tag{17.18}$$

Moreover, for each f in H^2, we have

$$V_u S f = i\mathbf{T}_u S f = T_u S f$$

and

$$S_{\mathcal{H}} V_u f = S_{\mathcal{H}} i \mathbf{T}_u f = S T_u f.$$

Since $T_u S = S T_u$, we deduce that

$$V_u S = S_{\mathcal{H}} V_u.$$

This identity reveals that $S_{\mathcal{H}}$ is similar to S. This is summarized in the following diagram.

$$\begin{tikzcd} H^2 \arrow[r, "S"] \arrow[d, "V_u"'] & H^2 \arrow[d, "V_u"] \\ \mathcal{H} \arrow[r, "S_{\mathcal{H}}"'] & \mathcal{H} \end{tikzcd} \tag{17.19}$$

This completes the proof. □

Note that, if $S_{\mathcal{H}}$ is similar to S, then clearly $S_{\mathcal{H}}$ is polynomially bounded and thus, by Theorem 9.18, $\mathfrak{Mult}(\mathcal{H}) = H^\infty$. But, in general, the converse is not true. We will see two particular cases where the converse holds, i.e.

(i) $\mathcal{H} = \mathcal{H}(b)$, with b a nonextreme point in the closed unit ball of H^∞;
(ii) $\mathcal{H} = \mathcal{M}(\bar{u})$, with u an outer function in H^∞.

If Θ is an inner function, then $\mathcal{M}(\Theta)$ is a closed subspace of H^2 and, since $\mathcal{H}$ is boundedly contained in H^2, then $\mathcal{M}(\Theta) \cap \mathcal{H}$ is closed in $\mathcal{H}$. Moreover, surely $\mathcal{M}(\Theta) \cap \mathcal{H}$ is invariant under $S_{\mathcal{H}}$. In the case where the operator $S_{\mathcal{H}}$ is similar to S, these subspaces describe all of the lattice of closed $S_{\mathcal{H}}$-invariant subspaces of $\mathcal{H}$.

Theorem 17.19 *Suppose that $\mathcal{H}$ is a reproducing kernel Hilbert space of analytic functions on $\mathbb{D}$. Assume that there is an outer function u in H^∞ such that $\mathcal{H} = \mathcal{M}(u)$ and let E be a closed subspace of $\mathcal{H}$. Then the following are equivalent.*

(i) *E is invariant under $S_{\mathcal{H}}$.*
(ii) *There is an inner function Θ such that $E = \mathcal{M}(\Theta) \cap \mathcal{H}$.*

Proof The implication (ii) $\Longrightarrow$ (i) is trivial and has already been noted.

Let us show the other implication. According to Theorem 17.18, the operator $S_{\mathcal{H}}$ is similar to S. Moreover, let $\mathbf{T}_u$, i and $V_u = i\mathbf{T}_u$ be as in the proof of Theorem 17.18. Hence, we know that V_u is an isomorphism such that $S_{\mathcal{H}} V_u = V_u S$. Now, let E be a closed $S_{\mathcal{H}}$-invariant subspace. It is easy to check that $V_u^{-1} E$ is a closed S-invariant subspace of H^2 and thus, by Beurling's theorem, there exists an inner function Θ such that $V_u^{-1} E = \mathcal{M}(\Theta)$. Hence, $E = V_u \mathcal{M}(\Theta)$ and the proof is complete if we can show that

$$V_u \mathcal{M}(\Theta) = \mathcal{M}(\Theta) \cap \mathcal{H}.$$

Since $\mathcal{H} = \mathcal{M}(u)$, the above identity is equivalent to

$$\mathbf{T}_u \mathcal{M}(\Theta) = \mathcal{M}(\Theta) \cap \mathcal{M}(u).$$

The inclusion $\mathbf{T}_u \mathcal{M}(\Theta) \subset \mathcal{M}(\Theta) \cap \mathcal{M}(u)$ is trivial because $T_u T_\Theta = T_\Theta T_u$. For the reverse inclusion, let $f \in \mathcal{M}(\Theta) \cap \mathcal{M}(u)$, i.e. $f = \Theta h = ug$, with some $h, g \in H^2$. Then we have

$$\frac{h}{u} = \bar{\Theta} g,$$

a.e. on $\mathbb{T}$, and then $h/u \in L^2$. Since u is an outer function, Corollary 4.28 implies that $h/u \in H^2$. Therefore, $g = \Theta h/u$ belongs to $\mathcal{M}(\Theta)$. This implies that $f = ug \in T_u \mathcal{M}(\Theta)$ and we get the reverse inclusion. $\square$

The space $\mathcal{M}(\bar{u})$, with u an outer function in H^∞, is invariant under the forward shift operator S. For simplicity of notation, we denote the restricted operator

$$\begin{array}{rccc} S_{\mathcal{M}(\bar{u})}: & \mathcal{M}(\bar{u}) & \longrightarrow & \mathcal{M}(\bar{u}) \\ & f & \longmapsto & Sf \end{array}$$

by $Z_{\bar{u}}$.

Theorem 17.20 *Let u be an outer function in H^∞. Then the following are equivalent.*

(i) *$Z_{\bar{u}}$ is similar to S.*
(ii) *$Z_{\bar{u}}$ is polynomially bounded.*
(iii) *$Z_{\bar{u}}$ is power bounded.*
(iv) *Every function in H^∞ is a multiplier of $\mathcal{M}(\bar{u})$.*
(v) *$\mathcal{M}(\bar{u}) = \mathcal{M}(u)$.*
(vi) *The operator $T_{u/\bar{u}}$ is invertible.*

Proof The implications (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii) are trivial. The equivalence between (v) and (vi) is proved in Theorem 17.17, and the equivalence between (ii) and (iv) is proved in Theorem 9.18. (Note that $\mathcal{M}(\bar{u})$ is a reproducing kernel Hilbert space that is contractively contained in H^2; moreover, by Theorems 17.16 and 17.15, the space $\mathcal{M}(\bar{u})$ is invariant under S and S^*.) Finally the implication (v) $\Longrightarrow$ (i) follows from Theorem 17.18.

We shall complete the proof of Theorem 17.20 by showing that (iii) $\Longrightarrow$ (vi). Saying that $Z_{\bar{u}}$ is power bounded means that there exists a constant $c > 0$ such that

$$\|Z_{\bar{u}}^n\|_{\mathcal{L}(\mathcal{M}(\bar{u}))} \le C,$$

or equivalently

$$\|S^n T_{\bar{u}} g\|_{\mathcal{M}(\bar{u})} \le C\, \|T_{\bar{u}} g\|_{\mathcal{M}(\bar{u})},$$

for every $n \ge 0$ and for every $g \in H^2$. For each fixed $g \in H^2$, since $S^n T_{\bar{u}} g \in \mathcal{M}(\bar{u})$, there exists $h_n \in H^2$ such that $S^n T_{\bar{u}} g = T_{\bar{u}} h_n$, and since, by Theorem 12.19 (recall that u is outer), the operator $T_{\bar{u}}$ is one-to-one, the function h_n is unique. Knowing this fact, define

$$\begin{array}{rccc} R_n: & H^2 & \longrightarrow & H^2 \\ & g & \longmapsto & h_n. \end{array}$$

It is clear that R_n is linear and we have

$$\begin{aligned} \|R_n g\|_2 = \|h_n\|_2 = \|T_{\bar{u}} h_n\|_{\mathcal{M}(\bar{u})} &= \|S^n T_{\bar{u}} g\|_{\mathcal{M}(\bar{u})} \\ &\le C\, \|T_{\bar{u}} g\|_{\mathcal{M}(\bar{u})} = C\, \|g\|_2. \end{aligned}$$

The identity $S^n T_{\bar{u}} g = T_{\bar{u}} h_n$ is now written as $S^n T_{\bar{u}} = T_{\bar{u}} R_n$, where the operators R_n fulfill

$$\sup_{n \geq 1} \|R_n\| \leq C.$$

By Douglas's criterion (Theorem 7.11), this is equivalent to

$$S^n T_{\bar{u}} T_u S^{*n} \leq C^2 T_{\bar{u}} T_u.$$

In other words, for every function f in H^2, we have

$$\|u S^{*n} f\|_2 = \|T_u S^{*n} f\|_2 \leq C \, \|T_u f\|_2 = C \, \|uf\|_2. \tag{17.20}$$

Let q be any trigonometric polynomial and choose n so that $z^n q$ is in H^2. The inequality (17.20) applied to the function $f = z^n q$ gives

$$\int_0^{2\pi} |u(e^{i\theta})|^2 \, |P_+(q)(e^{i\theta})|^2 \, d\theta \leq C^2 \int_0^{2\pi} |q(e^{i\theta})|^2 \, |u(e^{i\theta})|^2 \, d\theta.$$

Hence, P_+ is bounded in $L^2(|u|^2 \, dm)$ and thus, by Theorem 12.39 and Corollary 12.43, the operator $T_{u/\bar{u}}$ is invertible. □

17.9 Invariant subspaces of $Z_{\bar{u}}$ and $X_{\bar{u}}$

Recall that $\mathcal{M}(\bar{u})$ is invariant under S and S^*, and that we wrote $Z_{\bar{u}}$ and $X_{\bar{u}}$ respectively for the restriction of S and S^* to $\mathcal{M}(\bar{u})$.

Lemma 17.21 *Let u be an outer function in H^∞, and let E be a closed subspace of $\mathcal{M}(\bar{u})$ that is invariant under $X_{\bar{u}}$. Then, with respect to the norm of $\mathcal{M}(\bar{u})$, $T_{\bar{u}} E$ is dense in E.*

Proof By Theorem 17.15, for each $\varphi \in H^\infty$, we know that the space $\mathcal{M}(\bar{u})$ is invariant under $T_{\bar{\varphi}}$, and for the restricted operator

$$\begin{array}{rccc} \mathbf{T}_{\bar{\varphi}}: & \mathcal{M}(\bar{u}) & \longrightarrow & \mathcal{M}(\bar{u}) \\ & f & \longmapsto & T_{\bar{\varphi}} f \end{array}$$

we have

$$\|\mathbf{T}_{\bar{\varphi}}\|_{\mathcal{L}(\mathcal{M}(\bar{u}))} \leq \|\varphi\|_\infty.$$

Corollary 12.50 also ensures that $T_{\bar{u}} E \subset E$. Now, let $f \in E$, $f \perp T_{\bar{u}} E$ and let g be the (unique) function in H^2 such that $f = T_{\bar{u}} g$. Since, by (12.3),

$$S^{*n} T_{\bar{u}} f = T_{\bar{u}} S^{*n} f = T_{\bar{u}} X_{\bar{u}}^n f \in T_{\bar{u}} E \qquad (n \geq 0),$$

the function f is orthogonal to the sequence $S^{*n}T_{\bar{u}}f$, $n \geq 0$. This means that

$$\begin{aligned} 0 &= \langle f, S^{*n}T_{\bar{u}}f\rangle_{\bar{u}} \\ &= \langle T_{\bar{u}}g, T_{\bar{u}}S^{*n}f\rangle_{\bar{u}} \\ &= \langle g, S^{*n}f\rangle_2 \\ &= \langle S^n g, f\rangle_2 \\ &= \langle S^n g, T_{\bar{u}}g\rangle_2 \\ &= \langle S^n g, \bar{u}g\rangle_2 \\ &= \frac{1}{2\pi}\int_0^{2\pi} u(e^{i\theta})|g(e^{i\theta})|^2 e^{in\theta}\, d\theta. \end{aligned}$$

Since the last equation is valid for every nonnegative integer n, the function $u|g|^2$ belongs to the space H_0^1. But, since u is outer and $|g|^2 \in L^1(\mathbb{T})$, Corollary 4.28 implies that $|g|^2$ also belongs to H_0^1. By (4.12), we conclude that $|g|^2 \equiv 0$, and thus $g \equiv 0$ or $f \equiv 0$. Therefore, $T_{\bar{u}}E$ is dense in E. □

Theorem 17.22 *Let u be an outer function in H^∞ and let E be a closed subspace of $\mathcal{M}(\bar{u})$. Then the following are equivalent.*

(i) *E is invariant under $X_{\bar{u}}$.*
(ii) *There is an inner function Θ such that $E = K_\Theta \cap \mathcal{M}(\bar{u})$.*

Proof (i) $\Longrightarrow$ (ii) Let E be an invariant subspace of $X_{\bar{u}}$ and let $\mathcal{E}$ be the closure of E in H^2. Then $\mathcal{E}$ is an invariant subspace of S^*. By Beurling's theorem (Theorem 8.32), there exists an inner function Θ such that $\mathcal{E} = K_\Theta$. To conclude the proof, we should verify that $E = \mathcal{E} \cap \mathcal{M}(\bar{u})$.

In fact, the first inclusion $E \subset \mathcal{E} \cap \mathcal{M}(\bar{u})$ is trivial. For the second inclusion, since $\mathcal{E} \cap \mathcal{M}(\bar{u})$ is a closed invariant subspace of $X_{\bar{u}}$, then, by Lemma 17.21, $T_{\bar{u}}(\mathcal{E} \cap \mathcal{M}(\bar{u}))$ is dense in $\mathcal{E} \cap \mathcal{M}(\bar{u})$. We claim that $T_{\bar{u}}\mathcal{E} \subset E$. Indeed, let $f \in \mathcal{E}$. Then there exists a sequence $(f_n)_{n\geq 1}$ in E converging to f in the H^2 norm. Hence, $T_{\bar{u}}f_n \longrightarrow T_{\bar{u}}f$, as $n \longrightarrow \infty$, in the norm of $\mathcal{M}(\bar{u})$. But it follows from Corollary 12.50 that $T_{\bar{u}}f_n \in E$, whence we conclude that the function $T_{\bar{u}}f$ also belongs to E. Therefore, we have

$$T_{\bar{u}}(\mathcal{E} \cap \mathcal{M}(\bar{u})) \subset E.$$

Since $T_{\bar{u}}(\mathcal{E} \cap \mathcal{M}(\bar{u}))$ is dense in $\mathcal{E} \cap \mathcal{M}(\bar{u})$, we finally get

$$\mathcal{E} \cap \mathcal{M}(\bar{u}) \subset E.$$

(ii) $\Longrightarrow$ (i) This follows immediately from Theorem 17.15. □

The next theorem is an analogous result for the operator $Z_{\bar{u}}$ in the case where this operator is similar to S.

Theorem 17.23 *Let u be an outer function in H^∞ and assume that $T_{u/\bar{u}}$ is invertible. Let E be a closed subspace of $\mathcal{M}(\bar{u})$. Then the following are equivalent.*

(i) *E is invariant under $Z_{\bar{u}}$.*
(ii) *There is an inner function Θ such that $E = \mathcal{M}(\Theta) \cap \mathcal{M}(\bar{u})$.*

Proof Since $T_{u/\bar{u}}$ is invertible, then we know from Theorem 17.20 that $\mathcal{M}(\bar{u}) = \mathcal{M}(u)$ and now the result is a particular case of Theorem 17.19. □

17.10 An extension of Beurling's theorem

In this section, we give an extension of Beurling's theorem, and this extension characterizes the $\mathcal{M}(b)$ spaces. First we recall the well-known Wold–Kolmogorov decomposition theorem, which says that any isometry on a Hilbert space can be written as a direct sum of a unitary operator and copies of the unilateral shift (see Section 7.4). It is obvious that $\mathcal{M}(b)$ is S-invariant and that S acts as an isometry on $\mathcal{M}(b)$. Indeed, if $f \in \mathcal{M}(b)$, then $f = bg$ for some $g \in H^2$ and thus $Sf = zf = bzg = T_b(zg)$ is in $\mathcal{M}(b)$ and we have

$$\|Sf\|_{\mathcal{M}(b)} = \|T_b(zg)\|_{\mathcal{M}(b)} = \|zg\|_2 = \|g\|_2 = \|T_b(g)\|_{\mathcal{M}(b)} = \|f\|_{\mathcal{M}(b)}.$$

Note that we implicitly used the fact $\ker T_b = \{0\}$. That the converse also holds is a deeper result.

Theorem 17.24 *Let $\mathcal{M}$ be a Hilbert space contained contractively in H^2 such that $S\mathcal{M} \subset \mathcal{M}$ and that S acts as an isometry on $\mathcal{M}$. Then there is a function b, unique up to a unimodular constant, in the closed unit ball of H^∞ such that $\mathcal{M} \doteq \mathcal{M}(b)$.*

Proof Let $\mathbf{S}$ denote the restriction of S to $\mathcal{M}$, i.e.

$$\begin{array}{rccc} \mathbf{S}: & \mathcal{M} & \longrightarrow & \mathcal{M} \\ & f & \longmapsto & Sf. \end{array}$$

By hypothesis, $\mathbf{S}$ is an isometry and we have

$$\bigcap_{n=0}^{\infty} \mathbf{S}^n \mathcal{M} \subset \bigcap_{n=0}^{\infty} S^n H^2 = \{0\}. \tag{17.21}$$

The last equality stems from the fact that, if a function f is in $S^n H^2$, for all $n \geq 0$, then $f^{(n)}(0) = 0$, $n \geq 0$, which obviously implies that $f \equiv 0$. Put

$$\mathcal{N} = \mathcal{M} \ominus \mathbf{S}\mathcal{M}.$$

Note that $\mathcal{N} \neq \{0\}$, since otherwise we would have $\mathbf{S}\mathcal{M} = \mathcal{M}$, and then $\mathbf{S}^n\mathcal{M} = \mathcal{M}$, for all $n \geq 0$, which would contradict (17.21). Applying Theorem 7.21, with $A = \mathbf{S}$ and $\mathcal{H} = E = \mathcal{M}$, we obtain

$$\mathcal{M} = \bigoplus_{n \geq 0} \mathbf{S}^n \mathcal{N}. \tag{17.22}$$

Let b be any unit vector in $\mathcal{N}$. Then $(\mathbf{S}^n b)_{n\geq 0}$ is an orthonormal sequence in $\mathcal{M}$. We claim that the map $T : f \longmapsto bf$ is an isometry of H^2 into $\mathcal{M}$. Indeed, if $p(z) = \sum_{k=0}^{N} a_k z^k$ is an analytic polynomial, then, since

$$pb = \sum_{k=0}^{N} a_k S^k b = \sum_{k=0}^{N} a_k \mathbf{S}^k b,$$

the function pb is in $\mathcal{M}$ and using Parseval's equality we have

$$\|pb\|_{\mathcal{M}}^2 = \left\| \sum_{k=0}^{N} a_k \mathbf{S}^k b \right\|_{\mathcal{M}}^2 = \sum_{k=0}^{N} |a_k|^2 = \|p\|_2^2. \tag{17.23}$$

For an arbitrary $f \in H^2$, there exists a sequence $(p_n)_{n\geq 1}$ in $\mathcal{P}_+$ such that $\|p_n - f\|_2 \longrightarrow 0$ as $n \longrightarrow \infty$. In particular, $(p_n)_{n\geq 1}$ is a Cauchy sequence in H^2 and, using (17.23), we see that $(p_n b)_{n\geq 0}$ is also a Cauchy sequence in $\mathcal{M}$, whence it converges, say, to a function $g \in \mathcal{M}$. On the one hand, since $\mathcal{M}$ is contractively contained in H^2, $\|p_n b - g\|_2 \longrightarrow 0$ as $n \longrightarrow \infty$, and, in particular, at any point $\lambda \in \mathbb{D}$, we have $p_n(\lambda)b(\lambda) \longrightarrow g(\lambda)$. On the other hand, we also have $p_n(\lambda) \longrightarrow f(\lambda)$. Thus, $bf = g$, which proves that $bf \in \mathcal{M}$ and also

$$\|bf\|_{\mathcal{M}} = \|g\|_{\mathcal{M}} = \lim_{n\to\infty} \|p_n b\|_{\mathcal{M}} \lim_{n\to\infty} \|p_n\|_2 = \|f\|_2.$$

Therefore, T is an isometry of H^2 into $\mathcal{M}$. Using one more time that $\mathcal{M}$ is contractively contained in H^2, we deduce that the map $f \longmapsto bf$ is a contraction of H^2 into itself. This means that b is a multiplier of H^2 and $\|M_b\|_{\mathcal{L}(H^2)} \leq 1$. It follows from Corollary 9.7 and Lemma 9.6 that b is in H^∞ and

$$\|b\|_\infty \leq \|M_b\|_{\mathcal{L}(H^2)} \leq 1.$$

This fact implies that $\mathcal{M}(b)$ is contained isometrically in $\mathcal{M}$. Indeed, by the definition of the norm in $\mathcal{M}(b)$, we have

$$\|bf\|_{\mathcal{M}(b)} = \|f\|_2 = \|bf\|_{\mathcal{M}} \qquad (f \in H^2).$$

It remains to show that $\mathcal{M} = \mathcal{M}(b)$. First, we have

$$\mathcal{M}(b) = \bigoplus_{n\geq 0} \mathbf{S}^n(\mathbb{C}b). \tag{17.24}$$

To verify (17.24), take $f = bg \in \mathcal{M}(b)$ with $f \perp \mathbf{S}^n b$. Remember that $\mathbf{S}^n b \perp \mathbf{S}^k b$, $k \neq n$. Then we have $f \perp pb$, for any analytic polynomial p. But we saw that, in $\mathcal{M}$, $f = bg = \lim_{n\to\infty} bp_n$, where $(p_n)_{n\geq 1}$ is any sequence of analytic polynomials that converges to g in H^2. Thus,

$$\|f\|_{\mathcal{M}}^2 = \langle f, f\rangle_{\mathcal{M}} = \lim_{n\to+\infty} \langle f, bp_n\rangle_{\mathcal{M}} = 0,$$

which proves (17.24).

Second, we show that $\mathcal{N} = \mathbb{C}b$. Arguing by absurdity, assume that we have $\mathbb{C}b \subsetneqq \mathcal{N}$. Then pick any unit vector b_1 in $\mathcal{N} \ominus \mathbb{C}b$. We can apply the same argument as before, replacing b by b_1, and deduce that $\mathcal{M}(b_1)$ is also contained isometrically in $\mathcal{M}$. Moreover, since $b_1 \perp b$ and $b, b_1 \perp \mathbf{S}\mathcal{M}$, it is easily seen that

$$\langle \mathbf{S}^k b_1, \mathbf{S}^n b\rangle_{\mathcal{M}} = 0 \qquad (k, n \geq 0).$$

Using (17.24) (for b and b_1), we get $\mathcal{M}(b) \perp \mathcal{M}(b_1)$. In other words,

$$\langle bf, b_1 g\rangle_{\mathcal{M}} = 0, \tag{17.25}$$

for any f, g in H^2. Therefore, the mapping

$$\begin{array}{cccc} \mathcal{G}: & H^2 \oplus H^2 & \longrightarrow & \mathcal{M} \\ & f \oplus g & \longmapsto & bf + b_1 g \end{array}$$

is an isometry. Indeed, using (17.25) and the fact that $\mathcal{M}(b)$ and $\mathcal{M}(b_1)$ are contained isometrically in $\mathcal{M}$, we have

$$\begin{aligned} \|bf + b_1 g\|_{\mathcal{M}}^2 &= \|bf\|_{\mathcal{M}}^2 + \|b_1 g\|_{\mathcal{M}}^2 \\ &= \|bf\|_{\mathcal{M}(b)}^2 + \|b_1 g\|_{\mathcal{M}(b_1)}^2 \\ &= \|f\|_2^2 + \|g\|_2^2 \\ &= \|f \oplus g\|_{H^2 \oplus H^2}^2. \end{aligned}$$

But the mapping $\mathcal{G}$ can never be an isometry because $\mathcal{G}(-b_1 \oplus b) = -bb_1 + b_1 b = 0$. Thus, we have $\mathcal{N} = \mathbb{C}b$ and, by (17.22),

$$\mathcal{M} = \bigoplus_{n\geq 0} \mathbf{S}^n(\mathbb{C}b).$$

It remains to use (17.24) to conclude that $\mathcal{M} = \mathcal{M}(b)$.

The uniqueness of b (up to a unimodular constant) follows from Theorem 17.1(iii). $\square$

It is simple to construct an S-invariant Hilbert space $\mathcal{M}$ contained contractively in H^2 and such that S acts as a contraction but not as an isometry on $\mathcal{M}$. For example, take $(\varepsilon_n)_{n\geq 0}$ to be any nonconstant sequence of real numbers, such that $\varepsilon_n \geq 1$, $n \geq 0$, and such that $(\varepsilon_n)_{n\geq 0}$ is strictly decreasing. For an analytic polynomial $p(z) = \sum_{n=0}^{N} a_n z^n$, set

$$\|\!|p|\!\| = \sum_{n=0}^{N} |a_n|^2 \varepsilon_n^2.$$

Then we define $\mathcal{M}$ to be the completion of the set of analytic polynomials with respect to this norm. Note that $\|p\|_2 \leq \|\!|p|\!\|$ (because $\varepsilon_n \geq 1$), whence $\mathcal{M}$ is contractively contained in H^2. Surely, $\mathcal{M}$ is S-invariant and, if $p \neq 0$, we have

$$\|\!|Sp|\!\|^2 = \sum_{n=1}^{N+1} |a_{n-1}|^2 \varepsilon_n^2 = \sum_{n=0}^{N} |a_n|^2 \varepsilon_{n+1}^2 < \sum_{n=0}^{N} |a_n|^2 \varepsilon_n^2 = \|\!|p|\!\|^2,$$

because $\varepsilon_{n+1} < \varepsilon_n$.

Notes on Chapter 17

In this chapter, we introduce the main object of this book, the $\mathcal{H}(b)$ spaces. These spaces have a rich and fascinating structure, whose exploration is the aim of the present book. The spaces $\mathcal{H}(b)$ were introduced and studied by de Branges and Rovnyak in their book [65]. Subsequent work has been done by Sarason [159–161, 163, 164, 166, 167], Suárez [180–182], Hartmann, Sarason and Seip [97], Jury [112], Lotto [120], Lotto and Sarason [123–125], Baranov, Fricain and Mashreghi [29], Bolotnikov and Kheifets [36], Fricain [77], Fricain and Mashreghi [80, 81], Chevrot, Fricain and Timotin [53], Guyker [96], Costara and Ransford [58], Chevrot, Guillot and Ransford [54] and Fricain and Hartmann [78]. The vector-valued versions of $\mathcal{H}(b)$ were investigated by de Branges and Rovnyak [65]. This vector-valued context is very important in the theory of model operators (see [25, 26, 139]).

Section 17.1

Theorem 17.1 is due to Rovnyak [155, theorem 9]. In [139, theorem 7.9.1], Nikolskii and Vasyunin give a vector-valued analog of Lemma 17.2.

Section 17.2

A version of Theorem 17.4 can be found in [159]. Theorem 17.5 seems to be new.

Section 17.4

The characterization of the density of the overlapping space given by Theorem 17.7 is due to Lotto and Sarason [123, corollary 2.5]. Theorem 17.8 is also due to Lotto and Sarason [123, lemma 2.2]. Theorem 17.9 is taken from [166, chap. II].

Section 17.6

The idea of Theorem 17.15 is taken from [166, sec. II.7] even if it is not explicitly stated there. Theorem 17.16 is new.

Section 17.8

Theorem 17.20 is due to Sarason [159, theorem 4]. A similar result for $\mathcal{H}(b)$ spaces will be discussed in Section 28.4. The idea of Theorems 17.18 and 17.19 comes from [159]

Section 17.9

Lemma 17.21 is from [159, lemma 7]. The idea of Theorem 17.22 find its roots in [159] but this result as well as Theorem 17.23 are new.

Section 17.10

Theorem 17.24 is due to de Branges and Rovnyak [65, theorem 3]. See also [155]. It is a beautiful extension of Beurling's classic theorem. It is originally stated and proved in the context of square-summable power series. This scalar version is taken from [160], where only a sketch of the proof is given. It should be noted that the method used here to prove the result is based on a general method of Halmos for the analysis of Hilbert space isometries.

18

The structure of $\mathcal{H}(b)$ and $\mathcal{H}(\bar{b})$

In this chapter, we study the general properties of $\mathcal{H}(b)$ spaces and their cousins, $\mathcal{H}(\bar{b})$ spaces. In Sections 18.1 and 18.2, we characterize when these spaces are closed or dense subspaces of H^2. We also study when these spaces reduce to H^2 or to the trivial space $\{0\}$. In Section 18.3, we give some interesting decomposition of $\mathcal{H}(b)$ spaces. In Sections 18.4 and 18.6, we introduce the important family of reproducing kernels of $\mathcal{H}(b)$ and the associated family of difference quotients. In Section 18.5, we prove the crucial property of invariance of $\mathcal{H}(b)$ under Toeplitz operators with co-analytic symbols. In particular, this leads to the contraction X_b, which is the restriction of S^* to $\mathcal{H}(b)$. This operator X_b is certainly the most important operator of the theory and it will accompany us throughout this book. In Section 18.7, the elementary properties of X_b are given. In particular, we compute the defect operator of X_b. We establish the so-called inequality of difference quotient in Section 18.8. A nice characterization of membership to $\mathcal{H}(b)$ is given in Section 18.9. We also derive a formula to compute the norm of elements of $\mathcal{H}(b)$ spaces.

18.1 When is $\mathcal{H}(b)$ a closed subspace of H^2?

If b is an inner function, as the following result shows, $\mathcal{M}(b)$ and $\mathcal{H}(b)$ have rich structures.

Theorem 18.1 *The complementary space $\mathcal{H}(b)$ is a closed subspace of H^2 and inherits its Hilbert space structure if and only if b is inner or $b \equiv 0$. Moreover, in this case,*

$$\mathcal{M}(b) \doteq bH^2 \quad \textit{and} \quad \mathcal{H}(b) \doteq H^2 \ominus bH^2 = (bH^2)^{\perp}.$$

Proof By Lemma 16.14, $\mathcal{H}(b)$ is a closed subspace of H^2 and inherits its Hilbert space structure if and only if T_b is a partial isometry. But, by

Theorem 12.18, this happens if and only if b is inner or $b \equiv 0$. The latter case leads to $\mathcal{H}(b) = H^2$, and the result is a triviality.

To treat the former case, suppose that b is inner. The relation $\mathcal{M}(b) = bH^2$ is in fact valid for any b in the closed unit ball of H^∞. But, for an inner b, T_b is an isometry and thus, by Theorem 16.21(iv), $\mathcal{M}(b)$ and $\mathcal{H}(b)$ are orthogonal complements of each other. □

By tradition, we write Θ instead of b when b is an inner function. As we saw, since Θ is unimodular, T_Θ is an isometry, and by (17.2) we have

$$\|\Theta f\|_{\mathcal{M}(\Theta)} = \|f\|_{H^2} = \|\Theta f\|_{H^2} \qquad (f \in H^2).$$

We can equivalently write

$$\|g\|_{\mathcal{M}(\Theta)} = \|g\|_{H^2} \qquad (g \in \mathcal{M}(\Theta)). \tag{18.1}$$

In other words, ΘH^2 is a closed subspace of H^2 and

$$\mathcal{M}(\Theta) \doteq \Theta H^2.$$

As we discussed in Section 8.8, they are also called Beurling subspaces of H^2 and they are the only closed subspaces of H^2 that are invariant under the forward shift operator S (Theorem 8.32). Moreover, according to Theorem 18.1, we know that $\mathcal{H}(\Theta)$ is a closed subspace of H^2 that corresponds to the orthogonal complement of ΘH^2 in H^2. We have already denoted these spaces by K_Θ and called them model subspaces of H^2. In other words, we have

$$\mathcal{H}(\Theta) \doteq K_\Theta,$$

which means that $\mathcal{H}(\Theta) = (\Theta H^2)^\perp$ and, for each $f \in \mathcal{H}(\Theta)$, we have

$$\|f\|_{\mathcal{H}(\Theta)} = \|f\|_{H^2} \qquad (f \in \mathcal{H}(\Theta)). \tag{18.2}$$

Hence, the spaces $\mathcal{H}(\Theta)$, with Θ inner, are the only closed subspaces of H^2 that are invariant under the backward shift operator S^*. It is worthwhile to emphasize that we only considered closed subspaces of H^2 in the preceding discussion. Theorem 18.13 below shows that H^2 has plenty of other linear manifolds that are invariant under S^*.

We will examine in the next result some trivial situations where $\mathcal{H}(b)$ or $\mathcal{H}(\bar{b})$ reduce to trivial subspaces of H^2.

Theorem 18.2 *Let b be a function in the closed unit ball of H^∞. Then the following hold:*

(i) $\mathcal{H}(b) = \{0\}$ *if and only if b is a unimodular constant;*
(ii) $\mathcal{H}(\bar{b}) = \{0\}$ *if and only if b is inner;*
(iii) $\mathcal{H}(b) \doteq H^2$ *if and only if $b \equiv 0$;*
(iv) $\mathcal{H}(\bar{b}) \doteq H^2$ *if and only if $b \equiv 0$.*

Proof (i) If $b \equiv \lambda$, $\lambda \in \mathbb{T}$, then we have

$$I - T_b T_{\bar{b}} = (1 - |\lambda|^2)I = 0,$$

which gives $\mathcal{H}(b) = \{0\}$. Reciprocally, assume that $\mathcal{H}(b) = \{0\}$. Then, in particular, we must have

$$(I - T_b T_{\bar{b}})1 = 0.$$

But, we easily see that

$$((I - T_b T_{\bar{b}})1)(z) = 1 - \overline{b(0)}b(z) \qquad (z \in \mathbb{D}).$$

Therefore, $\overline{b(0)}b(z) = 1$, for every $z \in \mathbb{D}$, which implies that b is constant and, moreover, $|b(0)| = 1$. In other words, b is a unimodular constant.

(ii) If b is inner, then, by (17.10), $\mathcal{H}(\bar{b}) = \{0\}$. If b is not inner, in the light of the trivial set inclusion

$$\mathcal{M}(T_{1-|b|^2}) \subset \mathcal{M}(T^{1/2}_{1-|b|^2}),$$

Corollary 12.29 ensures that $\mathcal{H}(\bar{b})$ is infinite-dimensional.

(iii) This follows from Theorem 18.1.

(iv) Assume that $\mathcal{H}(\bar{b}) = H^2$. Then it follows from Lemma 16.14 that $T_{\bar{b}}$ is a partial isometry and, by Theorem 7.22, that T_b is also a partial isometry. Hence, Theorem 12.18 implies that $b = 0$ or b is inner. The latter is absurd, since it implies that $\mathcal{H}(\bar{b}) = \{0\}$. The converse is trivial, because if $b \equiv 0$, then $I - T_{\bar{b}} T_b = I$. □

18.2 When is $\mathcal{H}(b)$ a dense subset of H^2?

In Theorem 18.1, we saw that, if b is inner, then we do not obtain a new Hilbert space structure and $\mathcal{H}(b)$ is a proper closed subspace of H^2. The following result shows that the other alternative is dramatically different.

Theorem 18.3 *Relative to the norm $\|\cdot\|_2$, the set $\mathcal{H}(b)$ is dense in H^2 if and only if b is not an inner function.*

Proof Since the operator $I - T_b T_{\bar{b}}$ is self-adjoint, the set $\mathcal{H}(b)$ is dense in H^2 if and only if $\ker(I - T_b T_{\bar{b}})^{1/2} = \ker(I - T_b T_{\bar{b}}) = \{0\}$. Hence, we assume that b is not inner and deduce that $\ker(I - T_b T_{\bar{b}}) = \{0\}$.

Let $f \in \ker(I - T_b T_{\bar{b}})$. Then, by Corollary 17.6,

$$\|f\|^2_{H^2} - \|T_{\bar{b}} f\|^2_{H^2} = \|(I - T_b T_{\bar{b}})f\|^2_{\mathcal{H}(b)} = 0.$$

Hence,

$$\int_{\mathbb{T}} |f|^2 \, dm = \int_{\mathbb{T}} |T_{\bar{b}} f|^2 \, dm \le \int_{\mathbb{T}} |bf|^2 \, dm.$$

Thus, we must have $(1-|b|^2)|f|^2 = 0$, a.e. on $\mathbb{T}$. Since b is assumed not to be inner, then the Borel set $E = \{\zeta \in \mathbb{T} : |b(\zeta)| \neq 1\}$ is of positive Lebesgue measure, and, on that set, $f \equiv 0$. Therefore, by the uniqueness theorem for H^2 functions (Lemma 4.30), $f \equiv 0$. On the contrary, if b is an inner function, then, by Theorem 18.1, $\mathcal{H}(b) = H^2 \ominus bH^2$ is a closed subspace of H^2 and thus $\mathrm{Clos}_{H^2}\mathcal{H}(b) = \mathcal{H}(b) \subsetneqq H^2$. $\square$

We know that $\mathcal{H}(\bar{b})$ is contractively contained in $\mathcal{H}(b)$ (Theorem 17.9), and that, if b is inner, $\mathcal{H}(\bar{b})$ reduces to the trivial subspace $\{0\}$. However, $\mathcal{H}(b) = K_b$ is still a proper closed subspace of H^2 with many interesting properties and a very rich structure. On the contrary, if b is not inner, Theorem 18.3 says that $\mathcal{H}(b)$ is a *large set* in H^2. In the following, we show that, in this situation, $\mathcal{H}(\bar{b})$ is also a large set.

Theorem 18.4 *Relative to the norm $\|\cdot\|_2$, the set $\mathcal{H}(\bar{b})$ is dense in H^2 if and only if b is not an inner function.*

Proof Assume that b is not inner. The space $\mathcal{H}(\bar{b})$ is dense in H^2 if and only if $\ker(T^{1/2}_{1-|b|^2}) = \ker(T_{1-|b|^2}) = \{0\}$. To verify the latter fact, let $f \in \ker(T_{1-|b|^2})$. Then, by Corollary 17.10,

$$\int_{\mathbb{T}} (1-|b|^2)\,|f|^2\,dm = \|(I - T_{\bar{b}}T_b)f\|^2_{\mathcal{H}(\bar{b})} = 0.$$

Since $1-|b|^2 \geq 0$, a.e. on $\mathbb{T}$, and $1-|b|^2 > 0$ on a set of positive Lebesgue measure, with a similar argument that was used in the proof of Theorem 18.3, we deduce that $f \equiv 0$. On the contrary, if b is an inner function, then $\mathcal{H}(\bar{b}) = \{0\}$, and thus $\mathcal{H}(\bar{b})$ is far away from being dense in H^2. $\square$

To compare the above two results with Theorems 18.1 and 18.2, we combine them in the following corollary.

Corollary 18.5 *Let b be a function in the closed unit ball of H^∞. Then the following are equivalent:*

(i) *b is not an inner function;*
(ii) *the set $\mathcal{H}(b)$ is dense in H^2;*
(iii) *the set $\mathcal{H}(\bar{b})$ is dense in H^2.*

The following result examines another trivial situation that completes Theorem 18.2.

Corollary 18.6 *Let b be in the closed unit ball of H^∞. Then the following assertions are equivalent:*

(i) $\mathcal{H}(b) = H^2$;
(ii) $\mathcal{H}(\bar{b}) = H^2$;
(iii) $\|b\|_\infty < 1$.

Proof (i) $\Longrightarrow$ (ii) Let $f \in H^2$. Then $T_{\bar{b}} f \in H^2 = \mathcal{H}(b)$. By Theorem 17.8, we deduce that $f \in \mathcal{H}(\bar{b})$. Hence $\mathcal{H}(\bar{b}) = H^2$.

(ii) $\Longrightarrow$ (iii) The operator $(I - T_{\bar{b}} T_b)^{1/2} : H^2 \longrightarrow H^2$ is onto and, since it is self-adjoint, it is also automatically one-to-one. Hence it is an isomorphism and there exists $c > 0$ such that

$$c\|f\|^2 \le \|(I - T_{\bar{b}} T_b)^{1/2} f\|_2^2 \qquad (f \in H^2).$$

Since $\|(I - T_{\bar{b}} T_b)^{1/2} f\|_2^2 = \|f\|_2^2 - \|bf\|_2^2$, we get that

$$\|bf\|_2^2 \le (1 - c)\|f\|_2^2 \qquad (f \in H^2).$$

Theorem 12.2 now implies that $\|b\|_\infty \le \sqrt{1 - c} < 1$.

(iii) $\Longrightarrow$ (i) Using Theorem 12.2 once more, we see that, if $\|b\|_\infty < 1$, then $\|T_b T_{\bar{b}}\| < 1$ and the operator $I - T_b T_{\bar{b}}$ is invertible. Thus, $(I - T_b T_{\bar{b}})^{1/2}$ is also invertible and $\mathcal{H}(b) = H^2$. □

18.3 Decomposition of $\mathcal{H}(b)$ spaces

The following result is a direct consequence of Theorem 16.23.

Theorem 18.7 *Let b_1 and b_2 be two functions in the closed unit ball of H^∞, and let $b = b_1 b_2$. Then the following hold.*

(i) *The space $\mathcal{H}(b)$ decomposes as*

$$\mathcal{H}(b) = \mathcal{H}(b_2) + b_2 \mathcal{H}(b_1). \tag{18.3}$$

(ii) *For any representation $f = f_2 + b_2 f_1$, where $f_i \in \mathcal{H}(b_i)$, $i = 1, 2$, and $f \in \mathcal{H}(b)$, we have*

$$\|f\|_b^2 \le \|f_1\|_{b_1}^2 + \|f_2\|_{b_2}^2.$$

(iii) *For each $f \in \mathcal{H}(b)$, there is a unique pair of functions $f_1 \in \mathcal{H}(b_1)$ and $f_2 \in \mathcal{H}(b_2)$ such that $f = f_2 + b_2 f_1$ and*

$$\|f\|_b^2 = \|f_1\|_{b_1}^2 + \|f_2\|_{b_2}^2.$$

(iv) *The operator T_{b_2} acts as a contraction from $\mathcal{H}(b_1)$ into $\mathcal{H}(b)$.*

Proof It suffices to apply Theorem 16.23 to $A_1 = T_{b_1}$, $A_2 = T_{b_2}$ and $A = T_b = A_1 A_2$. □

It is natural to ask if we could have an orthogonal decomposition in (18.3). The following result gives a sufficient condition.

Theorem 18.8 *Let b_1 and b_2 be two functions in the closed unit ball of H^∞, and let $b = b_1 b_2$. If $\mathcal{H}(b_1) \cap \mathcal{H}(\bar{b}_2) = \{0\}$, then the following hold.*

(i) *The space $\mathcal{H}(b)$ decomposes as*

$$\mathcal{H}(b) = \mathcal{H}(b_2) \oplus b_2\mathcal{H}(b_1), \tag{18.4}$$

and the sum is orthogonal.

(ii) *The space $\mathcal{H}(b_2)$ is contained isometrically in $\mathcal{H}(b)$, that is*

$$\|f\|_{b_2} = \|f\|_b \qquad (f \in \mathcal{H}(b_2)).$$

(iii) *The operator T_{b_2} acts as an isometry from $\mathcal{H}(b_1)$ into $\mathcal{H}(b)$, that is*

$$\begin{array}{rccc} \mathbf{T}_{b_2}: & \mathcal{H}(b_1) & \longrightarrow & \mathcal{H}(b) \\ & f & \longmapsto & b_2 f \end{array}$$

and

$$\|b_2 f\|_b = \|f\|_{b_1} \qquad (f \in \mathcal{H}(b_1)).$$

Proof According to Corollary 16.24, we know that the decomposition

$$\mathcal{H}(b) = \mathcal{H}(b_2) + b_2\mathcal{H}(b_1)$$

given by Theorem 18.7 is an orthogonal decomposition if and only if

$$\mathcal{H}(b_2) \cap b_2\mathcal{H}(b_1) = \{0\}. \tag{18.5}$$

To prove (18.5), let us take $f \in \mathcal{H}(b_2) \cap b_2\mathcal{H}(b_1)$. On the one hand, we can write $f = b_2 g$, with $g \in \mathcal{H}(b_1)$. On the other hand, since $\mathcal{H}(b_2) \cap b_2\mathcal{H}(b_1) \subset \mathcal{H}(b_2) \cap \mathcal{M}(b_2)$, by (17.11), there exists $h \in \mathcal{H}(\bar{b}_2)$ such that $f = b_2 h$. Thus $g = h$ and $g \in \mathcal{H}(b_1) \cap \mathcal{H}(\bar{b}_2)$. By hypothesis, $g \equiv 0$ and thus $f \equiv 0$, which proves (18.5). The rest follows from Corollary 16.24. □

An immediate corollary is the following.

Corollary 18.9 *Let $b = b_1\Theta$, where b_1 is in the closed unit ball of H^∞ and Θ is an inner function. Then the complementary space $\mathcal{H}(b)$ is the orthogonal direct sum of K_Θ and $\Theta\mathcal{H}(b_1)$, i.e.*

$$\mathcal{H}(b) = K_\Theta \oplus \Theta\mathcal{H}(b_1).$$

The model space K_Θ is contained isometrically in $\mathcal{H}(b)$, i.e.

$$\|f\|_{K_\Theta} = \|f\|_{\mathcal{H}(b)} \qquad (f \in K_\Theta).$$

The operator T_Θ act as an isometry from $\mathcal{H}(b_1)$ into $\mathcal{H}(b)$, i.e.

$$\begin{array}{rccc} T_\Theta: & \mathcal{H}(b_1) & \longrightarrow & \mathcal{H}(b) \\ & f & \longmapsto & \Theta f \end{array}$$

and

$$\|\Theta f\|_b = \|f\|_{b_1} \qquad (f \in \mathcal{H}(b_1)).$$

Proof Since Θ is inner, then $\mathcal{H}(\bar{\Theta}) = \{0\}$ and thus it suffices to apply Theorem 18.8. □

Corollary 18.9 implies that $\mathcal{H}(b_1\Theta)\cap\Theta H^2 = \Theta\mathcal{H}(b_1)$. This fact is exploited in the following result.

Corollary 18.10 *Let Θ be an inner function, let b_1 be a nonconstant function in the closed unit ball of H^∞, and let $P_\Theta : L^2(\mathbb{T}) \longrightarrow K_\Theta$ denote the orthogonal projection of $L^2(\mathbb{T})$ onto K_Θ. Put*

$$\begin{array}{rccc} P: & \mathcal{H}(\Theta b_1) & \longrightarrow & K_\Theta \\ & f & \longmapsto & P_\Theta f. \end{array}$$

Then we have

$$\ker P = \Theta\mathcal{H}(b_1).$$

Proof Since $\ker P_\Theta = L^2(\mathbb{T}) \ominus K_\Theta = \Theta H^2 \oplus \overline{H_0^2}$, the result follows the identity $\mathcal{H}(\Theta b_1) \cap \Theta H^2 = \Theta\mathcal{H}(b_1)$, which itself follows from Corollary 18.9. □

In Section 25.6, we will discuss in further detail another situation where the orthogonal decomposition (18.4) happens. In particular, we will obtain a generalization of Corollary 18.9.

Exercises

Exercise 18.3.1 Let b be a function in the closed unit ball of H^∞ and let Θ be an inner function. Show that $\mathfrak{Mult}(\mathcal{H}(\Theta b)) \subset \mathfrak{Mult}(\mathcal{H}(b))$.
Hint: Decompose $\mathcal{H}(\Theta b)$ using Corollary 18.9.

Exercise 18.3.2 Let $b = b_1 b_2$, $b_j \in H^\infty$, $\|b_j\|_\infty \le 1$, $j = 1, 2$. Show that, for each $f \in \mathcal{H}(b_1)$, we have

$$\|f\|_{b_1}^2 = \sup_{g\in\mathcal{H}(b_2)} (\|f + b_1 g\|_b^2 - \|g\|_{b_2}^2).$$

Hint: Use Exercise 16.9.1 with $B = T_b$, $A = T_{b_1}$ and $C = T_{b_2}$.

18.4 The reproducing kernel of $\mathcal{H}(b)$

The reproducing kernel for the Hardy space H^2 is

$$k_z(w) = \frac{1}{1 - \bar{z}w} \qquad (z, w \in \mathbb{D}).$$

In other words, for each $f \in H^2$, we have the representation

$$f(z) = \langle f, k_z \rangle_{H^2} = \frac{1}{2\pi} \int_0^{2\pi} \frac{f(e^{it})}{1 - e^{-it} z} \, dt \qquad (z \in \mathbb{D}). \tag{18.6}$$

Theorem 16.13 enables us to find the reproducing kernel of $\mathcal{H}(b)$.

Theorem 18.11 *The reproducing kernel of $\mathcal{H}(b)$ is*

$$k_z^b = (I - T_b T_{\bar{b}}) k_z = (1 - \overline{b(z)} b) k_z \qquad (z \in \mathbb{D}), \tag{18.7}$$

or equivalently

$$k_z^b(w) = \frac{1 - \overline{b(z)} b(w)}{1 - \bar{z} w} \qquad (z, w \in \mathbb{D}).$$

Moreover, the norm of the evaluation functional

$$f(z) = \langle f, k_z^b \rangle_b \qquad (f \in \mathcal{H}(b), \ z \in \mathbb{D}),$$

is equal to

$$\|k_z^b\|_b = (k_z^b(z))^{1/2} = \left(\frac{1 - |b(z)|^2}{1 - |z|^2} \right)^{1/2}. \tag{18.8}$$

Proof Since $\mathcal{H}(b) = \mathcal{M}((I - T_b T_{\bar{b}})^{1/2})$, by Theorem 16.13 and (18.6), we have

$$k_z^b = (I - T_b T_{\bar{b}}) k_z.$$

But, by (12.7),

$$T_{\bar{b}} k_z = \overline{b(z)} k_z,$$

and clearly $T_b k_z = b k_z$. Hence, we obtain

$$k_z^b = (1 - \overline{b(z)} b) k_z.$$

The formula for the norm of k_z^b is a special case of (9.2). □

By the same token, the reproducing kernel of $\mathcal{H}(\bar{b})$ is

$$k_z^{\bar{b}} = (I - T_{\bar{b}} T_b) k_z = T_{1 - |b|^2} k_z \qquad (z \in \mathbb{D}).$$

However, the above identities do not provide an explicit formula for $k_z^{\bar{b}}$.

Exercise

Exercise 18.4.1 Show that the reproducing kernel of $\mathcal{M}(b)$ is

$$\kappa_z(w) = \frac{\overline{b(z)} b(w)}{1 - \bar{z} w} \qquad (z, w \in \mathbb{D}).$$

Moreover, the norm of the evaluation functional

$$f(z) = \langle f, \kappa_z \rangle_{\mathcal{M}(b)} \qquad (f \in \mathcal{M}(b),\ z \in \mathbb{D})$$

is given by

$$\|\kappa_z\|_{\mathcal{M}(b)} = (\kappa_z(z))^{1/2} = \left(\frac{|b(z)|^2}{1-|z|^2} \right)^{1/2}.$$

18.5 $\mathcal{H}(b)$ and $\mathcal{H}(\bar{b})$ are invariant under $T_{\bar{\varphi}}$

In Section 18.1, we saw that $\mathcal{H}(\Theta) = K_\Theta$, whenever Θ is an inner function. We recall that K_Θ was originally defined as a closed subspace of H^2 that is invariant under the backward shift operator $S^* = T_{\bar{z}}$. Hence, $\mathcal{H}(\Theta)$ is invariant under $T_{\bar{z}}$. In this section we further explore this property and show that each $\mathcal{H}(b)$ is invariant under a large family of operators, which includes S^*.

Lemma 18.12 *Let $\varphi \in H^\infty$. Then $\mathcal{H}(\bar{b})$ is invariant under $T_{\bar{\varphi}}$. In particular, $\mathcal{H}(\bar{b})$ is invariant under the backward shift operator $S^* = T_{\bar{z}}$. Moreover, the norm of the operator*

$$\begin{array}{ccc} \mathcal{H}(\bar{b}) & \longrightarrow & \mathcal{H}(\bar{b}) \\ f & \longmapsto & T_{\bar{\varphi}} f \end{array}$$

does not exceed $\|\varphi\|_\infty$.

Proof Without loss of generality, assume that $\|\varphi\|_\infty = 1$. By Corollary 16.10, the inequality $RAA^*R^* \le AA^*$ implies that $R\mathcal{M}(A) \subset \mathcal{M}(A)$ and that R acts contractively from $\mathcal{M}(A)$ into itself. In our setting, take $R = T_{\bar{\varphi}}$ and $A = (I - T_{\bar{b}} T_b)^{1/2}$. Hence, we need to show that

$$T_{\bar{\varphi}} (I - T_{\bar{b}} T_b) T_\varphi \le I - T_{\bar{b}} T_b.$$

But, by Theorem 12.4, this inequality is equivalent to

$$T_{(1-|b|^2)(1-|\varphi|^2)} \ge 0,$$

which, in the light of Theorem 12.3(ii), is in fact true. □

The preceding result also holds for $\mathcal{H}(b)$. However, for its proof, we exploit the fact that $\mathcal{H}(\bar{b})$ is invariant under $T_{\bar{\varphi}}$.

Theorem 18.13 *Let $\varphi \in H^\infty$. Then $\mathcal{H}(b)$ is invariant under $T_{\bar{\varphi}}$. In particular, $\mathcal{H}(b)$ is invariant under the backward shift operator $S^* = T_{\bar{z}}$. Moreover, the norm of the operator*

$$\begin{array}{ccc} \mathcal{H}(b) & \longrightarrow & \mathcal{H}(b) \\ f & \longmapsto & T_{\bar{\varphi}} f \end{array}$$

does not exceed $\|\varphi\|_\infty$.

Proof Without loss of generality, assume that $\|\varphi\|_\infty = 1$. To prove the statement for $\mathcal{H}(b)$, we apply the Lotto–Sarason theorem (Theorem 16.18) twice to go back and forth between $\mathcal{H}(b)$ and $\mathcal{H}(\bar{b})$. Let $f \in \mathcal{H}(b)$. Hence $T_{\bar{b}}f \in \mathcal{H}(\bar{b})$. According to Lemma 18.12, we thus have $T_{\bar{\varphi}}T_{\bar{b}}f \in \mathcal{H}(\bar{b})$. But, by (12.3), $T_{\bar{\varphi}}T_{\bar{b}} = T_{\bar{b}}T_{\bar{\varphi}}$. Hence $T_{\bar{b}}T_{\bar{\varphi}}f \in \mathcal{H}(\bar{b})$. Another application of the Lotto–Sarason theorem implies that $T_{\bar{\varphi}}f \in \mathcal{H}(\bar{b})$. Moreover, by the same theorem, we have

$$\|T_{\bar{\varphi}}f\|_b^2 = \|T_{\bar{\varphi}}f\|_2^2 + \|T_{\bar{b}}T_{\bar{\varphi}}f\|_{\bar{b}}^2$$

and

$$\|f\|_b^2 = \|f\|_2^2 + \|T_{\bar{b}}f\|_{\bar{b}}^2.$$

However, by Lemma 18.13 and Theorem 12.2,

$$\|T_{\bar{b}}T_{\bar{\varphi}}f\|_{\bar{b}} = \|T_{\bar{\varphi}}T_{\bar{b}}f\|_{\bar{b}} \le \|T_{\bar{b}}f\|_{\bar{b}} \quad \text{and} \quad \|T_{\bar{\varphi}}f\|_2 \le \|f\|_2.$$

Thus, $\|T_{\bar{\varphi}}f\|_b \le \|f\|_b$. □

The fact that $\mathcal{H}(b)$ is invariant under S^* can also be easily deduced from Theorem 16.29 (see Section 18.8).

In the following, we will abuse notation and still write $T_{\bar{\varphi}}$ for the restriction operators considered in Lemma 18.12 or in Theorem 18.13. One must be careful that the adjoint of these operators is not T_φ. In Theorem 18.13, we saw that

$$\|T_{\bar{\varphi}}\|_{\mathcal{L}(\mathcal{H}(b))} \le \|\varphi\|_\infty.$$

It is easy to construct examples for which the strict inequality holds. For example, if $b = b_w$ is the single Blaschke factor with a simple zero at $w \in \mathbb{D}$, then, by Theorem 14.7, $\mathcal{H}(b_w) = K_{b_w} = \mathbb{C}k_w$. Moreover, by (12.7),

$$T_{\bar{\varphi}}k_w = \overline{\varphi(w)}k_w.$$

Hence, we immediately conclude that

$$\|T_{\bar{\varphi}}\|_{\mathcal{L}(\mathcal{H}(b_w))} = |\varphi(w)|.$$

For the next result, we recall that, for each $w \in \mathbb{D}$, the operator Q_w is defined by

$$Q_w = (1 - wS^*)^{-1}S^* \in \mathcal{L}(H^2).$$

Corollary 18.14 *Let $w \in \mathbb{D}$. Then $\mathcal{H}(b)$ and $\mathcal{H}(\bar{b})$ are both invariant under Q_w. Moreover, the norm of*

$$\begin{array}{ccc} \mathcal{H}(b) & \longrightarrow & \mathcal{H}(b) \\ f & \longmapsto & Q_w f \end{array}$$

and that of

$$\begin{array}{ccc} \mathcal{H}(\bar{b}) & \longrightarrow & \mathcal{H}(\bar{b}) \\ f & \longmapsto & Q_w f \end{array}$$

do not exceed $1/(1-|w|)$.

Proof Since, by Theorem 18.13, $S^* = T_{\bar{z}} \in \mathcal{L}(\mathcal{H}(b))$ with $\|S^*\|_{\mathcal{L}(\mathcal{H}(b))} \le 1$, and

$$Q_w = \sum_{n=1}^{\infty} w^{n-1} S^{*n},$$

we immediately see that $Q_w \in \mathcal{L}(\mathcal{H}(b))$. Moreover,

$$\|Q_w\|_{\mathcal{L}(\mathcal{H}(b))} \le \sum_{n=1}^{\infty} |w|^{n-1} \, \|S^*\|^n_{\mathcal{L}(\mathcal{H}(b))} \le \sum_{n=1}^{\infty} |w|^{n-1} = \frac{1}{1-|w|}.$$

Lemma 18.12 shows that the same proof works for $\mathcal{H}(\bar{b})$. □

According to Theorem 18.13, the space $\mathcal{H}(b)$ is invariant under the backward shift operator $S^* = T_{\bar{z}}$, and the restriction of S^* is a contraction. We write $X_b = S^*|\mathcal{H}(b)$. This operator will be discussed in Section 18.7. We finish this section by giving a simple relation between cyclic vectors for X_b and those of S^*.

Corollary 18.15 *Let $b \in H^\infty$, $\|b\|_\infty \le 1$. Assume that b is not an inner function. Let $f \in \mathcal{H}(b)$. If f is a cyclic vector for X_b, then it is also a cyclic vector for S^*.*

Proof Assume that f is a cyclic vector for X_b but not for S^*. Denote by

$$E = \mathrm{Lin}\{S^{*n} f : n \ge 0\}$$

the linear manifold created by $S^{*n} f$, $n \ge 0$, and by F_2 the closure of E with respect to the H^2 norm and by F_b the closure of E with respect to the $\mathcal{H}(b)$ norm. On the one hand, we have $F_b = \mathcal{H}(b)$ and $F_2 \subsetneq H^2$, and on the other, according to Lemma 16.1, we have

$$\mathrm{Clos}_{H^2}(F_b) = F_2.$$

Hence $\mathrm{Clos}_{H^2}(\mathcal{H}(b)) \subsetneq H^2$. In other words, $\mathcal{H}(b)$ is not dense in H^2. By Theorem 18.3, that contradicts the fact that b is not an inner function. □

In Corollary 24.32, we will see that the converse of Corollary 18.15 is true in the nonextreme case.

Exercise

Exercise 18.5.1 Let b_1 and b_2 be two functions in the closed unit ball of H^∞. An analytic function m on $\mathbb{D}$ is called a *multiplier from* $\mathcal{H}(b_1)$ *onto* $\mathcal{H}(b_2)$ if $m\mathcal{H}(b_1) = \mathcal{H}(b_2)$. In the following, m is a multiplier from $\mathcal{H}(b_1)$ onto $\mathcal{H}(b_2)$.

(i) Show that m has no zeros on $\mathbb{D}$.
Hint: Note that $\mathcal{H}(b_2)$ contains outer functions (for example, the kernel functions k_λ^b) and so the functions in $\mathcal{H}(b_2)$ have no common inner factor and thus no common zeros.

(ii) Deduce that m is a multiplier from $\mathcal{H}(b_1)$ onto $\mathcal{H}(b_2)$ if and only if m^{-1} is a multiplier from $\mathcal{H}(b_2)$ onto $\mathcal{H}(b_1)$.

(iii) Show that the map $f \longmapsto mf$ is an isomorphism from $\mathcal{H}(b_1)$ onto $\mathcal{H}(b_2)$. In the following we denote this operator by M_m.

(iv) Show that

$$M_m^* k_\lambda^{b_2} = \overline{m(\lambda)} k_\lambda^{b_1} \qquad (\lambda \in \mathbb{D}).$$

(v) Show that the function $m_1 = (m - m(0))/z$ belongs to H^2.
Hint: Note that $(mf - m(0)f(0))/z = m(f - f(0))/z + f(0)m_1$ and use the fact that $S^*(m\mathcal{H}(b_1)) \subset m\mathcal{H}(b_1)$ and $S^*\mathcal{H}(b_1) \subset \mathcal{H}(b_1)$.

(vi) Deduce that $m \in H^2$.

(vii) Deduce that m is an outer function.
Hint: Note that we also have $m^{-1} \in H^2$.

(viii) Show that, if b_1 is not an inner function and m is a multiplier from $\mathcal{H}(b_1)$ onto $\mathcal{H}(b_2)$, then $m \in H^\infty$.
Hint: Use (iii) and the density of $\mathcal{H}(b_1)$ in H^2.

18.6 Some inhabitants of $\mathcal{H}(b)$

While inner functions are important entities in the theory of Hardy spaces, there are some properties in which inner factors play no role. For example, see Lemmas 17.3 and 17.11, and Theorems 17.5(iii) and 17.12(iii). Another classic example is the norm of a function in H^p. If $f = \Theta g$, where Θ is inner and $g \in H^p$, then $f \in H^p$ and moreover $\|f\|_p = \|g\|_g$.

Reciprocally, if $f \in H^p$ is given and Θ is the Blaschke product formed with the zeros of f and we put $g = f/\Theta$, then, according to a result of F. Riesz, $g \in H^p$, $\|g\|_p = \|f\|_p$ and, more importantly, g is zero-free on $\mathbb{D}$. This last property of g is vital in many applications. A similar phenomenon is discussed below in the context of $\mathcal{H}(b)$ spaces.

Theorem 18.16 *Let $f \in \mathcal{H}(b)$ and let $f = \Theta g$ be a decomposition of f with Θ an inner function and g an element of H^2. Then $g \in \mathcal{H}(b)$. Moreover, we have $\|g\|_b \le \|f\|_b$.*

Proof Since $|\Theta| = 1$, a.e. on $\mathbb{T}$, we can write $g = \bar{\Theta} f$ and thus $g = P_+(\bar{\Theta} f) = T_{\bar{\Theta}} f$. Now it suffices to apply Theorem 18.13 to conclude the result. □

As an immediate consequence of the previous theorem, we have the following result.

Corollary 18.17 *Let $f \in \mathcal{H}(b)$. Then the outer part of f belongs to $\mathcal{H}(b)$.*

We need to be careful in applying the preceding two results. They do not say that, if $g \in \mathcal{H}(b)$ and Θ is inner, then $f = \Theta g \in \mathcal{H}(b)$. Even if g is outer, such a statement is invalid. Examples of this type are easy to make. For example, if B is a finite Blaschke product with distinct zeros at $z_1, \dots, z_n$, then, by Theorem 14.7, K_B is the linear span of $k_{z_1}, \dots, k_{z_n}$. If we pick any point $p \in \mathbb{D} \setminus \{z_1, \dots, z_n\}$, and form the Blaschke factor b_p, and then we take any function $g \in K_B$, we surely have $b_p g \notin K_B$.

If we apply Lemma 18.12, we immediately see that Theorem 18.16 and Corollary 18.17 are also valid for the elements of $\mathcal{H}(\bar{b})$ spaces.

By Theorem 18.11, the only explicit elements in $\mathcal{H}(b)$ that we know up to now are the kernel functions

$$k_w^b(z) = \frac{1 - \overline{b(w)} b(z)}{1 - \bar{w} z} \qquad (z \in \mathbb{D}),$$

where w acts as a parameter and runs over $\mathbb{D}$. However, Theorem 18.13 enables us to distinguish more inhabitants of $\mathcal{H}(b)$.

Theorem 18.18 *For each $w \in \mathbb{D}$, we have*

$$Q_w b \in \mathcal{H}(b).$$

In particular, $S^* b \in \mathcal{H}(b)$. Moreover, we have

$$\|Q_w b\|_b \le \frac{\|S^* b\|_b}{1 - |w|} \qquad (w \in \mathbb{D}).$$

Proof We start with the special case $Q_0 = S^*$. Let $f = (I - T_{\bar{b}} T_b)\, 1$. Clearly, by definition, $f \in \mathcal{H}(\bar{b})$. Hence, by Theorem 18.13, $-S^* f \in \mathcal{H}(\bar{b})$. But a simple calculation shows that

$$-S^* f = S^* T_{\bar{b}} b = T_{\bar{b}} S^* b.$$

Since $T_{\bar{b}} S^* b \in \mathcal{H}(\bar{b})$, the Lotto–Sarason theorem (Theorem 16.18) ensures that

$$S^* b \in \mathcal{H}(b). \tag{18.9}$$

This part can be proven more directly if $b(0) \neq 0$. It is enough to observe that $1 - \overline{b(0)}b = (I - T_b T_{\bar{b}})1 \in \mathcal{H}(b)$, and then apply S^*.

Now, by Theorem 8.10,

$$Q_w b = \sum_{n=0}^{+\infty} w^n S^{*n}(S^* b). \tag{18.10}$$

Therefore, by (18.9) and Theorem 18.13, we see that $Q_w b \in \mathcal{H}(b)$. Using (18.10), we have

$$\|Q_w b\|_b \leq \sum_{n=0}^{\infty} |w|^n \, \|S^{*n} S^* b\|_b.$$

According to Theorem 18.13, S^* acts as a contraction on $\mathcal{H}(b)$ and thus $\|S^{*n} S^* b\|_b \leq \|S^* b\|_b$. Then, we obtain

$$\|Q_w b\|_b \leq \|S^* b\|_b \sum_{n=0}^{\infty} |w|^n = \frac{\|S^* b\|_b}{1 - |w|}. \qquad \square$$

To apply the preceding result, note that, by Theorem 8.10,

$$(Q_w b)(z) = \frac{b(z) - b(w)}{z - w} \qquad (z \in \mathbb{D}). \tag{18.11}$$

Hence, in the following, we use the more familiar notation

$$\hat{k}_w^b(z) = (Q_w b)(z) = \frac{b(z) - b(w)}{z - w} \qquad (z, w \in \mathbb{D}),$$

which are known as the family of *difference quotients*. With this new notation, Theorem 18.18 says that

$$\hat{k}_w^b \in \mathcal{H}(b)$$

for all $w \in \mathbb{D}$ and we have

$$\|\hat{k}_w^b\|_b \leq \frac{\|S^* b\|}{1 - |w|} \qquad (w \in \mathbb{D}). \tag{18.12}$$

In Corollary 19.21 we will give a more precise estimate, but one that needs a contraction from $\mathcal{H}(b)$ into $\mathcal{H}(b^*)$ to be constructed. See also Remark 20.7 for a second method using a contractive antilinear map that transforms the reproducing kernels into the difference quotients.

Since the linear combinations of reproducing kernels k_w^b, $w \in \mathbb{D}$, are dense in $\mathcal{H}(b)$, one may expect the same to be true for difference quotients. However, this is not necessarily the case, as we will see later.

Theorem 18.19 *We have*

$$\overline{\operatorname{Span}}\{\hat{k}_w^b : w \in \mathbb{D}\} = \overline{\operatorname{Span}}\{S^{*n} b : n \geq 1\},$$

where the closure is taken in $\mathcal{H}(b)$.

Proof According to (18.10), we have $f \in \mathcal{H}(b) \ominus \mathrm{Span}(\hat{k}_w^b : w \in \mathbb{D})$ if and only if

$$\sum_{n=0}^{\infty} w^n \langle S^{*(n+1)} b, f \rangle_b = 0$$

for all $w \in \mathbb{D}$. By Theorem 18.13,

$$|\langle S^{*(n+1)} b, f \rangle_b| \leq \|S^{*(n+1)} b\|_b \, \|f\|_b \leq \|S^* b\|_b \, \|f\|_b.$$

Thus, the function

$$w \longmapsto \sum_{n=0}^{\infty} w^n \langle S^{*(n+1)} b, f \rangle_b$$

is analytic in $\mathbb{D}$. Therefore, $f \in \mathcal{H}(b) \ominus \mathrm{Span}(\hat{k}_w^b : w \in \mathbb{D})$ if and only if

$$\langle S^{*(n+1)} b, f \rangle_b = 0 \qquad (n \geq 0).$$

In other words, $f \in \mathcal{H}(b) \ominus \mathrm{Span}(\hat{k}_w^b : w \in \mathbb{D})$ if and only if $f \in \mathcal{H}(b) \ominus \mathrm{Span}(S^{*(n+1)} b : n \geq 0)$.

There is a second, and more direct, proof, which is similar to the proof given in Theorem 5.5. First, the representation (18.10) implies that

$$\mathrm{Span}(\hat{k}_w^b : w \in \mathbb{D}) \subset \mathrm{Span}(S^{*n} b : n \geq 1).$$

Second, fix $n \geq 1$. Note that, for $n = 0$, we have $\hat{k}_0^b = Q_0 b = S^* b$. Then, for each $0 < r < 1$, we have

$$\frac{\hat{k}_r^b + \hat{k}_{r\zeta}^b + \cdots + \hat{k}_{r\zeta^{n-1}}^b}{n} = \sum_{k=0}^{\infty} r^{kn} S^{*(kn+1)} b.$$

From this formula we obtain

$$\frac{\hat{k}_r^b + \hat{k}_{r\zeta}^b + \cdots + \hat{k}_{r\zeta^{n-1}}^b - n\hat{k}_0^b}{nr^n} = S^{*(n+1)} b + \sum_{k=2}^{\infty} r^{(k-1)n} S^{*(kn+1)} b,$$

for all $w \in \mathbb{T}$. But, by Theorem 18.13, we have

$$\|S^{*(kn+1)} b\|_b \leq \|S^* b\|_b.$$

Hence, as $r \longrightarrow 0$,

$$\sum_{k=2}^{\infty} r^{(k-1)n} S^{*(kn+1)} b \longrightarrow 0,$$

in $\mathcal{H}(b)$ norm. Therefore,

$$S^{*(n+1)} b \in \mathrm{Span}(\hat{k}_w^b : w \in \mathbb{D}) \qquad (n \geq 1),$$

and we thus have

$$\mathrm{Span}(S^{*n} b : n \geq 1) \subset \mathrm{Span}(\hat{k}_w^b : w \in \mathbb{D}). \qquad \square$$

Remember that a family $(x_i)_{i\in I}$ in a Hilbert space H is called *complete* provided that

$$\operatorname{Span}(x_i : i \in I) = H.$$

We emphasize that Span E means the closure, in H, of all finite linear combinations of elements of E. The following result immediately follows from Theorem 18.19. It characterizes the completeness of $(\hat{k}_w^b)_{w\in\mathbb{D}}$ in $\mathcal{H}(b)$ in terms of the completeness of a sequence of functions.

Corollary 18.20 *The following are equivalent:*

(i) *the family* $\{\hat{k}_w^b : w \in \mathbb{D}\}$ *is complete in* $\mathcal{H}(b)$*;*
(ii) *the sequence* $(S^{*n}b)_{n\geq 1}$ *is complete in* $\mathcal{H}(b)$*.*

If b is any inner function, say $b = \Theta$, on the one hand, by Theorem 18.1, we know that $\mathcal{H}(\Theta) = K_\Theta$, and, on the other, by Theorem 14.4, we have

$$K_\Theta = \operatorname{Span}\{S^*\Theta, S^{*2}\Theta, S^{*3}\Theta, \dots\}.$$

Hence, Corollary 18.20 ensures that the family $\{\hat{k}_w^\Theta : w \in \mathbb{D}\}$ is complete in K_Θ

In Sections 24.8 and 26.6, we will discuss the completeness of the family $\hat{k}_w^b$, $w \in \mathbb{D}$ (or equivalently the completeness of $S^{*n}b$, $n \geq 0$). As we will see, the answer depends on whether b is an extreme or nonextreme point of the closed unit ball of H^∞. See also Exercises 19.1.1 and 19.1.2 for a method based on the abstract functional embedding, and also Exercise 19.4.1 for a method based on the link between $\mathcal{H}(b)$ and $\mathcal{H}(b^*)$.

Exercises

Exercise 18.6.1 Let $f \in \mathcal{H}(b)$, and fix $w \in \mathbb{D}$. Show that the function

$$\frac{f(z) - f(w)}{z - w} \qquad (z \in \mathbb{D})$$

is in $\mathcal{H}(b)$.
Hint: Consider $Q_w f$.

Exercise 18.6.2 Let $f \in H^2$ be such that $S^*f \in \mathcal{H}(b)$. Show that $Q_w f \in \mathcal{H}(b)$ for each $w \in \mathbb{D}$.
Remark: According to Theorem 18.13, this is a generalization of Exercise 18.6.1.

Exercise 18.6.3 Let b be in the closed unit ball of H^∞. The aim of this exercise is to give a different proof of the fact that $S^*b \in \mathcal{H}(b)$.

(i) Show that $S^*T_b - T_bS^*$ is a rank-one operator, whose range is spanned by S^*b.

(ii) Show that

$$(S^*T_b - T_bS^*)(S^*T_b - T_bS^*)^* \le I - T_bT_{\bar{b}}.$$

(iii) Deduce that $S^*b \in \mathcal{H}(b)$.

18.7 The unilateral backward shift operators X_b and $X_{\bar{b}}$

As a special case of Theorem 18.13, the space $\mathcal{H}(b)$ is invariant under the backward shift operator $S^* = T_{\bar{z}}$, and the restriction of S^* is a contraction. Writing $S^* \in \mathcal{L}(\mathcal{H}(b))$ might sound reasonable, but for many reasons, in particular since the Hilbert space structure of $\mathcal{H}(b)$ is not necessarily inherited from H^2, this notation will cause certain difficulties. Therefore, we henceforth write

$$\begin{array}{rccc} X_b: & \mathcal{H}(b) & \longrightarrow & \mathcal{H}(b) \\ & f & \longmapsto & S^*f. \end{array}$$

According to Theorem 18.13, X_b is a contraction. In a similar manner, we define

$$\begin{array}{rccc} X_{\bar{b}}: & \mathcal{H}(\bar{b}) & \longrightarrow & \mathcal{H}(\bar{b}) \\ & f & \longmapsto & S^*f. \end{array}$$

According to Lemma 18.12, $\mathcal{H}(\bar{b})$ is invariant under the backward shift S^* and thus $X_{\bar{b}}$ is well defined and is a contraction.

The relation between the Cauchy kernel and the shift operator is crystallized in the identity

$$k_w = (1 - \bar{w}S)^{-1}k_0 \qquad (w \in \mathbb{D}).$$

In a similar manner, the following result reveals the relation between the reproducing kernel k_w^b and the operator X_b.

Note that, for each $w \in \mathbb{D}$, the operator $I - \bar{w}X_b^*$ is invertible. Indeed, since X_b is a contraction, we have $\sigma(X_b^*) \subset \bar{\mathbb{D}}$.

Theorem 18.21 *We have*

$$k_w^b = (I - \bar{w}X_b^*)^{-1}k_0^b \qquad (w \in \mathbb{D}).$$

Proof By (8.19), for each $f \in H^2$, we have

$$f(w) = \langle (I - wS^*)^{-1}f,\ k_0\rangle_2 \qquad (w \in \mathbb{D}).$$

Thus, if $f \in \mathcal{H}(b)$, we can write

$$f(w) = \langle (I - wX_b)^{-1}f,\ k_0\rangle_2 \qquad (w \in \mathbb{D}).$$

But, by Corollary 16.16,

$$\langle (I - wX_b)^{-1} f,\ k_0 \rangle_2 = \langle (I - wX_b)^{-1} f,\ (I - T_b T_b^*) k_0 \rangle_b,$$

and Theorem 18.11 says that $k_0^b = (I - T_b T_{\bar{b}}) k_0$. Hence,

$$f(w) = \langle (I - wX_b)^{-1} f,\ k_0^b \rangle_b = \langle f,\ (I - \bar{w} X_b^*)^{-1} k_0^b \rangle_b \qquad (w \in \mathbb{D}).$$

Since the last relation is valid for every function $f \in \mathcal{H}(b)$, we conclude that $(I - \bar{w} X_b^*)^{-1} k_0^b$ is precisely the reproducing kernel k_w^b. □

For the following result, one needs to look at the definition (1.39) of the tensor product of two vectors.

Theorem 18.22 *The adjoint of $X_b \in \mathcal{L}(\mathcal{H}(b))$ is given by the formula*

$$X_b^* = S - b \otimes S^* b.$$

Proof Let $f \in \mathcal{H}(b)$. By the definition of reproducing kernel,

$$(X_b^* f)(z) = \langle X_b^* f, k_z^b \rangle_b \qquad (z \in \mathbb{D}),$$

and, by the defining property of the adjoint,

$$\langle X_b^* f, k_z^b \rangle_b = \langle f, X_b k_z^b \rangle_b.$$

In Theorem 18.11, we saw that $k_z^b = (1 - \overline{b(z)} b) k_z$. Hence, by Corollary 8.11, and the fact that $k_z(0) = 1$, we obtain

$$\begin{aligned} X_b k_z^b &= S^*((1 - \overline{b(z)} b) k_z) \\ &= (1 - \overline{b(z)} b) S^* k_z + S^*(1 - \overline{b(z)} b) \\ &= (1 - \overline{b(z)} b) \bar{z} k_z - \overline{b(z)} S^* b \\ &= \bar{z} k_z^b - \overline{b(z)} S^* b. \end{aligned}$$

Hence, for the record,

$$X_b k_z^b = \bar{z} k_z^b - \overline{b(z)} S^* b. \tag{18.13}$$

(In Theorem 18.18, we showed that $S^* b \in \mathcal{H}(b)$. The above identity is another manifestation of this fact.) Therefore, for each $z \in \mathbb{D}$,

$$\begin{aligned} (X_b^* f)(z) &= \langle f,\ \bar{z} k_z^b - \overline{b(z)} S^* b \rangle_b \\ &= z \langle f, k_z^b \rangle_b - b(z) \langle f, S^* b \rangle_b \\ &= z f(z) - \langle f, S^* b \rangle_b b(z) \\ &= (Sf)(z) - \langle f, S^* b \rangle_b b(z) \\ &= (Sf)(z) - ((b \otimes S^* b) f)(z). \end{aligned}$$

This completes the proof. □

The formula $X_b^* = S - b \otimes S^*b$ could be misleading if it is not properly applied. Since $b \in H^2$ and $S^*b \in \mathcal{H}(b)$, by definition, $b \otimes S^*b$ is an operator in $\mathcal{L}(\mathcal{H}(b), H^2)$. We also considered S as an operator in $\mathcal{L}(\mathcal{H}(b), H^2)$. However, as the formula for X_b^* manifests, we incidentally see that $S - b \otimes S^*b$ can be considered as an object in $\mathcal{L}(\mathcal{H}(b))$. Indeed, it might be more appropriate to say that X_b^* is given by the formula

$$(X_b^* f)(z) = zf(z) - \langle f, S^*b\rangle_b b(z) \qquad (z \in \mathbb{D},\ f \in \mathcal{H}(b)), \tag{18.14}$$

or at most by

$$X_b^* f = Sf - \langle f, S^*b\rangle_b b \qquad (f \in \mathcal{H}(b)). \tag{18.15}$$

This is one of the occasions that we realize how misleading it might be to use S^* instead of X_b. If so, what do we expect $(S^*)^*$ to be?

Corollary 18.23 *We have*

$$X_b X_b^* = I - S^*b \otimes S^*b,$$

which implies that

$$\|X_b^* f\|_b^2 = \|f\|_b^2 - |\langle f, S^*b\rangle_b|^2 \qquad (f \in \mathcal{H}(b)). \tag{18.16}$$

In particular,

$$\|X_b^* f\|_b = \|f\|_b \quad \Longleftrightarrow \quad \langle f, S^*b\rangle_b = 0 \qquad (f \in \mathcal{H}(b)).$$

Proof According to (18.15), we have

$$\begin{aligned} X_b X_b^* f &= S^*(Sf - \langle f, S^*b\rangle_b b) \\ &= S^*Sf - \langle f, S^*b\rangle_b S^*b \\ &= f - \langle f, S^*b\rangle_b S^*b. \end{aligned}$$

The preceding identity can be rewritten as $X_b X_b^* = I - S^*b \otimes S^*b$, where the identity operator I and the rank-one operator $S^*b \otimes S^*b$ are elements of $\mathcal{L}(\mathcal{H}(b))$. Hence,

$$\|X_b^* f\|_b^2 = \langle X_b X_b^* f, f\rangle_b = \|f\|_b^2 - |\langle f, S^*b\rangle_b|^2. \qquad \square$$

Remark 18.24 If b is inner, then $\mathcal{H}(b) = K_b$ and $\langle\cdot,\cdot\rangle_b = \langle\cdot,\cdot\rangle_2$. Furthermore, using the fact that $|b| = 1$ a.e. on $\mathbb{T}$, we can write

$$\begin{aligned} X_b^* f &= b(\bar{b}zf - \langle Sf, b\rangle_2) \\ &= bP_-(\bar{b}zf) + b(P_+(\bar{b}zf) - \langle Sf, b\rangle_2). \end{aligned}$$

Note that $bP_-(\bar{b}zf) = \mathbf{P}_b(Sf)$, where $\mathbf{P}_b$ is the orthogonal projection of H^2 onto K_b, and since $f \in K_b$, according to Theorem 8.44, there exists $h \in H^2$ such that $f = b\bar{z}\bar{h}$. Hence, $P_+(\bar{b}zf) = P_+(\bar{h}) = \overline{h(0)}$ and

$$\langle Sf, b\rangle_2 = \langle z\bar{b}f, 1\rangle_2 = P_+(\bar{b}zf)(0) = \overline{h(0)}.$$

That gives

$$X_b^* f = \mathbf{P}_b(Sf) = \mathbf{M}_b f,$$

and we recover the model operator introduced in Section 14.6.

The property that b is inner is reflected in the dynamics of the operator X_b^*.

Corollary 18.25 *Let $b \in H^\infty$, $\|b\|_\infty \le 1$. Then the following assertions are equivalent.*

(i) *For each $f \in \mathcal{H}(b)$, we have*

$$\lim_{n\to\infty} \|X_b^{*n} f\|_b = 0.$$

(ii) *The function b is inner.*

Proof (ii) $\Longrightarrow$ (i) Assume that b is inner. Then, according to Remark 18.24, we know that $\mathcal{H}(b)$ coincides with K_b and X_b^* coincides with $\mathbf{M}_b$. Then (i) follows from Lemma 14.21.

(i) $\Longrightarrow$ (ii) If we iterate (18.16), we have

$$\|X_b^{*n} f\|_b^2 = \|f\|_b^2 - \sum_{\ell=1}^{n} |\langle f, S^{*\ell} b\rangle_b|^2.$$

Thus the hypothesis implies that, for each $f \in \mathcal{H}(b)$, we have

$$\|f\|_b^2 = \sum_{\ell=1}^{\infty} |\langle f, S^{*\ell} b\rangle_b|^2.$$

If we apply this equality to $f = k_0^b$, using (18.8), we get

$$1 - |b(0)|^2 = \sum_{\ell=1}^{\infty} |(S^{*\ell} b)(0)|^2,$$

which can be rewritten as

$$1 = |\hat{b}(0)|^2 + \sum_{\ell=1}^{\infty} |\hat{b}(\ell)|^2 = \|b\|_2^2.$$

In other words, we have

$$\int_{\mathbb{T}} |b|^2\, dm = 1,$$

and since $|b| \le 1$ a.e. on $\mathbb{T}$, we finally deduce that $|b| = 1$ a.e. on $\mathbb{T}$. That means exactly that b is an inner function. $\square$

Since X_b is a contraction, we certainly have $\sigma(X_b) \subset \bar{\mathbb{D}}$. In the first step to determine $\sigma(X_b)$, we show that X_b and its adjoint X_b^* do not have any eigenvalues on the unit circle $\mathbb{T}$.

Theorem 18.26 *We have*

$$\sigma_p(X_b) \subset \mathbb{D} \quad \textit{and} \quad \sigma_p(X_b^*) \subset \mathbb{D}.$$

Proof Since X_b is the restriction of S^* to the subclass $\mathcal{H}(b)$, we thus have $\sigma_p(X_b) \subset \sigma_p(S^*)$, and, according to Lemma 8.6, $\sigma_p(S^*) = \mathbb{D}$. This gives the first inclusion.

For the second inclusion, assume that there exists $\lambda \in \mathbb{T} \cap \sigma_p(X_b^*)$ and let $f \in \mathcal{H}(b)$, $f \not\equiv 0$, be such that $X_b^* f = \lambda f$. Hence,

$$\|X_b^* f\|_b = \|\lambda f\|_b = \|f\|_b,$$

and Corollary 18.23 implies that $\langle f, S^* b\rangle_b = 0$. Thus, going back to Theorem 18.22, we see that $X_b^* f = Sf = \lambda f$. This means that $\lambda \in \sigma_p(S)$, which, by Lemma 8.6, is absurd. □

We will discuss more precise results concerning the spectral properties of X_b. In particular, we will see that the spectral properties differ depending on whether b is an extreme or nonextreme point of the closed unit ball of H^∞. This will be done in Sections 24.3 and 26.1

The operator X_b is a contraction, and so it is natural and interesting to study its defect operators. Remember the definition from Section 7.2. The structure of D_{X_b} depends on whether b is an extreme or nonextreme point of the closed unit ball of H^∞ (see Sections 23.5 and 25.4). The structure of $D_{X_b^*}$ is easier to determine.

Corollary 18.27 *The defect operator* $D_{X_b^*} = (I - X_b X_b^*)^{1/2}$ *has rank one, its range is spanned by* $S^* b$ *and its nonzero eigenvalue equals* $\|S^* b\|_b$.

Proof According to Corollary 18.23, we have

$$D^2_{X_b^*} = S^* b \otimes S^* b.$$

Hence, the matrix of this operator relative to the decomposition $\mathcal{H}(b) = \mathbb{C}S^* b \oplus (\mathbb{C}S^* b)^\perp$ is

$$\begin{pmatrix} \|S^* b\|_b^2 & 0 \\ 0 & 0 \end{pmatrix}.$$

This immediately gives the result. □

Exercises

Exercise 18.7.1 Let $g \in \mathcal{H}(b)$, $f \in H^2$ and $h = (I - T_b T_{\bar{b}})f$.

(i) Show that

$$\langle X_b g, h\rangle_b = \langle g,\ (I - T_b T_{\bar{b}}) S f\rangle_b.$$

Hint: Use Corollary 17.6.

(ii) Deduce that $X_b^* h = (I - T_b T_{\bar{b}}) S f = Sh - T_b(T_{\bar{b}} S - S T_{\bar{b}}) f$.
(iii) Show that $S^* T_b - T_b S^* = S^* b \otimes 1$.
(iv) Deduce that

$$(T_{\bar{b}} S - S T_{\bar{b}}) f = \langle f, S^* b \rangle_2 1 = \langle h, S^* b \rangle_b 1.$$

(v) Conclude that $X_b^* h = Sh - \langle h, S^* b \rangle_b b$.
(vi) Conclude that

$$X_b^* = S - b \otimes S^* b.$$

Exercise 18.7.2 Let $b \in H^\infty$, $\|b\|_\infty \leq 1$, let $n \geq 1$ and let $f \in \mathcal{H}(b)$. Show that there exists a polynomial p of degree less than n such that $X_b^{*n} f = S^n f - bp$ and

$$\|z^n f - bp\|_b^2 = \|f\|_b^2 - \|p\|_2^2.$$

Hint: Use Theorem 18.22 and Corollary 18.23.

18.8 The inequality of difference quotients

Using the geometric description of $\mathcal{H}(b)$ (see Section 16.9), we can show the following important property.

Theorem 18.28 *For every $f \in \mathcal{H}(b)$, we have*

$$\|X_b f\|_b^2 \leq \|f\|_b^2 - |f(0)|^2. \tag{18.17}$$

Proof We remember that $\mathcal{H}(b)$ is contractively contained in H^2 and $\mathcal{M}(b)$ is the complementary space of $\mathcal{H}(b)$. Moreover, $\mathcal{M}(b)$ is clearly invariant under the forward shift operator S and, for each $g = T_b f \in \mathcal{M}(b)$, we have

$$\|Sg\|_{\mathcal{M}(b)} = \|T_b S f\|_{\mathcal{M}(b)} = \|Sf\|_2 = \|f\|_2 = \|g\|_{\mathcal{M}(b)}.$$

Thus, we can apply Theorem 16.29 and conclude that $\mathcal{H}(b)$ is S^*-invariant and (18.17) holds. $\square$

Even though it was established in Theorem 18.13, the preceding proof also shows that $\mathcal{H}(b)$ is S^*-invariant.

The inequality (18.17) is called the *inequality for difference quotients*. This is because X_b, given by

$$(X_b f)(z) = \frac{f(z) - f(0)}{z},$$

is a difference-quotient transformation. In the case where (18.17) is an equality for all f in $\mathcal{H}(b)$, we say that $\mathcal{H}(b)$ satisfies the *identity for difference quotients*. As we will see in Chapters 23 and 25, this happens if and only if b is an extreme point of the closed unit ball of H^∞.

Corollary 18.29 *Let $b \in H^\infty$, $\|b\|_\infty \le 1$. Then X_b has no isometric part, i.e. if $\mathcal{E}$ is an X_b-invariant subspace of $\mathcal{H}(b)$ and if $X_{b|\mathcal{E}}$ is an isometry, then $\mathcal{E} = \{0\}$.*

Proof Let $f \in \mathcal{E}$. If we apply (18.17), we get

$$\|X_b^{n+1} f\|_b^2 \le \|X_b^n f\|_b^2 - |(S^{*n} f)(0)|^2.$$

Since $\|X_b^n f\|_b = \|X_b^{n+1} f\|_b$, we obtain $(S^{*n} f)(0) = 0$ for any $n \ge 0$. This means that $f \equiv 0$. □

18.9 A characterization of membership in $\mathcal{H}(b)$

For $0 < r < 1$, we have $1 - r^2|b|^2 \ge 1 - r^2$. Hence, we can define a_r to be the unique outer function such that

$$|a_r| = (1 - r^2|b|^2)^{1/2} \qquad (\text{on } \mathbb{T})$$

and $a_r(0) > 0$. In Section 23.1, this phenomenon is studied even for $r = 1$. Since on $\mathbb{T}$,

$$|a_r|^2 = 1 - r^2|b|^2 \ge 1 - r^2,$$

we deduce that $1/a_r \in L^\infty(\mathbb{T})$. Then, according to Corollary 4.28, the function $1/a_r$ also belongs to H^∞ and we have

$$\left\| \frac{1}{a_r} \right\|_\infty \le (1 - r^2)^{-1/2}. \tag{18.18}$$

If it happens that $\log(1 - |b|^2) \in L^1(\mathbb{T})$, we also define a to be the unique outer function such that $|a|^2 = 1 - |b|^2$, $a(0) > 0$.

Lemma 18.30 *Let $0 < r < 1$. Then the following hold.*

(i) $(I - r^2 T_b T_{\bar{b}})^{-1} = I + T_{rb/a_r} T_{r\bar{b}/\bar{a}_r}$.
(ii) *For each $f_1, f_2 \in H^2$, we have*

$$\begin{aligned}\langle (I - r^2 T_b T_{\bar{b}})^{-1/2} f_1,\ &(I - r^2 T_b T_{\bar{b}})^{-1/2} f_2 \rangle_2 \\ &= \langle f_1, f_2 \rangle_2 + \langle r T_{\bar{b}/\bar{a}_r} f_1,\ r T_{\bar{b}/\bar{a}_r} f_2 \rangle_2.\end{aligned}$$

In particular, for each $f \in H^2$,

$$\|(I - r^2 T_b T_{\bar{b}})^{-1/2} f\|_2^2 = \|f\|_2^2 + \|r T_{\bar{b}/\bar{a}_r} f\|_2^2.$$

(iii) *For each function $f \in H^2$, the quantities $\|(I - r^2 T_b T_{\bar{b}})^{-1/2} f\|_2$ and $\|r T_{\bar{b}/\bar{a}_r} f\|_2$, as functions of r for $0 \le r < 1$, are both increasing.*

Proof (i) Since $\|r^2 T_b T_{\bar{b}}\| \le r^2 \|b\|_\infty^2 \le r^2 < 1$, the operator $I - r^2 T_b T_{\bar{b}}$ is invertible. Moreover, by Theorem 12.4, we have

$$\begin{aligned}(I - r^2 T_b T_{\bar{b}})^{-1} &= (I - T_{rb} T_{r\bar{b}})^{-1} \\ &= \sum_{n=0}^{\infty} (T_{rb} T_{r\bar{b}})^n \\ &= I + \sum_{n=1}^{\infty} (T_{rb} T_{r\bar{b}})^n \\ &= I + T_{rb} \sum_{n=1}^{\infty} (T_{r\bar{b}} T_{rb})^{n-1} T_{r\bar{b}} \\ &= I + T_{rb} \sum_{n=0}^{\infty} (T_{r^2|b|^2})^n T_{r\bar{b}} \\ &= I + T_{rb} (I - T_{r^2|b|^2})^{-1} T_{r\bar{b}}.\end{aligned}$$

As $I - T_{r^2|b|^2} = T_{1-r^2|b|^2} = T_{|a_r|^2} = T_{\bar{a}_r} T_{a_r}$, we have

$$(I - T_{r^2|b|^2})^{-1} = T_{1/a_r} T_{1/\bar{a}_r},$$

and thus

$$(I - r^2 T_b T_{\bar{b}})^{-1} = I + T_{rb} T_{1/a_r} T_{1/\bar{a}_r} T_{r\bar{b}} = I + T_{rb/a_r} T_{r\bar{b}/\bar{a}_r}.$$

Another method is to form the product $(I - r^2 T_b T_{\bar{b}})(I + T_{rb/a_r} T_{r\bar{b}/\bar{a}_r})$ and, after simplification, we arrive at the identity operator.

(ii) This follows easily from (i).

(iii) Let $f \in H^2$. Then we have

$$\begin{aligned}\|(I - r^2 T_b T_{\bar{b}})^{-1/2} f\|_2^2 &= \langle (I - r^2 T_b T_{\bar{b}})^{-1} f,\ f\rangle_2 \\ &= \left\langle \sum_{n=0}^{\infty} r^{2n} (T_b T_{\bar{b}})^n f,\ f \right\rangle_2 \\ &= \sum_{n=0}^{\infty} r^{2n}\, \|(T_b T_{\bar{b}})^{n/2} f\|_2^2.\end{aligned}$$

Note that $T_b T_{\bar{b}}$ is a positive operator. Hence, $\|(I - r^2 T_b T_{\bar{b}})^{-1/2} f\|_2$ increases with r. The assertion for $\|r T_{\bar{b}/\bar{a}_r} f\|_2$ follows from (ii) and the preceding argument. $\square$

The following result gives a useful and nice characterization of membership in $\mathcal{H}(b)$.

Theorem 18.31 *Let $f \in H^2$. Then the following are equivalent:*

(i) *the function f belongs to $\mathcal{H}(b)$;*
(ii) *we have*

$$\lim_{r \to 1} \|T_{\bar{b}/\bar{a}_r} f\|_2 < +\infty;$$

(iii) *we have*

$$\lim_{r\to 1} \|(I - r^2 T_b T_{\bar{b}})^{-1/2} f\|_2 < +\infty.$$

Moreover, if $f = (I - T_b T_{\bar{b}})^{1/2} g$ *with* $g \perp \ker(I - T_b T_{\bar{b}})^{1/2}$, *then*

$$\lim_{r\to 1} \|(I - r^2 T_b T_{\bar{b}})^{-1/2} f - g\|_2 = 0.$$

Furthermore if f_1 *and* f_2 *are two functions in* $\mathcal{H}(b)$, *then*

$$\langle f_1, f_2\rangle_b = \lim_{r\to 1} \langle (I - r^2 T_b T_{\bar{b}})^{-1/2} f_1,\ (I - r^2 T_b T_{\bar{b}})^{-1/2} f_2\rangle_2 \tag{18.19}$$

and

$$\langle f_1, f_2\rangle_b = \langle f_1, f_2\rangle_2 + \lim_{r\to 1} \langle T_{\bar{b}/\bar{a}_r} f_1, T_{\bar{b}/\bar{a}_r} f_2\rangle_2. \tag{18.20}$$

Proof The equivalence between (ii) and (iii) follows from Lemma 18.30, and the equivalence between (i) and (iii) follows from Theorem 16.17 (with $A = T_b$). That theorem also implies that, if $f = (I - T_b T_{\bar{b}})^{1/2} g$ with $g \perp \ker(I - T_b T_{\bar{b}})^{1/2}$, then

$$\lim_{r\to 1} \|(I - r^2 T_b T_{\bar{b}})^{-1/2} f - g\|_2 = 0.$$

Finally, (18.19) follows from (16.18) and then (18.20) follows from Lemma 18.30(ii). □

Exercises

Remark: The following exercises will be studied in Chapters 23 and 25, using different methods.

Exercise 18.9.1 Let $m \geq 0$ and assume that b has a zero of order m at the origin. Show that

$$(I - T_b T_{\bar{b}}) z^m = z^m - \overline{\hat{b}(m)} b.$$

Then deduce that

$$z^m - \overline{\hat{b}(m)} b \in \mathcal{H}(b)$$

and, for all functions in $\mathcal{H}(b)$, we have

$$\hat{f}(m) = \langle f,\ z^m - \overline{\hat{b}(m)} b\rangle_b.$$

Hint: Use Corollary 17.6.

Exercise 18.9.2

(i) Show that

$$\|(I - r^2 T_b T_{\bar{b}})^{-1/2} k_w\|_2^2 = (1 - |w|^2)^{-1} \left(1 + \frac{|rb(w)|^2}{|a_r(w)|^2}\right)$$

for every $w \in \mathbb{D}$.
Hint: Use Lemma 18.30(ii) and (12.7).

(ii) Assume that $b(w) \neq 0$. Deduce that k_w belongs to $\mathcal{H}(b)$ if and only if $\lim_{r\to 1} |a_r(w)| > 0$.
Hint: Use Theorem 18.31.

(iii) Show that $\lim_{r\to 1} |a_r(w)| > 0$ if and only if b is not an extreme point of the closed unit ball of H^∞.
Hint: We have

$$\log |a_r(w)| = \int_{\mathbb{T}} \frac{1 - |w|^2}{|\zeta - w|^2} \log |a_r(\zeta)| \, dm(\zeta).$$

Hence, by the monotone convergence theorem,

$$\log \left(\lim_{r\to 1} |a_r(w)| \right) = \int_{\mathbb{T}} \frac{1 - |w|^2}{|\zeta - w|^2} \log |a(\zeta)| \, dm(\zeta).$$

(iv) Assume that b is not an extreme point of the closed unit ball of H^∞. Show that $k_w \in \mathcal{H}(b)$ and

$$\|k_w\|_b^2 = (1 - |w|^2)^{-1} \left(1 + \frac{|b(w)|^2}{|a(w)|^2}\right).$$

Exercise 18.9.3 Let $m \geq 1$ and assume that b has a zero of order m at the origin.

(i) Prove that

$$(I - T_b T_{\bar{b}}) z^{m-1} = z^{m-1}$$

and then deduce that $z^{m-1} \in \mathcal{H}(b)$.

(ii) Show that

$$T_{\bar{b}/\bar{a}_r} z^m = \overline{\hat{b}(m)} / \overline{a_r(0)}.$$

(iii) Show that

$$\|(I - r^2 T_b T_{\bar{b}})^{-1/2} z^m\|_2^2 = 1 + \frac{|r\hat{b}(m)|^2}{|a_r(0)|^2}.$$

Hint: Use Lemma 18.30(ii) and part (ii) of the present exercise.

(iv) Deduce that $z^m \in \mathcal{H}(b)$ if and only if b is not an extreme point of the closed unit ball of H^∞.
Hint: Similar to Exercise 18.9.2(iii), we have

$$\log|a_r(0)| = \int_{\mathbb{T}} \log|a_r(\zeta)|\,dm(\zeta).$$

Hence, by the monotone convergence theorem,

$$\log\left(\lim_{r\to 1}|a_r(0)|\right) = \int_{\mathbb{T}} \log|a(\zeta)|\,dm(\zeta).$$

Now, use Theorem 18.31(iii) and part (iii) of the present exercise.

(v) Assume that b is not an extreme point of the closed unit ball of H^∞. Show that

$$\|z^m\|_b^2 = 1 + \frac{|\hat{b}(m)|^2}{|a(0)|^2}.$$

(vi) Deduce that $b \in \mathcal{H}(b)$ if and only if b is not an extreme point in the closed unit ball of H^∞.
Hint: Use Exercise 18.9.1.

Exercise 18.9.4 Let b be in the closed unit ball of H^∞. Then show that the family of analytic polynomials $\mathcal{P}$ is contained in $\mathcal{H}(b)$ if and only if b is not an extreme point of the closed unit ball of H^∞.
Hint: Use Exercise 18.9.3.

Exercise 18.9.5

(i) Show that

$$\begin{aligned}\|S^*a_r\|_2^2 &= 1 - |a_r(0)|^2 - r^2\|b\|_2^2\\ &= -r^2|b(0)|^2 + (1-|a_r(0)|^2) - r^2\|S^*b\|_2^2.\end{aligned}$$

(ii) Show that

$$rT_{r\bar{b}/\bar{a}_r}S^*b = -S^*a_r \qquad (0<r<1)$$

and

$$\|S^*b\|_b^2 = -|b(0)|^2 + \lim_{r\to 1}(r^{-2}(1-|a_r(0)|^2)).$$

Hint: Use Theorem 18.31(iii) and part (i) of the present exercise.

(iii) Deduce that

$$\|S^*b\|_2^2 = \begin{cases} 1-|b(0)|^2 & \text{if } b \text{ is an extreme point,}\\ 1-|b(0)|^2-|a(0)|^2 & \text{if } b \text{ is not an extreme point.}\end{cases}$$

Notes on Chapter 18

Section 18.1

Theorem 18.1 is due to de Branges and Rovnyak [65, p. 23]. The results of this section and the following one explain why the case when b is inner plays a special role in the theory.

Section 18.2

Theorems 18.3 and 18.4 seem to be new.

Section 18.3

Theorem 18.7 is due to de Branges and Rovnyak [65, problem 52] and [64, theorem 4].

Section 18.4

The explicit formula for reproducing kernels of $\mathcal{H}(b)$ given in Theorem 18.11 is due to de Branges and Rovnyak [65, problem 43].

In this section, we have identified the reproducing kernels of $\mathcal{H}(b)$, which are one of the most important inhabitants of $\mathcal{H}(b)$. It often happens that the computations in the theory use reproducing kernels. Moreover, we should say that we can also start with reproducing kernels

$$k_\lambda^b(z) = \frac{1 - \overline{b(z)}b(w)}{1 - \bar{z}w} \qquad (z, w \in \mathbb{D})$$

to define $\mathcal{H}(b)$ spaces. Indeed, using Theorem 9.10, we can show that the above formula defines a positive kernel, in the sense that

$$\sum_{1 \le i,j \le n} c_i \bar{c_j} k_{\lambda_i}^b(\lambda_j) \ge 0,$$

for any choice of complex constants $c_1, c_2, \dots, c_n$ and points $\lambda_1, \dots, \lambda_n \in \mathbb{D}$. See Section 9.4 and Exercise 9.4.2. Then, initially populate $\mathcal{H}(b)$ with finite linear combinations of reproducing kernel functions

$$\sum_{j=1}^n c_j k_{\lambda_j}^b,$$

and endow these linear combinations of kernel functions with the following inner product:

$$\left\langle \sum_{\lambda \in \Lambda} \alpha_\lambda k_\lambda^b, \sum_{\mu \in \Lambda'} \beta_\mu k_\mu^b \right\rangle_b = \sum_{\lambda \in \Lambda, \mu \in \Lambda'} \alpha_\lambda \overline{\beta_\mu} k_\lambda^b(\mu).$$

It remains then to take the completion. See [6, 22] for the standard technique to construct a reproducing kernel Hilbert space of functions starting from a kernel.

Section 18.5

The invariance of $\mathcal{H}(b)$ and $\mathcal{H}(\bar{b})$ under a Toeplitz operator $T_{\bar{\varphi}}$ whose symbol is co-analytic is a very important property and is due to Lotto and Sarason [123, lemma 2.6]. See also [166, sec. II.7]. The particular case when the Toeplitz operator is $S^* = T_{\bar{z}}$ has already been noted by de Branges and Rovnyak [65, problem 47]. Corollary 18.14 also appears in [65, problem 82]. Exercise 18.5.1 is taken from [61].

Section 18.6

Theorem 18.18 is due to de Branges and Rovnyak [65, problem 84]. The proof presented here is due to Sarason [166, sec. II.8]. Theorem 18.19 is due to Fricain [77], who gave the first proof, which is presented here. The second proof is new. Exercise 18.6.3 is taken from [160].

Section 18.7

The operator $X_b = S^*_{|\mathcal{H}(b)}$ introduced in this section plays a central role in the theory and particularly in the model developed by de Branges and Rovnyak. Indeed, it serves as a model for a large class of contractions. See Section 26.5. Moreover, a lot of properties of $\mathcal{H}(b)$ functions are reflected in the property of this operator. See for instance Theorems 20.13 and 21.26.

Theorem 18.21 is due to Sarason [161]. A version for b inner already exists in Ahern and Clark's paper [10]. Theorem 18.22 is due to de Branges and Rovnyak [65, theorem 13]. The proof here is from [166]. We can also find an alternative proof in [160]; see Exercise 18.7.1 where this proof is presented. A vector-valued analog is given by Nikolskii and Vasyunin in [139, theorem 8.4]. Corollary 18.23 is due to de Branges and Rovnyak [65, problem 70]. Corollary 18.25 is also due to de Branges and Rovnyak [64, appdx, theorem 12]. The second inclusion of Theorem 18.26 concerning the point spectrum of X_b^* comes from [77]. Corollary 18.27 is due to Sarason [160]. Exercise 18.7.2 is due to de Branges and Rovnyak [65, problem 80].

Section 18.8

Theorem 18.28 is due to de Branges and Rovnyak, who called the inequality (18.17) the inequality of difference quotients. The proof of the S^*-invariance of $\mathcal{H}(b)$ given at the end of this section comes from [65, problem 47] and [64, appdx, theorem 2]. In [139], Nikolskii and Vasyunin give an analog of Corollary 18.29 in the vector-valued setting. Exercises 18.9.1, 18.9.2 and 18.9.3 are taken from [160]. In Sections 23.8 and 25.3 we will see a different method to recover the results of these exercises. The method to compute the norm of S^*b proposed in Exercise 18.9.5 is due to Sarason [160]. See Sections 23.3 and 25.1 for a different method.

Section 18.9

Theorem 18.31 is due to Sarason [160]. Exercises 18.9.2 and 18.9.3 are also taken from [160].

19

Geometric representation of $\mathcal{H}(b)$ spaces

The main goal of this chapter is to introduce a representation of $\mathcal{H}(b)$ spaces related to Julia operators as described in Section 16.10 in the general framework.

In Section 19.1, we introduce the notion of abstract functional embedding. This notion enables us to give in Section 19.2 a geometric representation of $\mathcal{H}(b)$ spaces. More precisely, we introduce a new Hilbert space, the so-called model space $\mathbb{K}_b$, which is central in operator theory, and we explain how these two spaces $\mathbb{K}_b$ and $\mathcal{H}(b)$ are connected. This new representation will be useful in many natural questions. For instance, it is central in the study of geometric properties of some families of functions in $\mathcal{H}(b)$ spaces. Furthermore, it enables us to make some more explicit computations, as we will see in the rest of this chapter. In Section 19.3, we construct a unitary operator $\mathfrak{W}_b$ from the model space $\mathbb{K}_b$ onto the model space $\mathbb{K}_{b^*}$, where $b^*(z) = \overline{b(\bar{z})}$. This operator is used to show that, in the case when b is an extreme point of the closed unit ball of H^∞, then X_b and $X_{b^*}^*$ are unitarily equivalent. In Section 19.4, we construct a contraction from $\mathcal{H}(b)$ to $\mathcal{H}(b^*)$. This contraction is used to give an optimal estimate of the norm of difference quotients. In Section 19.5, we study the effect on an $\mathcal{H}(b)$ space when we change b into $b_\lambda \circ b$, where b_λ is an automorphism of the disk. Finally, in Sections 19.6 and 19.7, we give our first applications of $\mathcal{H}(b)$ theory. The first one is a new proof of the Littlewood subordination theorem based on the theory of reproducing kernels. The second one is a Hilbert space approach to some generalizations of the classic Schwarz–Pick estimates.

19.1 Abstract functional embedding

We introduce the notion of abstract functional embedding, which will enable us to have a more geometric approach to the $\mathcal{H}(b)$ spaces. An *abstract functional embedding* (AFE) is a linear mapping

$$\Pi = (\pi, \pi_*) : L^2(\mathbb{T}) \oplus L^2(\mathbb{T}) \longrightarrow \mathbb{H},$$

which satisfies the following properties.

(i) The mappings π and π_* are isometries.
(ii) $\pi H^2 \perp \pi_*(H^2_-)$.
(iii) The range of Π is dense in $\mathbb{H}$.
(iv) $\pi_*^*\pi$ commutes with the shift operator and maps H^2 into H^2.

The above properties ensure that $\pi_*^*\pi = b$, where b is a function in the closed unit ball of H^∞.

Put $\Delta(\zeta) = (1 - |b(\zeta)|)^{1/2}$, $\zeta \in \mathbb{T}$, and denote by $\mathrm{Clos}(\Delta H^2)$ (respectively $\mathrm{Clos}(\Delta L^2)$) the closure of ΔH^2 (respectively ΔL^2) in $L^2 = L^2(\mathbb{T})$.

Lemma 19.1 *For every $f \in L^2(\mathbb{T})$, we have*

$$\|(\pi - \pi_* b) f\|_{\mathbb{H}} = \|\Delta f\|_2 \quad \textit{and} \quad \|(\pi_* - \pi \bar{b}) f\|_{\mathbb{H}} = \|\Delta f\|_2.$$

Proof For the first equality, using the fact that π and π_* are isometries and $\pi_*^*\pi = b$, we have

$$\begin{aligned}\|(\pi - \pi_* b) f\|_{\mathbb{H}}^2 &= \|\pi f\|_{\mathbb{H}}^2 + \|\pi^* b f\|_{\mathbb{H}}^2 - 2\Re\langle \pi f, \pi_* b f\rangle_{\mathbb{H}} \\ &= \|f\|_2^2 + \|bf\|_2^2 - 2\|bf\|_2^2 \\ &= \|f\|_2^2 - \|bf\|_2^2 \\ &= \|\Delta f\|_2^2.\end{aligned}$$

The second equality follows along the same lines. □

For $f \in L^2(\mathbb{T})$, set

$$\tau(\Delta f) = (\pi - \pi_* b) f, \qquad \tau_*(\Delta f) = (\pi_* - \pi \bar{b}) f.$$

According to Lemma 19.1, τ and τ_* can be continued as partial isometries

$$\tau : L^2 \longrightarrow \mathbb{H}, \qquad \tau_* : L^2 \longrightarrow \mathbb{H},$$

with the initial space $\mathrm{Clos}(\Delta L^2)$.

Lemma 19.2 *With the above notation, we have*

$$\tau^*\pi = \Delta, \quad \tau^*\pi_* = 0, \quad \tau_*^*\pi = 0, \quad \tau_*^*\pi_* = \Delta, \quad \tau_*^*\tau = -b, \tag{19.1}$$

where the last relation holds only on $\mathrm{Clos}(\Delta L^2)$. *Moreover, we also have*

$$\pi\pi^* + \tau_*\tau_*^* = \pi_*\pi_*^* + \tau\tau^* = I. \tag{19.2}$$

Proof Let f and g be in $L^2(\mathbb{T})$. Then we have

$$\begin{aligned}\langle \tau^* \pi f, \Delta g\rangle_2 &= \langle \pi f, \tau \Delta g\rangle_{\mathbb{H}}\\ &= \langle \pi f,\ \pi g - \pi_* b g\rangle_{\mathbb{H}}\\ &= \langle f, g\rangle_2 - \langle bf, bg\rangle_2\\ &= \langle \Delta^2 f, g\rangle_2\\ &= \langle \Delta f, \Delta g\rangle_2.\end{aligned}$$

But $\ker \tau = (\mathrm{Clos}(\Delta L^2))^{\perp}$, whence $\mathcal{R}(\tau^*) = (\ker \tau)^{\perp} = \mathrm{Clos}(\Delta L^2)$. (Note that, since τ is a partial isometry, τ^* is also a partial isometry and, in particular, its range is closed.) We can therefore deduce that $\tau^* \pi f = \Delta f$, which gives the first relation of (19.1).

For the second relation, we have

$$\begin{aligned}\langle \tau^* \pi_* f, \Delta g\rangle_2 &= \langle \pi_* f, \tau \Delta g\rangle_{\mathbb{H}}\\ &= \langle \pi_* f,\ \pi g - \pi_* b g\rangle_{\mathbb{H}}\\ &= \langle f, bg\rangle_2 - \langle f, bg\rangle_2 = 0,\end{aligned}$$

which proves the claim.

For the third, we have

$$\begin{aligned}\langle \tau_*^* \pi f, \Delta g\rangle_2 &= \langle \pi f, \tau_* \Delta g\rangle_{\mathbb{H}}\\ &= \langle \pi f,\ \pi_* g - \pi \bar{b} g\rangle_{\mathbb{H}}\\ &= \langle bf, g\rangle_2 - \langle f, \bar{b} g\rangle_2 = 0,\end{aligned}$$

and using that we have also $\mathcal{R}(\tau_*^*) = \mathrm{Clos}(\Delta L^2)$, we get the desired relation.

For the fourth relation of (19.1), we have

$$\begin{aligned}\langle \tau_*^* \pi_* f, \Delta g\rangle_2 &= \langle \pi_* f, \tau_* \Delta g\rangle_{\mathbb{H}}\\ &= \langle \pi_* f,\ \pi_* g - \pi \bar{b} g\rangle_{\mathbb{H}}\\ &= \langle f, g\rangle_2 - \langle f, |b|^2 g\rangle_2\\ &= \langle \Delta f, \Delta g\rangle_2.\end{aligned}$$

Finally, for the last relation of (19.1), we have

$$\tau_*^* \tau \Delta f = \tau_*^* (\pi f - \pi_* b f) = \tau_*^* \pi f - \tau_*^* \pi_* b f = -\Delta b f = -b \Delta f.$$

It remains to prove (19.2). For each $f \in L^2(\mathbb{T})$, we have

$$(\pi \pi^* + \tau_* \tau_*^*) \pi f = \pi \pi^* \pi f + \tau_* \tau_*^* \pi f = \pi f$$

and

$$\begin{aligned}(\pi \pi^* + \tau_* \tau_*^*) \pi_* f &= \pi \pi^* \pi_* f + \tau_* \tau_*^* \pi_* f\\ &= \pi \bar{b} f + \tau_* \Delta f\end{aligned}$$

$$\begin{aligned} &= \pi\bar{b}f + \pi_* f - \pi\bar{b}f \\ &= \pi_* f. \end{aligned}$$

Therefore, $\pi\pi^* + \tau_*\tau_*^*$ coincide with I on $\pi L^2 \vee \pi_* L^2$, a dense set of $\mathbb{H}$, and then we get $\pi\pi^* + \tau_*\tau_*^* = I$. For the other relation of (19.2), we have

$$\begin{aligned} (\pi_*\pi_*^* + \tau\tau^*)\pi f &= \pi_*\pi_*^*\pi f + \tau\tau^*\pi f \\ &= \pi_* b f + \pi f - \pi_* b f \\ &= \pi f \end{aligned}$$

and

$$(\pi_*\pi_*^* + \tau\tau^*)\pi_* f = \pi_*\pi_*^*\pi_* f + \tau\tau^*\pi_* f = \pi_* f,$$

which gives that $\pi_*\pi_*^* + \tau\tau^* = I$ (because, as before, equality holds on a dense subset of $\mathbb{H}$). □

Corollary 19.3 *Using the above notation, the following decompositions hold:*

$$\mathbb{H} = \pi(L^2) \oplus \tau_*(L^2) = \pi_*(L^2) \oplus \tau(L^2). \tag{19.3}$$

Proof Since π and τ_* are partial isometries, the operators $\pi\pi^*$ and $\tau_*\tau_*^*$ are orthogonal projections onto $\pi(L^2)$ and $\tau_*(L^2)$, respectively. But, according to (19.1) and (19.2), it is easy to see that $\pi\pi^*$ and $\tau_*\tau_*^*$ are complementary projections. Hence, we get the first decomposition of $\mathbb{H}$. The proof of the second decomposition follows similarly. □

It is worth mentioning that, conversely, if we start with a function b in the closed unit ball of H^∞, then we can construct an AFE $\Pi = (\pi, \pi_*) : L^2(\mathbb{T}) \oplus L^2(\mathbb{T}) \longrightarrow \mathbb{H}$ such that $\pi_*^*\pi = b$. We give an example in the next section of an explicit transcription related to the construction of the Sz.-Nagy–Foiaş model for contractions on Hilbert spaces.

For a given AFE Π, we define

$$\mathbb{K} = \mathbb{H} \ominus (\pi(H^2) \oplus \pi_*(H^2_-)).$$

Thus

$$\mathbb{H} = \mathbb{K} \oplus \pi(H^2) \oplus \pi_*(H^2_-). \tag{19.4}$$

It will be useful to have a formula for the orthogonal projection onto $\mathbb{K}$.

Lemma 19.4 *With the above notation, we have*

$$P_{\mathbb{K}} = I - \pi P_+ \pi^* - \pi_* P_- \pi_*^*. \tag{19.5}$$

Proof The mappings πP_+ and $\pi_* P_-$ are partial isometries. Hence, $\pi P_+ \pi^*$ and $\pi_* P_- \pi_*^*$ are orthogonal projections onto πH^2 and $\pi_* H^2_-$, respectively, which are by (19.4) orthogonal subspaces. Therefore, $\pi P_+ \pi^* + \pi_* P_- \pi_*^*$ is the orthogonal projection onto $\pi(H^2) \oplus \pi_*(H^2_-)$, and using (19.4) once more, we get (19.5). □

The space $\mathbb{K}$ further decomposes as

$$\mathbb{K} = \mathbb{K}' \oplus \mathbb{K}'' = \mathbb{K}'_* \oplus \mathbb{K}''_*,$$

where

$$\mathbb{K}'' = \mathbb{K} \cap \tau(L^2) = \mathbb{K} \cap \tau(\text{Clos}(\Delta L^2)), \tag{19.6}$$

$$\mathbb{K}' = \mathbb{K} \ominus \mathbb{K}'', \tag{19.7}$$

$$\mathbb{K}''_* = \mathbb{K} \cap \tau_*(L^2) = \mathbb{K} \cap \tau_*(\text{Clos}(\Delta L^2)), \tag{19.8}$$

$$\mathbb{K}'_* = \mathbb{K} \ominus \mathbb{K}''_*. \tag{19.9}$$

Lemma 19.5 *We have*

$$\mathbb{K}'' = \mathbb{K} \cap (\pi_*(H^2))^\perp \quad \textit{and} \quad \mathbb{K}''_* = \mathbb{K} \cap (\pi(H^2_-))^\perp.$$

Proof Using (19.3), we have $\tau(L^2) = (\pi_*(L^2))^\perp$ and then $\mathbb{K}'' = \mathbb{K} \cap (\pi_*(L^2))^\perp$. It remains to note that, according to (19.4), $\mathbb{K} \perp \pi_*(H^2_-)$ and then $\mathbb{K}'' = \mathbb{K} \cap (\pi_*(H^2))^\perp$. The relation for $\mathbb{K}''_*$ is proved similarly. □

We will also denote, for further use,

$$\mathcal{R} = \text{Clos}(\Delta L^2) \ominus \text{Clos}(\Delta H^2), \qquad \mathcal{R}_* = \text{Clos}(\Delta L^2) \ominus \text{Clos}(\Delta H^2_-). \tag{19.10}$$

The following simple result will be used several times in the sequel.

Lemma 19.6 *Let* $\Pi = (\pi, \pi_*) : L^2(\mathbb{T}) \oplus L^2(\mathbb{T}) \longrightarrow \mathbb{H}$ *be an AFE, and let* $b = \pi_*^*\pi$ *be the function in the closed unit ball of* H^∞ *that is associated with* Π. *Then we have*

$$\mathbb{K}'' = \tau(\mathcal{R}), \qquad \mathbb{K}''_* = \tau_*(\mathcal{R}_*). \tag{19.11}$$

Consequently, the following assertions are equivalent.

(i) $\mathbb{K} = \mathbb{K}'$.
(ii) $\mathbb{K} = \mathbb{K}'_*$.
(iii) $\text{Clos}(\Delta H^2) = \text{Clos}(\Delta L^2(\mathbb{T}))$.
(iv) $\text{Clos}(\Delta H^2_-) = \text{Clos}(\Delta L^2(\mathbb{T}))$.
(v) $\Delta L^2 \cap H^2_- = \{0\}$.
(vi) b *is an extreme point of the closed unit ball of* H^∞.

Proof Let $\chi \in \mathbb{H}$ and suppose that $\chi = \tau g$ with $g \in \text{Clos}(\Delta L^2(\mathbb{T}))$. By (19.1), it follows that $\chi \perp \pi_* L^2(\mathbb{T})$. In particular, $\chi \perp \pi_*(H^2)^\perp$. Then, using (19.4), we have

$$\chi \in \mathbb{K}'' \quad\Longleftrightarrow\quad \chi \in \mathbb{K} \quad\Longleftrightarrow\quad \chi \perp \pi(H^2).$$

Using (19.1) again, we see that $\chi \perp \pi(H^2)$ is equivalent to

$$0 = \langle \tau g, \pi h \rangle = \langle g, \tau^* \pi h \rangle = \langle g, \Delta h \rangle \qquad (h \in H^2),$$

which proves the first identity of (19.11). The second identity follows from a similar argument.

(i) $\Longleftrightarrow$ (iii) We recall that τ is an isometry on $\mathrm{Clos}(\Delta L^2)$. Therefore, $\mathbb{K} = \mathbb{K}'$ if and only if $\mathbb{K}'' = \{0\}$, which is equivalent to $\mathcal{R} = \{0\}$, that is, $\mathrm{Clos}(\Delta H^2) = \mathrm{Clos}(\Delta L^2)$.

(ii) $\Longleftrightarrow$ (iv) This follows from a similar argument, using τ_*.

(iii) $\Longrightarrow$ (iv) Assume that $\mathrm{Clos}(\Delta H^2) = \mathrm{Clos}(\Delta L^2(\mathbb{T}))$ and let f be a function in $L^2(\mathbb{T})$. Given $\varepsilon > 0$, there exists $h \in H^2$ so that

$$\|\Delta \bar{z} \bar{f} - \Delta h\|_2 \le \varepsilon.$$

Thus, putting $\varphi = zh \in H_0^2$, we get

$$\|\Delta f - \Delta \bar{\varphi}\|_2 \le \varepsilon.$$

That proves that $\Delta f \in \mathrm{Clos}(\Delta H^2_-)$, and we get (iv).

(iv) $\Longrightarrow$ (iii) Assume that $\mathrm{Clos}(\Delta H^2_-) = \mathrm{Clos}(\Delta L^2(\mathbb{T}))$ and let f be a function in $L^2(\mathbb{T})$. Given $\varepsilon > 0$, there exists $h \in H^2_-$ so that

$$\|\Delta \bar{f} - \Delta h\|_2 \le \varepsilon.$$

Since $\bar{h} \in H^2$ and

$$\|\Delta f - \Delta \bar{h}\|_2 \le \varepsilon,$$

we get that $\Delta f \in \mathrm{Clos}(\Delta H^2)$. That gives (iii).

(iii) $\Longrightarrow$ (v) Let $f \in L^2$ such that $\Delta f \in H^2_-$. Then, for any $h \in H^2$, we have

$$0 = \langle \Delta f, h \rangle_2 = \langle f, \Delta h \rangle_2,$$

which means that $f \perp \mathrm{Clos}(\Delta H^2)$. By hypothesis, we deduce that $f \perp \mathrm{Clos}(\Delta L^2)$ and, in particular,

$$0 = \langle f, \Delta f \rangle_2 = \int_{\mathbb{T}} |f|^2 \Delta \, dm.$$

Since the function $|f|^2 \Delta$ is positive almost everywhere, we get that $\Delta f = 0$ a.e. on $\mathbb{T}$, and that proves that $\Delta L^2 \cap H^2_- = \{0\}$.

(v) $\Longrightarrow$ (iii) Conversely, let $f \in \mathrm{Clos}(\Delta L^2) \ominus \mathrm{Clos}(\Delta H^2)$. Then

$$\langle f, \Delta h \rangle_2 = 0$$

for any $h \in H^2$. This is equivalent to $\Delta f \in H^2_-$. Hence Δf belongs to the space $H^2_- \cap \Delta L^2$, which is assumed to be $\{0\}$. Hence $\Delta f = 0$, which gives that $\mathrm{Clos}(\Delta L^2) \ominus \mathrm{Clos}(\Delta H^2) = \{0\}$.

Now, it remains to prove that (iii) is equivalent to (vi). We denote by $\rho = 1 - |b|^2 = \Delta^2$. Then we clearly have

$$\|\Delta f\|_2 = \|f\|_{L^2(\rho)}, \tag{19.12}$$

for every function f in $L^2(\rho) = \Delta^{-1} L^2(\mathbb{T})$.

(iii) $\Longrightarrow$ (vi) Fix $\varepsilon > 0$. Since $\Delta\bar{z} \in \Delta L^2(\mathbb{T})$, we can find $f \in H^2$ such that

$$\|\Delta\bar{z} - \Delta f\|_2 \leq \varepsilon/2.$$

Using (19.12), we then get $\|\bar{z} - f\|_{L^2(\rho)} \leq \varepsilon/2$. Since f is in H^2, there is an analytic polynomial $p \in \mathcal{P}_+$ such that $\|f - p\|_2 \leq \varepsilon/2$, and then $\|f - p\|_{L^2(\rho)} \leq \varepsilon/2$ (note that ρ is bounded by 1). Finally, we have

$$\|\bar{z} - p\|_{L^2(\rho)} \leq \|\bar{z} - f\|_{L^2(\rho)} + \|f - p\|_{L^2(\rho)} \leq \varepsilon,$$

which proves that $\bar{z} \in H^2(\rho)$. It remains to apply Corollary 13.34 to obtain that b is an extreme point of the closed unit ball of H^∞.

(vi) $\Longrightarrow$ (iii) Fix $\varepsilon > 0$, and let $f \in L^2(\mathbb{T})$. There is a trigonometric polynomial q such that $\|f - q\|_2 \leq \varepsilon/2$, and thus also $\|\Delta f - \Delta q\|_2 \leq \varepsilon/2$. Since q is bounded, q belongs to $L^2(\rho)$ and then also in $H^2(\rho)$ by Corollary 13.34. This means that there is an analytic polynomial $p \in \mathcal{P}_+$ such that $\|p - q\|_{L^2(\rho)} \leq \varepsilon/2$, and thus $\|\Delta p - \Delta q\|_2 \leq \varepsilon/2$. Finally, we get

$$\|\Delta f - \Delta p\|_2 \leq \|\Delta f - \Delta q\|_2 + \|\Delta q - \Delta p\|_2 \leq \varepsilon,$$

which proves that $\Delta L^2 \in \operatorname{Clos}(\Delta H^2)$ and thus $\operatorname{Clos}(\Delta H^2) = \operatorname{Clos}(\Delta L^2)$. $\square$

Now, we explore the connection between the abstract functional embedding with $\mathcal{H}(b)$ spaces.

Lemma 19.7 *Let $\Pi = (\pi, \pi_*) : L^2(\mathbb{T}) \oplus L^2(\mathbb{T}) \longrightarrow \mathbb{H}$ be an AFE, and let $b = \pi_*^* \pi$ be the function in the closed unit ball of H^∞ associated with Π. Then*

$$I - T_b T_b^* = \pi_*^* P_{\mathbb{K}} \pi_{*|H^2} = \pi_*^* P_{\mathbb{K}'} \pi_{*|H^2}.$$

Proof Using (19.5) as well as the relations $\pi_*^* \pi_* = Id$ and $\pi_*^* \pi = b$, we obtain

$$\pi_*^* P_{\mathbb{K}} \pi_* = \pi_*^* \pi_* - \pi_*^* \pi P_+ \pi^* \pi_* - \pi_*^* \pi_* P_- \pi_*^* \pi_* = P_+ - b P_+ b^*.$$

Hence,

$$\pi_*^* P_{\mathbb{K}} \pi_{*|H^2} = (P_+ - b P_+ b^*)_{|H^2} = I - T_b T_b^*.$$

The second equality follows since, according to (19.1), $\mathbb{K} \ominus \mathbb{K}' = \mathbb{K}'' = \tau(\mathcal{R})$ is contained in the kernel of π_*^*. $\square$

Theorem 19.8 *Let $\Pi = (\pi, \pi_*) : L^2(\mathbb{T}) \oplus L^2(\mathbb{T}) \longrightarrow \mathbb{H}$ be an AFE, and let $b = \pi_*^* \pi$ be the function in the closed unit ball of H^∞ associated with Π. Then the operator $\pi_{*\,|\mathbb{K}}^*$ is a partial isometry from $\mathbb{K}$ onto $\mathcal{H}(b)$, with $\ker \pi_{*\,|\mathbb{K}}^* = \mathbb{K}''$. In particular, $\pi_*^* : \mathbb{K} \longrightarrow \mathcal{H}(b)$ is unitary if and only if b is an extreme point of the closed unit ball of H^∞.*

Proof First, take $f \in \mathbb{K}$ and $g \in (H^2)^\perp$. Then, using (19.4), we have

$$\langle \pi_*^* f, g \rangle_2 = \langle f, \pi_* g \rangle_{\mathbb{H}} = 0,$$

whence $\pi_*^* \mathbb{K} \subset H^2$. Then, if we put $A = \pi_*^* |\mathbb{K} : \mathbb{K} \longrightarrow H^2$ and $B = (Id - T_b T_b^*)^{1/2}$, Lemma 19.7 shows that $AA^* = B^2$ and Corollary 16.8 implies that $\mathcal{M}(A) = \mathcal{H}(b)$. Thus, $\pi_*^*|\mathbb{K}$ is a partial isometry from $\mathbb{K}$ onto $\mathcal{H}(b)$. Its kernel is

$$\ker \pi_*^* |\mathbb{K} = \mathbb{K} \cap \ker \pi_*^* = \mathbb{K} \cap (\pi_*(L^2))^\perp = \mathbb{K}'',$$

where we used (19.3). The proof is ended using Lemma 19.6. □

Exercises

Exercise 19.1.1 Let b be a function in the closed unit ball of H^∞, and let $\Pi = (\pi, \pi_*) : L^2(\mathbb{T}) \oplus L^2(\mathbb{T}) \longrightarrow \mathbb{H}$ be an AFE such that $\pi_*^* \pi = b$. Our goal is to show that the following assertions are equivalent.

(i) $\mathrm{Span}(\hat{k}_\lambda^b : \lambda \in \mathbb{D}) = \mathcal{H}(b)$.
(ii) $\mathbb{K}' \cap \mathbb{K}_*'' = \{0\}$.
(iii) $\mathbb{K}'' \vee \mathbb{K}_*' = \mathbb{K}$.
(iv) $\ker(P_{\mathcal{R}} \bar{b} | \mathcal{R}_*) = \{0\}$ ($\mathcal{R}$ and $\mathcal{R}_*$ defined by (19.10)).

To do so, take the following steps.

(I) Let $\xi_n = P_{\mathbb{K}} \pi(\bar{z}^{n+1})$, $n \geq 0$. Verify that

$$\mathrm{Span}(\xi_n : n \geq 0) = \mathrm{Clos}(P_{\mathbb{K}} \pi (H^2)^\perp).$$

(II) Put $\eta_n = P_{\mathbb{K}'} \xi_n$. Show that

$$\pi_*^* \eta_n = \pi_*^* P_{\mathbb{K}} \pi(\bar{z}^{n+1})$$

and that

$$\pi_*^* \eta_n = S^{*n+1} b.$$

Hint: For the second formula, use (19.5) to show that $\pi_*^* P_{\mathbb{K}} \pi = P_+ b - b P_+$.

(III) Conclude that π_*^* is a unitary operator from $\mathrm{Span}(\xi_n : n \geq 0)$ onto $\mathrm{Span}({S^*}^{n+1}b : n \geq 0)$ and then show that (i) $\Longleftrightarrow$ (ii).
Hint: Use Corollary 18.20

(IV) Prove that (ii) $\Longleftrightarrow$ (iii).

(V) Prove that (ii) $\Longleftrightarrow$ (iv).
Hint: Use Lemma 19.6.

Exercise 19.1.2 In this exercise, we obtain an answer to the problem of the completeness of a family of difference quotients based on the AFE. We will show in Theorem 24.35 and Corollary 26.18 how to recover this result by a direct method.

(i) Assuming that b is an extreme point of the closed unit ball of H^∞, use Exercise 19.1.1 to show that $\mathrm{Span}(\hat{k}_\lambda^b : \lambda \in \mathbb{D}) = \mathcal{H}(b)$.

(ii) For the rest of the exercise, we assume that b is not an extreme point of the closed unit ball of H^∞ and we would like to prove that $\mathrm{Span}(\hat{k}_\lambda^b : \lambda \in \mathbb{D}) = \mathcal{H}(b)$ if and only if b is not pseudocontinuable.

(a) Show that $\mathrm{Clos}(\Delta L^2(\mathbb{T})) = L^2(\mathbb{T})$.
Hint: Use the fact that b is not an extreme point of the closed unit ball of H^∞.

(b) Show that there is an outer function $\phi \in H^2$, with $|\phi| = \Delta$, such that

$$\mathrm{Clos}(\Delta H^2) = \frac{\Delta}{\phi} H^2.$$

Hint: Use Theorem 8.30.

(c) Show that

$$\mathcal{R} = L^2 \ominus \frac{\Delta}{\phi} H^2 \quad \text{and} \quad \mathcal{R}_* = L^2 \ominus \mathrm{Clos}(\Delta H_-^2).$$

(d) Assume that $\mathrm{Span}(\hat{k}_\lambda^b : \lambda \in \mathbb{D}) \neq \mathcal{H}(b)$. Using Exercise 19.1.1, show that there exists $h \in L^2(\mathbb{T})$, $h \neq 0$, such that $g = \Delta h \in H^2$ and $\bar{b}h = (\Delta/\phi)h_1$, with $h_1 \in H^2$.

(e) Deduce that $b = \bar{h}_1/(\overline{\phi g + b h_1})$ and then that b is pseudocontinuable.

(f) Conversely, assume that b is pseudocontinuable. Show that there is an inner function Θ such that $h_1 = \bar{b}\,\Theta \in H^2$.

(g) Let $g = (\Theta - b h_1)/\phi$. Show that g belongs to H^2.

(h) Put $h = g/\Delta$. Prove that $h \in \ker(P_{\mathcal{R}} \bar{b}_{|\mathcal{R}_*})$ and conclude that $\mathrm{Span}(\hat{k}_\lambda^b : \lambda \in \mathbb{D}) \neq \mathcal{H}(b)$.

19.2 A geometric representation of $\mathcal{H}(b)$

We recall that b is a function in the closed unit ball of H^∞, $\Delta = (1 - |b|^2)^{1/2}$ and $\mathrm{Clos}(\Delta H^2)$ is the closure of ΔH^2 in $L^2(\mathbb{T})$. By the same token, $\mathrm{Clos}(\Delta L^2)$ is the closure of ΔL^2 in $L^2(\mathbb{T})$. Put

$$\mathbb{H}_b = L^2 \oplus \mathrm{Clos}(\Delta L^2),$$

and define

$$\begin{array}{rccc} \pi: & L^2 & \longrightarrow & \mathbb{H}_b \\ & f & \longmapsto & bf \oplus \Delta f \end{array}$$

and

$$\begin{array}{rccc} \pi_*: & L^2 & \longrightarrow & \mathbb{H}_b \\ & g & \longmapsto & g \oplus 0. \end{array}$$

Theorem 19.9 *The linear mapping $\Pi = (\pi, \pi_*) : L^2 \oplus L^2 \longrightarrow \mathbb{H}_b$ is an AFE.*

Proof Since, for each $f \in L^2$, we have

$$\begin{aligned} \|bf \oplus \Delta f\|_2^2 &= \|bf\|_2^2 + \|\Delta f\|_2^2 \\ &= \int_{\mathbb{T}} |bf|^2 \, dm + \int_{\mathbb{T}} (1 - |b|^2)|f|^2 \, dm \\ &= \|f\|_2^2, \end{aligned}$$

we get that π is an isometry. The map π_* is also clearly an isometry and it is easy to check that

$$\pi_*^*(h_1 \oplus h_2) = h_1, \qquad h_1 \oplus h_2 \in L^2 \oplus \mathrm{Clos}(\Delta L^2). \tag{19.13}$$

Now, let $f \in H^2$ and $g \in H^2_-$. Then

$$\langle \pi f, \pi_* g \rangle_{\mathbb{H}_b} = \langle bf \oplus \Delta f, \ g \oplus 0 \rangle_{\mathbb{H}_b} = \langle bf, g \rangle_2,$$

and the last scalar product is zero because $bf \in H^2$ and $g \in H^2_-$. This observation reveals that $\pi H^2 \perp \pi_* H^2_-$. By (19.13), we also have

$$\pi_*^* \pi f = \pi_*^*(bf \oplus \Delta f) = bf.$$

Thus $\pi_*^* \pi$ is the multiplication operator by b and, in particular, it commutes with the shift operator and maps H^2 into H^2. Finally, it remains to note that the range of Π is dense in $\mathbb{H}_b$. $\square$

Let $\mathbb{K}_b$ be the subspace defined by (19.4), and let $\mathbb{K}_b'$ and $\mathbb{K}_b''$ be the subspaces defined by (19.7) and (19.6). It will be useful to have the following more explicit transcriptions.

Lemma 19.10 *Let b be a function in the closed unit ball of H^∞. Then we have*

$$\mathbb{K}_b = (H^2 \oplus \mathrm{Clos}(\Delta L^2)) \ominus \{bf \oplus \Delta f : f \in H^2\}, \tag{19.14}$$
$$\mathbb{K}_b'' = \{0\} \oplus (\mathrm{Clos}(\Delta L^2) \ominus \mathrm{Clos}(\Delta H^2)), \tag{19.15}$$
$$\mathbb{K}_b' = (H^2 \oplus \mathrm{Clos}(\Delta H^2)) \ominus \{bf \oplus \Delta f : f \in H^2\}. \tag{19.16}$$

Proof Recall that

$$\mathbb{K}_b = (L^2 \oplus \mathrm{Clos}(\Delta L^2)) \ominus (\pi(H^2) \oplus \pi_*(H^2_-)).$$

First note that

$$\{bf \oplus \Delta f : f \in H^2\} = \pi(H^2),$$

and since π is an isometry, this space is a closed subspace of $H^2 \oplus \mathrm{Clos}(\Delta L^2)$. Now let $\varphi \oplus \psi \in L^2 \oplus \mathrm{Clos}(\Delta L^2)$. Then $\varphi \oplus \psi \in \mathbb{K}_b$ if and only if

$$\varphi \oplus \psi \perp \{bf \oplus \Delta f : f \in H^2\}$$

and

$$\varphi \oplus \psi \perp \pi_*(H^2_-).$$

The second condition gives that, for any $h \in H^2_-$, we have

$$0 = \langle \varphi \oplus \psi,\ \pi_*(h)\rangle_{\mathbb{H}_b} = \langle \varphi \oplus \psi,\ h \oplus 0\rangle_{\mathbb{H}_b} = \langle \varphi, h\rangle_2.$$

This condition is thus equivalent to $\varphi \in H^2$, which concludes the proof of (19.14).

Let us prove now (19.15). According to Lemma 19.5, we have

$$\mathbb{K}_b'' = \mathbb{K}_b \cap (\pi_*(H^2))^\perp.$$

Then it is clear that $\{0\} \oplus (\mathrm{Clos}(\Delta L^2) \ominus \mathrm{Clos}(\Delta H^2)) \subset \mathbb{K}_b''$. Conversely, if $\varphi \oplus \psi \in \mathbb{K}_b''$, using (19.14), we first have $\varphi \in H^2$ and

$$\varphi \oplus \psi \perp bf \oplus \Delta f \qquad (\forall\, f \in H^2). \tag{19.17}$$

On the other hand, since $\varphi \oplus \psi \perp \pi_*(H^2)$, that gives $\varphi \oplus \psi \perp f \oplus 0$, for any $f \in H^2$. Hence, $\langle \varphi, f\rangle_2 = 0$, $f \in H^2$, which implies that $\varphi \perp H^2$. But, since φ also belongs to H^2, we get that $\varphi = 0$. Now, if we use (19.17), we obtain

$$\langle \psi, \Delta f\rangle_2 = 0 \qquad (f \in H^2),$$

which gives that $\psi \perp \mathrm{Clos}(\Delta H^2)$. Hence $\psi \in \mathrm{Clos}(\Delta L^2) \ominus \mathrm{Clos}(\Delta H^2)$, and this proves (19.15).

The equation (19.16) follows immediately from (19.14) and (19.15) and the fact that $\mathbb{K}_b = \mathbb{K}_b' \oplus \mathbb{K}_b''$. □

Recall that Z denotes the forward shift operator on $L^2 = L^2(\mathbb{T})$, that is

$$\begin{array}{rccc} Z: & L^2 & \longrightarrow & L^2 \\ & f & \longmapsto & \chi_1 f, \end{array}$$

where $\chi_1(\zeta) = \zeta$, $\zeta \in \mathbb{T}$. It is immediate to see that $\mathrm{Clos}(\Delta H^2)$ and $\mathrm{Clos}(\Delta L^2)$ are invariant with respect to Z. We will denote by V_Δ and Z_Δ the corresponding restrictions of Z to these spaces, that is

$$\begin{array}{rccc} V_\Delta: & \mathrm{Clos}(\Delta H^2) & \longrightarrow & \mathrm{Clos}(\Delta H^2) \\ & f & \longmapsto & \chi_1 f \end{array}$$

and

$$\begin{array}{rccc} Z_\Delta: & \mathrm{Clos}(\Delta L^2) & \longrightarrow & \mathrm{Clos}(\Delta L^2) \\ & f & \longmapsto & \chi_1 f. \end{array}$$

We can easily check that V_Δ is an isometry, Z_Δ is unitary and $Z_\Delta^* f = \chi_{-1} f$, where $\chi_{-1}(\zeta) = \bar{\zeta}$, $\zeta \in \mathbb{T}$.

Theorem 19.11 *Let b be a function in the closed unit ball of H^∞. Then the following hold.*

(i) *$S \oplus V_\Delta$ is an isometry on $H^2 \oplus \mathrm{Clos}(\Delta H^2)$.*
(ii) *The space $\mathbb{K}_b'$ is invariant with respect to $S^* \oplus V_\Delta^*$.*
(iii) *The projection $\pi_*^* : L^2 \oplus \mathrm{Clos}(\Delta L^2) \longrightarrow L^2$ on the first coordinate maps unitarily $\mathbb{K}_b'$ onto $\mathcal{H}(b)$, and if Q_b is the unitary map defined by*

$$\begin{array}{rccc} Q_b: & \mathbb{K}_b' & \longrightarrow & \mathcal{H}(b) \\ & f \oplus g & \longmapsto & f, \end{array}$$

then we have

$$Q_b(S^* \oplus V_\Delta^*) = X_b Q_b \qquad \text{on } \mathbb{K}_b'.$$

Proof (i) This is immediate.

(ii) Let $f \in H^2$. Then we have

$$(S \oplus V_\Delta)\pi f = (S \oplus V_\Delta)(bf \oplus \Delta f) = b(\chi_1 f) \oplus \Delta(\chi_1 f) = \pi(\chi_1 f).$$

Since $\chi_1 f \in H^2$, the space πH^2 is invariant under $S \oplus V_\Delta$, whence its orthogonal in $H^2 \oplus \mathrm{Clos}(\Delta H^2)$, which is $\mathbb{K}_b'$ by (19.16), is invariant with respect to $S^* \oplus V_\Delta^*$.

(iii) The fact that π_*^* maps unitarily $\mathbb{K}_b'$ onto $\mathcal{H}(b)$ follows immediately from Theorem 19.8. Now let $f \oplus g \in \mathbb{K}_b'$. Then

$$Q_b(S^* \oplus V_\Delta^*)(f \oplus g) = Q_b(P_+(\bar{z}f) \oplus V_\Delta^* g) = P_+(\bar{z}f).$$

But $f = Q_b(f \oplus g) \in \mathcal{H}(b)$ and then we can write $P_+(\bar{z}f) = X_b f = X_b Q_b(f \oplus g)$, which gives the desired relation. □

Part (iii) of Theorem 19.11 says that the operator X_b is unitarily equivalent to $(S^* \oplus V_\Delta^*)_{|\mathbb{K}_b'}$ and the unitary equivalence is provided by the unitary map Q_b, which can be rephrased in the following commutative diagram.

$$\begin{tikzcd} \mathbb{K}_b' \arrow[r, "Q_b"] \arrow[d, "S^*\oplus V_\Delta^*"'] & \mathcal{H}(b) \arrow[d, "X_b"] \\ \mathbb{K}_b' \arrow[r, "Q_b"'] & \mathcal{H}(b) \end{tikzcd} \tag{19.18}$$

As suggested by Theorem 19.11, the operator $\mathbf{M}_b \in \mathcal{L}(\mathbb{K}_b)$, defined by the formula

$$\mathbf{M}_b = P_{\mathbb{K}_b}(S \oplus Z_\Delta)_{|\mathbb{K}_b}, \tag{19.19}$$

plays an important role. It is easy to see that $\pi(H^2)$ is invariant with respect to $S \oplus Z_\Delta$, whence $\mathbb{K}_b$ is invariant with respect to $S^* \oplus Z_\Delta^*$ and we get that

$$\mathbf{M}_b^* = (S^* \oplus Z_\Delta^*)_{|\mathbb{K}_b}.$$

The operator $\mathbf{M}_b$ is called the *model operator* and the space $\mathbb{K}_b$ is called the *model space*.

It should be noted that, in the inner case, we have already introduced these objects in Chapter 14. Indeed, when b is inner, then $\Delta = 0$ a.e. on $\mathbb{T}$, and $\mathbb{K}_b$ reduces to $H^2 \ominus bH^2 = K_b$. In that case, $\mathbf{M}_b$ also reduces to $P_{K_b}S_{|K_b}$, an operator that was introduced in (14.21).

In the case when b is extreme, then $\operatorname{Clos}(\Delta H^2) = \operatorname{Clos}(\Delta L^2)$, $V_\Delta = Z_\Delta$ and $\mathbb{K}_b' = \mathbb{K}_b$. Then, Theorem 19.11 says exactly that X_b is unitarily equivalent to the adjoint of the model operator $\mathbf{M}_b$. In other words, in the case when b is extreme, (19.18) can be rephrased as follows:

$$\begin{tikzcd} \mathbb{K}_b \arrow[r, "Q_b"] \arrow[d, "\mathbf{M}_b^*"'] & \mathcal{H}(b) \arrow[d, "X_b"] \\ \mathbb{K}_b \arrow[r, "Q_b"'] & \mathcal{H}(b) \end{tikzcd} \tag{19.20}$$

Remark 19.12 The fact that Q_b is a unitary map from $\mathbb{K}_b'$ onto $\mathcal{H}(b)$ can also be deduced from Theorem 16.30. Indeed, let $A = T_b$. Then $\mathcal{H}(A) = \mathcal{H}(b)$ and we have

$$\mathfrak{J}(A) = \begin{bmatrix} T_b & D_{T_{\bar{b}}} \\ D_{T_b} & -T_{\bar{b}} \end{bmatrix} = \begin{bmatrix} T_b & (I - T_b T_{\bar{b}})^{1/2} \\ (I - T_{\bar{b}} T_b)^{1/2} & -T_{\bar{b}} \end{bmatrix}.$$

With the notation of Theorem 16.30, we have

$$\mathcal{X}_1 = \mathfrak{J}(A)(H^2 \oplus \{0\}) = \{T_b f \oplus (I - T_{\bar{b}} T_b)^{1/2} f : f \in H^2\}$$

and

$$\mathcal{X}_2 = (H^2 \mathcal{D}_{T_b}) \ominus \{T_b f \oplus (I - T_{\bar{b}} T_b)^{1/2} f : f \in H^2\}.$$

Theorem 16.30 says that the projection of $H^2 \oplus \mathcal{D}_{T_b}$ onto its first coordinate maps unitarily $\mathcal{X}_2$ onto $\mathcal{H}(b)$. Since, for each $f \in H^2$,

$$\begin{aligned}\|D_{T_b} f\|_2^2 &= \|f\|_2^2 - \|bf\|_2^2 \\ &= \frac{1}{2\pi} \int_0^{2\pi} (1 - |b(e^{i\vartheta})|^2)\, |f(e^{i\vartheta})|^2 \, d\vartheta = \|\Delta f\|_2^2,\end{aligned}$$

the map $D_{T_b} f \longmapsto \Delta f$ extends to a unitary map $U : \mathcal{D}_{T_b} \longrightarrow \mathrm{Clos}(\Delta H^2)$. Then $I_{H^2} \oplus U$ maps unitarily $H^2 \oplus \mathcal{D}_{T_b}$ onto $H^2 \oplus \mathrm{Clos}(\Delta H^2)$, $\mathcal{X}_1$ onto $\{bf \oplus \Delta f : f \in H^2\}$ and $\mathcal{X}_2$ onto $\mathbb{K}_b'$. Since the operator $I_{H^2} \oplus U$ commutes with the projection on the first coordinate, we deduce that Q_b maps unitarily $\mathbb{K}_b'$ onto $\mathcal{H}(b)$.

19.3 A unitary operator from $\mathbb{K}_b$ onto $\mathbb{K}_{b^*}$

In Section 14.8, we associated with any inner function Θ a conjugate inner function Θ^* and showed that the adjoint of the model operator $\mathbf{M}_\Theta$ is unitarily equivalent to the model operator $\mathbf{M}_{\Theta^*}$ (see Theorem 14.31). Our purpose is now to generalize this result to model operators $\mathbf{M}_b$ where b is any function in the closed unit ball of H^∞.

With each function f in $L^2(\mathbb{T})$, we associate the function f^* defined by

$$f^*(\zeta) = \overline{f(\bar{\zeta})} \qquad (\zeta \in \mathbb{T}).$$

It is easy to see that $f \longmapsto f^*$ is a conjugation on L^2 that maps H^2 onto H^2. Moreover, if $f \in H^2$, then

$$f^*(z) = \overline{f(\bar{z})} \qquad (z \in \mathbb{D}).$$

Now if b is a function in the closed unit ball of H^∞, then b^* is also a function in the closed unit ball of H^∞, and it is easily seen that b is an extreme point if and only if b^* is an extreme point. In the present section, we explain how to construct a unitary map from $\mathbb{K}_b$ onto $\mathbb{K}_{b^*}$.

We recall that $\Delta = (1 - |b|^2)^{1/2}$ and $\Delta_* = (1 - |b^*|^2)^{1/2}$. Then it is easy to see that $f \longmapsto f^*$ maps $\mathrm{Clos}(\Delta L^2)$ onto $\mathrm{Clos}(\Delta_* L^2)$.

To construct the unitary map from $\mathbb{K}_b$ onto $\mathbb{K}_{b^*}$, we need to introduce another operator on $L^2(\mathbb{T})$. For each function $f \in L^2(\mathbb{T})$, we define

$$(\mathfrak{I} f)(\zeta) = \bar{\zeta} f(\bar{\zeta}) \qquad (\zeta \in \mathbb{T}).$$

This operator is closely related to the conjugation J on L^2, which we introduced in Section 14.6. More precisely, we recall that

$$(Jf)(\zeta) = \overline{\zeta f(\zeta)} \qquad (f \in L^2,\ \zeta \in \mathbb{T}),$$

and then an immediate computation shows that

$$\mathfrak{J}f = (Jf)^* \qquad (f \in L^2).$$

In other words, $\mathfrak{J}$ is the composition of the two conjugations J and $f \longmapsto f^*$. We can easily check that $\mathfrak{J}$ is unitary on L^2, it interchanges H^2 and H^2_-, and

$$\mathfrak{J}P_- = P_+\mathfrak{J}. \tag{19.21}$$

Furthermore, $\mathfrak{J}$ maps $\mathrm{Clos}(\Delta L^2)$ onto $\mathrm{Clos}(\Delta_* L^2)$.

Now, for $b \in H^\infty$, $\|b\|_\infty \le 1$, we recall that M_b denotes the operator of multiplication by b on L^2, that is,

$$\begin{array}{cccc} M_b: & L^2 & \longrightarrow & L^2 \\ & f & \longmapsto & bf. \end{array}$$

The reader should be careful not to confuse this with the model operator $\mathbf{M}_b$ introduced in (19.19). As a consequence of the analyticity of b, we have $M_b H^2 \subset H^2$ and $M_{b|H^2} = T_b$. The operator M_b is a contraction on L^2 and its defect can be easily computed. Indeed, we have

$$D_{M_b} f = (1 - |b|^2)^{1/2} f = \Delta f \qquad (f \in L^2). \tag{19.22}$$

Hence, it follows that $\mathcal{D}_{M_b} = \mathrm{Clos}(\Delta L^2)$. Notice also that, since Δ is real-valued, then $D^*_{M_b} f = D_{M_b} f = \Delta f$, $f \in L^2$. Then, according to Theorem 16.30, the Julia operator $\mathfrak{J}(M_{b^*})$ is a unitary operator from $L^2 \oplus \mathrm{Clos}(\Delta_* L^2)$ onto $L^2 \oplus \mathrm{Clos}(\Delta_* L^2)$.

Now let us denote by $\mathfrak{W}_b : L^2 \oplus \mathrm{Clos}(\Delta L^2) \longrightarrow L^2 \oplus \mathrm{Clos}(\Delta_* L^2)$ the operator defined by

$$\mathfrak{W}_b(f \oplus g) = \mathfrak{J}(M_{b^*})(\mathfrak{J}f \oplus \mathfrak{J}g), \qquad f \oplus g \in L^2 \oplus \mathrm{Clos}(\Delta L^2).$$

In other words, we have the following diagram.

$$\begin{tikzcd} L^2 \oplus \mathrm{Clos}(\Delta L^2) \arrow[r, "\mathfrak{J}\oplus\mathfrak{J}"] \arrow[dr, "\mathfrak{W}_b"'] & L^2 \oplus \mathrm{Clos}(\Delta_* L^2) \arrow[d, "\mathfrak{J}(M_{b^*})"] \\ & L^2 \oplus \mathrm{Clos}(\Delta_* L^2) \end{tikzcd} \tag{19.23}$$

We can give a more explicit formula for $\mathfrak{W}_b$, which in fact leads to the desired unitarity between $\mathbb{K}_b$ and $\mathbb{K}_{b^*}$.

Theorem 19.13 *Let b be in the closed unit ball of H^∞. Then the following hold.*

(i) $\mathfrak{W}_b$ *is a unitary operator from* $L^2 \oplus \mathrm{Clos}(\Delta L^2)$ *onto* $L^2 \oplus \mathrm{Clos}(\Delta_* L^2)$.

(ii) *For any* $f \oplus g \in L^2 \oplus \mathrm{Clos}(\Delta L^2)$, *we have*

$$\mathfrak{W}_b(f \oplus g) = (b^*(\zeta)\bar{\zeta}f(\bar{\zeta}) + \Delta_*(\zeta)\bar{\zeta}g(\bar{\zeta})) \oplus (\Delta_*(\zeta)\bar{\zeta}f(\bar{\zeta}) - b(\bar{\zeta})\bar{\zeta}g(\bar{\zeta})).$$

(iii) $\mathfrak{W}_b$ *maps unitarily* $\mathbb{K}_b$ *onto* $\mathbb{K}_{b^*}$.

(iv) $\mathfrak{W}_b^* = \mathfrak{W}_{b^*}$.

Proof (i) Since $\mathfrak{J}(M_{b^*})$ is a unitary map on $L^2 \oplus \mathrm{Clos}(\Delta_* L^2)$ and $\mathfrak{J}$ is a unitary operator from $\mathrm{Clos}(\Delta L^2)$ onto $\mathrm{Clos}(\Delta_* L^2)$, the first property follows immediately from the definition of $\mathfrak{W}_b$.

(ii) Recall that

$$\mathfrak{J}(M_{b^*}) = \begin{bmatrix} M_{b^*} & D^*_{M_{b^*}} \\ D_{M_{b^*}} & -M_{\overline{b^*}} \end{bmatrix}.$$

Thus, for any $f \oplus g \in L^2 \oplus \mathrm{Clos}(\Delta L^2)$, it follows from (19.22) and (19.23) that

$$\mathfrak{W}_b(f \oplus g) = \begin{bmatrix} b^*(\zeta)\bar{\zeta}f(\bar{\zeta}) + \Delta_*(\zeta)\bar{\zeta}g(\bar{\zeta}) \\ \Delta_*(\zeta)\bar{\zeta}g(\bar{\zeta}) - b(\bar{\zeta})\bar{\zeta}g(\bar{\zeta}) \end{bmatrix},$$

which gives (ii).

(iii) Since the map $\mathfrak{J}$ is unitary from H^2_- onto H^2, we have

$$\mathfrak{W}_b(H^2_- \oplus \{0\}) = \{b^*h \oplus \Delta_* h : h \in H^2\}. \tag{19.24}$$

On the other hand, for any $h \in H^2$, we have

$$\mathfrak{W}_b(bh \oplus \Delta h) = \begin{bmatrix} b^*(\zeta)\bar{\zeta}b(\bar{\zeta})h(\bar{\zeta}) + \Delta_*(\zeta)\bar{\zeta}\Delta(\bar{\zeta})h(\bar{\zeta}) \\ \Delta_*(\zeta)\bar{\zeta}b(\bar{\zeta})h(\bar{\zeta}) - b(\bar{\zeta})\bar{\zeta}\Delta(\bar{\zeta})h(\bar{\zeta}) \end{bmatrix} = \begin{bmatrix} \bar{\zeta}h(\bar{\zeta}) \\ 0 \end{bmatrix},$$

which proves that

$$\mathfrak{W}_b(\{bf \oplus \Delta f : f \in H^2\}) = H^2_- \oplus \{0\}.$$

Using (19.14) and the fact that $\mathfrak{W}_b$ is unitary, we get

$$\mathfrak{W}_b\mathbb{K}_b = \mathbb{K}_{b^*}.$$

(iv) Since $\mathfrak{W}_b = \mathfrak{J}(M_{b^*})(\mathfrak{J} \oplus \mathfrak{J})$, we have

$$\mathfrak{W}_b^* = (\mathfrak{J}^* \oplus \mathfrak{J}^*)\mathfrak{J}(M_{b^*})^*.$$

But, by (16.42), we have

$$\mathfrak{J}(M_{b^*})^* = \mathfrak{J}(M_{\overline{b^*}}) = \begin{bmatrix} M_{\overline{b^*}} & D_{M_{b^*}} \\ D_{M_{b^*}} & -M_{b^*} \end{bmatrix}.$$

On the other hand, an obvious computation yields $\mathfrak{J}^* = \mathfrak{J}$. Hence, for any $f \oplus g \in L^2 \oplus \mathrm{Clos}(\Delta_* L^2)$, we have

$$\begin{aligned}\mathfrak{W}_b^*(f \oplus g) &= (\mathfrak{J} \oplus \mathfrak{J}) \begin{bmatrix} \overline{b^*(\zeta)} f(\zeta) + \Delta_*(\zeta) g(\zeta) \\ \Delta_*(\zeta) f(\zeta) - b^*(\zeta) g(\zeta) \end{bmatrix} \\ &= \begin{bmatrix} \bar{\zeta}\overline{b^*(\bar{\zeta})} f(\bar{\zeta}) + \Delta_*(\bar{\zeta})\bar{\zeta} g(\bar{\zeta}) \\ \bar{\zeta}\Delta_*(\bar{\zeta}) f(\bar{\zeta}) - b^*(\bar{\zeta})\bar{\zeta} g(\bar{\zeta}) \end{bmatrix} \\ &= \begin{bmatrix} \bar{\zeta} b(\zeta) f(\bar{\zeta}) + \Delta(\zeta)\bar{\zeta} g(\bar{\zeta}) \\ \bar{\zeta}\Delta(\zeta) f(\bar{\zeta}) - b^*(\bar{\zeta})\bar{\zeta} g(\bar{\zeta}) \end{bmatrix}.\end{aligned}$$

If we compare this with the expression found in (ii), then we immediately see that $\mathfrak{W}_b^* = \mathfrak{W}_{b^*}$. □

Notice that, when $b = \Theta$ is inner, then $\Delta = \Delta_* = 0$. As we already said, $\mathbb{K}_\Theta$ reduces to $K_\Theta = H^2 \ominus \Theta H^2$, and moreover we see that $\mathfrak{W}_\Theta$ reduces to the operator on L^2 defined by

$$(\mathfrak{W}_\Theta f)(\zeta) = \Theta^*(\zeta)\bar{\zeta} f(\bar{\zeta}),$$

and we recover the unitary operator from K_Θ onto K_{Θ^*} introduced in Lemma 14.30 (and which was denoted by V_Θ).

It turns out that the map $\mathfrak{W}_b$ implements a unitary equivalence between the two model operators $\mathbf{M}_b^*$ and $\mathbf{M}_{b^*}$.

Theorem 19.14 *Let b be a function in the closed unit ball of H^∞. Then we have*

$$\mathfrak{W}_b \mathbf{M}_b^* = \mathbf{M}_{b^*} \mathfrak{W}_b \qquad \text{on } \mathbb{K}_b,$$

which can be written as follows:

$$\begin{tikzcd} \mathbb{K}_b \arrow[r, "\mathbf{M}_b^*"] \arrow[d, "\mathfrak{W}_b"'] & \mathbb{K}_b \arrow[d, "\mathfrak{W}_b"] \\ \mathbb{K}_{b^*} \arrow[r, "\mathbf{M}_{b^*}"'] & \mathbb{K}_{b^*} \end{tikzcd} \tag{19.25}$$

Proof It is easy to see that, on $L^2 \oplus \mathrm{Clos}(\Delta L^2)$, we have

$$(Z \oplus Z_{\Delta_*})\mathfrak{W}_b = \mathfrak{W}_b(Z^* \oplus Z_\Delta^*),$$

where $Z\colon L^2 \longrightarrow L^2$, $Z_{\Delta_*}\colon \mathrm{Clos}(\Delta_* L^2) \longrightarrow \mathrm{Clos}(\Delta_* L^2)$ and $Z_\Delta\colon \mathrm{Clos}(\Delta L^2) \longrightarrow \mathrm{Clos}(\Delta L^2)$ denote the multiplication operators by ζ (the independent variable) on the corresponding spaces. Then

$$P_{\mathbb{K}_{b^*}}(Z \oplus Z_{\Delta_*})\mathfrak{W}_b P_{\mathbb{K}_b} = P_{\mathbb{K}_{b^*}}\mathfrak{W}_b(Z^* \oplus Z_\Delta^*)P_{\mathbb{K}_b}. \tag{19.26}$$

Since $\mathfrak{W}_b \mathbb{K}_b = \mathbb{K}_{b^*}$, we have

$$P_{\mathbb{K}_{b^*}}(Z \oplus Z_{\Delta_*})\mathfrak{W}_b P_{\mathbb{K}_b} = P_{\mathbb{K}_{b^*}}(Z \oplus Z_{\Delta_*})P_{\mathbb{K}_{b^*}}\mathfrak{W}_b P_{\mathbb{K}_b},$$

and we note that $Z \oplus Z_{\tilde{\Delta}}$ and $S \oplus Z_{\Delta_*}$ coincide on $\mathbb{K}_{b^*}$, whence

$$P_{\mathbb{K}_{b^*}}(Z \oplus Z_{\Delta_*})\mathfrak{W}_b P_{\mathbb{K}_b} = \mathbf{M}_{b^*}\mathfrak{W}_b P_{\mathbb{K}_b}. \tag{19.27}$$

On the other hand, we claim that

$$P_{\mathbb{K}_{b^*}}\mathfrak{W}_b(Z^* \oplus Z_\Delta^*)P_{\mathbb{K}_b} = P_{\mathbb{K}_{b^*}}\mathfrak{M}_b(S^* \oplus Z_\Delta^*)P_{\mathbb{K}_b}. \tag{19.28}$$

Indeed, on the one hand, for any $f \oplus g \in \mathbb{K}_b$, we have

$$\begin{aligned}(\mathfrak{W}_b(Z^* \oplus Z_\Delta^*) - \mathfrak{W}_b(S^* \oplus Z_\Delta^*))(f \oplus g) &= \mathfrak{W}_b((Z^* - S^*)f \oplus 0)\\ &= \mathfrak{W}_b(P_-\bar{z}f \oplus 0).\end{aligned}$$

On the other hand, according to (19.24), we have

$$\mathfrak{W}_b(H_-^2 \oplus \{0\}) = \{b^*h \oplus \Delta_* h : h \in H^2\} \subset \mathbb{K}_{b^*}^\perp,$$

which implies (19.28). Now, since $\mathbb{K}_b$ is invariant with respect to $S^* \oplus Z_\Delta^*$ and $\mathfrak{W}_b\mathbb{K}_b \subset \mathbb{K}_{b^*}$, we have

$$P_{\mathbb{K}_{b^*}}\mathfrak{M}_b(S^* \oplus Z_\Delta^*)P_{\mathbb{K}_b} = \mathfrak{W}_b\mathbf{M}_b^*,$$

which gives the result with (19.26) and (19.27). □

In the case when b is extreme, then X_b and $X_{b^*}^*$ are unitarily equivalent.

Theorem 19.15 *Let b be an extreme point of the closed unit ball of H^∞. Let $U = Q_{b^*}\mathfrak{W}_b Q_b^*$. Then U is a unitary map from $\mathcal{H}(b)$ onto $\mathcal{H}(b^*)$ and $UX_b = X_{b^*}^* U$. In other words, we have the following diagram.*

$$\begin{tikzcd} \mathcal{H}(b) \arrow[r, "X_b"] \arrow[d, "U"'] & \mathcal{H}(b) \arrow[d, "U"] \\ \mathcal{H}(b^*) \arrow[r, "X_{b^*}^*"'] & \mathcal{H}(b^*) \end{tikzcd} \tag{19.29}$$

Proof According to Theorems 19.11 and 19.13, we know that U is a unitary map from $\mathcal{H}(b)$ onto $\mathcal{H}(b^*)$. On the other hand, since b is extreme, then b^*

is also extreme, and applying (19.20) and (19.25) gives that the following diagram is commutative.

$$\begin{tikzcd}
\mathcal{H}(b) \arrow[r, "Q_b^*"] \arrow[d, "X_b"'] & \mathbb{K}_b \arrow[r, "\mathfrak{W}_b"] \arrow[d, "\mathbf{M}_b^*"] & \mathbb{K}_{b^*} \arrow[r, "Q_{b^*}"] \arrow[d, "\mathbf{M}_{b^*}"] & \mathcal{H}(b^*) \arrow[d, "X_{b^*}^*"] \\
\mathcal{H}(b) \arrow[r, "Q_b^*"'] & \mathbb{K}_b \arrow[r, "\mathfrak{W}_b"'] & \mathbb{K}_{b^*} \arrow[r, "Q_{b^*}"'] & \mathcal{H}(b^*)
\end{tikzcd} \tag{19.30}$$

This is exactly (19.29). □

Remark 19.16 It should be noted that, in the case when b is a nonextreme point of the closed unit ball of H^∞, then X_b is never unitarily equivalent to $X_{b^*}^*$. Indeed, assume that there exists a unitary map $U\mathcal{H}(b) \longrightarrow \mathcal{H}(b^*)$ such that $X_b = U^* X_{b^*}^* U$. Then we would immediately obtain

$$D_{X_b^*} = U^* D_{X_{b^*}} U.$$

But Corollary 18.27 implies that the operator $D_{X_b^*}$ has rank one, and, as we will see in Corollary 23.16, since b^* is nonextreme, the operator $D_{X_{b^*}}$ has rank two, and that contradicts the above relation.

Exercises

Exercise 19.3.1 Prove that $f \longmapsto f^*$ maps H^2 onto H^2 and $\mathrm{Clos}(\Delta_* L^2)$ onto $\mathrm{Clos}(\Delta L^2)$.

Exercise 19.3.2 Prove that b is extreme if and only if b^* is extreme.

19.4 A contraction from $\mathcal{H}(b)$ to $\mathcal{H}(b^*)$

We know that $Q_b^* : \mathcal{H}(b) \longrightarrow \mathbb{K}_b'$ is unitary. The following result gives its action on reproducing kernels.

Lemma 19.17 *Let $b \in H^\infty$, $\|b\|_\infty \le 1$. Then, for each $\lambda \in \mathbb{D}$, we have $Q_b^* k_\lambda^b = \mathfrak{F}_\lambda^b$, where*

$$\mathfrak{F}_\lambda^b = k_\lambda^b \oplus (-\overline{b(\lambda)}\Delta k_\lambda). \tag{19.31}$$

Proof Let us prove first that (19.31) defines a function in $\mathbb{K}_b'$. We obviously have $\mathfrak{F}_\lambda^b \in H^2 \oplus \mathrm{Clos}(\Delta H^2)$. According to (19.16), we then have

$$\mathfrak{F}_\lambda^b \in \mathbb{K}_b' \quad \Longleftrightarrow \quad \mathfrak{F}_\lambda^b \perp bf \oplus \Delta f \qquad (f \in H^2).$$

But

$$\langle \mathfrak{F}_\lambda^b,\ bf \oplus \Delta f\rangle_{L^2\oplus L^2} = \langle k_\lambda^b, bf\rangle_2 - \overline{b(\lambda)}\langle \Delta k_\lambda, \Delta f\rangle_2.$$

Putting $k_\lambda^b = (1-\overline{b(\lambda)}b)k_\lambda$ gives

$$\langle \mathfrak{F}_\lambda^b,\ bf \oplus \Delta f\rangle_{L^2\oplus L^2} = \overline{b(\lambda)f(\lambda)} - \overline{b(\lambda)}(\langle bk_\lambda, bf\rangle_2 + \langle \Delta k_\lambda, \Delta f\rangle_2).$$

Notice now that

$$\begin{aligned}\langle bk_\lambda, bf\rangle_2 + \langle \Delta k_\lambda, \Delta f\rangle_2 &= \langle |b|^2 k_\lambda, f\rangle_2 + \langle (1-|b|^2)k_\lambda, f\rangle_2\\ &= \langle k_\lambda, f\rangle_2 = \overline{f(\lambda)}.\end{aligned}$$

Hence,

$$\langle \mathfrak{F}_\lambda^b,\ bf \oplus \Delta f\rangle_{L^2\oplus L^2} = \overline{b(\lambda)f(\lambda)} - \overline{b(\lambda)f(\lambda)} = 0,$$

which implies that $\mathfrak{F}_\lambda^b \in \mathbb{K}_b'$. It is now immediate that $Q_b\mathfrak{F}_\lambda^b = k_\lambda^b$, and that gives the desired relation. □

For $\lambda \in \mathbb{D}$, we recall that $\hat{k}_\lambda^b$ denotes the function in $\mathcal{H}(b)$ defined by

$$(\hat{k}_\lambda^b)(z) = (Q_\lambda b)(z) = \frac{b(z)-b(\lambda)}{z-\lambda} \qquad (\lambda \in \mathbb{D}).$$

Theorem 19.18 *Let $b \in H^\infty$, $\|b\|_\infty \le 1$. Then, for each $z \in \mathbb{D}$ and $f \in \mathcal{H}(b)$, the formula*

$$(W_b f)(z) = \langle f, \hat{k}_{\bar{z}}^b\rangle_b \tag{19.32}$$

defines an analytic function with $W_b f \in \mathcal{H}(b^*)$ and, furthermore, the map $f \longmapsto W_b f$ is a contraction from $\mathcal{H}(b)$ into $\mathcal{H}(b^*)$. The function W_b also satisfies the relation

$$W_b k_\lambda^b = \hat{k}_\lambda^{b^*} \qquad (\lambda \in \mathbb{D}). \tag{19.33}$$

Proof We claim that

$$W_b = Q_{b^*} P_{\mathbb{K}_{b^*}'} \mathfrak{W}_b i_b Q_b^*, \tag{19.34}$$

where $P_{\mathbb{K}_{b^*}'}$ denotes the orthogonal projection from $\mathbb{K}_{b^*}$ onto $\mathbb{K}_{b^*}'$ and i_b is the canonical injection from $\mathbb{K}_b'$ into $\mathbb{K}_b$.

We first verify the formula (19.34) on reproducing kernels of $\mathcal{H}(b)$. Fix $\lambda \in \mathbb{D}$. Then Lemma 19.17 implies that

$$Q_{b^*} P_{\mathbb{K}_{b^*}'} \mathfrak{W}_b i_b Q_b^* k_\lambda^b = Q_{b^*} P_{\mathbb{K}_{b^*}'} \mathfrak{W}_b\, \mathfrak{F}_\lambda^b.$$

Using Theorem 19.13, we have $\mathfrak{W}_b\, \mathfrak{F}_\lambda^b = f \oplus g$, where

$$f(\zeta) = b^*(\zeta)\bar{\zeta}k_\lambda^b(\bar{\zeta}) - \Delta_*(\zeta)\bar{\zeta}\overline{b(\lambda)}\Delta(\bar{\zeta})k_\lambda(\bar{\zeta})$$

and

$$g(\zeta) = \Delta_*(\zeta)\bar{\zeta}k_\lambda^b(\bar{\zeta}) + b(\bar{\zeta})\bar{\zeta}\overline{b(\lambda)}\Delta(\bar{\zeta})k_\lambda(\bar{\zeta}).$$

Now straightforward computations give

$$f(\zeta) = \frac{b^*(\zeta) - b^*(\bar{\lambda})}{\zeta - \bar{\lambda}} = (\hat{k}_{\bar{\lambda}}^{b^*})(\zeta)$$

and

$$g(\zeta) = \frac{\Delta(\bar{\zeta})}{\zeta - \bar{\lambda}} = \frac{\Delta_*(\zeta)}{\zeta - \bar{\lambda}}.$$

Hence,

$$\mathfrak{W}_b \mathfrak{F}_\lambda^b = \hat{k}_{\bar{\lambda}}^{b^*} \oplus \frac{\Delta_*}{\zeta - \bar{\lambda}}. \tag{19.35}$$

Finally, we get

$$Q_{b^*} P_{\mathbb{K}'_{b^*}} \mathfrak{W}_b i_b Q_b^* k_\lambda^b = Q_b^* P_{\mathbb{K}'_{b^*}} \left(\hat{k}_{\bar{\lambda}}^{b^*} \oplus \frac{\Delta_*}{\zeta - \bar{\lambda}} \right). \tag{19.36}$$

Now, if we denote by $\widetilde{\pi}_* : L^2 \longrightarrow L^2 \oplus \mathrm{Clos}(\Delta_* L^2)$ the linear map defined by $\widetilde{\pi}_*(h) = h \oplus 0$, then, by Theorem 19.8, we know that $\ker \widetilde{\pi}_*^* | \mathbb{K}_{b^*} = \mathbb{K}''_{b^*}$ and, by definition, $\widetilde{\pi}_*^* | \mathbb{K}_{b^*} = Q_{b^*}$. Hence, since

$$\hat{k}_{\bar{\lambda}}^{b^*} \oplus \frac{\Delta_*}{\zeta - \bar{\lambda}} = \mathfrak{W}_b \mathfrak{F}_\lambda^b \in \mathbb{K}_{b^*},$$

we have

$$\begin{aligned} Q_{b^*} P_{\mathbb{K}'_{b^*}} \mathfrak{W}_b i_b Q_b^* k_\lambda^b &= \widetilde{\pi}_*^* P_{\mathbb{K}'_{b^*}} \left(\hat{k}_{\bar{\lambda}}^{b^*} \oplus \frac{\Delta_*}{\zeta - \bar{\lambda}} \right) \\ &= \widetilde{\pi}_*^* \left(\hat{k}_{\bar{\lambda}}^{b^*} \oplus \frac{\Delta_*}{\zeta - \bar{\lambda}} \right) \\ &= \hat{k}_{\bar{\lambda}}^{b^*}. \end{aligned}$$

On the other hand, if we apply (19.32) to k_λ^b, we obtain

$$\begin{aligned} (W_b k_\lambda^b)(z) &= \langle k_\lambda^b, \hat{k}_{\bar{z}}^b \rangle_b \\ &= \overline{\hat{k}_{\bar{z}}^b(\lambda)} \\ &= \overline{\left(\frac{b(\lambda) - b(\bar{z})}{\lambda - \bar{z}} \right)} \\ &= \frac{\overline{b(\bar{z})} - \overline{b(\lambda)}}{z - \bar{\lambda}} \\ &= \frac{b^*(z) - b^*(\bar{\lambda})}{z - \bar{\lambda}} = \hat{k}_{\bar{\lambda}}^{b^*}(z). \end{aligned}$$

Thus if we compare this with (19.36), we deduce that

$$W_b k_\lambda^b = Q_{b^*} P_{\mathbb{K}'_{b^*}} \mathfrak{W}_b i_b Q_b^* k_\lambda^b.$$

By linearity, $W_b = Q_{b^*} P_{\mathbb{K}'_{b^*}} \mathfrak{W}_b i_b Q_b^*$ on the linear span of reproducing kernels of $\mathcal{H}(b)$. Now, if f is an arbitrary function in $\mathcal{H}(b)$, then we may take a sequence f_n in the linear span of reproducing kernels of $\mathcal{H}(b)$ such that $f_n \longrightarrow f$ in $\mathcal{H}(b)$. Then obviously we have

$$Q_{b^*} P_{\mathbb{K}'_{b^*}} \mathfrak{W}_b i_b Q_b^* f_n \longrightarrow Q_{b^*} P_{\mathbb{K}'_{b^*}} \mathfrak{W}_b i_b Q_b^* f \qquad (\text{as } n \longrightarrow \infty),$$

while on the other hand

$$(W_b f_n)(z) = \langle f_n, \hat{k}_{\bar{z}}^b \rangle_b \longrightarrow \langle f, \hat{k}_{\bar{z}}^b \rangle_b = (W_b f)(z).$$

Since $W_b f_n = Q_{b^*} P_{\mathbb{K}'_{b^*}} \mathfrak{W}_b i_b Q_b^* f_n$, we finally get (19.34).

To finish the proof, it remains to note that, since Q_b^*, Q_{b^*} and $\mathfrak{W}_b$ are all unitary, i_b is an isometry and $P_{\mathbb{K}'_{b^*}}$ is a contraction, then W_b is a contraction from $\mathcal{H}(b)$ into $\mathcal{H}(b^*)$. □

The following commutative diagram summarizes the construction of W_b given by (19.34).

$$\begin{tikzcd}
\mathbb{K}'_b \arrow[r, "i_b"] \arrow[d, "Q_b"'] & \mathbb{K}_b \arrow[r, "\mathfrak{W}_b"] & \mathbb{K}_{b^*} \arrow[r, "P_{\mathbb{K}'_{b^*}}"] & \mathbb{K}'_{b^*} \arrow[r, "Q_{b^*}"] & \mathcal{H}(b^*) \\
\mathcal{H}(b) \arrow[urrrr, "W_b"'] & & & &
\end{tikzcd} \tag{19.37}$$

Remark 19.19 Note that when b is extreme, the operator W_b can be written in a simpler form because $\mathbb{K}_b = \mathbb{K}'_b$, $\mathbb{K}'_{b^*} = \mathbb{K}_{b^*}$ and $\mathfrak{W}_b \mathbb{K}_b \subset \mathbb{K}_{b^*}$, and thus

$$W_b = Q_{b^*} \mathfrak{W}_b Q_b^*.$$

Now, remember that Q_b^* maps unitarily $\mathcal{H}(b)$ onto $\mathbb{K}_b$, $\mathfrak{W}_b$ maps unitarily $\mathbb{K}_b$ onto $\mathbb{K}_{b^*}$ and Q_{b^*} maps unitarily $\mathbb{K}_{b^*}$ onto $\mathcal{H}(b^*)$. Hence, if b is an extreme point, then W_b maps unitarily $\mathcal{H}(b)$ onto $\mathcal{H}(b^*)$. Moreover, in that case, W_b is the operator U that appears in Theorem 19.15 and that intertwines unitarily X_b and $X_{b^*}^*$.

It turns out that the converse is true. In fact, one can say more in this case.

Theorem 19.20 *With the above notation, the following are equivalent.*

(i) *W_b is a unitary operator from $\mathcal{H}(b)$ onto $\mathcal{H}(b^*)$.*
(ii) *W_b is an isometry from $\mathcal{H}(b)$ into $\mathcal{H}(b^*)$.*
(iii) *b is an extreme point of the closed unit ball of H^∞.*

Proof The implication (i) $\Longrightarrow$ (ii) is trivial and the implication (iii) $\Longrightarrow$ (i) was proved above.

It remains to prove (ii) $\Longrightarrow$ (iii). So let us assume that W_b is an isometry from $\mathcal{H}(b)$ into $\mathcal{H}(b^*)$. Then we claim that

$$\mathfrak{W}_b\mathbb{K}_b' \subset \mathbb{K}_{b^*}'.$$

Indeed, according to (19.34), we should have

$$\|Q_{b^*}P_{\mathbb{K}_{b^*}'}\mathfrak{W}_b i_b Q_b^* f\|_{b^*} = \|f\|_b \qquad (f \in \mathcal{H}(b)).$$

Since Q_b^* maps unitarily $\mathcal{H}(b)$ onto $\mathbb{K}_b'$, and Q_{b^*} maps unitarily $\mathbb{K}_{b^*}'$ onto $\mathcal{H}(b^*)$, we find that

$$\|P_{\mathbb{K}_{b^*}'}\mathfrak{W}_b g\|_{\mathbb{K}_{b^*}} = \|g\|_{\mathbb{K}_b} \qquad (g \in \mathbb{K}_b').$$

Now, using the fact that $\mathfrak{W}_b$ is an isometry from $\mathbb{K}_b'$ onto $\mathbb{K}_{b^*}$, we get

$$\|P_{\mathbb{K}_{b^*}'}\mathfrak{W}_b g\|_{\mathbb{K}_{b^*}} = \|\mathfrak{W}_b g\|_{\mathbb{K}_{b^*}} \qquad (g \in \mathbb{K}_b').$$

This shows that $\mathfrak{W}_b\mathbb{K}_b' \subset \mathbb{K}_{b^*}'$. In particular, $\mathfrak{W}_b\,\mathfrak{F}_\lambda^b \in \mathbb{K}_{b^*}'$ for all $\lambda \in \mathbb{D}$. By (19.35) and (19.16), this implies that

$$\frac{\Delta_*}{\zeta - \bar\lambda} \in \operatorname{Clos}(\Delta_* H^2) \qquad (\lambda \in \mathbb{D}).$$

Since the functions $1/(\zeta - \bar\lambda)$ span $L^2 \ominus H^2$, we must have $\Delta_*(L^2 \ominus H^2) \subset \operatorname{Clos}(\Delta_* H^2)$. Therefore,

$$\operatorname{Clos}(\Delta_* L^2) = \operatorname{Clos}(\Delta_* H^2),$$

which means that b^*, and thus b, is an extreme point. □

The following result improves the estimate given in (18.12).

Corollary 19.21 *Let b be in the closed unit ball of H^∞. Then, for each $\lambda \in \mathbb{D}$, we have*

$$\|\hat{k}_\lambda^b\|_b^2 \le \frac{1 - |b(\lambda)|^2}{1 - |\lambda|^2}, \tag{19.38}$$

and if b is an extreme point, then equality holds.

Proof It follows from Theorem 19.18 that there exists a contraction $W_b : \mathcal{H}(b) \longrightarrow \mathcal{H}(b^*)$ such that $W_b k_\lambda^b = \hat{k}_{\bar\lambda}^{b^*}$, $\lambda \in \mathbb{D}$. Hence,

$$\|\hat{k}_{\bar\lambda}^{b^*}\|_{b^*}^2 = \|W_b k_\lambda^b\|_{b^*}^2 \le \|k_\lambda^b\|_b^2 = \frac{1 - |b(\lambda)|^2}{1 - |\lambda|^2}.$$

If we replace b by b^* and $\bar\lambda$ by λ, we get

$$\|\hat{k}_\lambda^b\|_b^2 \le \frac{1 - |b^*(\bar\lambda)|^2}{1 - |\lambda|^2} = \frac{1 - |b(\lambda)|^2}{1 - |\lambda|^2}.$$

Furthermore, according to Theorem 19.20, we know that, when b is extreme, then the operator W_b is isometric, which gives equality in (19.38). $\square$

See also Remark 20.7 for another proof of Corollary 19.21.

Remember that the formula

$$(Q_w f)(z) = \frac{f(z) - f(w)}{z - w} \qquad (z, w \in \mathbb{D}),$$

defines a bounded linear operator from $\mathcal{H}(b)$ into itself. It is interesting to see that there is a formula to compute the adjoint of $Q_{w,b} = Q_w|\mathcal{H}(b)$ in terms of the contraction W_b.

Theorem 19.22 *Let b be a point in the closed unit ball of H^∞, and let W_b be the contraction defined by* (19.32)*. Then the following assertions hold.*

(i) *For each $f \in \mathcal{H}(b)$, we have*

$$(Q_{w,b}^* f)(z) = \frac{zf(z) - b(z)(W_b f)(\bar{w})}{1 - z\bar{w}}. \tag{19.39}$$

(ii) *We have*

$$W_b Q_{w,b}^* = Q_{\bar{w},b^*} W_b.$$

(iii) $W_b^* = W_{b^*}$.

Proof (i) We first check (19.39) on reproducing kernels of $\mathcal{H}(b)$. For each $f \in \mathcal{H}(b)$ and each $\lambda \in \mathbb{D}$, on the one hand, we have

$$\begin{aligned}\langle f, Q_{w,b}^* k_\lambda^b \rangle_b &= \langle Q_{w,b} f, k_\lambda^b \rangle_b \\ &= (Q_{w,b} f)(\lambda) = \frac{f(\lambda) - f(w)}{\lambda - w}.\end{aligned}$$

On the other hand, we know from (19.33) that $W_b k_\lambda^b = \hat{k}_{\bar{\lambda}}^{b^*}$, and then

$$(W_b k_\lambda^b)(\bar{w}) = \frac{b^*(\bar{w}) - b^*(\bar{\lambda})}{\bar{w} - \bar{\lambda}} = \overline{\left(\frac{b(w) - b(\lambda)}{w - \lambda}\right)}.$$

Now, a straightforward but tedious computation yields

$$\frac{zk_\lambda^b(z) - b(z)(W_b k_\lambda^b)(\bar{w})}{1 - z\bar{w}} = \frac{1}{\bar{w} - \bar{\lambda}}(k_w^b(z) - k_\lambda^b(z)).$$

Thus, we get that

$$\begin{aligned}\left\langle f, \frac{zk_\lambda^b(z) - b(z)(W_b k_\lambda^b)(\bar{w})}{1 - z\bar{w}} \right\rangle_b &= \left\langle f, \frac{k_w^b - k_\lambda^b}{\bar{w} - \bar{\lambda}} \right\rangle_b \\ &= \frac{1}{w - \lambda}(f(w) - f(\lambda)).\end{aligned}$$

Finally, we get

$$\left\langle f,\ \frac{zk_\lambda^b(z) - b(z)(W_b k_\lambda^b)(\bar{w})}{1 - z\bar{w}} \right\rangle_b = \langle f, Q_{w,b}^* k_\lambda^b \rangle_b.$$

Since the last equality is true for any $f \in \mathcal{H}(b)$, we deduce that (19.39) is satisfied on reproducing kernels of $\mathcal{H}(b)$. An easy argument based on the linearity, density and continuity of $Q_{w,b}^*$ and W_b gives (19.39).

(ii) Let $f \in \mathcal{H}(b)$. Then, we have

$$\begin{aligned}
(W_b Q_{w,b}^* f)(z) &= \langle Q_{w,b}^* f, \hat{k}_{\bar{z}}^b \rangle_b \\
&= \langle f, Q_{w,b} \hat{k}_{\bar{z}}^b \rangle_b \\
&= \left\langle f,\ \frac{\hat{k}_{\bar{z}}^b(u) - \hat{k}_{\bar{z}}^b(w)}{u - w} \right\rangle_b .
\end{aligned}$$

But a straightforward computation yields

$$\begin{aligned}
\frac{\hat{k}_{\bar{z}}^b(u) - \hat{k}_{\bar{z}}^b(w)}{u - w} &= \frac{(b(u) - b(\bar{z}))/(u - \bar{z}) - (b(u) - b(w))/(u - w)}{\bar{z} - w} \\
&= \frac{1}{\bar{z} - w}(\hat{k}_{\bar{z}}^b(u) - \hat{k}_w^b(u)).
\end{aligned}$$

Thus,

$$\begin{aligned}
(W_b Q_{w,b}^* f)(z) &= \frac{1}{z - \bar{w}} \langle f,\ \hat{k}_{\bar{z}}^b - \hat{k}_w^b \rangle_b \\
&= \frac{1}{z - \bar{w}} (\langle f, \hat{k}_{\bar{z}}^b \rangle_b - \langle f, \hat{k}_w^b \rangle_b) \\
&= \frac{1}{z - \bar{w}} ((W_b f)(z) - (W_b f)(\bar{w})) \\
&= (Q_{\bar{w},b^*} W_b f)(z).
\end{aligned}$$

(iii) Using (19.34) and the facts that $i_b^* = P_{\mathbb{K}_b'}$ and $P_{\mathbb{K}_{b^*}'}^* = i_{b^*}$, we find that

$$W_b^* = Q_b P_{\mathbb{K}_b'} \mathfrak{W}_b^* i_{b^*} Q_{b^*}^*.$$

But remember that $\mathfrak{W}_b^* = \mathfrak{W}_{b^*}$, which gives

$$W_b^* = Q_b P_{\mathbb{K}_b'} \mathfrak{W}_{b^*} i_{b^*} Q_{b^*}^*.$$

Using (19.34) once more, but this time for b^*, yields $W_b^* = W_{b^*}$. □

The formula (19.39) can be written in a more condensed way as

$$Q_{w,b}^* f = (Sf - b\langle f, \hat{k}_w^b \rangle_b) k_w \qquad (f \in \mathcal{H}(b)).$$

Exercise

Exercise 19.4.1 Let b be an extreme point in the closed unit ball of H^∞. Prove that

$$\mathrm{Span}_{\mathcal{H}(b)}(\hat{k}_\lambda^b : \lambda \in \mathbb{D}) = \mathcal{H}(b).$$

Hint: Use W_b, equation (19.33) and Theorem 19.20.
Remark: See Exercise 19.1.2 and Corollary 26.18 for other methods to prove this result.

19.5 Almost conformal invariance

Let $\lambda \in \mathbb{D}$, let

$$b_\lambda(z) = \frac{\lambda - z}{1 - \bar{\lambda} z} \qquad (z \in \mathbb{D})$$

and let

$$\tilde{k}_\lambda(z) = \frac{(1 - |\lambda|^2)^{1/2}}{1 - \bar{\lambda} z} \qquad (z \in \mathbb{D}).$$

Then b_λ is an automorphism of the unit disk and $\tilde{k}_\lambda$ is just the normalized reproducing kernel of H^2, that is, $\tilde{k}_\lambda = k_\lambda / \|k_\lambda\|_2$, where $k_\lambda(z) = (1 - \bar{\lambda} z)^{-1}$.

Lemma 19.23 *Let b be a function in the closed unit ball of H^∞. Then, for each $\lambda \in \mathbb{D}$, we have*

$$T_{\tilde{k}_\lambda \circ b}(I - T_b T_{\bar{b}}) T^*_{\tilde{k}_\lambda \circ b} = I - T_{b_\lambda \circ b} T^*_{b_\lambda \circ b}.$$

Proof Note that

$$\tilde{k}_\lambda \circ b = (1 - |\lambda|^2)^{1/2} (1 - \bar{\lambda} b)^{-1} = (1 - |\lambda|^2)^{1/2} h,$$

where $h = (1 - \bar{\lambda} b)^{-1}$, and

$$b_\lambda \circ b = \frac{\lambda - b}{1 - \bar{\lambda} b} = (\lambda - b) h.$$

Thus

$$\begin{aligned} T_{\tilde{k}_\lambda \circ b}(I - T_b T_{\bar{b}}) T^*_{\tilde{k}_\lambda \circ b} &= (1 - |\lambda|^2) T_h (I - T_b T_{\bar{b}}) T_{\bar{h}} \\ &= (1 - |\lambda|^2)(T_h T_{\bar{h}} - T_{bh} T_{\bar{b}\bar{h}}) \\ &= (T_h - \bar{\lambda} T_{bh})(T_{\bar{h}} - \lambda T_{\bar{b}\bar{h}}) - (\lambda T_h - T_{bh})(\bar{\lambda} T_{\bar{h}} - T_{\bar{b}\bar{h}}) \\ &= T_{(1 - \bar{\lambda} b) h} T_{(1 - \lambda \bar{b}) \bar{h}} - T_{(\lambda - b) h} T_{(\bar{\lambda} - \bar{b}) \bar{h}}. \end{aligned}$$

Since $(1 - \bar{\lambda} b) h = 1$ and $(\lambda - b) h = b_\lambda \circ b$, we get

$$T_{\tilde{k}_\lambda \circ b}(I - T_b T_{\bar{b}}) T^*_{\tilde{k}_\lambda \circ b} = I - T_{b_\lambda \circ b} T^*_{b_\lambda \circ b}. \qquad \square$$

Since, for each $\lambda, z \in \mathbb{D}$, we have $0 < 1 - |\lambda| \le |1 - \bar{\lambda} b(z)| \le 2$, the formula

$$\kappa_\lambda(z) = \frac{(1 - |\lambda|^2)^{1/2}}{1 - \bar{\lambda} b(z)} \qquad (z \in \mathbb{D})$$

gives us an outer function with $\kappa_\lambda, 1/\kappa_\lambda \in H^\infty$. Note that $\kappa_\lambda = \tilde{k}_\lambda \circ b$.

The following result is a generalization of Theorem 14.20.

Theorem 19.24 *Let b be a function in the closed unit ball of H^∞ and let $\lambda \in \mathbb{D}$. Then*

$$\begin{array}{rccc} \mathbf{T}_{\kappa_\lambda} : & \mathcal{H}(b) & \longrightarrow & \mathcal{H}(b_\lambda \circ b) \\ & f & \longmapsto & \kappa_\lambda f \end{array}$$

is a unitary operator. In other words, the multiplication operator by κ_λ acts as an isometry from $\mathcal{H}(b)$ onto $\mathcal{H}(b_\lambda \circ b)$.

Proof According to Lemma 19.23, we have

$$T_{\kappa_\lambda}(I - T_b T_b^*) T_{\kappa_\lambda}^* = I - T_{b_\lambda \circ b} T_{b_\lambda \circ b}^*.$$

Now since T_{κ_λ} is injective, Corollary 16.11 implies that T_{κ_λ} is a unitary map from $\mathcal{H}(b)$ onto $\mathcal{H}(b_\lambda \circ b)$. □

As a consequence of Theorem 19.24, we have the set identity

$$(1 - \bar{\lambda} b)\mathcal{H}(b_\lambda \circ b) = \mathcal{H}(b). \tag{19.40}$$

Theorem 19.25 *Let b be a function in the closed unit ball of H^∞, and let $\lambda \in \mathbb{D}$. Then the following hold.*

(i) $\mathcal{H}(\overline{b_\lambda \circ b}) = \mathcal{H}(\bar{b})$.
(ii) $\mathfrak{Mult}(\mathcal{H}(b_\lambda \circ b)) = \mathfrak{Mult}(\mathcal{H}(b))$.

Proof According to Theorem 17.12, property (i) is equivalent to

$$0 < \inf_{\zeta \in \mathbb{T}} \frac{1 - |b(\zeta)|^2}{1 - |b_\lambda \circ b(\zeta)|^2} \le \sup_{\zeta \in \mathbb{T}} \frac{1 - |b(\zeta)|^2}{1 - |b_\lambda \circ b(\zeta)|^2} < \infty. \tag{19.41}$$

But an easy computation shows that

$$\frac{1 - |b|^2}{1 - |b_\lambda \circ b|^2} = \frac{|1 - \bar{\lambda} b|^2}{1 - |\lambda|^2},$$

and $(1 - |\lambda|)^2 \le |1 - \bar{\lambda} b|^2 \le 4$. Thus

$$\frac{1 - |\lambda|}{1 + |\lambda|} \le \frac{1 - |b|^2}{1 - |b_\lambda \circ b|^2} \le \frac{4}{1 - |\lambda|^2},$$

which gives (19.41) and (i).

Now let us prove (ii). Let $f \in \mathfrak{Mult}(\mathcal{H}(b_\lambda \circ b))$ and $h \in \mathcal{H}(b)$. Then, according to (19.40), we have $h = (1 - \bar{\lambda} b)g$ for some $g \in \mathcal{H}(b_\lambda \circ b)$, which gives

$$fh = f(1 - \bar{\lambda} b)g = (1 - \bar{\lambda} b)fg.$$

Since $fg \in \mathcal{H}(b_\lambda \circ b)$, we deduce once more from (19.40) that $fh \in \mathcal{H}(b)$. This proves that $f \in \mathfrak{Mult}(\mathcal{H}(b))$. Conversely, let $f \in \mathfrak{Mult}(\mathcal{H}(b))$ and let $g \in \mathcal{H}(b_\lambda \circ b)$. Then, by (19.40), we know that $(1 - \bar{\lambda} b)g \in \mathcal{H}(b)$. Thus $f(1 - \bar{\lambda} b)g \in \mathcal{H}(b)$. Using (19.40) once more, there exists a function $\varphi \in \mathcal{H}(b_\lambda \circ b)$ such that $f(1 - \bar{\lambda} b)g = (1 - \bar{\lambda} b)\varphi$. But $1 - \bar{\lambda} b$ does not vanish on $\mathbb{D}$, whence $fg = \varphi \in \mathcal{H}(b_\lambda \circ b)$. Hence $f \in \mathfrak{Mult}(\mathcal{H}(b_\lambda \circ b))$. □

Exercises

Exercise 19.5.1 Let b be a function in the closed unit ball of H^∞, let $\lambda \in \mathbb{D}$, and let

$$b_\lambda^\sharp = -b_\lambda \circ b = \frac{b - \lambda}{1 - \bar{\lambda} b}. \tag{19.42}$$

Denote by $\rho = 1 - |b|^2$, $\rho_\lambda = 1 - |b_\lambda^\sharp|^2$ and $\zeta = (1 + \lambda)/(1 + \bar{\lambda})$. Show that:

(i) $\rho_\lambda = \rho \dfrac{1 - |\lambda|^2}{|1 - \bar{\lambda} b|^2}$;

(ii) $1 - b_\lambda^\sharp = (1 + \lambda) \dfrac{1 - \bar{\zeta} b}{1 - \bar{\lambda} b}$;

(iii) $\dfrac{\rho_\lambda}{|1 - b_\lambda^\sharp|^2} = \dfrac{1 - |\lambda|^2}{|1 + \lambda|^2} \dfrac{\rho}{|\zeta - b|^2}$.

Exercise 19.5.2 Let b be a function in the closed unit ball of H^∞, $\lambda \in \mathbb{D}$, $\zeta = (1+\lambda)/(1+\bar{\lambda})$, and let $b_\lambda^\sharp$ be defined by (19.42). Denote by $\mu_{b_\lambda^\sharp}$ the Clark measure associated with $b_\lambda^\sharp$ and by $\mu_{\bar{\zeta} b}$ the measure associated with $\bar{\zeta} b$. Prove that

$$\mu_{b_\lambda^\sharp} = c\mu_{\bar{\zeta} b},$$

where $c = (1 - |\lambda|^2)/|1 + \lambda|^2$.
Hint: Use Exercise 19.5.1(iii) to show that there exists $t_0 \in \mathbb{R}$ such that

$$\frac{1 + b_\lambda^\sharp}{1 - b_\lambda^\sharp} = c \frac{1 - \bar{\zeta} b}{1 - \bar{\zeta} b} + it_0.$$

Deduce that there exist $\delta_1, \delta_2 \in \mathbb{R}$ such that

$$it_0 = \int_{\mathbb{T}} \frac{e^{i\vartheta} + z}{e^{i\vartheta} - z} \, d\mu(e^{i\vartheta}) + i(\delta_1 - c\delta_2) \qquad (z \in \mathbb{D}),$$

where $d\mu(e^{i\vartheta}) = d\mu_{b^\sharp_\lambda}(e^{i\vartheta}) - c\mu_{\bar{\zeta}b}(e^{i\vartheta})$. Evaluating the last identity at $z = 0$, prove that $t_0 = \delta_1 - c\delta_2$. Conclude that $\mu = 0$.

19.6 The Littlewood subordination theorem revisited

Recall that, if b is an analytic map of the unit disk into itself such that $b(0) = 0$ and if G is a subharmonic function in $\mathbb{D}$, then, for any $0 < r < 1$,

$$\int_0^{2\pi} G(\varphi(re^{i\theta}))\, d\theta \le \int_0^{2\pi} G(re^{i\theta})\, d\theta.$$

If we apply this inequality, known as the Littlewood subordination principle, then we obtain as a corollary that, if b is an analytic map of the unit disk into itself such that $b(0) = 0$, then the composition operator $C_b f = f \circ b$ is a contraction from H^2 into itself. In this section, we will obtain a proof of this standard fact just using the theory of kernels (and in particular the kernel of $\mathcal{H}(b)$).

The key result is the following.

Lemma 19.26 *Let $b \in H^\infty$, $\|b\|_\infty \le 1$, and let $f \in \mathcal{H}(b)$, $\|f\|_b \le 1$. Then the function*

$$K(z, w) = \frac{1}{1 - \bar{w}z} - \frac{\overline{f(w)}f(z)}{1 - \overline{b(w)}b(z)}$$

is a weak kernel on $\mathbb{D} \times \mathbb{D}$.

Proof By Theorem 9.11, we know that the function

$$K_1(z, w) = \frac{1 - \overline{b(w)}b(z)}{1 - \bar{w}z} - \overline{f(w)}f(z)$$

is a weak kernel. Moreover, Lemma 9.13 implies that the function

$$K_2(z, w) = \frac{1}{1 - \overline{b(w)}b(z)}$$

is also a weak kernel. Now, it follows from Lemma 9.12 that the function

$$K_1(z, w)K_2(z, w) = \frac{1}{1 - \bar{w}z} - \frac{\overline{f(w)}f(z)}{1 - \overline{b(w)}b(z)}$$

is a weak kernel. □

For any function $f : \mathbb{D} \longrightarrow \mathbb{D}$, there is a densely defined operator T_f^* on H^2, defined on the Szegő kernel k_w by

$$T_f^* k_w = \overline{f(w)} k_w$$

and extended linearly. Note that the adjoint notation here is only formal: if $f \in H^\infty$, then T_f^* is bounded and equal to the adjoint of the Toeplitz operator T_f. For a function b in the closed unit ball of H^∞, b nonconstant, we also define the operator C_b^* on the linear span of the Szegő kernels by

$$C_b^* k_w = k_{b(w)} \qquad (w \in \mathbb{D}).$$

If $f \in H^2$ and $f \circ b \in H^2$, then

$$\langle C_b f, k_w \rangle_2 = \langle f \circ b, k_w \rangle_2 = f(b(w)) = \langle f, k_{b(w)} \rangle_2 = \langle f, C_b^* k_w \rangle_2,$$

so the operator C_b^* is the formal adjoint of the composition operator C_b. Hence, to prove that C_b is bounded, it suffices to prove that C_b^* is bounded.

Lemma 19.27 *Let b be a point in the closed unit ball of H^∞ and let $f \in \mathcal{H}(b)$. Then, the operator $C_b^* T_f^*$ is bounded on H^2 and*

$$\|C_b^* T_f^*\| \leq \|f\|_b.$$

Proof First note that we can assume that $\|f\|_b = 1$. According to Lemma 19.26, the function

$$K(z,w) = \frac{1}{1 - \bar{w}z} - \frac{\overline{f(w)} f(z)}{1 - \overline{b(w)} b(z)}$$

is a weak kernel on $\mathbb{D} \times \mathbb{D}$. But observe that $C_b^* T_f^* k_z = \overline{f(z)} k_{b(z)}$, which gives

$$\frac{1}{1 - \bar{w}z} - \frac{\overline{f(w)} f(z)}{1 - \overline{b(w)} b(z)} = \langle k_w, k_z \rangle_2 - \langle C_b^* T_f^* k_w, C_b^* T_f^* k_z \rangle_2.$$

Hence, for any finite set $\{w_1, \ldots, w_n\}$ of points in $\mathbb{D}$ and all complex numbers $c_1, c_2, \ldots, c_n$, we get

$$\begin{aligned} 0 &\leq \sum_{i,j=1}^n c_i \overline{c_j} \langle k_{w_i}, k_{w_j} \rangle_2 - \sum_{i,j=1}^n c_i \overline{c_j} \langle C_b^* T_f^* k_{w_i}, C_b^* T_f^* k_{w_j} \rangle_2 \\ &= \|h\|_2^2 - \|C_b^* T_f^* h\|_2^2, \end{aligned}$$

where $h = \sum_{i=1}^n c_i k_{w_i}$. In other words, for any h in the linear manifold of all finite linear combinations of k_w, $w \in \mathbb{D}$, we have

$$\|C_b^* T_f^* h\|_2 \leq \|h\|_2.$$

We conclude by density. □

Theorem 19.28 *Let $b \in H^\infty$, $\|b\|_\infty \leq 1$, b nonconstant. Then the composition operator C_b is bounded from H^2 into itself and we have*

$$\|C_b\| \leq \sqrt{\frac{1 + |b(0)|}{1 - |b(0)|}}.$$

Proof We shall apply Lemma 19.27 to $f(z) = 1-\overline{b(0)}b(z) = k_0^b(z) \in \mathcal{H}(b)$. On the one hand, we have $\|f\|_b = (1-|b(0)|^2)^{1/2}$. On the other hand, since f and f^{-1} belong to H^∞, the Toeplitz operator $T_f^* = T_{\bar{f}}$ is invertible on H^2 and $(T_f^*)^{-1} = T_{1/\bar{f}}$. Since $C_b^* = C_b^* T_f^* (T_f^*)^{-1}$, we obtain using Lemma 19.27 that C_b^* is bounded. Moreover we have

$$\begin{aligned}\|C_b^*\| &\leq \|C_b^* T_f^*\| \, \|T_{1/\bar{f}}\| \\ &\leq \|f\|_b \, \|f^{-1}\|_\infty \\ &\leq \frac{(1-|b(0)|^2)^{1/2}}{1-|b(0)|} = \sqrt{\frac{1+|b(0)|}{1-|b(0)|}},\end{aligned}$$

which concludes the proof. □

19.7 The generalized Schwarz–Pick estimates

Recall that, by Pick's invariant form of Schwarz's lemma, a function b in the closed unit ball of H^∞ satisfies the inequality

$$|b'(\lambda)| \leq \frac{1-|b(\lambda)|^2}{1-|\lambda|^2} \qquad (\lambda \in \mathbb{D}). \tag{19.43}$$

Using a Hilbert space method based on $\mathcal{H}(b)$ spaces, we generalize this inequality for higher-order derivatives.

Theorem 19.29 *Let b be a function in the closed unit ball of H^∞. Then, for each $\lambda \in \mathbb{D}$ and $n \geq 1$, we have*

$$(1-|\lambda|)^{n-1} \left| \frac{b^{(n)}(\lambda)}{n!} \right| \leq \frac{1-|b(\lambda)|^2}{1-|\lambda|^2}. \tag{19.44}$$

Moreover, for any fixed $\lambda \in \mathbb{D}$ and $n \geq 1$, the inequality is sharp, in the sense that

$$\sup_b \left[(1-|\lambda|)^{n-1} \left| \frac{b^{(n)}(\lambda)}{n!} \right| \frac{1-|\lambda|^2}{1-|b(\lambda)|^2} \right] = 1, \tag{19.45}$$

where the supremum is taken over all nonconstant functions b in the closed unit ball of H^∞.

Proof Fix $\lambda \in \mathbb{D}$. We recall that

$$(Q_\lambda f)(z) = \frac{f(z)-f(\lambda)}{z-\lambda} \qquad (z \in \mathbb{D})$$

defines a map from $\mathcal{H}(b)$ into itself. Set $f_1 = \hat{k}_\lambda^b$ and $f_n = Q_\lambda^{n-1} f_1$, $n \geq 2$. In a neighborhood of λ, we can write

$$b(z) = \sum_{k=0}^{\infty} \frac{b^{(k)}(\lambda)}{k!} (z-\lambda)^k,$$

whence

$$f_1(z) = \frac{b(z) - b(\lambda)}{z - \lambda} = \sum_{k=1}^{\infty} \frac{b^{(k)}(\lambda)}{k!}(z-\lambda)^{k-1}.$$

By induction, we can easily see that, for each $n \geq 1$, we have

$$f_n(z) = \sum_{k=n}^{\infty} \frac{b^{(k)}(\lambda)}{k!}(z-\lambda)^{k-n}.$$

In particular, we get

$$f_n(\lambda) = \frac{b^{(n)}(\lambda)}{n!}.$$

Hence

$$\frac{b^{(n)}(\lambda)}{n!} = \langle f_n, k_\lambda^b \rangle_b = \langle Q_\lambda^{n-1} f_1, k_\lambda^b \rangle_b,$$

which, by the Cauchy–Schwarz inequality, implies that

$$\left| \frac{b^{(n)}(\lambda)}{n!} \right| \leq \|Q_\lambda\|_{\mathcal{L}(\mathcal{H}(b))}^{n-1} \|f_1\|_b \, \|k_\lambda^b\|_b.$$

Now Corollary 18.14 says that $\|Q_\lambda\|_{\mathcal{L}(\mathcal{H}(b))} \leq (1-|\lambda|)^{-1}$ and Corollary 19.21 implies that $\|f_1\|_b \leq \|k_\lambda^b\|_b$. Hence,

$$\left| \frac{b^{(n)}(\lambda)}{n!} \right| \leq \frac{1}{(1-|\lambda|)^{n-1}} \|k_\lambda^b\|_b^2 = \frac{1-|b(\lambda)|^2}{(1-|\lambda|^2)(1-|\lambda|)^{n-1}},$$

which gives (19.44).

To prove (19.45), it is enough to suppose that $n \geq 2$, because when $n = 1$ equality in (19.44) is attained for any automorphism of $\mathbb{D}$. Recall that, for $w \in \mathbb{D}$, b_w denotes the automorphism on $\mathbb{D}$ defined by

$$b_w(z) = \frac{w - z}{1 - \bar{w}z}.$$

Then, a short computation yields

$$b_w^{(n)}(z) = n!\, \bar{w}^{n-1}(|w|^2 - 1)(1 - \bar{w}z)^{-n-1}.$$

Hence,

$$\begin{aligned}
(1-|\lambda|)^{n-1} & \left| \frac{b_w^{(n)}(\lambda)}{n!} \right| \frac{1-|\lambda|^2}{1-|b_w(\lambda)|^2} \\
&= \frac{(1-|\lambda|)^{n-1}(1-|\lambda|^2)(1-|w|^2)|w|^{n-1}|1-\bar{w}z|^{-n-1}}{1-|w-z|^2/|1-\bar{w}z|^2} \\
&= \frac{(1-|\lambda|)^{n-1}(1-|\lambda|^2)(1-|w|^2)|w|^{n-1}|1-\bar{w}z|^{-n+1}}{|1-\bar{w}z|^2 - |w-z|^2}.
\end{aligned}$$

But $|1-\bar{w}z|^2 - |w-z|^2 = (1-|w|^2)(1-|\lambda|^2)$, which gives

$$(1-|\lambda|)^{n-1}\left|\frac{b_w^{(n)}(\lambda)}{n!}\right|\frac{1-|\lambda|^2}{1-|b_w(\lambda)|^2} = |w|^{n-1}\left(\frac{1-|\lambda|}{|1-\bar{w}\lambda|}\right)^{n-1}.$$

Let $\vartheta \in \mathbb{R}$ such that $e^{-i\vartheta}\lambda = |\lambda|$ and let $w \longrightarrow e^{i\vartheta}$, then the last expression tends to 1 and (19.45) follows. □

Notes on Chapter 19

Section 19.1

The abstract functional embedding has been introduced by Nikolskii and Vasyunin in [140], but the idea of the coordinate-free construction of model operators seems to have first appeared in a paper of Vasyunin in 1977 [188]. This functional embedding was introduced for the construction of a model to Hilbert space contractions and its application to spectral theory including invariant subspaces, generalized spectral decompositions, similarity to normal operators. See [138, 143] for more details. The connection of the de Branges–Rovnyak spaces to functional model contractions is studied more precisely in [139, 160] and is used for instance in [53]. Note that, in contrast to [140], we do not include the purity of $\pi_*^*\pi$ in the definition of an AFE since in this text we are not really interested in the correspondence with the model contraction.

The preliminary material of this section on AFE comes mainly from [140]. The equivalence of the first three conditions in Lemma 19.6 is taken from [53], and the equivalence between (iii), (iv) and (v) is from [139]. Note that in these two papers the vector-valued situation is studied. The connection between the abstract functional embedding and de Branges–Rovnyak spaces is described by Chevrot, Fricain and Timotin in [53]. Exercises 19.1.1 and 19.1.2 are also from [53], where the completeness problem of difference quotients is discussed in a vector-valued setting.

Section 19.2

In the extreme case, Theorem 19.11 is due to Sarason [160] and independently to Nikolskii and Vasyunin [139]. In his paper, Sarason describes the Sz.-Nagy–Foiaş model of X_b. He shows that, in the particular case when b is extreme, then the characteristic function of X_b, defined by

$$\begin{array}{rccl} \Theta_{X_b}: & \mathbb{D} & \longmapsto & \mathcal{L}(\mathcal{D}_{X_b}, \mathcal{D}_{X_b^*}) \\ & \lambda & \longrightarrow & \Theta_{X_b}(\lambda) = [-X_b + \lambda D_{X_b^*}(I-\lambda X_b^*)^{-1}D_{X_b}]_{|\mathcal{D}_{X_b}}, \end{array}$$

satisfies $\Theta_{X_b}(\lambda)k_0^b = b^*(\lambda)S^*b$. (We have proved in Section 18.7 that S^*b spans $\mathcal{D}_{X_b^*}$ and it will be proved in Section 25.4 that k_0^b spans $\mathcal{D}_{X_b}$.) In this context, (19.30) says that, in the case when b is extreme, then $\mathbf{M}_{b^*}$ is the Sz.-Nagy–Foiaş model of X_b and the map $\mathfrak{W}_b Q_b^*$ implements the unitary equivalence between the operator X_b and its Sz.-Nagy–Foiaş model $\mathbf{M}_{b^*}$. The presentation based on the AFE is due to Nikolskii and Vasyunin [139, 140, 143]. The proof of Theorem 19.11 (based on the Julia operator) given in Remark 19.12 is due to Timotin [187].

Section 19.3

Theorems 19.13 and 19.14 are taken from Timotin [186]. It should be noted that, in that paper, Timotin studied the vector-valued case and obtained similar results.

Section 19.4

Lemma 19.17 is from Chevrot, Fricain and Timotin [53]. Theorem 19.18 appears in de Branges and Rovnyak's paper [64, appdx, theorem 5]. For another proof, see [15]. The proof presented here and based on the relation between the de Branges–Rovnyak and the Sz.-Nagy–Foiaş models is due to Timotin [186]. Corollary 19.21 is due to Anderson and Rovnyak [18]. Theorem 19.22 is also due to de Branges and Rovnyak [64, appdx, theorem 5] but with a different proof.

Section 19.5

Lemma 19.23 and Theorem 19.24 are due to Crofoot [61], who studied the multipliers between two spaces $\mathcal{H}(b_1)$ and $\mathcal{H}(b_2)$. Theorem 19.25 is due to Suárez [180]. Exercises 19.5.1 and 19.5.2 are also taken from [180].

Section 19.6

Theorem 19.28 is classic, and the standard proof of this fact appeals to the Littlewood subordination principle in harmonic analysis. For further detail, see, for instance, the book of Cowen and MacCluer [59]. The proof presented here is due to Jury [112]. The idea of expressing the boundedness of certain weighted composition operators in terms of the positivity of some kernels gives a new and interesting perspective on this classic theorem. Moreover, in [112], Jury uses this idea to obtain new results in other spaces of analytic functions.

Section 19.7

Quite recently, several authors [32, 126, 127] have obtained general estimates for higher-order derivatives of functions $b \in H^\infty$, $\|b\|_\infty \le 1$. Theorem 19.29 is due to Ruscheweyh [157], but the Hilbert space method presented here is due to Anderson and Rovnyak [18]. Results for functions of several variables have also been obtained in [32] and [127]. The case $\lambda = 0$ in (19.44) asserts that, if $b(z) = a_0 + a_1 z + a_2 z^2 + \cdots$, then

$$|a_n| \le 1 - |a_0|^2,$$

for every $n \ge 1$. This classic result is due to Wiener (see e.g. [32]).

20

Representation theorems for $\mathcal{H}(b)$ and $\mathcal{H}(\bar{b})$

One of the most useful properties of the Hardy space H^2 is that we have an integral representation for functions in H^2. This is not directly the case for functions in $\mathcal{H}(b)$. Of course, since $\mathcal{H}(b)$ is contained in H^2, we can use the H^2 integral representation, but we would like to have another one that is more specific to $\mathcal{H}(b)$ and reveals better the intrinsic structure of $\mathcal{H}(b)$. This is the main goal of this chapter.

In Section 20.1, we give an integral representation for functions in $\mathcal{H}(\bar{b})$ by constructing a unitary operator between $\mathcal{H}(\bar{b})$ and $H^2(\rho)$, where $\rho = (1-|b|^2)$ on $\mathbb{T}$ and $H^2(\rho)$ is the closure of the (analytic) polynomials with respect to the $L^2(\rho)$ norm. In Section 20.2, we give an intertwining relation between S_ρ^* and $X_{\bar{b}} = S^*_{|\mathcal{H}(\bar{b})}$. In Section 20.3, through the construction of an explicit unitary operator from $H^2(\mu)$ onto $\mathcal{H}(b)$, we give an integral representation for functions $f \in \mathcal{H}(b)$. Here, μ is the Clark measure associated with b. It turns out that many properties of $\mathcal{H}(b)$ spaces are reflected by the properties of the corresponding Clark measure. In Section 20.4, we construct a contractive antilinear map from $\mathcal{H}(b)$ into itself that maps the reproducing kernels into the difference quotients. When b is an extreme point of the closed unit ball of H^∞, it appears that this map is an isometry and even a conjugation, as we will see in Chapter 26. In Section 20.5, we give a simple sufficient condition for the absolute continuity of the Clark measure.

In Section 20.6, we give one application of the integral representation of $\mathcal{H}(b)$ to the study of inner divisors of the Cauchy transform. An intertwining relation between S_μ^* and X_b is given in Section 20.7. Since $\mathcal{H}(b)$ is a space of analytic functions on $\mathbb{D}$, it is natural to study the behavior of functions in $\mathcal{H}(b)$ when we approach the boundary. The question of analytic continuation through an open arc on $\mathbb{T}$ is studied in Section 20.8. The heuristic philosophy of the phenomenon is that the boundary behavior of b controls the boundary behavior of all functions in $\mathcal{H}(b)$. This is one of the major differences between $\mathcal{H}(b)$ and H^2. In general, functions in $\mathcal{H}(b)$ spaces have a better behavior on the boundary than those of a general function in H^2.

In Section 20.9, we study first the properties of multipliers of $\mathcal{H}(b)$ spaces. In particular, we show that every multiplier of $\mathcal{H}(b)$ is also a multiplier of $\mathcal{H}(\bar{b})$. It should be mentioned here that despite many interesting results, the question of characterizing the multipliers of $\mathcal{H}(b)$ is still an open problem. We show in Section 20.10 an interesting connection between multipliers of $\mathcal{H}(b)$ and Toeplitz operators on $H^2(\mu)$. In particular, this connection is used to show that $\mathcal{H}(b)$ is invariant under the shift operator if and only if b is a nonextreme point of the closed unit ball of H^∞. This is the first result where we see that the properties of $\mathcal{H}(b)$ depend on the fact that b is or is not an extreme point of the closed unit ball of H^∞. The dichotomy between the extreme and nonextreme cases will accompany us throughout this book. We also study the invariance of the set of multipliers of $\mathcal{H}(b)$ when we apply some specific transformations on b.

The last section of this chapter is devoted to the following question. Let b be in the closed unit ball of H^∞, and let Θ be an inner function. Let μ and ν denote, respectively, the Clark measures of b and Θ. When is ν absolutely continuous with respect to μ and, moreover, $d\nu/d\mu \in L^2(\mu)$? We give a nice characterization whose proof uses the integral representation of $\mathcal{H}(b)$. This characterization is then applied to detect if a Clark measure has a Dirac mass at a given point of $\mathbb{T}$.

20.1 Integral representation of $\mathcal{H}(\bar{b})$

For b a nonconstant function in the closed unit ball of H^∞, define

$$\rho(\zeta) = 1 - |b(\zeta)|^2 \qquad (\zeta \in \mathbb{T}).$$

Clearly, the function ρ depends on b and a better notation is ρ_b. But, since throughout our discussion b is fixed, we are content with the first notation. Since $\rho \in L^\infty(\mathbb{T})$, as we saw in Section 13.4, the mapping

$$\begin{array}{rccc} K_\rho : & L^2(\rho) & \longrightarrow & H^2 \\ & f & \longmapsto & P_+(\rho f) \end{array}$$

is a bounded operator whose norm is at most $\|\rho\|_\infty^{1/2}$. Moreover, by Theorem 13.15,

$$K_\rho^* = J_\rho, \tag{20.1}$$

where

$$\begin{array}{rccc} J_\rho : & H^2 & \longrightarrow & L^2(\rho) \\ & f & \longmapsto & f \end{array}$$

is the canonical injection of H^2 into $L^2(\rho)$. Finally, according to Corollary 13.16, we have

$$K_\rho J_\rho = T_\rho. \tag{20.2}$$

In other words, we have the following commutative diagram.

$$\begin{tikzcd} H^2 \arrow[r, "T_\rho"] \arrow[d, "J_\rho"'] & H^2 \\ L^2(\rho) \arrow[ur, "K_\rho"'] & \end{tikzcd} \tag{20.3}$$

The following result gives an integral representation for functions in $\mathcal{H}(\bar{b})$.

Theorem 20.1 *The operator K_ρ is a partial isometry from $L^2(\rho)$ onto $\mathcal{H}(\bar{b})$ and* $\ker K_\rho = (H^2(\rho))^\perp$.

Proof Theorem 13.10 ensures that $\ker K_\rho = (H^2(\rho))^\perp$. Hence, if we define the restricted operator

$$\begin{array}{rccc} A: & H^2(\rho) & \longrightarrow & H^2 \\ & f & \longmapsto & K_\rho f, \end{array}$$

then A is injective. Moreover, by (1.37),

$$A^* f = K_\rho^* f \qquad (f \in H^2). \tag{20.4}$$

Using (20.1), (20.2) and Theorem 12.4, we have

$$I - T_{\bar{b}} T_b = T_{1-|b|^2} = T_\rho = K_\rho K_\rho^*.$$

Hence, by (20.4), we can write this identity as

$$I - T_{\bar{b}} T_b = AA^*. \tag{20.5}$$

Therefore, by Corollary 16.12 and (20.5), A is an isometry from $H^2(\rho)$ onto the space $\mathcal{M}((I - T_{\bar{b}} T_b)^{1/2}) = \mathcal{H}(\bar{b})$. This is equivalent to saying that K_ρ is a partial isometry from $L^2(\rho)$ onto $\mathcal{H}(\bar{b})$ and $\ker K_\rho = (H^2(\rho))^\perp$. □

Corollary 20.2 *Let $f \in H^2$. Then f belongs to $\mathcal{H}(\bar{b})$ if and only if there exists a function $g \in L^2(\rho)$ such that $T_{\bar{b}} f = P_+(\rho g)$.*

Proof It suffices to apply Theorems 20.1 and 17.8. □

Let us more explicitly explain what Theorem 20.1 says. This result says that the mapping

$$\begin{array}{rccc} \mathbf{K}_\rho: & H^2(\rho) & \longrightarrow & \mathcal{H}(\bar{b}) \\ & g & \longmapsto & P_+(\rho g) \end{array}$$

is a unitary operator, i.e.

$$\mathbf{K}_\rho \mathbf{K}_\rho^* = I_{\mathcal{H}(\bar{b})} \quad \text{and} \quad \mathbf{K}_\rho^* \mathbf{K}_\rho = I_{H^2(\rho)}. \tag{20.6}$$

Hence, for each $f \in \mathcal{H}(\bar{b})$, there is a unique $g \in H^2(\rho)$ such that $f = K_\rho g = P_+(\rho g)$, or equivalently

$$f(z) = \int_{\mathbb{T}} \frac{\rho(\zeta) g(\zeta)}{1 - z\bar{\zeta}} \, dm(\zeta) \qquad (z \in \mathbb{D}). \tag{20.7}$$

Moreover, we have $\|f\|_{\bar{b}} = \|K_\rho g\|_{\bar{b}} = \|g\|_{H^2(\rho)}$. In the light of polarization identity (1.16), the last identity implies that

$$\langle g_1, g_2 \rangle_{H^2(\rho)} = \langle K_\rho g_1, K_\rho g_2 \rangle_{\bar{b}} \qquad (g_1, g_2 \in H^2(\rho)).$$

Thus, the representation (20.7) is rewritten as

$$f(z) = \langle g, k_z \rangle_{H^2(\rho)} = \langle K_\rho g, K_\rho k_z \rangle_{\bar{b}} = \langle f, K_\rho k_z \rangle_{\bar{b}}.$$

This representation reveals that the reproducing kernel of $\mathcal{H}(\bar{b})$ is given by

$$k_z^{\bar{b}} = K_\rho k_z. \tag{20.8}$$

Note that, as we did for k_z^b in Theorem 18.11, the relation (20.8) also follows from Theorem 16.13. Since the flavor of this latter approach is slightly different, we mention it below.

Theorem 20.3 *The reproducing kernel of $\mathcal{H}(\bar{b})$ is*

$$k_z^{\bar{b}} = (I - T_{\bar{b}} T_b) k_z = P_+(\rho k_z) \qquad (z \in \mathbb{D}).$$

Moreover, the norm of the evaluation functional

$$f(z) = \langle f, k_z^{\bar{b}} \rangle_{\bar{b}} \qquad (f \in \mathcal{H}(\bar{b}),\ z \in \mathbb{D})$$

is equal to

$$\|k_z^{\bar{b}}\|_{\bar{b}} = (k_z^{\bar{b}}(z))^{1/2}.$$

As a consequence, we can also give another formula for the scalar product in $\mathcal{H}(b)$. Let $f_1, f_2 \in \mathcal{H}(b)$. Hence, there are two functions $g_1, g_2 \in H^2(\rho)$ such that $T_{\bar{b}} f_i = P_+(\rho g_i)$, $i = 1, 2$. Then, using the fact that K_ρ is unitary from $H^2(\rho)$ onto $\mathcal{H}(\bar{b})$, and according to Theorem 17.8, we get

$$\langle f_1, f_2 \rangle_b = \langle f_1, f_2 \rangle_2 + \langle g_1, g_2 \rangle_{L^2(\rho)}$$

or, in other words,

$$\langle f_1, f_2 \rangle_b = \langle f_1, f_2 \rangle_2 + \langle \rho g_1, g_2 \rangle_2. \tag{20.9}$$

Exercises

Exercise 20.1.1 Let $\varphi \in H^\infty$ and let $(\varphi_n)_n$ be a uniformly bounded sequence in H^∞ such that $\varphi_n \longrightarrow \varphi$ a.e. on $\mathbb{T}$. Let b be a function in the closed unit ball of H^∞. Show that, for each $f \in \mathcal{H}(b)$, we have

$$\lim_{n\to+\infty} \|T_{\bar{\varphi}_n} f - T_{\bar{\varphi}} f\|_b = 0.$$

Hint: Put $\psi_n = \varphi_n - \varphi$ and let $g \in L^2(\rho)$ such that $T_{\bar{b}} f = K_\rho g$. Use Theorem 13.21 to get

$$T_{\bar{b}} T_{\bar{\psi}_n} f = T_{\bar{\psi}_n} T_{\bar{b}} f = K_\rho(\bar{\psi}_n g).$$

Hence,

$$\|T_{\bar{\psi}_n} f\|_b^2 \le \|T_{\bar{\psi}_n} f\|_2^2 + \|\bar{\psi}_n g\|_{L^2(\rho)}^2 \le \|\bar{\psi}_n f\|_2^2 + \|\bar{\psi}_n g\|_{L^2(\rho)}^2.$$

Then use the dominated convergence theorem.

Exercise 20.1.2 Let b be a function in the closed unit ball of H^∞ and let E be a closed subspace of $\mathcal{H}(b)$ that is X_b invariant. Show that $T_{\bar{\varphi}} E \subset E$ for any $\varphi \in H^\infty$.
Hint: Use Exercise 20.1.1 or apply Corollary 12.50.

20.2 K_ρ intertwines S_ρ^* and $X_{\bar{b}}$

Consider the operator

$$\begin{array}{rccc} X_{\bar{b}}: & \mathcal{H}(\bar{b}) & \longrightarrow & \mathcal{H}(\bar{b}) \\ & f & \longmapsto & S^* f, \end{array}$$

which was defined in Section 18.7. Taking $\varphi = \rho$, (13.28) is rewritten as

$$K_\rho i_\rho S_\rho^* = S^* K_\rho i_\rho,$$

where i_ρ is the injection of $H^2(\rho)$ into $L^2(\rho)$. The operators $K_\rho i_\rho S_\rho^*$ and $S^* K_\rho i_\rho$ map $H^2(\rho)$ into H^2 and their image is $\mathcal{H}(\bar{b})$. If we restrict their ranges, the last identity implies that

$$\mathbf{K}_\rho S_\rho^* = X_{\bar{b}} \mathbf{K}_\rho, \tag{20.10}$$

where $\mathbf{K}_\rho \in \mathcal{L}(H^2(\rho), \mathcal{H}(\bar{b}))$ is given by

$$\begin{array}{rccc} \mathbf{K}_\rho: & H^2(\rho) & \longrightarrow & \mathcal{H}(\bar{b}) \\ & f & \longmapsto & K_\rho f. \end{array}$$

The identity means that $\mathbf{K}_\rho$ intertwines S_ρ^* and $X_{\bar{b}}$. However, by (20.6), $\mathbf{K}_\rho$ is a unitary operator, and thus we can say that S_ρ^* and $X_{\bar{b}}$ are unitarily equivalent, which can be rephrased in the following commutative diagram.

$$\begin{tikzcd} H^2(\rho) \arrow[r, "S_\rho^*"] \arrow[d, "K_\rho"'] & H^2(\rho) \arrow[d, "K_\rho"] \\ \mathcal{H}(\bar{b}) \arrow[r, "X_{\bar{b}}"'] & \mathcal{H}(\bar{b}) \end{tikzcd} \tag{20.11}$$

Exercise

Exercise 20.2.1 Let b be an extreme point of the closed unit ball of H^∞ and define

$$\begin{array}{rccc} T_B: & H^2 & \longrightarrow & H^2 \oplus L^2(\rho) \\ & f & \longmapsto & T_b f \oplus (-K_\rho^* f), \end{array}$$

where $\rho = \Delta^2 = 1 - |b|^2$.

(i) Show that T_B is an isometry from H^2 into $H^2 \oplus L^2(\rho)$ and check that

$$\mathcal{H}(T_B) = \{f \oplus K_\rho^{-1} T_{\bar{b}} f : f \in \mathcal{H}(b)\}.$$

(ii) Show that the map

$$\begin{array}{rccc} P: & H^2 \oplus L^2(\rho) & \longrightarrow & H^2 \\ & f \oplus g & \longmapsto & f \end{array}$$

is a unitary map from $\mathcal{H}(T_B)$ onto $\mathcal{H}(b)$.

(iii) Check that $S^* \oplus Z_\rho^*$ leaves $\mathcal{H}(T_B)$ invariant and that, for any $h \in \mathcal{H}(T_B)$, we have

$$X_b P h = P(S^* \oplus Z_\rho^*) h.$$

20.3 Integral representation of $\mathcal{H}(b)$

The positive Borel measure μ on $\mathbb{T}$ whose Poisson integral is the real part of $(1 + b)/(1 - b)$, the so-called Clark measure of b, was introduced in Section 13.7. Indeed, we recall that, by (13.43), we have

$$\frac{1 + b(z)}{1 - b(z)} = \int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z}\, d\mu(\zeta) + i\,\Im\left(\frac{1 + b(0)}{1 - b(0)}\right).$$

In this section, we will use this measure to give an integral representation for functions in $\mathcal{H}(b)$. The simple identity

$$\frac{1+Z}{1-Z}+\frac{1+W}{1-W}=\frac{2(1-ZW)}{(1-Z)(1-W)} \qquad (Z,W\in\mathbb{D}) \tag{20.12}$$

will be exploited several times below. The key point of the integral representation for functions in $\mathcal{H}(b)$ is the following result.

Lemma 20.4 *Let μ denote the Clark measure of b. Then*

$$\langle k_w,k_z\rangle_\mu=\frac{k_w^b(z)}{(1-\overline{b(w)})\,(1-b(z))} \qquad (z,w\in\mathbb{D}).$$

Proof Using (20.12), we have

$$\begin{aligned}
\langle k_w,k_z\rangle_\mu &= \int_{\mathbb{T}} k_w(\zeta)\,\overline{k_z(\zeta)}\,d\mu(\zeta)\\
&= \int_{\mathbb{T}} \frac{1}{(1-\zeta\bar{w})(1-\bar{\zeta}z)}\,d\mu(\zeta)\\
&= \frac{1}{2(1-\bar{w}z)}\left(\int_{\mathbb{T}}\frac{1+\zeta\bar{w}}{1-\zeta\bar{w}}\,d\mu(\zeta)+\int_{\mathbb{T}}\frac{1+\bar{\zeta}z}{1-\bar{\zeta}z}\,d\mu(\zeta)\right)\\
&= \frac{1}{2(1-\bar{w}z)}\left(\overline{\int_{\mathbb{T}}\frac{\zeta+w}{\zeta-w}\,d\mu(\zeta)}+\int_{\mathbb{T}}\frac{\zeta+z}{\zeta-z}\,d\mu(\zeta)\right).
\end{aligned}$$

Hence, by (13.43),

$$\langle k_w,k_z\rangle_\mu=\frac{1}{2(1-\bar{w}z)}\left(\frac{1+\overline{b(w)}}{1-\overline{b(w)}}+\frac{1+b(z)}{1-b(z)}\right).$$

Applying (20.12) once more, we obtain

$$\frac{1+\overline{b(w)}}{1-\overline{b(w)}}+\frac{1+b(z)}{1-b(z)}=\frac{2(1-\overline{b(w)}b(z))}{(1-\overline{b(w)})(1-b(z))}.$$

Thus, by the formula for k_w^b in Theorem 18.11, we have

$$\begin{aligned}
\langle k_w,k_z\rangle_\mu &= \frac{1-\overline{b(w)}b(z)}{1-\bar{w}z}\,\frac{1}{(1-\overline{b(w)})(1-b(z))}\\
&= \frac{k_w^b(z)}{(1-\overline{b(w)})(1-b(z))}.
\end{aligned}$$

This completes the proof. □

Since

$$k_w^b(z)=\langle k_w^b,k_z^b\rangle_b,$$

the relation

$$\langle k_w, k_z \rangle_\mu = \frac{k_w^b(z)}{(1-\overline{b(w)})(1-b(z))}$$

can be rewritten as

$$\langle k_w, k_z \rangle_\mu = \langle (1-\overline{b(w)})^{-1} k_w^b,\ (1-\overline{b(z)})^{-1} k_z^b \rangle_b. \tag{20.13}$$

This representation suggests that the mapping

$$\begin{array}{rcl} H^2(\mu) & \longrightarrow & \mathcal{H}(b) \\ k_w & \longmapsto & k_w^b/(1-\overline{b(w)}) \end{array}$$

should play an important role in the theory of $\mathcal{H}(b)$ spaces. We take a slightly indirect approach to arrive again at this operator. According to Lemma 13.9, the mapping

$$\mathbf{V}_b f(z) = (1-b(z))(K_\mu f)(z) \qquad (f \in L^2(\mu),\ z \in \mathbb{D}) \tag{20.14}$$

is a well-defined continuous operator from $L^2(\mu)$ into $\mathrm{Hol}(\mathbb{D})$. But we can say more about the kernel and range of this operator.

Theorem 20.5 *The mapping* $\mathbf{V}_b$ *is a partial isometry from* $L^2(\mu)$ *onto* $\mathcal{H}(b)$ *and* $\ker \mathbf{V}_b = (H^2(\mu))^\perp$. *Moreover, we have*

$$\mathbf{V}_b k_w = \frac{k_w^b}{1-\overline{b(w)}} \qquad (w \in \mathbb{D}). \tag{20.15}$$

Proof Let $z, w \in \mathbb{D}$. Using the definition of K_μ and Lemma 20.4, we have

$$\begin{aligned} (\mathbf{V}_b k_w)(z) &= (1-b(z))(K_\mu k_w)(z) \\ &= (1-b(z)) \int_{\mathbb{T}} \frac{k_w(\zeta)}{1-\bar{\zeta} z}\, d\mu(\zeta) \\ &= (1-b(z)) \int_{\mathbb{T}} \frac{1}{(1-\bar{w}\zeta)(1-\bar{\zeta} z)}\, d\mu(\zeta) \\ &= (1-b(z)) \langle k_w, k_z \rangle_\mu \\ &= (1-\overline{b(w)})^{-1} k_w^b(z). \end{aligned}$$

Hence, by (20.13), we obtain

$$\langle k_w, k_z \rangle_\mu = \langle \mathbf{V}_b k_w, \mathbf{V}_b k_z \rangle_b.$$

In particular,

$$\|g\|_{L^2(\mu)} = \|\mathbf{V}_b g\|_b \tag{20.16}$$

for all

$$g \in L = \{\alpha_1 k_{w_1} + \cdots + \alpha_n k_{w_n} : \alpha_i \in \mathbb{C},\ w_i \in \mathbb{D}\}.$$

Now, let $g \in H^2(\mu)$. Then, by Theorem 5.11, there exists a sequence $(g_n)_{n\geq 1}$, $g_n \in L$, that converges to g in $H^2(\mu)$. We have already mentioned that, according to Lemma 13.9, $\mathbf{V}_b$ is continuous as an operator from $L^2(\mu)$ into $\mathrm{Hol}(\mathbb{D})$. Therefore, $(\mathbf{V}_b g_n)_{n\geq 1}$ converges to $\mathbf{V}_b g$ in the topology of $\mathrm{Hol}(\mathbb{D})$. In particular,

$$\lim_{n\to\infty} (\mathbf{V}_b g_n)(z) = (\mathbf{V}_b g)(z) \qquad (z \in \mathbb{D}).$$

On the other hand, by (20.16), $(\mathbf{V}_b g_n)_{n\geq 1}$ is a Cauchy sequence in $\mathcal{H}(b)$, and hence it converges to a function $f \in \mathcal{H}(b)$. Since $\mathcal{H}(b)$ is a reproducing kernel Hilbert space, and thus the evaluation functionals are continuous on $\mathcal{H}(b)$, we have

$$\lim_{n\to\infty} (\mathbf{V}_b g_n)(z) = f(z) \qquad (z \in \mathbb{D}).$$

See also Theorem 18.11. Thus, we obtain $\mathbf{V}_b g = f$. Moreover, using (20.16) once more, we have

$$\|\mathbf{V}_b g\|_b = \|f\|_b = \lim_{n\to\infty} \|\mathbf{V}_b g_n\|_b = \lim_{n\to\infty} \|g_n\|_{L^2(\mu)} = \|g\|_{L^2(\mu)}.$$

This shows that $\mathbf{V}_b$ is an isometry from $H^2(\mu)$ into $\mathcal{H}(b)$. In particular, its range is a closed subspace of $\mathcal{H}(b)$. But, by (20.15), the range of $\mathbf{V}_b$ contains all elements k_w^b, $w \in \mathbb{D}$. Hence, by (9.7), $\mathbf{V}_b H^2(\mu) = \mathcal{H}(b)$.

That $\ker \mathbf{V}_b = (H^2(\mu))^\perp$ follows immediately from Theorem 13.10. □

Theorem 20.5 shows that the operator

$$\begin{array}{rccc} \mathbf{V}_b: & L^2(\mu) & \longrightarrow & \mathcal{H}(b) \\ & g & \longmapsto & (1-b)K_\mu g \end{array}$$

is a partial isometry, and thus the restriction operator

$$\begin{array}{rccc} V_b: & H^2(\mu) & \longrightarrow & \mathcal{H}(b) \\ & g & \longmapsto & (1-b)K_\mu g \end{array}$$

is a unitary operator between $H^2(\mu)$ and $\mathcal{H}(b)$. In other words, given $f \in \mathcal{H}(b)$, there is a unique $g \in H^2(\mu)$ such that $f = V_b g$, i.e.

$$f(z) = (1-b(z)) \int_{\mathbb{T}} \frac{g(\zeta)}{1-\bar{\zeta}\, z}\, d\mu(\zeta) \qquad (z \in \mathbb{D}),$$

and, moreover, $\|f\|_b = \|g\|_{L^2(\mu)}$.

Exercises

Exercise 20.3.1 Show that

$$X_b V_b k_w = \bar{w} \frac{1-\overline{b(w)}b}{1-\overline{b(w)}} k_w - \frac{\overline{b(w)}}{1-\overline{b(w)}} S^* b \qquad (w \in \mathbb{D}).$$

In particular, show that

$$X_b V_b k_0 = -\frac{\overline{b(0)}}{1-\overline{b(0)}} S^* b.$$

Hint: First use (20.15) to get

$$V_b k_w = \frac{k_w^b}{1-\overline{b(w)}} \qquad (w \in \mathbb{D}).$$

Remember $k_w^b = (1-\overline{b(w)}b)k_w$. Then apply $X_b = S^*$. The formula for $S^*(fg)$ in Corollary 8.11 will be useful.

Exercise 20.3.2 Show that

$$(K_\mu k_w)(z) = \frac{k_w^b(z)}{(1-\overline{b(w)})(1-b(z))} \qquad (z, w \in \mathbb{D}).$$

In particular, show that

$$(K_\mu k_w)(0) = \frac{1-\overline{b(w)}b(0)}{(1-\overline{b(w)})(1-b(0))} \qquad (w \in \mathbb{D}).$$

Hint: Use Lemma 20.4. Remember that $(K_\mu k_w)(z) = \langle k_w, k_z \rangle_\mu$.

Exercise 20.3.3 Let $f \in \mathcal{H}(\bar{b})$ and $g \in L^2(\rho)$ such that $f = K_\rho g$ (see Theorem 20.1). Let $\mu_1^{(a)}$ be the absolutely continuous component of the Clark measure associated with b.

(i) Show that

$$\frac{d\mu_1^{(a)}}{dm} = \frac{1-|b|^2}{|1-b|^2} \qquad \text{(a.e. on } \mathbb{T}).$$

(ii) Show that $(1-b)f = V_b(|1-b|^2\tilde{g})$, where $\tilde{g}$ vanishes on the singular component of μ_1 (if there is one) and coincides with g otherwise.

Exercise 20.3.4 The aim of this exercise is to give another proof of (19.40). Let b be a function in the closed unit ball of H^∞, $\lambda \in \mathbb{D}$, $\zeta = (1+\lambda)/(1+\bar{\lambda})$ and let $b_\lambda^\sharp$ be defined by (19.42). Denote by $\mu_{b_\lambda^\sharp}$ the Clark measure associated with $b_\lambda^\sharp$ and by $\mu_{\bar{\zeta}b}$ that associated with $\bar{\zeta}b$.

(i) Use Exercise 19.5.2 to prove that $L^2(\mu_{b_\lambda^\sharp}) = L^2(\mu_{\bar{\zeta}b})$.

(ii) Show that, for any $f \in L^2(\mu_{b_\lambda^\sharp})$, we have

$$V_{b_\lambda^\sharp} f = \frac{c(1+\lambda)}{1-\bar{\lambda}b} V_{\bar{\zeta}b} f.$$

Hint: Use Exercise 19.5.1.

(iii) Deduce that $(1-\bar{\lambda}b)\mathcal{H}(b_\lambda \circ b) = \mathcal{H}(\bar{\zeta}b) = \mathcal{H}(b)$.

20.4 A contractive antilinear map on $\mathcal{H}(b)$

Using the integral representation of $\mathcal{H}(\bar{b})$, we can construct a contractive antilinear map from $\mathcal{H}(b)$ into itself that maps the reproducing kernel of $\mathcal{H}(b)$ into the difference quotients.

Theorem 20.6 *Let b be a function in the closed unit ball of H^∞. For $f \in H^2$, define*

$$\mathfrak{C}_b((I - T_b T_{\bar{b}})f) = b\bar{z}\bar{f} - \bar{z}\,\overline{T_{\bar{b}} f}.$$

Then $\mathfrak{C}_b$ extends to a contractive antilinear map from $\mathcal{H}(b)$ into itself such that, for each $\lambda \in \mathbb{D}$, we have

$$\mathfrak{C}_b k_\lambda^b = \hat{k}_\lambda^b.$$

Furthermore, if b is extreme, then $\mathfrak{C}_b$ is isometric.

Proof Let $f \in H^2$ and $g = (I - T_b T_{\bar{b}})f$. Let us first verify that $\mathfrak{C}_b g \in H^2$. Remember the conjugation J on $L^2(\mathbb{T})$ (introduced in Section 14.6), which was defined by

$$(Jf)(\zeta) = \overline{\zeta f(\zeta)} \qquad (f \in L^2(\mathbb{T}),\ \zeta \in \mathbb{T}),$$

and which satisfies $JP_+ = P_- J$. Then write

$$\begin{aligned} \mathfrak{C}_b g &= b\bar{z}\bar{f} - JP_+(\bar{b}f) \\ &= b\bar{z}\bar{f} - P_- \bar{z} b\bar{f} \\ &= P_+(b\bar{z}\bar{f}). \end{aligned} \tag{20.17}$$

In particular, $\mathfrak{C}_b g \in H^2$. According to Theorem 17.8, $\mathfrak{C}_b g \in \mathcal{H}(b)$ if and only if $T_{\bar{b}}\mathfrak{C}_b g \in \mathcal{H}(\bar{b})$. But we have

$$\begin{aligned} T_{\bar{b}}\mathfrak{C}_b g &= T_{\bar{b}} P_+(\bar{z}\bar{f}b) \\ &= P_+(|b|^2 \bar{z}\bar{f}) \\ &= -P_+(\rho\bar{z}\bar{f}) = -K_\rho(\bar{z}\bar{f}). \end{aligned} \tag{20.18}$$

Since $\bar{z}\bar{f} \in L^2(\rho)$, it follows from Theorem 20.1 that $K_\rho(\bar{z}\bar{f}) \in \mathcal{H}(\bar{b})$, whence $\mathfrak{C}_b g \in \mathcal{H}(b)$.

Let us verify that $\mathfrak{C}_b$ is contractive. According to (20.9), (20.17) and (20.18), we have

$$\|\mathfrak{C}_b g\|_b^2 = \|P_+(b\bar{z}\bar{f})\|_2^2 + \|P_{H^2(\rho)}\bar{z}\bar{f}\|_{L^2(\rho)}^2,$$

where $P_{H^2(\rho)}$ denotes the orthogonal projection from $L^2(\rho)$ onto $H^2(\rho)$. Hence we get

$$\|\mathfrak{C}_b g\|_b^2 \le \|P_+(b\bar{z}\bar{f})\|_2^2 + \|\rho\bar{z}\bar{f}\|_2^2 = \|P_+(b\bar{z}\bar{f})\|_2^2 + \|f\|_2^2 - \|bf\|_2^2. \tag{20.19}$$

Now using the fact that J is isometric and $JP_+ = P_- J$, we have

$$\begin{aligned}
\|P_+ b\bar{z}\bar{f}\|_2^2 &= \|JP_+ b\bar{z}\bar{f}\|_2^2 \\
&= \|P_- Jb\bar{z}\bar{f}\|_2^2 \\
&= \|P_- \bar{b}f\|_2^2 \\
&= \|bf\|_2^2 - \|P_+\bar{b}f\|_2^2 \\
&= \|bf\|_2^2 - \|T_{\bar{b}}f\|_2^2.
\end{aligned}$$

That implies

$$\|\mathfrak{C}_b g\|_b^2 \le \|f\|_2^2 - \|T_{\bar{b}}f\|_2^2 = \|(I - T_b T_{\bar{b}})^{1/2} f\|_2^2 = \|g\|_b^2.$$

We have thus proved that $\mathfrak{C}_b$ is contractive on the set $\{(I - T_b T_{\bar{b}})f : f \in H^2\}$, which is a dense set according to Corollary 17.6. Hence it extends to a contractive map from $\mathcal{H}(b)$ into itself.

Let us now compute the action of $\mathfrak{C}_b$ on reproducing kernels. We use the fact that $k_\lambda^b = (I - T_b T_{\bar{b}})k_\lambda$. Hence,

$$\mathfrak{C}_b k_\lambda^b = b\bar{z}\overline{k_\lambda} - \bar{z}\,\overline{T_{\bar{b}}k_\lambda}.$$

But remember that $T_{\bar{b}}k_\lambda = \overline{b(\lambda)}k_\lambda$, which gives, for any $\zeta \in \mathbb{T}$,

$$\begin{aligned}
(\mathfrak{C}_b k_\lambda^b)(\zeta) &= b(\zeta)\bar{\zeta}\overline{k_\lambda(\zeta)} - \bar{\zeta}b(\lambda)\overline{k_\lambda(\zeta)} \\
&= (b(\zeta) - b(\lambda))\overline{\zeta k_\lambda(\zeta)} \\
&= \frac{b(\zeta) - b(\lambda)}{\zeta - \lambda}.
\end{aligned}$$

This proves that $\mathfrak{C}_b k_\lambda^b = \hat{k}_\lambda^b$.

It remains to note that, when b is extreme, then $H^2(\rho) = L^2(\rho)$ and then K_ρ is an isometry from $L^2(\rho)$ onto $\mathcal{H}(\bar{b})$. In particular, we have equality in (20.19), which gives $\|\mathfrak{C}_b g\|_b = \|g\|_b$, for any $g \in \mathcal{R}(I - T_b T_{\bar{b}})$. This proves that $\mathfrak{C}_b$ extends to an isometric map from $\mathcal{H}(b)$ into itself when b is extreme. □

Let Θ be an inner function. Then, according to Theorem 12.19, we know that, for any $f \in \mathcal{H}(\Theta) = K_\Theta$, we have $T_{\bar{\Theta}}f = 0$ and that gives

$$\mathfrak{C}_\Theta f = \mathfrak{C}_\Theta((I - T_\Theta T_{\bar{\Theta}})f) = \Theta\bar{z}\bar{f}.$$

Hence, we see that $\mathfrak{C}_\Theta$ coincides with the conjugation on K_Θ introduced in Section 14.4. We can in fact further generalize this. More precisely, in Section 26.6, we will see that, in the case where b is an extreme point, then $\mathfrak{C}_b$ is a conjugation on $\mathcal{H}(b)$, i.e. an antilinear map that is isometric and satisfies $\mathfrak{C}_b^2 = I$. To prove this result, we will use a different representation of $\mathfrak{C}_b$, which seems interesting in its own right.

Remark 20.7 Using the link between $\mathcal{H}(b)$ and $\mathcal{H}(b^*)$, we saw in Corollary 19.21 an estimate on the norm of $\hat{k}_\lambda^b$ and we observe, in the case when b is extreme, that this estimate gives exactly the norm. Using this contractive antilinear map $\mathfrak{C}_b$, we can give another proof of this result. Indeed, according to Theorem 20.6, we have

$$\|\hat{k}_\lambda^b\|_b^2 = \|\mathfrak{C}_b k_\lambda^b\|_b^2 \leq \|k_\lambda^b\|_b^2 = \frac{1-|b(\lambda)|^2}{1-|\lambda|^2}.$$

Moreover, if b is extreme, then we know that $\mathfrak{C}_b$ is isometric and that gives equality in the previous estimate.

20.5 Absolute continuity of the Clark measure

The following result provides an integral representation for the norm of the singular part of the Clark measure. This formula will provide a sufficient condition for the absolute continuity of the Clark measure.

Lemma 20.8 *Let b be a function in the closed unit ball of H^∞, and let $d\mu_b = h\,dm + d\mu_s$ be the Lebesgue decomposition of its Clark measure with respect to m. Then*

$$\|\mu_s\| = \lim_{\varepsilon\to 0}\lim_{r\to 1^-}\int_{|1-b(e^{i\vartheta})|<\varepsilon} \frac{1-r^2|b(e^{i\vartheta})|^2}{|1-rb(e^{i\vartheta})|^2}\,\frac{d\vartheta}{2\pi}.$$

Proof Since, for each $0 \leq r \leq 1$, the Poisson kernel

$$P_r(z) = \frac{1-r^2|z|^2}{|1-rz|^2} \qquad (z \in \mathbb{D})$$

is harmonic on $\mathbb{D}$, then

$$z \longmapsto P_r(b(z)) = \frac{1-r^2|b(z)|^2}{|1-rb(z)|^2}$$

is also harmonic on $\mathbb{D}$. Thus

$$\int_0^{2\pi} \frac{1-r^2|b(e^{i\vartheta})|^2}{|1-rb(e^{i\vartheta})|^2}\,\frac{d\vartheta}{2\pi} = \frac{1-r^2|b(0)|^2}{|1-rb(0)|^2}$$

and

$$\lim_{r\to 1}\frac{1-r^2|b(0)|^2}{|1-rb(0)|^2} = \frac{1-|b(0)|^2}{|1-b(0)|^2} = \int_0^{2\pi} d\mu_b(e^{i\vartheta}).$$

On the other hand, for $\varepsilon > 0$, since $P_r(b(e^{i\vartheta})) \longrightarrow P_1(b(e^{i\vartheta}))$, as $r \longrightarrow 1$, uniformly on $|1-b(e^{i\vartheta})| > \varepsilon$, we get

$$\lim_{r\to 1^-}\int_{|1-b(e^{i\vartheta})|\geq\varepsilon} P_r(b(e^{i\theta}))\,\frac{d\vartheta}{2\pi} = \int_{|1-b(e^{i\vartheta})|\geq\varepsilon} P_1(b(e^{i\theta}))\,\frac{d\vartheta}{2\pi}.$$

Thus

$$\lim_{r\to 1^-}\int_{|1-b(e^{i\vartheta})|<\varepsilon}\frac{1-r^2|b(e^{i\vartheta})|^2}{|1-rb(e^{i\vartheta})|^2}\,\frac{d\vartheta}{2\pi}=\|\mu_b\|-\int_{|1-b(e^{i\vartheta})|\geq\varepsilon}P_1(b(e^{i\theta}))\,\frac{d\vartheta}{2\pi}.$$

But

$$(P_1\circ b)(e^{i\vartheta})=\frac{1-|b(e^{i\vartheta})|^2}{|1-b(e^{i\vartheta})|^2}=h(e^{i\vartheta})$$

and $h\in L^1(\mathbb{T})$. Hence,

$$\lim_{\varepsilon\to 0}\int_{|1-b(e^{i\vartheta})|\geq\varepsilon}P_1(b(e^{i\vartheta}))\,\frac{d\vartheta}{2\pi}=\int_0^{2\pi}h(e^{i\vartheta})\,\frac{d\vartheta}{2\pi}.$$

Finally we obtain

$$\begin{aligned}\lim_{\varepsilon\to 0}\lim_{r\to 1^-}\int_{|1-b(e^{i\vartheta})|<\varepsilon}\frac{1-r^2|b(e^{i\vartheta})|^2}{|1-rb(e^{i\vartheta})|^2}\,\frac{d\vartheta}{2\pi}&=\|\mu_b\|-\int_0^{2\pi}h(e^{i\vartheta})\,\frac{d\vartheta}{2\pi}\\&=\int_0^{2\pi}d\mu_s(e^{i\vartheta})=\|\mu_s\|.\end{aligned}$$

This completes the proof. □

Corollary 20.9 *Let b be a function in the closed unit ball of H^∞. Assume that $(1-b)^{-1}\in L^2(\mathbb{T})$. Then the Clark measure μ_b associated with b is absolutely continuous with respect to Lebesgue measure on $\mathbb{T}$.*

Proof Note that $|1-b(e^{i\vartheta})|\leq 2|1-rb(e^{i\vartheta})|$ a.e. on $\mathbb{T}$, and then

$$\frac{1-r^2|b(e^{i\vartheta})|^2}{|1-rb(e^{i\vartheta})|^2}\leq\frac{4}{|1-b(e^{i\vartheta})|^2}.$$

Since $1/(1-b)\in L^2(\mathbb{T})$, we can apply the dominated convergence theorem to get

$$\lim_{r\to 1}\int_{|1-b(e^{i\vartheta})|<\varepsilon}\frac{1-r^2|b(e^{i\vartheta})|^2}{|1-rb(e^{i\vartheta})|^2}\,\frac{d\vartheta}{2\pi}=\int_{|1-b(e^{i\vartheta})|<\varepsilon}\frac{1-|b(e^{i\vartheta})|^2}{|1-b(e^{i\vartheta})|^2}\,\frac{d\vartheta}{2\pi}.$$

Since $(1-|b|^2)/(|1-b|^2)\in L^1(\mathbb{T})$, the last integral tends to 0 when $\varepsilon\longrightarrow 0$. Hence, by Lemma 20.8, we get that $\|\mu_s\|=0$, that is, μ_b is absolutely continuous. □

20.6 Inner divisors of the Cauchy transform

If μ is a finite complex Borel measure on $\mathbb{T}$, then, according to Theorem 13.4, its Cauchy transform C_μ belongs to H^p for $0<p<1$ and so has an inner–outer factorization. Using Theorem 20.5, we can easily prove the following result.

Theorem 20.10 *Let μ be a finite complex Borel measure on $\mathbb{T}$ and let Θ be an inner function that divides the inner factor of C_μ. Then, there exists a measure σ such that $C_\mu = \Theta C_\sigma$ and $\|\sigma\| \leq \|\mu\|$.*

Proof We can, of course, assume that $\|\mu\| = 1$ and we choose b so that its Clark measure μ_b is precisely $|\mu|$. In other words, define the analytic function b by the formula

$$\frac{1+b(z)}{1-b(z)} = \int_{\mathbb{T}} \frac{\zeta+z}{\zeta-z}\, d|\mu|(\zeta) \qquad (z \in \mathbb{D}).$$

We can easily check that b is in the closed unit ball of H^∞. A well-known consequence of the Radon–Nikodym theorem says that there exists a measurable function h on $\mathbb{T}$ such that $|h(\zeta)| = 1$ for all $\zeta \in \mathbb{T}$ and such that $d\mu = h\, d|\mu| = h\, d\mu_b$. Hence

$$(1-b(z))C_\mu(z) = (1-b(z)) \int_{\mathbb{T}} \frac{d\mu(\zeta)}{1-z\bar{\zeta}} = (1-b(z)) \int_{\mathbb{T}} \frac{h(\zeta)}{1-z\bar{\zeta}}\, d\mu_b(\zeta),$$

which means that $(1-b)C_\mu = V_b h$. Moreover, note that

$$\|h\|^2_{L^2(\mu_b)} = \int_{\mathbb{T}} |h|^2\, d\mu_b = \mu_b(\mathbb{T}) = \|\mu\| = 1.$$

According to Theorem 20.5, we deduce that $(1-b)C_\mu \in \mathcal{H}(b)$ and

$$\|(1-b)C_\mu\|_b = \|V_b h\|_b \leq \|h\|_{L^2(\mu_b)} = 1.$$

Now, if Θ is an inner function, Theorem 18.13 implies that $T_{\bar{\Theta}}((1-b)C_\mu) \in \mathcal{H}(b)$ and

$$\|T_{\bar{\Theta}}((1-b)C_\mu)\|_b \leq 1.$$

In particular, using Theorem 20.5 once more, there exists $g \in H^2(\mu_b)$ such that $(1-b)K_{\mu_b}(g) = T_{\bar{\Theta}}((1-b)C_\mu)$ and $\|g\|_{L^2(\mu_b)} \leq 1$. But if Θ divides the inner factor of C_μ, then it divides the inner factor of $(1-b)C_\mu$, and then

$$T_{\bar{\Theta}}((1-b)C_\mu) = \frac{(1-b)C_\mu}{\Theta}.$$

Hence

$$(1-b)K_{\mu_b}(g) = \frac{(1-b)C_\mu}{\Theta},$$

which gives that $C_\mu = \Theta K_{\mu_b}(g)$. Therefore, $d\sigma = g\, d\sigma_b$ satisfies the required properties. □

20.7 V_b intertwines S_μ^* and X_b

We recall that S_μ denotes the operator on $H^2(\mu)$ of multiplication by the independent variable, i.e.

$$\begin{array}{rccc} S_\mu : & H^2(\mu) & \longrightarrow & H^2(\mu) \\ & f & \longmapsto & \chi_1 f \end{array}$$

or, in short (see Section 8.6)

$$(S_\mu f)(\zeta) = \zeta\, f(\zeta) \qquad (\zeta \in \mathbb{T}).$$

Lemma 20.11 *Let μ denote the Clark measure of b, and let $f \in H^2(\mu)$. Then we have*

$$V_b S_\mu^* f = X_b V_b f + (K_\mu f)(0)\, S^* b.$$

Proof Using the definition of V_b (see (20.14)) and Lemma 13.17, for each $f \in H^2(\mu)$ and $z \in \mathbb{D}$, $z \neq 0$, we have

$$\begin{aligned} (V_b S_\mu^* f)(z) &= (1 - b(z))(K_\mu S_\mu^* f)(z) \\ &= (1 - b(z)) \frac{(K_\mu f)(z) - (K_\mu f)(0)}{z} \\ &= \frac{(1 - b(z))(K_\mu f)(z) - (1 - b(0))(K_\mu f)(0)}{z} \\ &\quad - (K_\mu f)(0) \frac{b(z) - b(0)}{z} \\ &= \frac{(V_b f)(z) - (V_b f)(0)}{z} - (K_\mu f)(0) \frac{b(z) - b(0)}{z} \\ &= (S^* V_b f)(z) - (K_\mu f)(0)(S^* b)(z). \end{aligned}$$

Since $V_b f \in \mathcal{H}(b)$, we can write $S^* V_b f = X_b V_b f$. □

Using the rank-one tensor product, we can write the result of Lemma 20.11 in a more compact way. In the first attempt, we can write

$$V_b S_\mu^* = X_b V_b + S^* b \otimes k_0,$$

where $S^* b \otimes 1 \in \mathcal{L}(H^2(\mu), \mathcal{H}(b))$. However, there are other ways to write this identity.

Theorem 20.12 *Let $k_0 \otimes k_0$ denote the rank-one operator on $H^2(\mu)$, and let $S^* b \otimes k_0^b$ denote the rank-one operator on $\mathcal{H}(b)$. Then the following hold.*

(i) $X_b V_b = V_b S_\mu^* I_b$, *where* $I_b = I - (1 - b(0)) k_0 \otimes k_0$.
(ii) $V_b S_\mu^* V_b^{-1} = X_b + (1 - b(0))^{-1} S^* b \otimes k_0^b$.

Proof (i) Let $f \in H^2(\mu)$. Then, by Lemma 20.11,

$$X_b V_b f = V_b S_\mu^* f - (K_\mu f)(0) S^* b. \tag{20.20}$$

In particular,

$$V_b S_\mu^* k_0 = S^* V_b k_0 + (K_\mu k_0)(0) S^* b. \tag{20.21}$$

We calculate $S^* b$ from the second identity, and plug it in to the first one to obtain the required result.

By (13.17),

$$(K_\mu f)(0) = \langle f, k_0 \rangle_\mu,$$

and, using (13.13) and (13.41), we have

$$(K_\mu k_0)(0) = \int_{\mathbb{T}} d\mu(e^{i\theta}) = \frac{1 - |b(0)|^2}{|1 - b(0)|^2}.$$

Since $k_0 = 1$, it follows from (20.15) and Theorem 18.11 that

$$V_b k_0 = (1 - \overline{b(0)})^{-1} k_0^b = (1 - \overline{b(0)})^{-1} (1 - \overline{b(0)} b).$$

Therefore, by (20.21),

$$\begin{aligned} V_b S_\mu^* k_0 &= (1 - \overline{b(0)})^{-1} S^* (1 - \overline{b(0)} b) + \frac{1 - |b(0)|^2}{|1 - b(0)|^2} S^* b \\ &= \left[\frac{-\overline{b(0)}}{1 - \overline{b(0)}} + \frac{1 - |b(0)|^2}{|1 - b(0)|^2} \right] S^* b \\ &= \frac{-(1 - b(0))\overline{b(0)} + 1 - |b(0)|^2}{|1 - b(0)|^2} S^* b \\ &= \frac{1 - \overline{b(0)}}{|1 - b(0)|^2} S^* b \\ &= (1 - b(0))^{-1} S^* b. \end{aligned}$$

(See also Exercise 20.3.1, where we derived the formula for $X_b V_b k_0$.) Hence, for the record,

$$V_b S_\mu^* k_0 = \frac{S^* b}{1 - b(0)}. \tag{20.22}$$

Thus it follows from (20.20) and (20.22) that

$$\begin{aligned} X_b V_b f &= V_b S_\mu^* f - \langle f, k_0 \rangle_\mu (1 - b(0)) V_b S_\mu^* k_0 \\ &= V_b S_\mu^* (f - (1 - b(0)) \langle f, k_0 \rangle_\mu k_0) \\ &= V_b S_\mu^* (I - (1 - b(0)) k_0 \otimes k_0) f, \end{aligned}$$

which proves the first relation of the theorem.

(ii) If we compose the relation in (i) by $V_b^{-1} = V_b^*$, then we get

$$X_b = V_b S_\mu^* V_b^{-1} - (1 - b(0)) V_b S_\mu^* (k_0 \otimes k_0) V_b^*.$$

Hence, by Theorem 1.37, we obtain

$$V_b S_\mu^* V_b^{-1} = X_b + (1 - b(0))((V_b S_\mu^* k_0) \otimes (V_b k_0)).$$

Thus, according to (20.15) (for $w = 0$) and (20.22), we obtain

$$V_b S_\mu^* V_b^{-1} = X_b + S^* b \otimes (1 - \overline{b(0)})^{-1} k_0^b = X_b + (1 - b(0))^{-1} S^* b \otimes k_0^b,$$

which concludes the proof. □

Exercise

Exercise 20.7.1 Show that

$$V_b S_\mu^* k_w = \bar{w} \frac{1 - \overline{b(w)} b}{1 - \overline{b(w)}} k_w + \frac{S^* b}{1 - b(0)} \qquad (w \in \mathbb{D}).$$

Hint: Use Exercises 20.3.1 and 20.3.2 and Lemma 20.11.
Remark: This is a generalization of (20.22).

20.8 Analytic continuation of $\mathcal{H}(b)$ functions

A general element of H^2 does not necessarily have analytic continuation along a given arc on the unit circle $\mathbb{T}$. In fact, in the worse-case scenario, we can construct a function in H^2 whose natural boundary is $\mathbb{T}$. Since $\mathcal{H}(b) \subset H^2$, we expect a similar situation to hold for members of $\mathcal{H}(b)$. The following result shows that the situation pretty much depends on the behavior of b on the boundary. We recall that b is assumed to be a nonconstant function in the closed unit ball of H^∞.

Theorem 20.13 *Let I be an open arc of $\mathbb{T}$. Then the following are equivalent.*

(i) *b has an analytic continuation across I and $|b| = 1$ on I.*
(ii) *I is contained in the resolvent of X_b^*, i.e. $I \cap \sigma(X_b^*) = \emptyset$.*
(iii) *Each function f in $\mathcal{H}(b)$ has an analytic continuation across I.*
(iv) *Each function f in $\mathcal{H}(b)$ has a continuous extension to $\mathbb{D} \cup I$.*
(v) *b has a continuous extension to $\mathbb{D} \cup I$ and $|b| = 1$ on I.*

Proof (i) $\Longrightarrow$ (ii) Let μ be the Clark measure associated with b by (13.41). Take any open subarc I_0 of $\mathbb{T}$ such that $\mathrm{Clos}(I_0) \subset I$. By Corollary 13.30, $\mu_{|I_0}$

is singular with respect to m, and moreover $\mu_{|I_0}$ is finitely purely atomic, with atoms situated at the point of I_0 where b takes precisely the value 1, i.e.

$$\mu_{|I_0} = \sum_{i=1}^{n} a_i \delta_{\zeta_i},$$

with $b(\zeta_i) = 1$ and $a_i > 0$, $1 \le i \le n$. Thus, by Corollary 8.5, we have

$$\sigma_{ess}(Z_\mu) \cap I_0 = \emptyset.$$

But, according to Theorem 20.12,

$$X_b^* = V_b S_\mu V_b^* - (1 - \overline{b(0)})^{-1} k_0^b \otimes S^* b, \tag{20.23}$$

and since $k_0^b \otimes S^* b$ is a rank-one operator and V_b is invertible, we deduce that $\sigma_{ess}(X_b^*) = \sigma_{ess}(S_\mu)$. Therefore, we obtain

$$\sigma_{ess}(X_b^*) \cap I_0 = \emptyset.$$

Since this is true for all subarcs I_0 with $\mathrm{Clos}(I_0) \subset I$, we have

$$\sigma_{ess}(X_b^*) \cap I = \emptyset.$$

Hence, by Theorem 1.30, we have

$$\sigma_{ess}(X_b) \cap \bar{I} = \emptyset, \tag{20.24}$$

where $\bar{I}$ denotes the complex conjugate of I, i.e. $\bar{I} = \{\zeta \in \mathbb{T} : \bar{\zeta} \in I\}$.

Now, let $w \in \bar{I} \cap \sigma(X_b)$. Since $\sigma(X_b) \subset \bar{\mathbb{D}}$ (X_b is a contraction), we necessarily have $w \in \partial\sigma(X_b)$. Then it follows from Theorem 2.8 that $w \in \sigma_{ess}(X_b) \cup \sigma_p(X_b)$ and, according to (20.24), we can only have $w \in \sigma_p(X_b)$. But, by Theorem 18.26, $\sigma_p(X_b) \subset \mathbb{D}$ and we get a contradiction. Thus, $\bar{I} \cap \sigma(X_b) = \emptyset$, which, once more according to Theorem 1.30, implies that $I \cap \sigma(X_b^*) = \emptyset$.

(ii) $\Longrightarrow$ (iii) The key to the proof is the representation

$$f(w) = \langle f, k_w^b \rangle_b = \langle f,\ (I - \bar{w} X_b^*)^{-1} k_0^b \rangle_b \qquad (w \in \mathbb{D}),$$

which was obtained in Theorem 18.21. We show that $\langle f,\ (I - \bar{w} X_b^*)^{-1} k_0^b \rangle_b$, as a function of w, can be analytically continued across I.

Assume that $I \subset \mathbb{T} \setminus \sigma(X_b^*)$ and let $w_0 \in I$. Writing

$$I - \bar{w} X_b^* = I - \overline{w_0} X_b^* - (\bar{w} - \overline{w_0}) X_b^*,$$

and since $I - \overline{w_0} X_b^* = \overline{w_0}(w_0 I - X_b^*)$ is invertible, we get

$$I - \bar{w} X_b^* = (I - \overline{w_0} X_b^*)(I - (\bar{w} - \overline{w_0})(I - \overline{w_0} X_b^*)^{-1} X_b^*).$$

Put $r_0 = \|(I - \overline{w_0}X_b^*)^{-1}X_b^*\|^{-1}$. Then, for $w \in D(w_0, r_0)$,

$$\|(\bar{w} - \overline{w_0})(I - \overline{w_0}X_b^*)^{-1}X_b^*\| \leq |w - w_0|\, \|(I - \overline{w_0}X_b^*)^{-1}X_b^*\| < 1.$$

Thus, the operator $I - (\bar{w} - \overline{w_0})(I - \overline{w_0}X_b^*)^{-1}X_b^*$ is invertible and

$$(I - (\bar{w} - \overline{w_0})(I - \overline{w_0}X_b^*)^{-1}X_b^*)^{-1} = \sum_{n=0}^{\infty}(\bar{w} - \overline{w_0})^n (I - \overline{w_0}X_b^*)^{-n}X_b^{*n},$$

where the series is convergent in the operator norm. We deduce that $I - \bar{w}X_b^*$ is invertible and

$$\begin{aligned}(I - \bar{w}X_b^*)^{-1} &= (I - (\bar{w} - \overline{w_0})(I - \overline{w_0}X_b^*)^{-1}X_b^*)^{-1}(I - \overline{w_0}X_b^*)^{-1}\\ &= \sum_{n=0}^{\infty}(\bar{w} - \overline{w_0})^n (I - \overline{w_0}X_b^*)^{-(n+1)}X_b^{*n},\end{aligned}$$

where the series is still convergent in operator norm. Therefore, for every function f in $\mathcal{H}(b)$ and every point w in the disk $D(w_0, r_0)$, we have

$$\langle f,\ (I - \bar{w}X_b^*)^{-1}k_0^b\rangle_b = \sum_{n=0}^{\infty}(w - w_0)^n \langle f,\ (I - \overline{w_0}X_b^*)^{-(n+1)}X_b^{*n}k_0^b\rangle_b.$$

This last equality means that the function $w \longmapsto \langle f,\ (I - \bar{w}X_b^*)^{-1}k_0^b\rangle_b$ is analytic in $D(w_0, r_0)$. By Theorem 18.21, we have

$$f(w) = \langle f,\ (I - \bar{w}X_b^*)^{-1}k_0^b\rangle_b$$

for every $w \in D(w_0, r_0) \cap \mathbb{D}$. This fact implies that the function f can be analytically continued in a neighborhood of w_0. But since w_0 is any point of I, we deduce that the function $f \in \mathcal{H}(b)$ has an analytic continuation across I.

(iii) $\Longrightarrow$ (iv) This is clear.

(iv) $\Longrightarrow$ (v) Fix any $w_0 \in \mathbb{D}$ such that $b(w_0) \neq 0$. Since

$$k_w^b(z) = \frac{1 - \overline{b(w_0)}b(z)}{1 - \bar{w}_0 z} \in \mathcal{H}(b)$$

has a continuous extension to $\mathbb{D} \cup I$, b also has a continuous extension to $\mathbb{D} \cup I$.

Now, let ζ_0 be any point of I. Our assumption says that

$$\lim_{\substack{z \to \zeta_0 \\ z \in \mathbb{D}}} f(z) = f(\zeta_0) \tag{20.25}$$

for each $f \in \mathcal{H}(b)$. In the first place, an application of the principle of uniform boundedness shows that the functional on $\mathcal{H}(b)$ of evaluation at ζ_0 is bounded. Second, if $k_{\zeta_0}^b$ denotes the corresponding kernel function, the relation (20.25) can be rewritten as

$$\lim_{\substack{z \to \zeta_0 \\ z \in \mathbb{D}}} \langle f, k_z^b\rangle_b = \langle f, k_{\zeta_0}^b\rangle_b \qquad (f \in \mathcal{H}(b)).$$

This means that the family k_w^b tends weakly to $k_{\zeta_0}^b$ as w tends to ζ_0 from within $\mathbb{D}$. Thus, for any $z \in \mathbb{D}$, we also have

$$\begin{aligned} k_{\zeta_0}^b(z) = \langle k_{\zeta_0}^b, k_z^b \rangle_b &= \lim_{\substack{w \to \zeta_0 \\ w \in \mathbb{D}}} \langle k_w^b, k_z^b \rangle_b \\ &= \lim_{\substack{w \to \zeta_0 \\ w \in \mathbb{D}}} \frac{1 - \overline{b(w)} b(z)}{1 - \bar{w} z} = \frac{1 - \overline{b(\zeta_0)} b(z)}{1 - \bar{\zeta}_0 z}. \end{aligned}$$

In particular, the function $(1 - \overline{b(\zeta_0)} b(z))/(z - \zeta_0)$ is in H^2, which is possible only if $|b(\zeta_0)| = 1$. Hence we get that $|b| = 1$ on I.

(v) $\Longrightarrow$ (i) This follows from standard facts based on the Schwarz reflection principle. □

The following result is somehow stated in the above theorem. However, we mention it explicitly as a corollary.

Corollary 20.14 *We have*

$$\sigma(b) \cap \mathbb{T} = \sigma(X_b^*) \cap \mathbb{T}.$$

The equivalent conditions stated in Theorem 20.13 imply, in particular, that b is an extreme point of the closed unit ball of $H^\infty(\mathbb{D})$. Thus, the continuity (and even the analytic continuity) of the elements of $\mathcal{H}(b)$ at a boundary point somehow depends on b being an extreme point or not. On the one hand, if b is not an extreme point of the unit ball of $H^\infty(\mathbb{D})$ and if I is any open subarc of $\mathbb{T}$, then certainly there exists a function $f \in \mathcal{H}(b)$ such that f does not have a continuous extension to $\mathbb{D} \cup I$. On the other hand, if b is an extreme point such that b has continuous extension to $\mathbb{D} \cup I$ with $|b| = 1$ on I, then all the functions $f \in \mathcal{H}(b)$ are continuous on I (and even can be continued analytically across I).

We also highlight another feature of Theorem 20.13. This result shows that the de Branges–Rovnyak spaces $\mathcal{H}(b)$ have a remarkable property, i.e. continuity on an open arc of $\mathbb{T}$ of all functions of $\mathcal{H}(b)$ is enough to imply the analyticity of these functions.

20.9 Multipliers of $\mathcal{H}(b)$

Reproducing kernel Hilbert spaces and the space of their multipliers were studied in Section 9.1. The multipliers for a reproducing kernel Hilbert space $H \subset H^2$ can be interpreted slightly differently. A multiplier is necessarily a bounded analytic function on $\mathbb{D}$. Indeed, if φ is a multiplier, the multiplication operator M_φ, which was introduced in Section 9.1, is exactly the restriction of T_φ to H. Hence, an analytic function $\varphi \in H^\infty$ is a multiplier of H, i.e. $\varphi \in \mathfrak{Mult}(H)$, if and only if H is invariant under the Toeplitz operator T_φ.

Lemma 20.15 *Let $\varphi \in \mathfrak{Mult}(\mathcal{H}(b))$, and assume that φ has the decomposition $\varphi = \Theta\,\psi$, where Θ is inner and $\psi \in H^\infty$. Then $\psi \in \mathfrak{Mult}(\mathcal{H}(b))$ and we have*

$$\|M_\psi\|_{\mathcal{L}(\mathcal{H}(b))} \le \|M_\varphi\|_{\mathcal{L}(\mathcal{H}(b))}.$$

Proof Let $f \in \mathcal{H}(b)$. Then, we have

$$\psi f = \bar{\Theta}\varphi f = P_+(\bar{\Theta}\varphi f) = T_{\bar{\Theta}}(\varphi f).$$

Now, since $\varphi \in \mathfrak{Mult}(\mathcal{H}(b))$, the function φf is in $\mathcal{H}(b)$ and thus, by Theorem 18.13, the function ψf remains in $\mathcal{H}(b)$. Moreover,

$$\|\psi f\|_b = \|T_{\bar{\Theta}}(\varphi f)\|_b \le \|T_{\bar{\Theta}}\| \times \|\varphi f\|_b \le \|M_\varphi\| \times \|f\|_b,$$

which proves the result. □

The reproducing kernel Hilbert spaces that are in the center of our discussion are $\mathcal{H}(b)$ and $\mathcal{H}(\bar{b})$. According to Theorem 9.2, if $\varphi \in \mathfrak{Mult}(\mathcal{H}(b))$, then

$$M_\varphi^* k_w^b = \overline{\varphi(w)} k_w^b \qquad (w \in \mathbb{D}).$$

Moreover, rephrasing Corollary 9.4 for this case, we obtain the following result.

Theorem 20.16 *Let $\varphi \in H^\infty$. Then $\varphi \in \mathfrak{Mult}(\mathcal{H}(b))$ if and only if the mapping*

$$k_w^b \longmapsto \overline{\varphi(w)} k_w^b$$

extends to a continuous linear operator on $\mathcal{H}(b)$.

The next result shows that $\mathfrak{Mult}(\mathcal{H}(\bar{b}))$ contains $\mathfrak{Mult}(\mathcal{H}(b))$ as a subset.

Theorem 20.17 *Every multiplier of $\mathcal{H}(b)$ is also a multiplier of $\mathcal{H}(\bar{b})$.*

Proof Let $\varphi \in \mathfrak{Mult}(\mathcal{H}(b))$, and let $f \in \mathcal{H}(\bar{b})$. By Theorem 17.8, $T_{\bar{b}} f \in \mathcal{H}(b)$. Hence, $T_\varphi T_{\bar{b}} f \in \mathcal{H}(b)$. Since $T_\varphi T_{\bar{b}} = T_{\bar{b}} T_\varphi$, another application of Theorem 17.8 shows that $T_\varphi f \in \mathcal{H}(\bar{b})$. This means that $\varphi \in \mathfrak{Mult}(\mathcal{H}(\bar{b}))$. □

The next result shows that the space of multipliers of $\mathcal{H}(b)$ is closed under the backward shift operator.

Theorem 20.18 *Let $\varphi \in \mathfrak{Mult}(\mathcal{H}(b))$. Then, for each $w \in \mathbb{D}$,*

$$Q_w\varphi \in \mathfrak{Mult}(\mathcal{H}(b)).$$

In particular, $S^*\varphi$ is a multiplier of $\mathcal{H}(b)$.

Proof By Corollary 8.11, for each $f \in \mathcal{H}(b)$,

$$(Q_w\varphi)f = Q_w(\varphi f) - \varphi(w)Q_w f.$$

By assumption, $\varphi f \in \mathcal{H}(b)$. By Corollary 18.14, $Q_w(\varphi f)$ and $Q_w f$ stay in $\mathcal{H}(b)$. Hence, $(Q_w\varphi)\, f \in \mathcal{H}(b)$. This means that $Q_w\varphi$ is a multiplier of $\mathcal{H}(b)$. $\square$

Exercises

Exercise 20.9.1 Let $A \in \mathcal{L}(\mathcal{H}(b))$ be such that each kernel function k_z^b, $z \in \mathbb{D}$, is an eigenvector of A^*. Show that there is a $\varphi \in \mathfrak{Mult}(\mathcal{H}(b))$ such that $A = M_\varphi$.
Hint: Use Theorem 20.16 or Theorem 9.3.

Exercise 20.9.2 Let $\varphi \in \mathfrak{Mult}(\mathcal{H}(\bar{b}))$. Show that, for each $w \in \mathbb{D}$,

$$Q_w\varphi \in \mathfrak{Mult}(\mathcal{H}(\bar{b})).$$

In particular, show that $S^*\varphi$ is a multiplier of $\mathcal{H}(\bar{b})$.
Hint: Mimic the proof of Theorem 20.18.

20.10 Multipliers and Toeplitz operators

In Section 12.7, we introduced the notion of Toeplitz operators on generalized Hardy spaces $H^2(\nu)$, $\nu \in \mathcal{M}(\mathbb{T})$. We use this notion to give a criterion for a function $\varphi \in H^\infty$ to be a multiplier of $\mathcal{H}(b)$.

We recall that, in Section 20.3, we showed that

$$\begin{array}{rccc} V_b : & H^2(\mu) & \longrightarrow & \mathcal{H}(b) \\ & g & \longmapsto & (1-b)K_\mu g \end{array}$$

is a unitary operator between $H^2(\mu)$ and $\mathcal{H}(b)$.

Theorem 20.19 *Let $\varphi \in H^\infty$. Let b be a nonconstant function in the closed unit ball of H^∞, and let μ be the corresponding Clark measure. Then φ is a multiplier of $\mathcal{H}(b)$ if and only if $T_{\bar{\varphi}}$ is bounded on $H^2(\mu)$. Moreover, in this case, we have*

$$T_{\bar{\varphi}} = V_b^* M_\varphi^* V_b.$$

In other words, we have the following diagram.

$$\begin{tikzcd} H^2(\mu) \arrow[r, "V_b"] \arrow[d, "T_{\bar\varphi}"'] & \mathcal{H}(b) \arrow[d, "M_\varphi^*"] \\ H^2(\mu) \arrow[r, "V_b"'] & \mathcal{H}(b) \end{tikzcd} \tag{20.26}$$

Proof According to Theorem 20.16, φ is a multiplier of $\mathcal{H}(b)$ if and only if the mapping

$$k_w^b \longmapsto \overline{\varphi(w)}k_w^b$$

extends to a continuous linear operator on $\mathcal{H}(b)$. Since, by Theorem 20.5, V_b is a unitary operator from $H^2(\mu)$ onto $\mathcal{H}(b)$ such that

$$V_b k_w = \frac{k_w^b}{1-\overline{b(w)}} \qquad (w \in \mathbb{D}),$$

we deduce that φ is a multiplier of $\mathcal{H}(b)$ if and only if the mapping

$$k_w \longmapsto \overline{\varphi(w)}k_w$$

extends to a continuous linear operator on $H^2(\mu)$. According to Theorem 12.36, the last assertion is a characterization of $T_{\bar\varphi} \in \mathcal{L}(H^2(\mu))$.

Using Theorems 9.2 and 20.5, we have

$$\begin{aligned} V_b^* M_\varphi^* V_b k_w &= (1-\overline{b(w)})^{-1} V_b^* M_\varphi^* k_w^b \\ &= (1-\overline{b(w)})^{-1}\overline{\varphi(w)} V_b^* k_w^b = \overline{\varphi(w)}k_w \end{aligned}$$

for every $w \in \mathbb{D}$. Remember that $V_b^* = V_b^{-1}$. Moreover, by Theorem 12.36, we also have

$$T_{\bar\varphi} k_w = \overline{\varphi(w)}k_w.$$

Therefore, the last two relations imply that $V_b^* M_\varphi^* V_b k_w = T_{\bar\varphi} k_w$, for all $w \in \mathbb{D}$. Since the linear span of reproducing kernels is dense in $H^2(\mu)$, we immediately get $T_{\bar\varphi} = V_b^* M_\varphi^* V_b$. □

An important multiplier is multiplication by the independent variable z. On H^2, this operator was called the forward shift operator and was denoted by S. In this case, by induction, the space under consideration is closed under multiplication by any analytic polynomial. Hence, *a priori* we anticipate obtaining further interesting properties. We have enough tools now to completely characterize those $\mathcal{H}(b)$ spaces that are invariant under the forward shift operator.

Corollary 20.20 *The space $\mathcal{H}(b)$ is invariant under the forward shift operator S if and only if b is a nonextreme point of the closed unit ball of H^∞, that is,*

$$\int_{\mathbb{T}} \log(1 - |b|^2)\, dm > -\infty.$$

Proof To say that $\mathcal{H}(b)$ is invariant under the forward shift operator S means that $\chi_1(z) = z$ is a multiplier of $\mathcal{H}(b)$. According to Theorem 20.19, the function χ_1 is a multiplier of $\mathcal{H}(b)$ if and only if $T_{\bar{\chi}_1}$ is bounded on $H^2(\mu)$. But if p is an analytic polynomial, we have

$$T_{\bar{\chi}_1} p(z) = \frac{p(z) - p(0)}{z} \qquad (z \in \mathbb{T}).$$

Write this identity as

$$p(0) = (I - M_{\chi_1} T_{\bar{\chi}_1})p,$$

where $M_{\chi_1} \in \mathcal{L}(L^2(\mu))$ is the operator of multiplication by χ_1. Hence, it follows that $T_{\bar{\chi}_1}$ is bounded on $H^2(\mu)$ if and only if the functional

$$\begin{array}{rcl} H^2(\mu) & \longrightarrow & \mathbb{C} \\ p & \longmapsto & p(0) \end{array}$$

is bounded on $H^2(\mu)$. By Riesz's theorem, this is equivalent to the existence of an element $g \in H^2(\mu)$ such that

$$\langle \chi_n, g \rangle_{H^2(\mu)} = 0 \qquad (n \geq 1)$$

and

$$\langle \chi_0, g \rangle_{H^2(\mu)} = 1.$$

Hence, at least, we have $\chi_0 \notin \operatorname{Span}(\chi_n : n \geq 1) = H^2_0(\mu)$. According to Corollary 13.34, this is equivalent to b being a nonextreme point of the closed unit ball of H^∞. Finally, apply Theorem 6.7. □

In Theorem 24.1 and Corollary 25.5, we will see a different method to prove directly this result of the invariance of $\mathcal{H}(b)$.

Corollary 20.21 *Let μ be a positive Borel measure on $\mathbb{T}$ such that $H^2(\mu) = L^2(\mu)$, and let φ be an analytic polynomial. If $T_{\bar{\varphi}}$ is bounded on $H^2(\mu)$, then φ is constant.*

Proof Let b be the function associated with μ by (13.52). Then, according to Theorem 20.19, we see that φ is a multiplier of $\mathcal{H}(b)$. We argue by absurdity, assuming that φ is not constant and let $d \geq 1$ be the degree of φ. By Theorem 20.18, we deduce that $S^{*(d-1)}\varphi$ is also a multiplier of $\mathcal{H}(b)$. But $S^{*(d-1)}\varphi$ is a nonzero polynomial of degree 1 and since $\mathfrak{Mult}(\mathcal{H}(b))$ is an algebra that contains $\chi_0 = 1$, we conclude that χ_1 is a multiplier of $\mathcal{H}(b)$.

Therefore, it follows from Corollary 20.20 that b is not an extreme point of the closed unit ball of H^∞. Then Corollary 13.34 implies that $H^2(\mu) \neq L^2(\mu)$, which is absurd. Thus, φ is constant. □

An equivalent way to describe the content of Corollary 20.21 is as follows. Note that Corollary 20.20 implies that, if b is a nonextreme point and φ is an analytic polynomial, then $\varphi \in \mathfrak{Mult}(\mathcal{H}(b))$. In the extreme case, this result dramatically fails.

Corollary 20.22 *Let b be an extreme point in the closed unit ball of H^∞ and let φ be an analytic polynomial such that $\varphi \in \mathfrak{Mult}(\mathcal{H}(b))$. Then φ is constant.*

Proof Apply Theorem 20.19 and Corollaries 20.21 and 13.34. □

Theorem 20.23 *If b is not an inner function, then every multiplier of $\mathcal{H}(\bar{b})$ differs by a constant from a function in $\mathcal{H}(\bar{b})$.*

Proof Assume first that b is not an extreme point of the closed unit ball of H^∞. Then, according to Corollary 20.20, we have $S\mathcal{H}(b) \subset \mathcal{H}(b)$. Now, if we apply Theorem 18.22, we get that

$$\|S^*b\|_b^2 b = X_b^* S^* b - SS^*b,$$

and this formula implies that $b \in \mathcal{H}(b)$. Since we also have $b \in \mathcal{M}(b)$, by (17.11), there exists $\varphi \in \mathcal{H}(\bar{b})$ such that $b = T_b\varphi = b\varphi$. Hence, $1 = \varphi \in \mathcal{H}(\bar{b})$. This trivially implies that $\mathfrak{Mult}(\mathcal{H}(\bar{b})) \subset \mathcal{H}(\bar{b})$.

Assume now that b is an extreme point of the closed unit ball of H^∞. Let m be a multiplier of $\mathcal{H}(\bar{b})$ and let $h \in \mathcal{H}(\bar{b})$ such that $h(0) \neq 0$. We know by Lemma 18.12 that both functions mS^*h and $S^*(mh)$ belong to $\mathcal{H}(\bar{b})$. Since

$$S^*(mh) = mS^*h + h(0)S^*m,$$

it follows that $S^*m \in \mathcal{H}(\bar{b})$. Since b is an extreme point, then $H^2(\rho) = L^2(\rho)$ (see Corollary 8.23), and the operator

$$\begin{array}{cccc} Z_\rho : & L^2(\rho) & \longrightarrow & L^2(\rho) \\ & f & \longmapsto & \chi_1 f \end{array}$$

is unitary. The intertwining relation (20.10) can be rewritten as

$$K_\rho Z_\rho^* K_\rho^* = X_{\bar{b}},$$

where K_ρ is a unitary operator from $L^2(\rho)$ onto $\mathcal{H}(\bar{b})$ and $X_{\bar{b}} = S^*_{|\mathcal{H}(\bar{b})}$. Thus $X_{\bar{b}}$ is a unitary operator and we have $S^*\mathcal{H}(\bar{b}) = \mathcal{H}(\bar{b})$. In particular, $S^*m \in S^*\mathcal{H}(\bar{b})$ and there exists $h \in \mathcal{H}(\bar{b})$ such that $S^*m = S^*h$. This means that $m = h + c$, which gives that $m \in \mathcal{H}(\bar{b}) + \mathbb{C}$, as desired. □

The next result identifies a class of inner functions Θ for which $\mathcal{H}(b)$ and $\mathcal{H}(\Theta b)$ have the same multipliers. We begin with an elementary result that will be useful.

Lemma 20.24 *Let b be in the closed unit ball of H^∞, and let Θ be an inner function. Then*

$$\|T_{\bar{b}|K_\Theta}\| = \|P_\Theta T_{b|K_\Theta}\| = \mathrm{dist}(\bar{\Theta}b, H^\infty).$$

Proof For the first equality, it is sufficient to note that $(T_{\bar{b}|K_\Theta})^* = P_\Theta T_{b|K_\Theta}$. For the second identity, by Corollary 14.13, we have $P_{\Theta|H^2} = \Theta P_- \bar{\Theta}_{|H^2}$, whence

$$\|P_\Theta T_{b|K_\Theta}\| = \|P_- \bar{\Theta} b_{|K_\Theta}\|.$$

But it is obvious that $P_- \bar{\Theta} b_{|\Theta H^2} = 0$ and then we get

$$\|P_\Theta T_{b|K_\Theta}\| = \|P_- \bar{\Theta} b_{|H^2}\| = \|H_{\bar{\Theta}b}\|,$$

and it remains to apply Nehari's theorem to conclude the proof. □

Theorem 20.25 *Let b be in the closed unit ball of H^∞, and let Θ be an inner function. Assume that*

$$\mathrm{dist}(\bar{\Theta}b, H^\infty) < 1. \tag{20.27}$$

Then we have

$$\mathfrak{Mult}(\mathcal{H}(\Theta b)) = \mathfrak{Mult}(\mathcal{H}(b)).$$

Proof We first show that (20.27) is equivalent to

$$\mathcal{H}(\Theta) \subset (I - T_\Theta T_{\bar{\Theta}})\mathcal{H}(b). \tag{20.28}$$

By Corollary 16.10, the inclusion (20.28) is equivalent to the operator inequality

$$I - T_\Theta T_{\bar{\Theta}} \le c(I - T_\Theta T_{\bar{\Theta}})(I - T_b T_{\bar{b}})(I - T_\Theta T_{\bar{\Theta}})$$

for some $c > 0$. This inequality means that

$$\langle (I - T_\Theta T_{\bar{\Theta}} h, h\rangle_2 \le c\langle (I - T_b T_{\bar{b}})(I - T_\Theta T_{\bar{\Theta}})h, (I - T_\Theta T_{\bar{\Theta}})h\rangle_2 \qquad (h \in H^2).$$

Since Θ is inner, the operator $(I - T_\Theta T_{\bar{\Theta}})$ is the orthogonal projection onto $\mathcal{H}(\Theta)$ and the last inequality can be rephrased as

$$\|h\|_2^2 \le c\langle (I - T_b T_{\bar{b}})h, h\rangle_2 \qquad (h \in \mathcal{H}(\Theta)).$$

In other words,

$$\|h\|_2^2 \le c(\|h\|_2^2 - \|T_{\bar{b}} h\|_2^2) \qquad (h \in \mathcal{H}(\Theta)).$$

Note that necessarily $c \geq 1$ and the inequality is equivalent to

$$\|T_{\bar{b}}h\|_2^2 \leq \frac{c-1}{c}\|h\|_2^2 \qquad (h \in \mathcal{H}(\Theta)).$$

Therefore, the inclusion (20.28) is equivalent to

$$\|T_{\bar{b}|\mathcal{H}(\Theta)}\| \leq \frac{c-1}{c} < 1.$$

The equivalence with (20.27) follows now from Lemma 20.24.

We now prove that $\mathfrak{Mult}(\mathcal{H}(b)) = \mathfrak{Mult}(\mathcal{H}(\Theta b))$. The first inclusion

$$\mathfrak{Mult}(\mathcal{H}(\Theta b)) \subset \mathfrak{Mult}(\mathcal{H}(b))$$

is always true. Indeed, let $\varphi \in \mathfrak{Mult}(\mathcal{H}(\Theta b))$ and $f \in \mathcal{H}(b)$. Remember that, according to Corollary 18.9, we have

$$\mathcal{H}(\Theta b) = \mathcal{H}(\Theta) \oplus \Theta\mathcal{H}(b), \tag{20.29}$$

and the function Θf is in $\mathcal{H}(\Theta b)$. Thus $\varphi\Theta f$ belongs to $\mathcal{H}(\Theta b)$ and we can decompose this function as

$$\varphi\Theta f = g_1 + \Theta g_2,$$

with $g_1 \in \mathcal{H}(\Theta)$ and $g_2 \in \mathcal{H}(b)$. Hence $g_1 = \varphi\Theta f - \Theta g_2 = \Theta(\varphi f - g_2)$. In particular, $g_1 \in \mathcal{M}(\Theta) \cap \mathcal{H}(\Theta)$ and, since Θ is inner, we know that $\mathcal{M}(\Theta) \cap \mathcal{H}(\Theta) = \{0\}$. In other words, $g_1 \equiv 0$ and we conclude that $\varphi\Theta f = \Theta g_2$, i.e. $\varphi f = g_2 \in \mathcal{H}(b)$. Thus we have proved that, for any $f \in \mathcal{H}(b)$, $\varphi f \in \mathcal{H}(b)$, which exactly means that $\varphi \in \mathfrak{Mult}(\mathcal{H}(b))$.

For the second inclusion, assume that $\varphi \in \mathfrak{Mult}(\mathcal{H}(b))$ and let us prove that $\varphi \in \mathfrak{Mult}(\mathcal{H}(\Theta b))$. First, using once more the decomposition (20.29), it is sufficient to prove that

$$\varphi\mathcal{H}(\Theta)) \subset \mathcal{H}(\Theta b). \tag{20.30}$$

Let $g \in \mathcal{H}(\Theta)$. Then, according to (20.28), there exists $h \in \mathcal{H}(b)$ such that $g = (I - T_\Theta T_{\bar{\Theta}})h$. Hence

$$\varphi(h-g) = \varphi T_\Theta T_{\bar{\Theta}}h = \Theta\varphi T_{\bar{\Theta}}h.$$

Recall that $\mathcal{H}(b)$ is invariant under $T_{\bar{\Theta}}$ (see Theorem 18.13). Thus, $T_{\bar{\Theta}}h \in \mathcal{H}(b)$ and then $\varphi(h-g) \in \Theta\mathcal{H}(b) \subset \mathcal{H}(\Theta b)$. But $\varphi h \in \mathcal{H}(b) \subset \mathcal{H}(\Theta b)$. Hence, $\varphi g \in \mathcal{H}(\Theta b)$. That proves (20.30) and concludes the proof. □

The next result shows that $\mathcal{H}(b)$ and $\mathcal{H}(b^2)$ have the same multipliers.

Theorem 20.26 *Let b be a function in the closed unit ball of H^∞. Then*

$$\mathfrak{Mult}(\mathcal{H}(b)) = \mathfrak{Mult}(\mathcal{H}(b^2)).$$

Proof By Theorem 18.7 we have

$$\mathcal{H}(b^2) = \mathcal{H}(b) + b\,\mathcal{H}(b).$$

Let us first prove that $\mathfrak{Mult}(\mathcal{H}(b)) \subset \mathfrak{Mult}(\mathcal{H}(b^2))$. So take $\varphi \in \mathfrak{Mult}(\mathcal{H}(b))$ and $f \in \mathcal{H}(b^2)$. Decompose f as $f = f_1 + bf_2$, with $f_1, f_2 \in \mathcal{H}(b)$. Then

$$\varphi f = \varphi f_1 + b\varphi f_2,$$

and since φf_1 and φf_2 belong to $\mathcal{H}(b)$, we get that $\varphi f \in \mathcal{H}(b^2)$ as desired.

Conversely, let $\varphi \in \mathfrak{Mult}(\mathcal{H}(b^2))$ and let $f \in \mathcal{H}(b)$. Then $bf \in \mathcal{H}(b^2)$ and thus $\varphi bf \in \mathcal{H}(b^2)$ and there exist $g_1, g_2 \in \mathcal{H}(b)$ such that

$$\varphi bf = g_1 + bg_2.$$

Note that $g_1 = b(\varphi f - g_2) \in \mathcal{M}(b) \cap \mathcal{H}(b)$. According to (17.11), there exists $g_3 \in \mathcal{H}(b)$ such that $g_1 = bg_3$ and we get $\varphi bf = bg_3 + bg_2$. Since $b \not\equiv 0$, we obtain that $\varphi f = g_3 + g_2$ and thus $\varphi f \in \mathcal{H}(b)$ as desired. □

When b is an outer function in the closed unit ball of H^∞, then it has square roots, and Theorem 20.26 immediately gives by an induction argument that

$$\mathfrak{Mult}(\mathcal{H}(b^{2^n})) = \mathfrak{Mult}(\mathcal{H}(b)) \tag{20.31}$$

for any integer $n \in \mathbb{Z}$.

Exercise

Exercise 20.10.1 Let b be a nonextreme point of the closed unit ball of H^∞, let $f \in \mathcal{H}(b)$ and let $w \in \mathbb{D}$. Assume that $f(w) = 0$ and $f'(w) \neq 0$. Show that, for every $\lambda \in \mathbb{D}$, $\lambda \neq w$, the function

$$g(z) = \frac{z-\lambda}{z-w} f(z) \qquad (z \in \mathbb{D})$$

belongs to $\mathcal{H}(b)$ and satisfies $g(\lambda) = 0$ and $g(w) \neq 0$.
Hint: Find $(S - \lambda I)Q_w f$. Note that one needs Theorem 18.18 and Corollary 20.20.

20.11 Comparison of measures

If ν and μ_s are singular measures and ν is absolutely continuous with respect to μ_s, naively speaking, this means that ν lives on the territory (support) of μ_s and moreover it is controlled by μ_s. Hence, if we add an absolutely continuous measure to μ_s, then ν must still be under the control of the new bigger measure. This result is stated clearly below.

Lemma 20.27 *Let ν and μ be positive finite Borel measures on $\mathbb{T}$ and let $\mu = \mu_a + \mu_s$ be the Lebesgue decomposition of μ with respect to the Lebesgue measure m. Suppose that ν is singular with respect to m and that ν is absolutely continuous with respect to μ_s with $d\nu/d\mu_s \in L^2(\mu_s)$. Then ν is absolutely continuous with respect to μ and $d\nu/d\mu \in L^2(\mu)$.*

Proof In this proof, we consider the σ-algebra of the Borel subsets of $\mathbb{T}$. Since ν is singular with respect to m, there is a measurable subset B of $\mathbb{T}$ such that, for all measurable subsets E of $\mathbb{T}$, we have $\nu(E) = \nu(E \cap B)$ and $m(B) = 0$. By assumption, there is a function $h = d\nu/d\mu_s \in L^2(\mu_s)$, $h \geq 0$, such that

$$\nu(E) = \int_E h \, d\mu_s$$

for all measurable subsets E of $\mathbb{T}$. Of course, the function h is not unique and is free to change within a Borel set of μ_s-measure zero. Among all the possible candidates, we choose one that is *appropriate* for our case. Since $\nu(E) = \nu(E \cap B)$, for every measurable subset E of $\mathbb{T}$,

$$\nu(E) = \nu(E \cap B) = \int_{E \cap B} h \, d\mu_s = \int_E h\chi_B \, d\mu_s.$$

Hence, we can still say $d\nu/d\mu_s = h\chi_B$. However, this choice has an advantage. Since $m(E \cap B) = 0$ we have $\mu_a(E \cap B) = 0$, and thus the above result shows that

$$\nu(E) = \int_E h\chi_B \, d\mu_s = \int_E h\chi_B \, d\mu.$$

This representation shows that ν is absolutely continuous with respect to μ and

$$\frac{d\nu}{d\mu} = h\chi_B.$$

Moreover, by a similar reasoning,

$$\int_{\mathbb{T}} |h\chi_B|^2 \, d\mu = \int_B |h|^2 \, d\mu = \int_B |h|^2 \, d\mu_s = \int_{\mathbb{T}} |h|^2 \, d\mu_s < \infty.$$

Hence,

$$\frac{d\nu}{d\mu} \in L^2(\mu). \qquad \square$$

Theorem 20.28 *Let b be a nonconstant function in the closed unit ball of H^∞, and let Θ be a nonconstant inner function. Let μ and ν denote, respectively, the Clark measures of b and Θ. Then the following are equivalent.*

(i) *ν is absolutely continuous with respect to μ and $d\nu/d\mu \in L^2(\mu)$.*

(ii) $\dfrac{1-b}{1-\Theta} k_0^\Theta \in \mathcal{H}(b)$.

Proof (i) $\Longrightarrow$ (ii) For simplicity, let us write $h = d\nu/d\mu$. Since $h \in L^2(\mu)$, by Theorem 20.5, we know that $\mathbf{V}_b h \in \mathcal{H}(b)$. But let us evaluate $\mathbf{V}_b h(z)$. For $z \in \mathbb{D}$, we have

$$\begin{aligned}\mathbf{V}_b h(z) &= (1 - b(z))(K_\mu h)(z)\\ &= (1 - b(z)) \int_{\mathbb{T}} \frac{h(\zeta)}{1 - \bar{\zeta} z}\, d\mu(\zeta)\\ &= (1 - b(z)) \int_{\mathbb{T}} \frac{1}{1 - \bar{\zeta} z}\, d\nu(\zeta)\\ &= (1 - b(z))(K_\nu 1)(z)\\ &= \frac{1 - b(z)}{1 - \Theta(z)} (\mathbf{V}_\Theta 1)(z).\end{aligned}$$

By (20.15), we have

$$\mathbf{V}_\Theta 1 = \mathbf{V}_\Theta k_0 = (1 - \overline{\Theta(0)})^{-1} k_0^\Theta$$

and hence

$$\mathbf{V}_b h = \frac{1 - b}{1 - \Theta} (1 - \overline{\Theta(0)})^{-1} k_0^\Theta.$$

Therefore, we deduce that

$$\frac{1 - b}{1 - \Theta} k_0^\Theta \in \mathcal{H}(b).$$

Note that $\Theta(0) \neq 1$, since otherwise Θ would be a constant function.

(ii) $\Longrightarrow$ (i) By Theorem 20.5, we know that there is $g \in L^2(\mu)$ such that

$$\frac{1 - b}{1 - \Theta} k_0^\Theta = \mathbf{V}_b g,$$

which is equivalent to

$$\frac{1 - \overline{\Theta(0)}\Theta}{1 - \Theta} = K_\mu g.$$

Therefore, for each $z \in \mathbb{D}$, we can write

$$\begin{aligned}\frac{1 - \overline{\Theta(0)}\Theta(z)}{1 - \Theta(z)} &= \int_{\mathbb{T}} \frac{g(\zeta)}{1 - \bar{\zeta} z}\, d\mu(\zeta)\\ &= \int_{\mathbb{T}} \frac{\zeta}{\zeta - z} g(\zeta)\, d\mu(\zeta)\\ &= \frac{1}{2} \int_{\mathbb{T}} \left(\frac{\zeta + z}{\zeta - z} + 1 \right) g(\zeta)\, d\mu(\zeta)\\ &= \frac{1}{2} \int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} g(\zeta)\, d\mu(\zeta) + \frac{1}{2} \int_{\mathbb{T}} g(\zeta)\, d\mu(\zeta).\end{aligned}$$

Now, if we put $z = 0$, we get

$$\frac{1 - |\Theta(0)|^2}{1 - \Theta(0)} = \int_{\mathbb{T}} g(\zeta)\, d\mu(\zeta),$$

and thus

$$\frac{1 - \overline{\Theta(0)}\Theta(z)}{1 - \Theta(z)} = \frac{1}{2}\int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} g(\zeta)\, d\mu(\zeta) + \frac{1 - |\Theta(0)|^2}{2(1 - \Theta(0))}.$$

Since

$$\frac{1 - \overline{\Theta(0)}\Theta(z)}{1 - \Theta(z)} = \overline{\Theta(0)} + \frac{1 - \overline{\Theta(0)}}{2}\left(\frac{1 + \Theta(z)}{1 - \Theta(z)} + 1\right)$$

and

$$\frac{1}{2}\frac{1 - |\Theta(0)|^2}{|1 - \Theta(0)|^2} - \frac{\overline{\Theta(0)}}{1 - \overline{\Theta(0)}} = \frac{1}{2}\left(1 + i\,\Im\left(\frac{1 + \Theta(0)}{1 - \Theta(0)}\right)\right),$$

we get

$$\frac{1 + \Theta(z)}{1 - \Theta(z)} = \frac{1}{1 - \overline{\Theta(0)}}\int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z}\, g(\zeta)\, d\mu(\zeta) + i\,\Im\left(\frac{1 + \Theta(0)}{1 - \Theta(0)}\right).$$

On the other hand, by (13.43), we have

$$\frac{1 + \Theta(z)}{1 - \Theta(z)} = \int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z}\, d\nu(\zeta) + i\,\Im\left(\frac{1 + \Theta(0)}{1 - \Theta(0)}\right),$$

and thus we obtain the identity

$$\frac{1}{1 - \overline{\Theta(0)}}\int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} g(\zeta)\, d\mu(\zeta) = \int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z}\, d\nu(\zeta)$$

for all $z \in \mathbb{D}$. Taking the conjugate and then replacing $\bar{z}$ by z, we can rewrite this equality as

$$\frac{1}{1 - \Theta(0)}\int_{\mathbb{T}} \frac{\bar{\zeta} + z}{\bar{\zeta} - z}\overline{g(\zeta)}\, d\mu(\zeta) = \int_{\mathbb{T}} \frac{\bar{\zeta} + z}{\bar{\zeta} - z}\, d\nu(\zeta).$$

Then it follows from (4.11) that the measure $(1 - \Theta(0))^{-1}\bar{g}\, d\mu - d\nu$ is absolutely continuous with respect to m. Therefore, since ν is singular with respect to m, we must have

$$(1 - \Theta(0))^{-1}\bar{g}\, d\mu_s = d\nu,$$

where μ_s is the singular part of the measure μ with respect to the Lebesgue measure m. In other words, ν is absolutely continuous with respect to μ_s and

$$\frac{d\nu}{d\mu_s} = (1 - \Theta(0))^{-1}\bar{g} \in L^2(\mu) \subset L^2(\mu_s).$$

The conclusion now follows from Lemma 20.27.

We also give a second proof of this part. However, a closer look at the details reveals that, despite their different appearances, the two proofs are actually the same. The arguments used to prove the implication (i) $\Longrightarrow$ (ii) are almost reversible. On the one hand, we have

$$\begin{aligned}\frac{1-b(z)}{1-\Theta(z)}(1-\overline{\Theta(0)})^{-1}k_0^\Theta(z) &= \frac{1-b(z)}{1-\Theta(z)}(\mathbf{V}_\Theta 1)(z)\\ &= (1-b(z))(K_\nu 1)(z)\\ &= (1-b(z))\int_{\mathbb{T}}\frac{1}{1-\bar{\zeta}z}\,d\nu(\zeta).\end{aligned}$$

On the other hand, by Theorem 20.5, there is $g \in L^2(\mu)$ such that

$$\frac{1-b}{1-\Theta}(1-\overline{\Theta(0)})^{-1}k_0^\Theta = \mathbf{V}_b g = (1-b)K_\mu g.$$

Hence,

$$\int_{\mathbb{T}}\frac{1}{1-\bar{\zeta}z}\,d\nu(\zeta) = \int_{\mathbb{T}}\frac{g(\zeta)}{1-\bar{\zeta}z}\,d\mu(\zeta) \qquad (z\in\mathbb{D}).$$

Either by expanding k_z and considering the power series of both sides, or by appealing to the fact that the span of Cauchy kernels is uniformly dense in the disk algebra (Theorem 5.5), we deduce that

$$\int_{\mathbb{T}}\zeta^n\,d\nu(\zeta) = \int_{\mathbb{T}}\zeta^n g(\zeta)\,d\mu(\zeta) \qquad (n\le 0).$$

Then Riesz's theorem ensures that $d\nu - g\,d\mu$ is absolutely continuous with respect to Lebesgue measure. The rest is the same as the first proof. □

The preceding result can be exploited to detect if a Clark measure has a Dirac mass at a given point of $\mathbb{T}$.

Corollary 20.29 *Let b be a nonconstant function in the closed unit ball of H^∞ and let μ be the corresponding Clark measure. Let $\zeta_0\in\mathbb{T}$. Then*

$$\frac{b(z)-1}{z-\zeta_0}\in\mathcal{H}(b)$$

if and only if $\mu(\{\zeta_0\})>0$.

Proof First let us prove that $\mu(\{\zeta_0\})>0$ if and only if δ_{ζ_0} is absolutely continuous with respect to μ and $d\delta_{\zeta_0}/d\mu\in L^2(\mu)$, where δ_{ζ_0} is the Dirac measure associated with ζ_0.

Assume that $\mu(\{\zeta_0\})>0$. Hence, we can write

$$\mu = \mu(\{\zeta_0\})\delta_{\zeta_0} + \mu_0,$$

where μ_0 is a positive Borel measure such that $\mu_0(\{\zeta_0\}) = 0$. This representation reveals that δ_{ζ_0} is absolutely continuous with respect to μ and

$$\frac{d\delta_{\zeta_0}}{d\mu} = \frac{1}{\mu(\{\zeta_0\})}\,\chi_{\{\zeta_0\}},$$

where χ_E denotes the characteristic function of the set E. Thus, $d\delta_{\zeta_0}/d\mu \in L^2(\mu)$.

Reciprocally, assume that δ_{ζ_0} is absolutely continuous with respect to μ and $h = d\delta_{\zeta_0}/d\mu \in L^2(\mu)$. Assume that $\mu(\{\zeta_0\}) = 0$. Then, since δ_{ζ_0} is absolutely continuous with respect to μ, we necessarily have

$$\delta_{\zeta_0}(\{\zeta_0\}) = \int_{\{\zeta_0\}} h\,d\mu = 0,$$

which is absurd. Therefore, $\mu(\{\zeta_0\}) > 0$ and we get the desired equivalence.

Now, according to (13.46), δ_{ζ_0} is the Clark measure of the inner function $\Theta(z) = z/\zeta_0$. Hence, by Theorem 20.28 and the above discussion, $\mu(\{\zeta_0\}) > 0$ if and only if

$$\frac{1-b}{1-\Theta}k_0^\Theta \in \mathcal{H}(b).$$

It remains to note that

$$\frac{1-b(z)}{1-\Theta(z)}k_0^\Theta(z) = \frac{1-b(z)}{1-z/\zeta_0} = \zeta_0\frac{b(z)-1}{z-\zeta_0}. \qquad \square$$

Notes on Chapter 20

Section 20.1

Theorem 20.1 is due to Lotto and Sarason [123, lemma 3.3] but the proof is taken from [166]. Exercises 20.1.1 and 20.1.2 are taken from [181].

Section 20.2

The intertwining relation (20.10) between S_ρ^* and $X_{\bar{b}}$ is due to Lotto and Sarason [123, lemma 3.4].

Section 20.3

The connection between $H^2(\mu)$ and $\mathcal{H}(b)$ was discovered by Clark [55] in the special case when μ is a singular measure (or equivalently when b is an inner function). A vector-valued version containing Theorem 20.5 as a special case can be found in Ball and Lubin [26]. Another vector-valued result encompassing this one appears in Alpay and Dym [16]. The proof presented here is due to Sarason [163]. Exercise 20.3.4 is taken from [180].

Section 20.4

Theorem 20.6 was suggested to us by D. Timotin in a private communication.

Section 20.5

Lemma 20.8 and Corollary 20.9 are due to Suárez [180].

Section 20.6

Theorem 20.10 is due to Vinogradov [189]. The simple proof presented here and based on the theory of $\mathcal{H}(b)$ spaces is due to Lotto and Sarason [123].

Section 20.7

Lemma 20.11 and Theorem 20.12 are due to Lotto and Sarason [123, lemma 3.2].

Section 20.8

Theorem 20.13 is due to Helson [100] for the inner case and to Sarason [166] in the general case, except for the equivalence with (iv) and (v), which was noted in Fricain and Mashreghi [80]. If we use the Sz.-Nagy–Foiaş theory, we can give another proof of (i) $\Longrightarrow$ (ii). Indeed, since $|b| = 1$ on an open interval, using Theorem 6.7, b is an extreme point of the unit ball of $H^\infty(\mathbb{D})$. In this case, Sarason proved [160] that the characteristic function of the operator X_b^* (in the theory of Sz.-Nagy and Foiaş) is b. But then this theory tells us that $\sigma(X_b^*) = \sigma(b)$.

Section 20.9

Theorem 20.17 is due to Lotto and Sarason [123, corollary 2.4].

Section 20.10

The multipliers of $\mathcal{H}(b)$ have been studied by Lotto [120, 121], Lotto and Sarason [123–125], Suárez [180], and Davis and McCarthy [62]. Despite all these efforts, the general solution is still unknown.

The connection between multipliers of $\mathcal{H}(b)$ spaces and Toeplitz operators on weighted Hardy spaces was discovered by Lotto. In particular, Theorem 20.19 and the proof of Corollary 20.20 come from [121]. Note that Corollary 20.20 will be proved using different methods in Theorem 24.1 and

Corollary 25.5. This corollary is due originally to de Branges and Rovnyak [65] and Sarason [160].

The criterion for the multipliers of $\mathcal{H}(b)$ given in Theorem 20.19 has been exploited by Davis and McCarthy in [62]. Theorem 20.23 is due to Lotto and Sarason [123, theorem 3.6]. Theorem 20.25 is also due to Lotto and Sarason [123, theorem 13.5].

Section 20.11

For the case where b is an inner function, Theorem 20.28 appears (in a slightly different form) in a paper of Sarason [163]. The result there is based on an observation of Hitt [104] and is taken from Sarason's book [166].

21

Angular derivatives of $\mathcal{H}(b)$ functions

In the previous chapter, we characterized the boundary points where functions in $\mathcal{H}(b)$ admit an analytic continuation. In this chapter, we pursue this study and we characterize boundary points where functions in $\mathcal{H}(b)$ admit an angular derivative up to a certain order.

In Section 21.1, we start by characterizing those points $\zeta \in \mathbb{T}$ such that, for all functions $f \in \mathcal{H}(b)$, the nontangential limit

$$f(\zeta) = \lim_{\substack{z \to \zeta \\ \sphericalangle}} f(z)$$

exists. As we will see, this is connected to the well-known Julia–Carathéodory theorem. In fact, we recover this result using a Hilbert space approach based on $\mathcal{H}(b)$ spaces. We also show how to deduce Julia's inequality from the Cauchy–Schwarz inequality. In Section 21.2, we study the connection between angular derivatives and Clark measures. In Section 21.3, we give a simple sufficient condition for a Blaschke product and its derivatives up to a fixed order to admit radial limits at a boundary point. Then, in Section 21.4, we generalize this result to arbitrary functions in the closed unit ball of H^∞. In Section 21.5, we study an approximation problem by Blaschke products that will be useful in our studies on boundary derivatives of functions in $\mathcal{H}(b)$.

In Section 21.6, we give some interesting formulas for the reproducing kernels of derivatives of functions in $\mathcal{H}(b)$. In Section 21.7, we establish the connection between the existence of boundary derivatives in $\mathcal{H}(B)$, where B is a Blaschke product, and an interpolation problem. In Section 21.8, we give a nice characterization for the existence of boundary derivatives for functions in $\mathcal{H}(b)$. This explicit characterization is expressed in terms of the zeros of b, the singular measure associated with b and $\log |b|$.

21.1 Derivative in the sense of Carathéodory

In Section 3.2, we studied the angular derivative of analytic functions on the open unit disk $\mathbb{D}$. In this section we consider the smaller class of analytic functions $f : \mathbb{D} \longrightarrow \bar{\mathbb{D}}$, i.e. the elements of the closed unit ball of $H^\infty(\mathbb{D})$. We say that such a function has *angular derivative* in the sense of *Carathéodory* at $\zeta_0 \in \mathbb{T}$ if it has an angular derivative at ζ_0 and moreover $|f(\zeta_0)| = 1$. By the maximum principle, for some $z \in \mathbb{D}$, $f(z) \in \mathbb{T}$ happens only if f is a constant function of modulus one. Hence, from now on, we consider functions that map $\mathbb{D}$ into $\mathbb{D}$.

Theorem 21.1 *Let $b : \mathbb{D} \longrightarrow \mathbb{D}$ be analytic, let $\zeta \in \mathbb{T}$, and put*

$$c = \liminf_{z \to \zeta} \frac{1 - |b(z)|}{1 - |z|}.$$

Then the following are equivalent.

(i) *The constant c is finite, i.e.*

$$c < \infty.$$

(ii) *There is $\lambda \in \mathbb{T}$ such that*

$$\frac{b(z) - \lambda}{z - \zeta} \in \mathcal{H}(b).$$

(iii) *For all functions $f \in \mathcal{H}(b)$,*

$$f(\zeta) = \lim_{\substack{z \to \zeta \\ \sphericalangle}} f(z)$$

exists.

(iv) *The function b has an angular derivative in the sense of Carathéodory at ζ.*

Moreover, under the preceding equivalent conditions, the following results hold.

(a) *The constant c is not zero, i.e.*

$$c > 0.$$

(b) *We have $|b(\zeta)| = 1$, $c = |b'(\zeta)|$ and*

$$b'(\zeta) = \frac{b(\zeta)}{\zeta} |b'(\zeta)|.$$

(c) *We have*

$$k_\zeta^b(z) = \frac{1 - \overline{b(\zeta)} b(z)}{1 - \bar{\zeta} z} \in \mathcal{H}(b).$$

(d) *For each $f \in \mathcal{H}(b)$,*

$$f(\zeta) = \langle f, k_\zeta^b \rangle_b.$$

(e) *We have*

$$\lim_{\substack{z \to \zeta \\ \sphericalangle}} \|k_z^b - k_\zeta^b\|_b = 0.$$

(f) *We have*

$$|b'(\zeta)| = k_\zeta^b(\zeta) = \|k_\zeta^b\|_b^2 = c.$$

(g) *We have*

$$c = \lim_{\substack{z \to \zeta \\ \sphericalangle}} \frac{1 - |b(z)|}{1 - |z|}.$$

Proof Our plan is to show that

$$\text{(i)} \implies \text{(ii)} \implies \text{(iii)} \implies \text{(i)}$$

and then

$$\text{(i)}, \text{(ii)}, \text{(iii)} \implies \text{(iv)} \implies \text{(i)}.$$

The properties (a)–(g) will be obtained at different steps of the proof.

(i) $\implies$ (ii) If $c < \infty$, then there is a sequence $(z_n)_{n \geq 1}$ in $\mathbb{D}$ converging to ζ such that

$$c = \lim_{n \to \infty} \frac{1 - |b(z_n)|}{1 - |z_n|} < \infty.$$

Hence, we necessarily have $\lim_{n \to \infty} |b(z_n)| = 1$. Therefore, we can write

$$c = \lim_{n \to \infty} \frac{1 - |b(z_n)|^2}{1 - |z_n|^2}.$$

In the light of Theorem 18.11, this means that

$$c = \lim_{n \to \infty} \|k_{z_n}^b\|_b^2.$$

This is the main observation, due to Sarason, that allows us to use Hilbert space techniques. By Theorem 1.27, $(k_{z_n}^b)_{n \geq 1}$ has a weakly convergent subsequence in $\mathcal{H}(b)$. Since $(b(z_n))_{n \geq 1}$ is bounded, it also has a convergent subsequence in the closed unit disk. Hence, replacing $(z_n)_{n \geq 1}$ by a subsequence if needed, we assume there are $\lambda \in \bar{\mathbb{D}}$ and $k \in \mathcal{H}(b)$ such that $b(z_n) \longrightarrow \lambda$ and that $k_{z_n}^b \xrightarrow{w} k$. Therefore, for each $z \in \mathbb{D}$,

$$\begin{aligned} k(z) &= \langle k, k_z^b \rangle_b \\ &= \lim_{n\to\infty} \langle k_{z_n}^b, k_z^b \rangle_b \\ &= \lim_{n\to\infty} k_{z_n}^b(z) \\ &= \lim_{n\to\infty} \frac{1-\overline{b(z_n)}b(z)}{1-\bar{z}_n z} \\ &= \frac{1-\bar{\lambda}b(z)}{1-\bar{\zeta}z}. \end{aligned}$$

Since $k \in H^2(\mathbb{D})$ and $1/(1-\bar{\zeta}\,z) \notin H^2(\mathbb{D})$, we must have $|\lambda| = 1$ and thus

$$\lambda\bar{\zeta}k(z) = \frac{b(z)-\lambda}{z-\zeta} \in \mathcal{H}(b).$$

Clearly $k \not\equiv 0$ and, by (1.30), the condition $k_{z_n}^b \xrightarrow{w} k$ implies that

$$0 < \|k\|_b^2 \le \liminf_{n\to\infty} \|k_{z_n}^b\|_b^2 = c. \tag{21.1}$$

This also establishes part (a).

(ii) $\Longleftrightarrow$ (iii) By assumption, $k \in \mathcal{H}(b)$. Hence,

$$b(z) = \lambda + \lambda\bar{\zeta}(z-\zeta)k(z),$$

which, by (4.15) and the fact that $k \in H^2(\mathbb{D})$, implies that

$$|b(z)-\lambda| \le |z-\zeta|\,\|k\|_2\,\|k_z\|_2 = \|k\|_2\,\frac{|z-\zeta|}{(1-|z|^2)^{1/2}}.$$

Thus, if $z \in S_C(\zeta)$, we have

$$|b(z)-\lambda| \le C\,\|k\|_2(1-|z|^2)^{1/2},$$

and the last quantity tends to zero when z tends to ζ from within $S_C(\zeta)$. Therefore,

$$\lim_{\substack{z\to\zeta \\ \sphericalangle}} b(z) = \lambda.$$

Let us write $b(\zeta)$ for λ, and k_ζ^b for k, i.e.

$$k_\zeta^b(z) = \frac{1-\overline{b(\zeta)}b(z)}{1-\bar{\zeta}z}.$$

With the new notation, we have $k_\zeta^b \in \mathcal{H}(b)$. This is part (c). We also have

$$k_\zeta^b(z) = \langle k_\zeta^b, k_z^b \rangle_b \qquad (z \in \mathbb{D}).$$

Moreover, by the Cauchy–Schwarz inequality,

$$|k_\zeta^b(z)| \le \|k_\zeta^b\|_b \, \|k_z^b\|_b.$$

Since

$$|k_\zeta^b(z)| = \frac{|1-\overline{b(\zeta)}b(z)|}{|1-\bar{\zeta}z|} \ge \frac{1-|b(z)|}{|z-\zeta|} = \frac{(1-|z|^2)\,\|k_z^b\|_b^2}{(1+|b(z)|)\,|z-\zeta|},$$

the preceding two inequalities imply that

$$\|k_z^b\|_b \le \|k_\zeta^b\|_b \frac{1+|b(z)|}{1+|z|} \, \frac{|z-\zeta|}{1-|z|}.$$

Hence, in each Stolz domain $S_C(\zeta)$,

$$\|k_z^b\|_b \le 2C\,\|k_\zeta^b\|_b \qquad (z \in S_C(\zeta)). \tag{21.2}$$

This inequality means that $\|k_z^b\|_b$ stays bounded as z tends nontangentially to ζ. This fact is exploited below.

For each fixed $w \in \mathbb{D}$,

$$\lim_{\substack{z\to\zeta \\ \sphericalangle}} k_z^b(w) = \lim_{\substack{z\to\zeta \\ \sphericalangle}} \frac{1-\overline{b(z)}b(w)}{1-\bar{z}w} = \frac{1-\overline{b(\zeta)}b(w)}{1-\bar{\zeta}w} = k_\zeta^b(w).$$

We can rewrite this relation in the form

$$\lim_{\substack{z\to\zeta \\ \sphericalangle}} \langle k_z^b, k_w^b \rangle_b = \langle k_\zeta^b, k_w^b \rangle_b.$$

Therefore,

$$\lim_{\substack{z\to\zeta \\ \sphericalangle}} \langle f, k_z^b \rangle_b = \langle f, k_\zeta^b \rangle_b, \tag{21.3}$$

where $f \in \mathcal{H}(b)$ is any element of the form $f = \alpha_1 k_{w_1}^b + \cdots + \alpha_n k_{w_n}^b$. But the collection of such elements is dense in $\mathcal{H}(b)$, and thus, by (21.2), the identity (21.3) holds for all $f \in \mathcal{H}(b)$. At the same time, (21.3) shows that

$$f(\zeta) = \lim_{\substack{z\to\zeta \\ \sphericalangle}} f(z) = \langle f, k_\zeta^b \rangle_b \qquad (f \in \mathcal{H}(b)).$$

This is part (d). In particular, with $f = k_\zeta^b$, we obtain $k_\zeta^b(\zeta) = \|k_\zeta^b\|_b^2$. This is partially part (f). The relation (21.3) also implies that $k_z^b \xrightarrow{w} k_\zeta^b$ as z tends nontangentially to ζ.

(iii) $\Longrightarrow$ (i) Fix any Stolz domain $S_C(\zeta)$. Consider k_z^b as an element of the dual space of $\mathcal{H}(b)$. Then the relation $f(z) = \langle f, k_z^b\rangle_b$ along with our assumption imply that

$$\sup_{z\in S_C(\zeta)} |\langle f, k_z^b\rangle_b| = C(f) < \infty.$$

Thus, by the uniform boundedness principle,

$$C' = \sup_{z\in S_C(\zeta)} \|k_z^b\|_b < \infty.$$

Take $z_n = (1-1/n)\zeta$, $n \geq 1$. Since $z_n \in S_C(\zeta)$ for sufficiently large n, we have

$$\frac{1-|b(z_n)|^2}{1-|z_n|^2} = \|k_{z_n}^b\|_b^2 \leq C'^2 \qquad (n \geq N),$$

which implies in particular that $\lim_{n\to\infty} |b(z_n)| = 1$. Moreover,

$$c \leq \liminf_{n\to\infty} \frac{1-|b(z_n)|^2}{1-|z_n|^2} = \liminf_{n\to\infty} \|k_{z_n}^b\|_b^2 \leq C'^2.$$

(i), (ii), (iii) $\Longrightarrow$ (iv) Since $k_\zeta^b \in \mathcal{H}(b)$ we have

$$\frac{b(z)-b(\zeta)}{z-\zeta} = \frac{k_\zeta^b(z)b(\zeta)}{\zeta} = \langle k_\zeta^b, k_z^b\rangle_b \frac{b(\zeta)}{\zeta} \qquad (z \in \mathbb{D}).$$

On the other hand, we know that $k_z^b \xrightarrow{w} k_\zeta^b$ as z tends nontangentially to ζ. Hence,

$$\lim_{\substack{z\to\zeta \\ \sphericalangle}} \frac{b(z)-b(\zeta)}{z-\zeta} = \|k_\zeta^b\|_b^2 \frac{b(\zeta)}{\zeta},$$

which, by Theorem 3.1, means that

$$b'(\zeta) = \|k_\zeta^b\|_b^2 \frac{b(\zeta)}{\zeta}. \tag{21.4}$$

Thus, $|b'(\zeta)| = \|k_\zeta^b\|_b$. This is partially part (f).

By (21.1), $c \geq \|k_\zeta^b\|_b^2$. To show the reverse inequality, we prove that

$$\|k_z^b\|_b \longrightarrow \|k_\zeta^b\|_b$$

as z tends nontangentially to ζ. This fact has three consequences. First, it implies $c \leq \|k_\zeta^b\|_b^2$, and thus we indeed have $c = \|k_\zeta^b\|_b^2$. This is partially part (f). Second, since $k_z^b \xrightarrow{w} k_\zeta^b$, as z tends nontangentially to ζ, we have $\|k_z^b - k_\zeta^b\|_b \longrightarrow 0$. This is part (e). Third,

$$\lim_{\substack{z\to\zeta \\ \sphericalangle}} \frac{1-|b(z)|}{1-|z|} = \lim_{\substack{z\to\zeta \\ \sphericalangle}} \frac{1-|b(z)|^2}{1-|z|^2} = \lim_{\substack{z\to\zeta \\ \sphericalangle}} \|k_z^b\|_b^2 = \|k_\zeta^b\|_b^2 = c.$$

This is part (g).

To prove that $\|k_z^b\|_b \longrightarrow \|k_\zeta^b\|_b$, as z tends nontangentially to ζ, let

$$g(z) = \frac{b(z) - b(\zeta)}{z - \zeta} - b'(\zeta) \qquad (z \in \mathbb{D}).$$

Thus

$$b(z) = b(\zeta) + b'(\zeta)(z - \zeta) + (z - \zeta)g(z) \qquad (z \in \mathbb{D})$$

and, by (21.4),

$$|b(z)|^2 = 1 - 2\,\|k_\zeta^b\|_b^2\,\Re(1 - \bar{\zeta}z) + h(z) \qquad (z \in \mathbb{D}),$$

where

$$h(z) = (|b'(\zeta)|^2 + |g(z)|^2)\,|z - \zeta|^2 \\ + 2\,\Re\Big(g(z)(z - \zeta)\overline{(b(\zeta) + b'(\zeta)(z - \zeta))}\Big).$$

The only important fact about h that we need is that

$$\lim_{\substack{z \to \zeta \\ \sphericalangle}} \frac{h(z)}{1 - |z|} = 0.$$

It is also elementary to verify that

$$\frac{\Re(1 - \bar{\zeta}z)}{1 - |z|^2} = \frac{1}{2} + \frac{1}{2}\frac{|z - \zeta|^2}{1 - |z|^2},$$

which immediately gives

$$\lim_{\substack{z \to \zeta \\ \sphericalangle}} \frac{\Re(1 - \bar{\zeta}z)}{1 - |z|^2} = \frac{1}{2}.$$

Therefore,

$$\lim_{\substack{z \to \zeta \\ \sphericalangle}} \|k_z^b\|_b^2 = \lim_{\substack{z \to \zeta \\ \sphericalangle}} \frac{1 - |b(z)|^2}{1 - |z|^2} = \|k_\zeta^b\|_b^2.$$

(iv) $\Longrightarrow$ (i) If b has an angular derivative in the sense of Carathéodory at ζ, then the inequality

$$\frac{1 - |b(r\zeta)|}{1 - r} \le \left| \frac{b(r\zeta) - b(\zeta)}{r\zeta - \zeta} \right|$$

implies that

$$c = \liminf_{z \to \zeta} \frac{1 - |b(z)|}{1 - |z|} \le \lim_{r \to 1} \left| \frac{b(r\zeta) - b(\zeta)}{r\zeta - \zeta} \right| = |b'(\zeta)| < \infty. \qquad \square$$

Remark 21.2 It is trivial to see that, if

$$d = \sup_{r<1} \frac{1 - |b(r\zeta)|}{1 - r} < \infty,$$

then the quantity

$$c = \liminf_{z \to \zeta} \frac{1 - |b(z)|}{1 - |z|}$$

is finite. The converse is also true. Indeed, if $c < \infty$, then, by Theorem 21.1, we know that k_z^b tends to k_ζ^b in norm as $z \longrightarrow \zeta$ nontangentially. In particular, we have that $\|k_{r\zeta}^b\|_b \longrightarrow \|k_\zeta^b\|_b$ as $r \longrightarrow 1$. Hence, the norms $\|k_{r\zeta}^b\|_b$ are uniformly bounded with respect to r, which precisely means that $d < \infty$.

Corollary 21.3 *Let $b_1, b_2 : \mathbb{D} \longrightarrow \mathbb{D}$ be analytic, let $\zeta \in \mathbb{T}$, and assume that b_1 and b_2 have angular derivatives in the sense of Carathéodory at ζ. Then $b = b_1 b_2$ also has an angular derivative in the sense of Carathéodory at ζ and, moreover,*

$$|b'(\zeta)| = |b_1'(\zeta)| + |b_2'(\zeta)|.$$

Proof In the proof, we repeatedly appeal to several parts of Theorem 21.1. Since b_1 and b_2 have unimodular nontangential limits at ζ, then so does b. Write

$$\frac{1 - |b(z)|}{1 - |z|} = \frac{1 - |b_1(z)|}{1 - |z|} + |b_1(z)| \frac{1 - |b_2(z)|}{1 - |z|}.$$

Upon letting $z = r\zeta \longrightarrow \zeta$, the result follows. □

According to Theorem 21.1, the condition

$$\lim_{\substack{z \to \zeta \\ \sphericalangle}} \frac{1 - |b(z)|}{1 - |z|} < \infty \tag{21.5}$$

is equivalent to $(b(z) - \lambda)/(z - \zeta) \in \mathcal{H}(b)$ for some $\lambda \in \mathbb{T}$. Knowing this fact, one may naturally wonder if the condition

$$\frac{b(z) - \lambda}{z - \zeta} \in H^2$$

is still strong enough to imply (21.5). The following example provides a negative answer. Fix a number $p \in (1/2, 2/3)$ and let

$$b(z) = 1 - 2^{-p}(1 - z)^p.$$

It is clear that b has the nontangential limit 1 at the point $\zeta = 1$ and, due to the assumption $p > 1/2$, that

$$\frac{1 - b(z)}{1 - z} \in H^2.$$

Moreover, since

$$\frac{1 - b(r)}{1 - r} = 2^{-p}(1 - r)^{p-1}$$

and thus the quotient tends to ∞ as $r \longrightarrow 1$, the condition (21.5) fails for b. It just remains to show that b is in the closed unit ball of H^∞. For that, it suffices to show that the mapping $z \longmapsto z^p$ (where we take the principal branch) sends the disk $|z-1/2| \le 1/2$ into the disk $|z-1| \le 1$. To verify this fact, note that the boundary of the disk $|z-1/2| \le 1/2$ is parameterized by

$$\begin{array}{rcl} [-\pi/2, \pi/2] & \longrightarrow & \mathbb{C} \\ t & \longmapsto & e^{it}\cos(t). \end{array}$$

Hence, its image under the mapping $z \longmapsto z^p$ is given by

$$\begin{array}{rcl} [-\pi/2, \pi/2] & \longrightarrow & \mathbb{C} \\ t & \longmapsto & e^{ipt}\cos^p(t). \end{array}$$

Therefore, we need to verify that

$$[1-\cos^p(t)\cos(pt)]^2 + [\cos^p(t)\sin(pt)]^2 \le 1 \qquad (0 \le t \le \pi/2).$$

This can be rewritten as

$$\cos^p(t) \le 2\cos(pt) \qquad (0 \le t \le \pi/2),$$

which is an elementary inequality. The assumption $p < 2/3$ is exploited here.

However, despite the above example for the general case, whenever $b = \Theta$ is an inner function, then the assumption

$$\frac{\Theta(z)-\lambda}{z-\zeta} \in H^2, \tag{21.6}$$

where $\lambda \in \mathbb{T}$, is enough to ensure (21.5). In fact, by (4.15), we easily see that condition (21.6) implies that $\Theta(z)$ tends to λ as z nontangentially tends to ζ. Hence, we can write $\lambda = \Theta(\zeta)$ and

$$\frac{\Theta(z)-\lambda}{z-\lambda} = \frac{\Theta(z)-\Theta(\zeta)}{z-\zeta} = \frac{\Theta(\zeta)}{\zeta}\,\frac{1-\overline{\Theta(\zeta)}\Theta(z)}{1-\bar{\zeta}z}$$

or equivalently

$$\frac{1-\overline{\Theta(\zeta)}\Theta(z)}{1-\bar{\zeta}z} = \zeta\,\overline{\Theta(\zeta)}\,\frac{\Theta(z)-\lambda}{z-\lambda} \in H^2.$$

For almost all $z \in \mathbb{T}$, we also have

$$\frac{1-\overline{\Theta(\zeta)}\Theta(z)}{1-\bar{\zeta}z} = \bar{z}\,\Theta(z)\overline{\left(\frac{\Theta(z)-\Theta(\zeta)}{z-\zeta}\right)}.$$

Therefore, the function

$$\frac{1-\overline{\Theta(\zeta)}\Theta(z)}{1-\bar{\zeta}z}$$

actually belongs to $K_\Theta = H^2 \cap \Theta\overline{H_0^2}$. We can now apply Theorem 21.1 (implication (ii) $\Longrightarrow$ (i)) to conclude that Θ satisfies the condition (21.5).

A function that has an angular derivative in the sense of Carathéodory has an interesting geometrical property, which was discovered by Julia.

Theorem 21.4 *Let $b : \mathbb{D} \longrightarrow \mathbb{D}$ be analytic, and let $\zeta \in \mathbb{T}$. Suppose that b has an angular derivative in the sense of Carathéodory at ζ. Then*

$$\frac{|b(z)-b(\zeta)|^2}{1-|b(z)|^2} \le |b'(\zeta)| \frac{|z-\zeta|^2}{1-|z|^2} \qquad (z \in \mathbb{D}).$$

Moreover, the equality holds if and only if b is a Möbius transformation.

Proof By the Cauchy–Schwarz inequality,

$$|\langle k_\zeta^b, k_z^b\rangle_b|^2 \le \|k_\zeta^b\|_b^2 \, \|k_z^b\|_b^2.$$

But, by Theorem 21.1, this is exactly the required inequality. To see when equality holds, note that Julia's inequality can be rewritten as

$$\Re\left(\frac{z+\zeta}{z-\zeta} - c\,\frac{b(z)+b(\zeta)}{b(z)-b(\zeta)}\right) \ge 0,$$

where $c = |b'(\zeta)|$. A positive harmonic function either identically vanishes or has no zeros. Hence, if equality holds even at one point inside $\mathbb{D}$, then we must have

$$\Re\left(\frac{z+\zeta}{z-\zeta} - c\,\frac{b(z)+b(\zeta)}{b(z)-b(\zeta)}\right) = 0 \qquad (z \in \mathbb{D}).$$

Therefore, we have

$$\frac{z+\zeta}{z-\zeta} - c\,\frac{b(z)+b(\zeta)}{b(z)-b(\zeta)} = i\gamma \qquad (z \in \mathbb{D}),$$

where $\gamma \in \mathbb{R}$. This identity shows that b is a Möbius transformation. That the equality holds for a Möbius transformation is easy to verify directly. □

Julia's inequality has a geometrical interpretation. The relation

$$\frac{|1-z|^2}{1-|z|^2} \le r \quad\Longleftrightarrow\quad \left|z - \frac{1}{1+r}\right| \le \left(\frac{r}{1+r}\right)^2$$

reveals that the set

$$\left\{z \in \mathbb{C} : \frac{|z-\zeta|^2}{1-|z|^2} \le r\right\}$$

is a disk of radius $r/(1+r)$ in $\mathbb{D}$ whose center is on the ray $[0, \zeta]$ and is tangent to the unit circle $\mathbb{T}$ at the point ζ. Julia's inequality say that this disk is mapped into a similar disk of radius $rc/(1+rc)$ that is tangent to $\mathbb{T}$ at the point $b(\zeta)$; see Figure 21.1.

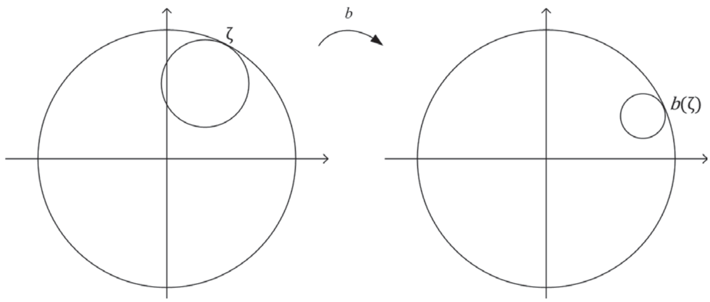

Figure 21.1 The geometric interpretation of Julia's inequality.

Exercises

Exercise 21.1.1 Let b be a function in the unit ball of H^∞ that is not the identity and not a constant. We say that a point $z_0 \in \bar{\mathbb{D}}$ is a fixed point of b if

$$\lim_{r \to 1^-} b(rz_0) = z_0.$$

Furthermore, a fixed point z_0 of b will be called a Denjoy–Wolff point of b if either $z_0 \in \mathbb{D}$ or $z_0 \in \mathbb{T}$ and b has an angular derivative at z_0 satisfying $b'(z_0) \leq 1$.

(i) Show that b can have at most one fixed point in $\mathbb{D}$ and furthermore that $|b'(z)| \leq 1$ at such a point.
Hint: Use the fact that we have equality in the Schwarz–Pick inequality if and only if b is a Möbius transformation.

(ii) Show that, if $\|b\|_\infty < 1$, then b does have a fixed point in $\mathbb{D}$.
Hint: Apply Rouché's theorem.

(iii) In this exercise, we would like to prove that b has at most one Denjoy–Wolff point.

(a) Assume first that b has two distinct Denjoy–Wolff points $z_0 \in \mathbb{T}$ and $z_1 \in \mathbb{D}$.

(1) Show that $k^b_{z_0}$ and $k^b_{z_1}$ are linearly dependent.
Hint: Show that

$$\det \begin{pmatrix} k^b_{z_0}(z_0) & k^b_{z_0}(z_1) \\ k^b_{z_1}(z_0) & k^b_{z_1}(z_1) \end{pmatrix} = b'(z_0) - 1.$$

(2) Prove that

$$\frac{1 - \bar{z}_1 b(z)}{1 - \bar{z}_1 z} = \frac{1 - \bar{z}_0 b(z)}{1 - \bar{z}_0 z}$$

and conclude that b is the identity function, which is a contradiction with the hypothesis.

(b) Assume now that b has two distinct Denjoy–Wolff points $z_0, z_1 \in \mathbb{T}$. Argue as before also to get a contradiction.
(c) Conclude that b has at most one Denjoy–Wolff point.

(iv) In this exercise, we would like to prove that b has a unique Denjoy–Wolff point. Assume that b has no fixed point in $\mathbb{D}$. For $0 < r < 1$, let z_r be the fixed point in $\mathbb{D}$ of the function rb (note that $\|rb\|_\infty < 1$).

(a) Show that there exists a sequence $z_n = z_{r_n}$ that converges to a point $z_0 \in \mathbb{T}$.
(b) Show that b has an angular derivative in the sense of Carathéodory at z_0 with $0 < b'(z_0) \le 1$.
Hint: Apply Theorem 21.1.
(c) Show that z_0 is a Denjoy–Wolff point of b.
(d) Conclude.

Exercise 21.1.2 Let b be a function in the unit ball of H^∞ that is not the identity and not a constant. Let $z_0, z_1, \dots, z_n$ be distinct fixed points of b in $\bar{\mathbb{D}}$. Assume that z_0 is the Denjoy–Wolff point of b and assume that b has an angular derivative at $z_1, \dots, z_n$.

(i) Justify that necessarily $b'(z_j) > 1$, $j = 1, \dots, n$.
(ii) Assume that $z_0 = 0$ and $|b'(z_0)| \le 1$. Show that

$$\sum_{j=1}^{n} \frac{1}{b'(z_j) - 1} \le \Re\left(\frac{1 + b'(0)}{1 - b'(0)}\right). \tag{21.7}$$

Hint: Define $B(z) = b(z)/z$, $z \in \mathbb{D} \setminus \{0\}$ and $B(0) = b'(0)$. For $j \ge 1$, the functions $k_{z_j}^B$ are mutually orthogonal in $\mathcal{H}(B)$. Also verify that $\|k_{z_j}^B\|_B^2 = b'(z_j) - 1$, $\|k_{z_0}^B\|_B^2 = 1 - |b'(0)|^2$ and $\langle k_{z_0}^B, k_{z_j}^B \rangle_B = 1 - \overline{b'(0)}$. Then apply Bessel's inequality.

(iii) Assume that $z_0 = 1$ and $b'(1) < 1$. Show that

$$\sum_{j=1}^{n} \frac{1}{b'(z_j) - 1} \le \frac{b'(1)}{1 - b'(1)}. \tag{21.8}$$

Hint: Note that all the fixed points are on $\mathbb{T}$ and $\|k_{z_j}^b\|_b^2 = b'(z_j)$. Also check that, for $j \ne \ell$, $\langle k_{z_j}^b, k_{z_\ell}^b \rangle_b = 1$. Denote by $G(k_{z_0}^b, \dots, k_{z_n}^b)$ the determinant of the Gram matrix whose (i, j) entry equals $\langle k_{z_i}^b, k_{z_j}^b \rangle_b$. Then show that

$$\begin{aligned} &G(k_{z_0}^b, \dots, k_{z_n}^b) \\ &\quad = (1 - b'(1)) \prod_{i=1}^{n} (b'(z_i) - 1) \left(\frac{b'(1)}{1 - b'(1)} - \sum_{j=1}^{n} \frac{1}{b'(z_j) - 1} \right). \end{aligned}$$

(iv) Assume that $z_0 = 1$ and $b'(1) = 1$. Show that

$$\sum_{j=1}^{n} \frac{|1 - z_j|^2}{b'(z_j) - 1} \le 2\,\Re\left(\frac{1}{b(0)} - 1\right). \tag{21.9}$$

Hint: Note that

$$\begin{aligned} &G(k_0^b, k_{z_0}^b, \dots, k_{z_n}^b) \\ &\quad = |b(0)|^2 \prod_{i=1}^{n} (b'(z_i) - 1) \left(2\,\Re\left(\frac{1}{b(0)} - 1\right) - \sum_{j=1}^{n} \frac{|1 - z_j|^2}{b'(z_j) - 1} \right). \end{aligned}$$

21.2 Angular derivatives and Clark measures

In this section, we explore the connection between the angular derivative in the sense of Carathéodory and the Clark measures μ_λ, which were introduced in Section 13.7. See also Section 20.11.

Theorem 21.5 *The function b has an angular derivative in the sense of Carathéodory at the point $z_0 \in \mathbb{T}$ if and only if there is a point $\lambda \in \mathbb{T}$ such that the Clark measure μ_λ has an atom at z_0. In that case, we necessarily have $\lambda = b(z_0)$ and $\mu_\lambda(\{z_0\}) = 1/|b'(z_0)|$.*

Proof According to Theorem 21.1, b has an angular derivative in the sense of Carathéodory at z_0 if and only if there is $\lambda \in \mathbb{T}$ such that

$$\frac{b(z) - \lambda}{z - \zeta} \in \mathcal{H}(b).$$

Since $\mathcal{H}(b) = \mathcal{H}(\bar{\lambda} b)$, we can say that b has an angular derivative in the sense of Carathéodory at z_0 if and only if there is $\lambda \in \mathbb{T}$ such that

$$\frac{\bar{\lambda} b(z) - 1}{z - \zeta} \in \mathcal{H}(\bar{\lambda} b).$$

By Corollary 20.29, this happens if and only if $\mu_\lambda(\{z_0\}) > 0$. Under the above conditions, Theorem 21.1 also says that $\lambda = b(z_0)$.

It remains to show that $\mu_\lambda(\{z_0\}) = 1/|b'(z_0)|$. Put

$$g_r = (1 - r) k_{r z_0} \qquad (0 < r < 1).$$

For each $\zeta \in \mathbb{T}$, we have

$$|g_r(\zeta)| = \frac{1 - r}{|1 - r\bar{z}_0 \zeta|} \le 1 \qquad (0 < r < 1).$$

Moreover,

$$g_r(z_0) = \frac{1 - r}{1 - r\bar{z}_0 z_0} = 1,$$

while, for each $\zeta \in \mathbb{T} \setminus \{z_0\}$,

$$\lim_{r\to 1} |g_r(\zeta)| = \lim_{r\to 1} \frac{1-r}{|1 - r\bar{z}_0\zeta|} = 0.$$

In short, we can write

$$\lim_{r\to 1} |g_r(\zeta)| = \chi_{\{z_0\}}(\zeta) \qquad (\zeta \in \mathbb{T}).$$

Hence, by the dominated convergence theorem,

$$\lim_{r\to 1} \|g_r\|^2_{L^2(\mu)} = \lim_{r\to 1} \int_{\mathbb{T}} |g_r(\zeta)|^2 \, d\mu_\lambda(\zeta) = \int_{\mathbb{T}} \chi_{\{z_0\}}(\zeta)\, d\mu_\lambda(\zeta) = \mu_\lambda(\zeta). \tag{21.10}$$

But, according to Theorem 20.5, the mapping $\mathbf{V}_{\bar{\lambda}b}$ is a partial isometry from $L^2(\mu_\lambda)$ onto $\mathcal{H}(\bar{\lambda}b)$. Moreover,

$$\begin{aligned}
\mathbf{V}_{\bar{\lambda}b} g_r &= (1-r)\mathbf{V}_{\bar{\lambda}b} k_{rz_0} \\
&= (1-r)\frac{k^{\bar{\lambda}b}_{rz_0}}{1 - \overline{\bar{\lambda}b(rz_0)}} \\
&= \frac{1-r}{1-\lambda\overline{b(rz_0)}}\, \frac{1-\lambda\overline{b(rz_0)}\bar{\lambda}b}{1 - r\bar{z}_0 z} \\
&= \frac{1-r}{1-\lambda\overline{b(rz_0)}} k^b_{rz_0}.
\end{aligned}$$

Remember that $\mathcal{H}(\bar{\lambda}b) = \mathcal{H}(b)$ and that $\lambda = b(z_0)$. Hence,

$$\begin{aligned}
\|\mathbf{V}_{\bar{\lambda}b} g_r\|^2_{\bar{\lambda}b} &= \|\mathbf{V}_{\bar{\lambda}b} g_r\|^2_b \\
&= \frac{(1-r)^2}{|1-\lambda\overline{b(rz_0)}|^2}\, \|k^b_{rz_0}\|^2_b \\
&= \left| \frac{rz_0 - z_0}{b(rz_0) - b(z_0)} \right|^2 \frac{1 - |b(rz_0)|^2}{1-r^2}.
\end{aligned}$$

Theorem 21.1 ensures that $|b'(z_0)| > 0$ and, by (21.10), that

$$\begin{aligned}
\mu_\lambda(\zeta) &= \lim_{r\to 1} \|g_r\|^2_{L^2(\mu)} \\
&= \lim_{r\to 1} \|\mathbf{V}_{\bar{\lambda}b} g_r\|^2_{\bar{\lambda}b} \\
&= |b'(z_0)|^{-2}\, |b'(z_0)| = 1/|b'(z_0)|.
\end{aligned}$$

This completes the proof. □

Assuming that b has an angular derivative in the sense of Carathéodory at the point $z_0 \in \mathbb{T}$, then it follows immediately from Theorem 21.5 that the measure μ_λ, where $\lambda \in \mathbb{T}$, has an atom at z_0 if and only if $\lambda = b(z_0)$. Now, we show that, for other values of λ, a certain integrability condition holds.

Theorem 21.6 *Assume that b has an angular derivative in the sense of Carathéodory at the point $z_0 \in \mathbb{T}$. Then, for every $\lambda \in \mathbb{T} \setminus \{b(z_0)\}$, we have*

$$\int_{\mathbb{T}} |e^{i\theta} - z_0|^{-2}\, d\mu_\lambda(e^{i\theta}) = \frac{|b'(z_0)|}{|\lambda - b(z_0)|^2}.$$

Proof Put $h_r = rz_0 k_{rz_0}$, where $0 < r < 1$. Hence, for each fixed $\zeta \in \mathbb{T}$,

$$|h_r(\zeta)|^2 = \frac{r^2}{|1 - r\bar{z}_0\zeta|^2} = \frac{r^2}{1 + r^2 - 2rC},$$

where $C = \Re(\bar{z}_0\zeta)$. A simple computation shows that

$$\frac{d}{dr}|h_r(\zeta)|^2 = \frac{2r(1 - rC)}{(1 + r^2 - 2rC)^2} \geq 0.$$

Therefore, the mapping $r \longmapsto |h_r(\zeta)|^2$ is increasing. Knowing this fact, by the monotone convergence theorem, we deduce that

$$\begin{aligned}
\lim_{r\to 1} \|h_r\|^2_{L^2(\mu_\lambda)} &= \lim_{r\to 1} \int_{\mathbb{T}} |h_r(\zeta)|^2\, d\mu_\lambda(\zeta)\\
&= \int_{\mathbb{T}} \lim_{r\to 1} |h_r(\zeta)|^2\, d\mu_\lambda(\zeta)\\
&= \int_{\mathbb{T}} \frac{1}{|1 - \bar{z}_0\zeta|^2}\, d\mu_\lambda(\zeta)\\
&= \int_{\mathbb{T}} \frac{1}{|e^{i\theta} - z_0|^2}\, d\mu_\lambda(e^{i\theta}).
\end{aligned}$$

Now, we use the same techniques as in the proof of Theorem 21.5. According to Theorem 20.5, the mapping $\mathbf{V}_{\bar{\lambda}b}$ is a partial isometry from $L^2(\mu_\lambda)$ onto $\mathcal{H}(\bar{\lambda}b)$. Moreover,

$$\begin{aligned}
\mathbf{V}_{\bar{\lambda}b} h_r &= rz_0 \mathbf{V}_{\bar{\lambda}b} k_{rz_0}\\
&= rz_0 \frac{k^{\bar{\lambda}b}_{rz_0}}{1 - \overline{\bar{\lambda}b(rz_0)}}\\
&= \frac{rz_0}{1 - \lambda\overline{b(rz_0)}}\, \frac{1 - \lambda\overline{b(rz_0)}\bar{\lambda}b}{1 - r\bar{z}_0 z}\\
&= \frac{rz_0}{1 - \lambda\overline{b(rz_0)}}\, k^b_{rz_0}.
\end{aligned} \tag{21.11}$$

Remember that $\mathcal{H}(\bar{\lambda}b) = \mathcal{H}(b)$. Hence,

$$\begin{aligned}\|h_r\|^2_{L^2(\mu_\lambda)} &= \|\mathbf{V}_{\bar{\lambda}b} h_r\|^2_{\bar{\lambda}b}\\ &= \|\mathbf{V}_{\bar{\lambda}b} h_r\|^2_b\\ &= \frac{r^2}{|1-\lambda\overline{b(rz_0)}|^2}\,\|k^b_{rz_0}\|^2_b\\ &= \frac{r^2}{|\lambda - b(rz_0)|^2}\,\|k^b_{rz_0}\|^2_b.\end{aligned}$$

Theorem 21.1 ensures that $|b'(z_0)| = \lim_{r\to 1}\|k^b_{rz_0}\|^2_b$. Moreover, if b has a derivative in the sense of Carathéodory at z_0, it surely has a radial limit at this point too. Thus,

$$\lim_{r\to 1}|\lambda - b(rz_0)| = |\lambda - b(z_0)| \qquad (\lambda\in\mathbb{T}\setminus\{b(z_0)\}).$$

Therefore, we finally deduce that

$$\lim_{r\to 1}\|h_r\|^2_{L^2(\mu_\lambda)} = \frac{|b'(z_0)|}{|\lambda - b(z_0)|^2}. \qquad \square$$

Theorem 21.7 *Let $z_0\in\mathbb{T}$. Suppose that there exists a point $\lambda\in\mathbb{T}$ such that*

$$\int_{\mathbb{T}} |e^{i\theta} - z_0|^{-2}\, d\mu_\lambda(e^{i\theta}) < \infty.$$

Then b has an angular derivative in the sense of Carathéodory at z_0.

Proof As in the proof of Theorem 21.6, put $h_r = rz_0 k_{rz_0}$. Then, by (21.11),

$$rz_0 k^b_{rz_0} = (1-\lambda\overline{b(rz_0)})^2 V_{\bar{\lambda}b} h_r.$$

Hence,

$$\begin{aligned}r^2\|k^b_{rz_0}\|^2_b &= |1-\lambda\overline{b(rz_0)}|^2\,\|V_{\bar{\lambda}b}h_r\|^2_b\\ &= |\lambda - b(rz_0)|^2\,\|h_r\|^2_{L^2(\mu_\lambda)}\\ &= |\lambda - b(rz_0)|^2\int_{\mathbb{T}}\frac{r^2}{|1-r\bar{z}_0\zeta|^2}\,d\mu_\lambda(\zeta). \qquad (21.12)\end{aligned}$$

In the proof of Theorem 21.6, we also saw that the mapping $r\longmapsto |h_r(\zeta)|^2$ is increasing. Therefore, since, by assumption,

$$\int_{\mathbb{T}}\frac{d\mu_\lambda(\zeta)}{|\zeta - z_0|^2} < \infty,$$

by the monotone convergence theorem, we deduce that

$$\sup_{0\le r<1} \int_{\mathbb{T}} \frac{r^2}{|1 - r\bar{z}_0\zeta|^2}\, d\mu_\lambda(\zeta) < \infty.$$

Since $|b(rz_0) - \lambda| \le 2$, by (21.12), the above growth restriction actually implies that

$$\sup_{0\le r<1} \|k^b_{rz_0}\|_b^2 < \infty.$$

In particular,

$$\liminf_{z\to z_0} \|k^b_z\|_b^2 \sup_{0\le r<1} \|k^b_{rz_0}\|_b^2 < \infty.$$

But a simple computation shows that

$$\|k^b_z\|_b^2 = \frac{1 - |b(z)|^2}{1 - |z|^2} \ge \frac{1}{2}\,\frac{1 - |b(z)|}{1 - |z|}.$$

Hence, we can say

$$\liminf_{z\to z_0} \frac{1 - |b(z)|}{1 - |z|} < \infty.$$

Therefore, by Theorem 21.1, b has an angular derivative in the sense of Carathéodory at z_0. □

21.3 Derivatives of Blaschke products

Let $(a_n)_{n\ge1}$ be a Blaschke sequence in $\mathbb{D}$, and let B be the corresponding Blaschke product. Fix a point ζ on the boundary $\mathbb{T}$. If ζ is not an accumulation point of the sequence $(a_n)_{n\ge1}$, then B is actually analytic at this point, and hence, in particular, for any value of $j \ge 0$, both limits

$$\lim_{r\to1^-} B^{(j)}(r\zeta) \quad \text{and} \quad \lim_{R\to1^+} B^{(j)}(R\zeta)$$

exist and are equal. What is more interesting is that ζ might be an accumulation point of the sequence $(a_n)_{n\ge1}$ and yet some of the above properties still hold.

Theorem 21.8 *Let $(a_n)_{n\ge1}$ be a Blaschke sequence in $\mathbb{D}$, and let B be the corresponding Blaschke product. Assume that, for an integer $N \ge 0$ and a point $\zeta \in \mathbb{T}$, we have*

$$\sum_{n=1}^{\infty} \frac{1 - |a_n|}{|\zeta - a_n|^{N+1}} \le A. \tag{21.13}$$

Then the following hold.

(i) *For each* $0 \le j \le N$*, both limits*

$$B^{(j)}(\zeta) := \lim_{r\to 1^-} B^{(j)}(r\zeta) \quad \textit{and} \quad \lim_{R\to 1^+} B^{(j)}(R\zeta)$$

exist and are equal.

(ii) *There is a constant* $C = C(N, A)$ *such that the estimation*

$$|B^{(j)}(r\zeta)| \le C$$

uniformly holds for $r \in [0,1]$ *and* $0 \le j \le N$.

Proof The essential case is $N = 0$. The rest follows by induction.

Case $N = 0$. Our strategy is to show that, under the proposed condition, $|B(r\zeta)|$ and $\arg B(r\zeta)$ have both finite limits as r tends to 1^-. For the simplicity of notation, without loss of generality, assume that $\zeta = 1$.

In the course of the proof, we repeatedly use the inequalities

$$|1-\bar{a}_n r| > 1-r \quad \text{and} \quad |1-\bar{a}_n r| > \tfrac{1}{2}|1-a_n|,$$

for $r \in (0,1)$, which are elementary to establish. As the first application, note that

$$\frac{(1-r^2)(1-|a_n|^2)}{|1-\bar{a}_n r|^2} \le 2\frac{(1-r^2)(1-|a_n|^2)}{(1-r)|1-a_n|} \le 8\frac{1-|a_n|}{|1-a_n|}.$$

Therefore, the Weierstrass M-test shows that the series

$$\sum_{n\ge 1} \frac{(1-r^2)(1-|a_n|^2)}{|1-\bar{a}_n r|^2}$$

converges uniformly in $r \in [0,1]$, and thus

$$\lim_{r\to 1^-} \sum_{n\ge 1} \frac{(1-r^2)(1-|a_n|^2)}{|1-\bar{a}_n r|^2} = 0.$$

But we have

$$\begin{aligned}
|B(r)|^2 &= \prod_{n\ge 1} \frac{|a_n - r|^2}{|1-\bar{a}_n r|^2} \\
&= \prod_{n\ge 1} \left(1 - \frac{(1-r^2)(1-|a_n|^2)}{|1-\bar{a}_n r|^2}\right) \\
&\ge 1 - \sum_{n\ge 1} \frac{(1-r^2)(1-|a_n|^2)}{|1-\bar{a}_n r|^2},
\end{aligned}$$

and this estimation enables us to deduce that

$$\liminf_{r\to 1^-} |B(r)|^2 \ge 1 - \lim_{r\to 1^-} \sum_{n\ge 1} \frac{(1-r^2)(1-|a_n|^2)}{|1-\bar{a}_n r|^2} = 1.$$

Since $|B(z)| < 1$, we conclude that

$$\lim_{r \to 1^-} |B(r)| = 1.$$

To deal with the argument, write

$$\frac{\bar{a}_n}{|a_n|} \frac{a_n - r}{1 - \bar{a}_n r} = \frac{1}{|a_n|} \frac{|a_n|^2 - 1 + 1 - r\bar{a}_n}{1 - \bar{a}_n r} = \frac{1}{|a_n|} \left(1 - \frac{1 - |a_n|^2}{1 - \bar{a}_n r}\right).$$

Thus

$$\arg\left(\frac{\bar{a}_n}{|a_n|} \frac{a_n - r}{1 - \bar{a}_n r}\right) = \arg\left(1 - \frac{1 - |a_n|^2}{1 - \bar{a}_n r}\right),$$

and, for large enough n for which the combination $(1 - |a_n|)/(|1 - a_n|)$ is small, we have

$$\left|\arg\left(1 - \frac{1 - |a_n|^2}{1 - \bar{a}_n r}\right)\right| \le M \frac{1 - |a_n|^2}{|1 - \bar{a}_n r|} \le 4M \frac{1 - |a_n|}{|1 - a_n|},$$

where M is a positive constant. Thus the series

$$\arg B(r) = \sum_{n \ge 1} \arg\left(1 - \frac{1 - |a_n|^2}{1 - \bar{a}_n r}\right)$$

converges absolutely and uniformly on $[0, 1]$, which proves that $\lim_{r \to 1^-} \arg B(r)$ exists.

The preceding two discussions together show that $L = \lim_{r \to 1^-} B(r)$ exists and has modulus one, i.e $|L| = 1$. The estimation in part (ii) trivially holds with $C = 1$. Finally, the Blaschke product satisfies the functional equation

$$B(z)\overline{B(1/\bar{z})} = 1.$$

Therefore,

$$\lim_{R \to 1^+} B(R) = \frac{1}{\lim_{R \to 1^+} \overline{B(1/R)}} = \frac{1}{\lim_{r \to 1^-} \overline{B(r)}} = \frac{1}{\bar{L}} = L.$$

This argument also shows that, if $\varepsilon > 0$ is such that $[1 - \varepsilon, 1)$ is free from the zeros of B, then B is actually continuous on $[1 - \varepsilon, 1 + \varepsilon]$.

Case $N \ge 1$. Fix $1 \le j \le N$, and suppose that the result holds for $0, 1, \dots, j - 1$. Using the formula for B and taking the logarithmic derivative of both sides gives us

$$\frac{B'(z)}{B(z)} = \sum_{n \ge 1} \frac{(1 - |a_n|^2)}{(z - a_n)(1 - \bar{a}_n z)}. \tag{21.14}$$

Thus,

$$B'(z) = \sum_{n \ge 1} B_n(z) \frac{(1 - |a_n|^2)}{(1 - \bar{a}_n z)^2}, \tag{21.15}$$

where

$$B_n(z) = \frac{B(z)(1-\bar{a}_n z)}{(z-a_n)} \qquad (n \geq 1) \tag{21.16}$$

is the subproduct formed with all zeros except a_n. Now, we use the formula for B' and take the derivative of both sides $j-1$ times. Then Leibniz's formula tells us that

$$B^{(j)}(z) = \sum_{k=0}^{j-1} \binom{j-1}{k} \sum_{n \geq 1} B_n^{(j-1-k)}(z)\, \frac{(k+1)!\, \bar{a}_n^k (1-|a_n|^2)}{(1-\bar{a}_n z)^{k+2}}.$$

Note that, on the right-hand side, we have $B_n^{(\ell)}$, where ℓ runs between 0 and $j-1$. Hence, the induction hypothesis applies. To deal with the other term, we consider $r<1$ and $R>1$ separately.

If $r<1$, then

$$\begin{aligned}
\left| \frac{(k+1)!\, \bar{a}_n^k (1-|a_n|^2)}{(1-\bar{a}_n r)^{k+2}} \right| &\leq \frac{(k+1)!\, (1-|a_n|^2)}{|(1-a_n)/2|^{k+2}} \\
&\leq \frac{2(k+1)!\, (1-|a_n|)}{|(1-a_n)/2|^{N+1}} \\
&= 2^{N+2}(k+1)!\, \frac{(1-|a_n|)}{|1-a_n|^{N+1}}.
\end{aligned}$$

But, for $R>1$, we have

$$\begin{aligned}
\left| \frac{(k+1)!\, \bar{a}_n^k (1-|a_n|^2)}{(1-\bar{a}_n R)^{k+2}} \right| &\leq \frac{(k+1)!\, (1-|a_n|^2)}{|R^{-1}-a_n|^{k+2}} \\
&\leq M\, \frac{(1-|a_n|)}{|1-a_n|^{N+1}},
\end{aligned}$$

where M is a constant. This is because the condition (21.13) ensures that any Stolz domain anchored at ζ can only contain a finite number of zeros a_n. Take any of these domains anchored at $\zeta = 1$, e.g. the one with opening $\pi/2$ or more explicitly the domain $|\Im z| \leq 1 - \Re z$. Then, for a_n that are not in this domain but are close to $\zeta = 1$, say at a distance at most 1, we have

$$|R^{-1} - a_n| \leq |1-a_n|/\sqrt{2}.$$

Thus,

$$\begin{aligned}
\frac{(k+1)!\, (1-|a_n|^2)}{|R^{-1}-a_n|^{k+2}} &\leq \frac{2^{(k+2)/2}(k+1)!\, (1-|a_n|^2)}{|1-a_n|^{k+2}} \\
&\leq \frac{2^{(N+1)/2} N!\, (1-|a_n|^2)}{|1-a_n|^{N+1}}.
\end{aligned}$$

The other points rest at a uniform positive distance from $\zeta = 1$.

Based on the above discussion and the induction hypothesis, if $\delta > 0$ is such that $[1-\delta, 1)$ is free from the zeros of B, then all the series

$$\sum_{n\geq 1} B_n^{(j-1-k)}(z)\, \frac{(k+1)!\, \bar{a}_n^k (1-|a_n|^2)}{(1-\bar{a}_n z)^{k+2}} \qquad (0 \leq k \leq j-1)$$

are uniformly and absolutely convergent for $z \in [1-\delta, 1+\delta]$. Hence, $B^{(j)}(z)$ is also a continuous function on this interval, which can be equally stated as in the theorem based on the right and left limits at $\zeta = 1$.

Appealing to the induction hypothesis, assume that the estimation in part (ii) holds for derivatives up to order $j-1$. Then the above calculation for $r < 1$ shows that

$$\begin{aligned} |B^{(j)}(r)| &\leq \sum_{k=0}^{j-1} \binom{j-1}{k} \sum_{n\geq 1} |B_n^{(j-1-k)}(r)|\, \frac{2^{N+2}(k+1)!\,(1-|a_n|)}{|1-a_n|^{N+1}} \\ &\leq \left(\sum_{k=0}^{j-1} \binom{j-1}{k} 2^{N+2}(k+1)! \right) CA. \end{aligned}$$

Hence, with a bigger constant, the result holds for the derivative of order j. We choose the largest constant corresponding to the derivative of order N as the constant C. This completes the proof of Theorem 21.8. □

The mere usefulness of the estimation in Theorem 21.8(ii) is that the constant C does not depend on the distribution of zeros. It just depends on the upper bound A and the integer N. Hence, it is equally valid for all the subproducts of B.

Theorem 21.8 is also valid if $\zeta \in \mathbb{D}$. In fact, the proof is simpler in this case, since part (i) is trivial. Hence, we can say that, if $\zeta \in \bar{\mathbb{D}}$ and

$$\sum_{n=1}^{\infty} \frac{1-|a_n|}{|1-\bar{a}_n \zeta|^{N+1}} \leq A,$$

then there is a constant $C = C(N, A)$ such that the estimation

$$|B^{(j)}(r\zeta)| \leq C$$

uniformly holds for $r \in [0, 1]$ and $0 \leq j \leq N$.

Corollary 21.9 *Let $(a_n)_{n\geq 1}$ be a Blaschke sequence in $\mathbb{D}$, and let B be the corresponding Blaschke product. Let $\zeta \in \mathbb{T}$ be such that*

$$\sum_{n=1}^{\infty} \frac{1-|a_n|}{|\zeta - a_n|^2} < \infty.$$

Then B has a derivative in the sense of Carathéodory at ζ and

$$|B'(\zeta)| = \sum_{n=1}^{\infty} \frac{1-|a_n|^2}{|\zeta - a_n|^2}.$$

Proof That B has a derivative in the sense of Carathéodory at ζ is a direct consequence of Theorems 21.1 and 21.8. To obtain the formula for $|B'(\zeta)|$, we use (21.15). Note that our condition implies that the subproducts B_n have radial limits at ζ. Hence, we can let $r \longrightarrow 1$ in

$$B'(r\zeta) = \sum_{n=1}^{\infty} B_n(r\zeta)\, \frac{(1-|a_n|^2)}{(1-\bar{a}_n r\zeta)^2}$$

to obtain

$$B'(\zeta) = \sum_{n=1}^{\infty} B_n(\zeta)\, \frac{(1-|a_n|^2)}{(1-\bar{a}_n \zeta)^2}.$$

The upper bound

$$\left| \frac{(1-|a_n|^2)}{(1-\bar{a}_n r\zeta)^2} \right| \le \frac{4(1-|a_n|)}{|\zeta - a|^2} \qquad (0<r<1)$$

allows one to pass to the limit inside the sum. But, according to (21.16), we have

$$B_n(\zeta) = \frac{B(\zeta)(1-\bar{a}_n\zeta)}{(\zeta - a_n)} \qquad (n \ge 1).$$

Plugging this back in to the formula for $B'(\zeta)$ gives

$$B'(\zeta) = \bar{\zeta} B(\zeta) \sum_{n=1}^{\infty} \frac{1-|a_n|^2}{|a_n - \zeta|^2}.$$

By taking the absolute values of both sides, the result follows. □

21.4 Higher derivatives of *b*

According to the canonical factorization theorem, b can be decomposed as

$$b(z) = B(z)S(z)O(z) \qquad (z \in \mathbb{D}), \tag{21.17}$$

where

$$B(z) = \gamma \prod_n \left(\frac{|a_n|}{a_n} \frac{a_n - z}{1-\bar{a}_n z} \right),$$

$$S(z) = \exp\left(- \int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z}\, d\sigma(\zeta) \right)$$

and

$$O(z) = \exp\left(\int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} \log|b(\zeta)|\, dm(\zeta)\right).$$

We can also extend the function b outside the unit disk by the identity (21.17) and the formulas provided for B, S and O. The extended function is analytic for $|z| > 1$, $z \neq 1/\bar{a}_n$. At $1/\bar{a}_n$ it has a pole of the same order as a_n, as a zero of B. We denote this function also by b, and it is easily verified that it satisfies the functional identity

$$b(z)\,\overline{b(1/\bar{z})} = 1. \tag{21.18}$$

One should be careful in dealing with function b inside and outside the unit disk. For example, if

$$b(z) = \tfrac{1}{2} z^n \qquad (|z| < 1),$$

it is natural to use the same nice formula for $|z| > 1$. However, the functional equation (21.18) says that

$$b(z) = 2z^n \qquad (|z| > 1).$$

Hence, b and its derivatives up to order n show a different behavior if we approach a point $\zeta_0 \in \mathbb{T}$ from within $\mathbb{D}$ or from outside. In Theorem 21.10 below, we show that, under certain circumstances, this can be avoided.

For our application in this section, we can merge $S(z)$ and $O(z)$ and write

$$b(z) = B(z) f(z), \tag{21.19}$$

where

$$f(z) = \exp\left(-\int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z}\, d\mu(\zeta)\right) \tag{21.20}$$

and μ is the positive measure $d\mu(\zeta) = -\log|b(\zeta)|\, dm(\zeta) + d\sigma(\zeta)$. Now, Leibniz's formula says that

$$b^{(j)}(z) = \sum_{k=0}^{j} B^{(k)}(z) f^{(j-k)}(z).$$

For the derivatives of B on a ray, we have already established Theorem 21.8. However, a similar result holds for function f, and thus similar statements actually hold for b, i.e. for any function in the closed unit ball of H^∞.

Theorem 21.10 *Let b be in the closed unit ball of H^∞ with the decomposition (21.17). Assume that, for an integer $N \geq 0$ and a point $\zeta_0 \in \mathbb{T}$, we have*

$$\sum_{n=1}^{\infty} \frac{1 - |a_n|}{|\zeta_0 - a_n|^{N+1}} + \int_{\mathbb{T}} \frac{d\sigma(\zeta)}{|\zeta_0 - \zeta|^{N+1}} + \int_0^{2\pi} \frac{\big|\log|b(\zeta)|\big|}{|\zeta_0 - \zeta|^{N+1}}\, dm(\zeta) \leq A. \tag{21.21}$$

Then the following hold.

(i) *For each* $0 \le j \le N$, *both limits*

$$b^{(j)}(\zeta_0) := \lim_{r \to 1^-} b^{(j)}(r\zeta_0) \quad \textit{and} \quad \lim_{R \to 1^+} b^{(j)}(R\zeta)$$

exist and are equal.

(ii) *There is a constant* $C = C(N, A)$ *such that the estimation*

$$|b^{(j)}(r\zeta_0)| \le C$$

uniformly holds for $r \in [0, 1]$ *and* $0 \le j \le N$.

Proof As discussed before the statement of the theorem, it is enough to establish the result just for the function $f = SO$ given by (21.20). The proof has the same flavor as the proof of Theorem 21.8. We first consider the case $N = 0$, and then the rest follows by induction.

Case $N = 0$. We show that, under the condition (21.21), which now translates as

$$\int_{\mathbb{T}} \frac{d\mu(\zeta)}{|\zeta_0 - \zeta|} \le A, \tag{21.22}$$

$|f(r\zeta_0)|$ and $\arg f(r\zeta_0)$ have both finite limits as r tends to 1^-. For simplicity of notation, without loss of generality, assume that $\zeta_0 = 1$.

A simple computation shows that

$$f(r) = \exp\left(-\int_{\mathbb{T}} \frac{1-r^2}{|\zeta - r|^2} d\mu(\zeta)\right) \exp\left(-i\int_{\mathbb{T}} \frac{r\,\Im(\zeta)}{|\zeta - r|^2}\, d\mu(\zeta)\right).$$

Therefore, we have explicit formulas for $|f(r)|$ and $\arg f(r)$.

The assumption (21.22) implies that there is no Dirac mass at $\zeta_0 = 1$, i.e. $\mu(\{1\}) = 0$. Therefore,

$$\lim_{r \to 1^-} \frac{1-r^2}{|\zeta - r|^2} = 0$$

for μ-almost every $\zeta \in \mathbb{T}$. Moreover, we have the upper bound estimation

$$\frac{1-r^2}{|\zeta - r|^2} \le \frac{2}{|1-\zeta|} \qquad (\zeta \in \mathbb{T}),$$

which holds uniformly for all values of the parameter $r \in (0, 1)$. The condition (21.22) means that the function on the right-hand side belongs to $L^1(\mu)$. Hence, by the dominated convergence theorem, we get

$$\lim_{r \to 1^-} \int_{\mathbb{T}} \frac{1-r^2}{|\zeta - r|^2}\, d\mu(\zeta) = 0.$$

In return, this observation implies that

$$\lim_{r \to 1^-} |f(r)| = 1.$$

In a similar manner,

$$\lim_{r\to 1^-} \frac{r\,\Im(\zeta)}{|\zeta - r|^2} = \frac{\Im(\zeta)}{|1-\zeta|^2}$$

for μ-almost all $\zeta \in \mathbb{T}$. We also have the upper bound estimation

$$\frac{r|\Im(\zeta)|}{|\zeta - r|^2} \le \frac{2}{|1-\zeta|} \qquad (\zeta \in \mathbb{T}),$$

which holds uniformly for all values of the parameter $r \in (0,1)$. Finally, again by the dominated convergence theorem, we see that the limit

$$\lim_{r\to 1^-} \int_{\mathbb{T}} \frac{r\,\Im(\zeta)}{|\zeta - r|^2}\, d\mu(\zeta) = \int_{\mathbb{T}} \frac{\Im(\zeta)}{|\zeta - 1|^2}\, d\mu(\zeta)$$

exists and is a finite real number. In return, this implies that

$$\lim_{r\to 1^-} \arg f(r)$$

also exists and is a finite real number. Therefore, $L := \lim_{r\to 1^-} f(r)$ exists and, moreover, $|L| = 1$.

Put $L = \lim_{r\to 1^-} f(r)$. By (21.18), the function f satisfies the functional equation

$$f(z)\overline{f(1/\bar{z})} = 1.$$

Therefore,

$$\lim_{R\to 1^+} f(R) = \frac{1}{\lim_{R\to 1^+} \overline{f(1/R)}} = \frac{1}{\lim_{r\to 1^-} \overline{f(r)}} = \frac{1}{\bar{L}} = L.$$

This argument also shows that f is actually bounded on $[0,+\infty)$. The estimation in part (ii) trivially holds with $C = 1$.

Case $N \ge 1$. Fix $1 \le j \le N$, and suppose that the result holds for $0, 1, \dots, j-1$. The condition (21.21) is rewritten as

$$\int_{\mathbb{T}} \frac{d\mu(\zeta)}{|1-\zeta|^{N+1}} \le A. \tag{21.23}$$

Using the formula for f and taking the derivative of both sides gives us

$$f'(z) = \left(\int_{\mathbb{T}} \frac{-2\zeta}{(\zeta - z)^2}\, d\mu(\zeta) \right) f(z). \tag{21.24}$$

Now, take the derivative of both sides $j-1$ times. Then Leibniz's formula tells us that

$$f^{(j)}(z) = \sum_{k=0}^{j-1} \left(\int_{\mathbb{T}} \frac{-2(k+1)!\,\zeta}{(\zeta - z)^{k+2}}\, d\mu(\zeta) \right) f^{(j-1-k)}(z). \tag{21.25}$$

On the right-hand side, we have $f^{(\ell)}$, where ℓ runs between 0 and $j-1$. Hence, the induction hypothesis applies. To deal with the other term, note that, for $z = r < 1$ and also $z = R > 1$, we have

$$\frac{1}{|\zeta - z|} \le \frac{2}{|\zeta - 1|} \qquad (\zeta \in \mathbb{T}).$$

Thus, for all $z \in (0, \infty) \setminus \{1\}$ and all k with $0 \le k \le j-1 \le N-1$,

$$\begin{aligned} \left| \frac{-2(k+1)!\,\zeta}{(\zeta - z)^{k+2}} \right| &\le \frac{2(k+1)!}{|(\zeta-1)/2|^{k+2}} \\ &\le \frac{2N!}{|(\zeta-1)/2|^{N+1}} \\ &= \frac{2^{N+2}N!}{|\zeta - 1|^{N+1}} \qquad (\zeta \in \mathbb{T}). \end{aligned} \tag{21.26}$$

Therefore, by (21.23), (21.26) and the dominated convergence theorem,

$$\lim_{r \to 1^\pm} \int_{\mathbb{T}} \frac{-2(k+1)!\,\zeta}{(\zeta - z)^{k+2}}\, d\mu(\zeta) = \int_{\mathbb{T}} \frac{-2(k+1)!\,\zeta}{(\zeta - 1)^{k+2}}\, d\mu(\zeta).$$

Note that again we have implicitly used the fact that $\mu(\{1\}) = 0$. Thus, by the induction hypothesis and (21.25), part (i) follows. Moreover, again by the induction hypothesis, assume that the estimation in part (ii) holds for derivatives up to order $j-1$. Then, by (21.25) and (21.26),

$$\begin{aligned} |f^{(j)}(r)| &\le \sum_{k=0}^{j-1} \left(\int_{\mathbb{T}} \left| \frac{-2(k+1)!\,\zeta}{(\zeta - r)^{k+2}} \right| d\mu(\zeta) \right) |f^{(j-1-k)}(r)| \\ &\le \left(\int_{\mathbb{T}} \frac{2^{N+2}N!}{|\zeta - 1|^{N+1}}\, d\mu(\zeta) \right) \sum_{k=0}^{j-1} |f^{(j-1-k)}(r)| \le j 2^{N+2} N!\, AC. \end{aligned}$$

Hence, with a bigger constant, the result holds for the derivative of order j. We choose the largest constant corresponding to the derivative of order N as the constant C. This completes the proof of Theorem 21.10. □

We highlight one property that was explicitly mentioned in the proof of Theorem 21.8 for Blaschke products, but also holds for an arbitrary b. Under the hypothesis of Theorem 21.10, there is a $\delta > 0$ (which depends on b) such that $b^{(j)}(z)$, for $0 \le j \le N$, is a continuous function on the ray $[(1-\delta)\zeta_0, (1+\delta)\zeta_0]$.

Corollary 21.11 *Let b be in the closed unit ball of H^∞ with the decomposition (21.17). Let $\zeta_0 \in \mathbb{T}$ be such that*

$$\sum_{n=1}^{\infty} \frac{1 - |a_n|}{|\zeta_0 - a_n|^2} + \int_{\mathbb{T}} \frac{d\sigma(\zeta)}{|\zeta_0 - \zeta|^2} + \int_0^{2\pi} \frac{\big|\log|b(\zeta)|\big|}{|\zeta_0 - \zeta|^2}\, dm(\zeta) < \infty.$$

Then b has a derivative in the sense of Carathéodory at ζ_0 and

$$|b'(\zeta_0)| = \sum_{n=1}^{\infty} \frac{1-|a_n|^2}{|\zeta_0 - a_n|^2} + \int_{\mathbb{T}} \frac{2\, d\sigma(\zeta)}{|\zeta_0 - \zeta|^2} + \int_0^{2\pi} \frac{2\big|\log|b(\zeta)|\big|}{|\zeta_0 - \zeta|^2}\, dm(\zeta).$$

Proof As we did in (21.19), write $b = Bf$. Corollary 21.9 treats the Blaschke product B and gives a formula (the first term appearing in $|b'(\zeta_0)|$ above). Hence, in the light of Corollary 21.3, we just need to study f and prove that $|f'(\zeta_0)|$ is precisely the remaining two terms in the formula for $|b'(\zeta_0)|$.

That f has a derivative in the sense of Carathéodory at ζ_0 is a direct consequence of Theorems 21.1 and 21.10. To obtain the formula for $|f'(\zeta_0)|$, we use (21.24), i.e.

$$f'(r\zeta_0) = \left(\int_{\mathbb{T}} \frac{-2\zeta}{(\zeta - r\zeta_0)^2}\, d\mu(\zeta) \right) f(r\zeta_0).$$

Now, let $r \longrightarrow 1$ to obtain

$$f'(\zeta_0) = \left(\int_{\mathbb{T}} \frac{-2\zeta}{(\zeta - \zeta_0)^2}\, d\mu(\zeta) \right) f(\zeta_0).$$

The upper bound

$$\frac{1}{|\zeta - r\zeta_0|} \le \frac{2}{|\zeta - \zeta_0|} \qquad (\zeta \in \mathbb{T},\ 0 < r < 1)$$

allows one to pass to the limit inside the integral. Now, note that

$$\frac{-2\zeta}{(\zeta - \zeta_0)^2} = \frac{2\zeta}{(\zeta - \zeta_0)(\bar{\zeta} - \bar{\zeta}_0)\zeta\zeta_0} = \frac{2\bar{\zeta}_0}{|\zeta - \zeta_0|^2}.$$

Hence, we rewrite the formula for $f'(\zeta_0)$ as

$$f'(\zeta_0) = \left(\int_{\mathbb{T}} \frac{2}{|\zeta - \zeta_0|^2}\, d\mu(\zeta) \right) \bar{\zeta}_0 f(\zeta_0).$$

By taking the absolute values of both sides, the result follows. □

21.5 Approximating by Blaschke products

According to (21.19), an arbitrary element of the closed unit ball of H^∞ may be decomposed as $b = Bf$, where B is a Blaschke product and f is a nonvanishing function given by (21.20). Generally speaking, since B is given by a product of some simple fraction of the form $(az+b)/(cz+d)$, it is easy to handle and study its properties. That is why in this section we explore the possibility of approximating f by some Blaschke products. This will enable us to establish certain properties for the family of Blaschke products first, and then extend them to the whole closed unit ball of H^∞.

Given a Blaschke product B with zeros $(a_n)_{n\geq 1}$, we define the measure σ_B on $\bar{\mathbb{D}}$ by

$$\sigma_B = \sum_{n=1}^{\infty} (1-|a_n|)\, \delta_{\{a_n\}},$$

where $\delta_{\{z\}}$ is the Dirac measure anchored at the point z. We consider σ_B as an element of $\mathcal{M}(\bar{\mathbb{D}})$, the space of finite complex Borel measures on $\bar{\mathbb{D}}$. This space is the dual of $\mathcal{C}(\bar{\mathbb{D}})$. Hence, we equip it with the weak-star topology. Since $\mathcal{C}(\bar{\mathbb{D}})$ is separable, this topology is first countable on $\mathcal{M}(\bar{\mathbb{D}})$. More specifically, this means that each measure has a countable local basis. Naively speaking, this implies that we just need to consider sequences of measures to study the properties of this topology.

In the following, we assume that the Blaschke products are normalized so that $B(0) > 0$.

Theorem 21.12 *Let f be given by (21.20), and let $(B_n)_{n\geq 1}$ be a sequence of Blaschke products. Then B_n converges uniformly to f on compact subsets of $\mathbb{D}$ if and only if $\sigma_{B_n} \longrightarrow \mu$ in the weak-star topology of $\mathcal{M}(\bar{\mathbb{D}})$.*

Proof Assume that $\sigma_{B_n} \longrightarrow \mu$ in the weak-star topology of $\mathcal{M}(\bar{\mathbb{D}})$, and denote the zeros of B_n by $(a_{nm})_{m\geq 1}$. Since μ is supported on $\mathbb{T}$, the zeros of B_n must tend to $\mathbb{T}$. In fact, fix any $r < 1$ and consider a continuous positive function φ that is identically 1 on $|z| \leq r$, and identically 0 on $|z| > (1+r)/2$. In between, it has a continuous transition from 1 to 0. Since $\sigma_{B_n} \longrightarrow \mu$ in the weak-star topology, we have

$$\int_{\bar{\mathbb{D}}} \varphi\, d\sigma_{B_n} \longrightarrow \int_{\bar{\mathbb{D}}} \varphi\, d\mu = 0.$$

But we also have

$$\begin{aligned}
\int_{\bar{\mathbb{D}}} \varphi\, d\sigma_{B_n} &\geq \int_{|z|\leq r} \varphi\, d\sigma_{B_n} \\
&= \sum_{|a_{nm}|\leq r} (1-|a_{nm}|) \\
&\geq (1-r) \times \text{Card}\{m : |a_{nm}| \leq r\}.
\end{aligned}$$

Therefore, for each $r < 1$, there is an $N = N(r)$ such that

$$|a_{nm}| > r \qquad (n \geq N,\ m \geq 1). \tag{21.27}$$

We now further explore our assumption to show that

$$B_n(0) \longrightarrow f(0). \tag{21.28}$$

Since $\sigma_{B_n} \longrightarrow \mu$ in the weak-star topology, we have

$$\int_{\bar{\mathbb{D}}} d\sigma_{B_n} \longrightarrow \int_{\bar{\mathbb{D}}} d\mu.$$

By (21.20), $f(0)$ is a positive real number and

$$\int_{\bar{\mathbb{D}}} d\mu = -\log f(0).$$

But the left-hand side is

$$\int_{\bar{\mathbb{D}}} d\sigma_{B_n} = \sum_{m=1}^{\infty} (1 - |a_{nm}|),$$

which is not precisely $-\log B_n(0)$. The actual formula is

$$-\log B_n(0) = -\sum_{m=1}^{\infty} \log |a_{nm}|.$$

However, thanks to (21.27), this difference can be handled. It is elementary to verify that

$$0 \le t - 1 - \log t \le (1-t)^2 \qquad (1/2 \le t \le 1).$$

Hence, for $n \ge N(r)$,

$$(1 - |a_{nm}|) \le -\log |a_{nm}| \le (1+r)(1 - |a_{nm}|) \qquad (m \ge 1).$$

Summing over m gives

$$\int_{\bar{\mathbb{D}}} d\sigma_{B_n} \le -\log B_n(0) \le (1+r) \int_{\bar{\mathbb{D}}} d\sigma_{B_n} \qquad (n \ge N(r)). \tag{21.29}$$

Let $n \longrightarrow \infty$ to deduce that

$$\begin{aligned} -\log f(0) &\le \liminf_{n\to\infty} -\log B_n(0) \\ &\le \limsup_{n\to\infty} -\log B_n(0) \le -(1+r) \log f(0). \end{aligned}$$

Now, let $r \longrightarrow 1$ to conclude that

$$\lim_{n\to\infty} \log B_n(0) = \log f(0).$$

The next step is to show that B_n actually uniformly converges to f on any compact subset of $\mathbb{D}$. In the language of σ_{B_n}, formula (21.14) can be rewritten as

$$\frac{B_n'(z)}{B_n(z)} = \int_{\bar{\mathbb{D}}} \frac{(1+|\zeta|)}{(z-\zeta)(1-\bar{\zeta}z)} \, d\sigma_{B_n}(\zeta).$$

At first glance, it seems that the function

$$\zeta \longmapsto \frac{(1+|\zeta|)}{(z-\zeta)(1-\bar{\zeta}z)}$$

is not continuous on $\bar{\mathbb{D}}$ and thus we cannot appeal to the weak-star convergence. However, we fix a compact set $|z| \leq r$ and, as we saw above, after a finite number of indices, the support of σ_{B_n} is in $|z| > (1+r)/2$. Hence, we can multiply the above function by a transient function that is 1 on $|z| \geq (1+r)/2$ and 0 on $|z| \leq 1$. This operation, on the one hand, will not change the value of the integrals and, on the other, will create a genuine continuous function on $\bar{\mathbb{D}}$. Therefore, we can surely say

$$\begin{aligned}\int_{\bar{\mathbb{D}}} \frac{(1+|\zeta|)}{(z-\zeta)(1-\bar{\zeta}z)}\, d\sigma_{B_n}(\zeta) \longrightarrow &\int_{\bar{\mathbb{D}}} \frac{(1+|\zeta|)}{(z-\zeta)(1-\bar{\zeta}z)}\, d\mu(\zeta)\\ = &\int_{\mathbb{T}} \frac{-2\zeta}{(\zeta-z)^2}\, d\mu(\zeta),\end{aligned}$$

which translates as

$$\frac{B_n'(z)}{B_n(z)} \longrightarrow \frac{f'(z)}{f(z)} \tag{21.30}$$

as $n \longrightarrow \infty$.

Since $(B_n)_{n\geq 1}$ is uniformly bounded by 1 on $\mathbb{D}$, it is a normal family. Let g be any pointwise limit of a subsequence of $(B_n)_{n\geq 1}$. Then, by (21.28) and (21.30), we must have

$$g(0) = f(0) \quad \text{and} \quad \frac{f'(z)}{f(z)} = \frac{f'(z)}{f(z)} \qquad (z \in \mathbb{D}).$$

Thus, $g = f$, which means that the whole sequence converges uniformly to f on compact sets.

To prove the other way around, assume that B_n converges uniformly to f on compact sets. Thus, $B_n(0) \longrightarrow f(0)$ and, since f has no zeros on $\mathbb{D}$, for each r, (21.27) must hold. Hence, if we let $n \longrightarrow \infty$ in (21.29), we obtain

$$\limsup_{n\to\infty} \int_{\bar{\mathbb{D}}} d\sigma_{B_n} \leq -\log f(0) \leq (1+r) \liminf_{n\to\infty} \int_{\bar{\mathbb{D}}} d\sigma_{B_n}.$$

Let $r \longrightarrow 1$ to deduce that

$$\int_{\bar{\mathbb{D}}} d\sigma_{B_n} \longrightarrow -\log f(0).$$

Hence, $(\sigma_{B_n})_{n\geq 1}$ is a bounded sequence in $\mathcal{M}(\bar{\mathbb{D}})$, and any weak-star limit of this sequence must be a positive measure supported on $\mathbb{T}$. But the sequence has just one weak-star limit, i.e. μ. This is because, if ν is any weak-star limit of the sequence, the first part of the proof shows that a subsequence of B_n converges to f_ν, where f_ν is given by (21.20) (with μ replaced by ν). Therefore, $f_\nu = f$ on $\mathbb{D}$ and, using the uniqueness theorem for Fourier coefficients of measures, we conclude that $\nu = \mu$. $\square$

To establish our next approximation theorem, we need a result of Frostman, which by itself is interesting and has numerous other applications. Let Θ be an inner function and, for each $w \in \mathbb{D}$, define

$$\Theta_w(z) = \frac{w - \Theta(z)}{1 - \bar{w}\Theta(z)}.$$

The function Θ_w is called a *Frostman shift* of Θ. It is easy to verify that Θ_w is an inner function for each $w \in \mathbb{D}$. However, a lot more is true. Define the *exceptional set* of Θ to be

$$\mathcal{E}(\Theta) = \{w \in \mathbb{D} : \Theta_w \text{ is not a Blaschke product}\}.$$

Frostman showed that $\mathcal{E}(u)$ is a very small set.

Lemma 21.13 *Let Θ be a nonconstant inner function, and let $0 < \rho < 1$. Define*

$$\mathcal{E}_\rho(\Theta) = \{\zeta \in \mathbb{T} : \Theta_{\rho\zeta} \textit{ is not a Blaschke product}\}.$$

Then $\mathcal{E}_\rho(\Theta)$ has one-dimensional Lebesgue measure zero.

Proof For each $\alpha \in \mathbb{D}$, we have

$$\int_{\mathbb{T}} \log \left| \frac{\rho\xi - \alpha}{1 - \rho\bar{\xi}\alpha} \right| dm(\xi) = \max(\log \rho, \log|\alpha|). \tag{21.31}$$

This is an easy consequence of the mean value property of harmonic functions. In (21.31) replace α by $\Theta(r\zeta)$ and then integrate with respect to ζ to get

$$\int_{\mathbb{T}} \left(\int_{\mathbb{T}} \log|\Theta_{\rho\xi}(r\zeta)|\, dm(\xi) \right) dm(\zeta) = \int_{\mathbb{T}} \max(\log\rho, \log|\Theta(r\zeta)|)\, dm(\zeta).$$

The collection

$$f_r(\zeta) = \max(\log\rho, \log|\Theta(r\zeta)|), \qquad r \in [0,1),$$

satisfies $\log\rho \le f_r(\zeta) \le 0$ and, moreover,

$$\lim_{r\to1} f_r(\zeta) = \max\left(\log\rho, \lim_{r\to1} \log|\Theta(r\zeta)|\right) = \max(\log\rho, 0) = 0$$

for almost every $\zeta \in \mathbb{T}$. Therefore, by the dominated convergence theorem,

$$\lim_{r\to1} \int_{\mathbb{T}} f_r(\zeta)\, dm(\zeta) = 0.$$

We rewrite this identity as

$$\lim_{r\to1} \int_{\mathbb{T}} \left(\int_{\mathbb{T}} -\log|\Theta_{\rho\xi}(r\zeta)|\, dm(\xi) \right) dm(\zeta) = 0.$$

Since the integrand $-\log|\Theta_{\rho\xi}|$ is positive, Fubini's theorem can be applied. The outcome is

$$\lim_{r\to 1}\int_{\mathbb{T}}\left(\int_{\mathbb{T}} -\log|\Theta_{\rho\xi}(r\zeta)|\,dm(\zeta)\right)dm(\xi)=0.$$

Now, Fatou's lemma implies that

$$\int_{\mathbb{T}}\left(\liminf_{r\to 1}\int_{\mathbb{T}} -\log|\Theta_{\rho\xi}(r\zeta)|\,dm(\zeta)\right)dm(\xi)=0.$$

Hence, we must have

$$\liminf_{r\to 1}\int_{\mathbb{T}} \log|\Theta_{\rho\xi}(r\zeta)|\,dm(\zeta)=0 \tag{21.32}$$

for almost all $\xi\in\mathbb{T}$, and this precisely means that, for such values of ξ, the Frostman shift $\Theta_{\rho\xi}$ is a Blaschke product. Indeed, if we consider the canonical factorization $\Theta_{\rho\xi}=BS$, where B is a Blaschke product and S is the singular measure

$$S(z)=\exp\left(-\int_{\mathbb{T}}\frac{w+z}{w-z}\,d\sigma(w)\right),$$

then

$$\int_{\mathbb{T}}\log|\Theta_{\rho\xi}(r\zeta)|\,dm(\zeta)\le\int_{\mathbb{T}}\log|S(r\zeta)|\,dm(\zeta)=-\sigma(\mathbb{T}).$$

which, by (21.32), implies that $\sigma\equiv 0$. □

Among numerous applications of Lemma 21.13, we single out the one which states that Blaschke products are uniformly dense in the family of inner functions. A variation of the technique used in the proof of the following result will be exploited in establishing Theorem 21.15.

Corollary 21.14 *Let Θ be an inner function, and let $\varepsilon>0$. Then there is a Blaschke product B such that*

$$\|\Theta-B\|_\infty<\varepsilon.$$

Proof We have

$$\Theta(z)+\Theta_w(z)=\Theta(z)+\frac{w-\Theta(z)}{1-\bar{w}\Theta(z)}=\frac{w-\bar{w}\Theta^2(z)}{1-\bar{w}\Theta(z)},$$

and thus

$$|\Theta(z)+\Theta_w(z)|\le\frac{2|w|}{1-|w|}.$$

On the one hand, this shows that $-\Theta_w\longrightarrow\Theta$ in the H^∞ norm as $w\longrightarrow 0$ and, on the other, Lemma 21.13 ensures that there are numerous choices of w for which $-\Theta_w$ is a Blaschke product. □

In the following result, we again use $\mathcal{M}(\bar{\mathbb{D}})$, equipped with the weak-star topology. We recall that it is first countable, i.e. each point has a countable local basis of open neighborhood.

Theorem 21.15 *Let $\lambda \in \bar{\mathbb{D}}$, let $N \geq 1$, and let μ be a positive measure on $\mathbb{T}$ such that*

$$\int_{\mathbb{T}} \frac{d\mu(\zeta)}{|1 - \bar{\lambda}\zeta|^N} < \infty.$$

Then there is a sequence of Blaschke products $(B_n)_{n\geq 1}$ such that $\sigma_{B_n} \longrightarrow \mu$ in the weak-star topology of $\mathcal{M}(\bar{\mathbb{D}})$ and, moreover,

$$\sum_{m=1}^{\infty} \frac{1 - |a_{nm}|^2}{|1 - \bar{\lambda}a_{nm}|^N} \longrightarrow \int_{\mathbb{T}} \frac{2\, d\mu(\zeta)}{|1 - \bar{\lambda}\zeta|^N}$$

as $n \longrightarrow \infty$.

Proof First, note that the growth restriction on μ implies that μ cannot have a Dirac mass at $1/\bar{\lambda}$. Our strategy is to prove the theorem for discrete measures with finitely many Dirac masses and then appeal to a limiting argument to extend it for the general case.

Assume that $\sigma = \alpha\delta_{\{1\}}$, where $\alpha > 0$. Construct f according to the recipe (21.20), i.e.

$$f(z) = \exp\left(-\alpha\,\frac{1+z}{1-z}\right).$$

By Lemma 21.13, the function

$$B_c(z) = \gamma_c\,\frac{f(z) - c}{1 - \bar{c}f(z)}$$

is a Blaschke product for values of c through a sequence that tends to zero and avoids the exceptional set of f. The unimodular constant

$$\gamma_c = \frac{f(0) - \bar{c}}{|f(0) - c|}\,\frac{|1 - \bar{c}f(0)|}{1 - cf(0)}$$

is added to ensure that $B_c(0) > 0$. The precise value of γ_c is not used below. We just need to know that $\gamma_c \longrightarrow 1$ as $c \longrightarrow 0$. The formula for B_c implies that

$$|f(z) - B_c(z)| \leq |1 - \gamma_c| + \frac{2|c|}{1 - |c|} \qquad (z \in \mathbb{D}).$$

Thus, B_c converges uniformly to f on $\mathbb{D}$ (even uniform convergence on compact sets is enough for us). Therefore, by Theorem 21.12, σ_{B_c} tends to μ in the weak-star topology. The zeros of B_c are

$$\{z : f(z) = c\} = \left\{a_{cm} = \frac{\alpha + \log c + i2\pi m\alpha}{-\alpha + \log c + i2\pi m\alpha} : m \in \mathbb{Z}\right\},$$

which clearly cluster at 1 as $c \longrightarrow 0$. In this case, $\lambda \neq 1$, and thus the function

$$\zeta \longmapsto \frac{1+|\zeta|}{|1-\bar{\lambda}\zeta|^N}$$

can be considered as a continuous function on $\bar{\mathbb{D}}$ when we deal with measures μ and σ_{B_c} (at least for small values of c). Hence,

$$\int_{\bar{\mathbb{D}}} \frac{1+|\zeta|}{|1-\bar{\lambda}\zeta|^N}\, d\sigma_{B_c}(\zeta) \longrightarrow \int_{\bar{\mathbb{D}}} \frac{1+|\zeta|}{|1-\bar{\lambda}\zeta|^N}\, d\mu(\zeta),$$

but

$$\int_{\bar{\mathbb{D}}} \frac{1+|\zeta|}{|1-\bar{\lambda}\zeta|^N}\, d\sigma_{B_c}(\zeta) = \sum_{m=1}^{\infty} \frac{1-|a_{cm}|^2}{|1-\bar{\lambda}a_{cm}|^N}$$

and

$$\int_{\bar{\mathbb{D}}} \frac{1+|\zeta|}{|1-\bar{\lambda}\zeta|^N}\, d\mu(\zeta) = \int_{\mathbb{T}} \frac{2}{|1-\bar{\lambda}\zeta|^N}\, d\mu(\zeta).$$

Therefore, the result follows.

If μ consists of a finite sum of Dirac masses, the result still holds by induction. Now, we turn to the general situation. Assume that μ is an arbitrary positive Borel measure on $\mathbb{T}$, fulfilling the above-mentioned growth restriction. Put

$$d\tau(z) = \frac{d\mu(z)}{|1-\bar{\lambda}z|^N}.$$

Again, note that μ cannot have a Dirac mass at $1/\bar{\lambda}$, and this property persists for all measures considered below. The family of discrete measures with finite number of Dirac masses is dense in $\mathcal{M}(\bar{\mathbb{D}})$. Hence, there is a sequence τ_n of such measures so that $\tau_n \longrightarrow \tau$ in the weak-star topology of $\mathcal{M}(\bar{\mathbb{D}})$. Therefore, for each $f \in \mathcal{C}(\bar{\mathbb{D}})$,

$$\int_{\bar{\mathbb{D}}} f(z)|1-\bar{\lambda}z|^N\, d\tau_n(z) \longrightarrow \int_{\bar{\mathbb{D}}} f(z)|1-\bar{\lambda}z|^N\, d\tau(z) = \int_{\bar{\mathbb{D}}} f(z)\, d\mu(z).$$

This means that $\sigma_n \longrightarrow \mu$ in the weak-star topology of $\mathcal{M}(\bar{\mathbb{D}})$, where σ_n is the discrete measure

$$d\sigma_n(z) = |1-\bar{\lambda}z|^N\, d\tau_n(z) \qquad (n \geq 1).$$

We appeal to the first part and find a Blaschke product B_n such that σ_{B_n} is close enough to σ_n in the weak-star topology and also

$$\left| \int_{\bar{\mathbb{D}}} \frac{1+|z|}{|1-\bar{\lambda}z|^N}\, d\sigma_{B_n}(z) - \int_{\mathbb{T}} \frac{2}{|1-\bar{\lambda}\zeta|^N}\, d\sigma_n(\zeta) \right| < \frac{1}{n}.$$

The result thus follows. Our choice of B_n also implies that $\sigma_{B_n} \longrightarrow \mu$ in the weak-star topology of $\mathcal{M}(\bar{\mathbb{D}})$. $\square$

21.6 Reproducing kernels for derivatives

Let $\mathcal{H}$ be a reproducing kernel of functions that are analytic on the domain Ω. The kernels of evaluation at point $z \in \Omega$ form a two-parameter family of functions $k_z^{\mathcal{H}}(w)$, where z and w run through Ω and $k_z^{\mathcal{H}}(w)$ is analytic with respect to w and conjugate analytic with respect to z. The essential property of $k_z^{\mathcal{H}}(z)$ is

$$f(z) = \langle f, k_z^{\mathcal{H}} \rangle_{\mathcal{H}} \qquad (f \in \mathcal{H},\ z \in \Omega). \tag{21.33}$$

For further information, see Chapter 9.

If we successively take the derivative of f with respect to z, we see that the evaluation functional $f \longmapsto f^{(n)}(z)$ is given by

$$f^{(n)}(z) = \langle f, \partial^n k_z^{\mathcal{H}} / \partial \bar{z}^n \rangle_{\mathcal{H}} \qquad (f \in \mathcal{H}). \tag{21.34}$$

But we need to show that $\partial^n k_z^{\mathcal{H}} / \partial \bar{z}^n \in \mathcal{H}$ and also that taking the derivative operator inside the inner product is legitimate. We verify this for $n = 1$. For higher derivatives, a similar argument works.

For simplicity, write k_z for $k_z^{\mathcal{H}}$. Put $\delta = (1 - |z|)/2$. Then, for each $f \in \mathcal{H}$ and each Δ with $0 < |\Delta| < \delta$, we have

$$\left| \left\langle f,\ \frac{k_{z+\Delta} - k_z}{\bar{\Delta}} \right\rangle_{\mathcal{H}} \right| = \left| \frac{f(z+\Delta) - f(z)}{\Delta} \right| = \left| \frac{1}{\Delta} \int_z^{z+\Delta} f'(\zeta)\, d\zeta \right| \le C_f,$$

where C_f is the maximum of f' on the disk with center z and radius δ. Therefore, by the uniform boundedness principle (Theorem 1.19), there is a constant C such that

$$\left\| \frac{k_{z+\Delta} - k_z}{\bar{\Delta}} \right\| \le C \qquad (0 < |\Delta| < \delta).$$

Let $g \in \mathcal{H}$ be a weak limit of this fraction as $\Delta \longrightarrow 0$. Then, on the one hand, for each $f \in \mathcal{H}$ we have

$$\langle f, g \rangle = \lim_{\Delta \to 0} \left\langle f,\ \frac{k_{z+\Delta} - k_z}{\bar{\Delta}} \right\rangle = \lim_{\Delta \to 0} \frac{f(z+\Delta) - f(z)}{\Delta} = f'(z).$$

On the other hand,

$$\begin{aligned} g(\zeta) = \langle g, k_\zeta \rangle &= \lim_{\Delta \to 0} \left\langle \frac{k_{z+\Delta} - k_z}{\bar{\Delta}},\ k_\zeta \right\rangle \\ &= \lim_{\Delta \to 0} \frac{k_{z+\Delta}(\zeta) - k_z(\zeta)}{\bar{\Delta}} = \frac{\partial k_z}{\partial \bar{z}}(\zeta). \end{aligned}$$

In short, $g = \partial k_z / \partial \bar{z}$.

In the light of relation (21.34), we define the notation

$$k_{z,n}^{\mathcal{H}} = \partial^n k_z^{\mathcal{H}} / \partial \bar{z}^n, \tag{21.35}$$

i.e. the kernel of the evaluation functional of the nth derivative at $z \in \Omega$. The relation (21.34) can be rewritten as

$$f^{(n)}(z) = \langle f, k_{z,n}^{\mathcal{H}} \rangle_{\mathcal{H}} \qquad (f \in \mathcal{H}). \tag{21.36}$$

In the above formula, if we replace f by $k_{z,n}^{\mathcal{H}}$, we obtain

$$(k_{z,n}^{\mathcal{H}})^{(n)}(z) = \|k_{z,n}^{\mathcal{H}}\|_{\mathcal{H}}^2. \tag{21.37}$$

There are some other formulas for $k_{z,n}^{\mathcal{H}}$ and each has its merits and uses in applications. We treat some of them below.

For the space $\mathcal{H}(b)$, instead of $k_{z,n}^{\mathcal{H}(b)}$ we will write $k_{z,n}^b$. Our first formula for $k_{z,n}^b$ is based on the operator X_b (see Section 18.7).

Lemma 21.16 *We have*

$$k_{z,n}^b = n!(I - \bar{z}X_b^*)^{-(n+1)} X_b^{*n} k_0^b.$$

Proof According to Theorem 18.21,

$$k_z^b = (I - \bar{z}X_b^*)^{-1} k_0^b.$$

Hence, using the definition (21.35), we get

$$\begin{aligned} k_{z,n}^b &= \frac{\partial^n k_z^b}{\partial \bar{z}^n} \\ &= \frac{\partial^n}{\partial \bar{z}^n}((I - \bar{z}X_b^*)^{-1} k_0^b) \\ &= n!(I - \bar{z}X_b^*)^{-(n+1)} X_b^{*n} k_0^b. \end{aligned}$$

This completes the proof. □

According to Theorem 18.11, the formula for $k_z^b = k_{z,0}^b$ is

$$k_z^b(w) = \frac{1 - \overline{b(z)}b(w)}{1 - \bar{z}w} \qquad (z, w \in \mathbb{D}).$$

Using Leibniz's rule, by straightforward computations, we obtain

$$k_{z,n}^b(w) = \frac{\partial^n k_z^b(w)}{\partial \bar{z}^n} = \frac{h_{z,n}^b(w)}{(1 - \bar{z}w)^{n+1}}, \tag{21.38}$$

where $h_{z,n}^b$ is the function

$$h_{z,n}^b(w) = n!\, w^n - b(w) \sum_{j=0}^{n} \binom{n}{j} \overline{b^{(j)}(z)} (n-j)!\, w^{n-j} (1 - \bar{z}w)^j. \tag{21.39}$$

Lemma 21.17 *Let $z_0 \in \mathbb{D}$ with $b(z_0) \neq 0$. Then*

$$h^b_{z_0,n}(1/\bar{z}_0) = (h^b_{z_0,n})'(1/\bar{z}_0) = \cdots = (h^b_{z_0,n})^{(n)}(1/\bar{z}_0) = 0.$$

Proof The functional 21.18 shows that b is analytic in a neighborhood of the point $1/\bar{z}_0$. (If $b(z_0) = 0$, then b has pole at $1/\bar{z}_0$ of the same order as the order of b at z_0.) Therefore, the formula for $k^b_{z_0}(w)$ shows that this kernel is a meromorphic function on $|w| > 1$ with poles as described above and a possible pole at $1/\bar{z}_0$. However, again by (21.18), we have $b(z_0)\overline{b(1/\bar{z}_0)} = 1$ and thus the pole is removable. In short, $w \longmapsto k^b_{z_0}(w)$ is analytic at $1/\bar{z}_0$. Therefore, the same is true for the application $w \longmapsto k^b_{z_0}(w)$. Respecting this property, the representation (21.38) implies that $h^b_{z_0,n}$ must have a zero of order $n+1$ at $1/\bar{z}_0$. □

We finish this section by studying $k^B_{z,n}$, where B is a Blaschke product formed with zeros $(a_n)_{n\geq 1}$. We recall that, by Theorem 14.7,

$$h_j(z) = \left(\prod_{k=1}^{j-1} \frac{a_k - z}{1 - \bar{a}_k z}\right) \frac{(1-|a_j|^2)^{1/2}}{1-\bar{a}_j z} \qquad (j \geq 1) \tag{21.40}$$

is an orthonormal basis for $K_B = \mathcal{H}(B)$. Sometimes, we will write

$$h_j(z) = (1-|a_j|^2)^{1/2} \frac{B_{j-1}(z)}{1-\bar{a}_j z} \qquad (j \geq 1), \tag{21.41}$$

where B_j is the finite product formed with the first j zeros.

Lemma 21.18 *Let B be a Blaschke product with zeros $(a_n)_{n\geq 1}$. Let $z \in \mathbb{D}$. Then*

$$k^B_{z,n} = \sum_{j=1}^{\infty} \overline{h_j^{(n)}(z)} h_j.$$

The series converges in H^2 norm.

Proof Since $(h_j)_{j\geq 1}$ forms an orthonormal basis for K_B, there are coefficients c_j, $j \geq 1$, such that

$$k^B_{z,n} = \sum_{j=1}^{\infty} c_j h_j,$$

where the series converges in H^2 norm. Moreover, thanks to orthonormality, c_j is given by

$$c_j = \langle k^B_{z,n}, h_j \rangle_2.$$

But the formula 21.34 immediately implies that $\bar{c}_j = h_j^{(n)}(z)$. □

For a Blaschke product, $X_B = S^*|K_B$ and $k_0^B = P_B 1$. Thus, $X_B^* = P_B S = M_B$ is the compressed shift on K_B. Therefore, by Lemma 21.16, we have

$$k_{z,n}^B = n!(I - \bar{z} M_B)^{-(n+1)} M_B^n P_B 1. \tag{21.42}$$

Lemma 21.19 *Let $z_0 \in \mathbb{D}$ and $N \geq 0$. Let B be a Blaschke product with zeros $(a_n)_{n\geq 1}$. Assume that there are functions $f, g \in H^2$ such that*

$$z^N = (1 - \bar{z}_0 z)^{N+1} f(z) + B(z) g(z) \qquad (z \in \mathbb{D}).$$

Then we have $P_B f = k_{z_0,N}^B / N!$.

Proof We write the above equation for f and g as

$$S^N 1 = (1 - \bar{z}_0 S)^{N+1} f + Bg.$$

Since M_B is the compression of S, if we apply P_B to both sides, we obtain

$$M_B^N P_B 1 = (1 - \bar{z}_0 M_B)^{N+1} P_B f,$$

and thus

$$P_B f = (1 - \bar{z}_0 M_B)^{-N-1} M_B^N P_B 1,$$

and the result follows from (21.42). □

21.7 An interpolation problem

There is a close relation between the existence of derivatives of elements of $\mathcal{H}(b)$ at the boundary and the containment of $X_b^{*N} k_0^b$ to the range of $(I - \overline{\zeta_0} X_b^*)^{N+1}$. This is fully explored in Theorem 21.26. But, to reach that general result, we need to pave the road by studying some special cases. We start doing this by considering Blaschke products. First, a technical lemma.

Lemma 21.20 *Let $S, (S_n)_{n\geq 1} \in \mathcal{L}(\mathcal{H})$ with the following properties.*

(i) *Each S_n is invertible.*
(ii) *S is injective.*
(iii) *$S_n \longrightarrow S$ in the norm topology.*
(iv) *There is a constant M such that*

$$\|S_n^{-1} S\| \leq M \qquad (n \geq 1).$$

Let $y \in \mathcal{H}$. Then $(S_n^{-1} y)_{n\geq 1}$ is a bounded sequence in $\mathcal{H}$ if and only if $y \in \mathcal{R}(S)$. Moreover, if this holds, we actually have $S_n^{-1} y \longrightarrow S^{-1} y$ in the weak topology.

Proof Assume that $(S_n^{-1}y)_{n\geq 1}$ is a bounded sequence in $\mathcal{H}$. Hence, it has at least one weak limit point in $\mathcal{H}$. Let $x \in \mathcal{H}$ be a weak limit point of the sequence. Since $S_n \longrightarrow S$ in the norm topology, we surely have (at least for a subsequence) $S_n S_n^{-1} y \longrightarrow Sx$. Therefore, $y = Sx$, i.e. $y \in \mathcal{R}(S)$. But, since S is injective, the above argument shows that the sequence has precisely one weak point (if x' is another weak limit point, we would have $y = Sx = Sx'$). In other words, the whole sequence tends weakly to x.

To prove the other (easy) direction, assume that $y \in \mathcal{R}(S)$, i.e. $y = Sx$ for some $x \in \mathcal{H}$. Then

$$\|S_n^{-1}y\| \leq \|S_n^{-1}Sx\| \leq \|S_n^{-1}S\|\,\|x\| \leq M\|x\| \qquad (n \geq 1). \qquad \square$$

The following corollary is a realization of the preceding lemma. The assumptions are adjusted to fit our application in the study of derivatives of $\mathcal{H}(b)$ functions.

Corollary 21.21 *Let $T_k \in \mathcal{L}(\mathcal{H})$, $\zeta_k \in \mathbb{T}$ and $\lambda_{k,n} \in \mathbb{D}$, for $n \geq 1$ and $1 \leq k \leq p$, with the following properties.*

(i) *Each T_k is a contraction.*
(ii) *Each $I - \zeta_k T_k$ is one-to-one.*
(iii) *$T_k T_{k'} = T_{k'} T_k$ for $k, k' \in \{1, \dots, p\}$.*
(iv) *For each k, $\lambda_{k,n}$ tends nontangentially to ζ_k as $n \longrightarrow \infty$.*

Let $y \in \mathcal{H}$. Then the sequence

$$((I - \lambda_{1,n}T_1)^{-1} \cdots (I - \lambda_{p,n}T_p)^{-1} y)_{n\geq 1}$$

is uniformly bounded if and only if y belongs to the range of the operator $(I - \zeta_1 T_1) \cdots (I - \zeta_p T_p)$, in which case,

$$(I - \lambda_{1,n}T_1)^{-1} \cdots (I - \lambda_{p,n}T_p)^{-1} y \longrightarrow (I - \zeta_1 T_1)^{-1} \cdots (I - \zeta_p T_p)^{-1} y$$

in the weak topology.

Proof We apply Lemma 21.20 with

$$S_n = (I - \lambda_{1,n}T_1) \cdots (I - \lambda_{p,n}T_p)$$

and

$$S = (I - \zeta_1 T_1) \cdots (I - \zeta_p T_p).$$

The only nontrivial property is the boundedness of $S_n^{-1}S$. Since T_k are commuting, it is enough to verify that the sequence

$$((I - \lambda_{k,n}T_k)^{-1}(I - \zeta_k T_k))_{n\geq 1}$$

is bounded. But we have

$$\begin{aligned}\|(I-\lambda_{k,n}T_k)^{-1}(I-\zeta_k T_k)\| &= \|I+(\lambda_{k,n}-\zeta_k)(I-\lambda_{k,n}T_k)^{-1}T_k\| \\ &\le 1+|\lambda_{k,n}-\zeta_k|\,\|(I-\lambda_{k,n}T_k)^{-1}\| \\ &\le 1+|\lambda_{k,n}-\zeta_k|\,(1-|\lambda_{k,n}|)^{-1} \\ &\le 1+M_k.\end{aligned}$$

The last estimation holds since $\lambda_{k,n}$ tends nontangentially to ζ_k. The result thus follows. □

In fact, we even need a special case of Corollary 21.21 in which $T_1 = \cdots = T_p$.

Corollary 21.22 *Let $T \in \mathcal{L}(\mathcal{H})$ be a contraction, $\zeta \in \mathbb{T}$ and $(\lambda_n)_{n\ge 1} \subset \mathbb{D}$, with the following properties.*

(i) *$I - \zeta T$ is one-to-one.*
(ii) *λ_n tends nontangentially to ζ as $n \longrightarrow \infty$.*

Let $y \in \mathcal{H}$. Then the sequence

$$((I-\lambda_n T)^{-p} y)_{n\ge 1}$$

is uniformly bounded if and only if y belongs to the range of the operator $(I-\zeta T)^p$, in which case

$$(I-\lambda_n T)^{-p} y \longrightarrow (I-\zeta T)^{-p} y$$

in the weak topology.

Now we are ready to establish the connection between the existence of boundary derivatives in K_B and an interpolation problem.

Theorem 21.23 *Let $\zeta \in \mathbb{T}$, and let $N \ge 0$. Let B be a Blaschke product with zeros $(a_n)_{n\ge 1}$ such that*

$$\sum_{n=1}^{\infty} \frac{1-|a_n|}{|\zeta - a_n|^{2N+2}} \le A.$$

Then there are functions $f, g \in H^2$ such that

$$z^N = (1-\bar{\zeta} z)^{N+1} f(z) + B(z) g(z) \qquad (z \in \mathbb{D})$$

with

$$\|f\|_2 \le C,$$

where $C = C(N, A)$ is a constant.

Proof According to Lemma 21.18,

$$k_{z,N}^{B} = \sum_{j=1}^{\infty} \overline{h_j^{(N)}(z)} h_j,$$

and hence

$$\|k_{z,n}^{B}\|^2 = \sum_{j=1}^{\infty} |h_j^{(n)}(z)|^2. \tag{21.43}$$

We rewrite the formula in (21.40) for h_j as

$$h_j(z) = (1 - |a_j|^2)^{1/2} \frac{B_{j-1}(z)}{1 - \bar{a}_j\, z}.$$

Hence, by Leibniz's formula,

$$h_j^{(N)}(z) = (1 - |a_j|^2)^{1/2} \sum_{k=1}^{N} \binom{N}{k} B_{j-1}^{(N-k)}(z) \frac{k!(-\bar{a}_j)^k}{(1 - \bar{a}_j z)^{k+1}}.$$

Therefore, by Theorem 21.8 and denoting the constant $C(N, A)$ of this theorem by C,

$$\begin{aligned}
|h_j^{(N)}(r\zeta)| &\le (1 - |a_j|^2)^{1/2} \sum_{k=1}^{N} \binom{N}{k} C \frac{k!}{|1 - \bar{a}_j r\zeta|^{k+1}} \\
&\le C(1 - |a_j|^2)^{1/2} \sum_{k=1}^{N} \binom{N}{k} \frac{2^{k+1} k!}{|1 - \bar{a}_j \zeta|^{k+1}} \\
&\le C(1 - |a_j|^2)^{1/2} \sum_{k=1}^{N} \binom{N}{k} \frac{2^{N+1} k!}{|1 - \bar{a}_j \zeta|^{N+1}} \\
&= \left(2^{N+1} C \sum_{k=1}^{N} k! \binom{N}{k} \right) \frac{(1 - |a_j|^2)^{1/2}}{|\zeta - a_j|^{N+1}} \\
&= C' \frac{(1 - |a_j|^2)^{1/2}}{|\zeta - a_j|^{N+1}}.
\end{aligned}$$

Considering (21.43), we conclude that

$$\|k_{r\zeta,N}^{B}\|^2 \le 2AC'^2 \qquad (0 < r < 1). \tag{21.44}$$

The next step is to appeal to the formula (21.42) and Corollary 21.22. Theorem 14.28 ensures that $\sigma_p(M_B) \subset \mathbb{D}$, and thus the operator $I - \bar{\zeta} M_B$ is injective. Hence, with $T = M_B$, $p = N + 1$ and $y = M_B^N P_B 1$, we see that

$M_B^N P_B 1$ belongs to the range of $(I - \bar{\zeta} M_B)^{N+1}$. This means that there is a function $f \in H^2$ such that

$$M_B^N P_B 1 = (I - \bar{\zeta} M_B)^{N+1} f.$$

Since M_B is the compressed shift, we can rewrite the preceding identity as

$$P_B(z^N) = P_B((1 - \bar{\zeta} z)^{N+1} f).$$

Hence, $z^N - (1 - \bar{\zeta} z)^{N+1} f \perp K_B$, or equivalently $z^N - (1 - \bar{\zeta} z)^{N+1} f \in BH^2$. Therefore, there is $g \in H^2$ such that

$$z^N - (1 - \bar{\zeta} z)^{N+1} f = Bg.$$

Finally, Corollary 21.22 also says that

$$f = (I - \bar{\zeta} M_B)^{-N-1} M_B^N P_B 1 = \lim_{r \to 1} (I - \overline{r\zeta} M_B)^{-N-1} M_B^N P_B 1.$$

Hence, by (21.42)

$$f = (I - \bar{\zeta} M_B)^{-N-1} M_B^N P_B 1 = \frac{1}{N!} \lim_{r \to 1} k_{r\zeta, N}^B,$$

and, by (21.44), the latter is uniformly bounded by a constant. □

The above result can be referred to as an interpolation problem since the equation

$$z^n = (1 - \bar{\zeta} z) f(z) + B(z) g(z)$$

has a solution if and only if there is a function $f \in H^2$ such that

$$f(a_n) = \frac{a_n}{(1 - \bar{\zeta} a_n)^{N+1}} \qquad (n \geq 1).$$

Since

$$\sum_{n=1}^{\infty} \left| \frac{a_n}{(1 - \bar{\zeta} a_n)^{N+1}} \right|^2 (1 - |a_n|^2) < \infty,$$

if $(a_n)_{n \geq 1}$ was an interpolation sequence, then the function f trivially exists (see Section 15.6). The surprising feature of Theorem 21.23 is that it ensures that a solution, even with an additional growth restriction, always exists.

Theorem 21.23, in a sense, is reversible. Indeed, this is the version that we need in the proof of Theorem 21.26.

Theorem 21.24 *Let $N \geq 0$. Let B be a Blaschke product with zeros $(a_n)_{n \geq 1}$. Assume that there are functions $f, g \in H^2$ such that*

$$z^N = (1 - \bar{z}_0 z)^{N+1} f(z) + B(z) g(z) \qquad (z \in \mathbb{D}),$$

with

$$\|f\|_2 \leq C \quad \text{and} \quad \left(1 - \frac{1}{2C^2}\right)^{1/2} \leq |z_0| < 1,$$

where $C > 1$ is a constant. Then there is a constant $A = A(N, C)$ such that

$$\sum_{n=1}^{\infty} \frac{1 - |a_n|}{|1 - \bar{a}_n z_0|^{2N+2}} \leq A.$$

Proof Since we appeal to induction, the functions f and g that appear in the Nth step will be denoted by f_N and g_N. Note that, by Lemma 21.19,

$$P_B f_N = \frac{k_{z_0,N}^B}{N!}. \tag{21.45}$$

Case $N = 0$. By Lemma 21.18,

$$k_{z_0}^B = \sum_{j=1}^{\infty} \overline{h_j(z_0)} h_j.$$

Hence, by (21.45), our condition $\|f_0\|_2 \leq C$ translates as

$$\sum_{j=1}^{\infty} |h_j(z_0)|^2 \leq C^2.$$

We use (21.41) to rewrite this estimation as

$$\sum_{j=1}^{\infty} |B_{j-1}(z_0)|^2 \frac{1 - |a_j|^2}{|1 - \bar{a}_j z_0|^2} \leq C^2. \tag{21.46}$$

We just need to get rid of $|B_{j-1}(z_0)|^2$ to establish the result. To do so, just note that, since B_j is a subproduct of B, we have

$$P_{B_j} k_{z_0}^B(z) = k_{z_0}^{B_j}(z) = \frac{1 - \overline{B_j(z_0)} B_j(z)}{1 - \bar{z}_0 z}.$$

Hence,

$$\frac{1 - |B_j(z_0)|^2}{1 - |z_0|^2} = k_{z_0}^{B_j}(z_0) = \|k_{z_0}^{B_j}\|^2 \leq \|k_z^{B_j}\|^2 \leq C^2.$$

The restriction $1 - 1/(2C^2) \leq |z_0|^2 < 1$ now implies that $|B_j(z_0)|^2 \geq 1/2$. Therefore, from (21.46), we conclude that

$$\sum_{j=1}^{\infty} \frac{1 - |a_j|^2}{|1 - \bar{a}_j z_0|^2} \leq 2C^2.$$

This settles the case $N = 0$.

Case $N \geq 1$. Assume that the result holds for $N - 1$. Our assumption is that there are functions $f_N, g_N \in H^2$ such that

$$z^N = (1 - \bar{z}_0 z)^{N+1} f_N(z) + B(z) g_N(z) \qquad (z \in \mathbb{D}) \tag{21.47}$$

with $\|f_N\|_2 \le C$. Write

$$1-(1-\bar{z}_0 z)^N = -\sum_{k=1}^{N} \binom{N}{k} (-\bar{z}_0)^k z^k.$$

Multiply by z^{N-1} to get

$$z^{N-1} = (1-\bar{z}_0 z)^N z^{N-1} - \left(\sum_{k=1}^{N} \binom{N}{k} (-\bar{z}_0)^k z^{k-1} \right) z^N.$$

Comparing this to (21.47) written for $N-1$ rather than N gives

$$z^{N-1} = (1-\bar{z}_0 z)^N f_{N-1}(z) + B(z) g_{N-1}(z) \qquad (z \in \mathbb{D}),$$

where

$$f_{N-1}(z) = z^{N-1} - \left(\sum_{k=1}^{N} \binom{N}{k} (-\bar{z}_0)^k z^{k-1} \right) (1-\bar{z}_0 z) f_N(z).$$

Hence,

$$\|f_{N-1}\|_2 \le 1 + 2^{N+1} C.$$

This means that all the required conditions are fulfilled and we can apply the induction for $N-1$. Thus, there is a constant A such that

$$\sum_{n=1}^{\infty} \frac{1-|a_n|}{|1-\bar{a}_n z_0|^{2N}} \le A. \tag{21.48}$$

Theorem 21.8 now ensures that $B_j^{(k)}(z_0)$, $0 \le k \le 2N-1$, exist and are uniformly bounded by a constant A', where B_j is any subproduct of B.

If we take N times the derivative of both sides in (21.41), we obtain

$$h_j^{(N)}(z) = (1-|a_j|^2)^{1/2} \sum_{k=0}^{N} \binom{N}{k} B_{j-1}^{(k)}(z) \frac{(N-k)!(-\bar{a}_j)^{N-k}}{(1-\bar{a}_j z)^{N-k+1}} \qquad (j \ge 1).$$

We rewrite this as

$$\begin{aligned} &(1-|a_j|^2)^{1/2} B_{j-1}(z) \frac{N!(-\bar{a}_j)^N}{(1-\bar{a}_j z)^{N+1}} \\ &\quad = h_j^{(N)}(z) - (1-|a_j|^2)^{1/2} \sum_{k=1}^{N} \binom{N}{k} B_{j-1}^{(k)}(z) \frac{(N-k)!(-\bar{a}_j)^{N-k}}{(1-\bar{a}_j z)^{N-k+1}}. \end{aligned} \tag{21.49}$$

As we saw above, for $1 \le k \le N$,

$$\left| \binom{N}{k} B_{j-1}^{(k)}(z_0) \frac{(N-k)!(-\bar{a}_j)^{N-k}}{(1-\bar{a}_j z)^{N-k+1}} \right| \le \frac{A' N!}{|1-\bar{a}_j z_0|^{N-k+1}}$$
$$\le \frac{A' N!\, 2^N}{|1-\bar{a}_j z_0|^N}.$$

Thus, the right-hand side of (21.49) is bounded above by

$$|h_j^{(N)}(z)| + A' N N!\, 2^N \frac{(1-|a_j|^2)^{1/2}}{|1-\bar{a}_j z_0|^N}.$$

The left-hand side of (21.49) is bounded below by

$$\frac{N!}{2^{N+1}} \frac{(1-|a_j|^2)^{1/2}}{|1-\bar{a}_j z_0|^{N+1}}$$

for zeros $|a_j| \ge 1/2$. Hence, for such j, we have

$$\frac{(1-|a_j|^2)^{1/2}}{|1-\bar{a}_j z_0|^{N+1}} \le 2^{N+1} |h_j^{(N)}(z_0)| + A' N 2^{2N+1} \frac{(1-|a_j|^2)^{1/2}}{|1-\bar{a}_j z_0|^N}.$$

Hence, by Minkowski's inequality, Lemmas 21.18 and 21.19, and (21.48), we find

$$\left(\sum_{|a_j| \ge 1/2} \frac{1-|a_j|^2}{|1-\bar{a}_j z_0|^{2N+2}} \right)^{1/2}$$
$$\le 2^{N+1} \left(\sum_{j=1}^{\infty} |h_j^{(N)}(z_0)|^2 \right)^{1/2} + A' N 2^{2N+1} \left(\sum_{j=1}^{\infty} \frac{1-|a_j|^2}{|1-\bar{a}_j z_0|^{2N}} \right)^{1/2}$$
$$\le 2^{N+1} \|k_{z_0,N}^B\| + A' A N 2^{2N+1}$$
$$\le 2^{N+1} N!\, \|P_B f_N\| + A' A N 2^{2N+1}$$
$$\le 2^{N+1} N!\, C + A' A N 2^{2N+1}.$$

For zeros with $|a_j| < 1/2$, we have

$$\sum_{|a_j|<1/2} \frac{1-|a_j|^2}{|1-\bar{a}_j z_0|^{2N+2}} \le 4 \sum_{|a_j|<1/2} \frac{1-|a_j|^2}{|1-\bar{a}_j z_0|^{2N}} \le 4A.$$

Hence, the result follows. □

21.8 Derivatives of $\mathcal{H}(b)$ functions

In Theorem 20.13, we saw the connection between the analytic continuation of b across a subarc of $\mathbb{T}$, on the one hand, and the analytic continuation of all functions of $\mathcal{H}(b)$ across the same subarc, on the other hand. In this section, we treat a similar result. While we are studying the derivative of elements in $\mathcal{H}(b)$, we are content with the existence of nontangential boundary values.

We begin with a simple lemma, which is a simple exercise from calculus and is interesting in its own right. We do not prove it in such a generality since in our application even the derivative of order $n+m+1$ exists at all points. However, the proof for the general case is essentially the same.

Lemma 21.25 *Let I be an open interval, and let $a, b \in I$. Suppose that the function $h : I \longrightarrow \mathbb{C}$ satisfies the following properties:*

(i) *h has $n+m$ continuous derivatives on I;*
(ii) *$h^{(n+m+1)}$ is continuous and bounded on $I \setminus \{a\}$;*
(iii) *$h(b) = h'(b) = \cdots = h^{(n-1)}(b) = 0$.*

Put

$$k(x) = \frac{h(x)}{(x-b)^n} \qquad (x \in I).$$

Then k is $m+1$ times differentiable on I and, moreover,

$$k^{(m+1)}(x) = \int_0^1 \cdots \int_0^1 h^{(m+n+1)}(b + t_1 \cdots t_n(x-b)) v(t)\, dt_1 \cdots dt_n,$$

where $v(t) = t_1^{p_1} \cdots t_n^{p_n}$ is some monomial.

Proof Since $h(b) = 0$, the fundamental theorem of calculus says

$$\begin{aligned} h(x) &= \int_0^1 \frac{d}{dt_1}[h(b + t_1(x-b))]\, dt_1 \\ &= (x-b) \int_0^1 h'(b + t_1(x-b))\, dt_1. \end{aligned}$$

Applying the same result to the function $x \longmapsto h'(b + t_1(x-b))$ gives

$$h'(b + t_1(x-b)) = t_1(x-b) \int_0^1 h''(b + t_1 t_2(x-b))\, dt_2.$$

Therefore,

$$h(x) = (x-b)^2 \int_0^1 \int_0^1 t_1 h''(b + t_1 t_2(x-b))\, dt_1\, dt_2.$$

Continuing this process n times gives

$$k(x) = \int_0^1 \cdots \int_0^1 t_1^{n-1} t_2^{n-2} \cdots t_{n-1} h^{(n)}(b + t_1 \cdots t_n(x-b))\, dt_1 \cdots dt_n.$$

Write $m(t) = t_1^{n-1} t_2^{n-2} \cdots t_{n-1}$.

Since h has $n+m$ continuous derivatives on I, and $h^{(n+m+1)}$ is continuous and bounded on $I \setminus \{a\}$, the function k has $m+1$ continuous derivatives on I and

$$\begin{aligned} k^{(m+1)}(x) &= \int_0^1 \cdots \int_0^1 m(t) \frac{\partial^{m+1}}{\partial x^{m+1}} h^{(n)}(b + t_1 \cdots t_n(x-b))\, dt_1 \cdots dt_n \\ &= \int_0^1 \cdots \int_0^1 v(t) h^{(m+n+1)}(b + t_1 \cdots t_n(x-b))\, dt_1 \cdots dt_n, \end{aligned}$$

where $v(t) = t_1^{m+n} t_2^{m+n-1} \cdots t_{n-1}^{m+2} t_n^{m+1}$. □

The following result gives a criterion for the existence of the derivatives for functions of $\mathcal{H}(b)$.

Theorem 21.26 *Let b be a point in the closed unit ball of $H^\infty(\mathbb{D})$ with the canonical factorization (21.17), let $\zeta_0 \in \mathbb{T}$ and let N be a nonnegative integer. Then the following are equivalent.*

- (i) *For every $f \in \mathcal{H}(b)$, the functions $f(z), f'(z), \ldots, f^{(N)}(z)$ have finite limits as z tends radially to ζ_0.*
- (ii) *For every $f \in \mathcal{H}(b)$, the function $|f^{(N)}(z)|$ remains bounded as z tends radially to ζ_0.*
- (iii) *$\|k^b_{z,N}\|_b$ is bounded on the ray $z \in [0, \zeta_0]$.*
- (iv) *$X_b^{*N} k_0^b$ belongs to the range of $(I - \overline{\zeta_0} X_b^*)^{N+1}$.*
- (v) *We have*

$$\sum_n \frac{1-|a_n|^2}{|\zeta_0 - a_n|^{2N+2}} + \int_{\mathbb{T}} \frac{d\mu(\zeta)}{|\zeta_0 - \zeta|^{2N+2}} + \int_{\mathbb{T}} \frac{\big|\log|b(\zeta)|\big|}{|\zeta_0 - \zeta|^{2N+2}}\, dm(\zeta) < \infty.$$

Moreover, we have

$$(I - \overline{\zeta_0} X_b^*)^{N+1} k^b_{\zeta_0,N} = N! X_b^{*N} k_0^b, \tag{21.50}$$

where $k^b_{\zeta_0,N} \in \mathcal{H}(b)$ and satisfies

$$f^{(N)}(\zeta_0) = \langle f, k^b_{\zeta_0,N} \rangle_b \qquad (f \in \mathcal{H}(b)).$$

Proof (i) $\Longrightarrow$ (ii) This is trivial.

(ii) $\Longrightarrow$ (iii) In the light of representation (21.36), this implication follows from the principle of uniform boundedness (Theorem 1.19).

(iii) $\Longrightarrow$ (iv) By Lemma 21.16,

$$k^b_{z,N} = N!(I - \bar{z} X_b^*)^{-(N+1)} X_b^{*N} k_0^b. \tag{21.51}$$

Theorem 18.26 ensures that $\sigma_p(X_b^*) \subset \mathbb{D}$, and thus the operator $I - \overline{\zeta_0} X_b^*$ is injective. By assumption, $(I - \overline{z_n} X_b^*)^{-(N+1)} X_b^{*N} k_0^b$ is uniformly bounded for any sequence $z_n \in \mathbb{D}$ tending radially to ζ_0. Now, we apply Corollary 21.22

with $T = X_b^*$, $p = N+1$ and $y = X_b^{*N} k_0^b$ to conclude that $X_b^{*N} k_0^b$ belongs to the range of $(I - \overline{\zeta}_0 X_b^*)^{N+1}$.

(iv) $\Longrightarrow$ (i) Using once more Corollary 21.22, we see that

$$(I - \overline{z_n} X_b^*)^{-(N+1)} X_b^{*N} k_0^b \longrightarrow (I - \overline{\zeta}_0 X_b^*)^{-(N+1)} X_b^{*N} k_0^b$$

in the weak topology, for any sequence $z_n \in \mathbb{D}$ tending radially to ζ. But (21.51) says that the left-hand side is precisely $(1/N!) k_{z_n,N}^b$. Hence, in the light of (21.36), for every function f in $\mathcal{H}(b)$, the Nth derivative $f^{(N)}(z)$ has a finite limit as z tends radially to ζ_0. Moreover, the linear functional $f \longmapsto f^{(N)}(\zeta_0)$ is continuous on $\mathcal{H}(b)$ and thus it is induced by a kernel function $k_{\zeta_0,N}^b$, which should satisfy

$$(I - \overline{\zeta}_0 X_b^*)^{-(N+1)} X_b^{*N} k_0^b = \frac{1}{N!}\, k_{\zeta_0,N}^b.$$

That proves (21.50).

The rest is by induction. We have

$$I - (I - \overline{\zeta}_0 X_b^*)^N = -\sum_{\ell=1}^{N} \binom{N}{\ell} (-\zeta_0)^\ell X_b^{*\ell}.$$

Applying to both sides the function $X_b^{*(N-1)} k_0^b$ we get

$$\begin{aligned} X_b^{*(N-1)} k_0^b &= (I - \overline{\zeta}_0 X_b^*)^N X_b^{*(N-1)} k_0^b \\ &\quad - \sum_{\ell=1}^{N} \binom{N}{\ell} (-\zeta_0)^\ell X_b^{*(\ell-1)} X_b^{*N} k_0^b. \end{aligned}$$

Hence, $X_b^{*(N-1)} k_0^b$ belongs to the range of $(I - \overline{\zeta}_0 X_b^*)^N$. The above argument applies with N replaced by $N-1$. We continue this process N times. Therefore, for every function f in $\mathcal{H}(b)$, $f^{(j)}(z)$, $0 \le j \le N$, has a finite limit as z tends radially to ζ_0.

(v) $\Longrightarrow$ (iii) Without loss of generality, we assume that $\zeta_0 = 1$. By Theorem 21.10, the condition (v) implies that

$$\lim_{r \to 1^-} b^{(j)}(r) \quad \text{and} \quad \lim_{R \to 1^+} b^{(j)}(R)$$

exist and are equal for $0 \le j \le 2N+1$. Moreover, since b can have only a finite number of real zeros, we can take $\delta > 0$ such that the interval $[1-\delta, 1)$ is free of zeros of b. Therefore, b has $2N+1$ continuous bounded derivatives on $[1-\delta, 1+\delta]$. Now, fix r in the interval $(1-\delta, 1)$.

We recall that, by (21.38) and (21.39), $k_{r,N}^b(x) = h_{r,N}^b(x)/(1-rx)^{N+1}$, where

$$h_{r,N}^b(x) = N!\, x^N - b(x) \sum_{j=0}^{N} \binom{N}{j} \overline{b^{(j)}(r)} (N-j)!\, x^{N-j} (1-rx)^j.$$

Hence, $h^b_{r,N}$ has $2N+1$ continuous bounded derivatives on $(1-\delta, 1+\delta)$. Moreover, by Lemma 21.17, we have

$$h^b_{r,N}(1/r) = (h^b_{r,N})'(1/r) = \cdots = (h^b_{r,n})^{(N)}(1/r) = 0.$$

We now apply Lemma 21.25 with $I = (1-\delta, 1+\delta)$, $a = 1$, $b = 1/r$, $n = N+1$, $m = N$ and $h = h^b_{r,N}$. Note that

$$\frac{h(x)}{(x-b)^n} = \frac{h^b_{r,N}(x)}{(x-1/r)^{N+1}} = (-r)^{N+1} k^b_{N,r}(x).$$

Thus, the lemma says that $(-r)^{N+1}(k^b_{N,r})^{(N)}(x)$ is equal to

$$\int_0^1 \cdots \int_0^1 (h^b_{r,N})^{(2N+1)} \left(\frac{1}{r} + t_1 \cdots t_{N+1} \left(x - \frac{1}{r} \right) \right) v(t)\, dt_1 \cdots dt_{N+1}.$$

Since there is an M such that

$$|(h^b_{r,N})^{(2N+1)}(s)| \le M \qquad (1-\delta < s < 1+\delta),$$

we deduce that

$$|(k^b_{r,N})^{(N)}(x)| \le M\delta^{-N-1} \qquad (1-\delta < x < 1).$$

In particular, $(k^b_{r,N})^{(N)}(r)$ is bounded as $r \longrightarrow 1^-$. But, according to (21.37),

$$\|k^b_{z,N}\|_b^2 = (k^b_{z,N})^{(N)}(z).$$

Thus, $\|k^b_{r,N}\|_b$ remains bounded as $r \longrightarrow 1^-$.

(iii) $\Longrightarrow$ (v) Again, without loss of generality, assume that $\zeta_0 = 1$. Fix $r \in (0,1)$. Considering the canonical factorization of b, since b is in the closed unit ball of H^∞, we have

$$\sum_n \frac{1-|a_n|^2}{|1-a_n r|^{2N+2}} + \int_{\mathbb{T}} \frac{d\mu(\zeta)}{|r-\zeta|^{2N+2}} + \int_{\mathbb{T}} \frac{\big|\log|b(\zeta)|\big|}{|r-\zeta|^{2N+2}}\, dm(\zeta) < \infty.$$

For simplicity of formulas, denote the left-hand side by Δ_r. According to Theorem 21.15, there is a sequence $(B_j)_{j\ge1}$ of Blaschke products, with zeros $(a_{jk})_{k\ge1}$, converging uniformly to b on compact subsets of $\mathbb{D}$ and such that

$$\sum_{k=1}^{\infty} \frac{1-|a_{jk}|^2}{|1-ra_{jk}|^{2N+2}} \longrightarrow \Delta_r$$

as $j \longrightarrow \infty$. Hence, the formulas (21.38) and (21.39) show that $k^{B_j}_{w,N}$ tends to $k^b_{w,N}$ uniformly on compact subsets of $\mathbb{D}$. In particular, we must have

$$\lim_{j\to\infty} (k^{B_j}_{w,N})^{(N)}(w) = (k^b_{w,N})^{(N)}(w).$$

In the light of (21.37), we can rewrite this identity as

$$\lim_{j\to\infty} \|k_{w,N}^{B_j}\|_2 = \|k_{w,N}^{b}\|_b.$$

The assumption (iii) implies that there is a $C > 0$ such that

$$\|k_{r,N}^{b}\|_b \le C \qquad (0 < r < 1).$$

Therefore, there is an index j_r such that

$$\|k_{r,N}^{B_j}\|_2 \le C + 1 \qquad (j \ge j_r).$$

The formulas (21.38) and (21.39) also show that

$$(1 - rz)^{N+1} k_{r,N}^{B_j}(z) = N!\, z^N - B_j(z) g_j(z),$$

where $g_j \in H^2$. Hence, it follows from Theorem 21.24 that there is a constant $A = A(C, N)$ (independent of r) such that

$$\sum_k \frac{1 - |a_{jk}|^2}{|1 - r a_{jk}|^{2N+2}} \le A \qquad (j \ge j_r).$$

Letting $j \longrightarrow \infty$, we obtain $\Delta_r \le A$ for all $r \in (0,1)$. Finally, we let $r \longrightarrow 1^-$ to get the desired condition (v). This completes the proof of Theorem 21.26. □

The identity (21.38) provides an explicit formula for the kernel of the functional for a derivative at the point $z \in \mathbb{D}$. Using Theorem 21.26, it is easy to see that this formula can be extended for the kernel of the functional for a derivative at the point $\zeta_0 \in \mathbb{T}$ that satisfies one of the equivalent conditions (i)–(v); see Lemma 22.4.

Theorem 21.26 implies also a sufficient condition for the existence of the derivatives for functions in the range of a Toeplitz operator with co-analytic symbol.

Corollary 21.27 *Let a be a nonextreme point of the closed unit ball of H^∞, let $\zeta_0 \in \mathbb{T}$, and assume that there is a neighborhood I_{ζ_0} of ζ_0 on $\mathbb{T}$, a constant $c > 0$ and an integer $N \ge 0$ such that*

$$|a(\zeta)| \le c\,|\zeta - \zeta_0|^N \qquad (\zeta \in I_{\zeta_0}).$$

Then every function $f \in \mathcal{M}(\bar{a})$, as well as its derivatives up to order $N-1$, have finite radial limits at ζ_0.

Proof Thanks to Lemma 17.3, we can assume that a is an outer function with $a(0) > 0$. Consider the outer function b such that $|a|^2 + |b|^2 = 1$ a.e. on $\mathbb{T}$. Then (a, b) is a pair and we have $\mathcal{M}(\bar{a}) \subset \mathcal{H}(b)$. Thus it is sufficient to prove that every function $f \in \mathcal{H}(b)$, as well as its derivatives up to order $N-1$, has a finite radial limit at ζ_0. For this purpose, we may check the sufficient condition (v) of Theorem 21.26. Since b is outer, this condition is simply

$$\int_{\mathbb{T}} \frac{\big|\log|b(\zeta)|\big|}{|\zeta-\zeta_0|^{2N}}\,dm(\zeta) < \infty. \tag{21.52}$$

Pick $\zeta \in I_{\zeta_0}$. Then,

$$\big|\log|b(\zeta)|\big| \asymp \big|\log|b(\zeta)|^2\big| = \big|\log(1-|a(\zeta)|^2)\big| \lesssim |a(\zeta)|^2 \le c^2|\zeta-\zeta_0|^{2N}.$$

Hence,

$$\int_{I_{\zeta_0}} \frac{\big|\log|b(\zeta)|\big|}{|\zeta-\zeta_0|^{2N}}\,dm(\zeta) \le c^2 m(I_{\zeta_0}) < \infty.$$

On the other hand, since b is nonextreme, we have

$$\int_{\mathbb{T}\setminus I_{\zeta_0}} \frac{\big|\log|b(\zeta)|\big|}{|\zeta-\zeta_0|^{2N}}\,dm(\zeta) \lesssim \int_{\mathbb{T}} \big|\log|b(\zeta)|\big|\,dm(\zeta) < \infty,$$

which proves (21.52). □

If we combine Theorems 21.1 and 21.26 and Remark 21.2, we get immediately the following result, which will be useful in Chapter 31.

Corollary 21.28 *Let $\zeta_0 \in \mathbb{T}$. Then the following assertions are equivalent:*

(i) *b has an angular derivative in the sense of Carathéodory at ζ_0;*
(ii) *k_0^b belongs to the range of $I - \overline{\zeta_0}X_b^*$.*

Moreover, in this case, we have $(I-\overline{\zeta_0}X_b^*)k_{\zeta_0}^b = k_0^b$.

Notes on Chapter 21

Section 21.1

Theorems 21.1 and 21.4 can be found under different names in the literature, e.g. the Julia–Carathéodory theorem, the Julia–Wolff–Carathéodory theorem and even the Julia–Wolff theorem. These results combine some celebrated results of Julia [111], Carathéodory [40–42] and also Wolff's boundary version of the Schwarz lemma [191]. The proof here is due to Sarason [161], who applied Hilbert space techniques to prove the existence of angular derivatives. Using the hyperbolic Poincaré metric, P. R. Mercer [132] gave a strengthened version of Julia's result. Potapov [145] extended Julia's result to matrix-valued holomorphic mappings of a complex variable. His results were generalized by Fan and Ando [20, 71–73] to operator-valued holomorphic mappings, and to holomorphic mappings of proper contractions on the unit Hilbert ball acting in the sense of functional calculus. Different generalizations of Theorem 21.1 for bounded domains in $\mathbb{C}^n$ are known, e.g. for the unit ball [103, 148, 156], for the polydisk [3, 110], and for strongly convex and strongly pseudoconvex domains [1, 2]. Abate and Tauraso [5] used the Kobayashi metric on a bounded domain in $\mathbb{C}^n$ to obtain a generalized version. There are various versions and proofs of this concept – see for example [40, 59, 95, 115, 132, 161, 173, 175, 192]. For a survey of work in higher dimensions, see [4, 5, 94, 131, 147, 156]. The proof of the existence and uniqueness of a Denjoy–Wolff point for a function b in the unit ball of H^∞ which is not the identity is known as the Denjoy–Wolff theorem. The proof of this fact presented in Exercise 21.1.1 is taken from [166]. Exercise 21.1.2 is taken from Li [118]. The inequalities (21.7), (21.8) and (21.9) are due to Cowen and Pommerenke [60], who established many inequalities for fixed points of holomorphic functions. For the proof of these inequalities, Cowen and Pommerenke used deep complex analysis and some Grunsky-type inequalities. In his paper, Li employed a new method (which is presented in Exercise 21.1.2) based on $\mathcal{H}(b)$ spaces. This new method not only provides simpler proofs but also leads to some improvements.

Section 21.2

The connection between angular derivatives and mass points on the boundary has a long history. It can be traced back to Nevanlinna [135]. The connection between angular derivatives and square summability is due to M. Riesz [153].

Section 21.3

The cases $N = 0$ and $N = 1$ of Theorem 21.8(i) are due to Frostman [83]. Frostman's results were generalized by Cargo [43]. The version presented here was obtained by Ahern and Clark [10, 11]. In fact, Ahern and Clark systematically studied the boundary behavior of analytic functions in a series of papers [7–13]. Some of their results are addressed in this chapter; see also [44, 45]. The monograph [129] treats a systematic study of this subject.

Section 21.4

A special case of Theorem 21.10 for $N = 0$ is given in [44] without proof. The general version was mentioned in [10, 11], again without proof.

Section 21.5

The approximation Theorems 21.12 and 21.15 are taken from [10]. Frostman shifts, exceptional sets, Lemma 21.13 and Corollary 21.14 were introduced in [84]. This result has several applications, in particular in Carleson's proof of the corona conjecture; see also [108, 109, 130, 133].

Section 21.6

The results presented in this section are very general and considered as common knowledge. For example, Lemmas 21.18 and 21.19 are implicitly used in [10].

Section 21.7

Lemma 21.20 can be found in [10, 81]. Theorems 21.23 and 21.24 are due to Ahern and Clark [10].

Section 21.8

In the case where b is an inner function, Helson [100] studied the problem of analytic continuation across the boundary for functions in the model space K_b. Then, still when b is an inner function, Ahern and Clark [8] characterized those points x_0 of $\mathbb{R}$ where every function f of K_b and all its derivatives up to order n have a radial limit. These results were generalized in the form of Theorem 21.26 for an arbitrary element of the closed unit ball by Fricain and Mashreghi [81]. We also mention that Sarason has obtained another criterion in terms of the Clark measure μ_λ associated with b; see following theorem.

Theorem 21.29 (Sarason [166]) *Let ζ_0 be a point of $\mathbb{T}$ and let ℓ be a non-negative integer. The following conditions are equivalent.*

(i) *Each function in $\mathcal{H}(b)$ and all its derivatives up to order ℓ have non-tangential limits at ζ_0.*
(ii) *There is a point $\lambda \in \mathbb{T}$ such that*

$$\int_{\mathbb{T}} |e^{i\theta} - \zeta_0|^{-2\ell-2}\, d\mu_\lambda(e^{i\theta}) < \infty. \tag{21.53}$$

(iii) *The last inequality holds for all $\lambda \in \mathbb{T} \setminus \{b(\zeta_0)\}$.*
(iv) *There is a point $\lambda \in \mathbb{T}$ such that μ_λ has a point mass at ζ_0 and*

$$\int_{\mathbb{T}\setminus\{z_0\}} |e^{i\theta} - \zeta_0|^{-2\ell}\, d\mu_\lambda(e^{i\theta}) < \infty.$$

Recently, Bolotnikov and Kheifets [36] gave a third criterion (in some sense more algebraic) in terms of the Schwarz–Pick matrix. Recall that, if b is a function in the closed unit ball of H^∞, then the matrix $\mathbf{P}_\ell^\omega(z)$, which will be referred to as to a Schwarz–Pick matrix and defined by

$$\mathbf{P}_\ell^b(z) := \left[\frac{1}{i!j!}\frac{\partial^{i+j}}{\partial z^i \partial \bar{z}^j}\frac{1-|b(z)|^2}{1-|z|^2}\right]_{i,j=0}^{\ell},$$

is positive semidefinite for every $\ell \geq 0$ and $z \in \mathbb{D}$. One can extend this notion to boundary points as follows: given a point $\zeta_0 \in \mathbb{T}$, the boundary Schwarz–Pick matrix is

$$\mathbf{P}_\ell^b(\zeta_0) = \lim_{\substack{z \to \zeta_0 \\ \sphericalangle}} \mathbf{P}_\ell^b(z) \qquad (\ell \geq 0),$$

provided this nontangential limit exists; see following theorem.

Theorem 21.30 *Let b be a point in the closed unit ball of H^∞, let $\zeta_0 \in \mathbb{T}$ and let ℓ be a nonnegative integer. Assume that the boundary Schwarz–Pick matrix $\mathbf{P}_\ell^b(\zeta_0)$ exists. Then each function in $\mathcal{H}(b)$ and all its derivatives up to order ℓ have nontangential limits at ζ_0.*

Further it is shown in [36] that the boundary Schwarz–Pick matrix $\mathbf{P}_\ell^b(\zeta_0)$ exists if and only if

$$\lim_{\substack{z \to \zeta_0 \\ \sphericalangle}} d_{b,\ell}(z) < \infty, \tag{21.54}$$

where

$$d_{b,\ell}(z) := \frac{1}{(\ell!)^2} \frac{\partial^{2\ell}}{\partial z^\ell \partial \bar{z}^\ell} \frac{1 - |b(z)|^2}{1 - |z|^2}.$$

We should mention that, to date, there is no clear direct connection between conditions (21.53), (21.54) and condition (v) of Theorem 21.26.

22

Bernstein-type inequalities

In simple language, the classic Bernstein inequality provides a way to control the norm of the derivative of a function f in the Paley–Wiener space by the norm of f itself. In this chapter, we study a similar problem in $\mathcal{H}(b)$ spaces.

Since the formulas are easier to get in the upper half-plane $\mathbb{C}_+$, we explain, in Section 22.1, the passage from $\mathbb{D}$ to $\mathbb{C}_+$. In particular, we rewrite in this context some of the results of previous chapters. In Section 22.2, we give an important integral representation for the derivatives of functions in $\mathcal{H}(b)$ at point w. First we start with a point w in the upper half-plane. Then we explain how to get a similar formula for a point on the boundary. This integral representation will be our main tool to obtain Bernstein inequalities for functions in $\mathcal{H}(b)$. Our Bernstein inequalities involve a certain weight $w_{p,n}$, which is needed to control the growth of the derivatives when we approach the boundary. This weight is introduced in Section 22.3. Some auxiliary singular integral operators are studied in Section 22.4. In Section 22.5, we finally obtain our Bernstein-type inequality for functions in $\mathcal{H}(b)$. Using a link between the weight $w_{p,n}$ and the distance to the level sets of b, we give in Section 22.6 another interesting Bernstein-type inequality. Then, in Section 22.7, we study the embedding problem for $\mathcal{H}(b)$. In particular, we prove that, if a measure μ on the closed upper half-plane satisfies Carleson's condition on squares, which intersects a certain set where b is small, then $\mathcal{H}(b)$ embeds into $L^2(\mu)$. For a special class of b, we show that this necessary condition is also sufficient. The main tool in the study of the embedding problem is the Bernstein inequality. In Chapter 31, we will see the second application of the Bernstein inequality to the problem of stability of basis. In Section 22.8, we establish a nice formula of combinatorics, which will be useful in the last section for giving an important property of norm convergence for the reproducing kernels of derivatives.

22.1 Passage between $\mathbb{D}$ and $\mathbb{C}_+$

Let $\mathbb{C}_+$ denote the upper half-plane of the complex plane and let $H^2(\mathbb{C}_+)$ denote the usual Hardy space consisting of analytic functions f on $\mathbb{C}_+$ that satisfy the growth restriction

$$\|f\|_2 = \sup_{y>0} \left(\int_{\mathbb{R}} |f(x+iy)|^2 \, dx \right)^{1/2} < \infty.$$

The upper half-plane version of Theorem 4.1 says that, for each function $f \in H^2(\mathbb{C}_+)$ and for almost all $x_0 \in \mathbb{R}$,

$$f^*(x_0) = \lim_{t \to 0^+} f(x_0 + it)$$

exists. Moreover, we have $f^* \in L^2(\mathbb{R})$ and $\mathcal{F} f^* = 0$ on $(-\infty, 0)$, where $\mathcal{F}$ is the Fourier–Plancherel transformation, and $\|f^*\|_2 = \|f\|_2$.

A particularly interesting class of subspaces of $H^2(\mathbb{C}_+)$ is the $\mathcal{H}(b)$ classes of the upper half-plane. The definitions are similar to those for the open unit disk. However, for the sake of completeness, we mention them below. We also mention some facts without proof.

For $\varphi \in L^\infty(\mathbb{R})$, let T_φ stand for the Toeplitz operator defined on $H^2(\mathbb{C}_+)$ by

$$T_\varphi(f) = P_+(\varphi f) \qquad (f \in H^2(\mathbb{C}_+)),$$

where P_+ is the orthogonal projection of $L^2(\mathbb{R})$ onto $H^2(\mathbb{C}_+)$. Then, for $b \in H^\infty(\mathbb{C}_+)$, with $\|b\|_\infty \le 1$, the de Branges–Rovnyak space $\mathcal{H}(b)$ consists of those $H^2(\mathbb{C}_+)$ functions which are in the range of the operator $(I - T_b T_{\bar{b}})^{1/2}$. As before, $\mathcal{H}(b)$ is a Hilbert space when equipped with the inner product

$$\langle (I - T_b T_{\bar{b}})^{1/2} f, \; (I - T_b T_{\bar{b}})^{1/2} g \rangle_b = \langle f, g \rangle_2,$$

where $f, g \in H^2(\mathbb{C}_+) \ominus \ker\, (I - T_\varphi T_{\bar{b}})^{1/2}$. For each $w \in \mathbb{C}_+$, the function

$$k_w^b(z) = \frac{i}{2\pi} \frac{1 - \overline{b(w)} b(z)}{z - \bar{w}} \qquad (z \in \mathbb{C}_+)$$

is the reproducing kernel of $\mathcal{H}(b)$, that is,

$$f(w) = \langle f, k_w^b \rangle_b \qquad (f \in \mathcal{H}(b)). \tag{22.1}$$

In particular, with $f = k_w^b(z)$, we obtain

$$\|k_w^b\|_b^2 = k_w^b(w) = \frac{1}{2\pi} \frac{1 - |b(w)|^2}{2\, \Im\, w} \qquad (f \in \mathcal{H}(b)). \tag{22.2}$$

Let $\rho(t) = 1 - |b(t)|^2$, $t \in \mathbb{R}$, and let $L^2(\rho)$ stand for the usual Hilbert space of measurable functions $f : \mathbb{R} \longrightarrow \mathbb{C}$ with $\|f\|_\rho < \infty$, where

$$\|f\|_\rho^2 = \int_{\mathbb{R}} |f(t)|^2 \rho(t) \, dt.$$

For each $w \in \mathbb{C}_+$, the Cauchy kernel k_w belongs to $L^2(\rho)$, where $k_w(z) = (i/2\pi)(z - \bar{w})^{-1}$. Hence, we define $H^2(\rho)$ to be the span in $L^2(\rho)$ of the functions k_w ($w \in \mathbb{C}_+$). If g is a function in $L^2(\rho)$, then $g\rho$ is in $L^2(\mathbb{R})$, being the product of $g\rho^{1/2} \in L^2(\mathbb{R})$ and the bounded function $\rho^{1/2}$. Thus, we define the operator $C_\rho : L^2(\rho) \longrightarrow H^2(\mathbb{C}_+)$ by

$$C_\rho(g) = P_+(g\rho).$$

Then C_ρ is a partial isometry from $L^2(\rho)$ onto $\mathcal{H}(\bar{b})$ whose initial space is equal to $H^2(\rho)$ and it is an isometry if and only if b is an extreme point of the unit ball of $H^\infty(\mathbb{C}_+)$.

Write $k_w^b = k_w - \overline{b(w)} b k_w$. Since $T_{\bar{b}} k_w = \overline{b(w)} k_w$, we obtain

$$\begin{aligned} T_{\bar{b}} k_w^b &= \overline{b(w)}(k_w - P_+(|b|^2 k_w)) \\ &= \overline{b(w)} P_+((1 - |b|^2)k_w) \\ &= \overline{b(w)} P_+(\rho k_w) \\ &= \overline{b(w)} C_\rho(k_w). \end{aligned} \tag{22.3}$$

In Section 21.8, we have studied the boundary behavior of the derivatives of functions in $\mathcal{H}(b)$ spaces of the open unit disk $\mathbb{D}$. We mention some parts of Theorems 21.10 and 21.26, modified for the upper half-plane $\mathbb{C}_+$, that are needed below.

Theorem 22.1 *Let b be in the closed unit ball of $H^\infty(\mathbb{C}_+)$ and let $b = BI_\mu O_b$ be its canonical factorization, where*

$$B(z) = \prod_k e^{i\alpha_k} \frac{z - z_k}{z - \bar{z}_k}$$

is a Blaschke product, the singular inner function I_μ is given by

$$I_\mu(z) = \exp\left(iaz - \frac{i}{\pi} \int_{\mathbb{R}} \left(\frac{1}{z - t} + \frac{t}{t^2 + 1} \right) d\mu(t) \right),$$

with a positive singular measure μ and $a \geq 0$, and O_b is the outer function

$$O_b(z) = \exp\left(\frac{i}{\pi} \int_{\mathbb{R}} \left(\frac{1}{z - t} + \frac{t}{t^2 + 1} \right) \log|b(t)|\, dt \right).$$

Put

$$S_n(x) = \sum_{k=1}^{\infty} \frac{\Im z_k}{|x - z_k|^n} + \int_{\mathbb{R}} \frac{d\mu(t)}{|x - t|^n} + \int_{\mathbb{R}} \frac{\big|\log|b(t)|\big|}{|x - t|^n}\, dt. \tag{22.4}$$

Then, for $x \in \mathbb{R}$ and for $n \geq 0$, the following hold.

(i) *If $S_{n+1}(x) < \infty$, then the limits*

$$b^{(j)}(x) = \lim_{t \to 0^+} b^{(j)}(x + it) \qquad (0 \leq j \leq n)$$

exist.

(ii) *For every function $f \in \mathcal{H}(b)$, the limits*

$$f^{(j)}(x) = \lim_{t \to 0^+} f^{(j)}(x+it) \qquad (0 \le j \le n)$$

exist if and only if $S_{2n+2}(x) < \infty$.

Theorem 22.1 suggests to define

$$E_n(b) = \{x \in \mathbb{R} : S_n(x) < \infty\}.$$

The upper half-plane version of Corollary 21.11 says that, if $x \in E_2(b)$, then the modulus of the angular derivative of b at a point x is given by

$$|b'(x)| = a + \sum_k \frac{2\,\Im z_k}{|x - z_k|^2} + \frac{1}{\pi}\int_{\mathbb{R}} \frac{d\mu(t)}{|x-t|^2} + \frac{1}{\pi}\int_{\mathbb{R}} \frac{\big|\log|b(t)|\big|}{|x-t|^2}\,dt. \tag{22.5}$$

We will need the following simple estimate.

Lemma 22.2 *For any $x \in \mathbb{R}$, $y > 0$, we have $|b'(x+iy)| \le |b'(x)|$.*

Proof Let $z = x + iy$, $y > 0$. We consider the decomposition $b = BI_\mu O_b$ and use (22.5) repeatedly.

Case $b = B$. We have

$$b(z) = \prod_k e^{i\alpha_k} \frac{z - z_k}{z - \bar{z}_k}$$

and then

$$b'(z) = \sum_k b_k(z) \frac{2i\,\Im z_k}{(z - \bar{z}_k)^2}.$$

Hence, by (22.5),

$$|b'(z)| \le \sum_k \frac{2\,\Im z_k}{|z - \bar{z}_k|^2} \le \sum_k \frac{2\,\Im z_k}{|x - z_k|^2} = |b'(x)|.$$

Case $b = I_\mu$. We have

$$b(z) = \exp\left(iaz - \frac{i}{\pi}\int_{\mathbb{R}} \left(\frac{1}{z-t} + \frac{t}{t^2+1} \right) d\mu(t) \right)$$

and then

$$b'(z) = b(z)\left(ia - \frac{i}{\pi}\int_{\mathbb{R}} \frac{d\mu(t)}{(t-z)^2} \right).$$

Hence, by (22.5),

$$|b'(z)| \le \frac{1}{\pi}\int_{\mathbb{R}} \frac{d\mu(t)}{|t-z|^2} \le \frac{1}{\pi}\int_{\mathbb{R}} \frac{d\mu(t)}{|t-x|^2} = |b'(x)|.$$

Case $b = O_b$. We have

$$b(z) = \exp\left(\frac{i}{\pi}\int_{\mathbb{R}}\left(\frac{1}{z-t} + \frac{t}{t^2+1}\right)\log|b(t)|\,dt\right)$$

and then

$$b'(z) = -b(z)\frac{i}{\pi}\int_{\mathbb{R}}\frac{\log|b(t)|}{(t-z)^2}\,dt.$$

Hence, by (22.5),

$$|b'(z)| \le \frac{1}{\pi}\int_{\mathbb{R}}\frac{\big|\log|b(t)|\big|}{|t-z|^2}\,dt \le \frac{1}{\pi}\int_{\mathbb{R}}\frac{\big|\log|b(t)|\big|}{|t-x|^2}\,dt = |b'(x)|.$$

To obtain the general case as the combination of the above three cases, we just need to apply the identity (22.5) one more time. □

22.2 Integral representations for derivatives

In this section, our goal is to prove an integral representation for the derivatives of elements of $\mathcal{H}(b)$. We start by finding such a formula at a point w in the upper half-plane. Since w is away from the boundary, the representation is easy to establish. In order to get an integral representation for the nth derivative of f at point w for functions in the de Branges–Rovnyak spaces, we need to introduce the following kernels:

$$k^b_{w,n}(z) = -\frac{n!}{2\pi i}\,\frac{1 - b(z)\sum_{p=0}^{n}[\overline{b^{(p)}(w)}/p!](z-\bar{w})^p}{(z-\bar{w})^{n+1}} \qquad (z \in \mathbb{C}_+) \tag{22.6}$$

and

$$k^\rho_{w,n}(t) = -\frac{n!}{2\pi i}\,\frac{\sum_{p=0}^{n}[\overline{b^{(p)}(w)}/p!](t-\bar{w})^p}{(t-\bar{w})^{n+1}} \qquad (t \in \mathbb{R}). \tag{22.7}$$

For $n = 0$, we see that $k^b_{w,0} = k^b_w$ and $k^\rho_{w,0} = \overline{b(w)}k_w$. In fact, (22.6) is the upper half-plane version of (21.38), and we have

$$f^{(n)}(w) = \langle f, k^b_{w,n}\rangle_b \qquad (f \in \mathcal{H}(b)).$$

In the following lemma, we obtain a more friendly representation for $f^{(n)}(w)$.

Lemma 22.3 *Let b be a point in the closed unit ball of $H^\infty(\mathbb{C}_+)$, let $f \in \mathcal{H}(b)$ and let $g \in H^2(\rho)$ be such that $T_{\bar{b}}f = C_\rho(g)$. Then, for all $w \in \mathbb{C}_+$ and for any integer $n \ge 0$, we have $k^b_{w,n} \in \mathcal{H}(b)$, $k^\rho_{w,n} \in H^2(\rho)$ and*

$$f^{(n)}(w) = \int_{\mathbb{R}} f(t)\overline{k^b_{w,n}(t)}\,dt + \int_{\mathbb{R}} g(t)\rho(t)\overline{k^\rho_{w,n}(t)}\,dt. \tag{22.8}$$

Proof According to (22.1) and Theorem 17.8, we have

$$f(w) = \langle f, k_w^b \rangle_b = \langle f, k_w^b \rangle_2 + \langle T_{\bar{b}} f, T_{\bar{b}} k_w^b \rangle_{\bar{b}}.$$

Hence, by (22.3),

$$f(w) = \langle f, k_w^b \rangle_2 + b(w) \langle C_\rho(g), C_\rho(k_w) \rangle_{\bar{b}}.$$

Since C_ρ is a partial isometry from $L^2(\rho)$ onto $\mathcal{H}(\bar{b})$, with initial space equal to $H^2(\rho)$, we conclude that

$$f(w) = \langle f, k_w^b \rangle_2 + b(w) \langle g, k_w \rangle_\rho = \langle f, k_w^b \rangle_2 + \langle \rho g, k_w^\rho \rangle_2.$$

We rewrite this identity as

$$f(w) = \langle f, k_{w,0}^b \rangle_2 + \langle \rho g, k_{w,0}^\rho \rangle_2,$$

which is precisely the representation 22.8) for $n = 0$.

Now straightforward computations show that

$$\frac{\partial^n k_{w,0}^b}{\partial \bar{w}^n} = k_{w,n}^b \quad \text{and} \quad \frac{\partial^n k_{w,0}^\rho}{\partial \bar{w}^n} = k_{w,n}^\rho.$$

Since $k_{w,0}^b \in \mathcal{H}(b)$ and $k_{w,0}^\rho \in H^2(\rho)$, as we justified similarly in Section 21.6, we have $k_{w,n}^b \in \mathcal{H}(b)$ and $k_{w,n}^\rho \in H^2(\rho)$, $n \geq 0$. The representation (22.8) follows now by induction and by differentiating under the integral sign. □

The next step is to show that (22.8) is still valid at the boundary points x_0 that satisfy $S_{2n+2}(x_0) < \infty$. To do so, we need the boundary analogs of the kernels (22.6) and (22.7). In fact, both formulas make sense if we simply replace w by x_0 and assume that $S_{n+1}(x_0) < \infty$. However, we see that, under the stronger condition $S_{2n+2}(x_0) < \infty$, $k_{x_0,n}^b$ is actually the kernel function in $\mathcal{H}(b)$ for the functional of the nth derivative at x_0.

Lemma 22.4 *Let b be a point in the closed unit ball of $H^\infty(\mathbb{C}_+)$, let $n \geq 0$ and let $x_0 \in \mathbb{R}$. Assume that x_0 satisfies the condition $S_{2n+2}(x_0) < \infty$. Then $k_{x_0,n}^b \in \mathcal{H}(b)$ and, for every function $f \in \mathcal{H}(b)$, we have*

$$f^{(n)}(x_0) = \langle f, k_{x_0,n}^b \rangle_b. \tag{22.9}$$

Proof According to Theorem 22.1, the condition $S_{2n+2}(x_0) < \infty$ guarantees that, for every function $f \in \mathcal{H}(b)$, $f^{(n)}(w)$ tends to $f^{(n)}(x_0)$ as w tends radially to x_0. Therefore, an application of the uniform boundedness principle shows that the functional $f \longmapsto f^{(n)}(x_0)$ is bounded on $\mathcal{H}(b)$. Hence, by Riesz's theorem, there exists $\varphi_{x_0,n} \in \mathcal{H}(b)$ such that

$$f^{(n)}(x_0) = \langle f, \varphi_{x_0,n} \rangle_b \qquad (f \in \mathcal{H}(b)).$$

Now, the formula

$$f^{(n)}(w) = \langle f, k_{w,n}^b \rangle_b \qquad (f \in \mathcal{H}(b))$$

implies that $k^b_{w,n}$ tends weakly to $\varphi_{x_0,n}$ as w tends radially to x_0. Thus, for $z \in \mathbb{C}_+$, we can write

$$\begin{aligned}
\varphi_{x_0,n}(z) &= \langle \varphi_{x_0,n}, k^b_z \rangle_b \\
&= \lim_{t\to 0^+} \langle k^b_{x_0+it,n}, k^b_z \rangle_b \\
&= \lim_{t\to 0^+} k^b_{x_0+it,n}(z) \\
&= \lim_{t\to 0^+} \left(-\frac{n!}{2\pi i} \, \frac{1 - b(z) \sum_{p=0}^{n} [\overline{b^{(p)}(x_0+it)}/p!](z - x_0 + it)^p}{(z - x_0 + it)^{n+1}} \right) \\
&= -\frac{n!}{2\pi i} \, \frac{1 - b(z) \sum_{p=0}^{n} [\overline{b^{(p)}(x_0)}/p!](z - x_0)^p}{(z - x_0)^{n+1}} \\
&= k^b_{x_0,n}(z).
\end{aligned}$$

Hence, $k^b_{x_0,n} \in \mathcal{H}(b)$ and, for every function $f \in \mathcal{H}(b)$, (22.9) holds. $\square$

The next result gives a (standard) Taylor formula at a point on the boundary.

Lemma 22.5 *Let h be a holomorphic function in the upper half-plane $\mathbb{C}_+$, let $n \geq 0$ and let $x_0 \in \mathbb{R}$. Assume that $h^{(n)}$ has a radial limit at x_0. Then $h, h', \ldots, h^{(n-1)}$ have radial limits at x_0 and*

$$h(w) = \sum_{p=0}^{n} \frac{h^{(p)}(x_0)}{p!} (w - x_0)^p + (w - x_0)^n \varepsilon(w) \qquad (w \in \mathbb{C}_+),$$

with $\lim_{t\to 0^+} \varepsilon(x_0 + it) = 0$.

Proof The proof is by induction.

Case $n = 0$. This is trivial.

Case $n \geq 1$. Assume that the property is true for $n-1$. Applying the induction hypothesis to $g = h'$, we see that $g = h',\ g' = h^{(2)},\ \ldots,\ g^{(n-1)} = h^{(n)}$ have a radial limit at x_0 and

$$h'(w) = \sum_{p=0}^{n-1} \frac{h^{(p+1)}(x_0)}{p!} (w - x_0)^p + (w - x_0)^{n-1} \varepsilon_1(w),$$

with $\lim_{t\to 0^+} \varepsilon_1(x_0 + it) = 0$.

Since h' has a radial limit at x_0, we can define $h(x_0)$ by

$$h(x_0) = h(x_0 + it) - \int_{[x_0, x_0+it]} h'(u)\, du.$$

By Cauchy's theorem, $h(x_0)$ is well defined and its value does not depend on t. However, this freedom in choosing t shows that $h(x_0) = \lim_{t\to 0} h(x_0 + it)$.

Then another application of Cauchy's theorem reveals that we can even write

$$h(w) = h(x_0) + \int_{\Gamma_w} h'(u)\, du,$$

for all $w \in \mathbb{C}_+$. The path Γ_w is from x_0 to w. To have the uniform continuity of h on the path, we assume that in the beginning Γ_w is a vertical segment starting at x_0 (the height of the segment is not important), and then it goes to w via a rectifiable path in $\mathbb{C}_+$. Hence, we have

$$\begin{aligned} h(w) &= h(x_0) + \int_{\Gamma_w} \left(\sum_{p=0}^{n-1} \frac{h^{(p+1)}(x_0)}{p!} (u - x_0)^p + (u - x_0)^{n-1} \varepsilon_1(u) \right) du \\ &= \sum_{p=0}^{n} \frac{h^{(p)}(x_0)}{p!} (w - x_0)^p + \int_{\Gamma_w} (u - x_0)^{n-1} \varepsilon_1(u)\, du. \end{aligned}$$

The natural choice for ε is

$$\varepsilon(w) = \frac{1}{(w - x_0)^n} \int_{\Gamma_w} (u - x_0)^{n-1} \varepsilon_1(u)\, du \qquad (w \in \mathbb{C}_+).$$

Thus,

$$\varepsilon(x_0 + it) = \frac{1}{(it)^n} \int_{s=0}^{t} (is)^{n-1} \varepsilon_1(x_0 + is)\, ds.$$

Therefore, based on the induction hypothesis on ε_1, we have

$$\begin{aligned} |\varepsilon(x_0 + it)| &\le \frac{1}{t^n} \int_{s=0}^{t} s^{n-1} |\varepsilon_1(x_0 + is)|\, ds \\ &\le \frac{1}{t} \int_{s=0}^{t} |\varepsilon_1(x_0 + is)|\, ds \longrightarrow 0 \end{aligned}$$

as $t \longrightarrow 0^+$. □

If x_0 satisfies the condition $S_{2n+2}(x_0) < \infty$, we also have $k^{\rho}_{x_0,n} \in L^2(\rho)$. Indeed, according to (22.7), it suffices to prove that $(t - x_0)^{-j} \in L^2(\rho)$, for $1 \le j \le n+1$. Since $\rho \le 1$, it is enough to verify this fact in a neighborhood of x_0, say $I_{x_0} = [x_0 - 1, x_0 + 1]$. But according to the condition $S_{2n+2}(x_0) < \infty$, we have

$$\int_{I_{x_0}} \frac{1 - |b(t)|^2}{|t - x_0|^{2j}}\, dt \le 2 \int_{I_{x_0}} \frac{\big|\log|b(t)|\big|}{|t - x_0|^{2j}}\, dt \le 2 \int_{I_{x_0}} \frac{\big|\log|b(t)|\big|}{|t - x_0|^{2n+2}}\, dt < \infty.$$

Theorem 22.6 *Let b be a point in the closed unit ball of $H^\infty(\mathbb{C}_+)$, let $n \ge 0$, let $f \in \mathcal{H}(b)$, and let $g \in H^2(\rho)$ be such that $T_{\bar{b}} f = C_\rho(g)$. Then, for every point $x_0 \in \mathbb{R}$ satisfying the condition $S_{2n+2}(x_0) < \infty$, we have*

$$f^{(n)}(x_0) = \int_{\mathbb{R}} f(t) \overline{k^b_{x_0,n}(t)}\, dt + \int_{\mathbb{R}} g(t)\rho(t) \overline{k^{\rho}_{x_0,n}(t)}\, dt. \qquad (22.10)$$

Proof Recall that the condition $S_{2n+2}(x_0) < \infty$ guarantees that $b^{(j)}(x_0)$ exists for $0 \le j \le 2n+1$. Moreover, Lemma 22.4 implies that $k^b_{x_0,p} \in \mathcal{H}(b)$, for $0 \le p \le n$.

Put

$$h_{x_0,n}(z) = \frac{b(z) - \sum_{p=0}^{n} [b^{(p)}(x_0)/p!](z-x_0)^p}{(z-x_0)^{n+1}} \qquad (z \in \mathbb{C}_+).$$

Let us verify that $h_{x_0,n}$ satisfies

$$h_{x_0,n} = 2\pi i \sum_{p=0}^{n} \frac{b^{(n-p)}(x_0)}{(n-p)!\,p!} k^b_{x_0,p}. \tag{22.11}$$

To simplify the following computations a little, we put $a_p := b^{(p)}(x_0)/p!$, for $0 \le p \le n$. According to (22.6), we have

$$\begin{aligned}
&2\pi i \sum_{p=0}^{n} a_{n-p} \frac{k^b_{x_0,p}(z)}{p!} \\
&\quad = \sum_{p=0}^{n} a_{n-p} \left(b(z) \frac{\sum_{j=0}^{p} \overline{a_j}(z-x_0)^j - 1}{(z-x_0)^{p+1}} \right) \\
&\quad = \frac{1}{(z-x_0)^{n+1}} \left[\sum_{p=0}^{n} a_{n-p}(z-x_0)^{n-p} \left(b(z) \sum_{j=0}^{p} \overline{a_j}(z-x_0)^j - 1 \right) \right] \\
&\quad = \frac{1}{(z-x_0)^{n+1}} \left[b(z) \left(\sum_{p=0}^{n} \sum_{j=0}^{p} a_{n-p}\overline{a_j}(z-x_0)^{n-p+j} \right) \right. \\
&\qquad\qquad \left. - \sum_{k=0}^{n} a_k (z-x_0)^k \right].
\end{aligned}$$

Therefore, we see that (22.11) is equivalent to

$$\sum_{p=0}^{n} \sum_{j=0}^{p} a_{n-p}\overline{a_j}(z-x_0)^{n-p+j} = 1. \tag{22.12}$$

But, putting $j = \ell - n + p$, we obtain

$$\begin{aligned}
\sum_{p=0}^{n} \sum_{j=0}^{p} a_{n-p}\overline{a_j}(z-x_0)^{n-p+j} &= \sum_{\ell=0}^{n} \left(\sum_{p=n-\ell}^{n} a_{n-p}\overline{a_{\ell-n+p}} \right)(z-x_0)^\ell \\
&= \sum_{\ell=0}^{n} \left(\sum_{q=0}^{\ell} a_{\ell-q}\overline{a_q} \right)(z-x_0)^\ell.
\end{aligned}$$

Consequently, (22.12) is equivalent to

$$|a_0|^2 = |b(x_0)|^2 = 1 \qquad (\text{for } \ell = 0) \tag{22.13}$$

and

$$\sum_{q=0}^{\ell} a_{\ell-q}\overline{a_q} = 0 \qquad (1 \le \ell \le n). \tag{22.14}$$

To establish (22.13) and (22.14), put

$$\varphi(z) = 1 - b(z)\sum_{p=0}^{n} \overline{a_p}(z-x_0)^p \qquad (z \in \mathbb{C}_+).$$

Then φ is holomorphic in $\mathbb{C}_+$ and φ, and its derivatives up to order $2n+1$ have radial limits at x_0. An application of Lemma 22.5 shows that we can write

$$\varphi(z) = \sum_{p=0}^{n} \frac{\varphi^{(p)}(x_0)}{p!}(z-x_0)^p + o((z-x_0)^n)$$

as z tends radially to x_0. Assume that there exists $p \in \{0,\dots,n\}$ such that $\varphi^{(p)}(x_0) \neq 0$ and put

$$p_0 = \min\{0 \le p \le n : \varphi^{(p)}(x_0) \neq 0\}.$$

Hence, as $t \longrightarrow 0^+$,

$$|k^b_{x_0,n}(x_0+it)| \sim \frac{1}{2\pi}\,\frac{|\varphi^{(p_0)}(x_0)|}{p_0!}\,t^{p_0-(n+1)},$$

which implies that $\lim_{t\to 0^+}|k^b_{x_0,n}(x_0+it)| = \infty$. This is a contradiction with the fact that $k^b_{x_0,n}$ belongs to $\mathcal{H}(b)$ and has a finite radial limit at x_0. Therefore, we necessarily have $\varphi^{(p)}(x_0) = 0$, for $0 \le p \le n$. But

$$\varphi(x_0) = 1 - b(x_0)\overline{b(x_0)} = 1 - |b(x_0)|^2,$$

and if we use Leibniz's rule to compute the derivative of φ, for $1 \le \ell \le n$, we get

$$\varphi^{(\ell)}(x_0) = -\sum_{p=0}^{\ell} \overline{a_p}\binom{\ell}{p} p!\, b^{(\ell-p)}(x_0) = -\ell!\sum_{p=0}^{\ell} \overline{a_p} a_{\ell-p}.$$

Thus, we have established (22.13) and (22.14), which in turn proves (22.11). According to Lemma 22.4, (22.11) implies that $h_{x_0,n} \in \mathcal{H}(b)$.

Now, for almost all $t \in \mathbb{R}$, we have

$$\begin{aligned}
&\overline{b(t)}k^b_{x_0,n}(t)\\
&\quad = -\frac{n!}{2\pi i}\,\frac{\overline{b(t)} - |b(t)|^2\sum_{p=0}^{n}\overline{a_p}(t-x_0)^p}{(t-x_0)^{n+1}}\\
&\quad = -\frac{n!}{2\pi i}\,(1-|b(t)|^2)\frac{\sum_{p=0}^{n}\overline{a_p}(t-x_0)^p}{(t-x_0)^{n+1}}
\end{aligned}$$

$$- \frac{n!}{2\pi i} \frac{\overline{b(t)} - \sum_{p=0}^{n} \overline{a_p}(t-x_0)^p}{(t-x_0)^{n+1}}$$
$$= \rho(t) k^{\rho}_{x_0,n}(t) - \frac{n!}{2\pi i} \overline{h_{x_0,n}(t)}.$$

Since $h_{x_0,n} \in \mathcal{H}(b) \subset H^2(\mathbb{C}_+)$, we get that $P_+(\bar{b} k^b_{x_0,n}) = P_+(\rho k^{\rho}_{x_0,n})$, which can be written as $T_{\bar{b}} k^b_{x_0,n} = C_\rho k^{\rho}_{x_0,n}$. It follows from Theorem 17.8 and Lemma 22.4 that

$$\begin{aligned} f^{(n)}(x_0) &= \langle f, k^b_{x_0,n} \rangle_b \\ &= \langle f, k^b_{x_0,n} \rangle_2 + \langle T_{\bar{b}} f, T_{\bar{b}} k^b_{x_0,n} \rangle_{\bar{b}} \\ &= \langle f, k^b_{x_0,n} \rangle_2 + \langle g, k^{\rho}_{x_0,n} \rangle_\rho \\ &= \int_{\mathbb{R}} f(t) \overline{k^b_{x_0,n}(t)}\, dt + \int_{\mathbb{R}} g(t) \rho(t) \overline{k^{\rho}_{x_0,n}(t)}\, dt, \end{aligned}$$

which proves the relation (22.10). □

If b is inner, then it is clear that the second integral in (22.10) is zero and we obtain the easier formula

$$f^{(n)}(w) = \int_{\mathbb{R}} f(t) \overline{k^b_{w,n}(t)}\, dt. \tag{22.15}$$

We now provide a variation of Theorem 22.6 that is more suitable for obtaining Bernstein-type inequalities. To do so, we need the new kernel

$$\mathfrak{K}^{\rho}_{z_0,n}(t) = \overline{b(z_0)}\, \frac{\sum_{p=0}^{n} \binom{n+1}{p+1} (-1)^p \overline{b^p(z_0)} b^p(t)}{(t-\overline{z_0})^{n+1}} \qquad (t \in \mathbb{R}), \tag{22.16}$$

which is well defined for all $z_0 \in \mathbb{C}_+$. It also makes sense whenever $z_0 = x_0 \in E_1(b)$. We highlight that b^p is the p-power of b and should not be mistaken for, nor confused with, the pth derivative.

Corollary 22.7 *Let b be in the closed unit ball of $H^\infty(\mathbb{C}_+)$, let $z_0 \in \mathbb{C}_+ \cup E_{2n+2}(b)$ and let $n \geq 0$. Then $(k^b_{z_0})^{n+1} \in H^2(\mathbb{C}_+)$ and $\mathfrak{K}^{\rho}_{z_0,n} \in L^2(\rho)$. Moreover, for every function $f \in \mathcal{H}(b)$, we have*

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \left(\int_{\mathbb{R}} f(t) \overline{(k^b_{z_0})^{n+1}(t)}\, dt + \int_{\mathbb{R}} g(t) \rho(t) \overline{\mathfrak{K}^{\rho}_{z_0,n}(t)}\, dt \right), \tag{22.17}$$

where $g \in H^2(\rho)$ is such that $T_{\bar{b}} f = C_\rho g$.

Proof

Step 1. To show that the first integral in (22.17) is well defined, we show that $(k^b_{z_0})^{n+1} \in H^2(\mathbb{C}_+)$.

Let $a_p = b^{(p)}(z_0)/p!$. Fix $0 \le j \le n$. Then we rewrite (22.6) as

$$-\frac{2\pi i}{j!}k^b_{z_0,j}(z) = \frac{1-\overline{b(z_0)}b(z)}{(z-\overline{z_0})^{j+1}} - b(z)\sum_{p=1}^{j}\frac{\overline{a_p}}{(z-\overline{z_0})^{j+1-p}}.$$

Hence, multiplying by $(1-\overline{b(z_0)}b(z))^j$ and rearranging the terms, we obtain

$$\begin{aligned}(k^b_{z_0}(z))^{j+1} &= (1-\overline{b(z_0)}b(z))^j\left(-\frac{2\pi i}{j!}k^b_{z_0,j}(z)\right)\\ &\quad + b(z)\sum_{p=1}^{j}\overline{a_{j+1-p}}(1-\overline{b(z_0)}b(z))^{j-p}(k^b_{z_0}(z))^p. \qquad (22.18)\end{aligned}$$

Since $z_0 \in \mathbb{C}_+ \cup E_{2n+2}(b)$, according to Lemmas 22.3 and 22.4, the functions $k^b_{z_0}$ and $k^b_{z_0,j}$ $(1 \le j \le n)$ belong to $\mathcal{H}(b)$. Hence, using the recurrence relation (22.18) and the fact that $1-\overline{b(z_0)}b(z) \in H^\infty(\mathbb{C}_+)$, we see immediately by induction that $(k^b_{z_0})^{n+1} \in H^2(\mathbb{C}_+)$.

Step 2. To show that the second integral in (22.17) is well defined, we show that $\mathfrak{K}^\rho_{z_0,n} \in L^2(\rho)$.

We have $\mathfrak{K}^\rho_{z_0,n}(t) = (t-\overline{z_0})^{-(n+1)}\varphi(t)$, with

$$\varphi(t) = \overline{b(z_0)}\sum_{j=0}^{n}\binom{n+1}{j+1}(-1)^j\overline{b^j(z_0)}b^j(t).$$

Since $|\varphi(t)| \le 2^{n+1}$ $(t \in \mathbb{R})$, it is sufficient to prove that $(t-z_0)^{-(n+1)} \in L^2(\rho)$. If $z_0 \in \mathbb{C}_+$, this fact is trivial; and if $z_0 \in E_{2n+2}(b)$, the inequality $1-x \le |\log x|$, $x \in [0,1]$, implies that

$$\int_{\mathbb{R}}\frac{\rho(t)}{|t-z_0|^{2n+2}}\,dt \le \int_{\mathbb{R}}\frac{1-|b(t)|^2}{|t-z_0|^{2n+2}}\,dt \le 2\int_{\mathbb{R}}\frac{\big|\log|b(t)|\big|}{|t-z_0|^{2n+2}}\,dt < \infty,$$

which is the required result.

Step 3. It remains to prove that (22.17) holds.

Let ψ be any element of $H^2(\mathbb{C}_+)$. According to (22.10), we have

$$\begin{aligned}f^{(n)}(z_0) &= \langle f, k^b_{z_0,n}\rangle_2 + \langle \rho g, k^\rho_{z_0,n}\rangle_2\\ &= \langle f,\ k^b_{z_0,n} - b\psi\rangle_2 + \langle \bar{b}f, \psi\rangle_2 + \langle \rho g, k^\rho_{z_0,n}\rangle_2.\end{aligned}$$

But we have $T_{\bar{b}}f = C_\rho g$, which means that $\bar{b}f - \rho g \perp H^2(\mathbb{C}_+)$. Since $\psi \in H^2(\mathbb{C}_+)$, it follows that $\langle \bar{b}f, \psi\rangle_2 = \langle \rho g, \psi\rangle_2$. Hence, the identity

$$f^{(n)}(z_0) = \langle f,\ k^b_{z_0,n} - b\psi\rangle_2 + \langle \rho g,\ k^\rho_{z_0,n} + \psi\rangle_2 \qquad (22.19)$$

holds for each $\psi \in H^2(\mathbb{C}_+)$. A very specific ψ gives us the required representation. To find the appropriate ψ, note that, on the one hand, we have

$$\begin{aligned}
&-\frac{2\pi i}{n!}\, k^b_{z_0,n}(t) - (k^b_{z_0})^{n+1}(t) \\
&\quad = \frac{1 - b(t)\sum_{p=0}^{n} \overline{a_p}(t-\overline{z}_0)^p - (1-\overline{b(z_0)}b(t))^{n+1}}{(t-\overline{z}_0)^{n+1}} \\
&\quad = \frac{1-(1-\overline{b(z_0)}b(t))^{n+1}}{(t-\overline{z}_0)^{n+1}} - b(t)\frac{\sum_{p=0}^{n} \overline{a_p}(t-\overline{z}_0)^p}{(t-\overline{z}_0)^{n+1}} \\
&\quad = b(t)\psi(t),
\end{aligned}$$

where

$$\psi(t) = \frac{\sum_{p=1}^{n+1}(-1)^{p+1}\binom{n+1}{p}(\overline{b(z_0)})^p(b(t))^{p-1}}{(t-\overline{z}_0)^{n+1}} - \frac{\sum_{p=0}^{n} \overline{a_p}(t-\overline{z}_0)^p}{(t-\overline{z}_0)^{n+1}}.$$

On the other hand, we easily see that

$$\begin{aligned}
&-\frac{2\pi i}{n!}\, k^\rho_{z_0,n}(t) + \psi(t) \\
&\quad = \frac{\sum_{p=1}^{n+1}(-1)^{p+1}\binom{n+1}{p}(\overline{b(z_0)})^p(b(t))^{p-1}}{(t-\overline{z}_0)^{n+1}} \\
&\quad = \overline{b(z_0)}\,\frac{\sum_{p=0}^{n}(-1)^{p}\binom{n+1}{p+1}(\overline{b(z_0)})^p(b(t))^{p}}{(t-\overline{z}_0)^{n+1}} \\
&\quad = \mathfrak{K}^\rho_{z_0,n}(t).
\end{aligned}$$

Therefore, (22.17) follows immediately from (22.19). $\square$

22.3 The weight $w_{p,n}$

Let $1 < p \leq 2$ and let q be its conjugate exponent. Let $n \geq 0$. Then, for $z \in \overline{\mathbb{C}_+}$, we define

$$w^{(1)}_{p,n}(z) := \|(k^b_z)^{n+1}\|_q^{-pn/(pn+1)}$$

and

$$w^{(2)}_{p,n}(z) := \|\rho^{1/q}\mathfrak{K}^\rho_{z,n}\|_q^{-pn/(pn+1)},$$

where we assume that $w^{(i)}_{p,n}(z) = 0$ if the corresponding integrand is not in $L^q(\mathbb{R})$, and

$$w_{p,n}(z) = \min\{w^{(1)}_{p,n}(z),\ w^{(2)}_{p,n}(z)\}.$$

In what follows we will write w_p for $w_{p,1}$. The choice of the weight is motivated by representation (22.17), which shows that the quantity

$$(w_{2,n}(z))^{-(2n+1)/2n} = \max\{\|(k_z^b)^{n+1}\|_2,\ \|\rho^{1/2}\mathfrak{K}^{\rho}_{z,n}\|_2\}$$

is related to the norm of the functional $f \longmapsto f^{(n)}(z)$ on $\mathcal{H}(b)$.

The weight $w_{p,n}$ is designed such that $g = f^{(n)}w_{p,n}$ is well defined at all points of $\mathbb{R}$ for any $f \in \mathcal{H}(b)$ and $1 < p \le 2$. In fact, if $S_{2n+2}(x) < \infty$, then, by Corollary 22.7, $f^{(n)}(x)$ and $w_{p,n}(x)$ are both well defined and hence so is g. But if $S_{2n+2}(x) = \infty$, then $\|(k_x^b)^{n+1}\|_2 = \infty$. Hence, $\|(k_x^b)^{n+1}\|_q = \infty$, which, by definition, implies that $w_{p,n}(x) = 0$, and thus we may assume $g(x) = 0$.

Lemma 22.8 *For $1 < p \le 2$, $n \ge 0$, there is a constant $A = A(p,n) > 0$ such that*

$$w_{p,n}(z) \ge A\,\frac{(\Im z)^n}{(1-|b(z)|)^{pn/(q(pn+1))}}, \qquad (z \in \mathbb{C}_+).$$

Proof On the one hand, by (22.2),

$$\begin{aligned}
\|(k_z^b)^{n+1}\|_q^q &= \int_{\mathbb{R}} \left| \frac{1-\overline{b(z)}b(t)}{t-\bar{z}} \right|^{(n+1)q} dt\\
&\le \frac{C}{(\Im z)^{(n+1)q-2}} \int_{\mathbb{R}} \left| \frac{1-\overline{b(z)}b(t)}{t-\bar{z}} \right|^2 dt\\
&= \frac{C}{(\Im z)^{(n+1)q-2}} \|k_z^b\|_b^2\\
&\le C\,\frac{1-|b(z)|}{(\Im z)^{(n+1)q-1}}.
\end{aligned}$$

Hence,

$$\|(k_z^b)^{n+1}\|_q^{-pn/(pn+1)} \ge C\,\frac{(\Im z)^n}{(1-|b(z)|)^{pn/(q(pn+1))}} \qquad (z \in \mathbb{C}_+).$$

On the other hand, we have

$$\begin{aligned}
\|\rho^{1/q}\mathfrak{K}^{\rho}_{z,n}\|_q^q &= \int_{\mathbb{R}} \left| \frac{b(z)\sum_{j=0}^{n}\binom{n+1}{j+1}(-1)^j\overline{b(z)}^j b^j(t)}{(t-\bar{z})^{n+1}} \right|^q (1-|b(t)|^2)\,dt\\
&\le \frac{C}{(\Im z)^{(n+1)q-2}} \int_{\mathbb{R}} \frac{1-|b(t)|}{|t-z|^2}\,dt.
\end{aligned} \tag{22.20}$$

If $|b(z)| < 1/2$, then we obviously have

$$\int_{\mathbb{R}} \frac{1-|b(t)|}{|t-z|^2}\,dt \le C\,\frac{1-|b(z)|}{\Im z},$$

and if $|b(z)| \geq 1/2$, using $1 - |b(t)| \leq \big|\log|b(t)|\big|$, we get

$$\begin{aligned}\Im z \int_{\mathbb{R}} \frac{1-|b(t)|}{|t-z|^2}\,dt &\leq \Im z \int_{\mathbb{R}} \frac{\big|\log|b(t)|\big|}{|t-z|^2}\,dt \\ &= \pi \log 1/|O_b(z)| \\ &\asymp 1 - |O_b(z)|,\end{aligned}$$

since $|O_b(z)| \geq |b(z)| \geq 1/2$. We recall that O_b is the outer part of b. Therefore, in any case we have

$$\int_{\mathbb{R}} \frac{1-|b(t)|}{|t-z|^2}\,dt \leq C\,\frac{1-|b(z)|}{\Im z}.$$

Hence, by (22.20), we get

$$\|\rho^{1/q}\mathfrak{K}^{\rho}_{z,n}\|_q^q \leq C\,\frac{1-|b(z)|}{(\Im z)^{(n+1)q-1}}.$$

Therefore,

$$\|\rho^{1/q}\mathfrak{K}^{\rho}_{z,n}\|_q^{-pn/(pn+1)} \geq C\,\frac{(\Im z)^n}{(1-|b(z)|)^{pn/(q(pn+1))}} \qquad (z \in \mathbb{C}_+).$$

The result now follows. □

22.4 Some auxiliary integral operators

In this section, we introduce some integral operators and study their boundedness. They will play a key role in proving the Bernstein-type inequality in Section 22.5. Throughout this section $\mu, \nu \in \mathcal{C}$, and $h : \overline{\mathbb{C}_+} \longrightarrow [0,\infty)$ is measurable with respect to these measures and satisfies

$$h(z) \geq A\,\Im z \qquad (z \in \overline{\mathbb{C}_+}),$$

where A is a positive constant. In the following, A is reserved for this purpose and is not used to denote any other constant. Put

$$E_z = \{w \in \overline{\mathbb{C}_+} : |w-z| \geq h(z)\}$$

and

$$E_z^c = \{w \in \overline{\mathbb{C}_+} : |w-z| < h(z)\}.$$

All integrals are taken in $\overline{\mathbb{C}_+}$, or on its subdomains.

Lemma 22.9 *There is a constant $C = C(\mu, A)$ such that*

$$\int_{E_z} \frac{d\mu(w)}{|w-z|^2} \leq \frac{C}{h(z)} \qquad (z \in \overline{\mathbb{C}_+}).$$

Proof Since $\mu \in \mathcal{C}$, we have

$$\|f\|_{L^2(\mu)} \leq C_\mu \|f\|_2 \qquad (f \in H^2(\mathbb{C}_+)).$$

Choosing $f = k_z$, the reproducing kernel at $z \in \mathbb{C}_+$, we get

$$\int_{\overline{\mathbb{C}_+}} \frac{d\mu(w)}{|w-\bar{z}|^2} \leq \frac{\pi C_\mu^2}{\Im z} \qquad (z \in \mathbb{C}_+).$$

Note that this estimation trivially holds if $z \in \mathbb{R}$. If $w \in E_z$, then

$$\frac{|w - \overline{(z+ih(z))}|}{|w-z|} \leq \frac{\Im z + h(z)}{h(z)} \leq \frac{A+1}{A}.$$

Hence,

$$\begin{aligned}
\int_{E_z} \frac{d\mu(w)}{|w-z|^2} &\leq \frac{(A+1)^2}{A^2} \int_{\overline{\mathbb{C}_+}} \frac{d\mu(w)}{|w - \overline{(z+ih(z))}|^2} \\
&\leq \frac{(A+1)^2}{A^2} \frac{\pi C_\mu^2}{\Im z + h(z)} \\
&\leq \frac{C}{h(z)} \qquad (z \in \overline{\mathbb{C}_+}).
\end{aligned}$$

This completes the proof. □

We recall that an operator $S\colon L^1(\nu) \longrightarrow L^1(\mu)$ is of weak type $(1,1)$ if there is a constant $C = C(\nu,\mu)$ such that

$$\mu(\{|Sf| > t\}) \leq C \frac{\|f\|_{L^1(\nu)}}{t} \qquad (t > 0),$$

for all $f \in L^1(\nu)$.

Lemma 22.10 *Put*

$$Sf(z) = \frac{1}{h(z)} \int_{E_z^c} f(w)\, d\nu(w).$$

Then $S : L^1(\nu) \longrightarrow L^1(\mu)$ *is of weak type* $(1,1)$*, and* $S : L^p(\nu) \longrightarrow L^p(\mu)$ *is bounded for any* $p > 1$.

Proof
Case $p = 1$. Let $t > 0$, and let

$$M(t) = \{z \in \overline{\mathbb{C}_+} : |Sf(z)| > t\}.$$

We can write $E_z^c = B(z, h(z))$, the disk of center z and radius $h(z)$. Hence, for each $z \in M(t)$,

$$\int_{B(z,h(z))} |f(w)|\, d\nu(w) \geq \left| \int_{E_z^c} f(w)\, d\nu(w) \right| \geq t h(z). \tag{22.21}$$

We use this estimation twice. First, it shows that

$$h(z) \le \frac{\|f\|_{L^1(\nu)}}{t} \qquad (z \in M(t)).$$

Therefore, we can apply the Vitali covering lemma to obtain a countable collection of disjoint disks $B(z_n, h(z_n))$, $n \ge 1$, such that

$$M(t) \subset \bigcup_{n=1}^{\infty} B(z_n, 5h(z_n)).$$

Since $\mu \in \mathcal{C}$ and $h(z) \ge A\,\Im z$, we have

$$\mu(B(z_n, 5h(z_n))) \le Ch(z_n) \qquad (n \ge 1).$$

Thus, by (22.21),

$$\begin{aligned} \mu(M(t)) &\le C \sum_{n=1}^{\infty} h(z_n) \\ &\le \frac{C}{t} \sum_{n=1}^{\infty} \int_{B(z_n, h(z_n))} |f(w)|\, d\nu(w) \\ &= \frac{C}{t} \int_{\bigcup_{n=1}^{\infty} B(z_n, h(z_n))} |f(w)|\, d\nu(w) \\ &\le C\, \frac{\|f\|_{L^1(\nu)}}{t}. \end{aligned}$$

Case $p = \infty$. Fix $z \in \overline{\mathbb{C}}_+$. Since $\nu \in \mathcal{C}$ and $h(z) \ge A\,\Im z$, we have

$$\nu(B(z, h(z))) \le Ch(z).$$

Thus,

$$|Sf(z)| \le \frac{1}{h(z)} \int_{E_z^c} |f(w)|\, d\nu(w) \le \frac{\nu(E_z')}{h(z)} \|f\|_{L^\infty(\nu)} \le C\, \|f\|_{L^\infty(\nu)},$$

or equivalently

$$\|Sf\|_{L^\infty(\mu)} \le C\, \|f\|_{L^\infty(\nu)}.$$

Case $1 < p < \infty$. This is an immediate consequence of the Marcinkiewicz interpolation theorem. □

Lemma 22.11 *Let k be a nonnegative $\nu \times \nu$-measurable function such that*

$$\int k(z, w) k(z, w')\, d\nu(z) \le c(k(w, w') + k(w', w) + K(w, w')),$$

where $K \ge 0$ is the kernel of a bounded operator on $L^2(\nu)$ and c is a constant. For $f \ge 0$, write

$$Af(z) = \int k(z, w) f(w)\, d\nu(w)$$

and

$$Bf(z) = \int K(z,w)f(w)\,d\nu(w).$$

Then $\|Af\|_{L^2(\nu)} \leq C\,\|f\|_{L^2(\nu)}$ *whenever* $\|Af\|_{L^2(\nu)} < \infty$*. The constant* C *depends on just* c *and* $\|B\|$*.*

Proof Assume $\|f\|_{L^2(\nu)} \leq 1$. Then

$$\begin{aligned}
&\|Af\|^2_{L^2(\nu)} \\
&= \int (Af(z))^2\,d\nu(z) \\
&= \int \left(\int k(z,w)f(w)\,d\nu(w)\right)\left(\int k(z,w')f(w')\,d\nu(w')\right)d\nu(z) \\
&= \iint f(w)f(w')\left(\int k(z,w)k(z,w')\,d\nu(z)\right)d\nu(w)\,d\nu(w') \\
&= c\iint f(w)f(w')[k(w,w') + k(w',w) + K(w,w')]\,d\nu(w)\,d\nu(w') \\
&= 2c\int f(w)Af(w)\,d\nu(w) + c\int f(w)Bf(w)\,d\mu(w) \\
&\leq 2c\,\|Af\|_{L^2(\nu)}\|f\|_{L^2(\nu)} + c\,\|Bf\|_{L^2(\nu)}\|f\|_{L^2(\nu)} \\
&\leq 2c\,\|Af\|_{L^2(\nu)} + c\,\|B\|.
\end{aligned}$$

Hence,

$$\|Af\|_{L^2(\nu)} \leq c + \sqrt{c^2 + c\|B\|},$$

provided that $\|Af\|_{L^2(\nu)} < \infty$. □

Corollary 22.12 *Let* k *be a nonnegative* $\nu\times\nu$*-measurable function such that*

$$\int k(z,w)k(z',w)\,d\nu(w) \leq C(k(z,z') + k(z',z) + K(z,z')),$$

where $K \geq 0$ *is the kernel of a bounded operator on* $L^2(\nu)$ *and* C *is a constant. Let* A *and* B *be as in Lemma 22.11. Then* $\|Af\|_{L^2(\nu)} \leq C\,\|f\|_{L^2(\nu)}$*, whenever* $\|Af\|_{L^2(\nu)} < \infty$*. The constant* C *depends on just* c *and* $\|B\|$*.*

Proof Consider $L(z,w) = \overline{k(w,z)} = k(w,z)$. Lemma 22.11 ensures that the operator with the kernel L is bounded on $L^2(\nu)$. But the adjoint of this operator is precisely the operator with kernel k. Hence, the result follows. □

Note that, in Lemma 22.11, or Corollary 22.12, if $\|Af\|_{L^2(\nu)} < \infty$ holds for a dense subset of $L^2(\nu)$, then we immediately conclude that A is a bounded operator on $L^2(\nu)$. It is usually straightforward to verify this fact for $f = \chi_E$, where E is a subset with $\nu(E) < \infty$. This would be enough for the boundedness of A.

We apply Corollary 22.12 to the operator

$$Tf(z) = h(z)\int_{E_z} \frac{f(w)}{|w-z|^2}\, d\nu(w) \tag{22.22}$$

defined on $L^2(\nu)$. Then we make an extra step and consider it as an operator from $L^p(\nu)$ to $L^p(\mu)$.

Lemma 22.13 *Let T be defined by (22.22). Then T is a bounded operator on $L^2(\nu)$.*

Proof To apply Corollary 22.12, write

$$I(z,z') = \int k(z,w)k(z',w)\, d\nu(w).$$

The kernel of T is

$$k(z,w) = h(z)|z-w|^{-2}\chi_{E_z}(w). \tag{22.23}$$

Thus, with $w, w' \in E_z$,

$$I(z,z') = h(z)h(z')\int_{E_z\cap E_{z'}} \frac{d\nu(w)}{|z-w|^2|z'-w|^2}.$$

Now, we consider three cases.

Case $|z-z'| \geq \max\{h(z), h(z')\}$. This assumption means that $z' \in E_z$ and $z \in E_{z'}$, or equivalently

$$\chi_{E_z}(z') = \chi_{E_{z'}}(z) = 1,$$

which implies that

$$k(z,z') = \frac{h(z)}{|z-z'|^2} \quad \text{and} \quad k(z',z) = \frac{h(z')}{|z-z'|^2}.$$

Since

$$\frac{1}{|z-w|^2\,|z'-w|^2} \leq \frac{2}{|z-z'|^2}\left(\frac{1}{|z-w|^2} + \frac{1}{|z'-w|^2}\right),$$

we have

$$I(z,z') \leq \frac{2h(z)h(z')}{|z-z'|^2}\left(\int_{E_z} \frac{d\nu(w)}{|z-w|^2} + \int_{E_{z'}} \frac{d\nu(w)}{|z'-w|^2}\right).$$

Hence, by Lemma 22.9,

$$I(z,z') \leq C\frac{h(z)+h(z')}{|z-z'|^2} = C(k(z,z') + k(z',z)).$$

Case $|z-z'|<h(z)$. Again, by Lemma 22.9, we have

$$\begin{aligned} I(z,z') &\le \frac{h(z')}{h(z)} \int_{E_z \cap E_{z'}} \frac{1}{|z'-w|^2}\, d\nu(w) \\ &\le \frac{h(z')}{h(z)} \int_{E_{z'}} \frac{1}{|z'-w|^2}\, d\nu(w) \\ &\le \frac{C}{h(z)} \le CK(z,z'), \end{aligned}$$

where

$$K(z,z') = \frac{1}{h(z)} \chi_{E_z^c}(z').$$

By Lemma 22.10, K is the kernel of a bounded operator on $L^2(\nu)$.

Case $|z-z'|<h(z')$. This is similar to the previous case and gives

$$I(z,z') \le CK(z',z).$$

Putting together the previous three cases gives

$$I(z,z') \le C(k(z,z') + k(z',z) + K(z,z') + K(z',z)). \tag{22.24}$$

Therefore, by Corollary 22.12, T is a bounded operator on $L^2(\nu)$ if we can show that $T\chi_A \in L^2(\nu)$. Here we may even assume that A is bounded.

First, by Lemma 22.9, we see that $T\chi_A$ is a bounded function on $L^2(\nu)$. Second, for R large enough, we have

$$\begin{aligned} T\chi_A(z) &= \int_{A\cap E_z} \frac{h(z)}{|z-w|^2}\, d\mu(w) \\ &\le \int_A \frac{d\mu(w)}{|z-w|} \\ &\le C\frac{\mu(A)}{|z+i|}. \end{aligned}$$

Since $1/(z+i) \in H^2$ and $\nu \in \mathcal{C}$, we conclude that $T\chi_A \in L^2(\nu)$. □

In the following, we consider T as an operator from $L^p(\nu)$ to $L^p(\mu)$. Since, in the proof, it also appears as an operator on $L^p(\nu)$, we denote the former by $\mathbf{T}$.

Lemma 22.14 *Let* $\mathbf{T}$ *be defined by* (22.22)*, and let* $2 \le p \le \infty$*. Then* $\mathbf{T}$ *is a bounded operator from* $L^p(\nu)$ *to* $L^p(\mu)$*.*

Proof The case $p=\infty$ was actually established in Lemma 22.9. We prove the case $p=2$, and then the rest follows from the Riesz–Torin interpolation theorem.

Since the kernel of $\mathbf{T}$ is given by (22.23), its adjoint $\mathbf{T}^*\colon L^2(\mu) \longrightarrow L^2(\nu)$ is

$$\mathbf{T}^* f(w) = \int \frac{h(z)f(z)}{|z-w|^2}\, \chi_{E_z}(w)\, d\mu(z).$$

Therefore, by direct calculation,

$$\begin{aligned}
&\|\mathbf{T}^* f\|^2_{L^2(\nu)} \\
&\quad = \int |\mathbf{T}^* f(w)|^2\, d\nu(w) \\
&\quad = \int \left(\int \frac{h(z)f(z)}{|z-w|^2}\, \chi_{E_z}(w)\, d\mu(z) \right) \\
&\qquad \times \left(\int \frac{h(z')\overline{f(z')}}{|z'-w|^2}\, \chi_{E_{z'}}(w)\, d\mu(z') \right) d\nu(w) \\
&\quad = \iint f(z)\overline{f(z')} \left(h(z)h(z') \int_{E_z \cap E_{z'}} \frac{d\nu(w)}{|z-w|^2\, |z'-w|^2} \right) d\mu(z)\, d\mu(z') \\
&\quad = \iint f(z)\overline{f(z')} I(z,z')\, d\mu(z)\, d\mu(z')
\end{aligned}$$

Hence, by (22.24), the last integral is bounded above by

$$\iint |f(z)|\, |f(z')|\, [k(z,z') + k(z',z) + K(z,z') + K(z',z)]\, d\mu(z)\, d\mu(z'),$$

which can be rewritten as

$$2 \int |f(z)|\, [|T_1 f(z)| + T_2 f(z)]\, d\mu(z),$$

with

$$T_1 f(z) = \int k(z',z)\, |f(z)|\, d\mu(z) \quad \text{and} \quad T_2 f(z) = \int 2(z',z)\, |f(z)|\, d\mu(z).$$

According to Lemma 22.13, both T_1 and T_2 are bounded operators on $L^2(\mu)$. Hence, the result follows. □

We now apply the above tools to provide two important results, which play an essential role in proving the Bernstein-type inequality in Section 22.5.

Theorem 22.15 *Let T be defined by (22.22), with $\nu = m$, and let $2 \leq p \leq \infty$. Then $T : L^1(\mathbb{R}) \longrightarrow L^1(\mu)$ is of weak type $(1,1)$, and $T : L^p(\mathbb{R}) \longrightarrow L^p(\mu)$ is bounded for any $p > 1$.*

Proof The case $2 \leq p \leq \infty$ was studied in Lemma 22.14, in even more generality. Hence, assume that $1 \leq p < 2$. Let $\widetilde{T} = T + S$, where S is the operator introduced in Lemma 22.10 (with $\nu = m$). More explicitly, $\widetilde{T}$ is given by

$$\widetilde{T}f(z) = h(z) \int_{E_z} \frac{f(t)}{|t-z|^2}\, dt + \frac{1}{h(z)} \int_{E_z^c} f(t)\, dt.$$

Thus, the kernel of $\widetilde{T}$ is

$$\tilde{k}(z,t) = \frac{h(z)}{|t-z|^2}\, \chi_{E_z}(t) + \frac{1}{h(z)}\, \chi_{E_z^c(t)}.$$

This kernel satisfies

$$|\tilde{k}(z,t) - \tilde{k}(z,t')| \le 3 \frac{|t-t'|}{|z-t|^2} \tag{22.25}$$

whenever $|t-t'| \le |z-t|/2$. For the justification of this fact, we need to consider four cases. In all cases, we use the following elementary estimations. Since $|z-t| \le |t'-t| + |z-t'| \le \frac{1}{2}|z-t| + |z-t'|$, we have $|z-t| \le 2|z-t'|$. Similarly, since $|z-t'| \le |t'-t| + |z-t| \le \frac{1}{2}|z-t| + |z-t|$, we have $|z-t'| \le \frac{3}{2}|z-t'|$. In brief,

$$\frac{1}{2} \le \frac{|z-t'|}{|z-t|} \le \frac{3}{2}.$$

Case $t, t' \in E_z$. We have

$$\begin{aligned}
|\tilde{k}(z,t) - \tilde{k}(z,t')| &= \left| \frac{h(z)}{|z-t|^2} - \frac{h(z)}{|z-t'|^2} \right| \\
&= \frac{\big||z-t|^2 - |z-t|^2\big|}{|z-t|^2\, |z-t'|^2}\, h(z) \\
&\le \frac{\big||z-t| - |z-t|\big|}{|z-t|^2}\, \frac{|z-t| + |z-t'|}{|z-t'|}\, \frac{h(z)}{|z-t'|} \\
&\le 3 \frac{|t-t'|}{|z-t|^2}.
\end{aligned}$$

Case $t, t' \in E_z^c$. In this case, $\tilde{k}(z,t) = \tilde{k}(z,t') = 1/h(z)$ and the estimation is trivial.

Case $t \in E_z,\ t' \in E_z^c$. We have

$$\begin{aligned}
|\tilde{k}(z,t) - \tilde{k}(z,t')| &= \left| \frac{h(z)}{|z-t|^2} - \frac{1}{h(z)} \right| \\
&= \frac{|z-t|^2 - h^2(z)}{|z-t|^2\, h(z)} \\
&\le \frac{|z-t|^2 - |z-t'|^2(z)}{|z-t|^2\, h(z)}
\end{aligned}$$

$$\begin{aligned}&\leq \frac{\big||z-t|-|z-t|\big|}{|z-t|^2}\,\frac{|z-t|+|z-t'|}{h(z)}\\&\leq 3\frac{|t-t'|}{|z-t|^2}.\end{aligned}$$

Case $t\in E_z^c,\ t'\in E_z$. We have

$$\begin{aligned}|\tilde{k}(z,t)-\tilde{k}(z,t')| &= \left|\frac{1}{h(z)}-\frac{h(z)}{|z-t'|^2}\right|\\&=\frac{|z-t'|^2-h^2(z)}{|z-t'|^2\,h(z)}\\&\leq\frac{|z-t'|^2-|z-t|^2(z)}{|z-t|^2\,h(z)}\\&\leq \frac{\big||z-t|-|z-t|\big|}{|z-t|^2}\,\frac{|z-t|+|z-t'|}{h(z)}\\&\leq \frac{5}{2}\,\frac{|t-t'|}{|z-t|^2}.\end{aligned}$$

Now, any kernel that satisfies (22.25) is of weak type $(1,1)$ by a classic result of Calderon–Zygmund theory, which is done via the so-called Calderon–Zygmund decomposition. □

For the next result, we consider operators with diagonal kernels. Let $k(z,w)$ be a $\mu\times\nu$-measurable function, where, as before $\mu,\nu\in\mathcal{C}$. Let $p>1$, and let q be its conjugate exponent. For $z\in\overline{\mathbb{C}_+}$, put

$$\Delta_z=\{w\in\overline{\mathbb{C}_+}:|w-z|<\|k(z,\cdot)\|_{L^q(\nu)}^{-p}\}.$$

We recall the convention $\|k(z,\cdot)\|_{L^q(\nu)}^{-p}=0$ whenever $k(z,\cdot)\notin L^q(\nu)$. We also assume that

$$\|k(z,\cdot)\|_{L^q(\nu)}^{-p}\geq A\,\Im\,z.$$

Then define

$$T_pf(z)=\int_{\Delta_z}k(z,w)f(w)\,d\nu(w).$$

Theorem 22.16 *The operator $T_p:L^p(\nu)\longrightarrow L^p(\mu)$ is of weak type (p,p), and $T_p:L^r(\nu)\longrightarrow L^r(\mu)$ is bounded for any $r>p$.*

Proof The proof is similar to that for Lemma 22.10. We establish the cases $r=\infty$ and $r=p$, and then appeal to the Marcinkiewicz interpolation theorem for $p<r<\infty$.

Case $r = \infty$. To show that $T_p : L^\infty(\nu) \longrightarrow L^\infty(\mu)$ is bounded, let $f \in L^\infty(\nu)$. Then, by Hölder's inequality,

$$\begin{aligned}|T_p f(z)| &= \left| \int_{\Delta_z} k(z,w) f(w)\, d\nu(w) \right| \\ &\leq \|f\|_{L^\infty(\nu)} \, \|k(z,\cdot)\|_{L^q(\nu)} \, (\nu(\Delta_z))^{1/p}.\end{aligned}$$

But, since $\nu \in \mathcal{C}$ and $\|k(z,\cdot)\|^{-p}_{L^q(\nu)} \geq A\, \Im z$,

$$\nu(\Delta_z) = \nu(\{w \in \overline{\mathbb{C}_+} : |w - z| < \|k(z,\cdot)\|^{-p}_{L^q(\nu)}\}) \leq C\, \|k(z,\cdot)\|^{-p}_{L^q(\nu)}.$$

Hence,

$$|T_p f(z)| \leq C\, \|f\|_{L^\infty(\nu)}.$$

In other words, $T_p : L^\infty(\nu) \longrightarrow L^\infty(\mu)$ is bounded.

Case $r = p$. Let

$$M(t) = \{z \in \overline{\mathbb{C}_+} : |T_p(z)| > t\}.$$

We cover $M(t)$ with balls of the form Δ_z, where $z \in M(t)$. Then by the Vitali covering lemma, we can choose a sequence of disjoint balls $\Delta_{z_n}(p)$ such that

$$M(t) \subset \bigcup_{n=1}^{\infty} \widetilde{\Delta}_{z_n}, \tag{22.26}$$

where

$$\widetilde{\Delta}_z = \{w \in \overline{\mathbb{C}_+} : |w - z| < 5\, \|k(z,\cdot)\|^{-p}_{L^q(\nu)}\}.$$

Since $\mu \in \mathcal{C}$ and $\|k(z,\cdot)\|^{-p}_{L^q(\nu)} \geq A\, \Im z$, we have

$$\mu(\widetilde{\Delta}_z) \leq C_1\, \|k(z,\cdot)\|^{-p}_{L^q(\nu)}. \tag{22.27}$$

Also, by Hölder's inequality,

$$|T_p f(z)| \leq \|k(z,\cdot)\|_{L^q(\nu)} \left(\int_{\Delta_z} |f(w)|^p \, d\nu(w) \right)^{1/p}.$$

Thus,

$$\|k(z_n,\cdot)\|_{L^q(\nu)} \left(\int_{\Delta_{z_n}} |f(w)|^p \, d\nu(w) \right)^{1/p} > t. \tag{22.28}$$

Therefore, by (22.26) and (22.27),

$$\mu(M(t)) \leq C_1 \sum_{n=1}^{\infty} \|k(z_n,\cdot)\|^{-p}_{L^q(\nu)},$$

and by (22.28), we conclude that

$$\mu(M(t)) \leq C_1 \sum_{n=1}^{\infty} t^{-p} \int_{\Delta_{z_n}} |f(w)|^p \, d\nu(w) \leq C_1 t^{-p} \, \|f\|_{L^p(\nu)}^p.$$

Thus, T_p is of weak type (p, p). □

22.5 The operator $T_{p,n}$

Representation formulas for derivatives turn the study of differentiation in $\mathcal{H}(b)$ spaces into the study of certain integral operators. More specifically, we consider the mapping $T_{p,n}$ defined on $\mathcal{H}(b)$ by

$$(T_{p,n} f)(z) = f^{(n)}(z) w_{p,n}(z) \qquad (f \in \mathcal{H}(b))$$

for $n \geq 0$ and $1 < p \leq 2$. We discuss this mapping in this section.

Let us recall that a Borel measure μ in the closed upper half-plane $\overline{\mathbb{C}_+}$ is said to be a Carleson measure if there is a constant $C_\mu > 0$ such that

$$\mu(S(x, h)) \leq C_\mu h \tag{22.29}$$

for all squares $S(x, h) = [x, x + h] \times [0, h]$, $x \in \mathbb{R}$, $h > 0$, with the lower side on the real axis. We denote the class of Carleson measures by $\mathcal{C}$. Recall that, according to a classic theorem of Carleson, $\mu \in \mathcal{C}$ if and only if $H^p(\mathbb{C}_+) \subset L^p(\mu)$ for some (all) $p > 0$; see Section 5.6.

The following Bernstein-type inequality is essential for some future applications.

Theorem 22.17 *Let $\mu \in \mathcal{C}$, let $n \geq 0$ and let $1 < p \leq 2$. If $1 < p < 2$, then $T_{p,n}$ is a bounded operator from $\mathcal{H}(b)$ to $L^2(\mu)$, that is, there is a constant $C = C(\mu, p, n) > 0$ such that*

$$\|f^{(n)} w_{p,n}\|_{L^2(\mu)} \leq C \, \|f\|_b \qquad (f \in \mathcal{H}(b)). \tag{22.30}$$

If $p = 2$, then $T_{2,n}$ is of weak type $(2, 2)$ as an operator from $\mathcal{H}(b)$ to $L^2(\mu)$.

Proof Put $h_k(z) = (w_{p,n}^{(k)}(z))^{1/n}$, $k = 1, 2$. Note that, by Lemma 22.8,

$$h_k(z) \geq A \, \Im z \qquad (z \in \mathbb{C}_+, \ k = 1, 2). \tag{22.31}$$

The definition of $w_{p,n}$ also shows that

$$w_{p,n}(z) \leq h_k^n(z) \qquad (z \in \mathbb{C}_+, \ k = 1, 2).$$

Fix $f \in \mathcal{H}(b)$. We multiply both sides of the representation (22.17) by $w_{p,n}(z)$ and then split each of the two integrals on the right-hand side into two parts, i.e.

$$\frac{2\pi i}{n!} f^{(n)}(z) w_{p,n}(z) = I_1 f(z) + I_2 f(z) + I_3 g(z) + I_4 g(z),$$

where

$$I_1 f(z) = w_{p,n}(z) \int_{|t-z| \geq h_1(z)} f(t) \overline{(k_z^b(t))^{n+1}}\, dt,$$

$$I_2 f(z) = w_{p,n}(z) \int_{|t-z| < h_1(z)} f(t) \overline{(k_z^b(t))^{n+1}}\, dt,$$

$$I_3 g(z) = w_{p,n}(z) \int_{|t-z| \geq h_2(z)} g(t) \rho(t) \overline{\mathfrak{K}_{z,n}^{\rho}(t)}\, dt,$$

$$I_4 g(z) = w_{p,n}(z) \int_{|t-z| < h_2(z)} g(t) \rho(t) \overline{\mathfrak{K}_{z,n}^{\rho}(t)}\, dt.$$

Now, we treat each integral separately.

Case I_1. We may even assume that $f \in L^2(\mathbb{R})$. By (22.31),

$$\begin{aligned} |I_1 f(z)| &\leq C h_1^n(z) \int_{|t-z| \geq h_1(z)} \frac{|f(t)|}{|t-z|^{n+1}}\, dt \\ &\leq C h_1(z) \int_{|t-z| \geq h_1(z)} \frac{|f(t)|}{|t-z|^2}\, dt. \end{aligned}$$

By Theorem 22.15, we see that $I_1 : L^2(\mathbb{R}) \longrightarrow L^2(\mu)$ is a bounded operator.

Case I_3. We may even assume that $g \in L^2(\rho)$. By (22.31),

$$\begin{aligned} |I_3 g(z)| &\leq C h_2^n(z) \int_{|t-z| \geq h_2(z)} \frac{|g(t)| \rho^{1/2}(t)}{|t-z|^{n+1}}\, dt \\ &\leq C h_2(z) \int_{|t-z| \geq h_2(z)} \frac{|g(t)| \rho^{1/2}(t)}{|t-z|^2}\, dt. \end{aligned}$$

By Theorem 22.15, we see that $I_3 : L^2(\rho) \longrightarrow L^2(\mu)$ is a bounded operator.

Case I_2. To estimate the integral $I_2 f$, put

$$K(z,t) = h_1^n(z)\, |(k_z^b)^{n+1}(t)|.$$

Then

$$\begin{aligned} \|K(z,\cdot)\|_q^{-p} &= (h_1(z))^{-pn} \|(k_z^b)^{n+1}\|_q^{-p} \\ &= (h_1(z))^{-pn} (w_{p,n}^{(1)}(z))^{(pn+1)/n} = h_1(z). \end{aligned}$$

Thus,

$$\begin{aligned} |I_2 f(z)| &\leq h_1^n(z) \int_{|t-z| < h_1(z)} |f(t)|\, |(k_z^b)^{n+1}(t)|\, dt \\ &= \int_{|t-z| < \|K(z,\cdot)\|_q^{-p}} |f(t)| K(z,t)\, dt. \end{aligned}$$

Since $\|K(z,\cdot)\|_q^{-p} = h_1(z) \geq A\,\Im z$, we may apply Theorem 22.16. Therefore, the operator I_2 is of weak type $(2,2)$ as an operator from $L^2(\mathbb{R})$ to $L^2(\mu)$ if $p=2$ and it is a bounded operator from $L^2(\mathbb{R})$ to $L^2(\mu)$ if $1<p<2$.

Case I_4. We use the same techniques. Put

$$\kappa(z,t) = \frac{\rho^{1/q}(t)|\mathfrak{K}_{z,n}^{\rho}(t)|}{\|\rho^{1/q}\mathfrak{K}_{z,n}^{\rho}\|_q^{pn/(pn+1)}}.$$

In other words, $\kappa(z,t) = w_{p,n}^{(2)}(z)\rho^{1/q}(t)|\mathfrak{K}_{z,n}^{\rho}(t)|$. Thus,

$$\begin{aligned}|I_4 g(z)| &\leq w_{p,n}^{(2)}(z)\int_{|t-z|<h_2(z)} |g(t)|\rho(t)|\mathfrak{K}_{z,n}^{\rho}(t)|\,dt\\ &= \int_{|t-z|<h_2(z)} |g(t)|\rho^{1/p}(t)\kappa(z,t)\,dt.\end{aligned}$$

But,

$$\|\kappa(z,\cdot)\|_q^{-p} = (w_{p,n}^{(2)}(z))^{-p}\|\rho^{1/q}\mathfrak{K}_{z,n}^{\rho}\|_q^{-p} = h_2(z).$$

Hence, we get

$$|I_4 g(z)| \leq \int_{|t-z|<\|\kappa(z,\cdot)\|_q^{-p}} |g(t)|\rho^{1/p}(t)\kappa(z,t)\,dt.$$

Since $p\leq 2$ and $\rho(t)\leq 1$, we have

$$|I_4 g(z)| \leq \int_{|t-z|<\|\kappa(z,\cdot)\|_q^{-p}} |g(t)|\rho^{1/2}(t)\kappa(z,t)\,dt,$$

and since $\|\kappa(z,\cdot)\|_q^{-p} = h_2(z) \geq A\,\Im z$, we may again apply Theorem 22.16. Therefore, the operator I_4 is of weak type $(2,2)$ as an operator from $L^2(\rho)$ to $L^2(\mu)$ if $p=2$ and it is a bounded operator from $L^2(\rho)$ to $L^2(\mu)$ if $1<p<2$.

To conclude it remains to note that

$$\|f\|_b^2 = \|f\|_2^2 + \|g\|_\rho^2,$$

which implies that the operators $f \longmapsto f$ from $\mathcal{H}(b)$ to $H^2(\mathbb{C}_+)$ and $f \longmapsto g$ from $\mathcal{H}(b)$ to $L^2(\rho)$ are contractions. □

We emphasize that the constants in the Bernstein-type inequalities corresponding to Theorem 22.17 depend only on p, n and the Carleson constant C_μ of the measure μ, but not on b. The properties of b are hidden in the weight $w_{p,n}$ on the left-hand side of (22.30).

22.6 Distances to the level sets

To apply Theorem 22.17, one should have effective estimates for the weight $w_{p,n}$. In this section we relate the weight $w_{p,n}$ to the distances to the level sets of $|b|$.

Denote by $\sigma(b)$ the boundary spectrum of b, i.e.

$$\sigma(b) = \left\{ x \in \mathbb{R} : \liminf_{\substack{z \to x \\ z \in \mathbb{C}_+}} |b(z)| < 1 \right\}.$$

Considering the decomposition $b = BI_\mu O_b$, we see that $\mathrm{Clos}_{\mathbb{R}}\, \sigma(b)$ is the smallest closed subset of $\mathbb{R}$ containing the limit points of the zeros of the Blaschke product B and the supports of the measures μ and $\log|b(t)|\, dt$. We have already seen that b and all element of $\mathcal{H}(b)$ have analytic continuation through any interval from the open set $\mathbb{R} \setminus \mathrm{Clos}_{\mathbb{R}}\, \sigma(b)$.

For $\varepsilon \in (0,1)$, we put

$$\Omega(b,\varepsilon) = \{z \in \mathbb{C}_+ : |b(z)| < \varepsilon\}$$

and

$$\widetilde{\Omega}(b,\varepsilon) = \sigma(b) \cup \Omega(b,\varepsilon).$$

Finally, for $x \in \mathbb{R}$, we introduce the following three distances:

$$\begin{aligned} d_0(x) &= \mathrm{dist}(x, \sigma(b)), \\ d_\varepsilon(x) &= \mathrm{dist}(x, \Omega(b,\varepsilon)), \\ \tilde{d}_\varepsilon(x) &= \mathrm{dist}(x, \widetilde{\Omega}(b,\varepsilon)). \end{aligned}$$

Note that, whenever $b = \Theta$ is an inner function, for all $x \in \sigma(\Theta)$, we have

$$\liminf_{\substack{z \to x \\ z \in \mathbb{C}_+}} |\Theta(z)| = 0,$$

and thus $d_\varepsilon(t) = \tilde{d}_\varepsilon(t)$, $t \in \mathbb{R}$. However, for an arbitrary function b in the unit ball of $H^\infty(\mathbb{C}_+)$, we have to distinguish between the two distance functions d_ε and $\tilde{d}_\varepsilon$.

Lemma 22.18 *Let $0 < \varepsilon < 1$. Then there exists a positive constant $C = C(\varepsilon)$ such that*

$$|b'(x)| \le \frac{C}{\tilde{d}_\varepsilon(x)}$$

for all $x \in \mathbb{R} \setminus \sigma(b)$.

Proof Let $b = I_b O_b$ be the inner–outer factorization of b. Since, by Corollary 21.11,

$$|b'(x)| = |I_b'(x)| + |O_b'(x)| \qquad (x \in \mathbb{R} \setminus \sigma(b)),$$

we may treat the inner and outer cases separately.

Case 1, b outer. In this case

$$|b'(x)| = \frac{1}{\pi}\int_{\mathbb{R}} \frac{\big|\log|b(t)|\big|}{|t-x|^2}\,dt \qquad (x \in \mathbb{R}\setminus\sigma(b)).$$

Fix $x \in \mathbb{R}\setminus\sigma(b)$ and $0 < y \le d_0(x)$. Then, for $z = x+iy$,

$$\begin{aligned}\log\frac{1}{|b(z)|} &= \frac{y}{\pi}\int_{\mathbb{R}} \frac{\big|\log|b(t)|\big|}{|t-z|^2}\,dt\\ &= \frac{y}{\pi}\int_{t\in\sigma(b)} \frac{\big|\log|b(t)|\big|}{|t-z|^2}\,dt\\ &= \frac{y}{\pi}\int_{|t-x|\ge d_0(x)} \frac{\big|\log|b(t)|\big|}{|t-z|^2}\,dt.\end{aligned}$$

Since $|t-z| \le |t-x| + y \le 2|t-x|$ whenever $|t-x| \ge d_0(x)$, we have

$$\log\frac{1}{|b(z)|} \ge \frac{y}{4\pi}\int_{|t-x|\ge d_0(x)} \frac{\big|\log|b(t)|\big|}{|t-x|^2}\,dt = \frac{y|b'(x)|}{4}.$$

Hence, we obtain the essential estimation

$$|b(x+iy)| \le \exp(-y|b'(x)|/4) \qquad (x \in \mathbb{R}\setminus\sigma(b),\ 0 < y \le d_0(x)),$$

which we rewrite as

$$|b'(x)| \le \frac{4}{y}\log\frac{1}{|b(x+iy)|} \qquad (x \in \mathbb{R}\setminus\sigma(b),\ 0 < y \le d_0(x)). \tag{22.32}$$

If $|b(x+id_0(x))| \ge \varepsilon$, then we have

$$|b'(x)| \le \frac{4}{d_0(x)}\log\frac{1}{\varepsilon} \le \frac{4\log(1/\varepsilon)}{\tilde{d}_\varepsilon(x)},$$

since $\tilde{d}_\varepsilon(x) \le d_0(x)$. But, if $|b(x+id_0(x))| < \varepsilon$, since $|b(x)| = 1$, by continuity, there is $y_\varepsilon \in (0, d_0(x))$ such that $|b(x+iy_\varepsilon)| = \varepsilon$. The existence of this point also reveals that $\tilde{d}_\varepsilon(x) \le y_\varepsilon$. Hence, by (22.32), we get

$$|b'(x)| \le \frac{4}{y_\varepsilon}\log\frac{1}{\varepsilon} \le \frac{4\log(1/\varepsilon)}{\tilde{d}_\varepsilon(x)}.$$

Case 2, b inner. First assume that b is a Blaschke product, i.e.

$$b(z) = \prod_k e^{i\alpha_k}\frac{z-z_k}{z-\bar{z}_k}.$$

Then

$$|b'(x)| = \sum_k \frac{2\,\Im z_k}{|x-\bar{z}_k|^2}$$

and, since $-\log(1-t) \geq t$,

$$\begin{aligned}
\log \frac{1}{|b(z)|^2} &= \sum_k -\log\left(1 - \frac{4yy_k}{|z-\bar{z}_k|^2}\right) \\
&\geq \sum_k \frac{4yy_k}{|z-\bar{z}_k|^2}.
\end{aligned}$$

For $z = x+iy$, with $0 < y \leq \tilde{d}_\varepsilon(x)/2$, we have

$$\begin{aligned}
\frac{|x-\bar{z}_k|}{|z-\bar{z}_k|} &\geq \frac{|x-\bar{z}_k|}{|z-x|+|x-\bar{z}_k|} \\
&\geq \frac{|x-\bar{z}_k|}{|x-\bar{z}_k|+\tilde{d}_\varepsilon(x)/2} \\
&\geq \frac{\tilde{d}_\varepsilon(x)}{\tilde{d}_\varepsilon(x)+\tilde{d}_\varepsilon(x)/2} = \frac{2}{3}.
\end{aligned}$$

Hence,

$$\log \frac{1}{|b(z)|} \geq 2y \sum_k \frac{4}{9}\,\frac{y_k}{|x-\bar{z}_k|^2} = \frac{8y}{9}|b'(x)|.$$

This is similar to the formula (22.32) and the rest of the argument is the same.

For an arbitrary inner function b, let

$$b_\alpha(z) = \frac{b(z)-\alpha}{1-\bar{\alpha}b(z)}.$$

Clearly $b_\alpha(z) \longrightarrow b(z)$ and $b'_\alpha(x) \longrightarrow b'(x)$ as $\alpha \longrightarrow 0$. But, as the proof of Corollary 21.14 shows, apart from a certain small exceptional set of values for α, b_α is a Blaschke product. According to the preceding paragraph, for these values of α, the above estimation holds for b_α. Then, by letting $\alpha \longrightarrow 0$, we see that it also holds for b. □

Lemma 22.19 *For each $p > 1$, $n \geq 1$ and $\varepsilon \in (0,1)$, there exists a constant $C = C(\varepsilon,p,n) > 0$ such that*

$$(\tilde{d}_\varepsilon(x))^n \leq C w_{p,n}(x+iy) \tag{22.33}$$

for all $x \in \mathbb{R}$ and $y \geq 0$.

Proof If $x \in \sigma(b)$ then $\tilde{d}_\varepsilon(x) = 0$, and thus (22.33) is trivial. From now on, fix $z = x+iy$, with $x \in \mathbb{R}\backslash\sigma(b)$ and $y \geq 0$. Since $(n+1)q-1 = q(np+1)/p$, the estimate (22.33) is equivalent to

$$\int_{\mathbb{R}} \left| \frac{1-\overline{b(z)}b(t)}{t-\bar{z}} \right|^{(n+1)q} dt \leq \frac{C}{(\tilde{d}_\varepsilon(x))^{(n+1)q-1}} \tag{22.34}$$

and

$$\int_{\mathbb{R}} \left| \frac{\overline{b(z)} \sum_{j=0}^{n} \binom{n+1}{j+1} (-1)^j \overline{b(z)}^j b^j(t)}{(t-\bar{z})^{n+1}} \right|^q \rho(t)\, dt \le \frac{C}{(\tilde{d}_\varepsilon(x))^{(n+1)q-1}}. \tag{22.35}$$

If $|t-x| < \tilde{d}_\varepsilon(x)$, then t remains in $\mathbb{R} \setminus \sigma(b)$, and thus $\rho(t) = 0$. Therefore, in (22.35) the integration is actually over values of t with $|t-x| \ge \tilde{d}_\varepsilon(x)$. Therefore, the inequality (22.35) follows rather trivially.

To prove (22.34), we estimate separately the integrals over the complementary sets

$$E_x = \{t : |t-x| \le \tilde{d}_\varepsilon(x)/2\} \quad \text{and} \quad F_x = \{t : |t-x| > \tilde{d}_\varepsilon(x)/2\}.$$

As in the previous paragraph, the estimation

$$\int_{F_x} \left| \frac{1-\overline{b(z)}b(t)}{t-\bar{z}} \right|^{(n+1)q} dt \le \frac{C}{(\tilde{d}_\varepsilon(x))^{(n+1)q-1}}$$

trivially holds. Since $|b(t)| = 1$ if $t \in E_x$, for the integral over E_x, we have

$$\begin{aligned} \int_{E_x} \left| \frac{1-\overline{b(z)}b(t)}{t-\bar{z}} \right|^{(n+1)q} dt &= \int_{E_x} \left| \frac{b(t)-b(z)}{t-z} \right|^{(n+1)q} dt \\ &\le \tilde{d}_\varepsilon(x) \max_u |b'(u)|^{(n+1)q}, \end{aligned}$$

where the maximum is taken over the line segment $u \in [t,z]$ with $t \in E_x$. For any such u, we have $|\Re u - x| \le \tilde{d}_\varepsilon(x)/2$, i.e. $\Re u \in E_x$. By Lemma 22.2, $|b'(u)| \le |b'(\Re u)|$, and hence

$$\int_{E_x} \left| \frac{1-\overline{b(z)}b(t)}{t-\bar{z}} \right|^{(n+1)q} dt \le \tilde{d}_\varepsilon(x) \max_{s \in E_x} |b'(s)|^{(n+1)q}.$$

But, according to Lemma 22.18,

$$|b'(s)| \le \frac{C_1}{\tilde{d}_\varepsilon(s)} \le \frac{C_2}{\tilde{d}_\varepsilon(x)},$$

whenever $s \in E_x$, and this leads to the required estimate. □

Corollary 22.20 *For each* $\varepsilon \in (0,1)$ *and* $n \ge 1$, *there exists a constant* $C = C(\varepsilon, n)$ *such that*

$$\|f^{(n)} \tilde{d}_\varepsilon^n\|_2 \le C\, \|f\|_b \qquad (f \in \mathcal{H}(b)).$$

Proof The statement follows immediately from Lemma 22.19 and Theorem 22.17. □

We conclude this section with a corollary concerning regularity on the boundary for functions in $\mathcal{H}(b)$. This technical result will be used is studying Carleson measures.

Corollary 22.21 *Let $I = [x_0, x_0 + y_0]$ be a bounded closed interval on $\mathbb{R}$ such that*

$$\int_I \frac{dx}{(w_p(x))^2} < \infty \tag{22.36}$$

for some $1 < p < 2$. Then the following hold.

(i) *We have $(x_0, x_0 + y_0) \cap \sigma(b) = \emptyset$. In particular, each $f \in \mathcal{H}(b)$ is differentiable on the open interval $(x_0, x_0 + y_0)$.*
(ii) *The function b is continuous on the Carleson square $S(I) = [x_0, x_0 + y_0] \times [0, y_0]$.*

Proof (i) According to Theorem 22.17, there is a constant $C > 0$ such that

$$\int_{\mathbb{R}} |f'(x) w_p(x)|^2 \, dx \le C \, \|f\|_b^2 \qquad (f \in \mathcal{H}(b)).$$

Then, using (22.36) and the Cauchy–Schwarz inequality, we get $f' \in L^1(I)$ for any $f \in \mathcal{H}(b)$. Now choose $z \in \mathbb{C}_+$ such that $b(z) \neq 0$ and take $f = k_z^b$. Thus $(x - \bar{z}) f(x) = 1 - \overline{b(z)}\, b(x)$, and differentiation with respect to x gives

$$f'(x) + (x - \bar{z}) f(x) = -\overline{b(z)} b'(x).$$

Thus, we conclude that

$$\int_{x_0}^{x_0 + y_0} |b'(x)| \, dx < \infty. \tag{22.37}$$

We now appeal to the formula (22.5) for $|b'(x)|$ to deduce that (22.37) implies $(x_0, x_0 + y_0)$ is free from the points of $\sigma(b)$. In fact, this is obvious for the outer and the singular inner factors since $\int_I (x - t)^{-2} \, dt = \infty$ for any $x \in I$. Hence, $\log |b| \equiv 0$ on this interval and also the singular measure μ lives outside this interval. For Blaschke product with zeros z_k, if there is an accumulation point inside $(x_0, x_0 + y_0)$, then, for sufficiently large values of k,

$$\int_{x_0}^{x_0 + y_0} \frac{2 \Im z_k}{|x - z_k|^2} \, dx \ge \pi,$$

and so the integral in (22.37) diverges. Hence, the accumulation points of the zeros are also outside this interval. In short, $(x_0, x_0 + y_0) \cap \sigma(b) = \emptyset$.

(ii) By part (i), b is continuous on $S(I)$ except possibly at the end points x_0 and $x_0 + y_0$. It remains to show that b is continuous at x_0 and $x_0 + y_0$. Fix $x_1 \in (x_0, x_0 + y_0)$ and define $b(x_0)$ via the formula

$$b(x_0) = b(x_1) - \int_{x_0}^{x_1} b'(x) \, dx.$$

Since b is differentiable on the interval $(x_0, x_0 + y_0)$, this definition does not depend on the choice of x_1 and we see from (22.37) that $b(x)$ tends to $b(x_0)$

as $x \longrightarrow x_0$ along I. Now let $z = x + iy \in S(I)$, with $x \in [x_0, x_0 + y_0/2)$, $y \in (0, y_0/2)$. Write $b(z) - b(x_0) = b(x+iy) - b(x+y) + b(x+y) - b(x_0)$. Using the continuity of b at x_0 along I, we have $b(x + y) - b(x_0) \longrightarrow 0$, as $x \longrightarrow x_0$ and $y \longrightarrow 0$. Moreover, since b is analytic on $\mathbb{C}_+ \cup (x_0, x_0 + y_0)$, we can write

$$b(x+y) - b(x+iy) = (1-i)y \int_0^1 b'(t(x+y) + (1-t)(x+iy))\, dt.$$

Applying Lemma 22.2, we get

$$|b(x+y) - b(x+iy)| \le \sqrt{2} \int_x^{x+y} |b'(u)|\, du.$$

According to (22.37), we deduce that $b(x+y) - b(x+iy) \longrightarrow 0$, as $x \longrightarrow x_0$ and $y \longrightarrow 0$. Therefore, $b(z) \longrightarrow b(x_0)$, as $z \longrightarrow x_0$, $z \in S(I)$. □

22.7 Carleson-type embedding theorems

Weighted Bernstein-type inequalities are an efficient tool for the study of the so-called Carleson-type embedding theorems for $\mathcal{H}(b)$ spaces, and in particular for model spaces K^p_Θ. In this section, we develop this method.

A Carleson measure for the closed upper half-plane is called a *vanishing Carleson measure* if $\mu(S(x,h))/h \longrightarrow 0$ whenever $h \longrightarrow 0$ or $\operatorname{dist}(S(x,h), 0) \longrightarrow \infty$. It is not difficult to see that this is equivalent to

$$\int_{\overline{\mathbb{C}_+}} \frac{\Im z}{|w - \bar{z}|^2}\, d\mu(w) \longrightarrow 0,$$

whenever either $\Im z \longrightarrow 0$ or $|z| \longrightarrow \infty$. We recall that the embedding $H^p(\mathbb{C}_+) \subset L^p(\mu)$ is compact if and only if μ is a vanishing Carleson measure.

In what follows, if I is a segment in $\mathbb{C}$, the restriction of the (one-dimensional) Lebesgue measure to this segment will be denoted by m_I.

By a *closed square* in $\overline{\mathbb{C}}_+$, we mean a set of the form

$$S(x_0, y_0, h) = \{x + iy : x_0 \le x \le x_0 + h,\ y_0 \le y \le y_0 + h\}, \qquad (22.38)$$

where $x_0 \in \mathbb{R}$, $y_0 \ge 0$ and $h > 0$. By the *lower side* of the closed square $S(x_0, y_0, h)$, we mean the interval $\{x + iy_0 : x_0 \le x \le x_0 + h\}$.

Theorem 22.22 *Let $\{S_k\}_{k\ge1}$ be a sequence of closed squares in $\overline{\mathbb{C}_+}$, and let I_k denote the closed lower side of the square S_k. Assume that*

$$\sum_{k=1}^{\infty} m_{I_k} \in \mathcal{C} \qquad (22.39)$$

and that, for some p with $1 < p < 2$,

$$\sup_{k\geq 1,\, y\geq 0} |I_k| \int_{S_k\cap\{\Im z=y\}} \frac{|du|}{w_p^2(u)} < \infty. \tag{22.40}$$

Let μ be a Borel measure with $\operatorname{supp}\mu \subset \bigcup_k S_k$. *Then the following hold.*

(i) *If*

$$\mu(S_k) = O(|I_k|) \qquad (k \longrightarrow \infty),$$

then $\mathcal{H}(b) \subset L^2(\mu)$.

(ii) *If, moreover,*

$$\mu(S_k) = o(|I_k|) \qquad (k \longrightarrow \infty)$$

and

$$I_k \cap \operatorname{Clos}_{\mathbb{R}} \sigma(b) = \emptyset \qquad (k \geq 1),$$

then the embedding $\mathcal{H}(b) \subset L^2(\mu)$ is compact.

Proof (i) The idea of the proof is to replace the measure μ with some Carleson measure ν, and to estimate the difference between the norms $\|f\|_{L^2(\mu)}$ and $\|f\|_{L^2(\nu)}$ using the Bernstein-type inequality of Section 22.5.

It follows from (22.40) and Corollary 22.21(ii) that the set of functions $f \in \mathcal{H}(b)$ that are continuous on each of S_k is dense in $\mathcal{H}(b)$. For example, we may take all the reproducing kernels k_z^b, with $z \in \mathbb{C}^+$. Thus, it is sufficient to prove the estimate $\|f\|_{L^2(\mu)} \leq C\,\|f\|_b$ only for those $f \in \mathcal{H}(b)$ which are continuous on $\bigcup_k S_k$.

Now, fix an $f \in \mathcal{H}(b)$ that is continuous on each of S_k. Then there exist $w_k \in S_k$ such that

$$\begin{aligned} \|f\|_{L^2(\mu)}^2 &\leq \sum_{k=1}^{\infty} |f(w_k)|^2 \mu(S_k) \\ &\leq \left(\sup_{k\geq 1} \frac{\mu(S_k)}{|I_k|}\right) \sum_{k=1}^{\infty} |f(w_k)|^2 |I_k|. \end{aligned} \tag{22.41}$$

Statement (i) will be proved as soon as we show that

$$\sum_{k=1}^{\infty} |f(w_k)|^2 |I_k| \leq C\,\|f\|_b^2, \tag{22.42}$$

where the constant C does not depend on f and on the choice of $w_k \in S_k$.

Consider the intervals

$$J_k = S_k \cap \{z : \Im z = \Im w_k\} \qquad (k \geq 1)$$

and define

$$\nu = \sum_{k=1}^{\infty} m_{J_k}.$$

Then it follows from (22.39) that $\nu \in \mathcal{C}$ (and the Carleson constants C_ν of such measures ν are uniformly bounded). Since

$$\left(\sum_{k=1}^{\infty} |f(w_k)|^2 |I_k| \right)^{1/2} \le \|f\|_{L^2(\nu)} + \left(\sum_{k=1}^{\infty} \int_{J_k} |f(z) - f(w_k)|^2 \, |dz| \right)^{1/2}$$

and $\|f\|_{L^2(\nu)} \le C_1 \|f\|_2 \le C_1 \|f\|_b$, we need to estimate the last term in the above inequality.

For $z \in J_k$, denote by $[z, w_k]$ the straight-line interval with the end points z and w_k. Then

$$f(z) - f(w_k) = \int_{[z,w_k]} f'(u) \, du.$$

In the case $J_k \subset \mathbb{R}$, note that, by Corollary 22.21(i), any $f \in \mathcal{H}(b)$ is differentiable on J_k except possibly at the end points. However, by continuity, the identity still holds. Moreover, we have

$$|f(z) - f(w_k)| \le \int_{J_k} |f'(u)| \, |du| \qquad (z \in J_k).$$

Hence, by the Cauchy–Schwarz inequality and (22.40),

$$\begin{aligned} |f(z) - f(w_k)|^2 &\le \left(\int_{J_k} |f'(u)| \, |du| \right)^2 \\ &\le \left(\int_{J_k} w_p^{-2}(u) \, |du| \right) \left(\int_{J_k} |f'(u)|^2 w_p^2(u) \, |du| \right) \\ &\le \frac{C_2}{|J_k|} \left(\int_{J_k} |f'(u)|^2 w_p^2(u) \, |du| \right) \qquad (z \in J_k). \end{aligned}$$

Thus,

$$\int_{J_k} |f(z) - f(w_k)|^2 \, |dz| \le C_2 \int_{J_k} |f'(u)|^2 w_p^2(u) \, |du|,$$

and consequently

$$\begin{aligned} \sum_{k=1}^{\infty} \int_{J_k} |f(z) - f(w_k)|^2 \, |dz| &\le C_2 \sum_{k=1}^{\infty} \int_{J_k} |f'(u)|^2 w_p^2(u) \, |du| \\ &= C_2 \|f' w_p\|_{L^2(\nu)}^2 \\ &\le C_3 \|f\|_b^2, \end{aligned}$$

where the last inequality follows from the crucial Theorem 22.17.

(ii) For a Borel set $E \subset \overline{\mathbb{C}_+}$ define the operator

$$\begin{array}{rccc} \mathcal{I}_E: & \mathcal{H}(b) & \longrightarrow & L^2(\mu) \\ & f & \longmapsto & \chi_E f, \end{array}$$

where χ_E is the characteristic function of E. For $N \geq 1$ put $F_N = \bigcup_{k=1}^N S_k$ and $\widehat{F}_N = \overline{\mathbb{C}_+} \setminus F_N$. As above, we assume that $f \in \mathcal{H}(b)$ is continuous on $\bigcup_k S_k$. Then it follows from (22.41) and (22.42) that

$$\int_{\widehat{F}_N} |f|^2 \, d\mu \leq C \left(\sup_{k>N} \frac{\mu(S_k)}{|I_k|} \right) \|f\|_b^2,$$

and so $\|\mathcal{I}_{\widehat{F}_N}\| \longrightarrow 0$ as $N \longrightarrow \infty$. Since our embedding operator is $\mathcal{I}_{F_N} + \mathcal{I}_{\widehat{F}_N}$, statement (ii) will be proved as soon as we show that $\mathcal{I}_{F_N}$ is a compact operator for any N. Clearly, it suffices to prove the compactness of $\mathcal{I}_{S_k}$ for each fixed k. To establish this fact, we approximate $\mathcal{I}_{S_k}$ by finite-rank operators.

For a given $\epsilon > 0$, partition the square S_k into a finite union of squares $\{S_{kl}\}_{l=1}^L$ with pairwise disjoint interiors so that

$$\left(\int_{[\zeta,z]} \frac{|du|}{(w_p(u))^2} \right) < \epsilon \tag{22.43}$$

for any $\zeta, z \in S_{kl}$, $1 \leq l \leq L$. Such a partition exists since $I_k \cap \mathrm{Clos}_{\mathbb{R}}\, \sigma(b) = \emptyset$, $k \geq 1$. Thus, b is analytic in a neighborhood of S_k, and the norms involved in the definition of $w_p(z)$ are continuous on S_k.

Now, fix $\zeta_l \in S_{kl}$ and consider the finite-rank operator $T : \mathcal{H}(b) \longrightarrow L^2(\mu)$ defined by

$$(Tf)(z) = \sum_{l=1}^{L} f(\zeta_l) \chi_{S_{kl}}(z).$$

We show that $\|\mathcal{I}_{S_k} - T\|^2 \leq C\epsilon$. As in the proof of (i), we have

$$\|(\mathcal{I}_{S_K} - T)f\|_{L^2(\mu)}^2 = \sum_{l=1}^{L} \int_{S_{kl}} |f(z) - f(\zeta_l)|^2 \, d\mu(z).$$

But

$$\begin{aligned} |f(z) - f(\zeta_l)|^2 &\leq \left(\int_{[\zeta_l,z]} |f'(u)| \, |du| \right)^2 \\ &\leq \left(\int_{[\zeta_l,z]} |f'(u)|^2 w_p^2(u) \, |du| \right) \left(\int_{[\zeta_l,z]} w_p^{-2}(u) \, |du| \right), \end{aligned}$$

and, by Theorem 22.17,

$$\int_{[\zeta_l,z]} |f'(u)|^2 w_p^2(u) \, |du| \leq C_1 \|f\|_b^2,$$

where C_1 does not depend on $f \in \mathcal{H}(b)$, $z \in S_{kl}$ and l. Hence, by (22.43),

$$\|(\mathcal{I}_{S_K} - T)f\|^2_{L^2(\mu)} \le C_1\epsilon \, \|f\|^2_b \sum_{l=1}^{L} \mu(S_{kl}) = C_1\epsilon\mu(S_k)\|f\|^2_b.$$

We conclude that $\mathcal{I}_{S_K}$ may be approximated by finite-rank operators. This means that $\mathcal{I}_{S_K}$ is compact. □

The following result follows as a special case.

Corollary 22.23 *Let μ be a Borel measure in $\overline{\mathbb{C}_+}$, and let $\varepsilon \in (0,1)$.*

(i) *Assume that $\mu(S(x,h)) \le Kh$ for all the Carleson squares $S(x,h)$ satisfying*

$$S(x,h) \cap \widetilde{\Omega}(b,\varepsilon) \ne \emptyset.$$

Then $\mathcal{H}(b) \subset L^2(\mu)$, that is, there is a constant $C > 0$ such that

$$\|f\|_{L^2(\mu)} \le C\,\|f\|_b \qquad (f \in \mathcal{H}(b)).$$

(ii) *If μ is a vanishing Carleson measure for $\mathcal{H}(b)$, that is, $\mu(S(x,h))/h \to 0$ whenever $S(x,h) \cap \widetilde{\Omega}(b,\varepsilon) \ne \emptyset$ and $h \to 0$ or dist$(S(x,h),0) \to +\infty$, then the embedding $\mathcal{H}(b) \subset L^2(\mu)$ is compact.*

Proof (i) Consider the open set $E = \mathbb{R} \setminus \mathrm{Clos}_{\mathbb{R}}\, \widetilde{\Omega}(b,\varepsilon)$. If $E = \emptyset$, then μ is a Carleson measure and $\mathcal{H}(b) \subset H^2(\mathbb{C}_+) \subset L^2(\mu)$. So we may assume that $E \ne \emptyset$ and we can write it as a union of disjoint intervals Δ_l. Note that

$$\int_{\Delta_l} \frac{dt}{\tilde{d}_\varepsilon(t)} = \infty.$$

Hence, partitioning the intervals Δ_l, we may represent E as a union of intervals I_k with mutually disjoint interiors such that

$$\int_{I_k} \frac{dt}{\tilde{d}_\varepsilon(t)} = \frac{1}{2}.$$

It follows that there exists $x_k \in I_k$ such that $\tilde{d}_\varepsilon(x_k) = 2|I_k|$. Hence, for any $x \in I_k$, $\tilde{d}_\varepsilon(x) \ge \tilde{d}_\varepsilon(x_k) - |I_k| = |I_k|$ and $\tilde{d}_\varepsilon(x) \le 3|I_k|$. This implies that

$$|I_k| \int_{I_k} \frac{dt}{(\tilde{d}_\varepsilon(t))^2}\, dt \le 1,$$

and, using Lemma 22.19, we conclude that the intervals I_k satisfy (22.40). The condition (22.39) is trivial.

Let $S_k = S(I_k)$ be the Carleson square with the lower side I_k, let $F = \bigcup_k S_k$ and let $G = \overline{\mathbb{C}_+} \setminus F$. Put $\mu_1 = \mu|_F$ and $\mu_2 = \mu|_G$. We show that the measure μ_1 satisfies the conditions of Theorem 22.22, whereas μ_2 is a usual Carleson measure and thus $\mathcal{H}(b) \subset H^2(\mathbb{C}_+) \subset L^2(\mu_2)$.

Let us show that $\mu_1(S_k) \le C_2|I_k|$. Indeed, it follows from the estimate $|I_k| \le \tilde{d}_\varepsilon(x) \le 3|I_k|$, $x \in I_k$, that $S(6I_k) \cap \widetilde{\Omega}(b,\varepsilon) \ne \emptyset$ (by $6I_k$ we denote the six times larger interval with the same center as I_k). By the hypothesis, $\mu_1(S_k) \le \mu(S(6I_k)) \le C|I_k|$. Hence, μ_1 satisfies the conditions of Theorem 22.22(i), and so $\mathcal{H}(b) \subset L^2(\mu_1)$.

Now, we show that $\mu_2 \in \mathcal{C}$. Assume that $S(I) \cap G \ne \emptyset$ for some interval $I \subset \mathbb{R}$, and let $z = x + iy \in S(I) \cap G$. If $x \in \mathrm{Clos}_{\mathbb{R}}\, \widetilde{\Omega}(b,\varepsilon)$, then $S(2I) \cap \widetilde{\Omega}(b,\varepsilon) \ne \emptyset$. Otherwise, if $x \in I_k$ for some k, then $\tilde{d}_\varepsilon(x) \le 3|I_k| \le 3|I|$ since $z \in S(I) \setminus S(I_k)$. Thus

$$S(6I) \cap \widetilde{\Omega}(b,\varepsilon) \ne \emptyset. \tag{22.44}$$

By the hypothesis, $\mu_2(S(I)) \le \mu(S(6I)) \le C|I|$, and so μ_2 is a Carleson measure.

(ii) Let F, G, μ_1 and μ_2 be the same as above. We show that μ_1 satisfies the conditions of Theorem 22.22(ii), whereas μ_2 is a vanishing Carleson measure. Indeed, we can split the family $\{S_k\}$ into two families $\{S_k\}_{k\in K_1}$ and $\{S_k\}_{k\in K_2}$ such that $|I_k| \longrightarrow 0$, $k \longrightarrow \infty$, $k \in K_1$, whereas $\mathrm{dist}(I_k, 0) \longrightarrow \infty$ when $k \longrightarrow \infty$, $k \in K_2$. Since $S(6I_k) \cap \widetilde{\Omega}(b,\varepsilon) \ne \emptyset$, we conclude that Theorem 22.22(ii) applies to μ_1 and the embedding $\mathcal{H}(b) \subset L^2(\mu_1)$ is compact. Finally, any Carleson square $S(I)$ with $S(I) \cap G \ne \emptyset$ satisfies (22.44), and so, by the assumptions of Corollary 22.23(ii), μ_2 is a vanishing Carleson measure. □

In Corollary 22.23 we need to verify the Carleson condition only on a *special* subclass of squares. Geometrically, this means that when we are far from the spectrum $\sigma(b)$, the measure μ in Corollary 22.23 can be essentially larger than standard Carleson measures. The reason is that functions in $\mathcal{H}(b)$ have much more regularity at the points $x \in \mathbb{R} \setminus \mathrm{Clos}\, \sigma(b)$ where $|b(x)| = 1$. On the other hand, if $|b(x)| \le \delta < 1$, almost everywhere on some interval $I \subset \mathbb{R}$, then the functions in $\mathcal{H}(b)$ behave on I essentially the same as a general element of $H^2(\mathbb{C}_+)$ on that interval, and for any Carleson measure for $\mathcal{H}(b)$ its restriction to the square $S(I)$ is a standard Carleson measure.

We will see that, for a class of functions b, the sufficient condition of Corollary 22.23 is also necessary. However, it may be far from being necessary for certain functions b even in the model space setting. We say that b satisfies the *connected level set condition* if the set $\Omega(b,\varepsilon)$ is connected for some $\varepsilon \in (0,1)$.

Theorem 22.24 *Let b satisfy the connected level set condition for some $\varepsilon \in (0,1)$. Assume that $\Omega(b,\varepsilon)$ is unbounded and $\sigma(b) \subset \mathrm{Clos}_{\mathbb{R}}\, \Omega(b,\varepsilon)$. Let μ be a Borel measure on $\overline{\mathbb{C}_+}$. Then the following statements are equivalent.*

(i) $\mathcal{H}(b) \subset L^2(\mu)$.
(ii) *There exists $C > 0$ such that $\mu(S(x,h)) \le Ch$ for all Carleson squares $S(x,h)$ such that $S(x,h) \cap \widetilde{\Omega}(b,\varepsilon) \ne \emptyset$.*

(iii) *There exists* $C > 0$ *such that*

$$\int_{\bar{\mathbb{C}}_+} \frac{\Im z}{|\zeta - \bar{z}|^2}\, d\mu(\zeta) \le \frac{C}{1 - |b(z)|} \qquad (z \in \mathbb{C}_+). \tag{22.45}$$

Proof (ii) $\Longrightarrow$ (i) By Corollary 22.23, this implication actually holds for any b.

(i) $\Longrightarrow$ (iii) This is trivial (apply the inequality $\|f\|_{L^2(\mu)} \le C\,\|f\|_b$ to $f = k_z^b$).

(iii) $\Longrightarrow$ (ii) Let $S(x,h)$ be a Carleson square such that $S(x,h) \cap \widetilde{\Omega}(b,\varepsilon) \neq \emptyset$. Since $\sigma(b) \subset \mathrm{Clos}_{\mathbb{R}}\, \Omega(b,\varepsilon)$ it follows that $S(x,2h) \cap \Omega(b,\varepsilon) \neq \emptyset$. Choose $z_1 \in S(x,2h) \cap \mathbb{C}_+$ with $|b(z_1)| < \varepsilon$. Now consider $S(x,3h)$. Since $\Omega(b,\varepsilon)$ is connected and unbounded, there exists a point z_2 on the boundary of $S(x,3h)$ such that $|b(z_2)| < \varepsilon$. Hence, there exists a continuous curve γ connecting z_1 and z_2 and such that $|b| < \varepsilon$ on γ. Now let $z = x + ih$. Applying the theorem on two constants to the domain $\mathrm{Int}\, S(x,3h) \setminus \gamma$ we conclude that $|b(z)| \le \delta$ where $\delta \in (0,1)$ depends only on ε. Then inequality (22.45) implies that

$$h \int_{S(x,h)} \frac{d\mu(\zeta)}{|\zeta - \bar{z}|^2} \le C(1-\delta)^{-1}.$$

It remains to note that $|\zeta - \bar{z}| \le C_1 h$, for $\zeta \in S(x,h)$, to obtain that $\mu(S(x,h)) \le C_2 h$. □

Examples are known of inner functions satisfying the connected level set condition. We would like to emphasize that there are also many outer functions satisfying the conditions of Theorem 22.24. For example, let $b(z) = \exp((i/\pi)\log z)$, where $\log z$ is the main branch of the logarithm in $\mathbb{C} \setminus (-\infty, 0]$.

We see that, if b satisfies the conditions of Theorem 22.24, then it suffices to verify the inequality $\|f\|_{L^2(\mu)} \le C\,\|f\|_b$ for the reproducing kernels of the space $\mathcal{H}(b)$ to get it for all functions f in $\mathcal{H}(b)$. This is no longer true in the general case.

22.8 A formula of combinatorics

We first recall some well-known facts concerning hypergeometric series. The ${}_2F_1$ hypergeometric series is a power series in z defined by

$${}_2F_1\left[\begin{matrix} a, b \\ c \end{matrix}; z\right] = \sum_{p=0}^{+\infty} \frac{(a)_p (b)_p}{p!\,(c)_p} z^p, \tag{22.46}$$

where $a, b, c \in \mathcal{C}$, $c \neq 0, -1, -2, \dots$, and

$$(t)_p := \begin{cases} 1, & \text{if } p = 0, \\ t(t+1)\cdots(t+p-1), & \text{if } p \geq 1. \end{cases}$$

The hypergeometric series reduces to a polynomial of degree n in z when a or b is equal to $-n$, $n \geq 0$. It is clear that the radius of convergence of the ${}_2F_1$ series is equal to 1. One can show that when $\Re(c-a-b) \leq -1$ this series is divergent on the entire unit circle, when $-1 < \Re(c-a-b) \leq 0$ this series converges on the unit circle except for $z = 1$, and when $0 < \Re(c-a-b)$ this series is (absolutely) convergent on the entire unit circle.

We note that a power series $\sum_p \alpha_p z^p$ $(\alpha_0 = 1)$ can be written as a hypergeometric series ${}_2F_1\left[\begin{matrix} a, b \\ c \end{matrix}; z\right]$ if and only if

$$\frac{\alpha_{p+1}}{\alpha_p} = \frac{(p+a)(p+b)}{(p+1)(p+c)}. \tag{22.47}$$

Finally, we recall two useful well-known formulas for the hypergeometric series. The first one is Euler's formula

$${}_2F_1\left[\begin{matrix} a, b \\ c \end{matrix}; z\right] = (1-z)^{c-a-b}\, {}_2F_1\left[\begin{matrix} c-a, c-b \\ c \end{matrix}; z\right] \tag{22.48}$$

and the second is Pfaff's formula

$${}_2F_1\left[\begin{matrix} a, b \\ c \end{matrix}; \tfrac{1}{2}\right] = 2^a\, {}_2F_1\left[\begin{matrix} a, c-b \\ c \end{matrix}; -1\right], \qquad \Re(b-a) > -1. \tag{22.49}$$

Lemma 22.25 *For $m \geq 0$, we have*

$$\sum_{k=0}^{m} \binom{m}{k} (z-1)^{-k}\, {}_2F_1\left[\begin{matrix} a, b-k \\ c \end{matrix}; z\right] = \frac{(c-a)_m}{(c)_m} \left(\frac{z}{z-1}\right)^m {}_2F_1\left[\begin{matrix} a, b \\ c+m \end{matrix}; z\right].$$

Proof Rewrite the above identity as $L = R$, where

$$L = \sum_{k=0}^{m} \binom{m}{k} (1-z)^{-k} (-1)^{m-k}\, {}_2F_1\left[\begin{matrix} a, b-k \\ c \end{matrix}; z\right]$$

and

$$R = \frac{(c-a)_m}{(c)_m} \left(\frac{z}{1-z}\right)^m {}_2F_1\left[\begin{matrix} a, b \\ c+m \end{matrix}; z\right].$$

Then, by applying the formula (22.48), we obtain

$$L = \sum_{k=0}^{m} \binom{m}{k} (-1)^{m-k} (1-z)^{c-a-b} \, {}_2F_1\!\left[\begin{matrix} c-a, c-b+k \\ c \end{matrix}; z\right].$$

Now, consider the difference operator Δ defined by $\Delta f(x) = f(x+1) - f(x)$. It is well known and easy to verify that

$$\Delta^m f(x) = \sum_{k=0}^{m} \binom{m}{k} (-1)^{m-k} f(x+k).$$

Using this formula, we see that $L = (1-z)^{c-a-b} \Delta^m f(c-b)$, with

$$f(x) = {}_2F_1\!\left[\begin{matrix} c-a, x \\ c \end{matrix}; z\right].$$

But now we can compute $\Delta^m f(x)$. Indeed, we have

$$\begin{aligned} \Delta f(x) &= \sum_{k=0}^{\infty} \frac{(c-a)_k}{(c)_k} ((x+1)_k - (x)_k) \frac{z^k}{k!} \\ &= \sum_{k=1}^{\infty} \frac{(c-a)_k}{(c)_k} (x+1)_{k-1} \frac{z^k}{(k-1)!} \\ &= \frac{(c-a)}{c} z \, {}_2F_1\!\left[\begin{matrix} c-a+1, x+1 \\ c+1 \end{matrix}; z\right], \end{aligned}$$

and, by induction, it follows that

$$\Delta^m f(x) = \frac{(c-a)_m}{(c)_m} z^m \, {}_2F_1\!\left[\begin{matrix} c-a+m, x+m \\ c+m \end{matrix}; z\right].$$

Therefore, we get

$$L = (1-z)^{c-a-b} \frac{(c-a)_m}{(c)_m} z^m \, {}_2F_1\!\left[\begin{matrix} c-a+m, c-b+m \\ c+m \end{matrix}; z\right].$$

Applying Euler's formula once more, we obtain the result. □

Corollary 22.26 *Let $n \geq 0$, let $0 \leq r \leq 2n+1$, and define*

$$A_{n,r} = (-1)^{r+1} \sum_{p=0}^{n} \sum_{\ell=0}^{n} (-2)^{p-\ell} \binom{r}{n-\ell} \binom{2n+1-r}{p} \binom{n-p+\ell}{\ell}.$$

Then

$$A_{n,r} = \begin{cases} -2^n, & 0 \leq r \leq n, \\ 2^n, & n+1 \leq r \leq 2n+1. \end{cases}$$

Proof Changing ℓ into $n-\ell$ in the second sum of $A_{n,r}$, we see that

$$A_{n,r} = (-1)^{r+1} \sum_{p=0}^{n} \sum_{\ell=0}^{n} (-2)^{p+\ell-n} \binom{r}{\ell} \binom{2n+1-r}{p} \binom{2n-p-\ell}{n-\ell}.$$

Hence,

$$\begin{aligned}
&A_{n,2n+1-r} \\
&\quad = (-1)^{2n+1-r+1} \sum_{p=0}^{n} \sum_{\ell=0}^{n} (-2)^{p+\ell-n} \binom{2n+1-r}{\ell} \binom{r}{p} \binom{2n-p-\ell}{n-\ell} \\
&\quad = -(-1)^{r+1} \sum_{p=0}^{n} \sum_{\ell=0}^{n} (-2)^{p+\ell-n} \binom{2n+1-r}{\ell} \binom{r}{p} \binom{2n-p-\ell}{n-p} \\
&\quad = -A_{n,r}.
\end{aligned}$$

Therefore, it is sufficient to show that $A_{n,r} = -2^n$ for $0 \le r \le n$ and then the result for $n+1 \le r \le 2n+1$ will follow immediately.

We will now assume that $0 \le r \le n$. Changing p to $n-p$ in the first sum of $A_{n,r}$ (the above new formula), and permuting the two sums, we get

$$A_{n,r} = (-1)^{r+1} \sum_{\ell=0}^{n} (-2)^{l} \binom{r}{\ell} \sum_{p=0}^{n} (-2)^{-p} \binom{2n+1-r}{n-p} \binom{n+p-\ell}{n-\ell}.$$

According to (22.46) and (22.47), we see that

$$\begin{aligned}
&\sum_{p=0}^{n} (-2)^{-p} \binom{2n+1-r}{n-p} \binom{n+p-\ell}{n-\ell} \\
&\qquad = \binom{2n+1-r}{n} {}_2F_1\left[\begin{matrix} n-\ell+1, -n \\ n+2-r \end{matrix}; \tfrac{1}{2}\right],
\end{aligned}$$

which implies that

$$A_{n,r} = (-1)^{r+1} \binom{2n+1-r}{n} \sum_{\ell=0}^{n} (-2)^{\ell} \binom{r}{\ell} {}_2F_1\left[\begin{matrix} n-\ell+1, -n \\ n+2-r \end{matrix}; \tfrac{1}{2}\right].$$

Since $r \le n$ and since $\binom{r}{\ell} = 0$ if $r < \ell$, the sum (in the last equation) ends at $\ell = r$ and we can apply Lemma 22.25 with $m = r$, $a = -n$, $b = n+1$, $c = n+2-r$ and $z = \frac{1}{2}$. Therefore,

$$\begin{aligned}
A_{n,r} &= (-1)^{r+1} \binom{2n+1-r}{n} \frac{(2n+2-r)_r}{(n+2-r)_r} (-1)^r \, {}_2F_1\left[\begin{matrix} -n, n+1 \\ n+2 \end{matrix}; \tfrac{1}{2}\right] \\
&= -\frac{(2n+1)!}{(n+1)!n!} \, {}_2F_1\left[\begin{matrix} -n, n+1 \\ n+2 \end{matrix}; \tfrac{1}{2}\right].
\end{aligned}$$

We now use the formula (22.49), which gives

$$\begin{aligned}
{}_2F_1\left[\begin{matrix}-n, n+1\\ n+2\end{matrix}; \tfrac{1}{2}\right] &= 2^{-n}\, {}_2F_1\left[\begin{matrix}-n, 1\\ n+2\end{matrix}; -1\right]\\
&= 2^{-n} \sum_{i=0}^{\infty} \frac{(-n)_i (1)_i}{(n+2)_i} \frac{(-1)^i}{i!}\\
&= 2^{-n} \sum_{i=0}^{n} \binom{n}{i} \frac{i!(n+1)!}{(n+i+1)!},
\end{aligned}$$

where we have used

$$(1)_i = i!, \qquad \frac{(-n)_i(-1)^i}{i!} = \binom{n}{i} \quad \text{and} \quad (n+2)_i = \frac{(n+i+1)!}{(n+1)!}.$$

Hence,

$${}_2F_1\left[\begin{matrix}-n, n+1\\ n+2\end{matrix}; \tfrac{1}{2}\right] = 2^{-n}(n+1)!n! \sum_{i=0}^{n} \frac{1}{(n+i+1)!(n-i)!}$$

which implies that

$$A_{n,r} = -2^{-n} \sum_{i=0}^{n} \binom{2n+1}{i}.$$

But

$$\sum_{i=0}^{n} \binom{2n+1}{i} = \frac{1}{2} \sum_{i=0}^{2n+1} \binom{2n+1}{i} = 2^{2n}.$$

This ends the proof. □

22.9 Norm convergence for the reproducing kernels

In Section 22.2, we saw that, if $x_0 \in \mathbb{R}$ satisfies $S_{2n+2}(x_0) < \infty$, then $k^b_{w,n}$ tends *weakly* to $k^b_{x_0,n}$ in $\mathcal{H}(b)$ as w approaches radially to x_0. It is natural to ask if this weak convergence can be replaced by norm convergence. In other words, is it true that $\|k^b_{w,n} - k^b_{x_0,n}\|_b \longrightarrow 0$ as w tends radially to x_0? In this section, we provide an affirmative answer.

Since we already have weak convergence, to prove the norm convergence, a standard Hilbert space technique says that it is sufficient to prove that $\|k^b_{w,n}\|_b \longrightarrow \|k^b_{x_0,n}\|_b$ as w tends radially to x_0. Hence, to apply this method, we need to compute $\|k^b_{x_0,n}\|_b$.

Lemma 22.27 *Let b be a point in the closed unit ball of $H^\infty(\mathbb{C}_+)$, let $n \ge 0$ and let $x_0 \in \mathbb{R}$ satisfy the condition $S_{2n+2}(x_0) < \infty$. Then*

$$\|k^b_{x_0,n}\|_b^2 = \frac{(n!)^2}{2\pi i} \sum_{p=0}^{n} \frac{\overline{b^{(p)}(x_0)}}{p!} \frac{b^{(2n+1-p)}(x_0)}{(2n+1-p)!}.$$

Proof Following the notation of Section 22.2, let

$$\varphi(z) = 1 - b(z) \sum_{p=0}^{n} \frac{\overline{b^{(p)}(x_0)}}{p!} (z - x_0)^p. \tag{22.50}$$

Then, by Lemma 22.5, as z tends radially to x_0, we have

$$\begin{aligned} &k^b_{x_0,n}(z) \\ &= -\frac{n!}{2\pi i}(z - x_0)^{-n-1} \left(\sum_{p=0}^{2n+1} \frac{\varphi^{(p)}(x_0)}{p!}(z - x_0)^p + o((z - x_0)^{2n+1}) \right). \end{aligned}$$

As we saw in the proof of Theorem 22.6, $\varphi^{(k)}(x_0) = 0$ for $0 \le k \le n$. Hence, the preceding formula simplifies to

$$k^b_{x_0,n}(z) = -\frac{n!}{2\pi i} \sum_{p=0}^{n} \frac{\varphi^{(p+n+1)}(x_0)}{(p + n + 1)!}(z - x_0)^p + o((z - x_0)^n).$$

Using Lemma 22.5 once more, we can also write

$$k^b_{x_0,n}(z) = \sum_{p=0}^{n} \frac{(k^b_{x_0,n})^{(p)}(x_0)}{p!}(z - x_0)^p + o((z - x_0)^n).$$

Thus, the uniqueness of the coefficients in a Taylor series representation implies that

$$(k^b_{x_0,n})^{(p)}(x_0) = -\frac{n!}{2\pi i} \frac{p!}{(p + 1 + n)!} \varphi^{(p+n+1)}(x_0) \qquad (0 \le p \le n).$$

But, according to Lemma 22.4, we have $\|k^b_{x_0,n}\|_b^2 = (k^b_{x_0,n})^{(n)}(x_0)$. Therefore,

$$\|k^b_{x_0,n}\|_b^2 = -\frac{(n!)^2}{2\pi i} \frac{\varphi^{(2n+1)}(x_0)}{(2n + 1)!}.$$

Now, we appeal again to formula (22.50) to compute $\varphi^{(2n+1)}(x_0)$. Using Leibniz's rule, we have

$$\frac{\varphi^{(2n+1)}(z)}{(2n + 1)!} = -\sum_{p=0}^{n} \frac{\overline{b^{(p)}(x_0)}}{p!} \frac{b^{(2n+1-p)}(x_0)}{(2n + 1 - p)!} + O(z - x_0).$$

Hence, upon letting $z = x_0 + iy \longrightarrow x_0$, the result follows. Note that in writing $O(z - x_0)$, we have implicitly used the fact that b has finite radial derivatives up to order $2n + 1$ at x_0. □

The next result provides an affirmative answer to the question of norm convergence.

Theorem 22.28 *Let b be a point in the closed unit ball of $H^\infty(\mathbb{C}_+)$, let $n \geq 0$ and let $x_0 \in \mathbb{R}$ satisfy the condition $S_{2n+2}(x_0) < \infty$. Then*

$$\|k^b_{w,n} - k^b_{x_0,n}\|_b \longrightarrow 0$$

as w tends radially to x_0.

Proof Put $a_p(w) = b^{(p)}(w)/p!$ and $a_p = a_p(x_0)$. We have

$$\frac{\partial^n}{\partial z^n}\left(\frac{1}{(z-\bar{w})^{n+1}}\right) = (-1)^n \frac{(2n)!}{n!} \frac{1}{(z-\bar{w})^{2n+1}},$$

and by Leibniz's rule

$$\begin{aligned}&\frac{\partial^n}{\partial z^n}((z-\bar{w})^{p-n-1}b(z))\\ &\quad= \sum_{\ell=0}^{n}\binom{n}{\ell}(-1)^\ell \frac{(n-p+\ell)!}{(n-p)!}(z-\bar{w})^{p-n-\ell-1}b^{(n-\ell)}(z).\end{aligned}$$

According to (21.37), we have $\|k^b_{w,n}\|_b^2 = (k^b_{w,n})^{(n)}(w)$, which implies

$$\begin{aligned}&\|k^b_{w,n}\|_b^2\\ &\quad= -\frac{n!}{2\pi i}\frac{1}{(w-\bar{w})^{2n+1}}\Bigg((-1)^n\frac{(2n)!}{n!}\\ &\qquad - \sum_{p=0}^{n}\sum_{\ell=0}^{n}\binom{n}{\ell}(-1)^\ell\frac{(n-p-\ell)!}{(n-p)!}(w-\bar{w})^{n+p-\ell}\overline{a_p(w)}b^{(n-\ell)}(w)\Bigg).\end{aligned} \tag{22.51}$$

For $0 \leq s \leq n$, the function $b^{(s)}$ is analytic in the upper half-plane and its derivative of order $2n+1-s$, which coincides with $b^{(2n+1)}$, has a radial limit at x_0. According to Lemma 22.5, as w tends radially to x_0, we have

$$b^{(s)}(w) = \sum_{r=s}^{2n+1} a_r \frac{r!}{(r-s)!}(w-x_0)^{r-s} + o((w-x_0)^{2n+1-s}).$$

Hence, if we put $w = x_0 + it$, we get

$$(w-\bar{w})^s b^{(s)}(w) = 2^s \sum_{r=s}^{2n+1} a_r \frac{r!}{(r-s)!} i^r t^r + o(t^{2n+1})$$

and thus

$$\begin{aligned}
&(w-\bar{w})^{n+p-\ell}\,\overline{a_p(w)}b^{(n-\ell)}(w)\\
&\quad=\frac{(-1)^p}{p!}2^{n+p-\ell}\left(\sum_{r=n-\ell}^{2n+1}a_r\frac{r!}{(r-n+\ell)!}i^rt^r+o(t^{2n+1})\right)\\
&\qquad\times\left(\sum_{j=p}^{2n+1}\overline{a_j}\frac{j!}{(j-p)!}(-i)^jt^j+o(t^{2n+1})\right).
\end{aligned}$$

We deduce from (22.51) that

$$\begin{aligned}
\|k_{w,n}^b\|_b^2=\frac{(-1)^nn!}{2^{2n+2}\pi}t^{-2n-1}\Bigg\{&(-1)^n\frac{(2n)!}{n!}\\
&-n!\sum_{p=0}^{n}\sum_{\ell=0}^{n}\left[(-1)^{p+\ell}2^{n+p-\ell}\binom{n-p+\ell}{\ell}\right.\\
&\times\left.\left(\sum_{r=n-\ell}^{2n+1}a_r\binom{r}{n-\ell}i^rt^r\right)\left(\sum_{j=p}^{2n+1}\overline{a_j}\binom{j}{p}(-i)^jt^j\right)\right]\\
&+o(t^{2n+1})\Bigg\}.
\end{aligned}$$

Put

$$c_n=\frac{(-1)^nn!}{2^{2n+2}\pi}$$

and

$$\begin{aligned}
\lambda_{s,n}=i^s\sum_{p=0}^{n}\sum_{\ell=0}^{n}&\left[(-1)^{p+\ell}2^{n+p-\ell}\binom{n-p+\ell}{\ell}\right.\\
&\times\left.\sum_{r=0}^{s}\binom{r}{n-\ell}\binom{s-r}{p}(-1)^{s-r}a_r\overline{a_{s-r}}\right],
\end{aligned}$$

where we assumed that $\binom{a}{b}=0$ if $a<b$ or $b<0$. Then we can write

$$\|k_{w,n}^b\|_b^2=c_nt^{-2n-1}\left[(-1)^n\frac{(2n)!}{n!}-n!\sum_{s=0}^{2n+1}\lambda_{s,n}t^s+o(t^{2n+1})\right].$$

Now, we recall that $k_{w,n}^b$ is weakly convergent as w tends radially to x_0, and thus $\|k_{w,n}^b\|_b$ remains bounded. Therefore, we necessarily have

$$(-1)^n\frac{(2n)!}{n!}-n!\lambda_{0,n}=0\qquad(\text{for } s=0)$$

and

$$\lambda_{s,n}=0\qquad(1\le s\le 2n).$$

This implies that

$$\|k_{w,n}^b\|_b^2 = -n!c_n\lambda_{2n+1,n} + o(1) \tag{22.52}$$

as w tends radially to x_0. But

$$\begin{aligned}
&\lambda_{2n+1,n} \\
&\quad = (-1)^n i \sum_{r=0}^{2n+1} (-1)^{r+1} a_r \overline{a_{2n+1-r}} \\
&\qquad \times \left[\sum_{p=0}^{n} \sum_{\ell=0}^{n} (-1)^{p+\ell} 2^{n+p-\ell} \binom{n-p+\ell}{\ell} \binom{r}{n-\ell} \binom{2n+1-r}{p} \right] \\
&\quad = (-1)^n i 2^n \sum_{r=0}^{2n+1} A_{n,r} a_r \overline{a_{2n+1-r}}.
\end{aligned}$$

According to Corollary 22.26, we have $A_{n,r} = -2^n$ if $0 \le r \le n$ and $A_{n,r} = 2^n$ if $n+1 \le r \le 2n+1$. Thus, we obtain

$$\lambda_{2n+1,n} = (-1)^n i 2^{2n} \left(\sum_{r=n+1}^{2n+1} a_r \overline{a_{2n+1-r}} - \sum_{r=0}^{n} a_r \overline{a_{2n+1-r}} \right).$$

Since

$$\sum_{r=n+1}^{2n+1} a_r \overline{a_{2n+1-r}} = \sum_{r=0}^{n} \overline{a_r} a_{2n+1-r},$$

we can rewrite the formula as

$$\lambda_{2n+1,n} = (-1)^{n+1} 2^{2n+1} \, \Im \left(\sum_{r=0}^{n} \overline{a_r} a_{2n+1-r} \right).$$

In the light of Lemma 22.27, we again rewrite the preceding formula as

$$\lambda_{2n+1,n} = (-1)^{n+1} 2^{2n+2} \frac{\pi}{(n!)^2} \|k_{x_0,n}^b\|_b^2.$$

Hence, by (22.52) and the definition of c_n, we obtain

$$\|k_{w,n}^b\|_b^2 = \|k_{x_0,n}^b\|_b^2 + o(1),$$

which proves that $\|k_{w,n}^b\|_b \longrightarrow \|k_{x_0,n}^b\|_b$ as w tends radially to x_0. Since $k_{w,n}^b$ also tends weakly to $k_{x_0,n}^b$ in $\mathcal{H}(b)$ as w tends radially to x_0, we get the desired conclusion. $\square$

Notes on Chapter 22

Section 22.2

Lemma 22.3 and Theorem 22.6 were obtained by Fricain and Mashreghi [81]. Corollary 22.7 was obtained in [29] to pave the way for Bernstein-type inequalities.

Section 22.3

The generalized weights $w_{p,n}$ and Lemma 22.8 are from [29]. For the case of inner functions, they have been considered before by Baranov [27].

Section 22.4

The contents of this section are of independent interest and, with the exception of Lemma 22.11, have been extracted from [27]. Lemma 22.11 is known as the Vinogradov–Senichkin test [137]. More information on this topic can be found in [178, 179]

Section 22.5

Theorem 22.17, the so-called Bernstein-type inequality, is the main result of this chapter, and is due to Baranov, Fricain and Mashreghi [29]. This topic has a long and rich history. The famous classic Bernstein inequality asserts that

$$\|f'\|_p \leq a\|f\|_p$$

for all f in the Paley–Wiener space PW_a^p, where $1 \leq p \leq \infty$. Since PW_a^p is a translated version of K_Θ^p with $\Theta(z) = \exp(iaz)$, it is natural to generalize this inequality to the model spaces K_Θ^p for other choices of the inner function Θ. One of the earliest results is due to Levin [117], who showed that, under certain conditions,

$$\|f'/\Theta'\|_\infty \leq \|f\|_\infty$$

for all $f \in K_\Theta^\infty$. For the case of finite Blaschke products, this inequality was recently rediscovered by Borwein and Erdélyi [38, 39] and Li *et al.* [119]. Differentiation in model subspaces K_Θ^p, $1 < p < \infty$, was also studied by Dyakonov [68–70]. Briefly speaking, he showed that differentiation is bounded as an operator from K_Θ^p to $L^p(\mathbb{R})$, i.e.

$$\|f'\|_p \leq C\,\|f\|_p \qquad (f \in K_\Theta^p),$$

provided that $\Theta' \in H^\infty$. Moreover, this operator is compact if and only if $\Theta'(x) \longrightarrow 0$ as $x \longrightarrow \pm\infty$ [69]. The first important step was taken by Baranov in [27] when he introduced the Bernstein-type inequalities for model spaces. Theorem 22.17 is in a sense a general version of all these results.

Section 22.6

The results of this section are taken from [29]. The main goal is to set the stage for the next section in which Carleson measures are discussed.

Section 22.7

The classic Carleson embedding theorem gives a simple geometrical condition on a measure μ in the closed upper half-plane such that the embedding $H^p(\mathbb{C}_+) \subset L^p(\mu)$ holds; see Section 5.6. A similar question for model subspaces K^p_Θ was studied by Cohn [56, 57] and then by Volberg and Treil [190]. An approach based on the Bernstein-type inequalities for model subspaces was suggested in [27]. Theorems 22.22 and 22.24 and Corollary 22.23 are due to Baranov, Fricain and Mashreghi [29]. These results generalize the previous results of Cohn, Volberg and Treil, and Baranov *et al*. We mention also the recent work of Blandignères *et al.* [34, 35], who studied some reverse embeddings for $\mathcal{H}(b)$ spaces.

Section 22.8

This section is of independent interest and is taken from [81]. Its main goal is to obtain the combinatoric identity in Corollary 22.26. This formula plays the major role in establishing the norm convergence of reproducing kernels in the next section.

The hypergeometric series we need in this section are very standard and are a major tool in combinatorics. The interested reader may consult the books [21, 177] for further information. In particular, formulas (22.48) and (22.49) can be found (with proofs) in [21, p. 68].

Section 22.9

Theorem 22.28 was first established by Sarason for the case $n = 0$ [166, chap. V]. The general version presented here via a formula of combinatorics is due to Fricain and Mashreghi [81].

23

$\mathcal{H}(b)$ spaces generated by a nonextreme symbol b

As we have already said, many properties of $\mathcal{H}(b)$ depend on whether b is or is not an extreme point of the closed unit ball of H^∞. Recall that, by the de Leeuw–Rudin theorem (Theorem 6.7), b is a nonextreme point of the closed unit ball of H^∞ if and only if $\log(1-|b|^2) \in L^1(\mathbb{T})$, i.e.

$$\int_{\mathbb{T}} \log(1-|b|^2)\, dm > -\infty. \tag{23.1}$$

In this chapter, we study some specific properties of the space $\mathcal{H}(b)$ when b is a nonextreme point. Roughly speaking, when b is a nonextreme point, the space $\mathcal{H}(b)$ looks like the Hardy space H^2.

In this situation, an important property is the existence of an outer function a such that $a(0) > 0$ and which satisfies $|a|^2 + |b|^2 = 1$ a.e. on $\mathbb{T}$. This function a is introduced in Section 23.1 and we will see that $\mathcal{H}(\bar{b}) \doteq \mathcal{M}(\bar{a})$. In Section 23.2, we characterize the inclusion $\mathcal{M}(u) \subset \mathcal{H}(b)$ where $u \in H^\infty$. An important object in the nonextreme case is the associated function f^+ introduced in Section 23.3. This function, which is defined via the equation $T_{\bar{b}} f = T_{\bar{a}} f^+$, enables us to give a useful formula for the scalar product in $\mathcal{H}(b)$. We also show, in Section 23.3, that $b \in \mathcal{H}(b)$ and we compute its norm. It turns out that the analytic polynomials belong to and are dense in $\mathcal{H}(b)$. This is the content of Section 23.4. Then, in Section 23.5, we give a formula for $\|X_b f\|_b$, $f \in \mathcal{H}(b)$, and we compute the defect operator D_{X_b}. Recall that, in Section 19.2, we gave a geometric representation of $\mathcal{H}(b)$ space based on the abstract functional embedding. In Section 23.6, we obtain another representation, which corresponds to the Sz.-Nagy–Foiaş model for the contraction X_b. In Section 23.7, we characterize $\mathcal{H}(b)$ spaces when b is a nonextreme point. The analog for the extreme case will be done in Section 25.8. In Section 23.8, we exhibit some new inhabitants of $\mathcal{H}(b)$. In the last section, we finally show that the $\mathcal{H}(b)$ space can be viewed as the domain of the adjoint of an unbounded Toeplitz operator with symbol in the Smirnov class.

23.1 The pair (a, b)

If b satisfies the condition (23.1), then we define a to be the unique outer function whose modulus on $\mathbb{T}$ is $(1 - |b|^2)^{1/2}$ and is positive at the origin. Hence, on the open unit disk, a is given by the formula

$$a(z) = \exp\left(\int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} \, \log(1 - |b(\zeta)|^2)^{1/2} \, dm(\zeta) \right) \qquad (z \in \mathbb{D}). \quad (23.2)$$

Clearly, $a \in H^\infty$ with $\|a\|_\infty \le 1$ and

$$|a|^2 + |b|^2 = 1 \qquad \text{(a.e. on } \mathbb{T}). \quad (23.3)$$

Whenever we use the pair (a, b), we mean that they are related as described above. We sometimes say that a is the Pythagorean mate associated with b.

Theorem 23.1 *For each pair* (a, b), *we have*

$$\frac{a}{1 - b} \in H^2.$$

Proof By Corollary 4.26, $1/(1 - b)$ is an outer function in H^p for each $0 < p < 1$. Since a is an outer function in H^∞, then $a/(1 - b)$ is also an outer function in H^p for each $0 < p < 1$. But, by (13.50) and (23.3),

$$\frac{|a|^2}{|1 - b|^2} = \frac{1 - |b|^2}{|1 - b|^2} \in L^1(\mathbb{T}),$$

or equivalently $a/(1 - b) \in L^2(\mathbb{T})$. Hence, Corollary 4.28 ensures that $a/(1 - b) \in H^2$. ◻

Theorem 23.2 *Let b be a nonextreme point of the closed unit ball of H^∞. Then*

$$\mathcal{M}(\bar{a}) \doteq \mathcal{H}(\bar{b}).$$

Moreover,

$$\mathcal{M}(a) \hookrightarrow \mathcal{M}(\bar{a}) \hookrightarrow \mathcal{H}(b),$$

i.e. both inclusions are contractive. In particular, $\mathcal{M}(a)$ is contractively contained in $\mathcal{H}(b)$.

Proof The relation $\mathcal{M}(a) \hookrightarrow \mathcal{M}(\bar{a})$ follows from Theorem 17.17. Using Theorem 12.4 and (23.3), we see that

$$T_{\bar{a}} T_a = T_{|a|^2} = T_{1-|b|^2} = I - T_{\bar{b}} T_b.$$

Hence, Corollary 16.8 implies that $\mathcal{M}(\bar{a}) = \mathcal{M}(T_{\bar{a}}) \doteq \mathcal{M}((I - T_{\bar{b}} T_b)^{1/2}) = \mathcal{H}(\bar{b})$. The contractive inclusion $\mathcal{H}(\bar{b}) \hookrightarrow \mathcal{H}(b)$ is contained in Theorem 17.9. ◻

Theorem 23.2 ensures that $\mathcal{M}(\bar{a})$ embeds contractively in $\mathcal{H}(b)$. The following result provides another contraction between these spaces.

Theorem 23.3 *Let b be a nonextreme point of the closed unit ball of H^∞. Then the operator T_b maps $\mathcal{M}(\bar{a})$ contractively into $\mathcal{H}(b)$.*

Proof According to Lemma 16.20, the operator T_b acts as a contraction from $\mathcal{H}(\bar{b})$ into $\mathcal{H}(b)$. The result follows since, by Theorem 23.2, we have $\mathcal{H}(\bar{b}) \doteq \mathcal{M}(\bar{a})$. □

According to Theorem 23.2, $\mathcal{H}(\bar{b}) \doteq \mathcal{M}(\bar{a})$, and thus, if $f \in \mathcal{H}(\bar{b})$, then there exists a unique $g \in H^2$ such that

$$f = T_{\bar{a}} g. \tag{23.4}$$

The uniqueness of g follows from the fact that $T_{\bar{a}}$ is injective; see Theorem 12.19(ii). In other words, $T_{\bar{a}}$ is an isometry from H^2 onto $\mathcal{M}(\bar{a})$. Therefore, if $f_1 = T_{\bar{a}} g_1$ and $f_2 = T_{\bar{a}} g_2$, with $g_1, g_2 \in H^2$, then

$$\langle f_1, f_2 \rangle_{\bar{b}} = \langle T_{\bar{a}} g_1, T_{\bar{a}} g_2 \rangle_{\mathcal{M}(\bar{a})} = \langle g_1, g_2 \rangle_2. \tag{23.5}$$

We recall that k_w denotes the Cauchy kernel.

Theorem 23.4 *Let (a, b) be a pair. Then*

$$k_w \in \mathcal{H}(\bar{b}) \qquad (w \in \mathbb{D})$$

and, for every function $f \in \mathcal{H}(\bar{b})$, we have

$$\langle f, k_w \rangle_{\bar{b}} = \frac{g(w)}{a(w)},$$

where $g \in H^2$ is related to f via (23.4). Moreover, we have

$$\|k_w\|_{\bar{b}} = \frac{1}{|a(w)|\,(1 - |w|^2)^{1/2}}. \tag{23.6}$$

Proof According to (12.7), we have $T_{\bar{a}} k_w = \overline{a(w)} k_w$. Since a is outer, then $a(w) \neq 0$ and we can write the last identity as

$$k_w = T_{\bar{a}} \left(\frac{k_w}{\overline{a(w)}} \right). \tag{23.7}$$

This representation shows that $k_w \in \mathcal{M}(\bar{a}) = \mathcal{H}(\bar{b})$ and the function corresponding to k_w via (23.4) is equal to $k_w / \overline{a(w)}$. Therefore, for each $f \in \mathcal{H}(\bar{b})$, by (23.5), we have

$$\langle f, k_w \rangle_{\bar{b}} = a(w)^{-1} \langle g, k_w \rangle_2 = a(w)^{-1} g(w).$$

In particular, if we take $f = k_w$, we obtain

$$\|k_w\|_{\bar{b}}^2 = a(w)^{-1}k_w(w)/\overline{a(w)} = |a(w)|^{-2}(1-|w|^2)^{-1}.$$

Remember, as we established in (4.19), that $k_w(w) = 1/(1-|w|^2)$. □

Recall that, in Section 17.5, we studied the question of inclusion of different $\mathcal{H}(\bar{b})$ spaces. In the case when b is nonextreme, we can state the condition (17.12) in terms of the associated function a.

Corollary 23.5 *Let (a_1, b_1) and (a_2, b_2) be two pairs. Then the following are equivalent:*

(i) $\mathcal{H}(\bar{b}_2) \subset \mathcal{H}(\bar{b}_1)$;
(ii) $a_2/a_1 \in H^\infty$.

Proof (i) $\Longrightarrow$ (ii) By Theorem 17.12, there is a constant $c > 0$ such that

$$1 - |b_2(\zeta)|^2 \le c(1 - |b_1(\zeta)|^2) \qquad (\text{a.e. on } \mathbb{T}).$$

Hence,

$$|a_2|^2 \le c\,|a_1|^2 \qquad (\text{a.e. on } \mathbb{T}).$$

This means that $a_2/a_1 \in L^\infty(\mathbb{T})$. But, since a_1 is outer, the function a_2/a_1 in fact belongs to H^∞.

(ii) $\Longrightarrow$ (i) Assume that $a_2 = a_1 g$, with some function $g \in H^\infty$. Then we have $T_{\bar{a}_2} = T_{\bar{a}_1}T_{\bar{g}}$, which trivially implies that $\mathcal{M}(\bar{a}_2) \subset \mathcal{M}(\bar{a}_1)$. The conclusion follows now from Theorem 23.2, because we have $\mathcal{H}(\bar{b}_k) = \mathcal{M}(\bar{a}_k)$, $k = 1, 2$. □

Exercises

Exercise 23.1.1 Let (a, b) be a pair. Show that

$$|a(\lambda)|^2 + |b(\lambda)|^2 \le 1 \qquad (\lambda \in \mathbb{D}).$$

Moreover, if b is not constant, the inequality is strict.
Hint: (First method) Note that $|a|^2 + |b|^2$ is harmonic and apply the maximum principle for harmonic functions.

(Second method) By Theorem 12.10, we know that, for any $\varphi \in H^\infty$, we have $T_\varphi T_{\bar{\varphi}} \le T_{\bar{\varphi}}T_\varphi$. Apply this inequality to get $\|T_{\bar{a}}k_\lambda\|_2^2 + \|T_{\bar{b}}k_\lambda\|_2^2 \le \|k_\lambda\|_2^2$.

Exercise 23.1.2 Let b be a nonextreme point of the closed unit ball of H^∞, and let a be the associated outer function. Show that $a/b \in H^\infty$ if and only if $\|b\|_\infty < 1$.

23.2 Inclusion of $\mathcal{M}(u)$ into $\mathcal{H}(b)$

Theorem 23.2 reveals that $\mathcal{M}(a)$ is a linear manifold in $\mathcal{H}(b)$. Generally speaking, it is important to distinguish a submanifold of $\mathcal{H}(b)$ that is of the form $\mathcal{M}(u)$ for a certain bounded analytic function u. The following result is a characterization of this type.

Theorem 23.6 *Let (a, b) be a pair, and let u be a function in H^∞. Then the following are equivalent:*

(i) $u/a \in H^\infty$;
(ii) $\mathcal{M}(u) \subset \mathcal{M}(a)$;
(iii) $\mathcal{M}(u) \subset \mathcal{H}(b)$.

Proof (i) $\Longleftrightarrow$ (ii) This is already contained in Theorem 17.1.

(ii) $\Longrightarrow$ (iii) This follows from Theorem 23.2.

(iii) $\Longrightarrow$ (i) According to Lemma 16.6, there is a constant $c > 0$ such that

$$\|f\|_b \le c\,\|f\|_{\mathcal{M}(u)}, \tag{23.8}$$

for every function $f \in \mathcal{M}(u)$. Now applying Theorem 16.7 gives

$$T_u T_{\bar{u}} \le c^2(I - T_b T_{\bar{b}}). \tag{23.9}$$

Applying (23.9) to k_w, $w \in \mathbb{D}$, gives

$$\|T_{\bar{u}} k_w\|_2^2 \le c(\|k_w\|_2^2 - \|T_{\bar{b}} k_w\|_2^2).$$

But, by (12.7), $T_{\bar{u}} k_w = \overline{u(w)} k_w$ and $T_{\bar{b}} k_w = \overline{b(w)} k_w$, and thus we obtain

$$|u(w)|^2 \le c(1 - |b(w)|^2) \qquad (w \in \mathbb{D}).$$

In particular, we deduce from this inequality that

$$|u(\zeta)|^2 \le c(1 - |b(\zeta)|^2) \qquad (\text{a.e. } \zeta \in \mathbb{T}).$$

By definition, we have $|a|^2 = 1 - |b|^2$ almost everywhere on $\mathbb{T}$ and thus we get

$$|u(\zeta)|^2 \le c|a(\zeta)|^2 \qquad (\text{a.e. } \zeta \in \mathbb{T}).$$

Hence, u/a belongs to $L^\infty(\mathbb{T})$. But, since a is outer, Corollary 4.28 ensures that u/a belongs to H^∞. □

Considering the set-theoretic inclusion, Theorem 23.6 also reveals that among spaces $\mathcal{M}(u)$, $u \in H^\infty$, that fulfill $\mathcal{M}(u) \subset \mathcal{H}(b)$, the space $\mathcal{M}(a)$ is the largest one.

Exercise

Exercise 23.2.1 Let (a, b) be a pair, and let u be a function in H^∞. Show that the following are equivalent.

(i) $u/a \in H^\infty$ and $\|u/a\|_\infty \le 1$.
(ii) $\mathcal{M}(u) \hookrightarrow \mathcal{M}(a)$.
(iii) $\mathcal{M}(u) \hookrightarrow \mathcal{H}(b)$.

Hint: See the proof of Theorem 23.6.

23.3 The element f^+

Let $f \in \mathcal{H}(b)$. Thus, using Theorems 17.8 and 23.2, we know that $T_{\bar{b}} f \in \mathcal{H}(\bar{b}) = \mathcal{M}(\bar{a})$. Theorem 12.19(ii) says that $T_{\bar{a}}$ is injective. Therefore, there is a unique element of H^2, henceforth denoted by f^+, such that

$$T_{\bar{b}} f = T_{\bar{a}} f^+. \tag{23.10}$$

It is also useful to mention that, if a function $f \in H^2$ satisfies $T_{\bar{b}} f = T_{\bar{a}} g$, for some function $g \in H^2$, then it follows from Theorems 17.8 and 23.2 that f surely belongs to $\mathcal{H}(b)$ and $g = f^+$. The element f^+ is a useful tool in studying the properties of $f \in \mathcal{H}(b)$. In this section, we study some elementary properties of f^+.

Looking at the definition in (23.10), it is no wonder that this operation is invariant under a Toeplitz operator with a conjugate-analytic symbol.

Lemma 23.7 *Let b be a nonextreme point of the closed unit ball of H^∞, let $f \in \mathcal{H}(b)$ and let $\varphi \in H^\infty$. Then*

$$(T_{\bar{\varphi}} f)^+ = T_{\bar{\varphi}} f^+.$$

Proof We know from Theorem 18.13 that $\mathcal{H}(b)$ is invariant under $T_{\bar{\varphi}}$. Consequently, we have $T_{\bar{\varphi}} f \in \mathcal{H}(b)$. Then, according to Theorem 12.4,

$$T_{\bar{b}} T_{\bar{\varphi}} f = T_{\bar{\varphi}} T_{\bar{b}} f = T_{\bar{\varphi}} T_{\bar{a}} f^+ = T_{\bar{a}} T_{\bar{\varphi}} f^+.$$

Hence, remembering the uniqueness of $(T_{\bar{\varphi}} f)^+$, the identity $T_{\bar{b}}(T_{\bar{\varphi}} f) = T_{\bar{a}}(T_{\bar{\varphi}} f^+)$ means that $(T_{\bar{\varphi}} f)^+ = T_{\bar{\varphi}} f^+$. □

Theorem 23.8 *Let $f_1, f_2 \in \mathcal{H}(b)$. Then we have*

$$\langle f_1, f_2 \rangle_b = \langle f_1, f_2 \rangle_2 + \langle f_1^+, f_2^+ \rangle_2.$$

In particular, for each $f \in \mathcal{H}(b)$,

$$\|f\|_b^2 = \|f\|_2^2 + \|f^+\|_2^2.$$

Proof Using Theorem 17.8, we can write

$$\begin{aligned}\langle f_1, f_2\rangle_b &= \langle f_1, f_2\rangle_2 + \langle T_{\bar b} f_1, T_{\bar b} f_2\rangle_{\bar b}\\ &= \langle f_1, f_2\rangle_2 + \langle T_{\bar a} f_1^+, T_{\bar a} f_2^+\rangle_{\bar b}.\end{aligned}$$

Since $\mathcal{H}(\bar b) = \mathcal{M}(\bar a)$, we have

$$\langle T_{\bar a} f_1^+, T_{\bar a} f_2^+\rangle_{\bar b} = \langle T_{\bar a} f_1^+, T_{\bar a} f_2^+\rangle_{\mathcal{M}(\bar a)}.$$

Since, according to Theorem 12.19(ii), $T_{\bar a}$ is injective, it follows that

$$\langle T_{\bar a} f_1^+, T_{\bar a} f_2^+\rangle_{\mathcal{M}(\bar a)} = \langle f_1^+, f_2^+\rangle_2,$$

and this implies

$$\langle f_1, f_2\rangle_b = \langle f_1, f_2\rangle_2 + \langle f_1^+, f_2^+\rangle_2. \qquad \square$$

Theorem 23.8 is very useful in computing the norm of elements of $\mathcal{H}(b)$. Two such computations are discussed below.

Corollary 23.9 *Let b be a nonextreme point of the closed unit ball of H^∞. Then $b \in \mathcal{H}(b)$, with*

$$b^+ = \frac{1}{\overline{a(0)}} - a,$$

and, moreover, we have

$$\begin{aligned}\|b\|_b^2 &= |a(0)|^{-2} - 1\\ \|S^*b\|_b^2 &= 1 - |b(0)|^2 - |a(0)|^2.\end{aligned}$$

Proof According to Theorems 18.1 and 23.2, we have $b \in \mathcal{H}(b)$ if and only if $T_{\bar b} b \in \mathcal{H}(\bar b) = \mathcal{M}(\bar a)$. But

$$T_{\bar b} b = P_+|b|^2 = P_+(1 - |a|^2) = 1 - T_{\bar a} a,$$

and we can write $1 = P_+(\bar a/\overline{a(0)}) = T_{\bar a}(1/\overline{a(0)})$. Therefore, we obtain

$$T_{\bar b} b = T_{\bar a}\left(\frac{1}{\overline{a(0)}} - a\right) \in \mathcal{M}(\bar a).$$

This fact ensures that $b \in \mathcal{H}(b)$. Moreover, the last identity also reveals that

$$b^+ = \frac{1}{\overline{a(0)}} - a. \tag{23.11}$$

A simple calculation shows that

$$\|b^+\|_2^2 = \|a\|_2^2 + \frac{1}{|a(0)|^2} - 2.$$

Hence, by Theorem 23.8 and the fact that $\|a\|_2^2 + \|b\|_2^2 = 1$, we obtain

$$\begin{aligned} \|b\|_b^2 &= \|b\|_2^2 + \|b^+\|_2^2 \\ &= \|b\|_2^2 + \|a\|_2^2 + \frac{1}{|a(0)|^2} - 2 \\ &= \frac{1}{|a(0)|^2} - 1. \end{aligned}$$

By Lemma 23.7 and (23.11), we see that

$$(S^*b)^+ = -S^*a. \tag{23.12}$$

According to Theorem 23.8 and (8.16), we thus have

$$\begin{aligned} \|S^*b\|_b^2 &= \|S^*b\|_2^2 + \|S^*a\|_2^2 \\ &= \|b\|_2^2 + \|a\|_2^2 - |b(0)|^2 - |a(0)|^2 \\ &= 1 - |b(0)|^2 - |a(0)|^2. \end{aligned}$$

This completes the proof. □

By Theorem 23.2, we know that $\mathcal{M}(\bar{a}) = \mathcal{H}(\bar{b}) \subset \mathcal{H}(b)$. The following result reveals that, in a sense, $\mathcal{M}(\bar{a})$ is a large subset of $\mathcal{H}(b)$. In the extreme case, this is far from being true. For example, if b is inner, then $\mathcal{H}(\bar{b}) = \{0\}$.

Corollary 23.10 *Let b be a nonextreme point of the closed unit ball of H^∞. Then, relative to the topology of $\mathcal{H}(b)$, the space $\mathcal{H}(\bar{b})$ is a dense submanifold of $\mathcal{H}(b)$.*

Proof By Theorem 23.2, $\mathcal{M}(\bar{a}) \overset{=}{=} \mathcal{H}(\bar{b}) \hookrightarrow \mathcal{H}(b)$. Let $f \in \mathcal{H}(b)$ and assume that, relative to the inner product of $\mathcal{H}(b)$, f is orthogonal to $\mathcal{M}(\bar{a})$. Thus, in particular, we have

$$\langle f, T_{\bar{a}} S^{*n} f \rangle_b = 0 \tag{23.13}$$

for all $n \geq 0$. Using Theorem 12.4, we can write

$$T_{\bar{a}} S^{*n} f = T_{\bar{a}} T_{\bar{z}^n} f = T_{\bar{a}\bar{z}^n} f.$$

Again, since $z^n a(z) \in H^\infty$, by Lemma 23.7,

$$(T_{\bar{a}} S^{*n} f)^+ = T_{\bar{a}\bar{z}^n} f^+.$$

Therefore, according to Lemma 4.8 and Theorem 23.8, we have

$$\begin{aligned} \langle f, T_{\bar{a}} S^{*n} f \rangle_b &= \langle f, T_{\bar{a}\bar{z}^n} f \rangle_2 + \langle f^+, T_{\bar{a}\bar{z}^n} f^+ \rangle_2 \\ &= \langle T_{az^n} f, f \rangle_2 + \langle T_{az^n} f^+, f^+ \rangle_2 \\ &= \langle az^n f, f \rangle_2 + \langle az^n f^+, f^+ \rangle_2 \\ &= \frac{1}{2\pi} \int_0^{2\pi} a(e^{i\theta}) [|f(e^{i\theta})|^2 + |f^+(e^{i\theta})|^2] \, e^{in\theta} \, d\theta \\ &= \hat{\varphi}(-n), \end{aligned}$$

where φ denotes the L^1 function defined by $\varphi = (|f|^2 + |f^+|^2)a$ (the function φ belongs to $L^1(\mathbb{T})$ since it is the product of the H^∞ function a and the L^1 function $(|f|^2 + |f^+|^2)$). Thus, (23.13) and the previous computation imply that $\hat{\varphi}(n) = 0$ for all $n \leq 0$. This precisely means that $\varphi \in H_0^1$. Since a is an outer function and $|f|^2 + |f^+|^2 \in L^1(\mathbb{T})$, we deduce from Corollary 4.28 that $|f|^2 + |f^+|^2 \in H_0^1$. Since this function is real-valued, (4.12) implies that $|f|^2 + |f^+|^2 \equiv 0$. In particular, $f \equiv 0$. Therefore, $\mathcal{M}(\bar{a})$ is dense in $\mathcal{H}(b)$. □

Recall that, if $0 < r < 1$, then, by definition, a_r is the unique outer function whose modulus on $\mathbb{T}$ is $(1 - r^2|b|^2)^{1/2}$ and $a_r(0) > 0$. In other words, (a_r, rb) is a pair. Note that, on $\mathbb{T}$, we have

$$|a|^2 = 1 - |b|^2 \leq 1 - r^2|b|^2 = |a_r|^2,$$

which implies that $a/a_r \in L^\infty(\mathbb{T})$. Then, according to Corollary 4.28, the function a/a_r belongs to H^∞ and we have

$$\left\| \frac{a}{a_r} \right\|_\infty \leq 1. \tag{23.14}$$

A similar argument shows that a_r^{-1} belongs to H^∞.

Given a function f in $\mathcal{H}(b)$, the next result gives a method to find the associated function f^+. To give the motivation for the following result, note that, if incidentally $bf/a \in L^2(\mathbb{T})$, then

$$f^+ = P_+(\bar{b}f/\bar{a}). \tag{23.15}$$

Indeed, we have

$$T_{\bar{a}}P_+(\bar{b}f/\bar{a}) = P_+(\bar{a}P_+(\bar{b}f/\bar{a})) = P_+(\bar{a}\bar{b}f/\bar{a}) = T_{\bar{b}}f,$$

which, by uniqueness of f^+, gives the formula (23.15). However, if bf/a does not belong to $L^2(\mathbb{T})$, we appeal to a limiting process to get a similar result.

Theorem 23.11 *Let $f \in \mathcal{H}(b)$. Then*

$$\lim_{r \to 1} \|T_{\bar{b}/\bar{a}_r} f - f^+\|_2 = 0.$$

Proof Since $a_r^{-1} \in H^\infty$, multiplying both sides of $T_{\bar{b}}f = T_{\bar{a}}f^+$ by $T_{1/\bar{a}_r}$ gives

$$T_{\bar{b}/\bar{a}_r} f = T_{\bar{a}/\bar{a}_r} f^+.$$

Hence, by (23.14), we have

$$\|T_{\bar{b}/\bar{a}_r} f\|_2 = \|T_{\bar{a}/\bar{a}_r} f^+\|_2 \leq \left\| \frac{a}{a_r} \right\|_\infty \|f^+\|_2 \leq \|f^+\|_2 \tag{23.16}$$

for all $r \in (0,1)$. Let us now prove that a/a_r tends to 1, as $r \longrightarrow 1$, in the weak-star topology of H^∞. According to Theorem 4.16, this is equivalent to saying that

$$\sup_{0 \le r < 1} \left\| \frac{a}{a_r} \right\|_\infty < +\infty$$

and

$$\lim_{r \to 1} \frac{a(z)}{a_r(z)} = 1 \qquad (z \in \mathbb{D}).$$

The first fact follows immediately from (23.14). To verify the second fact, recall that

$$a_r(z) = \exp\left(\int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} \log |a_r(\zeta)| \, dm(\zeta) \right),$$

and then an application of the dominated convergence theorem gives the result. Consequently, for every $\phi \in L^1(\mathbb{T})$, we have

$$\lim_{r \to 1} \int_{\mathbb{T}} \frac{\bar{a}}{\bar{a}_r} \, \phi \, dm = \int_{\mathbb{T}} \phi \, dm.$$

Now, let $u, v \in H^2$. Since $u\bar{v} \in L^1(\mathbb{T})$, the last identity gives

$$\begin{aligned} \lim_{r \to 1} \langle T_{\bar{a}/\bar{a}_r} u, v \rangle_2 &= \lim_{r \to 1} \langle \bar{a} u / \bar{a}_r, v \rangle_2 \\ &= \lim_{r \to 1} \int_{\mathbb{T}} \frac{\bar{a}}{\bar{a}_r} u\bar{v} \, dm = \int_{\mathbb{T}} u\bar{v} \, dm = \langle u, v \rangle_2. \end{aligned}$$

This means that $T_{\bar{a}/\bar{a}_r} u$ is weakly convergent to u in H^2. Therefore, $T_{\bar{b}/\bar{a}_r} f = T_{\bar{a}/\bar{a}_r} f^+$ weakly converges to f^+ in H^2, as $r \longrightarrow 1$. But, according to (23.16), we have

$$\begin{aligned} \|T_{\bar{b}/\bar{a}_r} f - f^+\|_2^2 &= \|T_{\bar{b}/\bar{a}_r} f\|_2^2 + \|f\|_2^2 - 2\,\Re \langle T_{\bar{b}/\bar{a}_r} f, f^+ \rangle_2 \\ &\le 2\|f^+\|_2^2 - 2\,\Re \langle T_{\bar{b}/\bar{a}_r} f, f^+ \rangle_2. \end{aligned}$$

Hence, we get

$$\limsup_{r \to 1} \|T_{\bar{b}/\bar{a}_r} f - f^+\|_2^2 \le 2\|f^+\|_2^2 - 2 \lim_{r \to 1} \Re \langle T_{\bar{b}/\bar{a}_r} f, f^+ \rangle_2 = 0,$$

from which we deduce that $T_{\bar{b}/\bar{a}_r} f$ actually converges to f^+ in H^2 norm, as $r \longrightarrow 1$. $\square$

Using this fact and Theorem 23.8, we can give another proof of formula (18.20) in the nonextreme case.

Theorem 23.12 *The map* $\mathfrak{G} : h \longmapsto h^+$ *is a partial isometry of* $\mathcal{H}(b)$ *onto* $\mathcal{H}(a)$, *and its kernel is* $\ker T_{\bar{b}} \cap \mathcal{H}(b)$.

Proof Let $h \in \mathcal{H}(b)$. Note that $h^+ \in H^2$ and then $h^+ \in \mathcal{H}(a)$ if and only if $T_{\bar{a}} h^+ \in \mathcal{H}(\bar{a})$. By applying Theorem 23.2 to a (which is of course also a nonextreme point of the closed unit ball of H^∞), then $\mathcal{M}(\bar{b}) = \mathcal{H}(\bar{a})$ and we deduce that

$$T_{\bar{a}} h^+ = T_{\bar{b}} h \in \mathcal{H}(\bar{a}).$$

Hence $h^+ \in \mathcal{H}(a)$. Now, let $\varphi \in \mathcal{H}(a)$. Then $T_{\bar{a}}\varphi \in \mathcal{H}(\bar{a})$. Using Theorem 23.2 once more, there exists $h \in H^2$ such that $T_{\bar{a}}\varphi = T_{\bar{b}} h$. Since $T_{\bar{b}} h \in \mathcal{M}(\bar{a}) = \mathcal{H}(\bar{b})$, we deduce that $h \in \mathcal{H}(b)$ and the last equation gives that $h^+ = \varphi$. That means that $\mathfrak{G}$ is a surjective map from $\mathcal{H}(b)$ onto $\mathcal{H}(a)$.

Let $h \in \mathcal{H}(b)$. Since $T_{\bar{a}}$ is one-to-one, we have

$$\begin{aligned} \mathfrak{G}(h) = 0 &\iff h^+ = 0 \\ &\iff T_{\bar{a}} h^+ = 0 \\ &\iff T_{\bar{b}} h = 0 \\ &\iff h \in \ker T_{\bar{b}}. \end{aligned}$$

Hence $\ker \mathfrak{G} = \ker T_{\bar{b}} \cap \mathcal{H}(b)$.

It remains to check that $\mathfrak{G}$ is a partial isometry. So let $h \in \mathcal{H}(b)$, $h \perp \ker T_{\bar{b}}$. On the one hand, we have

$$\|h\|_b^2 = \|h\|_2^2 + \|h^+\|_2^2,$$

and on the other,

$$\|h^+\|_a^2 = \|h^+\|_2^2 + \|T_{\bar{a}} h^+\|_{\bar{a}}^2 = \|h^+\|_2^2 + \|T_{\bar{b}} h\|_{\mathcal{M}(\bar{b})}^2.$$

Since $h \in \ker T_{\bar{b}}$, we have $\|T_{\bar{b}} h\|_{\mathcal{M}(\bar{b})}^2 = \|h\|_2^2$, which gives

$$\|h^+\|_a^2 = \|h^+\|_2^2 + \|h\|_2^2 = \|h\|_b^2.$$

In other words, $\mathfrak{G}$ is a partial isometry. □

Exercises

Exercise 23.3.1 Assume that b is not an extreme point of the closed unit ball of H^∞.

(i) Prove that

$$r T_{r\bar{b}/\bar{a}_r} b = \overline{a_r^{-1}(0)} - a_r.$$

(ii) Deduce that

$$\|b\|_b^2 = |a(0)|^{-2} - 1.$$

(iii) Prove that, for $n \geq 1$, we have

$$rT_{r\bar{b}/\bar{a}_r}X^n b = -S^{*n}a_r.$$

(iv) Show that $T_{\bar{b}/\bar{a}_r}1 = \overline{b(0)}/\overline{a_r(0)}$.

(v) Deduce that

$$\langle X^n b, 1\rangle_b = \hat{b}(n) - b(0)\hat{a}(n)a(0)^{-1} \qquad (n \geq 1).$$

Hint: Use (iii) and (iv).

Exercise 23.3.2 Assume that b is not an extreme point of the closed unit ball of H^∞ and assume that b has a zero of order m at the origin. Show that

$$\langle X^n b, z^m\rangle_b = \hat{b}(n+m) - \hat{b}(m)\hat{a}(n)a(0)^{-1} \qquad (n \geq 1).$$

Hint: Use Exercise 23.3.1(iii) and Exercise 18.9.3(ii).

Exercise 23.3.3 Assume that b has a zero of order m (possibly 0) at the origin and assume that b is not an extreme point of the closed unit ball of H^∞. Show that

$$\langle X^n b, b\rangle_b = -\hat{a}(n)/a(0) \qquad (n \geq 1).$$

Hint: Use Exercise 18.9.1 with $f = X^n b$ and Exercise 23.3.2.

23.4 Analytic polynomials are dense in $\mathcal{H}(b)$

Theorem 17.4 tells us that the analytic polynomials are dense in $\mathcal{M}(\bar{a})$. Then Theorem 23.2 says that the latter linear manifold is dense and contractively contained in $\mathcal{H}(b)$. Hence, it is natural to deduce some result about the family of analytic polynomials in $\mathcal{H}(b)$.

Theorem 23.13 *Let b be a nonextreme point of the closed unit ball of H^∞, and let $\mathcal{P}$ denote the linear manifold of analytic polynomials. Then the following hold.*

(i) $\mathcal{P} \subset \mathcal{M}(\bar{a}) \subset \mathcal{H}(b)$.
(ii) *$\mathcal{P}$ is a dense manifold in $\mathcal{M}(\bar{a})$.*
(iii) *$\mathcal{P}$ is a dense manifold in $\mathcal{H}(b)$.*

Proof (i) The inclusion $\mathcal{P} \subset \mathcal{M}(\bar{a})$ was shown in Theorem 17.4, and $\mathcal{M}(\bar{a}) \subset \mathcal{H}(b)$ was established in Theorem 23.2.

(ii) This is also from Theorem 17.4.

(iii) Let $f \in \mathcal{H}(b)$ and let $\varepsilon > 0$. According to Corollary 23.10, there exists $g \in \mathcal{M}(\bar{a})$ such that

$$\|f - g\|_b \leq \frac{\varepsilon}{2},$$

and, appealing to part (ii), there is a $p \in \mathcal{P}$ such that

$$\|g - p\|_{\mathcal{M}(\bar{a})} \le \frac{\varepsilon}{2}.$$

But, by Theorem 23.2,

$$\|g - p\|_b \le \|g - p\|_{\mathcal{M}(\bar{a})}.$$

The three inequalities above imply that $\|f - p\|_b \le \varepsilon$. □

Let u_o be the inner part and b_o be the outer part of a function b in the closed unit ball of H^∞. Since $|b_o| = |b|$ a.e. on $\mathbb{T}$, if b is nonextreme, then b_o is also nonextreme. In particular, we will have, according to Theorems 23.13 and 18.7,

$$\mathcal{P} \subset \mathcal{H}(b_o) \subset \mathcal{H}(b).$$

Since $\mathcal{P}$ is dense in $\mathcal{H}(b)$, we immediately get that $\mathcal{H}(b_o)$ is also dense in $\mathcal{H}(b)$. The situation in the extreme case is dramatically different because we will see in Section 25.6 that $\mathcal{H}(b_o)$ is a closed subspace of $\mathcal{H}(b)$ and, if u_o is not a finite Blaschke product, the orthogonal complement of $\mathcal{H}(b_o)$ in $\mathcal{H}(b)$ is of infinite dimension.

23.5 A formula for $\|X_b f\|_b$

We recall that $\mathcal{H}(b)$ is invariant under the backward shift S^* and that the restriction of S^* to $\mathcal{H}(b)$ was denoted by X_b. In this section, we give a formula for $\|X_b f\|_b$.

Theorem 23.14 *Assume that b is a nonextreme point of the closed unit ball of H^∞. Then we have*

$$X_b^* X_b = I - k_0^b \otimes k_0^b - |a(0)|^2 b \otimes b.$$

Moreover, for every $f \in \mathcal{H}(b)$, we have

$$\|X_b f\|_b^2 = \|f\|_b^2 - |f(0)|^2 - |a(0)|^2 |\langle f, b\rangle_b|^2. \tag{23.17}$$

Proof According to Corollary 18.23, we have

$$\begin{aligned} X_b^* X_b f &= SS^* f - \langle X_b f, S^* b\rangle_b b \\ &= f - f(0) - \langle X_b f, X_b b\rangle_b b \\ &= f - f(0) - \langle f, X_b^* X_b b\rangle_b b \end{aligned} \tag{23.18}$$

for every $f \in \mathcal{H}(b)$. By Corollary 23.9, $b \in \mathcal{H}(b)$, and thus by setting $f = b$ in (23.18), we obtain

$$X_b^* X_b b = b - b(0) - \langle b, X_b^* X_b b\rangle_b b = b - b(0) - \|X_b b\|_b^2\, b.$$

Using Corollary 23.9 again and the formula for $X_b b = S^* b$, we simplify the preceding identity to get

$$X_b^* X_b b = (|b(0)|^2 + |a(0)|^2) b - b(0).$$

Plugging the preceding expression for $X_b^* X_b b$ and the formula $f(0) = \langle f, k_0^b \rangle_b$ into (23.18) gives

$$\begin{aligned} X_b^* X_b f &= f - \langle f, k_0^b \rangle_b - (|b(0)|^2 + |a(0)|^2) \langle f, b \rangle_b b + \overline{b(0)} \langle f, 1 \rangle_b \\ &= f - \langle f, k_0^b \rangle_b - |a(0)|^2 \langle f, b \rangle_b b + \overline{b(0)} (\langle f, 1 \rangle_b - b(0) \langle f, b \rangle_b) b \\ &= f - \langle f, k_0^b \rangle_b - |a(0)|^2 \langle f, b \rangle_b b + \overline{b(0)} \langle f, 1 - \overline{b(0)} b \rangle_b b \\ &= f - |a(0)|^2 \langle f, b \rangle_b b - \langle f, k_0^b \rangle_b k_0^b \\ &= (I - k_0^b \otimes k_0^b - |a(0)|^2 b \otimes b) f. \end{aligned}$$

Using this formula for $X_b^* X_b$, we can write

$$\begin{aligned} \|X_b f\|_b^2 &= \langle X_b f, X_b f \rangle_b \\ &= \langle X_b^* X_b f, f \rangle_b \\ &= \big\langle f - \langle f, k_0^b \rangle_b k_0^b - |a(0)|^2 \langle f, b \rangle_b b,\ f \big\rangle_b \\ &= \|f\|_b^2 - |\langle f, k_0^b \rangle_b|^2 - |a(0)|^2 |\langle f, b \rangle_b|^2 \\ &= \|f\|_b^2 - |f(0)|^2 - |a(0)|^2 |\langle f, b \rangle_b|^2. \end{aligned}$$

This completes the proof. □

We recall that, in Corollary 18.27, we proved that the defect operator $D_{X_b^*} = (I - X_b X_b^*)^{1/2}$ has rank one, its range is spanned by $S^* b$ and its nonzero eigenvalue equals $\|S^* b\|_b$. The analogous result for D_{X_b} depends on whether b is an extreme or nonextreme point of the closed unit ball of H^∞.

Corollary 23.15 *Let b be a nonextreme point of the closed unit ball of H^∞. The operator $D_{X_b}^2 = I - X_b^* X_b$ has rank two. It has two eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 1 - |b(0)|^2 - |a(0)|^2$. Moreover, if $e_1 = 1$ and $e_2 = -b(0) k_0^b + |a(0)|^2 b$, then*

$$\ker(D_{X_b}^2 - \lambda_1 I) = \mathbb{C} e_1 \quad \textit{and} \quad \ker(D_{X_b}^2 - \lambda_2 I) = \mathbb{C} e_2.$$

Proof Using Theorem 23.17, we have

$$D_{X_b}^2 = k_0^b \otimes k_0^b + |a(0)|^2 b \otimes b.$$

Since b and k_0^b are linearly independent, $D_{X_b}^2$ has rank two, and it is sufficient to study its restriction to the two-dimensional space $\mathbb{C} k_0^b \oplus \mathbb{C} b$. Relative to the basis (k_0^b, b), this restriction has the following matrix:

$$A = \begin{pmatrix} \|k_0^b\|_b^2 & \langle b, k_0^b \rangle_b \\ |a(0)|^2 \langle k_0^b, b \rangle_b & |a(0)|^2 \|b\|_b^2 \end{pmatrix}.$$

According to (18.8), Theorem 18.11 and Corollary 23.9, we have

$$\|k_0^b\|_b^2 = 1 - |b(0)|^2, \quad \langle b, k_0^b\rangle_b = b(0) \quad \text{and} \quad |a(0)|^2\|b\|_b^2 = 1 - |a(0)|^2.$$

Hence,

$$A = \begin{pmatrix} 1 - |b(0)|^2 & b(0) \\ \overline{b(0)}|a(0)|^2 & 1 - |a(0)|^2 \end{pmatrix}.$$

It is now easy to compute the eigenvalue and eigenvectors of this matrix. The characteristic polynomial is given by

$$\det(A - \lambda I) = \lambda^2 - \lambda(2 - |a(0)|^2 - |b(0)|^2) + 1 - |a(0)|^2 - |b(0)|^2.$$

As already noted, we have $1 - |a(0)|^2 - |b(0)|^2 > 0$. Hence, there are two real roots, which are 1 and $1 - |a(0)|^2 - |b(0)|^2$. Therefore, $\lambda_1 = 1$ and $\lambda_2 = 1 - |a(0)|^2 - |b(0)|^2$ are the two eigenvalues. To compute the eigenvectors, we need to solve linear systems. Let $u = \alpha k_0^b + \beta b$, $\alpha, \beta \in \mathbb{C}$. Then $u \in \ker(D_{X_b}^2 - \lambda_1 I)$ if and only if

$$\begin{pmatrix} 1 - |b(0)|^2 & b(0) \\ \overline{b(0)}|a(0)|^2 & 1 - |a(0)|^2 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}.$$

This equivalent to

$$\begin{cases} \alpha|b(0)|^2 = \beta b(0), \\ \alpha\overline{b(0)}|a(0)|^2 = \beta|a(0)|^2. \end{cases}$$

Since $a(0) \neq 0$, this equivalent to $\beta = \alpha\overline{b(0)}$ and we get that $u \in \ker(D_{X_b}^2 - \lambda_1 I)$ if and only if $u = \alpha k_0^b + \alpha\overline{b(0)}b = \alpha$. This proves that

$$\ker(D_{X_b}^2 - \lambda_1) = \mathbb{C}1.$$

Similarly, $u \in \ker(D_{X_b}^2 - \lambda_2 I)$ if and only if

$$\begin{pmatrix} 1 - |b(0)|^2 & b(0) \\ \overline{b(0)}|a(0)|^2 & 1 - |a(0)|^2 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \lambda_2 \begin{pmatrix} \alpha \\ \beta \end{pmatrix},$$

which is equivalent to

$$\begin{cases} \beta b(0) = \alpha(\lambda_2 - 1 + |b(0)|^2), \\ \alpha\overline{b(0)}|a(0)|^2 = \beta(\lambda_2 - 1 + |a(0)|^2). \end{cases}$$

Using the fact that $\lambda_2 = 1 - |a(0)|^2 - |b(0)|^2$, we see that the system is equivalent to $\alpha = -\beta b(0)/|a(0)|^2$. Hence, $u \in \ker(D_{X_b}^2 - \lambda_1 I)$ if and only if

$$u = -\beta\frac{b(0)}{|a(0)|^2}k_0^b + \beta b = \frac{\beta}{|a(0)|^2}(-b(0)k_0^b + |a(0)|^2 b),$$

which gives

$$\ker(D_{X_b}^2 - \lambda_2 I) = \mathbb{C}(-b(0)k_0^b + |a(0)|^2 b). \qquad \square$$

We are now ready to explicitly determine the defect operator D_{X_b}.

Corollary 23.16 *Let* b *be a nonextreme point of the closed unit ball of* H^∞. *Then the following hold.*

(i) *The operator* D_{X_b} *has rank two and it has two eigenvalues* $\mu_1 = 1$ *and* $\mu_2 = (1 - |b(0)|^2 - |a(0)|^2)^{1/2}$.
(ii) *If* $e_1 = 1$ *and* $e_2 = -b(0)k_0^b + |a(0)|^2 b$, *then we have*

$$\ker(D_{X_b} - \mu_1 I) = \mathbb{C}e_1 \quad \textit{and} \quad \ker(D_{X_b} - \mu_2 I) = \mathbb{C}e_2.$$

(iii) *We have*

$$D_{X_b} = \frac{1}{|a(0)|^2 + |b(0)|^2}\left(|a(0)|^2 e_1 \otimes e_1 + \frac{1}{\mu_2}\, e_2 \otimes e_2\right).$$

Proof Parts (i) and (ii) follow immediately from Corollary 23.15 and the fact that $\mu_\ell = \sqrt{\lambda_\ell}$, $\ell = 1, 2$.

To prove (iii), note that $\langle e_1, e_2 \rangle_b = 0$ since they correspond to eigenvectors associated with different eigenvalues of a self-adjoint operator. With respect to the orthogonal basis (e_1, e_2), the operator D_{X_b} can then be written as

$$D_{X_b} = \frac{1}{\|e_1\|_b^2}\, e_1 \otimes e_1 + \frac{\mu_2}{\|e_2\|_b^2}\, e_2 \otimes e_2.$$

It remains to compute $\|e_1\|_b$ and $\|e_2\|_b$. First, note that $e_1^+ = \overline{b(0)}/\overline{a(0)}$, which gives, using Theorem 23.8,

$$\|e_1\|_b^2 = \|e_1\|_2^2 + \|e_1^+\|_2^2 = 1 + \frac{|b(0)|^2}{|a(0)|^2} = \frac{|a(0)|^2 + |b(0)|^2}{|a(0)|^2}.$$

On the other hand, using Corollary 23.9, we have

$$\begin{aligned}
\|e_2\|_b^2 &= |b(0)|^2 \|k_0^b\|_b^2 + |a(0)|^4 \|b\|_b^2 - 2|a(0)|^2\, \Re(b(0)\langle k_0^b, b\rangle_b) \\
&= |b(0)|^2(1 - |b(0)|^2) + |a(0)|^4 \left(\frac{1}{|a(0)|^2} - 1\right) - 2|a(0)|^2|b(0)|^2 \\
&= (1 - |b(0)|^2 - |a(0)|^2)(|b(0)|^2 + |a(0)|^2) \\
&= \mu_2^2(|b(0)|^2 + |a(0)|^2).
\end{aligned}$$

Finally, we get

$$D_{X_b} = \frac{|a(0)|^2}{|a(0)|^2 + |b(0)|^2}\, e_1 \otimes e_1 + \frac{1}{\mu_2(|b(0)|^2 + |a(0)|^2)}\, e_2 \otimes e_2. \qquad \square$$

23.6 Another representation of $\mathcal{H}(b)$

In Section 19.2, we saw a representation of the $\mathcal{H}(b)$ space based on an abstract functional embedding. In the nonextreme case, we can also give a slightly different representation. Let b be a nonextreme point of the closed unit ball of H^∞ and let a be the outer function defined by (23.2). Denote $\mathbb{H}_b = L^2 \oplus L^2$ along with

$$\begin{array}{rccl} \pi: & L^2 & \longrightarrow & \mathbb{H}_b \\ & f & \longmapsto & bf \oplus (-af), \end{array}$$

and

$$\begin{array}{rccl} \pi_*: & L^2 & \longrightarrow & \mathbb{H}_b \\ & g & \longmapsto & g \oplus 0. \end{array}$$

Theorem 23.17 *The linear mapping $\Pi = (\pi, \pi_*) : L^2 \oplus L^2 \longrightarrow \mathbb{H}_b$ is an abstract functional embedding (AFE).*

Proof For any $f \in L^2$, we have

$$\begin{aligned} \|bf \oplus (-af)\|_{\mathbb{H}_b}^2 &= \|bf\|_2^2 + \|af\|_2^2 \\ &= \int_{\mathbb{T}} (|b|^2 + |a|^2)|f|^2 \, dm \\ &= \|f\|_2^2, \end{aligned}$$

the last equality following from the fact that $|a|^2 + |b|^2 = 1$ a.e. on $\mathbb{T}$. Thus π is an isometry. The map π_* is also clearly an isometry and one can easily check that

$$\pi_*^*(h_1 \oplus h_2) = h_1, \qquad h_1 \oplus h_2 \in L^2 \oplus L^2. \tag{23.19}$$

Now let $f \in H^2$ and $g \in H^2_-$. We have

$$\langle \pi f, \pi_* g\rangle_{\mathbb{H}_b} = \langle bf \oplus (-af),\ g \oplus 0\rangle_{\mathbb{H}_b} = \langle bf, g\rangle_2 = 0,$$

because $bf \in H^2$ and $g \in H^2_-$. That proves that $\pi H^2 \perp \pi_* H^2_-$. By (23.19), we also clearly have

$$\pi_*^* \pi f = \pi_*^*(bf \oplus (-af)) = bf.$$

Thus $\pi_*^*\pi$ is the multiplication operator by b and, in particular, it commutes with the shift operator and maps H^2 into H^2.

Finally, note that $\mathrm{Clos}(aL^2)$ is a reducing invariant subspace for the multiplication operator by z on L^2. Hence, it follows from Theorem 8.29 that there exists a measurable set $E \subset \mathbb{T}$ such that $\mathrm{Clos}(aL^2) = \chi_E L^2$. Since $a \in \chi_E L^2$, a should vanish a.e. on $\mathbb{T} \setminus E$ and then necessarily $m(\mathbb{T} \setminus E) = 0$. That implies that $\mathrm{Clos}(aL^2) = L^2$ and then the range of Π is dense in $\mathbb{H}_b$. $\square$

Let $\mathbb{K}_b$ be the subspace defined by (19.4), and let $\mathbb{K}_b'$ and $\mathbb{K}_b''$ the subspaces defined by (19.7) and (19.6). It will be useful to have the following more explicit transcriptions.

Lemma 23.18 *Let* b *be a nonextreme point of the closed unit ball of* H^∞*. We have:*

(i) $\mathbb{K}_b = (H^2 \oplus L^2) \ominus \{bf \oplus (-af) : f \in L^2\}$;
(ii) $\mathbb{K}_b'' = 0 \oplus H^2_-$;
(iii) $\mathbb{K}_b' = (H^2 \oplus H^2) \ominus \{bf \oplus (-af) : f \in H^2\}$.

Proof (i) Recall that

$$\mathbb{K}_b = \mathbb{H}_b \ominus (\pi(H^2) \oplus \pi_*(H^2_-)).$$

First note that

$$\{bf \oplus (-af) : f \in H^2\} = \pi(H^2),$$

and since π is an isometry, this space is a closed subspace of $H^2 \oplus L^2$. Now let $\varphi \oplus \psi \in L^2 \oplus L^2$. Then $\varphi \oplus \psi \in \mathbb{K}_b$ if and only if

$$\varphi \oplus \psi \perp \{bf \oplus (-af) : f \in H^2\}$$

and

$$\varphi \oplus \psi \perp \pi_*(H^2_-).$$

The second condition gives that, for any $h \in H^2_-$, we have

$$0 = \langle \varphi \oplus \psi,\ \pi_*(h)\rangle_{\mathbb{H}_b} = \langle \varphi \oplus \psi,\ h \oplus 0\rangle_{\mathbb{H}_b} = \langle \varphi, h\rangle_2.$$

This condition is thus equivalent to $\varphi \in H^2$. Thus, we get that

$$\mathbb{K}_b = \{\varphi \oplus \psi : \varphi \in H^2,\ \psi \in L^2 \text{ and } \varphi \oplus \psi \perp bf \oplus (-af),\ f \in H^2\}.$$

(ii) According to Lemma 19.5, we have

$$\mathbb{K}_b'' = \mathbb{K}_b \cap (\pi_*(H^2))^\perp.$$

Then it is clear that $0 \oplus H^2_- \subset \mathbb{K}_b''$. Conversely, if $\varphi \oplus \psi \in \mathbb{K}_b''$, using (i), we first have $\varphi \in H^2$ and

$$\varphi \oplus \psi \perp bf \oplus (-af) \qquad (\forall\, f \in H^2). \tag{23.20}$$

On the other hand, since $\varphi \oplus \psi \perp \pi_*(H^2)$, that gives $\varphi \oplus \psi \perp f \oplus 0$, for any $f \in H^2$. Hence, $\langle \varphi, f\rangle_2 = 0$, $f \in H^2$, which implies that $\varphi \perp H^2$. But, since φ also belongs to H^2, we get that $\varphi = 0$. Now, if we use (23.20), we obtain

$$\langle \psi, af\rangle_2 = 0 \qquad (f \in H^2).$$

Since a is outer, aH^2 is dense in H^2. Hence, $\psi \perp H^2$. We thus obtain that $\varphi \oplus \psi \in 0 \oplus H^2_-$.

(iii) Recall that $\mathbb{K}_b' = \mathbb{K}_b \ominus \mathbb{K}_b''$. Hence, $\varphi \oplus \psi \in \mathbb{K}_b'$ if and only if $\varphi \in H^2$, $\varphi \oplus \psi \perp bf \oplus (-af)$, $f \in H^2$ and $\varphi \oplus \psi \perp 0 \oplus g$, $g \in H^2_-$. The last condition is equivalent to $\psi \perp H^2_-$, which means that $\psi \in H^2$ and that gives the desired description of $\mathbb{K}_b'$. □

According to Theorem 19.8, we know that the map

$$Q_b = \pi^*_{*|\mathbb{K}_b'} : \mathbb{K}_b' \longrightarrow \mathcal{H}(b)$$

is a unitary map. It could be useful to compute its adjoint. We have the following lemma.

Lemma 23.19 *Let b be a nonextreme point of the closed unit ball of H^∞. For any $h \in \mathcal{H}(b)$, we have*

$$Q_b^* h = h \oplus h^+,$$

where we recall that h^+ is the unique function in H^2 such that $T_{\bar{b}} h = T_{\bar{a}} h^+$.

Proof Let $\varphi \oplus \psi \in \mathbb{K}_b'$ and let $h \in \mathcal{H}(b)$. According to Lemma 23.18, $\varphi, \psi \in H^2$ and

$$\langle \varphi, bf \rangle_2 = \langle \psi, af \rangle_2 \qquad (f \in H^2). \tag{23.21}$$

Using Theorem 23.8, we have

$$\begin{aligned} \langle \varphi \oplus \psi,\ Q_b^* h \rangle_{\mathbb{K}_b'} &= \langle Q_b(\varphi \oplus \psi),\ h \rangle_b \\ &= \langle \varphi, h \rangle_b = \langle \varphi, h \rangle_2 + \langle \varphi^+, h^+ \rangle_2. \end{aligned}$$

Let us check that $\varphi^+ = \psi$. Using (23.21), for any $f \in H^2$, we have

$$\langle \bar{b}\varphi, f \rangle_2 = \langle \bar{a}\psi, f \rangle_2,$$

which means that $\bar{b}\varphi - \bar{a}\psi \perp H^2$. In other words, $P_+(\bar{b}\varphi) = P_+(\bar{a}\psi)$. By the uniqueness of φ^+, we get that $\varphi^+ = \psi$. Thus,

$$\langle \varphi \oplus \psi,\ Q_b^* h \rangle_{\mathbb{K}_b'} = \langle \varphi, h \rangle_2 + \langle \psi, h^+ \rangle_2 = \langle \varphi \oplus \psi,\ h \oplus h^+ \rangle_{\mathbb{H}_b}.$$

It remains to note that $h \oplus h^+ \in \mathbb{K}_b'$. We have $h \oplus h^+ \in H^2 \oplus H^2$. Moreover, for any $f \in H^2$, we have

$$\begin{aligned} \langle h \oplus h^+,\ bf \oplus (-af) \rangle_{\mathbb{H}_b} &= \langle h, bf \rangle_2 - \langle h^+, af \rangle_2 \\ &= \langle P_+(\bar{b}h), f \rangle_2 - \langle P_+(\bar{a}h^+), f \rangle_2, \end{aligned}$$

and since $P_+(\bar{b}h) = P_+(\bar{a}h^+)$, we get that $h \oplus h^+ \perp bf \oplus (-af)$ for any $f \in H^2$. According to Lemma 23.18, we can conclude that $h \oplus h^+ \in \mathbb{K}_b'$ and $Q_b^* h = h \oplus h^+$. □

Let

$$\begin{aligned} W :\quad H^2 \oplus H^2 &\longmapsto H^2 \oplus H^2 \\ f \oplus g &\longmapsto zf \oplus zg. \end{aligned}$$

Then W defines a bounded and linear operator on $H^2 \oplus H^2$ and it is clear that W leaves the (closed) subspace $\{bf \oplus (-af) : f \in H^2\}$ invariant. Hence, W^* leaves $\mathbb{K}_b'$ invariant. Furthermore, it is easy to check that

$$\begin{array}{rrcl} W^* : & H^2 \oplus H^2 & \longmapsto & H^2 \oplus H^2 \\ & f \oplus g & \longmapsto & P_+(\bar{z}f) \oplus P_+(\bar{z}g). \end{array}$$

In other words, $W^* = S^* \oplus S^*$.

Theorem 23.20 *Let* b *be a nonextreme point of the closed unit ball of* H^∞. *Then the following diagram is commutative.*

$$\begin{tikzcd} \mathbb{K}_b' \arrow[r, "Q_b"] \arrow[d, "S^* \oplus S^*"'] & \mathcal{H}(b) \arrow[d, "X_b"] \\ \mathbb{K}_b' \arrow[r, "Q_b"'] & \mathcal{H}(b) \end{tikzcd} \tag{23.22}$$

In particular, X_b *is unitarily equivalent to* $(S^* \oplus S^*)_{|\mathbb{K}_b'}$.

Proof Let $f \oplus g \in \mathbb{K}_b'$. Then

$$\begin{aligned} Q_b W^*(f \oplus g) &= Q_b(S^* f \oplus S^* g) \\ &= S^* f \\ &= X_b f \\ &= X_b Q_b(f \oplus g). \end{aligned}$$

This completes the proof. □

In Theorem 19.11, we have given a different representation of $\mathcal{H}(b)$ and a different model for X_b. It is interesting to explore the link between these two representations. This will be done in Exercise 23.6.2.

Exercises

Exercise 23.6.1 Let b be a nonextreme point of the closed unit ball of H^∞ and define

$$\begin{array}{rrcl} T_B : & H^2 & \longrightarrow & H^2 \oplus H^2 \\ & f & \longmapsto & bf \oplus (-af). \end{array}$$

Show that T_B is an isometry and check that $\mathcal{H}(T_B) = \mathbb{K}_b'$.

Exercise 23.6.2 Let b be a nonextreme point of the closed unit ball of H^∞, let $\Delta = (1-|b|^2)^{1/2}$ on $\mathbb{T}$, let $\mathbb{K}'_b$ be defined as in Lemma 23.18, and let

$$\mathcal{K}'_b := H^2 \oplus \text{Clos}(\Delta H^2) \ominus \{bf \oplus \Delta f : f \in H^2\}.$$

For $f, g \in H^2$, define

$$\Omega(f \oplus (-ag)) = f \oplus \Delta g.$$

(i) Show that Ω can be extended into a unitary operator from $H^2 \oplus H^2$ onto $H^2 \oplus \text{Clos}(\Delta H^2)$.
(ii) Show that $\Omega \mathbb{K}'_b = \mathcal{K}'_b$.
(iii) Show that $(S^* \oplus S^*)_{|\mathbb{K}'_b}$ and $(S^* \oplus V_\Delta^*)_{\mathcal{K}'_b}$ are unitarily equivalent and the unitary equivalence is given by Ω.

This result explains the link between the models of X_b given by Theorem 19.11 and Theorem 23.20.

23.7 A characterization of $\mathcal{H}(b)$

In this section, we treat an analog of Theorem 17.24 that characterizes $\mathcal{H}(b)$ spaces when b is a nonextreme point of the closed unit ball of H^∞. To give the motivation, we gather some properties of S^* on $\mathcal{H}(b)$.

Lemma 23.21 *Let b be a nonextreme point of the closed unit ball of H^∞, and $b \not\equiv 0$. Then the following assertions hold.*

(i) *$\mathcal{H}(b)$ is S^*-invariant (we recall that the restriction of S^* to $\mathcal{H}(b)$ was denoted by X_b).*
(ii) *$I - X_b X_b^*$ and $I - X_b^* X_b$, respectively, are operators of rank one and rank two.*
(iii) *For every $f \in \mathcal{H}(b)$,*

$$\|X_b f\|_b^2 \le \|f\|_b^2 - |f(0)|^2.$$

(iv) *There is an element $f \in \mathcal{H}(b)$, with $f(0) \neq 0$, such that*

$$\|X_b f\|_b^2 = \|f\|_b^2 - |f(0)|^2.$$

Proof (i) This was established in Theorem 18.13.

(ii) This follows from Corollaries 18.23 and 23.15.

(iii) According to Theorem 23.14, for every function $f \in \mathcal{H}(b)$, we have

$$\|X_b f\|_b^2 = \|f\|_b^2 - |f(0)|^2 - |a(0)|^2 |\langle f, b\rangle_b|^2. \tag{23.23}$$

This gives the required inequality.

(iv) Define

$$f = \|b\|_b^2 k_0^b - \overline{b(0)}b.$$

By Corollary 23.9, this function belongs to $\mathcal{H}(b)$. Moreover, we have

$$\langle b, f\rangle_b = \|b\|_b^2 \langle b, k_0^b\rangle_b - b(0)\langle b, b\rangle_b = \|b\|_b^2 b(0) - b(0)\|b\|_b^2 = 0,$$

and thus, by (23.23),

$$\|X_b f\|_b^2 = \|f\|_b^2 - |f(0)|^2.$$

It remains to check that $f(0) \neq 0$. Remembering that $\|b\|_b^2 = |a(0)|^{-2} - 1$ (Corollary 23.9), an easy computation shows that

$$f(0) = \frac{1 - |a(0)|^2 - |b(0)|^2}{|a(0)|^2},$$

and thus $f(0) \neq 0$, because $|a(0)|^2 + |b(0)|^2 < 1$. In fact,

$$a(0) = \int_{\mathbb{T}} a(\zeta)\, dm(\zeta) \quad \text{and} \quad b(0) = \int_{\mathbb{T}} b(\zeta)\, dm(\zeta),$$

and thus, using the Cauchy–Schwarz inequality, we get

$$|a(0)|^2 + |b(0)|^2 \leq \int_{\mathbb{T}} (|a(\zeta)|^2 + |b(\zeta)|^2)\, dm(\zeta) = 1.$$

Hence, we have $|a(0)|^2 + |b(0)|^2 = 1$ if and only if

$$\left|\int_{\mathbb{T}} a(\zeta)\, dm(\zeta)\right|^2 = \int_{\mathbb{T}} |a(\zeta)|^2\, dm(\zeta)$$

and

$$\left|\int_{\mathbb{T}} b(\zeta)\, dm(\zeta)\right|^2 = \int_{\mathbb{T}} |b(\zeta)|^2\, dm(\zeta).$$

The last two identities hold provided that b is a constant function, which is absurd. □

Lemma 23.21 provides the motivation for the following characterization of $\mathcal{H}(b)$ spaces.

Theorem 23.22 *Let $\mathcal{H}$ be a Hilbert space contained in H^2. Assume that the following hold.*

(i) *$\mathcal{H}$ is S^*-invariant (and denote the restriction of S^* to $\mathcal{H}$ by T).*
(ii) *The operators $I - TT^*$ and $I - T^*T$, respectively, are of rank one and rank two.*
(iii) *For each $f \in \mathcal{H}$,*

$$\|Tf\|_{\mathcal{H}}^2 \leq \|f\|_{\mathcal{H}}^2 - |f(0)|^2. \tag{23.24}$$

(iv) *There is an element $f \in \mathcal{H}$, with $f(0) \neq 0$, such that*

$$\|Tf\|_{\mathcal{H}}^2 = \|f\|_{\mathcal{H}}^2 - |f(0)|^2.$$

Then there is a nonextreme point b in the closed unit ball of H^∞, unique up to a unimodular constant, such that $\mathcal{H} = \mathcal{H}(b)$.

Proof According to Theorem 16.29, we know that $\mathcal{H}$ is contained contractively in H^2 and, if $\mathcal{M}$ denotes its complementary space, then S acts as a contraction on $\mathcal{M}$ (note that the notation is different in this theorem, and in fact the roles of $\mathcal{M}$ and $\mathcal{H}$ are exchanged). Our strategy is quite simple. We show that S acts as an isometry on $\mathcal{M}$. Then we apply Theorem 17.24 to deduce that there exists a function b in the closed unit ball of H^∞ such that $\mathcal{M} = \mathcal{M}(b)$, and then Corollary 16.27 enables us to conclude that $\mathcal{H} = \mathcal{H}(b)$. However, the proof is very long. To show that S acts as an isometry, we decompose the proof into several steps, 14 in all.

Step 1: T is onto.

This is equivalent to saying that $\ker T^* = \{0\}$ and T has a closed range. Assume that $\ker T^* \neq \{0\}$. Since $\ker T^* \subset \mathcal{R}(I - TT^*)$, by an argument of dimension, we get $\ker T^* = \mathcal{R}(I - TT^*)$. It follows from Theorem 7.22 that T^* is a partial isometry and $\ker T = \mathcal{R}(I - T^*T)$. Hence, by hypothesis, $\dim \ker T = 2$. But, this is impossible because $\ker T \subset \ker S^* = \mathbb{C}$. Thus, $\ker T^* = \{0\}$.

Now, we show that T^*T has a closed range. Indeed, according to the decomposition $\mathcal{H} = \ker(I - T^*T) \oplus \mathcal{R}(I - T^*T)$, the operator T^*T admits the matrix representation

$$T^*T = \begin{pmatrix} I & 0 \\ 0 & T^*T \end{pmatrix},$$

where T^*T is restricted to $\mathcal{R}(I - T^*T)$. But, since $\mathcal{R}(I - T^*T)$ is of finite dimension, the operator $T^*T_{|\mathcal{R}(I-T^*T)}$ has a closed range and then, by Lemma 1.38, the operator T^*T also has a closed range. Then Corollary 1.35 ensures that T is onto.

Step 2: $1 \in \mathcal{H}$ and $f \in \mathcal{H} \implies Sf \in \mathcal{H}$. In particular, all analytic polynomials belong to $\mathcal{H}$.

Argue by absurdity and assume that $1 \notin \mathcal{H}$. Then we would have

$$\ker T = \ker S^* \cap \mathcal{H} = \mathbb{C} \cap \mathcal{H} = \{0\},$$

i.e. T is a bijection. But, since $T(I - T^*T) = (I - TT^*)T$, we would obtain $\dim \mathcal{R}(I - T^*T) = \dim \mathcal{R}(I - TT^*)$, which is a contradiction. Therefore, $1 \in \mathcal{H}$. Furthermore, if $f \in \mathcal{H}$, then $S^*Sf = f - f(0) \in \mathcal{H}$. Since T is onto, there exists $h \in \mathcal{H}$ such that $S^*Sf = Th = S^*h$. This is equivalent to $Sf - h \in \ker S^* = \mathbb{C}$. Thus, $Sf = h - h(0)$, which implies that $Sf \in \mathcal{H}$.

Step 3: The set

$$\mathcal{D} = \{f \in \mathcal{H} : \|Tf\|_{\mathcal{H}}^2 = \|f\|_{\mathcal{H}}^2 - |f(0)|^2\}$$

is a closed subspace of $\mathcal{H}$*. Moreover,* $\ker(I - T^*T) \subset \{f \in \mathcal{D} : f(0) = 0\}$.

It is clear that, if $f \in \mathcal{D}$ and $\lambda \in \mathbb{C}$, then $\lambda f \in \mathcal{D}$. Now, let $f, g \in \mathcal{D}$. We use the parallelogram law twice below. First,

$$\|Tf + Tg\|_{\mathcal{H}}^2 + \|T(f-g)\|_{\mathcal{H}}^2 = 2\|Tf\|_{\mathcal{H}}^2 + 2\|Tg\|_{\mathcal{H}}^2.$$

Second, by the definition of $\mathcal{D}$,

$$\begin{aligned}
&2\|Tf\|_{\mathcal{H}}^2 + 2\|Tg\|_{\mathcal{H}}^2 \\
&\quad = 2\|f\|_{\mathcal{H}}^2 - 2|f(0)|^2 + 2\|g\|_{\mathcal{H}}^2 - 2|g(0)|^2 \\
&\quad = \|f+g\|_{\mathcal{H}}^2 + \|f-g\|_{\mathcal{H}}^2 - |(f+g)(0)|^2 - |(f-g)(0)|^2.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\|Tf + Tg\|_{\mathcal{H}}^2 - \|f+g\|_{\mathcal{H}}^2 + |(f+g)(0)|^2 \\
&\quad = \|f-g\|_{\mathcal{H}}^2 - |(f-g)(0)|^2 - \|T(f-g)\|_{\mathcal{H}}^2.
\end{aligned}$$

According to (23.24), on the one hand, we have

$$\|f-g\|_{\mathcal{H}}^2 - |(f-g)(0)|^2 - \|T(f-g)\|_{\mathcal{H}}^2 \ge 0$$

and, on the other,

$$\|Tf + Tg\|_{\mathcal{H}}^2 = \|T(f+g)\|_{\mathcal{H}}^2 \le \|f+g\|_{\mathcal{H}}^2 - |(f+g)(0)|^2,$$

which is equivalent to

$$\|Tf + Tg\|_{\mathcal{H}}^2 - \|f+g\|_{\mathcal{H}}^2 + |(f+g)(0)|^2 \le 0.$$

Hence, we get

$$\|T(f+g)\|_{\mathcal{H}}^2 = \|f+g\|_{\mathcal{H}}^2 - |(f+g)(0)|^2,$$

which means that $f + g \in \mathcal{D}$. Therefore, $\mathcal{D}$ is a vector subspace of $\mathcal{H}$.

We proceed to prove that $\mathcal{D}$ is closed. Let $f \in \overline{\mathcal{D}}$. Then there exists a sequence $(f_n)_{n\ge1}$ in $\mathcal{D}$ that converges to f in $\mathcal{H}$. Since T is continuous (in fact, according to (23.24), it is a contraction), the sequence $(Tf_n)_{n\ge1}$ converges to Tf in $\mathcal{H}$ and, since $\mathcal{H}$ is contractively contained in H^2, the sequence $(f_n)_{n\ge1}$ is also convergent to f in H^2. In particular, since evaluations at points of $\mathbb{D}$ are continuous on $\mathbb{D}$, the scalar sequence $(f_n(0))_{n\ge1}$ converges to $f(0)$. Since $f_n \in \mathcal{D}$, we have

$$\|Tf_n\|_{\mathcal{H}}^2 = \|f_n\|_{\mathcal{H}}^2 - |f_n(0)|^2.$$

Letting n tend to ∞, we thus get

$$\|Tf\|_{\mathcal{H}}^2 = \|f\|_{\mathcal{H}}^2 - |f(0)|^2,$$

which means that $f \in \mathcal{D}$. Therefore, $\mathcal{D}$ is a closed subspace of $\mathcal{H}$.

It remains to check that $\ker(I - T^*T) \subset \{f \in \mathcal{D} : f(0) = 0\}$. Fix an element $f \in \ker(I - T^*T)$. Then we have $f = T^*Tf$, which implies that

$$\|f\|_{\mathcal{H}}^2 = \langle f, T^*Tf\rangle_{\mathcal{H}} = \|Tf\|_{\mathcal{H}}^2 \leq \|f\|_{\mathcal{H}}^2 - |f(0)|^2 \leq \|f\|_{\mathcal{H}}^2.$$

Thus, $\|Tf\|_{\mathcal{H}}^2 = \|f\|_{\mathcal{H}}^2$ and $f(0) = 0$. In particular, $f \in \mathcal{D}$.

Step 4: There exists $f_0 \in \mathcal{D}$ with $f_0(0) \neq 0$ and $f_0 \perp \ker(I - T^*T)$.

By hypothesis, we know that there is a function $f \in \mathcal{D}$ such that $f(0) \neq 0$. Decompose $f = f_0 + f_1$ such that $f_0 \perp \ker(I - T^*T)$ and $f_1 \in \ker(I - T^*T)$. Using Step 3, we know that $f_1 \in \mathcal{D}$ and $f_1(0) = 0$. Thus, $f_0 \in \mathcal{D}$ and $f_0(0) = f(0) \neq 0$. The function f_0 satisfies the required conditions.

To prove that S acts as an isometry on $\mathcal{M}$, we now consider two situations: $1 \notin \mathcal{D}$ and $1 \in \mathcal{D}$. The verification of the latter is longer (Steps 6–13).

Step 5: S acts as an isometry on $\mathcal{M}$ (case $1 \notin \mathcal{D}$).

Denote by $\mathsf{V}(1, f_0)$ the vector space generated by 1 and f_0. This vector space is of dimension 2 because 1 and f_0 are linearly independent ($1 \notin \mathcal{D}$ and $f_0 \in \mathcal{D}$). Moreover, since $1 = (I - T^*T)1$, the inclusion $\mathsf{V}(1, f_0) \subset \mathcal{R}(I - T^*T)$ holds. Then, with an argument on dimension, we get

$$\mathsf{V}(1, f_0) = \mathcal{R}(I - T^*T),$$

and this implies that

$$\mathcal{H} = \ker(I - T^*T) \oplus \mathsf{V}(1, f_0). \tag{23.25}$$

Using Steps 3 and 4, we have

$$\ker(I - T^*T) \oplus \mathbb{C}f_0 \subset \mathcal{D}.$$

Thus, appealing to Step 1 and (23.25)), we deduce that

$$\mathcal{H} = T\mathcal{H} = T(\ker(I - T^*T) \oplus \mathbb{C}f_2) = T\mathcal{D}.$$

Now, for each $g \in \mathcal{M}$, we have

$$\begin{aligned}
\|g\|_{\mathcal{M}}^2 &= \sup_{f \in \mathcal{H}} (\|g + f\|_2^2 - \|f\|_{\mathcal{H}}^2) \\
&= \sup_{f \in \mathcal{D}} (\|g + Tf\|_2^2 - \|Tf\|_{\mathcal{H}}^2) \\
&= \sup_{f \in \mathcal{D}} (\|Sg + STf\|_2^2 - \|Tf\|_{\mathcal{H}}^2) \\
&= \sup_{f \in \mathcal{D}} (\|Sg + f - f(0)\|_2^2 - \|Tf\|_{\mathcal{H}}^2).
\end{aligned}$$

But, for each $f \in \mathcal{D}$,

$$\begin{aligned}\|Sg + f - f(0)\|_2^2 &= \|Sg + f\|_2^2 + |f(0)|^2 - 2\,\Re\langle Sg + f,\ f(0)\rangle \\ &= \|Sg + f\|_2^2 - |f(0)|^2 \\ &= \|Sg + f\|_2^2 + \|Tf\|_{\mathcal{H}}^2 - \|f\|_{\mathcal{H}}^2.\end{aligned}$$

Thus, we obtain

$$\begin{aligned}\|g\|_{\mathcal{M}}^2 &= \sup_{f\in\mathcal{D}}(\|Sg + f\|_2^2 - \|f\|_{\mathcal{H}}^2) \\ &\le \sup_{f\in\mathcal{H}}(\|Sg + f\|_2^2 - \|f\|_{\mathcal{H}}^2) = \|Sg\|_{\mathcal{M}}^2.\end{aligned}$$

But, from Theorem 16.29, we already know that S acts as a contraction on $\mathcal{M}$ and hence we conclude that S acts as an isometry on $\mathcal{M}$.

For the rest of proof, we assume that $1 \in \mathcal{D}$ and our goal is to show that S still acts as an isometry on $\mathcal{M}$.

Step 6: Suppose that there exists an integer $n \ge 1$ such that $z^m \in \mathcal{D}$, with $0 \le m \le n-1$. Then

$$\|z^m\|_{\mathcal{H}} = 1 \qquad (0 \le m \le n-1).$$

In particular, $i_{\mathcal{H}}^*(z^m) = z^m$, for all $0 \le m \le n-1$, where $i_{\mathcal{H}}$ is the canonical injection from $\mathcal{H}$ into H^2.

We argue by induction. For $m = 0$, since $1 \in \mathcal{D}$, we have

$$\|T1\|_{\mathcal{H}}^2 = \|1\|_{\mathcal{H}}^2 - 1.$$

But, $T1 = S^*1 = 0$, which gives $\|1\|_{\mathcal{H}} = 1$. Assume that, for some m_0 with $0 \le m_0 < n-1$, the identity $\|z^m\|_{\mathcal{H}} = 1$ holds for all $0 \le m \le m_0$. Then, using the fact that $z^{m_0+1} \in \mathcal{D}$, we get

$$\|Tz^{m_0+1}\|_{\mathcal{H}} = \|z^{m_0+1}\|_{\mathcal{H}}.$$

However, $Tz^{m_0+1} = z^{m_0}$, and we deduce that $\|z^{m_0+1}\|_{\mathcal{H}} = \|z^{m_0}\|_{\mathcal{H}} = 1$. Hence, the identity $\|z^m\|_{\mathcal{H}} = 1$ holds for all $0 \le m \le m_0 + 1$. Therefore, by induction, it holds for all $0 \le m \le n-1$.

In the trivial decomposition $z^m = z^m + 0$, we have $z^m \in \mathcal{H}$, $0 \in \mathcal{M}$ and $\|z^m\|_2^2 = \|z^m\|_{\mathcal{H}}^2 + \|0\|_{\mathcal{M}}^2$. Thus, by Corollary 16.28, we have $i_{\mathcal{H}}^* z_m = z_m$ for all $0 \le m \le n-1$.

Step 7: There exists an integer $n \ge 1$ such that $z^m \in \mathcal{D}$, for all $0 \le m \le n-1$, but $z^n \notin \mathcal{D}$.

Assume on the contrary that, for all $k \ge 0$, $z^k \in \mathcal{D}$. Then, according to Step 6, we get $i_{\mathcal{H}}^* z_k = z_k$, for all $k \ge 0$. Therefore, $i_{\mathcal{H}} i_{\mathcal{H}}^* z^k = z^k$, for all $k \ge 0$.

But, z^k is an orthonormal basis of H^2 and thus $i_{\mathcal{H}} i_{\mathcal{H}}^* = I_{H^2}$. In particular, using Corollary 16.8, we get

$$\mathcal{H} = \mathcal{M}(i_{\mathcal{H}}) = \mathcal{M}((i_{\mathcal{H}} i_{\mathcal{H}}^*)^{1/2}) = \mathcal{M}(I_{H^2}) = H^2.$$

Thus, we have $T = S^*$, or equivalently $T^* = S$, which gives $I - TT^* = 0$. This is absurd.

Step 8: Let n be as in Step 7. Then $(I - TT^*)z^{n-1} \neq 0$ and $T^{*n}1 \neq z^n$. Moreover, if $n > 1$, we also have

$$\begin{aligned} T^* z^{m-1} &= z^m, \\ (I - TT^*)z^{m-1} &= 0, \\ T^{*k} z^{m-k} &= z^m, \end{aligned}$$

for all $1 \leq m \leq n-1$ and $0 \leq k \leq m$.

To prove the first relation, we again argue by absurdity. Assume that $(I - TT^*)z^{n-1} = 0$. Since

$$(I - TT^*)z^{n-1} = (I - TT^*)Tz^n = T((I - T^*T)z^n),$$

it would imply that $(I - T^*T)z^n \in \ker T$. But the function $(I - T^*T)z^n$ is also orthogonal to the kernel of T. Indeed, we have $\ker T = \ker S^* \cap \mathcal{H} = \mathbb{C}1$ and, since $n \geq 1$,

$$\begin{aligned} \langle (I - T^*T)z^n,\ 1\rangle_{\mathcal{H}} &= \langle z^n,\ (I - T^*T)1\rangle_{\mathcal{H}} \\ &= \langle z^n, 1\rangle_{\mathcal{H}} \\ &= \langle z^n, i_{\mathcal{H}}^* 1\rangle_{\mathcal{H}} \\ &= \langle i_{\mathcal{H}}(z^n), 1\rangle_2 \\ &= \langle z^n, 1\rangle_2 \\ &= 0. \end{aligned}$$

Thus, $(I - T^*T)z^n \perp \ker T$, which is equivalent to $(I - T^*T)z^n = 0$. This means that $z^n \in \ker(I - T^*T)$. But, by Step 3, we conclude that $z^n \in \mathcal{D}$, a contradiction with the definition of n. Therefore, $(I - TT^*)z^{n-1} \neq 0$.

If $n = 1$, then $(I - TT^*)1 \neq 0$, that is $1 \neq TT^*1$. Hence, $z \neq T^*1$. Now, assume that $n > 1$. We first prove that

$$T^* z^{m-1} = z^m, \qquad \text{for every } 1 \leq m \leq n-1. \tag{23.26}$$

We have

$$\|T^* z^{m-1} - z^m\|_{\mathcal{H}}^2 = \|T^* z^{m-1}\|_{\mathcal{H}}^2 + \|z^m\|_{\mathcal{H}}^2 - 2\,\Re\langle T^* z^{m-1}, z^m\rangle_{\mathcal{H}}$$

and

$$\langle T^* z^{m-1}, z^m\rangle_{\mathcal{H}} = \langle z^{m-1}, Tz^m\rangle_{\mathcal{H}} = \|z^{m-1}\|_{\mathcal{H}}^2.$$

Hence, using Step 6, we get

$$\|T^* z^{m-1} - z^m\|_{\mathcal{H}}^2 = \|T^* z^{m-1}\|_{\mathcal{H}}^2 + 1 - 2 = \|T^* z^{m-1}\|_{\mathcal{H}}^2 - 1. \quad (23.27)$$

But, since T is a contraction on $\mathcal{H}$, we have

$$\|T^* z^{m-1}\|_{\mathcal{H}} \le \|T^*\| \, \|z^{m-1}\| \le 1.$$

Thus (23.27) implies that $\|T^* z^{m-1} - z^m\|_{\mathcal{H}} \le 0$, which gives (23.26).

Since $T^* z^{m-1} = z^m$, we have $TT^* z^{m-1} = z^{m-1}$, and thus

$$(I - TT^*) z^{m-1} = 0 \qquad (1 \le m \le n-1).$$

To prove that $T^{*n} 1 \ne z^n$, we argue by absurdity. Assume that $T^{*n} 1 = z^n$. Then

$$\|z^n\|_{\mathcal{H}}^2 = \langle z^n, z^n \rangle_{\mathcal{H}} = \langle z^n, T^{*n} 1 \rangle_{\mathcal{H}} = \langle T^n z^n, 1 \rangle_{\mathcal{H}}.$$

But, $T^n z^n = 1$, whence

$$\|z^n\|_{\mathcal{H}}^2 = \|1\|_{\mathcal{H}}^2 = 1.$$

In particular, we deduce that

$$\|z^n\|_{\mathcal{H}} = \|z^{n-1}\|_{\mathcal{H}} = \|T z^n\|_{\mathcal{H}}.$$

This means that $z^n \in \mathcal{D}$, which is a contradiction. Thus, we have $T^{*n} 1 \ne z^n$.

Finally, it remains to prove that

$$T^{*k} z^{m-k} = z^m \qquad (0 \le k \le m). \quad (23.28)$$

We argue by induction. For $k = 0$, it is obvious. Now, assume that, for some $0 \le k < m$, we have $T^{*k} z^{m-k} = z^m$. Then using (23.26), we have

$$T^{*(k+1)} z^{m-(k+1)} = T^{*k} (T^* z^{m-k-1}) = T^{*k} z^{m-k} = z^m,$$

which proves (23.28).

Step 9: Let $f \in \mathcal{H}$ and write

$$f(z) = \sum_{m=0}^{n-1} a_m z^m + z^m T^m f(z) \qquad (z \in \mathbb{D}).$$

Then

$$\|f\|_{\mathcal{H}}^2 = \sum_{m=0}^{n-1} |a_m|^2 + \|z^n T^n f\|_{\mathcal{H}}^2.$$

We have

$$\|f\|_{\mathcal{H}}^2 = \left\| \sum_{m=0}^{n-1} a_m z^m \right\|_{\mathcal{H}}^2 + \|z^n T^n f\|_{\mathcal{H}}^2 + 2 \sum_{m=0}^{n-1} \Re(a_m \langle z^m, z^n T^n f\rangle_{\mathcal{H}}).$$

But, using Step 6,

$$\begin{aligned} \langle z^k, z^\ell\rangle_{\mathcal{H}} &= \langle i_{\mathcal{H}}^*(z^k), z^\ell\rangle_{\mathcal{H}} \\ &= \langle z^k, i_{\mathcal{H}}(z^\ell)\rangle_2 \\ &= \langle z^k, z^\ell\rangle_2 \\ &= \delta_{k,\ell} \qquad (0 \le k, \ell \le n-1). \end{aligned}$$

Hence,

$$\left\| \sum_{m=0}^{n-1} a_m z^m \right\|_{\mathcal{H}}^2 = \sum_{m=0}^{n-1} |a_m|^2.$$

Moreover,

$$\begin{aligned} \langle z^m, z^n T^n f\rangle_{\mathcal{H}} &= \langle i_{\mathcal{H}}^*(z^m), z^n T^n f\rangle_{\mathcal{H}} \\ &= \langle z^m, i_{\mathcal{H}}(z^n T^n f)\rangle_2 \\ &= \langle z^m, z^n T^n f\rangle_2 = 0 \qquad (0 \le m \le n-1). \end{aligned}$$

This proves Step 9.

Step 10: For every $f \in \mathcal{H}$ and $g \in \mathcal{M}$, we have

$$\|g + f\|_2^2 - \|f\|_{\mathcal{H}}^2 = \|g + z^n T^n f\|_2^2 - \|z^n T^n f\|_{\mathcal{H}}^2.$$

Write

$$f = \sum_{m=0}^{n-1} a_m z^m + z^n T^n f.$$

Then

$$\begin{aligned} \|g + f\|_2^2 - \|f\|_{\mathcal{H}}^2 &= \left\| g + z^n T^n f + \sum_{m=0}^{n-1} a_m z^m \right\|_2^2 - \|f\|_{\mathcal{H}}^2 \\ &= \|g + z^n T^n f\|_2^2 + \sum_{m=0}^{n-1} |a_m|^2 - \|f\|_{\mathcal{H}}^2 \\ &\quad + 2 \sum_{m=0}^{n-1} \Re(a_m \langle z^m,\ g + z^n T^n f\rangle_2). \end{aligned}$$

Using Step 9, we get

$$\begin{aligned}
&\|g+f\|_2^2-\|f\|_{\mathcal{H}}^2\\
&\quad=\|g+z^nT^nf\|_2^2-\|z^nT^nf\|_{\mathcal{H}}^2+2\sum_{m=0}^{n-1}\Re(a_m\langle z^m,\ g+z^nT^nf\rangle_2).
\end{aligned}$$

But, for every $0\le m\le n-1$, we have

$$\begin{aligned}
\langle z^m,\ g+z^nT^nf\rangle_2&=\langle z^m,g\rangle_2\\
&=\langle z^m,i_{\mathcal{M}}(g)\rangle_2\\
&=\langle i_{\mathcal{M}}^*(z^m),g\rangle_{\mathcal{M}}=0,
\end{aligned}$$

because $i_{\mathcal{M}}^*(z^m)=z^m-i_{\mathcal{H}}^*(z^m)=z^m-z^m=0$. This proves Step 10.

Step 11. For every $f\in\mathcal{H}$, there exists $\hat f\in\ker(I-T^nT^{*n})$ such that

$$\|g+f\|_2^2-\|f\|_{\mathcal{H}}^2=\|g+\hat f\|_2^2-\|\hat f\|_{\mathcal{H}}^2\qquad(g\in\mathcal{M}).$$

Let $f\in\mathcal{H}$, and define the constants $c_0,c_1,\dots,c_{n-1}$ recursively by the formulas

$$\alpha_n=\langle z^{n-1},\ (I-TT^*)z^{n-1}\rangle_{\mathcal{H}},$$

$$c_{n-1}=-\langle f,\ (I-TT^*)z^{n-1}\rangle_{\mathcal{H}}/\alpha_n$$

and, if $n>1$,

$$c_{n-k}=-\left\langle f+\sum_{m=n-k+1}^{n-1}c_mz^m,\ T^{k-1}(I-TT^*)T^{*k-1}z^{n-k}\right\rangle_{\mathcal{H}}\Big/\alpha_n,$$

for $2\le k\le n$. Note that $\alpha_n\ne0$ and thus the sequence $c_0,c_1,\dots,c_{n-1}$ is well defined. Indeed, since $I-TT^*$ is a self-adjoint operator of rank one, there exists an element $g\in\mathcal{H}$ such that $I-TT^*=g\otimes g$, and thus $\alpha_n=|\langle z^{n-1},g\rangle_{\mathcal{H}}|^2$. If $\alpha_n=0$, then it would imply that $\langle z^{n-1},g\rangle_{\mathcal{H}}=0$ and that $(I-TT^*)z^{n-1}=0$, a contradiction with Step 8.

Then we define

$$\hat f=f+\sum_{m=0}^{n-1}c_mz^m,$$

and we show that $\hat f$ satisfies the required properties. We obviously have $T^n\hat f=T^nf$, whence, according to Step 10, we have

$$\begin{aligned}
\|g+f\|_2^2-\|f\|_{\mathcal{H}}^2&=\|g+z^nT^nf\|_2^2-\|z^nT^nf\|_{\mathcal{H}}^2\\
&=\|g+z^nT^n\hat f\|_2^2-\|z^nT^n\hat f\|_{\mathcal{H}}^2\\
&=\|g+\hat f\|_2^2-\|\hat f\|_{\mathcal{H}}^2\qquad(g\in\mathcal{M}).
\end{aligned}$$

Thus, it remains to check that $\hat{f} \in \ker(I - T^n T^{*n})$, which is equivalent to $\hat{f} \perp \mathcal{R}(I - T^n T^{*n})$. But

$$I - T^n T^{*n} = \sum_{k=1}^{n} T^{k-1}(I - TT^*){T^*}^{k-1},$$

whence it is sufficient to prove that $\hat{f} \perp \mathcal{R}(T^{k-1}(I - TT^*){T^*}^{k-1})$. Define $u_k = T^{k-1}(I - TT^*){T^*}^{k-1} z^{n-k}$ and note that $u_k \neq 0$. In fact, according to Step 8, we have

$$\begin{aligned}
\langle z^{n-k}, u_k \rangle_{\mathcal{H}} &= \langle {T^*}^{k-1} z^{n-k},\ (I - TT^*){T^*}^{k-1} z^{n-k} \rangle_{\mathcal{H}} \\
&= \langle z^{n-1},\ (I - TT^*) z^{n-1} \rangle_{\mathcal{H}} \\
&= \alpha_n \neq 0.
\end{aligned}$$

Hence, $T^{k-1}(I - TT^*){T^*}^{k-1}$ is an operator of rank one and its range is generated by u_k. Therefore, $\hat{f} \perp \mathcal{R}(T^{k-1}(I - TT^*){T^*}^{k-1})$ is equivalent to $\hat{f} \perp u_k$, $1 \leq k \leq n$. Now, note that

$$\langle \hat{f}, u_k \rangle_{\mathcal{H}} = \langle f, u_k \rangle_{\mathcal{H}} + \sum_{m=0}^{n-1} c_m \langle z^m, u_k \rangle_{\mathcal{H}}.$$

But, according to the definitions of c_m, we have

$$c_{n-k}\alpha_n = -\left\langle f + \sum_{m=n-k+1}^{n-1} c_m z^m,\ u_k \right\rangle_{\mathcal{H}},$$

whence

$$\langle f, u_k \rangle_{\mathcal{H}} = -c_{n-k}\alpha_n - \sum_{m=n-k+1}^{n-1} c_m \langle z^m, u_k \rangle_{\mathcal{H}} = -\sum_{m=n-k}^{n-1} c_m \langle z^m, u_k \rangle_{\mathcal{H}}.$$

Thus, we get

$$\langle \hat{f}, u_k \rangle_{\mathcal{H}} = \sum_{m=0}^{n-k-1} c_m \langle z^m, u_k \rangle_{\mathcal{H}}.$$

For every $0 \leq m \leq n - k - 1$, we have

$$\begin{aligned}
\langle z^m, u_k \rangle_{\mathcal{H}} &= \langle z^m,\ T^{k-1}(I - TT^*)T^{*(k-1)} z^{n-k} \rangle_{\mathcal{H}} \\
&= \langle z^{m+k-1},\ (I - TT^*) z^{n-1} \rangle_{\mathcal{H}} \\
&= \langle (I - TT^*) z^{m+k-1},\ z^{n-1} \rangle_{\mathcal{H}},
\end{aligned}$$

and, according to Step 8, we have $(I - TT^*) z^{m+k-1} = 0$ (and note that $m + k - 1 \leq n - 2$). Thus, $\langle z^m, u_k \rangle_{\mathcal{H}} = 0$ and $\langle \hat{f}, u_k \rangle_{\mathcal{H}} = 0$, for every $1 \leq k \leq n$. This proves Step 11.

Step 12: If $h \in \ker(I - T^n T^{*n})$, *then*

$$\|h\|_{\mathcal{H}} = \|z^n h\|_{\mathcal{H}}. \tag{23.29}$$

Moreover, for every $g \in \mathcal{M}$, *we have*

$$\|g\|_{\mathcal{M}}^2 = \sup\{\|g+f\|_2^2 - \|z^n f\|_{\mathcal{H}}^2 : f \in \mathcal{H} \text{ and } (I - T^n T^{*n})f = 0\}. \tag{23.30}$$

Take any $h \in \ker(I - T^n T^{*n})$. Then, for every $0 \le m \le n-1$, we have

$$\begin{aligned}
\langle (I - T^{*n}T^n)(z^n h),\ z^m\rangle_{\mathcal{H}} &= \langle z^n h,\ (I - T^{*n}T^n)(z^m)\rangle_{\mathcal{H}}\\
&= \langle z^n h, z^m\rangle_{\mathcal{H}}\\
&= \langle z^n h, i_{\mathcal{H}}^*(z^m)\rangle_{\mathcal{H}}\\
&= \langle z^n h, z^m\rangle_2 = 0.
\end{aligned}$$

This proves that $(I - T^{*n}T^n)(z^n h) \perp \ker T^n$. Moreover,

$$T^n((I - T^{*n}T^n)(z^n h)) = (I - T^n T^{*n})(T^n z^n h) = (I - T^n T^{*n})h = 0.$$

Therefore, $(I - T^{*n}T^n)(z^n h) = 0$, that is $z^n h = T^{*n}T^n(z^n h)$. Thus,

$$\begin{aligned}
\|z^n h\|_{\mathcal{H}}^2 &= \langle z^n h, T^{*n}T^n(z^n f)\rangle_{\mathcal{H}}\\
&= \|T^n(z^n h)\|_{\mathcal{H}}^2 = \|h\|_{\mathcal{H}}^2.
\end{aligned}$$

Now, using Step 11 and (23.29), we get

$$\begin{aligned}
\|g\|_{\mathcal{M}}^2 &= \sup\{\|g+f\|_2^2 - \|f\|_{\mathcal{H}}^2 : f \in \mathcal{H}\}\\
&= \sup\{\|g+f\|_2^2 - \|f\|_{\mathcal{H}}^2 : f \in \mathcal{H} \text{ and } f \in \ker(I - T^n T^{*n})\}\\
&= \sup\{\|g+f\|_2^2 - \|z^n f\|_{\mathcal{H}}^2 : f \in \mathcal{H} \text{ and } f \in \ker(I - T^n T^{*n})\},
\end{aligned}$$

which proves (23.30).

Step 13: S *acts as an isometry on* $\mathcal{M}$ *(case* $1 \in \mathcal{D}$*).*

Since $\|zg\|_{\mathcal{M}} \le \|g\|_{\mathcal{M}}$, for every $g \in \mathcal{M}$ and $\mathcal{H} = T^n\mathcal{H}$, using Step 12, we have

$$\|z^n g\|_{\mathcal{M}}^2 \le \|zg\|_{\mathcal{M}}^2 \le \|g\|_{\mathcal{M}}^2.$$

But

$$\begin{aligned}
\|g\|_{\mathcal{M}}^2 &= \sup_{\substack{f\in\mathcal{H},\\ (I-T^nT^{*n})(T^nf)=0}} \|g + T^n f\|_2^2 - \|z^n T^n f\|_{\mathcal{H}}^2\\
&= \sup_{\substack{f\in\mathcal{H},\\ (I-T^nT^{*n})(T^nf)=0}} \|z^n g + z^n T^n f\|_2^2 - \|z^n T^n f\|_{\mathcal{H}}^2\\
&\le \|z^n g\|_{\mathcal{M}}^2.
\end{aligned}$$

Hence, $\|zg\|_{\mathcal{M}} = \|g\|_{\mathcal{M}}$, which proves Step 13.

Step 14: There is a nonextreme point b in the closed unit ball of H^∞, unique up to a unimodular constant, such that $\mathcal{H} \doteq \mathcal{H}(b)$.

According to Steps 5 and 13, S acts as an isometry on $\mathcal{M}$. Therefore, Theorem 17.24 implies that there exists a function b in the closed unit ball of H^∞ such that $\mathcal{M} \doteq \mathcal{M}(b)$. Now Corollary 16.27 implies that $\mathcal{H} \doteq \mathcal{H}(b)$. Finally, b cannot be an extreme point of the closed unit ball of H^∞, since for instance the analytic polynomials belongs to $\mathcal{H}(b)$ (see Exercise 18.9.4).

This completes the proof of Theorem 23.22. □

23.8 More inhabitants of $\mathcal{H}(b)$

In Section 18.6, we showed that

$$Q_w b \in \mathcal{H}(b) \qquad (w \in \mathbb{D}).$$

It is trivial that the reproducing kernel k_w^b is also in $\mathcal{H}(b)$. In Section 23.4, we saw that the analytic polynomials form a dense manifold in $\mathcal{H}(b)$. Now, we use this information to find more objects in $\mathcal{H}(b)$. Moreover, we also discuss some properties on the newly found elements.

Theorem 23.23 *Let b be a nonextreme point of the closed unit ball of H^∞, and let $w \in \mathbb{D}$. Then*

$$k_w \in \mathcal{H}(b) \quad \textit{and} \quad bk_w \in \mathcal{H}(b).$$

Moreover, for every $f \in \mathcal{H}(b)$, we have

$$\langle f, k_w \rangle_b = f(w) + \frac{b(w)}{a(w)}\, f^+(w) \tag{23.31}$$

and

$$\langle f, bk_w \rangle_b = \frac{f^+(w)}{a(w)}. \tag{23.32}$$

Proof According to Theorems 17.8 and 23.2, the Cauchy kernel k_w belongs to $\mathcal{H}(b)$ if and only if $T_{\bar b} k_w$ belongs to $\mathcal{M}(\bar a)$. But, by (12.7), we have

$$T_{\bar b} k_w = \overline{b(w)} k_w \quad \text{and} \quad T_{\bar a} k_w = \overline{a(w)} k_w,$$

which implies that

$$T_{\bar b} k_w = T_{\bar a}\left(\frac{\overline{b(w)}}{\overline{a(w)}}\, k_w \right).$$

This identity shows that $k_w \in \mathcal{H}(b)$ and, moreover, that

$$k_w^+ = \frac{\overline{b(w)}}{\overline{a(w)}}\, k_w. \tag{23.33}$$

Thus, by Theorem 23.8, for every $f \in \mathcal{H}(b)$, we have

$$\begin{aligned}\langle f, k_w\rangle_b &= \langle f, k_w\rangle_2 + \langle f^+, k_w^+\rangle_2\\ &= \langle f, k_w\rangle_2 + \frac{b(w)}{a(w)}\,\langle f^+, k_w\rangle_2\\ &= f(w) + \frac{b(w)}{a(w)}\,f^+(w).\end{aligned}$$

Remember that k_w is the reproducing kernel of H^2.

Similarly, the function bk_w belongs to $\mathcal{H}(b)$ if and only if the function $T_{\bar{b}}(bk_w)$ belongs to $\mathcal{M}(\bar{a})$. But, once more using $T_{\bar{a}}k_w = \overline{a(w)}k_w$, we obtain

$$\begin{aligned}T_{\bar{b}}(bk_w) &= P_+(|b|^2 k_w)\\ &= P_+((1-|a|^2)k_w)\\ &= k_w - T_{\bar{a}}(ak_w)\\ &= T_{\bar{a}}\left(\frac{k_w}{\overline{a(w)}} - ak_w\right),\end{aligned}$$

which shows that $bk_w \in \mathcal{H}(b)$ and, moreover, that

$$(bk_w)^+ = \left(\frac{1}{\overline{a(w)}} - a\right)k_w. \tag{23.34}$$

Thus, by Theorem 23.8, for every $f \in \mathcal{H}(b)$, we have

$$\begin{aligned}\langle f, bk_w\rangle_b &= \langle f, bk_w\rangle_2 + \langle f^+, (bk_w)^+\rangle_2\\ &= \langle f, bk_w\rangle_2 + \frac{1}{a(w)}\,\langle f^+, k_w\rangle_2 - \langle f^+, ak_w\rangle_2\\ &= \langle f, bk_w\rangle_2 - \langle f^+, ak_w\rangle_2 + \frac{f^+(w)}{a(w)}.\end{aligned}$$

To finish the proof and get the equality (23.32), it remains to notice that, by Lemma 4.8,

$$\begin{aligned}\langle f, bk_w\rangle_2 &= \langle \bar{b}f, k_w\rangle_2\\ &= \langle T_{\bar{b}}f, k_w\rangle_2\\ &= \langle T_{\bar{a}}f^+, k_w\rangle_2\\ &= \langle f^+, ak_w\rangle_2.\end{aligned}$$

This completes the proof. □

If we take $w = 0$ in Theorem 23.23, we obtain the following special case. However, note that the first conclusion was already obtained in Corollary 23.9.

Corollary 23.24 *Let b be a nonextreme point of the closed unit ball of H^∞. Then*

$$b \in \mathcal{H}(b).$$

Moreover, for every $f \in \mathcal{H}(b)$, we have

$$\langle f, 1\rangle_b = f(0) + \frac{b(0)}{a(0)}\, f^+(0)$$

and

$$\langle f, b\rangle_b = \frac{f^+(0)}{a(0)}.$$

Corollary 23.25 *Let $z, w \in \mathbb{D}$. Then we have*

$$\langle k_z, k_w\rangle_b = \left(1 + \frac{\overline{b(z)}b(w)}{\overline{a(z)}a(w)}\right)k_z(w), \tag{23.35}$$

$$\langle k_z, bk_w\rangle_b = \frac{\overline{b(z)}}{\overline{a(z)}a(w)}k_z(w), \tag{23.36}$$

$$\langle bk_z, bk_w\rangle_b = \left(\frac{1}{\overline{a(z)}a(w)} - 1\right)k_z(w). \tag{23.37}$$

Proof Using (23.31) with $f = k_z$, we get

$$\langle k_z, k_w\rangle_b = k_z(w) + \frac{b(w)}{a(w)}\, k_z^+(w).$$

Now, apply (23.33) to obtain (23.35).

If we put $f = k_z$ in (23.32), we obtain

$$\langle k_z, bk_w\rangle_b = \frac{k_z^+(w)}{a(w)} = \frac{\overline{b(z)}}{\overline{a(z)}a(w)}\, k_z(w).$$

Finally, to prove (23.37), we apply (23.32) with $f = bk_z$ and use (23.34). Hence, we have

$$\langle bk_z, bk_w\rangle_b = \frac{(bk_z)^+(w)}{a(w)} = \frac{1}{a(w)}\left(\frac{1}{\overline{a(z)}} - a(w)\right)k_z(w). \qquad \square$$

Note that if we take $z = w$ in (23.35), then we get

$$\|k_w\|_b^2 = \frac{1}{1-|w|^2}\left(1 + \frac{|b(w)|^2}{|a(w)|^2}\right). \tag{23.38}$$

In Theorem 23.13, we showed that analytic polynomials form a dense manifold in $\mathcal{H}(b)$. Knowing that Cauchy kernels are also in $\mathcal{H}(b)$ (Theorem 23.23), we expect to have a similar result for the manifold they create. The following result provides an affirmative answer.

Corollary 23.26 *Let b be a nonextreme point of the closed unit ball of H^∞. Then*

$$\mathrm{Span}(k_w : w \in \mathbb{D}) = \mathcal{H}(b).$$

Proof Let $f \in \mathcal{H}(b)$ be such that $f \perp \mathrm{Span}(k_w : w \in \mathbb{D})$. Then, according to Theorem 23.23, we have

$$f(w) + \frac{b(w)}{a(w)} f^+(w) = 0 \qquad (w \in \mathbb{D}).$$

This is equivalent to $fa = -bf^+$ on $\mathbb{T}$. Multiplying this equality by $\bar{b}$ and using the identity $|a|^2 + |b|^2 = 1$, we obtain

$$a(\bar{b}f - \bar{a}f^+) = -f^+. \tag{23.39}$$

The relation $T_{\bar{b}}f = T_{\bar{a}}f^+$ can be rewritten as $P_+(\bar{b}f - \bar{a}f^+) = 0$, which means that the function $\bar{b}f - \bar{a}f^+$ belongs to $\overline{H_0^2}$. In particular, by (23.39), we deduce that f^+/a belongs to L^2. Now, on the one hand, it follows from Corollary 4.28 that f^+/a belongs to H^2, because a is outer. On the other hand, (23.39) also implies that f^+/a belongs to $\overline{H_0^2}$, whence $f^+/a = 0$. That is, $f^+ = 0$ and then $f = 0$, which proves that the linear span of Cauchy kernels k_w, $w \in \mathbb{D}$, is dense in $\mathcal{H}(b)$. □

Exercise

Exercise 23.8.1 Let (a, b) be a pair. Show that

$$(k_w^b)^+ = \overline{b(w)} a k_w \qquad (w \in \mathbb{D}).$$

Hint: Note that $k_w^b = k_w - \overline{b(w)} b k_w$. Then use (23.33) and (23.34).

23.9 Unbounded Toeplitz operators and $\mathcal{H}(b)$ spaces

In this section, we explain the close relation between $\mathcal{H}(b)$ spaces and unbounded Toeplitz operators with symbols in the Smirnov class. We first recall that the Nevanlinna class $\mathcal{N}$ consists of holomorphic functions in $\mathbb{D}$ that are quotients of functions in H^∞, and the Smirnov class $\mathcal{N}^+$ consists of such quotients in which the denominators are outer functions; see Section 5.1. The representation of such functions as the quotient of two H^∞ functions, even if we assume the denominator is outer, is not unique. However, if we impose some extra conditions, then the representation becomes unique.

Lemma 23.27 *Let φ be a nonzero function in the Smirnov class $\mathcal{N}^+$. Then there exists a unique pair (a, b) such that $\varphi = b/a$.*

Proof By definition, we can write φ as $\varphi = \psi_1/\psi_2$, where $\psi_1, \psi_2 \in H^\infty$, $\psi_1 \neq 0$ and ψ_2 is outer. If the required pair (a, b) exists then, because $|a|^2 + |b|^2 = 1$ a.e. on $\mathbb{T}$, the function a must satisfy the identity

$$\frac{1 - |a|^2}{|a|^2} = \frac{|\psi_1|^2}{|\psi_2|^2} \qquad \text{(a.e. on } \mathbb{T}\text{)},$$

that is,

$$|a|^2 = \frac{|\psi_2|^2}{|\psi_1|^2 + |\psi_2|^2} \qquad \text{(a.e. on } \mathbb{T}\text{)}. \tag{23.40}$$

Since $\psi_2 \in H^\infty$, the function $|\psi_2|^2$ is log-integrable on $\mathbb{T}$ and hence $|\psi_1|^2 + |\psi_2|^2$ is also log-integrable on $\mathbb{T}$. Thus there is a unique function $a \in H^\infty$ that satisfies (23.40) and is positive at the origin. For the function $b = a\varphi$, then we have

$$|a|^2 + |b|^2 = \frac{|\psi_2|^2}{|\psi_1|^2 + |\psi_2|^2} + \frac{|\psi_2|^2}{|\psi_1|^2 + |\psi_2|^2} \frac{|\psi_1|^2}{|\psi_2|^2} = 1 \qquad \text{(a.e. on } \mathbb{T}\text{)}.$$

Hence (a, b) is a pair and the existence of the desired representation of φ is established. The uniqueness holds because the outer function a is uniquely determined by (23.40) and $a(0) > 0$. $\square$

The representation of $\varphi \in \mathcal{N}^+$ given by Lemma 23.27 is called the *canonical representation* of φ.

We start now with a function φ that is holomorphic in $\mathbb{D}$ and define T_φ to be the operator of multiplication by φ on the domain

$$\mathcal{D}(T_\varphi) = \{f \in H^2 : \varphi f \in H^2\}.$$

It is easily seen that T_φ is a closed operator; see Section 7.7. Indeed, let $f_n \in \mathcal{D}(T_\varphi)$ such that $f_n \longrightarrow f$ in H^2 and $\varphi f_n \longrightarrow g$ in H^2. In particular, for each $z \in \mathbb{D}$, we have $f_n(z) \longrightarrow f(z)$ and $(\varphi f_n)(z) \longrightarrow g(z)$. Since $(\varphi f_n)(z)$ also tends to $\varphi(z) f(z)$, we deduce that $\varphi f = g$. In other words, $f \in \mathcal{D}(T_\varphi)$ and $T_\varphi f = g$. Hence, the graph of T_φ, $\mathcal{G}(T_\varphi) = \{f \oplus \varphi f : f \in H^2, \varphi f \in H^2\}$, is closed in $H^2 \oplus H^2$, which means that T_φ is a closed operator.

Lemma 23.28 *Let φ be a function holomorphic on $\mathbb{D}$. Then the following are equivalent:*

(i) $\mathcal{D}(T_\varphi) \neq \{0\}$;
(ii) *φ is in the Nevanlinna class $\mathcal{N}$.*

Proof Suppose that there exists a function $f \neq 0$ that belongs to $\mathcal{D}(T_\varphi)$. Thus $\varphi = \varphi f / f$ is the quotient of two H^2 functions, hence the quotient of two functions in $\mathcal{N}$. Thus, $\varphi \in \mathcal{N}$. Conversely, if φ is in the Nevanlinna class, then we can write $\varphi = \psi_1/\psi_2$, where ψ_1 and ψ_2 are in H^∞. Then $\mathcal{D}(T_\varphi)$ contains the set $\psi_2 H^2$. $\square$

Lemma 23.29 *Let φ be a function holomorphic on $\mathbb{D}$. Then the following are equivalent:*

(i) *$\mathcal{D}(T_\varphi)$ is dense in H^2;*
(ii) *φ is in the Smirnov class $\mathcal{N}^+$.*

Proof (i) $\Longrightarrow$ (ii) Since $\mathcal{D}(T_\varphi)$ is dense, it is in particular not reduced to $\{0\}$. Hence, according to Lemma 23.28, φ is in the Nevanlinna class. Write $\varphi = \psi/\chi$, where ψ and χ are functions in H^∞, whose inner factors are relatively prime. Assume that f is in $\mathcal{D}(T_\varphi)$ and let $g = \varphi f$. Then $\psi f = \chi g$. Write $\psi = \psi_i \psi_o$, $f = f_i f_o$, $\chi = \chi_i \chi_o$ and $g = g_i g_o$, where $\psi_i,\ f_i,\ \chi_i,\ g_i$ are inner and $\psi_o,\ f_o,\ \chi_o,\ g_o$ are outer. By the uniqueness of the canonical factorization for the inner and outer parts, we have $\psi_i f_i = \chi_i g_i$. Since $GCD(\psi_i, \chi_i) = 1$, then χ_i divides f_i, which means that there is an inner function θ_i such that $f_i = \theta_i \chi_i$. Hence, $\psi_o f = \psi_o f_i f_o = \chi_i \theta_i \psi_o f_o$. We get from this relation that $\psi_o f \in \chi H^2$. Using once more the uniqueness of the canonical factorization, we deduce that $f \in \chi_i H^2$. Thus $\mathcal{D}(T_\varphi) \subset \chi_i H^2$. Now, since $\mathcal{D}(T_\varphi)$ is dense in H^2, we conclude by Theorem 8.16 that χ_i must be a constant. In other words, χ must be outer and then $\varphi \in \mathcal{N}^+$.

(ii) $\Longrightarrow$ (i) If $\varphi = \psi/\chi$, where ψ and χ are in H^∞ and χ is outer, then, as noted above, $\mathcal{D}(T_\varphi)$ contains χH^2, which is dense in H^2 by Theorem 8.16. Hence $\mathcal{D}(T_\varphi)$ is also dense in H^2. □

We just have seen that, when $\varphi \in \mathcal{N}^+$, then the domain of T_φ is dense in H^2. Using the canonical representation of φ, we can precisely identify $\mathcal{D}(T_\varphi)$.

Lemma 23.30 *Let φ be a nonzero function in $\mathcal{N}^+$ with canonical representation $\varphi = b/a$. Then*

$$\mathcal{D}(T_\varphi) = aH^2.$$

Proof The inclusion $aH^2 \subset \mathcal{D}(T_\varphi)$ is clear (as noted above). Suppose now that $f \in \mathcal{D}(T_\varphi)$. Then we have

$$|\varphi f|^2 = \frac{|b|^2 |f^2|}{|a|^2} = \left|\frac{f}{a}\right|^2 - |f|^2 \qquad \text{(a.e. on } \mathbb{T}),$$

which implies that f/a is in $L^2(\mathbb{T})$. Since a is outer, Corollary 4.28 implies that f/a is in H^2, giving the inclusion $\mathcal{D}(T_\varphi) \subset aH^2$. □

Since, whenever $\varphi \in \mathcal{N}^+$, the operator T_φ is densely defined and closed, its adjoint T_φ^* is also densely defined and closed. The next result shows that de Branges–Rovnyak spaces naturally occur as the domain of the adjoint of Toeplitz operators with symbols in $\mathcal{N}^+$.

Theorem 23.31 *Let φ be a nonzero function in $\mathcal{N}^+$ with canonical representation $\varphi = b/a$. Then the following assertions hold.*

(i) $\mathcal{D}(T_\varphi^*) = \mathcal{H}(b)$.
(ii) *For each $f \in \mathcal{H}(b)$, we have $T_\varphi^* f = f^+$ and*

$$\|f\|_b^2 = \|f\|_2^2 + \|T_\varphi^* f\|_2^2. \tag{23.41}$$

Proof (i) By definition, a function $f \in H^2$ belongs to $\mathcal{D}(T_\varphi^*)$ if and only if there is a function $g \in H^2$ such that

$$\langle T_\varphi h, f\rangle_2 = \langle h, g\rangle_2 \tag{23.42}$$

for all $h \in \mathcal{D}(T_\varphi)$. By Lemma 23.30, $\mathcal{D}(T_\varphi) = aH^2$, which means that $f \in \mathcal{D}(T_\varphi^*)$ if and only if there is $g \in H^2$ such that

$$\langle T_\varphi(a\psi), f\rangle_2 = \langle a\psi, g\rangle_2 \tag{23.43}$$

for all $\psi \in H^2$. But

$$\langle T_\varphi(a\psi), f\rangle_2 = \langle b\psi, f\rangle_2.$$

Hence, (23.43) is equivalent to

$$\langle b\psi, f\rangle_2 = \langle a\psi, g\rangle_2 \qquad (\psi \in H^2),$$

which can be written as

$$\langle \psi, \bar{b}f - \bar{a}g\rangle_2 = 0 \qquad (\psi \in H^2).$$

In other words, $f \in \mathcal{D}(T_\varphi^*)$ if and only if there exists a function $g \in H^2$ such that

$$T_{\bar{b}} f = T_{\bar{a}} g. \tag{23.44}$$

It follows from Theorems 17.8 and 23.2 that this is equivalent to saying that $f \in \mathcal{H}(b)$.

(ii) If we compare (23.44) and (23.42), we have

$$f^+ = g = T_\varphi^* f.$$

Then, (23.41) follows from Theorem 23.8. □

Exercises

Exercise 23.9.1 Let φ be a rational function in the Smirnov class. Show that the functions a and b in the canonical representation of φ are rational functions. Hint: Assume that $\varphi = p/q$, where p and q are polynomials with GCD $(p, q) = 1$, q has no roots in $\mathbb{D}$ and $q(0) > 0$. Note that the function $|p|^2 + |q|^2$ is a nonnegative trigonometric polynomial. Apply the Fejér–Riesz theorem to get a polynomial r without roots in $\mathbb{D}$, with $r(0) > 0$ and such that $|r|^2 = |p|^2 + |q|^2$; see Theorem 27.19. Note now that $a = q/r$ is a rational function and $b = a\varphi = p/r$ is also a rational function. Verify that (a, b) is a pair and $\varphi = b/a$.

Exercise 23.9.2 Let $\varphi \in \mathcal{N}^+$ and $\psi \in H^\infty$. We denote $T_{\bar{\varphi}} = T_\varphi^*$.

(i) Show that $\mathcal{D}(T_\varphi) \subset \mathcal{D}(T_{\bar{\varphi}})$.
Hint: Use Theorem 23.31 and Lemma 23.30.

(ii) Show that, for any $g \in \mathcal{D}(T_\varphi)$, we have

$$T_{\bar{\varphi}} g = P_+(\bar{\varphi} g).$$

Hint: Note that, for any $f \in \mathcal{D}(T_\varphi)$,

$$\langle T_{\bar{\varphi}} g, f\rangle_2 = \langle g, \varphi f\rangle_2 = \langle \bar{\varphi} g, f\rangle_2 = \langle P_+(\bar{\varphi} g), f\rangle_2.$$

Exercise 23.9.3 Let $\varphi \in \mathcal{N}^+$ and $\psi \in H^\infty$. Show that, for any $f \in \mathcal{D}(T_{\bar{\varphi}})$, we have

$$T_{\bar{\varphi}} T_{\bar{\psi}} f = T_{\bar{\varphi}\bar{\psi}} f = T_{\bar{\psi}} T_{\bar{\varphi}} f.$$

Hint: Note that, if $\varphi = a/b$ is the canonical representation of φ, then $\mathcal{D}(T_{\bar{\varphi}}) = \mathcal{H}(b)$ is invariant under $T_{\bar{\psi}}$. Hence $T_{\bar{\psi}} f \in \mathcal{D}(T_{\bar{\varphi}})$. For $g \in \mathcal{D}(T_\varphi)$, we have

$$\begin{aligned}
\langle T_{\bar{\varphi}} T_{\bar{\psi}} f, g\rangle_2 &= \langle T_{\bar{\psi}} f, \varphi g\rangle_2 \\
&= \langle f, \psi\varphi g\rangle_2 \\
&= \langle T_{\bar{\psi}\bar{\varphi}} f, g\rangle_2,
\end{aligned}$$

which shows that $T_{\bar{\varphi}} T_{\bar{\psi}} f = T_{\bar{\varphi}\bar{\psi}} f$. Argue similarly to prove that $T_{\bar{\psi}} T_{\bar{\varphi}} f = T_{\bar{\varphi}\bar{\psi}} f$.

Notes on Chapter 23

Section 23.1

Theorems 23.2 and 23.3 are due to Sarason [159, lemmas 3, 4 and 5].

Section 23.3

Theorem 23.8 is due to Sarason [159, lemma 2]. The idea of using the element f^+ to compute the norm is very useful and has also been introduced by Sarason in [159]. The power of the method is illustrated by Corollary 23.9. It illustrates very well that the computation of the norm of an element $f \in \mathcal{H}(b)$ is transformed into the resolution of a system $T_{\bar{b}} f = T_{\bar{a}} g$, where we are looking for a solution $g \in H^2$. For instance, the norm of $S^* b$ has been computed by Sarason in [160] using another more difficult method; see Exercise 18.9.5. The computation presented here and based on f^+ is from Sarason's book [166].

In [159], Sarason proved the density of $\mathcal{H}(\bar{b})$ in $\mathcal{H}(b)$, when b is nonextreme; see Corollary 23.10.

The formula of Theorem 23.11 to find the element f^+ by a limiting process is due to Sarason [159].

Exercises 23.3.1, 23.3.2 and 23.3.3 come also from [159].

Section 23.4

The density of polynomials in $\mathcal{M}(\bar{a})$ and $\mathcal{H}(b)$ (in the nonextreme case) proved in Theorem 23.13 is due to Sarason [159, corollary 1].

Section 23.5

Theorem 23.14 and Corollary 23.15 are due to Sarason [160]. In that paper, he is motivated by relating de Branges and Rovnyak's model theory with that of Sz.-Nagy and Foiaş. Thus, he constructs the Sz.-Nagy–Foiaş model of X_b and, for that, he needs to determine the defect operators of the contraction X_b.

Section 23.6

Lemma 23.19 is from [166]. Theorem 23.20 is also due to Sarason [160] and can be rephrased in the context of Sz.-Nagy–Foiaş theory. Indeed, in the case when b is nonextreme, then $\dim \mathcal{D}_{X_b} = 2$ and $\dim \mathcal{D}_{X_b^*} = 1$. Let u_1 and u_2 be a pair of orthogonal unit vectors in $\mathcal{D}_{X_b}$ and let $v = \|S^*b\|_b^{-1} S^*b$ be the unit vector spanning $\mathcal{D}_{X_b^*}$. Then, the operator function Θ_{X_b} is determined by the 1×2 matrix function (θ_1, θ_2), where θ_j is defined by

$$\Theta_{X_b}(\lambda) u_j = \theta_j(\lambda) v \qquad (j = 1, 2).$$

If we replace u_1, u_2 by another orthonormal basis for $\mathcal{D}_{X_b}$, then it will multiply the matrix function (θ_1, θ_2) from the right by a constant 2×2 unit matrix. In [160], Sarason shows that there is a choice of basis (u_1, u_2) such that $\theta_1(\lambda) = \overline{b(\bar{\lambda})}$ and $\theta_2(\lambda) = \overline{a(\bar{\lambda})}$. In this context, Theorem 23.20 says exactly that $S^* \oplus S^*_{|\mathbb{K}'_b}$ is the Sz.-Nagy–Foiaş model of X_b and the projection Q_b implements the unitary equivalence between the operator X_b and its Sz.-Nagy–Foiaş model.

Section 23.7

Theorem 23.22 is due to Guyker [96]. It answers a question raised by de Branges and Rovnyak [65, p. 39]. See also the paper of Leech [116], who obtained other equivalent conditions for a Hilbert space $\mathcal{H}$ to coincide with a de Branges–Rovnyak space $\mathcal{H}(b)$ for some nonextreme function b.

Section 23.8

The fact that the Cauchy kernel k_w belongs to $\mathcal{H}(b)$ when b is nonextreme, as well as the computation of the norm of k_w, are due to Sarason [160, proposition 1]. The two formulas (23.31) and (23.32) that appear in Theorem 23.23 are also due to Sarason [164, proposition].

Corollary 23.26 is from [159], but we have given a different proof.

Section 23.9

Unbounded Toeplitz operators on the Hardy space H^2 arise often with symbols belonging to $L^2(\mathbb{T})$. However, there are natural questions that lead to Toeplitz operators having more restrictive symbols, in particular with symbols in the Smirnov class. We mention interesting works of Helson [101], Suárez [182] and Seubert [174]. The links between $\mathcal{H}(b)$ spaces and the domain of the adjoint of Toeplitz operators with symbols in the Smirnov class are due to Sarason [170].

24

Operators on $\mathcal{H}(b)$ spaces with b nonextreme

In this chapter, we pursue our studies of $\mathcal{H}(b)$ spaces when b is a nonextreme point. We focus on some important operators acting on $\mathcal{H}(b)$. It turns out that, when b is a nonextreme point, then $\mathcal{H}(b)$ is invariant with respect to the forward shift S. This gives rise to the operator S_b that is the restriction of S to $\mathcal{H}(b)$. This operator is introduced and studied in Section 24.1. In particular, we compute its norm and its spectrum, and we show that any point in $\mathbb{D}$ is an eigenvalue for S_b^*. Since $\mathcal{H}(b)$ is invariant under S, it immediately gives that any polynomial is a multiplier of $\mathcal{H}(b)$. In fact, we prove that any function that is analytic on a neighborhood of $\bar{\mathbb{D}}$ is a multiplier of $\mathcal{H}(b)$. In Section 24.2, we get a characterization of the inclusion $H^\infty \subset \mathcal{H}(b)$. We have already seen that, for a function b in the closed unit ball of H^∞, we have $\sigma_p(X_b) \subset \mathbb{D}$. In the case when b is nonextreme we show, in Section 24.3, that equality holds, i.e. $\sigma_p(X_b) = \mathbb{D}$. We also show that X_b^* has no eigenvalues and that $\sigma(X_b^*) = \sigma(X_b) = \bar{\mathbb{D}}$. In Section 24.4, we pursue the study started in Theorem 20.28 that concerns the relations between two elements of the closed unit ball of H^∞, on the one hand, and their Clark measures, on the other. In Section 24.5, we introduce the second important object in the nonextreme case, i.e. the outer function $F_\lambda = a/(1 - \bar{\lambda}b)$, $\lambda \in \mathbb{T}$. We connect this function with the Clark measure μ_λ associated with the function $\bar{\lambda}b$. In Section 24.6, we use F_λ to construct a family of isometries from H^2 into $\mathcal{H}(b)$. These isometries are products of Toeplitz operators (one of them possibly unbounded, but the product is always bounded). Remember that Beurling's theorem says that any nontrivial closed invariant subspace of S^* is of the form $K_\Theta = \mathcal{H}(\Theta)$ for some inner function Θ. In Section 24.7, we give an analog of this result for the operator X_b in the case when b is a nonextreme point. This enables us to characterize cyclic vectors for X_b. They are in fact the same as those of S^*. We also describe the commutants of X_b. In the last section, we study the problem of completeness of the difference quotient.

24.1 The unilateral forward shift operator S_b

In Section 18.5, we saw that $\mathcal{H}(b)$ is invariant under $T_{\bar{\varphi}}$, where $\varphi \in H^\infty$. In particular, it is invariant under $S^* = T_{\bar{z}}$. This fact enabled us to define the operator X_b as the restriction of S^* to $\mathcal{H}(b)$ (see Section 18.7). However, $\mathcal{H}(b)$ is not necessarily invariant under the forward shift operator. Using the theory of multipliers, we have seen in Corollary 20.20 that $\mathcal{H}(b)$ is invariant under the forward shift operator if and only if b is a nonextreme point of the closed unit ball of H^∞. Owing to the importance of this result, we now give a more direct proof of one implication of this result. The other implication also will be proved directly in Section 25.2.

Theorem 24.1 *Let b be a nonextreme point of the closed unit ball of H^∞. Then the space $\mathcal{H}(b)$ is invariant under the unilateral forward shift S. Moreover, we have*

$$\|Sf\|_b \le C\,\|f\|_b \qquad (f \in \mathcal{H}(b)),$$

where $C = C_b$ is a positive constant.

Proof We know from Theorems 18.13 and 18.22 that X_b is a contraction on $\mathcal{H}(b)$ and

$$X_b^* = S - (b \otimes S^*b).$$

Therefore, for each $f \in \mathcal{H}(b)$, we have

$$Sf = X_b^* f + \langle f, S^*b\rangle_b b.$$

It follows from (24.1) and Corollary 23.9 (or Corollary 23.24) that $Sf \in \mathcal{H}(b)$, which means that $\mathcal{H}(b)$ is invariant under S and, moreover, that

$$\begin{aligned}\|Sf\|_b &\le \|X_b^* f\|_b + |\langle f, S^*b\rangle_b|\,\|b\|_b\\ &\le (1 + \|S^*b\|_b\,\|b\|_b)\,\|f\|_b.\end{aligned}$$

This completes the proof. $\square$

Theorem 24.1 shows that the mapping

$$\begin{array}{rccc} S_b: & \mathcal{H}(b) & \longrightarrow & \mathcal{H}(b)\\ & f & \longmapsto & Sf \end{array}$$

gives a well-defined operator on $\mathcal{H}(b)$. Moreover, it is implicitly shown in the proof that

$$\|S_b\|_{\mathcal{L}(\mathcal{H}(b))} \le 1 + \|S^*b\|_b\,\|b\|_b.$$

Moreover, we have

$$S_b = X_b^* + b \otimes S^*b \quad \text{and} \quad S_b^* = X_b + S^*b \otimes b. \tag{24.1}$$

The tensor products are taken in $\mathcal{H}(b)$. In the literature, the operator S_b is also denoted by Y.

The following result will be useful to compute exactly the norm of S_b.

Lemma 24.2 *Let (a, b) be a pair. Then we have*

$$S_b^* S_b = I + |a(0)|^{-2} S^* b \otimes S^* b,$$

where I stands for the identity operator on $\mathcal{H}(b)$.

Proof According to (24.1), we have

$$\begin{aligned} S_b^* S_b &= (X_b + S^* b \otimes b) S_b \\ &= X_b S_b + S^* b \otimes S_b^* b \\ &= I + S^* b \otimes S_b^* b. \end{aligned}$$

Moreover, by Corollary 23.9,

$$\begin{aligned} S_b^* b &= X_b b + (S^* b \otimes b) b \\ &= S^* b + \|b\|_b^2 S^* b \\ &= (1 + \|b\|_b^2) S^* b \\ &= |a(0)|^{-2} S^* b. \end{aligned}$$

Plugging back into the previous set of identities, we get the result. □

Theorem 24.3 *Let b be a nonextreme point of the closed unit ball of H^∞ with the Pythagorean mate a. Then*

$$\|S_b\| = \sqrt{1 + |a(0)|^{-2} \|S^* b\|_b^2} = \frac{\sqrt{1 - |b(0)|^2}}{|a(0)|}.$$

Proof Let $f \in \mathcal{H}(b)$. Using Lemma 24.2, we get

$$\|S_b f\|_b^2 = \langle S_b^* S_b f, f \rangle_b = \|f\|_b^2 + |a(0)|^{-2} |\langle f, S^* b \rangle_b|^2. \tag{24.2}$$

By the Cauchy–Schwarz inequality, we deduce that

$$\|S_b f\|_b^2 \le \|f\|_b^2 + |a(0)|^{-2} \|f\|_b^2 \, \|S^* b\|_b^2,$$

which gives that $\|S_b\| \le \sqrt{1 + |a(0)|^{-2} \|S^* b\|_b^2}$.

To get the reverse inequality, note that, if we apply (24.2) to $f = S^* b$, we obtain

$$\|S_b\|^2 \|S^* b\|_b^2 \ge \|S_b S^* b\|_b^2 = \|S^* b\|_b^2 + |a(0)|^{-2} \|S^* b\|_b^4.$$

Dividing by $\|S^* b\|_b^2$, we deduce that $\|S_b\| \ge \sqrt{1 + |a(0)|^{-2} \|S^* b\|_b^2}$.

For the second identity, remember that $\|S^*b\|_b^2 = 1 - |b(0)|^2 - |a(0)|^2$ (see Corollary 23.9). Hence

$$\begin{aligned}1 + |a(0)|^{-2}\|S^*b\|_b^2 &= 1 + |a(0)|^{-2}(1 - |b(0)|^2 - |a(0)|^2)\\ &= |a(0)|^{-2}(1 - |b(0)|^2),\end{aligned}$$

which gives the second equality. □

In particular, we see that, according to Theorem 24.3, $\|S_b\| > 1$ and thus the shift operator S does not act as an isometry, or even as a contraction, on $\mathcal{H}(b)$. This is a major difference with the shift operator on H^2 (which is obviously an isometry). The reader should keep in mind this difference.

We know that $\sigma(S) = \bar{\mathbb{D}}$. The following result shows that S_b has the same property. However, the proof is somewhat more delicate than the classic case for S.

Theorem 24.4 *Let b be a nonextreme point of the closed unit ball of H^∞. Then*

$$\sigma(S_b) = \bar{\mathbb{D}}.$$

Proof Suppose that $|\lambda| > 1$. Since X_b^* is a contraction on $\mathcal{H}(b)$, the operator $X_b^* - \lambda I$ is invertible. Hence, by (24.1),

$$S_b - \lambda I = (X_b^* - \lambda I) + (b \otimes S^*b).$$

Lemma 7.33 now ensures that $S_b - \lambda I$ is a Fredholm operator with

$$\operatorname{ind}(S_b - \lambda I) = 0.$$

In other words, $S_b - \lambda I$ has a closed range and

$$\dim\ker(S_b - \lambda I) = \dim\ker(S_b^* - \bar{\lambda} I).$$

But we know from Lemma 8.6 that

$$\ker(S_b - \lambda I) \subset \ker(S - \lambda I) = \{0\},$$

which implies that $\ker(S_b^* - \bar{\lambda} I) = \{0\}$. In the light of Theorem 1.30, this latter fact shows that the range of $S_b - \lambda I$ is dense in $\mathcal{H}(b)$. But the range of $S_b - \lambda I$ is also closed in $\mathcal{H}(b)$, and thus we have $\mathcal{R}(S_b - \lambda I) = \mathcal{H}(b)$. Therefore, $S_b - \lambda I$ is invertible. This means that $\sigma(S_b) \subset \bar{\mathbb{D}}$.

To prove the converse inclusion, let $\lambda \in \mathbb{D}$. It is easy to see that

$$\bigcap_{n=0}^{\infty} (S_b - \lambda I)^n \mathcal{H}(b) = \{0\}. \tag{24.3}$$

In fact,

$$\bigcap_{n=0}^{\infty} (S_b - \lambda I)^n \mathcal{H}(b) \subset \bigcap_{n=0}^{\infty} (S - \lambda I)^n H^2,$$

and, if f belongs to the right set, then, for each $n \geq 0$, there exists $f_n \in H^2$ such that $f = (z - \lambda)^n f_n$. In particular, we must have $f^{(n)}(\lambda) = 0$, $n \geq 0$. Since f is analytic on $\mathbb{D}$, we conclude that $f \equiv 0$.

If $S_b - \lambda I$, with $\lambda \in \mathbb{D}$, is invertible on $\mathcal{H}(b)$, then, for each $n \geq 0$, $(S_b - \lambda I)^n$ is also invertible on $\mathcal{H}(b)$. In particular, we have $(S_b - \lambda I)^n \mathcal{H}(b) = \mathcal{H}(b)$, which contradicts (24.3). Thus, $\lambda \in \sigma(S_b)$.

Up to now, we have proved that

$$\mathbb{D} \subset \sigma(S_b) \subset \bar{\mathbb{D}}.$$

Since $\sigma(S_b)$ is a closed set in $\mathbb{D}$, we thus have $\sigma(S_b) = \bar{\mathbb{D}}$. □

Theorem 24.4 says that $\sigma(S_b) = \bar{\mathbb{D}}$, and according to Lemma 8.6, we have

$$\sigma_p(S_b) = \emptyset. \tag{24.4}$$

We now deduce the spectrum of S_b^* from the general theory of analytic reproducing kernel Hilbert spaces developed in Chapter 9.

Theorem 24.5 *Let b be a nonextreme point in the closed unit ball of H^∞. Then*

$$\sigma(S_b^*) = \bar{\mathbb{D}} \quad \textit{and} \quad \mathbb{D} \subset \sigma_p(S_b^*).$$

Moreover, for every $w \in \mathbb{D}$, we have

$$\ker(S_b^* - \bar{w} I) = \mathbb{C}\, k_w^b.$$

Proof The equality $\sigma(S_b^*) = \bar{\mathbb{D}}$ follows immediately from Theorems 24.4 and 1.30. The result on the point spectrum of S_b^* follows from Theorem 9.15 and Corollary 18.14. □

We will find the boundary eigenvalues of S_b^* in Section 28.8.

By Theorem 24.1, $\mathcal{H}(b)$ is invariant under the forward shift operator S. In other words, we have

$$f \in \mathcal{H}(b) \quad \Longrightarrow \quad zf \in \mathcal{H}(b).$$

By induction, we deduce that

$$f \in \mathcal{H}(b) \quad \Longrightarrow \quad pf \in \mathcal{H}(b),$$

where p is any analytic polynomial. This means that p is a multiplier of $\mathcal{H}(b)$. This observation is generalized below.

For the following result, we recall that $\mathfrak{Mult}(\mathcal{H}(b))$ denotes the space of multipliers of $\mathcal{H}(b)$ (see Section 9.1), and we also recall that the family $\mathrm{Hol}(\bar{\mathbb{D}})$ consists of all functions that are analytic on a domain containing $\bar{\mathbb{D}}$.

Theorem 24.6 *Let b be a nonextreme point of the closed unit ball of H^∞ with Pythagorean mate a. Then*

$$\mathrm{Hol}(\bar{\mathbb{D}}) \subset \mathfrak{Mult}(\mathcal{H}(b)) \subset \mathfrak{Mult}(\mathcal{M}(\bar{a})).$$

In particular, we also have

$$\mathrm{Hol}(\bar{\mathbb{D}}) \subset \mathcal{M}(\bar{a}) \subset \mathcal{H}(b).$$

Proof Let $f \in \mathrm{Hol}(\bar{\mathbb{D}})$. Then there exists $R > 1$ such that f is analytic on $D(0,R)$. Since $\sigma(S_b) = \bar{\mathbb{D}} \subset D(0,R)$, we can use the Riesz–Dunford functional calculus and define

$$f(S_b) = \frac{1}{2\pi} \int_0^{2\pi} f(re^{i\vartheta}) re^{i\vartheta} (S_b - re^{i\vartheta} I)^{-1}\, d\vartheta \qquad (1 < r < R).$$

Thus $f(S_b)$ is a well-defined bounded operator on $\mathcal{H}(b)$. In particular, for every $g \in \mathcal{H}(b)$, we have $f(S_b)g \in \mathcal{H}(b)$. Since f is analytic on $D(0,R)$, we can write

$$f(re^{i\vartheta}) = \sum_{n=0}^{\infty} a_n r^n e^{in\vartheta}.$$

Then we have

$$\begin{aligned} f(S_b)g &= \sum_{n=0}^{\infty} a_n r^{n+1} \int_0^{2\pi} e^{i(n+1)\vartheta} (S_b - re^{i\vartheta} I)^{-1} g\, d\vartheta \\ &= \sum_{n=0}^{\infty} a_n \chi_n(S_b) g, \end{aligned}$$

where we recall that $\chi_n(z) = z^n$, $n \geq 0$. Since the Riesz–Dunford calculus extends the polynomial calculus, we have $\chi_n(S_b)g = \chi_n g$, which gives

$$f(S_b)g = \sum_{n=0}^{\infty} a_n \chi_n g.$$

In other words, $f(S_b)g = fg$. This exactly means that f is a multiplier of $\mathcal{H}(b)$.

Theorem 20.17 says that every multiplier of $\mathcal{H}(b)$ is also a multiplier of $\mathcal{H}(\bar{b})$. But, by Theorem 23.2, $\mathcal{H}(\bar{b}) = \mathcal{M}(\bar{a})$. Hence, each $f \in \mathrm{Hol}(\bar{\mathbb{D}})$ is also a multiplier of $\mathcal{M}(\bar{a})$. Finally, it remains to note that, by Theorem 23.13, the constant function 1 belongs to $\mathcal{M}(\bar{a})$ and thus we have $f = f1 \in \mathcal{M}(\bar{a})$. □

If $\varphi \in \mathfrak{Mult}(\mathcal{H}(b))$, then M_φ is a bounded operator on $\mathcal{H}(b)$, i.e. $M_\varphi \in \mathcal{L}(\mathcal{H}(b))$. Since $S_b = M_z$, for each multiplier φ, we must have

$$S_b M_\varphi = M_\varphi S_b.$$

But, more interestingly, we show that the multiplication operators are the only ones that commute with S_b. This follows again from the general theory on analytic reproducing kernel Hilbert spaces.

Theorem 24.7 *Let b be a nonextreme point in the closed unit ball of H^∞, and let*

$$A \in \mathcal{L}(\mathcal{H}(b)).$$

Then the following are equivalent.

(i) $AS_b = S_b A$.
(ii) *There exists $\varphi \in \mathfrak{Mult}(\mathcal{H}(b))$ such that $A = M_\varphi$.*

Proof According to Theorem 24.5, we can apply Theorem 9.16, which immediately gives the result. □

If we apply Corollary 9.17, then we get the following result.

Corollary 24.8 *Let b be a nonextreme point of the closed unit ball of H^∞ such that $\mathfrak{Mult}(\mathcal{H}(b)) = H^\infty$. Then, relative to the norm topology,*

$$\mathcal{F} = \{M_\varphi : \varphi \in H^\infty\}$$

is a closed subspace of $\mathcal{L}(\mathcal{H}(b))$. Moreover, there is a constant c, not depending on φ, such that

$$\|M_\varphi\| \le c\,\|\varphi\|_\infty \qquad (\varphi \in H^\infty).$$

Exercises

Exercise 24.1.1 Let (a, b) be a pair. Show that

$$\|S_b b\|_b^2 = |a(0)|^{-2} - 1 + |a(0)|^{-4}|a'(0)|^2.$$

Hint: Use Lemma 24.2 to get

$$\|S_b b\|_b^2 = \|b\|_b^2 + |a(0)|^{-2}|\langle b, S^* b\rangle_b|^2.$$

Apply Corollary 23.9 to compute $\|b\|_b^2$. To compute $\langle b, S^* b\rangle_b$, use Theorem 23.8 and the facts that $b^+ = -a + \overline{a(0)}^{-1}$, $(S^* b)^+ = -S^* a$ and $\langle Sb, b\rangle_2 + \langle Sa, a\rangle_2 = 0$.

Exercise 24.1.2 Let b be a nonextreme point of the closed unit ball of H^∞. Assume that

$$\|S_b f\|_b = \|f\|_b \qquad (f \in \mathcal{H}(b)).$$

Show that b is constant.

Hint: Use Lemma 24.2 to get $\|S_b f\|_b^2 = \|f\|_b^2 + |a(0)|^{-2}|\langle f, S^*b\rangle_b|^2$. Hence, if $\|Sf\|_b = \|f\|_b$, then $\langle f, S^*b\rangle_b = 0$ for every $f \in \mathcal{H}(b)$, which gives that $S^*b = 0$.

24.2 A characterization of $H^\infty \subset \mathcal{H}(b)$

We recall that a_r denotes the unique outer function such that

$$|a_r| = (1 - r^2|b|^2)^{1/2} \qquad (\text{on } \mathbb{T})$$

and $a_r(0) > 0$; see Section 18.9. To study the possibility of $H^\infty \subset \mathcal{H}(b)$, we start with the following lemma.

Lemma 24.9 *Let b be a nonextreme point of the closed unit ball of H^∞ with the Pythagorean mate a. Let $f \in \mathcal{H}(b)$. Then the following assertions are equivalent:*

(i) $\sup_{n\geq 0} \|S_b^n f\|_b < \infty$;
(ii) $f \in \mathcal{M}(a)$;
(iii) $T_{\bar{b}} f \in \mathcal{M}(a)$.

In this case, we have

$$\sup_{n\geq 0} \|S_b^n f\|_b = \|f/a\|_2. \tag{24.5}$$

Proof (i) $\Longrightarrow$ (ii) According to Theorem 23.11 and Lemma 18.30(iii), for all $n \geq 0$ and $0 < r < 1$, we have

$$r \left\| P_+ \left(\frac{\bar{b}}{\bar{a}_r} z^n f \right) \right\|_2 \leq \|(z^n f)^+\|_2. \tag{24.6}$$

By Lemma 12.1,

$$\lim_{n\to\infty} \|P_+(z^n g)\|_2 = \|g\|_2 \qquad (g \in L^2(\mathbb{T})).$$

Thus, letting $n \longrightarrow \infty$ in (24.6) gives

$$r \left\| \frac{\bar{b}}{\bar{a}_r} f \right\|_2 \leq \sup_{n\geq 0} \|(z^n f)^+\|_2,$$

which is equivalent to

$$\left\| \frac{\bar{b}}{\bar{a}_r} f \right\|_2 \leq \frac{1}{r} \sup_{n\geq 0} \|(z^n f)^+\|_2.$$

Now, letting $r \longrightarrow 1$, we conclude by Fatou's lemma that $\bar{b}f/\bar{a} \in L^2(\mathbb{T})$ and

$$\left\| \frac{\bar{b}}{\bar{a}} f \right\|_2 \leq \sup_{n\geq 0} \|(z^n f)^+\|_2.$$

Hence, since $|a|^2 + |b|^2 = 1$ a.e. on $\mathbb{T}$, we deduce that

$$\begin{aligned}
\left\| \frac{f}{a} \right\|_2^2 &= \|f\|_2^2 + \left\| \frac{\bar{b}}{\bar{a}} f \right\|_2^2 \\
&\leq \sup_{n\geq 0} (\|z^n f\|_2^2 + \|(z^n f)^+\|_2^2) \\
&= \sup_{n\geq 0} \|z^n f\|_b^2.
\end{aligned}$$

Thus $f/a \in L^2(\mathbb{T})$. Corollary 4.28 now implies that $f/a \in H^2$ and

$$\left\| \frac{f}{a} \right\|_2 \leq \sup_{n\geq 0} \|z^n f\|_b.$$

(ii) $\Longrightarrow$ (iii) This is trivial.

(iii) $\Longrightarrow$ (i) Assume that there is a function $g \in H^2$ such that

$$T_{\bar{b}} f = T_{\bar{a}} g.$$

According to Theorem 12.4, we have,

$$\begin{aligned}
T_{\bar{b}} S_b^n f &= T_{\bar{b}/b} T_b S^n f \\
&= T_{\bar{b}/b} S^n T_b f \\
&= T_{\bar{b}/b} S^n T_a g \\
&= T_{\bar{b}/b} T_a S^n g \\
&= T_{\bar{b}a/b} S^n g \\
&= T_{\bar{a}} T_{a\bar{b}/\bar{a}b} S^n g \qquad (n \geq 0).
\end{aligned}$$

In other words,

$$(S_b^n f)^+ = T_{a\bar{b}/\bar{a}b} S^n g \qquad (n \geq 0).$$

Therefore, by Theorem 23.8, we get

$$\|S_b^n f\|_b^2 = \|S^n f\|_2^2 + \|T_{a\bar{b}/\bar{a}b} S^n g\|_2^2 \leq \|f\|_2^2 + \|g\|_2^2.$$

To conclude the proof, it remains to note that

$$\|f\|_2^2 + \|g\|_2^2 = \|f\|_2^2 + \left\| \frac{bf}{a} \right\|_2^2 = \left\| \frac{f}{a} \right\|_2^2. \qquad \square$$

The next result gives a set of necessary and sufficient conditions for $\mathcal{H}(b)$ to contain H^∞.

Theorem 24.10 *Let (a, b) be a pair. Then the following assertions are equivalent:*

(i) $H^\infty \subset \mathcal{H}(b)$;

(ii) $\sup_{n\geq 0} \|z^n\|_b < \infty$;

(iii) $b/a \in H^2$;
(iv) $(1 - |b|^2)^{-1} \in L^1(\mathbb{T})$.

Proof (i) $\Longrightarrow$ (ii) Using the closed graph theorem, there exists a constant $c > 0$ such that

$$\|f\|_b \le c\,\|f\|_\infty$$

for every function $f \in H^\infty$. In particular, we have

$$\|z^n\|_b \le c\,\|z^n\|_\infty = c \qquad (n \ge 0).$$

(ii) $\Longrightarrow$ (iii) If $\sup_{n\ge 0} \|z^n\|_b < +\infty$, then we apply Lemma 24.9 with $f = 1$ to conclude that $T_{\bar b}1 \in \mathcal{M}(a) = \mathcal{R}(T_a) = T_a H^2$. In other words, there exists a $g \in H^2$ such that $b = T_{\bar b}1 = T_{\bar a} g = ag$. Hence, $b/a = g \in H^2$.

(iii) $\Longrightarrow$ (iv) By the definition of a, we have

$$(1 - |b|^2)^{-1} = |a|^{-2} = 1 + \frac{|b|^2}{|a|^2}.$$

Hence, if b/a is in H^2, then the preceding equality implies that $(1 - |b|^2)^{-1}$ is integrable on $\mathbb{T}$.

(iv) $\Longrightarrow$ (i) Since $(1 - |b|^2)^{-1}$ is integrable on $\mathbb{T}$, then the function a^{-1} is in $L^2(\mathbb{T})$. But a is outer and then Corollary 4.28 implies that $1/a$ belongs to H^2. Now if f belongs to H^∞, the function f/a belongs to H^2 and thus $f = af/a = T_a(f/a)$ belongs to $\mathcal{M}(a)$. In other words $H^\infty \subset \mathcal{M}(a)$ and the conclusion follows from Theorem 23.2. □

In Theorem 24.10, we saw that b/a belongs to H^2 if and only if $\sup_{n\ge 0} \|z^n\|_b < \infty$. To exploit this equivalence, we give a formula that expresses the norm of z^n in $\mathcal{H}(b)$ in terms of the Taylor coefficient of the function b/a. To obtain the formula for $\|z^n\|_b^2$, we need the following result.

Lemma 24.11 *Let (a, b) be a pair, and suppose that*

$$\frac{b(z)}{a(z)} = \sum_{n=0}^{\infty} c_n z^n \qquad (z \in \mathbb{D}).$$

Then

$$\langle S^* b, z^n\rangle_b = a(0) c_{n+1} \qquad (n \ge 0).$$

Proof According to (12.3), we have

$$\begin{aligned} r^2 T_{\bar b/\bar a_r} S^* b = r^2 T_{\bar z \bar b/\bar a_r} b &= P_+\left(\frac{r^2 \bar z |b|^2}{\bar a_r}\right) \\ &= P_+\left(\frac{\bar z(1 - |a_r|^2)}{\bar a_r}\right) = P_+\left(\bar z/\bar a_r - \bar z a_r\right). \end{aligned}$$

But $a_r^{-1} \in H^\infty$, which implies that $\bar{z}/\bar{a}_r \in \overline{H_0^2}$, and thus

$$r^2 T_{\bar{b}/\bar{a}_r} S^* b = -P_+(\bar{z}a_r) = -\bar{z}(a_r - a_r(0)).$$

Therefore,

$$\begin{aligned}
T_{rb/a_r} T_{r\bar{b}/\bar{a}_r} S^* b &= -\frac{\bar{z}b}{a_r}(a_r - a_r(0)) \\
&= -\bar{z}b + \frac{\bar{z}b}{a_r}\, a_r(0) \\
&= -\bar{z}(b - b(0)) + a_r(0)\bar{z}\left(\frac{b}{a_r} - \frac{b(0)}{a_r(0)}\right) \\
&= -S^* b + a_r(0) S^*(b/a_r).
\end{aligned}$$

Applying (18.20), we get

$$\begin{aligned}
\langle S^* b, z^n \rangle_b &= \langle S^* b, z^n \rangle_2 + \lim_{r \to 1} \langle T_{rb/a_r} T_{r\bar{b}/\bar{a}_r} S^* b, z^n \rangle_2 \\
&= \langle S^* b, z^n \rangle_2 + \lim_{r \to 1} (-\langle S^* b, z^n \rangle_2 + a_r(0) \langle S^*(b/a_r), z^n \rangle_2) \\
&= \lim_{r \to 1} (a_r(0) \langle b/a_r, z^{n+1} \rangle_2) \\
&= a(0) c_{n+1}.
\end{aligned}$$

Note that, as $r \longrightarrow 1$, the family $a_r(z)$ converges uniformly to $a(z)$ on compact subsets of $\mathbb{D}$. In the first place, this shows that $a_r(0) \longrightarrow a(0)$. Second, since a is outer and thus nonzero, $b(z)/a_r(z)$ converges uniformly to $b(z)/a(z)$ on compact subsets of $\mathbb{D}$. This latter fact ensures that $\langle b/a_r, z^{n+1} \rangle_2$, interpreted as the Taylor coefficient of b/a_r, tends to the corresponding Taylor coefficient of b/a, which is c_{n+1}. In the above calculations, since b/a is not necessarily in $L^2(\mathbb{T})$, an expression of the form $\langle b/a, z^{n+1} \rangle_2$ could be meaningless. That is why we had to bring a_r into the game. □

Theorem 24.12 *Let (a, b) be a pair, and suppose that*

$$\frac{b(z)}{a(z)} = \sum_{n=0}^{\infty} c_n z^n \qquad (z \in \mathbb{D}). \tag{24.7}$$

Then

$$\|z^n\|_b^2 = 1 + \sum_{j=0}^{n} |c_j|^2 \qquad (n \ge 0) \tag{24.8}$$

and

$$\langle z^{n+k}, z^n \rangle_b = \sum_{j=0}^{n} \overline{c_{j+k}}\, c_j \qquad (n \ge 0,\ k \ge 1). \tag{24.9}$$

Proof We would like to apply Theorem 23.8 and so we need to compute χ_n^+, where $\chi_n(z) = z^n$, $n \geq 0$. First note that the function g defined by

$$g(z) = b(z) - a(z) \sum_{j=0}^{n} c_j z^j \qquad (z \in \mathbb{D})$$

belongs to H^∞ and, according to (24.7), we have $g^{(\ell)}(0) = 0$, $0 \leq \ell \leq n$. In particular, the function

$$h(z) = \bar{z}^n \left(b(z) - a(z) \sum_{j=0}^{n} c_j z^j \right) \qquad (z \in \mathbb{T})$$

belongs to H_0^2. Hence, $P_+ \bar{h} = 0$, which can be written as

$$P_+(\bar{b} z^n) = P_+ \left(\bar{a} \sum_{j=0}^{n} \overline{c_j} z^{n-j} \right).$$

By the uniqueness of χ_n^+, we thus deduce that

$$\chi_n^+(z) = \sum_{j=0}^{n} \overline{c_j} z^{n-j} \qquad (n \geq 0). \tag{24.10}$$

Now (24.8) follows from Theorem 23.8 because we have

$$\|\chi_n\|_b^2 = \|\chi_n\|_2^2 + \|\chi_n^+\|_2^2 = 1 + \sum_{j=0}^{n} |c_j|^2.$$

To prove (24.9), we use once more Theorem 23.8 to get

$$\langle \chi_{n+k}, \chi_n \rangle_b = \langle \chi_{n+k}, \chi_n \rangle_2 + \langle \chi_{n+k}^+, \chi_n^+ \rangle_2.$$

But, since $k \geq 1$, $\langle \chi_{n+k}, \chi_n \rangle_2 = 0$ and using (24.10), we have

$$\begin{aligned} \langle \chi_{n+k}^+, \chi_n^+ \rangle_2 &= \sum_{\substack{0 \leq j \leq n+k \\ 0 \leq \ell \leq n}} \overline{c_j} c_\ell \langle \chi_{n+k-j}, \chi_{n-\ell} \rangle_2 \\ &= \sum_{\substack{0 \leq j \leq n+k \\ 0 \leq \ell \leq n}} \overline{c_j} c_\ell \delta_{n+k-j, n-\ell}, \end{aligned}$$

where $\delta_{m,n}$ denotes as usual the Kronecker symbol. Hence, we get

$$\langle \chi_{n+k}, \chi_n \rangle_b = \sum_{\ell=0}^{n} \overline{c_{k+\ell}} c_\ell. \qquad \square$$

Exercise

Exercise 24.2.1 In this exercise, we give another proof of Theorem 24.12. Let (a, b) be a pair, and suppose that

$$\frac{b(z)}{a(z)} = \sum_{n=0}^{\infty} c_n \, z^n \qquad (z \in \mathbb{D}).$$

(i) Using Corollary 23.24, verify the formula (24.8) for $n = 0$.
(ii) For $n \geq 1$, use Lemma 24.2 to get (24.8) by induction.
(iii) Use the same method to prove (24.9).

24.3 Spectrum of X_b and X_b^*

In Theorem 18.26, We showed that

$$\sigma_p(X_b) \subset \mathbb{D} \quad \text{and} \quad \sigma_p(X_b^*) \subset \mathbb{D}.$$

Now, we have enough tools to completely determine $\sigma_p(X_b)$ and $\sigma_p(X_b^*)$.

Theorem 24.13 *Let b be a nonextreme point of the closed unit ball of H^∞. Then*

$$\sigma_p(X_b) = \mathbb{D}$$

and, for every $\lambda \in \mathbb{D}$, we have

$$\ker(X_b - \bar{\lambda} I) = \mathbb{C} k_\lambda.$$

Proof We know that $\sigma_p(X_b) \subset \mathbb{D}$. The key step in proving the equality is that, by Theorem 23.23, in the nonextreme case we have

$$k_\lambda \in \mathcal{H}(b) \qquad (\lambda \in \mathbb{D}).$$

Then, according to Lemma 8.6,

$$X_b k_\lambda = S^* k_\lambda = \bar{\lambda} k_\lambda.$$

This identity shows that $\bar{\lambda} \in \sigma_p(X_b)$, and thus $\mathbb{D} \subset \sigma_p(X_b)$. Therefore, in fact, we have $\sigma_p(X_b) = \mathbb{D}$. Finally, again by Lemma 8.6,

$$\mathbb{C} k_\lambda \subset \ker(X_b - \bar{\lambda} I) \subset \ker(S^* - \bar{\lambda} I) = \mathbb{C} k_\lambda. \qquad \square$$

The operator X_b^* has no eigenvalues. However, the verification of this fact is more technical.

Theorem 24.14 *Let b be a nonextreme point of the closed unit ball of H^∞. Then $\sigma_p(X_b^*) = \emptyset$.*

Proof Suppose, on the contrary, that $\sigma_p(X_b^*) \neq \emptyset$ and pick a $\lambda \in \sigma_p(X_b^*)$. Hence, there is an $f \in \mathcal{H}(b)$, $f \not\equiv 0$, such that $X_b^* f = \lambda f$. Remember that, since b is a nonextreme point of the closed unit ball of H^∞, the polynomials belong to $\mathcal{H}(b)$ and generate a dense subspace. In particular, for every $n \geq 0$, we have $\langle X_b^* f, z^n \rangle_b = \lambda \langle f, z^n \rangle_b$, and thus

$$\langle f, X_b z^n \rangle_b = \lambda \langle f, z^n \rangle_b.$$

But $X_b 1 = 0$ and, for $n \geq 1$, $X_b z^n = z^{n-1}$. Thus

$$\lambda \langle f, 1 \rangle_b = 0 \quad \text{and} \quad \langle f, z^{n-1} \rangle_b = \lambda \langle f, z^n \rangle_b \qquad (n \geq 1).$$

We easily see that the two relations imply that

$$\langle f, z^n \rangle_b = 0 \qquad (n \geq 0).$$

By the density of the polynomials in $\mathcal{H}(b)$, we then deduce that $f \equiv 0$, which is absurd. □

Despite the fact that $\sigma_p(X_b^*) = \emptyset$, the spectrum of X_b^* is the maximum possible set for a contraction.

Theorem 24.15 *Let b be a nonextreme point of the closed unit ball of H^∞. Then*

$$\sigma(X_b^*) = \sigma(X_b) = \bar{\mathbb{D}}.$$

Proof Since X_b^* acts as a contraction in $\mathcal{H}(b)$, we have $\sigma(X_b^*) \subset \bar{\mathbb{D}}$. Now, let $\lambda \in \mathbb{D} \setminus \sigma(X_b^*)$. It follows from Theorem 18.22 that

$$X_b^* - \lambda I = (S_b - \lambda I) - (b \otimes S^* b)$$

and then Lemma 7.33 implies that $S_b - \lambda I$ is a Fredholm operator with

$$\operatorname{ind}(S_b - \lambda I) = 0.$$

In particular, we have

$$\dim \ker(S_b - \lambda I) = \dim \ker(S_b^* - \bar{\lambda} I)$$

and, according to Theorem 24.5, this identity implies $\dim \ker(S_b - \lambda I) = 1$. But $\ker(S_b - \lambda I) \subset \ker(S - \lambda I)$ and we get a contradiction with Lemma 8.6 (remember that $\sigma_p(S) = \emptyset$). Therefore, $\mathbb{D} \subset \sigma(X_b^*)$ and since the spectrum of an operator is a closed subset, we get the desired equality, i.e. $\sigma(X_b^*) = \bar{\mathbb{D}}$.

By Theorem 1.30, we have $\sigma(X_b) = \overline{\sigma(X_b^*)}$, where the bar stands for the complex conjugate. Hence, we also have $\sigma(X_b) = \bar{\mathbb{D}}$.

Another simple, but indirect, proof is based on Theorem 24.13 and the fact that, for any contraction A,

$$\sigma_p(A) \subset \sigma(A) \subset \bar{\mathbb{D}}.$$ □

It is easily checked that $S^{*n} \longrightarrow 0$ strongly as $n \longrightarrow \infty$. The same is true for X_b in the case where b is a nonextreme point of the closed unit ball of H^∞. But in the case where b is an extreme point of the closed unit ball of H^∞, this is not true (see Corollary 25.14).

Theorem 24.16 *Let b be a nonextreme point of the closed unit ball of H^∞. Then*

$$X_b^n f \longrightarrow 0 \qquad (f \in \mathcal{H}(b)),$$

as $n \longrightarrow \infty$.

Proof According to Theorems 23.23 and 24.13, we have $k_w \in \mathcal{H}(b)$, for every $w \in \mathbb{D}$, and $X_b k_w = \bar{w} k_w$. Hence, we get $X_b^n k_w = \bar{w}^n k_w$, which by induction gives

$$\|X_b^n k_w\|_b = |w|^n \|k_w\|_b.$$

Since $|w|^n \longrightarrow 0$, we deduce that $\|X_b^n k_w\|_b \longrightarrow 0$ as $n \longrightarrow \infty$. Now note that $\|X_b^n\| \leq 1$ and, by Corollary 23.26, the functions k_w span $\mathcal{H}(b)$. Thus $\|X_b^n f\|_b \longrightarrow 0$, as $n \longrightarrow \infty$, for every $f \in \mathcal{H}(b)$. □

24.4 Comparison of measures

In Theorem 20.28, we studied the relations between two elements of the closed unit disk of H^∞, on the one hand, and their Clark measures, on the other. To apply this result, we need to verify whether a function belongs to $\mathcal{H}(b)$, a task that is usually nontrivial and difficult to fulfill. In the nonextreme case, we are able to replace this condition with an easier one.

Theorem 24.17 *Let b be a nonextreme point of the closed unit ball of H^∞ with Pythagorean mate a, let Θ be a nonconstant inner function, and let μ and ν be the Clark measures associated respectively with b and Θ. Then the following are equivalent.*

(i) *ν is absolutely continuous with respect to μ and $d\nu/d\mu$ is in $L^2(\mu)$.*
(ii) *The function $(1-b)/(1-\Theta)$ is in $\mathcal{H}(b)$.*
(iii) *The functions $(1-b)/(1-\Theta)$ and $a/(1-\Theta)$ are in H^2.*

Proof (i) $\Longleftrightarrow$ (ii) By Theorem 20.28, we know that condition (i) holds if and only if

$$\left(\frac{1-b}{1-\Theta}\right) k_0^\Theta \in \mathcal{H}(b).$$

But

$$\left(\frac{1-b}{1-\Theta}\right) k_0^\Theta = (1-\overline{\Theta(0)}\Theta)\,\frac{1-b}{1-\Theta}$$
$$= (1-\overline{\Theta(0)})\,\frac{1-b}{1-\Theta} + \overline{\Theta(0)}(1-b).$$

By Corollary 23.9 and Theorem 23.13, in the nonextreme case, we have $1-b \in \mathcal{H}(b)$. Hence, the previous computation shows that

$$\left(\frac{1-b}{1-\Theta}\right) k_0^\Theta \in \mathcal{H}(b) \quad \Longleftrightarrow \quad \frac{1-b}{1-\Theta} \in \mathcal{H}(b).$$

This shows that (i) $\Longleftrightarrow$ (ii).

To prove that (ii) $\Longleftrightarrow$ (iii), we need the following identities. First,

$$\frac{|a|^2}{1-\Theta} = \frac{1-|b|^2}{1-\Theta} = \bar{b}\left(\frac{1-b}{1-\Theta}\right) - \overline{\Theta\left(\frac{1-b}{1-\Theta}\right)}. \tag{24.11}$$

Second, if we assume that $(1-b)/(1-\Theta) \in H^2$, then certainly the function $\Theta(1-b)/(1-\Theta)$ is also in H^2 and thus the function $|a|^2/(1-\Theta)$ stays in $L^2(\mathbb{T})$. Therefore, in this case, (24.11) implies that

$$P_+\left(\frac{|a|^2}{1-\Theta}\right) = P_+\left(\bar{b}\,\frac{1-b}{1-\Theta}\right) - c, \tag{24.12}$$

where c is a constant, since it is the projection of the conjugate of an H^2 function.

(ii) $\Longleftrightarrow$ (iii) It is trivial that $(1-b)/(1-\Theta) \in \mathcal{H}(b)$ implies that $(1-b)/(1-\Theta) \in H^2$. We need to prove that the function $a/(1-\Theta)$ is also in H^2.

According to Theorems 17.8 and 23.2, there is an $f \in H^2$ such that

$$T_{\bar{b}}\left(\frac{1-b}{1-\Theta}\right) = T_{\bar{a}} f.$$

Thus,

$$P_+\left(\bar{b}\frac{1-b}{1-\Theta}\right) = P_+(\bar{a}f),$$

which, by (24.12) and $P_+\bar{a} = \overline{a(0)}$, becomes

$$P_+\left(\bar{a}f - \frac{|a|^2}{1-\Theta} - \frac{c}{\overline{a(0)}}\,\bar{a}\right) = 0.$$

If we put $g = f - c/\overline{a(0)}$, then from the preceding identity we deduce that

$$\overline{a\left(g - \frac{a}{1-\Theta}\right)} \in H_0^2. \tag{24.13}$$

By Theorem 23.1, and writing

$$\frac{a}{1-\Theta} = \frac{a}{1-b} \times \frac{1-b}{1-\Theta},$$

we see that $a/(1-\Theta)$ is the product of two H^2 functions and thus at least we have that $a/(1-\Theta)$ is in H^1. Hence, since $g \in H^2 \subset H^1$,

$$g - \frac{a}{1-\Theta} \in H^1.$$

This identity implies that

$$\overline{\left(g - \frac{a}{1-\Theta}\right)} \in L^1(\mathbb{T}),$$

and thus by (24.13) and Corollary 4.28, we have

$$\overline{\left(g - \frac{a}{1-\Theta}\right)} \in H^1_0.$$

Therefore,

$$g - \frac{a}{1-\Theta} \in H^1 \cap \overline{H^1_0} = \{0\},$$

which reveals that

$$\frac{a}{1-\Theta} \in H^2.$$

(iii) $\Longrightarrow$ (ii) Since the function $a/(1-\Theta)$ is in H^2, we can write

$$T_{\bar{a}}\left(\frac{a}{1-\Theta}\right) = P_+\left(\frac{|a|^2}{1-\Theta}\right),$$

which reveals that $P_+(|a|^2/(1-\Theta))$ belongs to $\mathcal{M}(\bar{a})$. With a similar reasoning, we have

$$T_{\bar{b}}\left(\frac{1-b}{1-\Theta}\right) = P_+\left(\bar{b}\,\frac{1-b}{1-\Theta}\right).$$

According to Theorem 23.13, the constant functions belong to $\mathcal{M}(\bar{a})$. Hence, by (24.12), we see that

$$T_{\bar{b}}\left(\frac{1-b}{1-\Theta}\right) \in \mathcal{M}(\bar{a}) = \mathcal{H}(\bar{b}).$$

Therefore, according to Theorem 17.8, we conclude that

$$\frac{1-b}{1-\Theta} \in \mathcal{H}(b).$$

□

A byproduct of the above proof is that

$$f = g + \frac{c}{\overline{a(0)}} = \frac{a}{1-\Theta} + \frac{c}{\overline{a(0)}},$$

where

$$c = P_+\left(\overline{\Theta\left(\frac{1-b}{1-\Theta}\right)}\right) = \overline{\Theta(0)}\left(\frac{1-\overline{b(0)}}{1-\overline{\Theta(0)}}\right).$$

Hence, under the conditions of Theorem 24.17, we have

$$\left(\frac{1-b}{1-\Theta}\right)^+ = \frac{a}{1-\Theta} + \frac{\overline{\Theta(0)}}{\overline{a(0)}}\,\frac{1-\overline{b(0)}}{1-\overline{\Theta(0)}}.$$

24.5 The function F_λ

Let (a, b) be a pair and put

$$F = \frac{a}{1-b}.$$

In Theorem 23.1, we saw that F is an outer function in H^2. Moreover, for almost every $\zeta \in \mathbb{T}$, we have

$$|F(\zeta)|^2 = \frac{1-|b(\zeta)|^2}{|1-b(\zeta)|^2} = \Re\left(\frac{1+b(\zeta)}{1-b(\zeta)}\right) = \frac{d\mu_a}{dm}(\zeta), \tag{24.14}$$

where μ_a is the continuous part of the Clark measure μ associated with b.

Lemma 24.18 *We have*

$$\|F\|_2^2 \le \frac{1-|b(0)|^2}{|1-b(0)|^2}. \tag{24.15}$$

Moreover, in (24.15) equality holds if and only if the measure μ is absolutely continuous with respect to the Lebesgue measure.

Proof Write the Radon–Nikodym decomposition for the measure μ (with respect to the Lebesgue measure m) as

$$d\mu = h\,dm + d\mu_s,$$

where μ_s is a positive measure, singular with respect to m, and h is a positive function in $L^1(\mathbb{T})$. Then, according to (24.14), we have $|F|^2 = h$, almost everywhere on $\mathbb{T}$. Therefore,

$$\|F\|_2^2 = \int_{\mathbb{T}} |F(\zeta)|^2\,dm(\zeta) = \int_{\mathbb{T}} h(\zeta)\,dm(\zeta) \le \int_{\mathbb{T}} d\mu(\zeta) = \mu(\mathbb{T}).$$

On the other hand, according to (13.41), we have

$$\frac{1-|b(0)|^2}{|1-b(0)|^2} = \int_{\mathbb{T}} d\mu(\zeta) = \mu(\mathbb{T}).$$

Comparing the last two relations gives us

$$\|F\|_2^2 \le \frac{1-|b(0)|^2}{|1-b(0)|^2}.$$

Moreover, we easily see that we have equality in (24.15) if and only if

$$\int_{\mathbb{T}} h(\zeta)\,dm(\zeta) = \int_{\mathbb{T}} d\mu(\zeta),$$

which is equivalent to $\mu_s = 0$, that is, μ is absolutely continuous with respect to the Lebesgue measure m. □

If λ is a point of $\mathbb{T}$ and we replace the function b by the function $\bar{\lambda}b$, then the space $\mathcal{H}(b)$ remains the same. In fact, since λ is unimodular we have

$$I - T_{\bar{\lambda}b}T_{\lambda\bar{b}} = I - |\lambda|^2 T_b T_{\bar{b}} = I - T_b T_{\bar{b}},$$

and thus, according to Corollary 16.8, $\mathcal{H}(\bar{\lambda}b) = \mathcal{H}(b)$. Knowing this, denote by F_λ the corresponding function associated with $\bar{\lambda}b$ and by μ_λ the corresponding Clark measure. In other words, since the outer function corresponding to $\bar{\lambda}b$ is also a, we have

$$F_\lambda = \frac{a}{1-\bar{\lambda}b} \qquad (\lambda \in \mathbb{T}), \tag{24.16}$$

and the positive Borel measure μ_λ is such that

$$\frac{1-|b(z)|^2}{|\lambda - b(z)|^2} = \int_{\mathbb{T}} \frac{1-|z|^2}{|z-\zeta|^2}\,d\mu_\lambda(\zeta) \qquad (z \in \mathbb{D},\ \lambda \in \mathbb{T}). \tag{24.17}$$

Therefore, if $\mu_\lambda^{(a)}$ denotes the continuous part of the measure μ_λ, we have

$$|F_\lambda(\zeta)|^2 = \frac{d\mu_\lambda^{(a)}}{dm}(\zeta) = \frac{1-|b(\zeta)|^2}{|\lambda - b(\zeta)|^2}, \tag{24.18}$$

almost everywhere on $\mathbb{T}$.

For a nonextreme point of the closed unit ball of H^∞, we surely have $|b| < 1$ almost everywhere on $\mathbb{T}$ (this is not a sufficient condition). Hence, the following result works in particular in the nonextreme case.

Theorem 24.19 *Let b be a point in the closed unit ball of H^∞. Assume that $|b| < 1$ almost everywhere on $\mathbb{T}$. Then, for almost all $\lambda \in \mathbb{T}$, the measure μ_λ is absolutely continuous with respect to the Lebesgue measure.*

Proof According to Lemma 24.18 applied to the function $\bar{\lambda}b$, we see that the function

$$\lambda \longmapsto \frac{1-|b(0)|^2}{|\lambda - b(0)|^2} - \|F_\lambda\|_2^2$$

is measurable and positive on $\mathbb{T}$. Moreover, it vanishes if and only if the measure μ_λ is absolutely continuous. Therefore, the measure μ_λ is absolutely continuous almost everywhere on $\mathbb{T}$ if and only if

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{1-|b(0)|^2}{|e^{i\theta}-b(0)|^2}\,d\theta = \frac{1}{2\pi}\int_0^{2\pi} \|F_{e^{i\theta}}\|_2^2\,d\theta. \tag{24.19}$$

But, on the one hand, it follows from (3.19) that

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{1-|b(0)|^2}{|e^{i\theta}-b(0)|^2}\,d\theta = 1.$$

On the other hand, by (24.18), we have

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi} \|F_{e^{i\theta}}\|_2^2\,d\theta &= \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{1}{2\pi}\int_0^{2\pi}\frac{1-|b(e^{it})|^2}{|e^{i\theta}-b(e^{it})|^2}\,dt\right)d\theta\\ &= \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{1}{2\pi}\int_0^{2\pi}\frac{1-|b(e^{it})|^2}{|e^{i\theta}-b(e^{it})|^2}\,d\theta\right)dt.\end{aligned}$$

Since $|b| < 1$ almost everywhere on $\mathbb{T}$, we can use once more (3.19) with $z = b(e^{it})$ (which is in $\mathbb{D}$ for almost all t) to get

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{1-|b(e^{it})|^2}{|e^{i\theta}-b(e^{it})|^2}\,d\theta = 1,$$

and thus

$$\frac{1}{2\pi}\int_0^{2\pi} \|F_{e^{i\theta}}\|_2^2\,d\theta = 1.$$

Therefore, since the two integrals in (24.19) are equal to 1, we conclude that the measure μ_λ is absolutely continuous almost everywhere on $\mathbb{T}$. □

The next corollary follows immediately from Lemma 24.18 and Theorem 24.19.

Corollary 24.20 *Let b be a nonextreme point in the closed unit ball of H^∞. Then*

$$\|F_\lambda\|_2 \le \frac{1-|b(0)|^2}{|1-\bar{\lambda}b(0)|^2},$$

and we have equality for almost all $\lambda \in \mathbb{T}$. In particular, if $b(0) = 0$, then $\|F_\lambda\|_2 \le 1$ and we have $\|F_\lambda\|_2 = 1$ for almost all $\lambda \in \mathbb{T}$.

The above discussion helps to show that the operator $T_{1-\bar{\lambda}b}T_{\bar{F}_\lambda}T_{F_\lambda/\bar{F}_\lambda}$ is bounded.

Lemma 24.21 *Let b be a nonextreme point in the closed unit ball of H^∞ and let $\lambda \in \mathbb{T}$. Then the operator $T_{1-\bar{\lambda}b}T_{\bar{F}_\lambda}T_{F_\lambda/\bar{F}_\lambda}$ is a bounded operator on H^2 and we have*

$$T_{1-\bar{\lambda}b}T_{\bar{F}_\lambda}T_{F_\lambda/\bar{F}_\lambda} = T_a.$$

Proof First, using Theorem 13.23, we have $T_{\bar{F}_\lambda} T_{F_\lambda/\bar{F}_\lambda} = T_{F_\lambda}$. Fix $f \in H^2$. Using the fact that $F_\lambda f \in H^1$ and (13.14) and (24.16), we obtain

$$T_{1-\bar{\lambda}b} T_{\bar{F}_\lambda} T_{F_\lambda/\bar{F}_\lambda} f = T_{1-\bar{\lambda}b} T_{F_\lambda} f = P_+((1-\bar{\lambda}b)F_\lambda f) = P_+(af) = T_a f.$$

This reveals that the operator $T_{1-\bar{\lambda}b} T_{\bar{F}_\lambda} T_{F_\lambda/\bar{F}_\lambda}$ coincides with T_a, and in particular it is a bounded operator on H^2. □

Exercise

Exercise 24.5.1 Let (a, b) be a pair. Let $h \in H^2$ and let $\alpha \in \mathbb{T}$ such that μ_α is absolutely continuous with respect to the Lebesgue measure on $\mathbb{T}$.

(i) Show that $h_\alpha(\zeta) := h(\zeta)\overline{F_\alpha(\zeta)}^{-1}$ is in $L^2(\mu_\alpha)$.

(ii) Show that $V_{\bar{\alpha}b}(h_\alpha) = ah$ and deduce that $\mathcal{M}(a) \hookrightarrow \mathcal{H}(b)$.
Hint: Use Theorem 20.5 to get that $\mathcal{M}(a)$ is contractively contained in $\mathcal{H}(b)$.

24.6 The operator W_λ

In this section, we exploit the outer function $F = a/(1-b)$ to construct an isometry from H^2 into $\mathcal{H}(b)$. In fact, since $\mathcal{H}(b) = \mathcal{H}(\lambda b)$, $\lambda \in \mathbb{T}$, this method exhibits a family of isometries from H^2 into $\mathcal{H}(b)$.

Lemma 24.22 *Let b be a nonextreme point of the closed unit ball of H^∞, and let μ denote the corresponding Clark measure. Then the map $A : f \longrightarrow f/F$ defines an isometry from H^2 onto $H^2(\mu_a)$, where μ_a is the continuous part of the measure μ.*

Proof Let $f \in H^2$. Then, using (24.14), we have

$$\begin{aligned} \|f\|_2^2 &= \int_{\mathbb{T}} |f(\zeta)|^2 \, dm(\zeta) \\ &= \int_{\mathbb{T}} |F(\zeta)|^{-2} |f(\zeta)|^2 |F(\zeta)|^2 \, dm(\zeta) \\ &= \int_{\mathbb{T}} |F(\zeta)|^{-2} |f(\zeta)|^2 \, d\mu_a(\zeta). \end{aligned}$$

This means that $f/F \in L^2(\mu_a)$ and

$$\|F^{-1} f\|_{L^2(\mu_a)} = \|f\|_2 \,. \tag{24.20}$$

Up to now, we have shown that our mapping isometrically sends H^2 into $L^2(\mu_a)$.

Since F is an outer function in H^2, according to Corollary 8.17, there exists a sequence of polynomials $(p_n)_{n\geq 1}$ such that p_nF tends to f in H^2. But, by (24.20), we have

$$\|p_n - F^{-1}f\|_{L^2(\mu_a)} = \|F^{-1}(p_nF - f)\|_{L^2(\mu_a)} = \|p_nF - f\|_2,$$

and thus p_n tends to f/F in $L^2(\mu_a)$. Therefore, for each f in H^2,

$$F^{-1}f \in \operatorname{Clos}_{L^2(\mu_a)} \mathcal{P} = H^2(\mu_a).$$

This shows that the map $A : f \longrightarrow F^{-1}f$ is an isometry of H^2 into $H^2(\mu_a)$.

To prove that this isometry A is surjective, note that, for each polynomial p, we have $A(pF) = p$, which implies that

$$\mathcal{P} \subset \mathcal{R}(T) \subset H^2(\mu_a).$$

Since, by definition, the set of polynomials $\mathcal{P}$ is dense in $H^2(\mu_a)$, it follows that $\operatorname{Clos}_{L^2(\mu_a)} \mathcal{R}(T) = H^2(\mu_a)$. But, since A is an isometry, it has a closed range, and thus $\mathcal{R}(A) = H^2(\mu_a)$. □

Now recall that V_b is the map from $H^2(\mu)$ onto $\mathcal{H}(b)$ defined by

$$V_b(f) = (1-b)K_\mu(f),$$

where $K_\mu(f)$ is the Cauchy transform of the measure $f\,d\mu$ defined by

$$K_\mu(f)(z) = \int_{\mathbb{T}} \frac{f(\zeta)}{1-\bar{\zeta}z}\,d\mu(\zeta) \qquad (z \in \mathbb{D}).$$

We know from Theorem 20.5 that V_b is an isometry from $H^2(\mu)$ onto $\mathcal{H}(b)$. Since $H^2(\mu_a)$ is a closed subspace of $H^2(\mu)$, we can consider the map

$$A : f \longrightarrow f/F$$

as an isometry of H^2 into $H^2(\mu)$. Combining the previous two operators, we see that V_bA is an isometry from H^2 into $\mathcal{H}(b)$. The following result gives another representation for this mapping. We recall that, for φ in $L^2(\mathbb{T})$ and ψ in H^2, then the product $T_\psi T_\varphi$ is interpreted as ψT_φ.

Theorem 24.23 *Let b be a nonextreme point of the closed unit ball of H^∞, and let μ denote the corresponding Clark measure. Let A be the operator introduced in Lemma 24.22. Then we have*

$$V_bAf = T_{1-b}T_{\bar{F}}f \qquad (f \in H^2),$$

which reveals that the map $T_{1-b}T_{\bar{F}}$ is an isometry from H^2 into $\mathcal{H}(b)$. Moreover, $T_{1-b}T_{\bar{F}}$ is surjective if and only if the measure μ is absolutely continuous.

Proof We have already noted that $V_b A$ is an isometry from H^2 into $\mathcal{H}(b)$. Moreover, it is surjective if and only if A is onto. But, according to Lemma 24.22, $AH^2 = H^2(\mu_a)$ and thus $V_b A$ is surjective if and only if $H^2(\mu_a) = H^2(\mu)$. Hence, by Lemma 8.19, we see that this last condition is equivalent to $L^2(\mu_s) = \{0\}$, that is, $\mu_s = 0$. In other words, A is surjective if and only if μ is absolutely continuous. It remains to show that $V_b A f = T_{1-b} T_{\bar{F}} f$.

Let $f \in H^2$. Since

$$f/F = (1-b)f/a,$$

by Theorem 13.28, $f/F = 0$ on a carrier of μ_s. Thus, for each $z \in \mathbb{D}$, we have

$$\begin{aligned}
(V_b A f)(z) &= V_b(F^{-1} f)(z) \\
&= (1-b(z)) \int_{\mathbb{T}} \frac{F^{-1}(\zeta) f(\zeta)}{1-\bar{\zeta} z} \, d\mu(\zeta) \\
&= (1-b(z)) \int_{\mathbb{T}} \frac{F^{-1}(\zeta) f(\zeta)}{1-\bar{\zeta} z} \, d\mu_a(\zeta) \\
&= (1-b(z)) \int_{\mathbb{T}} \frac{F^{-1}(\zeta) f(\zeta)}{1-\bar{\zeta} z} \, |F(\zeta)|^2 \, dm(\zeta) \\
&= (1-b(z)) \int_{\mathbb{T}} \frac{f(\zeta)}{1-\bar{\zeta} z} \, \overline{F(\zeta)} \, dm(\zeta) \\
&= (1-b(z)) K_{\bar{F}}(f)(z) \\
&= (1-b(z)) (T_{\bar{F}} f)(z).
\end{aligned}$$

Hence, $V_b A f = (1-b) T_{\bar{F}} f = T_{1-b} T_{\bar{F}} f$. □

We saw that the operator $T_{1-b} T_{\bar{F}}$ maps isometrically H^2 into $\mathcal{H}(b)$. We denote this operator by W. In other words, the map

$$\begin{array}{cccc}
W: & H^2 & \longrightarrow & \mathcal{H}(b) \\
& f & \longmapsto & T_{1-b} T_{\bar{F}} f
\end{array}$$

is an isometry. Since $\mathcal{H}(b)$ is contractively included into H^2, we can define the contraction

$$\begin{array}{cccc}
\widetilde{W}: & H^2 & \longrightarrow & H^2 \\
& f & \longmapsto & T_{1-b} T_{\bar{F}} f.
\end{array}$$

It is interesting that, even if the operator $T_{\bar{F}}$ is possibly unbounded, the product $T_{1-b} T_{\bar{F}}$ is always a bounded operator on H^2.

Corollary 24.24 *Let b be a nonextreme point of the closed unit ball of H^∞, and let μ denote the corresponding Clark measure. Then we have*

$$\widetilde{W}^* = (T_{1-b} T_{\bar{F}})^* = T_F T_{1-\bar{b}}.$$

Moreover, if μ is absolutely continuous with respect to the Lebesgue measure, then

$$\widetilde{W}\widetilde{W}^* = (T_{1-b}T_{\bar{F}})(T_F T_{1-\bar{b}}) = I - T_b T_{\bar{b}}. \tag{24.21}$$

Proof Let $f \in H^\infty$ and $g \in H^2$. Then we have

$$\begin{aligned}
\langle f, \widetilde{W}^* g\rangle_2 &= \langle \widetilde{W} f, g\rangle_2 \\
&= \langle (T_{1-b}T_{\bar{F}}) f,\ g\rangle_2 \\
&= \langle (1-b)P_+(\bar{F}f),\ g\rangle_2 \\
&= \langle P_+(\bar{F}f),\ (1-\bar{b})g\rangle_2 \\
&= \langle \bar{F}f,\ P_+((1-\bar{b})g)\rangle_2 \\
&= \langle f, F T_{1-\bar{b}}\, g\rangle_2 \\
&= \langle f, T_F T_{1-\bar{b}}\, g\rangle_2.
\end{aligned}$$

Since, with respect to the $\|\cdot\|_2$ norm, H^∞ in dense in H^2, the identity

$$\langle f, \widetilde{W}^* g\rangle_2 = \langle f, T_F T_{1-\bar{b}}\, g\rangle_2 \qquad (f \in H^\infty,\ g \in H^2)$$

implies that $\widetilde{W}^* = T_F T_{1-\bar{b}}$. This proves the first part of the corollary.

Now, if μ is absolutely continuous, according to Theorem 24.23, the operator W is an isometry from H^2 onto $\mathcal{H}(b)$. Then Corollary 16.12 implies that

$$\widetilde{W}\widetilde{W}^* = (T_{1-b}T_{\bar{F}})(T_F T_{1-\bar{b}}) = I - T_b T_{\bar{b}}. \qquad \square$$

We can also directly verify (24.21). Indeed, let $z, w \in \mathbb{D}$. Then, using (20.14) and (24.14) and the fact that μ is absolutely continuous, we have

$$\begin{aligned}
\langle \widetilde{W}\widetilde{W}^* k_z, k_w\rangle_2 &= \langle \widetilde{W}^* k_z, \widetilde{W}^* k_w\rangle_2 \\
&= \langle T_F T_{1-\bar{b}} k_z, T_F T_{1-\bar{b}} k_w\rangle_2 \\
&= (1-\overline{b(z)})(1-b(w))\langle F k_z, F k_w\rangle_2 \\
&= (1-\overline{b(z)})(1-b(w)) \int_{\mathbb{T}} \frac{|F(\zeta)|^2}{(1-\bar{z}\zeta)(1-\bar{\zeta}w)}\, dm(\zeta) \\
&= (1-\overline{b(z)})(1-b(w)) \int_{\mathbb{T}} \frac{1}{(1-\bar{z}\zeta)(1-\bar{\zeta}w)}\, d\mu(\zeta) \\
&= (1-\overline{b(z)})(1-b(w)) \int_{\mathbb{T}} \frac{k_z(\zeta)}{1-\bar{\zeta}w}\, d\mu(\zeta) \\
&= (1-\overline{b(z)})(V_b k_z)(w).
\end{aligned}$$

Now, Theorems 18.11 and 20.5 imply that

$$\begin{aligned}
\langle (I - T_b T_{\bar{b}})k_z, k_w\rangle_2 &= \langle k_z^b, k_w\rangle_2 \\
&= k_z^b(w) \\
&= (1-\overline{b(z)})(V_b k_z)(w).
\end{aligned}$$

Hence, $\widetilde{W}\widetilde{W}^* k_z = (I - T_b T_{\bar{b}})k_z$ and, by linearity and continuity, we get (24.21).

We studied the contraction $W = T_{1-b}T_{\bar{F}} \in \mathcal{L}(H^2)$ and saw that its range is inside $\mathcal{H}(b)$ and, moreover, that

$$\|T_{1-b}T_{\bar{F}}f\|_b = \|f\|_2 \qquad (f \in H^2).$$

Now, we study a similar object for the space $\mathcal{H}(a)$.

Lemma 24.25 *Let* (a, b) *be a pair, and let* $f \in H^2$. *Then*

$$T_a T_{\bar{F}} f \in H^2$$

and we have

$$T_a T_{\bar{F}} f = f + (T_{1-b}T_{\bar{F}}f)^+.$$

Proof Fix $f \in H^2$. According to Theorem 24.23, we have $h = T_{1-b}T_{\bar{F}}f \in \mathcal{H}(b)$. Thus, by (23.10), there exists an $h^+ \in H^2$ such that

$$T_{\bar{b}}h = T_{\bar{b}}T_{1-b}T_{\bar{F}}f = T_{\bar{a}}h^+.$$

It is enough to show that $h^+ = -(I - T_a T_{\bar{F}})f$ to obtain both required results.

Let p be an analytic polynomial. Then $\bar{F}p \in L^2(\mathbb{T})$ and we have

$$\begin{aligned} T_{\bar{b}}T_{1-b}T_{\bar{F}}p &= P_+\bar{b}(1-b)P_+\bar{F}p \\ &= P_+\bar{b}P_+\bar{F}p - P_+|b|^2P_+\bar{F}p \\ &= P_+(1-|b|^2)P_+\bar{F}p - P_+(1-\bar{b})P_+\bar{F}p \\ &= P_+|a|^2P_+\bar{F}p - P_+(1-\bar{b})P_+\bar{F}p. \end{aligned} \tag{24.22}$$

On the one hand, we have

$$P_+|a|^2P_+\bar{F}p = T_{\bar{a}}T_aT_{\bar{F}}p, \tag{24.23}$$

and on the other, by Corollary 4.11, we have

$$P_+(1-\bar{b})P_+\bar{F}p = P_+(1-\bar{b})\bar{F}p = P_+\bar{a}p = T_{\bar{a}}p. \tag{24.24}$$

Hence, it follows from (24.22), (24.23) and (24.24) that

$$T_{\bar{b}}T_{1-b}T_{\bar{F}}p = T_{\bar{a}}T_aT_{\bar{F}}p - T_{\bar{a}}p = T_{\bar{a}}(I - T_aT_{\bar{F}})p$$

for all analytic polynomials p. In other words, we have

$$\big(T_{1-b}T_{\bar{F}}p\big)^+ = (I - T_aT_{\bar{F}})p. \tag{24.25}$$

For each $f \in H^2$, there exists a sequence $(p_n)_{n\geq 1}$ of analytic polynomials such that $p_n \longrightarrow f$ in H^2. Put $h_n = T_{1-b}T_{\bar{F}}p_n$, and as above let $h = T_{1-b}T_{\bar{F}}f$. Hence, according to Theorem 24.23,

$$h_n \longrightarrow h$$

in $\mathcal{H}(b)$. Theorem 23.12 now ensures that

$$h_n^+ \longrightarrow h^+$$

in $\mathcal{H}(a)$. But, by (24.25), we have $h_n^+ = -(I - T_a T_{\bar{F}})p_n$. Since the mapping $I - T_a T_{\bar{F}} : H^2 \longmapsto H(\mathbb{D})$ is continuous, and $p_n \longrightarrow f$ in H^2, thus h_n^+ tends to $-(I - T_a T_{\bar{F}})f$ uniformly on compact subsets of $\mathbb{D}$. In particular, we have

$$h^+ = -(I - T_a T_{\bar{F}})f. \qquad \square$$

Theorem 24.26 *Let (a, b) be a pair, and let μ denote the Clark measure corresponding to b. Then the operator $I - T_a T_{\bar{F}}$ is a contraction from H^2 into $\mathcal{H}(a)$. Moreover, if μ is absolutely continuous, then $I - T_a T_{\bar{F}}$ is a co-isometry from H^2 onto $\mathcal{H}(a)$.*

Proof According to Theorem 23.12, the map $\mathfrak{G} : h \longmapsto h^+$ is a partial isometry from $\mathcal{H}(b)$ onto $\mathcal{H}(a)$. Now it follows from Lemma 24.25 that

$$I - T_a T_{\bar{F}} = -\mathfrak{G} T_{1-b} T_{\bar{F}}.$$

Hence, by Theorem 24.23, $I - T_a T_{\bar{F}}$ is a partial isometry from H^2 into $\mathcal{H}(a)$. In particular, it is a contraction. Moreover, if μ is absolutely continuous, then, again by Theorem 24.23, we see that the partial isometry $I - T_a T_{\bar{F}}$ is surjective. Then the result follows from Corollary 7.23. $\square$

The following diagram summarizes the relation between all these applications.

$$\begin{tikzcd}
H^2 \arrow[rr, "I - T_a T_{\bar{F}}"] \arrow[dr, "T_{1-b} T_{\bar{F}}"'] & & \mathcal{H}(a) \\
& \mathcal{H}(b) \arrow[ur, "-\mathfrak{G}"'] &
\end{tikzcd} \tag{24.26}$$

Let $\lambda \in \mathbb{T}$ and consider $W_\lambda = T_{1-\bar{\lambda} b} T_{\bar{F}_\lambda}$, regarded as an operator from H^2 into $\mathcal{H}(b)$. In fact, this is precisely the operator W, which was introduced at the beginning of this section, corresponding to the function $\bar{\lambda} b$.

Lemma 24.27 *Let $\lambda, \eta \in \mathbb{T}$. Then the following hold.*

(i) *The operator W_λ^* is a co-isometry from $\mathcal{H}(b)$ onto H^2.*
(ii) *The operator $W_\lambda^* W_\eta$ is a contraction of H^2 into itself.*
(iii) *If both measures μ_λ and μ_η are absolutely continuous, then $W_\lambda^* W_\eta$ is a unitary operator on H^2.*

Proof All the assertions of the lemma follow immediately from Theorem 24.23. $\square$

Theorem 24.28 *Let $\lambda, \eta \in \mathbb{T}$, $\lambda \neq \eta$. Then the operator $T_{F_\lambda} T_{\overline{F}_\eta}$ is a bounded operator on H^2, whose norm is at most $2/(|\eta - \lambda|)$. Moreover, we have*

$$W_\lambda^* W_\eta = (1 - \lambda\bar{\eta}) T_{F_\lambda} T_{\overline{F}_\eta} + \lambda\bar{\eta} I.$$

Proof Let $z, w \in \mathbb{D}$. Then we have

$$\begin{aligned}
\langle W_\lambda^* W_\eta k_z, k_w \rangle_2 &= \langle W_\eta k_z, W_\lambda k_w \rangle_b \\
&= \langle T_{1-\bar{\eta} b} T_{\overline{F}_\eta} k_z, T_{1-\bar{\lambda} b} T_{\overline{F}_\lambda} k_w \rangle_b \\
&= \overline{F_\eta(z)} F_\lambda(w) \langle T_{1-\bar{\eta} b} k_z, T_{1-\bar{\lambda} b} k_w \rangle_b \\
&= \overline{F_\eta(z)} F_\lambda(w) \langle (1-\bar{\eta} b) k_z,\ (1-\bar{\lambda} b) k_w \rangle_b.
\end{aligned}$$

Put

$$\begin{aligned}
S_{\eta,\lambda}(z,w) &= \langle (1-\bar{\eta} b) k_z,\ (1-\bar{\lambda} b) k_w \rangle_b \\
&= \langle k_z, k_w \rangle_b - \bar{\eta} \langle b k_z, k_w \rangle_b - \lambda \langle k_z, b k_w \rangle_b + \lambda\bar{\eta} \langle b k_z, b k_w \rangle_b.
\end{aligned}$$

Using Corollary 23.25, we get

$$\begin{aligned}
S_{\eta,\lambda}(z,w) = {} & \left(1 + \frac{\overline{b(z)} b(w)}{\overline{a(z)} a(w)}\right) k_z(w) - \bar{\eta} \frac{b(w)}{a(w)\overline{a(z)}} \overline{k_w(z)} \\
& - \lambda \frac{\overline{b(z)}}{\overline{a(z)} a(w)} k_z(w) + \lambda\,\bar{\eta} \left(\frac{1}{\overline{a(z)} a(w)} - 1 \right) k_z(w).
\end{aligned}$$

Since $\overline{k_w(z)} = k_z(w)$, we deduce that

$$\begin{aligned}
S_{\eta,\lambda}(z,w) &= \left[1 + \frac{\overline{b(z)} b(w)}{\overline{a(z)} a(w)} - \bar{\eta} \frac{b(w)}{\overline{a(z)} a(w)} - \lambda \frac{\overline{b(z)}}{\overline{a(z)} a(w)} \right. \\
& \qquad \left. + \lambda\bar{\eta} \left(\frac{1}{\overline{a(z)} a(w)} - 1 \right) \right] k_z(w) \\
&= \left[1 - \lambda\bar{\eta} + \frac{\overline{b(z)} b(w) - \bar{\eta} b(w) - \lambda \overline{b(z)} + \lambda\bar{\eta}}{\overline{a(z)} a(w)} \right] k_z(w) \\
&= \left[1 - \lambda\bar{\eta} + \frac{\bar{\eta}\lambda (1 - \eta\overline{b(z)}) - \bar{\eta} b(w) (1 - \eta\overline{b(z)})}{\overline{a(z)} a(w)} \right] k_z(w) \\
&= \left[1 - \lambda\bar{\eta} + \frac{(\bar{\eta}\lambda - \bar{\eta} b(w))(1 - \eta\overline{b(z)})}{\overline{a(z)} a(w)} \right] k_z(w) \\
&= \left[1 - \lambda\bar{\eta} + \frac{\bar{\eta}\lambda (1 - \bar{\lambda} b(w))(1 - \eta\overline{b(w)})}{\overline{a(z)} a(w)} \right] k_z(w) \\
&= \left[1 - \lambda\bar{\eta} + \frac{\bar{\eta}\lambda}{\overline{F_\eta(z)} F_\lambda(w)} \right] k_z(w).
\end{aligned}$$

Consequently, we obtain

$$
\begin{aligned}
\langle W_\lambda^* W_\eta k_z, k_w \rangle_2 &= \overline{F_\eta(z)} F_\lambda(w) \left[1 - \lambda\bar{\eta} + \bar{\eta}\lambda \frac{1}{\overline{F_\eta(z)} F_\lambda(w)} \right] k_z(w) \\
&= (1 - \lambda\bar{\eta}) \overline{F_\eta(z)} F_\lambda(w) k_z(w) + \lambda\bar{\eta} k_z(w) \\
&= (1 - \lambda\bar{\eta}) \langle T_{\overline{F}_\eta} k_z, T_{\overline{F}_\lambda} k_w \rangle_2 + \lambda\bar{\eta} \langle k_z, k_w \rangle_2 \\
&= (1 - \lambda\bar{\eta}) \langle T_{F_\lambda} T_{\overline{F}_\eta} k_z, k_w \rangle_2 + \lambda\bar{\eta} \langle k_z, k_w \rangle_2 \\
&= \langle [(1 - \lambda\bar{\eta}) T_{F_\lambda} T_{\overline{F}_\eta} + \lambda\bar{\eta} I] k_z, \ k_w \rangle_2.
\end{aligned}
$$

Since $\operatorname{Span}(k_w : w \in \mathbb{D}) = H^2$, we have

$$
W_\lambda^* W_\eta k_z = [(1 - \lambda\bar{\eta}) T_{F_\lambda} T_{\overline{F}_\eta} + \lambda\bar{\eta} I] k_z
$$

and thus

$$
W_\lambda^* W_\eta f = [(1 - \lambda\bar{\eta}) T_{F_\lambda} T_{\overline{F}_\eta} + \lambda\bar{\eta} I] f, \tag{24.27}
$$

where f is any finite linear combination of k_z, $z \in \mathbb{D}$.

Now, let $\varphi \in H^2$. Then there is a sequence $(\varphi_n)_{n \geq 1}$, where each φ_n is a finite linear combination of k_z, $z \in \mathbb{D}$, such that $\varphi_n \longrightarrow \varphi$ in the norm of H^2. But the operators $T_{F_\lambda} T_{\overline{F}_\eta}$ and $W_\lambda^* W_\eta$ are continuous as operators from H^2 into $H(\mathbb{D})$. Therefore, $W_\lambda^* W_\eta \varphi_n$ (respectively $T_{F_\lambda} T_{\overline{F}_\eta} \varphi_n$) tends to $W_\lambda^* W_\eta \varphi$ (respectively $T_{F_\lambda} T_{\overline{F}_\eta} \varphi$) in the space $H(\mathbb{D})$. Therefore, the identity (24.27) also holds for each $\varphi \in H^2$.

According to Lemma 24.27, $W_\lambda^* W_\eta$ is a contraction on H^2. Thus, $T_{F_\lambda} T_{\overline{F}_\eta}$ is a bounded operator on H^2. Moreover, by (24.27),

$$
|1 - \lambda\bar{\eta}| \, \|T_{F_\lambda} T_{\overline{F}_\eta}\| = \|W_\lambda^* W_\eta - \lambda\bar{\eta} I\| \leq \|W_\lambda^* W_\eta\| + 1 \leq 2,
$$

which gives the required estimation. □

Theorem 24.29 *Let b be a nonextreme point in the closed unit ball of H^∞, and let Θ be a nonconstant inner function. Let μ and ν be the Clark measures associated respectively with b and Θ. Suppose that ν is absolutely continuous with respect to μ and that $d\nu/d\mu \in L^2(\mu)$. Then*

$$
\frac{F_\lambda}{1 - \Theta} \in H^2
$$

for all $\lambda \in \mathbb{T} \setminus \{1\}$.

Proof For simplicity, write $f = d\nu/d\mu$ and put $h = V_b f$. Then, according to Theorem 20.5, the function h belongs to $\mathcal{H}(b)$ and, more explicitly, is given by the formula

$$\begin{aligned} h(z) &= (1-b(z)) \int_{\mathbb{T}} \frac{f(e^{it})}{1-ze^{-it}}\, d\mu(e^{it}) \\ &= (1-b(z)) \int_{\mathbb{T}} \frac{d\nu(e^{it})}{1-ze^{-it}} \\ &= \frac{1-b(z)}{1-\Theta(z)} \times (1-\Theta(z)) \int_{\mathbb{T}} \frac{d\nu(e^{it})}{1-ze^{-it}} \\ &= \frac{1-b(z)}{1-\Theta(z)}\, (V_\Theta 1)(z). \end{aligned}$$

But it follows from (20.15) that

$$V_\Theta 1 = V_\Theta k_0 = (1-\overline{\Theta(0)})^{-1} k_0^\Theta = \frac{1-\overline{\Theta(0)}\Theta}{1-\overline{\Theta(0)}},$$

which implies that

$$h(z) = \frac{1-b(z)}{1-\Theta(z)}\, \frac{1-\overline{\Theta(0)}\Theta}{1-\overline{\Theta(0)}}. \tag{24.28}$$

We know from Theorem 24.19 that there is a Borel subset A of $\mathbb{T}$ such that $m(A) = 0$ and μ_λ is absolutely continuous with respect to m for every $\lambda \in \mathbb{T} \setminus A$. Let $\lambda \in \mathbb{T} \setminus (A \cup \{1\})$. Then, according to Theorem 24.23, $T_{1-\bar{\lambda}b} T_{\overline{F}_\lambda}$ is an isometry from H^2 onto $\mathcal{H}(b)$. In particular, there exists a $g_\lambda \in H^2$ such that $h = T_{1-\bar{\lambda}b} T_{\overline{F}_\lambda} g_\lambda$ and $\|g_\lambda\|_2 = \|h\|_b$. Therefore, it follows from (24.28) that

$$\begin{aligned} \frac{F_\lambda}{1-\Theta} &= \frac{(1-\overline{\Theta(0)})hF_\lambda}{(1-\overline{\Theta(0)}\Theta)(1-b)} \\ &= \frac{1-\overline{\Theta(0)}}{1-\overline{\Theta(0)}\Theta}\, \frac{(1-\bar{\lambda}b)F_\lambda}{1-b}\, T_{\overline{F}_\lambda} g_\lambda \\ &= \frac{1-\overline{\Theta(0)}}{1-\overline{\Theta(0)}\Theta}\, \frac{a}{1-b}\, T_{\overline{F}_\lambda} g_\lambda \\ &= \frac{1-\overline{\Theta(0)}}{1-\overline{\Theta(0)}\Theta}\, T_F T_{\overline{F}_\lambda} g_\lambda. \end{aligned}$$

On the one hand, since $\lambda \neq 1$, Theorem 24.28 ensures that $T_F T_{\overline{F}_\lambda} g_\lambda$ belongs to H^2, and we have

$$\|T_F T_{\overline{F}_\lambda} g_\lambda\|_2 \le \|T_F T_{\overline{F}_\lambda}\|\, \|g_\lambda\|_2 \le \frac{2}{|1-\lambda|}\, \|h\|_b.$$

On the other hand, $(1-\overline{\Theta(0)})/(1-\overline{\Theta(0)}\Theta)$ belongs to H^∞ and

$$\left\| \frac{1-\overline{\Theta(0)}}{1-\overline{\Theta(0)}\Theta} \right\|_\infty \leq \frac{1+|\Theta(0)|}{1-|\Theta(0)|}.$$

Therefore, $F_\lambda/(1-\Theta)$, as the product of an H^∞ function with an H^2 function, belongs to H^2. Moreover, we have

$$\left\| \frac{F_\lambda}{1-\Theta} \right\|_2 \leq \left\| \frac{1-\overline{\Theta(0)}}{1-\overline{\Theta(0)}\Theta} \right\|_\infty \|T_F T_{\overline{F}_\lambda} g_\lambda\|_2 \leq \frac{1+|\Theta(0)|}{1-|\Theta(0)|} \frac{2}{|1-\lambda|} \|h\|_b.$$

So far, we have proved that, for every $\lambda \in \mathbb{T} \setminus (A \cup \{1\})$, then $F_\lambda/(1-\Theta)$ is in H^2 and

$$\left\| \frac{F_\lambda}{1-\Theta} \right\|_2 \leq \frac{2(1+|\Theta(0)|)}{1-|\Theta(0)|} \frac{\|h\|_b}{|1-\lambda|}. \tag{24.29}$$

Now, suppose that $\lambda \in \mathbb{T} \setminus \{1\}$. Then there is a sequence $\lambda_n \in \mathbb{T} \setminus (A \cup \{1\})$ that tends to λ. Let $0 \leq r < 1$. Using Fatou's lemma and (24.29), we have

$$\begin{aligned}
\int_0^{2\pi} \liminf_{n\to\infty} \left| \frac{F_{\lambda_n}(re^{it})}{1-\Theta(re^{it})} \right|^2 \frac{dt}{2\pi} &\leq \liminf_{n\to\infty} \int_0^{2\pi} \left| \frac{F_{\lambda_n}(re^{it})}{1-\Theta(re^{it})} \right|^2 \frac{dt}{2\pi} \\
&= \liminf_{n\to\infty} \left\| \frac{F_{\lambda_n}}{1-\Theta} \right\|_2^2 \\
&\leq \liminf_{n\to\infty} \frac{2(1+|\Theta(0)|)}{1-|\Theta(0)|} \frac{\|h\|_b}{|1-\lambda_n|} \\
&= \frac{2(1+|\Theta(0)|)}{1-|\Theta(0)|} \frac{\|h\|_b}{|1-\lambda|}.
\end{aligned}$$

Since F_{λ_n} tends to F_λ on $\mathbb{D}$, we get

$$\int_0^{2\pi} \left| \frac{F_\lambda(re^{it})}{1-\Theta(re^{it})} \right|^2 \frac{dt}{2\pi} \leq \frac{2(1+|\Theta(0)|)}{1-|\Theta(0)|} \frac{\|h\|_b}{|1-\lambda|},$$

which proves that $F_\lambda/(1-\Theta)$ belongs to H^2. □

Exercises

Exercise 24.6.1 Let (a, b) be a pair and let $\alpha \in \mathbb{T}$.

(i) Show that $\|a\|_b = \|P_+(F_\alpha/\bar{F}_\alpha)\|_2$.
(ii) Let $b(z) = (1+z)/2$. Show that $a(z) = (1-z)/2$ and show that $F_1 = 1$. Deduce that $\|a\|_b = 1$.

Hint: To prove (i), use Lemma 24.21 to write $a = T_a 1 = T_{1-\bar{\alpha}b} T_{\bar{F}_\alpha} T_{F_\alpha/\bar{F}_\alpha} 1$ and with Theorem 24.23, we get $\|a\|_b = \|T_{F_\alpha/\bar{F}_\alpha} 1\|_2$.

Exercise 24.6.2 Show that, for each $\lambda, \eta \in \mathbb{T}$, $\lambda \neq \eta$, the function $(1-\lambda\bar{\eta})F_\lambda \overline{F_\eta} + \lambda\bar{\eta}$ is unimodular.

Exercise 24.6.3 Assume $b(0) = 0$. Let $\lambda, \eta \in \mathbb{T}$, $\lambda \neq \eta$, be such that $\|F_\lambda\|_2 = \|F_\eta\|_2 = 1$. Show that $(1-\lambda\bar{\eta})T_{F_\lambda}T_{\overline{F_\eta}} + \lambda\bar{\eta}I$ is a unitary operator on H^2.
Hint: Use Theorem 24.28 and Lemma 24.27.

24.7 Invariant subspaces of $\mathcal{H}(b)$ under X_b

By the celebrated theorem of Beurling (Theorem 8.32), the closed invariant subspaces of S^* are precisely the model subspaces $K_\Theta = \mathcal{H}(\Theta) = H^2 \ominus \Theta H^2$, where Θ is an inner function for the open unit disk $\mathbb{D}$. Since K_Θ is a closed subspace of H^2 and $\mathcal{H}(b)$ is contractively contained in H^2, the intersection $K_\Theta \cap \mathcal{H}(b)$ is a closed subspace of $\mathcal{H}(b)$. It is also clear that $K_\Theta \cap \mathcal{H}(b)$ is invariant under X_b. What is more interesting is that they are the only closed invariant subspaces of $\mathcal{H}(b)$, whenever b is a nonextreme point in the closed unit ball of H^∞.

To establish this fact, recall that, by Theorem 18.13, if $\varphi \in H^\infty$, then $\mathcal{H}(b)$ is invariant under $T_{\bar{\varphi}}$ and the norm of the restricted operator

$$\begin{array}{rccc} T_{\bar{\varphi},b}: & \mathcal{H}(b) & \longrightarrow & \mathcal{H}(b) \\ & f & \longmapsto & T_{\bar{\varphi}} f \end{array}$$

is at most $\|\varphi\|_\infty$. Therefore, in the light of Corollary 12.50, we see that, if $E \subset \mathcal{H}(b)$ is a closed invariant subspace of X_b, then, for every function $\varphi \in H^\infty$, the subspace E is also invariant under $T_{\bar{\varphi}}$, i.e.

$$T_{\bar{\varphi}} E \subset E \qquad (\varphi \in H^\infty). \tag{24.30}$$

Lemma 24.30 *Let (a, b) be a pair, and let $E \subset \mathcal{H}(b)$ be a closed invariant subspace of X_b. Then, with respect to the norm of $\mathcal{H}(b)$, the linear manifold $T_{\bar{a}} E$ is dense in E.*

Proof We know from (24.30) that $T_{\bar{a}} E \subset E$. Now, let $f \in E$ be such that $f \perp T_{\bar{a}} E$. Since, by (12.3),

$$S^{*n} T_{\bar{a}} f = T_{\bar{a}} S^{*n} f = T_{\bar{a}} X_b^n f \in T_{\bar{a}} E,$$

we must have

$$f \perp S^{*n} T_{\bar{a}} f \qquad (n \geq 0).$$

By Lemma 23.7, we also have

$$(S^{*n} T_{\bar{a}} f)^+ = S^{*n} T_{\bar{a}} f^+.$$

Hence, according to Theorem 23.8,

$$\begin{aligned}
0 &= \langle f, S^{*n} T_{\bar{a}} f \rangle_b \\
&= \langle f, S^{*n} T_{\bar{a}} f \rangle_2 + \langle f^+, S^{*n} T_{\bar{a}} f^+ \rangle_2 \\
&= \langle T_a S^n f, f \rangle_2 + \langle T_a S^n f^+, f^+ \rangle_2 \\
&= \langle a z^n f, f \rangle_2 + \langle a z^n f^+, f^+ \rangle_2 \\
&= \frac{1}{2\pi} \int_0^{2\pi} e^{in\theta} a(e^{i\theta}) [|f(e^{i\theta})|^2 + |f^+(e^{i\theta})|^2] \, d\theta.
\end{aligned}$$

Since the last set of identities hold for every $n \geq 0$, we conclude that the function $a(|f|^2 + |f^+|^2)$ belongs to the space H_0^1. But a is outer and $|f|^2 + |f^+|^2 \in L^1$, whence Corollary 4.28 implies that $|f|^2 + |f^+|^2$ also belongs to H_0^1. Now, it suffices to apply (4.12) to conclude that $|f|^2 + |f^+|^2 \equiv 0$, that is, $f \equiv 0$. □

Theorem 24.31 *Let b be a nonextreme point in the closed unit ball of H^∞, and let E be a closed subspace of $\mathcal{H}(b)$, $E \neq \mathcal{H}(b)$. Then the following are equivalent.*

(i) *E is invariant under X_b.*
(ii) *There exists an inner function Θ such that $E = K_\Theta \cap \mathcal{H}(b)$.*

Proof (ii) $\Longrightarrow$ (i) As discussed above, this follows immediately from Theorem 18.13.

(i) $\Longrightarrow$ (ii) Let E be an invariant subspace of X_b and let $\bar{E}$ denote the closure of E in H^2. It is clear that $\bar{E}$ is invariant under S^*. Hence, by Beurling's theorem (Theorem 8.32), there exists a function Θ that is either inner or $\Theta \equiv 0$ such that $\bar{E} = \mathcal{H}(\Theta)$. To conclude the proof, it is enough to verify that

$$E = \bar{E} \cap \mathcal{H}(b).$$

The inclusion $E \subset \bar{E} \cap \mathcal{H}(b)$ is trivial. For the reverse inclusion $\bar{E} \cap \mathcal{H}(b) \subset E$, since both E and $\bar{E} \cap \mathcal{H}(b)$ are closed subspaces of $\mathcal{H}(b)$, it is enough to show that a dense subset of $\bar{E} \cap \mathcal{H}(b)$ is in E. The candidate is $T_{\bar{a}}(\bar{E} \cap \mathcal{H}(b))$.

Since $\bar{E} \cap \mathcal{H}(b)$ is a closed subspace of $\mathcal{H}(b)$ that is invariant under X_b, Lemma 24.30 ensures that, with respect to the $\mathcal{H}(b)$ topology, $T_{\bar{a}}(\bar{E} \cap \mathcal{H}(b))$ is dense in $\bar{E} \cap \mathcal{H}(b)$. Clearly, we have $T_{\bar{a}}(\bar{E} \cap \mathcal{H}(b)) \subset T_{\bar{a}}\bar{E}$; let us establish $T_{\bar{a}}\bar{E} \subset E$. Indeed, let $f \in \bar{E}$. Then there exists a sequence $(f_n)_{n \geq 1}$ in E converging to f in H^2 norm. Hence, $T_{\bar{a}} f_n \longrightarrow T_{\bar{a}} f$, as $n \longrightarrow \infty$, in the norm of $\mathcal{M}(\bar{a})$ and thus, by Theorem 23.2, also in the norm of $\mathcal{H}(b)$. According to (24.30), $T_{\bar{a}} f_n \in E$, and thus the function $T_{\bar{a}} f$ also belongs to E. □

The cyclic vectors for S^* were characterized in Theorem 8.42. We showed that a function $f \in H^2$ is not cyclic for S^* if and only if it has a bounded-type meromorphic pseudocontinuation across $\mathbb{T}$ to $\mathbb{D}_e = \{z : 1 < |z| \le \infty\}$. This is also equivalent to the existence of two functions $g, h \in \bigcup_{p>0} H^p$ such that

$$f = \frac{\bar{h}}{\bar{g}} \qquad \text{(a.e. on } \mathbb{T}).$$

It turns out that cyclic vectors of X_b are precisely the cyclic vectors of S^* that live in $\mathcal{H}(b)$.

Corollary 24.32 *Let b be a nonextreme point of the closed unit ball of H^∞ and let $f \in \mathcal{H}(b)$. The following assertions are equivalent :*

(i) *The function f is cyclic for X_b.*
(ii) *The function f is cyclic for S^*.*

Proof The implication (i) $\Longrightarrow$ (ii) follows from Corollary 18.15. Conversely assume that f is cyclic for S^* and denote by $\mathcal{J}$ the closed subspace of $\mathcal{H}(b)$ defined by

$$\mathcal{J} = \overline{\text{Span}(X_b^n f : n \ge 0)}.$$

It is clear that $\mathcal{J}$ is a closed invariant subspace of X_b. Then, according to Theorem 24.31, there exists a function u that is either inner or $u \equiv 0$ such that

$$\overline{\text{Span}(X_b^n f : n \ge 0)} = \mathcal{H}(b) \cap \mathcal{H}(u).$$

If u is inner, since $f \in \mathcal{H}(u) = K_u$, then it is not cyclic for S^*, which is contrary to the hypothesis. Thus, $u \equiv 0$ and $\mathcal{H}(u) = H^2$, which implies that

$$\overline{\text{Span}(X_b^n f : n \ge 0)} = \mathcal{H}(b).$$

Therefore, the function f is cyclic for X_b. □

If we restrict the operators involved in the identity $S^* T_{\bar{\varphi}} = T_{\bar{\varphi}} S^*$ to the space $\mathcal{H}(b)$, this identity can be rewritten as

$$X_b T_{\bar{\varphi},b} = T_{\bar{\varphi},b} X_b.$$

In other words, $T_{\bar{\varphi},b}$ is in the commutant of X_b. We can now show that there is no other element in the commutant.

Corollary 24.33 *Let b be a nonextreme point in the closed unit ball of H^∞. Then the operators commuting with X_b are precisely the operators $T_{\bar{\varphi},b}$, with $\varphi \in H^\infty$.*

Proof As discussed above, by (12.3), it is clear that $T_{\bar{\varphi},b}$ is in the commutant of X_b.

Now, let A be an operator on $\mathcal{H}(b)$ that commutes with X_b. For each $\lambda \in \mathbb{D}$, we know from Theorem 24.13 that

$$\ker(X_b - \bar{\lambda} I) = \mathbb{C} k_\lambda.$$

Moreover,

$$X_b A k_\lambda = A X_b k_\lambda = A S^* k_\lambda = \bar{\lambda} A k_\lambda,$$

which implies that $Ak_\lambda \in \ker(X_b - \bar{\lambda} I) = \mathbb{C} k_\lambda$. Thus, there exists a complex number $\varphi(\lambda)$ such that

$$A k_\lambda = \overline{\varphi(\lambda)} k_\lambda. \tag{24.31}$$

Since A is a continuous operator on $\mathcal{H}(b)$ and, by Theorem 18.3, $\mathcal{H}(b)$ is densely and contractively contained in H^2, A can be extended to a continuous operator on H^2. Using Theorem 9.3, (9.13) and (24.31), we thus get that φ is a multiplier of H^2 and thus $\varphi \in H^\infty$. Now, using once more (24.31) and (12.7), we have

$$A k_\lambda = T_{\bar{\varphi}} k_\lambda \qquad (\lambda \in \mathbb{D}),$$

and it remains to apply Corollary 23.26 to get that $A = T_{\bar{\varphi}, b}$. □

Corollary 24.34 *Let b be a nonextreme point of the closed unit ball of H^∞, and let u be any inner function. Then $T_{\bar{u}} \mathcal{H}(b)$ is dense in $\mathcal{H}(b)$.*

Proof We argue by absurdity. Assume that $T_{\bar{u}}\mathcal{H}(b)$ is not dense in $\mathcal{H}(b)$. Then, with respect to the $\mathcal{H}(b)$ topology, the closure of $T_{\bar{u}}\mathcal{H}(b)$ is a proper X_b-invariant subspace of $\mathcal{H}(b)$. Hence, by Theorem 24.31, there exists an inner function Θ such that

$$\mathrm{Clos}_{\mathcal{H}(b)}(T_{\bar{u}}\mathcal{H}(b)) = \mathcal{H}(b) \cap K_\Theta.$$

In particular, we have

$$T_{\bar{u}}\mathcal{H}(b) \subset K_\Theta,$$

and this inclusion implies that, relative to the inner product of L^2, $T_{\bar{u}}\mathcal{H}(b)$ is orthogonal to ΘH^2. Hence, $\mathcal{H}(b)$ has to be orthogonal to $u\Theta H^2$. But this is a contradiction, since, by Theorem 18.3, $\mathcal{H}(b)$ is dense in H^2. □

24.8 Completeness of the family of difference quotients

We recall that

$$\hat{k}_\lambda^b(z) = (Q_\lambda b)(z) = \frac{b(z) - b(\lambda)}{z - \lambda} \qquad (z, \lambda \in \mathbb{D}).$$

See Sections 8.2, 14.4 and 18.6 where this notion was discussed.

Theorem 24.35 *Let b be a nonextreme point of the closed unit ball of H^∞. Then the following are equivalent.*

(i) $\mathrm{Span}(\hat{k}_\lambda^b : \lambda \in \mathbb{D}) = \mathcal{H}(b)$.

(ii) *b is not pseudocontinuable to a meromorphic function of bounded type across $\mathbb{T}$.*

Proof (i) $\Longrightarrow$ (ii) Assume that $\text{Span}\{\hat{k}_\lambda^b : \lambda \in \mathbb{D}\} = \mathcal{H}(b)$ but b is pseudocontinuable to a meromorphic function of bounded type across $\mathbb{T}$. Then, by Corollary 8.33 and Theorem 8.42, there exists a nonconstant inner function Θ such that $b \in K_\Theta$. Since K_Θ is S^*-invariant, $S^{*(n+1)}b \in K_\Theta$ for all $n \geq 0$. As $\mathcal{H}(b)$ is contained continuously in H^2, we deduce that

$$\text{Span}_{\mathcal{H}(b)}(S^{*(n+1)}b : n \geq 0) \subset \text{Clos}_{\mathcal{H}(b)} K_\Theta \subset K_\Theta,$$

and it follows from Theorem 18.19 that $\mathcal{H}(b) \subset K_\Theta$. Now since b is not an extreme point in the closed unit ball of H^∞, we know that the polynomials belong to $\mathcal{H}(b)$ and consequently to K_Θ. Hence $H^2 \subset K_\Theta$, which is absurd. Thus, if the difference quotients are complete in $\mathcal{H}(b)$, then b is not pseudocontinuable.

(ii) $\Longrightarrow$ (i) Conversely, assume b is not pseudocontinuable to a meromorphic function of bounded type across $\mathbb{T}$. It is clear that

$$\text{Span}_{\mathcal{H}(b)}(S^{*(n+1)}b : n \geq 0)$$

is a closed S^*-invariant subspace of $\mathcal{H}(b)$. But we know from Theorem 24.31 the description of these subspaces when b is not an extreme point: they are just the intersection of $\mathcal{H}(b)$ with the invariant subspaces of S^*. Hence, there is an inner function Θ such that

$$\text{Span}_{\mathcal{H}(b)}(S^{*(n+1)}b : n \geq 0) = \mathcal{H}(b) \cap K_\Theta.$$

But $S^*b \in K_\Theta$ implies that $b \in K_{z\Theta}$, which is absurd unless $\Theta \equiv 0$ (because b is not pseudocontinuable). Hence,

$$\text{Span}_{\mathcal{H}(b)}(S^{*(n+1)}b : n \geq 0) = \mathcal{H}(b),$$

and applying once more Theorem 18.19, we obtain that the difference quotients are complete in $\mathcal{H}(b)$. $\square$

Exercises

Exercise 24.8.1

(i) Show that

$$S^*T_b - T_bS^* = S^*b \otimes 1.$$

In particular, $S^*T_b - T_bS^*$ is a rank-one operator whose range is spanned by S^*b.

(ii) Show that $S^*b \in \mathcal{H}(b)$ if and only if

$$(S^*T_b - T_bS^*)(T_{\bar{b}}S - ST_{\bar{b}}) \leq I - T_bT_{\bar{b}}. \tag{24.32}$$

(iii) Prove (24.32) and thus $S^*b \in \mathcal{H}(b)$.

Exercise 24.8.2

(i) Show that $\mathcal{H}(b)$ is S^*-invariant and S^* acts as a contraction on $\mathcal{H}(b)$ if and only if

$$S^*(I - T_b T_{\bar{b}})S \le I - T_b T_{\bar{b}}. \tag{24.33}$$

(ii) Show that

$$I - T_b T_{\bar{b}} - S^*(I - T_b T_{\bar{b}})S = S^* T_b (I - SS^*) T_{\bar{b}} S,$$

and prove (24.33).

Exercise 24.8.3 Let $g \in \mathcal{H}(b)$, $f \in H^2$ and put $h = (I - T_b T_{\bar{b}})f$.

(i) Show that

$$\langle X_b g, h\rangle_b = \langle g,\ (I - T_b T_{\bar{b}})Sf\rangle_b.$$

(ii) Deduce that $X_b^* h = (I - T_b T_{\bar{b}})Sf$.

(iii) Show that

$$T_{\bar{b}} S - S T_{\bar{b}} = 1 \otimes S^* b.$$

Hint: Use Exercise 24.8.1(i).

(iv) Show that

$$(I - T_b T_{\bar{b}})Sf = Sh - T_b (T_{\bar{b}} S - S T_{\bar{b}} S)f.$$

(v) Using (iii), prove that

$$(T_{\bar{b}} S - S T_{\bar{b}})f = \langle h, S^* b\rangle_b 1.$$

(vi) Deduce that

$$X_b^* h = Sh - \langle h, S^* b\rangle_b b, \tag{24.34}$$

for every $h \in \mathcal{M}(I - T_b T_{\bar{b}})$.

(vii) Prove that formula (24.34) is valid for every $h \in \mathcal{H}(b)$.

Notes on Chapter 24

Section 24.1

Theorem 24.1 is due to de Branges and Rovnyak [65, problem 74]. In fact, they proved that, if $b \in \mathcal{H}(b)$, then $\mathcal{H}(b)$ is invariant under the shift operator S. But the link between the membership of b to $\mathcal{H}(b)$ and the fact that b is a nonextreme point is due to Sarason [160]. Lemma 24.2 is also due to Sarason [159, sec. 4]. The computation of the norm of S_b made in Theorem 24.3 appears to be new. The determination of the spectrum of S_b (Theorems 24.4 and 24.5) is made by Sarason [159], and he applied this information to obtain Theorem 24.6

on multipliers. It should be noted that multipliers of de Branges–Rovnyak spaces in the nonextreme case have been studied by Lotto and Sarason in [124]. In particular, they obtain a criterion for multipliers in terms of the boundedness of a certain product of (possibly) unbounded Hankel operators.

Theorem 24.7 on the commutant of S_b is taken from Sarason [159], but as noted it follows from a standard rather general theorem (Theorem 9.16) due to Shields and Wallen [176].

Section 24.2

The equivalence of (i) and (iii) in Lemma 24.9 is due to Sarason [159, lemma 6] whereas the equivalence of (i) and (ii) has been noticed by Costara and Ransford [58]. Theorem 24.10, Lemma 24.11 and Theorem 24.12 have been proved by Sarason in [159]. Note that, in the case when $b/a \in L^2(\mathbb{T})$, an alternative proof of Theorem 24.12 has been given by Blandignères, Fricain, Gaunard, Hartmann and Ross in [35].

Section 24.3

As already mentioned, Sarason [160] determined the characteristic operator function of the contraction X_b^*. More precisely, he showed that, in the nonextreme case, then the characteristic function of X_b^* is the two-by-one matrix function $\binom{b}{a}$. Then, we can recover Theorems 24.13, 24.14 and 24.15 using the Sz.-Nagy–Foiaş model theory. Here we choose to give a more elementary and direct proof.

Section 24.4

Theorem 24.17 is due to Sarason [166, chap. IV].

Section 24.5

Corollary 24.20 is due to Sarason [163, lemma 2]. It is also implicitly contained in a paper of Nakamura [134], where he studied the rank-one perturbations of the shift operator which are pure isometries.

Section 24.6

Lemma 24.22 and Theorem 24.23 are due to Sarason [163, lemma 3]. Theorem 24.28 is from [164, proposition 4]. Theorem 24.29 is from [166].

Section 24.7

The description of the invariant subspaces of X_b given in Theorem 24.31 is due to Sarason [159, theorem 5]. The situation for $S_b = S_{|\mathcal{H}(b)}$ is more difficult and, in fact, it remains an unsolved problem. See Section 28.5 for a discussion and some results in this direction. In [181], Suárez has classified the invariant subspaces of the operator X_b in the case where b is extreme, but the description is more complicated than in the nonextreme case. Corollary 24.32 is due to Fricain, Mashreghi and Seco [82]. It should be noted that, despite the work of Suárez, cyclic vectors for X_b in the extreme case are far from being understood. Corollary 24.34 is also due to Sarason [163, lemma 4].

Section 24.8

Theorem 24.35 is due to Chevrot, Fricain and Timotin [53]. In that paper, they studied the vector-valued situation using an approach based on the Sz.-Nagy–Foiaş model. An analog of this result in the case when b is extreme is discussed in Section 26.6 and appears in Fricain [77].

25

$\mathcal{H}(b)$ spaces generated by an extreme symbol b

In this chapter, we study the specific properties of $\mathcal{H}(b)$ spaces when b is an extreme point of the closed unit ball of H^∞. Thus, by Theorem 6.7, we assume

$$\int_{\mathbb{T}} \log(1 - |b(e^{i\theta})|^2)\, d\theta = -\infty.$$

In particular, this happens if $b = \Theta$ is an inner function. In this case, $\mathcal{H}(b)$ is precisely the model space K_Θ. Roughly speaking, when b is an extreme point, the space $\mathcal{H}(b)$ looks like the model space K_Θ.

In Section 25.1, we introduce a unitary operator from $L^2(\rho)$ onto $\mathcal{H}(\bar{b})$, where $\rho = 1 - |b|^2$ on $\mathbb{T}$. This unitary operator is important, in particular to compute the norm of functions $f \in \mathcal{H}(b)$. For example, we do this computation for $f = S^*b$. In Section 25.2, we prove that a nonzero element of $\mathcal{H}(\bar{b})$ cannot be analytically continued across all of $\mathbb{T}$. This fact is used to show that $b \notin \mathcal{H}(b)$. In contrast to the nonextreme case, we also show that the space $\mathcal{H}(b)$ is not invariant under the forward shift operator S. Despite the situation for $\mathcal{H}(\bar{b})$, some elements of $\mathcal{H}(b)$ can be extended across all of $\mathbb{T}$. In Section 25.3, we characterize such functions. This characterization is used to prove that $k_\lambda \in \mathcal{H}(b)$ if and only if $b(\lambda) = 0$. In Section 25.4, we give a formula for $\|X_b f\|_b$, $f \in \mathcal{H}(b)$, and show that the defect operator D_{X_b} has rank one. In Section 25.5, we show that the only function in $\mathcal{H}(\bar{b})$ that has a bounded-type meromorphic pseudocontinuation across $\mathbb{T}$ to the exterior disk $\mathbb{D}_e$ is the zero function. We also prove a similar result for $\mathcal{H}(b)$ functions. In Section 25.6, we exhibit an important orthogonal decomposition of $\mathcal{H}(b)$. We use this orthogonal decomposition to show, in Section 25.7, that the closure of $\mathcal{H}(\bar{b})$ in $\mathcal{H}(b)$ is $\mathcal{H}([b])$, where $[b]$ is the outer factor of b. This is dramatically different from the nonextreme case, where we have seen that $\mathcal{H}(\bar{b})$ is always dense in $\mathcal{H}(b)$. Finally, in Section 25.8, we give a characterization of $\mathcal{H}(b)$ spaces when b is an extreme point. The corresponding result for the nonextreme case was given in Section 23.7.

25.1 A unitary map between $\mathcal{H}(\bar{b})$ and $L^2(\rho)$

We recall that ρ is the function

$$\rho = 1 - |b|^2 \in L^\infty(\mathbb{T})$$

and K_ρ is the application defined on $L^2(\rho)$ by the formula $K_\rho(g) = P_+(\rho g)$ into H^2 (see Section 13.4). In this situation, we can give further information about K_ρ and its range.

Theorem 25.1 *Let b be an extreme point of the closed unit ball of H^∞. Then*

$$H^2(\rho) = L^2(\rho)$$

and K_ρ is an isometry from $L^2(\rho)$ onto $\mathcal{H}(\bar{b})$. If μ is the Clark measure associated with b, then we also have $H^2(\mu) = L^2(\mu)$.

Proof That $H^2(\rho) = L^2(\rho)$ and $H^2(\mu) = L^2(\mu)$ were established in Corollary 13.34. According to Theorem 20.1, K_ρ is a partial isometry from $L^2(\rho)$ onto $\mathcal{H}(\bar{b})$ and $\ker K_\rho = (H^2(\rho))^\perp$. Since $H^2(\rho) = L^2(\rho)$, we conclude that K_ρ is in fact an isometry from $L^2(\rho)$ onto $\mathcal{H}(\bar{b})$. □

Even though in Theorem 25.1 we assumed that b is an extreme point to deduce that $H^2(\rho) = L^2(\rho)$ and $H^2(\mu) = L^2(\mu)$, we emphasize that the last two identities occur precisely when b is an extreme point of the closed unit ball of H^∞. Hence, Theorem 25.1 can be rewritten in a proper way to give a characterization of the identity $H^2(\rho) = L^2(\rho)$.

Corollary 25.2 *Let b be an extreme point of the closed unit ball of H^∞ and let f be a function in $\mathcal{H}(\bar{b})$. Then there is a unique function $g \in L^2(\rho)$ such that*

$$f = P_+(\rho g).$$

Moreover, we have $\log|\rho g| \notin L^1(\mathbb{T})$.

Proof The first part follows immediately from Theorem 25.1.

For the second part, write

$$\log|\rho g| = \log|g\rho^{1/2}| + \tfrac{1}{2}\log\rho. \tag{25.1}$$

On the one hand, since b is an extreme point of the closed unit ball of H^∞, we have

$$\int_{\mathbb{T}} \log\rho \, dm = -\infty.$$

On the other hand, using Jensen's inequality, we see that

$$\begin{aligned}\int_{\mathbb{T}} \log |g\rho^{1/2}|\, dm &\le \log\left(\int_{\mathbb{T}} |g\rho^{1/2}|\, dm\right)\\ &\le \log\left(\int_{\mathbb{T}} |g|^2\, \rho\, dm\right)^{1/2}.\end{aligned}$$

Since $g \in L^2(\rho)$, we deduce that

$$\int_{\mathbb{T}} \log |g\rho^{1/2}|\, dm < +\infty.$$

Thus, the conclusion follows from (25.1). □

Corollary 25.3 *Let b be an extreme point of the closed unit ball of H^∞. Then*

$$\|S^*b\|_b^2 = 1 - |b(0)|^2.$$

Proof By Theorem 17.8, $T_{\bar{b}}S^*b \in \mathcal{H}(\bar{b})$ and

$$\|S^*b\|_b^2 = \|S^*b\|_2^2 + \|T_{\bar{b}}S^*b\|_{\bar{b}}^2. \tag{25.2}$$

To compute the norm of $T_{\bar{b}}S^*b$ in $\mathcal{H}(\bar{b})$, we use the operator Z_ρ, which was introduced in Section 8.1. Recall that Z_ρ denotes the operator on $L^2(\rho)$ of multiplication by the independent variable z, i.e.

$$(Z_\rho f)(z) = zf(z),$$

where $z \in \mathbb{T}$ and $f \in L^2(\rho)$. Hence, by Corollary 13.18, we have

$$\begin{aligned}K_\rho Z_\rho^* \chi_0 = S^*K_\rho\chi_0 &= S^*P_+\rho = S^*P_+(1-|b|^2)\\ &= -S^*P_+|b|^2 = -S^*T_{\bar{b}}b.\end{aligned}$$

Since $S^*T_{\bar{b}} = T_{\bar{b}}S^*$, we obtain

$$T_{\bar{b}}S^*b = -K_\rho Z_\rho^*\chi_0. \tag{25.3}$$

Now, using Theorem 25.1 and the fact that Z_ρ is a unitary operator, we can write

$$\|T_{\bar{b}}S^*b\|_{\bar{b}} = \|K_\rho Z_\rho^*\chi_0\|_{\bar{b}} = \|Z_\rho^*\chi_0\|_{L^2(\rho)} = \|\chi_0\|_{L^2(\rho)}.$$

Therefore, (25.2) becomes

$$\|S^*b\|_b^2 = \|S^*b\|_2^2 + \|\chi_0\|_{L^2(\rho)}^2. \tag{25.4}$$

But it is easy to see that

$$\|\chi_0\|_{L^2(\rho)}^2 = \int_{\mathbb{T}} \rho\, dm = \int_{\mathbb{T}} (1-|b|^2)\, dm = 1 - \|b\|_2^2$$

and, by (8.16), that

$$\|S^* b\|_b^2 = \|b\|_2^2 - |b(0)|^2.$$

Plug the last two identities into (25.4) to get the result. □

25.2 Analytic continuation of $f \in \mathcal{H}(\bar{b})$

A nonzero element of $\mathcal{H}(\bar{b})$ certainly has a singularity somewhere on $\mathbb{T}$. This fact is stated in the following form.

Theorem 25.4 *Let b be an extreme point in the closed unit ball of H^∞, and let $f \in \mathcal{H}(\bar{b})$. If f can be analytically continued across all of $\mathbb{T}$, then $f \equiv 0$.*

Proof Let f be a function in $\mathcal{H}(\bar{b})$ that can be analytically continued across all of $\mathbb{T}$. Hence, Theorem 5.7 implies that there is a $c > 0$ such that $|\hat{f}(n)| = O(e^{-cn})$, as $n \longrightarrow +\infty$. We know from 25.1 that there is a unique function $g \in L^2(\rho)$ such that $f = P_+(\rho g)$ and $\log|g\rho| \notin L^1(\mathbb{T})$. In particular, the condition $f = P_+(\rho g)$ implies that

$$\hat{f}(n) = \widehat{g\rho}(n) \qquad (n \geq 0).$$

Put $h = \overline{g\rho}$. We prove that h satisfies the hypotheses of Theorem 4.31. First note that h belongs to $L^2(\mathbb{T})$, since it is the product of the $L^2(\mathbb{T})$ function $g\rho^{1/2}$ and the $L^\infty(\mathbb{T})$ function $\rho^{1/2}$. Moreover, an easy computation shows that

$$\hat{h}(-n) = \overline{\hat{\bar{h}}(n)} = \overline{\widehat{g\rho}(n)} = \overline{\hat{f}(n)} \qquad (n \geq 0).$$

Thus, $|\hat{h}(-n)| = O(e^{-cn})$, as $n \longrightarrow +\infty$. But, since $\log|h| = \log|g\rho| \notin L^1(\mathbb{T})$, Theorem 4.31 ensures that $h \equiv 0$. Therefore, $f \equiv 0$. □

In Corollary 23.9, we saw that, if b is a nonextreme point of the closed unit ball of H^∞, then $b \in \mathcal{H}(b)$. But this is in fact the only case where the inclusion $b \in \mathcal{H}(b)$ is possible.

Corollary 25.5 *Let b be a point in the closed unit ball of H^∞. Then the following are equivalent.*

(i) $b \in \mathcal{H}(b)$.
(ii) *$\mathcal{H}(b)$ is invariant under the forward shift operator S.*
(iii) *$\mathcal{H}(\bar{b})$ has a nonzero element that is analytic on $\bar{\mathbb{D}}$.*
(ii) *b is a nonextreme point of the closed unit ball of H^∞.*

Proof (i) $\Longrightarrow$ (ii) According to Theorem 18.22, we have

$$X_b^* f = Sf - \langle f, S^*b\rangle_b b \qquad (f \in \mathcal{H}(b)). \tag{25.5}$$

Hence, the assumption $b \in \mathcal{H}(b)$ immediately implies that $Sf \in \mathcal{H}(b)$.

(ii) $\Longrightarrow$ (i) Take any function $f \in \mathcal{H}(b)$ such that $\langle f, S^*b\rangle_b \neq 0$. For instance, one can choose $f = S^*b \in \mathcal{H}(b)$ for which $\|S^*b\|_b^2 \neq 0$. Then we rewrite (25.5) as

$$b = \frac{Sf - X^*f}{\langle f, S^*b\rangle_b}$$

to deduce $b \in \mathcal{H}(b)$.

(i) $\Longrightarrow$ (iii) If $b \in \mathcal{H}(b)$, then, according to Theorem 17.8, we have $T_{\bar{b}}b \in \mathcal{H}(\bar{b})$. Since, by definition, the function $(I - T_{\bar{b}}T_b)\chi_0 = \chi_0 - T_{\bar{b}}b$ also belongs to $\mathcal{H}(\bar{b})$, we must have $\chi_0 \in \mathcal{H}(\bar{b})$. But the nonzero function χ_0 can obviously be analytically continued across all of $\mathbb{T}$.

(iii) $\Longrightarrow$ (iv) This follows from Theorem 25.4.

(iv) $\Longrightarrow$ (i) This was already proved in Corollary 23.9. □

We just saw that, if b is an extreme point of the closed unit ball of H^∞, then $\mathcal{H}(b)$ is not S-invariant. Nevertheless, it could be possible for some functions $f \in \mathcal{H}(b)$ that Sf remains in $\mathcal{H}(b)$. The following result characterizes this class of functions.

Corollary 25.6 *Let b be an extreme point of the closed unit ball of H^∞, and let h be a function in $\mathcal{H}(b)$. The following are equivalent.*

(i) $Sh \in \mathcal{H}(b)$.
(ii) $\langle h, S^*b\rangle_b = 0$.

Proof Remember that

$$X_b^* h = Sh - \langle h, S^*b\rangle_b b.$$

According to Corollary 25.5, we know that $b \notin \mathcal{H}(b)$. That gives the equivalence. □

25.3 Analytic continuation of $f \in \mathcal{H}(b)$

In contrast to the case of $\mathcal{H}(\bar{b})$, some elements of $\mathcal{H}(b)$ can be extended across all of $\mathbb{T}$. These elements are of a special form, which is described below.

Theorem 25.7 *Let b be an extreme point in the closed unit ball of H^∞, and let f be a function in H^2. Then the following are equivalent.*

(i) *$f \in \mathcal{H}(b)$ and can be analytically continued across all of $\mathbb{T}$.*
(ii) *f is rational and $T_{\bar{b}}f = 0$.*

Proof (i) $\Longrightarrow$ (ii) Assume that $f \in \mathcal{H}(b)$ and can be analytically continued across all of $\mathbb{T}$. Since f can be analytically continued across all of $\mathbb{T}$, Theorem 5.7 ensures that there are $c_1 > 0$ and $c_2 > 0$ such that

$$|\hat{f}(n)| \leq c_2 e^{-c_1 n} \qquad (n \geq 0). \tag{25.6}$$

We first show that $T_{\bar{b}} f$ can also be analytically continued across all of $\mathbb{T}$. In fact, for each $n \geq 1$, we have

$$\begin{aligned}\widehat{T_{\bar{b}} f}(n) &= \langle T_{\bar{b}} f, \chi_n \rangle_2 = \langle P_+(\bar{b} f), \chi_n \rangle_2 = \langle \bar{b} f, \chi_n \rangle_2 \\ &= \langle \chi_{-n} f, b \rangle_2 = \langle P_+(\chi_{-n} f), b \rangle_2 = \langle S^{*n} f, b \rangle_2.\end{aligned}$$

We repeatedly used Lemma 4.8. Therefore,

$$|\widehat{T_{\bar{b}} f}(n)| \leq \|S^{*n} f\|_2 \, \|b\|_2 \leq \|S^{*n} f\|_2 \, \|b\|_\infty \leq \|S^{*n} f\|_2.$$

But, by (8.16),

$$\|S^{*n} f\|_2^2 = \sum_{k=0}^{\infty} |\hat{f}(k+n)|^2.$$

Hence, by (25.6),

$$\|S^{*n} f\|_2^2 \leq c_2 \sum_{k=0}^{\infty} e^{-2c_1(k+n)} = c_2 e^{-2c_1 n} \sum_{k=0}^{\infty} e^{-2c_1 k} = c_2 \frac{e^{-2c_1 n}}{1 - e^{-2c_1}}.$$

Thus, $\widehat{T_{\bar{b}} f}(n) = O(e^{-c_1 n})$ as $n \longrightarrow +\infty$. Another application of Theorem 5.7 implies that $T_{\bar{b}} f$ can be analytically continued across all of $\mathbb{T}$. Since $f \in \mathcal{H}(b)$, we have $T_{\bar{b}} f \in \mathcal{H}(\bar{b})$ and thus it follows from Theorem 25.4 that $T_{\bar{b}} f = 0$, which means that f belongs to the kernel of $T_{\bar{b}}$.

It remains to show that f is a rational function. By Theorem 14.10, f is a cyclic vector for S^* if and only if so is $T_{\bar{b}} f$. But, since $T_{\bar{b}} f = 0$, $T_{\bar{b}} f$ is not a cyclic vector of S^*, and thus nor is f. Theorem 8.42 now ensures that f is a rational function.

(ii) $\Longrightarrow$ (i) Assume that f is a rational function in H^2 that belongs to $\ker T_{\bar{b}}$. Then $T_{\bar{b}} f = 0 \in \mathcal{H}(\bar{b})$ and Theorem 17.8 implies that $f \in \mathcal{H}(b)$. The fact that f can be analytically continued across all of $\mathbb{T}$ follows from Theorem 5.8. □

As the first application, we can characterize the Cauchy kernels that belong to $\mathcal{H}(b)$.

Corollary 25.8 *Let b be an extreme point of the closed unit ball of H^∞ and let $\lambda \in \mathbb{D}$. Then the following are equivalent:*

(i) $k_\lambda \in \mathcal{H}(b)$;

(ii) $b(\lambda) = 0$.

Proof The implication (ii) $\Longrightarrow$ (i) is rather obvious since, if $b(\lambda) = 0$, then we have $k_\lambda = k_\lambda^b \in \mathcal{H}(b)$. For the converse, first note that the function k_λ can be analytically continued across all of $\mathbb{T}$. Therefore, by Theorem 25.7, k_λ belongs to the kernel of $T_{\bar{b}}$. But, using (12.7), we have $T_{\bar{b}} k_\lambda = \overline{b(\lambda)} k_\lambda$ and thus we must have $b(\lambda) = 0$. □

If b is an extreme point in the closed unit ball of H^∞ and if f belongs to $\mathcal{H}(b)$ and can be analytically continued across all of $\mathbb{T}$, then $\|f\|_b = \|f\|_2$. In fact, by Theorem 25.7, we must have $T_{\bar{b}} f = 0$. Then, using Theorem 17.8, we get

$$\|f\|_b^2 = \|f\|_2^2 + \|T_{\bar{b}} f\|_b^2 = \|f\|_2^2. \tag{25.7}$$

This fact is used to detect the monomials that are in $\mathcal{H}(b)$.

Corollary 25.9 *Let b be an extreme point of the closed unit ball of H^∞. Let m be a nonnegative integer. Then the following assertions are equivalent.*

(i) *The monomial z^m belongs to $\mathcal{H}(b)$.*
(ii) *b has a zero of order at least $m+1$ at the origin.*

Moreover, if $z^m \in \mathcal{H}(b)$, then $\|z^m\|_b = 1$.

Proof According to Theorem 25.7, the function z^m belongs to $\mathcal{H}(b)$ if and only if it belongs to the kernel of $T_{\bar{b}}$. But, by (12.4),

$$T_{\bar{b}}\, z^m = \sum_{k=0}^{m} \overline{\hat{b}(k)} z^{m-k}.$$

Thus, z^m belongs to $\mathcal{H}(b)$ if and only if $\hat{b}(k) = 0$, $0 \le k \le m$, which means that b has a zero of order at least $m+1$ at the origin. The statement concerning the norm of z^m follows directly from (25.7). □

In fact, the above result shows that, if b has a zero of order $m+1$ at the origin, then the set of polynomials at $\mathcal{H}(b)$ is a linear manifold of dimension $m+1$ spanned by $1, z, \dots, z^m$. This fact is mentioned in more detail in the following corollary and it shows that we can add one extra item to the list given in Corollary 25.5.

Corollary 25.10 *Let b be in the closed unit ball of H^∞. Then the set of analytic polynomials $\mathcal{P}$ is contained in $\mathcal{H}(b)$ if and only if b is a nonextreme point of the closed unit ball of H^∞.*

Proof In Theorem 23.13, we have already proved that, if b is a nonextreme point of the closed unit ball of H^∞, then the set of analytic polynomials $\mathcal{P}$ is contained in $\mathcal{H}(b)$ (in fact, it is even dense in $\mathcal{H}(b)$). For the converse, we argue by absurdity. Assume that the set $\mathcal{P}$ is contained in $\mathcal{H}(b)$, but b is an extreme point of the closed unit ball of H^∞. Then, by Corollary 25.9, b has a zero of order infinity at the origin, which is absurd (remember that b is not a constant). □

Exercise

Exercise 25.3.1 Let b be an extreme point of the closed unit ball of H^∞ and assume further that b is an outer function. Show that, if a function f in $\mathcal{H}(b)$ can be analytically continued across all of $\mathbb{T}$, then $f \equiv 0$.
Hint: Use Theorems 12.19 and 25.7.

25.4 A formula for $\|X_b f\|_b$

In this section, we give a formula for $\|X_b f\|_b$, and then easily generalize it for $\|X_b^n f\|_b$. The result of this section should be compared with those in Section 23.5, which were about the nonextreme case.

Theorem 25.11 *Let b be an extreme point of the closed unit ball of H^∞. Then we have*

$$X_b^* X_b = I - (k_0^b \otimes k_0^b),$$

which implies that

$$\|X_b f\|_b^2 = \|f\|_b^2 - |f(0)|^2$$

for every function $f \in \mathcal{H}(b)$.

Proof According to (18.15), with f replaced by $X_b f = S^* f$, we have

$$\begin{aligned} X_b^*(X_b f) &= S(S^* f) - \langle X_b f, S^* b\rangle_b b \\ &= f - f(0) - \langle X_b f, X_b b\rangle_b b \\ &= f - f(0) - \langle f, X_b^* X_b b\rangle_b b. \end{aligned}$$

Hence, we look for a formula for $X_b^* X_b b$. Once more, by (18.15) with $f = S^* b = X_b b$, we obtain

$$X_b^* X_b b = X_b^*(S^* b) = SS^* b - \|S^* b\|_b^2 b = b - b(0) - \|S^* b\|_b^2\, b.$$

But we know from Corollary 25.3 that $\|S^* b\|_b^2 = 1 - |b(0)|^2$, and thus

$$X_b^* X_b b = b - b(0) - (1 - |b(0)|^2) b = -b(0)(1 - \overline{b(0)} b) = -b(0) k_0^b.$$

Back to the first relation above, we can now write

$$\begin{aligned} X_b^* X_b f &= f - f(0) - \langle f, X_b^* X_b b\rangle_b b \\ &= f - f(0) + \overline{b(0)} \langle f, k_0^b\rangle_b b \\ &= f - \langle f, k_0^b\rangle_b + \overline{b(0)} \langle f, k_0^b\rangle_b b \end{aligned}$$

$$= f - \langle f, k_0^b \rangle_b (1 - \overline{b(0)} b)$$
$$= f - \langle f, k_0^b \rangle_b k_0^b.$$

The preceding identity is rewritten as

$$X_b^* X_b = I - (k_0^b \otimes k_0^b).$$

Moreover, from this identity, we get

$$\begin{aligned}
\|X_b f\|_b^2 &= \langle X_b f, X_b f \rangle_b \\
&= \langle f, X_b^* X_b f \rangle_b \\
&= \langle f,\ f - f(0) k_0^b \rangle_b \\
&= \|f\|_b^2 - \overline{f(0)} \langle f, k_0^b \rangle_b \\
&= \|f\|_b^2 - |f(0)|^2.
\end{aligned}$$

This completes the proof. □

The following result should be compared with Corollary 23.16, i.e. the analogous result in the nonextreme case.

Corollary 25.12 *Let b be an extreme point of the closed unit ball of H^∞. The operator $D_{X_b} = (I - X_b^* X_b)^{1/2}$ has rank one, its range is spanned by k_0^b and its nonzero eigenvalue equals $\|k_0^b\|_b$.*

Proof This follows immediately from Theorem 25.11. □

It is straightforward to generalize the preceding formula for $\|X_b f\|_b^2$ to $\|X_b^n f\|_b^2$.

Corollary 25.13 *Let b be an extreme point of the closed unit ball of H^∞. Then we have*

$$\|X_b^n f\|_b^2 = \|f\|_b^2 - \sum_{k=0}^{n-1} |\hat{f}(k)|^2$$

for every function $f \in \mathcal{H}(b)$ and every integer $n \geq 1$.

Proof The proof is by induction on the integer n. For $n = 1$ the equality is precisely the one proved in Theorem 25.11. Just note that $f(0) = \hat{f}(0)$. Assume that the equality holds for some n. Then, using once again Theorem 25.11 and the induction hypothesis, we have

$$\begin{aligned}
\|X_b^{n+1} f\|_b^2 &= \|X_b (X_b^n f)\|_b^2 \\
&= \|X_b^n f\|_b^2 - |(X_b^n f)(0)|^2 \\
&= \|f\|_b^2 - \sum_{k=0}^{n-1} |\hat{f}(k)|^2 - |(X_b^n f)(0)|^2.
\end{aligned}$$

But

$$(X_b^n f)(0) = \langle X_b^n f, \chi_0\rangle_2 = \langle S^{*n} f, \chi_0\rangle_2 = \langle f, S^n \chi_0\rangle_2 = \langle f, \chi_n\rangle_2 = \hat{f}(n).$$

The proof is complete. □

Corollary 25.14 *Let b be an extreme point of the closed unit ball of H^∞. Then, for each function $f \in \mathcal{H}(b)$, we have*

$$\lim_{n\to\infty} \|X_b^n f\|_b^2 = \|f\|_b^2 - \|f\|_2^2.$$

Proof For each function $f \in H^2$, we have

$$\|f\|_2^2 = \sum_{k=0}^{\infty} |\hat{f}(k)|^2 = \lim_{n\to\infty} \sum_{k=0}^{n-1} |\hat{f}(k)|^2.$$

Now, let $n \longrightarrow \infty$ in the formula given in Corollary 25.13. □

Corollary 25.15 *Let b be in the closed unit ball of H^∞. Then, for every function $f \in \mathcal{H}(b)$, we have*

$$\|X_b f\|_b^2 \le \|f\|_b^2 - |f(0)|^2. \tag{25.8}$$

Moreover, the last inequality is an equality for all $f \in \mathcal{H}(b)$ if and only if b is an extreme point of the closed unit ball of H^∞.

Proof The inequality (25.8) has already been proved in Theorem 18.28. We have seen in Theorem 25.11 that (25.8) becomes an equality when b is extreme. Assume now that b is nonextreme. Then, according to (23.17) we have

$$\|X_b b\|_b^2 = \|b\|_b^2 - |b(0)|^2 - |a(0)|^2\|b\|_b^2 < \|b\|_b^2 - |b(0)|^2.$$

In other words, when b is nonextreme, then the inequality (25.8) can be strict. □

25.5 S^*-cyclic vectors in $\mathcal{H}(b)$ and $\mathcal{H}(\bar{b})$

The result of Douglas, Shapiro and Shields (Theorem 8.42) completely characterizes the cyclic vectors of S^* as an operator on H^2. We saw that $\mathcal{H}(b)$ and $\mathcal{H}(\bar{b})$ are invariant under S^*. In fact, the restriction of S^* on $\mathcal{H}(b)$ was denoted by X_b. In this section, we characterize the S^*-cyclic vectors that are in $\mathcal{H}(b)$ and $\mathcal{H}(\bar{b})$. The first result says that, except of course for the zero function, all other elements of $\mathcal{H}(\bar{b})$ are cyclic vectors for S^*.

Theorem 25.16 *Let b be an extreme point of the closed unit ball of H^∞. Then each $f \in \mathcal{H}(\bar{b})$, $f \not\equiv 0$, is a cyclic vector for S^*. Hence, the only function in $\mathcal{H}(\bar{b})$ that has a bounded-type meromorphic pseudocontinuation across $\mathbb{T}$ to $\mathbb{D}_e$ is the zero function.*

Proof Fix $f \in \mathcal{H}(\bar{b})$, $f \not\equiv 0$. It follows from Corollary 25.2 that there is a unique function $g \in L^2(\rho)$ such that

$$f = P_+(g\rho) \quad \text{and} \quad \log|g\rho| \notin L^1(\mathbb{T}).$$

Denote by M the closed invariant subspace of S^* generated by f. In other words, let

$$M = \operatorname{Span}_{H^2}(S^{*n} f : n \geq 0).$$

We need to show that $M = H^2$. We recall that Z is the bilateral forward shift operator on $L^2(\mathbb{T})$, i.e.

$$(Zg)(z) = zg(z) \qquad (z \in \mathbb{T},\ g \in L^2(\mathbb{T})).$$

The closed subspace $M \oplus \overline{H_0^2}$ of $L^2(\mathbb{T})$ is invariant under Z^*. In fact, if $f_1 \in M$ and $f_2 \in \overline{H_0^2}$, then we have

$$\begin{aligned} Z^*(f_1 + f_2) &= \bar{z} f_1 + \bar{z} f_2 \\ &= P_+(\bar{z} f_1) + P_-(\bar{z} h_1) + \bar{z} h_2 \\ &= S^* f_1 + P_-(\bar{z} f_1) + \bar{z} f_2, \end{aligned}$$

and $S^* f_1 \in M$ (remember, M is invariant under S^*) and $P_-(\bar{z} f_1) + \bar{z} f_2 \in \overline{H_0^2}$. Therefore, by Theorems 8.29 and 8.30, either $M \oplus \overline{H_0^2} = \Theta \overline{H^2}$ with Θ a unimodular function in $L^\infty(\mathbb{T})$, or $M \oplus \overline{H_0^2} = \chi_E L^2(\mathbb{T})$ with E a Borel subset of $\mathbb{T}$.

Let us show that the first case cannot occur. To do so, suppose that there is a unimodular function Θ such that $M \oplus \overline{H_0^2} = \Theta \overline{H^2}$. We find a contradiction. Since $f \in M$, the function $g\rho = f + P_-(g\rho)$ belongs to $M \oplus \overline{H_0^2}$, and thus there is an $h \in H^2$ such that $g\rho = \Theta \bar{h}$. Hence, $\log|g\rho| = \log|\Theta \bar{h}| = \log|h|$, which shows that $\log|h| \notin L^1(\mathbb{T})$. But then Lemma 4.30 implies that $h \equiv 0$, which in turn yields $g \equiv 0$ and $f \equiv 0$. This is absurd.

Therefore, for a proper Borel set E, we have $M \oplus \overline{H_0^2} = \chi_E L^2(\mathbb{T})$. Since $f \in M \subset \chi_E L^2(\mathbb{T})$, we deduce that $f \equiv 0$ almost everywhere on $\mathbb{T} \setminus E$. Then Lemma 4.30 implies that $m(\mathbb{T} \setminus E) = 0$. Hence, $\chi_E L^2(\mathbb{T}) = L^2(\mathbb{T})$, or equivalently $M \oplus \overline{H_0^2} = L^2(\mathbb{T})$. Finally, since $M \subset H^2$, we must have $M = H^2$. The second assertion follows immediately from Theorem 8.42. □

While the preceding result says that $\mathcal{H}(\bar{b})$ is filled with cyclic vectors and the only exception is the zero function, the space $\mathcal{H}(b)$ might have more noncyclic elements. More explicitly, the noncyclic vectors are precisely the elements of K_Θ, where Θ is the inner part of b.

Theorem 25.17 *Let b be an extreme point of the closed unit ball of H^∞, and let f be a function in H^2. Then the following are equivalent.*

(i) $f \in \mathcal{H}(b)$ *and* f *is not a cyclic vector for* S^*.

(ii) $f \in \mathcal{H}(b)$ *and* f *has a bounded-type meromorphic pseudocontinuation across* $\mathbb{T}$ *to* $\mathbb{D}_e$.

(iii) $T_{\bar{b}}f = 0$.

(iv) $f \in K_\Theta$, *where* Θ *is the inner part of* b.

Proof (i) $\Longleftrightarrow$ (ii) This follows immediately from Theorem 8.42.

(i) $\Longrightarrow$ (iii) Theorem 14.10 implies that $T_{\bar{b}}f$ is not a cyclic vector for S^*. But, by Theorem 17.8, $T_{\bar{b}}f \in \mathcal{H}(\bar{b})$. Hence, Theorem 25.16 implies that $T_{\bar{b}}f = 0$.

(iii) $\Longrightarrow$ (i) According to Theorem 17.8, we have $f \in \mathcal{H}(b)$. Since $T_{\bar{b}}f$ is obviously not a cyclic vector for S^*, it follows once more from Theorem 14.10 that f is not a cyclic vector for S^*.

(iii) $\Longleftrightarrow$ (iv) This follows from Theorem 12.19. □

The above result yields a statement similar to Theorem 25.16 for the S^*-cyclic elements of $\mathcal{H}(b)$, in the case where b is outer.

Corollary 25.18 *Let* b *be outer and an extreme point of the closed unit ball of* H^∞. *Then each* $f \in \mathcal{H}(b)$, $f \not\equiv 0$, *is a cyclic vector for* S^*. *Hence, the only function in* $\mathcal{H}(b)$ *that is pseudocontinuable across* $\mathbb{T}$ *is the zero function.*

If we combine Theorem 25.17 and Corollary 18.15, then we get the following necessary condition for cyclic vectors for X_b.

Corollary 25.19 *Let* b *be an extreme point of the closed unit ball of* H^∞, *let* $b = \Theta b_o$ *be the factorization of* b *into its inner part* Θ *and its outer part* b_o *and assume that* Θ *and* b_o *are not constant. Let* $f \in \mathcal{H}(b)$. *If* f *is a cyclic vector for* X_b, *then* $f \notin K_\Theta$.

25.6 Orthogonal decompositions of $\mathcal{H}(b)$

Remember that, if $b_1 = \Theta$ is an inner function and b_2 is a function in H^∞ and $b = b_1 b_2$, then, according to Corollary 18.9, the space $\mathcal{H}(b)$ can be decomposed as

$$\mathcal{H}(b) = \mathcal{H}(\Theta) \oplus \Theta\mathcal{H}(b_2),$$

and the sum is orthogonal. When b is extreme, we can give another orthogonal decomposition for $\mathcal{H}(b)$. In a sense, we can say that the roles of Θ and b_2 can be exchanged.

Theorem 25.20 *Let* $b = b_1 b_2$, $b_\ell \in H^\infty$, $\|b_\ell\|_\infty \le 1$ *and* b_ℓ *is nonconstant. Then the following assertions hold.*

(i) *The space* $\mathcal{H}(b)$ *decomposes as*

$$\mathcal{H}(b) = \mathcal{H}(b_1) + b_1\mathcal{H}(b_2). \tag{25.9}$$

(ii) *If* b_1 *is extreme and* b_2 *is inner, then the sum in* (25.9) *is orthogonal, the inclusion map of* $\mathcal{H}(b_1)$ *into* $\mathcal{H}(b)$ *is an isometry and the operator* T_{b_1} *acts as an isometry from* $\mathcal{H}(b_2)$ *into* $\mathcal{H}(b)$.

(iii) *If the sum in* (25.9) *is orthogonal, then necessarily* b_1 *is extreme.*

Proof Part (i) has already been proved in Theorem 18.7.

Let us now prove (ii). According to Theorem 18.8, it is sufficient to check that

$$\mathcal{H}(b_2) \cap \mathcal{H}(\bar{b}_1) = \{0\}.$$

Let $f \in \mathcal{H}(b_2) \cap \mathcal{H}(\bar{b}_1)$. On the one hand, since b_2 is inner, $\mathcal{H}(b_2)$ is a closed S^*-invariant subspace of H^2, and thus f cannot be a cyclic vector for S^*. On the other hand, since $|b| = |b_1|$ (once again because b_2 is inner), then b_2 is also an extreme point of the closed unit ball of H^∞. It now follows from Theorem 25.16 that $f \equiv 0$ and we conclude that $\mathcal{H}(b_2) \cap \mathcal{H}(\bar{b}_1) = \{0\}$. It remains to show that, if b_1 is nonextreme, then the sum in (25.9) is not orthogonal. Since b_1 is nonextreme, then we have

$$\mathcal{P} \subset \mathcal{H}(b_1) \subset \mathcal{H}(b).$$

Since b is nonextreme (note that $\log(1 - |b_1|) \leq \log(1 - |b|)$), Theorem 23.13 implies that $\mathcal{P}$ is dense in $\mathcal{H}(b)$. Therefore, we get that $\mathcal{H}(b_1)$ is also dense in $\mathcal{H}(b)$. In particular, its orthogonal complement is reduced to $\{0\}$ and thus the decomposition (25.9) cannot be orthogonal. □

25.7 The closure of $\mathcal{H}(\bar{b})$ in $\mathcal{H}(b)$

In Theorem 17.9, we saw that the space $\mathcal{H}(\bar{b})$, for any b, is contractively contained in $\mathcal{H}(b)$. Then, in Corollary 23.10, we showed that $\mathcal{H}(\bar{b})$ is a dense submanifold of $\mathcal{H}(b)$ whenever b is a nonextreme point of the closed unit ball of H^∞. The situation in the extreme case is different and is discussed below.

Theorem 25.21 *Let* b *be an extreme point in the closed unit ball of* H^∞ *and let* $b = \Theta[b]$ *be its canonical factorization, with* Θ *the inner part and* $[b]$ *the outer part of* b. *Then the closure of* $\mathcal{H}(\bar{b})$ *in* $\mathcal{H}(b)$ *is* $\mathcal{H}([b])$. *In particular,* $\mathcal{H}(\bar{b})$ *is dense in* $\mathcal{H}(b)$ *if and only if* b *is an outer function.*

Proof First note that, by Lemma Theorem 17.11, we have $\mathcal{H}(\bar{b}) = \mathcal{H}(\overline{[b]})$, and Theorem 17.9 implies that

$$\mathcal{H}(\bar{b}) = \mathcal{H}(\overline{[b]}) \subset \mathcal{H}([b]).$$

But, as a consequence of the orthogonal decomposition given in Theorem 25.20, we know that $\mathcal{H}([b])$ is a closed subspace of $\mathcal{H}(b)$, whence we conclude that the closure of $\mathcal{H}(\bar{b})$ in $\mathcal{H}(b)$ is contained in $\mathcal{H}([b])$. Using once more Theorem 25.20, it only remains to show that every function in $\mathcal{H}(b)$ that is orthogonal to $\mathcal{H}(\bar{b})$ belongs to $[b]\mathcal{H}(\Theta)$.

To continue the argument, let f be a function in $\mathcal{H}(b)$ and assume that f is orthogonal to $\mathcal{H}(\bar{b})$. Since $f \in \mathcal{H}(b)$, by Corollary 20.2, there exists a function $g \in L^2(\rho)$ such that

$$T_{\bar{b}} f = K_\rho(g) = P_+(\rho g).$$

Now, take any function $h \in H^2(\rho)$ and let $k = K_\rho(h)$. Then $k \in \mathcal{H}(\bar{b})$ and it follows from Theorem 13.21 that

$$T_{\bar{b}} k = T_{\bar{b}} K_\rho h = K_\rho(\bar{b}h).$$

Using the fact that f is orthogonal to $\mathcal{H}(\bar{b})$ and applying Theorems 17.8 and 25.1, we obtain

$$\begin{aligned} 0 &= \langle f, k \rangle_b \\ &= \langle f, k \rangle_2 + \langle T_{\bar{b}} f, T_{\bar{b}} k \rangle_{\bar{b}} \\ &= \langle f, P_+(\rho h) \rangle_2 + \langle K_\rho g, K_\rho(\bar{b}h) \rangle_{\bar{b}} \\ &= \langle f, \rho h \rangle_2 + \langle g, \bar{b}h \rangle_{L^2(\rho)} \\ &= \langle f, h \rangle_{L^2(\rho)} + \langle gb, h \rangle_{L^2(\rho)} \\ &= \langle f + gb, h \rangle_{L^2(\rho)}. \end{aligned}$$

But, since this relation holds for all functions $h \in H^2(\rho)$, and since $H^2(\rho) = L^2(\rho)$ (Theorem 25.1), we deduce that $f + gb = 0$ in $L^2(\rho)$. Therefore, we have

$$f(1 - |b|^2) + bg(1 - |b|^2) = 0$$

almost everywhere on $\mathbb{T}$, which implies that

$$\frac{f}{b} = \bar{b}f - (1 - |b|^2)g = \bar{b}f - \rho g,$$

because $b \neq 0$ almost everywhere on $\mathbb{T}$. The last equality implies that the function f/b belongs to $L^2(\mathbb{T})$ and, by the definition of g, we have

$$P_+\left(\frac{f}{b}\right) = P_+(\bar{b}f) - P_+(\rho g) = T_{\bar{b}} f - K_\rho g = 0. \tag{25.10}$$

Since

$$\frac{f}{[b]} = \frac{f}{b}\,\Theta,$$

the function $f/[b]$ also belongs to $L^2(\mathbb{T})$, and since $[b]$ is outer, Corollary 4.28 implies that $f/[b]$ is in H^2. Then, using (25.10), we get

$$T_{\bar{\Theta}}\left(\frac{f}{[b]}\right) = P_+\left(\bar{\Theta}\frac{f}{[b]}\right) = P_+\left(\frac{f}{b}\right) = 0,$$

which means that $f/[b]$ belongs to the kernel of $T_{\bar{\Theta}}$. It remains to apply Theorem 12.19 to deduce that $f/[b] \in K_\Theta$, which means that $f \in [b]K_\Theta$. That concludes the proof of the first assertion.

For the second assertion, note that $\mathcal{H}(\bar{b})$ is dense in $\mathcal{H}(b)$ if and only if $\mathcal{H}([b]) = \mathcal{H}(b)$, which is equivalent by Theorem 25.20 to $K_\Theta = \{0\}$. This last identity precisely means that Θ is a constant of modulus one, that is, b is outer. □

25.8 A characterization of $\mathcal{H}(b)$

In this section, we study an analog of Theorem 17.24, which characterizes the spaces $\mathcal{H}(b)$ when b is extreme.

Theorem 25.22 *Let $\mathcal{H}$ be a Hilbert space contained contractively in H^2. Then the following assertions are equivalent.*

(i) *$\mathcal{H}$ is S^*-invariant (and T denotes the restriction of S^* to $\mathcal{H}$), the operator $I - TT^*$ is an operator of rank one and we have*

$$\|Tf\|^2_{\mathcal{H}} = \|f\|^2_{\mathcal{H}} - |f(0)|^2 \qquad (f \in \mathcal{H}). \tag{25.11}$$

(ii) *There is an extreme point b in the closed unit ball of H^∞, unique up to a unimodular constant, such that $\mathcal{H} = \mathcal{H}(b)$.*

Proof (i) $\Longrightarrow$ (ii) According to Theorem 16.29, we know that $\mathcal{H}$ is contained contractively in H^2 and, if $\mathcal{M}$ denotes its complementary space, then S acts as a contraction on $\mathcal{M}$. Now the strategy of the proof is quite simple and quite similar to the strategy of the proof of Theorem 23.22. We show that S acts as an isometry on $\mathcal{M}$. Then we apply Theorem 17.24 to deduce that there exists a function b in the closed unit ball of H^∞ such that $\mathcal{M} = \mathcal{M}(b)$, and Corollary 16.27 enables us to conclude that $\mathcal{H} = \mathcal{H}(b)$. To show that S acts as an isometry, we decompose the proof into several steps, 10 in total.

Step 1: Let $k_0^{\mathcal{H}}$ be the unique vector in $\mathcal{H}$ such that

$$f(0) = \langle f, k_0^{\mathcal{H}}\rangle_{\mathcal{H}} \qquad (f \in \mathcal{H}).$$

Then $I - T^*T = k_0^{\mathcal{H}} \otimes k_0^{\mathcal{H}}$.

Let $f \in \mathcal{H}$. Then, according to Lemma 2.16 and (25.11), we have

$$\begin{aligned} f \in \ker(I - T^*T) &\iff \|Tf\|_{\mathcal{H}} = \|f\|_{\mathcal{H}} \\ &\iff f(0) = 0 \\ &\iff f \perp k_0^{\mathcal{H}}, \end{aligned}$$

whence $\ker(I - T^*T) = (\mathbb{C}k_0^{\mathcal{H}})^{\perp}$. Thus, we get $\mathcal{R}(I - T^*T) = \mathbb{C}k_0^{\mathcal{H}}$ and $I - T^*T$ is a rank-one operator whose range is spanned by $k_0^{\mathcal{H}}$. Since this operator is positive and self-adjoint, we get

$$I - T^*T = ck_0^{\mathcal{H}} \otimes k_0^{\mathcal{H}},$$

for some positive real constant c. It remains to show that $c = 1$. On the one hand, we have

$$\begin{aligned} \|I - T^*T\| &= \sup_{f \in \mathcal{H},\, \|f\|_{\mathcal{H}} \le 1} |\langle (I - T^*T)f,\ f\rangle_{\mathcal{H}}| \\ &= \sup_{f \in \mathcal{H},\, \|f\|_{\mathcal{H}} \le 1} (\|f\|_{\mathcal{H}}^2 - \|Tf\|_{\mathcal{H}}^2) \\ &= \sup_{f \in \mathcal{H},\, \|f\|_{\mathcal{H}} \le 1} |\langle f, k_0^{\mathcal{H}}\rangle_{\mathcal{H}}|^2 \\ &= \|k_0^{\mathcal{H}}\|_{\mathcal{H}}^2, \end{aligned}$$

and, on the other, we have $\|I - T^*T\| = c\|k_0^{\mathcal{H}}\|^2$, whence $c = 1$, which ends the proof of Step 1.

Step 2: Let f_0 be the unique vector in $\mathcal{H}$ such that $I - TT^* = f_0 \otimes f_0$. Then

$$T^*f_0 = \frac{\langle f_0, Tk_0^{\mathcal{H}}\rangle_{\mathcal{H}}}{\|f_0\|_{\mathcal{H}}^2} k_0^{\mathcal{H}}.$$

Using $T^*(I - TT^*) = (I - T^*T)T^*$, we have

$$T^*f_0 \otimes f_0 = (k_0^{\mathcal{H}} \otimes k_0^{\mathcal{H}})T^* = k_0^{\mathcal{H}} \otimes Tk_0^{\mathcal{H}}.$$

Thus, for every $f \in \mathcal{H}$, we have

$$\langle f, f_0\rangle_{\mathcal{H}} T^*f_0 = \langle f, Tk_0^{\mathcal{H}}\rangle_{\mathcal{H}} k_0^{\mathcal{H}}. \tag{25.12}$$

In particular, this equality with $f = f_0$ gives

$$\|f_0\|_{\mathcal{H}}^2 T^*f_0 = \langle f_0, Tk_0^{\mathcal{H}}\rangle_{\mathcal{H}} k_0^{\mathcal{H}},$$

which concludes the proof of Step 2.

Step 3: If $1 \notin \mathcal{H}$, then there exist nonzero constants $c_1, c_2 \in \mathbb{C}$ such that $Sf_0 = c_1 k_0^{\mathcal{H}} + c_2$.

Put $\alpha = \langle f_0, Tk_0^{\mathcal{H}}\rangle_{\mathcal{H}} k_0^{\mathcal{H}}/\|f_0\|_{\mathcal{H}}^2$. Then, according to Step 2, we have $T^*f_0 = \alpha k_0^{\mathcal{H}}$. Note that $\alpha \neq 0$. Indeed, applying (25.12) to $f = Tk_0^{\mathcal{H}}$ gives

$$\langle Tk_0^{\mathcal{H}}, f_0\rangle_{\mathcal{H}} T^*f_0 = \|Tk_0^{\mathcal{H}}\|_{\mathcal{H}}^2 k_0^{\mathcal{H}},$$

whence

$$|\langle Tk_0^{\mathcal{H}}, f_0\rangle_{\mathcal{H}}|^2 = \langle Tk_0^{\mathcal{H}}, f_0\rangle_{\mathcal{H}} \langle T^*f_0, k_0^{\mathcal{H}}\rangle_{\mathcal{H}} = \|Tk_0^{\mathcal{H}}\|_{\mathcal{H}}^2.$$

But, since $1 \notin \mathcal{H}$, we have $Tk_0^{\mathcal{H}} \neq 0$, which implies that $\langle Tk_0^{\mathcal{H}}, f_0\rangle_{\mathcal{H}} \neq 0$. Thus, $\alpha \neq 0$.

Now, using $(I - TT^*)f_0 = \|f_0\|_{\mathcal{H}}^2 f_0$, we get

$$(1 - \|f_0\|_{\mathcal{H}}^2) f_0 = TT^*f_0 = \alpha Tk_0^{\mathcal{H}}.$$

Since $\alpha Tk_0^{\mathcal{H}} \neq 0$, we necessarily have $\|f_0\|_{\mathcal{H}} \neq 1$. Hence,

$$f_0 = \frac{\alpha}{1 - \|f_0\|_{\mathcal{H}}^2} Tk_0^{\mathcal{H}} = c_1 Tk_0^{\mathcal{H}},$$

where $c_1 = \alpha/(1 - \|f_0\|_{\mathcal{H}}^2) \neq 0$. Thus,

$$\begin{aligned} Sf_0 &= S(c_1 Tk_0^{\mathcal{H}}) \\ &= SS^*(c_1 Tk_0^{\mathcal{H}}) \\ &= c_1 k_0^{\mathcal{H}} - c_1 k_0^{\mathcal{H}}(0) \\ &= c_1 k_0^{\mathcal{H}} + c_2, \end{aligned}$$

where $c_1 \neq 0$ and $c_2 = -c_1 k_0^{\mathcal{H}}(0) = -c_1 \|k_0^{\mathcal{H}}\|_{\mathcal{H}}^2 \neq 0$.

Step 4: S acts as an isometry on $\mathcal{M}$ (case $1 \notin \mathcal{H}$).

Let $f \in \mathcal{H}$ and $g \in \mathcal{M}$. Write

$$f = (I - TT^*)f + TT^*f = \langle f, f_0\rangle_{\mathcal{H}} f_0 + TT^*f = \lambda f_0 + TT^*f,$$

where $\lambda = \langle f, f_0\rangle_{\mathcal{H}}$. Then

$$\begin{aligned} \|g + f\|_2^2 &= \|g + \lambda f_0 + TT^*f\|_2^2 \\ &= \|g\|_2^2 + \|\lambda f_0 + TT^*f\|_2^2 + 2\,\Re\langle g, \lambda f_0 + TT^*f\rangle_2. \end{aligned}$$

But

$$\begin{aligned} \langle g, \lambda f_0 + TT^*f\rangle_2 &= \langle g, S^*T^*f\rangle_2 + \langle g, \lambda f_0\rangle_2 \\ &= \langle Sg, T^*f\rangle_2 + \langle g, \lambda f_0\rangle_2 \end{aligned}$$

and, using Step 3, we also have

$$\begin{aligned} \langle g, \lambda f_0\rangle_2 &= \langle zg, \lambda z f_0\rangle_2 = \langle zg, \lambda c_1 k_0^{\mathcal{H}} + \lambda c_2\rangle_2 \\ &= \langle zg, \lambda c_1 k_0^{\mathcal{H}}\rangle_2 = \langle zg, \lambda h_0\rangle_2, \end{aligned}$$

with $h_0 = c_1 k_0^{\mathcal{H}} \in \mathcal{H}$. Thus,

$$\langle g, \lambda f_0 + TT^* f\rangle_2 = \langle zg,\ T^* f + \lambda h_0\rangle_2$$

and we obtain

$$\|g + f\|_2^2 = \|zg + T^* f + \lambda h_0\|_2^2 + \|\lambda f_0 + TT^* f\|_2^2 - \|T^* f + \lambda f_0\|_2^2.$$

This gives

$$\begin{aligned}
&\|g + f\|_2^2 - \|f\|_{\mathcal{H}}^2 \\
&\quad = \|zg + T^* f + \lambda h_0\|_2^2 + \|\lambda f_0 + TT^* f\|_2^2 - \|T^* f + \lambda h_0\|_2^2 - \|f\|_{\mathcal{H}}^2.
\end{aligned}$$

Now, we prove that

$$\|\lambda f_0 + TT^* f\|_2^2 - \|T^* f + \lambda h_0\|_2^2 = \|f\|_{\mathcal{H}}^2 - \|T^* f + \lambda h_0\|_{\mathcal{H}}^2. \tag{25.13}$$

Using Step 3, we have

$$f_0 = S^* S f_0 = S^*(c_1 k_0^{\mathcal{H}} + c_2) = c_1 S^* k_0^{\mathcal{H}} = T h_0.$$

Thus, on the one hand, we have

$$\begin{aligned}
\|\lambda f_0 + TT^* f\|_2^2 - \|T^* f + \lambda h_0\|_2^2 &= \|T(\lambda h_0 + T^* f)\|_2^2 - \|\lambda f_0 + T^*\|_2^2 \\
&= |(\lambda h_0 + T^* f)(0)|^2,
\end{aligned}$$

and, on the other, we also have

$$\begin{aligned}
\|f\|_{\mathcal{H}}^2 - \|T^* f + \lambda h_0\|_{\mathcal{H}}^2 &= \|\lambda f_0 + TT^* f\|_{\mathcal{H}}^2 - \|T^* f + \lambda h_0\|_{\mathcal{H}}^2 \\
&= \|T(\lambda h_0 + T^* f)\|_{\mathcal{H}}^2 - \|\lambda h_0 + T^* f\|_{\mathcal{H}}^2 \\
&= |(\lambda h_0 + T^* f)(0)|^2.
\end{aligned}$$

The last equality follows from (25.11). This concludes the proof of (25.13). Hence,

$$\begin{aligned}
\|g + f\|_2^2 - \|f\|_{\mathcal{H}}^2 &= \|zg + T^* f + \lambda h_0\|_2^2 - \|T^* f + \lambda h_0\|_{\mathcal{H}}^2 \\
&\le \sup_{h \in \mathcal{H}} (\|zg + h\|_2^2 - \|h\|_{\mathcal{H}}^2) \\
&= \|zg\|_{\mathcal{M}}^2 \qquad (f \in \mathcal{H}).
\end{aligned}$$

This gives $\|g\|_{\mathcal{M}}^2 \le \|zg\|_{\mathcal{M}}^2$, which with Theorem 16.29 implies that S is an isometry on $\mathcal{M}$.

From now on, we assume that $1 \in \mathcal{H}$ and $n \ge 1$ is such that $f_0 = z^{n-1} \tilde{f}_0$, with $\tilde{f}_0 \in H^2$ and $\tilde{f}_0(0) \neq 0$.

Step 5: $\|1\|_{\mathcal{H}} = 1$ *and* $k_0^{\mathcal{H}} = 1$.

Since $1 \in \mathcal{H}$, using (25.11), we have

$$\|T1\|_{\mathcal{H}}^2 = \|1\|_{\mathcal{H}}^2 - 1.$$

Hence, $\|1\|_{\mathcal{H}} = 1$, because $T1 = 0$. But, by Corollary 16.28,

$$i^*_{\mathcal{H}}(1) = 1 \quad \text{and} \quad i^*_{\mathcal{M}}(1) = 0,$$

where $i_{\mathcal{H}}$ (respectively $i_{\mathcal{M}}$) denotes the canonical injection of $\mathcal{H}$ (respectively $\mathcal{M}$) into H^2. Thus, for each $f \in \mathcal{H}$, we get

$$\begin{aligned}
\langle f, 1\rangle_{\mathcal{H}} &= \langle i_{\mathcal{H}}(f), 1\rangle_{\mathcal{H}} \\
&= \langle f, i^*_{\mathcal{H}}(1)\rangle_2 \\
&= \langle f, 1\rangle_2 = f(0) \\
&= \langle f, k_0^{\mathcal{H}}\rangle_{\mathcal{H}}.
\end{aligned}$$

By the uniqueness of reproducing kernels, we deduce that $k_0^{\mathcal{H}} = 1$.

Step 6: We have

$$T^*h = S(h - \langle h, f_0\rangle_{\mathcal{H}} f_0) \qquad (h \in \mathcal{H}).$$

Since $I - f_0 \otimes f_0 = TT^*$, we get

$$\begin{aligned}
S(I - f_0 \otimes f_0)h = STT^*h &= SS^*T^*Th \\
&= T^*h - (T^*h)(0) \qquad (h \in \mathcal{H}).
\end{aligned}$$

But, according to Step 5,

$$(T^*h)(0) = \langle T^*h, 1\rangle_{\mathcal{H}} = \langle h, T1\rangle_{\mathcal{H}} = 0.$$

Thus,

$$S(I - f_0 \otimes f_0)h = T^*h.$$

Step 7: The function $\tilde{f}_0$ belongs to $\mathcal{H}$ and

$$T^{*n}1 = z^n(1 - \overline{\tilde{f}_0(0)}\tilde{f}_0). \tag{25.14}$$

Moreover, if $n \geq 2$, we have

$$T^{*k}1 = z^k \qquad (1 \leq k \leq n-1). \tag{25.15}$$

Since $\tilde{f}_0 = S^{*n-1}f_0 = T^{n-1}f_0$, we surely have $\tilde{f}_0 \in \mathcal{H}$. Now, using Step 6, we have

$$T^*1 = S(1 - \langle 1, f_0\rangle_{\mathcal{H}} f_0)$$

and

$$\langle 1, f_0\rangle_{\mathcal{H}} = \overline{f_0(0)},$$

because $1 = k_0^{\mathcal{H}}$. Hence,

$$T^*1 = z(1 - \overline{f_0(0)} f_0). \tag{25.16}$$

If $n = 1$, we have $f_0 = \tilde{f}_0$. This proves (25.14) in the case $n = 1$.

Now, assume that $n > 1$. We first prove (25.15) by induction. Since $n > 1$, we have $f_0(0) = 0$, and thus, by (25.16), we get $T^*1 = z$. Assume that $T^{*k}1 = z^k$, for some $k < n - 1$. Thus, using Step 6, we obtain

$$\begin{aligned} T^{*(k+1)}1 = T^* z^k &= z(z^k - \langle z^k, f_0\rangle_{\mathcal{H}} f_0) \\ &= z^{k+1} - \langle z^k, f_0\rangle_{\mathcal{H}} z f_0. \end{aligned}$$

But

$$\langle z^k, f_0\rangle_{\mathcal{H}} = \langle T^{*k}1, f_0\rangle_{\mathcal{H}} = \langle 1, T^k f_0\rangle_{\mathcal{H}} = \overline{(T^k f_0)(0)}$$

and

$$T^k f_0 = S^{*k} f_0 = P_+(\bar{z}^k z^{n-1} f_0) = P_+(z^{n-k-1} f_0) = z^{n-k-1} f_0.$$

Hence, $(T^k f_0)(0) = 0$, because $n - k - 1 > 0$, and then

$$T^{*(k+1)}1 = z^{k+1}.$$

Therefore,

$$T^{*k}1 = z^k \qquad (1 \le k \le n-1).$$

Using Step 6 once more, we have

$$\begin{aligned} T^{*n}1 = T^* z^{n-1} &= z^n - \langle z^{n-1}, f_0\rangle_{\mathcal{H}} z f_0 \\ &= z^n - \langle z^{n-1}, f_0\rangle_{\mathcal{H}} z^n \tilde{f}_0 \\ &= z^n (1 - \langle z^{n-1}, f_0\rangle_{\mathcal{H}} \tilde{f}_0). \end{aligned}$$

It just remains to note that

$$\langle z^{n-1}, f_0\rangle_{\mathcal{H}} = \langle T^{*n-1}1, f_0\rangle_{\mathcal{H}} = \langle 1, T^{n-1} f_0\rangle_{\mathcal{H}} = \langle 1, \tilde{f}_0\rangle_{\mathcal{H}} = \overline{\tilde{f}_0(0)}.$$

Step 8: $z^{n-1} \in \mathcal{H}$ *and* $\langle g, z^{n-1}\rangle_2 = 0$, *for every* $g \in \mathcal{M}$.

According to (25.15), we have $z^k \in \mathcal{H}$, for every $0 \le k \le n-1$. Moreover, $\|z^k\|_{\mathcal{H}} = 1$. Indeed, this is true for $k = 0$ by Step 5. Assume that, for some $0 \le k \le n-1$, $\|z^k\|_{\mathcal{H}} = 1$. Hence, considering (25.11), we get

$$\|z^{k+1}\|_{\mathcal{H}}^2 = \|T z^{k+1}\|_{\mathcal{H}}^2 = \|z^k\|_{\mathcal{H}}^2 = 1.$$

By induction, we then have $\|z^k\|_{\mathcal{H}} = 1$, for every $0 \le k \le n-1$. It follows from Corollary 16.28 that $i_{\mathcal{H}}^*(z^{n-1}) = z^{n-1}$ and $i_{\mathcal{M}}^*(z^{n-1}) = 0$. Hence, if $g \in \mathcal{M}$, we have

$$\langle g, z^{n-1}\rangle_2 = \langle i_{\mathcal{M}}(g), z^{n-1}\rangle_2 = \langle g, i_{\mathcal{M}}^*(z^{n-1})\rangle_{\mathcal{M}} = 0.$$

Step 9: S acts as an isometry on $\mathcal{M}$ (case $1 \in \mathcal{H}$).

Let $f \in \mathcal{H}$ and $g \in \mathcal{M}$. Argue as in Step 4, and write $f = \lambda f_0 + TT^*f$, with $\lambda = \langle f, f_0\rangle_{\mathcal{H}}$. Then

$$\begin{aligned}\|g+f\|_2^2 &= \|g + \lambda f_0 + TT^*f\|_2^2 \\ &= \|g\|_2^2 + \|\lambda f_0 + TT^*f\|_2^2 + 2\,\Re\langle g, \lambda f_0 + TT^*f\rangle_2.\end{aligned}$$

But

$$\langle g, \lambda f_0 + TT^*f\rangle_2 = \langle g, \lambda f_0\rangle_2 + \langle Sg, T^*f\rangle_2.$$

Write $h = -\overline{\tilde{f}_0(0)}^{(-1)} T^{*n}1$. According to Step 6, we have

$$h = -\overline{\tilde{f}_0(0)}^{(-1)} z^n + z^n \tilde{f}_0 = -\overline{\tilde{f}_0(0)}^{(-1)} z^n + z f_0.$$

Hence,

$$\langle zg, \lambda h\rangle_2 = -\tilde{f}_0(0)^{-1}\bar{\lambda}\langle zg, z^n\rangle_2 + \langle zg, \lambda z f_0\rangle_2 = \langle g, \lambda f_0\rangle_2,$$

because $\langle zg, z^n\rangle_2 = \langle g, z^{n-1}\rangle_2 = 0$, according to Step 8. Therefore,

$$\langle g, \lambda f_0 + TT^*f\rangle_2 = \langle zg, \lambda h\rangle_2 + \langle zg, T^*f\rangle_2 = \langle zg, \lambda h + T^*f\rangle_2.$$

We thus get

$$\|g+f\|_2^2 = \|zg + \lambda h + T^*f\|_2^2 + \|\lambda f_0 + TT^*f\|_2^2 - \|\lambda h + T^*f\|_2^2,$$

and then

$$\begin{aligned}&\|g+f\|_2^2 - \|f\|_{\mathcal{H}}^2 \\ &\quad = \|zg + \lambda h + T^*f\|_2^2 + \|\lambda f_0 + TT^*f\|_2^2 - \|\lambda h + T^*f\|_2^2 - \|f\|_{\mathcal{H}}^2.\end{aligned}$$

Now, we prove that

$$\|\lambda f_0 + TT^*f\|_2^2 - \|T^*f + \lambda h\|_2^2 = \|\lambda f_0 + TT^*f\|_{\mathcal{H}}^2 - \|T^*f + \lambda h\|_{\mathcal{H}}^2.$$

Denote the right-hand side by A and the left-hand side by B. Then, using Step 6, we obtain

$$\begin{aligned}A &= \|f\|_2^2 - \|z(f - \lambda f_0 - \lambda\overline{\tilde{f}_0(0)}^{(-1)} z^n + \lambda f_0)\|_2 \\ &= \|f\|_2^2 - \|f - \lambda\overline{\tilde{f}_0(0)}^{(-1)} z^n)\|_2 \\ &= \|f\|_2^2 - (\|f\|_2^2 + |\lambda|^2|\tilde{f}_0(0)|^{-2} - 2\,\Re(\bar{\lambda}\tilde{f}_0(0)^{-1}\langle f, z^{n-1}\rangle_2)) \\ &= -|\lambda|^2|\tilde{f}_0(0)|^{-2} + 2\,\Re(\bar{\lambda}\tilde{f}_0(0)^{-1}\langle f, z^{n-1}\rangle_2).\end{aligned}$$

Moreover, using (25.11), we have

$$\begin{aligned} B &= \|f\|_{\mathcal{H}}^2 - \|T^*f + \lambda h\|_{\mathcal{H}}^2 \\ &= \|f\|_{\mathcal{H}}^2 - \|TT^*f + \lambda Th\|_{\mathcal{H}}^2 - |(T^*f)(0) + \lambda h(0)|. \end{aligned}$$

But $(T^*f)(0) = h(0) = 0$ and $Th = S^*h = -\overline{\tilde{f}_0(0)}^{(-1)} z^{n-1} + f_0$. Hence,

$$TT^*f + \lambda Th = TT^*f + \lambda f_0 - \lambda \overline{\tilde{f}_0(0)}^{(-1)} z^{n-1},$$

which gives

$$\begin{aligned} B &= \|f\|_{\mathcal{H}}^2 - \|f - \lambda \overline{\tilde{f}_0(0)}^{(-1)} z^{n-1}\|_{\mathcal{H}}^2 \\ &= \|f\|_{\mathcal{H}}^2 - [\,\|f\|_{\mathcal{H}}^2 + |\lambda|^2 |\tilde{f}_0(0)|^{-2} \|z^{n-1}\|_{\mathcal{H}}^2 \\ &\quad -2\,\Re(\bar{\lambda}\overline{\tilde{f}_0(0)}^{-1} \langle f, z^{n-1}\rangle_{\mathcal{H}})\,] \\ &= -|\lambda|^2 |\tilde{f}_0(0)|^{-2} + 2\,\Re(\bar{\lambda}\overline{\tilde{f}_0(0)}^{-1} \langle f, z^{n-1}\rangle_{\mathcal{H}}). \end{aligned}$$

Considering

$$\begin{aligned} \langle f, z^{n-1}\rangle_{\mathcal{H}} &= \langle i_{\mathcal{H}}(f), z^{n-1}\rangle_{\mathcal{H}} \\ &= \langle f, i_{\mathcal{H}}^*(z^{n-1})\rangle_2 \\ &= \langle f, z^{n-1}\rangle_2, \end{aligned}$$

we deduce that $A = B$. This reveals that

$$\begin{aligned} \|g + f\|_2^2 - \|f\|_{\mathcal{H}}^2 &= \|zg + \lambda h + T^*f\|_2 - \|\lambda h + T^*f\|_{\mathcal{H}}^2 \\ &\le \sup_{u \in \mathcal{H}} (\|zg + u\|_2^2 - \|u\|_{\mathcal{H}}^2) \\ &= \|zg\|_{\mathcal{M}}^2 \qquad (f \in \mathcal{H}). \end{aligned}$$

This gives $\|g\|_{\mathcal{M}}^2 \le \|zg\|_{\mathcal{M}}^2$, which, in the light of Theorem 16.29, ensures that S is an isometry on $\mathcal{M}$.

Step 10: There is an extreme point b *in the closed unit ball of* H^∞*, unique up to a unimodular constant, such that* $\mathcal{H} \doteq \mathcal{H}(b)$*.*

According to Steps 4 and 9, S acts as an isometry on $\mathcal{M}$. Therefore, Theorem 17.24 implies that there exists a function b in the closed unit ball of H^∞ such that $\mathcal{M} \doteq \mathcal{M}(b)$. Now, Corollary 16.27 implies that $\mathcal{H} \doteq \mathcal{H}(b)$. Finally, b is an extreme point of the closed unit ball of H^∞, since $I - T^*T$ is an operator of rank one. Remember that, according to Theorem 23.14, if b is a nonextreme point of the closed unit ball of H^∞, then the operator $I - T^*T$ is of rank two.

That finishes the proof of the implication (i) $\Longrightarrow$ (ii).

(ii) $\Longrightarrow$ (i) This follows from Corollary 18.23 and Theorem 25.11.

This completes the proof of Theorem 25.22. □

Notes on Chapter 25

Section 25.1

The formula for the norm $\|S^*b\|_b$ that appears in Corollary 25.3 is due to Sarason [160]. However, the proof presented here comes from [166].

Section 25.2

Theorem 25.4 is due to Lotto and Sarason [123, theorem 5.1]. The equivalence between (i) and (iv) in Corollary 25.5 is due to Sarason [160] and the equivalence of (i) and (ii) is due to de Branges and Rovnyak [65]. Corollary 25.6 is due to Lotto and Sarason [123, lemma 8.1].

Section 25.3

Theorem 25.7 comes from Sarason's book [166], but the immediate corollary that is presented in Exercise 25.3.1 is due to Lotto and Sarason [123, corollary 5.2]. Corollaries 25.9 and 25.10 appear in a paper of Sarason [160], but the proofs come from his book.

Section 25.4

The determination of the defect operator of the contraction X_b made in Theorem 25.11 and Corollary 25.12 follows Sarason [160]. In that paper, he identifies the characteristic function (in the language of Sz.-Nagy and Foiaş) of X_b; see also [139]. Corollaries 25.13 and 25.14 are due to Lotto and Sarason [123].

Corollary 25.15 is due to de Branges and Rovnyak [65, theorem 16]. More precisely, they proved that (25.8) is an equality if and only if $b \notin \mathcal{H}(b)$, and we know this condition is equivalent to b being an extreme point of the closed unit ball of H^∞. See also Nikolskii and Vasyunin [139, corollary 8.8] for a generalization of this result in the vector-valued situation.

As already mentioned, de Branges called (25.8) the inequality for difference quotients.

Section 25.5

Theorem 25.16 has been proved by Lotto and Sarason [123, theorem 5.3], who applied this to the problem of multipliers of $\mathcal{H}(b)$. See Section 26.2 for results in this direction. It should be noted that Suárez [181] described the invariant subspaces of X_b but, as already mentioned, the problem of determining the cyclic vector of X_b (in the extreme case) is an open problem. Corollary 25.18 is also due to Lotto and Sarason [123, theorem 5.4].

Section 25.6

Theorem 25.20 is a slight generalization of a result of Lotto and Sarason [123, theorem 6.1].

Section 25.7

Theorem 25.21 has been proved by Lotto and Sarason [123, theorem 6.2].

Section 25.8

The characterization of the $\mathcal{H}(b)$ spaces in the extreme case obtained in Theorem 25.22 is due to de Branges and Rovnyak [64, appdx, theorem 6] and [65, theorem 15]. The proof of de Branges and Rovnyak is based on the construction of an auxiliary Hilbert space of analytic functions. Our proof here is different and is inspired by the analogous result in the nonextreme case due to Guyker [96]; see Theorem 23.22.

26

Operators on $\mathcal{H}(b)$ spaces with b extreme

In this chapter, we pursue our studies of $\mathcal{H}(b)$ spaces when b is an extreme point. We focus on some important operators acting on $\mathcal{H}(b)$.

In Section 26.1, we describe the spectrum of X_b and X_b^*. In Section 26.2, we study the multipliers of $\mathcal{H}(b)$ spaces when b is an extreme point. The situation for multipliers in the extreme case differs a lot from the nonextreme case. We remember that any function holomorphic in a neighborhood of $\bar{\mathbb{D}}$ is a multiplier of $\mathcal{H}(b)$ when b is a nonextreme point. In the extreme case, this is precisely the opposite: if φ is a nonconstant multiplier of $\mathcal{H}(b)$, then φ cannot be continued analytically across all of $\mathbb{T}$. Even more, it cannot have a bounded-type meromorphic pseudocontinuation across $\mathbb{T}$ to $\mathbb{D}_e$. When φ is a multiplier of $\mathcal{H}(b)$, we will see that, apart from the reproducing kernels k_λ^b, the operator M_φ^* has a lot of other unexpected eigenvectors. In Section 26.3, we pursue the study began in Theorem 20.28, which concerns the relations between two elements of the closed unit disk of H^∞, on the one hand, and their Clark measures, on the other.

Further characterizations of angular derivatives for b are given in Section 26.4. It should be mentioned that, in this section, we do not assume that b is an extreme point of the closed unit ball of H^∞. In Section 26.5, we show that, if T is a Hilbert space contraction such that its defect indices are 1 and which is completely nonisometric, then T is unitarily equivalent to X_b, for some extreme point b of the closed unit ball of H^∞. In other words, the contraction X_b is a model for a large class of contractions. In the last section, we introduce the important conjugation Ω_b on $\mathcal{H}(b)$ and we use this conjugation to show that the family of difference quotients is always complete in the extreme case. We also show that the conjugation Ω_b intertwines the operator X_b and its adjoint.

26.1 Spectrum of X_b and X_b^*

In Section 20.8, we characterized the intersection of the spectrum of X_b with the unit circle $\mathbb{T}$. We recall that $\bar{\lambda} \notin \sigma(X_b) \cap \mathbb{T}$ if and only if there exists a neighborhood I of λ such that b has an analytic continuation across I and, moreover, $|b| = 1$ on I. The next result completes the picture in the extreme case by giving the part of the spectrum that resides inside the open unit disk $\mathbb{D}$.

Theorem 26.1 *Let* b *be an extreme point of the closed unit ball of H^∞. Then*

$$\sigma(X_b) \cap \mathbb{D} = \sigma_p(X_b) = \{\bar{\lambda} : \lambda \in \mathbb{D},\ b(\lambda) = 0\}. \tag{26.1}$$

Moreover, for $\lambda \in \mathbb{D}$ with $b(\lambda) = 0$, we have

$$\ker(X_b - \bar{\lambda} I) = \mathbb{C} k_\lambda.$$

Proof First, according to Lemma 18.26, we have

$$\sigma_p(X_b) \subset \sigma(X_b) \cap \mathbb{D}.$$

Second, recall that, by Theorem 20.12, we have

$$V_b S^*_{H^2(\mu)} V_b^{-1} = X_b + (1 - b(0))^{-1} S^* b \otimes k_0^b. \tag{26.2}$$

But, since b is an extreme point of the closed unit ball of H^∞, it follows from Corollary 13.34 and Theorem 20.5 that $H^2(\mu) = L^2(\mu)$, V_b is an isometry from $L^2(\mu)$ onto $\mathcal{H}(b)$, and $S_{L^2(\mu)}$ is a unitary operator on $L^2(\mu)$. Hence, we can rewrite (26.2) as

$$V_b Z_\mu^* V_b^{-1} = X_b + (1 - b(0))^{-1} S^* b \otimes k_0^b, \tag{26.3}$$

where $Z_\mu = S_{L^2(\mu)}$.

Assume now that $\bar{\lambda} \in \sigma(X_b) \cap \mathbb{D}$. Since Z_μ is a unitary operator, we have $\sigma(Z_\mu^*) \subset \mathbb{T}$ and thus $Z_\mu^* - \bar{\lambda} I$ is invertible. But the relation (26.3) implies that

$$X_b - \bar{\lambda} I = V_b (Z_\mu^* - \bar{\lambda} I) V_b^{-1} - (1 - b(0))^{-1} S^* b \otimes k_0^b,$$

and then Lemma 7.33 ensures that $X_b - \bar{\lambda} I$ is a Fredholm operator with $\operatorname{ind}(X_b - \bar{\lambda} I) = 0$. We proceed to show that $\ker(X_b - \bar{\lambda} I) \neq \{0\}$. Assume on the contrary that $X_b - \bar{\lambda} I$ is injective. Then since

$$0 = \operatorname{ind}(X_b - \bar{\lambda} I) = \dim\, \ker(X_b - \bar{\lambda} I) - \dim\, \ker(X_b - \bar{\lambda} I)^*,$$

the operator $(X_b - \bar{\lambda} I)^*$ should also be injective. Thus, the range of the operator $X_b - \bar{\lambda} I$ is dense in $\mathcal{H}(b)$. But, since it has a closed range (remember, $X_b - \bar{\lambda} I$ is Fredholm), we conclude that $X_b - \bar{\lambda} I$ is surjective. Therefore, the operator $X_b - \bar{\lambda} I$ is invertible and this contradicts the assumption $\bar{\lambda} \in \sigma(X_b)$. Thus, $\ker(X_b - \bar{\lambda} I) \neq \{0\}$, which means that $\bar{\lambda} \in \sigma_p(X_b)$, and we get

$$\sigma(X_b) \cap \mathbb{D} \subset \sigma_p(X_b).$$

Since $X_b = S^*|\mathcal{H}(b)$, according to Lemma 8.6, we have

$$\ker(X_b - \bar{\lambda}I) \subset \ker(S^* - \bar{\lambda}I) = \mathbb{C}k_\lambda.$$

Hence, we must have $k_\lambda \in \mathcal{H}(b)$ and $\ker(X_b - \bar{\lambda}I) = \mathbb{C}k_\lambda$. Moreover, by Corollary 25.8, we get $b(\lambda) = 0$ and thus

$$\sigma(X_b) \cap \mathbb{D} \subset \{\bar{\lambda} : \lambda \in \mathbb{D},\, b(\lambda) = 0\}.$$

For the converse, assume that $\lambda \in \mathbb{D}$ is such that $b(\lambda) = 0$. Then surely $k_\lambda = k_\lambda^b \in \mathcal{H}(b)$. Moreover, since $S^* k_\lambda = \bar{\lambda} k_\lambda$, then we have

$$(X_b - \bar{\lambda}I)k_\lambda = S^* k_\lambda - \bar{\lambda}k_\lambda = 0,$$

which means that $k_\lambda \in \ker(X_b - \bar{\lambda}I)$. Therefore, $\bar{\lambda} \in \sigma_p(X_b) \subset \sigma(X_b)$ and that concludes the proof. □

Theorem 26.1 ensures that $X_b - \bar{\lambda}I$ in invertible whenever $b(\lambda) \neq 0$. In the following, we give an explicit formula for $(X_b - \bar{\lambda}I)^{-1}$.

Corollary 26.2 *Let b be an extreme point of the closed unit ball of H^∞, and let $\lambda \in \mathbb{D}$. Assume that $b(\lambda) \neq 0$. Then*

$$(X_b - \bar{\lambda}I)^{-1} = X_b^*(I - \bar{\lambda}X_b^*)^{-1} - \bar{\mu}k_\lambda^b \otimes Q_\lambda b,$$

where $Q_\lambda = (I - \lambda S^*)^{-1}S^*$ and $\mu = 1/b(\lambda)$.

Proof By Corollary 18.23, we have $X_b X_b^* = I - S^*b \otimes S^*b$. Since $X_b k_\lambda^b = \bar{\lambda}k_\lambda^b - \overline{b(\lambda)}S^*b$ (see the proof of Theorem 18.22), we have $(X_b - \bar{\lambda}I)k_\lambda^b = -\overline{b(\lambda)}S^*b$ and thus

$$X_b X_b^* = I + \bar{\mu}(X_b - \bar{\lambda}I)k_\lambda^b \otimes S^*b.$$

Writing $X_b X_b^* = (X_b - \bar{\lambda}I)X_b^* + \bar{\lambda}X_b^*$, we get

$$(X_b - \bar{\lambda}I)(X_b^* - \bar{\mu}k_\lambda^b \otimes S^*b) = I - \bar{\lambda}X_b^*.$$

Since X_b^* is a contraction, its spectrum is contained in the closed unit disk, and thus $I - \bar{\lambda}X_b^*$ is invertible. Then we can write

$$(X_b - \bar{\lambda}I)(X_b^* - \bar{\mu}k_\lambda^b \otimes S^*b)(I - \bar{\lambda}X_b^*)^{-1} = I,$$

which means that

$$(X_b - \bar{\lambda}I)^{-1} = (X_b^* - \bar{\mu}k_\lambda^b \otimes S^*b)(I - \bar{\lambda}X_b^*)^{-1}.$$

We can simplify this identity a little by writing

$$\begin{aligned}
(X_b - \bar{\lambda}I)^{-1} &= X_b^*(I - \bar{\lambda}X_b^*)^{-1} - \bar{\mu}k_\lambda^b \otimes (I - \lambda X_b)^{-1}S^*b \\
&= X_b^*(I - \bar{\lambda}X_b^*)^{-1} - \bar{\mu}k_\lambda^b \otimes (I - \lambda S^*b)^{-1}S^*b \\
&= X_b^*(I - \bar{\lambda}X_b^*)^{-1} - \bar{\mu}k_\lambda^b \otimes Q_\lambda b.
\end{aligned}$$

Here we used the formula $Q_\lambda b = (I - \lambda S^*)^{-1}S^*b$ from (8.20). □

Corollary 26.3 *Let b be an extreme point of the closed unit ball of H^∞. Then*

$$\sigma(X_b^*) \cap \mathbb{D} = \sigma_p(X_b^*) = \{\lambda : \lambda \in \mathbb{D},\ b(\lambda) = 0\}. \tag{26.4}$$

Moreover, for $\lambda \in \mathbb{D}$ with $b(\lambda) = 0$, we have

$$\ker(X_b^* - \bar{\lambda} I) = \mathbb{C} Q_\lambda b,$$

where

$$(Q_\lambda b)(z) = \frac{b(z)}{z - \lambda}.$$

Proof According to Lemma 18.26, we have $\sigma_p(X_b^*) \subset \sigma(X_b^*) \cap \mathbb{D}$ and by Theorems 26.1 and 1.30, we get

$$\sigma(X_b^*) \cap \mathbb{D} = \{\lambda : \lambda \in \mathbb{D},\ b(\lambda) = 0\}.$$

To prove (26.4), it remains to verify that $\sigma(X_b^*) \cap \mathbb{D} \subset \sigma_p(X_b^*)$. Hence, let $\lambda \in \sigma(X_b^*) \cap \mathbb{D}$. Then, by (26.3),

$$X_b^* - \lambda I = V_b^{-1}(Z_\mu - \lambda I) V_b - (1 - \overline{b(0)})^{-1} k_0^b \otimes S^* b.$$

But, since Z_μ is a unitary operator and $\lambda \in \mathbb{D}$, the operator $Z_\mu - \lambda I$ is invertible and thus, by Lemma 7.33, $X_b^* - \lambda I$ is a Fredholm operator with ind $(X_b^* - \lambda I) = 0$. Arguing as in the proof of Theorem 26.1 and using the fact that $\lambda \in \sigma(X_b^*)$, we deduce that $\lambda \in \sigma_p(X_b^*)$.

Now, let $\lambda \in \mathbb{D}$ with $b(\lambda) = 0$. By the above discussion, we know that $\lambda \in \sigma_p(X_b^*)$. Hence, let $f \in \ker(X_b^* - \lambda I)$, $f \not\equiv 0$. Then, using Theorem 18.22, we have

$$\lambda f = X_b^* f = Sf - \langle f, S^* b \rangle_b b,$$

which implies that

$$f(z) = \langle f, S^* b \rangle_b \frac{b(z)}{z - \lambda} = \langle f, S^* b \rangle_b (Q_\lambda b)(z),$$

whence $f \in \mathbb{C}\, Q_\lambda b$. The above identity shows that

$$\ker(X_b^* - \lambda I) \subset \mathbb{C} Q_\lambda b.$$

Considering the dimensions on both sides, we conclude that in fact the equality holds. □

If we combine Theorem 20.13 and Corollary 26.3, we have the following complete description of the spectrum of X_b^*.

Corollary 26.4 *Let b be an extreme point of the closed unit ball of H^∞ and let $\lambda \in \bar{\mathbb{D}}$. Then $\lambda \in \sigma(X_b^*)$ if and only if one of the following situations holds.*

(i) *$\lambda \in \mathbb{D}$ and $b(\lambda) = 0$.*
(ii) *$\lambda \in \mathbb{T}$ and b has an analytic continuation across a neighborhood I of λ and, moreover, $|b| = 1$ on I.*

26.2 Multipliers of $\mathcal{H}(b)$ spaces, extreme case, part I

In this section, we study the multipliers of $\mathcal{H}(b)$ in the extreme case. This topic will be continued in Section 28.3.

In Theorem 24.6, we have seen that, if b is a nonextreme point of the closed unit ball of H^∞, then every function f holomorphic in a neighborhood of $\bar{\mathbb{D}}$ is a multiplier of $\mathcal{H}(b)$. If b is an extreme point, exactly the opposite is true.

Theorem 26.5 *If b is an extreme point of the closed unit ball of H^∞, and φ is a nonconstant multiplier of $\mathcal{H}(b)$, then φ cannot be continued analytically across all of $\mathbb{T}$.*

Proof Since φ is supposed to be a nonconstant multiplier of $\mathcal{H}(b)$, then b cannot be an inner function; see Theorem 14.40. According to Theorem 20.17, the function φ is also a multiplier of $\mathcal{H}(\bar{b})$, and then Theorem 20.23 implies that φ can be written as

$$\varphi = h + c,$$

where c is a constant and $h \in \mathcal{H}(\bar{b})$. If φ can be continued analytically across all of $\mathbb{T}$, then so can h, and Theorem 25.4 implies that $h \equiv 0$. In particular, φ must be a constant, which is absurd. □

Theorem 26.6 *If b is an extreme point of the closed unit ball of H^∞, and φ is a nonconstant multiplier of $\mathcal{H}(b)$, then φ cannot have a bounded-type meromorphic pseudocontinuation across $\mathbb{T}$ to $\mathbb{D}_e$.*

Proof Arguing as in the previous theorem, we can write φ as $\varphi = h+c$, where c is a constant and $h \in \mathcal{H}(\bar{b})$. Now it remains to apply Theorem 25.16. □

Since an inner function has a bounded-type meromorphic pseudocontinuation across $\mathbb{T}$ to $\mathbb{D}_e$, we immediately get the following.

Corollary 26.7 *Let b be an extreme point of the closed unit ball of H^∞. Then no nonconstant inner function is a multiplier of $\mathcal{H}(b)$.*

When b is not an inner function, the reader may ask if $\mathcal{H}(b)$ always has nontrivial (i.e. nonconstant) multipliers. We have already seen that, when b is a nonextreme point, there are a lot. But when b is an extreme point, the situation is more delicate. However, it will be proved in Corollary 28.5 that, when b is not an inner function, then $\mathcal{H}(b)$ always has nonconstant multipliers.

When m is a multiplier of $\mathcal{H}(b)$, then we know (see Theorem 9.2) that the reproducing kernels k_λ^b, $\lambda \in \mathbb{D}$, are eigenvectors of the adjoint of the multiplication operator M_φ^*. More precisely, we have

$$M_\varphi^* k_\lambda^b = \overline{\varphi(\lambda)} k_\lambda^b. \tag{26.5}$$

When b is an extreme point of the closed unit ball of H^∞, it follows that the operator M_φ^* has a lot of other unexpected eigenvectors.

Theorem 26.8 *Let b be an extreme point of the closed unit ball of H^∞ and let φ be a multiplier of $\mathcal{H}(b)$. Then S^*b is an eigenvector of M_φ^*. If $\bar\alpha$ is the corresponding eigenvalue, then*

$$(\varphi - \alpha)b \in \mathcal{H}(b) \quad \textit{and} \quad \varphi - \alpha \in \mathcal{H}(\bar b).$$

Moreover, we have the following commutation relation:

$$M_\varphi^* X_b - X_b M_\varphi^* = S^*b \otimes (\varphi - \alpha)b.$$

Proof First, note that the orthogonal complement of the one-dimensional space $\mathbb{C}S^*b$ is invariant under the multiplication operator by φ, that is,

$$M_\varphi(\mathcal{H}(b) \ominus \mathbb{C}S^*b) \subset \mathcal{H}(b) \ominus \mathbb{C}S^*b. \tag{26.6}$$

Indeed, let $h \in \mathcal{H}(b) \ominus \mathbb{C}S^*b$. Then, by Corollary 25.6, we have $Sh \in \mathcal{H}(b)$ and $M_\varphi(h) \perp S^*b$ if and only if $SM_\varphi(h) \in \mathcal{H}(b)$. But $SM_\varphi(h) = M_\varphi(Sh)$ and the conclusion follows because φ is a multiplier of $\mathcal{H}(b)$.

Now, it is standard to see that (26.6) implies that $\mathbb{C}S^*b$ is invariant under M_φ^*, which shows that S^*b is an eigenvector of M_φ^*. To obtain the other part of the theorem, remember that the relation (18.13) says that, for each $\lambda \in \mathbb{D}$, we have

$$X_b k_\lambda^b = \bar\lambda k_\lambda^b - \overline{b(\lambda)} S^*b.$$

Combining this relation with (26.5) gives

$$\begin{aligned}(M_\varphi^* X_b - X_b M_\varphi^*)k_\lambda^b &= M_\varphi^*(\bar\lambda k_\lambda^b - \overline{b(\lambda)}S^*b) - X_b(\overline{\varphi(\lambda)}k_\lambda^b)\\ &= -\bar\alpha\overline{b(\lambda)}S^*b + \overline{b(\lambda)}\,\overline{\varphi(\lambda)}S^*b\\ &= \overline{b(\lambda)}(\overline{\varphi(\lambda)} - \bar\alpha)S^*b.\end{aligned}$$

As the functions k_λ^b span $\mathcal{H}(b)$, it follows that $M_\varphi^* X_b - X_b M_\varphi^*$ is an operator of rank one with range spanned by S^*b. More precisely, there is a function $\phi \in \mathcal{H}(b)$ such that

$$M_\varphi^* X_b - X_b M_\varphi^* = S^*b \otimes \phi.$$

By the preceding equality, we have $\overline{\phi(\lambda)} = \overline{b(\lambda)}(\overline{\varphi(\lambda)} - \bar\alpha)$; in other words,

$$\phi = b(\varphi - \alpha).$$

In particular, $(\varphi - \alpha)b \in \mathcal{H}(b)$ and

$$M_\varphi^* X_b - X_b M_\varphi^* = S^*b \otimes (\varphi - \alpha)b.$$

It remains to prove that $\varphi - \alpha \in \mathcal{H}(\bar{b})$. But we have $(\varphi - \alpha)b \in \mathcal{H}(b) \cap \mathcal{M}(b)$, and, according to (17.11), there exists $g \in \mathcal{H}(\bar{b})$ such that

$$(\varphi - \alpha)b = T_{\bar{b}}g = bg.$$

Since $b \neq 0$ a.e. on $\mathbb{T}$, we get the desired result. □

Remember that, for $\lambda \in \mathbb{D}$, $\hat{k}_\lambda^b$ is the function defined by

$$\hat{k}_\lambda^b(z) = (Q_\lambda b)(z) = \frac{b(z) - b(\lambda)}{z - \lambda}$$

and $\hat{k}_\lambda^b = (I - \lambda X_b)^{-1} S^* b$.

Corollary 26.9 *Let b be an extreme point of the closed unit ball of H^∞, and let φ be a multiplier of $\mathcal{H}(b)$. Then, for each $\lambda \in \mathbb{D}$, the function $\hat{k}_\lambda^b$ is an eigenvector of M_φ^*.*

Proof The case $\lambda = 0$ follows from Theorem 26.8. Now assume that $\lambda \neq 0$. By Theorem 26.8, we have

$$(I - \lambda X_b)M_\varphi^* - M_\varphi^*(I - \lambda X_b) = \lambda S^* b \otimes (\varphi - \alpha)b,$$

where α is such that $M_\varphi^* S^* b = \bar{\alpha} S^* b$. Applying $\hat{k}_\lambda^b$ to both sides, we get

$$\begin{aligned}(I - \lambda X_b)M_\varphi^* \hat{k}_\lambda^b &= M_\varphi^* S^* b + \lambda \langle \hat{k}_\lambda^b,\ (\varphi - \alpha)b \rangle_b S^* b \\ &= (\bar{\alpha} + \lambda \langle \hat{k}_\lambda^b,\ (\varphi - \alpha)b \rangle_b) S^* b.\end{aligned}$$

It follows that

$$M_\varphi^* \hat{k}_\lambda^b = (\bar{\alpha} + \lambda \langle \hat{k}_\lambda^b, (\varphi - \alpha)b \rangle_b) \hat{k}_\lambda^b,$$

which gives that $\hat{k}_\lambda^b$ is an eigenvector of M_φ^* associated with the eigenvalue $\bar{\alpha} + \lambda \langle \hat{k}_\lambda^b,\ (\varphi - \alpha)b \rangle_b$. □

The properties of M_φ given by Theorem 26.8 characterize multiplication operators in the extreme point case.

Theorem 26.10 *Let b be an extreme point of the closed unit ball of H^∞, and let $M \in \mathcal{L}(\mathcal{H}(b))$ such that*

$$M^* S^* b = \bar{\alpha} S^* b \tag{26.7}$$

and

$$M^* X_b - X_b M^* = S^* b \otimes \phi, \tag{26.8}$$

for some $\alpha \in \mathbb{C}$ and $\phi \in \mathcal{H}(b) \cap \mathcal{M}(b)$. Then there exists $\varphi \in \mathfrak{Mult}(\mathcal{H}(b))$ such that $M = M_\varphi$.

Proof According to Theorem 9.3, we need to prove that k_λ^b is an eigenvector of M^* for any $\lambda \in \mathbb{D}$. First assume that $b(\lambda) = 0$. Then $k_\lambda^b = k_\lambda$ and since $X_b k_\lambda = S^* k_\lambda = \bar{\lambda} k_\lambda$, using (26.8), we get

$$\begin{aligned}\bar{\lambda} M^* k_\lambda &= X_b M^* k_\lambda + \langle k_\lambda, \phi \rangle_b S^* b \\ &= X_b M^* k_\lambda + \overline{\phi(\lambda)} S^* b.\end{aligned}$$

Since $\phi \in \mathcal{M}(b)$, we must have $\phi(\lambda) = 0$, which gives that

$$M^* k_\lambda \ker(X_b - \bar{\lambda} I).$$

According to Theorem 26.1, there exists $\mu \in \mathbb{C}$ such that $M^* k_\lambda = \mu_\lambda k_\lambda$. In other words, $k_\lambda = k_\lambda^b$ is an eigenvector of M^*.

Assume now that $b(\lambda) \neq 0$. Applying relation (26.8) to k_λ^b gives

$$M^* X_b k_\lambda^b - X_b M^* k_\lambda^b = \overline{\phi(\lambda)} S^* b.$$

But $X_b k_\lambda^b = \bar{\lambda} k_\lambda^b - \overline{b(\lambda)} S^* b$. Hence using relation (26.7)

$$(\bar{\lambda} I - X_b) M^* k_\lambda^b = (\bar{\alpha} \overline{b(\lambda)} + \overline{\phi(\lambda)}) S^* b.$$

Since $(\bar{\lambda} I - X_b) k_\lambda^b = \overline{b(\lambda)} S^* b$, we obtain

$$(\bar{\lambda} I - X_b) M^* k_\lambda^b = (\bar{\lambda} I - X_b) \left(\frac{\bar{\alpha} \overline{b(\lambda)} + \overline{\phi(\lambda)}}{\overline{b(\lambda)}} k_\lambda^b \right).$$

Note that, by Theorem 26.1, $\bar{\lambda} I - X_b$ is injective because $b(\lambda) \neq 0$. Hence

$$M^* k_\lambda^b = \frac{\bar{\alpha} \overline{b(\lambda)} + \overline{\phi(\lambda)}}{\overline{b(\lambda)}} k_\lambda^b,$$

which proves the result. □

26.3 Comparison of measures

In Section 20.11, we studied the comparison of Clark measures corresponding to a general element b in the closed unit ball of H^∞, and an inner function u. In Section 24.4, we continued this study by considering the nonextreme elements b. In this section, we complete this study by considering the extreme case.

Theorem 26.11 *Let b be a nonconstant extreme function in the closed unit ball of H^∞, and let Θ be a nonconstant inner function. Let μ and ν denote the Clark measures of b and Θ, respectively. Then the following are equivalent.*

(i) *ν is absolutely continuous with respect to μ and $d\nu/d\mu$ is in $L^2(\mu)$.*
(ii) *$(1 - b)/(1 - \Theta) \in H^2$ and $1/(1 - \Theta) \in L^2(\rho)$.*

Proof According to Theorem 20.28, it is sufficient to prove that

$$\text{(ii)} \quad \Longleftrightarrow \quad \frac{1-b}{1-\Theta}k_0^\Theta \in \mathcal{H}(b).$$

We first show the implication (ii) $\Longrightarrow [(1-b)/(1-\Theta)]k_0^\Theta \in \mathcal{H}(b)$. To prove this fact, we use Corollary 20.2. Since $(1-b)/(1-\Theta) \in H^2$ (by hypothesis) and $k_0^\Theta \in H^\infty$, we surely have $[(1-b)/(1-\Theta)]k_0^\Theta \in H^2$. Write

$$\begin{aligned} T_{\bar{b}}\left(\frac{1-b}{1-\Theta}k_0^\Theta\right) &= P_+\left(\frac{\bar{b}-|b|^2}{1-\Theta}k_0^\Theta\right) \\ &= P_+\left(\frac{1-|b|^2}{1-\Theta}k_0^\Theta + \frac{\bar{b}-1}{1-\Theta}k_0^\Theta\right). \end{aligned} \tag{26.9}$$

Using the fact that $|\Theta| = 1$ almost everywhere on $\mathbb{T}$, we have

$$\begin{aligned} \frac{\bar{b}-1}{1-\Theta}k_0^\Theta &= \frac{\bar{b}-1}{1-\Theta}(1-\overline{\Theta(0)}\Theta) \\ &= \frac{\bar{b}-1}{\bar{\Theta}-1}(\bar{\Theta}-\overline{\Theta(0)}) \\ &= \overline{\left(\frac{b-1}{\Theta-1}(\Theta-\Theta(0))\right)}. \end{aligned} \tag{26.10}$$

Since $[(b-1)/(\Theta-1)](\Theta-\Theta(0)) \in H_0^2$, we have

$$P_+\left(\frac{\bar{b}-1}{1-\Theta}k_0^\Theta\right) = 0,$$

which implies that

$$T_{\bar{b}}\left(\frac{1-b}{1-\Theta}k_0^\Theta\right) = P_+\left(\frac{1-|b|^2}{1-\Theta}k_0^\Theta\right) = P_+\left(\frac{1-\overline{\Theta(0)}\Theta}{1-\Theta}\rho\right).$$

Since, by hypothesis, $1/(1-\Theta) \in L^2(\rho)$, and since $1-\overline{\Theta(0)}\Theta \in L^\infty(\mathbb{T})$, we have

$$\frac{1-\overline{\Theta(0)}\Theta}{1-\Theta} \in L^2(\rho).$$

Therefore, by Corollary 20.2, we conclude that $[(1-b)/(1-\Theta)]k_0^\Theta \in \mathcal{H}(b)$.

Now we show the reverse implication $[(1-b)/(1-\Theta)]k_0^\Theta \in \mathcal{H}(b) \Longrightarrow$ (ii). For every $z \in \mathbb{D}$, we have

$$|k_0^\Theta(z)| = |1-\overline{\Theta(0)}\Theta(z)| \geq 1-|\Theta(0)| > 0,$$

which implies that $1/k_0^\Theta \in H^\infty$. Therefore, the function $(1-b)/(1-\Theta)$, being the product of the H^2 function $[(1-b)/(1-\Theta)]k_0^\Theta$ and the H^∞ function $1/k_0^\Theta$, belongs to H^2. It remains to prove that $1/(1-\Theta)$ belongs to $L^2(\rho)$. Using once more Theorem 17.8 and Corollary 25.2, our assumption ensures that

$$T_{\bar{b}}\left(\frac{1-b}{1-\Theta}k_0^\Theta\right) \in \mathcal{H}(\bar{b})$$

and there exists a function $g \in L^2(\rho)$ such that

$$T_{\bar{b}}\left(\frac{1-b}{1-\Theta}k_0^\Theta\right) = K_\rho(g) \quad \text{and} \quad \log|\rho g| \notin L^1. \tag{26.11}$$

As before, using (26.9) and (26.10), we can write

$$T_{\bar{b}}\left(\frac{1-b}{1-\Theta}k_0^\Theta\right) = P_+\left(\frac{1-|b|^2}{1-\Theta}k_0^\Theta\right) = P_+\left(\frac{\rho}{1-\Theta}k_0^\Theta\right),$$

because $[(b-1)/(\Theta-1)](\Theta-\Theta(0)) \in H_0^2$. Therefore, according to (26.11), we obtain

$$P_+\left(\frac{\rho}{1-\Theta}k_0^\Theta\right) = P_+(\rho g),$$

that is,

$$P_+\left(\left(\frac{1-\overline{\Theta(0)}\Theta}{1-\Theta} - g\right)\rho\right) = 0.$$

In other words, the function

$$h = \left(\frac{1-\overline{\Theta(0)}\Theta}{1-\Theta} - g\right)\rho$$

belongs to $\overline{H_0^2}$. We prove that $\log|h| \notin L^1$. Write

$$|h| = \left|\left(\frac{(1-\overline{\Theta(0)}\Theta)\rho^{1/2}}{1-\Theta} - g\rho^{1/2}\right)\rho^{1/2}\right| \le \left(\frac{2}{|1-\Theta|} + |g|\rho^{1/2}\right)\rho^{1/2},$$

because $|1-\overline{\Theta(0)}\Theta| \le 2$ and $\rho \le 1$. Thus

$$\log|h| \le \log\left(\frac{2}{|1-\Theta|} + |g|\rho^{1/2}\right) + \frac{1}{2}\log\rho. \tag{26.12}$$

Define $E = \{\zeta \in \mathbb{T} : |g(\zeta)\rho^{1/2}(\zeta)| \ge 1\}$. On the Borel set E, we have

$$\frac{2}{|1-\Theta|} + |g\rho^{1/2}| \le \frac{4}{|1-\Theta|}|g\rho^{1/2}|,$$

because $|1-\Theta| \le 2$ and, if $a \ge 1$ and $b \ge 1$, then $a+b \le 2ab$. Therefore, on E, we have

$$\begin{aligned}\log\left(\frac{2}{|1-\Theta|} + |g|\rho^{1/2}\right) &\le \log\left(\frac{4}{|1-\Theta|}\right) + \log(|g\rho^{1/2}|)\\ &\le \log^+\left(\frac{4}{|1-\Theta|}\right) + \log^+(|g\rho^{1/2}|).\end{aligned} \tag{26.13}$$

On the other hand, on $\mathbb{T} \setminus E$, we have $\log^+(|g\rho^{1/2}|) = 0$ and

$$|g\rho^{1/2}| \le 1 \le \frac{2}{|1-\Theta|},$$

which implies that

$$\frac{2}{|1-\Theta|} + |g\rho^{1/2}| \le \frac{4}{|1-\Theta|}$$

and

$$\begin{aligned}\log\left(\frac{2}{|1-\Theta|} + |g|\rho^{1/2}\right) &\le \log\left(\frac{4}{|1-\Theta|}\right) \le \log^+\left(\frac{4}{|1-\Theta|}\right)\\ &= \log^+\left(\frac{4}{|1-\Theta|}\right) + \log^+(|g\rho^{1/2}|). \quad (26.14)\end{aligned}$$

Hence, by (26.13) and (26.14), we have

$$\log\left(\frac{2}{|1-\Theta|} + |g|\rho^{1/2}\right) \le \log^+\left(\frac{4}{|1-\Theta|}\right) + \log^+(|g\rho^{1/2}) \quad \text{(a.e. on } \mathbb{T}).$$

Therefore, by (26.12),

$$\log|h| \le \log^+\left(\frac{4}{|1-\Theta|}\right) + \log^+(|g\rho^{1/2}) + \frac{1}{2}\log\rho \qquad \text{(a.e. on } \mathbb{T}).$$

The function $4/(1-\Theta)$ is analytic on $\mathbb{D}$ and, being the quotient of two nonzero H^∞ functions, it is in the Nevanlinna class. Therefore,

$$\log^+\left(\frac{4}{|1-\Theta|}\right) \in L^1(\mathbb{T}).$$

In the proof of Corollary 25.2, we saw that $\log^+|g\rho^{1/2}| \in L^1(\mathbb{T})$. Finally, since b is an extreme point of the closed unit ball of H^∞, we have

$$\int_{\mathbb{T}} \log\rho\, dm = -\infty.$$

These facts ensure that $\log|h| \notin L^1(\mathbb{T})$. Using Lemma 4.30, we deduce that

$$h = \left(\frac{1-\overline{\Theta(0)}\Theta}{1-\Theta} - g\right)\rho = 0 \tag{26.15}$$

almost everywhere on $\mathbb{T}$. Define the Borel subset of $\mathbb{T}$ by $F = \{\zeta \in \mathbb{T} : \rho(\zeta) = 0\}$. Then the identity (26.15) implies that

$$\frac{1-\overline{\Theta(0)}\Theta}{1-\Theta} = g$$

almost everywhere on $\mathbb{T} \setminus F$. Since

$$
\begin{aligned}
(1-|\Theta(0)|)^2 \int_{\mathbb{T}} \frac{\rho(\zeta)}{|1-\Theta(\zeta)|^2}\, dm(\zeta) &\le \int_{\mathbb{T}} \left| \frac{1-\overline{\Theta(0)}\Theta(\zeta)}{1-\Theta(\zeta)} \right|^2 \rho(\zeta)\, dm(\zeta)\\
&= \int_{\mathbb{T}\setminus F} \left| \frac{1-\overline{\Theta(0)}\Theta(\zeta)}{1-\Theta(\zeta)} \right|^2 \rho(\zeta)\, dm(\zeta)\\
&= \int_{\mathbb{T}\setminus F} |g(\zeta)|^2 \rho(\zeta)\, dm(\zeta)\\
&= \int_{\mathbb{T}} |g(\zeta)|^2 \rho(\zeta)\, dm(\zeta) < +\infty,
\end{aligned}
$$

we conclude that

$$\int_{\mathbb{T}} \frac{\rho(\zeta)}{|1-\Theta(\zeta)|^2}\, dm(\zeta) < +\infty.$$

In other words,

$$\frac{1}{1-\Theta} \in L^2(\rho). \qquad \square$$

Corollary 26.12 *Let Θ and u be two nonconstant inner functions. Let ν and μ denote the Clark measures of Θ and u, respectively. Then the following assertions are equivalent:*

(i) *The measure ν is absolutely continuous with respect to μ and the function $d\nu/d\mu$ belongs to $L^2(\mu)$.*
(ii) *The function $(1-u)/(1-\Theta)$ belongs to H^2.*

Proof According to Theorem 26.11, we know that (i) is equivalent to the two conditions

$$\frac{1-u}{1-\Theta} \in H^2 \quad \text{and} \quad \frac{1}{1-\Theta} \in L^2(\rho),$$

where $\rho = 1-|u|^2$ on $\mathbb{T}$. But, since u is inner, $\rho = 0$ a.e. on $\mathbb{T}$, and then $1/(1-\Theta)$ always belongs to $L^2(\rho)$. $\square$

26.4 Further characterizations of angular derivatives for b

In Sections 21.1 and 21.2, we have already seen some equivalent conditions for the existence of angular derivatives in the sense of Carathéodory. In this section, we pursue this study to give some other equivalent conditions. The results of that section do not assume that b is an extreme point or not of the unit ball of H^∞.

Theorem 26.13 *Let $b \in H^\infty$, $\|b\|_\infty \le 1$, and $z_0 \in \mathbb{T}$. Then the following are equivalent.*

(i) *b has an angular derivative in the sense of Carathéodory at z_0.*
(ii) *There is $\lambda \in \mathbb{T}$ such that*

$$\frac{|b-\lambda|^2}{|z-z_0|^2} \in L^1(\mathbb{T}) \quad \textit{and} \quad \frac{1-|b|^2}{|z-z_0|^2} \in L^1(\mathbb{T}).$$

(iii) *There is $\lambda \in \mathbb{T}$ such that*

$$\frac{\Re(1-\bar{\lambda}b)}{\Re(1-\bar{z}_0 z)} \in L^1(\mathbb{T}).$$

Proof (i) $\Longleftrightarrow$ (ii) According to Theorem 21.5, b has an angular derivative in the sense of Carathéodory at the point $z_0 \in \mathbb{T}$ if and only if there is a point $\lambda \in \mathbb{T}$ such that the measure $\mu_\lambda(\{z_0\}) > 0$. The latter can be rewritten as $\delta_{z_0} \ll \mu_\lambda$. To continue, we consider two cases. We just recall that, as discussed in Section 13.7, the Clark measure associated with the inner function $\Theta(z) = \bar{z}_0 z$ is δ_{z_0}.

Nonextreme case. b is nonextreme if and only if so is $\bar{\lambda}b$. But $a_\lambda = a$. Hence, by Theorem 24.17,

$$\begin{aligned}
\delta_{z_0} \ll \mu_\lambda \quad &\Longleftrightarrow \quad \frac{1-\bar{\lambda}b}{1-\bar{z}_0 z} \in H^2 \quad \text{and} \quad \frac{a_\lambda}{1-\bar{z}_0 z} \in H^2,\\
&\Longleftrightarrow \quad \frac{b-\lambda}{z-z_0} \in H^2 \quad \text{and} \quad \frac{a}{z-z_0} \in H^2.
\end{aligned}$$

Since the function $z \longmapsto z - z_0$ is outer, by Corollary 4.28,

$$\begin{aligned}
&\frac{b-\lambda}{z-z_0} \quad \text{and} \quad \frac{a}{z-z_0} \in H^2\\
&\quad \Longleftrightarrow \quad \frac{|b-\lambda|}{|z-z_0|} \in L^2(\mathbb{T}) \quad \text{and} \quad \frac{|a|}{|z-z_0|} \in L^2(\mathbb{T}),\\
&\quad \Longleftrightarrow \quad \frac{|b-\lambda|^2}{|z-z_0|^2} \in L^1(\mathbb{T}) \quad \text{and} \quad \frac{|a|^2}{|z-z_0|^2} \in L^1(\mathbb{T}),\\
&\quad \Longleftrightarrow \quad \frac{|b-\lambda|^2}{|z-z_0|^2} \in L^1(\mathbb{T}) \quad \text{and} \quad \frac{1-|b|^2}{|z-z_0|^2} \in L^1(\mathbb{T}).
\end{aligned}$$

Extreme case. This case has a similar flavor. b is extreme if and only if so is $\bar{\lambda}b$. But, still $\rho_\lambda = 1 - |\bar{\lambda}b|^2 = 1 - |b|^2 = \rho$. Hence, by Theorem 26.11,

$$\begin{aligned}
\delta_{z_0} \ll \mu_\lambda \quad &\Longleftrightarrow \quad \frac{1-\bar{\lambda}b}{1-\bar{z}_0 z} \in H^2 \quad \text{and} \quad \frac{1}{1-\bar{z}_0 z} \in L^2(\rho_\lambda),\\
&\Longleftrightarrow \quad \frac{b-\lambda}{z-z_0} \in H^2 \quad \text{and} \quad \frac{1}{z-z_0} \in L^2(\rho).
\end{aligned}$$

Since the function $z \longmapsto z - z_0$ is outer, by Corollary 4.28,

$$\begin{aligned}
&\frac{b-\lambda}{z-z_0} \in H^2 \quad \text{and} \quad \frac{1}{z-z_0} \in L^2(\rho) \\
&\iff \quad \frac{|b-\lambda|}{|z-z_0|} \quad \text{and} \quad \frac{\rho^{1/2}}{|z-z_0|} \in L^2(\mathbb{T}), \\
&\iff \quad \frac{|b-\lambda|^2}{|z-z_0|^2} \quad \text{and} \quad \frac{\rho}{|z-z_0|^2} \in L^1(\mathbb{T}), \\
&\iff \quad \frac{|b-\lambda|^2}{|z-z_0|^2} \quad \text{and} \quad \frac{1-|b|^2}{|z-z_0|^2} \in L^1(\mathbb{T}).
\end{aligned}$$

(ii) $\iff$ (iii) Since the functions presented in part (ii) are positive, this condition is equivalent to

$$\frac{|b-\lambda|^2}{|z-z_0|^2} + \frac{1-|b|^2}{|z-z_0|^2} \in L^1(\mathbb{T}).$$

Moreover, for each $z \in \mathbb{T}$,

$$\begin{aligned}
|z-z_0|^2 &= |z|^2 + |z_0|^2 - 2\,\Re(\bar{z}_0 z) \\
&= 2 - 2\,\Re(\bar{z}_0 z) \\
&= 2\,\Re(1-\bar{z}_0 z),
\end{aligned}$$

and, for each $\lambda \in \mathbb{T}$,

$$\begin{aligned}
|b-\lambda|^2 + (1-|b|^2) &= |b|^2 + |\lambda|^2 - 2\,\Re(\bar{\lambda} b) + 1 - |b|^2 \\
&= 2 - 2\,\Re(\bar{\lambda} b) \\
&= 2\,\Re(1-\bar{\lambda} b).
\end{aligned}$$

Thus, we clearly see that (ii) and (iii) are equivalent. □

The following result is a direct consequence of Theorems 21.5, 21.6, 21.7 and 26.13.

Corollary 26.14 *Let $b \in H^\infty$, with $\|b\|_\infty \leq 1$, and let $z_0 \in \mathbb{T}$. Then the following are equivalent.*

(i) *b has an angular derivative in the sense of Carathéodory at z_0.*
(ii) *There is $\lambda \in \mathbb{T}$ such that*

$$\frac{|b-\lambda|^2}{|z-z_0|^2} \in L^1(\mathbb{T}) \quad \textit{and} \quad \frac{1-|b|^2}{|z-z_0|^2} \in L^1(\mathbb{T}).$$

(iii) *There is $\lambda \in \mathbb{T}$ such that*

$$\frac{\Re(1-\bar{\lambda} b)}{\Re(1-\bar{z}_0 z)} \in L^1(\mathbb{T}).$$

(iv) *There is $\lambda \in \mathbb{T}$ such that the Clark measure μ_λ has an atom at z_0.*

(v) *There is* $\eta \in \mathbb{T} \setminus \{b(z_0)\}$ *such that*

$$\int_{\mathbb{T}} |e^{i\theta} - z_0|^{-2} \, d\mu_\eta(e^{i\theta}) = \frac{|b'(z_0)|}{|\eta - b(z_0)|^2}.$$

(vi) *For every* $\eta \in \mathbb{T} \setminus \{b(z_0)\}$, *we have*

$$\int_{\mathbb{T}} |e^{i\theta} - z_0|^{-2} \, d\mu_\eta(e^{i\theta}) = \frac{|b'(z_0)|}{|\eta - b(z_0)|^2}.$$

Moreover, for parts (i)–(iv), *we necessarily have* $\lambda = b(z_0)$ *and* $\mu_\lambda(\{z_0\}) = 1/|b'(z_0)|$.

26.5 Model operator for Hilbert space contractions

In this section, we show that the operator X_b can serve as a model for a large class of Hilbert space contractions. Indeed, that was one of the motivations for de Branges and Rovnyak to introduce $\mathcal{H}(b)$ spaces. We first begin with a simple result.

Theorem 26.15 *Let* $T : \mathcal{H} \longrightarrow \mathcal{H}$ *be a Hilbert space contraction. Assume that*

$$\lim_{n\to\infty} \|T^n x\|_{\mathcal{H}} = 0 \qquad (x \in \mathcal{H})$$

and that the operator $I - T^*T$ *is of rank one. Then there exists an inner function* Θ *such that* T *is unitarily equivalent to* X_Θ.

Proof Let D_T be the unique positive square root of the positive operator $I - T^*T$, i.e. $D_T = (I - T^*T)^{1/2}$. The key point of the proof is the following simple observation. Let x be any vector in $\mathcal{H}$. Then we have

$$\begin{aligned} \|D_T T^n x\|_{\mathcal{H}}^2 &= \langle D_T^2 T^n x, T^n x \rangle_{\mathcal{H}} \\ &= \langle (I - T^*T) T^n x, \; T^n x \rangle_{\mathcal{H}} \\ &= \|T^n x\|_{\mathcal{H}}^2 - \|T^{n+1} x\|_{\mathcal{H}}^2, \end{aligned}$$

which, using the fact that $\|T^n x\| \longrightarrow 0$, implies the sequence $(D_T T^n x)_{n\ge 0}$ is in ℓ^2 and its ℓ^2 norm is equal to $\|x\|$. Note that $D_T T^n x$ belongs to the range of D_T, which is of dimension one and thus can be identified by $\mathbb{C}$.

Now, define a linear map U from $\mathcal{H}$ into H^2 by

$$(Ux)(z) = \sum_{n=0}^{\infty} (D_T T^n x) z^n \qquad (z \in \mathbb{D}).$$

Since the sequence $(D_T T^n x)_{n\geq 0}$ is in ℓ^2 and of norm $\|x\|$, the map U is well defined and isometric from $\mathcal{H}$ onto a closed subspace E of H^2. Then we have

$$\begin{aligned}(UTx)(z) &= \sum_{n=0}^{\infty}(D_T T^{n+1}x)z^n \\ &= S^*\left(D_T Tx + \sum_{n=0}^{\infty}(D_T T^{n+1}x)z^{n+1}\right) \\ &= S^*\left(\sum_{n=0}^{\infty}(D_T T^n x)z^n\right) \\ &= (S^*Ux)(z) \qquad (z \in \mathbb{D}),\end{aligned}$$

which reveals that $UT = S^*U$. This relations implies two things. First, E is a closed S^*-invariant subspace of H^2 and, second, T is unitarily equivalent to $S^*_{|E}$. Hence, by Beurling's result (Corollary 8.33), there exists an inner function Θ such that $E = K_\Theta$ and this proves that T is unitarily equivalent to X_Θ. □

We can generalize the previous result as follows.

Theorem 26.16 *Let $T : \mathcal{H} \longrightarrow \mathcal{H}$ be a Hilbert space contraction. Assume that the operators $I - TT^*$ and $I - T^*T$ are of rank one and that*

$$\|T^n f\|_{\mathcal{H}} = \|f\|_{\mathcal{H}} \quad (n \geq 1) \quad \Longrightarrow \quad f = 0.$$

Then there exists b, an extreme point of the closed unit ball of H^∞, such that T is unitarily equivalent to X_b.

Proof Let $k_0 \in \mathcal{H}$ be such that $I - T^*T = k_0 \otimes k_0$. We claim that

$$|\langle T^n f, k_0\rangle_{\mathcal{H}}|^2 = \|T^n f\|^2_{\mathcal{H}} - \|T^{n+1}f\|^2_{\mathcal{H}}, \tag{26.16}$$

for every $n \geq 0$ and every $f \in \mathcal{H}$. Indeed, we have

$$\begin{aligned}|\langle T^n f, k_0\rangle_{\mathcal{H}}|^2 &= \langle T^n f, k_0\rangle_{\mathcal{H}}\langle k_0, T^n f\rangle_{\mathcal{H}} \\ &= \big\langle \langle T^n f, k_0\rangle_{\mathcal{H}} k_0, T^n f\big\rangle_{\mathcal{H}} \\ &= \langle (k_0 \otimes k_0)T^n f,\ T^n f\rangle_{\mathcal{H}} \\ &= \langle (I - T^*T)T^n f,\ T^n f\rangle_{\mathcal{H}} \\ &= \|T^n f\|^2_{\mathcal{H}} - \|T^{n+1}f\|^2_{\mathcal{H}}.\end{aligned}$$

In particular, we have

$$\sum_{n=0}^{N} |\langle T^n f, k_0\rangle_{\mathcal{H}}|^2 = \|f\|^2_{\mathcal{H}} - \|T^{N+1}f\|^2_{\mathcal{H}} \leq \|f\|^2_{\mathcal{H}}.$$

Hence the sequence $(\langle T^n f, k_0\rangle_{\mathcal{H}})_{n\geq 0}$ is in ℓ^2. Hence, for each function $f \in \mathcal{H}$, the function

$$g(z) = (Uf)(z) = \sum_{n=0}^{\infty} \langle T^n f, k_0\rangle_{\mathcal{H}} z^n \qquad (z \in \mathbb{D})$$

belongs to H^2. Moreover, we have

$$\|g\|_2 \leq \|f\|_{\mathcal{H}}.$$

More precisely, we have

$$\|g\|_2^2 = \|f\|_{\mathcal{H}}^2 - \lim_{n\to\infty} \|T^n f\|_{\mathcal{H}}^2.$$

(Note that the sequence $(\|T^n f\|_{\mathcal{H}})_{n\geq 1}$ is convergent, because it is decreasing and positive). Thus, the map $U : f \longmapsto g$ is a continuous linear map from $\mathcal{H}$ into H^2. It is also one-to-one. Indeed, if $Uf = 0$, then we must have $\langle T^n f, k_0\rangle_{\mathcal{H}} = 0$, for every $n \geq 0$. But then (26.16) implies that $\|T^n\|_{\mathcal{H}} = \|f\|_{\mathcal{H}}$ for every $n \geq 0$. The hypothesis of theorem now ensures that $f \equiv 0$.

Put $\mathcal{K} = \mathcal{M}(U)$. We verify that $\mathcal{K}$ satisfies the hypotheses of Theorem 25.22. As the first step, to show that $\mathcal{K}$ is S^*-invariant, let $f \in \mathcal{H}$. Then

$$\begin{aligned}(S^*Uf)(z) &= S^*\left(\sum_{n=0}^{\infty} \langle T^n f, k_0\rangle_{\mathcal{H}} z^n\right)\\ &= \sum_{n=0}^{\infty} \langle T^{n+1} f, k_0\rangle_{\mathcal{H}} z^n\\ &= (UTf)(z) \qquad (z \in \mathbb{D}).\end{aligned}$$

In other words, we have

$$S^*U = UT.$$

This proves that $\mathcal{K}$ is S^*-invariant and, if $X := S^*|\mathcal{K}$, then we have

$$XU = UT. \tag{26.17}$$

Moreover, since U is one-to-one, it is a unitary operator from $\mathcal{H}$ onto $\mathcal{K}$ and then $X = UTU^*$. Therefore, we get

$$I - XX^* = I - UTU^*UT^*U = I - UTT^*U^* = U(I - TT^*)U^*,$$

which implies that $I - XX^*$ is an operator of rank one. It remains to prove that the identity

$$\|Xg\|_{\mathcal{K}}^2 = \|g\|_{\mathcal{K}}^2 - |g(0)|^2 \tag{26.18}$$

holds for every $g \in \mathcal{K}$. Take any $g = Uf \in \mathcal{K}$, with $f \in \mathcal{H}$. We have $Xg = UTU^*Uf = UTf$. Then

$$\begin{aligned}
\|Xg\|_{\mathcal{K}}^2 &= \|UTf\|_{\mathcal{K}}^2 \\
&= \|Tf\|_{\mathcal{H}}^2 = \langle Tf, Tf\rangle_{\mathcal{H}} \\
&= \langle T^*Tf, f\rangle_{\mathcal{H}} \\
&= \|f\|_{\mathcal{H}}^2 - \langle (I - T^*T)f, f\rangle_{\mathcal{H}} \\
&= \|f\|_{\mathcal{H}}^2 - |\langle f, k_0\rangle_{\mathcal{H}}|^2 \\
&= \|Uf\|_{\mathcal{K}}^2 - |(Uf)(0)|^2 \\
&= \|g\|_{\mathcal{K}}^2 - |g(0)|^2,
\end{aligned}$$

which proves (26.18). According to Theorem 25.22, there is b, an extreme point in the closed unit ball of H^∞, such that $\mathcal{K} \doteq \mathcal{H}(b)$. Moreover, (26.17) says that T is unitarily equivalent to $X = S^*|\mathcal{H}(b)$. □

26.6 Conjugation and completeness of difference quotients

Let $\mathcal{H}$ be a Hilbert space. We remember that a map $C : \mathcal{H} \longrightarrow \mathcal{H}$ is a *conjugation* if C is antilinear, isometric, surjective and $C^2 = Id$. Let Θ be an inner function, and let $K_\Theta = H^2 \ominus \Theta H^2$. Since $K_\Theta = H^2 \cap \Theta \overline{H_0^2}$, where $H_0^2 = zH^2$, then it is easy to see that the map Ω_Θ, defined on K_Θ,

$$\Omega_\Theta(f) = \bar{z}\bar{f}\Theta \qquad (f \in K_\Theta),$$

is a conjugation on K_Θ; see Section 14.4. Moreover, since $|\Theta| = 1$ a.e. on $\mathbb{T}$, we have

$$\begin{aligned}
\Omega_\Theta(k_\lambda^\Theta)(z) &= \bar{z}\Theta(z)\frac{1 - \Theta(\lambda)\overline{\Theta(z)}}{1 - \lambda\bar{z}} \\
&= \frac{\Theta(z) - \Theta(\lambda)}{z - \lambda} = \hat{k}_\lambda^\Theta(z)
\end{aligned}$$

for a.e. every $z \in \mathbb{T}$. In particular, the family of difference quotients $(\hat{k}_\lambda^\Theta)_{\lambda\in\mathbb{D}}$ is complete in K_Θ. We will see below that this result remains true for $\mathcal{H}(b)$ spaces in the case where b is an extreme point of the closed unit ball of H^∞.

In the following result, C denotes the conjugation on $L^2(\mu)$ defined by

$$C(f) = \overline{zf}.$$

Theorem 26.17 *Let b be an extreme point in the closed unit ball of H^∞. Then $\Omega_b = V_b C V_b^{-1}$ is a conjugation on $\mathcal{H}(b)$ and we have*

$$\Omega_b k_\lambda^b = \hat{k}_\lambda^b. \tag{26.19}$$

Proof Since b is an extreme point of the closed unit ball of H^∞, then $L^2(\mu) = H^2(\mu)$ and thus V_b is a unitary map from $L^2(\mu)$ onto $\mathcal{H}(b)$. Hence, $V_b C V_b^{-1}$ is clearly a conjugation on $\mathcal{H}(b)$. It remains to verify the formula (26.19). Using (20.15) and the definition of C, we have

$$\begin{aligned} V_b C V_b^{-1} k_\lambda^b &= V_b C((1-\overline{b(\lambda)})k_\lambda) \\ &= (1-b(\lambda)) V_b \left(\frac{e^{-i\theta}}{1-\lambda e^{-i\theta}} \right) \\ &= (1-b(\lambda)) V_b \left(\frac{1}{e^{i\theta}-\lambda} \right). \end{aligned}$$

Then, using the definition of V_b, we can write

$$\begin{aligned} (V_b C V_b^{-1}) k_\lambda^b(z) &= (1-b(\lambda))(1-b(z)) \int_{\mathbb{T}} \frac{1}{1-e^{-i\theta}z} \, \frac{1}{e^{i\theta}-\lambda} \, d\mu(e^{i\theta}) \\ &= (1-b(\lambda))(1-b(z)) \mathcal{I}, \end{aligned}$$

where

$$\mathcal{I} = \int_{\mathbb{T}} \frac{e^{i\theta}}{(e^{-i\theta}-z)(e^{-i\theta}-\lambda)} \, d\mu(e^{i\theta}).$$

An easy computation shows that

$$\frac{e^{i\theta}+z}{e^{i\theta}-z} - \frac{e^{i\theta}+\lambda}{e^{i\theta}-\lambda} = \frac{2(z-\lambda)e^{i\theta}}{(e^{i\theta}-\lambda)(e^{i\theta}-z)},$$

and using (13.43), we obtain

$$\begin{aligned} \mathcal{I} &= \frac{1}{2(z-\lambda)} \left[\int_{\mathbb{T}} \left(\frac{e^{i\theta}+z}{e^{i\theta}-z} - \frac{e^{i\theta}+\lambda}{e^{i\theta}-\lambda} \right) d\mu(e^{i\theta}) \right] \\ &= \frac{1}{2(z-\lambda)} \left[\frac{1+b(z)}{1-b(z)} - \frac{1+b(\lambda)}{1-b(\lambda)} \right] \\ &= \frac{b(z)-b(\lambda)}{(z-\lambda)(1-b(z))(1-b(\lambda))}. \end{aligned}$$

Therefore,

$$\begin{aligned} (V_b C V_b^{-1}) k_\lambda^b(z) &= (1-b(\lambda))(1-b(z)) \frac{b(z)-b(\lambda)}{(z-\lambda)(1-b(z))(1-b(\lambda))} \\ &= \frac{b(z)-b(\lambda)}{z-\lambda}, \end{aligned}$$

This completes the proof. □

It turns out that the conjugation Ω_b coincides with the map $\mathfrak{C}_b$ constructed in Section 20.4. Indeed, according to Theorems 20.6 and 26.17, we have

$$\mathfrak{C}_b k_\lambda^b = \Omega_b k_\lambda^b \qquad (\lambda \in \mathbb{D}).$$

Hence, by antilinearity and density, we get $\mathfrak{C}_b = \Omega_b$.

Corollary 26.18 *Let b be an extreme point of the closed unit ball of H^∞. Then*

$$\mathrm{Span}(\hat{k}_\lambda^b : \lambda \in \mathbb{D}) = \mathcal{H}(b).$$

Proof Since Ω_b is a conjugation, it maps a complete family into a complete family and the result follows from (9.7) and (26.19).

It seems interesting to give another proof of this result. According to Theorem 18.19, it is sufficient to prove that $\mathrm{Span}(S^{*n}b : n \geq 1) = \mathcal{H}(b)$. So, let $f \in \mathcal{H}(b) \ominus \mathrm{Span}(S^{*n}b : n \geq 1)$. Thus, using Theorem 17.8, we have

$$0 = \langle f, S^{*n}b\rangle_b = \langle f, S^{*n}b\rangle_2 + \langle T_{\bar{b}}f, T_{\bar{b}}S^{*n}b\rangle_{\bar{b}} \qquad (n \geq 1).$$

By Corollary 25.2, there is a unique $g \in L^2(\rho)$ such that $T_{\bar{b}}f = K_\rho g$ and $\log|\rho g| \notin L^1(\mathbb{T})$. Moreover,

$$T_{\bar{b}}S^{*n} = -K_\rho Z_\rho^{*n}1 \qquad (n \geq 1). \tag{26.20}$$

Indeed, for $n = 1$, this is precisely (25.3). By induction, assume that (26.20) is valid for n. Then, using (13.28) and (12.3), we have

$$T_{\bar{b}}S^{*(n+1)}b = S^*T_{\bar{b}}S^{*n}b = -S^*K_\rho Z_\rho^{*n}1 = -K_\rho Z_\rho^{*n+1}1.$$

We then get

$$\begin{aligned} 0 &= \langle f, S^{*n}b\rangle_2 - \langle K_\rho g, K_\rho Z_\rho^{*n}1\rangle_{\bar{b}} \\ &= \langle f, S^{*n}b\rangle_2 - \langle g, Z_\rho^{*n}1\rangle_{L^2(\rho)} \\ &= \langle f - \rho g,\ \bar{z}^n b\rangle_2. \end{aligned}$$

This implies that the function $f\bar{b} - \rho g$ belongs to H^2. But, since $P_+\bar{b}f = K_\rho g = P_+\rho g$, this function also belongs to $\overline{H_0^2}$. Thus, $f\bar{b} = \rho g$. In particular, the H^2 function fb is not log-integrable on the unit circle. By Lemma 4.30, this implies that $fb \equiv 0$, and therefore $f \equiv 0$, which concludes the proof. $\square$

The conjugation Ω_b intertwines the operator X_b and its adjoint.

Theorem 26.19 *Let b be an extreme point of the closed unit ball of H^∞. Then*

$$\Omega_b X_b \Omega_b = X_b^*.$$

Proof We have

$$\Omega_b X_b \Omega_b = V_b C V_b^{-1} X_b V_b C V_b^{-1}.$$

According to Theorem 20.12, we have $V_b^{-1}X_bV_b = S_\mu^* I_b$, where $I_b = I - (1 - b(0))k_0 \otimes k_0$. Hence,

$$\Omega_b X_b \Omega_b = V_b C S_\mu^* I_b C V_b^{-1}.$$

It is now easy to check that $CS_\mu^* = S_\mu C$. Remember that, since b is extreme, then $H^2(\mu) = L^2(\mu)$ and S_μ is a unitary operator. More precisely, $S_\mu = Z_\mu$

is the operator of multiplication by χ_1 on $L^2(\mu)$ and S_μ^* is the operator of multiplication by χ_{-1} on $L^2(\mu)$. We thus get

$$\Omega_b X_b \Omega_b = V_b S_\mu C I_b C V_b^{-1}.$$

Now, using the fact that C is a conjugation we have

$$\begin{aligned} C I_b C &= C(I - (1 - b(0))k_0 \otimes k_0)C \\ &= C^2 - (1 - \overline{b(0)})Ck_0 \otimes Ck_0 \\ &= I - (1 - \overline{b(0)})Ck_0 \otimes Ck_0, \end{aligned}$$

which gives

$$\Omega_b X_b \Omega_b = V_b S_\mu V_b^{-1} - (1 - \overline{b(0)})V_b S_\mu C k_0 \otimes V_b C k_0.$$

Since V_b is unitary, so $V_b^{-1} = V_b^*$ and $T(x \otimes y)U = Tx \otimes U^* y$, where T and U are linear operators on H and $x, y \in H$. According to (20.23), it remains to check that

$$(1 - \overline{b(0)})V_b S_\mu C k_0 \otimes V_b C k_0 = (1 - \overline{b(0)})^{-1} k_0^b \otimes S^* b. \tag{26.21}$$

But $Ck_0 = \chi_{-1}$ and according to (20.15), we have

$$V_b S_\mu C k_0 = V_b \chi_1 \chi_{-1} = V_b 1 = (1 - \overline{b(0)})^{-1} k_0^b \tag{26.22}$$

Moreover,

$$V_b C k_0 = \Omega_b V_b k_0 = \Omega_b((1 - \overline{b(0)})^{-1} k_0^b) = (1 - \overline{b(0)})^{-1} \Omega_b k_0^b,$$

and Theorem 26.17 implies that

$$\Omega_b k_0^b = \hat{k}_0^b = S^* b$$

and

$$V_b C k_0 = (1 - \overline{b(0)})^{-1} S^* b. \tag{26.23}$$

Now it is clear that (26.21) follows from (26.22) and (26.23). □

Exercises

Exercise 26.6.1 The purpose of this exercise is to give another construction of the conjugation involved in Theorem 26.17 using the tools of the abstract functional embedding introduced in Section 19.1.

Let b be an extreme point in the closed unit ball of H^∞. We will construct an antilinear surjective isometry from $\mathcal{H}(b)$ onto $\mathcal{H}(b)$ that maps k_λ^b onto $\hat{k}_\lambda^b$ in the language of AFE.

Let $\Pi = (\pi, \pi_*) : L^2 \oplus L^2 \longrightarrow \mathbb{H}$ be an AFE such that $\pi_*^*\pi = b$ and let $W : \mathbb{H} \longrightarrow \mathbb{H}$ be the operator defined (on a dense set) by

$$W(\pi f + \pi_* g) = \pi J g + \pi_* J f,$$

with $J : L^2 \longrightarrow L^2$ being the antilinear map defined by $Jf = \bar{z}\bar{f}$.

(i) Show that W is an antilinear surjective isometry satisfying $WP_{\mathbb{K}} = P_{\mathbb{K}}W$, $W(\pi H^2) = \pi_* H^2_-$ and $W(\pi_* H^2_-) = \pi H^2$.

(ii) For $\chi \in \mathbb{K}$, put

$$\Omega(\pi_*^*\chi) = \pi_*^*(W\chi).$$

Show that Ω defines an antilinear surjective isometry from $\mathcal{H}(b)$ onto $\mathcal{H}(b)$.

(iii) Show that $\pi_*^* P_{\mathbb{K}} \pi_* k_\lambda = k_\lambda^b$.

(iv) Show that

$$\Omega(k_\lambda^b) = \pi_*^* P_{\mathbb{K}} \pi \left(\frac{1}{z - \lambda} \right).$$

(v) Using (19.5), show that $\pi_*^* P_{\mathbb{K}} \pi = P_+ b - b P_+$.

(vi) Conclude that $\Omega(k_\lambda^b) = \hat{k}_\lambda^b$ and finally that $\mathrm{Span}(\hat{k}_\lambda^b : \lambda \in \mathbb{D}) = \mathcal{H}(b)$.

Exercise 26.6.2 Let b be an extreme point of the closed unit ball of H^∞, let $\varphi \in H^\infty$ and define

$$\begin{array}{rccc} T_{\bar{\varphi},b} : & \mathcal{H}(b) & \longrightarrow & \mathcal{H}(b) \\ & f & \longmapsto & T_{\bar{\varphi}} f. \end{array}$$

Prove that

$$T_{\bar{\varphi},b}^* = \Omega_b T_{\bar{\varphi},b} \Omega_b.$$

Hint: Let $(p_n)_n$ be a uniformly bounded sequence of polynomials tending to φ a.e. on $\mathbb{T}$. Note that, if $p_n(z) = \sum_k a_k z^k$, then

$$T_{\bar{p}_n,b} = \sum_k \bar{a}_k X_b^k \quad \text{and} \quad T_{\bar{p}_n,b}^* = \sum_k a_k X_b^{*k}.$$

Then use Theorem 26.19 to get that $T_{\bar{p}_n,b}^* = \Omega_b T_{\bar{p}_n,b} \Omega_b$. Use Exercise 20.1.1 to deduce the result. (Note that the convergence in the strong operator topology of a sequence of operators T_n to an operator T implies the convergence of T_n^* to T^* in the weak operator topology.)

Exercise 26.6.3 Let b be an extreme point of the closed unit ball of H^∞. Assume that $b = ub_1$, where u is an inner function. Then, according to Corollary 18.9 and Theorem 25.20, we know that

$$\mathcal{H}(b) = \mathcal{H}(u) \oplus u\mathcal{H}(b_1) = \mathcal{H}(b_1) \oplus b_1 \mathcal{H}(u). \tag{26.24}$$

Show that $\Omega_b(\mathcal{H}(u)) = b_1\mathcal{H}(u)$ and $\Omega_b(u\mathcal{H}(b_1)) = \mathcal{H}(b_1)$.
Hint: Remember that $\ker T_{\bar{u},b} = \ker T_{\bar{u}} \cap \mathcal{H}(b) = \mathcal{H}(u)$. Use Exercise 26.6.2 to obtain

$$\Omega_b(\mathcal{H}(u)) = \ker T^*_{\bar{u},b}.$$

Hence, $\Omega_b(\mathcal{H}(u)) = \mathcal{R}(T_{\bar{u},b})^\perp$. Then, use (26.24) to get $\mathcal{R}(T_{\bar{u},b}) = \mathcal{H}(b_1)$ and deduce that $\Omega_b(\mathcal{H}(u)) = b_1\mathcal{H}(u)$. Use once more (26.24) and the fact that Ω_b is a conjugation to deduce the second relation $\Omega_b(u\mathcal{H}(b_1)) = \mathcal{H}(b_1)$.

Exercise 26.6.4 Let b be an extreme point of the closed unit ball of H^∞, let $\varphi \in H^\infty$ and let $\psi = T_{\bar{\varphi},b}S^*b$. The purpose of this exercise is to show that

$$\mathrm{Span}(X_b^n\psi : n \geq 0) = \mathcal{H}(b_1),$$

where $b_1 = b/u$ and u is the greatest common inner divisor between the inner factors of φ and b.

(i) Let $h \in \mathcal{H}(b)$. Then show that $h \perp \mathrm{Span}(X_b^n\psi : n \geq 0)$ if and only if $T^*_{\bar{\varphi},b}h \perp \mathrm{Span}(S^{*n+1}b : n \geq 0)$.
(ii) Deduce that this is equivalent to $T^*_{\bar{\varphi},b}h = 0$.
Hint: Use Theorem 18.19 and Corollary 26.18.
(iii) Deduce that

$$\Omega_b(\mathrm{Span}(X_b^n\psi : n \geq 0))^\perp = \mathcal{H}(b) \cap \mathcal{H}(v),$$

where v is the inner factor of φ.
Hint: Use Exercise 26.6.2.
(iv) Let v_1 be the inner factor of b. Show that

$$\Omega_b(\mathrm{Span}(X_b^n\psi : n \geq 0))^\perp = \mathcal{H}(v) \cap \mathcal{H}(v_1) = \mathcal{H}(u).$$

Hint: Use the fact that, if $h \in \Omega_b(\mathrm{Span}(X_b^n\psi : n \geq 0))^\perp$, then $\Omega_b h \in \mathcal{H}(v)$ and, in particular, $\Omega_b h$ is a noncyclic vector of S^* in $\mathcal{H}(b)$. Then apply Theorem 25.17.
(v) Deduce the desired relation.
Hint: Use Exercise 26.6.3.

Notes on Chapter 26

Section 26.1

Theorem 26.1 and Corollary 26.2 are taken from Sarason's book [166]. Corollary 26.3 is due to Fricain [77].

Section 26.2

The contents of this section come from the paper of Lotto and Sarason [123]. Theorems 26.5 and 26.6 have been found independently by Davis and McCarthy [62] using the connection with Toeplitz operators on weighted Hardy spaces. The proofs presented here are due to Lotto and Sarason. Corollary 26.7 is due originally to Lotto [120] but with a different proof. Then, with Sarason, he recovered this result as an immediate consequence of Theorem 26.6.

Information about multipliers of $\mathcal{H}(b)$ for the nonextreme case can be found in [62, 120, 124, 159]; see Sections 24.1 and 28.4. The main source for the extreme case is the paper of Lotto and Sarason [123] already mentioned and the paper of Suárez [180].

Section 26.3

Theorem 26.11 is from Sarason's book. Corollary 26.12 is due to Hitt. See [164] for a direct proof of this result.

Section 26.4

The content of this section is taken from [166].

Section 26.5

Theorem 26.15 is due to Beurling [33]. Theorem 26.16 is due to de Branges and Rovnyak [64, appdx, theorem 1] and [65, problem 73]. We give here a simple version of general results on Hilbert spaces contractions. In particular, since we restrict ourselves to the scalar case, we need to impose restrictions on the defect index of the contractions. Nevertheless, the reader should know that there are more general results. See [138, 184] for an account of the whole theory.

Section 26.6

The notion of conjugation is linked with the notion of complex symmetric operators, which have recently been studied extensively by Balayan, Chevrot, Fricain, Garcia, Poore, Putinar, Ross, Tener, Timotin and Wogen [24, 52, 85–92].

The conjugation Ω_b on $\mathcal{H}(b)$ and Theorem 26.17 are due to Lotto and Sarason [123, sec. 9]. Corollary 26.18, due to Fricain, appears in [77], which contains the second proof given in this text.

Theorem 26.19 is due to Lotto and Sarason [123, lemma 9.2], but the proof is slightly different. Note that this result has been generalized by Suárez [181]. More precisely, he showed that Ω_b intertwines the operator $T_{\bar{\varphi}|\mathcal{H}(b)}$ and its adjoint. Exercise 26.6.1 is from [53]. Exercises 26.6.2, 26.6.3 and 26.6.4 are taken from [181]. When b is an inner function, the X_b-invariant subspaces of $\mathcal{H}(b) = K_b$ are given by Beurling's theorem, and, as we have seen, when b is not an extreme point, they have been characterized by Sarason and coincide with the intersection of Beurling's subspaces with $\mathcal{H}(b)$ (see Theorem 24.31). In [181], Suárez characterized the S^*-invariant subspaces when b is an extreme point, but the situation is dramatically more complicated.

27

Inclusion between two $\mathcal{H}(b)$ spaces

In this chapter, we discuss the inclusion (as sets) of an $\mathcal{H}(b)$ space into another. This question may appear simple at first sight. But it is a difficult question, the answer to which will be based on a geometric representation of $\mathcal{H}(b)$ related to the Sz.-Nagy–Foiaş model and on the commutant lifting theorem.

In Section 27.1, we introduce a new representation of $\mathcal{H}(b)$ spaces, which will be central in the inclusion problem between two $\mathcal{H}(b)$ spaces. In order to treat the nonextreme case and the extreme case simultaneously, we introduce a representation that reduces in the extreme case to the one given in Section 19.2 and in the nonextreme case to the one given in Section 23.6. The intrinsic reason why it is not possible to use in the nonextreme case the representation given in Section 19.2 is the fact that the characteristic function of X_b in the nonextreme case is not scalar, but it is a 2×1 matrix, and then, when we write the model, we should add a coordinate. Two important operators appear in this construction, the operator $\mathscr{S}_b$ and its minimal isometric dilation V_b. It will be proved that X_b is unitarily equivalent to $\mathscr{S}_b^*$.

In Section 27.2, given two functions b_1 and b_2 in the closed unit ball of H^∞, we completely describe the class $\mathscr{I}nt(V_{b_1}, V_{b_2})$ of all operators that intertwine the operators V_{b_1} and V_{b_2}. Then, using the abstract lifting theorem (Theorem 7.50), we deduce, in Section 27.3, a description of $\mathscr{I}nt(\mathscr{S}_{b_1}, \mathscr{S}_{b_2})$. In Section 27.4, we study some relations between different $\mathcal{H}(b)$ spaces. In particular, given two functions b_1 and b_2 in the closed unit ball of H^∞, we give a characterization of $\mathcal{H}(b_1) \subset \mathcal{H}(b_2)$. The proof of this characterization is based on the description obtained before for $\mathscr{I}nt(\mathscr{S}_{b_1}, \mathscr{S}_{b_2})$. In Section 27.5, we focus on the study of $\mathcal{H}(b)$ when b is a rational function in the closed unit ball of H^∞. In particular, we give a nice and simple description of $\mathcal{H}(b)$ in that case. In the last section, we discuss the question of the coincidence between $\mathcal{H}(b)$ spaces and some Dirichlet-type spaces.

27.1 A new geometric representation of $\mathcal{H}(b)$ spaces

Let b be a function in the closed unit ball of H^∞ and let a be defined as follows:

(a) if b is nonextreme, then a denotes the unique outer function whose modulus on $\mathbb{T}$ is $(1-|b|^2)^{1/2}$ and is positive at the origin;
(b) if b is extreme, then $a = 0$.

Recall that $\Delta = (1-|b|^2)^{1/2}$ and let us write

$$\nabla = (1-|b|^2-|a|^2)^{1/2} \qquad (\text{on } \mathbb{T}). \tag{27.1}$$

In particular, we see that, when b is nonextreme, then $\nabla = 0$, and when b is extreme, then $\nabla = \Delta = (1-|b|^2)^{1/2}$. Define

$$\mathscr{H}_b = H^2 \oplus \operatorname{Clos}(aH^2) \oplus \operatorname{Clos}(\nabla L^2) \tag{27.2}$$

equipped with the norm

$$\|(f_1, f_2, f_3)\|_{\mathscr{H}_b}^2 = \|f_1\|_2^2 + \|f_2\|_2^2 + \|f_3\|_2^2 \qquad (f_1, f_2, f_3) \in \mathscr{H}_b.$$

Then, we define the following subspace:

$$\mathscr{M}_b = \{(bf, af, \nabla f) : f \in H^2\}$$

and

$$\mathscr{K}_b = \mathscr{H}_b \ominus \mathscr{M}_b.$$

Note that $\mathscr{M}_b$ is a closed subspace of $\mathscr{H}_b$, and then $\mathscr{K}_b$ is well defined. It is easy to see that

$$\mathscr{K}_b = \{(u, v, w) \in \mathscr{H}_b : \bar{b}u + \bar{a}v + \nabla w \perp H^2\}. \tag{27.3}$$

Moreover, when b is extreme, then

$$\mathscr{H}_b = H^2 \oplus \{0\} \oplus \operatorname{Clos}(\Delta L^2)$$

and

$$\mathscr{K}_b = \{(u, 0, w) \in \mathscr{H}_b : \bar{b}u + \Delta w \perp H^2\}.$$

While, when b is nonextreme, then

$$\mathscr{H}_b = H^2 \oplus H^2 \oplus \{0\}$$

and

$$\mathscr{K}_b = \{(u, v, 0) \in \mathscr{H}_b : \bar{b}u + \bar{a}v \perp H^2\}.$$

Let us verify that

$$\mathfrak{k} = (S^*b, S^*a, \nabla\bar{z}) \in \mathscr{K}_b. \tag{27.4}$$

On the one hand, $\mathfrak{k} \in \mathscr{H}_b$, and on the other, we have

$$\begin{aligned}
&\bar{b}S^*b + \bar{a}S^*a + \nabla^2\bar{z} \\
&\quad = \bar{b}(\bar{z}(b - b(0))) + \bar{a}(\bar{z}(a - a(0))) + (1 - |a|^2 - |b|^2)\bar{z} \\
&\quad = \bar{z}(1 - b(0)\bar{b} - a(0)\bar{a}).
\end{aligned}$$

The last function belongs to H^2_-, which proves that $\mathfrak{k} \in \mathscr{K}_b$.

Define

$$\begin{array}{rccc}
V = V_b : & \mathscr{H}_b & \longrightarrow & \mathscr{H}_b \\
 & (u, v, w) & \longmapsto & (Su, Sv, Zw),
\end{array}$$

where $S : H^2 \longrightarrow H^2$ denotes the forward shift operator on H^2 and $Z : L^2(\mathbb{T}) \longrightarrow L^2(\mathbb{T})$ denotes the forward shift operator on $L^2(\mathbb{T})$. Finally, consider the operator $\mathscr{S}_b$ on $\mathcal{L}(\mathscr{K}_b)$ defined by

$$\mathscr{S}_b = P_{\mathscr{K}_b} V | \mathscr{K}_b. \tag{27.5}$$

In other words, $\mathscr{S}_b$ is the compression of V_b on the subspace $\mathscr{K}_b$. It is immediate to check that V_b is an isometry and $\mathscr{S}_b$ is a contraction. In fact, we can say more.

Lemma 27.1 *Let b be a function in the closed unit ball of H^∞ and let V_b and $\mathscr{S}_b$ be defined as above. Then, V_b is the minimal isometric dilation of $\mathscr{S}_b$.*

Proof It is clear that V_b is an isometric dilation of $\mathscr{S}_b$ because $V_b\mathscr{M}_b \subset \mathscr{M}_b$ and $\mathscr{H}_b = \mathscr{K}_b \oplus \mathscr{M}_b$. By Lemma 7.44, to prove that V_b is minimal, we must show that

$$\mathscr{H}_b = \bigvee_{n \geq 0} V_b^n \mathscr{K}_b. \tag{27.6}$$

Let $f \in \mathscr{H}_b$ with $f \perp V_b^n \mathscr{K}_b$, $n \geq 0$. Applying this property for $n = 0$ gives that $f \perp \mathscr{K}_b$, or equivalently $f \in \mathscr{M}_b$. Thus, there exists $g \in H^2$ such that $f = (bg, ag, \nabla g)$. Now using (27.4), we know that $f \perp V_b\mathfrak{k}$, which gives

$$\begin{aligned}
0 &= \langle bg, zS^*b\rangle_2 + \langle ag, zS^*a\rangle_2 + \langle \nabla g, z\nabla\bar{z}\rangle_2 \\
&= \langle |b|^2 g, 1\rangle_2 - |b(0)|^2 g(0) + \langle |a|^2 g, 1\rangle_2 - |a(0)|^2 g(0) + \langle \nabla^2 g, 1\rangle_2.
\end{aligned}$$

Using that $\nabla^2 + |b|^2 + |a|^2 = 1$, we obtain

$$(1 - |b(0)|^2 - |a(0)|^2) g(0) = 0.$$

But remembering that $1 - |b(0)|^2 - |a(0)|^2 = \|S^*b\|_b^2$ and that b is nonconstant, we must have $1 - |b(0)|^2 - |a(0)|^2 > 0$, whence the above equation implies that $g(0) = 0$. In other words, we can write $g = zg_1$, with $g_1 \in H^2$, and $f = V_b(bg_1, ag_1, \nabla g_1)$. Since V_b is an isometry, we have $(bg_1, ag_1, \nabla g_1) \perp V_b\mathfrak{k}$, and the preceding argument implies that $g_1(0) = 0$ or $g'(0) = 0$. By induction, we get that $g^{(n)}(0) = 0$ for any $n \geq 0$ and, since g is analytic on $\mathbb{D}$, this is possible only if $g \equiv 0$. Thus $f \equiv 0$ and that proves (27.6). $\square$

For each $\lambda \in \mathbb{D}$, we define

$$H_\lambda^b = (k_\lambda^b, -\overline{b(\lambda)} a k_\lambda, -\overline{b(\lambda)} \nabla k_\lambda), \tag{27.7}$$

where we recall that k_λ (respectively k_λ^b) denotes the reproducing kernel of H^2 (respectively $\mathcal{H}(b)$). This is a useful element of $\mathscr{H}_b$.

Lemma 27.2 *Let $\lambda \in \mathbb{D}$. Then $H_\lambda^b \in \mathscr{K}_b$ and, for any $(u,v,w) \in \mathscr{K}_b$, we have*

$$u(\lambda) = \langle (u,v,w), H_\lambda^b \rangle_{\mathscr{H}_b}.$$

Proof Let $f \in H^2$. Then, we have

$$\langle H_\lambda^b, (bf, af, \nabla f) \rangle_{\mathscr{H}_b} = \langle k_\lambda^b, bf \rangle_2 - \overline{b(\lambda)} \langle a k_\lambda, af \rangle_2 - \overline{b(\lambda)} \langle \nabla k_\lambda, \nabla f \rangle_2.$$

Since $k_\lambda^b = (1 - \overline{b(\lambda)} b) k_\lambda$, we get

$$\begin{aligned} \langle H_\lambda^b, (bf, af, \nabla f) \rangle_{\mathscr{H}_b} &= \langle k_\lambda, bf \rangle_2 - \overline{b(\lambda)} \langle b k_\lambda, bf \rangle_2 \\ &\quad - \overline{b(\lambda)} \langle a k_\lambda, af \rangle_2 - \overline{b(\lambda)} \langle \nabla k_\lambda, \nabla f \rangle_2. \end{aligned}$$

Since $\langle k_\lambda, bf \rangle_2 = \overline{b(\lambda) f(\lambda)} = \overline{b(\lambda)} \langle k_\lambda, f \rangle_2$, we can rewrite the last identity as

$$\langle H_\lambda^b, (bf, af, \nabla f) \rangle_{\mathscr{H}_b} = \overline{b(\lambda)} \langle k_\lambda, (1 - |b|^2 - |a|^2 - \nabla^2) f \rangle_2,$$

and the last term is zero by (27.1). Hence, $H_\lambda^b \perp \mathscr{M}_b$, which proves that $H_\lambda^b \in \mathscr{K}_b$. To prove the second part of the lemma, note that

$$(k_\lambda, 0, 0) = H_\lambda^b + \overline{b(\lambda)} (b k_\lambda, a k_\lambda, \nabla k_\lambda), \tag{27.8}$$

and the second term $(b k_\lambda, a k_\lambda, \nabla k_\lambda)$ belongs to $\mathscr{M}_b$. Thus, for any $(u,v,w) \in \mathscr{K}_b$, we have

$$\langle (u,v,w), H_\lambda^b \rangle_{\mathscr{H}_b} = \langle (u,v,w), (k_\lambda, 0, 0) \rangle_{\mathscr{H}_b} = u(\lambda). \qquad \square$$

Equation (27.8) says in particular that

$$H_\lambda^b = P_{\mathscr{K}_b}(k_\lambda, 0, 0).$$

Lemma 27.3 *We have*

$$\operatorname{Span}(H_\lambda^b : \lambda \in \mathbb{D}) = \mathscr{K}_b.$$

Proof Let $(u,v,w) \in \mathscr{K}_b$ be such that $(u,v,w) \perp H_\lambda^b$ for every $\lambda \in \mathbb{D}$. Then, according to Lemma 27.2, we get $u(\lambda) = 0$, for every $\lambda \in \mathbb{D}$, which means that $u \equiv 0$. It remains to prove that $v \equiv 0$ and $w \equiv 0$. Using (27.3), we obtain $\bar{a} v + \nabla w \perp H^2$ and $v \in H^2$. We now distinguish two cases.

First assume that b is extreme. Then, we know that $a = 0$ and $v = 0$. Thus, $\nabla w \perp H^2$, that is, $w \perp \nabla H^2$. But, since b is extreme, $\nabla = \Delta$ and $\operatorname{Clos}(\Delta H^2) = \operatorname{Clos}(\Delta L^2)$, and we finally get that $w = 0$, which concludes the proof in this case.

Assume now that b is nonextreme. Then, we know that $\nabla = 0$ and $w = 0$. Thus, $\bar{a}v \perp H^2$, or $v \perp aH^2$. This implies that $v \perp \mathrm{Clos}(aH^2)$ and, since a is outer, $\mathrm{Clos}(aH^2) = H^2$. Hence, $v \perp H^2$ and $v = 0$. ☐

Put

$$J: \begin{array}{ccc} \mathscr{K}_b & \longrightarrow & \mathcal{H}(b) \\ (u, v, w) & \longmapsto & u. \end{array}$$

We now show that this is a unitary operator.

Theorem 27.4 *The map J is a unitary map from $\mathscr{K}_b$ onto $\mathcal{H}(b)$.*

Proof Note that, by Lemma 27.2, we have

$$\langle H_\mu^b, H_\lambda^b \rangle_{\mathscr{K}_b} = k_\mu^b(\lambda) = \langle k_\mu^b, k_\lambda^b \rangle_b \qquad (\lambda, \mu \in \mathbb{D}). \tag{27.9}$$

This relation says that the map $H_\lambda^b \longmapsto k_\lambda^b$ can be extended by linearity and continuity to an isometry from the span of $\{H_\lambda^b : \lambda \in \mathbb{D}\}$ onto the span of $\{k_\lambda^b : \lambda \in \mathbb{D}\}$. But, since $\mathrm{Span}(H_\lambda^b : \lambda \in \mathbb{D}) = \mathscr{K}_b$ and $\mathrm{Span}(k_\lambda^b : \lambda \in \mathbb{D}) = \mathcal{H}(b)$, we deduce that this map can be extended to an isometry from $\mathscr{K}_b$ onto $\mathcal{H}(b)$. Since k_λ^b is the first component of H_λ^b, we see that this isometry can be expressed as the projection onto the first coordinate. ☐

It turns out that the map J intertwines the operators X_b and $\mathscr{S}_b^*$.

Theorem 27.5 *We have the commutative diagram*

$$\begin{tikzcd} \mathscr{K}_b \arrow[r, "J"] \arrow[d, "\mathscr{S}_b^*"'] & \mathcal{H}(b) \arrow[d, "X_b"] \\ \mathscr{K}_b \arrow[r, "J"'] & \mathcal{H}(b) \end{tikzcd}$$

and thus X_b and $\mathscr{S}_b^*$ are unitarily equivalent.

Proof First, let us check that

$$\mathscr{S}_b^*(u, v, w) = (S^*u, S^*v, \bar{z}w) \qquad (u, v, w) \in \mathscr{K}_b. \tag{27.10}$$

It is easy to see that

$$V_b^*(u, v, w) = (S^*u, S^*v, \bar{z}w) \qquad (u, v, w) \in \mathscr{H}_b.$$

Moreover, we have $\mathscr{M}_b \in \mathrm{Lat}(V_b)$, which gives $\mathscr{K}_b = \mathscr{M}_b^\perp \in \mathrm{Lat}(V_b^*)$. Thus, $\mathscr{S}_b^* = P_{\mathscr{K}_b} V_b^* | \mathscr{K}_b = V_b^* | \mathscr{K}_b$, which gives (27.10). Now, if $(u, v, w) \in \mathscr{K}_b$, then $u \in \mathcal{H}(b)$ and we have

$$J\mathscr{S}_b^*(u, v, w) = J(S^*u, S^*v, \bar{z}w) = S^*u = X_b u = X_b J(u, v, w). \qquad ☐$$

Exercise

Exercise 27.1.1 Let b be a function in the closed unit ball of H^∞, let

$$\mathbb{H}_b = L^2 \oplus \mathrm{Clos}(aL^2) \oplus \mathrm{Clos}(\nabla L^2)$$

and let $\pi : L^2 \longrightarrow \mathbb{H}_b$ and $\pi_* : L^2 \longrightarrow \mathbb{H}_b$ be the linear maps defined by

$$\pi(f) = bf \oplus af \oplus \nabla f \quad \text{and} \quad \pi_*(f) = f \oplus 0 \oplus 0 \qquad (f \in L^2).$$

(i) Show that $\Pi = (\pi, \pi_*) : L^2 \oplus L^2 \longrightarrow \mathbb{H}_b$ is an AFE; see Section 19.1.

(ii) With the notation of Section 19.1 verify that

$$\begin{aligned}\mathbb{K}_b &= H^2 \oplus \mathrm{Clos}(aL^2) \oplus \mathrm{Clos}(\nabla L^2) \ominus \{bf \oplus af \oplus \nabla f : f \in H^2\},\\ \mathbb{K}_b' &= H^2 \oplus \mathrm{Clos}(aH^2) \oplus \mathrm{Clos}(\nabla L^2) \ominus \{bf \oplus af \oplus \nabla f : f \in H^2\}.\end{aligned}$$

In particular, note that $\mathbb{K}_b' = \mathscr{K}_b$.

(iii) Deduce Theorem 27.5 from Theorem 19.8.

27.2 The class $\mathscr{I}nt(V_{b_1}, V_{b_2})$

Given two operators T_1 and T_2 on Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, let us write

$$\mathscr{I}nt(T_1, T_2) = \{X \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2) : XT_1 = T_2X\}$$

for the class of operators $X : \mathcal{H}_1 \longrightarrow \mathcal{H}_2$ that intertwine T_1 and T_2. In this section, given two functions b_1 and b_2 in the closed unit ball of H^∞, we will describe the set $\mathscr{I}nt(V_{b_1}, V_{b_2})$.

We first recall some notation. For $j = 1, 2$, we put:

(i) $\Delta_j = (1 - |b_j|^2)^{1/2}$ a.e. on $\mathbb{T}$;

(ii) (a) if b_j is nonextreme, then a_j denotes the unique outer function whose modulus on $\mathbb{T}$ is $(1 - |b_j|^2)^{1/2}$ and is positive at the origin;

(b) if b_j is extreme, then $a_j = 0$;

(iii) $\nabla_j = (1 - |b_j|^2 - |a_j|^2)^{1/2}$;

(iv) $\mathscr{K}_j = \mathscr{K}_{b_j} = H^2 \oplus \mathrm{Clos}(a_j H^2) \oplus \mathrm{Clos}(\nabla_j L^2)$.

We also recall that

$$\begin{array}{rccc} V_j = V_{b_j} : & \mathscr{K}_j & \longrightarrow & \mathscr{K}_j \\ & (u, v, w) & \longmapsto & (Su, Sv, Zw). \end{array}$$

To describe $\mathscr{I}nt(V_{b_1}, V_{b_2})$, it will be useful to introduce a certain class of matrices. Let

$$Y = \begin{pmatrix} h_{11} & h_{12} & 0 \\ h_{21} & h_{22} & 0 \\ g_1 & g_2 & g_3 \end{pmatrix},$$

where each h_{ij} is a function in H^∞ and each g_j is a function in L^∞ satisfying the following properties:

(I) If b_1 is extreme and b_2 is extreme, then $h_{12} = h_{22} = g_2 = h_{21} = 0$ and $g_1 H^2 \subset \mathrm{Clos}(\nabla_2 L^2)$, $g_3 \mathrm{Clos}(\nabla_1 L^2) \subset \mathrm{Clos}(\nabla_2 L^2)$ and $g_3 = 0$ a.e. on $\nabla_1^{-1}(\{0\})$.

(II) If b_1 is extreme and b_2 is nonextreme, then $g_1 = g_2 = g_3 = h_{12} = h_{22} = 0$.

(III) If b_1 is nonextreme and b_2 is extreme, then $h_{21} = h_{22} = g_3 = 0$ and $g_1 H^2 \subset \mathrm{Clos}(\nabla_2 L^2)$, $g_2 H^2 \subset \mathrm{Clos}(\nabla_2 L^2)$.

(IV) If b_1 is nonextreme and b_2 is nonextreme, then $g_1 = g_2 = g_3 = 0$.

Under the above conditions, we say that the matrix Y is *admissible* for (b_1, b_2). We have thus four cases of admissible matrices, which are restated more explicitly below.

(I) When b_1 and b_2 are extreme, then an admissible matrix is of the form

$$Y = \begin{pmatrix} h_{11} & 0 & 0 \\ 0 & 0 & 0 \\ g_1 & 0 & g_3 \end{pmatrix},$$

with $h_{11} \in H^\infty$, $g_j \in L^\infty(\mathbb{T})$ and $g_1 H^2 \subset \mathrm{Clos}(\nabla_2 L^2)$, $g_3 \mathrm{Clos}(\nabla_1 L^2) \subset \mathrm{Clos}(\nabla_2 L^2)$ and $g_3 = 0$ a.e. on $\nabla_1^{-1}(\{0\})$.

(II) When b_1 is extreme and b_2 is nonextreme, then an admissible matrix is of the form

$$Y = \begin{pmatrix} h_{11} & 0 & 0 \\ h_{21} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

with $h_{11}, h_{21} \in H^\infty$.

(III) When b_1 is nonextreme and b_2 is extreme, then an admissible matrix is of the form

$$Y = \begin{pmatrix} h_{11} & h_{12} & 0 \\ 0 & 0 & 0 \\ g_1 & g_2 & 0 \end{pmatrix},$$

with $h_{11}, h_{12} \in H^\infty$, $g_1, g_2 \in L^\infty(\mathbb{T})$ and $g_1 H^2 \subset \mathrm{Clos}(\nabla_2 L^2)$, $g_2 H^2 \subset \mathrm{Clos}(\nabla_2 L^2)$.

(IV) When b_1 and b_2 are nonextreme, then an admissible matrix is of the form

$$Y = \begin{pmatrix} h_{11} & h_{12} & 0 \\ h_{21} & h_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

with $h_{ij} \in H^\infty$.

Let Y be an admissible matrix and let us write T_Y for the multiplication operator from $\mathscr{H}_1$ to $\mathscr{H}_2$ given by

$$T_Y \begin{pmatrix} u \\ v \\ w \end{pmatrix} = Y \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} h_{11}u + h_{12}v \\ h_{21}u + h_{22}v \\ g_1 u + g_2 v + g_3 w \end{pmatrix}.$$

It is easy to see that T_Y is a linear bounded operator from $\mathscr{H}_1$ into $\mathscr{H}_2$. Note also that

$$T_Y = 0 \quad \Longrightarrow \quad Y = 0.$$

In fact, the condition $g_3 = 0$ a.e. on $\nabla_1^{-1}(\{0\})$ (in the case where b_1 and b_2 are extreme) is imposed to have this property.

Moreover, we claim that

$$T_Y V_1 = V_2 T_Y. \tag{27.11}$$

Indeed, let $(u, v, w) \in \mathscr{H}_1$. Then,

$$\begin{aligned} T_Y V_1 \begin{pmatrix} u \\ v \\ w \end{pmatrix} &= T_Y \begin{pmatrix} zu \\ zv \\ zw \end{pmatrix} \\ &= \begin{pmatrix} h_{11}zu + h_{12}zv \\ h_{21}zu + h_{22}zv \\ g_1 zu + g_2 zv + g_3 zw \end{pmatrix} \\ &= V_2 \begin{pmatrix} h_{11}u + h_{12}v \\ h_{21}u + h_{22}v \\ g_1 u + g_2 v + g_3 w \end{pmatrix} \\ &= V_2 T_Y \begin{pmatrix} u \\ v \\ w \end{pmatrix}. \end{aligned}$$

Equation (27.11) says that $T_Y \in \mathscr{I}nt(V_1, V_2)$ for any admissible matrix Y for (b_1, b_2). It turns out that the converse is also true. To prove it, we need first the following elementary result.

Lemma 27.6 *Let S (respectively Z) be the forward shift on H^2 (respectively L^2). Then*

$$\mathscr{I}nt(S, Z) = \{M_\varphi : \varphi \in L^\infty(\mathbb{T})\},$$

where $M_\varphi : H^2 \longrightarrow L^2$ is defined by $M_\varphi u = \varphi u$, $u \in H^2$.

Proof Let $T \in \mathscr{I}nt(S, Z)$ and define $\varphi = T\chi_0 \in L^2(\mathbb{T})$, where we recall that $\chi_n(z) = z^n$, $n \in \mathbb{Z}$, $z \in \mathbb{T}$. Using the fact that $TS = ZT$, we get that

$T\chi_1 = TS\chi_0 = ZT\chi_0 = Z\varphi = \chi_1\varphi$. By induction, we get $T\chi_n = \chi_n\varphi$, $n \geq 0$. By linearity we deduce that $Tp = p\,\varphi$ for any analytic polynomial p. In particular, we have

$$\|p\,\varphi\|_2 = \|Tp\|_2 \leq \|T\|\,\|p\|_2,$$

for any analytic polynomial p. Since $\|\chi_n f\|_2 = \|f\|_2$, $n \in \mathbb{Z}$, we see that the preceding inequality holds in fact for all trigonometric polynomials. Hence, using Lemma 8.12, $\varphi \in L^\infty(\mathbb{T})$. By continuity and density of the analytic polynomials in H^2, we conclude that $Tu = \varphi u$ for any $u \in H^2$. □

The following shows that the collection $\{T_Y : Y \text{ admissible for } (b_1, b_2)\}$ describes the set $\mathscr{I}nt(V_1, V_2)$.

Theorem 27.7 *Let b_1 and b_2 be two functions in the closed unit ball of H^∞. Let $T \in \mathscr{I}nt(V_1, V_2)$. Then, there exists an admissible matrix Y for (b_1, b_2) such that $T = T_Y$.*

Proof Write the operator $T \in \mathcal{L}(\mathscr{H}_1, \mathscr{H}_2)$ as a matrix with respect to the decompositions $\mathscr{H}_j = H^2 \oplus \mathrm{Clos}(a_j H^2) \oplus \mathrm{Clos}(\nabla_j L^2)$, for $j = 1, 2$, and

$$T = \begin{pmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{pmatrix} : \begin{pmatrix} H^2 \\ \mathrm{Clos}(a_1 H^2) \\ \mathrm{Clos}(\nabla_1 L^2) \end{pmatrix} \longrightarrow \begin{pmatrix} H^2 \\ \mathrm{Clos}(a_2 H^2) \\ \mathrm{Clos}(\nabla_2 L^2) \end{pmatrix},$$

where

$$\begin{aligned} T_{11} &\in \mathcal{L}(H^2, H^2), \\ T_{12} &\in \mathcal{L}(\mathrm{Clos}(a_1 H^2), H^2), \\ T_{13} &\in \mathcal{L}(\mathrm{Clos}(\nabla_1 L^2), H^2), \end{aligned}$$

$$\begin{aligned} T_{21} &\in \mathcal{L}(H^2, \mathrm{Clos}(a_2 H^2)), \\ T_{22} &\in \mathcal{L}(\mathrm{Clos}(a_1 H^2), \mathrm{Clos}(a_2 H^2)), \\ T_{23} &\in \mathcal{L}(\mathrm{Clos}(\nabla_1 L^2), \mathrm{Clos}(a_2 H^2)), \end{aligned}$$

and

$$\begin{aligned} T_{31} &\in \mathcal{L}(H^2, \mathrm{Clos}(\nabla_2 L^2)), \\ T_{32} &\in \mathcal{L}(\mathrm{Clos}(a_1 H^2), \mathrm{Clos}(\nabla_2 L^2)), \\ T_{33} &\in \mathcal{L}(\mathrm{Clos}(\nabla_1 L^2), \mathrm{Clos}(\nabla_2 L^2)). \end{aligned}$$

By hypothesis, for any $(u, v, w) \in H^2 \oplus \mathrm{Clos}(a_1 H^2) \oplus \mathrm{Clos}(\nabla_1 L^2)$, we have

$$TV_1 \begin{pmatrix} u \\ v \\ w \end{pmatrix} = V_2 T \begin{pmatrix} u \\ v \\ w \end{pmatrix},$$

which is equivalent to the following system:

$$(S) \qquad \begin{cases} T_{11}(zu) + T_{12}(zv) + T_{13}(zw) = zT_{11}u + zT_{12}v + zT_{13}w, \\ T_{21}(zu) + T_{22}(zv) + T_{23}(zw) = zT_{21}u + zT_{22}v + zT_{23}w, \\ T_{31}(zu) + T_{32}(zv) + T_{33}(zw) = zT_{31}u + zT_{32}v + zT_{33}w. \end{cases}$$

Taking $u \in H^2$ and $v = w = 0$ gives

$$(S_1) \qquad \begin{cases} T_{11}(zu) = zT_{11}u, \\ T_{21}(zu) = zT_{21}u, \\ T_{31}(zu) = zT_{31}u. \end{cases}$$

The first equation means that $T_{11}S = ST_{11}$. Then, according to Theorem 8.14, there exists $h_{11} \in H^\infty$ such that $T_{11} = T_{h_{11}}$, that is, $T_{11}u = h_{11}u$, $u \in H^2$. The same is true for T_{21}, i.e. there exists $h_{21} \in H^\infty$ such that $T_{21}u = h_{21}u$, $u \in H^2$. Moreover, since $T_{21} : H^2 \longrightarrow \text{Clos}(a_2H^2)$, we must have

$$h_{21}H^2 \subset \text{Clos}(a_2H^2).$$

The third equation in (S_1) gives, using Lemma 27.6, the existence of $g_1 \in L^\infty(\mathbb{T})$ such that $T_{31}u = g_1u$, $u \in H^2$. Moreover, since $T_{31} : H^2 \longrightarrow \text{Clos}(\nabla_2L^2)$, we must have

$$g_1H^2 \subset \text{Clos}(\nabla_2L^2).$$

Now if b_1 is extreme, then $a_1 = 0$ and $\text{Clos}(a_1H^2) = 0$ and we take $h_{12} = h_{22} = g_2 = 0$. If b_1 is nonextreme, then $\text{Clos}(a_1H^2) = H^2$ because a_1 is outer. Taking $u = 0$, $v \in H^2$ and $w = 0$ in (S) gives

$$(S_2) \qquad \begin{cases} T_{12}(zv) = zT_{12}v, \\ T_{22}(zv) = zT_{22}v, \\ T_{32}(zv) = zT_{32}v. \end{cases}$$

Arguing as previously, there exists $h_{12} \in H^\infty$, $h_{22} \in H^\infty$ and $g_2 \in L^\infty(\mathbb{T})$ such that

$$T_{12}v = h_{12}v, \quad T_{22}v = h_{22}v, \quad T_{32}v = g_2v, \quad v \in \text{Clos}(a_1H^2),$$

with

$$h_{22}H^2 \subset \text{Clos}(a_2H^2) \quad \text{and} \quad g_2H^2\text{Clos}(\nabla_2L^2).$$

Finally, if b_1 is nonextreme, then $\nabla_1 = 0$ and we take $h_{13} = h_{23} = g_3 = 0$. If b_1 is extreme, then $a_1 = 0$ and $\nabla_1 = \Delta_1 = (1-|b_1|^2)^{1/2}$. Taking $u = v = 0$ and $w \in \text{Clos}(\nabla_1L^2)$ in (S) gives

$$(S_3) \qquad \begin{cases} T_{13}(zw) = zT_{13}w, \\ T_{23}(zw) = zT_{23}w, \\ T_{33}(zw) = zT_{33}w. \end{cases}$$

Note that $\mathrm{Clos}(\nabla_1 L^2)$ is a reducing subspace of the backward shift Z on $L^2(\mathbb{T})$. Hence, by Lemma 8.15, there exists $h_{13}, h_{23}, g_3 \in L^\infty(\mathbb{T})$ such that

$$T_{13}w = h_{13}w, \quad T_{23}w = h_{23}w, \quad T_{33}w = g_3 w, \quad w \in \mathrm{Clos}(\nabla_1 L^2),$$

with $h_{13}\mathrm{Clos}(\nabla_1 L^2) \subset H^2$, $h_{23}\mathrm{Clos}(\nabla_1 L^2) \subset \mathrm{Clos}(a_2 H^2)$, $g_3\mathrm{Clos}(\nabla_1 L^2) \subset \mathrm{Clos}(\nabla_2 L^2)$. Moreover, it is clear that we can choose g_3 such that $g_3 = 0$ a.e. on $\nabla_1^{-1}(\{0\})$.

Now, let us show that $T_{13} = T_{23} = 0$. The function $\varphi_{13} = h_{13}\nabla_1$ must belong to H^2. Moreover, for any $k \in \mathbb{Z}$,

$$T_{13}(z^k\nabla_1) = h_{13}z^k\nabla_1 = z^k\varphi_{13}.$$

Hence, for any $k \in \mathbb{Z}$, $z^k\varphi_{13} \in H^2$. That necessarily implies that $\varphi_{13} = 0$, whence $T_{13}(z^k\nabla_1) = 0$ and by continuity $T_{13} = 0$. We argue similarly for T_{23}.

Then, take

$$Y = \begin{pmatrix} h_{11} & h_{12} & 0 \\ h_{21} & h_{22} & 0 \\ g_1 & g_2 & g_3 \end{pmatrix}.$$

All the preceding conditions show that the matrix Y is admissible for (b_1, b_2) and we have

$$\begin{aligned} T_Y \begin{pmatrix} u \\ v \\ w \end{pmatrix} &= \begin{pmatrix} h_{11}u + h_{12}v \\ h_{21}u + h_{22}v \\ g_1u + g_2v + g_3w \end{pmatrix} \\ &= \begin{pmatrix} T_{11}u + T_{12}v + T_{13}w \\ T_{21}u + T_{22}v + T_{23}w \\ T_{31}u + T_{32}v + T_{33}w \end{pmatrix} \\ &= T \begin{pmatrix} u \\ v \\ w \end{pmatrix}. \end{aligned}$$

This completes the proof of Theorem 27.7. □

27.3 The class $\mathscr{I}nt(\mathscr{S}_{b_1}, \mathscr{S}_{b_2})$

Given two functions b_1 and b_2 in the closed unit ball of H^∞, we will describe, in this section, the set $\mathscr{I}nt(\mathscr{S}_{b_1}, \mathscr{S}_{b_2})$. We adopt the notation of the preceding section and for $j = 1, 2$, and we put:

(i) $\mathscr{M}_j = \mathscr{M}_{b_j} = \{(b_j f, a_j f, \nabla_j f) : f \in H^2\}$;
(ii) $\mathscr{K}_j = \mathscr{K}_{b_j} = \mathscr{H}_j \ominus \mathscr{M}_j$;
(iii) $\mathscr{S}_j = \mathscr{S}_{b_j} = P_{\mathscr{K}_j} V_j | \mathscr{K}_j$.

Recall that V_j is the minimal isometric dilation of $\mathscr{S}_j$, and to describe $\mathscr{I}nt(\mathscr{S}_{b_1}, \mathscr{S}_{b_2})$, we will use Theorems 27.7 and 7.50. This enables us to introduce the set

$$\mathscr{A} = \{Y \text{ admissible matrix for } (b_1, b_2) : T_Y \mathscr{M}_1 \subset \mathscr{M}_2\}.$$

The following result gives a useful description of this set.

Lemma 27.8 *Let Y be an admissible matrix for (b_1, b_2). Then $Y \in \mathscr{A}$ if and only if there exists $q \in H^\infty$ such that*

$$Y \begin{pmatrix} b_1 \\ a_1 \\ \nabla_1 \end{pmatrix} = \begin{pmatrix} b_2 \\ a_2 \\ \nabla_2 \end{pmatrix} q. \tag{27.12}$$

Proof First assume that there exists $q \in H^\infty$ satisfying (27.12). Since

$$\mathscr{M}_j = \begin{pmatrix} b_j \\ a_j \\ \nabla_j \end{pmatrix} H^2,$$

and, using $qH^2 \subset H^2$, we get

$$T_Y \mathscr{M}_1 = Y \begin{pmatrix} b_1 \\ a_1 \\ \nabla_1 \end{pmatrix} H^2 = \begin{pmatrix} b_2 \\ a_2 \\ \nabla_2 \end{pmatrix} qH^2 \subset \begin{pmatrix} b_2 \\ a_2 \\ \nabla_2 \end{pmatrix} H^2 = \mathscr{M}_2.$$

Hence, $Y \in \mathscr{A}$. Conversely, if $Y \in \mathscr{A}$, then $T_Y \mathscr{M}_1 \subset \mathscr{M}_2$. In particular,

$$T_Y \begin{pmatrix} b_1 \\ a_1 \\ \nabla_1 \end{pmatrix} \in \mathscr{M}_2 = \begin{pmatrix} b_2 \\ a_2 \\ \nabla_2 \end{pmatrix} H^2.$$

Hence, there exists $q \in H^2$ satisfying (27.12). It remains to prove that $q \in L^\infty(\mathbb{T})$. Since, Y is admissible for (b_1, b_2), it has the form

$$Y = \begin{pmatrix} h_{11} & h_{12} & 0 \\ h_{21} & h_{22} & 0 \\ g_1 & g_2 & g_3 \end{pmatrix},$$

with $h_{ij} \in H^\infty$ and $g_\ell \in L^\infty(\mathbb{T})$, $1 \le i, j \le 2$ and $1 \le \ell \le 3$. By (27.12), we have the following system:

$$\begin{cases} b_2 q = h_{11} b_1 + h_{12} a_1, \\ a_2 q = h_{21} b_1 + h_{22} a_1, \\ \nabla_2 q = g_1 b_1 + g_2 a_1 + g_3 \nabla_1. \end{cases}$$

Note that $\varphi := h_{11} b_1 + h_{12} a_1$ and $\varphi_2 := h_{21} b_1 + h_{22} a_1$ belong to H^∞ and the function $\varphi_3 := g_1 b_1 + g_2 a_1 + g_3 \nabla_3 \in L^\infty(\mathbb{T})$. Moreover, we have

$$|b_2|^2 |q|^2 + |a_2|^2 |q|^2 + |\nabla_2|^2 |q|^2 = |\varphi_1|^2 + |\varphi_2|^2 + |\varphi_3|^2,$$

and since $|b_2|^2 + |a_2|^2 + |\nabla_2|^2 = 1$ a.e. on $\mathbb{T}$, we get

$$|q|^2 = |\varphi_1|^2 + |\varphi_2|^2 + |\varphi_3|^2 \qquad \text{(a.e. on } \mathbb{T}\text{)}.$$

From this last equation, we deduce that $q \in L^\infty(\mathbb{T})$. Since q also belongs to H^2, we finally get $q \in H^\infty$. $\square$

Put

$$\begin{array}{rccc} \pi: & \mathscr{A} & \longrightarrow & \mathcal{L}(\mathscr{K}_1, \mathscr{K}_2) \\ & Y & \longmapsto & P_{\mathscr{K}_2} T_Y | \mathscr{K}_1. \end{array}$$

The following result is true in a more general situation (see Exercise 7.11.3) and follows from the fact that V_j is the minimal isometric dilation of $\mathscr{S}_j$ and $T_Y \mathscr{M}_1 \subset \mathscr{M}_2$ when $Y \in \mathscr{A}$, but we will give a direct proof in our case.

Lemma 27.9 *We have*

$$\pi(\mathscr{A}) \subset \mathscr{I}nt(\mathscr{S}_1, \mathscr{S}_2).$$

In other words, for any $Y \in \mathscr{A}$, we have

$$\pi(Y)\mathscr{S}_1 = \mathscr{S}_2 \pi(Y).$$

Proof Let $Y \in \mathscr{A}$. Then Y is admissible for (b_1, b_2) and $T_Y \mathscr{M}_1 \subset \mathscr{M}_2$. Let $h_1 \in \mathscr{K}_1$. We have

$$\begin{aligned} \pi(Y)\mathscr{S}_1 h_1 &= P_{\mathscr{K}_2} T_Y \mathscr{S}_1 h_1 \\ &= P_{\mathscr{K}_2} T_Y P_{\mathscr{K}_1} V_1 h_1 \\ &= P_{\mathscr{K}_2} T_Y V_1 h_1, \end{aligned}$$

the last equality following from the fact that $P_{\mathscr{K}_2} T_Y P_{\mathscr{M}_1} = 0$. Now, since Y is admissible, (27.11) implies that $T_Y V_1 = V_2 T_Y$, whence

$$\begin{aligned} \pi(Y)\mathscr{S}_1 h_1 &= P_{\mathscr{K}_2} V_2 T_Y h_1 \\ &= P_{\mathscr{K}_2} V_2 P_{\mathscr{K}_2} T_Y h_1 + P_{\mathscr{K}_2} V_2 P_{\mathscr{M}_2} T_Y h_1. \end{aligned}$$

But $V_2 \mathscr{M}_2 \subset \mathscr{M}_2$, which gives that $P_{\mathscr{K}_2} V_2 P_{\mathscr{M}_2} = 0$. Therefore,

$$\pi(Y)\mathscr{S}_1 h_1 = P_{\mathscr{K}_2} V_2 P_{\mathscr{K}_2} T_Y h_1 = \mathscr{S}_2 \pi(Y) h_1. \qquad \square$$

The following result is a particular case of the abstract lifting theorem we saw in Chapter 7. It tells us that the range of the map π is indeed all $\mathscr{I}nt(\mathscr{S}_1, \mathscr{S}_2)$.

Theorem 27.10 *Let $X \in \mathscr{I}nt(\mathscr{S}_1, \mathscr{S}_2)$. Then there exists $Y \in \mathscr{A}$ such that*

$$X = \pi(Y) = P_{\mathscr{K}_2} T_Y | \mathscr{K}_1$$

with $\|X\| = \|T_Y\|$.

Proof We recall that V_j is the minimal isometric dilation of $\mathscr{S}_j$. Hence, by Theorem 7.50, there exists $T \in \mathscr{I}nt(V_1, V_2)$ such that $X = P_{\mathscr{K}_2} T|\mathscr{K}_1$, $\|X\| = \|T\|$ and $T\mathscr{M}_1 \subset \mathscr{M}_2$. Now, Theorem 27.7 gives the existence of an admissible matrix Y for (b_1, b_2) such that $T = T_Y$. Since $T_Y \mathscr{M}_1 \subset \mathscr{M}_2$, the matrix Y must belong to $\mathscr{A}$ and we have $X = \pi(Y)$. □

The reader should pay attention to the fact that the matrix Y, which represents an intertwining operator $X \in \mathscr{I}nt(\mathscr{S}_1, \mathscr{S}_2)$ (in Theorem 27.10), is not unique. The following result explicitly gives the kernel of the map π.

Lemma 27.11 *Let Y be an admissible matrix for (b_1, b_2). Then $Y \in \ker \pi$ if and only if there exists $q_1, q_2 \in H^\infty$ such that*

$$Y = \begin{pmatrix} b_2 q_1 & b_2 q_2 & 0 \\ a_2 q_1 & a_2 q_2 & 0 \\ \nabla_2 q_1 & \nabla_2 q_2 & 0 \end{pmatrix}. \tag{27.13}$$

Proof First assume that Y has the form (27.13). Then we easily check that

$$Y \begin{pmatrix} b_1 \\ a_1 \\ \nabla_1 \end{pmatrix} = \begin{pmatrix} b_2 \\ a_2 \\ \nabla_2 \end{pmatrix} h,$$

where $h = b_1 q_1 + q_2 a_1 \in H^\infty$. It thus follows from Lemma 27.8 that $Y \in \mathscr{A}$. Now, for any $h_1 = (u_1, v_1, w_1) \in \mathscr{K}_1$, we have

$$T_Y h_1 = \begin{pmatrix} b_2 q_1 u_1 + b_2 q_2 v_1 \\ a_2 q_1 u_1 + a_2 q_2 v_1 \\ \nabla_2 q_1 u_1 + \nabla_2 q_2 v_1 \end{pmatrix} = \begin{pmatrix} b_2 \\ a_2 \\ \nabla_2 \end{pmatrix} (q_1 u_1 + q_2 v_1).$$

Since $q_1 u_1 + q_2 v_1 \in H^2$, we deduce that $T_Y h_1 \in \mathscr{M}_2$. Hence, $P_{\mathscr{K}_2} T_Y h_1 = 0$, which says that $\pi(Y) = 0$. Conversely, let $Y \in \mathscr{A}$ be such that $\pi(Y) = 0$. Write

$$Y = \begin{pmatrix} h_{11} & h_{12} & 0 \\ h_{21} & h_{22} & 0 \\ g_1 & g_2 & g_3 \end{pmatrix}.$$

By hypothesis, on the one hand, we have $P_{\mathscr{K}_2} T_Y \mathscr{K}_1 = 0$, that is, $T_Y \mathscr{K}_1 \subset \mathscr{M}_2$, and on the other, $T_Y \mathscr{M}_1 \subset \mathscr{M}_2$. Hence, $T_Y \mathscr{H}_1 \subset \mathscr{M}_2$. In particular, for any $h = (u, v, w) \in H^2 \oplus \mathrm{Clos}(a_1 H^2) \oplus \mathrm{Clos}(\nabla_1 L^2)$, there exists $\varphi_h \in H^2$ such that

$$T_Y \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} b_2 \\ a_2 \\ \nabla_2 \end{pmatrix} \varphi_h. \tag{27.14}$$

Applying (27.14) to $u = 1$, $v = w = 0$ gives the existence of $q_1 \in H^2$ such that

$$\begin{cases} h_{11} = b_2 q_1, \\ h_{21} = a_2 q_1, \\ \ \ g_1 = \nabla_2 q_1. \end{cases}$$

Since $|b_2|^2 + |a_2|^2 + |\nabla_2|^2 = 1$ a.e. on $\mathbb{T}$, we have $|q_1|^2 = |h_{11}|^2 + |h_{21}|^2 + |g_1|^2$, which implies that $q_1 \in L^\infty(\mathbb{T})$. Hence, $q_1 \in H^\infty$. Now, if b_1 is extreme, then $h_{12} = h_{22} = g_2 = 0$ and we can take $q_2 = 0$. If b_1 is nonextreme, we have $\mathrm{Clos}(a_1 H^2) = H^2$ and we can apply (27.14) to $u = 0$, $v = 1$ and $w = 0$. There exists $q_2 \in H^2$ such that

$$\begin{cases} h_{12} = b_2 q_2, \\ h_{22} = a_2 q_2, \\ \ \ g_2 = \nabla_2 q_2. \end{cases}$$

As previously, we check that $q_2 \in L^\infty$ and then $q_2 \in H^\infty$. It remains to prove that $g_3 = 0$. By definition of admissible matrices, the only case to consider is when b_1 and b_2 are extreme. Then $g_3 \mathrm{Clos}(\nabla_1 L^2) \subset \mathrm{Clos}(\nabla_2 L^2)$. We can take $u = v = 0$ and $w = \nabla_1$ and apply (27.14). This gives $q_3 \in H^2$ such that

$$\begin{cases} b_2 q_3 = 0, \\ a_2 q_3 = 0, \\ \nabla_2 q_3 = g_3 \nabla_1. \end{cases}$$

Since $b_2 \not\equiv 0$, the first equation gives that $q_3 = 0$. Hence, $g_3 \nabla_1 = 0$. But $g_3 = 0$ a.e. on $\{\zeta \in \mathbb{T} : \nabla_1(\zeta) = 0\}$. Thus, $g_3 = 0$ a.e. We finally get

$$Y = \begin{pmatrix} b_2 q_1 & b_2 q_2 & 0, \\ a_2 q_1 & a_2 q_2 & 0, \\ \nabla_2 q_1 & \nabla_2 q_2 & 0. \end{pmatrix} \qquad \square$$

Exercise

Exercise 27.3.1 Let $Y \in \mathscr{A}$ and define

$$\|Y\|_\infty = \sup_{\zeta \in \mathbb{T}} \|Y(\zeta)\|,$$

where $\|Y(\zeta)\|$ is the norm operator of the 3×3 matrix $Y(\zeta)$.

(i) Show that $(\mathscr{A}, \|\cdot\|_\infty)$ is a Banach space.
(ii) Show that $\|T_Y\| = \|Y\|_\infty$.

27.4 Relations between different $\mathcal{H}(b)$ spaces

The characterization of contractive embeddings (and isometric equalities) of different $\mathcal{H}(b)$ spaces is rather simple.

Theorem 27.12 *Let b_1 and b_2 be two functions in the closed unit ball of H^∞. Then the following hold.*

(i) $\mathcal{H}(b_2) \hookrightarrow \mathcal{H}(b_1)$ *if and only if* $b_1 = b_2 b$ *with some b in the closed unit ball of H^∞.*
(ii) $\mathcal{H}(b_2) \overset{\frown}{=} \mathcal{H}(b_1)$ *if and only if* $b_1 = \gamma b_2$ *for some constant* $\gamma \in \mathbb{T}$.

Proof (i) We appeal to Theorem 16.7. By that result, $\mathcal{H}(b_2) \hookrightarrow \mathcal{H}(b_1)$ if and only if

$$I - T_{b_2} T_{\bar{b}_2} \le I - T_{b_1} T_{\bar{b}_1}.$$

Hence, applying this relation to k_w, $w \in \mathbb{D}$, gives

$$\|k_w\|_2^2 - \|T_{\bar{b}_2} k_w\|_2^2 \le \|k_w\|_2^2 - \|T_{\bar{b}_1} k_w\|_2^2.$$

But, by (12.7), $T_{\bar{b}_j} k_w = \overline{b_j(w)} k_w$, and thus we obtain

$$1 - |b_2(w)|^2 \le 1 - |b_1(w)|^2 \qquad (w \in \mathbb{D}).$$

In other words, we have

$$|b_1(w)| \le |b_2(w)| \qquad (w \in \mathbb{D}),$$

which exactly means that b_1/b_2 is in the closed unit ball of H^∞.

Conversely, assume that $b_1 = b_2 b$, with b in the closed unit ball of H^∞. Then using the fact that $T_b T_{\bar{b}} \le I$, we get that

$$I - T_{b_1} T_{\bar{b}_1} = I - T_{b_2} T_b T_{\bar{b}} T_{\bar{b}_2} \ge I - T_{b_2} T_{\bar{b}_2},$$

and, using Theorem 16.7 once more, we obtain that $\mathcal{H}(b_2) \hookrightarrow \mathcal{H}(b_1)$.

(ii) It follows immediately from (i) that $\mathcal{H}(b_2) \overset{\frown}{=} \mathcal{H}(b_1)$ if and only if $b_1 = b_2 b$, where b and b^{-1} both lie in the closed unit ball of H^∞. Hence, by Corollary 4.24, b must be an outer function and at the same time $|b| = 1$, a.e. on $\mathbb{T}$. Therefore, b is necessarily a constant of modulus one. Conversely, if $b_1 = \gamma b_1$, with $\gamma \in \mathbb{T}$, then, by part (i), we obviously have $\mathcal{H}(b_2) \overset{\frown}{=} \mathcal{H}(b_1)$. $\square$

The situation of set inclusion of different $\mathcal{H}(b)$ spaces appears dramatically more difficult. We first begin with a simple necessary condition.

Lemma 27.13 *Let b_2 and b_1 be two functions in the closed unit ball of H^∞. If $\mathcal{H}(b_2) \subset \mathcal{H}(b_1)$, then*

$$\sup_{z \in \mathbb{D}} \frac{1 - |b_2(z)|^2}{1 - |b_1(z)|^2} < +\infty.$$

Proof According to Lemma 16.6 and Theorem 16.7, the inclusion $\mathcal{H}(b_2) \subset \mathcal{H}(b_1)$ holds if and only if there exists a constant $c > 0$ such that

$$I - T_{b_2}T_{\bar{b}_2} \leq c^2(I - T_{b_1}T_{\bar{b}_1}).$$

Then, arguing as in the proof of Theorem 27.12, we get

$$1 - |b_2(z)|^2 \leq c^2(1 - |b_1(z)|^2) \qquad (z \in \mathbb{D}).$$

□

Corollary 27.14 *Let b_2 and b_1 be two functions in the closed unit ball of H^∞. Assume that $\mathcal{H}(b_2) \subset \mathcal{H}(b_1)$. If b_2 is a nonextreme point of the closed unit ball of H^∞, then b_1 is also a nonextreme point of the closed unit ball of H^∞.*

Proof If $\mathcal{H}(b_2) \subset \mathcal{H}(b_1)$ then it follows from Lemma 27.13 that there is a constant $c > 0$ such that

$$1 - |b_2(z)|^2 \leq c(1 - |b_1(z)|^2) \qquad (z \in \mathbb{D}).$$

This fact implies that

$$1 - |b_2(\zeta)|^2 \leq c(1 - |b_1(\zeta)|^2) \qquad (\text{a.e. } \zeta \in \mathbb{T}).$$

Therefore,

$$\int_{\mathbb{T}} \log(1 - |b_2|^2)\, dm \leq \log c + \int_{\mathbb{T}} \log(1 - |b_1|^2)\, dm$$

and now the result follows from Theorem 6.7. □

To get the complete characterization, we use the three coordinate transcription introduced in Section 27.1 and the commutant abstract lifting theorem proved in Section 27.3.

Given two functions b_1 and b_2 in the closed unit ball of H^∞, recall the notation at the start of Sections 27.1 and 27.2. With that notation, we have the following theorem.

Theorem 27.15 *Let b_2 and b_1 be two functions in the closed unit ball of H^∞. The following assertions are equivalent.*

(i) $\mathcal{H}(b_2) \subset \mathcal{H}(b_1)$;
(ii) *there exists $v, w \in H^\infty$ and $\gamma > 0$ such that*
   (a) $b_1 + va_1 = b_2 w$,
   (b) $\Delta_2^2 \leq \gamma \Delta_1^2$ *a.e. on* $\mathbb{T}$.

Proof (ii) $\Longrightarrow$ (i) First note that, according to Theorem 17.12, the condition (b) implies that $\mathcal{H}(\bar{b}_2) \subset \mathcal{H}(\bar{b}_1)$. Now, let $f \in \mathcal{H}(b_2)$ and let us prove that $f \in \mathcal{H}(b_1)$. Using Theorem 17.8, $f \in \mathcal{H}(b_1)$ if and only if $T_{\bar{b}_1} f \in \mathcal{H}(\bar{b}_1)$. Using condition (a), we can write

$$T_{\bar{b}_1} f = T_{\overline{b_2 w}} f - T_{\overline{va_1}} f.$$

By (12.3), this equation can be rewritten as

$$T_{\bar{b}_1} f = T_{\bar{w}} T_{\bar{b}_2} f - T_{\bar{a}_1} T_{\bar{v}} f.$$

Since $f \in \mathcal{H}(b_2)$, the function $T_{\bar{b}_2} f$ is in $\mathcal{H}(\bar{b}_2)$, a subspace that is invariant under $T_{\bar{w}}$, whence $T_{\bar{w}} T_{\bar{b}_2} f \in \mathcal{H}(\bar{b}_2)$. Since $\mathcal{H}(\bar{b}_2) \subset \mathcal{H}(\bar{b}_1)$, it remains to prove that $T_{\bar{a}_1} T_{\bar{v}} f \in \mathcal{H}(\bar{b}_1)$. But, if b_1 is extreme, then $a_1 = 0$ and we automatically get that $T_{\bar{a}_1} T_{\bar{v}} f \in \mathcal{H}(\bar{b}_1)$. In the other case, when b_1 is nonextreme, then we know that $\mathcal{H}(\bar{b}_1) = \mathcal{M}(\bar{a}_1)$ and we also get that $T_{\bar{a}_1} T_{\bar{v}} f \in \mathcal{H}(\bar{b}_1)$.

(i) $\Longrightarrow$ (ii) According to Lemma 27.13, the set inclusion $\mathcal{H}(b_2) \subset \mathcal{H}(b_1)$ implies that there exists $\gamma > 0$ such that

$$1 - |b_2(z)|^2 \le \gamma(1 - |b_1(z)|^2) \qquad (z \in \mathbb{D}).$$

If we let z tend to the boundary, then we immediately get

$$\Delta_2^2 \le \gamma \Delta_1^2 \qquad \text{(a.e. on } \mathbb{T}),$$

which gives the condition (b). To show the necessity of condition (a) is harder, and we use the three coordinate transcription of the AFE. First, note that, since $\mathcal{H}(b_2) \subset \mathcal{H}(b_1)$, by the closed graph theorem, the injection

$$\begin{array}{rccc} i: & \mathcal{H}(b_2) & \longrightarrow & \mathcal{H}(b_1) \\ & f & \longmapsto & f \end{array}$$

is bounded. In particular, the adjoint operator i^* satisfies

$$i^* k_\lambda^{b_1} = k_\lambda^{b_2} \qquad (\lambda \in \mathbb{D}).$$

Indeed, for any $f \in \mathcal{H}(b_2)$, we have

$$f(\lambda) = \langle f, k_\lambda^{b_1} \rangle_{b_1} = \langle i(f), k_\lambda^{b_1} \rangle_{b_1} = \langle f, i^* k_\lambda^{b_1} \rangle_{b_2},$$

and that proves that $i^* k_\lambda^{b_1} = k_\lambda^{b_2}$. For $\ell = 1, 2$, recall that the operator

$$\begin{array}{rccc} J_\ell: & \mathscr{K}_\ell & \longrightarrow & \mathcal{H}(b_\ell) \\ & (u, v, w) & \longmapsto & u \end{array}$$

is a unitary operator. Let us check that $J_1^* i J_2 \in \mathscr{I}nt(\mathscr{S}_{b_2}^*, \mathscr{S}_{b_1}^*)$. Using Theorem 27.5 twice, we have

$$\begin{aligned} J_1^* i J_2 \mathscr{S}_{b_2}^* &= J_1^* i X_{b_2} J_2 \\ &= J_1^* X_{b_1} i J_2 \\ &= \mathscr{S}_{b_1}^* J_1^* i J_2. \end{aligned}$$

This is equivalent to saying that $J_2^* i^* J_1 \in \mathscr{I}nt(\mathscr{S}_{b_1}, \mathscr{S}_{b_2})$. Hence, by Theorem 27.10, there exists $Y \in \mathscr{A}$ such that

$$J_2^* i^* J_1 = \pi(Y) = P_{\mathscr{K}_2} T_Y | \mathscr{K}_1$$

with $\|X\| = \|T_Y\|$. Now use the fact that $J_\ell H_\lambda^{b_\ell} = k_\lambda^{b_\ell}$, $\ell = 1, 2$, to get

$$J_2^* i^* J_1 H_\lambda^{b_1} = J_2^* i^* k_\lambda^{b_1} = J_2^* k_\lambda^{b_2} = H_\lambda^{b_2}.$$

Thus we obtain

$$P_{\mathscr{K}_2} T_Y H_\lambda^{b_1} = H_\lambda^{b_2} \qquad (\lambda \in \mathbb{D}),$$

which is equivalent to saying that

$$T_Y H_\lambda^{b_1} - H_\lambda^{b_2} \in \mathscr{M}_{b_2}.$$

Applying this with $\lambda = 0$ gives that there exists $m \in H^2$ such that

$$Y H_0^{b_1} = H_0^{b_2} + \begin{pmatrix} b_2 \\ a_2 \\ \nabla_2 \end{pmatrix} m. \tag{27.15}$$

Since the entries of Y are bounded functions and $|b_2|^2 + |a_2|^2 + |\nabla_2|^2 = 1$ a.e. on $\mathbb{T}$, we easily see that $m \in L^\infty(\mathbb{T})$, whence $m \in H^\infty$. Write $H_0^{b_\ell}$ in matrix form as

$$H_0^{b_\ell} = \begin{pmatrix} 1 - \overline{b_\ell(0)} b_\ell \\ -\overline{b_\ell(0)} a_\ell \\ -\overline{b_\ell(0)} \nabla_\ell \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} b_\ell \\ a_\ell \\ \nabla_\ell \end{pmatrix} \overline{b_\ell(0)}.$$

Using this in (27.15) gives

$$Y \left[\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} b_1 \\ a_1 \\ \nabla_1 \end{pmatrix} \overline{b_1(0)} \right] = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} b_2 \\ a_2 \\ \nabla_2 \end{pmatrix} \overline{b_2(0)} + \begin{pmatrix} b_2 \\ a_2 \\ \nabla_2 \end{pmatrix} m.$$

But remember that $Y \in \mathscr{A}$; in particular, $T_Y \mathscr{M}_{b_1} \subset \mathscr{M}_{b_2}$, which means that there exists $m_1 \in H^2$ such that

$$Y \begin{pmatrix} b_1 \\ a_1 \\ \nabla_1 \end{pmatrix} = \begin{pmatrix} b_2 \\ a_2 \\ \nabla_2 \end{pmatrix} m_1.$$

As previously, we have that $m_1 \in H^\infty$ and we obtain

$$(Y - I_3) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} b_2 \\ a_2 \\ \nabla_2 \end{pmatrix} m_2,$$

where $m_2 = \overline{b_1(0)} m_1 - \overline{b_2(0)} + m \in H^\infty$. Writing

$$Y = \begin{pmatrix} h_{11} & h_{12} & 0 \\ h_{21} & h_{22} & 0 \\ g_1 & g_2 & g_3 \end{pmatrix},$$

we get

$$\begin{pmatrix} h_{11}-1 & 0 & 0 \\ h_{21} & 0 & 0 \\ g_1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} b_2 m_2 & 0 & 0 \\ a_2 m_2 & 0 & 0 \\ \nabla_2 m_2 & 0 & 0 \end{pmatrix}.$$

According to Lemma 27.11, the matrix

$$Y_1 = \begin{pmatrix} b_2 m_2 & 0 & 0 \\ a_2 m_2 & 0 & 0 \\ \nabla_2 m_2 & 0 & 0 \end{pmatrix}$$

belongs to $\ker \pi$ and then in particular $Y_1 \in \mathscr{A}$. Hence, $Z = Y - Y_1$ belongs to $\mathscr{A}$. But

$$Z = \begin{pmatrix} h_{11} & 0 & 0 \\ h_{21} & 0 & 0 \\ g_1 & 0 & 0 \end{pmatrix} - \begin{pmatrix} b_2 m_2 & 0 & 0 \\ a_2 m_2 & 0 & 0 \\ \nabla_2 m_2 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & h_{12} & 0 \\ 0 & h_{22} & 0 \\ 0 & g_2 & g_3 \end{pmatrix}.$$

According to Lemma 27.8, there should exist $q \in H^\infty$ such that

$$Z \begin{pmatrix} b_1 \\ a_1 \\ \nabla_1 \end{pmatrix} = \begin{pmatrix} b_2 \\ a_2 \\ \nabla_2 \end{pmatrix} q,$$

which gives

$$\begin{cases} b_1 + h_{12} a_1 = b_2 q, \\ h_{22} a_1 = a_2 q, \\ g_2 a_1 + g_3 \nabla_1 = \nabla_2 q. \end{cases}$$

Hence, the condition (a) is satisfied with $v = h_{12}$ and $w = q$. □

Corollary 27.16 *Let b_1 and b_2 be two extreme points of the closed unit ball of H^∞. Then the following are equivalent.*

(i) *The two sets $\mathcal{H}(b_1)$ and $\mathcal{H}(b_2)$ coincide.*
(ii) *The following conditions hold:*
   (a) *$b_1 b_2^{-1}$ and $b_2 b_1^{-1}$ belong to H^∞;*
   (b) *there exist two constants $c, C > 0$ such that*

$$c(1-|b_1|^2) \le (1-|b_2|^2) \le C(1-|b_1|^2),$$

a.e. on $\mathbb{T}$.

Proof This follows immediately from Theorem 27.15. □

Corollary 27.17 *Let b_1 and b_2 be two nonextreme points of the closed unit ball of H^∞. Then the following are equivalent.*

(i) *The two sets* $\mathcal{H}(b_1)$ *and* $\mathcal{H}(b_2)$ *coincide.*

(ii) *The following conditions hold:*

(a) *there exist* $g_1, g_2, h_1, h_2 \in H^\infty$ *such that*

$$b_1 = g_1 b_2 + h_1 a_1 \quad \textit{and} \quad b_2 = g_2 b_1 + h_2 a_2;$$

(b) $a_1 a_2^{-1}$ *and* $a_2 a_1^{-1}$ *belong to* H^∞.

Proof This follows immediately from Theorem 27.15. □

If $a_1 = 0$ (that is, b_1 is an extreme point of the closed unit ball of H^∞), then the condition (ii)(a) in Theorem 27.15 says simply that $b_1/b_2 \in H^\infty$. However, if b_1 is a nonextreme point, (ii)(a) becomes a weaker requirement. To interpret this, we remember the notion of the spectrum $\sigma(\Theta)$ of an inner function Θ; see Section 5.2.

Corollary 27.18 *Suppose* b_1 *is a nonextreme point of the closed unit ball of* H^∞ *and assume that* b_1 *is continuous on the closed unit disk. Let* b_2 *be a function in the closed unit ball of* H^∞ *and let* Θ_2 *be the inner factor of* b_2. *Then the following are equivalent.*

(i) *We have the set inclusion* $\mathcal{H}(b_2) \subset \mathcal{H}(b_1)$.

(ii) *The following conditions hold:*

(a) *the set* $\{\zeta \in \mathbb{T} : |b_1(\zeta)| = 1\}$ *does not intersect* $\sigma(\Theta_2)$*;*

(b) *there exists* $\gamma > 0$ *such that*

$$\Delta_2^2 \le \gamma \Delta_1^2 \qquad (\textit{a.e. on } \mathbb{T}).$$

Proof Assume first that $\mathcal{H}(b_2) \subset \mathcal{H}(b_1)$. Then Theorem 27.15 implies the existence of two functions $v, w \in H^\infty$ and a constant $\gamma > 0$ such that

$$b_1 + v a_1 = b_2 w \tag{27.16}$$

and

$$\Delta_2^2 \le \gamma \Delta_1^2 \qquad (\text{a.e. on } \mathbb{T}).$$

Assume that there exists $\zeta_0 \in \mathbb{T} \cap \sigma(\Theta_2)$ such that $|b_1(\zeta_0)| = 1$. Note that

$$|a_1(z)|^2 = \exp\left(\int_{\mathbb{T}} \frac{1 - |z|^2}{|z - \zeta|^2} \log(1 - |b(\zeta)|^2)\, dm(\zeta) \right).$$

Since b_1 is continuous on the closed unit disk, it follows from a standard fact on harmonic functions that $|a_1|$ is also continuous on the closed unit disk. Moreover, we have $|a_1(\zeta_0)|^2 = 1 - |b_1(\zeta_0)|^2 = 0$. On the other hand, since $\zeta_0 \in \sigma(\Theta_2)$, there is a sequence $(z_n)_{n \ge 1}$ with $z_n \longrightarrow \zeta_0$ and such that $\Theta_2(z_n) \longrightarrow 0$. Since $|b_2(z_n)| \le |\Theta_2(z_n)|$, $|b_1(z_n)| \longrightarrow |b_1(\zeta_0)| = 1$ and $|a_1(z_n)| \longrightarrow |a_1(\zeta_0)| = 0$, we thus get a contradiction with (27.16). This proves that (i) implies (ii).

Conversely, suppose that (ii) holds. According to Theorem 27.15, it is sufficient to show that 1 is in the ideal of H^∞ generated by a_1 and b_2. If this fails, the corona theorem (see Theorem 5.20) will produce a sequence $(z_n)_n$ in $\mathbb{D}$ with $a_1(z_n) \longrightarrow 0$ and $b_2(z_n) \longrightarrow 0$ as $n \longrightarrow \infty$. Taking a subsequence if necessary, we can also assume that $z_n \longrightarrow \zeta_0$ for some $\zeta_0 \in \mathbb{T}$. We will check that

$$\zeta_0 \in \sigma(\Theta_2) \cap \{\zeta \in \mathbb{T} : |b_1(\zeta)| = 1\},$$

giving the desired contradiction. First, from the continuity of b_1 and the fact that $|a_1|^2 = 1 - |b_1|^2$ on $\mathbb{T}$, we see that $|b_1(\zeta_0)| = 1$, so $\Delta_1(\zeta_0) = 0$. Once again, by continuity of b_1, given $\delta > 0$, there exists an open arc I of $\mathbb{T}$ containing ζ_0 so that $\Delta_1^2 \leq \delta$ on I. Thus, using condition (b), we get

$$|b_2|^2 \geq 1 - \gamma\delta \qquad \text{a.e. on } I.$$

Let us prove that $\zeta_0 \in \sigma(\Theta_2)$. Assume, on the contrary, that $\zeta_0 \notin \sigma(\Theta_2)$. This means that Θ_2 can be analytically continued through a neighborhood of ζ_0 and $|\Theta_2| = 1$ on this neighborhood. Without loss of generality, we can assume that this neighborhood is I. In particular, if $\mathcal{O}_2$ is the outer part of b_2, we get

$$\mathcal{O}_2(z_n) \longrightarrow 0, \quad n \longrightarrow \infty \quad \text{and} \quad |\mathcal{O}_2|^2 \geq 1 - \gamma\delta \quad \text{a.e. on } I.$$

Now, we consider the outer functions $\mathcal{O}_2^{(1)}$ and $\mathcal{O}_2^{(1)}$ whose modulus on $\mathbb{T}$ satisfies

$$|\mathcal{O}_2^{(1)}(\zeta)| = \begin{cases} |\mathcal{O}_2(\zeta)| & \text{if } \zeta \in I, \\ 1 & \text{if } \zeta \notin I, \end{cases}$$

and

$$|\mathcal{O}_2^{(2)}(\zeta)| = \begin{cases} 1 & \text{if } \zeta \in I, \\ |\mathcal{O}_2(\zeta)| & \text{if } \zeta \notin I, \end{cases}$$

so that $\mathcal{O}_2 = \mathcal{O}_2^{(1)}\mathcal{O}_2^{(2)}$. Using again the fact that the Poisson integral Pf of a function $f \in L^1(\mathbb{T})$, which is continuous at a point $\zeta_0 \in \mathbb{T}$, is continuous at this point, we get that $|\mathcal{O}_2^{(2)}|$ is continuous on $\mathbb{D} \cup I$. In particular, we obtain

$$|\mathcal{O}_2^{(2)}(z_n)| \longrightarrow |\mathcal{O}_2^{(2)}(\zeta_0)| = 1.$$

We can conclude finally that

$$\mathcal{O}_2^{(1)}(z_n) \longrightarrow 0 \qquad (n \longrightarrow \infty) \tag{27.17}$$

and

$$|\mathcal{O}_2^{(1)}|^2 \geq 1 - \gamma\delta \qquad (\text{a.e. on } I).$$

On the other hand, since $|\mathcal{O}_2^{(1)}| = 1$ on $\mathbb{T} \setminus I$, we get

$$|\mathcal{O}_2(z)|^2 \geq 1 - \gamma\delta \qquad (z \in \mathbb{D}). \tag{27.18}$$

The property (27.17) and the inequality (27.18) give the desired contradiction. $\square$

Exercises

Exercise 27.4.1 Let b_1 and b_2 be two functions in the closed unit ball of H^∞. We say that an analytic function m on $\mathbb{D}$ is an *isometric multiplier from* $\mathcal{H}(b_1)$ *onto* $\mathcal{H}(b_2)$ if the following map

$$\begin{array}{ccc} \mathcal{H}(b_1) & \longrightarrow & \mathcal{H}(b_2) \\ f & \longmapsto & mf \end{array}$$

is a unitary map from $\mathcal{H}(b_1)$ onto $\mathcal{H}(b_2)$. Assume that $b_1(0) = b_2(0) = 0$ and assume that there exists an isometric multiplier m from $\mathcal{H}(b_1)$ onto $\mathcal{H}(b_2)$. Show that there exists $\lambda \in \mathbb{T}$ such that $b_2 = \lambda b_1$ and m is a unimodular constant.

Hint: Note that $1 \in \mathcal{H}(b_1) \cap \mathcal{H}(b_2)$ (because $b_1(0) = b_2(0) = 0$) and then $m \in \mathcal{H}(b_2)$ and $m^{-1} \in \mathcal{H}(b_1)$. Since multiplication by m is an isometry, we have

$$\langle m, 1 \rangle_{b_2} = \langle 1, m^{-1} \rangle_{b_1}$$

and deduce that $|m(0)| = 1$. Now $\|m\|_2 \leq \|m\|_{b_2} = \|1\|_{b_1} = 1$ and thus $m = m(0)$ is a unimodular constant. Hence, $\mathcal{H}(b_2) = \mathcal{H}(b_1)$, which gives that $b_2 = \lambda b_1$ for some constant $\lambda \in \mathbb{T}$.

Exercise 27.4.2 Let b be a function in the closed unit ball of H^∞ and let $\lambda \in \mathbb{D}$. Define

$$b_\lambda(z) = \frac{\lambda - z}{1 - \bar{\lambda} z} \quad \text{and} \quad \tilde{k}_\lambda = (1 - |\lambda|^2)^{1/2} k_\lambda,$$

where $k_\lambda(z) = (1 - \bar{\lambda} z)^{-1}$.

(i) Prove that $\kappa_\lambda = \tilde{k}_\lambda \circ b$ is an isometric multiplier from $\mathcal{H}(b)$ onto $\mathcal{H}(b_\lambda \circ b)$. (See Exercise 27.4.1 for the definition of isometric multiplier.)
Hint: Apply Theorem 19.24.

(ii) Let b_1 and b_2 be two functions in the closed unit ball of H^∞, neither of which is a unimodular constant. Assume that there exists an isometric multiplier from $\mathcal{H}(b_1)$ onto $\mathcal{H}(b_2)$. Show that $b_2 = \tau \circ b_1$, where τ is an automorphism of the unit disk. Furthermore, if m_1 and m_2 are two such multipliers, then show that $m_2 = \gamma m_1$ for some constant $\gamma \in \mathbb{T}$.

Hint: Let $\lambda_1 = b_1(0)$, $\lambda_2 = b_2(0)$. Then, by (i), $\kappa_1 = \tilde{k}_{\lambda_1} \circ b_1$ is an isometric multiplier from $\mathcal{H}(b_1)$ onto $\mathcal{H}(b_{\lambda_1} \circ b_1)$ and $\kappa_2 = \tilde{k}_{\lambda_2} \circ b_2$ is an isometric multiplier from $\mathcal{H}(b_2)$ onto $\mathcal{H}(b_{\lambda_2} \circ b_2)$. If m is an isometric multiplier from $\mathcal{H}(b_1)$ onto $\mathcal{H}(b_2)$, then $m\kappa_1^{-1}\kappa_2$ is an isometric multiplier from $\mathcal{H}(b_{\lambda_1} \circ b_1)$ onto $\mathcal{H}(b_{\lambda_2} \circ b_2)$. But $(b_{\lambda_1} \circ b_1)(0) = b_{\lambda_1}(\lambda_1) = 0$ and $(b_{\lambda_2} \circ b_2)(0) = b_{\lambda_2}(\lambda_2) = 0$. Then apply Exercise 27.4.1 to get that $b_{\lambda_2} \circ b_2 = \lambda b_{\lambda_1} \circ b_1$ for some constant $\lambda \in \mathbb{T}$ and the multiplier $m\kappa_1^{-1}\kappa_2$ is a unimodular constant.

27.5 The rational case

The study of the rational case is based on the classic Fejér–Riesz theorem.

Theorem 27.19 *Let*

$$w(e^{i\theta}) = \sum_{j=-n}^{n} c_j e^{ij\theta}$$

be a nonzero trigonometric polynomial that assumes nonnegative values for all values of θ. Then there is an analytic polynomial

$$p(z) = \sum_{j=0}^{n} a_j z^j$$

with no zeros on $\mathbb{D}$ such that

$$w(e^{i\theta}) = |p(e^{i\theta})|^2.$$

Proof As a function of the complex variable z, we see that, if

$$w(z) = \sum_{j=-n}^{n} c_j z^j,$$

then w should satisfy $\overline{w(1/\bar{z})} = w(z)$, $z \in \mathbb{T}$. Assuming that $c_{-n} \neq 0$, we see that $s(z) = z^n w(z)$, $z \in \mathbb{C}$, is a polynomial of degree $2n$ and the roots of s occur in pairs $\alpha, 1/\bar{\alpha}$ of equal multiplicity. It follows that

$$w(z) = c \prod_{j=1}^{n} (z - \alpha_j) \left(\frac{1}{z} - \overline{\alpha_j} \right)$$

for some positive constant c and where $\alpha_1, \ldots, \alpha_n$ satisfy $|\alpha_j| \geqslant 1$ for $1 \leq j \leq n$. The desired polynomial p is

$$p(z) = \sqrt{c} \prod_{j=1}^{n} (z - \alpha_j).$$

Note that p is zero-free in $\mathbb{D}$. $\square$

Let b be a rational function that is a nonextreme point of the closed unit ball of H^∞. Let us write $b = p/r$, where p, r are polynomials with $GCD(p, r) = 1$ and $r(0) > 0$. Note that, since $b \in H^\infty$, r has no zeros in $\bar{\mathbb{D}}$. Furthermore, we have

$$|r|^2 - |p|^2 \geq 0 \qquad (\text{on } \mathbb{T}),$$

and, by the Fejér–Riesz theorem, there exists a polynomial q with no zeros on $\mathbb{D}$ and $q(0) > 0$ such that

$$|r|^2 - |p|^2 = |q|^2 \qquad (\text{on } \mathbb{T}).$$

Setting $a = q/r$, we find that $a \in H^\infty$ (since r has no zeros in $\bar{\mathbb{D}}$). By Corollary 4.27, the function a is outer and $a(0) = q(0)/r(0) > 0$. Moreover,

$$|a|^2 + |b|^2 = \frac{|q|^2}{|r|^2} + \frac{|p|^2}{|r|^2} = 1 \qquad (\text{on } \mathbb{T}).$$

Hence, (a, b) is a pair. Note that a is also rational. We have the following useful description of $\mathcal{H}(b)$. We recall that $\mathcal{P}_N$ denotes the set of (analytic) polynomials of degree less than or equal to N.

Theorem 27.20 *Let b be a rational function that is a nonextreme point of the closed unit ball of H^∞. Let $\lambda_1, \dots, \lambda_s$ be the zeros of a on $\mathbb{T}$, and let m_j be the multiplicity of λ_j, $1 \leq j \leq s$. Then the following assertions hold.*

(i)

$$\mathcal{H}(b) = \prod_{j=1}^{s} (z - \lambda_i)^{m_j} H^2 \dot{+} \mathcal{P}_{n-1},$$

where $n = \sum_{j=1}^{s} m_j$ and the sum in the above decomposition is direct.

(ii) *We have*

$$\mathcal{M}(a) = \prod_{j=1}^{s} (z - \lambda_i)^{m_j} H^2.$$

(iii) *The space $\mathcal{M}(a)$ is closed in $\mathcal{H}(b)$.*

(iv) *There exist two constants $c_1, c_2 > 0$ such that, for any $f = \prod_{j=1}^{s} (z - \lambda_i)^{m_j} g + p$, with $g \in H^2$ and $p \in \mathcal{P}_{n-1}$, then*

$$c_1(\|g\|_2 + \|p\|_2) \leq \|f\|_b \leq c_2(\|g\|_2 + \|p\|_2).$$

Proof (i) By the Fejér–Riesz theorem, we can write

$$1 + \left| \prod_{j=1}^{s} (1 - \overline{\lambda}_i z)^{m_j} \right|^2 = |r(z)|^2 \qquad (z \in \mathbb{T}),$$

where r is a polynomial with no roots in $\bar{\mathbb{D}}$ and $r(0) > 0$. Define

$$b_1(z) = \frac{z^n}{r(z)} \quad \text{and} \quad a_1(z) = \frac{\prod_{j=1}^s (1 - \overline{\lambda}_i z)^{m_j}}{r(z)}.$$

Then (a_1, b_1) is a pair. Let us now prove that $\mathcal{H}(b) = \mathcal{H}(b_1)$. We will use Corollary 27.17. First, since a and a_1 have exactly the same zeros on $\mathbb{T}$ (with the same multiplicities), the functions aa_1^{-1} and $a_1 a^{-1}$ belong to H^∞. On the other hand, by Theorem 5.8, b, a, b_1 and a_1 should belong to the disk algebra $\mathcal{A}$. In particular, there exists $z_0 \in \bar{\mathbb{D}}$ such that

$$|b(z_0)| + |a_1(z_0)| = \inf_{z \in \bar{\mathbb{D}}} (|b(z)| + |a_1(z)|).$$

Note that $a_1(z_0) = 0$ if and only if there exists $1 \le \ell \le s$ such that $z_0 = \lambda_\ell$ and on the other hand $|b(\lambda_j)| = 1$ for all $j = 1, \dots, s$. Hence,

$$\inf_{z \in \bar{\mathbb{D}}} (|b(z)| + |a_1(z)|) > 0.$$

According to Theorem 5.20, there exist $g_1, h_1 \in H^\infty$ such that

$$b_1 = g_1 b + h_1 a_1.$$

Likewise there exist $g_2, h_2 \in H^\infty$ such that

$$b = g_2 b_1 + h_2 a.$$

Therefore, Corollary 27.17 implies that $\mathcal{H}(b) = \mathcal{H}(b_1)$. It remains to prove that

$$\mathcal{H}(b_1) = \prod_{j=1}^s (z - \lambda_i)^{m_j} H^2 + \mathcal{P}_{n-1}.$$

Let $f \in H^2$. By Theorems 17.8 and 23.2, we have $f \in \mathcal{H}(b_1)$ if and only if $T_{\bar{b}_1} f = T_{\bar{a}_1} H^2$. Now, for $g \in H^2$,

$$\begin{aligned}
T_{\bar{b}_1} f = T_{\bar{a}_1} g \quad &\iff \quad P_+(\bar{b}_1 f - \bar{a}_1 g) = 0 \\
&\iff \quad \frac{\bar{z}^n}{\bar{r}} f - \frac{\prod_{j=1}^s (1 - \lambda_j \bar{z})^{m_j}}{\bar{r}} g \perp H^2 \\
&\iff \quad \frac{\bar{z}^n}{\bar{r}} \left(f - \prod_{j=1}^s (z - \lambda_j)^{m_j} g \right) \perp H^2 \\
&\iff \quad f - \prod_{j=1}^s (z - \lambda_j)^{m_j} g \perp \frac{z^n}{r} H^2.
\end{aligned}$$

Since $1/r \in H^\infty$, we have $(z^n/r)H^2 = z^n H^2$ and then

$$T_{\bar{b}_1} f = T_{\bar{a}_1} g \quad \iff \quad f - \prod_{j=1}^s (z - \lambda_j)^{m_j} g \perp z^n H^2.$$

Now, note that $(z^n H^2)^\perp = \mathcal{P}_{n-1}$ and we get that $f \in \mathcal{H}(b_1)$ if and only if there exists $g \in H^2$ and $p \in \mathcal{P}_{n-1}$ so that

$$f = \prod_{j=1}^{s} (z - \lambda_j)^{m_j} g + p.$$

This proves the desired decomposition. It remains to prove now that this decomposition is direct. So let $q \in \mathcal{P}_{n-1} \cap \prod_{j=1}^{s}(z-\lambda_j)^{m_j} H^2$. This means that the polynomials q can be written as

$$q = \prod_{j=1}^{s} (z - \lambda_j)^{m_j} g$$

for some $g \in H^2$. But then the rational function $q / \prod_{j=1}^{s}(z-\lambda_j)^{m_j}$ belongs to H^2. This is clearly possible if and only if the poles of this rational function are outside $\bar{\mathbb{D}}$. In particular, this implies that the polynomial q should have a zero of order at least m_j at each point λ_j. Since the degree of q is less than or equal to $n-1 \le \sum_{j=1}^{s} m_j$, this necessarily implies that $q = 0$. Hence the sum is direct.

(ii) Note that a can be written as

$$a = \tilde{a} \prod_{j=1}^{s} (z - \lambda_j)^{m_j},$$

with $\tilde{a}$ being a rational function with no zeros on $\bar{\mathbb{D}}$. In particular, $\tilde{a}^{-1} \in H^\infty$ and

$$\mathcal{M}(a) = \prod_{j=1}^{s} (z - \lambda_j)^{m_j} H^2.$$

(iii) Let $\alpha \in \mathbb{T}$. According to Lemma 24.21, we have

$$\mathcal{M}(a) = T_{1-\bar{\alpha}b} T_{\bar{F}_\alpha} T_{F_\alpha/\bar{F}_\alpha} H^2,$$

where we recall that $F_\alpha = a/(1-\bar{\alpha}b)$. Moreover, by Theorem 24.23, the map $T_{1-\bar{\alpha}b} T_{\bar{F}_\alpha}$ is an isometry from H^2 into $\mathcal{H}(b)$. So, to prove that $\mathcal{M}(a)$ is a closed subspace of $\mathcal{H}(b)$, it is sufficient to prove that $T_{F_\alpha/\bar{F}_\alpha} H^2$ is a closed subspace of H^2. But

$$F_\alpha = \frac{a}{1-\bar{\alpha}b} = \prod_{j=1}^{s} (z - \lambda_j)^{m_j} \frac{\tilde{a}}{1-\bar{\alpha}b}.$$

Note that we actually have

$$\{\zeta \in \mathbb{T} : |b(\zeta)| = 1\} = \{\lambda_1, \dots, \lambda_s\}.$$

Hence, if we choose $\alpha \in \mathbb{T} \setminus \{b(\lambda_1), \dots, b(\lambda_s)\}$, then the function $1 - \bar{\alpha}b$, which is continuous on $\bar{\mathbb{D}}$, cannot vanish and the functions $f = \tilde{a}/(1-\bar{\alpha}b)$

and $1/f$ belong to H^∞. Now put $p(z) = \prod_{j=1}^{s}(z-\lambda_j)^{m_j}$. Then we easily see that

$$\frac{p}{\bar{p}} = \zeta z^n,$$

with $\zeta \in \mathbb{T}$. Therefore,

$$\frac{F_\alpha}{\bar{F}_\alpha} = \zeta \frac{f}{\bar{f}} z^n,$$

which gives

$$T_{F_\alpha/\bar{F}_\alpha} = \zeta T_{1/\bar{f}} T_f S^n.$$

Note now that T_f and $T_{1/\bar{f}}$ are invertible and S^n is an isometry. Consequently, the subspace $T_{1/\bar{f}} T_f S^n H^2$ is a closed subspace of H^2.

(iv) Let us consider the linear map

$$\begin{array}{rccc} \varphi: & \mathcal{M}(a) \times \mathcal{P}_{n-1} & \longrightarrow & \mathcal{H}(b) \\ & (af_1, p) & \longmapsto & f = af_1 + p. \end{array}$$

It follows from (i) that φ is one-to-one and onto. Equip $\mathcal{M}(a) \times \mathcal{P}_{n-1}$ with the norm

$$\|(af_1, p)\|_{\mathcal{M}(a)\times\mathcal{P}_{n-1}} = \|f_1\|_2 + \|p\|_2.$$

Then one easily check that $\mathcal{M}(a) \times \mathcal{P}_{n-1}$ is a Banach space. Moreover, using the fact that $\mathcal{M}(a)$ is contractively contained in $\mathcal{H}(b)$, we have

$$\begin{aligned} \|f\|_b = \|af_1 + p\|_b &\le \|af_1\|_b + \|p\|_b \\ &\le \|af_1\|_{\mathcal{M}(a)} + \|p\|_b = \|f_1\|_2 + \|p\|_b. \end{aligned}$$

But, since $\mathcal{P}_{n-1}$ is a finite-dimensional space, we can find a constant $c_2 \ge 1$ such that, for any polynomial $q \in \mathcal{P}_{n-1}$, we have $\|q\|_b \le c_2\|q\|_2$. Thus, we get

$$\|f\|_b \le c_2(\|f_1\|_2 + \|p\|_2).$$

This proves that φ is continuous. Now, the Banach isomorphism theorem ensures the existence of a constant $c_1 > 0$ such that

$$c_1(\|f_1\|_2 + \|p\|_2) \le \|f\|_b.$$

To conclude, note that, if $f = \prod_{j=1}^{s}(1-\lambda_j)^{m_j} g + p$, then we have $f = a(g/\tilde{a}) + p$, and we obtain that

$$\|f\|_b \asymp \|g/\tilde{a}\|_2 + \|p\|_2 \asymp \|g\|_2 + \|p\|_2,$$

because $1/\tilde{a}$ and $\tilde{a}$ belong to H^∞. ∎

In Chapter 29, we will slightly extend Theorem 27.20. The proof of Theorem 27.20 does not give the explicit decomposition of a given function f in $\mathcal{H}(b)$. In the following, we explain how to obtain this explicit decomposition and give in particular a formula for the polynomial p (which appears in the decomposition). For that purpose, we need the following lemma.

Lemma 27.21 *Let $f \in H^2$ and $\ell \in \mathbb{N}$. Then, there exists $c_\ell > 0$ such that*

$$|f^{(\ell)}(z)| \leq c_\ell \|f\|_2 \frac{1}{(1-|z|^2)^{\ell+1/2}}.$$

Proof For a point $z \in \mathbb{D}$, the function $\partial^\ell k_z / \partial \bar{z}^\ell$ is easily seen to be the kernel function in H^2 for the evaluation functional of the ℓth derivative at z, that is,

$$f^{(\ell)}(z) = \left\langle f, \frac{\partial^\ell k_z}{\partial \bar{z}^\ell} \right\rangle_2 \qquad (f \in H^2). \tag{27.19}$$

Hence, by the Cauchy–Schwarz inequality, we obtain

$$|f^{(\ell)}(z)| \leq \|f\|_2 \left\| \frac{\partial^\ell k_z}{\partial \bar{z}^\ell} \right\|_2.$$

It remains to prove that

$$\left\| \frac{\partial^\ell k_z}{\partial \bar{z}^\ell} \right\|_2 \leq c_\ell \frac{1}{(1-|z|^2)^{\ell+1/2}},$$

for some constant c_ℓ depending only on ℓ. By (27.19), we have

$$\left\| \frac{\partial^\ell k_z}{\partial \bar{z}^\ell} \right\|_2^2 = g^{(\ell)}(z),$$

where $g(u) = \partial^\ell k_z(u) / \partial \bar{z}^\ell$. Using the fact that $k_z(u) = (1-\bar{z}u)^{-1}$, we easily check that

$$g(u) = \frac{\partial^\ell k_z}{\partial \bar{z}^\ell}(u) = \frac{(-1)^\ell \ell! u^\ell}{(1-\bar{z}u)^{\ell+1}}.$$

Hence

$$g^{(\ell)}(u) = \frac{P_\ell(z,u)}{(1-\bar{z}u)^{2\ell+1}},$$

where $P_\ell(z,u)$ is a polynomial in $\bar{z}$ and u. Since $z, u \in \mathbb{D}$, we have

$$|P_\ell(\bar{z},u)| \leq c_\ell \qquad (z,u \in \mathbb{D}),$$

for some constant c_ℓ and then

$$|g^{(\ell)}(z)| \leq \frac{c_\ell}{(1-|z|^2)^{2\ell+1}},$$

which gives the desired estimate. □

Corollary 27.22 *Let b be a rational function that is a nonextreme point of the closed unit ball of H^∞, let $\lambda_1, \ldots, \lambda_s$ be the zeros of a on $\mathbb{T}$, and let $m_1, \ldots, m_s$ be the corresponding multiplicities. Denote $n = \sum_{j=1}^s m_j$. Then, each function f in $\mathcal{H}(b)$ and all its derivative up to order $m_j - 1$ have* *nontangential limits at λ_j, $1 \le j \le s$. Moreover, if $(r_{j,k})_{1\le j\le s,\, 0\le k\le m_j-1}$ denotes the Hermite interpolation polynomials of degree $n-1$ such that*

$$r_{j,k}^{(\ell)}(\lambda_i) = \begin{cases} 1 & \textit{if } j = i \textit{ and } k = \ell, \\ 0 & \textit{otherwise}, \end{cases}$$

then

$$f - \sum_{j=1}^{s} \sum_{k=0}^{m_j-1} f^{(k)}(\lambda_j) r_{j,k}$$

belongs to $\mathcal{M}(a)$.

Proof Let $f \in \mathcal{H}(b)$. Then, according to Theorem 27.20, there exist $g \in H^2$ and $p \in \mathcal{P}_{n-1}$ such that

$$f(z) = \prod_{j=1}^{s} (z-\lambda_j)^{m_j} g + p.$$

Let us prove that, for any $1 \le j \le s$ and any $0 \le k \le m_j - 1$, $f^{(k)}$ has a nontangential limit at λ_j and

$$p^{(k)}(\lambda_j) = f^{(k)}(\lambda_j).$$

We have

$$f(z) = p(z) + (z-\lambda_j)^{m_j} g(z) \prod_{r \ne j} (z-\lambda_r)^{m_r},$$

and, since $|g(z)| = O((1-|z|^2)^{1/2})$, and $|z-\lambda_j|/(1-|z|^2)$ stays bounded as $z \longrightarrow \lambda_j$ nontangentially, we have $|z-\lambda_j|^{m_j} g(z) = O(|z-\lambda_j|^{m_j-1/2})$. Since $m_j \ge 1$, then $|z-\lambda_j|^{m_j-1/2} \longrightarrow 0$ as $z \longrightarrow \lambda_j$. Hence

$$f(z) \longrightarrow p(\lambda_j) \qquad \text{as } z \longrightarrow \lambda_j \text{ nontangentially.}$$

Now, a straightforward computation shows that

$$\begin{aligned} f'(z) = p'(z) + \sum_{j=1}^{s} m_j (z-\lambda_j)^{m_j-1} g(z) \prod_{r\ne j,\, 1\le r\le s} (z-\lambda_r)^{m_r} \\ + \prod_{r=1}^{s} (z-\lambda_r)^{m_r} g'(z). \end{aligned}$$

Using Lemma 27.21, we see that, if $m_j \ge 2$, then

$$f'(z) \longrightarrow p'(\lambda_j) \qquad \text{as } z \longrightarrow \lambda_j \text{ nontangentially.}$$

By induction, we check that

$$f^{(k)}(z) \longrightarrow p^{(k)}(\lambda_j) \quad \text{as } z \longrightarrow \lambda_j \text{ nontangentially, for any } 0 \le k \le m_{j-1}.$$

Now, since p is a polynomial of degree less than or equal to $n-1$, we have

$$p = \sum_{j=1}^{s} \sum_{k=0}^{m_j-1} p^{(k)}(\lambda_j) r_{j,k},$$

and the result follows. □

Exercises

Exercise 27.5.1 Let b be a rational function that is a nonextreme point of the closed unit ball of H^∞, let $\lambda_1, \dots, \lambda_s$ be the zeros of a on $\mathbb{T}$, and let $m_1, \dots, m_s$ be the corresponding multiplicities. Show that, for any $1 \le j \le s$ and $0 \le k \le m_j - 1$, the linear functional

$$f \longmapsto f^{(k)}(\lambda_j)$$

is continuous on $\mathcal{H}(b)$. Then, show that there exist two constants $c_1, c_2 > 0$ such that

$$c_1 \|f\|_b \le \|g\|_2 + \sum_{j=1}^{s} \sum_{k=0}^{m_j-1} |f^{(k)}(\lambda_j)| \le c_2 \|f\|_b,$$

for any $f = ag + p \in \mathcal{H}(b)$, where $g \in H^2$ and $p \in \mathcal{P}_{n-1}$.

Exercise 27.5.2 Let $b(z) = (1 - z^2)/2$, $z \in \mathbb{D}$.

(i) Verify that $a(z) = \frac{1}{2}(z - i)(z + i)$,

(ii) Show that

$$\mathcal{H}(b) = (z - i)(z + i)H^2 \dot{+} \mathcal{P}_1.$$

Exercise 27.5.3 Let $b(z) = \frac{1}{4}(z + 1)^2$.

(i) Show that b is a nonextreme point of the closed unit ball of H^∞.

(ii) Use the Riesz–Fejér algorithm to compute the associated function a.

(iii) Describe $\mathcal{H}(b)$.

27.6 Coincidence between $\mathcal{H}(b)$ and $\mathcal{D}(\mu)$ spaces

In this section, we discuss the coincidence between de Branges–Rovnyak and Dirichlet-type spaces. We first review some basic facts on Dirichlet spaces that we need.

Let μ be a finite positive Borel measure on $\mathbb{T}$. We recall that $P\mu$ denotes the Poisson integral of μ, that is

$$(P\mu)(z) = \int_{\mathbb{T}} \frac{1-|z|^2}{|z-\zeta|^2}\, d\mu(\zeta)$$

(see Section 3.4). Then we define the Dirichlet-type space $\mathcal{D}(\mu)$ as the space of functions $f \in H^2$ such that

$$D_\mu(f) = \frac{1}{\pi} \int_{\mathbb{D}} |f'(z)|^2 (P\mu)(z)\, dA(z) < +\infty.$$

Here dA is the area measure $dA(z) = dx\,dy = r\,dr\,d\vartheta$, $z = x+iy = re^{i\vartheta}$. The space $\mathcal{D}(\mu)$ is equipped with the norm

$$\|f\|_\mu^2 = \|f\|_2^2 + D_\mu(f), \tag{27.20}$$

which makes it a Hilbert space. Note that, if $\mu = m$, the normalized Lebesgue measure on $\mathbb{T}$, then $Pm = 1$ and $\mathcal{D}(m)$ is the classic Dirichlet space. An important property of $\mathcal{D}(\mu)$ spaces is that the polynomials are dense in $\mathcal{D}(\mu)$.

For the following result, we recall that

$$\hat{\mu}(k) = \int_{\mathbb{T}} e^{-ikt}\, d\mu(e^{it}) \qquad (k \in \mathbb{Z}).$$

Lemma 27.23 *Let μ be a finite positive Borel measure on $\mathbb{T}$. Then we have*

$$\|z^n\|_\mu^2 = 1 + n\mu(\mathbb{T}) \qquad (n \geq 0)$$

and

$$\langle z^{n+k}, z^n \rangle_\mu = n\hat{\mu}(-k) \qquad (n \geq 0,\ k \geq 1).$$

Proof By (27.20) and the polarization identity, we have

$$\langle f, g\rangle_\mu = \langle f, g\rangle_2 + \frac{1}{\pi}\int_{\mathbb{D}} f'(z)\overline{g'(z)} P\mu(z)\, dA(z) \qquad (f, g \in \mathcal{D}(\mu)).$$

It is thus sufficient to prove that, for $f(z) = z^{n+k}$ and $g(z) = z^n$, then

$$\frac{1}{\pi}\int_{\mathbb{D}} f'(z)\overline{g'(z)} P\mu(z)\, dA(z) = n\hat{\mu}(-k) \qquad (n \geq 0,\ k \geq 0). \tag{27.21}$$

Write

$$\begin{aligned}
&\frac{1}{\pi}\int_{\mathbb{D}} f'(z)\overline{g'(z)} P\mu(z)\, dA(z) \\
&\quad = \frac{1}{\pi}\int_{\mathbb{D}} (n+k) z^{n+k-1} n \bar{z}^{n-1} P\mu(z)\, dA(z) \\
&\quad = \frac{n(n+k)}{\pi} \int_0^1 \int_0^{2\pi} r^{2(n-1)+k} e^{ik\vartheta} (P\mu)(re^{i\vartheta}) r\, dr\, d\vartheta \\
&\quad = \frac{n(n+k)}{\pi} \int_0^1 r^{2n+k-1} \left(\int_0^{2\pi} e^{ik\vartheta} (P\mu)(re^{i\vartheta})\, d\vartheta \right) dr.
\end{aligned}$$

Using (3.15), we have

$$\begin{aligned}
&\int_0^{2\pi} e^{ik\vartheta}(P\mu)(re^{i\vartheta})\,d\vartheta \\
&\quad = \hat{\mu}(0)\int_0^{2\pi} e^{ik\vartheta}\,d\vartheta + \sum_{n=1}^{+\infty}\hat{\mu}(-n)r^n\int_0^{2\pi} e^{i(k-n)\vartheta}\,d\vartheta \\
&\qquad + \sum_{n=1}^{+\infty}\hat{\mu}(n)r^n\int_0^{2\pi} e^{i(k+n)\vartheta}\,d\vartheta \\
&\quad = 2\pi\hat{\mu}(-k)r^k.
\end{aligned}$$

Hence

$$\begin{aligned}
\frac{1}{\pi}\int_{\mathbb{D}} f'(z)\overline{g'(z)}P\mu(z)\,dA(z) &= 2n(n+k)\hat{\mu}(-k)\int_0^1 r^{2(n+k)-1}\,dr \\
&= n\hat{\mu}(-k),
\end{aligned}$$

which proves (27.21) and the result. □

In the following result, for $\lambda \in \mathbb{T}$, we denote by δ_λ the Dirac measure at point λ, that is, for any Borel subset $A \subset \mathbb{T}$, we have $\delta_\lambda(A) = 1$ if $\lambda \in A$ and 0 otherwise.

Theorem 27.24 *Let b be a function in the closed unit ball of H^∞ and let μ be a finite positive Borel measure on $\mathbb{T}$. Then the following are equivalent:*

(i) $\mathcal{H}(b) \overset{=}{=} \mathcal{D}(\mu)$;
(ii) $\mu = c\delta_\lambda$ *and*

$$b(z) = \frac{\sqrt{\tau}\,\alpha\bar{\lambda}z}{1-\tau\bar{\lambda}z},$$

where $\lambda \in \mathbb{T}$, $c \geq 0$, $\alpha \in \mathbb{C}$ with $|\alpha|^2 = c$ and $0 < \tau \leq 1$ with $\tau + 1/\tau = 2 + c$.

Proof (i) $\Longrightarrow$ (ii) Note first that polynomials belong to $\mathcal{D}(\mu)$ and therefore also to $\mathcal{H}(b)$. By Corollary 25.10, this implies that b is not an extreme point of the closed unit ball of H^∞. Thus, let a be its Pythagorean mate and write

$$\phi(z) = \frac{b(z)}{a(z)} = \sum_{j=0}^{+\infty} c_j z^j \qquad (z \in \mathbb{D}).$$

We would like to determine the coefficients c_j. Since $\|z^n\|_b = \|z^n\|_\mu$ for all $n \geq 0$, Lemma 27.23 and Theorem 24.12 imply that

$$1 + \sum_{j=0}^{n} |c_j|^2 = 1 + n\mu(\mathbb{T}) \qquad (n \geq 0).$$

Hence $c_0 = 0$ and $|c_j|^2 = \mu(\mathbb{T})$ for all $j \geq 1$. Also, since $\langle z^{n+1}, z^n\rangle_b = \langle z^{n+1}, z^n\rangle_\mu$ for all $n \geq 0$, Lemma 27.23 and Theorem 24.12 also imply that

$$\sum_{j=0}^{n} \overline{c_{j+1}} c_j = n\hat{\mu}(-1) \qquad (n \geq 0).$$

Hence $\overline{c_{j+1}} c_j = \hat{\mu}(-1)$ for all $j \geq 1$. Putting these facts together, it follows that we can write c_j as

$$c_j = \alpha \bar{\lambda}^j \qquad (j \geq 1),$$

where $\lambda \in \mathbb{T}$ and $\alpha \in \mathbb{C}$ with $|\alpha|^2 = \mu(\mathbb{T})$. Hence

$$\phi(z) = \sum_{j=0}^{+\infty} c_j z^j = \alpha \sum_{j=1}^{+\infty} \bar{\lambda}^j z^j = \frac{\alpha\bar{\lambda}z}{1 - \bar{\lambda}z} \qquad (z \in \mathbb{D}).$$

Since $\phi = b/a$ and $|a|^2 + |b|^2 = 1$ a.e. on $\mathbb{T}$, it follows that

$$|a(z)|^2 = \frac{1}{1 + |\phi(z)|^2} = \frac{|1 - \bar{\lambda}z|^2}{|1 - \bar{\lambda}z|^2 + |\alpha|^2} \qquad \text{(a.e. on } \mathbb{T}).$$

A straightforward computation shows that

$$|1 - \bar{\lambda}z|^2 + |\alpha|^2 = \tau^{-1}|1 - \tau\bar{\lambda}z|^2 \qquad (z \in \mathbb{T}),$$

where $\tau \in (0, 1]$ is chosen so that $\tau + 1/\tau = 2 + c$ with $c = |\alpha|^2 = \mu(\mathbb{T})$. As a is an outer function, it follows that

$$a(z) = \sqrt{\tau}\frac{1 - \bar{\lambda}z}{1 - \tau\bar{\lambda}z} \qquad (z \in \mathbb{D}).$$

Hence, finally,

$$b(z) = a(z)\phi(z) = \frac{\sqrt{\tau}\,\alpha\bar{\lambda}z}{1 - \tau\bar{\lambda}z} \qquad (z \in \mathbb{D}).$$

It remains to determine μ. Using Lemma 27.23 and Theorem 24.12 once again, we have

$$\begin{aligned}\hat{\mu}(-k) &= \langle z^{k+1}, z\rangle_\mu = \langle z^{k+1}, z\rangle_b \\ &= \sum_{j=0}^{1} \overline{c_{j+k}} c_j = |\alpha|^2 \lambda^k = c\lambda^k\end{aligned}$$

for any $k \geq 1$. Since μ is a real measure, the same relation holds for all $k \leq -1$ and clearly it is also true for $k = 0$ because $c = \mu(\mathbb{T}) = \hat{\mu}(0)$. Thus

$$\hat{\mu}(-k) = c\lambda^k = \widehat{c\delta_\lambda}(-k) \qquad (k \in \mathbb{Z}).$$

Hence we conclude that $\mu = c\delta_\lambda$, which completes the proof of the first implication.

(ii) $\Longrightarrow$ (i) Note that, with the given choice of b and μ, working back through the computations above, we get

$$\langle z^{n+k}, z^n \rangle_b = \langle z^{n+k}, z^n \rangle_\mu \qquad (n, k \geq 0).$$

That clearly implies that, for any polynomials p, we have

$$\|p\|_b = \|p\|_\mu. \tag{27.22}$$

Now, if $f \in \mathcal{H}(b)$, using the density of the polynomials in $\mathcal{H}(b)$ (note that the given b is a nonextreme point), there is a sequence of polynomials p_n such that $p_n \longrightarrow f$ in $\mathcal{H}(b)$. By (27.22), it follows that the sequence $(p_n)_n$ is a Cauchy sequence in $\mathcal{D}(\mu)$. Hence, there is a function $g \in \mathcal{D}(\mu)$ such that $p_n \longrightarrow g$ in $\mathcal{D}(\mu)$. In particular, $p_n \longrightarrow g$ in H^2 and $p_n \longrightarrow f$ in H^2. We thus get that $f = g$ and $\mathcal{H}(b) \subset \mathcal{D}(\mu)$. Moreover, using (27.22), we also obtain that

$$\|f\|_b = \|g\|_\mu = \|f\|_\mu \qquad (f \in \mathcal{H}(b)).$$

Reversing the roles of $\mathcal{H}(b)$ and $\mathcal{D}(\mu)$ in the preceding argument gives finally that $\mathcal{D}(\mu) = \mathcal{H}(b)$. □

Notes on Chapter 27

The characterization of the inclusion between two $\mathcal{H}(b)$ spaces is due to Ball and Kriete. A large part of this chapter is taken from their paper [25], where they study the vector-valued situation. As we have seen, their method is based on the relation between $\mathcal{H}(b)$ spaces and the functional embedding. It would be interesting to have a direct proof of the implication (i) $\Longrightarrow$ (ii) in Theorem 27.15.

Section 27.1

The three coordinate transcription is due to Ball and Kriete [25], who gave an analog of Theorem 27.5 in the vector-valued case.

Section 27.2

Theorem 27.7 is due to Ball and Kriete and appears in [25] but without proof.

Section 27.3

Theorem 27.10 is a particular case of the abstract lifting theorem due to Sz.-Nagy and Foiaş [183] and Sarason [158]; see Section 7.11.

Section 27.4

Theorem 27.12 is due to de Branges and Rovnyak [65, theorem 10]. Theorem 27.15 is due to Ball and Kriete [25], who obtained this result in the more general context of vector-valued functions. Corollaries 27.16 and 27.18 are also due to Ball and Kriete. Exercises 27.4.1 and 27.4.2 are taken from [61].

Section 27.5

Fejér [74] was the first to realize the importance of the class of trigonometric polynomials that assume only nonnegative real values. His conjecture on the form of such functions was proved by F. Riesz, and Theorem 27.19 is known nowadays as the Fejér–Riesz theorem. The construction of outer functions in H^2, starting from a nonnegative function $w \in L^1(\mathbb{T})$ and such that $\log w \in L^1(\mathbb{T})$, can be viewed as a generalization of this fact. Operator extensions of the Fejér–Riesz theorem were also proved by several authors, the final form being that given by Rosenblum [154]. These extensions to operator-valued functions arise in linear prediction theory.

Theorem 27.20 is due to Costara and Ransford, who explore in [58] which de Branges–Rovnyak spaces coincide with another class of analytic reproducing kernel Hilbert space, the so-called Dirichlet-type spaces $\mathcal{D}(\mu)$. Another proof that does not use Corollary 27.16 is due to Blandignères, Fricain, Gaunard, Hartmann and Ross [35]. Let us also mention the paper of Chacón, Fricain and Shabankhah [49], who obtain a similar description of $\mathcal{D}(\mu)$ spaces when μ is a finite sum of dirac measures on $\mathbb{T}$. It turns out (due to the result of Costara and Ransford) that, in that case, the space $\mathcal{D}(\mu)$ coincides with a de Branges–Rovnyak space $\mathcal{H}(b)$ for some rational nonextreme function. A version of Theorem 27.20 in the case when $b(z) = (1+z)/2$ is due to Sarason [159]. The fact that the sum is direct in the decomposition of $\mathcal{H}(b)$ in Theorem 27.20 was noticed by Blandignères *et al.* [35]. The description of the orthogonal complement of $\mathcal{M}(a)$ in $\mathcal{H}(b)$ (in the case when b is rational) was recently done by Fricain, Hartmann and Ross [79].

Section 27.6

The Dirichlet-type spaces $\mathcal{D}(\mu)$, introduced in this section, arise naturally in the study of 2-isometries, which form a natural class of operators whose prominent member is the Dirichlet shift, the operator of multiplication by the independent variable on the classic Dirichlet space. In [149], Richter shows that the restriction of the shift to $\mathcal{D}(\mu)$ spaces is a model for the class of 2-isometries. The $\mathcal{D}(\mu)$ spaces have recently received a lot of attention in a series of papers by Chacón [46–48], Chacón, Fricain and Shabankhah [49], Chevrot, Guillot and Ransford [54], Costara and Ransford [58], Richter and Sundberg [150–152] and Sarason [167–169, 171].

Sarason [167] proved that, for $b(z) = \sqrt{\tau}\,\bar{\lambda}z/(1-\tau\bar{\lambda}z)$, with $\tau \in (0,1]$, such that $\tau + 1/\tau = 3$, then $\mathcal{H}(b) \doteq \mathcal{D}(\delta_\lambda)$. This result was then completed by Chevrot, Guillot and Ransford [54], who proved the converse of Sarason's result; see Theorem 27.24. In their paper, Chevrot *et al.* asked for which b and μ do we have $\mathcal{H}(b) = \mathcal{D}(\mu)$ without equality of norms. This natural question seems to be more delicate. Some recent results in this direction appeared in a paper of Costara and Ransford [58]. In particular, they proved that, if (a, b) is a rational pair and a has only simple zeros on $\mathbb{T}$, then one can find a measure μ, which is indeed a finite sum of Dirac measures, such that $\mathcal{H}(b) = \mathcal{D}(\mu)$. However, this question remains wide open.

28

Topics regarding inclusions $\mathcal{M}(a) \subset \mathcal{H}(\bar{b}) \subset \mathcal{H}(b)$

According to Theorem 17.8, the inclusion $\mathcal{H}(\bar{b}) \subset \mathcal{H}(b)$ always holds. Moreover, if b is a nonextreme point of the closed unit ball of H^∞ with Pythagorean mate a, then, by Theorem 23.2, we have

$$\mathcal{M}(a) \hookrightarrow \mathcal{M}(\bar{a}) \doteq \mathcal{H}(\bar{b}) \hookrightarrow \mathcal{H}(b).$$

Throughout this chapter, we explore some conditions that ensure $\mathcal{H}(\bar{b}) = \mathcal{H}(b)$ or $\mathcal{H}(b) = \mathcal{M}(a)$. In the case where $\|b\|_\infty < 1$, then $\mathcal{H}(\bar{b}) = \mathcal{H}(b) = H^2$ and the three norms are equivalent. On the contrary, in the case where b is a nonconstant inner function, then $\mathcal{H}(b) = H^2 \ominus bH^2 = K_b$ and $\mathcal{H}(\bar{b}) = \{0\}$, and these two spaces do not coincide (see Section 18.1).

In Sections 28.1 and 28.2, we characterize the equality $\mathcal{H}(\bar{b}) = \mathcal{H}(b)$ in terms of the function b and also in terms of the contraction X_b. As we will see, the situation is different depending on whether b is an extreme point of the closed unit ball of H^∞ or not. In the nonextreme case, we will see that there is a connection with the *corona problem*. As a by product of the identity $\mathcal{H}(\bar{b}) = \mathcal{H}(b)$, we will show that, when b is not an inner function, then $\mathcal{H}(b)$ has nonconstant multipliers. We also get that, when b is an outer function, then b is a multiplier of $\mathcal{H}(b)$ if and only if $\mathcal{H}(b^r) = \mathcal{H}(b)$ for all $r > 0$. In Section 28.3, we pursue the study, started in Section 26.2, of multipliers of $\mathcal{H}(b)$ when b is an extreme point of the closed unit ball of H^∞. In particular, we show that there is a connection between the spectrum of b (as an element of the algebra H^∞) and the multipliers of $\mathcal{H}(b)$. This study of multipliers uses a lot the result of the previous sections on the equality $\mathcal{H}(\bar{b}) = \mathcal{H}(b)$. We also prove that, when b is an extreme point that is not an inner function and such that b is continuous on $\mathbb{T}$, then the set of multipliers of $\mathcal{H}(b)$ coincide with the set of multipliers of $\mathcal{H}(\bar{b})$. In Section 28.4, we study the equality $\mathcal{H}(b) = \mathcal{M}(a)$, and we obtain different characterizations in terms of pairs (a, b), in terms of Toeplitz operators, in terms of multipliers and in terms of similarity for the operator S_b. In Section 28.5, we discuss the problem of invariant subspaces of $S_b = S_{|\mathcal{H}(b)}$, when b is a nonextreme point.

In Section 24.7, we described the invariant subspaces of X_b when b is a nonextreme point. The situation is dramatically more complicated for S_b. For a particular class of nonextreme points, we give a characterization of invariant subspaces of S_b, and then we show that this characterization cannot be valid in general by studying the case $b(z) = (1 + z)/2$. In Section 28.6, we give a characterization of the density of $\mathcal{M}(a)$ in $\mathcal{H}(b)$ in terms of quasi-similarity of S_b and S. The closedness of $\mathcal{M}(a)$ in $\mathcal{H}(b)$ is studied in Section 28.7. This is connected to the property of the Toeplitz operator $T_{F_\lambda/\bar{F}_\lambda}$, where $F_\lambda = a/(1-\bar{\lambda}b)$, $\lambda \in \mathbb{T}$. In Section 28.8, we completely describe the boundary eigenvalue of S_b^* and also the corresponding eigenvectors. In Section 28.9, when $\mathcal{M}(a)$ is not dense in $\mathcal{H}(b)$, we introduce the space $\mathcal{H}_0(b) = \mathcal{H}(b) \ominus \overline{\mathcal{M}(a)}$ and the compression of S_b to $\mathcal{H}_0(b)$, denoted by S_0. The spectrum of S_0 is studied in Section 28.10.

28.1 A necessary and sufficient condition for $\mathcal{H}(\bar{b}) = \mathcal{H}(b)$

The set identity $\mathcal{H}(\bar{b}) = \mathcal{H}(b)$ demands several restrictions. For example, according to Lemma 16.6, if $\mathcal{H}(\bar{b}) = \mathcal{H}(b)$, then there are two positive constants c and C such that

$$c\,\|f\|_{\bar{b}} \le \|f\|_b \le C\,\|f\|_{\bar{b}} \tag{28.1}$$

for all functions $f \in \mathcal{H}(b)$. The following result provides a simple sufficient condition to have $\mathcal{H}(\bar{b}) = \mathcal{H}(b)$.

Theorem 28.1 *Let b be such that $b^{-1} \in H^\infty$. Then $\mathcal{H}(\bar{b}) = \mathcal{H}(b)$.*

Proof Since b and b^{-1} are in H^∞, according to Theorem 12.4, we have

$$T_{\bar{b}^{-1}}T_{\bar{b}} = T_{\bar{b}^{-1}\bar{b}} = T_1 = I_{H^2}.$$

In particular, the identity $T_{\bar{b}^{-1}}T_{\bar{b}} = I_{H^2}$ implies that

$$T_{\bar{b}^{-1}}T_{\bar{b}}\mathcal{H}(b) = \mathcal{H}(b).$$

Now, on the one hand, Theorem 17.8 says that

$$T_{\bar{b}}\mathcal{H}(b) \subset \mathcal{H}(b)$$

and, on the other, since b is invertible in H^∞, Lemma 18.12 ensures that

$$T_{\bar{b}}\mathcal{H}(\bar{b}) \subset \mathcal{H}(\bar{b}).$$

Therefore, we obtain

$$\mathcal{H}(b) = T_{\bar{b}^{-1}} T_{\bar{b}} \mathcal{H}(b) \subset T_{\bar{b}^{-1}} \mathcal{H}(\bar{b}) \subset \mathcal{H}(\bar{b}).$$

Since, by Theorem 17.8, $\mathcal{H}(\bar{b}) \subset \mathcal{H}(b)$ is always true, we get the desired equality. □

In Theorem 28.4, we will see that the condition $b^{-1} \in H^\infty$ is also necessary whenever b is an extreme point of the closed unit ball of H^∞.

In Section 20.2, we saw that S_ρ^* and $X_{\bar{b}}$ are unitarily equivalent. This fact is needed below. We show that the set identity $\mathcal{H}(\bar{b}) = \mathcal{H}(b)$ implies that X_b is similar to A^*, where A is the multiplication by z on a particular space.

Theorem 28.2 *Let b be such that $\mathcal{H}(\bar{b}) = \mathcal{H}(b)$. Then the following hold.*

(i) *If b is an extreme point, then X_b is similar to the unitary operator Z_ρ^*.*
(ii) *If b is a nonextreme point, then X_b is similar to S^*.*

Proof Since $\mathcal{H}(\bar{b}) = \mathcal{H}(b)$, we know from (28.1) that the map

$$\begin{array}{rccc} i: & \mathcal{H}(b) & \longrightarrow & \mathcal{H}(\bar{b}) \\ & f & \longmapsto & f \end{array}$$

is an isomorphism from $\mathcal{H}(b)$ onto $\mathcal{H}(\bar{b})$. This operator makes the connection between X_b and $X_{\bar{b}}$ via the formula

$$X_b = i^{-1} X_{\bar{b}}\, i. \tag{28.2}$$

In other words, this formula means that X_b is similar to $X_{\bar{b}}$. This can be summarized by the following commutative diagram.

$$\begin{tikzcd} \mathcal{H}(b) \arrow[r, "X_b"] \arrow[d, "i"'] & \mathcal{H}(b) \\ \mathcal{H}(\bar{b}) \arrow[r, "X_{\bar{b}}"'] & \mathcal{H}(\bar{b}) \arrow[u, "i^{-1}"'] \end{tikzcd} \tag{28.3}$$

On the other hand, by (20.10), S_ρ^* and $X_{\bar{b}}$ are unitarily equivalent via the formula

$$X_{\bar{b}} = \mathbf{K}_\rho S_\rho^* \mathbf{K}_\rho^*.$$

Therefore, X_b is similar to S_ρ^*, and thus we have the following diagram.

$$\begin{tikzcd}
\mathcal{H}(b) \arrow[r, "X_b"] \arrow[d, "i"'] & \mathcal{H}(b) \\
\mathcal{H}(\bar{b}) \arrow[r, "X_{\bar{b}}"'] \arrow[d, "\mathbf{K}_\rho^*"] & \mathcal{H}(\bar{b}) \arrow[u, "i^{-1}"'] \\
H^2(\rho) \arrow[r, "S_\rho^*"'] & H^2(\rho) \arrow[u, "\mathbf{K}_\rho"']
\end{tikzcd} \tag{28.4}$$

(i) If b is an extreme point, then, by Corollary 13.34, we have $H^2(\rho) = L^2(\rho)$. Hence, as a consequence, we have $S_\rho = Z_\rho$. But Z_ρ is a unitary operator. Therefore, X_b is similar to the unitary operator Z_ρ^*.

(ii) If b is a nonextreme point, then, by Theorem 23.2,

$$\mathcal{H}(\bar{b}) \doteq \mathcal{M}(\bar{a}). \tag{28.5}$$

As a temporary notation, put

$$\begin{array}{rccc}\mathbf{S}_{\bar{a}}^* : & \mathcal{M}(\bar{a}) & \longrightarrow & \mathcal{M}(\bar{a}) \\ & f & \longmapsto & S^* f.\end{array}$$

By (28.2) and (28.5), X_b is similar to $\mathbf{S}_{\bar{a}}^*$. We show that $\mathbf{S}_{\bar{a}}^*$ is unitarily equivalent to S^*. To continue, we need another temporary notation. By definition, the operator

$$\begin{array}{rccc}\mathbf{T}_{\bar{a}} : & H^2 & \longrightarrow & \mathcal{M}(\bar{a}) \\ & f & \longmapsto & T_{\bar{a}} f\end{array}$$

is a partial isometry from H^2 onto $\mathcal{M}(\bar{a})$. But, since a is outer, it follows from Theorem 12.19(ii) that $\mathbf{T}_{\bar{a}}$ is injective. Therefore, $\mathbf{T}_{\bar{a}}$ is in fact an isometry from H^2 onto $\mathcal{M}(\bar{a})$. In other words, $\mathbf{T}_{\bar{a}}$ is a unitary map.

According to (12.3), we have

$$T_{\bar{a}} S^* = S^* T_{\bar{a}}.$$

Using the operators $\mathbf{S}_{\bar{a}}^*$ and $\mathbf{T}_{\bar{a}}$, this relation implies that

$$\mathbf{T}_{\bar{a}}\, S^* = \mathbf{S}_{\bar{a}}^*\, \mathbf{T}_{\bar{a}},$$

which can be summarized by the following diagram.

$$\begin{tikzcd}
H^2 \arrow[r, "S^*"] \arrow[d, "\mathbf{T}_{\bar{a}}"'] & H^2 \arrow[d, "\mathbf{T}_{\bar{a}}"] \\
\mathcal{M}(\bar{a}) \arrow[r, "\mathbf{S}_{\bar{a}}^*"'] & \mathcal{M}(\bar{a})
\end{tikzcd} \tag{28.6}$$

Thus, $\mathbf{S}_{\bar{a}}^*$ is unitarily equivalent to S^* and, as a consequence, X_b is similar to S^*. □

28.2 Characterizations of $\mathcal{H}(\bar{b}) = \mathcal{H}(b)$

In this section, we give several characterizations of $\mathcal{H}(\bar{b}) = \mathcal{H}(b)$. The first characterization is in terms of the multipliers of $\mathcal{H}(b)$. Recall that a function $\varphi \in H^\infty$ is called a multiplier of $\mathcal{H}(b)$ if $\varphi\mathcal{H}(b) \subset \mathcal{H}(b)$.

Theorem 28.3 *The following are equivalent:*

(i) $\mathcal{H}(\bar{b}) = \mathcal{H}(b)$;
(ii) *b is a multiplier of $\mathcal{H}(b)$.*

Proof (i) $\Longrightarrow$ (ii) By (17.11), we have

$$b\,\mathcal{H}(b) = b\,\mathcal{H}(\bar{b}) = \mathcal{H}(b) \cap \mathcal{M}(b) \subset \mathcal{H}(b).$$

(ii) $\Longrightarrow$ (i) By definition of $\mathcal{M}(b)$, we trivially have $b\,\mathcal{H}(b) \subset \mathcal{M}(b)$. Hence, using (17.11) once more, we get

$$b\,\mathcal{H}(b) \subset \mathcal{H}(b) \cap \mathcal{M}(b) = b\,\mathcal{H}(\bar{b}).$$

Since $b \neq 0$ almost everywhere on $\mathbb{T}$, the previous inclusion implies $\mathcal{H}(b) \subset \mathcal{H}(\bar{b})$. Moreover, according to Theorem 17.8, the other inclusion $\mathcal{H}(\bar{b}) \subset \mathcal{H}(b)$ always holds. Therefore, in fact we have $\mathcal{H}(\bar{b}) = \mathcal{H}(b)$. □

Theorem 28.1 ensures that the condition $b^{-1} \in H^\infty$ is sufficient to have $\mathcal{H}(\bar{b}) = \mathcal{H}(b)$, while Theorem 28.2 says that, if the set identity $\mathcal{H}(\bar{b}) = \mathcal{H}(b)$ holds and if b is an extreme point, then X_b is necessarily similar to the unitary operator Z_ρ^*. We now show that, in the extreme case, these conditions are in fact equivalent.

Theorem 28.4 *Let b be an extreme point of the closed unit ball of H^∞. Then the following are equivalent.*

(i) $\mathcal{H}(\bar{b}) = \mathcal{H}(b)$.

(ii) *X_b is similar to a unitary operator.*
(iii) $b^{-1} \in H^\infty$.
(iv) *b^{-1} is a multiplier of $\mathcal{H}(b)$.*

Moreover, under the above equivalent conditions, we have $b\,\mathcal{H}(b) = \mathcal{H}(b)$.

Proof (i) $\Longrightarrow$ (ii) This implication was established in Theorem 28.2(i).

(ii) $\Longrightarrow$ (iii) Our hypothesis says that there exists a Hilbert space $\mathcal{H}$, a unitary operator $U \in \mathcal{L}(\mathcal{H})$ and an isomorphism $A \in \mathcal{L}(\mathcal{H}, \mathcal{H}(b))$ such that

$$X_b A = AU.$$

Hence, by induction, we easily obtain

$$X_b^n = AU^n A^{-1} \qquad (n \geq 0).$$

Since U is a unitary operator on $\mathcal{H}$, we have

$$\|Ux\|_{\mathcal{H}} = \|x\|_{\mathcal{H}} \qquad (x \in \mathcal{H}).$$

Moreover, since $A \in \mathcal{L}(\mathcal{H}, \mathcal{H}(b))$ is an isomorphism, there are two positive constants c_1 and c_2 such that

$$c_1\|x\|_{\mathcal{H}} \leq \|Ax\|_b \leq c_2\|x\|_{\mathcal{H}} \qquad (x \in \mathcal{H}).$$

Thus, for every function $f \in \mathcal{H}(b)$, we get

$$\begin{aligned} \|X_b^n f\|_b = \|AU^n A^{-1} f\|_b \geq c_1\|U^n A^{-1} f\|_{\mathcal{H}} &= c_1\|A^{-1} f\|_{\mathcal{H}} \\ &\geq c\|f\|_b, \end{aligned} \tag{28.7}$$

where $c = c_1/c_2$. Remember that Corollary 25.14 says that

$$\lim_{n\to\infty} \|X_b^n f\|_b^2 = \|f\|_b^2 - \|f\|_2^2.$$

Hence, by (28.7),

$$\|f\|_b^2 - \|f\|_2^2 \geq c^2\|f\|_b^2,$$

which gives

$$\|f\|_2^2 \leq (1 - c^2)\,\|f\|_b^2 \qquad (f \in \mathcal{H}(b)). \tag{28.8}$$

This relation is needed to establish $b^{-1} \in H^\infty$.

Each $f \in \mathcal{H}(b)$ is of the form $f = (I - T_b T_{\bar{b}})^{1/2} g$, with $g \in H^2$ and $\|f\|_b \leq \|g\|_2$. Moreover,

$$\|f\|_2^2 = \langle (I - T_b T_{\bar{b}})g, g\rangle_2 = \|g\|_2^2 - \|T_{\bar{b}}\, g\|_2^2.$$

Hence, by (28.8),

$$\|g\|_2^2 - \|T_{\bar{b}}\, g\|_2^2 \leq (1 - c^2)\,\|g\|_2^2 \qquad (g \in H^2),$$

which simplifies to

$$\|T_{\bar{b}}\, g\|_2 \geq c\,\|g\|_2 \qquad (g \in H^2).$$

Therefore, by Corollary 12.14, b is invertible in H^∞.

(iii) $\Longrightarrow$ (iv) Using (17.11), we have

$$b\,\mathcal{H}(\bar{b}) = \mathcal{M}(b) \cap \mathcal{H}(b).$$

But, if $b^{-1} \in H^\infty$, we have $\mathcal{M}(b) = H^2$ and $b\,\mathcal{H}(\bar{b}) = \mathcal{H}(b)$. Now, it suffices to apply Theorem 28.1, which gives

$$b\,\mathcal{H}(b) = \mathcal{H}(b).$$

Hence, b and b^{-1} are multipliers of $\mathcal{H}(b)$.

(iv) $\Longrightarrow$ (i) If b^{-1} is a multiplier of $\mathcal{H}(b)$, then, if we combine Corollary 9.7 and Theorem 28.1, we get $\mathcal{H}(b) = \mathcal{H}(\bar{b})$. □

Recall that, if b is inner, then $\mathfrak{Mult}(\mathcal{H}(b)) = \mathbb{C}$. In the opposite case, we always have nonconstant multipliers.

Corollary 28.5 *If b is not an inner function, then $\mathcal{H}(b)$ has nonconstant multipliers.*

Proof According to Theorem 24.6, we only need to treat the case where b is an extreme point. Under the assumption that b is not an inner function, there is a factorization $b = b_1 b_2$, where $b_j \in H^\infty$, $\|b_j\|_\infty \leq 1$ and b_1 is nonconstant and $b_1^{-1} \in H^\infty$. (For instance, one can take $b_1 = [\max(|b|, 1/2)]$, that is, an outer function such that $|b_1| = \max(|b|, 1/2)$ a.e. on $\mathbb{T}$; see (4.52) for the definition.) By (18.3), we have

$$\mathcal{H}(b) = \mathcal{H}(b_1) + b_1\mathcal{H}(b_2) = \mathcal{H}(b_2) + b_2\mathcal{H}(b_1).$$

Thus

$$b_1\mathcal{H}(b_2) \subset \mathcal{H}(b) \quad \text{and} \quad b_2\mathcal{H}(b_1) \subset \mathcal{H}(b)$$

with

$$b_1\mathcal{H}(b) = b_1\mathcal{H}(b_2) + b_1 b_2\mathcal{H}(b_1). \tag{28.9}$$

Since $b_1^{-1} \in H^\infty$, then, by Theorems 28.4 and 28.3, b_1 is a multiplier of $\mathcal{H}(b_1)$, which means that $b_1\mathcal{H}(b_1) \subset \mathcal{H}(b_1)$. Hence (28.9) gives that

$$b_1\mathcal{H}(b) \subset \mathcal{H}(b).$$

In particular, b_1 is a nonconstant multiplier of $\mathcal{H}(b)$. □

Theorem 28.6 *Let b be an extreme point of the closed unit ball of H^∞. Let b_i denote the inner part and b_o the outer part of b. If $b_o^{-1} \in H^\infty$, then b_o and b_o^{-1} are multipliers of $\mathcal{H}(b)$ and the decompositions*

$$\mathcal{H}(b) = K_{b_i} + \mathcal{H}(b_o) = b_o K_{b_i} + b_i \mathcal{H}(b_o)$$

hold.

Proof Remember by (18.3) that

$$\mathcal{H}(b) = K_{b_i} + b_i \mathcal{H}(b_o) = \mathcal{H}(b_o) + b_o K_{b_i}.$$

If $b_o^{-1} \in H^\infty$, then Theorem 28.4 implies that $b_o \mathcal{H}(b_o) = \mathcal{H}(b_o)$, so that

$$b_o \mathcal{H}(b) = b_o K_{b_i} + b_i b_o \mathcal{H}(b_o) = b_o K_{b_i} + b_i \mathcal{H}(b_0) \subset \mathcal{H}(b)$$

and

$$b_o^{-1} \mathcal{H}(b) = b_o^{-1} \mathcal{H}(b_o) + b_o^{-1} b_o K_{b_i} = \mathcal{H}(b_o) + \mathcal{H}(u_0) \subset \mathcal{H}(b).$$

Thus b_o and b_o^{-1} are multipliers of $\mathcal{H}(b)$. In particular, we have $b_o \mathcal{H}(b) = \mathcal{H}(b)$ and $b_o^{-1} \mathcal{H}(b) = \mathcal{H}(b)$ and we can come back to the preceding inclusions, which gives the desired decomposition of $\mathcal{H}(b)$. $\square$

When b is a nonextreme point of the closed unit ball of H^∞, the set equality $\mathcal{H}(\bar{b}) = \mathcal{H}(b)$ is connected to the corona theorem. The contents of Section 5.8 are needed below.

Remember that two H^∞ functions (f, g) satisfy the condition (HCR), and we write $(f, g) \in (HCR)$, if there exists a $\delta > 0$ such that

$$|f(z)| + |g(z)| \geq \delta \qquad (z \in \mathbb{D}).$$

Theorem 28.7 *Let b be a nonextreme point of the closed unit ball of H^∞ with Pythagorean mate a. Then the following are equivalent.*

(i) $(a, b) \in (HCR)$.
(ii) $\mathcal{H}(\bar{b}) = \mathcal{H}(b)$.
(iii) *X_b is similar to S^*.*

Proof (i) $\Longrightarrow$ (ii) By Theorem 5.20, there are $u, v \in H^\infty$ such that

$$au + bv = 1.$$

Hence, for each $h \in \mathcal{H}(b)$, we have

$$h = h(\bar{u}\bar{a} + \bar{b}\bar{v}) = P_+(h(\bar{u}\bar{a} + \bar{b}\bar{v})) = P_+(\bar{u}\bar{a}h) + P_+(\bar{v}\bar{b}h).$$

According to Corollary 4.11, we can write the preceding identity as

$$h = P_+(\bar{u}P_+(\bar{a}h)) + P_+(\bar{v}P_+(\bar{b}h)) = T_{\bar{u}}T_{\bar{a}}h + T_{\bar{v}}T_{\bar{b}}h.$$

As we saw, by (23.10), there is a unique $h^+ \in H^2$ such that $T_{\bar{b}}h = T_{\bar{a}}h^+$. Thus,

$$h = T_{\bar{u}}T_{\bar{a}}h + T_{\bar{v}}T_{\bar{a}}h^+,$$

which, by (12.3), gives the representation

$$h = T_{\bar{a}}(T_{\bar{u}}h + T_{\bar{v}}h^+).$$

This identity reveals that $h \in \mathcal{M}(\bar{a})$. But, according to Theorem 23.2, $\mathcal{M}(\bar{a}) = \mathcal{H}(\bar{b})$. Therefore, we have proved that $\mathcal{H}(b) \subset \mathcal{H}(\bar{b})$. By Theorem 17.8, the other inclusion $\mathcal{H}(\bar{b}) \subset \mathcal{H}(b)$ is always true.

(ii) $\Longrightarrow$ (i) According to (28.1), there exists a constant $c > 0$ such that

$$c\,\|h\|_{\bar{b}} \le \|h\|_b \qquad (h \in \mathcal{H}(b)). \tag{28.10}$$

We apply this inequality to the reproducing kernels of H^2, i.e. k_w, $w \in \mathbb{D}$. Note that Theorem 23.23 guarantees that $k_w \in \mathcal{H}(b)$. By Corollary 23.25, we have

$$\|k_w\|_b^2 = \left(1 + \frac{|b(w)|^2}{|a(w)|^2}\right) k_w(w) \qquad (w \in \mathbb{D}).$$

Thus, taking into account (4.19), we obtain

$$\|k_w\|_b^2 = \frac{|a(w)|^2 + |b(w)|^2}{(1-|w|^2)|a(w)|^2} \qquad (w \in \mathbb{D}).$$

Moreover, it follows from Theorem 23.4 that

$$\|k_w\|_{\bar{b}} = \frac{1}{|a(w)|(1-|w|^2)^{1/2}} \qquad (w \in \mathbb{D}).$$

Therefore, plugging the last two identities into (28.10) gives

$$\frac{c^2}{|a(w)|^2(1-|w|^2)} \le \frac{|a(w)|^2 + |b(w)|^2}{(1-|w|^2)|a(w)|^2} \qquad (w \in \mathbb{D}),$$

which simplifies to

$$c^2 \le |a(w)|^2 + |b(w)|^2 \le |a(w)| + |b(w)| \qquad (w \in \mathbb{D}).$$

This means that (a,b) forms a corona pair.

(ii) $\Longrightarrow$ (iii) This is precisely the content of Theorem 28.2(ii).

(iii) $\Longrightarrow$ (ii) Let $A : H^2 \longrightarrow \mathcal{H}(b)$ be the isomorphism such that

$$X_b A = AS^*. \tag{28.11}$$

Since $\mathcal{H}(b)$ embeds continuously into H^2, the operator

$$\begin{array}{rccc} \mathbf{A}: & H^2 & \longrightarrow & H^2 \\ & f & \longmapsto & Af \end{array}$$

is bounded. With this new operator, the identity (28.11) is rewritten as

$$S^*\mathbf{A} = \mathbf{A}S^*.$$

The commutant lifting theorem (Theorem 12.27) now implies that there exists a function $u \in H^\infty$ such that $\mathbf{A} = T_{\bar{u}}$. Hence, we have

$$\mathcal{M}(\bar{u}) = T_{\bar{u}}H^2 = \mathbf{A}H^2 = AH^2 = \mathcal{H}(b),$$

where the above identities are understood only as set identities. By Theorem 17.17, we have $\mathcal{M}(u) \subset \mathcal{M}(\bar{u})$ and thus $\mathcal{M}(u) \subset \mathcal{H}(b)$. According to Theorem 23.6, the inclusion $\mathcal{M}(u) \subset \mathcal{H}(b)$ implies that $u/a \in H^\infty$. Put $v = u/a$. Then, by (12.3), we have

$$T_{\bar{u}} = T_{\bar{a}\bar{v}} = T_{\bar{a}}T_{\bar{v}}.$$

In particular, this gives $\mathcal{H}(b) = \mathcal{M}(\bar{u}) \subset \mathcal{M}(\bar{a})$. But, by Theorem 23.2, $\mathcal{M}(\bar{a}) = \mathcal{H}(\bar{b})$, and thus we finally obtain $\mathcal{H}(b) \subset \mathcal{H}(\bar{b})$. □

Example 28.8 Let (a, b) be a pair. Because $|a|^2 + |b|^2 = 1$ a.e. on $\mathbb{T}$, the reader might think for an instant that (a, b) would automatically form a corona pair, at least in the case when b is an outer function. That is not true however. Let us provide an example. Let b be the outer function satisfying

$$|b(e^{i\vartheta})|^2 = \frac{\vartheta}{2\pi} \qquad (0 \le \vartheta < 2\pi).$$

It is clear that $\log(1 - |b|^2) \in L^1(\mathbb{T})$, whence there is an outer function a satisfying $a(0) > 0$ and

$$|a(e^{i\vartheta})|^2 = 1 - |b(e^{i\vartheta})|^2 = 1 - \frac{\vartheta}{2\pi} \qquad (0 \le \vartheta < 2\pi).$$

Now it is clear that

$$\lim_{\vartheta \to 0^+} b(e^{i\vartheta}) = 0 \quad \text{and} \quad \lim_{\vartheta \to (2\pi)^-} a(e^{i\vartheta}) = 0.$$

Hence, a classic theorem of Lindelöf gives us that

$$\lim_{r \to 1} b(r) = \lim_{r \to 1} a(r) = 0.$$

In particular, (a, b) cannot satisfy the corona condition.

If we combine Theorems 28.7 and 28.3, then we deduce that, if $(a, b) \in (HCR)$, then $b \in \mathfrak{Mult}(\mathcal{H}(b))$. This fact can be slightly generalized.

Theorem 28.9 *Let (a, b) be a pair. Let $f \in H^\infty$ be such that $(c - |f|^2)/a \in L^\infty(\mathbb{T})$, for some constant $c > 0$, and $(a, f) \in (HCR)$. Then $f \in \mathfrak{Mult}(\mathcal{H}(b))$.*

Proof Since $(a, f) \in (HCR)$, by Theorem 5.20, there exists $\phi, \psi \in H^\infty$ such that

$$f\phi + a\psi = 1 \qquad (\text{on } \mathbb{D}). \tag{28.12}$$

Let $g \in \mathcal{H}(b)$. We have to show that $fg \in \mathcal{H}(b)$, which can be reduced to show that $T_{\bar{b}}(fg) \in \mathcal{M}(\bar{a})$. Using (28.12), we can write

$$\begin{aligned} T_{\bar{b}}(fg) &= P_+(\bar{b}fg) = P_+(\bar{b}fg(\bar{f}\bar{\phi} + \bar{a}\bar{\psi})) \\ &= P_+(\bar{b}g\bar{\phi}|f|^2 + \bar{a}\bar{b}fg\bar{\psi}). \end{aligned}$$

Note that

$$P_+(\bar{a}\bar{b}fg\bar{\psi}) = P_+(\bar{a}P_+(\bar{b}fg\bar{\psi})) = T_{\bar{a}}(P_+(\bar{b}fg\bar{\psi})) \in \mathcal{M}(\bar{a}).$$

So it remains to prove that $P_+(\bar{b}g\bar{\phi}\,|f|^2) \in \mathcal{M}(\bar{a})$. But

$$\begin{aligned} P_+(\bar{b}g\bar{\phi}\,|f|^2) &= P_+(-\bar{b}g\bar{\phi}(c - |f|^2)) + cP_+(\bar{b}g\bar{\phi}) \\ &= P_+(\bar{a}\chi) + cP_+(\bar{b}g\bar{\phi}), \end{aligned} \tag{28.13}$$

where $\chi = -\bar{b}g\bar{\phi}(c-|f|^2)/\bar{a}$. By assumption $\chi \in L^2(\mathbb{T})$, and since $P_+(\bar{a}\chi) = P_+(\bar{a}P_+\chi) = T_{\bar{a}}(P_+\chi)$, the first term of (28.13) belongs to $\mathcal{M}(\bar{a})$. For the second term, remember that $g \in \mathcal{H}(b)$, and thus we can write $\bar{b}g = \bar{a}g^+ + \bar{\psi}_1$, where $g^+ \in H^2$ and $\psi_1 \in H^2_0$. Hence

$$P_+(\bar{b}g\bar{\phi}) = P_+(\bar{a}g^+) + P_+(\bar{\phi}\bar{\psi}_1) = P_+(\bar{a}g^+) = T_{\bar{a}}g^+,$$

which proves also that $P_+(\bar{b}g\bar{\phi}) \in \mathcal{M}(\bar{a})$. □

Note that, if we take $f = b$ in Theorem 28.9, we recover that, if $(a, b) \in (HCR)$, then b is a multiplier of $\mathcal{H}(b)$.

If b is an outer function in the closed unit ball of H^∞, it has no zeros on $\mathbb{D}$ and so we can define b^r by considering any logarithm of b. Note that b^r is also an outer function in the closed unit ball of H^∞. We will see that in certain situations $\mathcal{H}(b^r)$ does not depend on $r > 0$. We first begin with a preliminary result.

Lemma 28.10 *Suppose b is an outer function in the closed unit ball of H^∞ and r is a positive real number. Then b and b^r are simultaneously nonextreme. Moreover, if $a^{(r)}$ is the Pythagorean mate for b^r, the pairs (a, b) and $(a^{(r)}, b^r)$ are simultaneously corona.*

Proof Since

$$\frac{1 - x^r}{1 - x} \asymp 1 \qquad (x \in [0, 1)), \tag{28.14}$$

we see that $1 - |b^r| \asymp 1 - |b|$, whenever b is in the closed unit ball of H^∞. This asymptotic formula implies the first part of the lemma.

Now observe that

$$\frac{|a|^2}{|a^{(r)}|^2} = \frac{1-|b|^2}{1-|b^2|^r} \asymp 1,$$

and since a and $a^{(r)}$ are outer, Corollary 4.28 shows that $a/a^{(r)}$ is in H^∞ and also invertible in H^∞. Thus the two expressions

$$\inf_{z\in\mathbb{D}}(|a(z)|+|b(z)|) \quad \text{and} \quad \inf_{z\in\mathbb{D}}(|a^{(r)}(z)|+|b^r(z)|)$$

are strictly positive (or not) simultaneously. Indeed, if there is a sequence $\{z_n\}_{n\geqslant 1}$ in $\mathbb{D}$ such that one expression goes to 0, then, since both $a(z_n)$ and $b(z_n)$ go to zero, the other expression will go to zero as well. □

Corollary 28.11 *Suppose b is an outer function in the closed unit ball of H^∞. Then the following are equivalent.*

(i) *For any $r>0$ we have $\mathcal{H}(b^r)=\mathcal{H}(b)$ (as sets).*
(ii) *$\mathcal{H}(b^2)=\mathcal{H}(b)$ (as sets).*
(iii) *b is a multiplier of $\mathcal{H}(b)$.*

If b is nonextreme with Pythagorean mate a, then conditions (i), (ii) *and* (iii) *are also equivalent to*

$$\inf\{|a(z)|+|b(z)| : z\in\mathbb{D}\} > 0. \tag{28.15}$$

But, if b is extreme, then conditions (i), (ii) *and* (iii) *are also equivalent to the condition*

$$b \text{ is invertible in } H^\infty. \tag{28.16}$$

Proof The implication (i) $\Longrightarrow$ (ii) is trivial.

To show (ii) $\Longrightarrow$ (iii), note that from (18.3) we have

$$\mathcal{H}(b^2) = \mathcal{H}(b) + b\,\mathcal{H}(b).$$

But, since we are assuming that $\mathcal{H}(b^2)=\mathcal{H}(b)$, it follows that $b\,\mathcal{H}(b)\subset\mathcal{H}(b)$.

For the implication (iii) $\Longrightarrow$ (28.15), we use the fact that b is nonextreme, and Theorems 28.3 and 28.7 to see that b being a multiplier of $\mathcal{H}(b)$ is equivalent to (a,b) being a corona pair.

To show that (28.15) $\Longrightarrow$ (i), we proceed as follows. By Lemma 28.10 we know that, since (a,b) is a corona pair, then so is (a_r, b^r). Thus from Theorem 28.7, we see that $\mathcal{H}(b)=\mathcal{H}(\bar{b})$ and $\mathcal{H}(b^r)=\mathcal{H}(\overline{b^r})$. But, by (28.14) and Theorem 17.12(i), we get that $\mathcal{H}(\bar{b}) = \mathcal{H}(\overline{b^r})$, which gives the desired conclusion.

For the implication (iii) $\Longrightarrow$ (28.16), we use the fact that b is extreme, and Theorems 28.3 and 28.4 to see that b being a multiplier of $\mathcal{H}(b)$ is equivalent to b being an invertible element of H^∞.

It remains to show (28.16) $\Longrightarrow$ (i). Assuming that b is invertible in H^∞, we use, once again, Theorem 28.4 to see that $\mathcal{H}(b) = \mathcal{H}(\bar{b})$. But, since b is invertible in H^∞, then so is b^r and thus we also have $\mathcal{H}(b^r) = \mathcal{H}(\overline{b^r})$. Remember now that

$$\frac{1-|b|^2}{1-|b^r|^2} \asymp 1$$

and Theorem 17.12(i) once again implies that $\mathcal{H}(\bar{b}) = \mathcal{H}(\overline{b^r})$, which concludes the proof. □

Exercise

Exercise 28.2.1 Let (a, b) be a pair and let n be a positive integer.

(i) Show that

$$\mathcal{H}(b^n) = \mathcal{H}(b) + b\,\mathcal{H}(b) + b^2\mathcal{H}(b) + \cdots + b^{n-1}\mathcal{H}(b).$$

Hint: Apply Theorem 16.23.

(ii) Assume that $(a, b) \in (HCR)$. Then, for any $n \geq 1$, show that

$$\mathcal{H}(b^n) = \mathcal{H}(b).$$

Hint: Apply Theorems 28.7 and 28.3.

(iii) Let b be a nonextreme rational function in the closed unit ball of H^∞. Show that

$$\mathcal{H}(b^n) = \mathcal{H}(b) \qquad (n \geq 1).$$

Hint: Show that the pair (a, b) is a corona pair.

28.3 Multipliers of $\mathcal{H}(b)$ spaces, extreme case, part II

Recall that H^∞ is a Banach algebra and, for a point b in the closed unit ball of H^∞, we will denote by $\sigma_{H^\infty}(b)$ the spectrum of b in H^∞, that is,

$$\sigma_{H^\infty}(b) = \{\lambda \in \mathbb{C} : b - \lambda \text{ is not invertible in } H^\infty\}.$$

Since $\|b\|_\infty \leq 1$, according to the general theory of Banach algebras, we know that $\sigma_{H^\infty}(b) \subset \bar{\mathbb{D}}$. There is a close connection between the spectrum of b in H^∞ and the multipliers of $\mathcal{H}(b)$.

Theorem 28.12 *Let b be an extreme point of the closed unit ball of H^∞ and let $\lambda \in \mathbb{D} \setminus \{0\}$. Then the following are equivalent.*

(i) $\lambda \in \mathbb{D} \setminus \sigma_{H^\infty}(b)$.

(ii) $(1-\bar{\lambda}b)\mathcal{H}(\bar{b}) = \mathcal{H}(b)$.
(iii) $(1-\bar{\lambda}b)^{-1} \in \mathfrak{Mult}(\mathcal{H}(b))$.

Proof (i) $\Longleftrightarrow$ (ii) Note that $\lambda \in \mathbb{D} \setminus \sigma_{H^\infty}(b)$ if and only if the function

$$b_\lambda \circ b = \frac{\lambda - b}{1-\bar{\lambda}b}$$

is invertible in H^∞. By (19.41), we see that b is extreme if and only if $b_\lambda \circ b$ is extreme. Thus, Theorem 28.4 implies that $b_\lambda \circ b$ is invertible in H^∞ if and only if

$$\mathcal{H}(\overline{b_\lambda \circ b}) = \mathcal{H}(b_\lambda \circ b). \tag{28.17}$$

It remains to prove that this last equality is equivalent to (ii). First assume that (28.17) holds. Then it follows from (19.40) and Theorem 19.25 that

$$(1-\bar{\lambda}b)\mathcal{H}(\bar{b}) = (1-\bar{\lambda}b)\mathcal{H}(\overline{b_\lambda \circ b}) = (1-\bar{\lambda}b)\mathcal{H}(b_\lambda \circ b) = \mathcal{H}(b).$$

Conversely, assume that (ii) holds. Then, using (19.40) once more, we have

$$(1-\bar{\lambda}b)\mathcal{H}(\bar{b}) = \mathcal{H}(b) = (1-\bar{\lambda}b)\mathcal{H}(b_\lambda \circ b),$$

and Theorem 19.25 implies that

$$\mathcal{H}(\overline{b_\lambda \circ b}) = \mathcal{H}(\bar{b}) = \mathcal{H}(b_\lambda \circ b),$$

which gives (28.17).

(ii) $\Longrightarrow$ (iii) It suffices to note that

$$(1-\bar{\lambda}b)^{-1}\mathcal{H}(b) = \mathcal{H}(\bar{b}) \subset \mathcal{H}(b).$$

(iii) $\Longrightarrow$ (ii) Let $f \in \mathcal{H}(b)$. By hypothesis, $g = (1-\bar{\lambda}b)^{-1} f \in \mathcal{H}(b)$. Since $g - \bar{\lambda}bg = (1-\bar{\lambda}b)g = f$, we deduce that $bg \in \mathcal{H}(b)$. Hence, Theorem 17.8 implies that $g \in \mathcal{H}(\bar{b})$ and $f = (1-\bar{\lambda}b)g \in (1-\bar{\lambda}b)\mathcal{H}(\bar{b})$. In other words, we have proved the inclusion

$$\mathcal{H}(b) \subset (1-\bar{\lambda}b)\mathcal{H}(\bar{b}).$$

Since the reverse inclusion is always true (see (17.11)) that concludes the proof. □

Note that the assumption that b is an extreme point of the closed unit ball of H^∞ has been used only for the implication (ii) $\Longrightarrow$ (i). In other words, the implication (i) $\Longrightarrow$ (ii) and the equivalence between (ii) and (iii) remain valid in the nonextreme case. Note also that, for $\lambda = 0$, the equivalence between (i) and (ii) is true and has been proved in Theorem 28.4.

Corollary 28.13 *Let b be an extreme point of the closed unit ball of H^∞ and let $\lambda \in \mathbb{D} \setminus \sigma_{H^\infty}(b)$. Then*

$$\frac{1}{b-\lambda} \in \mathfrak{Mult}(\mathcal{H}(b)).$$

Proof Since $\lambda \in \mathbb{D} \setminus \sigma_{H^\infty}(b)$, we know from Theorem 28.12 that $(1-\bar{\lambda}b)^{-1}$ is a multiplier of $\mathcal{H}(b)$. Since

$$\frac{1}{b-\lambda} = -\frac{1}{1-\bar{\lambda}b}\frac{1}{b_\lambda \circ b},$$

it is sufficient to prove that $1/b_\lambda \circ b$ is a multiplier of $\mathcal{H}(b)$. But, according to Theorem 28.4, we know that $1/b_\lambda \circ b$ is a multiplier of $\mathcal{H}(b_\lambda \circ b)$ and Theorem 19.25 implies that $1/b_\lambda \circ b$ is a multiplier of $\mathcal{H}(b)$. □

The following result should be compared with Theorem 20.25.

Corollary 28.14 *Let b be an extreme point of the closed unit ball of H^∞. If Θ is an inner function such that $\sigma_{H^\infty}(b\Theta) \neq \bar{\mathbb{D}}$, then*

$$\mathfrak{Mult}(\mathcal{H}(\Theta b)) = \mathfrak{Mult}(\mathcal{H}(b)) = \mathfrak{Mult}(\mathcal{H}(\bar{b})).$$

Proof Since $\sigma_{H^\infty}(b\Theta)$ is compact, there must be a point $\lambda \in \mathbb{D} \setminus \sigma_{H^\infty}(b\Theta)$ and such that $\lambda \neq 0$. By Theorem 28.12, we have

$$(1-\bar{\lambda}\Theta b)\mathcal{H}(\overline{\Theta b}) = \mathcal{H}(\Theta b). \tag{28.18}$$

We claim that

$$\mathfrak{Mult}(\mathcal{H}(\overline{\Theta b})) = \mathfrak{Mult}(\mathcal{H}(\Theta b)). \tag{28.19}$$

The inclusion $\mathfrak{Mult}(\mathcal{H}(\Theta b)) \subset \mathfrak{Mult}(\mathcal{H}(\overline{\Theta b}))$ follows from Theorem 20.17. For the reverse inclusion, let $f \in \mathfrak{Mult}(\mathcal{H}(\overline{\Theta b}))$ and let $g \in \mathcal{H}(\Theta b)$. Then, according to (28.18), there exists $h \in \mathcal{H}(\overline{\Theta b})$ such that $(1-\bar{\lambda}\Theta b)h = g$. Hence,

$$fg = f(1-\bar{\lambda}\Theta b)h = (1-\bar{\lambda}\Theta b)fh \in (1-\bar{\lambda}\Theta b)\mathcal{H}(\overline{\Theta b}) = \mathcal{H}(\Theta b).$$

We deduce that $f \in \mathfrak{Mult}(\mathcal{H}(\Theta b))$ and we get (28.19). Now, since by Lemma 17.11, $\mathcal{H}(\overline{\Theta b}) = \mathcal{H}(\bar{b})$, we thus get

$$\mathfrak{Mult}(\mathcal{H}(\Theta b)) = \mathfrak{Mult}(\mathcal{H}(\bar{b})).$$

The inclusion

$$\mathfrak{Mult}(\mathcal{H}(\Theta b)) \subset \mathfrak{Mult}(\mathcal{H}(b))$$

is always true – see the proof of Theorem 20.25. Hence, we obtain that

$$\mathfrak{Mult}(\mathcal{H}(\bar{b})) \subset \mathfrak{Mult}(\mathcal{H}(b)),$$

and the conclusion now follows from Theorem 20.17. □

We will now examine the situation when b is a continuous function that is an extreme point of the closed unit ball of H^∞. For this study, we need some preliminary results. The first one shows that the outer part of a continuous H^1 function is also continuous.

Lemma 28.15 *Let f be a function in H^1, f_0 its outer part and Θ its inner part. Assume that f is continuous on $\mathbb{T}$. Then f_0 and $\bar{\Theta} f_0$ are continuous on $\mathbb{T}$.*

Proof Let $K = \{e^{i\vartheta} \in \mathbb{T} : f(e^{i\vartheta}) = 0\}$ and write $\Theta = BS_\mu$, where B is the Blaschke product associated with the zeros of f and S_μ is its singular inner part. Since f is continuous on $\mathbb{T}$, K is a closed subset of $\mathbb{T}$. Since $|f_0| = |f|$, then $|f_0|$ is continuous on $\mathbb{T}$. This implies that f_0 is continuous and zero on K. On $\mathbb{T} \setminus K$, $|f| > 0$, so that B and S_μ cannot tend to zero at any point on $\mathbb{T} \setminus K$. Then B and S_μ can be analytically continued across $\mathbb{T} \setminus K$ and $f_0 = f/BS_\mu$ is continuous on $\mathbb{T} \setminus K$. Hence, f_0 is continuous on $\mathbb{T}$.

The argument above shows that $\bar{\Theta} f_0 = \bar{B}\bar{S}_\mu f_0$ is continuous on $\mathbb{T} \setminus K$. This function is also continuous and zero on K because $|\bar{\Theta} f_0| = |f_0| = |f|$. Hence, $\bar{\Theta} f_0$ is also continuous on $\mathbb{T}$. $\square$

The second result we need concerns an extremal problem.

Lemma 28.16 *Let $f \in L^\infty(\mathbb{T})$. Then the following assertions hold.*

(i) *We have*

$$\operatorname{dist}(f, H^\infty) = \sup\left\{\left|\int_{\mathbb{T}} fF\,\frac{d\vartheta}{2\pi}\right| : F \in H_0^1,\ \|F\|_1 \le 1\right\}.$$

(ii) *There exists $g \in H^\infty$ such that*

$$\operatorname{dist}(f, H^\infty) = \|f - g\|_\infty. \tag{28.20}$$

(iii) *If there exists $F \in H_0^1$, $\|F\|_1 \le 1$, such that*

$$\operatorname{dist}(f, H^\infty) = \int_{\mathbb{T}} fF\,\frac{d\vartheta}{2\pi},$$

then the solution g in (28.20) is unique and satisfies

$$|f - g| = \operatorname{dist}(f, H^\infty) \quad \text{a.e. on } \mathbb{T}.$$

Proof (i) Let X be a Banach space, X^* its dual space and Y a closed subspace of X. Then the Hahn–Banach theorem implies that, for $x^* \in X^*$, we have

$$\operatorname{dist}(x^*, Y^\perp) = \sup\{|\langle x^*, y\rangle| : y \in Y,\ \|y\| \le 1\}.$$

If we apply this to $X = L^1(\mathbb{T})$ and $Y = H_0^1$, then $X^* \simeq L^\infty(\mathbb{T})$ and $Y^\perp = H^\infty$, which gives exactly (i).

(ii) Let $(g_n)_n$ be a sequence of H^∞ functions satisfying

$$\|f - g_n\|_\infty \longrightarrow \operatorname{dist}(f, H^\infty) \qquad (\text{as } n \longrightarrow \infty).$$

A normal family argument shows that there exist a subsequence $(g_{n_\ell})_\ell$ and a function $g \in H^\infty$ such that $g_{n_\ell}(z) \longrightarrow g(z)$, $\ell \longrightarrow \infty$, uniformly on compact subsets of $\mathbb{D}$. On the other hand, we have

$$\begin{aligned} |f(z) - g(z)| &= \lim_{\ell \to \infty} |f(z) - g_{n_\ell}(z)| \\ &\leq \limsup_{\ell \to \infty} \|f - g_{n_\ell}\|_\infty = \operatorname{dist}(f, H^\infty). \end{aligned}$$

Hence $\|f - g\|_\infty \leq \operatorname{dist}(f, H^\infty)$. The reverse inequality is trivial, proving the existence of a solution to the problem (28.20).

(iii) Assume that there exists a function $F \in H_0^1$, $\|F\|_1 \leq 1$, that satisfies

$$\operatorname{dist}(f, H^\infty) = \int_{\mathbb{T}} fF \, \frac{d\vartheta}{2\pi}.$$

For any function $g \in H^\infty$, we have

$$\operatorname{dist}(f, H^\infty) = \int_{\mathbb{T}} (f - g)F \, \frac{d\vartheta}{2\pi} \leq \|f - g\|_\infty \|F\|_1 \leq \|f - g\|_\infty.$$

So, if g is a solution to (28.20), then we get equality in the preceding inequalities. The case of equality in the Hölder inequality thus gives that

$$\frac{f - g}{\operatorname{dist}(f, H^\infty)} = \frac{f - g}{\|f - g\|_\infty} = \frac{\bar{F}}{|F|} \quad \text{a.e.}$$

That proves the uniqueness of g and the fact that $|f - g| = \operatorname{dist}(f, H^\infty)$ a.e. on $\mathbb{T}$. □

When the function f is continuous, then the problem (28.20) always has a unique solution.

Corollary 28.17 *Let $f \in \mathcal{C}(\mathbb{T})$. Then there exists a unique $g \in H^\infty$ such that*

$$\operatorname{dist}(f, H^\infty) = \|f - g\|_\infty.$$

Moreover, the function g satisfies

$$|f - g| = \operatorname{dist}(f, H^\infty) \qquad (\textit{a.e. on } \mathbb{T}).$$

Proof By Lemma 28.16, there exists $F_n \in H_0^1$, $\|F_n\|_1 \leq 1$, so that

$$\frac{1}{2\pi} \int_{\mathbb{T}} fF_n \, d\vartheta \longrightarrow \operatorname{dist}(f, H^\infty) \qquad (n \longrightarrow \infty).$$

Taking a subsequence, we can assume that $F_n(z) \longrightarrow F(z)$, $z \in \mathbb{D}$, and $F \in H_0^1$, $\|F\|_1 \leq 1$. In particular, that gives the convergence of the Fourier coefficients, so that

$$\int_{\mathbb{T}} F_n p \, \frac{d\vartheta}{2\pi} \longrightarrow \int_{\mathbb{T}} F p \, \frac{d\vartheta}{2\pi} \qquad (n \longrightarrow \infty),$$

for all trigonometric polynomials p. Since $f \in \mathcal{C}(\mathbb{T})$, for all $\varepsilon > 0$, there exists a trigonometric polynomial p such that $\|f - p\|_\infty \leq \varepsilon$ and, for sufficiently large n, we have

$$\left| \int_{\mathbb{T}} (F_n p - F p) \, \frac{d\vartheta}{2\pi} \right| \leq \varepsilon.$$

Denote by

$$A = \operatorname{dist}(f, H^\infty) - \int_{\mathbb{T}} F f \, \frac{d\vartheta}{2\pi}$$

and write that

$$\begin{aligned} A = \operatorname{dist}(f, H^\infty) &- \int_{\mathbb{T}} f F_n \, \frac{d\vartheta}{2\pi} + \int_{\mathbb{T}} (f - p) F_n \, \frac{d\vartheta}{2\pi} \\ &+ \int_{\mathbb{T}} (p F_n - p F) \, \frac{d\vartheta}{2\pi} + \int_{\mathbb{T}} (p F - F f) \, \frac{d\vartheta}{2\pi}. \end{aligned}$$

Hence, we get

$$\begin{aligned} |A| &\leq \left| \operatorname{dist}(f, H^\infty) - \int_{\mathbb{T}} f F_n \, \frac{d\vartheta}{2\pi} \right| + 2\|f - p\|_\infty + \left| \int_{\mathbb{T}} (F_n p - F p) \, \frac{d\vartheta}{2\pi} \right| \\ &\leq 4\varepsilon, \end{aligned}$$

which gives

$$\operatorname{dist}(f, H^\infty) = \int_{\mathbb{T}} F f \, \frac{d\vartheta}{2\pi}.$$

It remains now to apply Lemma 28.16. □

Theorem 28.18 *Let b be an extreme point of the closed unit ball of H^∞ that is not an inner function and assume that b is continuous on $\mathbb{T}$. Then*

$$\mathfrak{Mult}(\mathcal{H}(b)) = \mathfrak{Mult}(\mathcal{H}(\bar{b})).$$

Proof Factorize $b = u b_0$, where u is the inner factor of b and b_0 is its outer factor. Since b is continuous on $\mathbb{T}$, then we get from Lemma 28.15 that $\bar{u} b_0$ is also continuous on $\mathbb{T}$. It follows from Corollary 28.17 that there exists a unique $g \in H^\infty$ such that

$$\operatorname{dist}(\bar{u} b_0, H^\infty) = \|\bar{u} b - g\|_\infty$$

and

$$|\bar{u}_0 - g| = \operatorname{dist}(\bar{u} b_0, H^\infty) \qquad \text{(a.e. on } \mathbb{T}). \tag{28.21}$$

Therefore, $\operatorname{dist}(\bar{u}b_0, H^\infty) < 1$. Indeed, if $\operatorname{dist}(\bar{u}b_0, H^\infty) = 1$, since $\|\bar{u}b_0\|_\infty = 1$ (note that, since b is extreme, then $\|b\|_\infty = \|b_0\|_\infty = 1$), we get by uniqueness that the best approximate for $\bar{u}b_0$ in H^∞ must be 0. Then, by (28.21), we should have $|\bar{u}b_0| = 1$ a.e. on $\mathbb{T}$. This is a contradiction (remember that b is assumed not to be an inner function). Hence, $\operatorname{dist}(\bar{u}b_0, H^\infty) < 1$. Now, it follows from Theorem 20.25 that

$$\mathfrak{Mult}(\mathcal{H}(b_0)) = \mathfrak{Mult}(\mathcal{H}(ub_0)) = \mathfrak{Mult}(\mathcal{H}(b)).$$

On the other hand, since $\mathcal{H}(\bar{b}) = \mathcal{H}(\bar{b}_0)$, we see that we can assume that $b = b_0$ is outer. Now using Corollary 17.14 and (20.31), we see that

$$\mathcal{H}(\bar{b}^{2^n}) = \mathcal{H}(\bar{b}) \quad \text{and} \quad \mathfrak{Mult}(\mathcal{H}(b^{2^n})) = \mathfrak{Mult}(\mathcal{H}(b))$$

for any $n \in \mathbb{Z}$. Hence, it will be enough to prove that there is an integer n such that

$$\mathfrak{Mult}(\mathcal{H}(b^{2^n})) = \mathfrak{Mult}(\mathcal{H}(\bar{b}^{2^n})). \tag{28.22}$$

Since the argument of b is continuous on the compact set $F = \{z \in \bar{\mathbb{D}} : |b(z)| = 1\}$, there is some negative integer n such that the argument of $b^{2^n}(z)$ lives in $(-\pi/4, \pi/4)$ for $z \in F$. Now the continuity of b^{2^n} implies that there exists $z_0 \in \bar{\mathbb{D}}$ such that

$$|1 + b^{2^n}(z)| \geq |1 + b^{2^n}(z_0)| \qquad (z \in \bar{\mathbb{D}}).$$

Note that necessarily $b^{2^n}(z_0) \neq -1$. Otherwise we would have $z_0 \in F$ and we would obtain a contradiction with the fact that the argument of $b^{2^n}(z_0)$ lives in $(-\pi/4, \pi/4)$. Hence we have proved the existence of $\delta > 0$ such that

$$|1 + b^{2^n}(z)| \geq \delta > 0,$$

for all $z \in \bar{\mathbb{D}}$. In particular, we see that $\sigma_{H^\infty}(b^{2^n}) \neq \bar{\mathbb{D}}$ and Corollary 28.14 implies (28.22). □

28.4 Characterizations of $\mathcal{M}(a) = \mathcal{H}(b)$

We recall that, by (24.17), for each $\lambda \in \mathbb{T}$, the Clark measure corresponding to $\bar{\lambda} b$ is defined such that

$$\frac{1 - |b(z)|^2}{|\lambda - b(z)|^2} = \int_{\mathbb{T}} \frac{1 - |z|^2}{|z - \zeta|^2} \, d\mu_\lambda(\zeta) \qquad (z \in \mathbb{D}, \ \lambda \in \mathbb{T}).$$

Moreover, if b is nonextreme, then we defined the function F_λ by (see (24.16))

$$F_\lambda = \frac{a}{1 - \bar{\lambda} b} \qquad (\lambda \in \mathbb{T}).$$

Theorem 28.19 *Let b be a nonextreme point of the closed unit ball of H^∞ with Pythagorean mate a. Then the following are equivalent.*

(i) $\mathcal{H}(b) = \mathcal{M}(a)$.

(ii) *For every $\lambda \in \mathbb{T}$, the measure μ_λ is absolutely continuous with respect to the Lebesgue measure m and $T_{F_\lambda/\bar{F}_\lambda}$ is invertible.*

(iii) *There exists a $\lambda \in \mathbb{T}$ such that the measure μ_λ is absolutely continuous with respect to the Lebesgue measure m and $T_{F_\lambda/\bar{F}_\lambda}$ is invertible.*

Proof (i) $\Longrightarrow$ (ii) Lemma 24.21 says that

$$\mathcal{M}(a) = (T_{1-\bar{\lambda}b}T_{\bar{F}_\lambda})(T_{F_\lambda/\bar{F}_\lambda}H^2) \subset T_{1-\bar{\lambda}b}T_{\bar{F}_\lambda}H^2 \subset \mathcal{H}(b).$$

By assumption, $\mathcal{H}(b) = \mathcal{M}(a)$, and thus we must have

$$(T_{1-\bar{\lambda}b}T_{\bar{F}_\lambda})(T_{F_\lambda/\bar{F}_\lambda}H^2) = T_{1-\bar{\lambda}b}T_{\bar{F}_\lambda}H^2 = \mathcal{H}(b). \tag{28.23}$$

But, according to Theorem 24.23, the operator $T_{1-\bar{\lambda}b}T_{\bar{F}_\lambda}$ is an isometry from H^2 into $\mathcal{H}(b)$ and it is surjective if and only if μ_λ is absolutely continuous with respect to m. Thus, μ_λ has to be absolutely continuous with respect to m. Since the operator $T_{1-\bar{\lambda}b}T_{\bar{F}_\lambda}$ is invertible, (28.23) also implies that

$$T_{F_\lambda/\bar{F}_\lambda}H^2 = H^2.$$

In other words, $T_{F_\lambda/\bar{F}_\lambda}$ is surjective. But we know from Theorem 12.24 that $T_{F_\lambda/\bar{F}_\lambda}$ is always injective. Therefore, $T_{F_\lambda/\bar{F}_\lambda}$ is invertible.

(ii) $\Longrightarrow$ (iii) This is trivial.

(iii) $\Longrightarrow$ (i) Fix the parameter $\lambda \in \mathbb{T}$ for which the measure μ_λ is absolutely continuous with respect to the Lebesgue measure m and $T_{F_\lambda/\bar{F}_\lambda}$ is invertible. Since μ_λ is absolutely continuous with respect to the Lebesgue measure m, by Theorem 24.23, we have

$$T_{1-\bar{\lambda}b}T_{\bar{F}_\lambda}H^2 = \mathcal{H}(b).$$

Moreover, since $T_{F_\lambda/\bar{F}_\lambda}$ is invertible, we also have

$$T_{F_\lambda/\bar{F}_\lambda}H^2 = H^2.$$

Therefore, using Lemma 24.21 once more, we obtain

$$\mathcal{M}(a) = (T_{1-\bar{\lambda}b}T_{\bar{F}_\lambda})(T_{F_\lambda/\bar{F}_\lambda}H^2) = T_{1-\bar{\lambda}b}T_{\bar{F}_\lambda}H^2 = \mathcal{H}(b). \qquad \square$$

In Theorem 28.7, we saw that the equality $\mathcal{H}(\bar{b}) = \mathcal{H}(b)$ is connected to the corona problem via the condition $(a, b) \in (HCR)$. The following theorem is a similar result in which the identity $\mathcal{H}(b) = \mathcal{M}(a)$ is addressed.

Theorem 28.20 *Let b be a nonextreme point of the closed unit ball of H^∞ with Pythagorean mate a. Then the following are equivalent.*

(i) $\mathcal{H}(b) = \mathcal{M}(a)$.
(ii) $(a, b) \in (HCR)$ *and* $T_{a/\bar{a}}$ *is invertible.*

Proof (i) $\Longrightarrow$ (ii) If $\mathcal{H}(b) = \mathcal{M}(a)$, then, in the light of Theorem 23.2, we must have

$$\mathcal{M}(a) = \mathcal{M}(\bar{a}) = \mathcal{H}(\bar{b}) = \mathcal{H}(b).$$

But then, according to Theorem 28.7, we have $(a, b) \in (HCR)$ and it follows from Theorem 17.17 that $T_{a/\bar{a}}$ is invertible.

(ii) $\Longrightarrow$ (i) Since $(a, b) \in (HCR)$, Theorem 28.7 ensures that

$$\mathcal{H}(\bar{b}) = \mathcal{H}(b).$$

Similarly, by Theorem 17.17, the invertibility of $T_{a/\bar{a}}$ implies that

$$\mathcal{M}(a) = \mathcal{M}(\bar{a}).$$

Finally, by Theorem 23.2, we always have $\mathcal{M}(\bar{a}) = \mathcal{H}(\bar{b})$. Therefore, we deduce that $\mathcal{H}(b) = \mathcal{M}(a)$. □

In Section 24.1, we saw that the space $\mathcal{H}(b)$ is invariant under the forward shift operator S and we denoted this restricted operator by S_b. In the following result, we characterize $\mathcal{H}(b) = \mathcal{M}(a)$ in terms of a similarity and boundedness of S_b. This result has the same flavor as Theorem 28.7.

Theorem 28.21 *Let b be a nonextreme point of the closed unit ball of H^∞ with Pythagorean mate a. Then the following are equivalent.*

(i) S_b *is similar to* S.
(ii) S_b *is polynomially bounded.*
(iii) S_b *is power bounded.*
(iv) $\mathcal{H}(b) = \mathcal{M}(a)$.

Proof The implications (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii) are trivial.

(iii) $\Longrightarrow$ (iv) Assume that S_b is power bounded. If $f \in \mathcal{H}(b)$, then we have

$$\sup_{n \geq 0} \|S_b^n f\|_b < \infty,$$

and thus Lemma 24.9 implies that $T_{\bar{b}} f \in T_{\bar{a}} H^2$. In other words, the function $\bar{b}f/\bar{a}$ is in H^2. Since

$$|\bar{b}f/\bar{a}|^2 + |f|^2 = |f/a|^2 \qquad \text{(a.e. on } \mathbb{T}),$$

it follows that f/a is in L^2. Now Corollary 4.28 implies that f/a is in fact in H^2. Hence, we conclude that $\mathcal{H}(b) \subset \mathcal{M}(a)$. But, by Theorem 23.2, we have the opposite inclusion and thus $\mathcal{H}(b) = \mathcal{M}(a)$.

(iv) $\Longrightarrow$ (i) Put

$$\begin{array}{rrcl} \mathbf{T}_a: & H^2 & \longrightarrow & \mathcal{M}(a) \\ & f & \longmapsto & T_a f \end{array}$$

and

$$\begin{array}{rrcl} \mathbf{S}_a: & \mathcal{M}(a) & \longrightarrow & \mathcal{M}(a) \\ & f & \longmapsto & Sf. \end{array}$$

In other words, $\mathbf{T}_a$ is the operator T_a whose range, for some technical reasons, is restricted to $\mathcal{M}(a)$, and $\mathbf{S}_a$ is the restriction of the forward shift operator to $\mathcal{M}(a)$. In fact, for each $f \in H^2$, we have

$$S(T_a f) = \chi_1 a f = T_a(Sf).$$

On the one hand, this relation shows that the operator $\mathbf{S}_a$ is well defined. On the other hand, it can be rewritten as

$$\mathbf{S}_a \mathbf{T}_a = \mathbf{T}_a S. \tag{28.24}$$

In other words, we have the following commutative diagram.

$$\begin{tikzcd} H^2 \arrow[r, "S"] \arrow[d, "\mathbf{T}_a"'] & H^2 \arrow[d, "\mathbf{T}_a"] \\ \mathcal{M}(a) \arrow[r, "\mathbf{S}_a"'] & \mathcal{M}(a) \end{tikzcd} \tag{28.25}$$

But, by Theorem 12.19, T_a is an injective operator. Hence, in the first place, $\mathbf{T}_a$ is bijective, and second,

$$\|\mathbf{T}_a f\|_a = \|T_a f\|_a = \|f\|_2 \qquad (f \in H^2).$$

Thus, $\mathbf{T}_a$ is a unitary operator from H^2 onto $\mathcal{M}(a)$. Therefore, (28.24) reveals that $\mathbf{S}_a$ is unitarily equivalent to S. Since $\mathcal{H}(b) = \mathcal{M}(a)$, by the closed graph theorem, the norms $\|\cdot\|_b$ and $\|\cdot\|_{\mathcal{M}(a)}$ are equivalent. Hence, S_b is similar to $\mathbf{S}_a$. Therefore, by (28.24), S_b is similar to S. $\square$

For a Hilbert space $\mathcal{H}$, we used $\mathfrak{Mult}(\mathcal{H})$ to denote the space of all multipliers of H; see Section 9.1.

Theorem 28.22 *Let b be a nonextreme point of the closed unit ball of H^∞ with Pythagorean mate a. Then the following are equivalent.*

(i) $\mathcal{H}(b) = \mathcal{M}(a)$.
(ii) $\mathfrak{Mult}(\mathcal{H}(b)) = H^\infty$.

Proof (i) $\Longrightarrow$ (ii) It is rather trivial that each $\varphi \in H^\infty$ is a multiplier of $\mathcal{M}(a)$. In fact, if $f \in \mathcal{M}(a)$, there exists a $g \in H^2$ such that $f = T_a g = ag$. Hence,

$$\varphi f = \varphi a g = P_+(a\varphi g) = T_a(\varphi g) \in \mathcal{M}(a).$$

Note that, since $\varphi \in H^\infty$, we have $\varphi g \in H^2$. Thus, according to the assumption $\mathcal{H}(b) = \mathcal{M}(a)$, we get $\varphi \in \mathfrak{Mult}(\mathcal{H}(b))$. The other inclusion $\mathfrak{Mult}(\mathcal{H}(b)) \subset H^\infty$ follows from Corollary 9.7.

(ii) $\Longrightarrow$ (i) By Corollary 24.8, $\mathcal{F} = \{M_\varphi : \varphi \in H^\infty\}$ is a closed subspace of $\mathcal{L}(\mathcal{H}(b))$, and there is a constant $c > 0$ such that

$$\|M_\varphi\| \le c\,\|\varphi\|_\infty \qquad (\varphi \in H^\infty). \tag{28.26}$$

Let μ be the Clark measure corresponding to b. We show that P_+ is continuous on $L^2(\mu)$. Recall that, according to Theorem 20.19, if $\varphi \in \mathfrak{Mult}(\mathcal{H}(b))$ then the Toeplitz operator $T_{\bar\varphi}$, defined primarily on the family of analytic polynomials, extends to a continuous operator on $H^2(\mu)$ and

$$T_{\bar\varphi} = V_b^* M_\varphi^* V_b.$$

Hence, by (28.26) and the fact that V_b is unitary, we get

$$\|T_{\bar\varphi}\|_{\mathcal{L}(H^2(\mu))} = \|M_\varphi^*\| = \|M_\varphi\| \le c\,\|\varphi\|_\infty.$$

Thus, for every function $\varphi \in H^\infty$ and every analytic polynomial p, we have

$$\|T_{\bar\varphi} p\|_{H^2(\mu)} \le c\,\|\varphi\|_\infty \|p\|_{H^2(\mu)}.$$

In particular, if $\varphi(z) = z^n$, $n \ge 0$, then we obtain

$$\|P_+(\bar z^n p)\|_{H^2(\mu)} \le c\,\|p\|_{H^2(\mu)}, \tag{28.27}$$

for every analytic polynomial p and every $n \ge 0$.

Now let q be any trigonometric polynomial. Then there exists an $n \ge 0$ such that $p = z^n q$ is an analytic polynomial. Using (28.27), we thus deduce that

$$\|P_+ q\|_{H^2(\mu)} = \|P_+(\bar z^n p)\|_{H^2(\mu)} \le c\,\|p\|_{H^2(\mu)} = c\,\|q\|_{L^2(\mu)}.$$

Since the family of trigonometric polynomials is dense in $L^2(\mu)$, the estimation

$$\|P_+ q\|_{H^2(\mu)} \le c\,\|q\|_{L^2(\mu)}$$

ensures that the Riesz projection P_+ is continuous on $L^2(\mu)$. Hence, by Theorem 12.39, the measure μ is absolutely continuous with respect to the Lebesgue measure and $d\mu = |f|^2\, dm$, where f is an outer function in H^2 such that

$$\operatorname{dist}(\bar f/f, H^\infty) < 1.$$

But, according to (24.14), we also have $d\mu_a = |F|^2\, dm$. Therefore, $|f| = |F|$ almost everywhere on $\mathbb{T}$. Since F and f are outer functions in H^2, this implies that there is a unimodular constant η such that $F = \eta f$. Hence,

$$\operatorname{dist}(\bar{F}/F, H^\infty) = \operatorname{dist}(\bar{f}/f, H^\infty) < 1,$$

and thus it follows from Corollary 12.43 that $T_{F/\bar{F}}$ is invertible. Theorem 28.19, with $\lambda = 1$, now shows that $\mathcal{H}(b) = \mathcal{M}(a)$. □

In Theorems 28.19, 28.20, 28.21 and 28.22 we saw different characterizations for $\mathcal{H}(b) = \mathcal{M}(a)$. For further reference, we gather all these results in the following theorem.

Theorem 28.23 *Let b be a nonextreme point of the closed unit ball of H^∞ with Pythagorean mate a. Then the following are equivalent.*

(i) $\mathcal{H}(b) = \mathcal{M}(a)$.
(ii) $(a, b) \in (HCR)$ *and* $T_{a/\bar{a}}$ *is invertible.*
(iii) S_b *is similar to* S.
(iv) $\mathfrak{Mult}(\mathcal{H}(b)) = H^\infty$.
(v) *For every $\lambda \in \mathbb{T}$, the measure μ_λ is absolutely continuous with respect to the Lebesgue measure m and $T_{F_\lambda/\bar{F}_\lambda}$ is invertible.*
(vi) *There exists a $\lambda \in \mathbb{T}$ such that the measure μ_λ is absolutely continuous with respect to the Lebesgue measure m and $T_{F_\lambda/\bar{F}_\lambda}$ is invertible.*

Let us give an example where $(a, b) \notin (HCR)$, but $T_{a/\bar{a}}$ is invertible.

Example 28.24 Let $a(z) = 2^{-\alpha}(1 - z)^\alpha$, $\alpha \in (0, 1/2)$. We first claim that $|a|^2 \in (HS)$, which means that there exist two bounded real functions u and v on $\mathbb{T}$, with $\|v\|_\infty < \pi/2$, such that $|a|^2 = \exp(u + \tilde{v})$. To verify this fact directly, consider $f(z) = 2i\alpha \log((1-z)/2)$, where log is the principal branch of the logarithm, i.e.

$$\log(w) = \log|w| + i \arg_{(-\pi,\pi)}(w) \qquad (w \in \mathbb{C}^*).$$

Then $f \in H^2$ and, for $z = e^{it}$, $0 < t < 2\pi$, an easy computation shows that we have

$$f(z) = v(t) + i \log|a(z)|^2,$$

where $v(t) = \alpha(\pi - t)$, $0 < t < 2\pi$. Hence, $v \in L^\infty(\mathbb{T})$, $\tilde{v} = \log|a|^2$ and $\|v\|_\infty = \alpha\pi < \pi/2$, which proves that $|a|^2 \in (HS)$. Hence, Corollary 12.43 implies that $T_{a/\bar{a}}$ is invertible. On the other hand, clearly the function $1-|a|^2$ is a bounded log-integrable function on $\mathbb{T}$, and so there is an outer function $b_0 \in H^\infty$ such that $|a|^2+|b_0|^2 = 1$ a.e. on $\mathbb{T}$. Thus (a, b_0) forms a Pythagorean pair. Now consider the Blaschke product B whose zeros are $\Lambda = \{1-2^{-n} : n \geq 1\}$

and set $b = Bb_0$. Then (a, b) is still a Pythagorean pair but $(a, b) \notin (HCR)$ since

$$|a(\lambda_n)| + |b(\lambda_n)| = |a(\lambda_n)| \longrightarrow 0 \qquad (n \longrightarrow \infty).$$

It should be noted that in this example the inclusion

$$\mathfrak{Mult}(\mathcal{H}(b)) \subset \mathfrak{Mult}(\mathcal{H}(\bar{b}))$$

is strict. Indeed, since $T_{a/\bar{a}}$ is invertible and $T_a = T_{\bar{a}} T_{a/\bar{a}}$, the operators T_a and $T_{\bar{a}}$ have the same range. In other words, we have

$$\mathcal{M}(a) = \mathcal{M}(\bar{a}) = \mathcal{H}(\bar{b}).$$

But any function in H^∞ clearly is a multiplier of $\mathcal{M}(a)$, whence

$$\mathfrak{Mult}(\mathcal{H}(\bar{b})) = H^\infty.$$

On the other hand, it follows from Theorem 28.23 that

$$\mathfrak{Mult}(\mathcal{H}(b)) \subsetneq H^\infty.$$

Exercises

Exercise 28.4.1 In Theorem 28.21, we saw that, if S_b is similar to S, then $\mathcal{H}(b) = \mathcal{M}(a)$. In this exercise, we give a direct proof of this implication.

Assume that S_b is similar to S and let $A : H^2 \longrightarrow \mathcal{H}(b)$ be the isomorphism such that $S_b A = AS$ – see the following commutative diagram.

$$\begin{tikzcd} H^2 \arrow[r, "S"] \arrow[d, "A"'] & H^2 \arrow[d, "A"] \\ \mathcal{H}(b) \arrow[r, "S_b"'] & \mathcal{H}(b) \end{tikzcd} \tag{28.28}$$

Since $\mathcal{H}(b)$ is embedded contractively into H^2, the operator

$$\begin{array}{rccc} \mathbf{A}: & H^2 & \longrightarrow & H^2 \\ & f & \longmapsto & Af \end{array}$$

is well defined and bounded.

(i) Show that $\mathbf{A}S = S\mathbf{A}$.
(ii) Deduce that there is a $u \in H^\infty$ such that $\mathbf{A} = T_u$.
(iii) Prove that $\mathcal{M}(u) = \mathcal{H}(b)$.
(iv) Prove that $\mathcal{H}(b) = \mathcal{M}(a)$.

Hint: For (ii), use Theorem 12.27, and for (iv) use Theorem 23.6; see also Theorem 23.2.

Exercise 28.4.2 Let b be a nonextreme point of the closed unit ball H^∞ with the Pythagorean mate a. Assume that the Clark measure μ_1 is absolutely continuous with respect to the Lebesgue measure m. Prove that the following are equivalent:

(i) The pair $(a, b) \in (HCR)$ and $|a|^2 \in (HS)$.
(ii) We have

$$\left|\frac{a}{1-b}\right|^2 \in (HS).$$

Hint: Use Theorem 28.23 and Corollary 12.43.

28.5 Invariant subspaces of S_b when $b(z) = (1 + z)/2$

In Theorem 24.31, in the case where b is a nonextreme point of the closed unit ball of H^∞, it was shown that the invariant subspaces of X_b are precisely the intersection of $\mathcal{H}(b)$ with the invariant subspaces of S^*. The lattice of invariant subspaces of X_b is thus isomorphic to the lattice $\{K_\Theta : \Theta \text{ an inner function}\}$. But the situation for the invariant subspaces of S_b is more complicated. If Θ is an inner function, then $\mathcal{H}(b) \cap \mathcal{M}(\Theta)$ is obviously an invariant subspace of S_b. If $(a, b) \in (HCR)$ and $T_{a/\bar{a}}$ is invertible, we will see that every nontrivial invariant subspace of S_b has the preceding form. But, in general, this is not the case, as we will see by studying the example when $b(z) = (1 + z)/2$.

Theorem 28.25 *Let b be a nonextreme point of the closed unit ball of H^∞ and assume that $(a, b) \in (HCR)$ and that $T_{a/\bar{a}}$ is invertible. Let E be a closed subspace of $\mathcal{H}(b)$. Then the following statements are equivalent.*

(i) *E is invariant under S_b.*
(ii) *There exists an inner function Θ such that $E = \mathcal{M}(\Theta) \cap \mathcal{H}(b)$.*

Proof Since $(a, b) \in (HCR)$ and $T_{a/\bar{a}}$ is invertible, then we know from Theorem 28.20 that $\mathcal{H}(b) = \mathcal{M}(a)$ and now the result is a special case of Theorem 17.19. □

We now examine a case for which we cannot apply Theorem 28.25. However, for this example, the invariant subspace lattice of S_b is determined below, and then we see that this lattice is more complicated than Beurling's lattice.

Put $b(z) = (1 + z)/2$. Then it is easy to verify that $a(z) = (1 - z)/2$. Moreover,

$$|a(z)|^2 + |b(z)|^2 = \frac{1 + |z|^2}{2} \geq \frac{1}{2} \qquad (z \in \mathbb{D}),$$

and thus the functions a and b form a corona pair. Hence, by Theorems 28.7 and 23.2, we have

$$\mathcal{M}(\bar{a}) = \mathcal{H}(\bar{b}) = \mathcal{H}(b).$$

But $a(z)/\overline{a(z)} = -z$, a.e. on $\mathbb{T}$, and thus

$$T_{a/\bar{a}} = -S,$$

which, in particular, implies that $T_{a/\bar{a}}$ is not invertible. We recall that the invertibility of $T_{a/\bar{a}}$ was one of the conditions of Theorem 28.25.

We now find the lattice of S_b and see that it is more complicated than Beurling's lattice. Since $\mathcal{H}(b) = \mathcal{M}(\bar{a})$, the operator S_b is similar to the restriction of S to $\mathcal{M}(\bar{a})$. It is actually more convenient (and obviously permissible) to work in $\mathcal{M}(2\bar{a})$ rather than $\mathcal{M}(\bar{a})$. We let e denote the function $2a$, that is, $e(z) = 1 - z$, and we denote the norm and inner product in the space $\mathcal{M}(\bar{e})$ respectively by $\|\cdot\|_{\bar{e}}$ and $\langle\cdot,\cdot\rangle_{\bar{e}}$. For simplicity, the operator $S_{\mathcal{M}(\bar{e})}$ will be denoted by $S_{\bar{e}}$. The following lemma describes the functions in $\mathcal{M}(\bar{e})$. A part of this result is a particular case of the more general Theorem 27.20. However, we still give another proof in this special case.

Lemma 28.26 *Let $f \in H^2$. Then $f \in \mathcal{M}(\bar{e})$ if and only if there exists a function $g \in H^2$ and a constant $c \in \mathbb{C}$ such that*

$$f = (S - I)g + c.$$

The function g and the constant c in this decomposition are unique. Moreover, if f_1 and f_2 are two functions in $\mathcal{M}(\bar{e})$ with $f_j = (S-1)g_j + c_j$, $j = 1, 2$, then

$$\langle f_1, f_2\rangle_{\bar{e}} = \langle g_1, g_2\rangle_2 + c_1\overline{c_2}. \tag{28.29}$$

Proof The operator $I - SS^* \in \mathcal{L}(H^2)$, which is the orthogonal projection on the one-dimensional subspace of constant functions, can be rewritten as

$$I - SS^* = 1 \otimes 1.$$

Equivalently, $I - S^* = (S - I)S^* + 1 \otimes 1$. Since $T_{\bar{e}} = I - S^*$, the previous identity becomes

$$T_{\bar{e}} = (S - I)S^* + 1 \otimes 1. \tag{28.30}$$

Let $f \in \mathcal{M}(\bar{e})$. Then using (28.30), there exists a function $h \in H^2$ such that

$$f = T_{\bar{e}}h = (S - I)S^*h + h(0).$$

This is the required representation with $g = S^*h$ and $c = h(0)$. Conversely, if f has the representation $f = (S - I)g + c$ with $g \in H^2$ and $c \in \mathbb{C}$, put $h = Sg + c$. Then $h \in H^2$ and we have

$$T_{\bar{e}}h = (I - S^*)h = h - S^*h = Sg + c - S^*Sg = Sg - g + c = f.$$

Thus, $f \in \mathcal{M}(\bar{e})$.

We now prove the uniqueness of the function g and constant c in the decomposition of f. Let g_1 and g_2 be two functions in H^2 and let c_1 and c_2 be two constants satisfying

$$f = (S - I)g_1 + c_1 = (S - I)g_2 + c_2.$$

Applying S^* to f gives

$$S^*f = S^*(S - I)g_1 = S^*(S - I)g_2,$$

whence, remembering that $S^*S = I$,

$$(I - S^*)g_1 = (I - S^*)g_2.$$

In other words, we must have $T_{\bar{e}}\, g_1 = T_{\bar{e}}\, g_2$. But, since e is outer, the operator $T_{\bar{e}}$ is one-to-one (Theorem 12.19) and the last identity implies that $g_1 = g_2$. That $c_1 = c_2$ is now obvious.

It remains to prove (28.29). Let f_1 and f_2 be two functions in $\mathcal{M}(\bar{e})$ with the representations $f_j = (S - 1)g_j + c_j$, $j = 1, 2$. Put $h_j = Sg_j + c_j$, $j = 1, 2$. By the preceding computation, we have $f_j = T_{\bar{e}}h_j$ and thus

$$\begin{aligned}
\langle f_1, f_2\rangle_{\bar{e}} &= \langle h_1, h_2\rangle_2 \\
&= \langle Sg_1 + c_1,\ Sg_2 + c_2\rangle_2 \\
&= \langle Sg_1, Sg_2\rangle_2 + \langle c_1, c_2\rangle_2 \\
&= \langle g_1, g_2\rangle_2 + c_1\overline{c_2}.
\end{aligned}$$

This completes the proof. □

Lemma 28.26 reveals that all constant functions belong to $\mathcal{M}(\bar{e})$. Hence, at least, we have

$$\chi_0 \in \mathcal{M}(\bar{e}).$$

Lemma 28.27 *Every function $f \in \mathcal{M}(\bar{e})$ has a nontangential limit at the point 1, denoted as usual by $f(1)$. Moreover, the functional*

$$\begin{array}{ccc}
\mathcal{M}(\bar{e}) & \longrightarrow & \mathbb{C} \\
f & \longmapsto & f(1)
\end{array}$$

is bounded and we have

$$f(1) = \langle f, \chi_0\rangle_{\bar{e}} \qquad (f \in \mathcal{M}(\bar{e})).$$

Proof Let f be any function in $\mathcal{M}(\bar{e})$. Then, by Lemma 28.26, there exists a function g in H^2 and a constant c such that $f = (S - I)g + c$. We then have

$$|f(z) - c| \leq |z - 1| \, \|g\|_2 (1 - |z|^2)^{-1/2} \qquad (z \in \mathbb{D}).$$

If z is in a Stolz domain anchored at the point 1, then we have

$$|f(z) - c| \leq K(1 - |z|)^{1/2} \|g\|_2,$$

and therefore

$$\lim_{\substack{z \to 1 \\ \sphericalangle}} f(z) = c.$$

Moreover, according to (28.29), we have

$$\|f\|_{\bar{e}}^2 = \|g\|_2^2 + |f(1)|^2 \geq |f(1)|^2,$$

which proves that $\|f(1)\| \leq \|f\|_{\bar{e}}$ and the functional $f \longmapsto f(1)$ is bounded on $\mathcal{M}(\bar{e})$. Finally, if f is in $\mathcal{M}(\bar{e})$ (using the fact that $1 = (S - 1)0 + 1$), we get from (28.29) that

$$\langle f, 1 \rangle_{\bar{e}} = \langle f, 0 \rangle_2 + f(1) = f(1). \qquad \square$$

Corollary 28.28 *The space $\mathcal{M}(e)$ is a closed subspace of $\mathcal{M}(\bar{e})$ and we have*

$$\mathcal{M}(\bar{e}) = \mathcal{M}(e) \oplus^{\perp} \mathbb{C}1. \tag{28.31}$$

Proof That $\mathcal{M}(e)$ is a subspace of $\mathcal{M}(\bar{e})$ follows from Lemma 28.26 and the observation that $\mathcal{M}(e) = (S - I)H^2$. Using Lemma 28.26 once more, we easily see that $(\mathbb{C}1)^{\perp} = \mathcal{M}(e)$. That proves that $\mathcal{M}(e)$ is closed and also we have the orthogonal decomposition (28.31). $\square$

Corollary 28.29 *We have*

$$\mathfrak{Mult}(\mathcal{M}(\bar{e})) = H^\infty \cap \mathcal{M}(\bar{e}).$$

Proof Let $u \in H^\infty \mathcal{M}(\bar{e})$ and let $f \in \mathcal{M}(\bar{e})$. Write $f = (S - I)g + c$, $u = (S - I)g_1 + c_1$, with $g, g_1 \in H^2$ and $c, c_1 \in \mathbb{C}$. Then

$$\begin{aligned} u(z)f(z) &= u(z)((z - 1)g(z) + c) = (z - 1)u(z)g(z) + cu(z) \\ &= (z - 1)u(z)g(z) + c((z - 1)g_1(z) + c_1) \\ &= (z - 1)(u(z)g(z) + cg_1(z)) + cc_1, \end{aligned}$$

which can be rewritten as $uf = (S - I)(ug + cg_1) + cc_1$. Since $ug + cg_1 \in H^2$, we get using Lemma 28.26 that $uf \in \mathcal{M}(\bar{e})$ and then $u \in \mathfrak{Mult}(\mathcal{M}(\bar{e}))$.

Conversely, let $u \in \mathfrak{Mult}(\mathcal{M}(\bar{e}))$. Then, by Corollary 9.7, $u \in H^\infty$ and, since $1 \in \mathcal{M}(\bar{e})$, we have $u = u1 \in \mathcal{M}(\bar{e})$. $\square$

The following result gives more precise information for an inner function to belong to $\mathcal{M}(\bar{e})$.

Theorem 28.30 *Let Θ be an inner function with zero sequence $(z_j)_j$ and singular measure μ. The following are equivalent.*

(i) *The function Θ is in $\mathcal{M}(\bar{e})$.*
(ii) *We have $1 - |\Theta(r)|^2 = O(1 - r^2)$, as $r \longrightarrow 1^-$.*
(iii) *The function Θ has an angular derivative in the sense of Carathéodory at the point 1.*
(iv) *We have*

$$\sum_{j=1}^{\infty} \frac{1 - |z_j|^2}{|z_j - 1|^2} + \int_{\mathbb{T}} \frac{d\mu(z)}{|z - 1|^2} < +\infty.$$

Proof The equivalence between (ii) and (iii) is a special case of Theorem 21.1.

The implication (iv) $\Longrightarrow$ (iii) follows from Corollary 21.11.

The implication (ii) $\Longrightarrow$ (iv) follows from Theorem 21.26 (apply for $N = 0$, then (ii) says that $\|k_r^{\Theta}\|_2^2 = (1 - |\Theta(r)|^2)/(1 - r^2)$ is bounded on $[0, 1)$.)

To complete the proof, we shall establish the equivalence of (i) and (ii).

Suppose first that the inner function Θ belongs to $\mathcal{M}(\bar{e})$. Then the function g defined by

$$g(z) := \frac{\Theta(z) - \Theta(1)}{z - 1} \qquad (z \in \mathbb{D})$$

belongs to H^2 by Lemma 28.26. Since $|\Theta| = 1$ a.e. on $\mathbb{T}$, we must have $|\Theta(1)| = 1$. Without loss of generality, we will assume that $\Theta(1) = 1$. For $0 < r < 1$, we have $|e^{i\theta} - 1/r| \geq |e^{i\theta} - 1|$ and then

$$\begin{aligned}
\|g\|_2^2 &= \frac{1}{2\pi} \int_0^{2\pi} \left| \frac{\Theta(e^{i\theta}) - 1}{e^{i\theta} - 1} \right|^2 d\theta \\
&\geq \frac{1}{2\pi} \int_0^{2\pi} \left| \frac{\Theta(e^{i\theta}) - 1}{e^{i\theta} - 1/r} \right|^2 d\theta \\
&= \frac{r^2}{2\pi} \int_0^{2\pi} \left| \frac{\Theta(e^{i\theta}) - 1}{re^{i\theta} - 1} \right|^2 d\theta \\
&= \frac{r^2}{2\pi} \int_0^{2\pi} \frac{2 - 2\,\Re(\Theta(e^{i\theta}))}{|e^{i\theta} - r|^2}\, d\theta \\
&= \frac{r^2}{\pi(1 - r^2)} \int_0^{2\pi} (1 - \Re(\Theta(e^{i\theta}))) \frac{1 - r^2}{|e^{i\theta} - r|^2}\, d\theta.
\end{aligned}$$

Since $1 - \Re(\Theta(z))$ is harmonic, we obtain

$$\begin{aligned}\|g\|_2^2 &\geq \frac{2r^2}{1-r^2}(1-\Re(\Theta(r))) = \frac{r^2}{1-r^2}(|1-\Theta(r)|^2 + 1 - |\Theta(r)|^2)\\ &\geq r^2\frac{1-|\Theta(r)|^2}{1-r^2},\end{aligned}$$

which gives the estimate $1 - |\Theta(r)|^2 = O(1-r^2)$, as $r \longrightarrow 1^-$. So the implication (i) $\Longrightarrow$ (ii) is established.

To prove the converse implication, assume that Θ satisfies (ii). Let ϑ be the inner function $\Theta\Theta^*$, where $\Theta^*(z) = \overline{\Theta(\bar{z})}$. Condition (ii) for Θ implies that ϑ has 1 as a radial limit at the point 1 and $1 - \vartheta(r) = 1 - |\Theta(r)|^2 = O(1-r^2)$, as $r \longrightarrow 1^-$. If we replace Θ by ϑ in the string of equalities above, we get, for $0 < r < 1$,

$$\frac{1}{2\pi}\int_0^{2\pi}\left|\frac{\vartheta(e^{i\theta})-1}{e^{i\theta}-1/r}\right|^2 d\theta = \frac{2r^2(1-\vartheta(r))}{1-r^2}.$$

The left-hand side is therefore bounded as $r \longrightarrow 1^-$, so, applying the monotone convergence theorem, we can conclude that the function $(\vartheta - 1)/(z-1)$ is in L^2. But since $z - 1$ is an outer function, the function $(\vartheta - 1)/(z-1)$ is indeed in H^2 and hence, by Lemma 28.26, ϑ is in $\mathcal{M}(\bar{e})$. As $\Theta = T_{\overline{\Theta^*}}\vartheta$ and since $\mathcal{M}(\bar{e})$ is invariant under $T_{\overline{\Theta^*}}$ (see Theorem 17.15), we conclude that Θ is also in $\mathcal{M}(\bar{e})$. □

Corollary 28.31 *Let Θ be an inner function and assume that Θ is in $\mathcal{M}(\bar{e})$. Then*

$$\|\Theta\|_{\bar{e}} = 1 + |\Theta'(1)|,$$

where $\Theta'(1)$ denotes the angular derivative of Θ at 1.

Proof According to Theorem 28.30, we know that the inner function Θ has an angular derivative in the sense of Carathéodory at 1 and we have

$$\Theta(z) = (z-1)g(z) + \Theta(1),$$

with $g \in H^2$ and $|\Theta(1)| = 1$. But then

$$g(z) = \frac{\Theta(z)-\Theta(1)}{z-1} = \Theta(1)\frac{1-\overline{\Theta(1)}\Theta(z)}{1-z} = \Theta(1)k_1^\Theta(z).$$

Hence by Theorem 21.1(f), we have

$$\|g\|_2^2 = \|k_1^\Theta\|_2^2 = |\Theta'(1)|,$$

and Lemma 28.26 implies that

$$\|\Theta\|_{\bar{a}}^2 = \|g\|_2^2 + |\Theta(1)|^2 = 1 + |\Theta'(1)|.$$ □

Lemma 28.32 *Let Θ be an inner function. Then $\mathcal{M}(\bar{e}) \cap \mathcal{M}(\Theta)$ and $\mathcal{M}(e) \cap \mathcal{M}(\Theta)$ are closed invariant subspaces for $S_{\bar{e}} = S_{\mathcal{M}(\bar{e})}$. Moreover, we have*

$$\mathcal{M}(e) \cap \mathcal{M}(\Theta) \subset \mathcal{M}(\bar{e}) \cap \mathcal{M}(\Theta), \tag{28.32}$$

and the inclusion in (28.32) is an equality if and only if $\Theta \notin \mathcal{M}(\bar{e})$.

Proof Since Θ is an inner function, $\mathcal{M}(\Theta)$ is a closed subspace of H^2 and then $\mathcal{M}(\bar{e}) \cap \mathcal{M}(\Theta)$ is a closed subspace of $\mathcal{M}(\bar{e})$, because $\mathcal{M}(\bar{e})$ is boundedly contained in H^2. It follows from Corollary 28.28 that the same is true for $\mathcal{M}(e) \cap \mathcal{M}(\Theta)$. Moreover, the inclusion (28.32) also follows immediately from Corollary 28.28. It remains to check that the inclusion is an equality if and only if $\Theta \notin \mathcal{M}(\bar{e})$. So assume first that $\Theta \in \mathcal{M}(\bar{e})$. Then we have $\Theta \in \mathcal{M}(\bar{e}) \cap \mathcal{M}(\Theta)$. But we also have $\Theta \notin \mathcal{M}(e)$ because, if Θ belongs to $\mathcal{M}(e)$, then there exists h in H^2 such that $\Theta = eh$, but, since Θ is inner and e is outer, we get a contradiction. That proves that $\mathcal{M}(e) \cap \mathcal{M}(\Theta) \subsetneq \mathcal{M}(\bar{e}) \cap \mathcal{M}(\Theta)$.

Now assume that $\Theta \notin \mathcal{M}(\bar{e})$ and let us prove that we have equality in (28.32). Let $f \in \mathcal{M}(\bar{e}) \cap \mathcal{M}(\Theta)$. Write

$$f(z) = (z-1)g(z) + f(1) = \Theta(z)h(z), \tag{28.33}$$

with g, h in H^2. First note that

$$T_{\bar{\Theta}} f = T_{\bar{\Theta}} \Theta h = P_+ h = h,$$

because Θ is an inner function. Since $\mathcal{M}(\bar{e})$ is invariant under $T_{\bar{\Theta}}$, we get that $h \in \mathcal{M}(\bar{e})$ and we can also decompose h as

$$h(z) = (z-1)h_1(z) + h(1).$$

Plugging in (28.33) gives

$$(z-1)g(z) + f(1) = (z-1)\Theta(z)h_1(z) + \Theta(z)h(1).$$

Since $\Theta \notin \mathcal{M}(\bar{e})$, the preceding relation implies necessarily that $h(1) = 0$ and then

$$(z-1)g(z) + f(1) = (z-1)\Theta(z)h_1(z). \tag{28.34}$$

But because $\lim_{r \to 1}(r-1)\varphi(r) = 0$, for every φ in H^2, we get from (28.34) that $f(1) = 0$. Therefore $f(z) = (z-1)g(z) = ((S-I)g)(z)$ and then $f \in \mathcal{M}(e)$. That proves that $\mathcal{M}(\bar{e}) \cap \mathcal{M}(\Theta) \subset \mathcal{M}(e) \cap \mathcal{M}(\Theta)$. □

The next result shows that the lattice of $S_{\bar{e}}$ is described by the subspaces of Lemma 28.32.

Theorem 28.33 *The closed invariant subspaces of* $S_{\bar{e}} = S_{\mathcal{M}(\bar{e})}$*, besides* $\{0\}$*, are the subspaces* $\mathcal{M}(\bar{e})\cap\mathcal{M}(\Theta)$ *and* $\mathcal{M}(e)\cap\mathcal{M}(\Theta)$*, with* Θ *an inner function.*

Proof That the subspaces mentioned in the theorem are closed invariant subspaces under $S_{\bar{e}}$ follows from Lemma 28.32. It remains to show that there are no others. Let $\mathfrak{J}$ be a closed $S_{\bar{e}}$-invariant subspace. We will distinguish three cases.

First case: $\mathfrak{J}$ *is contained in* $\mathcal{M}(e)$*.*

Let

$$\begin{array}{rccc} \mathbf{T}_e: & H^2 & \longrightarrow & \mathcal{M}(e) \\ & f & \longmapsto & f. \end{array}$$

Then the operator $\mathbf{T}_e$ is clearly a unitary operator. Moreover, we have

$$Z_{|\mathcal{M}(e)}\mathbf{T}_e = \mathbf{T}_e S,$$

whence $Z_{|\mathcal{M}(e)}$ is unitarily equivalent to S. This, together with Beurling's theorem, implies that $\mathfrak{J} = T_e\mathcal{M}(\Theta)$, with Θ an inner function. Let us prove that

$$T_e\mathcal{M}(\Theta) = \mathcal{M}(\Theta) \cap \mathcal{M}(e). \tag{28.35}$$

The inclusion $T_e\mathcal{M}(\Theta) \subset \mathcal{M}(\Theta) \cap \mathcal{M}(e)$ is trivial (because $T_eT_\Theta = T_\Theta T_e$). For the reverse inclusion, let $f \in \mathcal{M}(\Theta) \cap \mathcal{M}(e)$, that is, $f = u\varphi = eh$, with $\varphi, h \in H^2$. Note that, on $\mathbb{T}$, the function $\varphi/e = \bar{\Theta}h \in L^2$. Since e is outer, Corollary 4.28 implies that φ/e is in H^2. Thus $h = u\varphi/e \in \mathcal{M}(u)$. Hence $f = eh$ belongs to $T_e\mathcal{M}(u)$, which proves (28.35). Therefore $\mathfrak{J} = \mathcal{M}(\Theta) \cap \mathcal{M}(e)$.

Second case: $\mathfrak{J}$ *is not contained in* $\mathcal{M}(e)$ *and the greatest common inner divisor of the functions in* $\mathfrak{J}$ *is* 1.

Let θ be the greatest common inner divisor of the functions in $\mathfrak{J}\cap\mathcal{M}(e)$. First, note that $\theta = 1$. Indeed, let f be in $\mathfrak{J}$. Then, since $\mathfrak{J}$ is Z-invariant, the function $(z-1)f$ is in $\mathfrak{J}\cap\mathcal{M}(e)$. Therefore, there exists h in H^2 such that

$$(z-1)f(z) = \theta(z)h(z). \tag{28.36}$$

But since θ is inner and $z-1$ is outer, equality (28.36) implies that θ must divide f and, since this holds for every function in $\mathfrak{J}$, we conclude that $\theta = 1$. Now $\mathfrak{J}\cap\mathcal{M}(e)$ is a closed $S_{\bar{e}}$-invariant subspace contained in $\mathcal{M}(e)$ and then, by the first case, there exists an inner function Θ such that

$$\mathfrak{J}\cap\mathcal{M}(e) = \mathcal{M}(\Theta)\cap\mathcal{M}(e).$$

But since the greatest common inner divisor of the functions in $\mathfrak{J}\cap\mathcal{M}(e)$ is 1, necessarily we have $\Theta = 1$ and thus

$$\mathfrak{J}\cap\mathcal{M}(e) = \mathcal{M}(e),$$

that is, $\mathcal{M}(e) \subset \mathfrak{J}$. Now since $\mathfrak{J}$ is not contained in $\mathcal{M}(e)$, there exists a function h in $\mathfrak{J}$ that is not in $\mathcal{M}(e)$. Then $h = (S-I)g + h(1)$, with g in H^2, and, since h is not in $\mathcal{M}(e)$, we must have $h(1) \neq 0$. Then $h(1) = h-(S-I)g$ belongs to $\mathfrak{J}$ (because $(S-I)g \in \mathcal{M}(e) \subset \mathfrak{J}$). Thus $\mathfrak{J}$ contains the constant and then $\mathfrak{J} = \mathcal{M}(\bar{e})$.

Third case: $\mathfrak{J}$ is not contained in $\mathcal{M}(e)$ and Θ, the greatest common inner divisor of the functions in $\mathfrak{J}$, is not 1.

Since $\mathfrak{J}$ is not contained in $\mathcal{M}(e)$, there exists a function f in $\mathfrak{J}$ with $f(1) \neq 0$. Moreover, by definition of Θ, we have $f = \Theta g$, where $g = T_{\bar{\Theta}} f$ is in $\mathcal{M}(\bar{e})$ (recall that $\mathcal{M}(\bar{e})$ is $T_{\bar{\varphi}}$-invariant for any φ in H^∞). Now write $f = (S-I)f_1 + f(1)$, $g = (S-I)g_1 + g(1)$, with f_1, g_1 in H^2. Then plugging these two relations into $f = \Theta g$ gives

$$(z-1)f_1(z) + f(1) = \Theta(z)(z-1)g_1(z) + \Theta(z)g(1). \tag{28.37}$$

Using the fact that $|\varphi(z)| = o((1-|z|)^{-1})$, $|z| \longrightarrow 1$, for any function φ in H^2, we see that (28.37) implies that $g(1) \neq 0$ and also that Θ has a (nonzero) radial limit $\Theta(1)$ at the point 1. We then have

$$\frac{f(z)-f(1)}{z-1} = u(z)\,\frac{g(z)-g(1)}{z-1} + g(1)\,\frac{\Theta(z)-\Theta(1)}{z-1}.$$

The term on the left and the first term on the right belong to H^2 (because f and g belong to $\mathcal{M}(\bar{e})$) and therefore so does the second term on the right. Since $g(1) \neq 0$, that means that Θ is in $\mathcal{M}(\bar{e})$.

Now let $\mathfrak{K} = T_{\bar{\Theta}}\mathfrak{J}$. Then $\mathfrak{K}$ is contained in $\mathcal{M}(\bar{e})$ and we have

$$S_{\bar{e}}\mathfrak{K} \subset \mathfrak{K}.$$

Indeed, let $h \in \mathfrak{K}$, which means that there is a function h_1 in $\mathfrak{J}$ such that $h = T_{\bar{\Theta}} h_1$. But, by the definition of Θ, since h_1 belongs to $\mathfrak{J}$, Θ divides h_1, that is, there is a function h_2 in H^2 such that $h_1 = \Theta h_2$. Then $h = T_{\bar{\Theta}} h_1 = h_2$ and we have

$$\begin{aligned} S_{\bar{e}} h &= Z T_{\bar{\Theta}} h_1 = Z h_2 \\ &= T_{\bar{\Theta}}(\Theta Z h_2) \\ &= T_{\bar{\Theta}}(Z\Theta h_2) = T_{\bar{\Theta}}(Z h_1). \end{aligned}$$

Since $h_1 \in \mathfrak{J}$ and $\mathfrak{J}$ is Z-invariant, we obtain that $Zh \in T_{\bar{\Theta}}\mathfrak{J} = \mathfrak{K}$. Therefore, $Z\mathfrak{K} \subset \mathfrak{K}$ and, by continuity, the closure of $\mathfrak{K}$ in $\mathcal{M}(\bar{e})$, say $\widetilde{\mathfrak{K}}$, is a closed Z-invariant subspace. Now note that, if v is the greatest common inner divisor of the functions in $\widetilde{\mathfrak{K}}$, then $v = 1$. Indeed, for every function h in $\mathfrak{J}$, we have $T_{\bar{\Theta}} h = v\varphi$ for some function φ in H^2. But, since h belongs to $\mathfrak{J}$, there is a function h_1 in H^2 such that $h = \Theta h_1$ and then $h_1 = T_{\bar{\Theta}} h = v\varphi$, whence

$h = \Theta v \varphi$. That means that the inner function Θv divides every function h in $\mathfrak{J}$. But, by the definition of Θ, this is possible only if $v = 1$. Therefore, we can apply the second case to $\widetilde{\mathfrak{K}}$ and we get that $\widetilde{\mathfrak{K}} = \mathcal{M}(\bar{e})$ (note that $g = T_{\bar{\Theta}} f$ is in $\mathfrak{K}$ and g does not belong to $\mathcal{M}(e)$ because $g(1) \neq 0$ as we have seen). We shall complete the proof by showing that

$$\mathfrak{J} = \mathcal{M}(\bar{e}) \cap \mathcal{M}(\Theta).$$

The first inclusion $\mathfrak{J} \subset \mathcal{M}(\bar{e}) \cap \mathcal{M}(\Theta)$ is trivial by the definition of Θ. For the opposite inclusion, first note that

$$\Theta \mathfrak{K} = \mathfrak{J}. \tag{28.38}$$

Indeed, let h be in $\mathfrak{K}$, which means that $h = T_{\bar{\Theta}} h_1$, for some h_1 in $\mathfrak{J}$. Since h_1 is in $\mathfrak{J}$, we have $h_1 = \Theta h_2$, for some h_2 in H^2. Then $T_{\bar{\Theta}} h_1 = h_2$ and

$$\Theta h = \Theta T_{\bar{\Theta}} h_1 = \Theta h_2 = h_1,$$

which implies that $\Theta h \in \mathfrak{J}$. Conversely, let $g_1 \in \mathfrak{J}$. Then $g_1 = \Theta g_2$, for some g_2 in H^2. Then $g_2 = T_{\bar{\Theta}} g_1 \in T_{\bar{\Theta}} \mathfrak{J} = \mathfrak{K}$, whence $g_1 \in \Theta \mathfrak{K}$, which concludes the proof of (28.38). Now we will show that

$$\Theta \mathcal{M}(\bar{e}) \subset \mathfrak{J}.$$

Note that, since Θ is in $\mathcal{M}(\bar{e})$, it follows from Corollary 28.29 that $\Theta \in \mathcal{M}(\mathcal{M}(\bar{e}))$. So, let $h \in \mathcal{M}(\bar{e}) = \widetilde{\mathfrak{K}}$. Then, by definition, there exists h_n in $\mathfrak{K}$ such that $h_n \longrightarrow h$ in $\mathcal{M}(\bar{e})$, as $n \longrightarrow +\infty$. Using the fact that $\Theta \in \mathcal{M}(\mathcal{M}(\bar{e}))$, we obtain that $\Theta h_n \longrightarrow \Theta h$ in $\mathcal{M}(\bar{e})$, as $n \longrightarrow +\infty$. But $\Theta h_n \in \Theta \mathfrak{K} = \mathfrak{J}$, by (28.38), and thus $\Theta h \in \mathfrak{J}$ (recall that $\mathfrak{J}$ is a closed subspace of $\mathcal{M}(\bar{e})$). Therefore $\Theta \mathcal{M}(\bar{e}) \subset \mathfrak{J}$. To conclude, we will show that

$$\Theta \mathcal{M}(\bar{e}) = \mathcal{M}(\bar{e}) \cap \mathcal{M}(\Theta).$$

The inclusion $\Theta \mathcal{M}(\bar{e}) \subset \mathcal{M}(\bar{e}) \cap \mathcal{M}(\Theta)$ follows easily because Θ is a multiplier of $\mathcal{M}(\bar{e})$. For the converse inclusion, let $h \in \mathcal{M}(\bar{e}) \cap \mathcal{M}(\Theta)$. Then $h = \Theta h_1 = T_{\bar{e}} h_2$, with $h_1, h_2 \in H^2$. Then

$$h_1 = T_{\bar{\Theta}} h = T_{\bar{\Theta}} T_{\bar{e}} h_2 = T_{\bar{e}} T_{\bar{\Theta}} h_2,$$

whence $h_1 \in \mathcal{M}(\bar{e})$ and $h = \Theta h_1 \in \Theta \mathcal{M}(\bar{e})$. Finally, we obtain that

$$\mathcal{M}(\bar{e}) \cap \mathcal{M}(\Theta) \subset \mathfrak{J},$$

which gives the desired inclusion and concludes the proof. □

Exercises

Exercise 28.5.1 The purpose of this exercise is to give another proof of Theorem 28.25. Let b be a nonextreme point of the closed unit ball of H^∞ and assume that $(a, b) \in (HCR)$ and $T_{a/\bar{a}}$ is invertible. Recall that, in this case, by Theorems 28.20 and 28.22, we have $\mathfrak{Mult}(\mathcal{H}(b)) = H^\infty$ and $\mathcal{H}(b) = \mathcal{M}(a)$.

First, let E be a closed invariant subspace of S_b.

(i) Show that $T_a \subset E$.

(ii) Let $f \in E$, $f \perp T_a E$ (with respect to the scalar product of $\mathcal{M}(a)$).

   (a) Justify that $f \perp S^n T_a f$, for every $n \geq 0$.

   (b) Writing $f = T_a g$, $g \in H^2$, show that

$$\langle f, S^n T_u f \rangle_{\mathcal{M}(a)} = \frac{1}{2\pi} \int_0^{2\pi} a(e^{i\theta}) |g(e^{i\theta})|^2 e^{in\theta} \, d\theta.$$

   (c) Show that $f \equiv 0$ and that $T_a E$ is dense in E with respect to the norm of $\mathcal{H}(b)$.

Second, let E be a closed S_b-invariant subspace and let $\mathfrak{K}$ be the closure of E in H^2.

(iii) Show that there exists an inner function Θ such that $\mathfrak{K} = \mathcal{M}(\Theta)$.

(iv) Show that $E \subset \mathcal{M}(\Theta) \cap \mathcal{H}(b)$.

(v) Show that $T_a \mathfrak{K} \subset E$.
Hint: Use Theorem 9.19.

(vi) Using the first part, conclude that $E = \mathcal{M}(\Theta) \cap \mathcal{H}(b)$.

Exercise 28.5.2 Let $b(z) = (1 + z)/2$ and let $a(z) = (1 - z)/2$ be its Pythagorean pair.

(i) Prove that a function $\varphi \in H^\infty$ is a multiplier of $\mathcal{H}(b)$ if and only if it belongs to $\mathcal{H}(b)$.
Hint: Note that $\mathcal{H}(b) = \mathcal{M}(\bar{a}) = \mathcal{M}(\bar{e})$ where $e(z) = 1 - z$ and apply Corollary 28.29.

(ii) Let B be a Blaschke product whose zeros cluster at 1 and which has an angular derivative at 1. Show that $B \in \mathfrak{Mult}(\mathcal{H}(b))$.
Hint: Use Theorem 28.30.

(iii) Show that (a, B) do not form a corona pair and conclude that the multiplier criterion given by Theorem 28.9 is just a sufficient condition.

28.6 Characterization of $\overline{\mathcal{M}(a)}^b = \mathcal{H}(b)$

In this section, we assume that b is a nonextreme point in the closed unit ball of H^∞ and as usual we let a denote the outer function such that $|a| = (1-|b|^2)^{1/2}$ almost everywhere on $\mathbb{T}$ and $a(0) > 0$. In the preceding sections, we obtained some characterizations for the equality $\mathcal{H}(b) = \mathcal{M}(a)$. We continue this trend and give a characterization of the density of $\mathcal{M}(a)$ in $\mathcal{H}(b)$. Furthermore, we will also study the case where $\mathcal{M}(a)$ is not dense in $\mathcal{H}(b)$. In particular, we investigate the conditions under which the orthogonal complement of $\mathcal{M}(a)$ in $\mathcal{H}(b)$ is of finite dimension.

We recall that, in the nonextreme case, the space $\mathcal{H}(b)$ is invariant under the forward shift operator S and we use S_b to denote the restriction of S to $\mathcal{H}(b)$; see Section 24.1.

Theorem 28.34 *Let b be a nonextreme point in the closed unit ball of H^∞. Then the following are equivalent.*

(i) *$\mathcal{M}(a)$ is dense in $\mathcal{H}(b)$.*
(ii) *S_b and S are quasi-similar.*

Proof (i) $\Longrightarrow$ (ii) Put

$$\begin{array}{cccc} \mathcal{T}_a : & H^2 & \longrightarrow & \mathcal{H}(b) \\ & f & \longmapsto & T_a f. \end{array}$$

Since, by Theorem 12.19, T_a is injective, the mapping $\mathcal{T}_a$ is a quasi-affinity. Moreover, by (12.3), we have $ST_a = T_a S$, which, using the new notation, can be rewritten as

$$S_b \mathcal{T}_a = \mathcal{T}_a S.$$

In other words, we have the following commutative diagram.

$$\begin{tikzcd} H^2 \arrow[r, "S"] \arrow[d, "\mathcal{T}_a"'] & H^2 \arrow[d, "\mathcal{T}_a"] \\ \mathcal{H}(b) \arrow[r, "\mathbf{S}_b"'] & \mathcal{H}(b) \end{tikzcd} \tag{28.39}$$

On the other hand, by Theorem 18.3, $\mathcal{H}(b)$ is dense in H^2 and thus the canonical injection $i : \mathcal{H}(b) \longrightarrow H^2$ is a quasi-affinity. Moreover, we have

$$iS_b = Si.$$

Hence, S_b and S are quasi-similar.

(ii) $\Longrightarrow$ (i) Let $A \in \mathcal{L}(H^2, \mathcal{H}(b))$ be the quasi-affinity such that

$$S_b A = AS \tag{28.40}$$

(see (28.28)). Since $\mathcal{H}(b)$ is embedded contractively into H^2, the operator

$$\begin{array}{rccc} \mathbf{A}: & H^2 & \longrightarrow & H^2 \\ & f & \longmapsto & Af \end{array}$$

is well defined and bounded, and (28.40) implies that

$$S\mathbf{A} = \mathbf{A}S.$$

Therefore, by Theorem 12.27, there is a $u \in H^\infty$ such that $A = T_u$. Hence, $\mathcal{M}(u) = AH^2 \subset \mathcal{H}(b)$. Theorem 23.6 now implies that $\mathcal{M}(u) \subset \mathcal{M}(a)$. Then, according to Theorem 23.2 and the fact that A is a quasi-affinity, we obtain

$$\mathcal{H}(b) = \overline{AH^2}^b = \overline{\mathcal{M}(u)}^b \subset \overline{\mathcal{M}(a)}^b \subset \mathcal{H}(b),$$

which proves that $\mathcal{M}(a)$ is dense in $\mathcal{H}(b)$. □

28.7 Characterization of the closedness of $\mathcal{M}(a)$ in $\mathcal{H}(b)$

In this section, we still assume that b is a nonextreme point in the closed unit ball of H^∞ and as usual we let a denote the outer function such that $|a| = (1-|b|^2)^{1/2}$ almost everywhere on $\mathbb{T}$ and $a(0) > 0$. We give a characterization of the closedness of $\mathcal{M}(a)$ in $\mathcal{H}(b)$.

Theorem 28.35 *Let (a, b) be a pair. Then the following assertions are equivalent.*

(i) *$\mathcal{M}(a)$ is closed in $\mathcal{H}(b)$.*
(ii) *For any $\lambda \in \mathbb{T}$, the operator $T_{F_\lambda/\bar{F}_\lambda}$ is left semi-Fredholm.*
(iii) *There exists $\lambda \in \mathbb{T}$ such that the operator $T_{F_\lambda/\bar{F}_\lambda}$ is left semi-Fredholm.*

Proof (i) $\Longrightarrow$ (ii) Lemma 24.21 says that

$$\mathcal{M}(a) = (T_{1-\bar{\lambda}b} T_{\bar{F}_\lambda})(T_{F_\lambda/\bar{F}_\lambda} H^2),$$

and, according to Theorem 24.23, the operator $T_{1-\bar{\lambda}b} T_{\bar{F}_\lambda}$ is an isometry from H^2 into $\mathcal{H}(b)$. Hence, the subspace $T_{F_\lambda/\bar{F}_\lambda} H^2$ should be closed in H^2. Since $T_{F_\lambda/\bar{F}_\lambda}$ is always injective, we get that $T_{F_\lambda/\bar{F}_\lambda}$ should be a left semi-Fredholm operator.

The implication (ii) $\Longrightarrow$ (iii) is trivial.

It remains to prove that (iii) $\Longrightarrow$ (i). Since $T_{F_\lambda/\bar{F}_\lambda}$ is left semi-Fredholm, the subspace $T_{F_\lambda/\bar{F}_\lambda} H^2$ should be closed in H^2 and, using one more time Lemma 24.21 and Theorem 24.23, we deduce that $\mathcal{M}(a)$ is closed in $\mathcal{H}(b)$. □

28.8 Boundary eigenvalues and eigenvectors of S_b^*

We recall that, according to Theorem 24.23, if λ is a point on $\mathbb{T}$ such that the measure μ_λ is absolutely continuous, then the operator

$$W_\lambda = T_{1-\bar{\lambda}b} T_{\bar{F}_\lambda}$$

is a unitary operator from H^2 onto $\mathcal{H}(b)$. Exploiting this operator, we obtain a useful intertwining relation.

Lemma 28.36 *Let λ be a point on $\mathbb{T}$ such that the measure μ_λ is absolutely continuous. Then*

$$W_\lambda^* S_b^* W_\lambda = S^* - F_\lambda(0)^{-1}(S^* F_\lambda \otimes 1).$$

Proof Recall that F_λ is an outer function in H^2. Hence, according to Corollary 8.18, it is sufficient to check the equality on functions $F_\lambda k_w$, $w \in \mathbb{D}$. More explicitly, if the identity

$$W_\lambda^* S_b^* W_\lambda (F_\lambda k_w) = (S^* - F_\lambda(0)^{-1}(S^* F_\lambda \otimes 1))(F_\lambda k_w) \tag{28.41}$$

holds for every $w \in \mathbb{D}$, then we are done.

On the one hand, using Corollary 8.11, we have

$$\begin{aligned}
&(S^* - F_\lambda(0)^{-1}(S^* F_\lambda \otimes 1))(F_\lambda k_w) \\
&\quad = S^*(F_\lambda k_w) - F_\lambda(0)^{-1}\langle F_\lambda k_w, k_0\rangle S^* F_\lambda \\
&\quad = F_\lambda S^* k_w + k_w(0) S^* F_\lambda - F_\lambda(0)^{-1} F_\lambda(0) k_w(0) S^* F_\lambda \\
&\quad = F_\lambda S^* k_w \\
&\quad = F_\lambda \bar{w} k_w.
\end{aligned}$$

On the other hand, by Theorems 24.23 and 20.5, we have

$$W_\lambda(F_\lambda k_w) = V_{\bar{\lambda}b} k_w = (1 - \lambda\overline{b(w)})^{-1} k_w^b.$$

Note that $k_w^b = k_w^{\bar{\lambda}b}$. Hence, using Theorem 24.5 and the fact that $W_\lambda^* W_\lambda = id$, we get

$$W_\lambda^* S_b^* W_\lambda (F_\lambda k_w) = (1 - \lambda\overline{b(w)})^{-1} \bar{w} W_\lambda^* k_w^b = \bar{w} F_\lambda k_w$$

Therefore, we have established (28.41). □

Theorem 28.37 *Let $z_0 \in \mathbb{T}$ and let $\lambda \in \mathbb{T}$ be such that the measure μ_λ is absolutely continuous. Then the following are equivalent.*

(i) *$\bar{z}_0$ is an eigenvalue for the operator S_b^*.*
(ii) *The function $F_\lambda(z)/(1 - \bar{z}_0 z)$ belongs to H^2.*
(iii) *The function b has an angular derivative at z_0.*

Proof By (24.14), since μ_λ is absolutely continuous, we have

$$\frac{d\mu_\lambda}{dm} = |F_\lambda|^2.$$

Moreover, note that $1 - \bar{z}_0 z$ is an outer function and thus Corollary 4.28 implies that the function $F_\lambda/(1 - \bar{z}_0 z)$ is in H^2 if and only if it is in L^2. But $F_\lambda/(1 - \bar{z}_0 z)$ is in L^2 if and only if $|F_\lambda|^2/|1 - \bar{z}_0 z|^2$ is in L^1, which is equivalent to

$$\int_{\mathbb{T}} \frac{d\mu_\lambda(e^{i\theta})}{|e^{i\theta} - z_0|^2} = \int_{\mathbb{T}} \frac{|F_\lambda(e^{i\theta})|^2}{|z_0 - e^{i\theta}|^2}\, dm(e^{i\theta}) < \infty. \tag{28.42}$$

(ii) $\Longrightarrow$ (iii) Since $F_\lambda/(1 - \bar{z}_0 z)$ is in H^2, according to the previous discussion, the condition (28.42) is satisfied and then Theorem 21.7 ensures that b has an angular derivative at z_0.

(iii) $\Longrightarrow$ (ii) If b has an angular derivative at z_0, then it follows from Theorem 21.6 that, for every $\zeta \in \mathbb{T} \setminus \{b(z_0)\}$, we have

$$\int_{\mathbb{T}} \frac{d\mu_\zeta(e^{i\theta})}{|e^{i\theta} - z_0|^2} < \infty. \tag{28.43}$$

But, since μ_λ is absolutely continuous, we have $\mu_\lambda(\{z_0\}) = 0$ and, according to Theorem 21.5, we get $\lambda \neq b(z_0)$. Now, (28.43) implies that (28.42) holds. Hence, the preliminary discussion given above shows that the function $F_\lambda/(1 - \bar{z}_0 z)$ is in H^2.

To show that (i) is equivalent to (ii), recall that, according to Lemma 28.36, we have

$$W_\lambda S_b^* W_\lambda^* = A_\lambda,$$

where A_λ is the operator on H^2 defined by $A_\lambda = S^* - F_\lambda(0)^{-1}(S^* F_\lambda \otimes 1)$. Moreover, since μ_λ is absolutely continuous, the operator W_λ is unitary. Therefore, $\bar{z}_0$ is an eigenvalue of S_b^* if and only if $\bar{z}_0$ is an eigenvalue of A_λ. As another observation, if g is an H^2 function, according to (8.14), a direct computation gives

$$((A_\lambda - \bar{z}_0 I)g)(z) = \frac{(1 - \bar{z}_0 z)g(z) - g(0)F_\lambda(0)^{-1}F_\lambda(z)}{z} \qquad (z \in \mathbb{D}). \tag{28.44}$$

Thus g belongs to $\ker(A_\lambda - \bar{z}_0 I)$ if and only if g satisfies

$$(1 - \bar{z}_0 z)g(z) = g(0)F_\lambda(0)^{-1}F_\lambda(z) \qquad (z \in \mathbb{D}). \tag{28.45}$$

(i) $\Longrightarrow$ (ii) If z_0 is an eigenvalue of S_b^*, from the previous discussion, we know that z_0 is also an eigenvalue of A_λ. Hence, there exists a g in H^2, $g \neq 0$, that satisfies (28.45). We must have $g(0) \neq 0$ (since otherwise $g \equiv 0$) and thus we can write

$$\frac{F_\lambda}{1-\bar{z}_0 z} = \frac{F_\lambda(0)}{g(0)} g,$$

which implies that the function $F_\lambda/(1-\bar{z}_0 z)$ belongs to H^2.

(ii) $\Longrightarrow$ (i) If $F_\lambda/(1-\bar{z}_0 z)$ belongs to H^2, then so does

$$h(z) = F_\lambda(0)^{-1} \frac{F_\lambda}{1-\bar{z}_0 z}$$

and $h \neq 0$. Since $h(0) = 1$, we easily see that h satisfies (28.45) and thus we can conclude that $h \in \ker(A_\lambda - \bar{z}_0 I)$. Therefore, $\bar{z}_0$ is an eigenvalue of A_λ and then $\bar{z}_0$ is an eigenvalue of S_b^*. □

The proof of Theorem 28.37 also implicitly contains the following result.

Corollary 28.38 *Let $z_0 \in \mathbb{T}$ and let $\lambda \in \mathbb{T}$ be such that the measure μ_λ is absolutely continuous. Denote by A_λ the operator on H^2 defined by*

$$A_\lambda = S^* - F_\lambda(0)^{-1}(S^* F_\lambda \otimes 1).$$

Assume that $\bar{z}_0$ is an eigenvalue for the operator S_b^*. Then $\bar{z}_0$ is also an eigenvalue for the operator A_λ and we have

$$\ker(A_\lambda - \bar{z}_0 I) = \mathbb{C}\frac{F_\lambda(z)}{1-\bar{z}_0 z}.$$

In Theorem 24.5, we saw that

$$\sigma(S_b^*) = \bar{\mathbb{D}} \quad \text{and} \quad \mathbb{D} \subset \sigma_p(S_b^*).$$

Even more explicitly, for every $w \in \mathbb{D}$, we have

$$\ker(S_b^* - \bar{w} I) = \mathbb{C} k_w^b,$$

i.e. the eigenvector corresponding to $\bar{w} \in \mathbb{D}$ is precisely the kernel k_w^b. In this section, we find the eigenvectors for eigenvalues on $\mathbb{T}$.

Theorem 28.39 *Let $z_0 \in \mathbb{T}$ and let $\lambda \in \mathbb{T}$ be such that the measure μ_λ is absolutely continuous. Suppose that there exists an integer $n \geq 1$ such that*

$$\frac{F_\lambda(z)}{(1-\bar{z}_0 z)^n} \in H^2$$

but

$$\frac{F_\lambda(z)}{(1-\bar{z}_0 z)^{n+1}} \notin H^2.$$

Then

$$\ker(A_\lambda - \bar{z}_0 I)^k = \operatorname{Lin}\left(\frac{F_\lambda(z)}{(1-\bar{z}_0 z)}, \frac{F_\lambda(z)}{(1-\bar{z}_0 z)^2}, \ldots, \frac{F_\lambda(z)}{(1-\bar{z}_0 z)^{\min(k,n)}} \right) \tag{28.46}$$

and

$$\dim \ker(S_b^* - \bar{z}_0 I)^k = \min(k, n). \tag{28.47}$$

Proof According to Lemma 28.36, we have $S_b^* = W_\lambda A_\lambda W_\lambda^*$. Hence,

$$(S_b^* - \bar{z}_0 I)^k = W_\lambda (A_\lambda - \bar{z}_0 I)^k W_\lambda^*.$$

In particular, since W_λ is unitary, we get

$$\dim \ker(S_b^* - \bar{z}_0 I)^k = \dim \ker(A_\lambda - \bar{z}_0 I)^k. \tag{28.48}$$

Moreover, if $\mathcal{P}_\ell$ denotes the analytic polynomials of degree less than or equal to $\ell - 1$, then, for every integer $\ell \geq 1$, it is easy to check that

$$\text{Lin}\left(\frac{F_\lambda(z)}{(1 - \bar{z}_0 z)}, \frac{F_\lambda(z)}{(1 - \bar{z}_0 z)^2}, \dots, \frac{F_\lambda(z)}{(1 - \bar{z}_0 z)^\ell}\right) = \frac{F_\lambda(z)}{(1 - \bar{z}_0 z)^\ell} \mathcal{P}_\ell. \tag{28.49}$$

Therefore, (28.47) follows immediately from (28.46), (28.49) and (28.48). Thus, we proceed to prove (28.46) by induction on k.

For $k = 1$, the identity (28.46) follows from Corollary 28.38. Now, assume that the identity holds for some $k - 1$. Put $j = \min(k - 1, n)$ and let $g \in \ker(A_\lambda - \bar{z}_0 I)^k$. Then $(A_\lambda - \bar{z}_0 I)g \in \ker(A_\lambda - \bar{z}_0 I)^{k-1}$ and by induction there exists a polynomial $p \in \mathcal{P}_j$ such that

$$((A_\lambda - \bar{z}_0 I)g)(z) = (1 - \bar{z}_0 z)^{-j} F_\lambda(z) p(z).$$

According to (28.44), the preceding identity is rewritten as

$$(1 - \bar{z}_0 z)g(z) - g(0)F_\lambda(0)^{-1} F_\lambda(z) = z(1 - \bar{z}_0 z)^{-j} F_\lambda(z) p(z),$$

or equivalently as

$$(1 - \bar{z}_0 z)g(z) = (1 - \bar{z}_0 z)^{-j} F_\lambda(z) q(z),$$

where $q(z) = g(0)F_\lambda(0)^{-1}(1 - \bar{z}_0 z)^j + zp(z)$. Hence, we obtain the representation

$$g(z) = \frac{F_\lambda(z)}{(1 - \bar{z}_0 z)^{j+1}} q(z) \tag{28.50}$$

with $q \in \mathcal{P}_{j+1}$. At this point, we distinguish two cases.

Case I: $k - 1 < n$. Then $j = k - 1$, which gives that $\min(k, n) = k = j + 1$. According to (28.50), we then have

$$g(z) = \frac{F_\lambda(z)}{(1 - \bar{z}_0 z)^k} q(z)$$

with $q \in \mathcal{P}_k$ and this means that

$$\ker(A_\lambda - \bar{z}_0 I)^k \subset \mathrm{Lin}\left(\frac{F_\lambda(z)}{(1-\bar{z}_0 z)}, \frac{F_\lambda(z)}{(1-\bar{z}_0 z)^2}, \dots, \frac{F_\lambda(z)}{(1-\bar{z}_0 z)^k}\right).$$

For the converse, First, by the induction hypothesis, we have

$$\begin{aligned}\mathrm{Lin}\left(\frac{F_\lambda(z)}{(1-\bar{z}_0 z)}, \frac{F_\lambda(z)}{(1-\bar{z}_0 z)^2}, \dots, \frac{F_\lambda(z)}{(1-\bar{z}_0 z)^{k-1}}\right) &\subset \ker(A_\lambda - \bar{z}_0 I)^{k-1}\\ &\subset \ker(A_\lambda - \bar{z}_0 I)^k.\end{aligned}$$

Second, we verify that $F_\lambda(z)/(1-\bar{z}_0 z)^k \in \ker(A_\lambda - \bar{z}_0 I)^k$, which is equivalent to

$$(A_\lambda - \bar{z}_0 I)(F_\lambda(z)/(1-\bar{z}_0 z)^k) \in \ker(A_\lambda - \bar{z}_0 I)^{k-1}. \tag{28.51}$$

Applying (28.44), we have

$$(A_\lambda - \bar{z}_0 I)(F_\lambda(z)/(1-\bar{z}_0 z)^k) = \frac{F_\lambda(z)}{(1-\bar{z}_0 z)^{k-1}}\, q(z),$$

where $q(z) = z^{-1}(1-(1-\bar{z}_0 z)^{k-1})$. We easily check that q is a polynomial of degree less than or equal to $k-2$, i.e. $q \in \mathcal{P}_{k-1}$. Thus

$$(A_\lambda - \bar{z}_0 I)(F_\lambda/(1-\bar{z}_0 z)^k) \in \frac{F_\lambda(z)}{(1-\bar{z}_0 z)^{k-1}}\, \mathcal{P}_{k-1},$$

and, using (28.49) and the induction hypothesis, we obtain (28.51).

In conclusion, we have proved that

$$\ker(A_\lambda - \bar{z}_0 I)^k = \mathrm{Lin}\left(\frac{F_\lambda(z)}{(1-\bar{z}_0 z)}, \frac{F_\lambda(z)}{(1-\bar{z}_0 z)^2}, \dots, \frac{F_\lambda(z)}{(1-\bar{z}_0 z)^k}\right).$$

Case II: $k-1 \geq n$. Then $j = \min(k,n) = n$, and the identity (28.50) becomes

$$g(z) = \frac{F_\lambda(z)}{(1-\bar{z}_0 z)^{n+1}}\, q(z)$$

with $q \in \mathcal{P}_{n+1}$. By Taylor's formula, we can write

$$q(z) = \sum_{p=0}^{n} a_p (1-\bar{z}_0 z)^p$$

where $a_p \in \mathbb{C}$, $0 \leq p \leq n$. That gives

$$g(z) = \sum_{p=0}^{n} a_p \frac{F_\lambda(z)}{(1-\bar{z}_0 z)^{n+1-p}}.$$

Remembering that g and $F_\lambda(z)/(1-\bar{z}_0 z)^j$ belong to H^2, for $1 \leq j \leq n$, but $F_\lambda(z)/(1-\bar{z}_0 z)^{n+1}$ does not belong to H^2, we see that the last representation necessarily implies that $a_0 = 0$. Hence, we obtain

$$g(z) = \sum_{p=1}^{n} a_p \frac{F_\lambda(z)}{(1-\bar{z}_0 z)^{n+1-p}},$$

that is,

$$\ker(A_\lambda - \bar{z}_0 z)^k \subset \operatorname{Lin}\left(\frac{F_\lambda(z)}{(1-\bar{z}_0 z)}, \frac{F_\lambda(z)}{(1-\bar{z}_0 z)^2}, \dots, \frac{F_\lambda(z)}{(1-\bar{z}_0 z)^n} \right).$$

For the converse, just note that, by the induction hypothesis, we have

$$\begin{aligned} \operatorname{Lin}\left(\frac{F_\lambda(z)}{(1-\bar{z}_0 z)}, \frac{F_\lambda(z)}{(1-\bar{z}_0 z)^2}, \dots, \frac{F_\lambda(z)}{(1-\bar{z}_0 z)^n} \right) &\subset \ker(A_\lambda - \bar{z}_0 I)^{k-1} \\ &\subset \ker(A_\lambda - \bar{z}_0 I)^k, \end{aligned}$$

and that concludes the proof. □

Corollary 28.40 *Let $z_0 \in \mathbb{T}$ and let $\lambda \in \mathbb{T}$ be such that the measure μ_λ is absolutely continuous. Suppose that, for every integer $n \geq 1$, the function $F_\lambda(z)/(1-\bar{z}_0 z)^n$ is in H^2. Then*

$$\dim \ker(S_b^* - \bar{z}_0 I)^n = n \qquad (n \geq 1).$$

Proof The proof follows along the same lines as the proof of Theorem 28.39, but the second case in that proof does not occur. □

28.9 The space $\mathcal{H}_0(b)$

In the case where $\mathcal{M}(a)$ is not dense in $\mathcal{H}(b)$, we denote by $\mathcal{H}_0(b)$ the orthogonal complement of $\mathcal{M}(a)$ in $\mathcal{H}(b)$, i.e. the closed subspace of $\mathcal{H}(b)$ defined by

$$\mathcal{H}_0(b) = \mathcal{H}(b) \ominus \overline{\mathcal{M}(a)}.$$

Moreover, let $P_0 \in \mathcal{L}(\mathcal{H}(b))$ denote the orthogonal projection of $\mathcal{H}(b)$ onto $\mathcal{H}_0(b)$ and S_0 denote the compression of S_b to $\mathcal{H}_0(b)$, i.e.

$$\begin{array}{rccc} S_0: & \mathcal{H}_0(b) & \longrightarrow & \mathcal{H}_0(b) \\ & f & \longmapsto & P_0 S_b f. \end{array}$$

In other words, we have

$$S_0 = P_0 S_b i_0, \tag{28.52}$$

where $i_0 : \mathcal{H}_0(b) \longrightarrow \mathcal{H}(b)$ is the inclusion operator. Note that

$$i_0 i_0^* = P_0 \quad \text{and} \quad i_0^* i_0 = I_{\mathcal{H}_0(b)}. \tag{28.53}$$

The manifold $\mathcal{M}(a)$ is invariant under the operator S_b, and thus, by continuity, so is $\overline{\mathcal{M}(a)}$. Hence, by Lemma 1.39, $\mathcal{H}_0(b) = \mathcal{M}(a)^\perp$ is invariant under the operator S_b^*. In fact, for each $f, g \in \mathcal{H}_0(b)$, we have

$$\langle g, S_0^* f\rangle_b = \langle S_0 g, f\rangle_b = \langle P_0 S_b g, f\rangle_b = \langle S_b g, f\rangle_b = \langle g, S_b^* f\rangle_b,$$

and, since $S_b^* f \in \mathcal{H}_0(b)$, we get

$$S_0^* f = S_b^* f \qquad (f \in \mathcal{H}_0(b)), \tag{28.54}$$

i.e. $S_0^* = S^*_{b\,|\mathcal{H}_0(b)}$.

Lemma 28.41 *For each $k \geq 0$ and $z_0 \in \mathbb{C}$ with $|z_0| \geq 1$, we have*

$$\overline{(S_b - z_0 I)^k \mathcal{M}(a)}^{\mathcal{M}(a)} = \mathcal{M}(a).$$

Proof Let $\mathbf{T}_a$ and $\mathbf{S}_a$ be the operators introduced in the proof of Theorem 28.21. Then, by (28.24), we have

$$\mathbf{S}_a \mathbf{T}_a = \mathbf{T}_a S.$$

Thus, by induction,

$$(\mathbf{S}_a - z_0 I)^k \mathbf{T}_a = \mathbf{T}_a (S - z_0 I)^k.$$

Since $\mathbf{T}_a$ is an isometry from H^2 onto $\mathcal{M}(a)$, we get

$$\overline{(\mathbf{S}_a - z_0 I)^k \mathbf{T}_a H^2}^{\mathcal{M}(a)} = \overline{\mathbf{T}_a (S - z_0 I)^k H^2}^{\mathcal{M}(a)} = \mathbf{T}_a\big(\overline{(S - z_0 I)^k H^2}^{H^2}\big).$$

But

$$(S - z_0 I)^k H^2 = (z - z_0)^k H^2$$

and, by Corollary 4.27, $(z - z_0)^k$ is an outer function. Thus, Theorem 8.16 implies that $\overline{(S - z_0 I)^k H^2}^{H^2} = H^2$. Therefore,

$$\overline{(S_b - z_0 I)^k \mathcal{M}(a)}^{\mathcal{M}(a)} = \overline{(\mathbf{S}_a - z_0 I)^k \mathcal{M}(a)}^{\mathcal{M}(a)} = \mathbf{T}_a H^2 = \mathcal{M}(a). \quad \square$$

Theorem 28.42 *For each $k \geq 0$ and $z_0 \in \mathbb{T}$, we have*

$$\ker(S_b^* - \bar{z}_0 I)^k \subset \mathcal{H}_0(b).$$

In particular,

$$\ker(S_b^* - \bar{z}_0 I)^k = \ker(S_0^* - \bar{z}_0 I)^k.$$

Proof By Lemma 28.41, we have

$$\overline{(S_b - z_0 I)^k \mathcal{M}(a)}^{\mathcal{M}(a)} = \mathcal{M}(a).$$

Hence, it follows from Lemma 16.1 that

$$\overline{(S_b - z_0 I)^k \mathcal{M}(a)}^b = \overline{\mathcal{M}(a)}^b.$$

Therefore,

$$\begin{aligned}\ker(S_b^* - \bar{z}_0 I)^k &= \mathcal{H}(b) \ominus (S_b - z_0 I)^k \mathcal{H}(b)\\ &\subset \mathcal{H}(b) \ominus (S_b - z_0 I)^k \mathcal{M}(a)\\ &= \mathcal{H}(b) \ominus \overline{(S_b - z_0 I)^k \mathcal{M}(a)}^b\\ &= \mathcal{H}(b) \ominus \overline{\mathcal{M}(a)}^b\\ &= \mathcal{H}(b) \ominus \mathcal{M}(a)\\ &= \mathcal{H}_0(b).\end{aligned}$$

The equality $\ker(S_b^* - \bar{z}_0 I)^k = \ker(S_0^* - \bar{z}_0 I)^k$ follows immediately from the preceding inclusion and (28.54). □

We end this section with a result about the relation between the multipliers of $\mathcal{H}(b)$.

Theorem 28.43 *The following assertions hold.*

(i) *If $\varphi \in \mathfrak{Mult}(\mathcal{H}(b))$, then $\mathcal{M}(a)$ is invariant under the multiplication operator M_φ.*

(ii) *If $\varphi, \psi \in \mathfrak{Mult}(\mathcal{H}(b))$, then we have*

$$P_0 M_\varphi P_0 M_\psi P_0 = P_0 M_{\varphi\psi} P_0,$$

where $P_0 \in \mathcal{L}(\mathcal{H}(b))$ is the orthogonal projection of $\mathcal{H}(b)$ onto $\mathcal{H}_0(b)$.

Proof (i) Let $g \in H^2$. By Lemma 9.6, $\varphi \in H^\infty$, and thus $\varphi g \in H^2$. Hence, we have

$$M_\varphi(T_a g) = \varphi a g = T_a(\varphi g) \in \mathcal{M}(a),$$

which proves that $M_\varphi \mathcal{M}(a) \subset \mathcal{M}(a)$.

(ii) We have

$$(P_0 M_\varphi P_0 M_\psi P_0)^* = P_0 M_\psi^* P_0 M_\varphi^* P_0.$$

But, since $\mathcal{M}(a)$ is invariant under the operator M_φ, then $\mathcal{H}_0(b)$ is invariant under the operator M_φ^*. This fact translates as

$$P_0 M_\varphi^* P_0 = M_\varphi^* P_0$$

and thus

$$(P_0 M_\varphi P_0 M_\psi P_0)^* = P_0 M_\psi^* M_\varphi^* P_0.$$

Taking the adjoint of this equality gives us

$$P_0 M_\varphi P_0 M_\psi P_0 = P_0 M_\varphi M_\psi P_0.$$

But it is obvious that $M_\varphi M_\psi = M_\psi M_\varphi = M_{\varphi\psi}$ and the result follows. □

28.10 The spectrum of S_0

If $\mathcal{M}(a)$ is not dense in $\mathcal{H}(b)$, then $\mathcal{H}_0(b)$ is not the trivial space. Hence, the forward compression of the forward shift operator on this space, i.e. S_0, is not the zero operator. Knowing this, we proceed to find its spectrum.

Theorem 28.44 *Assume that $\mathcal{M}(a)$ is not dense in $\mathcal{H}(b)$. Then $\sigma(S_0) \subset \mathbb{T}$.*

Proof Let $|w| > 1$, let $1 < r < |w|$, and let $\varphi(z) = 1/(z-w)$, $|z| < r$. Then it is clear that φ is analytic in the disk $D(0,r)$ that contains the closed unit disk. Thus, it follows from Theorem 24.6 that φ is a multiplier of $\mathcal{H}(b)$. The function $\psi(z) = z - w$ is also a multiplier of $\mathcal{H}(b)$ and we have $S_b - wI = M_\psi$. Moreover,

$$M_\varphi M_\psi = M_\psi M_\varphi = I,$$

which gives

$$P_0 M_\varphi M_\psi P_0 = P_0 M_\psi M_\varphi P_0 = P_0.$$

Hence, Theorem 28.43 implies that

$$P_0 M_\varphi P_0 M_\psi P_0 = P_0 M_\psi P_0 M_\varphi P_0 = P_0,$$

that is,

$$P_0 M_\varphi P_0 (S_b - wI) P_0 = P_0 (S_b - wI) P_0 M_\varphi P_0 = P_0. \tag{28.55}$$

Put

$$A = i_0^* M_\varphi i_0 \in \mathcal{L}(\mathcal{H}_0(b)).$$

Then, by (28.52) and (28.53), (28.55) implies that

$$A(S_0 - wI) = (S_0 - wI)A = I,$$

where $I = I_{\mathcal{H}_0(b)}$. This identity says that $S_0 - wI$ is invertible in $\mathcal{L}(\mathcal{H}_0(b))$. Hence, $\sigma(S_0) \subset \bar{\mathbb{D}}$.

Now, let $|w| < 1$ and define

$$\varphi(z) = \frac{1}{a(w)} \frac{a(w) - a(z)}{z - w} = -\frac{1}{a(w)} (Q_w a)(z) \qquad (z \in \mathbb{D}).$$

Note that $a \in \mathfrak{Mult}(\mathcal{H}(b))$, because, if $f \in \mathcal{H}(b)$, then $af = T_a f \in \mathcal{M}(a) \subset \mathcal{H}(b)$. Hence, it follows from Theorem 20.18 that $\varphi \in \mathfrak{Mult}(\mathcal{H}(b))$. Moreover, for every function $f \in \mathcal{H}(b)$, we have

$$\begin{aligned}(M_\varphi(S_b - wI)f)(z) &= \varphi(z)(z-w)f(z)\\ &= \frac{1}{a(w)}(a(w) - a(z))f(z)\\ &= f(z) - \frac{a(z)}{a(w)} f(z),\end{aligned}$$

or equivalently

$$M_\varphi(S_b - wI) = I - a(w)^{-1} M_a,$$

where M_a is the multiplication by a on the space $\mathcal{H}(b)$. The same argument also shows that

$$(S_b - wI)M_\varphi = I - a(w)^{-1} M_a.$$

Since P_0 is the orthogonal projection of $\mathcal{H}(b)$ onto $\mathcal{H}_0(b) = \mathcal{M}(a)^\perp$, we have $P_0 M_a = 0$. Thus,

$$P_0 M_\varphi (S_b - wI) P_0 = P_0 (S_b - wI) M_\varphi P_0 = P_0.$$

It thus follows from Theorem 28.43 that

$$P_0 M_\varphi P_0 (S_b - wI) P_0 = P_0 (S_b - w) P_0 M_\varphi P_0 = P_0. \tag{28.56}$$

Put

$$B = i_0^* M_\varphi i_0.$$

Then, by (28.52) and (28.53), (28.56) implies that

$$B(S_0 - wI) = (S_0 - wI)B = I,$$

where $I = I_{\mathcal{H}_0(b)}$. This relation shows that $S_0 - wI$ is invertible in $\mathcal{H}_0(b)$. Hence, $\sigma(S_0) \subset \mathbb{C} \setminus \mathbb{D}$.

Therefore, we deduce that

$$\sigma(S_0) \subset \bar{\mathbb{D}} \cap (\mathbb{C} \setminus \mathbb{D}) = \mathbb{T}. \qquad \square$$

Notes on Chapter 28

Section 28.1

Theorem 28.1 and assertion (i) of Theorem 28.2 are due to Lotto and Sarason [123]. Assertion (ii) of Theorem 28.2 comes from [159].

Section 28.2

Theorem 28.3 in the nonextreme case is due to Sarason [159]. The general version presented here, as well as Theorem 28.4, were proved by Lotto and Sarason in [123]. Corollary 28.5 and Theorem 28.6 are also due to Lotto and Sarason [123]. Theorem 28.7 is taken from [159]. Theorem 28.9 is also obtained by Lotto and Sarason in [124] but the proof they gave is different. In their paper, Lotto and Sarason deduced this fact from a criterion on multipliers of $\mathcal{H}(b)$. More precisely, they proved that, if $\varphi \in \mathcal{M}(\bar{a}) \cap H^\infty$ and $\varphi = T_{\bar{a}}\psi$, $\psi \in H^2$, then $\varphi \in \mathfrak{Mult}(\mathcal{H}(b))$ if and only if both operators $H^*_{\bar{\psi}} H_{\bar{a}}$ and $H^*_{\bar{\psi}} H_{\bar{b}}$ are bounded on H^2. It should be noted here that it is perfectly possible for the former two operators to be bounded even though the latter one is unbounded. It is an open problem to characterize those pairs of L^2 functions f and g for which $H^*_f H_g$ is bounded.

Lemma 28.10 and Corollary 28.11 are due to Fricain, Hartmann and Ross [79]. In that paper, the authors give some concrete examples of $\mathcal{H}(b)$ spaces. More precisely, they explicitly describe de Branges–Rovnyak spaces $\mathcal{H}(b)$ when b is of the form q^r, where q is a rational outer function in the closed unit ball of H^∞ and r is a positive number.

Section 28.3

Suárez [180] studied the multipliers of $\mathcal{H}(b)$ in the extreme case. The results of this section are taken from that paper. Lemma 28.15 is taken from Hoffman [105]. The results on the extremal problem in H^∞ (Lemma 28.16 and Corollary 28.17) are taken from Garnett [93]. As we have seen in Theorem 20.17, we have

$$\mathfrak{Mult}(\mathcal{H}(b)) \subset \mathfrak{Mult}(\mathcal{H}(\bar{b})).$$

Theorem 28.18 says that, if b is an extreme point in the closed unit ball of H^∞ that is continuous up to the boundary, the equality holds in the formula above. In Example 28.24, we construct a nonextreme function b where the multipliers of $\mathcal{H}(b)$ and $\mathcal{H}(\bar{b})$ are different. In [159], Sarason even gives an example where the function b is outer (still nonextreme). However, it is unknown if the multipliers of $\mathcal{H}(b)$ and $\mathcal{H}(\bar{b})$ can be different when b is outer and extreme.

Section 28.4

Theorems 28.20, 28.21 and 28.22 are due to Sarason [159]. The equivalence between (iv) and (vi) in Theorem 28.23 is due to Davis and McCarthy [62]. More precisely, they proved that, if the Clark measure $\mu = \mu_1$ associated with an extreme point b of the closed unit ball of H^∞ is absolutely continuous and its Radon–Nikodym derivative is a Helson–Szegő weight (which is equivalent by Corollary 12.43 to assertion (vi)), then every function in H^∞ is a multiplier of $\mathcal{H}(b)$ and conversely. Lotto and Sarason gave an analogous result for extreme points whose corresponding measures are made in a simple way from Helson–Szegő weights; see [123, theorem 14.1].

In [62], Davis and McCarthy studied the problem of multipliers of de Branges–Rovnyak spaces using the link between multipliers of $\mathcal{H}(b)$ and of $H^2(\mu_1)^*$, where μ_1 is the Clark measure associated with b. In particular, they proved the following interesting result. Let $\varphi \in H^\infty$. Then, $\varphi \in \mathfrak{Mult}(\mathcal{H}(b))$ for any nonextreme b if and only if there exists $c > 0$ such that

$$\hat{\varphi}(n) = O(\exp(-c\sqrt{n})).$$

Remember that, according to Theorem 5.7, if $\hat{\varphi}(n) = O(\exp(-cn))$, then $\varphi \in \mathrm{Hol}(\bar{\mathbb{D}})$ and we know by Theorem 24.6 that $\varphi \in \mathfrak{Mult}(\mathcal{H}(b))$ for any nonextreme b. The result of Davis and McCarthy is thus an improvement of this fact. Exercise 28.4.2 is from [62, corollary 2.5].

Section 28.5

This section is taken from Sarason [159]. Corollary 28.28 has recently been generalized by Fricain, Hartmann and Ross [79]. In that paper, the authors identify the orthogonal complement of $\mathcal{M}(a)$ in $\mathcal{M}(\bar{a})$ when (a, b) is a rational pair. Corollary 28.29 is due to Sarason [159], but Lotto [120] gave another proof of this result based on the following criterion. Let b be in the closed unit ball of H^∞, Θ be an inner function and $c > 1$. Then $\Theta \in \mathfrak{Mult}(\mathcal{H}(b))$ and $\|M_\Theta\|_{\mathcal{L}(\mathcal{H}(b)} \le c$ if and only if the following holds:

$$\|P_\Theta f\|_2^2 - \|P_\Theta(bf)\|_2^2 \le c^2 \int_{\mathbb{T}} (1 - |b|^2)|f|^2 \, dm \qquad (f \in H^2),$$

where we recall that P_Θ denotes the orthogonal projection of H^2 onto $K_\Theta = (\Theta H^2)^\perp$.

The formula of Corollary 28.31 is announced in [159] but without proof. Exercise 28.5.2 comes from [124].

Sections 28.6–28.10

All the results of these sections are due to Sarason and come from his book [166].

29

Rigid functions and strongly exposed points of H^1

In Section 6.8, we introduced rigid functions and gave some sufficient conditions to create such elements (Theorems 6.22 and 6.23). In Section 12.6, we explored the relation between rigid functions (exposed points) and Toeplitz operators. This investigation enabled us to find some new rigid functions that are not covered by Theorems 6.22 and 6.23. Now we can use this relation to connect the problem of rigid functions with $\mathcal{H}(b)$ spaces. We will also investigate the relation between $\mathcal{H}(b)$ spaces and strongly exposed points.

In Section 29.1, we introduce the notion of admissible and special pairs associated with any outer function $F \in H^2$. This construction is the key to connecting the problem of rigid functions and $\mathcal{H}(b)$ spaces. In Section 29.2, we give a characterization of a rigid function (or an exposed point of the closed unit ball of H^1) expressed in the language of $\mathcal{H}(b)$ spaces. We give, in Section 29.3, a characterization under which $\mathcal{H}_0(b)$ is finite-dimensional (this space was introduced in Section 28.9). This characterization is connected with the rigidity property. As we have already said, the description of the S_b-invariant subspaces is far from being known. In Section 28.5, we have seen some results in this direction. We pursue this study in Section 29.4. Then, we apply these results to give, in Section 29.5, a necessary condition for nonrigidity. In Section 29.6, we study the problem of strongly exposed points of the unit ball of H^1. We use an approach based on $\mathcal{H}(b)$ spaces. It enables us to give a characterization in terms of the Helson–Szegő condition.

29.1 Admissible and special pairs

Given an outer function F in H^2, we construct three auxiliary holomorphic functions that make the connection to $\mathcal{H}(b)$ spaces. Put

$$G(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} \, |F(e^{it})|^2 \, dt \qquad (z \in \mathbb{D})$$

and

$$b(z) = \frac{G(z) - 1}{G(z) + 1} \quad \text{and} \quad a(z) = \frac{2F(z)}{G(z) + 1}.$$

Observe that we can recover the functions F and G from a and b by

$$F = \frac{a}{1 - b} \quad \text{and} \quad G = \frac{1 + b}{1 - b}.$$

The fact that

$$\Re(G(z)) \geq |F(z)|^2 \geq 0 \qquad (z \in \mathbb{D}) \tag{29.1}$$

implies that G is an outer function contained in H^p for all $0 < p < 1$ (see Corollary 4.25). Using (29.1), one can also easily check that a and b are in the closed unit ball of H^∞. Also, as a quotient of two outer functions, a is again outer. Finally, the fact that $\Re G = |F|^2$ a.e. on $\mathbb{T}$ gives that $|a|^2 + |b|^2 = 1$ a.e. on $\mathbb{T}$. This implies that a and b are nonextreme points of H^∞. We shall call (a, b) the *admissible pair* for F. Given a pair (a, b), we will say that (a, b) is *admissible* if (a, b) is the admissible pair for an outer function F, which is necessarily given by $F = a/(1 - b)$.

Note that $G(0) = \|F\|_2^2$, whence $b(0)$ is always real and $\|F\|_2 = 1$ if and only if $b(0) = 0$. Moreover, if (a, b) is the admissible pair for F, then we have

$$\frac{1 + b(z)}{1 - b(z)} = \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} \, |F(e^{it})|^2 \, dt,$$

whence we see that the Clark measure associated with b is absolutely continuous.

Lemma 29.1 *Let (a, b) be a pair and let μ be the Clark measure associated with b. Then the following are equivalent.*

(i) (a, b) *is admissible.*
(ii) $b(0)$ *is real and μ is absolutely continuous.*

Proof The implication (i) $\Longrightarrow$ (ii) has already been verified.

So let us assume now that $b(0)$ is real and μ is absolutely continuous. We denote by $(a^\sharp, b^\sharp)$ the admissible pair for the function F defined by $F = a/(1 - b)$.

One the one hand, since μ is the Clark measure associated with (a, b) and it is absolutely continuous, we have

$$\frac{d\mu(e^{i\theta})}{d\theta} = \frac{1 - |b(e^{i\theta})|^2}{|1 - b(e^{i\theta})|^2} = \frac{|a(e^{i\theta})|^2}{|1 - b(e^{i\theta})|^2} = |F(e^{i\theta})|^2.$$

On the other hand, we have

$$\Re\left(\frac{1 + b^\sharp(z)}{1 - b^\sharp(z)}\right) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - |z|^2}{|e^{i\theta} - z|^2} \, |F(e^{i\theta})|^2 \, d\theta$$

and

$$\Re\left(\frac{1+b(z)}{1-b(z)}\right) = \int_0^{2\pi} \frac{1-|z|^2}{|e^{i\theta}-z|^2}\, d\mu(e^{i\theta}).$$

Thus,

$$\Re\left(\frac{1+b^\sharp(z)}{1-b^\sharp(z)}\right) = \Re\left(\frac{1+b(z)}{1-b(z)}\right) \qquad (z \in \mathbb{D}),$$

which implies that there exists a constant $c \in \mathbb{R}$ such that

$$\frac{1+b^\sharp(z)}{1-b^\sharp(z)} = \frac{1+b(z)}{1-b(z)} + ic \qquad (z \in \mathbb{D}).$$

Since $b^\sharp(0)$ and $b(0)$ are real, we get in particular that $c = 0$ and then $b^\sharp = b$. Now,

$$|a^\sharp(\zeta)|^2 = 1 - |b^\sharp(\zeta)|^2 = 1 - |b(\zeta)|^2 = |a(\zeta)|^2,$$

almost everywhere on $\mathbb{T}$. Therefore, $a^\sharp = a$ and $(a, b) = (a^\sharp, b^\sharp)$ is an admissible pair. □

It is important to note that, in general, the admissibility of a given pair (a, b) does not imply the admissibility of the pair $(a, \bar{\lambda} b)$, where $\lambda \in \mathbb{T}$. Indeed, if $b(0) \neq 0$, write $b(0) = re^{i\theta}$. Then, for any $\lambda = e^{i\alpha}$, where $\alpha - \theta$ is not an integer multiple of π, the combination $\bar{\lambda} b(0)$ is not real. Thus, according to Lemma 29.1, the pair $(a, \bar{\lambda} b)$ cannot be admissible. In the case where $b(0) = 0$, we can produce an example where the pair (a, b) is admissible whereas $(a, \bar{\lambda} b)$ is not admissible for a certain choice of λ.

Example 29.2 Let $b(z) = z(z-1)/2$ and $a(z) = (1+z)/2$. Then (a, b) is a pair. Since the function $G = (1+b)/(1-b)$ is equal to

$$G(z) = \frac{2+z-z^2}{2-z+z^2},$$

it is easily seen that G is bounded on $\mathbb{D}$ (for instance, the roots of $2 - z + z^2$ are of modulus $\sqrt{2}$ and thus they are outside the closed unit disk). This implies (see Section 3.4) that the Clark measure associated with b has no singular component. Thus, according to Lemma 29.1, (a, b) is an admissible pair. Now, we check that $(a, -b)$ is not an admissible pair. Indeed, still according to Lemma 29.1, this is equivalent to saying that the Clark measure μ_{-1} is absolutely continuous with respect to the Lebesgue measure. But b has an angular derivative in the sense of Carathéodory at point $\zeta_0 = -1$, whence, using Theorem 21.5, we conclude that μ_{-1} has an atom at $\zeta_0 = -1$. Therefore, μ_{-1} cannot be absolutely continuous and then $(a, -b)$ is not an admissible pair.

Given a pair (a, b), we shall say that (a, b) is *special* if the Clark measure μ associated with b is absolutely continuous. According to Lemma 29.1, the pair (a, b) is admissible if and only if it is special and $b(0)$ is real.

Exercise

Exercise 29.1.1 Let F be an outer function in H^2 and let (a, b) be the admissible pair for F. Assume that there is no nonconstant inner function Θ such that the function $F/(1-\Theta)$ is in H^2. Prove that, for every $\lambda \in \mathbb{T}$, the pair $(a, \bar{\lambda}b)$ is special.
Hint: Argue by absurdity. Assume that there is a $\lambda_0 \in \mathbb{T}$ such that μ_{λ_0} is not absolutely continuous. Show that $\lambda_0 \neq 1$. Then consider the singular part $\mu_{\lambda_0}^{(s)}$ of μ_{λ_0} and let Θ be the inner function associated with $\mu_{\lambda_0}^{(s)}$ by (13.41). To conclude, use Theorem 24.29.

29.2 Rigid functions of H^1 and $\mathcal{H}(b)$ spaces

In this section, we investigate the connection between the rigid functions of H^1 and $\mathcal{H}(b)$ spaces. We recall that, according to (24.16) and (24.17), for every $\lambda \in \mathbb{T}$, we denote by F_λ the outer function in H^2 defined by

$$F_\lambda = \frac{a}{1-\bar{\lambda}b},$$

and by μ_λ the positive and finite Borel measure on $\mathbb{T}$ given by

$$\frac{1-|b(z)|^2}{|\lambda - b(z)|^2} = \int_0^{2\pi} \frac{1-|z|^2}{|z-e^{i\theta}|^2}\, d\mu_\lambda(e^{i\theta}) \qquad (z \in \mathbb{D}).$$

Theorem 29.3 *Let (a, b) be a pair. Then the following are equivalent.*

(i) *$\mathcal{M}(a)$ is a dense manifold in $\mathcal{H}(b)$.*
(ii) *For every $\lambda \in \mathbb{T}$, the pair $(a, \bar{\lambda}b)$ is special and the function F_λ^2 is rigid.*
(iii) *For some $\lambda \in \mathbb{T}$, the pair $(a, \bar{\lambda}b)$ is special and the function F_λ^2 is rigid.*

Proof In the following, the symbols $\overline{M}^b$ and $\overline{M}^2$, respectively, stand for the closure of M in the norm of $\mathcal{H}(b)$ and H^2.

(i) $\Longrightarrow$ (ii) According to Theorem 24.23, for each $\lambda \in \mathbb{T}$, the operator $T_{1-\bar{\lambda}b}T_{\bar{F}_\lambda}$ is an isometry from H^2 into $\mathcal{H}(b)$, which is onto if and only if the measure μ_λ is absolutely continuous. Moreover, it follows from Lemma 24.21 that

$$T_{1-\bar{\lambda}b}T_{\bar{F}_\lambda}T_{F_\lambda/\bar{F}_\lambda} = T_a.$$

Hence, by assumption, we have

$$\overline{T_{1-\bar{\lambda}b}T_{\bar{F}_\lambda}T_{F_\lambda/\bar{F}_\lambda}H^2}^b = \overline{\mathcal{M}(a)}^b = \mathcal{H}(b).$$

But, since $T_{1-\bar{\lambda}b}T_{\bar{F}_\lambda}$ is an isometry from H^2 into $\mathcal{H}(b)$, we get

$$\begin{aligned}\mathcal{H}(b) &= \overline{T_{1-\bar{\lambda}b}T_{\bar{F}_\lambda}T_{F_\lambda/\bar{F}_\lambda}H^2}^b \\ &= T_{1-\bar{\lambda}b}T_{\bar{F}_\lambda}\left(\overline{T_{F_\lambda/\bar{F}_\lambda}H^2}^2\right) \subset T_{1-\bar{\lambda}b}T_{\bar{F}_\lambda}H^2 \subset \mathcal{H}(b).\end{aligned}$$

Therefore, we have

$$T_{1-\bar{\lambda}b}T_{\bar{F}_\lambda}\left(\overline{T_{F_\lambda/\bar{F}_\lambda}H^2}^2\right) = T_{1-\bar{\lambda}b}T_{\bar{F}_\lambda}H^2 = \mathcal{H}(b). \tag{29.2}$$

In particular, the operator $T_{1-\bar{\lambda}b}T_{\bar{F}_\lambda}$ is onto, which implies that μ_λ is absolutely continuous. Moreover, using once again that $T_{1-\bar{\lambda}b}T_{\bar{F}_\lambda}$ is an isometry, we get from (29.2) that

$$\overline{T_{F_\lambda/\bar{F}_\lambda}H^2}^2 = H^2.$$

But since $T^*_{F_\lambda/\bar{F}_\lambda} = T_{\bar{F}_\lambda/F_\lambda}$, this is equivalent to $\ker(T_{\bar{F}_\lambda/F_\lambda}) = \{0\}$. It remains to apply Theorem 12.30 to deduce that F_λ^2 is rigid.

(ii) $\Longrightarrow$ (iii) This is trivial.

(iii) $\Longrightarrow$ (i) Let $\lambda \in \mathbb{T}$ be such that the measure μ_λ is absolutely continuous with respect to the Lebesgue measure m and the function F_λ^2 is rigid. Since μ_λ is absolutely continuous, the operator $T_{1-\bar{\lambda}b}T_{\bar{F}_\lambda}$ is onto. Moreover, since F_λ^2 is rigid, Theorem 12.30 ensures that $\ker(T_{\bar{F}_\lambda/F_\lambda}) = \{0\}$, which is equivalent to

$$\overline{T_{F_\lambda/\bar{F}_\lambda}H^2}^2 = H^2.$$

Hence, we have

$$\begin{aligned}\overline{\mathcal{M}(a)}^b &= \overline{T_{1-\bar{\lambda}b}T_{\bar{F}_\lambda}T_{F_\lambda/\bar{F}_\lambda}H^2}^b \\ &= T_{1-\bar{\lambda}b}T_{\bar{F}_\lambda}\left(\overline{T_{F_\lambda/\bar{F}_\lambda}H^2}^2\right) \\ &= T_{1-\bar{\lambda}b}T_{\bar{F}_\lambda}H^2 = \mathcal{H}(b).\end{aligned}$$

This completes the proof. □

Corollary 29.4 *Let F be an outer function and let (a, b) be the admissible pair for F. Then the following are equivalent.*

(i) *F^2 is rigid.*
(ii) *$\mathcal{M}(a)$ is dense in $\mathcal{H}(b)$.*

Proof Apply Theorem 29.3 and Lemma 29.1 to prove this. □

In Theorem 29.3, we saw that, if (a, b) is a pair such that $\mathcal{M}(a)$ is dense in $\mathcal{H}(b)$, then, for any $\lambda \in \mathbb{T}$, the pair $(a, \bar{\lambda} b)$ is special and F_λ^2 is rigid. The following result extends this property. For an inner function Θ, we let F_Θ denote the outer function $a/(1 - \Theta b)$. Note that F_Θ belongs to H^2.

Theorem 29.5 *Let (a, b) be a special pair, let $F = a/(1 - b)$ and assume that F^2 is rigid. Then, for any inner function Θ, the pair $(a, \Theta b)$ is special and F_Θ^2 is rigid.*

Proof Since Θ is inner, then $|a|^2 + |\Theta b|^2 = 1$ almost everywhere on $\mathbb{T}$ and thus the corresponding outer function associated with Θb is also a. In particular, we have $\mathcal{M}(a) \subset \mathcal{H}(\Theta b)$. In fact, $\mathcal{M}(a)$ is a dense manifold in $\mathcal{H}(\Theta b)$. To verify this claim, by (18.3), we know that

$$\mathcal{H}(b) \subset \mathcal{H}(\Theta b).$$

Now, since $\mathcal{M}(a)$ is dense in $\mathcal{H}(b)$, we can apply Lemma 16.1 and get

$$\overline{\mathcal{M}(a)}^{\Theta b} = \overline{\mathcal{H}(b)}^{\Theta b}.$$

Moreover, using Theorem 23.13, we have

$$\mathcal{P}_+ \subset \mathcal{H}(b) \subset \mathcal{H}(\Theta b),$$

where $\mathcal{P}_+$ is the set of analytic polynomials. Since $\mathcal{P}_+$ is dense in $\mathcal{H}(\Theta b)$, we get that $\overline{\mathcal{H}(b)}^{\Theta b} = \mathcal{H}(\Theta b)$. Therefore,

$$\overline{\mathcal{M}(a)}^{\Theta b} = \mathcal{H}(\Theta b).$$

Finally, we apply Theorem 29.3 and conclude that the pair $(a, \Theta b)$ is special and $a^2/(1 - \Theta b)^2$ is rigid. □

In the following example, we show that the converse of Theorem 29.5 is not true. More precisely, there are F and Θ such that F^2 is not rigid whereas F_Θ^2 is rigid.

Example 29.6 We follow a pattern similar to the one in Example 29.2. Let $b(z) = z(1 - z)/2$, which gives $a(z) = (1 + z)/2$. Then

$$F(z) = \frac{a(z)}{1 - b(z)} = \frac{1 + z}{2 - z + z^2}$$

and

$$G(z) = \frac{1 + b(z)}{1 - b(z)} = \frac{2 + z - z^2}{2 - z + z^2} \in H^\infty(\mathbb{D}).$$

Thus, μ, the Clark measure associated with b, has no singular component and then, for almost every $\zeta \in \mathbb{T}$, we have

$$|F(\zeta)|^2 = \frac{1 - |b(\zeta)|^2}{|1 - b(\zeta)|^2} = \frac{d\mu}{dm}(\zeta)$$

and

$$\|F\|_2^2 = \mu(\mathbb{T}) = \frac{1 - |b(0)|^2}{|1 - b(0)|^2} = 1.$$

Since F is divisible in H^2 by the function $1 + z$, Lemma 6.19 implies that F^2 is not a rigid function (once again, we use the fact that the roots of $2 - z + z^2$ are outside the closed unit disk and then $(2 - z + z^2)^{-1}$ is bounded on $\mathbb{D}$).

Now, let Θ be a Blaschke product with zeros $(r_n)_{n\geq 1}$ lying in $(-1, 0)$ and tending to -1. We claim that F_Θ^2 is a rigid function, and, to verify this, we show that $\mathcal{M}(a)$ is dense in $\mathcal{H}(\Theta b)$ (note that $(a, \Theta b)$ is a pair and then $\mathcal{M}(a) \subset \mathcal{H}(\Theta b)$). To start, note that the functions $(1 + z)k_{r_n}$ are in $\mathcal{H}(z\Theta) \cap \mathcal{M}(a)$. Indeed, first, if h is in H^2, then

$$\begin{aligned} \langle z\Theta h,\ (1 + z)k_{r_n}\rangle_2 &= \langle (1 + z)\Theta h,\ k_{r_n}\rangle_2 \\ &= (1 + r_n)\Theta(r_n)h(r_n) = 0, \end{aligned}$$

whence $(1 + z)k_{r_n} \in H^2 \ominus z\Theta H^2 = \mathcal{H}(z\Theta)$. Second,

$$(1 + z)k_{r_n} = 2ak_{r_n} = T_a(2k_{r_n}) \in \mathcal{M}(a)$$

and thus $(1 + z)k_{r_n} \in \mathcal{H}(z\Theta) \cap \mathcal{M}(a)$. Now, note that

$$(1 + z)k_{r_n}(z) - 1 = \frac{(1 + r_n)z}{1 - r_n z} = (1 + r_n)zk_{r_n},$$

whence

$$\|(1 + z)k_{r_n} - 1\|_2^2 = \frac{(1 + r_n)^2}{1 - r_n^2} \longrightarrow 0 \qquad (n \longrightarrow \infty),$$

which proves that $(1 + z)k_{r_n}$ converges to the constant function 1 in H^2 norm, as $n \longrightarrow \infty$. Since $\mathcal{H}(z\Theta)$ is contained isometrically in H^2 (because $z\Theta$ is inner), these functions also converge to 1 in $\mathcal{H}(z\Theta)$ norm. But, by Corollary 18.9, we have

$$\mathcal{H}(\Theta b) = \mathcal{H}(z\Theta) \oplus z\Theta\mathcal{H}((1 - z)/2),$$

and $\mathcal{H}(z\Theta)$ is contained isometrically in $\mathcal{H}(\Theta b)$. Therefore, $(1 + z)k_{r_n}$ converges to the constant function 1 in $\mathcal{H}(\Theta b)$ norm, showing that

$$1 \in \overline{\mathcal{M}(a)}^{\mathcal{H}(\Theta b)}.$$

Since b is a nonextreme point of the closed unit ball of H^∞, then so is Θb. We know that, in this situation, the analytic polynomials are multipliers of $\mathcal{H}(\Theta b)$. Since the polynomials are also obviously multipliers of $\mathcal{M}(a)$, it follows that

$$\mathcal{P}_+ \subset \overline{\mathcal{M}(a)}^{\mathcal{H}(\Theta b)} \subset \mathcal{H}(\Theta b).$$

Finally, it remains to apply Theorem 23.13 to conclude that

$$\overline{\mathcal{M}(a)}^{\mathcal{H}(\Theta b)} = \mathcal{H}(\Theta b).$$

Hence, by Theorem 29.3, F_Θ^2 is rigid.

Theorem 29.7 *Let (a,b) be a pair. If the function a^2 is rigid, then $\mathcal{M}(a)$ is dense in $\mathcal{H}(b)$.*

Proof According to Theorem 24.19, we can choose $\lambda \in \mathbb{T}$ such that the measure μ_λ is absolutely continuous. Moreover, the rough estimate $|1-\bar{\lambda}b| \le 2$ gives $|F_\lambda| \ge |a/2|$, where $F_\lambda = a/(1-\bar{\lambda}b)$. Since a^2 is rigid, the function $a^2/4$ is also rigid and, remembering that F_λ is outer, we can apply Corollary 12.32 to conclude that F_λ^2 is rigid. It remains to use Theorem 29.3 to see that $\mathcal{M}(a)$ is dense in $\mathcal{H}(b)$. □

Theorems 29.7 and 29.3 reveal that, if (a,b) is a pair such that a^2 is rigid, then F^2 is also rigid. The converse is not true, as shown in Example 29.6, because $a(z) = (1+z)/2$ is not rigid (by Lemma 6.19) whereas $F_{\Theta b}^2$ is rigid.

We can translate Corollary 29.4 and Theorem 29.5 into the language of exposed points of the unit ball of H^1. Indeed, let f be an outer function in H^1, having unit norm. (Of course, it is not restrictive for the problem of exposed points, because, if f is an exposed point, then, in particular, f is an extreme point of the closed unit ball of H^1, and thus f must be an outer function of unit norm.) Then we can consider $F = f^{1/2}$, where we may take any branch of $f^{1/2}$. The function F is now an outer function in H^2 and, according to Theorem 6.15, F^2 is rigid if and only if f is an exposed point of the closed unit ball of H^1. Now, let (a,b) be the admissible pair for F. Note that, since $\|F\|_2 = 1$, then we have $b(0) = 0$.

Corollary 29.8 *Let f be an outer function in H^1, with $\|f\|_1 = 1$, and let (a,b) be the admissible pair for $f^{1/2}$. Then f is an exposed point of the closed unit ball of H^1 if and only if $\mathcal{M}(a)$ is dense in $\mathcal{H}(b)$.*

Proof To show this, apply Corollary 29.4. □

Assuming the conditions of Corollary 29.8, let f_Θ be the outer function defined by

$$f_\Theta = F_\Theta^2 = \left(\frac{a}{1-\Theta b}\right)^2.$$

Corollary 29.9 *Assume that f is an exposed point of the closed unit ball of H^1 and let Θ be an inner function. Then $\|f_\Theta\|_1 = 1$ and f_Θ is an exposed point of the closed unit ball of H^1.*

Proof Since f is an exposed point of the closed unit ball of H^1, F^2 is rigid and the pair (a, b) is special (because it is admissible). Then, Theorem 29.5 implies that $(a, \Theta b)$ is special and F_Θ^2 is rigid. Since $(a, \Theta b)$ is special, we have

$$\frac{1 - |\Theta(z)b(z)|^2}{|1 - \Theta(z)b(z)|^2} = \int_{\mathbb{T}} \frac{1 - |z|^2}{|\zeta - z|^2} \, |F_\Theta(\zeta)|^2 \, dm(\zeta).$$

Taking $z = 0$ and using the fact that $b(0) = 0$, we get $\|f_\Theta\|_1 = \|F_\Theta^2\|_2^2 = 1$. Now, we apply Theorem 6.15 to deduce that f_Θ is exposed. $\square$

Exercise

Exercise 29.2.1 Let F be a nonconstant outer function in H^2 such that $\|F\|_2 = 1$. Assume that there is a $\lambda \in \mathbb{T}$ such that $\|F_\lambda\|_2 < 1$. Show that there is a nonconstant inner function u such that $F/(1 - u) \in H^2$.

We suggest the following method. Let (a, b) be the admissible pair for F.

(i) Show that $\mu_{\lambda b}$ is a probabilistic measure, i.e. $\|\mu_{\lambda b}\| = 1$.

(ii) Deduce that $\mu_{\lambda b}$ has a nonzero singular component, say ν.

(iii) Let u be the function in the closed unit ball of H^∞ such that $(1 + u)/(1 - u)$ is the Herglotz integral of ν. Show that u is inner.

(iv) Show that $F/(1 - u) \in H^2$.
Hint: Use Theorem 24.29.

29.3 Dimension of $\mathcal{H}_0(b)$

If $\mathcal{M}(a)$ is not dense in $\mathcal{H}(b)$, we naturally wonder how big its complement $\mathcal{H}_0(b)$ could be. In this section, we obtain conditions under which $\mathcal{H}_0(b)$ is finite-dimensional.

Lemma 29.10 *Assume that $\mathcal{M}(a)$ is not dense in $\mathcal{H}(b)$. Let $\lambda \in \mathbb{T}$ be such that the Clark measure μ_λ is absolutely continuous with respect to the Lebesgue measure m. Then we have*

$$\mathcal{H}_0(b) = W_\lambda \ker T_{\bar{F}_\lambda / F_\lambda},$$

where $W_\lambda = T_{1 - \bar{\lambda} b} T_{\bar{F}_\lambda}$.

Proof According to (1.33), we can write

$$H^2 = \ker T_{\bar{F}_\lambda / F_\lambda} \oplus \overline{T_{F_\lambda / \bar{F}_\lambda} H^2}.$$

Then, using the fact that W_λ is a unitary map from H^2 onto $\mathcal{H}(b)$ (because μ_λ is absolutely continuous), we have

$$\begin{aligned}\mathcal{H}(b) &= W_\lambda H^2\\ &= W_\lambda \ker T_{\bar{F}_\lambda/F_\lambda} \oplus W_\lambda \overline{T_{F_\lambda/\bar{F}_\lambda} H^2}\\ &= W_\lambda \ker T_{\bar{F}_\lambda/F_\lambda} \oplus \overline{W_\lambda T_{F_\lambda/\bar{F}_\lambda} H^2}^b.\end{aligned}$$

But, by Lemma 24.21, $W_\lambda T_{F_\lambda/\bar{F}_\lambda} = T_a$, and thus we obtain

$$\mathcal{H}(b) = W_\lambda \ker T_{\bar{F}_\lambda/F_\lambda} \oplus \overline{\mathcal{M}(a)}^b.$$

This identity reveals that $\mathcal{H}_0(b) = \mathcal{M}(a)^\perp = W_\lambda \ker T_{\bar{F}_\lambda/F_\lambda}$. □

Theorem 29.11 *Let N be a positive integer and let $\lambda \in \mathbb{T}$ be such that the Clark measure μ_λ is absolutely continuous with respect to the Lebesgue measure m. Then the following are equivalent.*

(i) $\dim \mathcal{H}_0(b) = N$.
(ii) $\dim \ker T_{\bar{F}_\lambda/F_\lambda} = N$.
(iii) *There exists a polynomial p of degree N having all its roots on the unit circle $\mathbb{T}$ and a function $f \in H^2$, with f^2 being rigid, such that $F_\lambda = pf$.*

Proof For $N = 0$, the result follows immediately from Theorem 29.3. Hence, we can assume that $N \geq 1$, i.e. $\mathcal{M}(a)$ is not dense in $\mathcal{H}(b)$. Then it follows from Lemma 29.10 that $\mathcal{H}_0(b) = W_\lambda \ker T_{\bar{F}_\lambda/F_\lambda}$ and, since W_λ is a unitary operator from H^2 onto $\mathcal{H}(b)$, we have

$$\dim \mathcal{H}_0(b) = \dim \ker T_{\bar{F}_\lambda/F_\lambda}, \tag{29.3}$$

which gives the equivalence between (i) and (ii).

(i) $\implies$ (iii) Assume that $\dim \mathcal{H}_0(b) = N$. According to Theorem 28.44, the operator $S_0 : \mathcal{H}_0(b) \longrightarrow \mathcal{H}_0(b)$ has its spectrum contained in $\mathbb{T}$ and, since $\mathcal{H}_0(b)$ is finite-dimensional, its spectrum coincides with its point spectrum. Let $z_1, z_2, \dots, z_s$ be distinct eigenvalues of S_0, with multiplicities $m_1, m_2, \dots, m_s$, respectively. Then we have

$$\dim \ker (S_0 - z_i I)^{m_i} = m_i \qquad (1 \leq i \leq s)$$

and

$$\mathcal{H}_0(b) = \bigoplus_{i=1}^{s} \ker (S_0 - z_i I)^{m_i}$$

and surely $m_1 + \cdots + m_s = N$. Note that, according to Theorem 28.42 and Lemma 1.32, we have

$$\begin{aligned}\dim \ker (S_b^* - \bar{z}_i I)^{m_i} &= \dim \ker (S_0^* - \bar{z}_i I)^{m_i}\\ &= \dim \ker (S_0 - z_i I)^{m_i} = m_i.\end{aligned} \tag{29.4}$$

In particular, $\bar{z}_i$ is an eigenvalue of S_b^*. Now, consider the polynomial p defined by

$$p(z) = (1 - \bar{z}_1 z)^{m_1} \cdots (1 - \bar{z}_s z)^{m_s}.$$

Clearly, p is a polynomial of degree N and its roots $\{z_1, z_2, \dots, z_s\}$ are all located on $\mathbb{T}$. It is enough to show that $f = F_\lambda/p$ is an outer H^2 function such that f^2 is rigid.

We can write

$$\frac{1}{p(z)} = \sum_{i=1}^{s} \sum_{j=1}^{m_i} \frac{a_{i,j}}{(1 - \bar{z}_i z)^j},$$

where $a_{i,j}$ are complex numbers, and thus

$$f(z) = \frac{F_\lambda(z)}{p(z)} = \sum_{i=1}^{s} \sum_{j=1}^{m_i} a_{i,j} \frac{F_\lambda(z)}{(1 - \bar{z}_i z)^j}. \tag{29.5}$$

Let us check that $F_\lambda/(1 - \bar{z}_i z)^{m_i}$ belongs to H^2, for every $1 \le i \le s$. Argue by absurdity, assuming that there is some $i_0 \in \{1, 2, \dots, s\}$ such that $F_\lambda/(1 - \bar{z}_{i_0} z)^{m_{i_0}}$ does not belong to H^2. Let

$$k_{i_0} = \max \left\{ k \ge 0 : \frac{F_\lambda(z)}{(1 - \bar{z}_{i_0} z)^k} \in H^2 \right\}.$$

We have $k_{i_0} < m_{i_0}$ and, since $\bar{z}_{i_0}$ is an eigenvalue of S_b^*, by Theorem 28.37, we also have $k_{i_0} \ge 1$. Thus, using Theorem 28.39 and (29.4), we obtain

$$m_{i_0} = \dim \ker(S_b^* - \bar{z}_{i_0})^{m_{i_0}} = \min(k_{i_0}, m_{i_0}) = k_{i_0},$$

which is absurd. Therefore, $F_\lambda(z)/(1 - \bar{z}_i z)^{m_i}$ belongs to H^2, for every $1 \le i \le s$, and we easily deduce from (29.5) that $f \in H^2$. Then Corollary 4.24 implies that f is outer.

To show that f^2 is rigid, we exploit Theorem 12.30 and verify that $T_{\bar{f}/f}$ is one-to-one. Note that

$$\frac{\bar{f}}{f} = \frac{\bar{F}_\lambda}{F_\lambda} \frac{p}{\bar{p}},$$

and, using the fact that

$$\frac{(1 - \bar{z}_i z)}{\overline{(1 - \bar{z}_i z)}} = -\bar{z}_i z \qquad (z \in \mathbb{T}),$$

we get

$$\frac{\bar{f}}{f} = c\,\chi_N \frac{\bar{F}_\lambda}{F_\lambda},$$

where c is a unimodular constant. Let $g \in \ker T_{\bar{f}/f}$. Then

$$0 = T_{\bar{f}/f} g = P_+\left(\frac{\bar{f} g}{f}\right) = c\,P_+\left(\frac{\bar{F}_\lambda}{F_\lambda} \chi_N g\right) = c\,T_{\bar{F}_\lambda/F_\lambda}(\chi_N g),$$

which reveals that $\chi_N g \in \ker T_{\bar{F}_\lambda/F_\lambda} \cap S^N H^2$. But, by (29.3), we have that $\dim \ker T_{\bar{F}_\lambda/F_\lambda} = N$, and thus Corollary 12.22 implies that $\chi_N g = 0$, that is, $g = 0$. Therefore, $T_{\bar{f}/f}$ is one-to-one and Theorem 12.30 implies that f^2 is rigid.

(iii) $\Longrightarrow$ (i) Assume that $F_\lambda = fp$, where p is a polynomial of degree N, having all its roots on the unit circle, and f is an H^2 function such that f^2 is rigid. According to (29.3), it is sufficient to prove that $\dim \ker T_{\bar{F}_\lambda/F_\lambda} = N$. Since $f = F_\lambda/p$, Corollary 4.24 ensures that f is outer. Moreover, by hypothesis, f^2 is rigid and thus, by Theorem 12.30, the operator $T_{\bar{f}/f}$ is one-to-one. Therefore,

$$\dim(\ker T_{\bar{F}_\lambda/F_\lambda}) = \dim(T_{\bar{f}/f} \ker T_{\bar{F}_\lambda/F_\lambda}). \tag{29.6}$$

Now, we show that

$$T_{\bar{f}/f} \ker T_{\bar{F}_\lambda/F_\lambda} = \mathcal{P}_{N-1}, \tag{29.7}$$

where $\mathcal{P}_{N-1}$ denotes the family of all analytic polynomials of degree at most $N-1$. Since p has all its roots on the unit circle, we can write

$$p(z) = c \prod_{i=1}^{s} (1 - \bar{z}_i z)^{m_i},$$

where $c \in \mathbb{C}$, $c \neq 0$, $z_1, z_2, \dots, z_s \in \mathbb{T}$ and $\sum_{i=1}^{s} m_i = N$. Then we easily see that

$$\frac{\overline{p(z)}}{p(z)} = \lambda z^N \qquad (z \in \mathbb{T}),$$

where $\lambda \in \mathbb{T}$. We have $\bar{F}_\lambda/F_\lambda = f\bar{p}/\bar{f}p$, and thus

$$\frac{\overline{F_\lambda(z)}}{F_\lambda(z)} = \lambda \bar{z}^N \frac{\overline{f(z)}}{f(z)} \qquad (z \in \mathbb{T}).$$

Hence, for every function g in H^2, we have

$$T_{\bar{F}_\lambda/F_\lambda} g = \lambda P_+ \left(\bar{z}^N \frac{\bar{f}}{f} g \right) = \lambda P_+ \left(\bar{z}^N P_+ \left(\frac{\bar{f}}{f} g \right) \right) = \lambda S^{*N} T_{\bar{f}/f} g,$$

or equivalently

$$T_{\bar{F}_\lambda/F_\lambda} = \lambda S^{*N} T_{\bar{f}/f}.$$

This relation implies that

$$T_{\bar{f}/f} \ker T_{\bar{F}_\lambda/F_\lambda} = \mathcal{R}(T_{\bar{f}/f}) \cap \ker S^{*N}$$

and, using Lemma 8.7, we get

$$T_{\bar{f}/f} \ker T_{\bar{F}_\lambda/F_\lambda} = \mathcal{R}(T_{\bar{f}/f}) \cap \mathcal{P}_{N-1}.$$

To prove (29.7), it remains to verify that $\mathcal{P}_{N-1} \subset \mathcal{R}(T_{\bar{f}/f})$. For each $j \le N-1$, we have

$$T_{\bar{f}/f}(z^j f) = P_+(z^j \bar{f}) = \sum_{k=0}^{j} \overline{\hat{f}(k)}\, z^{j-k},$$

and thus, in particular, we have $T_{\bar{f}/f}(\chi_p f) \in \mathcal{P}_{N-1}$. Therefore, we can regard $T_{\bar{f}/f}$ as an operator from $f\mathcal{P}_{N-1}$ into $\mathcal{P}_{N-1}$. Since this operator is injective (as already noticed) and since $\dim(f\mathcal{P}_{N-1}) = \dim \mathcal{P}_{N-1} = N$, it is an isomorphism. In particular, it is onto, i.e.

$$\mathcal{P}_{N-1} = T_{\bar{f}/f}(f\mathcal{P}_{N-1}) \subset \mathcal{R}(T_{\bar{f}/f}).$$

Thus, in fact, we have

$$T_{\bar{f}/f} \ker T_{\bar{F}_\lambda/F_\lambda} = \mathcal{P}_{N-1}$$

and, according to (29.6), we obtain

$$\dim \ker T_{\bar{F}_\lambda/F_\lambda} = N. \qquad \square$$

Theorem 29.12 *Let N be a positive integer and let $\lambda \in \mathbb{T}$ be such that the Clark measure μ_λ is absolutely continuous with respect to the Lebesgue measure m. Then the following are equivalent.*

(i) *$\mathcal{M}(a)$ is a closed subspace of $\mathcal{H}(b)$ and $\dim(\mathcal{H}(b) \ominus \mathcal{M}(a)) = N$.*
(ii) *The operator $T_{\bar{F}_\lambda/F_\lambda}$ is a Fredholm operator of index N.*
(iii) *The operator $T_{\bar{F}_\lambda/F_\lambda}$ is a left semi-Fredholm operator of index N.*
(iv) *There exists a polynomial p of degree N having all its roots on the unit circle $\mathbb{T}$ and an outer function f in H^2 satisfying $|f|^2 \in (HS)$ such that $F_\lambda = pf$.*

In this case, we have

$$\mathcal{H}(b) = \mathcal{M}(a) \dotplus \mathcal{P}_{N-1}, \tag{29.8}$$

where the symbol $\dotplus$ denotes a direct sum (not necessarily orthogonal). Moreover, there are two constants $c_1, c_2 > 0$ such that, for any $f = ag + p$, with $g \in H^2$ and $p \in \mathcal{P}_{N-1}$, we have

$$c_1(\|g\|_2 + \|p\|_2) \le \|f\|_b \le c_2(\|g\|_2 + \|p\|_2).$$

Proof According to Lemma 24.21 and Theorem 24.23, we have

$$\mathcal{M}(a) = W_\lambda T_{F_\lambda/\bar{F}_\lambda} H^2,$$

where $W_\lambda = T_{1-\bar{\lambda}b} T_{\bar{F}_\lambda}$ is a unitary operator from H^2 onto $\mathcal{H}(b)$. Therefore, we see that (i) is equivalent to saying that $T_{F_\lambda/\bar{F}_\lambda} H^2$ is closed in H^2 and

$\dim(H^2 \ominus T_{F_\lambda/\bar{F}_\lambda} H^2) = N$. But $H^2 \ominus T_{F_\lambda/\bar{F}_\lambda} H^2 = \ker(T^*_{F_\lambda/\bar{F}_\lambda})$ and, moreover, we know from Theorem 12.24 that $T_{F_\lambda/\bar{F}_\lambda}$ is always one-to-one. Hence, (i) is equivalent to saying that the operator $T_{F_\lambda/\bar{F}_\lambda}$ is a Fredholm operator of index $-N$. The equivalence between (i) and (ii) follows now from Theorem 7.32.

The equivalence between (ii) and (iii) follows from the fact that $T_{F_\lambda/\bar{F}_\lambda} = (T_{\bar{F}_\lambda/F_\lambda})^*$ is one-to-one.

(ii) $\implies$ (iv) According to Theorem 29.11, the only part to prove is that $|f|^2 \in (HS)$. But, as in the proof of Theorem 29.11, we have that

$$T_{f/\bar{f}} = \gamma S^{*N} T_{F_\lambda/\bar{F}_\lambda},$$

where γ is a unimodular constant. Then, by Lemma 8.7 and Theorem 7.39, $T_{f/\bar{f}}$ is a Fredholm operator with

$$\operatorname{ind}(T_{f/\bar{f}}) = \operatorname{ind}(S^{*N}) + \operatorname{ind}(T_{F_\lambda/\bar{F}_\lambda}) = N - N = 0.$$

But, by Theorem 12.24, the operator $T_{f/\bar{f}}$ is always injective. Hence, it is an isomorphism and Corollary 12.43 implies that $|f|^2 \in (HS)$.

(iv) $\implies$ (ii) Since $|f|^2 \in (HS)$, Corollary 12.43 shows that $T_{f/\bar{f}}$ is invertible and, in particular, it is a Fredholm operator of index 0. Moreover, by Lemma 8.7, S^N is a Fredholm operator of index $-N$. But $T_{F_\lambda/\bar{F}_\lambda} = \gamma S^N T_{f/\bar{f}}$ and then Theorem 7.39 implies that $T_{F_\lambda/\bar{F}_\lambda}$ is a Fredholm operator of index $-N$, which, by Theorem 7.32, is equivalent to (ii).

It remains to prove the decomposition of the space $\mathcal{H}(b)$ as well as the equivalence of norms. Write

$$p(z) = \prod_{j=1}^{s} (z - \zeta_j)^{m_j},$$

where, by hypothesis, $\zeta_j \in \mathbb{T}$ and $N = m_1 + m_2 + \cdots + m_s$. First, note that, since b is nonextreme, the polynomials belong to $\mathcal{H}(b)$. Now let $q \in \mathcal{P}_{N-1} \cap \mathscr{M}(a)$. This means that the polynomial q can be written as $q = ag$ for some $g \in H^2$. But then, since

$$pf = \frac{a}{1 - \bar{\lambda} b},$$

we see that the rational function

$$\frac{q}{p} = (1 - \bar{\lambda} b) f g$$

belongs to H^1. This is clearly possible if and only if the poles of q/p are outside $\mathbb{D}^-$. In particular, we see that the polynomial q should have a zero of order at least m_i at each point ζ_i. Since the degree of q is less than or equal to

$N-1$, this necessarily implies that $q=0$. Hence, the sum $\mathscr{M}(a) \oplus \mathcal{P}_{N-1}$ is direct. Now since

$$\dim \mathcal{P}_{N-1} = N = \operatorname{codim} \mathscr{M}(a),$$

we obtain (29.8).

In particular, the angle between the subspaces $\mathscr{M}(a)$ and $\mathcal{P}_{N-1}$ is strictly positive, which means that

$$\|f\|_b \asymp \|ag\|_b + \|p\|_b$$

for every $f = ag + p \in \mathcal{H}(b)$, where $ag \in \mathscr{M}(a)$ and $p \in \mathcal{P}_{N-1}$. Moreover, since $\mathscr{M}(a)$ is a closed subspace of $\mathcal{H}(b)$, contractively embedded ($\|ag\|_b \le \|ag\|_{\mathscr{M}(a)}$, $g \in H^2$), the open mapping theorem yields a constant $c > 0$ such that

$$\|g\|_2 = \|ag\|_{\mathscr{M}(a)} \le c\,\|ag\|_b \qquad (g \in H^2).$$

It remains to note that, since $\mathcal{P}_{N-1}$ is a finite-dimensional space, the norms $\|\cdot\|_2$ and $\|\cdot\|_b$ are equivalent on $\mathcal{P}_{N-1}$. Hence, we can conclude that there are two constants $c_1, c_2 > 0$ such that, for any $f = ag + p$, with $g \in H^2$ and $p \in \mathcal{P}_{N-1}$, we have

$$c_1(\|g\|_2 + \|p\|_2) \le \|f\|_b \le c_2(\|g\|_2 + \|p\|_2). \qquad \square$$

Exercise

Exercise 29.3.1 Let b be a rational and nonextreme function in the closed unit ball of H^∞. Let a be the associated outer function, which is also rational; see Section 27.5. Write $a = q/r$, where q and r are two polynomials with $GCD(q,r) = 1$. Let us denote by ζ_j, $1 \le j \le N$, the zeros of q on $\mathbb{T}$, repeated according to their multiplicities.

(i) Show that there exists $\lambda \in \mathbb{T} \setminus \{b(\zeta_1), \dots, b(\zeta_N)\}$ such that the Clark measure σ_λ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{T}$.

(ii) Justify that

$$\inf_{z \in \bar{\mathbb{D}}} |r(z)(1 - \bar{\lambda} b(z))| > 0.$$

(iii) Denote by $q_1(z) = \prod_{j=1}^N (z - \zeta_j)$. Show that $F_\lambda = q_1 f$, where f is an outer function in H^2 satisfying $|f|^2 \in (HS)$.

(iv) Using Theorem 29.12, recover Theorem 27.20.

29.4 S_b-invariant subspaces of $\mathcal{H}(b)$

Let F be an outer function in H^2 of unit norm and such that $F(0) > 0$. Let (a, b) be the admissible pair associated with F and assume that F^2 is rigid. Then, according to Theorem 29.3, the pair $(a, \bar{\lambda} b)$ is special, for every $\lambda \in \mathbb{T}$. To establish a partial converse of this result, in this section we study some facts about the S_b-invariant subspace of $\mathcal{H}(b)$.

Lemma 29.13 *Assume that $\mathcal{M}(a)$ is not dense in $\mathcal{H}(b)$ and put*

$$\mathcal{H}_0(b) = \mathcal{H}(b) \ominus \overline{\mathcal{M}(a)}^b.$$

Let $h \in \mathcal{H}(b)$. Then the function h belongs to $\mathcal{H}_0(b)$ if and only

$$T_{\bar{a}} h + T_{b\bar{a}/a} h^+ = 0.$$

Proof For each $g \in H^2$, we have

$$(ag)^+ = T_{\bar{b}a/\bar{a}} g. \tag{29.9}$$

Indeed, by Theorem 12.4,

$$T_{\bar{a}} T_{\bar{b}a/\bar{a}} g = T_{\bar{b}a} g = T_{\bar{b}}(ag),$$

which, by definition of $(ag)^+$, leads to (29.9). Therefore,

$$\begin{aligned} \langle h, ag \rangle_b &= \langle h, ag \rangle_2 + \langle h^+, T_{\bar{b}a/\bar{a}} g \rangle_2 \\ &= \langle T_{\bar{a}} h, g \rangle_2 + \langle T_{\bar{a}b/a} h^+, g \rangle_2 \\ &= \langle T_{\bar{a}} h + T_{b\bar{a}/a} h^+, g \rangle_2. \end{aligned}$$

Clearly, h belongs to $\mathcal{H}_0(b)$ if and only if h is orthogonal in $\mathcal{H}(b)$ to ag for any function g in H^2, and the last identity shows that this happens if and only if $T_{\bar{a}} h + T_{b\bar{a}/a} h^+ = 0$. □

Lemma 29.14 *The function $h \in \mathcal{H}(b)$ is orthogonal to the S_b-invariant subspace generated by b if and only if $h^+ = 0$.*

Proof Suppose first that h is orthogonal to $S_b^n b$, $n \geq 0$. Then, using (24.1), we have

$$\begin{aligned} S_b^* h &= X_b h + \langle h, b \rangle_b S^* b \\ &= X_b h = S^* h. \end{aligned}$$

To argue by induction, assume that $S_b^{*n}h = S^{*n}h$ holds for some nonnegative integer n. Then, using (24.1) once more, we have

$$\begin{aligned}
S_b^{*(n+1)}h &= S_b^*(S_b^{*n}h)\\
&= X_b S_b^{*n}h + \langle S_b^{*n}h, b\rangle_b S^*b\\
&= X_b S_b^{*n}h + \langle h, S_b^{n}b\rangle_b S^*b\\
&= X_b S_b^{*n}h = S^{*(n+1)}h.
\end{aligned}$$

Thus, by induction,

$$S_b^{*n}h = S^{*n}h$$

holds for all nonnegative integers n. Now, by (23.11), Lemma 23.7 and Theorem 23.8, we have

$$\begin{aligned}
0 &= \langle h, S_b^n b\rangle_b\\
&= \langle S_b^{*n}h, b\rangle_b\\
&= \langle S^{*n}h, b\rangle_b\\
&= \langle S^{*n}h, b\rangle_2 + \langle (S^{*n}h)^+, b^+\rangle_2\\
&= \langle S^{*n}h, b\rangle_2 + \langle S^{*n}h^+, b^+\rangle_2\\
&= \langle \bar{z}^n h, b\rangle_2 + \langle \bar{z}^n h^+,\ (1/\overline{a(0)} - a)\rangle_2\\
&= \langle T_{\bar{b}}h, z^n\rangle_2 + \frac{1}{a(0)}\langle h^+, z^n\rangle_2 - \langle T_{\bar{a}}h^+, z^n\rangle_2.
\end{aligned}$$

Since $T_{\bar{b}}h = T_{\bar{a}}h^+$, we obtain

$$\langle h^+, z^n\rangle_2 = 0 \qquad (n \geq 0).$$

Thus, $h^+ = 0$.

Conversely, let $h \in \mathcal{H}(b)$ and assume that $h^+ = 0$. Then it follows from the preceding computation that

$$\langle S^{*n}h, b\rangle_b = 0 \qquad (n \geq 0). \tag{29.10}$$

By (24.1) and (29.10), we have

$$S_b^*h = X_b h + \langle h, b\rangle_b S^*b = X_b h = S^*h.$$

Assume now that $S_b^{*n}h = S^{*n}h$, for some nonnegative integer n. Then, using (24.1) once more, we have

$$S_b^{*(n+1)}h = S_b^*(S_b^{*n}h) = X_b S^{*n}h + \langle S^{*n}h, b\rangle_b S^*b = S^{*(n+1)}h.$$

Hence, by induction, $S_b^{*n}h = S^{*n}h$ holds for all nonnegative integers n. Therefore, we deduce that

$$\begin{aligned} 0 = \langle S^{*n}h, b\rangle_b &= \langle S_b^{*n}h, b\rangle_b \\ &= \langle h, S_b^n b\rangle_b, \end{aligned}$$

which proves that h is orthogonal to the S_b-invariant subspace generated by b. □

Theorem 29.15 *Assume that $b \in \overline{\mathcal{M}(a)}^b$. Then $\mathcal{M}(a)$ is dense in $\mathcal{H}(b)$.*

Proof Let $h \in \mathcal{M}(b)$, $h \perp \mathcal{M}(a)$. The manifold $\mathcal{M}(a)$ is invariant under the operator S_b, and thus, by continuity, so is $\overline{\mathcal{M}(a)}^b$. Therefore, for every $k \geq 1$, $S_b^k b \in \overline{\mathcal{M}(a)}^b$. Thus h is orthogonal to the S_b-invariant subspace generated by b. Then Lemma 29.14 implies that $h^+ = 0$. But, by Lemma 29.13, we must have

$$T_{\bar{a}}h + T_{b\bar{a}/a}h^+ = 0,$$

whence $T_{\bar{a}}h = 0$. Now, since a is outer, Theorem 12.19 implies that $h \equiv 0$. This proves that $\mathcal{M}(a)$ is dense in $\mathcal{H}(b)$. □

Theorem 29.16 *Given $\zeta \in \mathbb{T}$, denote the S_b-invariant subspace of $\mathcal{H}(b)$ generated by $1 - \bar{\zeta}b$ by $\mathcal{M}_{\zeta,b}$. Then the following hold.*

(i) $T_{1-\bar{\zeta}b}T_{\bar{g}}\mathcal{P}_+ \subset \mathcal{M}_{\zeta,b}$, *for each function* $g \in H^2$.
(ii) $\overline{\mathcal{M}(a)}^b \subset \mathcal{M}_{\zeta,b}$.
(iii) $T_{1-\bar{\zeta}b}T_{\bar{F}_\zeta}H^2 = \mathcal{M}_{\zeta,b}$.

Proof (i) Write

$$g(z) = \sum_{k=0}^{\infty} a_k z^k.$$

Then, for each nonnegative integer n, we have

$$\begin{aligned} P_+(\bar{g}z^n) &= P_+\left(z^n \sum_{k=0}^{\infty} \overline{a_k}\bar{z}^k\right) \\ &= P_+\left(\sum_{k=0}^{\infty} \overline{a_k}z^{n-k}\right) \\ &= \sum_{k=0}^{n} \overline{a_k}z^{n-k} \\ &= \sum_{k=0}^{n} \overline{a_k}S_b^{n-k}1. \end{aligned}$$

Therefore, using the fact that $T_{1-\bar{\zeta}b}S_b 1 = S_b T_{1-\bar{\zeta}b}1$, we get

$$T_{1-\bar{\zeta}b}T_{\bar{g}}z^n = \sum_{k=0}^{n} \overline{a_k} S_b^{n-k}(1-\bar{\zeta}b),$$

which implies that $T_{1-\bar{\zeta}b}T_{\bar{g}}z^n \in \mathcal{M}_{\zeta,b}$. The result now follows by linearity.

(ii) Let $h \in H^2$. Then, by Lemma 24.21, we have

$$ah = T_a h = T_{1-\bar{\zeta}b}T_{\bar{F}_\zeta}T_{F_\zeta/\bar{F}_\zeta}h.$$

Now, there exists a sequence $(p_n)_{n\geq 1}$ of polynomials in $\mathcal{P}_+$ that converges to $T_{F_\zeta/\bar{F}_\zeta}h$. Thus, by continuity, we get

$$T_a h = \lim_{n\to\infty} T_{1-\bar{\zeta}b}T_{\bar{F}_\zeta}p_n,$$

and part (i) implies that $T_a h \in \mathcal{M}_{\zeta,b}$. Thus $\mathcal{M}(a) \subset \mathcal{M}_{\zeta,b}$ and we get the conclusion.

(iii) Let $g \in H^2$. Then there exists a sequence $(p_n)_{n\geq 1}$ in $\mathcal{P}_+$ that converges to g. Thus,

$$\lim_{n\to+\infty} T_{1-\bar{\zeta}b}T_{\bar{F}_\zeta}p_n = T_{1-\bar{\zeta}b}T_{\bar{F}_\zeta}g.$$

But part (i) implies that $T_{1-\bar{\zeta}b}T_{\bar{F}_\zeta}p_n \in \mathcal{M}_{\zeta,b}$ and then we obtain

$$T_{1-\bar{\zeta}b}T_{\bar{F}_\zeta}H^2 \subset \mathcal{M}_{\zeta,b}.$$

Denote now by $\mathcal{P}_+^n$ the set of analytic polynomials of degree less than or equal to n. Then, by a computation made in part (i), we see that $T_{\bar{F}_\zeta}$ maps $\mathcal{P}_+^n$ into itself. Since F_ζ is outer, we know furthermore that $T_{\bar{F}_\zeta}$ is one-to-one. Hence, by an argument of dimension, we see that $T_{\bar{F}_\zeta}$ is onto. So if $p \in \mathcal{P}_+$, then there exists $q \in \mathcal{P}_+$ such that

$$T_{\bar{F}_\zeta}(q) = p.$$

Thus,

$$(1-\bar{\zeta}b)p = (1-\bar{\zeta}b)T_{\bar{F}_\zeta}q = T_{1-\bar{\zeta}b}T_{\bar{F}_\zeta}q,$$

which gives $(1-\bar{\zeta}b)p \in T_{1-\bar{\zeta}b}T_{\bar{F}_\zeta}H^2$. But the functions $(1-\bar{\zeta}b)p$, $p \in \mathcal{P}_+$, span the space $\mathcal{M}_{\zeta,b}$, which implies that $\mathcal{M}_{\zeta,b} \subset T_{1-\bar{\zeta}b}T_{\bar{F}_\zeta}H^2$ and concludes the proof. □

29.5 A necessary condition for nonrigidity

Let F be an outer function in H^2 of unit norm and such that $F(0) > 0$. Let (a, b) be the admissible pair associated with F and assume that F^2 is rigid.

Then, according to Theorem 29.3, the pair $(a, \bar{\lambda}b)$ is special, for every $\lambda \in \mathbb{T}$. In this section, we establish a partial converse of this result. We recall that

$$\mathcal{H}_0(b) = \mathcal{H}(b) \ominus \overline{\mathcal{M}(a)}^b.$$

Theorem 29.17 *Let F be an outer function in H^2, with $\|F\|_2 = 1$ and $F(0) > 0$. Let (a, b) be the admissible pair associated with F and assume that F^2 is not rigid. Let P_0 be the orthogonal projection of $\mathcal{H}(b)$ onto $\mathcal{H}_0(b)$. Suppose that $P_0 1$ and $P_0 b$ are linearly dependent. Then there is a unique $\lambda \in \mathbb{T}$ such that the pair $(a, \bar{\lambda}b)$ is not special.*

Proof Since F^2 is not rigid, we know from Corollary 29.4 that $\mathcal{M}(a)$ is not dense in $\mathcal{H}(b)$. Therefore, $\mathcal{H}_0(b) \neq \{0\}$.

The constant function 1 is not in $\overline{\mathcal{M}(a)}^b$. Indeed, argue by absurdity and assume that $1 \in \overline{\mathcal{M}(a)}^b$. The manifold $\mathcal{M}(a)$ is invariant under the operator $S_b = S|\mathcal{H}(b)$, and thus, by continuity, so is $\overline{\mathcal{M}(a)}^b$. Therefore, for every $k \geq 1$, $z^k \in \mathcal{M}(a)$, which implies, by Theorem 23.13, that $\mathcal{H}(b) \subset \overline{\mathcal{M}(a)}^b$. We thus get a contradiction.

By Theorem 29.15, we also know that b is not in $\mathcal{M}(a)$. The hypothesis of the theorem ensures the existence of a unique nonzero scalar λ such that $P_0 1 = \bar{\lambda} P_0 b$. This means that $1 - \bar{\lambda}b$ is in the closure of $\mathcal{M}(a)$. Let us show that $|\lambda| = 1$.

Remembering that the spectrum of the operator S_b is the closed unit disk (Theorem 24.4), we have

$$(1 - \bar{w}S_b)^{-1}(1 - \bar{\lambda}b) = \sum_{n=0}^{\infty} \bar{w}^n S_b^n (1 - \bar{\lambda}b) \qquad (w \in \mathbb{D}).$$

Consequently, each function $(1 - \bar{w}S_b)^{-1}(1 - \bar{\lambda}b)$ belongs to the closure of $\mathcal{M}(a)$. But

$$\begin{aligned}((I - \bar{w}b)(1 - \bar{\lambda}b)k_w)(z) &= (1 - \bar{w}z)(1 - \bar{\lambda}b(z))k_w(z) \\ &= 1 - \bar{\lambda}b(z) \\ &= (1 - \bar{\lambda}b)(z),\end{aligned}$$

which gives $(I - \bar{w}S_b)(1 - \bar{\lambda}b)k_w = (1 - \bar{\lambda}b)$, that is,

$$(I - \bar{w}S_b)^{-1}(1 - \bar{\lambda}b) = (1 - \bar{\lambda}b)k_w.$$

Therefore, for each $w \in \mathbb{D}$, $(1 - \bar{\lambda}b)k_w$ belongs to the closure of $\mathcal{M}(a)$. In particular, $(1 - \bar{\lambda}b)k_w$ is orthogonal to $\mathcal{H}_0(b)$. Thus, for each function $h \in \mathcal{H}_0(b)$, and using Theorem 23.23, we get

$$
\begin{aligned}
0 &= \langle h,\ (1-\bar{\lambda}b)k_w\rangle_b \\
&= \langle h, k_w\rangle_b - \lambda\langle h, bk_w\rangle_b \\
&= h(w) + \frac{b(w)}{a(w)}h^+(w) - \lambda\frac{h^+(w)}{a(w)} \qquad (w \in \mathbb{D}).
\end{aligned}
$$

Therefore,

$$
ah = (\lambda - b)h^+ \qquad (h \in \mathcal{H}_0(b)). \tag{29.11}
$$

To show that $|\lambda|1$, we show that $|\lambda| > 1$ and $|\lambda| < 1$ are not possible.

The case $|\lambda| > 1$. Let $h \in \mathcal{H}_0(b)$. The function $g = (\lambda - b)^{-1}h$ is then in H^2 (note that $|\lambda - b| \geq |\lambda| - 1 > 0$) and, by (29.11), we get $h^+ = ag$. Now, Lemma 29.13 implies that

$$
0 = T_{\bar{a}}h + T_{b\bar{a}/a}h^+ = T_{\bar{a}}h + T_{b\bar{a}/a}(ag) = T_{\bar{a}}g + T_{\bar{a}}(bg).
$$

Since $T_{\bar{a}}$ is one-to-one (remember that a is outer), we obtain $h = -bg$. But we also have $h = (\lambda - b)g$. Thus, $\lambda g = 0$, that is, $g = 0$, which implies that $h = 0$, in contradiction with $\mathcal{H}_0(b) \neq \{0\}$.

The case $|\lambda| < 1$. Let $h \in \mathcal{H}_0(b)$. By (29.11), we have

$$
\begin{aligned}
-\bar{b}h + \bar{a}h^+ &= -\bar{b}h + \bar{a}\frac{ah}{\lambda - b} \\
&= \frac{-\lambda\bar{b}h + |b|^2h + |a|^2h}{\lambda - b} \\
&= \frac{(1-\lambda\bar{b})h}{\lambda - b},
\end{aligned}
$$

and thus

$$
\frac{h}{\lambda - b} = \frac{-\bar{b}h + \bar{a}h^+}{1 - \lambda\bar{b}}.
$$

Since $T_{\bar{b}}h = T_{\bar{a}}h^+$, the function $-\bar{b}h + \bar{a}h^+$ is in $\overline{H_0^2}$. Since $|\lambda| < 1$, we conclude that the function $h/(\lambda - b)$ is also in $\overline{H_0^2}$. On the other hand, because $(\lambda - b)h^+ = ah$ and a is outer, the inner factor of $\lambda - b$ must divide the inner factor of h. Since $h/(\lambda - b)$ is in L^2, we conclude, by Corollary 4.28, that $h/(\lambda - b)$ is in H^2. Since $h/(\lambda - b)$ is in both H^2 and $\overline{H_0^2}$, it is 0. So $h \equiv 0$, and again we get a contradiction with the nontriviality of $\mathcal{H}_0(b)$.

The above discussion shows that $|\lambda| = 1$. According to Theorem 29.16, we have

$$
T_{1-\bar{\lambda}b}T_{\bar{F}_\lambda}H^2 \subset \mathcal{M}_{\zeta,b} \subset \overline{\mathcal{M}(a)}^b.
$$

Theorem 24.23 implies that the measure μ_λ is not absolutely continuous, that is, $(a, \bar{\lambda}b)$ is not special. It remains to prove the uniqueness of λ. Let η be any point on $\mathbb{T}$ different from λ. Then the S_b-invariant subspace generated by $1-\bar{\lambda}b$ and $1-\bar{\eta}b$ together is all of $\mathcal{H}(b)$, since it contains the constant function 1. Now, let $h \in \mathcal{H}(b)$, $h \perp \mathcal{M}_{\eta,b}$. Hence, by Theorem 29.16, h is orthogonal to $\mathcal{M}(a)$. Since $\mathcal{M}_{\zeta,b} \subset \mathcal{M}(a)$, we get that $h \perp \mathcal{M}_{\zeta,b}$. Thus h is orthogonal to the S_b-invariant subspace generated by $1-\bar{\lambda}b$ and $1-\bar{\eta}b$ together. Therefore, $h \equiv 0$. This shows that $\mathcal{M}_{\eta,b} = \mathcal{H}(b)$ and with Theorem 29.16, we obtain

$$T_{1-\bar{\eta}b} T_{\bar{F}_\eta} H^2 = \mathcal{H}(b).$$

Theorem 24.23 now implies that $(a, \bar{\eta}b)$ is special. □

Let F be an outer function in H^2 of unit norm and such that $F(0) > 0$. Let (a, b) be the admissible pair associated with F and assume that F^2 is not rigid. If there exists a nonzero scalar λ such that $1-\bar{\lambda}b$ is in $\overline{\mathcal{M}(a)}^b$, then it follows from the proof of Theorem 29.17 that necessarily $\lambda \in \mathbb{T}$ and $(a, \bar{\lambda}b)$ is not special, whereas $(a, \bar{\eta}b)$ is special for all $\eta \in \mathbb{T} \setminus \{\lambda\}$. In other words, we have

$$\|F_\lambda\|_2 < 1 \quad \text{and} \quad \|F_\eta\|_2 = 1,$$

for all $\eta \in \mathbb{T} \setminus \{\lambda\}$.

Corollary 29.18 *Let (a, b) be a pair such that $b(0) = 0$. Assume that there exists $\mu \in \mathbb{T}$ such that $1-\bar{\mu}b \in \overline{\mathcal{M}(a)}^b$. Let λ_0 be any point in $\mathbb{T}$ such that $\|F_{\lambda_0}\|_2 = 1$. Then, for each $\eta \in \mathbb{T} \setminus \{\bar{\lambda}_0\mu\}$, the pair $(a, \bar{\eta}b)$ is special. In particular, the pair $(a, -\bar{\mu}b)$ is special.*

Proof It follows from Corollary 24.20 that $\|F_\lambda\|_2 = 1$ for almost all $\lambda \in \mathbb{T}$. So take λ_0 to be any point in $\mathbb{T}$ such that $\|F_{\lambda_0}\|_2 = 1$. It follows from Lemma 29.1 that $(a, \bar{\lambda}_0 b)$ is the admissible pair for F_{λ_0}. If $F_{\lambda_0}^2$ is rigid, then it follows from Theorem 29.3 that F_λ^2 is rigid and $(a, \bar{\lambda}b)$ is special, for any point λ in $\mathbb{T}$. The result is proved in this case. So we may now assume that $F_{\lambda_0}^2$ is not rigid. Then, as we saw after the proof of Theorem 29.17, $(a, \lambda_0\bar{\mu}b)$ is not special, whereas $(a, \bar{\eta}b)$ is special for all $\eta \in \mathbb{T} \setminus \{\bar{\lambda}_0\mu\}$. In particular, if we choose $\lambda_0 \neq -1$, then we can take $\eta = -\mu$ and conclude that $(a, -\bar{\mu}b)$ is special. □

Exercise

Exercise 29.5.1 Let u be any inner function vanishing at the origin and let $b = -\frac{1}{2}u(1+u)$.

(i) Show that $a = (1-u)/2$ and $F = (1-u)/(2+u+u^2)$.
(ii) Show that $1+b \in \mathcal{M}(a)$.

(iii) Show that $\|F\|_2 = 1$.
(iv) Conclude that there is $\lambda \in \mathbb{T}$ such that

$$\|F_\lambda\|_2 < 1 \quad \text{and} \quad \|F_\eta\|_2 = 1$$

for each $\eta \in \mathbb{T} \setminus \{\lambda\}$.

Hint: Use Corollary 29.18.

29.6 Strongly exposed points and $\mathcal{H}(b)$ spaces

We now give an analog of Corollary 29.8 for strongly exposed points. It is a characterization of strongly exposed points of the closed unit ball of H^1 in the language of $\mathcal{H}(b)$ spaces.

Theorem 29.19 *Let f be an outer function in H^1, $\|f\|_1 = 1$, let (a, b) be the admissible pair for $f^{1/2}$, and let $F = f^{1/2}$. Then, the following assertions are equivalent.*

(i) *The function f is a strongly exposed point of the closed unit ball of H^1.*
(ii) *We have $\mathcal{M}(a) = \mathcal{H}(b)$.*
(iii) *The Toeplitz operator $T_{F/\bar{F}}$ is invertible.*

Proof First, note that, since (a, b) is the admissible pair for $f^{1/2}$, the Clark measure μ is absolutely continuous (Lemma 29.1) and

$$F = \frac{a}{1-b}.$$

(ii) $\Longleftrightarrow$ (iii) This follows from Theorem 28.23.

(ii) $\Longrightarrow$ (i) It follows from Corollary 12.43 that

$$\operatorname{dist}(\bar{F}/F, H^\infty) < 1.$$

Since

$$\frac{\bar{F}}{F} = \frac{\bar{F}^2}{|F|^2} = \frac{\bar{f}}{|f|},$$

we thus have

$$\operatorname{dist}(\bar{f}/|f|, H^\infty) < 1.$$

By Corollary 29.8, the function f is also an exposed point of the closed unit ball of H^1. The conclusion now follows from Corollary 6.29. It ensures that f is a strongly exposed point.

(i) $\Longrightarrow$ (iii) Assume that f is strongly exposed. Using one more time Corollary 6.29, we get that

$$\operatorname{dist}\left(\frac{\bar{f}}{|f|},\ H^\infty + C(\mathbb{T})\right) < 1.$$

Thus, Theorem 12.46 implies that $T_{\bar{f}/|f|}$ is left semi-Fredholm. In particular, $T_{\bar{F}/F} = T_{\bar{f}/|f|}$ has a closed range in H^2. But remember that its adjoint $T_{F/\bar{F}}$ is always injective. Then the range of $T_{\bar{F}/F}$ must be all of H^2. Moreover, since f is an exposed point, we know from Corollary 12.33 that $T_{\bar{F}/F} = T_{\bar{f}/|f|}$ is also injective. Hence, $T_{F/\bar{F}} = T^*_{\bar{F}/F}$ must be invertible. □

Recall that a positive $L^1(\mathbb{T})$ function w is called a Helson–Szegő weight if there are $x, y \in L^\infty(\mathbb{T})$ (real-valued) with $\|y\|_\infty < \pi/2$ such that

$$w = e^{x+\tilde{y}},$$

where $\tilde{y}$ denotes the Hilbert transform of y; see Sections 3.6 and 12.8.

The following result says that strongly exposed points of the closed unit ball of H^1 are induced by Helson–Szegő weights. It will lead to several interesting properties of this class of functions.

Theorem 29.20 *Let f be an outer function in H^1, $\|f\|_1 = 1$. Then f is a strongly exposed point of the closed unit ball of H^1 if and only if $|f|$ is a Helson–Szegő weight.*

Proof It is sufficient to combine Theorem 29.19 and Corollary 12.43. □

Corollary 29.21 *An outer function $f \in H^1$ of norm 1 is a strongly exposed point of the closed unit ball of H^1 if and only if f can be written as $f = gh$, where g is an invertible function in H^∞ and h is a function in H^1 such that $\|\arg h\|_\infty < \pi/2$.*

Proof According to Theorem 29.20, the function f is a strongly exposed point if and only if we can write

$$|f| = e^{x+\tilde{y}},$$

where $x, y \in L^\infty(\mathbb{T})$ and $\|y\|_\infty < \pi/2$. Hence, we get

$$f = e^{x+i\tilde{x}+\tilde{y}-iy} = gh,$$

where $g = e^{x+i\tilde{x}}$ and $h = e^{\tilde{y}-iy}$. But $\|\arg h\|_\infty = \|y\|_\infty < \pi/2$ and g is outer and

$$|g^{-1}| = e^{-x} \in L^\infty(\mathbb{T}).$$

Thus g^{-1} belongs to H^∞.

Conversely, assume that $f = gh$, where g is an invertible function in H^∞ and h is a function in H^1 such that $\|\arg h\|_\infty < \pi/2$. If we choose log to be the principal determination of the logarithm (defined and analytic on $\mathbb{C} \setminus \mathbb{R}_-$), then $\log h$ is a well-defined analytic function and

$$\log(h) = \log|h| + i\,\widetilde{\log|h|} \qquad \text{(a.e. on } \mathbb{T}),$$

with $\left|\widetilde{\log|h|}\right| < \pi/2$. Put $y = -\widetilde{\log|h|}$. Then $y \in L^\infty(\mathbb{T})$ and $\|y\|_\infty < \pi/2$. Moreover, note that $\log|h| \in L^1(\mathbb{T})$, and thus, by Theorem 3.14, we have

$$\tilde{y} = \log|h| + c,$$

where c is some constant. Hence, $|h| = e^{\tilde{y}-c}$. On the other hand, since g is invertible in H^∞, we should have $|g| \geq \delta$, for some constant $\delta > 0$. In particular, $x = \log|g| \in L^\infty(\mathbb{T})$. Therefore,

$$|f| = |g|\,|h| = e^{x-c}e^{\tilde{y}},$$

which proves that $|f| \in (HS)$. □

We will give a result that can be used to rule out extreme points that are not strongly exposed. Before we do so, we need a simple lemma on Helson–Szegő weights.

Lemma 29.22 *Let ω be a positive and measurable function on $\mathbb{T}$. If $\omega \in (HS)$, then there exists $\varepsilon > 0$ such that $\omega \in L^{1+\varepsilon}(\mathbb{T})$ and $\omega^{-1} \in L^{1+\varepsilon}(\mathbb{T})$.*

Proof Since $\omega \in (HS)$, there exist two bounded real functions u, v on $\mathbb{T}$, with $\|v\|_\infty < \pi/2$, such that

$$\omega = e^{u+\tilde{v}}.$$

If $\varepsilon > 0$ is chosen so that

$$(1+\varepsilon)\|v\|_\infty < \frac{\pi}{2},$$

the result follows immediately from Theorem 3.15. □

Corollary 29.23 *Let f be an outer function in H^1, $\|f\|_1 = 1$. If f is a strongly exposed point of the closed unit ball of H^1, then, for all sufficiently small $\varepsilon > 0$, f and f^{-1} belong to $H^{1+\varepsilon}$.*

Proof By Theorem 29.20, we know that $|f| \in (HS)$. Then Lemma 29.22 implies that, for some $\varepsilon > 0$, the functions f and f^{-1} belong to $L^{1+\varepsilon}(\mathbb{T})$. The result follows immediately from the fact that f is an outer function. □

In Theorem 6.22, we prove that, if f is an outer function in H^1, $\|f\|_1 = 1$, such that $1/f \in H^1$, then f is an exposed point of the closed unit ball of H^1. On the other hand, we know from Example 6.24 that the condition $1/f \in H^1$ is not necessary. But it seems natural to ask whether there is a $p < 1$ such that, if f is an exposed point of the closed unit ball of H^1, then we have $1/f \in H^p$. The following example shows that this is not the case.

Example 29.24 We could modify Example 29.6 a little to show that there is no H^p class with the property suggested above. Fix a positive integer m and set $a(z) = ((1+z)/2)^m$. Let b_0 be an outer function with modulus $(1-|a|^2)^{1/2}$ on $\mathbb{T}$, and let $b(z) = zb_0(z)$, $z \in \mathbb{D}$. Owing to the smoothness of $|a|^2$, the function b is continuous in $\bar{\mathbb{D}}$. Its modulus is less than 1 except at the point -1, and we can arrange to have $b(-1) \neq 1$. Then the function $F = a/(1-b)$ has unit norm in H^2 by the same reasoning as was used in the original Example 29.6. For Θ, we take a Blaschke product with zeros r_n as above, but such that each zero is of multiplicity at least m. The previous reasoning goes through without change, provided we replace the functions $(1+z)k_{r_n}$ used earlier by their mth powers. The function $f_\Theta = F_\Theta^2$ is thus an exposed point of the closed unit ball of H^1 and satisfies

$$|f_\Theta(r_n)| = |a(r_n)|^2 = \left(\frac{1-|r_n|}{2}\right)^{2m}. \tag{29.12}$$

But if f_Θ^{-1} was in $H^{1/(4m)}$, then we would obtain that

$$|f_\Theta^{-1}(z)| = o((1-|z|)^{-4m}),$$

which contradicts (29.12). Thus $f_\Theta^{-1} \notin H^{1/(4m)}$ for any $m \geq 0$, which suffices to conclude that $1/f_\Theta \notin H^p$, for any $p > 0$.

Let f be a strongly exposed point. Then, it follows immediately from Corollary 29.23 that $1/f \in H^1$. This property can also be deduced directly from Theorem 29.19. Indeed, if f is a strongly exposed point, then Theorem 29.19 implies that $\mathcal{M}(a) = \mathcal{H}(b)$. But, since $1 \in \mathcal{H}(b)$, we get that $1/a \in H^2$. In particular, the function

$$\frac{1}{f} = \left(\frac{1-b}{a}\right)^2$$

belongs to H^1. As f^{-1} is actually in H^p for p slightly larger than 1, a^{-1} is actually in H^p for p slightly larger than 2.

Corollary 29.25 *Let f be an outer function in H^1, $\|f\|_1 = 1$, and let (a, b) be the admissible pair for $f^{1/2}$. If f is a strongly exposed point of the closed unit ball of H^1, then, for all sufficiently small $\varepsilon > 0$, a^{-1} belongs to $H^{2+\varepsilon}$ and $(1-b)^{-1}$ belongs to $H^{1+\varepsilon}$.*

Proof We apply Theorem 29.19 to deduce that $\mathcal{M}(a) = \mathcal{H}(b)$. Then, Theorem 28.23 implies that $(a, b) \in (HCR)$ and $T_{a/\bar{a}}$ is invertible. It follows from Corollary 12.43 that $|a|^2 \in (HS)$, and Lemma 29.22 gives the existence of some $\varepsilon > 0$ such that $|a|^{-2} \in L^{1+\varepsilon}(\mathbb{T})$. In other words, $a^{-1} \in L^{2+2\varepsilon}(\mathbb{T})$ and, since a is outer, we get that a^{-1} belongs to $H^{2+2\varepsilon}$. Now, we also know that $f \in L^{1+\varepsilon}(\mathbb{T})$ for sufficiently small $\varepsilon > 0$. Hence, by the Cauchy–Schwarz inequality, we have

$$\int_{\mathbb{T}} \frac{1}{|1-b|^{1+\varepsilon}}\, dm \le \left(\int_{\mathbb{T}} \left| \frac{a}{1-b} \right|^{2(1+\varepsilon)} dm \right)^{1/2} \left(\int_{\mathbb{T}} \frac{dm}{|a|^{2(1+\varepsilon)}} \right)^{1/2}$$
$$= \left(\int_{\mathbb{T}} |f|^{1+\varepsilon}\, dm \right)^{1/2} \left(\int_{\mathbb{T}} \frac{dm}{|a|^{2(1+\varepsilon)}} \right)^{1/2},$$

and the right-hand side is finite for sufficiently small $\varepsilon > 0$. Hence we get $(1-b)^{-1} \in H^{1+\varepsilon}$. □

Corollary 29.26 *Let f be an outer function in H^1, $\|f\|_1 = 1$. The following assertions are equivalent.*

(i) *The function f is a strongly exposed point of the closed unit ball of H^1.*
(ii) *The function $1/f$ belongs to H^1 and we have*

$$\operatorname{dist}\left(\frac{\bar{f}}{|f|},\ H^\infty + C(\mathbb{T}) \right) < 1.$$

Proof The implication (i) $\Longrightarrow$ (ii) follows from Corollary 6.29 and Corollary 29.23. The implication (ii) $\Longrightarrow$ (i) follows from Theorem 6.22 and Corollary 6.29. □

Let f be an outer function in H^1, $\|f\|_1 = 1$, let (a, b) be the admissible pair for $f^{1/2}$, and let $F = f^{1/2}$. Assume that f satisfies

$$\operatorname{dist}(\bar{f}/|f|,\ H^\infty + C(\mathbb{T})) < 1.$$

Using Theorem 12.46, we see that $T_{\bar{F}/F} = T_{\bar{f}/|f|}$ is left semi-Fredholm. But remember that $T_{F/\bar{F}}$ is always one-to-one; see Theorem 12.24. Hence, $T_{\bar{F}/F}$ is a Fredholm operator. Then, by Theorem 29.12, we get that $F = pg$, where p is a polynomial having all its roots on the unit circle and g is an outer function in H^2 satisfying $|g|^2 \in (HS)$. Therefore, $f = F^2 = p^2 g^2$. Put

$$q = p\|g^2\|_1^{1/2} \quad \text{and} \quad f_1 = g^2/\|g^2\|_1.$$

Then q is a polynomial having all its roots on the unit circle, f_1 is an outer function in H^1, $\|f_1\|_1 = 1$ and $f = q^2 f_1$. Since $|f_1| \in (HS)$, we deduce from Theorem 29.20 that f_1 is a strongly exposed point of the closed unit ball of H^1.

Conversely, let us assume that f can be written as $f = q^2 f_1$, where q is a polynomial having all its roots on the unit circle and f_1 is a strongly exposed point of the closed unit ball of H^1. Then, using the fact that

$$\frac{\overline{q(z)}}{q(z)} = \lambda z^N \qquad (z \in \mathbb{T}),$$

where N is the degree of the polynomial q and λ is some unimodular constant, we get

$$\operatorname{dist}\left(\frac{\bar{f}}{|f|},\ H^\infty + \mathcal{C}(\mathbb{T})\right) = \operatorname{dist}\left(z^N \frac{\bar{f}_1}{|f_1|},\ H^\infty + \mathcal{C}(\mathbb{T})\right).$$

Since $H^\infty + C(\mathbb{T})$ is an algebra (see Theorem 5.10), we obtain

$$\operatorname{dist}\left(\frac{\bar{f}}{|f|},\ H^\infty + \mathcal{C}(\mathbb{T})\right) = \operatorname{dist}\left(\frac{\bar{f}_1}{|f_1|},\ H^\infty + \mathcal{C}(\mathbb{T})\right),$$

and, by Corollary 6.29,

$$\operatorname{dist}\left(\frac{\bar{f}}{|f|}, H^\infty + \mathcal{C}(\mathbb{T})\right) < 1.$$

On the other hand, if the degree N of the polynomial q is positive, then

$$\operatorname{dist}(\bar{f}/|f|, H^\infty) = 1.$$

Indeed, if $\operatorname{dist}(\bar{f}/f|, H^\infty) < 1$, then it follows from Corollary 12.43 that $T_{f/\bar{f}}$ is invertible. Since

$$\frac{f}{\bar{f}} = \frac{q^2}{\bar{q}^2}\frac{f_1}{\bar{f}_1} = \bar{\lambda}\bar{z}^{2N}\frac{f_1}{\bar{f}_1},$$

we obtain

$$T_{f/\bar{f}} = \bar{\lambda} S^{*2N} T_{f_1/\bar{f}_1}.$$

Since f_1 is a rigid function, the operator $T_{f_1/\bar{f}_1}$ is invertible. Hence, we deduce that S^{*2N} is invertible, which is absurd if $N > 0$.

This reasoning serves to illustrate that, if, for a given extreme point f of the closed unit ball of H^1, we have $\operatorname{dist}(\bar{f}/|f|, H^\infty, H^\infty + C(\mathbb{T})) < 1$, the only thing that can prevent f from being exposed (and hence strongly exposed) is the divisibility of f in H^1 by functions of the form $(1-u)^2$, with $u(z) = \lambda z$ $(\lambda \in \mathbb{T})$ a particularly simple inner function. Functions in H^1 that lack this divisibility property are called *strong outer functions*. In other words, a function f in H^1 is called a strong outer function if, for any $\lambda \in \mathbb{T}$,

$$f/(1-\lambda z)^2 \notin H^1.$$

We can therefore make Corollary 6.29 a little more precise in the following way.

Corollary 29.27 *Let f be a function in H^1, $\|f\|_1 = 1$. Then the following are equivalent.*

(i) *f is a strongly exposed point of the closed unit ball of H^1.*
(ii) *f is a strong outer function and* $\mathrm{dist}(\bar{f}/|f|, H^\infty, H^\infty + C(\mathbb{T})) < 1$.

Proof The implication (i) $\implies$ (ii) follows from Corollary 6.29 and Lemma 6.19.

For the reverse implication, since $\mathrm{dist}(\bar{f}/|f|, H^\infty, H^\infty + C(\mathbb{T})) < 1$, we know from the preceding discussion that f can be written as $f = p^2 g$, where g is a strongly exposed point of the closed unit ball of H^1 and p is a polynomial of degree N that has all its zeros on $\mathbb{T}$. By definition of a strong outer function, we see that N should be necessarily 0. Hence p is constant and then f is a strongly exposed point. □

Exercises

Exercise 29.6.1 Let p be a polynomial of unit norm in H^1.

(i) Show that p is a strongly exposed point of the closed unit ball of H^1 if and only if all its zeros are outside $\bar{\mathbb{D}}$.
Hint: Argue by absurdity, consider $1/p$ and use Corollary 29.23.
(ii) Construct a polynomial P that is an exposed point of the closed unit ball of H^1 but not a strongly exposed point.

Exercise 29.6.2 Let $f(z) = c(z-1)\log^2(z-1)$, where c is a constant such that $\|f\|_1 = 1$.

(i) Show that $f^{-1} \in H^1$ and deduce that f is an exposed point of the closed unit ball of H^1.
(ii) Show that there is no $\varepsilon > 0$ such that $f^{-1} \in H^{1+\varepsilon}$ and then deduce that f is not an strongly exposed point.

Exercise 29.6.3 Let $f = (a/(1-b))^2$ and assume that f is a strongly exposed point of the closed unit ball of H^1.

(i) Show that, if $F_\lambda = a/(1-\bar{\lambda}b)$, $\lambda \in \mathbb{T}$, then F_λ^2 is also a strongly exposed point of the closed unit ball of H^1.
(ii) Let v be a nonconstant finite Blaschke product. Show that the function

$$\left(\frac{a}{1-vb}\right)^2$$

is not a strongly exposed point of H^1. Compare with Theorem 29.5.

Hint: Use Theorem 29.19.

Notes on Chapter 29

Although the exposed points of the closed unit ball of H^1 turn up in several connections, as already mentioned and despite Helson's criterion [102] (see Theorem 6.20), a structural and explicit description of them is still lacking. Indeed, this criterion seems difficult to check in practice. The approach based on $\mathcal{H}(b)$ spaces is due to Sarason [163, 164], and a large part of this chapter comes from those two papers. This approach sheds some new light on the structure of exposed points of the closed unit ball of H^1 and also permits one to obtain the nice characterization of strongly exposed points given in this chapter. The interested reader could also refer to the survey of Beneker and Wiegerinck [31] on the problem of extreme, exposed and strongly exposed points of the unit ball in various H^1-type spaces.

Section 29.1

The notions of admissible and special pairs are introduced by Sarason in [163]. Nevertheless, the reader should be aware that we have not adopted exactly the same definition as Sarason.

Section 29.2

The characterization of the density of $\mathcal{M}(a)$ in $\mathcal{H}(b)$ in terms of rigid functions given in Theorem 29.3 and Corollary 29.4 is due to Sarason [163]. The invariance of rigidity when we pass from F to F_Θ given in Theorem 29.5, as well as Example 29.6, also come from [163]. Exercise 29.2.1 is taken from [164].

We also mention a result of Younis [194], who proved that, if f is rigid in H^1 and Θ is a nonconstant inner function, then $f \circ \Theta$ is rigid in H^1. See also [122] for an alternative proof and a slight generalization.

Section 29.3

Theorems 29.11 and 29.12 are due to Sarason [166], but the decomposition (29.8) of $\mathcal{H}(b)$ space (in the framework of Theorem 29.12) was noticed for the first time by Blandignères, Fricain, Gaunard, Hartmann and Ross in [35].

Section 29.4

The criterion for membership in $\mathcal{H}_0(b)$ given by Lemma 29.13 is due to Sarason [164].

Section 29.5

According to Corollary 29.9, we know that, if f is an exposed point of the closed unit ball of H^1, then $\|f_\lambda\|_2 = 1$ for all $\lambda \in \mathbb{T}$. Sarason conjectured in [163] that the converse is true. Theorem 29.17 is a partial result in this direction and has been proved by Sarason in [164]. However, this conjecture has been disproved by Kapustin in [114]. Exercise 29.5.1 comes also from [164].

Section 29.6

A large part of this section is due to Beneker [30]. Example 29.24 is due to Sarason [163]. Temme and Wiegerinck [185] showed directly using a clever trick that a polynomial (of unit norm) is strongly exposed if all its zeros are outside $\bar{\mathbb{D}}$. The approach based on Corollary 29.23 is due to Beneker [30].

The terminology of strong outer functions for functions in H^1 lacking the divisibility property by functions of the forms $(1-\lambda z)^2$, $\lambda \in \mathbb{T}$, was introduced by de Leeuw and Rudin in [66]. The reader should pay attention that, for some authors, such as Helson or Nakazi, a strong outer function is what we call in this text a rigid function.

30

Nearly invariant subspaces and kernels of Toeplitz operators

The aim of this chapter is to give some characterizations of the closed subspaces of H^2 that are kernels of Toeplitz operators. Using an approach based on $\mathcal{H}(b)$ spaces, we first describe certain subspaces of H^2, which, modulo one dimension, are invariant under the backward shift operator. These subspaces are called *nearly* S^**-invariant subspaces*. Then, using this description, we present a characterization of the kernels of Toeplitz operators. In this characterization, the notion of a rigid functions of H^1 will play the central role.

In Section 30.1, we introduce the nearly S^*-invariant subspaces and show through a simple example that there is a close link between kernels of Toeplitz operators and rigid functions. In Section 30.2, we introduce a crucial operator for our study, the operator R_f. In particular, we will prove an intertwining relation between R_f and X_b, which will be essential for future applications. In Section 30.3, we introduce and solve an extremal problem related to the problem of the description of nearly invariant subspaces. Using the operator R_f and this extremal problem, we then give, in Section 30.4, a nice description of nearly S^*-invariant subspaces. In Section 30.5, we apply this result to describe the closed subspaces of H^2 that are the kernels of Toeplitz operators. As we will see, this description is expressed in terms of rigidity. In Section 30.6, we obtain a necessary and sufficient condition for a noninjective Toeplitz operator on H^2 to be surjective. This condition involves the solution of the extremal problem related to the kernel of the Toeplitz operator and introduced in Section 30.3. In the last section, we give an explicit formula for the right-inverse of a Toeplitz operator that is surjective but not one-to-one.

30.1 Nearly invariant subspaces and rigid functions

We say that a closed subspace $\mathcal{M}$ of H^2 is *nearly* S^**-invariant* if

$$h \in \mathcal{M},\ h(0) = 0 \quad \Longrightarrow \quad S^*h \in \mathcal{M}.$$

As usual, we say that $\mathcal{M}$ is *nontrivial* if $\mathcal{M}$ is neither the null subspace nor the whole space H^2. It is trivial that an S^*-invariant subspace is also nearly S^*-invariant *a priori*. However, the latter is a larger family. To see this, let Θ be a nonconstant inner function. Since K_Θ is a closed S^*-invariant subspace, as we said, K_Θ is also a nearly invariant S^*-invariant subspace. Moreover, its orthogonal complement $\mathcal{M}(\Theta) = \Theta H^2$ is also a nearly invariant S^*-invariant subspace if $\Theta(0) \neq 0$. Indeed, let $f = \Theta f_1 \in \mathcal{M}(\Theta)$ with $f_1 \in H^2$ and assume that $f(0) = 0$. Then, since $\Theta(0) \neq 0$, we necessarily have $f_1(0) = 0$. This means that $f_1 = z f_2$ with $f_2 \in H^2$. Therefore, $S^* f = S^*(z\Theta f_2) = \Theta f_2 \in \mathcal{M}(\Theta)$, which proves that $\mathcal{M}(\Theta) = \Theta H^2$ is a nearly S^*-invariant subspace. Since $\mathcal{M}(\Theta)$ is not an S^*-invariant subspace, the family of nearly invariant S^*-invariant subspaces is strictly bigger than the family of S^*-invariant subspaces.

Corollary 12.23 says that the kernel of a Toeplitz operator is a nearly S^*-invariant subspace. In Section 30.4, we give a description of nearly S^*-invariant subspaces. For the present, the following simple result shows that there is a close link between kernels of Toeplitz operators and rigid functions.

Theorem 30.1 *Let $\mathcal{M}$ be a one-dimensional subspace of H^2 spanned by the function f. Then $\mathcal{M}$ is the kernel of a Toeplitz operator if and only if f^2 is rigid.*

Proof We know $\mathcal{M} = \mathbb{C} f$. First, assume that there exists a function $\varphi \in L^\infty(\mathbb{T})$ such that $\mathcal{M} = \ker T_\varphi$. Then, as a consequence of Theorem 12.21, f is outer. Since $T_\varphi f = 0$, the function φf belongs to $\bar{z}\overline{H^2}$. In other words, there exists an inner function Θ and an outer function h such that $\varphi f = \bar{z}\bar{\Theta}\bar{h}$. Then

$$T_\varphi(\Theta f) = P_+(\bar{z}\bar{h}) = 0,$$

and thus the assumption $\mathcal{M} = \ker T_\varphi$ implies that $\Theta f \in \mathcal{M}$. Hence, there is a constant $\lambda \in \mathbb{C}$ such that $\Theta f = \lambda f$. This means that Θ is constant and $\varphi f = \bar{z}\bar{g}$, with g being outer. Now, take $a = g/f$. We have

$$a = \frac{\bar{g}}{f}\frac{g}{\bar{g}} = z\varphi\frac{g}{\bar{g}},$$

almost everywhere on $\mathbb{T}$. Thus, $a \in L^\infty(\mathbb{T})$, and since f is outer, Corollary 4.28 ensures that a is in H^∞. Write

$$\varphi = \bar{a}\frac{\bar{g}}{\bar{f}}\frac{\bar{f}}{f} = \bar{a}\bar{z}\frac{\bar{f}}{f},$$

whence $T_\varphi = T_{\bar{a}} T_{\bar{z}\bar{f}/f}$. But, since a is an outer function, the operator $T_{\bar{a}}$ is one-to-one. Thus, $\ker T_\varphi = \ker T_{\bar{z}\bar{f}/f}$. Therefore, $\dim(\ker T_{\bar{z}\bar{f}/f}) = 1$ and Theorem 12.30 implies that f^2 is a rigid function.

Conversely, let $\mathcal{M} = \mathbb{C} f$, with f^2 being a rigid function. We trivially have $f \in \ker T_{\bar{z}\bar{f}/f}$, which implies $\mathcal{M} \subset \ker T_{\bar{z}\bar{f}/f}$. Then, again appealing to

Theorem 12.30 and doing an argument of dimension, we see that equality holds in the last inclusion, i.e. $\mathcal{M}$ is the kernel of the Toeplitz operator $T_{\bar{z}\bar{f}/f}$. $\square$

30.2 The operator R_f

Let f be a function of unit norm in H^2. Denote by F the outer part of f, normalized such that $F(0) > 0$, and denote its inner part by Θ, i.e. the canonical factorization of f is

$$f = \Theta F.$$

Let (a, b) be the admissible pair for F. Since $\|F\|_2 = \|f\|_2 = 1$, we have $b(0) = 0$. Moreover, recall that the Clark measure associated with b is absolutely continuous, which means that (a, b) is a special pair; see Section 29.1. In this case, Theorem 24.23 tells us that $T_{1-b}T_{\bar{F}}$ is an isometry from H^2 onto $\mathcal{H}(b)$. For the characterization of nearly S^*-invariant subspaces, we need some of the properties of the operator $T_{1-b}T_{\bar{f}}$. According to Theorem 13.23, we have

$$T_{1-b}T_{\bar{f}} = T_{1-b}T_{\bar{F}}T_{\bar{\Theta}}. \tag{30.1}$$

In particular, it follows from this relation and Theorem 24.23 that the operator $T_{1-b}T_{\bar{f}}$ is a bounded operator from H^2 into $\mathcal{H}(b)$.

Lemma 30.2 *Let f be a function of unit norm in H^2, with the canonical factorization $f = \Theta F$, and let (a, b) be the admissible pair for F. Then the operator $T_{1-b}T_{\bar{f}}$ is a partial isometry from H^2 onto $\mathcal{H}(b)$, whose kernel is K_Θ.*

Proof Let $g \in H^2$. Then $g \in \ker T_{1-b}T_{\bar{f}}$ if and only if $T_{\bar{\Theta}}g \in \ker T_{1-b}T_{\bar{F}}$. But, remembering that $T_{1-b}T_{\bar{F}}$ is an isometry, we deduce that $T_{\bar{\Theta}}g = 0$, which means that $f \in \ker T_{\bar{\Theta}} = K_\Theta$ by Theorem 12.19. This proves that $\ker T_{1-b}T_{\bar{f}} = K_\Theta$.

Now, let $g \in H^2 \ominus K_\Theta = \Theta H^2$ and write $g = \Theta g_1$, with $g_1 \in H^2$. Then, using (30.1), we have $T_{1-b}T_{\bar{f}}g = T_{1-b}T_{\bar{F}}T_{\bar{\Theta}}g = T_{1-b}T_{\bar{F}}g_1$. In particular, we see that

$$T_{1-b}T_{\bar{f}}H^2 = T_{1-b}T_{\bar{f}}(\Theta H^2) = T_{1-b}T_{\bar{F}}H^2 = \mathcal{H}(b).$$

Moreover, we have

$$\|T_{1-b}T_{\bar{f}}g\|_b = \|T_{1-b}T_{\bar{F}}g_1\|_b = \|g_1\|_2 = \|\Theta g_1\|_2 = \|g\|_2,$$

which proves that $T_{1-b}T_{\bar{f}}$ is a surjective partial isometry. $\square$

Since $T_{1-b}T_{\bar{f}}$ is a bounded operator from H^2 into $\mathcal{H}(b)$ that is contractively contained in H^2, it can be viewed as a bounded operator on H^2. Then the following result gives its adjoint and another useful formula.

Lemma 30.3 *Let f be a function of unit norm in H^2, with the canonical factorization $f = \Theta F$, and let (a, b) be the admissible pair for F. If we consider $T_{1-b}T_{\bar{f}}$ as on operator on H^2, then we have*

$$(T_{1-b}T_{\bar{f}})^* = T_f T_{1-\bar{b}}$$

and

$$(T_{1-b}T_{\bar{f}})(T_{1-b}T_{\bar{f}})^* = I - T_b T_b^*.$$

Proof Using (30.1) and Corollary 24.24, we have

$$\begin{aligned}(T_{1-b}T_{\bar{f}})^* &= (T_{1-b}T_{\bar{F}}T_{\bar{\Theta}})^* \\ &= T_{\bar{\Theta}}^*(T_{1-b}T_{\bar{F}})^* \\ &= T_\Theta T_F T_{1-\bar{b}} = T_f T_{1-\bar{b}},\end{aligned}$$

which proves the first relation. Then, remembering that the Clark measure associated with b is absolutely continuous, Corollary 24.24 also implies that

$$(T_{1-b}T_{\bar{F}})(T_F T_{1-\bar{b}}) = I - T_b T_b^*.$$

Therefore, using the identity $T_{\bar{\Theta}}T_\Theta = I$, we have

$$\begin{aligned}(T_{1-b}T_{\bar{f}})(T_{1-b}T_{\bar{f}})^* &= T_{1-b}T_{\bar{F}}T_{\bar{\Theta}}T_\Theta T_F T_{1-\bar{b}} \\ &= T_{1-b}T_{\bar{F}}T_F T_{1-\bar{b}} \\ &= I - T_b T_b^*.\end{aligned}$$

This completes the proof. □

Lemma 30.4 *Let f be a function of unit norm in H^2, with the canonical factorization $f = \Theta\, F$, and let (a, b) be the admissible pair for F. Then*

$$T_{1-b}T_{\bar{f}}S^*f = S^*b.$$

Proof Using Theorems 20.5 and 24.23, we have

$$(T_{1-b}T_{\bar{F}})F = T_{1-b}T_{\bar{F}}(Fk_0) = V_b k_0 = (1 - \overline{b(0)})^{-1}k_0^b.$$

Since $b(0) = 0$, we get $(T_{1-b}T_{\bar{F}})F = 1$. Therefore,

$$(T_{1-b}T_{\bar{f}})f = (T_{1-b}T_{\bar{F}}T_{\bar{\Theta}})(F\Theta) = (T_{1-b}T_{\bar{F}})F = 1,$$

because $T_{\bar{\Theta}}\Theta = 1$. Recall that X denotes the extension of S^* to the space $H(\mathbb{D})$ of all analytic functions on $\mathbb{D}$, that is, $(Xh)(z) = (h(z) - h(0))/z$,

$h \in H(\mathbb{D})$, $z \in \mathbb{D}$; see Section 13.6. Then, using Corollary 13.24 and Lemma 13.25, we get

$$\begin{aligned}(T_{1-b}T_{\bar{f}})S^*f &= T_{1-b}XT_{\bar{f}}f \\ &= XT_{1-b}T_{\bar{f}}f + (XT_{b-1} - T_{b-1}X)T_{\bar{f}}f \\ &= X1 + (T_{\bar{f}}f)(0)S^*b \\ &= (T_{\bar{f}}f)(0)S^*b.\end{aligned}$$

But

$$(T_{\bar{f}}f)(0) = (K_{\bar{f}}f)(0) = \int_{\mathbb{T}} |f(\zeta)|^2\, dm(\zeta) = \|f\|_2^2 = 1,$$

which gives the result. □

Given $f \in H^2$, with $\|f\|_2 = 1$, define

$$R_f = S^*(I - f \otimes f) \in \mathcal{L}(H^2). \tag{30.2}$$

The following result shows that R_f is a contraction on H^2.

Lemma 30.5 *Let f be a function of unit norm in H^2 and let R_f be given by* (30.2)*. Then $\|R_f\| \leq 1$.*

Proof Clearly, $R_f^* = (I - f \otimes f)S$. Since f is a function of unit norm, the operator $I - f \otimes f$ is the orthogonal projection onto $(\mathbb{C}f)^\perp$. Therefore, we get $\|R_f\| = \|R_f^*\| \leq \|S\| = 1$. □

Theorem 30.6 *Let f be a function of unit norm in H^2, with the canonical factorization $f = \Theta F$, and let (a, b) be the admissible pair for F. Then we have*

$$T_{1-b}T_{\bar{f}}R_f = X_bT_{1-b}T_{\bar{f}}.$$

Proof We have $R_fK_\Theta \subset K_\Theta$. Indeed, if $g \in K_\Theta = H^2 \ominus \Theta H^2$, then

$$(f \otimes f)g = \langle g, f\rangle_2 f = \langle g, \Theta F\rangle_2 f = 0,$$

whence $R_fg = S^*g \in K_\Theta$, because K_Θ is S^*-invariant. Therefore, since $K_\Theta = \ker(T_{1-b}T_{\bar{f}})$ (see Lemma 30.2), we see that both operators $T_{1-b}T_{\bar{f}}R_f$ and $X_bT_{1-b}T_{\bar{f}}$ vanish on K_Θ. It remains to show that these operators also coincide on ΘH^2. Since the multiplication by Θ is an isometry on H^2 (the function Θ is inner) and since $\mathrm{Span}(k_w : w \in \mathbb{D}) = H^2$, it is sufficient to show that

$$T_{1-b}T_{\bar{f}}R_f(\Theta k_w) = X_bT_{1-b}T_{\bar{f}}(\Theta k_w) \qquad (w \in \mathbb{D}). \tag{30.3}$$

Using Corollary 8.11 and Lemma 8.6, we have

$$\begin{aligned} R_f(\Theta k_w) &= S^*(\Theta k_w) - \langle \Theta k_w, f\rangle S^* f \\ &= \Theta S^* k_w + S^*\Theta - \langle k_w, \bar{\Theta} f\rangle_2 S^* f \\ &= \bar{w}\Theta k_w + S^*\Theta - \overline{F(w)}\, S^* f. \end{aligned}$$

Hence,

$$\begin{aligned} T_{1-b}T_{\bar{f}}R_f(\Theta k_w) &= T_{1-b}T_{\bar{f}}(\bar{w}\Theta k_w + S^*\Theta - \overline{F(w)}S^* f) \\ &= \bar{w}T_{1-b}T_{\bar{f}}(\Theta k_w) - \overline{F(w)}\, T_{1-b}T_{\bar{f}}S^* f, \end{aligned}$$

because $S^*\Theta \in K_\Theta$ and then $T_{1-b}T_{\bar{f}}S^*\Theta = 0$. But, on the one hand, using Lemma 13.26, we have

$$T_{1-b}T_{\bar{f}}(\Theta k_w) = T_{1-b}T_{\bar{F}}T_{\bar{\Theta}}(\Theta k_w) = T_{1-b}T_{\bar{F}}(k_w) = (1-b)\overline{F(w)}k_w,$$

and, on the other hand, we have $T_{1-b}T_{\bar{f}}S^* f = S^* b$. Therefore, we obtain

$$T_{1-b}T_{\bar{f}}R_f(\Theta k_w) = \overline{wF(w)}(1-b)k_w - \overline{F(w)}S^* b. \tag{30.4}$$

Now, using Lemma 13.26 once more, note that

$$\begin{aligned} X_b T_{1-b}T_{\bar{f}}(\Theta k_w) &= S^* T_{1-b}T_{\bar{F}}k_w \\ &= \overline{F(w)}S^*(1-b)k_w \\ &= \overline{wF(w)}k_w - \overline{F(w)}S^*(bk_w). \end{aligned}$$

According to Corollary 8.11, we have

$$S^*(bk_w) = bS^* k_w + S^* b = \bar{w}bk_w + S^* b,$$

whence

$$\begin{aligned} X_b T_{1-b}T_{\bar{f}}(\Theta k_w) &= \overline{wF(w)}k_w - \overline{F(w)}(\bar{w}bk_w + S^* b) \\ &= \overline{wF(w)}k_w(1-b)k_w - \overline{F(w)}S^* b, \end{aligned}$$

and, by (30.4), we conclude that (30.3) is satisfied. □

Theorem 30.7 *Let f be a function of unit norm in H^2 and let R_f be given by* (30.2). *Then $(R_f^n)_{n\geq 1}$ strongly tends to zero, i.e.*

$$\lim_{n\to\infty} \|R_f^n g\|_2 = 0 \qquad (g \in H^2).$$

Proof As usual, we denote by F the outer part of f, by Θ its inner part and by (a, b) the admissible pair for F. Fix a function $g \in H^2$ and $\varepsilon > 0$. Then we know from Theorem 24.16 that there exists $m \in \mathbb{N}$ such that

$$\|X_b^m T_{1-b}T_{\bar{f}}g\|_b < \varepsilon.$$

Write $R_f^n g = g_0 + g_1$, with $g_0 \in K_\Theta$ and $g_1 \in \Theta H^2$. According to Theorem 30.6, we have $X_b^m T_{1-b} T_{\bar{f}} = T_{1-b} T_{\bar{f}} R_f^m$, whence

$$X_b^m T_{1-b} T_{\bar{f}} g = T_{1-b} T_{\bar{f}} R_f^m g = T_{1-b} T_{\bar{f}} g_1,$$

because $T_{1-b} T_{\bar{f}} g_0 = 0$ (see Lemma 30.2). Thus,

$$\|g_1\|_2 = \|T_{1-b} T_{\bar{f}} g_1\|_b = \|X_b^m T_{1-b} T_{\bar{f}} g\|_b < \varepsilon.$$

Since $R_f | K_\Theta = S^* | K_\Theta$, we get

$$\begin{aligned} \|R_f^{m+n} g\|_2 &= \|R_f^n (g_0 + g_1)\|_2 \\ &\leq \|R_f^n g_0\|_2 + \|R_f^n g_1\|_2 \\ &\leq \|S^{*n} g_0\|_2 + \|g_1\|_2 \\ &\leq \|S^{*n} g_0\|_2 + \varepsilon. \end{aligned}$$

But S^{*n} tends strongly to zero as $n \longrightarrow \infty$. Therefore, there exists $n \in \mathbb{N}$ such that $\|S^{*n} g_0\|_2 \leq \varepsilon$, whence

$$\|R_f^{m+n} g\|_2 \leq 2\varepsilon,$$

which proves that $\|R_f^k g\|_2 \longrightarrow 0$, as $k \longrightarrow \infty$. □

30.3 Extremal functions

Let $\mathcal{M}$ be a nontrivial closed subspace of H^2. In this section, we study the extremal problem

$$\sup\{\Re(f(0)) : f \in \mathcal{M}, \ \|f\|_2 \leq 1\}.$$

The solution to this extremal problem is called the *extremal function* for $\mathcal{M}$. As $\mathcal{M}$ is a closed subspace of H^2, the linear functional $f \longmapsto f(0)$ is bounded on $\mathcal{M}$ and thus, by Riesz's theorem (Theorem 1.24), there exists a unique function $k_0^{\mathcal{M}} \in \mathcal{M}$ such that

$$f(0) = \langle f, k_0^{\mathcal{M}} \rangle_2 \qquad (f \in \mathcal{M}).$$

We show that the function

$$\varphi_0 = \frac{k_0^{\mathcal{M}}}{\|k_0^{\mathcal{M}}\|_2}$$

is the unique solution of the above extremal problem. Note that $k_0^{\mathcal{M}} = P_{\mathcal{M}} 1$, where $P_{\mathcal{M}}$ is the orthogonal projection of H^2 onto $\mathcal{M}$.

Theorem 30.8 *Let $\mathcal{M}$ be a nontrivial closed subspace of H^2. Then the extremal problem*

$$\sup\{\Re(f(0)) : f \in M,\ \|f\|_2 \le 1\}$$

has a unique solution given by $f = k_0^{\mathcal{M}}/\|k_0^{\mathcal{M}}\|_2$, for which the supremum is attained and is equal to $\|k_0^{\mathcal{M}}\|_2$.

Proof Take any function $f \in \mathcal{M}$, with $\|f\|_2 \le 1$. Then we have

$$\Re(f(0)) = \Re(\langle f, k_0^{\mathcal{M}}\rangle_2) \le \|f\|_2\,\|k_0^{\mathcal{M}}\|_2 \le \|k_0^{\mathcal{M}}\|_2.$$

Moreover, since we have $\varphi_0 = k_0^{\mathcal{M}}/\|k_0^{\mathcal{M}}\|_2 \in \mathcal{M}$, $\|\varphi_0\|_2 = 1$ and $\varphi_0(0) = k_0^{\mathcal{M}}(0)/\|k_0^{\mathcal{M}}\|_2 = \|k_0^{\mathcal{M}}\|_2$, we thus deduce that

$$\sup\{\Re(f(0)) : f \in M,\ \|f\|_2 \le 1\} = \varphi_0(0) = \|k_0^{\mathcal{M}}\|_2.$$

In other words, the function φ_0 is a solution to the extremal problem.

To establish the uniqueness of the solution, assume that there exists another function $\varphi \in \mathcal{M}$, $\|\varphi\|_2 \le 1$, such that $\Re(\varphi(0)) = \|k_0^{\mathcal{M}}\|_2$. Then

$$\|\varphi - \varphi_0\|_2^2 = \|\varphi\|_2^2 + \|\varphi_0\|_2^2 - 2\,\Re(\langle \varphi, \varphi_0\rangle_2).$$

But

$$\langle \varphi, \varphi_0\rangle_2 = \frac{1}{\|k_0^{\mathcal{M}}\|_2}\langle \varphi, k_0^{\mathcal{M}}\rangle_2 = \frac{\varphi(0)}{\|k_0^{\mathcal{M}}\|_2},$$

which gives

$$\Re(\langle \varphi, \varphi_0\rangle_2) = \frac{\Re(\varphi(0))}{\|k_0^{\mathcal{M}}\|_2} = 1$$

and thus

$$\|\varphi - \varphi_0\|_2^2 = \|\varphi\|_2^2 + 1 - 2 = \|\varphi\|_2^2 - 1 \le 0.$$

Therefore, we must have $\varphi = \varphi_0$. □

We now give another characterization of the extremal function for a nontrivial nearly S^*-invariant subspace of H^2. We begin with a useful lemma.

Lemma 30.9 *Let $\mathcal{M}$ be a nontrivial nearly S^*-invariant subspace of H^2. Then*

$$\dim(\mathcal{M} \ominus (\mathcal{M} \cap H_0^2)) = 1.$$

Proof The subspace $\mathcal{M}$ is not contained in H_0^2. Since, otherwise, by Beurling's theorem (Theorem 8.32), there would exist a nonconstant inner function Θ such that $\mathcal{M} = K_\Theta = H^2 \ominus \Theta H^2$. But $\varphi = 1 - \overline{\Theta(0)}\Theta = k_0^\Theta \in K_\Theta$ and thus we must have $\varphi(0) = 0$. Therefore, $|\Theta(0)| = 1$ and this is absurd.

Now, since $\mathcal{M}$ is not contained in H_0^2, there exists a function $g \in \mathcal{M}$, $g \not\equiv 0$, such that $g \perp \mathcal{M} \cap H_0^2$. We prove that $\mathcal{M} \ominus (M \cap H_0^2) = \mathbb{C}g$. Take any function

$f \in \mathcal{M}$, $f \not\equiv 0$ and $f \perp \mathcal{M} \cap H^2_0$. Note that $g(0) \neq 0$, since otherwise $g \in H^2_0 \cap \mathcal{M}$, which would imply that $g \equiv 0$. Therefore, the function

$$h = f - \frac{f(0)}{g(0)}\, g$$

is well defined. This function h belongs to $\mathcal{M} \cap H^2_0$ (because $h(0) = 0$) and also $h \perp \mathcal{M} \cap H^2_0$. Hence, $h \equiv 0$, which means that $f = (f(0)/g(0))g$. □

According to Lemma 30.9, if $\mathcal{M}$ is a nontrivial nearly S^*-invariant subspace of H^2, then $\dim(\mathcal{M} \ominus (\mathcal{M} \cap H^2_0)) = 1$. Hence, there is a unique function $g \in \mathcal{M}$, with $\|g\|_2 = 1$, which is orthogonal to $\mathcal{M} \cap H^2_0$ and is positive at the origin. There is a close connection between this function and $k_0^{\mathcal{M}}$, the reproducing kernel of $\mathcal{M}$ at the origin.

Corollary 30.10 *Let $\mathcal{M}$ be a nontrivial nearly S^*-invariant subspace of H^2, and let g be the unique function of unit norm in $\mathcal{M}$ that is orthogonal to $M \cap H^2_0$ and positive at the origin. Then*

$$k_0^{\mathcal{M}} = g(0)g.$$

Proof We need to show that

$$f(0) = \langle f, g(0)g \rangle_2 \tag{30.5}$$

for all functions $f \in \mathcal{M}$. According to Lemma 30.9, we have

$$\mathcal{M} = (\mathcal{M} \cap H^2_0) \oplus \mathbb{C}g.$$

First, the equality (30.5) holds for $f = g$, since $\|g\|_2 = 1$ and $g(0) > 0$. Second, if $f \in \mathcal{M} \cap H^2_0$, then we have $f(0) = 0$ and $\langle f, g \rangle_2 = 0$ because $g \perp M \cap H^2_0$. Thus the equality (30.5) is also satisfied for any function $f \in \mathcal{M} \cap H^2_0$. □

The next result shows that g is the extremal function for $\mathcal{M}$.

Corollary 30.11 *Let $\mathcal{M}$ be a nontrivial nearly S^*-invariant subspace of H^2, and let g be the unique function of unit norm in $\mathcal{M}$ that is orthogonal to $\mathcal{M} \cap H^2_0$ and positive at the origin. Then*

$$\sup\{\Re(f(0)) : f \in \mathcal{M},\ \|f\|_2 \le 1\} = g(0),$$

and the function g is the unique solution to this extremal problem.

Proof According to Theorem 30.8, we know that the extremal problem above has a unique solution given by $\varphi_0 = k_0^{\mathcal{M}}/\|k_0^{\mathcal{M}}\|_2$. Moreover, Corollary 30.10 says that $k_0^{\mathcal{M}} = g(0)g$. Hence,

$$\varphi_0 = \frac{g(0)g}{|g(0)|\,\|g\|_2} = g.$$

□

30.4 A characterization of nearly invariant subspaces

We recall that a closed subspace $\mathcal{M}$ of H^2 is called nearly S^*-invariant if

$$h \in \mathcal{M},\ h(0) = 0 \quad \Longrightarrow \quad S^*h \in \mathcal{M}$$

(see Section 30.1).

Lemma 30.12 *Let $\mathcal{M}$ be a nontrivial nearly S^*-invariant subspace of H^2 and let g be the extremal function for $\mathcal{M}$. Then $\mathcal{M}$ is invariant under the operator R_g.*

Proof Let $h \in \mathcal{M}$. Then, according to Corollary 30.10, we have

$$\begin{aligned} R_g h &= S^*(I - g \otimes g)h \\ &= S^*h - \langle h, g\rangle S^*g \\ &= S^*h - h(0)g(0)^{-1}S^*g = S^*\varphi, \end{aligned}$$

with $\varphi = h - h(0)g(0)^{-1}g \in H^2$. The function φ belongs to $\mathcal{M}$ and $\varphi(0) = 0$. Thus $S^*\varphi \in \mathcal{M}$, which proves that $R_g h \in \mathcal{M}$. □

Theorem 30.13 *Let $\mathcal{M}$ be a nontrivial closed subspace of H^2. The following are equivalent.*

(i) *$\mathcal{M}$ is a nearly S^*-invariant subspace of H^2.*
(ii) *There exists a unique function g of unit norm in $\mathcal{M}$ such that $g(0) > 0$ and a unique S^*-invariant subspace $\mathcal{M}'$ of H^2 on which T_g acts isometrically such that*

$$\mathcal{M} = T_g \mathcal{M}'.$$

Proof (i) $\Longrightarrow$ (ii) Let g be the extremal function of $\mathcal{M}$. Let h be any function in $\mathcal{M}$, and let $c_0 = \langle h, g\rangle_2$. We have $R_g h = S^*(h - c_0 g)$ and, by Corollary 30.10, the function $h - c_0 g$ vanishes at the origin. Therefore

$$SR_g h = SS^*(h - c_0 g) = h - c_0 g,$$

whence

$$h = c_0 g + SR_g h.$$

The function $SR_g h$ belongs to $\mathcal{M} \cap H_0^2$ and thus it is orthogonal to g. This implies that

$$\|h\|_2^2 = |c_0|^2 \|g\|_2^2 + \|SR_g h\|_2^2 = |c_0|^2 + \|R_g h\|_2^2.$$

Similarly, if $c_n = \langle R_g^n h, g\rangle_2$, $n \geq 0$, we have

$$R_g^n h = c_n g + SR_g^{n+1} h$$

and

$$\|R_g^n h\|_2^2 = |c_n|^2 + \|R_g^{n+1} h\|_2^2.$$

By induction, we obtain

$$h = (c_0 + c_1 S + \cdots + c_n S^n)g + S^{n+1} R_g^{n+1} h \tag{30.6}$$

and

$$\|h\|_2^2 = |c_0|^2 + |c_1|^2 + \cdots + |c_n|^2 + \|R_g^{n+1} h\|_2^2, \tag{30.7}$$

for any positive integer n. From (30.7), it follows in particular that

$$\sum_{k=0}^{\infty} |c_k|^2 < \infty,$$

which proves that the function q, defined on $\mathbb{D}$ by

$$q(z) = \sum_{k=0}^{\infty} c_k z^k,$$

belongs to H^2. Now, using Theorem 30.7 and letting $n \longrightarrow \infty$ in (30.6) and (30.7), we get

$$h = gq \quad \text{and} \quad \|h\|_2 = \|q\|_2.$$

This proves that $\mathcal{M}' = \{h/g : h \in \mathcal{M}\}$ is a closed subspace of H^2 and T_g maps $\mathcal{M}'$ isometrically onto $\mathcal{M}$. It remains to show that $\mathcal{M}'$ is S^*-invariant. Let $q = h/g$ with $h \in \mathcal{M}$. Thus, $q(0) = h(0)/g(0) = c_0$ and we have

$$\begin{aligned} R_g h &= S^*(h - c_0 g) \\ &= S^*(qg) - q(0) S^* g \\ &= g S^* q + q(0) S^* g - q(0) S^* g = g S^* q. \end{aligned}$$

Therefore, $S^* q = (R_g h)/g$. Since by Lemma 30.12, $R_g h \in M$, we get that $S^* q \in \mathcal{M}'$, showing that $\mathcal{M}'$ is S^*-invariant.

(ii) $\Longrightarrow$ (i) Assume that $\mathcal{M} = T_g \mathcal{M}'$, where g is a function of unit norm in H^2 that is positive at the origin, and $\mathcal{M}'$ is an S^*-invariant subspace on which T_g acts isometrically. Note that, since T_g is an isometry on $\mathcal{M}'$, then $\mathcal{M}$ is a closed subspace of H^2. Now, let $h \in \mathcal{M}$, $h(0) = 0$, and write $h = g h_1$ with $h_1 \in \mathcal{M}'$. Since $g(0) > 0$, we necessarily have $h_1(0) = 0$. Then, according to Corollary 8.11, we have

$$S^* h = g S^* h_1 + h_1(0) S^* g = g S^* h_1.$$

Since $h_1 \in \mathcal{M}'$ and $\mathcal{M}'$ is S^*-invariant, we deduce that $S^* h \in T_g \mathcal{M}' = \mathcal{M}$. Hence, $\mathcal{M}$ is a nearly S^*-invariant subspace.

It remains to prove the uniqueness of function g and subspace $\mathcal{M}'$ in the decomposition of the subspace $\mathcal{M}$. So, let us assume that $\mathcal{M} = T_{g_1}\mathcal{M}'_1$, where g_1 is a function of unit norm in $\mathcal{M}$ that is positive at the origin, and $\mathcal{M}'_1$ is an S^*-invariant subspace on which T_{g_1} acts isometrically. Let us prove first that g_1 is the extremal function for $\mathcal{M}$. Indeed, let $f \in \mathcal{M}$, $\|f\|_2 \le 1$. Then $f = g_1\varphi$ with $\varphi \in \mathcal{M}'$. Since T_{g_1} is an isometry on $\mathcal{M}'$, we have $\|\varphi\|_2 = \|f\|_2 \le 1$, whence

$$\Re(f(0)) = \Re(g_1(0)\varphi(0)) = g_1(0)\Re(\varphi(0)) \le g(0)\|\varphi\|_2 = g(0).$$

Since $g_1 \in \mathcal{M}$ and $\|g\|_2 = 1$, we get that

$$\sup\{\Re(h(0)) : h \in \mathcal{M},\ \|h\|_2 \le 1\} = g_1(0),$$

and the function g_1 is the solution of the extremal problem. Therefore, by the uniqueness of the solution, we get that $g_1 = g$ and then $T_g\mathcal{M} = T_g\mathcal{M}'$, which gives $\mathcal{M} = \mathcal{M}'$ and concludes the proof. □

One can distinguish two particular cases in Theorem 30.13. If $\mathcal{M}' = H^2$, then T_g acts as an isometry on H^2 and, in particular, it should be bounded. Therefore, Theorem 13.22 implies that g is in $H^2 \cap L^\infty(\mathbb{T}) = H^\infty$ and then it follows from Theorem 12.18 that g must be inner and $\mathcal{M} = gH^2$. In the more interesting case, $\mathcal{M}'$ is a proper subspace of H^2 and then, by Beurling's theorem, there exists a unique inner function Θ (up to a constant of modulus one) such that $\mathcal{M}' = K_\Theta$. Note that, since $g \in \mathcal{M} = T_gK_\Theta$, the constant function 1 belongs to K_Θ and thus Θ must vanish at the origin.

The question of how g and Θ must be related so that T_g acts isometrically on K_Θ now arises. The answer is given by the next result.

Theorem 30.14 *Let g be a function of unit norm in H^2, let G be the outer factor of g and let (a, b) be the admissible pair associated with G. Let Θ be an inner function with $\Theta(0) = 0$. Then the following are equivalent.*

(i) *T_g acts isometrically on K_Θ.*
(ii) *Θ divides b (in the algebra H^∞).*
(iii) *K_Θ is contained isometrically in $\mathcal{H}(b)$.*

Proof (i) $\Longrightarrow$ (ii) Let $h \in K_\Theta$. According to Theorem 14.9, we have $T_{1-\bar{b}}h \in K_\Theta$ and then

$$\|T_{1-\bar{b}}h\|_2 = \|T_gT_{1-\bar{b}}h\|_2.$$

Note that $T_gT_{1-\bar{b}}h = (T_{1-b}T_{\bar{g}})^*h \perp \ker(T_{1-b}T_{\bar{g}})$ (see Lemma 30.3). According to Lemma 30.2, the operator $T_{1-b}T_{\bar{g}}$ is a partial isometry of H^2 onto $\mathcal{H}(b)$, whence

$$\|T_{1-\bar{b}}h\|_2 = \|T_{1-b}T_{\bar{g}}T_gT_{1-\bar{b}}h\|_b.$$

Applying once more Lemma 30.3 and Corollary 17.6, we get

$$\begin{aligned}\|T_{1-\bar{b}}h\|_2 &= \|(I - T_bT_{\bar{b}})h\|_b \\ &= \|(I - T_bT_{\bar{b}})^{1/2}h\|_2.\end{aligned}$$

But

$$\begin{aligned}\|T_{1-\bar{b}}h\|_2^2 &= \langle T_{1-b}T_{1-\bar{b}}h, h\rangle_2 \\ &= \langle (I - T_b)(I - T_{\bar{b}})h,\ h\rangle_2 \\ &= \langle h - T_bh - T_{\bar{b}}h + T_bT_{\bar{b}}h,\ h\rangle_2 \\ &= \|h\|^2 - \langle h, T_{\bar{b}}h\rangle_2 - \langle T_{\bar{b}}h\rangle_2 + \|T_{\bar{b}}h\|_2^2,\end{aligned}$$

and, since $\|(I - T_bT_{\bar{b}})^{1/2}h\|_2^2 = \|h\|_2^2 - \|T_{\bar{b}}h\|_2^2$, we obtain

$$2\|T_{\bar{b}}h\|_2^2 = \langle h, T_{\bar{b}}h\rangle_2 + \langle T_{\bar{b}}h, h\rangle_2 \qquad (h \in K_\Theta).$$

Remembering now that $\Theta(0) = 0$, whence $1 \in K_\Theta$ and since $b(0) = 0$, we get that $T_{\bar{b}}1 = 0$. Therefore, for each constant $c \in \mathbb{C}$, we have

$$\begin{aligned}2\|T_{\bar{b}}h\|_2^2 &= \langle h + c,\ T_{\bar{b}}h\rangle_2 + \langle T_{\bar{b}}h,\ h + c\rangle_2 \\ &= \langle h, T_{\bar{b}}h\rangle_2 + \langle T_{\bar{b}}h, h\rangle_2 + 2\,\Re\left(\bar{c}(T_{\bar{b}}h)(0)\right).\end{aligned}$$

This is possible only if $(T_{\bar{b}}h)(0) = 0$. Thus, we have proved that, for every function $h \in K_\Theta$, we have $(T_{\bar{b}}h)(0) = 0$. Using the fact that $S^*T_{\bar{b}} = T_{\bar{b}}S^*$ and that $S^*K_\Theta \subset K_\Theta$, we obtain

$$(S^{*n}T_{\bar{b}}h)(0) = 0 \qquad (n \geq 0,\ h \in K_\Theta).$$

It is easily seen that this implies $T_{\bar{b}}h = 0$. Hence, $K_\Theta \subset \ker T_{\bar{b}} = \mathcal{H}(b_0)$, where b_0 is the inner part of b. In other words, $b_0H^2 \subset \Theta H^2$, that is, Θ divides b_0 and then also b.

(ii) $\Longrightarrow$ (i) Assume that Θ divides b, which means that there is a function $b_1 \in H^\infty$ such that $b = \Theta b_1$. Let $h \in K_\Theta$. Then, according to (12.3) and Theorem 12.19, we have

$$T_{\bar{b}}h = \Theta_{\bar{b}_1}T_{\bar{\Theta}}h = 0.$$

Now, using Lemma 30.3, we obtain

$$\begin{aligned}\|T_gh\|_2^2 &= \|T_gT_{1-\bar{b}}h\|_2^2 \\ &= \langle T_{1-b}T_{\bar{g}}(T_{1-b}T_{\bar{g}})^*h,\ h\rangle_2 \\ &= \langle (I - T_bT_{\bar{b}})h,\ h\rangle_2 = \|h\|_2^2,\end{aligned}$$

which proves that T_g acts isometrically on K_Θ.

(ii) $\Longrightarrow$ (iii) This follows immediately from Corollary 18.9.

(iii) $\Longrightarrow$ (ii) This follows from Theorem 27.12(ii), because, if K_Θ is contained isometrically in $\mathcal{H}(b)$, then, in particular, it is contained contractively in $\mathcal{H}(b)$. □

The hypothesis $\Theta(0) = 0$ in Theorem 30.14 has been used only for the implication (i) $\Longrightarrow$ (ii). This implication could be false without this hypothesis, as will be shown in the following example. Fix $w \in \mathbb{D}$, $w \neq 0$, and let Θ be the inner function defined by

$$\Theta(z) = \frac{z - w}{1 - \bar{w}z} \qquad (z \in \mathbb{D}).$$

Then $K_\Theta = \mathbb{C}k_w$. Now, let g be an outer function, let F be the function associated with g by

$$F(z) = \int_{-\pi}^{\pi} \frac{e^{it} + z}{e^{it} - z} \, |g(e^{it})|^2 \, \frac{dt}{2\pi}$$

and let (a, b) be the admissible pair associated with g, which means, in particular, that $b = (F-1)/(F+1)$. We see that T_g acts isometrically on K_Θ if and only if $\|gk_w\|_2 = \|k_w\|_2$, which is equivalent to

$$\int_0^{2\pi} \frac{1 - |w|^2}{|1 - \bar{w}e^{it}|^2} \, |g(e^{it})|^2 \, \frac{dt}{2\pi} = 1.$$

Therefore, we get that T_g acts isometrically on K_Θ if and only if $\Re(F(w)) = 1$. On the other hand, Θ divides b if and only if $b(w) = 0$, which is equivalent to $F(w) = 1$.

We now study another example. Let $\mathcal{M} = K_\Theta$, with Θ an inner function such that $\Theta(0) \neq 0$. Then $\mathcal{M}$ is a nearly S^*-invariant subspace. Then we know from Theorems 30.13 and 30.14 that there exist a function g and an inner function u such that $\mathcal{M} = T_g K_u$ and $g(0) > 0$, $u(0) = 0$ and u divides the function b, where (a, b) is the admissible pair associated with the outer part of g. Since $\Theta(0) \neq 0$, we cannot take $g = 1$ and $u = \Theta$. So the question arises of how to determine the functions g and u.

First, let us determine the function g. Recall that, according to Corollary 30.10, we should have $g \in K_\Theta$ and

$$h(0) = \langle h, g(0)g \rangle_2 \qquad (h \in K_\Theta).$$

But we also should have $h(0) = \langle h, k_0^\Theta \rangle_2$, whence we get that

$$g(0)g = k_0^\Theta = 1 - \overline{\Theta(0)}\Theta.$$

In particular, evaluating the last equality at $z = 0$, we obtain $g(0) = (1 - |\Theta(0)|^2)^{1/2}$ (remembering that $g(0) > 0$), whence

$$g = (1 - |\Theta(0)|^2)^{-1/2} \big(1 - \overline{\Theta(0)}\Theta\big).$$

Hence, we see that g is an outer function. Now, let u be the inner function defined by

$$u(z) = \frac{\Theta(0) - \Theta(z)}{1 - \overline{\Theta(0)}\Theta(z)}.$$

It follows from Theorem 14.20 that T_g acts isometrically from K_u onto K_Θ, that is, $\mathcal{M} = T_g K_u$. Note that $u(0) = 0$ and, by Theorem 30.14, we know that u divides the function b, where (a, b) is the admissible pair associated with g. It is also interesting to determine b. For this purpose, we should first compute the function F defined by

$$F(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{it}+z}{e^{it}-z} (1 - |\Theta(0)|^2)^{-1} \big|1 - \overline{\Theta(0)}\Theta(e^{it})\big|^2 \, dt.$$

This formula can be rewritten as

$$\begin{aligned} F(z) = (1 - |\Theta(0)|^2)^{-1} \int_{-\pi}^{\pi} \frac{e^{it}+z}{e^{it}-z} \\ \times \left(1 + |\Theta(0)|^2 - \overline{\Theta(0)}\Theta(e^{it}) - \Theta(0)\overline{\Theta(e^{it})}\right) \frac{dt}{2\pi}. \end{aligned}$$

Now, using the fact that

$$\frac{e^{it}+z}{e^{it}-z} = 1 + 2\sum_{n\geq 1} z^n e^{-int},$$

we easily see that

$$\int_{-\pi}^{\pi} \frac{e^{it}+z}{e^{it}-z} \frac{dt}{2\pi} = 1$$

and

$$\int_{-\pi}^{\pi} \frac{e^{it}+z}{e^{it}-z} \, \overline{\Theta(e^{it})} \, \frac{dt}{2\pi} = \overline{\Theta(0)}.$$

Thus, we have

$$F(z) = (1 - |\Theta(0)|^2)^{-1} \left(1 - \overline{\Theta(0)} \int_{-\pi}^{\pi} \frac{e^{it}+z}{e^{it}-z} \, \Theta(e^{it}) \, \frac{dt}{2\pi}\right).$$

For the last integral, write

$$\begin{aligned} \int_{-\pi}^{\pi} \frac{e^{it}+z}{e^{it}-z} \, \Theta(e^{it}) \, \frac{dt}{2\pi} &= \int_{-\pi}^{\pi} \frac{e^{it}+z}{e^{it}-z} \, (\Theta(e^{it}) - \Theta(0)) \, \frac{dt}{2\pi} + \Theta(0) \\ &= \int_{-\pi}^{\pi} \frac{e^{it}+z}{1 - ze^{-it}} \, \frac{\Theta(e^{it}) - \Theta(0)}{e^{it}} \, \frac{dt}{2\pi} + \Theta(0) \\ &= \int_{-\pi}^{\pi} \frac{\varphi(e^{it})}{1 - ze^{-it}} \, \frac{dt}{2\pi} + \Theta(0), \end{aligned}$$

with

$$\varphi(w) = \frac{\Theta(w) - \Theta(0)}{w}(w + z).$$

Since the function φ belongs to H^2, we have

$$\int_{-\pi}^{\pi} \frac{\varphi(e^{it})}{1 - ze^{-it}} \frac{dt}{2\pi} = \langle \varphi, k_z \rangle_2 = \varphi(z),$$

which gives

$$\int_{-\pi}^{\pi} \frac{e^{it} + z}{e^{it} - z} \Theta(e^{it}) \frac{dt}{2\pi} = \varphi(z) + \Theta(0) = 2\Theta(z) - \Theta(0).$$

Finally, we obtain

$$\begin{aligned} F(z) &= (1 - |\Theta(0)|^2)^{-1}\big(1 - \overline{\Theta(0)(2\Theta(z) - \Theta(0))}\big) \\ &= (1 - |\Theta(0)|^2)^{-1}(1 + |\Theta(0)|^2 - 2\overline{\Theta(0)}\Theta(z)). \end{aligned}$$

Since $b(z) = (F(z) - 1)/(F(z) + 1)$, an elementary computation shows that

$$b(z) = \frac{\overline{\Theta(0)}(\Theta(0) - \Theta(z))}{1 - \overline{\Theta(0)}\Theta(z)},$$

that is, $b = \overline{\Theta(0)}\, u$. In particular, u is the inner part of b.

Based on the above observations, we have the following complete description of nearly S^*-invariant subspaces.

Theorem 30.15 *Let $\mathcal{M}$ be a nontrivial closed subspace of H^2. Then the following are equivalent.*

(i) *$\mathcal{M}$ is a nearly S^*-invariant subspace.*
(ii) *$\mathcal{M}$ has one of the following forms:*

  (a) either there exists a unique inner function Θ such that $\Theta(0) \neq 0$ and

$$\mathcal{M} = \Theta H^2,$$

  (b) or there exist a unique function g of unit norm in $\mathcal{M}$ such that $g(0) > 0$ and a unique inner function u satisfying $u(0) = 0$ and u divides the function b, where (a, b) is the admissible pair associated with the outer factor of g, such that

$$\mathcal{M} = T_g K_u.$$

Proof Apply Theorems 30.13 and 30.14 and also use the remark given in the paragraph after Theorem 30.13. □

Let $\mathcal{M}$ be a nontrivial closed subspace of H^2 that is nearly S^*-invariant and let g be the unique function of unit norm in $\mathcal{M}$ that is orthogonal to $\mathcal{M} \cap H_0^2$ and positive at the origin. If the outer part of g is not constant, then we are necessarily in case (ii) of Theorem 30.15. Indeed, we know from Theorem 30.13 that $\mathcal{M} = T_g \mathcal{M}'$, where $\mathcal{M}'$ is a closed S^*-invariant subspace and T_g acts isometrically on $\mathcal{M}$. Now, assume that there exists an inner function Θ such that $\mathcal{M} = \Theta H^2$. Then g divides Θ, but this is absurd if the outer part of g is not constant. If g is an inner function, say Θ, then either $\mathcal{M} = \Theta H^2$ or $\mathcal{M} = T_\Theta K_u$ for some inner function u such that $u(0) = 0$. Note that, in this case, the function b associated with g is identically zero, so any inner function u divides b.

30.5 Description of kernels of Toeplitz operators

In this section, we give a complete description of the kernels of Toeplitz operators. More precisely, given a closed nontrivial subspace $\mathcal{M}$ of H^2, we are looking for necessary and sufficient conditions so that there exists a function $\varphi \in L^\infty(\mathbb{T})$ such that $\mathcal{M} = \ker T_\varphi$. According to Corollary 12.23, we can, without loss of generality, assume that $\mathcal{M}$ is a nearly S^*-invariant subspace of H^2. Then Theorem 30.13 implies that $\mathcal{M} = T_g \mathcal{M}'$, where g is a function of unit norm in $\mathcal{M}$, such that $g(0) > 0$, and $\mathcal{M}'$ is a closed S^*-invariant subspace of H^2 and T_g acts isometrically on $\mathcal{M}'$. But, if $\mathcal{M} = \ker T_\varphi$, then necessarily g must be an outer function (because $g \in \mathcal{M}$ and the kernel of a Toeplitz operator is invariant under division by inner functions). According to the remark given in the paragraph after Theorem 30.15, we must have $\mathcal{M} = T_g K_u$, where u is an inner function that vanishes at the origin.

Therefore, in the description of kernels of Toeplitz operators, we may suppose, without loss of generality, that $\mathcal{M} = T_f K_u$, where f is an outer function of unit norm in H^2, u is an inner function that vanishes at the origin and T_f acts isometrically on K_u. Let (a, b) be the admissible pair associated with f. According to Theorem 30.14, the inner function u divides b. We let $b_0 = b/u$ and $f_0 = a/(1 - b_0)$. Then (a, b_0) is also a pair.

Lemma 30.16 *Let $\mathcal{M} = T_f K_u$, where f is an outer function of unit norm in H^2, u is an inner function that vanishes at the origin and T_f acts isometrically on K_u. If $\mathcal{M}$ is the kernel of a Toeplitz operator, say T_φ, then necessarily*

$$\varphi = \bar{\psi}\,\bar{u}\frac{\bar{f}}{f} \quad \textit{and} \quad \mathcal{M} = \ker T_{\bar{u}\bar{f}/f},$$

where ψ is an outer function such that $|\psi| = |\varphi|$.

Proof Let $\varphi \in L^\infty(\mathbb{T})$ be such that $\mathcal{M} = \ker T_\varphi$. Since $f \in \mathcal{M}$, we have $\varphi f \in (H^2)^\perp$. In other words, there exists a function $h \in H^2$, $h \not\equiv 0$, such that $\varphi f = \overline{zh}$ almost everywhere on $\mathbb{T}$. Hence, $\log|\varphi| + \log|f| = \log|h|$ and, since $\log|f|$ and $\log|h|$ are integrable on $\mathbb{T}$, the function $\log|\varphi|$ is also integrable on $\mathbb{T}$. Then consider the outer function ψ_0 in H^∞, positive at the origin and such that $|\psi_0| = |\varphi|$ on $\mathbb{T}$, and define $\chi = \varphi/\overline{\psi_0}$. Note that $\chi \in L^\infty(\mathbb{T})$ and $|\chi| = 1$. Since $\varphi = \overline{\psi_0}\chi$, Theorem 12.4 implies that $T_\varphi = T_{\bar\psi_0} T_\chi$. But, since the function ψ_0 is outer, the operator $T_{\bar\psi_0}$ is one-to-one (see Theorem 12.19) and then $\mathcal{M} = \ker T_\varphi = \ker T_\chi$.

Using once more that $f \in \mathcal{M}$, we have $T_\chi f = 0$, that is, $f\chi = \overline{zh}$ with $h \in H^2$, $h \not\equiv 0$. In particular, since $|\chi| = 1$, we have $|f| = |h|$. Remembering that f is an outer function, we deduce that $h = fu_1$, where u_1 is an inner function. Then

$$\chi = \frac{\bar z \bar u_1 \bar f}{f} = \frac{\bar u_2 \bar f}{f},$$

where $u_2 = zu_1$. Let us show that $u_2 = \lambda u$, for some constant λ of modulus one. Let $g \in K_u$. Then $fg \in \mathcal{M} = \ker T_\chi$. But $\chi f g = \overline{u_2 f} g$, whence $u_2 f \bar g \in H_0^2$. Since f is outer, we get $u_2 \bar g \in H_0^2$, that is, $\overline{u_2} g \in (H^2)^\perp$. Thus, $g \in u_2 H_-^2 \cap H^2 = K_{u_2}$. We have thus proved that $K_u \subset K_{u_2}$ and the function u divides the function u_2 (see Theorem 14.1). Now, let $g \in K_{u_2} \cap L^\infty(\mathbb{T})$. Then

$$T_\chi(fg) = P_+(\chi f g) = P_+(\overline{u_2} \bar f g) = 0,$$

because $\overline{u_2} g \in H_-^2 \cap L^\infty(\mathbb{T})$. Hence, $fg \in \ker T_\chi = T_f K_u$, which implies that $g \in K_u$. Thus $K_{u_2} \cap L^\infty(\mathbb{T}) \subset K_u$. Since the subspace $K_{u_2} \cap L^\infty(\mathbb{T})$ is dense in K_{u_2} (it contains, for instance, the reproducing kernels of K_{u_2}), we get that $K_{u_2} \subset K_u$ and u_2 divides u. We have thus proved that u_2 divides u and u divides u_2 and then there exists a constant λ of modulus one such that $u_2 = \lambda u$. Finally, we obtain $\chi = \bar\lambda \bar u \bar f / f$, whence

$$\mathcal{M} = \ker T_\chi = \ker T_{\bar\lambda \bar u \bar f/f} = \ker T_{\bar u \bar f/f}$$

and

$$\varphi = \overline{\psi_0}\chi = \overline{\psi_0}\bar\lambda \bar u \bar f/f = \bar\psi \bar u \bar f/f,$$

where $\psi = \lambda\psi_0$. □

Lemma 30.17 *Let f be an outer function of unit norm in H^2, and let u be an inner function that vanishes at the origin and such that T_f acts isometrically on K_u. Then*

$$T_{1-b} T_{\bar f}(fh) = h \qquad (h \in K_u).$$

Proof Let $h \in K_u$. Since T_f acts isometrically on K_u, it follows from Theorem 30.14 that the inner function u divides b, that is, $b = ub_0$, with $b_0 \in H^\infty$. Then

$$T_{\bar{b}}h = P_+(\bar{b}_0\bar{u}h) = 0,$$

because $\bar{u}h \in (H^2)^\perp$. Hence,

$$T_{1-b}T_{\bar{f}}(fh) = T_{1-b}T_{\bar{f}}T_fT_{1-\bar{b}}h.$$

But, by Corollary 24.24, we have $T_{1-b}T_{\bar{f}}T_fT_{1-\bar{b}} = I - T_bT_{\bar{b}}$, whence

$$T_{1-b}T_{\bar{f}}(fh) = (I - T_bT_{\bar{b}})h = h. \qquad \square$$

Lemma 30.18 *Let f be an outer function of unit norm in H^2, let (a,b) be the admissible pair associated with f and let Θ be an inner function. Then the operator $T_{1-b}T_{\bar{f}}$ maps the range of the operator $T_{\Theta f/\bar{f}}$ onto ΘaH^2.*

Proof According to Theorem 13.23, we have

$$T_{1-b}T_{\bar{f}}T_{\Theta f/\bar{f}} = T_{1-b}T_{\Theta f} = T_{(1-b)\Theta f} = T_{a\Theta},$$

which proves the assertion. $\square$

Theorem 30.19 *Let $\mathcal{M} = T_fK_u$, where f is an outer function of unit norm in H^2, u is an inner function that vanishes at the origin and T_f acts isometrically on K_u. Let (a,b) be the admissible pair associated with f and let $b_0 = b/u$. Then the following are equivalent.*

(i) *$\mathcal{M}$ is the kernel of a Toeplitz operator.*
(ii) *The pair (a,b_0) is special and $(a/(1-b_0))^2$ is rigid.*
(iii) *The pair (a,zb_0) is special and $(a/(1-zb_0))^2$ is rigid.*

Proof (i) $\Longrightarrow$ (ii) First, assume that $\mathcal{M} = T_fK_u$ is the kernel of a Toeplitz operator. Then Lemma 30.16 implies that $\mathcal{M} = \ker T_{\bar{u}\bar{f}/f}$. Remember that the operator $T_{1-b}T_{\bar{f}}$ is an isometry from H^2 onto $\mathcal{H}(b)$ (the pair (a,b) is special). Moreover, Lemma 30.18 implies that this operator maps the range of $T_{uf/\bar{f}}$ onto uaH^2 and, by Lemma 30.17, it maps the subspace $\mathcal{M}$ onto K_u. Since

$$H^2 = \mathcal{M} \oplus \overline{\mathcal{R}(T_{uf/\bar{f}})}^2,$$

we get

$$\mathcal{H}(b) = K_u \oplus \overline{uaH^2}^b.$$

But, by Corollary 18.9, we have $\mathcal{H}(b) = K_u \oplus u\mathcal{H}(b_0)$, which gives $\overline{uaH^2}^b = u\mathcal{H}(b_0)$. Since T_u acts as an isometry of $\mathcal{H}(b_0)$ into $\mathcal{H}(b)$, aH^2 must be dense in $\mathcal{H}(b_0)$. Hence, Theorem 29.3 implies that (a,b_0) is special and f_0^2 is a rigid function.

(ii) $\Longrightarrow$ (iii) It follows from Theorem 29.5.

(iii) $\Longrightarrow$ (i) According to Theorem 29.3, the space $\mathcal{M}(a)$ is dense in $\mathcal{H}(zb_0)$. Moreover, since $u(0)=0$, there exists an inner function u_1 such that $u = zu_1$ and $b = ub_0 = u_1 z b_0$. Then Corollary 18.9 implies that

$$\mathcal{H}(b) = \mathcal{H}(u_1) \oplus u_1 \mathcal{H}(zb_0) \tag{30.8}$$

and T_{u_1} acts as an isometry from $\mathcal{H}(zb_0)$ into $\mathcal{H}(b)$. Using Lemma 30.18, we know that the operator $T_{1-b}T_{\bar{f}}$ (which is an isometry from H^2 onto $\mathcal{H}(b)$) maps the range of the operator $T_{u_1 f/\bar{f}}$ onto $u_1 a H^2$. Thus,

$$\begin{aligned} T_{1-b}T_{\bar{f}}\big(\overline{\mathcal{R}(T_{u_1 f/\bar{f}})}^2\big) &= \overline{a u_1 H^2}^{\mathcal{H}(b)} \\ &= u_1 \overline{\mathcal{M}(a)}^{\mathcal{H}(zb_0)} = u_1 \mathcal{H}(zb_0). \end{aligned}$$

Using once more that $T_{1-b}T_{\bar{f}}$ is an isometry from H^2 onto $\mathcal{H}(b)$, we get from (30.8) that

$$T_{1-b}T_{\bar{f}}\big(\ker T_{\bar{u}_1 \bar{f}/f}\big) = \mathcal{H}(u_1).$$

Now note that $\mathcal{H}(u_1) \subset K_u$ and then it follows from Lemma 30.17 that

$$T_{1-b}T_{\bar{f}}\left(T_f \mathcal{H}(u_1)\right) = \mathcal{H}(u_1),$$

and, by injectivity, we obtain

$$\ker T_{\bar{u}_1 \bar{f}/f} = T_f \mathcal{H}(u_1). \tag{30.9}$$

Let us prove that $\mathcal{M} = T_f K_u = \ker T_{\bar{u}\bar{f}/f}$. First, note that

$$T_f K_u \subset \ker T_{\bar{u}\bar{f}/f}.$$

Indeed, if $h \in K_u$, then

$$T_{\bar{u}\bar{f}/f}(fh) = P_+(\bar{u}\bar{f}h) = 0,$$

because $\bar{u}h \in (H^2)^\perp$. Now, let $\varphi \in \ker T_{\bar{u}\bar{f}/f}$. Since $T_{\bar{u}\bar{f}/f} = S^* T_{\bar{u}_1 \bar{f}/f}$, we get $T_{\bar{u}_1 \bar{f}/f}(\varphi) \in \ker S^* = \mathbb{C}$. Note also that

$$T_{\bar{u}_1 \bar{f}/f}(f u_1) = P_+(\bar{f}) = \overline{f(0)}.$$

Hence, there exists a constant $\lambda \in \mathbb{C}$ such that

$$T_{\bar{u}_1 \bar{f}/f}(\varphi) = T_{\bar{u}_1 \bar{f}/f}(\lambda f u_1).$$

In other words, $\varphi - \lambda f u_1 \in \ker T_{\bar{u}_1 \bar{f}/f}$, which, using (30.9), gives

$$\varphi = \lambda f u_1 + fg,$$

for some $g \in K_{u_1}$. Then $\varphi = T_f(\lambda u_1 + g)$, $u_1 \in K_u$ and $g \in K_{u_1} \subset K_u$. Therefore, $\varphi \in T_f K_u$, which concludes the proof. $\square$

There is a recipe for constructing the general nontrivial proper subspace of H^2 that is the kernel of a Toeplitz operator. Start with an outer function $f_0 \in H^2$ such that f_0^2 is rigid and let u be an inner function that vanishes at the origin. Consider then (a, b_0), the admissible pair associated with f_0, and define $b = ub_0$ and $f = a/(1-b)$. According to Theorem 29.5, the function f^2 is rigid and the pair (a, b) is special. In particular, we have

$$\frac{1-|b(z)|^2}{|1-b(z)|^2} = \int_0^{2\pi} \frac{1-|z|^2}{|z-e^{i\theta}|^2} \, |f(e^{i\theta})|^2 \, \frac{d\theta}{2\pi}.$$

Taking $z = 0$ and since $b(0) = 0$, we deduce that $\|f\|_2 = 1$ and $f(0) > 0$. It then follows from Theorem 30.15 that $T_f K_u$ is a nontrivial nearly S^*-invariant subspace. Now, Theorem 30.19 and Lemma 30.18 enable us to conclude that $T_f K_u = \ker T_{\bar{u}\bar{f}/f}$. Moreover, f is the extremal function for $\ker T_{\bar{u}\bar{f}/f}$. One obtains in this manner every nontrivial proper subspace of H^2 that is the kernel of a Toeplitz operator.

Now, we study an example. Let $\mathcal{M}$ be a one-dimensional subspace of H^2 spanned by an outer function g of unit norm in H^2. Then we know from Theorem 30.1 that $\mathcal{M}$ is the kernel of a Toeplitz operator if and only if g^2 is rigid. We can recover this result as a simple consequence of Theorem 30.19. Indeed, let $u(z) = z$. Then the function u is inner and vanishes at the origin and, since $K_u = \mathbb{C}$, we can write

$$\mathcal{M} = T_g K_u.$$

Moreover, since $\|g\|_2 = 1$, it is clear that T_g acts as an isometry on K_u. Let $b_0 = b/u$, where (a, b) is the admissible pair for g. Then, according to Theorem 30.19, $\mathcal{M}$ is the kernel of a Toeplitz operator if and only if the pair $(a, zb_0) = (a, b)$ is special and the function $(a/(1-zb_0))^2 = g^2$ is rigid. Since (a, b) is special, we recover Theorem 30.1.

Corollary 30.20 *Let $\varphi \in L^\infty(\mathbb{T})$, such that $\varphi \not\equiv 0$ and $|\varphi| = 1$.*

(i) *Assume that $\ker T_\varphi \neq \{0\}$ and let g be the extremal function for $\ker T_\varphi$. Then g is an outer function such that g^2 is rigid. Moreover, if (a, b) is the admissible pair associated with g, then there is a unique inner function u that vanishes at the origin and divides the function b such that*

$$\ker T_\varphi = T_g K_u \quad \textit{and} \quad \varphi = \bar{u}\frac{\bar{g}}{g}.$$

(ii) *Assume that $\varphi = \bar{u}\bar{g}/g$, where g is in H^2 and u is an inner function that vanishes at the origin. Then $\ker T_\varphi \neq \{0\}$.*

Proof (i) Let $\mathcal{M} = \ker T_\varphi$. Then, according to Theorems 30.13 and 30.14, the function g is outer and there is a unique inner function u that vanishes at the origin and divides the function b such that $\mathcal{M} = T_g K_u$. Let $b_0 = b/u$.

Then Theorem 30.19 implies that (a, b_0) is special and $(a/(1-b_0))^2$ is rigid. Thus, it follows from Theorem 29.5 that g^2 is also rigid. Finally, Lemma 30.16 ends the proof of part (i).

(ii) Let $\varphi = \bar{u}\bar{g}/g$, where g is in H^2 and u is an inner function that vanishes at the origin. Then we have

$$T_\varphi(g) = P_+(\bar{u}\bar{g}) = 0,$$

because $ug \in H_0^2$. Hence, the kernel of T_φ is not trivial. □

Corollary 30.21 *Let $\varphi = \bar{u}\bar{g}/g$, where g is an outer function of unit norm in H^2, positive at the origin, and u is an inner function that vanishes at the origin and divides the function b, where (a, b) is the admissible pair for g. Then we have*

$$T_g K_u \subset \ker T_\varphi,$$

and we have equality if and only if g is the extremal function for $\ker T_\varphi$. In particular, in the case of equality, the function g^2 is rigid.

Proof It follows from Theorem 30.14 that T_g acts isometrically on K_u and, in particular, $T_g K_u$ is a closed subspace of H^2. Now, take any function $f_1 \in K_u$. We have

$$T_\varphi(T_g f_1) = T_\varphi(g f_1) = P_+(\varphi g f_1) = P_+(\bar{g}\bar{u} f_1) = 0,$$

because $\bar{u} f_1 \in \overline{H_0^2}$. Hence $T_g K_u \subset \ker T_\varphi$. According to Corollary 30.20 and the uniqueness in Theorem 30.13, we have equality if and only if g is the extremal function for $\ker T_\varphi$. □

If $\varphi = \bar{u}\bar{g}/g$, where g is an outer function of unit norm in H^2, positive at the origin, such that g^2 is not rigid and u is an inner function that vanishes at the origin and divides the function b, where (a, b) is the admissible pair for g, then Corollary 30.21 implies that $T_g K_u$ is strictly contained in $\ker T_\varphi$. We now explain how to construct a function h that belongs to $\ker T_\varphi$ but not to $T_g K_u$. First, since g^2 is not rigid, according to Lemma 6.17, there is an outer function $f_e \in H^1$ such that $\arg f_e = \arg g^2$ and $f_e \neq \lambda g^2$, for any positive constant λ. Consider then the function $f = f_e^{1/2}$. Clearly, f is an outer function in H^2. Moreover, we have

$$\arg f^2 = \arg f_e = \arg g^2.$$

This means that the unimodular functions $g/\bar{g}$ and $f/\bar{f}$ have the same argument and thus

$$\frac{g}{\bar{g}} = \frac{f}{\bar{f}}. \tag{30.10}$$

Put

$$h = \overline{g(0)} f - \overline{f(0)} g.$$

This function is defined such that $h \neq 0$ and $h \in \ker T_{\bar{f}/f}$. Indeed, if $h = 0$, then $f = \alpha g$, with $\alpha = \overline{f(0)}/\overline{g(0)}$. The equality (30.10) says that α is a real constant. Hence, $f_e = f^2 = \alpha^2 g^2$, with α^2 a positive real number, and this is impossible by the construction of f_e. On the other hand, we have

$$\begin{aligned}
T_{\bar{g}/g} h &= P_+(h\bar{g}/g) \\
&= \overline{g(0)} P_+(f\bar{g}/g) - \overline{f(0)} P_+(g\bar{g}/g) \\
&= \overline{g(0)} P_+(\bar{f}) - \overline{f(0)} P_+(\bar{g}) \\
&= \overline{g(0)}\,\overline{f(0)} - \overline{g(0)}\,\overline{g(0)} = 0.
\end{aligned}$$

Then the function h belongs to $\ker T_{\bar{f}/f}$ and, since $T_{\bar{u}\bar{f}/f} = T_{\bar{u}} T_{\bar{f}/f}$, it also belongs to $\ker T_\varphi$. Let us show that h does not belong to $T_g K_u$. Argue by absurdity and assume that $h \in T_g K_u$. That means that h/g belongs to K_u. But

$$\frac{h}{g} = \overline{g(0)} \frac{f}{g} - \overline{f(0)},$$

and, since the constants belong to K_u (remember that $u(0) = 0$), we get that

$$\frac{f}{g} \in K_u.$$

Now, it follows from (30.10) that $f/g \in \overline{H^2} \cap H^2$, that is, f/g is a real constant, say λ. Thus we obtain $f_e = \lambda^2 g^2$ and this is absurd by construction of f_e.

Exercises

Exercise 30.5.1 Let

$$b(z) = \frac{z(1-z)}{2} \quad \text{and} \quad a(z) = \frac{1+z}{2}.$$

Let Θ be a Blaschke product with zeros $(r_n)_{n \geq 1}$ lying in $(-1, 0)$ and tending to -1, and let $f_\Theta = a/(1 - \Theta b)$.

(i) Show that f_Θ is an outer function of unit norm in H^2 and positive at the origin.
(ii) Prove that T_{f_Θ} acts as an isometry on $\mathcal{H}(z\Theta)$.
(iii) Let $\mathcal{M} = T_{f_\Theta} \mathcal{H}(z\Theta)$. Show that $\mathcal{M}$ is a nearly S^*-invariant subspace that is not the kernel of a Toeplitz operator.

Hint: See Example 29.6.

Exercise 30.5.2 Let $\varphi \in L^\infty$, $|\varphi| = 1$ and assume that $\ker T_\varphi$ is not trivial. Take any outer function $h \in \ker T_\varphi$. Show that there exists an inner function Θ such that

$$\varphi = \bar{\Theta}\frac{\bar{h}}{h}.$$

Exercise 30.5.3 Construct an outer function g of unit norm in H^2 and such that g^2 is rigid, and an inner function u that vanishes at the origin and divides the function b, where (a, b) is the admissible pair for g and such that g is not the extremal function for $\ker T_\varphi$, with $\varphi = \bar{u}\bar{g}/g$.
Hint: Use Exercise 30.5.2 and Corollary 30.21.

30.6 A characterization of surjectivity for Toeplitz operators

Let $\varphi \in L^\infty$, $|\varphi| = 1$ a.e. Let us assume that $\ker T_\varphi \neq \{0\}$ and let g be the extremal function for $\ker T_\varphi$. Then, according to Corollary 30.20, we know that g is an outer function such that g^2 is rigid and, if (a, b) is the admissible pair associated with g, then there is a unique inner function u that vanishes at the origin and divides the function b such that

$$\ker T_\varphi = T_g K_u \quad \text{and} \quad \varphi = \bar{u}\frac{\bar{g}}{g}.$$

Moreover, if $b = ub_0$, then we know from Theorem 30.19 that (a, b_0) is special and g_0^2 is rigid, where $g_0 = a/(1 - b_0)$.

Lemma 30.22 *With the notation above, let $\varphi_0 = \overline{g_0}/g_0$. Then*

$$T_{\varphi_0} T_{\bar{\varphi}_0} = T_\varphi T_{\bar{\varphi}}.$$

Proof Since $T_{\varphi_0} T_{\bar{\varphi}_0}$ and $T_\varphi T_{\bar{\varphi}}$ are positive, it will be sufficient to prove that

$$\|T_{\bar{\varphi}_0} f\|_2 = \|T_{\bar{\varphi}} f\|_2 \qquad (f \in H^2).$$

Since (a, b) is an admissible pair, it is special by Lemma 29.1, and Theorem 24.23 implies that $T_{1-b} T_{\bar{g}}$ is an isometry of H^2 onto $\mathcal{H}(b)$. Therefore,

$$\|T_{\bar{\varphi}} f\|_2 = \|T_{1-b} T_{\bar{g}} T_{\bar{\varphi}} f\|_b.$$

But, by Theorem 13.23, we have

$$\begin{aligned} T_{1-b} T_{\bar{g}} T_{\bar{\varphi}} &= T_{1-b} T_{\bar{g}} T_{ug/\bar{g}} \\ &= T_{1-b} T_{ug} \\ &= T_{(1-b)ug} = T_{au}. \end{aligned}$$

Hence,

$$\|T_{\bar{\varphi}} f\|_2 = \|auf\|_b.$$

Now, since $b = ub_0$, by Corollary 18.9, we have $\mathcal{H}(b) = \mathcal{H}(ub_0) = K_u \oplus u\mathcal{H}(b_0)$ and T_u acts as an isometry from $\mathcal{H}(b_0)$ into $\mathcal{H}(b)$. Thus,

$$\|auf\|_b = \|T_u(af)\|_b = \|af\|_{b_0}.$$

Note that, since (a, b_0) is a pair, we have $\mathcal{M}(a) \subset \mathcal{H}(b_0)$. Therefore, we deduce

$$\|T_{\bar{\varphi}} f\|_2 = \|af\|_{b_0} \qquad (f \in H^2). \tag{30.11}$$

Now, since (a, b_0) is special, we can make similar computations. Then, according to Theorem 24.23, the operator $T_{1-b_0} T_{\bar{g}_0}$ is an isometry of H^2 onto $\mathcal{H}(b_0)$. Therefore,

$$\|T_{\bar{\varphi}_0} f\|_2 = \|T_{1-b_0} T_{\bar{g}_0} T_{\bar{\varphi}_0} f\|_{b_0}.$$

But

$$T_{1-b_0} T_{\bar{g}_0} T_{\bar{\varphi}_0} = T_{1-b_0} T_{\overline{g_0 \varphi_0}} = T_{(1-b_0)g_0} = T_a,$$

whence

$$\|T_{\bar{\varphi}_0} f\|_2 = \|af\|_{b_0}.$$

Finally, considering (30.11), we obtain the desired result. □

Theorem 30.23 *With the notation above, assume that* $\ker T_\varphi \neq \{0\}$. *Then the following are equivalent.*

(i) T_φ *is surjective.*
(ii) $|g_0|^2 \in (HS)$.

Proof By Lemma 30.22, T_φ is surjective if and only if T_{φ_0} is surjective. But we know that $\varphi_0 = \bar{g}_0 / g_0$ and g_0^2 is rigid. Therefore, Lemma 12.30 implies that T_{φ_0} is one-to-one. Hence, T_{φ_0} is surjective if and only if it is invertible, which is equivalent to $|g_0|^2 \in (HS)$, by Corollary 12.43. □

Corollary 30.24 *Assume that* T_φ *is surjective. Then* $|a|^2 \in (HS)$.

Proof By Theorem 30.23 and Corollary 12.43, we know that $|g_0|^2 \in (HS)$ and then T_{φ_0} is invertible, where $\varphi_0 = \bar{g}_0 / g_0$. Since (a, b_0) is special, Theorem 28.23 implies that $T_{\bar{a}/a}$ is invertible. Another application of Corollary 12.43 gives that $|a|^2 \in (HS)$. □

If $|g|^2 \in (HS)$, then $T_{\bar{g}/g}$ is invertible by Corollary 12.43, and it follows that $T_\varphi = T_{\bar{u}} T_{\bar{g}/g}$ is invertible. More precisely, if A denotes the bounded operator defined by $A = (T_{\bar{g}/g})^{-1} T_u$, then we easily check that $T_\varphi A = I$. In Example 30.26, we show that the reverse implication fails. The next result gives a necessary and sufficient condition for $|g|^2 \in (HS)$.

Theorem 30.25 *Let T_φ be a surjective Toeplitz operator that is not injective. Let g be the extremal function for $\ker T_\varphi$, let (a,b) be the admissible pair associated with g and let u be the unique inner function that vanishes at the origin, divides the function b and is such that $\varphi = \bar{u}\bar{g}/g$. Then the following are equivalent.*

(i) $|g|^2 \in (HS)$.
(ii) (a,u) *forms a corona pair.*

Proof (i) $\Longrightarrow$ (ii) Assume that $|g|^2 \in (HS)$. Then, by Corollary 12.43, the operator $T_{\bar{g}/g}$ is invertible and, since (a,b) is admissible, henceforth special, it follows from Theorem 28.23 that (a,b) forms a corona pair. This means that there exists a constant $\delta > 0$ such that $|a(z)|+|b(z)| \geq \delta$, for all $z \in \mathbb{D}$. Using the fact that $|b_0| \leq 1$, we obtain

$$\delta \leq |a(z)|+|b(z)| = |a(z)|+|ub_0(z)| \leq |a(z)|+|u(z)| \qquad (z \in \mathbb{D}),$$

which proves that (a,u) forms also a corona pair.

(ii) $\Longrightarrow$ (i) Assume that there exists $\delta > 0$ such that

$$\delta \leq |a(z)|+|u(z)| \qquad (z \in \mathbb{D}).$$

Using the surjectivity of T_φ, by Theorem 30.23, we know that $|g_0|^2 \in (HS)$ and then $T_{\bar{g}_0/g_0}$ is invertible. Now, since (a,b_0) is special, Theorem 28.23 implies that (a,b_0) forms a corona pair and $T_{\bar{a}/a}$ is invertible. Then there exists $\delta_1 > 0$ such that

$$\delta_1 \leq |a(z)|+|b_0(z)| \qquad (z \in \mathbb{D}).$$

We thus obtain

$$(|a(z)|+|b_0(z)|)(|a(z)|+|u(z)|) \geq \delta\delta_1.$$

Since

$$\begin{aligned}
&(|a(z)|+|b_0(z)|)(|a(z)|+|u(z)|)\\
&\quad = |a(z)|(|a(z)|+|u(z)|+|b_0(z)|)+|b(z)|\\
&\quad \leq 3(|a(z)|+|b(z)|),
\end{aligned}$$

we conclude that (a,b) forms a corona pair. Now, using the fact that $T_{\bar{a}/a}$ is invertible, we deduce from Theorem 30.23 that $T_{\bar{g}/g}$ is invertible, that is, $|g|^2 \in (HS)$. □

We give now an example in which T_φ is surjective but $|g|^2$ is not a Helson–Szegő weight. Note that, in this case, the inner function u associated with $\ker T_\varphi$ cannot serve as one of the inner functions J appearing in the surjectivity criterion given in Theorem 12.48.

Example 30.26 Let us start with $a(z) = \frac{1}{4}(1-z)^{1/2-\varepsilon}$, $\varepsilon \in (0,1/2)$. Then it is easily checked that a is an outer function in H^∞ such that $\|a\|_\infty < 1$ and $|a|^2$ is a Helson–Szegő weight. Let b_0 be the normalized outer function such that $|b_0|^2 = 1 - |a|^2$ on $\mathbb{T}$ and let $g_0 = a/(1-b_0)$. Now take an increasing sequence $(\lambda_n)_{n\geq 1}$ of real numbers in $[0,1)$ with $\lambda_1 = 0$ and such that $\sum_n (1-|\lambda_n|) < +\infty$ and consider the Blaschke product B associated with this sequence. Finally, let $b = Bb_0$, $g = a/(1-b)$ and $\varphi = \bar{B}\bar{g}/g$.

First, note that $|g_0|^2 \in (HS)$. Indeed, since $|a|^2 \in (HS)$, the operator $T_{\bar{a}/a}$ is invertible. Moreover, we easily see that b_0 is invertible in H^∞ and then (a, b_0) forms a corona pair. Therefore, Theorem 28.23 implies that the pair (a, b_0) is admissible and $T_{\bar{g}_0/g_0}$ is invertible. Thus, Corollary 12.43 gives that $|g_0|^2 \in (HS)$. In particular, g_0^2 is a rigid function.

Then, according to the remark given after Theorem 30.19, the function g is the extremal function for $\ker T_\varphi$ and $\ker T_\varphi = T_g \mathcal{H}(B)$. Now, Theorem 30.23 implies that T_φ is surjective (because $|g_0|^2 \in (HS)$). It is easy to see that $|a(\lambda_n)| + |B(\lambda_n)| \longrightarrow 0$, as $n \longrightarrow +\infty$, whence (a, B) do not form a corona pair. It then follows from Theorem 30.25 that $|g|^2$ is not a Helson–Szegő weight.

30.7 The right-inverse of a Toeplitz operator

As before, we assume that φ is a unimodular function such that T_φ is surjective and $\ker T_\varphi \neq \{0\}$. Let g be the extremal function for $\ker T_\varphi$, let (a,b) be the admissible pair associated with g and let u be the unique inner function that vanishes at the origin, divides the function b and is such that $\varphi = \bar{u}\bar{g}/g$. Our objective now is to determine the right-inverse of T_φ whose range is orthogonal to $\ker T_\varphi$. The first step is to determine the inverse image of k_λ under T_φ that is orthogonal to $\ker T_\varphi$.

Lemma 30.27 *For each $\lambda \in \mathbb{D}$, we have*

$$T_\varphi(ugk_\lambda) = \overline{g(\lambda)}k_\lambda.$$

Proof For each $\lambda \in \mathbb{D}$,

$$\begin{aligned} T_\varphi(ugk_\lambda) &= P_+(\bar{u}\bar{g}ugk_\lambda) \\ &= P_+(\bar{g}k_\lambda) = \overline{g(\lambda)}k_\lambda, \end{aligned}$$

where the last equality follows from Lemma 13.26. □

Thus, to find the function in $T_\varphi^{-1}(k_\lambda)$ that is orthogonal to $\ker T_\varphi$, we need to project the function $ugk_\lambda/g(\lambda)$ onto $(\ker T_\varphi)^\perp$.

Lemma 30.28 *The orthogonal projection Q on H^2, with range $\ker T_\varphi$, is given by*

$$Q = T_g P_u T_{1-b} T_{\bar{g}},$$

where P_u stands for the orthogonal projection on H^2 with range K_u.

Proof First, by Theorem 24.23 and Lemma 30.17, the operator $T_{1-b}T_{\bar{g}}$ is an isometry of H^2 onto $\mathcal{H}(b)$ and it sends $\ker T_\varphi$ onto K_u, acting on $\ker T_\varphi$ as division by g. Therefore, for each $h \in \ker T_\varphi$, writing $h = gh_1$, with $h_1 \in K_u$, we have

$$\begin{aligned} Qh &= T_g P_u T_{1-b} T_{\bar{g}}(gh_1) \\ &= T_g P_u(h_1) \\ &= T_g h_1 = gh_1 = h. \end{aligned}$$

Moreover, for $h \in \mathcal{R}(T_{\bar{\varphi}})$, writing $h = T_{\bar{\varphi}}(h_1)$, with $h_1 \in H^2$, we have

$$\begin{aligned} Qh &= T_g P_u T_{1-b} T_{\bar{g}}(h) \\ &= T_g P_u T_{1-b} T_{\bar{g}} T_{\bar{\varphi}} h_1 \\ &= T_g P_u T_{1-b} T_{\bar{g}\bar{\varphi}} h_1 \\ &= T_g P_u T_{1-b} T_{ug} h_1 \\ &= T_g P_u T(1-b) guh_1 \\ &= T_g P_u(auh_1) = 0, \end{aligned}$$

because $auh_1 \in uH^2 = (K_u)^\perp$. We thus deduce that

$$Q_{|\ker T_\varphi} = I_{|\ker T_\varphi} \quad \text{and} \quad Q_{|\overline{\mathcal{R}(T_\varphi^*)}} = 0.$$

Since $(\mathcal{R}(T_\varphi^*))^\perp = \ker T_\varphi$, this reveals that Q is the orthogonal projection onto $\ker T_\varphi$. □

Lemma 30.29 *We have*

$$P_+(|g|^2 uk_\lambda) = \frac{uk_\lambda}{1-b} + \frac{\overline{b_0(\lambda)}k_\lambda}{1-\overline{b(\lambda)}} \tag{30.12}$$

and

$$P_+(|g|^2 k_\lambda) = \frac{k_\lambda}{1-b} + \frac{\overline{b(\lambda)}k_\lambda}{1-\overline{b(\lambda)}}. \tag{30.13}$$

Proof Let us first prove (30.12). On $\mathbb{T}$, we have

$$|g|^2 = \frac{|a|^2}{|1-b|^2} = \frac{1-|b|^2}{|1-b|^2} = \frac{1}{1-b} + \frac{\bar{b}}{1-\bar{b}}.$$

Since $b = ub_0$, we get

$$|g|^2 uk_\lambda = \frac{uk_\lambda}{1-b} + \frac{\bar{b}_0 k_\lambda}{1-\bar{b}}.$$

Now, suppose for a moment that $\|b\|_\infty < 1$. Then $u/(1-b)$ and $b_0/(1-b)$ are in H^∞, which, using (12.7), gives

$$P_+(|g|^2 uk_\lambda) = \frac{uk_\lambda}{1-b} + \frac{\overline{b_0(\lambda)} k_\lambda}{1-\overline{b(\lambda)}}.$$

This is the desired equality. Now, if $\|b\|_\infty = 1$, then apply the preceding case to rb in place of b (with $0 < r < 1$). Denoting by $g_r = a_r/(1-rb)$, where a_r is the normalized outer function such that $|a_r|^2 = 1 - |rb|^2$ on $\mathbb{T}$, we thus obtain

$$P_+(|g_r|^2 uk_\lambda) = \frac{uk_\lambda}{1-rb} + \frac{r\overline{b_0(\lambda)} k_\lambda}{1-r\overline{b(\lambda)}}. \tag{30.14}$$

By (13.51), the pair (a_r, rb) is special, that is,

$$\frac{1-|rb(z)|^2}{|1-rb(z)|^2} = \int_{\mathbb{T}} \frac{1-|z|^2}{|z-\zeta|^2} |g_r(\zeta)|^2 \, dm(\zeta).$$

Hence, taking $z = 0$, remembering that $b(0) = 0$, we get $\|g\|_2 = \|g_r\|_2 = 1$, for all $0 < r < 1$. Moreover we have

$$|g_r|^2 = \frac{1-|rb|^2}{|1-rb|^2}$$

and thus $|g_r|^2$ tends to $|g|^2$ almost everywhere on $\mathbb{T}$. Therefore, by Lemma 1.2, $|g_r|^2$ tends to $|g|^2$ in $L^1(\mathbb{T})$. Now, using the continuity of P_+ as a map from $L^1(\mathbb{T})$ to $\mathrm{Hol}(\mathbb{D})$, we get that $P_+(|g_r|^2 uk_\lambda)$ tends to $P_+(|g|^2 uk_\lambda)$ in $\mathrm{Hol}(\mathbb{D})$. That gives (30.12) by letting r tend to 1 in (30.14).

For the proof of (30.13), we argue analogously. On $\mathbb{T}$, we have

$$|g|^2 k_\lambda = \frac{k_\lambda}{1-b} + \frac{\bar{b} k_\lambda}{1-\bar{b}}.$$

If $\|b\|_\infty < 1$, then $1/(1-b)$ and $b/(1-b)$ are in H^∞, which, using (12.7), gives

$$P_+(|g|^2 k_\lambda) = \frac{k_\lambda}{1-b} + \frac{\overline{b(\lambda)} k_\lambda}{1-\overline{b(\lambda)}}.$$

If $\|b\|_\infty = 1$, then we apply an analogous reasoning as before using rb and letting r tend to 1. $\square$

We can now identify the function in $T_\varphi^{-1}(k_\lambda)$ belonging to $(\ker T_\varphi)^\perp$.

Lemma 30.30 *Let*

$$d_\lambda = \frac{(u - \overline{b_0(\lambda)})k_\lambda g}{\overline{a(\lambda)}}.$$

Then $d_\lambda \in H^2 \ominus \ker T_\varphi$ *and* $T_\varphi(d_\lambda) = k_\lambda$.

Proof Denote by x_λ the unique vector in $(\ker T_\varphi)^\perp$ such that $T_\varphi x_\lambda = k_\lambda$. According to Lemma 30.27, we have

$$T_\varphi\left(\frac{ugk_\lambda}{\overline{g(\lambda)}}\right) = k_\lambda$$

and, by Lemma 30.28, the operator $T_g P_u T_{1-b} T_{\bar{g}}$ is the orthogonal projection onto $\ker T_\varphi$. Hence, we deduce that

$$x_\lambda = (I - T_g P_u T_{1-b} T_{\bar{g}})\left(\frac{ugk_\lambda}{\overline{g(\lambda)}}\right).$$

It remains to prove that $x_\lambda = d_\lambda$. For this purpose, using Lemma 30.29, we have

$$\begin{aligned}
T_g P_u T_{1-b} T_{\bar{g}}(ugk_\lambda) &= T_g P_u T_{1-b} P_+(|g|^2 uk_\lambda)\\
&= T_g P_u T_{1-b}\left(\frac{uk_\lambda}{1-b} + \frac{\overline{b_0(\lambda)}}{1-\overline{b(\lambda)}}k_\lambda\right)\\
&= T_g P_u(uk_\lambda) + \frac{\overline{b_0(\lambda)}}{1-\overline{b(\lambda)}}\, T_g P_u((1-b)k_\lambda).
\end{aligned}$$

Since both functions uk_λ and bk_λ belong to $uH^2 = K_u^\perp$, we have $T_g P_u(uk_\lambda) = 0$ and $T_g P_u(bk_\lambda) = 0$, whence, using Theorem 18.11,

$$\begin{aligned}
T_g P_u T_{1-b} T_{\bar{g}}(ugk_\lambda) &= \frac{\overline{b_0(\lambda)}}{1-\overline{b(\lambda)}}\, T_g P_u k_\lambda\\
&= \frac{\overline{b_0(\lambda)}}{1-\overline{b(\lambda)}}\, (1 - \overline{u(\lambda)}u)k_\lambda.
\end{aligned}$$

Finally, we divide by $\overline{g(\lambda)}$, which gives

$$T_g P_u T_{1-b} T_{\bar{g}}\left(\frac{ugk_\lambda}{\overline{g(\lambda)}}\right) = \frac{\overline{b_0(\lambda)}}{\overline{a(\lambda)}}\, g\left(1 - \frac{\overline{u(\lambda)}}{u}\right)k_\lambda.$$

Therefore, we get

$$\begin{aligned} x_\lambda &= \frac{ugk_\lambda}{\overline{g(\lambda)}} - \frac{\overline{b_0(\lambda)}}{\overline{a(\lambda)}}\, g\left(1 - \frac{u(\lambda)}{u}\right)k_\lambda \\ &= gk_\lambda\left(\frac{u}{\overline{g(\lambda)}} - \frac{\overline{b_0(\lambda)}}{\overline{a(\lambda)}} + \frac{\overline{b(\lambda)}}{\overline{a(\lambda)}}\, u\right) \\ &= \frac{gk_\lambda}{\overline{a(\lambda)}}\left(u\frac{\overline{a(\lambda)}}{\overline{g(\lambda)}} - \overline{b_0(\lambda)} + \overline{b(\lambda)}u\right) \\ &= \frac{gk_\lambda}{\overline{a(\lambda)}}\big(u(1 - \overline{b(\lambda)}) - \overline{b_0(\lambda)} + \overline{b(\lambda)}u\big) \\ &= \frac{gk_\lambda}{\overline{a(\lambda)}}\big(u - \overline{b_0(\lambda)}\big) = d_\lambda. \end{aligned}$$

This completes the proof. □

Theorem 30.31 *Let T_φ be surjective and* $\ker T_\varphi \neq \{0\}$. *Then the right-inverse of T_φ whose range is orthogonal to* $\ker T_\varphi$ *equals* $T_g T_{u-\overline{b_0}} T_{1/\bar{a}}$.

Proof Let A be the right-inverse of T_φ whose range is orthogonal to $\ker T_\varphi$ and let $B = T_g T_{u-\overline{b_0}} T_{1/\bar{a}}$. First, note that B is continuous as an operator from H^2 into $\mathrm{Hol}(\mathbb{D})$. Indeed, Corollary 30.24 implies that $|a|^2 \in (HS)$. In particular, $1/\bar{a}$ belongs to $L^2(\mathbb{T})$, so the operator $T_{1/\bar{a}}$ is continuous as an operator from H^2 into $\mathrm{Hol}(\mathbb{D})$. The same is true for $T_{\overline{b_0}/\bar{a}}$ (because $\overline{b_0}/\bar{a} \in L^2(\mathbb{T})$). Therefore, since $T_{u-\overline{b_0}} T_{1/\bar{a}} = T_u T_{1/\bar{a}} - T_{\overline{b_0}/\bar{a}}$, the operator $T_{u-\overline{b_0}} T_{1/\bar{a}}$ is continuous as an operator from H^2 into $\mathrm{Hol}(\mathbb{D})$. Finally, we get that B is continuous from H^2 into $\mathrm{Hol}(\mathbb{D})$.

Now, by Lemma 30.30, we have $T_\varphi d_\lambda = k_\lambda$, where

$$d_\lambda = \frac{g(u - \overline{b_0(\lambda)})k_\lambda}{\overline{a(\lambda)}}$$

and $d_\lambda \in (\ker T_\varphi)^\perp$. Hence, we must have $Ak_\lambda = d_\lambda$. But, using Lemma 13.26, we have

$$\begin{aligned} Bk_\lambda &= T_g T_{u-\overline{b_0}} T_{1/\bar{a}} k_\lambda \\ &= \frac{1}{\overline{a(\lambda)}}\, T_g T_{u-\overline{b_0}} k_\lambda \\ &= \frac{g}{\overline{a(\lambda)}}(u - \overline{b_0(\lambda)})k_\lambda = d_\lambda. \end{aligned}$$

Hence, $Ak_\lambda = Bk_\lambda$, for each $\lambda \in \mathbb{D}$. The operators A and B thus agree on a dense subset of H^2 and continuity guarantees that $A = B$. □

The following result is immediate.

Corollary 30.32 *If T_φ is surjective, then the left-inverse of $T_{\bar{\varphi}}$ with the same kernel as T_φ is equal to $T_{1/a} T_{\bar{u}-b_0} T_{\bar{g}}$.*

Notes on Chapter 30

Section 30.1

Nearly invariant subspaces arose in the work of Hitt [104], who was interested in the invariant subspaces of the Hardy space of an annulus. Hitt used the terminology of "weakly invariant" rather than "nearly invariant". The terminology of "nearly invariant" is due to Sarason [162]. Theorem 30.1 is taken from [165].

Section 30.2

The operator R_f was introduced and studied by Sarason and all the results of this section come from [162]. Theorem 30.6 on the intertwining relation between R_f and X_b is the key part of Sarason's approach in his study of nearly invariant subspaces.

Section 30.3

Theorem 30.13 is due to Hitt [104], but the approach presented here and based on $\mathcal{H}(b)$ spaces (through Theorem 30.7) is due to Sarason [162]. This new approach not only offers a new perspective on Hitt's result but also clarifies the relation between the function g and the subspace $\mathcal{M}'$ that arise in Theorem 30.13. This question, discussed in Theorems 30.14 and 30.15, is also taken from [162] and the key (as already mentioned) is provided by the operator R_f and the link with $\mathcal{H}(b)$ spaces. Using different methods, Hayashi [98] had independently obtained the above description in the special case that $\mathcal{M} = \ker T_\varphi$ for $\varphi \in L^\infty(\mathbb{T})$.

Section 30.5

The description of the kernels of a Toeplitz operator given in Theorem 30.19 is due to Hayashi [99]. The approach given here and based on Hitt's description of nearly invariant subspaces is taken from Sarason [165]. Corollary 30.21 answers a question posed by Sarason in [162].

Section 30.6

The characterization of surjectivity of noninjective Toeplitz operators given in Theorem 30.23 is due to Hartmann, Sarason and Seip [97]. Theorem 30.25 and Example 30.26 are also taken from [97]. It should be noted that the characterization of surjectivity given in this section is interesting from a theoretical point of view. However, it is not clear how it might be used in concrete situations because it is not easy to compute explicitly in terms of the symbol φ the extremal function g of $\ker T_\varphi$ and its associated inner function u. It would probably be useful to make further investigations in this direction, as was noted in [97].

Section 30.7

The computation of the right-inverse of a surjective and noninjective Toeplitz operator given in Theorem 30.31 is also due to Hartmann, Sarason and Seip. The contents of this section are taken from [97].

31

Geometric properties of sequences of reproducing kernels

In Chapter 15, we studied the geometric properties of sequences of reproducing kernels of H^2. In particular, we characterized sequences that form a Riesz basis of their closed linear span. We now pursue this line of research by providing a comprehensive treatment of geometric properties of reproducing kernels of $\mathcal{H}(b)$.

In Section 31.1, we first discuss the problem of completeness and minimality for sequences of reproducing kernels of $\mathcal{H}(b)$. In Section 31.2, we study the rank-one perturbations of X_b^* that are isometries on $\mathcal{H}(b)$. We describe the eigenvalue of these perturbations and show a connection with the angular derivative of b. Then we compute the spectrum of these perturbations with a dichotomy depending on whether b is or is not an extreme point of the closed unit ball of H^∞. In Section 31.3, we apply the spectral results obtained in the previous section to give a criterion for the existence of an orthogonal basis of $\mathcal{H}(b)$ formed by reproducing kernels. In particular, we show that, if $\mathcal{H}(b)$ has an orthogonal basis formed by reproducing kernels, then necessarily b is inner. In Section 31.4, we show that the problem of the Riesz basis can be reduced in some situations to the invertibility of a certain operator, called the distortion operator. In Section 31.5, using the abstract functional embedding and the connection with $\mathcal{H}(b)$ spaces (obtained in Section 19.1), we solve this question of invertibility and get, in particular, the conclusion that, if b is not inner, then $\mathcal{H}(b)$ does not admit a Riesz basis of reproducing kernels. In Section 31.6, we apply our results on $\mathcal{H}(b)$ to deduce some results for $H^2(\mu)$ and $\mathcal{H}(\bar{b})$ spaces. In Section 31.7, we focus on the problem of asymptotically orthonormal sequences. In Section 31.8, we give some stability results for the completeness and asymptotically orthonormal basis properties for a sequence $(k^b_{\lambda_n})_n$ under some small perturbations of the sequence $(\lambda_n)_n$. Finally, in Section 31.9, using our Bernstein inequalities (obtained in Chapter 22), we give some stability results for the Riesz basis/sequences.

In the following, when we consider a sequence $(k^b_{\lambda_n})_{n\geq 1}$ of reproducing kernels in $\mathcal{H}(b)$, we always assume implicitly that the sequence of points λ_n are pairwise distinct.

31.1 Completeness and minimality in $\mathcal{H}(b)$ spaces

We first give a result with two necessary conditions for minimality.

Lemma 31.1 *Let b be a point in the closed unit ball of H^∞, and let $(\lambda_n)_{n\geq 1}$ be a sequence of points in $\mathbb{D}$. Assume that $(k^b_{\lambda_n})_{n\geq 1}$ is a minimal family. Then the following hold.*

(i) $\lim_{n\to\infty} \operatorname{dist}(\lambda_n, \sigma(b)) = 0$.
(ii) $(\lambda_n)_{n\geq 1}$ *is a Blaschke sequence.*

Proof By minimality, there exists $\psi_1 \in \mathcal{H}(b)$ such that

$$\Psi(\lambda_n) = \langle \psi_1, k^b_{\lambda_n}\rangle_b = \begin{cases} 0 & \text{if } n = 1, \\ 1 & \text{if } n \geq 2. \end{cases}$$

(i) Assume that $(\lambda_n)_{n\geq 1}$ has a limit point λ belonging to $\mathbb{T} \setminus \operatorname{Clos}(\sigma(b))$. By Theorem 20.13, ψ_1 is analytic in $\mathbb{D} \cup D(\lambda, \varepsilon)$, for a sufficiently small ε, and then it follows from the uniqueness principle for analytic functions that $\psi_1 \equiv 0$, which is absurd.

(ii) Since $\mathcal{H}(b) \subset H^2$, the sequence $(\lambda_n)_{n\geq 2}$ is contained in the zero sequence of a nonnull function in H^2 and then must satisfy the Blaschke condition. □

The following lemma shows that, if a system of reproducing kernels is not complete, then it is minimal, and we can add one reproducing kernel, keeping the minimality.

Lemma 31.2 *Let $b \in H^\infty$, $\|b\|_\infty \leq 1$. Let Λ be a subset of $\mathbb{D}$ such that*

$$\operatorname{Span}(k^b_\lambda : \lambda \in \Lambda) \neq \mathcal{H}(b), \tag{31.1}$$

and let $\mu \in \mathbb{D} \setminus \Lambda$. Then the following hold.

(i) $k^b_\mu \notin \operatorname{Span}(k^b_\lambda : \lambda \in \Lambda)$*; in particular* $(k^b_\lambda : \lambda \in \Lambda)$ *is a minimal family.*
(ii) *For each $\lambda_0 \in \Lambda$, we have*

$$\operatorname{Span}(k^b_\lambda,\ \lambda \in \Lambda \cup \{\mu\},\ \lambda \neq \lambda_0) \neq \mathcal{H}(b).$$

(iii) $(k^b_\lambda)_{\lambda\in\Lambda\cup\{\mu\}}$ *is a minimal family.*

Proof It follows from (31.1) that there exists $f \in \mathcal{H}(b)$, $f \not\equiv 0$, such that $f(\lambda) = 0$, $\lambda \in \Lambda$. For each $\lambda \in \mathbb{D}$, denote by $d(\lambda)$ the multiplicity of the point λ as a zero of f (by convention, if $f(\lambda) \neq 0$, we set $d(\lambda) = 0$) and we also set $\varphi_\lambda = b_\lambda^{d(\lambda)}$, where we recall that b_λ denotes the Blaschke factor with a simple zero at λ.

(i) Define $g = T_{\bar{\varphi}_\mu} f$. Then, according to Theorem 18.13, the function g is in $\mathcal{H}(b)$ and we have (using the fact that φ_μ is an inner function and f/φ_μ is in H^2)

$$g = P_+(\bar{\varphi}_\mu f) = \frac{f}{\varphi_\mu}.$$

Therefore, $g(\lambda) = 0$, $\lambda \in \Lambda$ and $g(\mu) \neq 0$. Hence, $k_\mu^b \notin \mathrm{Span}(k_\lambda^b : \lambda \in \Lambda)$. To prove the minimality, let $\lambda_0 \in \Lambda$. Then applying the preceding result, we have

$$k_{\lambda_0}^b \notin \mathrm{Span}(k_\lambda^b : \lambda \in \Lambda,\ \lambda \neq \lambda_0),$$

and, since this property holds for every $\lambda_0 \in \Lambda$, we get the minimality.

(ii) Take any $\lambda_0 \in \Lambda$ and set

$$h = \frac{\varphi_{\lambda_0} - \varphi_{\lambda_0}(\mu)}{\varphi_{\lambda_0}} f.$$

Using Theorem 18.13 once more, it is easy to see that h is in $\mathcal{H}(b)$. Moreover, we have $h(\mu) = 0$ and $h(\lambda) = 0$, for every $\lambda \in \Lambda \setminus \{\lambda_0\}$. Since $h \not\equiv 0$, this implies that $\mathrm{Span}(k_\lambda^b,\ \lambda \in \Lambda \cup \{\mu\} \setminus \{\lambda_0\}) \neq \mathcal{H}(b)$.

(iii) According to (i), there exists $g_\mu \in \mathcal{H}(b)$ such that $g_\mu(\mu) = 1$ and $g_\mu(\lambda) = 0$, for $\lambda \in \Lambda$. Fix $\lambda \in \Lambda$. We get from part (ii) that

$$\mathrm{Span}(k_\nu^b,\ \nu \in \Lambda \cup \{\mu\} \setminus \{\lambda\}) \neq \mathcal{H}(b),$$

which implies by (i) that there exists $g_\lambda \in \mathcal{H}(b)$ such that $g_\lambda(\lambda) = 1$ and $g_\lambda(\nu) = 0$, $\nu \in \Lambda \cup \{\mu\} \setminus \{\lambda\}$. It follows that $(g_\lambda : \lambda \in \Lambda \cup \{\mu\})$ is a biorthogonal system to $(k_\lambda^b, \lambda \in \Lambda \cup \{\mu\})$, which proves the minimality. □

In a particular case, Lemma 31.2 permits us to complete a Riesz sequence up to a Riesz basis.

Corollary 31.3 *Let $(k_\lambda^b)_{\lambda \in \Lambda}$ be a Riesz sequence and let*

$$\dim(\mathcal{H}(b) \ominus \mathrm{Span}(k_\lambda^b : \lambda \in \Lambda)) = N,$$

with $0 < N \leq \infty$. Then, a finitely extended family $(k_\lambda)_{\lambda \in \Lambda'}$, with $\Lambda' = \Lambda \cup \{\mu_1, \mu_2, \dots, \mu_m\}$, is a Riesz sequence whenever $m \leq N$ distinct points $\mu_i \in \mathbb{D} \setminus \Lambda$ are added ($m < \infty$). In particular, for $m = N < \infty$, we get a Riesz basis of the entire space $\mathcal{H}(b)$.

Proof By Lemma 31.2(i), we have

$$\dim(\mathcal{H}(b) \ominus \mathrm{Span}(k_\lambda^b : \lambda \in \Lambda \cup \{\mu_1\})) = N - 1,$$

and, by Lemma 31.2(iii), the family $(k_\lambda^b : \lambda \in \Lambda \cup \{\mu_1\})$ is minimal. According to Corollary 10.23, we know that it is still a Riesz sequence and we end the proof by induction. □

In Lemma 31.2, we saw that the noncompleteness of a system of reproducing kernels implies its minimality. In general, the converse is not true. For instance, take $\Lambda = (\lambda_n)_{n\geq 1}$ to be a Blaschke sequence and take $b = B$ to be the Blaschke product associated with Λ. Then we know that the sequence $(k_{\lambda_n}^b)_{n\geq 1}$ is minimal and complete in $\mathcal{H}(b) = K_B$. Nevertheless, in the case where b is a nonextreme point of the closed unit ball of H^∞, the minimality implies the noncompleteness. The key point is that, in this case, the space $\mathcal{H}(b)$ is invariant under the shift.

Theorem 31.4 *Let $b \in H^\infty$, $\|b\|_\infty \leq 1$, and let $\Lambda = (\lambda_n)_{n\geq 1} \subset \mathbb{D}$. Assume that b is nonextreme. Then the following statements are equivalent.*

(i) *$(k_{\lambda_n}^b)_{n\geq 1}$ is minimal.*
(ii) *$(k_{\lambda_n}^b)_{n\geq 1}$ is not complete in $\mathcal{H}(b)$.*

Moreover, in this case, we have

$$\dim(\mathcal{H}(b) \ominus \operatorname{Span}(k_{\lambda_n}^b : n \geq 1)) = \infty.$$

Proof According to Lemma 31.2, it suffices to prove that, if $(k_{\lambda_n}^b)_{n\geq 1}$ is minimal, then $\dim(\mathcal{H}(b) \ominus \operatorname{Span}(k_{\lambda_n}^b : n \geq 1)) = \infty$. We argue by absurdity, assuming that

$$\dim(\mathcal{H}(b) \ominus \operatorname{Span}(k_{\lambda_n}^b : n \geq 1)) = N < \infty.$$

This then implies the existence of a sequence of reproducing kernels that is minimal and complete in $\mathcal{H}(b)$. Indeed, we can assume that $N \geq 1$. Applying Lemma 31.2 repeatedly, we get that, if $(\mu_i)_{1\leq i\leq N}$ is a sequence of distinct points in $\mathbb{D}$ such that $\mu_i \notin \Lambda$, $1 \leq i \leq N$, then the system $\{k_\lambda^b : \lambda \in \Lambda'\}$ is minimal and complete in $\mathcal{H}(b)$, where $\Lambda' = \Lambda \cup \{\mu_1, \dots, \mu_N\}$. In particular, it implies the existence of a function $h \in \mathcal{H}(b)$ such that $h(\lambda_1) = 0$ and $h(\lambda) = 0$, $\lambda \in \Lambda' \setminus \{\lambda_1\}$. Now, consider $f = (z - \lambda_1)h$. Recall that, according to Theorem 24.1, $\mathcal{H}(b)$ is invariant under the shift, whence f belongs to $\mathcal{H}(b)$ and, moreover,

$$f \in \mathcal{H}(b) \ominus \operatorname{Span}(k_\lambda^b : \lambda \in \Lambda').$$

Since $h \not\equiv 0$, we have $f \not\equiv 0$, which contradicts the completeness of $\{k_\lambda^b : \lambda \in \Lambda'\}$. □

We now give another variation of Theorem 31.4, which is in fact a more detailed version.

Theorem 31.5 *Let $b \in H^\infty$, $\|b\|_\infty \le 1$, and let $(\lambda_n)_{n\ge1} \in \mathbb{D}$.*

(i) *If b is a nonextreme point of the closed unit ball of H^∞ and b has a bounded-type meromorphic pseudocontinuation across $\mathbb{T}$, then the following statements are equivalent:*

  (a) *$(k^b_{\lambda_n})_{n\ge1}$ is complete in $\mathcal{H}(b)$;*
  (b) $\sum_{n\ge1}(1-|\lambda_n|) = \infty$.

(ii) *If b is a nonextreme point of the closed unit ball of H^∞ and b does not have a bounded-type meromorphic pseudocontinuation across $\mathbb{T}$, then the following statements are equivalent:*

  (a) *$(k^b_{\lambda_n})_{n\ge1}$ is complete in $\mathcal{H}(b)$;*
  (b) $S^*b \in \mathrm{Span}(k^b_{\lambda_n} : n \ge 1)$.

(iii) *If b is an extreme point of the closed unit ball of H^∞, then the following statements are equivalent:*

  (a) *$(k^b_{\lambda_n})_{n\ge1}$ is complete in $\mathcal{H}(b)$;*
  (b) $S^*b \in \mathrm{Span}(k^b_{\lambda_n} : n \ge 1)$.

Proof (i) Assume that b is a nonextreme point of the closed unit ball of H^∞ and b has a bounded-type meromorphic pseudocontinuation across $\mathbb{T}$.

(b) $\Longrightarrow$ (a) This follows from the fact that $\mathcal{H}(b) \subset H^2$.

(a) $\Longrightarrow$ (b) We argue by absurdity, assuming that $(k^b_{\lambda_n})_{n\ge1}$ is complete in $\mathcal{H}(b)$ and that $(\lambda_n)_{n\ge1}$ is a Blaschke sequence. Denote by B the Blaschke product associated with $(\lambda_n)_{n\ge1}$. Since b has a bounded-type meromorphic pseudocontinuation across $\mathbb{T}$, by Theorem 8.42 and Corollary 8.33, there exists a nonconstant inner function Θ such that $b \in K_\Theta$. Then, using the fact that $K_\Theta = H^2 \cap \Theta\overline{zH^2}$, we can write $b = \bar{z}\bar{h}\Theta$, where $h \in H^2$. Moreover, since $|h| = |b|$ a.e. on $\mathbb{T}$, the function h is actually in H^∞. Now, for any function $f \in H^2$, we have

$$\begin{aligned}
\langle k^b_{\lambda_n}, \Theta B f\rangle_2 &= \langle k_{\lambda_n}, \Theta B f\rangle_2 - \overline{b(\lambda_n)}\langle b k_{\lambda_n}, \Theta B f\rangle_2 \\
&= -\overline{b(\lambda_n)}\langle \bar{z}\bar{h}\Theta k_{\lambda_n}, \Theta B f\rangle_2 \\
&= -\overline{b(\lambda_n)}\langle k_{\lambda_n}, zhBf\rangle_2 = 0,
\end{aligned}$$

which proves that $k^b_{\lambda_n} \in \mathcal{H}(\Theta B)$, $n \ge 1$. Given $\varepsilon > 0$ and $f \in \mathcal{H}(b)$, there exists a function g that is a finite linear combination of $k^b_{\lambda_n}$ such that $\|f-g\|_b \le \varepsilon$. On the one hand, since $\mathcal{H}(b)$ is contained contractively in H^2, we have $\|f-g\|_2 \le \varepsilon$. On the other hand, the function $g \in \mathcal{H}(\Theta b)$. Hence we get that $\mathcal{H}(b) \subset \mathcal{H}(\Theta B)$, which is absurd by Corollary 27.14, because b is nonextreme and ΘB is extreme.

We can deal with the proofs of (ii) and (iii) in one shot. So, let us assume either that b is an extreme point of the closed unit ball of H^∞, or that b is not an extreme point of the closed unit ball of H^∞ but b does not have a bounded-type meromorphic pseudocontinuation across $\mathbb{T}$. In any case, the implication (a) $\Longrightarrow$ (b) is trivial. For the reverse implication, let us assume that S^*b belongs to $\operatorname{Span}(k^b_{\lambda_n} : n \geq 1)$. Recall that, by (18.13),

$$X_b k^b_\lambda = \bar{\lambda} k^b_\lambda - \overline{b(\lambda)} S^* b,$$

for every $\lambda \in \mathbb{D}$ (see the proof of Theorem 18.22). Then we get that $X_b k_{\lambda_n} \in \operatorname{Span}(k^b_{\lambda_n} : n \geq 1)$, whence by linearity and continuity, we deduce that $\operatorname{Span}(k^b_{\lambda_n} : n \geq 1)$ is invariant under X_b. In particular, we have

$$S^{*p+1} b = X_b^p S^* b \in \operatorname{Span}(k^b_{\lambda_n} : n \geq 1) \qquad (p \geq 0).$$

It suffices now to use Theorem 18.19, Theorem 24.35 and Corollary 26.18 to conclude that

$$\operatorname{Span}(k^b_{\lambda_n} : n \geq 1) = \mathcal{H}(b). \qquad \square$$

Consider the following assertions:

(i) $\sum_{n \geq 1} (1 - |\lambda_n|) = +\infty$.
(ii) $\operatorname{Span}(k^b_{\lambda_n} : n \geq 1) = \mathcal{H}(b)$.
(iii) $S^* b \in \operatorname{Span}(k^b_{\lambda_n} : n \geq 1)$.

Without any extra hypotheses on b, we always have (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii). Moreover, Theorem 31.5 gives sufficient conditions on b to have the reverse implications. Now, we give two examples to show that none of the reverse implications is true in general.

First, let us consider $b(z) = (1+z)/2$ and $\lambda_n = 1 - 1/n^2$, $n \geq 1$. The sequence $(\lambda_n)_{n \geq 1}$ is a Blaschke sequence, that is, (i) is not satisfied. On the other hand, we verify that (ii) is not satisfied whereas (iii) is satisfied. To show that (ii) is not satisfied, just notice that b is a nonextreme point of the closed unit ball of H^∞ which has a bounded-type meromorphic pseudocontinuation across $\mathbb{T}$. Then it follows from Theorem 31.5(i) that the systems of reproducing kernels $\{k^b_{\lambda_n} : n \geq 1\}$ is not complete in $\mathcal{H}(b)$. Now we check that

$$\lim_{n \to \infty} \|S^* b - k^b_{\lambda_n}\|_b = 0. \tag{31.2}$$

Note that $S^* b = 1/2$. Then, using Corollary 23.9, we get

$$\begin{aligned} \|S^* b - k^b_{\lambda_n}\|_b^2 &= \|k^b_{\lambda_n}\|_b^2 + \|S^* b\|_b^2 - 2\,\Re(\langle S^* b, k^b_{\lambda_n} \rangle_b) \\ &= \frac{1 - |b(\lambda_n)|^2}{1 - |\lambda_n|^2} + \|S^* b\|_b^2 - 2(S^* b)(\lambda_n) \\ &= \frac{1 - |b(\lambda_n)|^2}{1 - |\lambda_n|^2} - \frac{1}{2}. \end{aligned}$$

An easy computation gives

$$\frac{1-|b(\lambda_n)|^2}{1-|\lambda_n|^2}=\frac{1}{2}\,\frac{2-1/(2n^2)}{2-1/n^2},$$

which obviously implies (31.2). In particular, $S^*b \in \mathrm{Span}(k^b_{\lambda_n} : n \geq 1)$.

As another example, let $\Lambda = (\lambda_n)_{n\geq 1}$ be any Blaschke sequence, and let $b = B$ be the Blaschke product associated with Λ. Then, we see that (i) is not satisfied whereas (ii) is satisfied because

$$\mathrm{Span}(k^b_{\lambda_n} : n \geq 1) = \mathrm{Span}(k_{\lambda_n} : n \geq 1) = K_B = \mathcal{H}(b).$$

Hence, in general (ii) does not imply (i).

We saw in Exercise 15.3.1 that $(k_{\lambda_n})_{n\geq 1}$ is a summation basis of its closed linear span. If we make an assumption on multipliers of $\mathcal{H}(b)$, we can give an analog of this result.

Theorem 31.6 *Let b be a nonextreme point of the closed unit ball of H^∞, let $(\lambda_n)_{n\geq 1} \subset \mathbb{D}$ be a Blaschke sequence of distinct points of $\mathbb{D}$, and let B be the Blaschke product associated with Λ. Assume that B is a multiplier of $\mathcal{H}(b)$. Then the sequence $(k^b_{\lambda_n})_{n\geq 1}$ is minimal. Moreover, if $(\varphi_n)_{n\geq 1}$ is a biorthogonal of $(k^b_{\lambda_n})_{n\geq 1}$, then, for all functions $f \in \mathrm{Span}(k^b_{\lambda_n} : n \geq 1)$, we have*

$$f = \lim_{p\to\infty} \sum_{n\geq 1} \overline{B^p(\lambda_n)}\langle f, \varphi_n\rangle_b k^b_{\lambda_n},$$

where $B^{(p)} = \prod_{n\geq p} b_{\lambda_n}$.

Proof Whenever b is a nonextreme point of the closed unit ball of H^∞, the analytic polynomials belong to $\mathcal{H}(b)$ and, in particular, $1 \in \mathcal{H}(b)$. Since B is a multiplier of $\mathcal{H}(b)$, then we get $B \in \mathcal{H}(b)$. Now, it follows from Theorem 18.16 that $B_n \in \mathcal{H}(b)$ and we have

$$\left\langle \frac{B_n}{B_n(\lambda_n)}, k^b_{\lambda_p} \right\rangle_b = \frac{B_n(\lambda_p)}{B_n(\lambda_n)} = \delta_{n,p},$$

which implies that $(B_n/B_n(\lambda_n))_{n\geq 1}$ is a biorthogonal of $(k^b_{\lambda_n})_{n\geq 1}$. Thus, the sequence $(k^b_{\lambda_n})_{n\geq 1}$ is minimal.

On the other hand, according to Lemma 20.15, $B^{(p)}$ is a multiplier of $\mathcal{H}(b)$ and we have

$$\|M_{B^{(p)}}\| \leq \|M_B\| \qquad (p \geq 1).$$

Moreover, since $B^{(p)}$ is a multiplier of $\mathcal{H}(b)$,

$$M^*_{B^{(p)}}(k^b_{\lambda_n}) = \overline{B^{(p)}(\lambda_n)}k^b_{\lambda_n},$$

and thus $\lim_{p\to\infty} M^*_{B^{(p)}}(k^b_{\lambda_n}) = k^b_{\lambda_n}$, $n \geq 1$. Using the fact that $\|M^*_{B^{(p)}}\| = \|M_{B^{(p)}}\| \leq \|M_B\|$, we thus obtain

$$\lim_{p\to\infty} M^*_{B^{(p)}} f = f,$$

for all $f \in \operatorname{Span}(k^b_{\lambda_n} : n \geq 1)$. But it is easy to see that, for all $f \in \operatorname{Lin}\{k^b_{\lambda_n} : n \geq 1\}$, we have

$$M^*_{B^{(p)}} f = \sum_{n=1}^{p-1} \overline{B^{(p)}(\lambda_n)} \langle f, \varphi_n \rangle_b k^b_{\lambda_n}.$$

By a density argument, we get this equality for all $f \in \operatorname{Span}(k^b_{\lambda_n} : n \geq 1)$, which implies, by letting $p \longrightarrow \infty$, that

$$f = \lim_{p\to\infty} \sum_{n\geq 1} \overline{B^p(\lambda_n)} \langle f, \varphi_n \rangle_b k^b_{\lambda_n}. \qquad \square$$

We obviously get from Theorem 31.6 a result of completeness for the biorthogonal system. In general, this is not true. There exist complete and minimal systems in a Hilbert space $\mathcal{H}$, whose biorthogonal system is not complete. But this property is always true for reproducing kernels of H^2. There is an inner function Θ and a sequence $(\lambda_n)_{n\geq 1}$ such that the system $(k^\Theta_{\lambda_n})_{n\geq 1}$ is complete and minimal in K_Θ, but the biorthogonal is not complete.

Corollary 31.7 *Let b be a nonextreme point of the closed unit ball of H^∞, let $(\lambda_n)_{n\geq 1} \subset \mathbb{D}$ be a Blaschke sequence of distinct points of $\mathbb{D}$, and let B be the Blaschke product associated with Λ. Assume that B is a multiplier of $\mathcal{H}(b)$ and let $(\varphi_n)_{n\geq 1}$ be the unique biorthogonal of $(k^\Theta_{\lambda_n})_{n\geq 1}$ that is in* $\operatorname{Span}(k^\Theta_{\lambda_n} : n \geq 1)$. *Then we have*

$$\operatorname{Span}(\varphi_n : n \geq 1) = \operatorname{Span}(k^b_{\lambda_n} : n \geq 1).$$

Proof The inclusion $\operatorname{Span}(\varphi_n : n \geq 1) \subset \operatorname{Span}(k^b_{\lambda_n} : n \geq 1)$ is trivial. For the reverse inclusion, let $f \in \operatorname{Span}(k^b_{\lambda_n} : n \geq 1)$ be such that $f \perp \varphi_n$, for all $n \geq 1$. Then it follows from Theorem 31.6 that $f \equiv 0$. $\square$

Theorem 31.6 and Corollary 31.7 can be applied in the situation where b is a nonextreme point of the closed unit ball of H^∞ and such that $(a, b) \in (HCR)$ and $T_{a/\bar{a}}$ is invertible. Indeed, in this case, we know that $\mathfrak{Mult}(\mathcal{H}(b)) = H^\infty$ and thus, for any Blaschke sequence $\Lambda = (\lambda_n)_{n\geq 1}$, if B is the associated Blaschke product, we always have $B \in \mathfrak{Mult}(\mathcal{H}(b))$. In particular, in this situation, the minimality is equivalent to the Blaschke condition. But if we are just interested in the equivalence between minimality and the Blaschke condition, it is still true under the weaker assumption that $b/a \in H^2$.

Theorem 31.8 *Let b be a nonextreme point of the closed unit ball of H^∞ and let $\Lambda = (\lambda_n)_{n\geq 1}$ be a sequence of distinct points of $\mathbb{D}$. Assume that b/a is in H^2. Then the following assertions are equivalent.*

(i) *The sequence $(k_{\lambda_n}^b)_{n\geq 1}$ is minimal in $\mathcal{H}(b)$.*
(ii) *The sequence $(k_{\lambda_n}^b)_{n\geq 1}$ is not complete in $\mathcal{H}(b)$.*
(iii) *The sequence $(\lambda_n)_{n\geq 1}$ satisfies the Blaschke condition.*

Proof The equivalence between (i) and (ii) was established in Theorem 31.4.

The implication (ii) $\Longrightarrow$ (iii) is always true since $\mathcal{H}(b) \subset H^2$.

To prove the remaining implication (iii) $\Longrightarrow$ (i), since $b/a \in H^2$, we know from Theorem 24.10 that $H^\infty \subset \mathcal{H}(b)$. In particular, $B \in \mathcal{H}(b)$, where B is the Blaschke product associated with Λ. Then, by Theorem 18.16, we also have $B_n \in \mathcal{H}(b)$, where $B_n = B/b_{\lambda_n}$, and then we get that $(B_n/B_n(\lambda_n))_n$ is a biorthogonal to $(k_{\lambda_n}^b)_n$. □

It seems interesting to give some examples showing that Theorem 31.8 and Theorem 31.5 are completely independent.

As the first example, let $b(z) = \exp(1/(z-2))$. Then, owing to the essential singularity at $z = 2$, b does not have a bounded-type meromorphic pseudocontinuation across $\mathbb{T}$. We also easily check that $\|b\|_\infty < 1$, which implies that b is a nonextreme point of the closed unit ball of H^∞ and $1/a$ belongs to H^∞. Therefore, in particular, b/a is in H^2.

For the second example, take $b(z) = (1+z)/2$ and $a(z) = (1-z)/2$. Then b/a is not in H^2, but b has of course a bounded-type meromorphic pseudocontinuation across $\mathbb{T}$.

Finally, for the third example, let $b(z) = 1/(z-3)$. Then b has a bounded-type meromorphic pseudocontinuation across $\mathbb{T}$ and, since $\|b\|_\infty = 1/2$, we get that b/a is in H^2.

31.2 Spectral properties of rank-one perturbation of X_b^*

In this section, we investigate the spectral properties of rank-one perturbations of X_b^*. For our discussion, given any complex number $\lambda \in \mathbb{T}$, define the operator U_λ on $\mathcal{H}(b)$ by

$$U_\lambda = X_b^* + \lambda(1 - \lambda\overline{b(0)})^{-1} k_0^b \otimes S^* b. \tag{31.3}$$

Theorem 31.9 *The operator U_λ is an isometry of $\mathcal{H}(b)$. Moreover, it is a unitary operator on $\mathcal{H}(b)$ if and only if b is an extreme point of the closed unit ball of H^∞. In the latter case, U_λ, $\lambda \in \mathbb{T}$, are the only rank-one perturbations of X_b^* that are unitary.*

Proof Recall that μ_λ is the measure on $\mathbb{T}$ whose Poisson integral is the real part of $(1+\bar\lambda b)/(1-\bar\lambda b)$, $\mathbf{V}_{\bar\lambda b}$ denotes the transformation defined on $L^2(\mu_\lambda)$ by $V_{\bar\lambda b} f(z) = (1-\bar\lambda b(z))K_{\mu_\lambda} f(z)$, and finally S_{μ_λ} is the operator of multiplication by the independent variable on $H^2(\mu_\lambda)$. We also recall that, from Theorem 20.12, we have

$$U_\lambda = \mathbf{V}_{\bar\lambda b} S_{\mu_\lambda} \mathbf{V}_{\bar\lambda b}^{-1} \tag{31.4}$$

and, moreover, $\mathbf{V}_{\bar\lambda b}$ is an isometry of $H^2(\mu_\lambda)$ onto $\mathcal{H}(b)$. Hence, U_λ is clearly an isometry on $\mathcal{H}(b)$. We see also that this isometry is onto if and only if S_{μ_λ} is onto, which is equivalent to $H^2(\mu_\lambda) = L^2(\mu_\lambda)$. But Corollary 13.34 says that this is exactly equivalent to the condition that b be an extreme point of the closed unit ball of H^∞.

Now, assume that b is an extreme point of the closed unit ball of H^∞ and that the operator $U = X^* + h\otimes k$, $h, k \in \mathcal{H}(b)$, is a unitary operator. If $f \perp k$, then we have $Uf = X^* f$, which gives $\|X^* f\|_b = \|f\|_b$. Corollary 18.23 implies that $f \perp S^* b$. It follows that there exists $c \in \mathbb{C}$ such that $k = cS^*b$, which gives $U = X^* + h_1 \otimes S^* b$, with $h_1 = \bar c h$. Taking the adjoint of this relation, we see that, if $f \perp h_1$, then $\|Xf\|_b = \|f\|_b$. Now recall (see Theorem 25.11) that $\|Xf\|_b^2 = \|f\|_b^2 - |f(0)|^2$, which gives $f(0) = 0$, that is, $f \perp k_0^b$. It follows that there exists $c_1 \in \mathbb{C}$ such that $h_1 = c_1 k_0^b$ and thus $U = X^* + c_1 k_0^b \otimes S^* b$. It remains to show that there exists $\lambda \in \mathbb{T}$ such that $c_1 = \lambda(1-\lambda\overline{b(0)})^{-1}$. Notice that, for all $f \in \mathcal{H}(b)$, we have

$$\begin{aligned}\|f\|_b^2 &= \|Uf\|_b^2 \\ &= \|X^* f\|_b^2 + |c_1|^2 |\langle f, S^* b\rangle_b|^2 \|k_0^b\|_b^2 + 2\,\Re(c_1 \langle f, S^* b\rangle_b \langle k_0^b, X^* f\rangle_b).\end{aligned}$$

In particular, for $f = S^* b$, using once more Corollary 18.23 and $\|k_0^b\|_b^2 = 1 - |b(0)|^2$, we get

$$0 = -\|S^* b\|_b^2 + |c_1|^2 \|S^* b\|_b^2 (1-|b(0)|^2) + 2\,\Re(c_1 \overline{(X^* S^* b)(0)}).$$

But, by Theorem 18.22, $X^* S^* b = SS^* b - \|S^* b\|_b^2 b$, whence $(X^* S^* b)(0) = -\|S^* b\|_b^2 b(0)$, which implies that

$$0 = -1 + |c_1|^2 (1-|b(0)|^2) - 2\,\Re(c_1 \overline{b(0)}).$$

Now, define $\lambda := \overline{c_1^{-1}} + b(0)$. Using the previous equality, we see that $\lambda \in \mathbb{T}$ and $c_1 = \lambda(1-\lambda\overline{b(0)})^{-1}$, which ends the proof of the theorem. $\square$

Since U_λ is an isometry, its point spectrum is located on the unit circle. The notion of angular derivative will lead us to a characterization.

Theorem 31.10 *Let $\lambda \in \mathbb{T}$. Then a complex number $\zeta \in \mathbb{T}$ is an eigenvalue of U_λ if and only if b has an angular derivative in the sense of Carathéodory at ζ and $b(\zeta) = \lambda$. Moreover, we have*

$$\ker(U_\lambda - \zeta I) = \mathbb{C} k_\zeta^b. \tag{31.5}$$

Proof Assume that b has an angular derivative in the sense of Carathéodory at ζ and $b(\zeta) = \lambda$. Using Corollary 21.28, we have $k_0^b = (I - \bar{\zeta}X^*)k_\zeta^b$. Hence, by (31.3),

$$\begin{aligned}(U_\lambda - \zeta I)k_\zeta^b &= (X^* - \zeta I)k_\zeta^b + \lambda(1 - \lambda\overline{b(0)})^{-1}\langle k_\zeta^b, S^*b\rangle_b k_0^b \\ &= -\zeta k_0^b + \lambda(1 - \lambda\overline{b(0)})^{-1}\langle k_\zeta^b, S^*b\rangle_b k_0^b.\end{aligned}$$

Now take a sequence $(z_n)_{n\geq 1}$ that tends nontangentially to ζ. Then

$$\langle S^*b, k_\zeta^b\rangle_b = \lim_{n\to\infty}\langle S^*b, k_{z_n}^b\rangle_b = \lim_{n\to\infty}\frac{b(z_n) - b(0)}{z_n} = \frac{\lambda - b(0)}{\zeta}.$$

That implies

$$(U_\lambda - \zeta I)k_\zeta^b = -\zeta k_0^b + \lambda(1 - \lambda\overline{b(0)})^{-1}\zeta(\bar{\lambda} - \overline{b(0)})k_0^b = 0,$$

which proves that $\zeta \in \sigma_p(U_\lambda)$ and

$$\mathbb{C}k_\zeta^b \subset \ker(U_\lambda - \zeta I). \tag{31.6}$$

Reciprocally, let $\zeta \in \sigma_p(U_\lambda)$ and let $f \in \mathcal{H}(b)$, $f \not\equiv 0$, be such that $(U_\lambda - \zeta I)f = 0$. Then we have $(X^* - \zeta I)f = -\lambda(1 - \lambda\overline{b(0)})^{-1}\langle f, S^*b\rangle_b k_0^b$. Notice that, if $\langle f, S^*b\rangle_b = 0$, then $\zeta \in \sigma_p(X^*)$, which, according to Theorem 18.26, is absurd. Hence, $\langle f, S^*b\rangle_b \neq 0$, and there exists $c \in \mathbb{C}$, $c \neq 0$, such that $k_0^b = (I - \bar{\zeta}X^*)(cf)$. Now, Corollary 21.28 implies that b has an angular derivative in the sense of Carathéodory at ζ and $k_0^b = (I - \bar{\zeta}X^*)k_\zeta^b$. We deduce that $k_\zeta^b - cf \in \ker(X^* - \bar{\zeta}I)$ and using Theorem 18.26 once more, we have $k_\zeta^b = cf$. Hence, $k_\zeta^b \in \ker(U_\lambda - \zeta I)$. But previous computations show that

$$(U_\lambda - \zeta I)k_\zeta^b = \left(-\zeta + \lambda(1 - \lambda\overline{b(0)})^{-1}\frac{\overline{b(\zeta)} - \overline{b(0)}}{\bar{\zeta}}\right)k_0^b,$$

which implies that $\lambda(\overline{b(\zeta)} - \overline{b(0)})(1 - \lambda\overline{b(0)})^{-1} = 1$ and $b(\zeta) = \lambda$. Moreover, as $k_\zeta^b = cf$, we have

$$\ker(U_\lambda - \zeta I) \subset \mathbb{C}k_\zeta^b,$$

which, with (31.6), gives (31.5). □

We can now give a complete description of the spectrum of U_λ.

Corollary 31.11 *Let* $\lambda \in \mathbb{T}$.

(i) *If b is an extreme point of the closed unit ball of H^∞, then $\sigma(U_\lambda) \subset \mathbb{T}$ and*

$$\zeta \in \sigma(U_\lambda) \iff \begin{array}{l}\text{(a)}\ \ \zeta \in \sigma(b) \cap \mathbb{T},\ \ \textit{or} \\ \text{(b)}\ \ \zeta \in \mathbb{T}\setminus\sigma(b)\ \ \textit{and}\ \ b(\zeta) = \lambda.\end{array}$$

(ii) *If b is a nonextreme point of the closed unit ball of H^∞, then $\sigma(U_\lambda) = \bar{\mathbb{D}}$.*

Proof (i) Assume that b is an extreme point of the closed unit ball of H^∞. Then Theorem 31.9 shows that U_λ is unitary and thus $\sigma(U_\lambda) \subset \mathbb{T}$. Let $\zeta \in \sigma(U_\lambda)$ and assume that $\zeta \in \mathbb{T} \setminus \sigma(b)$. Using the fact that $\sigma(b) \cap \mathbb{T} = \sigma(X^*) \cap \mathbb{T}$ and that U_λ is a rank-one perturbation of X^*, we deduce that $U_\lambda - \zeta I$ is a Fredholm operator of index 0. As $\zeta \in \sigma(U_\lambda)$, we get that $\zeta \in \sigma_p(U_\lambda)$ and Theorem 31.10 implies that $b(\zeta) = \lambda$.

Reciprocally, let $\zeta \in \sigma(b) \cap \mathbb{T}$ and assume that $\zeta \notin \sigma(U_\lambda)$. Using once more the fact that U_λ is a rank-one perturbation of X^*, we get that $X^* - \zeta I$ is a Fredholm operator of index 0. According to Theorem 18.26, we have that $\ker(X^* - \zeta I) = \{0\}$. Hence, $X^* - \zeta I$ is invertible, which gives $\zeta \in \mathbb{T} \setminus \sigma(X^*) = \mathbb{T} \setminus \sigma(b)$. This is absurd.

On the other hand, let $\zeta \in \mathbb{T} \setminus \sigma(b)$ and $b(\zeta) = \lambda$. By definition, there exists an open arc I, $\zeta \in I$, such that b can be continued analytically across I and $|b| = 1$ on I. In particular, b has an angular derivative in the sense of Carathéodory at ζ and, since $b(\zeta) = \lambda$, by Theorem 31.10, we deduce that $\zeta \in \sigma_p(U_\lambda) \subset \sigma(U_\lambda)$.

(ii) Assume that b is a nonextreme point of the closed unit ball of H^∞. Since U_λ is an isometry, we clearly have $\sigma(U_\lambda) \subset \bar{\mathbb{D}}$. Now, let $\zeta \in \mathbb{D}$ and suppose that $\zeta \in {}^c\sigma(U_\lambda)$. Recall that, when b is a nonextreme point of the closed unit ball of H^∞, then $b \in \mathcal{H}(b)$ and the space $\mathcal{H}(b)$ is invariant under the unilateral shift S (see Corollary 23.9 and Theorem 24.1). Hence, using (31.3) and (18.22), we get

$$\begin{aligned} U_\lambda &= X^* + \lambda(1 - \lambda\overline{b(0)})^{-1} k_0^b \otimes S^* b \\ &= Y - b \otimes S^* b + \lambda(1 - \lambda\overline{b(0)})^{-1} k_0^b \otimes S^* b. \end{aligned}$$

Thus, $Y - \zeta I$ is a Fredholm operator of index 0 (because it is a rank-two perturbation of an invertible operator). Since $\ker(Y - \zeta I) = \{0\}$ (by Lemma 8.6), we get a contradiction, because we know by Theorem 24.4 that $\sigma(Y) = \bar{\mathbb{D}}$. Hence, $\mathbb{D} \subset \sigma(U_\lambda)$. □

31.3 Orthonormal bases in $\mathcal{H}(b)$ spaces

It is clear that, for any sequence $(\lambda_n)_{n\geq 1} \subset \mathbb{D}$, then the family $(k_{\lambda_n}^b)_{n\geq 1}$ cannot be orthogonal in $\mathcal{H}(b)$. In some cases, it is possible, however, to consider reproducing kernels with poles on the unit circle. To explore this possibility, let

$$b(z) = z^N \prod_n \frac{|a_n|}{a_n} \frac{a_n - z}{1 - \overline{a_n} z} \exp\left(- \int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} \, d\mu(\zeta) \right)$$

be the canonical factorization of b, where $\sum_n (1-|a_n|) < \infty$ and where μ is a positive Borel measure on $\mathbb{T}$. Then remember that

$$E_2(b) = \left\{\zeta \in \mathbb{T} : \sum_n \frac{1-|a_n|^2}{|\zeta - a_n|^2} + \int_{\mathbb{T}} \frac{d\mu(t)}{|t-\zeta|^2} < +\infty\right\},$$

and b has an angular derivative in the sense of Carathéodory at $\lambda \in \mathbb{T}$ if and only if $\lambda \in E_2(b)$. Moreover, in this case, each function $f \in \mathcal{H}(b)$ has a nontangential limit at λ, denoted by $f(\lambda)$. Corresponding to such a point, define

$$k_\lambda^b(z) = \frac{1-\overline{b(\lambda)}b(z)}{1-\bar{\lambda}} \qquad (z \in \mathbb{D}).$$

Then $k_\lambda \in \mathcal{H}(b)$ and we have

$$f(\lambda) = \langle f, k_\lambda^b \rangle_b \qquad (f \in \mathcal{H}(b)). \tag{31.7}$$

Now, let $\lambda, \lambda' \in E_2(b)$, $\lambda \neq \lambda'$. Then using (31.7), we get

$$\langle k_\lambda^b, k_{\lambda'}^b \rangle_b = k_\lambda^b(\lambda') = \frac{1-\overline{b(\lambda)}b(\lambda')}{1-\bar{\lambda}\lambda'}.$$

Since $|b(\lambda)| = |b(\lambda')| = 1$, we conclude that

$$\langle k_\lambda^b, k_{\lambda'}^b \rangle_b = 0 \quad \Longleftrightarrow \quad b(\lambda) = b(\lambda').$$

Thus, if we want to get an orthonormal basis of reproducing kernels $(k_{\lambda_n}^b)_{n\geq 1}$, then we have to choose a sequence $(\lambda_n)_{n\geq 1}$ such that $(\lambda_n)_{n\geq 1} \subset E_2(b)$ and $b(\lambda_n) = \alpha$, $n \geq 1$, where $\alpha \in \mathbb{T}$ is a fixed constant.

If $\zeta \in E_2(b)$, $b(\zeta) = \lambda$, then Theorem 31.10 implies that $U_\lambda k_\zeta^b = \zeta k_\zeta^b$. Therefore, the system $\{k_\zeta^b : \zeta \in E_2(b),\ b(\zeta) = \lambda\}$ forms an orthogonal system of eigenvectors of U_λ in $\mathcal{H}(b)$. Note that the set $\{\zeta \in E_2(b) : b(\zeta) = \lambda\}$ is at most countable (possibly empty) because $\mathcal{H}(b)$ is separable.

The following result says that the system $\{k_\zeta^b : \zeta \in E_2(b),\ b(\zeta) = \lambda\}$ is complete (and then U_λ has an orthogonal basis of eigenvectors) if and only if the Clark measure μ_λ is purely atomic.

Theorem 31.12 *Let $\lambda \in \mathbb{T}$. Then the following assertions are equivalent.*

(i) *The family $\{k_\zeta^b : \zeta \in E_2(b),\ b(\zeta) = \lambda\}$ forms an orthogonal basis of $\mathcal{H}(b)$.*
(ii) *The Clark measure μ_λ is purely atomic.*

Proof (i) $\Longrightarrow$ (ii) Write $\{\zeta \in E_2(b) : b(\zeta) = \lambda\} = (\zeta_n)_{n\geq 1}$. Since $\mathbf{V}_{\bar{\lambda}b}$ is an isometry from $H^2(\mu_\lambda)$ onto $\mathcal{H}(b)$, the family $(\mathbf{V}_{\bar{\lambda}b}^{-1} k_{\zeta_n}^b)_{n\geq 1}$ is an orthogonal basis of $H^2(\mu_\lambda)$. Moreover, using (31.4) and Theorem 31.10, we have

$$S_{\mu_\lambda} \mathbf{V}_{\bar{\lambda}b}^{-1} k_{\zeta_n}^b = \mathbf{V}_{\bar{\lambda}b}^{-1} U_\lambda k_{\zeta_n}^b = \zeta_n \mathbf{V}_{\bar{\lambda}b}^{-1} k_{\zeta_n}^b.$$

This means that $H^2(\mu_\lambda)$ has an orthogonal basis of eigenvectors of S_{μ_λ}. Applying Corollary 8.28 gives $\mu_\lambda = \sum_{n\geq 1} a_n \delta_{\{\zeta_n\}}$, with $a_n = \mu_\lambda(\zeta_n)$.

(ii) $\Longrightarrow$ (i) Assume that μ_λ is purely atomic, that is, $\mu_\lambda = \sum_{n\geq 1} a_n \delta_{\{\zeta_n\}}$, with $a_n = \mu_\lambda(\{\zeta_n\}) > 0$. Then we know from Corollary 8.28 that $(\chi_{\{\zeta_n\}})_{n\geq 1}$ forms an orthogonal basis of eigenvectors of S_μ, with $S_\mu \chi_{\{\zeta_n\}} = \zeta_n \chi_{\{\zeta_n\}}$. Therefore, we get that $(\mathbf{V}_{\bar{\lambda}b}\chi_{\{\zeta_n\}})_{n\geq 1}$ is an orthogonal basis of $\mathcal{H}(b)$. Using (31.4) once more, we have $U_\lambda(\mathbf{V}_{\bar{\lambda}b}\chi_{\{\zeta_n\}}) = \mathbf{V}_{\bar{\lambda}b} S_{\mu_\lambda}\chi_{\{\zeta_n\}} = \zeta_n \mathbf{V}_{\bar{\lambda}b}\chi_{\{\zeta_n\}}$. Theorem 31.10 implies that $\zeta_n \in E_2(b)$, $b(\zeta_n) = \lambda$, and there exists $c_n \in \mathbb{C}^*$ such that $\mathbf{V}_{\bar{\lambda}b}\chi_{\{\zeta_n\}} = c_n k^b_{\zeta_n}$. Hence, $(k^b_{\zeta_n})_{n\geq 1}$ is an orthogonal basis of $\mathcal{H}(b)$. It remains to note that $\{\zeta \in E_2(b) : b(\zeta) = \lambda\} = (\zeta_n)_{n\geq 1}$. The inclusion $(\zeta_n)_{n\geq 1} \subset \{\zeta \in E_2(b) : b(\zeta) = \lambda\}$ has already been proved. Assume that there exists $\zeta \in E_2(b)$, $b(\zeta) = \lambda$, $\zeta \neq \zeta_n$, $n \geq 1$. Theorem 21.1 implies that $k^b_\zeta \in \mathcal{H}(b)$ and

$$\langle k^b_{\zeta_n}, k^b_\zeta \rangle = \frac{1 - \overline{b(\zeta_n)}b(\zeta)}{1 - \bar{\zeta}_n \zeta} = 0.$$

Hence, $k^b_\zeta \in \mathcal{H}(b) \ominus \mathrm{Span}(k^b_{\zeta_n} : n \geq 1)$, which is absurd. □

Corollary 31.13 *Let $b \in H^\infty$, $\|b\|_\infty \leq 1$. Then the following are equivalent.*

(i) *The space $\mathcal{H}(b)$ has an orthogonal basis of reproducing kernels.*
(ii) *There is $\lambda \in \mathbb{T}$ such that the measure μ_λ is purely atomic.*

In particular, in this case, b is necessarily an inner function.

Proof The equivalence between (i) and (ii) follows immediately from Theorem 31.12. Now, if there is $\lambda \in \mathbb{T}$ such that the measure μ_λ is purely atomic, then, according to Theorem 13.28, we must have

$$\frac{1 - |b(e^{it})|^2}{|\lambda - b(e^{it})|^2} = 0,$$

for almost all e^{it} on $\mathbb{T}$, which implies that $|b(e^{it})| = 1$, a.e. on $\mathbb{T}$, that is, b is an inner function. □

Corollary 31.14 *Let Θ be an inner function. Assume that $\mathbb{T} \setminus E_2(\Theta)$ is at most countable. Then, for every $\alpha \in \mathbb{T}$, the family*

$$\{k^\Theta_\lambda : \lambda \in E_2(\Theta),\ \Theta(\lambda) = \alpha\}$$

is an orthogonal basis in K_Θ. In particular, this is the case (i.e. $\mathbb{T} \setminus E_2(\Theta)$ is at most countable) if Θ is a singular inner function associated with a measure μ such that supp μ *is at most countable.*

Proof Since Θ is an inner function, the measures μ_α are singular, for every $\alpha \in \mathbb{T}$. Moreover, μ_α is carried on the set

$$\{\zeta \in \mathbb{T} : \Theta(\zeta) = \alpha\}.$$

We have already noticed that the set $\{\zeta \in E_2(\Theta) : \Theta(\zeta) = \alpha\}$ is at most countable. Since $\mathbb{T}\backslash E_2(\Theta)$ is at most countable, we finally get that the measure μ_α is carried on a set that is also at most countable. Thus the measure μ_α has to be purely atomic. The conclusion now follows from Theorem 31.13. □

31.4 Riesz sequences of reproducing kernels in $\mathcal{H}(b)$

Let b be a function in the closed unit ball of H^∞, and let $\Lambda = (\lambda_n)_{n\geq 1}$ be a sequence of points in $\mathbb{D}$. In this section, we discuss the problem of finding a criterion for the sequence $(k^b_{\lambda_n})_{n\geq 1}$ to be an unconditional basis.

If $(k^b_{\lambda_n})_{n\geq 1}$ is minimal, then $(\lambda_n)_{n\geq 1}$ is a Blaschke sequence of distinct points. Thus, from now on, we assume that $(\lambda_n)_{n\geq 1}$ is a Blaschke sequence of distinct points in the unit disk and we denote by B the Blaschke product associated with $(\lambda_n)_{n\geq 1}$. For a point $\lambda \in \mathbb{D}$, we also denote the normalized reproducing kernel of H^2 at point λ by $\tilde{k}_\lambda$. Similarly, the normalized reproducing kernel of $\mathcal{H}(b)$ is denoted by $\tilde{k}^b_\lambda$. More explicitly, we have

$$\|k_\lambda\|_2^2 = \frac{1}{1-|\lambda|^2}, \qquad \|k^b_\lambda\|_b^2 = \frac{1-|b(\lambda)|^2}{1-|\lambda|^2}, \tag{31.8}$$

whence

$$\tilde{k}_\lambda(z) = \frac{(1-|\lambda|^2)^{1/2}}{1-\bar{\lambda}z} \qquad (z \in \mathbb{D}) \tag{31.9}$$

and

$$\tilde{k}^b_\lambda(z) = \left(\frac{1-|\lambda|^2}{1-|b(\lambda)|^2}\right)^{1/2} \frac{1-\overline{b(\lambda)}b(z)}{1-\bar{\lambda}z} \qquad (z \in \mathbb{D}). \tag{31.10}$$

Recall that, according to the Köthe–Toeplitz theorem (Theorem 10.30), $(\tilde{k}^b_{\lambda_n})_{n\geq 1}$ is a Riesz sequence (respectively, Riesz basis) of $\mathcal{H}(b)$ if and only if it is an unconditional basis of its closed linear span (respectively, of $\mathcal{H}(b)$). Using (31.8), we immediately get the following result.

Lemma 31.15 *Let $(\lambda_n)_{n\geq 1} \subset \mathbb{D}$. Then the norms of k_{λ_n} and $k^b_{\lambda_n}$ are equivalent, i.e. there exist two constants c_1 and c_2 such that*

$$c_1\|k_{\lambda_n}\|_2 \leq \|k^b_{\lambda_n}\|_b \leq c_2\|k_{\lambda_n}\|_2, \tag{31.11}$$

for every $n \geq 1$, if and only if

$$\sup_{n\geq 1} |b(\lambda_n)| < 1. \tag{31.12}$$

However, note that the second inequality in (31.11) is always true with $c_2 = 1$ without any extra assumption.

Now, we prove that the Carleson condition (C) is necessary for reproducing kernels to form a Riesz sequence. To this end, we employ the Gram matrices introduced in Section 10.7.

Theorem 31.16 *Let b be a function in the closed unit ball of H^∞, and let $(\tilde{k}^b_{\lambda_n})_{n\geq 1}$ be a sequence of reproducing kernels in $\mathcal{H}(b)$, which is a Riesz sequence. Then $(\lambda_n)_{n\geq 1} \in (C)$.*

Proof Let $\Gamma^b = (\Gamma^b_{n,p})_{n,p\geq 1}$ and $\Gamma = (\Gamma_{n,p})_{n,p\geq 1}$ be respectively the Gram matrices of the normalized sequences $(\tilde{k}^b_\lambda)_{n\geq 1}$ and $(\tilde{k}_{\lambda_n})_{n\geq 1}$, that is,

$$\Gamma^b_{n,p} = \langle \tilde{k}^b_{\lambda_n}, \tilde{k}^b_{\lambda_p}\rangle_b \quad \text{and} \quad \Gamma_{n,p} = \langle \tilde{k}_{\lambda_n}, \tilde{k}_{\lambda_p}\rangle_2 \qquad (n,p \geq 1).$$

Applying (31.9) and (31.10), we have

$$|\Gamma^b_{n,p}|^2 = |\Gamma_{n,p}|^2 \frac{|1-\overline{b(\lambda_n)}b(\lambda_p)|^2}{(1-|b(\lambda_n)|^2)(1-|b(\lambda_p)|^2)}.$$

Now, using the formula

$$\frac{(1-|\lambda|^2)(1-|\mu|^2)}{|1-\bar{\lambda}\mu|^2} = 1-|b_\lambda(\mu)|^2 \qquad (\lambda,\mu \in \mathbb{D}), \tag{31.13}$$

we see that

$$\frac{|1-\overline{b(\lambda_n)}b(\lambda_p)|^2}{(1-|b(\lambda_n)|^2)(1-|b(\lambda_p)|^2)} \geq 1,$$

which implies that

$$|\Gamma^b_{n,p}|^2 \geq |\Gamma_{n,p}|^2 \qquad (\text{for } n,p \geq 1). \tag{31.14}$$

According to Corollary 10.22, we then have

$$\sup_{n\neq p} |\Gamma_{n,p}| < 1 \tag{31.15}$$

and

$$\sup_{p\geq 1} \sum_{n\geq 1} |\Gamma_{n,p}|^2 < +\infty. \tag{31.16}$$

But, using (31.13) once more, we have

$$|\Gamma_{n,p}|^2 = \frac{(1-|\lambda_n|^2)(1-|\lambda_p|^2)}{|1-\bar{\lambda}_n\lambda_p|^2} = 1-|b_{\lambda_n}(\lambda_p)|^2,$$

and thus (31.15) implies that

$$\inf_{n\neq p} |b_{\lambda_n}(\lambda_p)| > 0.$$

On the other hand, (31.16) means that

$$\sup_{p\geq 1} \sum_{n\geq 1} \frac{(1-|\lambda_n|^2)(1-|\lambda_p|^2)}{|1-\bar{\lambda}_n\lambda_p|^2} < \infty.$$

Set $\mu = \sum_{n\geq 1}(1-|\lambda_n|^2)\delta_{\{\lambda_n\}}$. Then the last inequality can be rewritten as

$$\sup_{p\geq 1}\big((1-|\lambda_p|^2)\|k_{\lambda_p}\|^2_{L^2(\mu)}\big) < \infty,$$

which, by Theorem 5.15, is equivalent to saying that μ is a Carleson measure. Therefore, by Theorem 15.7, $(\lambda_n)_{n\geq 1}$ is a Carleson sequence. □

In the case where condition (31.12) is fulfilled, we can say more.

Theorem 31.17 *Let b be in the closed unit ball of H^∞, let $(\lambda_n)_{n\geq 1}$ be a Blaschke sequence of points in the unit disk, and let B be the associated Blaschke product. Assume that $(\lambda_n)_{n\geq 1}$ satisfies (31.12). Then the sequence of normalized reproducing kernels $(\tilde{k}^b_{\lambda_n})_{n\geq 1}$ is a Riesz sequence in $\mathcal{H}(b)$ (respectively, a Riesz basis of $\mathcal{H}(b)$) if and only if the following two conditions hold:*

(i) $(\lambda_n)_{n\geq 1} \in (C)$, *and*
(ii) *the restriction* $(I - T_bT_{\bar{b}})_{|\mathcal{H}(B)} : \mathcal{H}(B) \longrightarrow \mathcal{H}(b)$ *is an isomorphism onto its range (respectively, onto* $\mathcal{H}(b)$*).*

Proof For the necessity, assume that $(\tilde{k}^b_{\lambda_n})_{n\geq 1}$ is a Riesz sequence in $\mathcal{H}(b)$. By Theorem 31.16, we get that $(\lambda_n)_{n\geq 1} \in (C)$, and, by Theorem 15.7, that $(\tilde{k}_{\lambda_n})_{n\geq 1}$ is a Riesz basis of $\mathcal{H}(B)$. Now using Theorem 18.11, (31.9) and (31.10), we have

$$(I - T_bT_{\bar{b}})\tilde{k}_{\lambda_n} = (1-|\lambda_n|^2)^{1/2}k^b_{\lambda_n} = (1-|b(\lambda_n)|^2)^{1/2}\tilde{k}^b_{\lambda_n}, \tag{31.17}$$

whence

$$\begin{aligned}\Big\|(I - T_bT_{\bar{b}})\sum_{n\geq 1} a_n\tilde{k}_{\lambda_n}\Big\|_b^2 &= \Big\|\sum_{n\geq 1} a_n(1-|b(\lambda_n)|^2)^{1/2}\tilde{k}^b_{\lambda_n}\Big\|_b^2\\ &\asymp \sum_{n\geq 1}|a_n|^2(1-|b(\lambda_n)|^2)\\ &\asymp \sum_{n\geq 1}|a_n|^2\\ &\asymp \Big\|\sum_{n\geq 1} a_n\tilde{k}_{\lambda_n}\Big\|_2^2,\end{aligned}$$

for every finite sequence of complex numbers $a_n \in \mathbb{C}$. Therefore, $I - T_bT_{\bar{b}}$ is an isomorphism from $\mathrm{Span}(\tilde{k}_{\lambda_n} : n \geq 1) = \mathcal{H}(B)$ onto its range. If the sequence $(\tilde{k}^b_{\lambda_n})_{n\geq 1}$ is a Riesz basis of $\mathcal{H}(b)$, then the range of $(I - T_bT_{\bar{b}})_{|\mathcal{H}(B)}$ contains the dense set $(\tilde{k}^b_{\lambda_n})_{n\geq 1}$ and thus it is onto.

For the sufficiency, by Theorem 15.7, (i) implies that $(\tilde{k}_{\lambda_n})_{n\geq 1}$ is a Riesz basis of $\mathcal{H}(B)$ and consequently its isomorphic image, given by $(I - T_bT_{\bar{b}})\tilde{k}_{\lambda_n}$, is a Riesz basis of $(I - T_bT_{\bar{b}})\mathcal{H}(B)$. Then using (31.17) and $1 - |b(\lambda_n)|^2 \asymp 1$, we conclude that $(\tilde{k}^b_{\lambda_n})_{n\geq 1}$ is a Riesz basis of $(I - T_b T_{\bar{b}})\mathcal{H}(B)$, which gives the result. $\square$

Theorem 31.17 reduces the study of Riesz bases (or Riesz sequences) to the study of the invertibility (or left-invertibility) of the operator

$$(I - T_b T_{\bar{b}})_{|\mathcal{H}(B)} : \mathcal{H}(B) \longrightarrow \mathcal{H}(b).$$

We will call $(I - T_b T_{\bar{b}})_{|\mathcal{H}(B)}$ the *distortion operator*. We now examine the invertibility of the distortion operator

$$(I - T_b T_b^*)_{|K_\Theta} : K_\Theta \longrightarrow \mathcal{H}(b)$$

for a general nonconstant inner function Θ and a general function b in the closed unit ball of H^∞. We highlight that it is not assumed that Θ is a Blaschke product. We now give a criterion for left-invertibility.

Theorem 31.18 *Let b be in the closed unit ball of H^∞ and let Θ be an inner function. Then the following are equivalent.*

(i) *The distortion operator $(I - T_b T_{\bar{b}})_{|K_\Theta}$ is an isomorphism onto its range.*
(ii) $\operatorname{dist}(\bar{\Theta} b, H^\infty) < 1$.
(iii) $\|P_\Theta T_{b|K_\Theta}\| < 1$.

Proof The operator $(I - T_b T_{\bar{b}})_{|K_\Theta}$ is an isomorphism of K_Θ onto its range if and only if there exists $c > 0$ such that

$$c\|f\|_2 \leq \|(I - T_b T_{\bar{b}})f\|_b,$$

for every f in K_Θ. But according to Corollary 17.6, we have

$$\|(I - T_b T_{\bar{b}})f\|_b^2 = \|f\|_2^2 - \|T_{\bar{b}} f\|_2^2, \tag{31.18}$$

whence we get that the distortion operator is an isomorphism onto its range if and only if

$$\|T_{\bar{b}} f\|_2^2 \leq (1 - c^2)\|f\|_2^2, \tag{31.19}$$

for every f in K_Θ. But (31.19) is equivalent to

$$\|T_{\bar{b}|K_\Theta}\| < 1,$$

which ends the proof thanks to Lemma 20.24. $\square$

We now give a criterion for the invertibility of the distortion operator assuming that $b = \Theta_1$ is also inner. In that case, $\mathcal{H}(b) = K_{\Theta_1}$ and $I - T_b T_{\bar{b}}$ is the orthogonal projection of H^2 onto K_{Θ_1}.

Theorem 31.19 *Let Θ and Θ_1 be two inner functions. Then the following are equivalent.*

(i) *The distortion operator $P_{\Theta_1|K\Theta} : K_\Theta \longrightarrow K_{\Theta_1}$ is an isomorphism onto K_{Θ_1}.*
(ii) *The Toeplitz operator $T_{\Theta_1\bar{\Theta}}$ is invertible.*
(iii) $\operatorname{dist}(\Theta_1\bar{\Theta}, H^\infty) < 1$ *and* $\operatorname{dist}(\Theta\bar{\Theta}_1, H^\infty) < 1$.
(iv) $\operatorname{dist}(\bar{\Theta}\Theta_1, H^\infty) < 1$ *and $T_{\Theta\bar{\Theta}_1}$ is left-invertible.*
(v) *There exists an outer function h in H^∞ such that $\|\Theta_1\bar{\Theta} - h\|_\infty < 1$.*
(vi) *There exist real-valued bounded functions a, b and a constant $c \in \mathbb{R}$ such that $\Theta_1\bar{\Theta} = e^{i(a+\tilde{b}+c)}$ and $\|a\|_\infty < \pi/2$.*

Proof The equivalence between the assertions (ii), (iii), (iv), (v) and (vi) follows from Theorem 12.42. To prove the equivalence between (i) and (ii), recall that, according to Lemma 14.19, on the space $\Theta(H^2)^\perp = (H^2)^\perp \oplus K_\Theta$, we have

$$\Theta_1 J T_{\Theta_1\bar{\Theta}} J\bar{\Theta} = I_{(H^2)^\perp} \oplus P_{\Theta_1|K_\Theta}, \tag{31.20}$$

where $Jg = \bar{z}\bar{g}$, $g \in L^2(\mathbb{T})$. But, since J and the multiplications by Θ, $\bar{\Theta}$ and Θ_1 act as isometry on $L^2(\mathbb{T})$, it follows that $P_{\Theta_1|K\Theta}$ is an isomorphism onto K_{Θ_1} if and only if the Toeplitz operator $T_{\Theta_1\bar{\Theta}}$ is invertible. □

The key point in the proof of Theorem 31.19 was (31.20). This equation is no longer true if the function Θ_1 is not an inner function. To go beyond the case where both functions are inner, we need to use the method of abstract functional embedding and its link with $\mathcal{H}(b)$ spaces.

31.5 The invertibility of distortion operator and Riesz bases

We use the AFE to get a criterion for the invertibility of the distortion operator. The result for the left-invertibility has already been proved in Section 31.4, but it is interesting to give another proof using the language of the AFE.

Theorem 31.20 *Let b be a function in the closed unit ball of H^∞, and let Θ be an inner function. Then the distortion operator $(I - T_bT_b^*)_{|K_\Theta} : K_\Theta \longrightarrow \mathcal{H}(b)$ is an isomorphism onto its range if and only if*

$$\operatorname{dist}(\bar{\Theta}b, H^\infty) < 1.$$

Proof Let $\Pi = (\pi, \pi_*) : L^2(\mathbb{T}) \oplus L^2(\mathbb{T}) \longrightarrow \mathbb{H}$ be an AFE such that $\pi_*^*\pi = b$. Recall that, according to Lemma 19.7, we have

$$I - T_bT_b^* = \pi_*^* P_{\mathbb{K}} \pi_{*|H^2}.$$

Moreover, by Theorem 19.8, π_*^* is a partial isometry from $\mathbb{K}$ onto $\mathcal{H}(b)$ with kernel equal to $\mathbb{K}''$. Since $P_{\mathbb{K}}\pi_* L^2(\mathbb{T}) \subset (\ker \pi^*_{*|\mathbb{K}})^\perp$, we deduce that $I - T_b T_b^* : K_\Theta \longrightarrow \mathcal{H}(b)$ is an isomorphism onto its range if and only if $P_{\mathbb{K}|\pi_* K_\Theta}$ is bounded below. Applying Lemma 1.36, the last assertion is equivalent to

$$\|P_{\mathbb{H}\ominus\mathbb{K}|\pi_* K_\Theta}\| < 1.$$

Now, $\mathbb{H} \ominus \mathbb{K} = \pi(H^2) \oplus \pi_*(H^2)^\perp$, and the second term in the orthogonal sum is orthogonal to $\pi_* K_\Theta$. Thus, the condition is equivalent to $\|P_{\pi(H^2)|\pi_* K_\Theta}\| < 1$, or, passing to the adjoint, $\|P_{\pi_* K_\Theta|\pi(H^2)}\| < 1$.

But we have

$$\|P_{\pi_* K_\Theta|\pi(H^2)}\| = \|\pi_* P_{K_\Theta} \pi_*^* \pi_{|H^2}\| = \|P_{K_\Theta} b_{|H^2}\|,$$

while, using Nehari's result (Corollary 11.4) and Corollary 14.13,

$$\|P_{K_\Theta} b_{|H^2}\| = \|\Theta P_- \Theta^* b_{|H^2}\| = \|H_{\Theta^* b}\| = \operatorname{dist}(\Theta^* b, H^\infty),$$

where H_φ denotes the Hankel operator with symbol φ. These strings of equalities prove the theorem. □

Now, we can give the result for invertibility.

Theorem 31.21 *The distortion operator $(I - T_b T_b^*)_{|K_\Theta} : K_\Theta \longrightarrow \mathcal{H}(b)$ is an isomorphism onto $\mathcal{H}(b)$ if and only if* $\operatorname{dist}(\bar{\Theta} b, H^\infty) < 1$ *and the operator*

$$\Gamma_b := (P_+ \bar{b}\Theta \quad P_+\Delta) : \begin{matrix} H^2 \\ \oplus \\ \operatorname{Clos}(\Delta H^2) \end{matrix} \longrightarrow H^2$$

is bounded below.

Proof Let $\Pi = (\pi, \pi_*) : L^2(\mathbb{T}) \oplus L^2(\mathbb{T}) \longrightarrow \mathbb{H}$ be an AFE such that $\pi_*^*\pi = b$. It follows from Theorem 19.8 that the operator π_*^* is an isometry from $\mathbb{K}'$ onto $\mathcal{H}(b)$. Therefore, using Lemma 19.7, we get

$$I - T_b T_b^* = \pi_*^* P_{\mathbb{K}'} \pi_{*|H^2}$$

and $I - T_b T_b^* : K_\Theta \longrightarrow \mathcal{H}(b)$ is an isomorphism onto $\mathcal{H}(b)$ if and only if $P_{\mathbb{K}'|\pi_* K_\Theta}$ is bounded below and surjective. According to Theorem 31.20, $P_{\mathbb{K}'|\pi_* K_\Theta}$ is bounded below if and only if $\operatorname{dist}(\bar{\Theta} b, H^\infty) < 1$. Thus, it remains to show that $P_{\mathbb{K}'|\pi_* K_\Theta}$ is surjective if and only if Γ_b is bounded below.

Now, since

$$\mathbb{K}' \oplus \pi_*(H^2_-) = \mathbb{H} \ominus [\pi(H^2) \oplus \mathbb{K}''],$$
$$\pi_*(K_\Theta) \oplus \pi_*(H^2_-) = \pi_*(L^2) \ominus \pi_*(\Theta H^2),$$

it follows that $P_{\mathbb{K}'|\pi_* K_\Theta}$ is surjective if and only if

$$P_{\mathbb{H}\ominus[\pi(H^2)\oplus\mathbb{K}'']|\pi_*(L^2)\ominus\pi_*(\Theta H^2)}$$

is surjective. Then, applying Lemma 1.36 to

$$\begin{aligned}\mathfrak{X}_1 &= \pi_*(L^2) \ominus \pi_*(\Theta H^2),\\ \mathfrak{X}_2 &= \pi_*(\Theta H^2) \oplus \pi_*(L^2)^\perp,\\ \mathfrak{Y}_1 &= \mathbb{H} \ominus [\pi(H^2) \oplus \mathbb{K}''],\\ \mathfrak{Y}_2 &= [\pi(H^2) \oplus \mathbb{K}''],\end{aligned}$$

the surjectivity of $P_{\mathfrak{Y}_1|\mathfrak{X}_1}$ is equivalent to $P_{\mathfrak{Y}_2|\mathfrak{X}_2}$ being bounded below.

Since $\mathbb{K}'' \subset \pi_*(L^2)^\perp$, this last condition is equivalent to

$$P_{\pi(H^2)|\pi_*(\Theta H^2)} \oplus [\pi_*(L^2)^\perp \ominus \mathbb{K}'']$$

being bounded below. Now, note that $P_{\pi H^2} = \pi P_+ \pi^*$ and, according to (19.3) and Lemma 19.6, we have

$$\pi_*(L^2)^\perp \ominus \mathbb{K}'' = \tau(L^2) \ominus \mathbb{K}'' = \tau(\mathrm{Clos}(\Delta L^2)) \ominus \mathbb{K}'' = \tau(\mathrm{Clos}(\Delta H^2)).$$

Therefore, $P_{\mathbb{K}'|\pi_* K_\Theta}$ is surjective if and only if

$$\pi P_+ \pi^* (\pi_*\Theta \quad \tau) : H^2 \oplus \mathrm{Clos}(\Delta H^2) \longrightarrow \mathbb{H}$$

is bounded below. But it follows from (19.1) that

$$\pi P_+ \pi^* (\pi_*\Theta \quad \tau) = \pi(P_+\pi^*\pi_*\Theta \quad P_+\pi^*\tau) = \pi(P_+\bar{b}\,\Theta \quad P_+\Delta) = \pi\Gamma_b.$$

Since π is an isometry, we obtain the desired conclusion. □

Theorem 31.21 will be used below to show that, if the distortion operator is an isomorphism onto $\mathcal{H}(b)$, then necessarily the function b is inner. This will be an important consequence for the Riesz basis, because it shows that, if b is not an inner function, then $\mathcal{H}(b)$ has neither orthonormal basis nor Riesz basis of reproducing kernels.

Corollary 31.22 *Let b be a function in the closed unit ball of H^∞, and let Θ be an inner function. Assume that the distortion operator $(I - T_b T_b^*)_{|K_\Theta} : K_\Theta \longrightarrow \mathcal{H}(b)$ is an isomorphism onto $\mathcal{H}(b)$. Then b is inner.*

Proof If the distortion operator is an isomorphism onto $\mathcal{H}(b)$, then it follows from Theorem 31.21 that Γ_b is bounded below, which implies that $(\bar{b}\,\Theta \quad \Delta)$ is bounded below as an operator from $H^2 \oplus \mathrm{Clos}(\Delta H^2)$ to $L^2(\mathbb{T})$. This means that there is a constant $c > 0$ such that, for every f in H^2 and every g in $\mathrm{Clos}(\Delta H^2)$, we have

$$\|\bar{b}\,\Theta f + \Delta g\|^2 \ge c^2(\|f\|_2^2 + \|g\|_2^2). \tag{31.21}$$

We now show that this relation implies that $(\bar{b}\,\Theta \quad \Delta)$ is bounded below as an operator from $L^2(\mathbb{T}) \oplus \mathrm{Clos}(\Delta L^2(\mathbb{T}))$ to $L^2(\mathbb{T})$. Hence, let $f, g \in L^2(\mathbb{T})$ be such that $\|f\|_2^2 + \|\Delta g\|_2^2 = 1$, and let $\varepsilon > 0$. Then there are $f_1, g_1 \in H^2$ and an integer N sufficiently large such that

$$\|z^N f - f_1\|_2 \le \varepsilon, \quad \|z^N g - g_1\|_2 \le \varepsilon. \tag{31.22}$$

Thus, we have

$$\begin{aligned}
&\|(\bar{b}\,\Theta \quad \Delta)(f \oplus \Delta g)\|_2 \\
&\quad = \|\bar{b}\,\Theta f + \Delta^2 g\|_2 \\
&\quad = \|\bar{b}\,\Theta(z^N f - f_1) + \Delta^2(z^N g - g_1) + \bar{b}\,\Theta f_1 + \Delta^2 g_1\|_2 \\
&\quad \le \|\bar{b}\,\Theta f_1 + \Delta^2 g_1\|_2 - \|\bar{b}\,\Theta(z^N f - f_1)\|_2 - \|\Delta^2(z^N g - g_1)\|_2,
\end{aligned}$$

and, by (31.21) and (31.22), we get

$$\|(\bar{b}\,\Theta \quad \Delta)(f \oplus \Delta g)\|_2 \ge c(\|f_1\|_2^2 + \|\Delta g_1\|_2^2)^{1/2} - 2\varepsilon.$$

Using the relation $|a| + |b| \le \sqrt{2}(|a|^2 + |b|^2)^{1/2}$, $a, b \in \mathbb{R}$, we then get

$$\|(\bar{b}\,\Theta \quad \Delta)(f \oplus \Delta g)\|_2 \ge \frac{c}{\sqrt{2}}(\|f_1\|_2 + \|\Delta g_1\|_2) - 2\varepsilon.$$

Using (31.22) once more, we have

$$\|f_1\|_2 = \|f_1 - z^N f + z^N f\|_2 \ge \|z^N f\|_2 - \|f_1 - z^N f\|_2 \ge \|f\|_2 - \varepsilon$$

and

$$\begin{aligned}
\|\Delta g_1\|_2 &= \|\Delta g_1 - \Delta z^N g + \Delta z^N g\|_2 \\
&\ge \|\Delta z^N g\|_2 - \|\Delta(g_1 - z^N g)\|_2 \ge \|\Delta g\|_2 - \varepsilon,
\end{aligned}$$

and then

$$\|f_1\|_2 + \|\Delta g_1\|_2 \ge \|f\|_2 + \|\Delta g\|_2 - 2\varepsilon \ge 1 - 2\varepsilon.$$

Hence,

$$\|(\bar{b}\,\Theta \quad \Delta)(f \oplus \Delta g)\|_2 \ge \frac{c}{\sqrt{2}} - \varepsilon\left(\frac{2c}{\sqrt{2}} + 1\right),$$

and if we take $\varepsilon > 0$ sufficiently small, we get that

$$\delta = \frac{c}{\sqrt{2}} - \varepsilon\left(\frac{2c}{\sqrt{2}} + 1\right) > 0,$$

which proves that

$$\|(\bar{b}\,\Theta \quad \Delta)(f \oplus \Delta g)\|_2 \ge \delta,$$

for every $f, g \in L^2(\mathbb{T})$ such that $\|f \oplus \Delta g\|_2 = 1$. Therefore, this proves that $(\bar{b}\,\Theta \quad \Delta)$ is bounded below as an operator from $L^2(\mathbb{T}) \oplus \mathrm{Clos}(\Delta L^2(\mathbb{T}))$ to $L^2(\mathbb{T})$. But, since Θ is inner, the multiplication by Θ is a unitary operator on $L^2(\mathbb{T})$ and we obtain that $(\bar{b} \quad \Delta)$ is bounded below as an operator from $L^2(\mathbb{T}) \oplus \mathrm{Clos}(\Delta L^2(\mathbb{T}))$ to $L^2(\mathbb{T})$. Note that the adjoint of this operator is an isometry. Hence, this operator must be a unitary operator. In particular, the multiplication by $\bar{b}$ on $L^2(\mathbb{T})$ must be an isometry. But this implies that b must be an inner function, as desired. □

We can also recover Theorem 31.19.

Corollary 31.23 *Let $b = \Theta_1$ and Θ be two inner functions. Then the distortion operator $P_{\Theta_1|K_\Theta} : K_\Theta \longrightarrow K_{\Theta_1}$ is an isomorphism onto K_{Θ_1} if and only if* $\operatorname{dist}(\Theta\bar{\Theta}_1, H^\infty) < 1$ *and* $\operatorname{dist}(\bar{\Theta}\Theta_1, H^\infty) < 1$.

Proof If $b = \Theta_1$ is an inner function, then $\Delta = 0$ and the operator Γ_b is just the operator $P_+\bar{\Theta}_1\Theta : H^2 \longrightarrow H^2$. In other words, $\Gamma_b = T_{\bar{\Theta}_1\Theta}$. Now, Theorem 31.21 implies that the distortion operator $P_{\Theta_1|K_\Theta} : K_\Theta \longrightarrow K_{\Theta_1}$ is an isomorphism onto K_{Θ_1} if and only if $\operatorname{dist}(\Theta\bar{\Theta}_1, H^\infty) < 1$ and $T_{\bar{\Theta}_1\Theta}$ is bounded below. The conclusion now follows from Lemma 12.11. □

Theorem 31.24 *Let b be a function in the closed unit ball of H^∞, let $(\lambda_n)_{n\geq 1}$ be a Blaschke sequence of distinct points of the unit disk $\mathbb{D}$, and let B be the associated Blaschke product. Assume that*

$$\sup_{n\geq 1} |b(\lambda_n)| < 1.$$

Then the following are equivalent.

(i) *$(\tilde{k}^b_{\lambda_n})_{n\geq 1}$ forms a Riesz sequence in $\mathcal{H}(b)$.*
(ii) *$(\lambda_n)_{n\geq 1} \in (C)$ and* $\operatorname{dist}(\bar{B}b, H^\infty) < 1$.

Proof It suffices to combine Theorems 31.17 and 31.18. □

The following result can be viewed as an analog of Corollary 31.13 and shows that, if b is not an inner function, then there cannot exist a Riesz basis of reproducing kernels of the entire space $\mathcal{H}(b)$.

Theorem 31.25 *Let b be a function in the closed unit ball of H^∞, and let $\Lambda = (\lambda_n)_{n\geq 1}$ be a Blaschke sequence of distinct points of the unit disk $\mathbb{D}$. Assume that b is not an inner function and*

$$\sup_{n\geq 1} |b(\lambda_n)| < 1.$$

If the sequence $(\tilde{k}^b_{\lambda_n})_{n\geq 1}$ forms a Riesz sequence of $\mathcal{H}(b)$, then we have

$$\dim(\mathcal{H}(b) \ominus \operatorname{Span}(\tilde{k}^b_{\lambda_n} : n \geq 1)) = \infty.$$

Proof Arguing by absurdity, assume that

$$N = \dim(\mathcal{H}(b) \ominus \operatorname{Span}(\tilde{k}^b_{\lambda_n} : n \geq 1)) < \infty.$$

Then choose N distinct points $\mu_1, \mu_2, \dots, \mu_N$ in $\mathbb{D}$ such that $\mu_i \notin \Lambda$ and set $\Lambda' = \Lambda \cup \{\mu_i : 1 \le i \le N\}$. According to Corollary 31.3, the family $(\tilde{k}^b_\lambda)_{\lambda \in \Lambda'}$ is a Riesz basis of $\mathcal{H}(b)$. Note that we obviously still have

$$\sup_{\lambda \in \Lambda'} |\Theta(\lambda)| < 1,$$

and then Theorem 31.17 implies that the distortion operator $(I - T_b T_{\bar{b}})_{|\mathcal{H}(B_{\Lambda'})}$ is an isomorphism from $\mathcal{H}(B_{\Lambda'})$ onto $\mathcal{H}(b)$, where $B_{\Lambda'}$ is the Blaschke product associated with Λ'. Now, Corollary 31.22 reveals that there is a contradiction because b is not an inner function. □

In the case where both functions b and Θ are inner, we can give a criterion for the Riesz basis property.

Theorem 31.26 *Let Θ_1 be an inner function, let $(\lambda_n)_{n\ge1}$ be a Blaschke sequence of distinct points of the unit disk $\mathbb{D}$, and let B be the associated Blaschke product. Assume that*

$$\sup_{n\ge1} |\Theta_1(\lambda_n)| < 1.$$

Then the following are equivalent.

(i) $(\tilde{k}^{\Theta_1}_{\lambda_n})_{n\ge1}$ *forms a Riesz basis of* $\mathcal{H}(b)$.
(ii) $(\lambda_n)_{n\ge1} \in (C)$ *and the Toeplitz operator* $T_{\Theta_1 \bar{B}}$ *is invertible.*
(iii) $(\lambda_n)_{n\ge1} \in (C)$, $\operatorname{dist}(\Theta_1 \bar{B}, H^\infty) < 1$ *and* $\operatorname{dist}(B\bar{\Theta}_1, H^\infty) < 1$.
(iv) $(\lambda_n)_{n\ge1} \in (C)$ *and there exists an outer function* $h \in H^\infty$ *such that* $\|\Theta_1 \bar{B} - h\|_\infty < 1$.
(v) $(\lambda_n)_{n\ge1} \in (C)$ *and there exist real-valued bounded functions* γ, τ *and a constant* $c \in \mathbb{R}$ *such that* $\Theta_1 \bar{B} = e^{i(\gamma + \tilde{\tau} + c)}$ *and* $\|a\|_\infty < \pi/2$.

Proof This is a straightforward consequence of Theorem 31.17 and Theorem 31.19. □

It follows from the definition of the spectrum $\sigma(b)$ that the condition (31.12) means that the points λ_n, $n \ge 1$, are *not far* from the spectrum. In any case, the relation $\lim_{n\to\infty} b(\lambda_n) = 0$ implies that $\lim_{n\to\infty} \operatorname{dist}(\lambda_n, \sigma(b)) = 0$. As we have already proved in Lemma 31.1, if $(k^b_{\lambda_n})_{n\ge1}$ is minimal in $\mathcal{H}(b)$ (and, *a fortiori*, if $(\tilde{k}^b_{\lambda_n})_{n\ge1}$ is a Riesz sequence in $\mathcal{H}(b)$), then we have

$$\lim_{n\to\infty} \operatorname{dist}(\lambda_n, \sigma(b)) = 0.$$

On the other hand, the basis property of $(\tilde{k}^b_{\lambda_n})_{n\ge1}$ does not imply that $\lim_{n\to\infty} b(\lambda_n) = 0$. We will produce later an example where we even have $\lim_{n\to\infty} b(\lambda_n) = 1$. The following result shows that, in a certain sense, condition (31.12) is necessary for our method, which is based on projecting a basis from $\mathcal{H}(B)$.

Lemma 31.27 *Let b be a function in the closed unit ball of H^∞, let $(\lambda_n)_{n\geq 1}$ be a Blaschke sequence of distinct points of the unit disk $\mathbb{D}$, and let B be the associated Blaschke product. Assume that $(I - T_b T_{\bar{b}})_{|\mathcal{H}(B)}$ is an isomorphism onto its range. Then we have*

$$\sup_{n\geq 1} |b(\lambda_n)| < 1.$$

Proof Using Theorem 31.18, we have $\operatorname{dist}(\bar{B}b, H^\infty) < 1$, whence there is a function $h \in H^\infty$ such that $\|\bar{B}b - h\|_\infty < 1$. In particular, for every $n \geq 1$, we have

$$|b(\lambda_n)| = |b(\lambda_n) - B(\lambda_n)h(\lambda_n)| \leq \|b - Bh\|_\infty = \|\bar{B}b - h\|_\infty < 1. \qquad \square$$

Before giving a sufficient condition for the left-invertibility of the distortion operator, we state a useful estimate concerning the norm of $P_B T_{b|K_B}$.

Lemma 31.28 *Let $(\lambda_n)_{n\geq 1}$ be a Blaschke sequence of distinct points in $\mathbb{D}$, let B be the corresponding Blaschke product, and let b be a function in the closed unit ball of H^∞. Suppose that $(\tilde{k}_{\lambda_n})_{n\geq 1}$ is a Riesz basis of K_B and denote its corresponding constants by c and C. Then*

$$\|P_B T_{b|K_B}\| \leq \sqrt{\frac{C}{c}} \sup_{n\geq 1} |b(\lambda_n)|.$$

Proof The subspace K_B is spanned by the eigenvectors $\tilde{k}_{\lambda_n}$, $n \geq 1$, of T_b^*, and $T_b^* \tilde{k}_{\lambda_n} = \overline{b(\lambda_n)} \tilde{k}_{\lambda_n}$. Take any sum (with a finite number of nonzero terms) $\sum_{n\geq 1} a_n \tilde{k}_{\lambda_n}$. Then we have

$$\left\| \sum_{n\geq 1} a_n \tilde{k}_{\lambda_n} \right\|^2 \geq c \sum_{n\geq 1} |a_n|^2$$

and

$$\begin{aligned} \left\| T_b^* \left(\sum_{n\geq 1} a_n \tilde{k}_{\lambda_n} \right) \right\|^2 &= \left\| \sum_{n\geq 1} \overline{b(\lambda_n)} a_n \tilde{k}_{\lambda_n} \right\|^2 \\ &\leq C \sum_{n\geq 1} |b(\lambda_n)|^2 |a_n|^2 \\ &\leq C \left(\sup_{n\geq 1} |b(\lambda_n)| \right)^2 \sum_{n\geq 1} |a_n|^2, \end{aligned}$$

whence

$$\left\| T_b^* \left(\sum_{n\geq 1} a_n \tilde{k}_{\lambda_n} \right) \right\|^2 \leq \frac{C}{c} \sup_{n\geq 1} |b(\lambda_n)|^2 \left\| \sum_{n\geq 1} a_n \tilde{k}_{\lambda_n} \right\|^2.$$

Since $(P_B T_{b|K_B})^* = T^*_{b\,|K_B}$, the lemma is proved. $\square$

Theorem 31.29 *Let b be a function in the closed unit ball of H^∞, let $(\lambda_n)_{n\geq 1}$ be a Blaschke sequence of distinct points of the unit disk $\mathbb{D}$ that satisfies the Carleson condition, and let B be the associated Blaschke product. Assume that*

$$\|V\|\,\|V^{-1}\|\sup_{n\geq 1}|b(\lambda_n)|<1,$$

where V is an orthogonalizer of the family $(\tilde{k}_{\lambda_n})$. Then the distortion operator $(I-T_bT_{\bar{b}})_{|\mathcal{H}(B)}$ is an isomorphism onto its range and then $(\tilde{k}^b_{\lambda_n})_{n\geq 1}$ is a Riesz sequence in $\mathcal{H}(b)$.

Proof By definition of the orthogonalizer, we have

$$\|V\|^{-2}\sum_n |a_n|^2 \leq \left\|\sum_n a_n\tilde{k}_{\lambda_n}\right\|^2 \leq \|V^{-1}\|^2\sum_n |a_n|^2,$$

for every finite complex sequence $(a_n)_{n\geq 1}$. Therefore, applying Lemma 31.28 gives

$$\|P_BT_{b|K_B}\| \leq \|V\|\,\|V^{-1}\|\sup_{n\geq 1}|b(\lambda_n)|<1.$$

Now, Theorem 31.18 implies that the distortion operator $(I-T_bT_{\bar{b}})_{|\mathcal{H}(B)}$ is an isomorphism onto its range. To conclude that $(\tilde{k}^b_{\lambda_n})_{n\geq 1}$ is a Riesz sequence in $\mathcal{H}(b)$, it remains to apply Lemma 31.27 and Theorem 31.17. □

This result shows that, if we increase the space, the Carleson condition becomes necessary and sufficient.

Corollary 31.30 *Let b be a function in the closed unit ball of H^∞, and let $(\lambda_n)_{n\geq 1}$ be a Blaschke sequence of distinct points of $\mathbb{D}$ satisfying*

$$\sup_{n\geq 1}|b(\lambda_n)|<1.$$

Then the following statements are equivalent.

(i) *There exists $p\in\mathbb{N}$ sufficiently large such that $(\tilde{k}^{b^p}_{\lambda_n})_{n\geq 1}$ forms a Riesz sequence in $\mathcal{H}(b^p)$.*
(ii) $(\lambda_n)_{n\geq 1}\in(C)$.

Proof The implication (i) $\Longrightarrow$ (ii) follows from Theorem 31.16.

For the reverse implication, let $\delta=\sup_{n\geq 1}|b(\lambda_n)|$ and let V be an orthogonalizer of the family $(\tilde{k}_{\lambda_n})$ (this orthogonalizer exists because $(\lambda_n)_n$ satisfies the Carleson condition and then, by Theorem 15.7, we know that $(\tilde{k}_{\lambda_n})_n$ is a Riesz sequence). Since

$$\|V\|\,\|V^{-1}\|\,|b^p(\lambda_n)|\leq \delta^p\|V\|\,\|V^{-1}\|,$$

and $\delta<1$, we can choose p sufficiently large such that the right term in the last inequality becomes strictly less than 1. Then it remains to apply Theorem 31.29 to deduce that $(\tilde{k}^{b^p}_{\lambda_n})_{n\geq 1}$ forms a Riesz sequence in $\mathcal{H}(b^p)$. □

Corollary 31.31 *Let b be a function in the closed unit ball of H^∞, and let $(\lambda_n)_{n\geq 1}$ be a Blaschke sequence of distinct points of $\mathbb{D}$ satisfying*

$$\lim_{n\to\infty} b(\lambda_n) = 0.$$

Then the following statements are equivalent.

(i) *There exists $N \in \mathbb{N}$ sufficiently large such that $(\tilde{k}^b_{\lambda_n})_{n\geq N}$ forms a Riesz sequence in $\mathcal{H}(b)$.*

(ii) $(\lambda_n)_{n\geq 1} \in (C)$.

Proof We argue as in the proof of Corollary 31.30. The implication (i) $\Longrightarrow$ (ii) follows from Theorem 31.16. For the inverse implication, let V be an orthogonalizer of the family $(\tilde{k}_{\lambda_n})$. Then we can find an integer N sufficiently large to have

$$\|V\|\,\|V^{-1}\| \sup_{n\geq N} |b(\lambda_n)| < 1.$$

Now, it remains to apply Theorem 31.29 to get that $(\tilde{k}^b_{\lambda_n})_{n\geq N}$ forms a Riesz sequence in $\mathcal{H}(b)$. □

Under the condition $\lim_{n\to\infty} |b(\lambda_n)| = 0$, it turns out that uniform minimality and the property of being a Riesz sequence are equivalent.

Corollary 31.32 *Let b be a function in the closed unit ball of H^∞, and let $(\lambda_n)_{n\geq 1}$ be a Blaschke sequence of distinct points of $\mathbb{D}$ satisfying*

$$\lim_{n\to\infty} b(\lambda_n) = 0.$$

Then the following statements are equivalent.

(i) $(\tilde{k}^b_{\lambda_n})_{n\geq 1}$ *forms a Riesz sequence.*

(ii) $(\tilde{k}^b_{\lambda_n})_{n\geq 1}$ *is uniformly minimal.*

Proof (i) $\Longrightarrow$ (ii) This implication is always true.

(ii) $\Longrightarrow$ (i) Since $\lim_{n\to\infty} |b(\lambda_n)| = 0$, we have $\sup_{n\geq 1} |b(\lambda_n)| < 1$, which, by Lemma 31.15, gives $\|k^b_{\lambda_n}\|_b \asymp \|k_{\lambda_n}\|_2$. Then, using Lemma 10.2 and the uniform minimality of $(k^b_{\lambda_n})_{n\geq 1}$, we see that $(k_{\lambda_n})_{n\geq 1}$ is uniformly minimal and thus, by Theorem 15.7, $(\lambda_n)_{n\geq 1} \in (C)$. Now, thanks to Corollary 31.31, there exists $N \in \mathbb{N}$ such that $(\tilde{k}^b_{\lambda_n})_{n\geq N}$ forms a Riesz sequence. Using the minimality of the whole family, we can add the missing terms, still keeping the property of Riesz sequence (see Corollary 10.23). □

In the case where b is not a Blaschke product, the result of Corollary 31.31 can be improved.

Theorem 31.33 *Let b be a point in the closed unit ball of H^∞, and let $(\lambda_n)_{n\geq 1}$ be a Blaschke sequence of distinct points in $\mathbb{D}$. Assume that b is not a Blaschke product and*

$$\lim_{n\to\infty} b(\lambda_n) = 0.$$

Then the following statements are equivalent.

(i) $(\tilde{k}^b_{\lambda_n})_{n\geq 1}$ *forms a Riesz sequence.*
(ii) $(\tilde{k}^b_{\lambda_n})_{n\geq 1}$ *is uniformly minimal.*
(iii) $(\lambda_n)_{n\geq 1} \in (C)$.

Moreover, in this case, we have $\dim(\mathcal{H}(b) \ominus \mathrm{Span}(\tilde{k}^b_{\lambda_n} : n \geq 1\}) = \infty$.

Proof The equivalence of (i) and (ii) is contained in Corollary 31.32. The implication (i) $\Longrightarrow$ (iii) is always true and follows from Theorem 31.16.

It remains to prove the implication (iii) $\Longrightarrow$ (i). We distinguish two cases.

Case I. Assume that b is not an inner function.

Then, by Corollary 31.31, we see that there exists an integer $N \geq 1$ such that $(\tilde{k}^b_{\lambda_n})_{n\geq N}$ is a Riesz sequence in $\mathcal{H}(b)$. Since b is not an inner function, we apply Theorem 31.25 and get

$$\dim(\mathcal{H}(b) \ominus \mathrm{Span}(\tilde{k}^b_{\lambda_n} : n \geq N)) = \infty.$$

Now, by Corollary 31.3, we conclude that $(\tilde{k}^b_{\lambda_n})_{n\geq 1}$ is still a Riesz sequence, which proves the result in this case.

Case II. Suppose that $b = \Theta$ is an inner function, but not a Blaschke product.

Write $\Theta = B_1 I_\mu$, where B_1 is a Blaschke product and I_μ is a singular inner function and set, for $0 < \alpha < 1$, $\Theta_\alpha = B_1 I_\mu^\alpha$. Then we have

$$\lim_{n\to\infty} \Theta_\alpha(\lambda_n) = 0$$

and, applying Corollary 31.31 to Θ, we see that there exists an integer $N \geq 1$ such that $(\tilde{k}^\Theta_{\lambda_n})_{n\geq N}$ is a Riesz sequence in K_Θ. Applying the same result to Θ_α shows that there exists an integer $N \geq 1$ such that $(\tilde{k}^{\Theta_\alpha}_{\lambda_n})_{n\geq N}$ is a Riesz sequence in K_{Θ_α}. Moreover, using the fact that $\lim_{n\to\infty} \Theta(\lambda_n) = \lim_{n\to\infty} \Theta^\alpha(\lambda_n) = 0$, we see that $\|k^\Theta_{\lambda_n}\|_2^2 \asymp (1-|\lambda_n|^2)^{-1}$ and $\|k^{\Theta_\alpha}_{\lambda_n}\|_2^2 \asymp (1-|\lambda_n|^2)^{-1}$, whence

$$\|k^\Theta_{\lambda_n}\|_2 \asymp \|k^{\Theta_\alpha}_{\lambda_n}\|_2.$$

On the other hand, since $K_{\Theta_\alpha} \subset K_\Theta$, we have $k^{\Theta_\alpha}_{\lambda_n} = P_{\Theta_\alpha} k^\Theta_{\lambda_n}$. Therefore, $P_{\Theta_\alpha|K_\Theta}$ maps the Riesz sequence $(\tilde{k}^\Theta_{\lambda_n})_{n\geq N}$ into another Riesz sequence, and then we appeal to Lemma 10.8 to get

$$\dim(K_\Theta \ominus \mathrm{Span}(\tilde{k}^\Theta_{\lambda_n} : n \geq N)) \geq \dim\ker(P_{\Theta_\alpha|K_\Theta}).$$

Now, Corollary 18.10 implies that $\ker(P_{\Theta_\alpha | K_\Theta}) = \Theta^\alpha K_{\Theta^{1-\alpha}}$ and thus we get

$$\dim(K_\Theta \ominus \operatorname{Span}(\tilde{k}_{\lambda_n}^\Theta : n \geq N)) = \infty.$$

Finally, by Corollary 31.3, we conclude that $(\tilde{k}_{\lambda_n}^b)_{n\geq 1}$ is a Riesz sequence. $\square$

It is worth mentioning that, for a Blaschke product b, this property fails. Indeed, let b be a Blaschke product whose zero set $(\lambda_n)_{n\geq 2}$ satisfies the Carleson condition (C). Then $(\tilde{k}_{\lambda_n}^b)_{n\geq 2} = (\tilde{k}_{\lambda_n})_{n\geq 2}$ is a Riesz basis of $\mathcal{H}(b)$. Adding an arbitrary point $\lambda_1 \in \mathbb{D} \setminus \{\lambda_n : n \geq 2\}$, then we have $\lim_{n\to\infty} b(\lambda_n) = 0$ and $(\lambda_n)_{n\geq 1} \in (C)$, but since we have $\tilde{k}_{\lambda_1}^b \in \mathcal{H}(b) = \operatorname{Span}(\tilde{k}_{\lambda_n}^b : n \geq 2)$, the sequence $(\tilde{k}_{\lambda_n}^b)_{n\geq 1}$ is even not minimal.

For the next result, we recall that, given $\mathfrak{X} = (x_n)_{n\geq 1}$ a complete and minimal sequence in a Banach space $\mathcal{X}$, a sequence $\mu = (\mu_n)_{n\geq 1}$ of complex numbers is a multiplier of $(x_n)_{n\geq 1}$ if there exists an operator $M_\mu \in \mathcal{L}(\mathcal{X})$ such that $M_\mu(x_n) = \mu_n x_n$, $n \geq 1$. In this case, we write $\mu \in \mathfrak{Mult}(\mathfrak{X})$; see Section 10.3.

Theorem 31.34 *Let b be a nonextreme point of the closed unit ball of H^∞. Assume that $(a,b) \in (HCR)$, and that $T_{a/\bar{a}}$ is invertible. Let $\Lambda = (\Lambda_n)_{n\geq 1}$ be a Blaschke sequence of distinct points in $\mathbb{D}$. Then the following assertions are equivalent.*

(i) *The sequence $(\tilde{k}_{\lambda_n}^b)_{n\geq 1}$ is a Riesz sequence in $\mathcal{H}(b)$.*
(ii) *$(\lambda_n)_{n\geq 1} \in (C)$.*

Proof (i) $\Longrightarrow$ (ii) This implication follows from Theorem 31.16.

(ii) $\Longrightarrow$ (i) By Theorem 10.14, $(k_{\lambda_n}^b)_{n\geq 1}$ forms an unconditionnal basis in its closed linear span if and only if $\mathfrak{Mult}((k_{\lambda_n}^b)_{n\geq 1}) = \ell^\infty$. To prove this set equality, note that the inclusion $\mathfrak{Mult}((k_{\lambda_n}^b)_{n\geq 1}) \subset \ell^\infty$ is always true. For the other inclusion, fix any sequence $(\mu_n)_{n\geq 1} \in \ell^\infty$. Since $(\lambda_n)_{n\geq 1} \in (C)$, according to Theorem 15.11, there exists $f \in H^\infty$ such that $f(\lambda_n) = \overline{\mu_n}$, $n \geq 1$. Now, Theorem 28.23 implies that f must be a multiplier of $\mathcal{H}(b)$, which implies that

$$M_f^* k_{\lambda_n}^b = \overline{f(\lambda_n)} k_{\lambda_n}^b = \mu_n k_{\lambda_n}^b.$$

Hence, $(\mu_n)_{n\geq 1} \in \mathfrak{Mult}((k_{\lambda_n}^b)_{n\geq 1})$. $\square$

Lemma 31.35 *Let b be in the closed unit ball of H^∞, and let $(\lambda_n)_{n\geq 1}$ be a sequence of points in $\mathbb{D}$. Assume that $(\tilde{k}_{\lambda_n}^b)_{n\geq 1}$ is a Riesz sequence in $\mathcal{H}(b)$. Then*

$$\sum_{n\geq 1} \frac{1 - |\lambda_n|^2}{1 - |b(\lambda_n)|^2} < \infty.$$

Proof Since $(\tilde{k}^b_{\lambda_n})_{n\geq 1}$ is a Riesz sequence in $\mathcal{H}(b)$, there exists a constant $C > 0$ such that

$$\sum_{n\geq 1} |\langle f, \tilde{k}^b_{\lambda_n}\rangle_b|^2 \leq C\,\|f\|_b^2 \qquad (f \in \mathcal{H}(b)).$$

Hence,

$$\sum_{n\geq 1} \frac{1-|\lambda_n|^2}{1-|b(\lambda_n)|^2}\,|f(\lambda_n)|^2 \leq C\,\|f\|_b^2.$$

Applying this inequality to $f = k_0^b = 1-\overline{b(0)}b$ and using the fact that $|f(z)| \geq 1-|b(0)| > 0$, for every $z \in \mathbb{D}$, we obtain the result. □

Theorem 31.36 *Let b be in the closed unit ball of H^∞, let $(\lambda_n)_{n\geq 1}$ be a Blaschke sequence of points in $\mathbb{D}$, and let B be the associated Blaschke product. Then the following assertions are equivalent.*

(i) *The sequence $(\tilde{k}^b_{\lambda_n})_{n\geq 1}$ is a Riesz sequence (respectively, Riesz basis) of $\mathcal{H}(b)$.*
(ii) *$(\lambda_n)_{n\geq 1} \in (C)$ and there exists a function $G \in H^2$ such that*

$$|G(\lambda_n)| \asymp (1-|b(\lambda_n)|^2)^{1/2}$$

and the operator

$$(I - T_bT_{\bar{b}})T_{\bar{G}} : K_B \longrightarrow \mathcal{H}(b)$$

can be extended up to an isomorphism from K_B onto its range (respectively, onto $\mathcal{H}(b)$).

Proof (i) $\Longrightarrow$ (ii) According to Theorem 31.16, $(\lambda_n)_{n\geq 1} \in (C)$. It follows from Lemma 31.35 that

$$\sum_{n\geq 1} \frac{1-|\lambda_n|^2}{1-|b(\lambda_n)|^2} < \infty,$$

and, using the fact that $(\tilde{k}_{\lambda_n})_{n\geq 1}$ is a Riesz sequence, we can find a function $G \in H^2$ such that $G(\lambda_n) = (1-|b(\lambda_n)|^2)^{-1/2}$, for every $n \geq 1$. Then

$$(I - T_bT_{\bar{b}})T_{\bar{G}}k_{\lambda_n} = (I - T_bT_{\bar{b}})(\overline{G(\lambda_n)}k_{\lambda_n}) = \overline{G(\lambda_n)}k^b_{\lambda_n}$$

and

$$(I - T_bT_{\bar{b}})T_{\bar{G}}\tilde{k}_{\lambda_n} = c_n\tilde{k}^b_{\lambda_n}, \tag{31.23}$$

where $c_n = \overline{G(\lambda_n)}\,\|k^b_{\lambda_n}\|_b\|k_{\lambda_n}\|_2^{-1}$. Since $|c_n| = 1$, for every $n \geq 1$, the operator $(I - T_bT_{\bar{b}})T_{\bar{G}|K_B}$ maps a Riesz basis of K_B into a Riesz sequence (or basis) of $\mathcal{H}(b)$. Therefore, this operator has to be an isomorphism onto its range (or onto $\mathcal{H}(b)$).

(ii) $\Longrightarrow$ (i) Since $(\lambda_n)_{n\geq 1} \in (C)$, we get from Theorem 15.7 that $(\tilde{k}_{\lambda_n})_{n\geq 1}$ is a Riesz basis of K_B. Hence, under the operator $(I - T_b T_{\bar{b}})T_{\bar{G}|K_B}$ it is mapped into a Riesz sequence (or basis if this operator is onto). The conclusion follows from (31.23) (Note that, since $|G(\lambda_n)| \asymp (1 - |b(\lambda_n)|^2)^{1/2}$, we have $|c_n| \asymp 1$). $\square$

Exercises

Exercise 31.5.1 Let $\Lambda = (\lambda_n)_{n\geq 1}$ be a Blaschke sequence of points in $\mathbb{D}$, let B be the associated Blaschke product and assume that $\Lambda \in (C)$. Let b be a function in the closed unit ball of H^∞. Assume that $\|b - B\|_\infty < 1$.

(i) Show that $(\tilde{k}^b_{\lambda_n})_{n\geq 1}$ is a Riesz sequence in $\mathcal{H}(b)$.
(ii) Assume further that $b = \Theta$ is an inner function. Show that $(\tilde{k}^\Theta_{\lambda_n})_{n\geq 1}$ is a Riesz basis in K_Θ.

Exercise 31.5.2 Let $a > 0$ and let $(\mu_n)_{n\geq 1}$ be a sequence of complex numbers satisfying $\lim_{n\to\infty} \Im \mu_n = \infty$. Show that the following are equivalent.

(i) The family $(e^{i\mu_n t}\chi_{(0,a)})_{n\geq 1}$ is a Riesz sequence in $L^2(0, a)$.
(ii) The sequence $(\mu_n + i\alpha)_{n\geq 1} \in (C)$ for a suitable $\alpha > 0$.

Exercise 31.5.3 Let $(\mu_n)_{n\geq 1} \subset \mathbb{C}$ be such that $\inf_{n\geq 1} \Im \mu_n > -\alpha > -\infty$. Show that there is $\tau > 0$ such that $(e^{i\mu_n t}\chi_{(0,\tau)})_{n\geq 1}$ is a Riesz sequence in $L^2(0, \tau)$ if and only if $(\mu_n + i\alpha)_{n\geq 1} \in (C)$.

Exercise 31.5.4 Let $(\mu_n)_{n\geq 1} \subset \mathbb{C}$ be a Blaschke sequence in $\mathbb{C}$ such that $\inf_{n\geq 1} \Im \mu_n > -\alpha > -\infty$, let B be the Blaschke product associated with $(\mu_n + i\alpha)_{n\geq 1}$ and let $a > 0$. Show that the following assertions are equivalent.

(i) The system $(e^{i\mu_n t}\chi_{(0,a)})_{n\geq 1}$ is a Riesz basis.
(ii) $(\mu_n + i\alpha)_{n\geq 1} \in (C)$, $\mathrm{dist}(e^{iax}\bar{B}, H^\infty_+) < 1$ and $\mathrm{dist}(Be^{-iax}, H^\infty_+) < 1$, where $H^\infty_+ = H^2(\mathbb{C}_+) \cap L^\infty(\mathbb{R})$.
(iii) $(\mu_n + i\alpha)_{n\geq 1} \in (C)$ and there exist a real-valued function $b \in L^\infty(\mathbb{R})$ and a constant $c \in \mathbb{R}$ such that

$$\|ax - \arg B(x) - \tilde{b} - c\|_\infty < \pi/2.$$

Exercise 31.5.5 Let $\mu_n \in \mathbb{R}$ satisfy $|\mu_n - n| \leq \delta < 1/4$, $n \in \mathbb{Z}$. Then $(e^{i\mu_n t})_{n\in\mathbb{Z}}$ is a Riesz basis in $L^2(-\pi, \pi)$.

(i) Let $\alpha > 0$ and let us consider B_α, the Blaschke product associated with $\mathbb{Z} + i\alpha$. Then prove that

$$B_\alpha(x) = \frac{\sin(\pi(x - i\alpha))}{\sin(\pi(x + \alpha))} \qquad (x \in \mathbb{R})$$

and

$$\arg B_\alpha(x) = 2\pi x + \pi + c_\alpha(x) \qquad (x \in \mathbb{R}),$$

where $|c_\alpha(x)| \le Ce^{-\pi\alpha}$, for every $x \in \mathbb{R}$.

(ii) Now let $\mu_n = n + \delta_n$, $|\delta_n| \le \delta$, $n \in \mathbb{Z}$. Denote by $B_\alpha^{(1)}$ the Blaschke product associated with $(\mu_n + i\alpha)_{n\in\mathbb{Z}}$. Show that there is a constant $C > 0$ such that

$$\|2\pi x - \arg B_\alpha^{(1)} - c\|_\infty \le Ce^{-\pi\alpha} + 2\pi\delta.$$

(iii) Conclude that $(e^{i\mu_n t})_{n\in\mathbb{Z}}$ is a Riesz basis in $L^2(-\pi,\pi)$.

31.6 Riesz sequences in $H^2(\mu)$ and in $\mathcal{H}(\bar{b})$

Recall that, if μ is the Clark measure associated with a function b in the closed unit ball of H^∞, and if $\mathbf{V}_b$ is the linear map defined by $\mathbf{V}_b f(z) = (1 - b(z))K_\mu f(z)$, $f \in L^2(\mu)$, then V_b is an isometry from $H^2(\mu)$ onto $\mathcal{H}(b)$ and $V_b k_\lambda = (1-\overline{b(\lambda)})^{-1} k_\lambda^b$ (see Theorem 20.5). Hence, we have

$$V_b\left(\frac{k_\lambda}{\|k_\lambda\|_{L^2(\mu)}}\right) = (1-\overline{b(\lambda)})^{-1}\frac{\|k_\lambda^b\|_b}{\|k_\lambda\|_{L^2(\mu)}}\,\tilde{k}_\lambda^b = \alpha_\lambda \tilde{k}_\lambda^b,$$

with $\alpha_\lambda \in \mathbb{T}$. Therefore, we can translate all the results we have obtained for sequences of reproducing kernels in $\mathcal{H}(b)$ into results for sequences of Cauchy kernels in $H^2(\mu)$.

In particular, using Theorems 31.24 and 31.25, we immediately obtain the following result.

Theorem 31.37 *Let $(\lambda_n)_{n\ge1} \subset \mathbb{D}$, and let μ be a positive Borel measure on $\mathbb{T}$. Let $b \in H^\infty$, $\|b\|_\infty \le 1$, be such that*

$$\frac{1-|b(z)|^2}{|1-b(z)|^2} = \int_{\mathbb{T}} \frac{1-|z|^2}{|1-ze^{-i\theta}|^2}\, d\mu(e^{i\theta}) \qquad (z \in \mathbb{D}).$$

Assume that

$$\sup_{n\ge1} |b(\lambda_n)| < 1.$$

Then the following statements are equivalent.

(i) $(k_{\lambda_n}/\|k_{\lambda_n}\|_{L^2(\mu)})_{n\ge1}$ *is a Riesz sequence in* $H^2(\mu)$.

(ii) $(\lambda_n)_{n\ge1} \in (C)$ *and* $\operatorname{dist}(\bar{B}b, H^\infty) < 1$.

Moreover, in this case, if μ is not a singular measure, then we have

$$\dim(H^2(\mu) \ominus \operatorname{Span}(k_{\lambda_n} : n \ge 1)) = \infty.$$

Similarly, using the operator K_ρ, we could obtain similar results for reproducing kernels in $\mathcal{H}(\bar{b})$. Recall that ρ denotes the function $1-|b|^2$ on $\mathbb{T}$, and, for $q \in L^2(\rho)$, K_ρ is the linear map defined by

$$K_\rho(q)(z) = K(q\rho)(z) = \int_{\mathbb{T}} \frac{q(e^{i\theta})\rho(e^{i\theta})}{1-e^{-i\theta}z}\, dm(e^{i\theta}) \qquad (z \in \mathbb{D}).$$

Then K_ρ is an isometry from $H^2(\rho)$ onto $\mathcal{H}(\bar{b})$ and we have $K_\rho k_\lambda = k_\lambda^{\bar{b}}$, $\lambda \in \mathbb{D}$, where $k_\lambda^{\bar{b}}$ denotes the reproducing kernel of $\mathcal{H}(\bar{b})$ (see Section 20.1). Let $g \in \operatorname{Hol}(\mathbb{D})$ be such that

$$\Re g(z) = \int_{\mathbb{T}} \frac{1-|z|^2}{|1-ze^{-i\theta}|^2}\, \rho(e^{i\theta})\, dm(e^{i\theta}) \qquad (z \in \mathbb{D}),$$

and define $b_1(z) = (g(z)-1)/(g(z)+1)$. Then it is easy to see that $b_1 \in H^\infty$, $\|b_1\|_\infty \le 1$, and

$$\frac{1-|b_1(z)|^2}{|1-b_1(z)|^2} = \int_{\mathbb{T}} \frac{1-|z|^2}{|1-ze^{-i\theta}|^2}\, \rho(e^{i\theta})\, dm(e^{i\theta}) \qquad (z \in \mathbb{D}).$$

Applying Theorem 31.37 to $\mu = \rho\, dm$ gives immediately the following result.

Theorem 31.38 *Let $(\lambda_n)_{n\ge1} \subset \mathbb{D}$, $b \in H^\infty$, $\|b\|_\infty \le 1$, and let b_1 be the function associated with b as above. Assume that*

$$\sup_{n\ge1} |b_1(\lambda_n)| < 1.$$

Then the following assertions are equivalent.

(i) $(k_{\lambda_n}^{\bar{b}}/\|k_{\lambda_n}^{\bar{b}}\|_{\bar{b}})_{n\ge1}$ *is a Riesz sequence in* $\mathcal{H}(\bar{b})$.
(ii) $(\lambda_n)_{n\ge1} \in (C)$ *and* $\operatorname{dist}(\bar{B}b_1, H^\infty) < 1$.

Moreover, we have

$$\dim(\mathcal{H}(\bar{b}) \ominus \operatorname{Span}(k_{\lambda_n}^{\bar{b}} : n \ge 1)) = \infty.$$

31.7 Asymptotically orthonormal sequences and bases in $\mathcal{H}(b)$

We recall that a (Blaschke) sequence of points $\Lambda = (\lambda_n)_{n\ge1}$ in $\mathbb{D}$ is called a "thin sequence", and we write $\Lambda \in (T)$, if

$$\lim_{n\to\infty} |B_n(\lambda_n)| = 1,$$

where $B_n = B/b_{\lambda_n}$, B is the Blaschke product associated with Λ and b_{λ_n} is the elementary Blaschke factor associated with λ_n. We start with an analog of Theorem 31.16 showing that condition (T) is necessary to have an asymptotically orthonormal sequence.

Theorem 31.39 *If $(\tilde{k}^b_{\lambda_n})_{n\geq 1}$ is an asymptotically orthonormal sequence (AOS), then $(\lambda_n)_{n\geq 1} \in (T)$.*

Proof By applying formula (4.41), we have

$$\begin{aligned}|\Gamma^b_{n,p}|^2 &= |\Gamma_{n,p}|^2 \frac{|1-\overline{b(\lambda_n)}b(\lambda_p)|^2}{(1-|b(\lambda_n)|^2)(1-|b(\lambda_p)|^2)}\\ &= \frac{|\Gamma_{n,p}|^2}{1-|b_{b(\lambda_n)}(b(\lambda_p))|^2} \geq |\Gamma_{n,p}|^2.\end{aligned}$$

Since Theorem 10.32 implies that $\sum_{p\neq n} |\Gamma^b_{n,p}|^2 = \|(\Gamma^b - I)e_n\|^2 \longrightarrow 0$, it follows that $\|(\Gamma - I)e_n\|^2 = \sum_{p\neq n} |\Gamma_{n,p}|^2 \longrightarrow 0$. Then Theorem 15.19 implies that $(\lambda_n)_{n\geq 1} \in (T)$. □

We are interested in partial converses to Theorem 31.39. In order to obtain more satisfactory results, it is natural, in view of the theory of Riesz bases, to work under the supplementary condition $\sup_{n\geq 1} |b(\lambda_n)| < 1$.

Theorem 31.40 *Suppose $\sup_{n\geq 1} |b(\lambda_n)| < 1$. If $(\lambda_n)_{n\geq 1} \in (T)$ then either*

(i) *$(\tilde{k}^b_{\lambda_n})_{n\geq 1}$ is an asymptotically orthonormal basis (AOB), or*
(ii) *there exists $p \geq 2$ such that $(\tilde{k}^b_{\lambda_n})_{n\geq p}$ is a complete AOB in $\mathcal{H}(b)$.*

Proof According to Theorem 15.18, the condition on $(\lambda_n)_{n\geq 1}$ implies the existence of positive constants $(c_N)_{N\geq N_0}$ and $(C_N)_{N\geq N_0}$, tending to 1, such that

$$c_N \sum_{n\geq N} |a_n|^2 \leq \Big\| \sum_{n\geq N} a_n \tilde{k}_{\lambda_n} \Big\|^2 \leq C_N \sum_{n\geq N} |a_n|^2. \tag{31.24}$$

According to (31.9), (31.10) and Corollary 17.6, and applying (31.24), we get

$$\begin{aligned}&\Big\| \sum_{n\geq N} a_n \tilde{k}^b_{\lambda_n} \Big\|_b^2\\ &= \Big\| (I - T_b T_{\bar{b}}) \sum_n \frac{a_n}{\sqrt{1-|b(\lambda_n)|^2}} \tilde{k}_{\lambda_n} \Big\|_b^2\\ &= \Big\| \sum_{n\geq N} \frac{a_n}{\sqrt{1-|b(\lambda_n)|^2}} \tilde{k}_{\lambda_n} \Big\|^2 - \Big\| \sum_{n\geq N} \frac{a_n \overline{b(\lambda_n)}}{\sqrt{1-|b(\lambda_n)|^2}} \tilde{k}_{\lambda_n} \Big\|^2\\ &\leq C_N \sum_{n\geq N} \frac{|a_n|^2}{1-|b(\lambda_n)|^2} - c_N \sum_{n\geq N} \frac{|a_n|^2 |b(\lambda_n)|^2}{1-|b(\lambda_n)|^2}\\ &= C_N \sum_{n\geq N} |a_n|^2 + (C_N - c_N) \sum_{n\geq N} \frac{|a_n|^2 |b(\lambda_n)|^2}{1-|b(\lambda_n)|^2}\\ &\leq C_N \sum_{n\geq N} |a_n|^2 + (C_N - c_N) \sup_n \frac{|b(\lambda_n)|^2}{1-|b(\lambda_n)|^2} \sum_{n\geq N} |a_n|^2.\end{aligned}$$

Since $C_N \longrightarrow 1$, $C_N - c_N \longrightarrow 0$, while $\sup_n |b(\lambda_n)|^2/(1-|b(\lambda_n)|^2) < \infty$, we can find constants $C'_N \longrightarrow 1$, such that

$$\left\| \sum_{n\geq N} a_n \tilde{k}^b_{\lambda_n} \right\|_b^2 \leq C'_N \sum_{n\geq N} |a_n|^2.$$

A similar argument shows the existence of $c'_N \longrightarrow 1$, such that

$$\left\| \sum_{n\geq N} a_n \tilde{k}^b_{\lambda_n} \right\|_b^2 \geq c'_N \sum_{n\geq N} |a_n|^2.$$

It follows that $(\tilde{k}^b_{\lambda_n})_{n\geq 1}$ is an AOS. Hence, there exists $m \geq 1$ such that $(\tilde{k}^b_{\lambda_n})_{n\geq m}$ is an AOB.

Let p be the smallest positive integer with the property that $(\tilde{k}^b_{\lambda_n})_{n\geq p}$ is an AOB. If $p = 1$ we are in case (i) of the statement. Otherwise, remembering that a sequence is an AOB if and only if it is minimal and an AOS, Lemma 31.2 implies that $(\tilde{k}^b_{\lambda_n})_{n\geq p}$ is complete in $\mathcal{H}(b)$. The theorem is thus proved. □

Since case (ii) in Theorem 31.40 corresponds to $(\tilde{k}^b_{\lambda_n})_{n\geq 1}$ being not minimal, we immediately get the following result.

Corollary 31.41 *Suppose that* $\sup_{n\geq 1} |b(\lambda_n)| < 1$. *Then the following are equivalent.*

(i) $(\tilde{k}^b_{\lambda_n})_{n\geq 1}$ *is an AOB.*
(ii) $\Lambda \in (T)$ *and* $(\tilde{k}^b_{\lambda_n})_{n\geq 1}$ *is minimal.*

An example can be obtained by taking b to be a proper inner divisor of B. We can say more in the case when b is not a Blaschke product.

Theorem 31.42 *Let b be in the closed unit ball of H^∞, and let $(\lambda_n)_{n\geq 1}$ be a Blaschke sequence of distinct points in $\mathbb{D}$ such that* $\sup_{n\geq 1} |b(\lambda_n)| < 1$. *If b is not a Blaschke product and the sequence $(\tilde{k}^b_{\lambda_n})_{n\geq 1}$ is an AOB in $\mathcal{H}(b)$, then its span has infinite codimension.*

Proof First, assume that b is not an inner function. Then Theorem 31.25 implies that

$$\dim(\mathcal{H}(b) \ominus \mathrm{Span}(\tilde{k}^b_{\lambda_n} : n \geq 1)) = \infty,$$

and the result is proved in this case. Now, we assume that $b = \Theta$ is an inner function that is not a Blaschke product. Let $\sup_{n\geq 1} |\Theta(\lambda_n)| = \eta < 1$. We shall write $\Theta = \beta S$, with β a Blaschke product and S singular, nonconstant. Let us also denote $B^{(N)} = \prod_{n\geq N} b_{\lambda_n}$.

By Theorem 31.39, $(\tilde{k}_{\lambda_n})_{n\geq 1}$ is an AOB (and in particular a Riesz sequence). If c_N and C_N are the constants in (10.51), then, applying Lemma 31.28 to Θ and $B^{(N)}$, it follows that $\|P_{B^{(N)}} T_{\Theta|K_{B^{(N)}}}\| \leq (C_N/c_N)^{1/2}\eta$.

Since $C_N/c_N \longrightarrow 1$, we may find $N \in \mathbb{N}$ such that $\|P_{B^{(N)}} T_{\Theta|K_{B^{(N)}}}\| < 1$, which, according to Theorem 31.18, implies that $P_{\Theta|K_{B^{(N)}}}$ is an isomorphism on its image. (Note that, since Θ is inner, then $I - T_\Theta T_{\bar{\Theta}} = P_{\Theta|H^2}$.)

Now, if we define $\Theta' = \beta S^{1/2}$, Θ' is also an inner function, and

$$|\Theta'(\lambda_n)| \le |\beta(\lambda_n)|^{1/2}|S(\lambda_n)|^{1/2} = |\Theta(\lambda_n)|^{1/2} \le \eta^{1/2}.$$

If we apply the same argument to Θ', it follows that we can find $N \in \mathbb{N}$, such that both $P_{\Theta|K_{B(N)}}$ and $P_{\Theta'|K_{B(N)}}$ are isomorphisms on their images.

But we have

$$P_{\Theta'|K_{B(N)}} = P_{\Theta'|K_\Theta} P_{\Theta|K_{B(N)}}.$$

The operator on the left is one-to-one, while the image of $P_{\Theta|K_{B(N)}}$ is closed. Therefore, this image cannot intersect $\ker(P_{\Theta'|K_\Theta})$, which is infinite-dimensional. But the image of $P_{\Theta|K_{B(N)}}$ is the space spanned by $h^{\Theta}_{\lambda_n}$ for $n \ge N$. It follows that the space spanned by all the $h^{\Theta}_{\lambda_n}$, $n \ge 1$, also has infinite codimension. □

Under the conditions of Theorem 31.42, one can improve Corollary 31.41. This result is an analog of Theorem 31.33.

Corollary 31.43 *Let b be in the closed unit ball of H^∞, and let $(\lambda_n)_{n\ge1}$ be a Blaschke sequence of distinct points in $\mathbb{D}$ such that $\sup_{n\ge1}|b(\lambda_n)| < 1$. If b is not a Blaschke product, then the following assertions are equivalent.*

(i) $\Lambda \in (T)$.
(ii) $(\tilde{k}^b_{\lambda_n})_{n\ge1}$ *is an AOB.*

Moreover, in this case, $\operatorname{Span}(\tilde{k}^b_{\lambda_n} : n \ge 1)$ *has infinite codimension in* $\mathcal{H}(b)$.

Proof If $\Lambda \in (T)$, Theorem 31.42 shows that we are in case (i) of Theorem 31.40. Consequently, $(\tilde{k}^b_{\lambda_n})_{n\ge1}$ is an AOB. The converse was contained in Theorem 31.39. □

31.8 Stability of completeness and asymptotically orthonormal basis

In this section and the following, we study the stability of geometric properties (completeness, Riesz sequences and bases, asymptotically orthonormal sequences and bases) with respect to small perturbations. We begin with an elementary result concerning completeness.

Theorem 31.44 *Let $\Lambda = (\lambda_n)_{n\ge1}$ be a sequence of points in $\mathbb{D}$, let b be a point in the closed unit ball of H^∞ and assume that $(k^b_{\lambda_n})_{n\ge1}$ is complete in*

$\mathcal{H}(b)$. *Then, for every sequence* $\Lambda' = (\lambda'_n)_{n\geq 1}$ *satisfying*

$$\sum_{n\geq 1} \left| \frac{\lambda_n - \lambda'_n}{1 - \overline{\lambda_n}\lambda'_n} \right| < \infty, \tag{31.25}$$

the system $(k^b_{\lambda'_n})_{n\geq 1}$ *is complete in* $\mathcal{H}(b)$.

Proof Arguing by absurdity, assume that $(k^b_{\lambda'_n})_{n\geq 1}$ is not complete in $\mathcal{H}(b)$. Then, by the Hahn–Banach theorem, there is a function $f \in \mathcal{H}(b)$, $f \not\equiv 0$, such that $f(\lambda'_n) = 0$. Define by induction a sequence of functions $(\phi_n)_{n\geq 1}$ in H^2 by $\phi_0 = f$ and

$$\phi_n = \frac{b_{\lambda'_n} - b_{\lambda'_n}(\lambda_n)}{b_{\lambda'_n}} \phi_{n-1} \qquad (n \geq 1).$$

Using Theorem 18.16, we see that $\phi_n \in \mathcal{H}(b)$, $n \geq 1$. Moreover, $\phi_n(\lambda_k) = 0$, $k \leq n$, and $\phi_n(\lambda'_k) = 0$, $k > n$. Using Theorem 18.16 once more, we have

$$\|\phi_n - \phi_{n-1}\|_b \leq |b_{\lambda'_n}(\lambda_n)| \times \|\phi_{n-1}\|_b. \tag{31.26}$$

Therefore,

$$(1 - |b_{\lambda'_n}(\lambda_n)|)\, \|\phi_{n-1}\|_b \leq \|\phi_n\|_b \leq (1 + |b_{\lambda'_n}(\lambda_n)|)\, \|\phi_{n-1}\|_b,$$

and, by induction, we get

$$\prod_{k=1}^{n} (1 - |b_{\lambda'_k}(\lambda_k)|)\, \|f\|_q \leq \|\phi_n\|_q \leq \prod_{k=1}^{n} (1 + |b_{\lambda'_k}(\lambda_k)|)\, \|f\|_q. \tag{31.27}$$

It follows from (31.25) that the two infinite products

$$\prod_{k=1}^{\infty} (1 - |b_{\lambda'_k}(\lambda_k)|) \quad \text{and} \quad \prod_{k=1}^{\infty} (1 + |b_{\lambda'_k}(\lambda_k)|)$$

are converging. Put

$$c_i = \prod_{k=1}^{\infty} (1 + \varepsilon_i |b_{\lambda'_k}(\lambda_k)|) \qquad (i = 1, 2),$$

with $\varepsilon_1 = 1$ and $\varepsilon_2 = -1$. Then (31.27) implies that

$$\|\phi_n\|_b \leq c_1 \|f\|_b \qquad (n \geq 1). \tag{31.28}$$

Using the triangle inequality and (31.26) and (31.28), we then get

$$\|\phi_{n+p} - \phi_n\|_b \leq c_1 \|f\|_b \sum_{k=1}^{p} |b_{\lambda'_{n+k}}(\lambda_{n+k})|.$$

By (31.25), we finally deduce that $(\phi_n)_{n\geq 1}$ is a Cauchy sequence in $\mathcal{H}(b)$. Thus, there is $\phi \in \mathcal{H}(b)$ such that $\lim_{n\to\infty} \phi_n = \phi$, in $\mathcal{H}(b)$. Moreover, we get from (31.27) that $c_2 \|f\|_b \leq \|\phi\|_b$, which proves, in particular, that $\phi \not\equiv 0$.

Since $\phi(\lambda_n) = \lim_{p\to\infty} \phi_p(\lambda_n) = 0$, the sequence $(\tilde{k}^b_{\lambda_n})_{n\geq 1}$ is not complete in $\mathcal{H}(b)$, which contradicts the hypothesis. □

Theorem 31.45 *Suppose that* $\sup_{n\geq 1} |b(\lambda_n)| < 1$ *and* $(\tilde{k}^b_{\lambda_n})_{n\geq 1}$ *is an AOB. If* $\lambda' = (\lambda'_n)_{n\geq 1}$ *is a sequence of distinct points in* $\mathbb{D}$ *that satisfies*

$$\limsup_{n\to\infty} |b_{\lambda_n}(\lambda'_n)| < \frac{1 - \operatorname{dist}(b\bar{B}, H^\infty)}{1 + \operatorname{dist}(b\bar{B}, H^\infty)}, \tag{31.29}$$

then $(\tilde{k}^b_{\lambda'_n})_{n\geq 1}$ *is an AOB.*

Proof Fix $N \geq 1$, and define

$$\gamma_n = \begin{cases} \lambda_n & \text{if } n < N, \\ \lambda'_n & \text{if } n \geq N. \end{cases}$$

Let Φ be the Blaschke product associated with $(\gamma_n)_{n\geq 1}$. Theorem 31.39 implies that $\Lambda \in (T)$, whence, by Lemma 15.22, $\Lambda' = (\lambda'_n)_{n\geq 1}$ and $(\gamma_n)_{n\geq 1}$ are both thin sequences. If $g, h \in H^\infty$, then the equality $b\,\bar{\Phi} - gh = b\bar{B}(B\bar{\Phi} - g) + (b\bar{B} - h)g$ implies that

$$\|b\,\bar{\Phi} - gh\|_\infty \leq \|B\bar{\Phi} - g\|_\infty + \|(b\bar{B} - h)g\|_\infty,$$

which shows that

$$\operatorname{dist}(b\,\bar{\Phi}, H^\infty) \leq \operatorname{dist}(b\bar{B}, H^\infty) + (1 + \operatorname{dist}(b\bar{B}, H^\infty)) \operatorname{dist}(B\bar{\Phi}, H^\infty).$$

Now, if $B^{(N)} = \prod_{n\geq N} b_{\lambda_n}$ and $\Phi^{(N)} = \prod_{n\geq N} b_{\lambda'_n}$, then $B\bar{\Phi} = B^{(N)}\overline{\Phi^{(N)}}$. Suppose C_N and c_N are the constants associated with λ' as in (10.51), while $\varepsilon_N = \sup_{n>N} |b_{\lambda_n}(\lambda'_n)|$. Then obviously $\sup_{n\geq N} |B^{(N)}(\lambda'_n)| \leq \varepsilon_N$. Applying Lemmas 20.24 and 31.28, it follows that

$$\begin{aligned} \operatorname{dist}(B\bar{\Phi}, H^\infty) &= \operatorname{dist}(B^{(N)}\overline{\Phi^{(N)}}, H^\infty) \\ &= \|P_{\Phi^{(N)}} T_{B^{(N)}} | K_{\Phi^{(N)}}\| \leq \varepsilon_N (C_N/c_N)^{1/2}. \end{aligned}$$

Consequently,

$$\operatorname{dist}(b\,\bar{\Phi}, H^\infty) \leq \operatorname{dist}(b\bar{B}, H^\infty) + (1 + \operatorname{dist}(b\bar{B}, H^\infty))\varepsilon_N (C_N/c_N)^{1/2}.$$

The hypothesis implies that, for N sufficiently large,

$$\varepsilon_N (C_N/c_N)^{1/2} < \frac{1 - \operatorname{dist}(b\bar{B}, H^\infty)}{1 + \operatorname{dist}(b\bar{B}, H^\infty)}$$

and therefore $\operatorname{dist}(b\,\bar{\Phi}, H^\infty) < 1$. There exists thus $f \in H^\infty$, $f \neq 0$, such that $\|b - \Phi f\|_\infty < 1$, and therefore $\sup_{n\geq 1} |b(\gamma_n)| \leq \operatorname{dist}(b\,\bar{\Phi}, H^\infty) < 1$. It follows from Theorem 31.24 that $(\tilde{k}^b_{\gamma_n})_{n\geq 1}$ is a Riesz sequence in $\mathcal{H}(b)$ and, in particular, it is minimal. Now, according to Corollary 31.41, we get that $(\tilde{k}^b_{\gamma_n})_{n\geq 1}$ is an AOB. Applying Lemma 31.2 repeatedly, we obtain that $(\tilde{k}^b_{\lambda'_n})_{n\geq 1}$ is an AOB. □

In the particular case where b is not a Blaschke product, we can improve the stability constant in Theorem 31.45.

Theorem 31.46 *Suppose that b is not a Blaschke product,* $\sup_{n\geq 1}|b(\lambda_n)|<1$ *and $(\tilde{k}^b_{\lambda_n})_{n\geq 1}$ is an AOB. If $\Lambda' = (\lambda'_n)_{n\geq 1}$ is a sequence of distinct points in $\mathbb{D}$ that satisfies*

$$\limsup_{n\to\infty}|b_{\lambda_n}(\lambda'_n)| < \frac{1-\limsup_{n\to\infty}|b(\lambda_n)|}{1+\limsup_{n\to\infty}|b(\lambda_n)|},$$

then $(\tilde{k}^b_{\lambda'_n})_{n\geq 1}$ is an AOB.

Proof By Lemma 15.22, $\Lambda' \in (T)$. On the other hand,

$$\left|\frac{b(\lambda'_n)-b(\lambda_n)}{b_{\lambda_n}(\lambda'_n)}\right| \leq \left\|\frac{b-b(\lambda_n)}{b_{\lambda_n}}\right\|_\infty \leq 1+|b(\lambda_n)|,$$

whence

$$|b(\lambda'_n)| \leq |b(\lambda_n)| + (1+|b(\lambda_n)|)\,|b_{\lambda_n}(\lambda'_n)| < 1.$$

Corollary 31.43 implies that $(\tilde{k}^b_{\lambda'_n})_{n\geq 1}$ is an AOB. □

Using Corollary 31.14, we may obtain a large class of examples of complete AOBs in K_Θ formed by reproducing kernels.

Corollary 31.47 *If $\mathbb{T}\backslash E_2(\Theta)$ is at most countable, then there exist sequences $(\lambda_n)_{n\geq 1}$ in $\mathbb{D}$ such that $(k^\Theta_{\lambda_n}/\|k^\Theta_{\lambda_n}\|)_{n\geq 1}$ is a complete AOB in K_Θ.*

Proof By Corollary 31.14, take a sequence $(\zeta_n)_{n\geq 1} \subset E_2(\Theta)$ such that $(k^\Theta_{\zeta_n}/\|k^\Theta_{\zeta_n}\|)_{n\geq 1}$ is an orthonormal basis of K_Θ. By (i) of the same result, it follows that, if $\zeta \in E_2(\Theta)$, and $\lambda \longrightarrow \zeta$ nontangentially, then $k^\Theta_\lambda/\|k^\Theta_\lambda\| \longrightarrow k^\Theta_\zeta/\|k^\Theta_\zeta\|$. Then choose $\lambda_n \in \mathbb{D}$, $\lambda_n/\|\lambda_n\| = \zeta_n$, such that

$$\sum_{n\geq 1}\left\|\frac{k^\Theta_{\lambda_n}}{\|k^\Theta_{\lambda_n}\|} - \frac{k^\Theta_{\zeta_n}}{\|k^\Theta_{\zeta_n}\|}\right\|^2 < 1.$$

The required conclusion follows then by applying Corollary 10.34. □

In the above proof, note that the choice of λ_n can be made such that $|\Theta(\lambda_n)-\Theta(\zeta_n)| \longrightarrow 0$. It then follows that $|\Theta(\lambda_n)| \longrightarrow 1$.

In the case $\Theta = \Theta_a$, we have $E_2(\Theta_a) = \mathbb{T}\setminus\{1\}$, and thus the hypotheses of Corollary 31.14 and Corollary 31.47 are fulfilled. Actually, Clark's paper [55] indeed has the bases of exponentials as the starting point.

The previous example showed a complete AOB obtained by perturbing an orthonormal basis. Obviously reproducing kernels corresponding to points in $\mathbb{D}$ cannot be orthonormal, but Clark's result shows that orthonormal bases can be obtained by using reproducing kernels corresponding to points on $\mathbb{T}$, whenever the corresponding functional is continuous on K_Θ, that is, to points in $E_2(\Theta)$.

In the next example, we construct a complete AOB in a case where $E_2(\Theta) = \emptyset$. First, take a sequence of positive integers q_n, $n \geq 1$, such that $q_{n+1} - q_n \longrightarrow \infty$. Then choose another sequence of positive integers p_n, $n \geq 1$, subject to the conditions

$$\sum_{n\geq 1} \frac{p_n}{2^{q_n}} < \infty \quad \text{and} \quad \sum_{n\geq 1} \frac{p_n \log p_n}{2^{q_n}} = \infty.$$

On the circle centered at the origin and having radius $1 - 2^{-q_n}$, we choose p_n equidistant points. The union of all these points (for $n \geq 1$) is denoted by Λ. We also write $r_n = 1 - 2^{-q_n}$, and remark that in the estimates below the letter C will denote a constant that might not be the same in the different formulas.

We have $\sum_{\lambda\in\Lambda}(1 - |\lambda|) = \sum_{n\geq 1} p_n/2^{q_n} < \infty$. Thus, Λ satisfies the Blaschke condition and we may form the corresponding product B. We intend to show that the family $(k_\lambda)_{\lambda\in\Lambda}$ is an AOB in K_B, but that $E_2(B) = \emptyset$.

In order to show that $(k_\lambda)_{\lambda\in\Lambda}$ is an AOB in K_B, we apply Theorem 15.19. We have to prove that $(\Gamma - I)e_n \longrightarrow 0$, which is equivalent to

$$\sum_{\mu\neq\lambda} \frac{(1-|\lambda|)(1-|\mu|)}{|1-\bar{\lambda}\mu|^2} \longrightarrow 0$$

when $\lambda \in \Lambda$, $|\lambda| \longrightarrow 1$. This sum can be written as

$$(1-|\lambda|) \sum_{n\geq 1} \sum_{\substack{|\mu|=r_n,\\ \mu\neq\lambda}} \frac{1-|\mu|}{|1-\bar{\lambda}\mu|^2}.$$

If we suppose that $|\lambda| = r_N$, we can decompose the above sum into three terms, S_1, S_2 and S_3, corresponding to $n < N$, $n = N$ and $n > N$, respectively. We shall estimate these three terms separately.

(i) The first term is

$$S_1 = \frac{1}{2^{q_N}} \sum_{n<N} \frac{1}{2^{q_n}} \sum_{|\mu|=r_n} \frac{1}{|1-\bar{\lambda}\mu|^2}.$$

In turn, we may decompose S_1 into two terms $S_1 = S_1' + S_1''$. The first of these, S_1', contains, for each $n < N$, the (at most) two terms that have arguments closest to the argument of λ. For these we may estimate the last denominator (see (4.42)) as being larger than $(1 - |\lambda\mu|)^2$, which is of order $1/(2^{2q_n})$. Thus,

$$S_1' \leq C\frac{1}{2^{q_N}} \sum_{n<N} \frac{1}{2^{q_n}}\, 2^{2q_n} \leq C\frac{2^{q_{N-1}}}{2^{q_N}} = C2^{q_{N-1}-q_N} \longrightarrow 0 \qquad (31.30)$$

when $N \longrightarrow \infty$. The second, S_1'', contains the rest of the terms in S_1. For a fixed n, the arguments of the numbers $\bar{\lambda}\mu$ are comparable to integer multiples of $1/p_n$, and so are their sines. We will therefore use the second term on the left-hand side of (4.42) to estimate $|1-\bar{\lambda}\mu|^2$, obtaining

$$S_1'' \leq C\frac{1}{2^{q_N}}\sum_{n<N}\frac{1}{2^{q_n}}\sum_{j=1}^{p_n}\frac{1}{j^2/p_n^2} \leq C\sum_{n<N}\frac{p_n^2}{2^{2q_n}}\,\frac{1}{2^{q_N-q_n}}. \tag{31.31}$$

Since $\sum_{n\geq 1} p_n^2/2^{2q_n}$ is also convergent, the last term is smaller than $1/(2^{q_N-q_{N-1}})$, and thus tends to 0 for $N\longrightarrow\infty$.

(ii) For the second term, we have

$$S_2 = \frac{1}{2^{2q_N}}\sum_{\substack{|\mu|=r_n,\\ \mu\neq\lambda}}\frac{1}{|1-\bar{\lambda}\mu|^2}.$$

As in the argument for S_1', we have, by using (4.42),

$$S_2 \leq C\frac{1}{2^{2q_N}}\sum_{j=1}^{p_N}\frac{1}{j^2/p_N^2} \leq C\frac{p_N^2}{2^{2q_N}} \longrightarrow 0 \tag{31.32}$$

when $N\longrightarrow\infty$.

(iii) We decompose the third term, S_3, further as $S_3 = S_3' + S_3''$, keeping in S_3', for each $n<N$, the (at most) two terms that have arguments closest to the argument of λ. A similar argument as that for S_1' yields now

$$S_3' \leq C\frac{1}{2^{q_N}}\sum_{n>N}\frac{1}{2^{q_n}}\,2^{2q_N} \leq C2^{q_N-q_{N+1}} \longrightarrow 0. \tag{31.33}$$

The last term, and most delicate to estimate, is S_3''. We have, using (4.42),

$$S_3'' = \sum_{n>N}\sum_{j=1}^{p_n}\frac{1}{2^{q_N+q_n}}\,\frac{1}{1/(2^{2q_N})+j^2/p_n^2}. \tag{31.34}$$

We can write

$$\begin{aligned}
&\sum_{j=1}^{p_n}\frac{1}{2^{q_N+q_n}}\,\frac{1}{1/(2^{2q_N})+j^2/p_n^2}\\
&\quad= \sum_{j\leq p_n/2^{q_N}}\frac{1}{2^{q_N+q_n}}\,\frac{1}{1/(2^{2q_N})+j^2/p_n^2}\\
&\qquad+ \sum_{j>p_n/2^{q_N}}\frac{1}{2^{q_N+q_n}}\,\frac{1}{1/(2^{2q_N})+j^2/p_n^2}\\
&\quad\leq \sum_{j\leq p_n/2^{q_N}}\frac{1}{2^{q_N+q_n}}\,\frac{1}{1/(2^{2q_N})} + \frac{1}{2^{q_N+q_n}}\sum_{j>p_n/2^{q_N}}\frac{p_n^2}{j^2}.
\end{aligned}$$

The first sum contains approximately $p_n/2^{q_N}$ terms, all equal to $2^{q_N-q_n}$. It can therefore be estimated by $p_n/2^{q_n}$. As for the second, we note that $\sum_{j>J} 1/j^2$ is of order $1/J$, and therefore

$$\sum_{j>p_n/2^{q_N}} \frac{p_n^2}{j^2} \le C\frac{p_n^2 2^{q_N}}{p_n} = p_n 2^{q_N},$$

and thus the second sum can also be estimated by $p_n/2^{q_n}$. Finally, we obtain, using (31.34),

$$S_3'' \le C \sum_{n>N} \frac{p_n}{2^{q_n}} \longrightarrow 0 \tag{31.35}$$

when $N \longrightarrow \infty$.

Now, adding together the estimates (31.30), (31.31), (31.32), (31.33) and (31.35), we obtain $(\Gamma - I)e_n \longrightarrow 0$ as desired. Therefore Λ is a thin sequence.

It remains now to show that $E_2(B) = \emptyset$. In fact, more can be proved, namely that $\sum_{\lambda\in\Lambda}(1-|\lambda|)/(|\zeta-\lambda|) = \infty$ for all $\zeta \in \mathbb{T}$. Fix $\zeta \in \mathbb{T}$. We have

$$\sum_{\lambda\in\Lambda} \frac{1-|\lambda|}{|\zeta-\lambda|} = \sum_{n\ge1} \frac{1}{2^{q_n}} \sum_{|\lambda|=r_n} \frac{1}{|\zeta-\lambda|}.$$

For each fixed n, if $|\lambda| = r_n$, then, with the possible exception of two points, $|\zeta-\lambda|$ is comparable to $|r_n\zeta-\lambda|$. The other points λ on this circle are at distances to ζ comparable to $j \times 2\pi/p_n$, with $j = 1, 2, \dots, p_n - 2$. Therefore,

$$\sum_{|\lambda|=r_n} \frac{1}{|\zeta-\lambda|} \ge C \sum_{1}^{p_n-2} \frac{1}{j/p_n} \le C p_n \log p_n.$$

Then

$$\sum_{\lambda\in\Lambda} \frac{1-|\lambda|}{|\zeta-\lambda|} \ge C \sum_{n\ge1} \frac{1}{2^{q_n}}\, p_n \log p_n = \infty,$$

as required.

Exercises

Exercise 31.8.1 Let $(\lambda_n)_{n\ge1}$ and $(\lambda_n')_{n\ge1}$ be two sequences in $\mathbb{D}$. Assume that

$$\sup_{n\ge1} |b_{\lambda_n}(\lambda_n')| < 1. \tag{31.36}$$

(i) Show that $(1-|\lambda_n|) \asymp (1-|\lambda_n'|)$.

(ii) Deduce that $(k_{\lambda_n})_{n\ge1}$ is complete in H^2 if and only if $(k_{\lambda_n'})_{n\ge1}$ is complete in H^2.

Exercise 31.8.2 Let $(\mu_n)_{n\geq 1}$ and $(\mu'_n)_{n\geq 1}$ be two sequences in the upper half-plane $\mathbb{C}_+ = \{z \in \mathbb{C} : \Im(z) > 0\}$ and assume that

$$\sum_{n\geq 1} \frac{|\mu_n - \mu'_n|}{1 + \Im\,\mu_n + \Im\,\mu'_n} < \infty.$$

Let $a > 0$. Show that the two exponential systems $(e^{i\mu_n t})_{n\geq 1}$ and $(e^{i\mu'_n t})_{n\geq 1}$ are simultaneously complete or not in $L^2(-a, a)$.
Hint: Use Theorem 31.44.

Exercise 31.8.3 Let $(\lambda_n)_{n\geq 1}$ be a Blaschke sequence of points in $\mathbb{D}$, let B be the associated Blaschke product, and let b be a function in the closed unit ball of H^∞.

(i) Show that

$$\mathcal{H}(b) \ominus \mathrm{Span}(k^b_{\lambda_n} : n \geq 1) = \mathcal{H}(b) \cap B\mathcal{H}(b),$$

and deduce that the system $(k^b_{\lambda_n})_{n\geq 1}$ is complete in $\mathcal{H}(b)$ if and only if $\mathcal{H}(b) \cap B\mathcal{H}(b) = \{0\}$.

(ii) Show that

$$B \ker T_{\bar{b}B} \subset \mathcal{H}(b) \cap B\mathcal{H}(b). \tag{31.37}$$

(iii) Show that, if $b = \Theta$ is an inner function, then we have equality in (31.37). Deduce that the system $(k^\Theta_{\lambda_n})_{n\geq 1}$ is complete in K_Θ if and only if the Toeplitz operator $T_{\bar{\Theta}B}$ is one-to-one.

(iv) Construct an example of a function b and Blaschke product B such that the inclusion in (31.37) is strict.

31.9 Stability of Riesz bases

We now give some results concerning the stability of the Riesz sequences and bases of reproducing kernels $(k^b_{\lambda_n})$ under small perturbations of the points λ_n. Our method uses the Bernstein inequalities that we obtained in Chapter 22 and also the following general result, which is of independent interest.

Lemma 31.48 *Let $(h_n)_{n\geq 1}$ be a Riesz basis of a Hilbert space $\mathcal{H}$. Then there exists a constant $\varepsilon > 0$ with the following property: if $(h'_n)_{n\geq 1}$ is a sequence in $\mathcal{H}$ satisfying*

$$\sum_{n=1}^{\infty} |\langle f, h_n - h'_n\rangle_{\mathcal{H}}|^2 \leq \varepsilon \|f\|^2_{\mathcal{H}} \qquad (f \in \mathcal{H}), \tag{31.38}$$

then $(h'_n)_{n\geq 1}$ is also a Riesz basis of $\mathcal{H}$.

Proof Let J_1 and J_2 be the two linear mappings from $\mathcal{H}$ into the linear space of complex sequences defined by

$$J_1 f = (\langle f, h_n \rangle_{\mathcal{H}})_{n\ge 1} \quad \text{and} \quad J_2 f = (\langle f, h'_n \rangle_{\mathcal{H}})_{n\ge 1} \qquad (f \in \mathcal{H}).$$

Since $(h_n)_{n\ge1}$ is a Riesz basis, then we know from Lemma 10.20 that J_1 is an isomorphism from $\mathcal{H}$ onto ℓ^2. Moreover, inequality (31.38) means that

$$\|(J_1 - J_2)f\|^2_{\ell^2} \le \varepsilon \|f\|^2_{\mathcal{H}} \qquad (f \in \mathcal{H}).$$

In other words, we get that J_2 is a bounded operator from $\mathcal{H}$ into ℓ^2 and satisfies

$$\|J_1 - J_2\| \le \sqrt{\varepsilon}.$$

Since the set of invertible operators forms an open set in the space of bounded operators, one can choose $\varepsilon > 0$ sufficiently small so that J_2 will also be an isomorphism from $\mathcal{H}$ onto ℓ^2. In particular, $(h'_n)_{n\ge1}$ will be a complete sequence (because J_2 is one-to-one) and minimal (because $(\delta_{m,n})_{m\ge1}$ belongs to the range of J_2 for all $n \ge 1$). Now Theorem 10.21 implies that $(h'_n)_{n\ge1}$ is a Riesz basis of $\mathcal{H}$. □

To use the Bernstein inequalities that we obtained in Chapter 22, we now translate our setting into the upper half-plane. Let b be in the closed unit ball of $H^\infty(\mathbb{C}_+)$. We first need a technical result, which follows from the Schwarz–Pick inequality. Denote by $\rho(z,\omega)$ the pseudohyperbolic distance between two points z and ω in $\mathbb{C}_+$, i.e.

$$\rho(z,\omega) = \left| \frac{z-\omega}{z-\bar{\omega}} \right|.$$

Lemma 31.49 *Let $b \in H^\infty(\mathbb{C}_+)$ with $\|b\|_\infty \le 1$, and let $\varepsilon_0 \in (0,1)$. Then there exist constants $C_1, C_2 > 0$ (depending only on ε_0) such that, for any $z, \omega \in \mathbb{C}_+$ satisfying $\rho(z,\omega) < \varepsilon_0$, we have*

$$C_1 \le \frac{1-|b(z)|}{1-|b(\omega)|} \le C_2. \tag{31.39}$$

Proof If we translate (4.1) for function b in the closed unit ball of $H^\infty(\mathbb{C}_+)$, we get that

$$\left| \frac{b(z)-b(\omega)}{1-\overline{b(z)}b(\omega)} \right| \le \left| \frac{z-\omega}{\bar{z}-\omega} \right| < \varepsilon_0.$$

So it is sufficient to prove that, if $\lambda, \mu \in \mathbb{D}$ and satisfies

$$\left| \frac{\lambda-\mu}{1-\bar{\lambda}\mu} \right| \le \varepsilon_0,$$

then

$$\frac{1-\varepsilon_0}{1+\varepsilon_0} \le \frac{1-|\lambda|}{1-|\mu|} \le \frac{1+\varepsilon_0}{1-\varepsilon_0}. \tag{31.40}$$

To establish this inequality, first we have

$$\begin{aligned}\left|\frac{\lambda-\mu}{1-\bar{\lambda}\mu}\right|^2 &= 1 - \frac{(1-|\lambda|^2)(1-|\mu|^2)}{|1-\bar{\lambda}\mu|^2} \\ &\ge 1 - \frac{(1-|\lambda|^2)(1-|\mu|^2)}{(1-|\lambda|\,|\mu|)^2} = \left(\frac{|\mu|-|\lambda|}{1-|\mu|\,|\lambda|}\right)^2.\end{aligned}$$

Hence we get

$$\frac{|\lambda|-|\mu|}{1-|\lambda|\,|\mu|} \le \varepsilon_0.$$

Using that

$$\frac{(1-|\lambda|)-(1-|\mu|)}{(1-|\lambda|)+(1-|\mu|)} \le \frac{(1-|\lambda|)-(1-|\mu|)}{1-|\lambda|+|\lambda|(1-|\mu|)} = \frac{|\lambda|-|\mu|}{1-|\lambda|\,|\mu|},$$

we deduce that $1-|\lambda|-(1-|\mu|) \le \varepsilon_0(1-|\lambda|)+\varepsilon_0(1-|\mu|)$, which gives

$$\frac{1-|\lambda|}{1-|\mu|} \le \frac{1+\varepsilon_0}{1-\varepsilon_0}.$$

Since λ and μ are interchangeable, we get the other side of (31.40). □

Now, we recall that a real point x_0 belongs to $E_2(b)$ if and only if

$$\sum_{k=1}^{\infty} \frac{\Im z_k}{|x_0-z_k|^2} + \int_{\mathbb{R}} \frac{d\mu(t)}{|x_0-t|^2} + \int_{\mathbb{R}} \frac{\big|\log|b(t)|\big|}{|x_0-t|^2}\,dt < +\infty, \tag{31.41}$$

where $(z_k)_k$ are the zeros of the function b in $\mathbb{C}_+$ and μ is the singular measure associated with b. Now for $w \in \mathbb{C}_+ \cup E_2(b)$ we recall that the reproducing kernel of $\mathcal{H}(b)$ is defined as

$$k_w^b(z) = \frac{i}{2\pi}\,\frac{1-\overline{b(w)}b(z)}{1-\bar{w}z} \qquad (z \in \mathbb{C}_+),$$

and $\tilde{k}_w^b = k_w^b/\|k_w^b\|_b$; see Section 22.1.

Let $(\lambda_n)_{n\ge1}$ be a sequence of distinct points in $\mathbb{C}_+$ and assume that $(\tilde{k}_{\lambda_n}^b)_{n\ge1}$ is a Riesz basis in $\mathcal{H}(b)$. Our stability result will be based on sets G_n that satisfy the following properties:

(i) For any $n \ge 1$, $\lambda_n \in G_n$.
(ii) There exist positive constants c and C such that

$$c \le \frac{\|k_{z_n}^b\|_b}{\|k_{\lambda_n}^b\|_b} \le C \qquad (z_n \in G_n).$$

(iii) For any $z_n \in G_n$, the measure $\nu = \sum_n \delta_{[\lambda_n, z_n]}$ is a Carleson measure and, moreover, the Carleson constants C_ν of such measures are uniformly bounded with respect to z_n. Here $[\lambda_n, z_n]$ is the straight-line interval with the end points λ_n and z_n, and $\delta_{[\lambda_n, z_n]}$ is the Lebesgue measure on the interval.

It should be noted that there always exist nontrivial sets G_n satisfying (i), (ii) and (ii). More precisely, we can take

$$G_n := \{z \in \mathbb{C}_+ : |z - \lambda_n| < r \,\Im \lambda_n\},$$

for sufficiently small $r > 0$. Indeed, we know that, since $(\tilde{k}^b_{\lambda_n})_{n\geq 1}$ is a Riesz basis in $\mathcal{H}(b)$, then $(\lambda_n)_{n\geq 1}$ is a Carleson sequence, that is,

$$\inf_{k\geq 1} \prod_{n\neq k} \left| \frac{\lambda_n - \lambda_k}{\lambda_n - \bar{\lambda}_k} \right| > 0.$$

In particular, the measure $\nu := \sum_n \Im \lambda_n \, \delta_{\lambda_n}$ is a Carleson measure. Therefore, we see that G_n satisfies (iii). Moreover, Lemma 31.49 below reveals that G_n also satisfies the condition (ii).

We recall from Section 22.3 that

$$w_p(z) = w_{p,1}(z) = \min\left(\|(k^b_z)^2\|_q^{-p/(p+1)}, \; \|\rho^{1/q} \mathfrak{K}^\rho_{z,1}\|_q^{-p/(p+1)} \right).$$

Theorem 31.50 *Let* $(\lambda_n)_{n\geq 1} \subset \mathbb{C}_+ \cup E_2(b)$ *be such that* $(\tilde{k}^b_{\lambda_n})_{n\geq 1}$ *is a Riesz basis in* $\mathcal{H}(b)$ *and let* $p \in [1, 2)$*. Then, for any set* $G = \bigcup_n G_n$ *satisfying* (i), (ii) *and* (iii)*, there is* $\varepsilon > 0$ *such that the system of reproducing kernels* $(\tilde{k}^b_{\mu_n})_{n\geq 1}$ *is a Riesz basis whenever* $\mu_n \in G_n$ *and*

$$\sup_{n\geq 1} \frac{1}{\|k^b_{\lambda_n}\|_b^2} \int_{[\lambda_n, \mu_n]} w_p(z)^{-2}\, |dz| < \varepsilon. \tag{31.42}$$

Proof Since $\mu_n \in G_n$, the condition (ii) implies that $\|k^b_{\mu_n}\|_b \asymp \|k^b_{\lambda_n}\|_b$ and thus $(\tilde{k}^b_{\mu_n})_{n\geq 1}$ is a Riesz basis if and only if $(\kappa^b_{\mu_n})_{n\geq 1}$ is a Riesz basis, where

$$\kappa^b_{\mu_n} = \frac{k^b_{\mu_n}}{\|k^b_{\lambda_n}\|_b}.$$

In view of Lemma 31.48, it suffices to check the estimate

$$\sum_{n=1}^{\infty} |\langle f, \, \tilde{k}^b_{\lambda_n} - \kappa^b_{\mu_n} \rangle_b|^2 \leq \varepsilon \|f\|_b^2 \qquad (f \in \mathcal{H}(b)), \tag{31.43}$$

for sufficiently small $\varepsilon > 0$. But it follows from (31.42) and Corollary 22.21(i) that any f in $\mathcal{H}(b)$ is differentiable in $]\lambda_n, \mu_n[$. Moreover, the set of functions in $\mathcal{H}(b)$ that are continuous on $[\lambda_n, \mu_n]$ is dense in $\mathcal{H}(b)$ (e.g. take the set of reproducing kernels). Therefore, we can prove (31.43) only for functions $f \in \mathcal{H}(b)$ continuous on $[\lambda_n, \mu_n]$. Then

$$|\langle f,\, \tilde{k}^b_{\lambda_n} - \kappa^b_{\mu_n}\rangle_b|^2 = \frac{|f(\lambda_n) - f(\mu_n)|^2}{\|k^b_{\lambda_n}\|_b^2} = \frac{1}{\|k^b_{\lambda_n}\|_b^2}\left|\int_{[\lambda_n,\mu_n]} f'(z)\,dz\right|^2.$$

By the Cauchy–Schwarz inequality and (31.42), we get

$$|\langle f,\, \tilde{k}^b_{\lambda_n} - \kappa^b_{\mu_n}\rangle_b|^2 \le \varepsilon \int_{[\lambda_n,\mu_n]} |f'(z)w_p(z)|^2\,|dz|.$$

It follows from assumption (ii) that $\nu := \sum_n \delta_{[\lambda_n,\mu_n]}$ is a Carleson measure with a constant C_ν that does not exceed some absolute constant depending only on G. Hence, according to Theorem 22.17, we have

$$\begin{aligned}\sum_{n=1}^{\infty} |\langle f,\, \tilde{k}^b_{\lambda_n} - \kappa^b_{\mu_n}\rangle_b|^2 &\le \varepsilon \sum_{n=1}^{\infty} \int_{[\lambda_n,\mu_n]} |f'(z)w_p(z)|^2\,|dz| \\ &= \varepsilon\,\|f'w_p\|^2_{L^2(\nu)} \le C\varepsilon\,\|f\|_b^2,\end{aligned}$$

for a constant C that depends on G, (λ_n) and p. Then Lemma 31.48 implies that we can choose a sufficiently small $\varepsilon > 0$ such that $(\kappa^b_{\mu_n})_{n\ge1}$ is a Riesz basis in $\mathcal{H}(b)$. $\square$

Corollary 31.51 *Let $(\lambda_n) \subset \mathbb{C}_+$, let $(\tilde{k}^b_{\lambda_n})_{n\ge1}$ be a Riesz basis in $\mathcal{H}(b)$, and let $\gamma > 1/3$. Then there is $\varepsilon > 0$ such that the system $(\tilde{k}^b_{\mu_n})_{n\ge1}$ is a Riesz basis whenever*

$$\left|\frac{\lambda_n - \mu_n}{\lambda_n - \overline{\mu_n}}\right| \le \varepsilon(1 - |b(\lambda_n)|)^\gamma. \tag{31.44}$$

Proof As we pointed out before, for sufficiently small $r > 0$, the sets $G_n = \{z : |z - \lambda_n| \le r\,\Im\lambda_n\}$ satisfy the conditions (i), (ii) and (iii). Let $(\mu_n)_{n\ge1}$ satisfy (31.44). Then, we have

$$|\lambda_n - \mu_n| \le \frac{2\varepsilon}{1-\varepsilon}(1 - |b(\lambda_n)|)^\gamma\,\Im\lambda_n. \tag{31.45}$$

Therefore, if ε is sufficiently small, then $\mu_n \in G_n$. Without loss of generality, we can assume that $\gamma < 1$, and since $\gamma > 1/3$, there exists $1 < p < 2$ such that $2(p-1)/(p+1) = 1 - \gamma$. Let q be the conjugate exponent of p and note that $2p/(q(p+1)) = 1 - \gamma$.

Then it follows from Lemma 22.8 that there is a constant $C = C(p) > 0$ such that

$$w_p(z) \ge C\frac{\Im z}{(1 - |b(z)|)^{p/(q(p+1))}} \qquad (z \in \mathbb{C}_+).$$

Therefore, by Lemma 31.49, we have

$$w_p^{-2}(z) \le C_1\frac{(1 - |b(\lambda_n)|)^{1-\gamma}}{(\Im\lambda_n)^2}$$

for $z \in [\lambda_n, \mu_n]$. Hence,

$$\frac{1}{\|k^b_{\lambda_n}\|_b^2} \int_{[\lambda_n,\mu_n]} w_p(z)^{-2} \, |dz| \le C_2 \frac{\Im \lambda_n}{1 - |b(\lambda_n)|} \, |\lambda_n - \mu_n| \, \frac{(1 - |b(\lambda_n)|)^{1-\gamma}}{(\Im \lambda_n)^2}$$

and using (31.45), we obtain

$$\frac{1}{\|k^b_{\lambda_n}\|_b^2} \int_{[\lambda_n,\mu_n]} w_p(z)^{-2} \, |dz| \le C_3 \varepsilon.$$

To complete the proof, take a sufficiently small ε and apply Theorem 31.50. □

It should be noted that all the above statements remain valid if we are interested in the stability of Riesz sequences of reproducing kernels, that is, of systems of reproducing kernels that constitute Riesz bases in their closed linear spans.

Notes on Chapter 31

Starting with the work of Hruščëv, Nikolskii and Pavlov [106], a whole direction of research has investigated geometric properties of reproducing kernels in $\mathcal{H}(b)$ spaces. One of the motivations is the link with nontrigonometric exponential systems. Recall that, in the special case where $b(z) = \exp(a(z+1)/(z-1))$, $a > 0$, the reproducing kernels k^b_λ, with $\lambda \in \mathbb{D}$, arise as the range of the exponential functions $\exp(-i\bar{\mu}w)\chi_{(0,a)}$, with $\mu = i(1+\lambda)/(1-\lambda)$, under a natural unitary map going from $L^2(0,a)$ to $\mathcal{H}(b)$. Geometric properties of the family of exponentials arise in many problems such as scattering theory, controllability and analysis of convolution equations; see [23] and [106] for details. The approach proposed by Hruščëv, Nikolskii and Pavlov has proved fruitful, leading to all the classic results being recaptured and to several generalizations, including many results in the vector-valued setting; see [50, 51, 53, 75–77, 106, 137].

Section 31.1

The completeness problem of reproducing kernels in $\mathcal{H}(b)$ is a difficult and extremely open problem. Very few facts are known even in the inner case; see [76, 106, 128, 137]. A version of Theorem 31.6 in the case when b is inner has appeared in [137]. We also mention the paper of Boricheva [37], who studied the minimality and uniform minimality of $(k^\Theta_{\lambda_n})_n$, where Θ is inner, via some interactions between Λ and Θ involving Schur–Nevanlinna coefficients.

Section 31.2

The unitary rank-one perturbations of $S^*_{|\mathcal{H}(\Theta)}$ were first studied by Clark [55] in the inner case. In the general case, the study is due to Fricain [77].

Section 31.3

A version of Theorem 31.12 appears in [55] in the case when the function b is inner. The general version has been established by Fricain [77]. Corollary 31.14 is due to Clark [55].

Section 31.4

Theorem 31.16 is due to Fricain [77] and it extends a result of Nikolskii [137] in the inner case. The idea of seeing a family of reproducing kernels $(k^{\Theta}_{\lambda_n})_n$ as a distortion of the family of Cauchy kernels $(k_{\lambda_n})_n$ is due to Nikolskii [136]. Theorem 31.17 appears in [77].

Section 31.5

The idea of using the abstract functional embedding (AFE) to study geometric properties of reproducing kernels of $\mathcal{H}(b)$ spaces is due to Chevrot, Fricain and Timotin [53]. Theorem 31.21 and Corollary 31.22 are proved in [53]. Theorem 31.26 is due to Nikolskii [136]; see also [106, 137]. Corollaries 31.30, 31.31 and 31.32 appear in [77] but without proof. They are generalizations of corresponding results for the inner case due to Hruščëv, Nikolskii and Pavlov [106]. Exercise 31.5.5 is a result of Kadec.

As we have seen in this section, if b is not an inner function, then $\mathcal{H}(b)$ has no Riesz basis of reproducing kernels. However, it should be noted that, if b is an inner function, then it is not known if $\mathcal{H}(b)$ has a Riesz basis of reproducing kernels. This problem is stated by Nikolskii in [137] and it is still an open problem, which certainly merits attention.

Section 31.7

The results of this section are inspired by the work of Chalendar, Fricain and Timotin [51] for the inner case.

Section 31.8

The question of stability of completeness and bases finds its roots in the work of Paley and Wiener [144]. Then a lot of stability results appear for exponential systems. For the completeness problem, see the work of Alexander and Redheffer [14], Redheffer [146] and Sedleckiĭ [172]. See also the book of Young [193] for an account of results in this direction.

Theorem 31.44, in the inner case, is due to Fricain [76]. The general case presented here follows exactly along the same lines. It is inspired by an analogous result for exponential systems [193]. Theorem 31.45 is due to Chalendar, Fricain and Timotin [51] in the inner case. The example given at the end of this section follows a construction given by Hruščëv, Nikolskii and Pavlov [106].

Section 31.9

For the basis property, the most well-known stability result is probably the result of Ingham [107] and Kadec [113]. It deals with perturbations of trigonometric exponential systems; see Exercise 31.5.5. Theorem 31.50 is due to Baranov [28] in the inner case and to Baranov, Fricain and Mashreghi [29] in the general case. In the case where $\sup_{n\geq 1} |b(\lambda_n)| < 1$, the stability condition (31.44) is equivalent to

$$\left|\frac{\lambda_n - \mu_n}{\lambda_n - \overline{\mu_n}}\right| \leq \varepsilon,$$

and we essentially get the result of stability obtained in the inner case in [75].

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Symbol index

$\mathfrak{C}_b$	A contraction on $\mathcal{H}(b)$	144
$D_{X_b^*}$	The defect operator of X_b^*	89
D_{X_b}	The defect operator of X_b	286
Δ	The function $\Delta = (1 - \lvert b\rvert^2)^{1/2}$	100
Δ_*	The function $\Delta_* = (1 - \lvert b^*\rvert^2)^{1/2}$	112
$E_n(b)$	The set $E_n(b)$	227
f^*	The conjugation f^* on $L^2(\mathbb{T})$	112
f^+	The element f^+ associated to each function $f \in \mathcal{H}(b)$	278
F	The function $a/(1-b)$ in the nonextreme case	332
F_λ	The function $a/(1-\bar{\lambda}b)$ in the nonextreme case	333
F_Θ	The function $a/(1-\Theta b)$ in the nonextreme case	494
f_Θ	The function F_Θ^2	496
$\mathcal{H}(A)$	The complementary space $\mathcal{H}(A)$	15
$\mathcal{H}(b)$	The de Branges–Rovnyak space $\mathcal{H}(b)$	49
$\mathcal{H}_0(b)$	The space $\mathcal{H}(b) \ominus \overline{\mathcal{M}(a)}$	482
$\mathcal{H}(\bar{b})$	The space $\mathcal{H}(\bar{b})$	51
$\mathscr{H}_b$	The space $\mathscr{H}_b$	403
$H^2(\rho)$	The space $H^2(\rho)$	226
$\mathscr{I}nt(T_1, T_2)$	The operators that interwine T_1 and T_2	407
$\mathfrak{J}$	The unitary operator $\mathfrak{J}$ on $L^2(\mathbb{T})$	112
$\mathfrak{J}(A)$	The Julia operator $\mathfrak{J}(A)$	39
$\mathbf{K}_\rho$	The partial isometry from $L^2(\rho)$ onto $\mathcal{H}(\bar{b})$	138
$\mathbb{K}$	The space $\mathbb{K}$ associated to an (AFE)	102
$\mathbb{K}'$	The space $\mathbb{K}'$ associated to an (AFE)	103
$\mathbb{K}''$	The space $\mathbb{K}''$ associated to an (AFE)	103
$\mathbb{K}_b$	A transcription of the space $\mathbb{K}$ for $b \in H^\infty$, $\lVert b\rVert_\infty \le 1$	109
$\mathbb{K}_b'$	A transcription of the space $\mathbb{K}$ for $b \in H^\infty$, $\lVert b\rVert_\infty \le 1$	109
$\mathbb{K}_b''$	A transcription of the space $\mathbb{K}$ for $b \in H^\infty$, $\lVert b\rVert_\infty \le 1$	109

$\mathscr{K}_b$	The space $\mathscr{K}_b$	403
$k_{z,n}^{\mathcal{H}}$	The kernel of the n-th derivative at z for functions in $\mathcal{H}$	205
k_z^b	Reproducing kernel of $\mathcal{H}(b)$	76
$\widetilde{k}_z^b$	The normalized reproducing kernel of $\mathcal{H}(b)$	567
$k_{z,n}^b$	The kernel of the n-th derivative at z for functions in $\mathcal{H}(b)$	205
$k_{w,n}^\rho$	The kernel $k_{w,n}^\rho$	228
$\mathfrak{K}_{z_0,n}^\rho$	The kernel $\mathfrak{K}_{z_0,n}^\rho$	234
$\hat{k}_z^b$	Difference quotient of $\mathcal{H}(b)$	82
$\mathcal{M}(A)$	The range space equipped with the range norm	3
$\mathbf{M}_b$	The model operator on $\mathbb{K}_b$	111
$\mathscr{M}_b$	The space $\mathscr{M}_b$	403
$\mathcal{M}(u)$	The space $\mathcal{M}(u)$, $u \in L^\infty(\mathbb{T})$	45
$\mathcal{M}(u)$	The space $\mathcal{M}(\bar{u})$, $u \in H^\infty$	47
$\mathfrak{W}_b$	The unitary operator $\mathfrak{W}_b$ between the model spaces $\mathbb{K}_b$ and $\mathbb{K}_{b^*}$	113
Ω_b	A conjugation on $\mathcal{H}(b)$	394
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