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PREFACE

There are several ways of producing new functions under certain conditions when two functions f and T are given, and one of them is to compose them, *i.e.*, to evaluate the old function f at the new points $T(x)$ whenever the range of T is a subset of the domain of f. This new function, as is well known, is called the composite of f and T and is denoted by the symbol $f \circ T$. If f varies in a linear space of functions on the range of T with pointwise linear operations, then the mapping taking f into $f \circ T$ is a linear transformation. This correspondence is known as the composition transformation induced by the function T and we denote it by the symbol C_T. Under certain conditions on T and the linear space on which C_T acts it turns out that C_T is a nice and well behaved operator. Another way of producing a new function when two functions π and f are given is to multiply them whenever it makes sense. This gives rise to a linear transformation known as the multiplication transformation induced by π. The multiplication transformations and the composition transformations under suitable situations breed another class of transformations, known as the weighted composition transformations which we denote by $W_{\pi,T}$ and define as

$$W_{\pi,T}(f) = \pi \cdot f \circ T.$$

This monograph presents a study of these operators on different spaces of functions. So far as we think, the study of the composition operators was initiated with two goals in mind, namely, to have concrete examples of bounded operators on Hilbert spaces or Banach spaces of functions, and to attack the "Invariant Subspace Problem" of functional analysis from a different angle. These goals, as I think had their origin in the thinking of P.R. Halmos and were grasped by E.A. Nordgren who started a systematic study of these operators in 1968. During the last 25 years or so quite a bit has been achieved in respect of the first goal; many concrete examples of the composition operators on L^p–spaces, H^p–spaces or topological vector spaces of functions have been obtained giving a place to these operators in 1991 mathematics subject classification of Mathematical Reviews (47B38). It is interesting to note that in certain situations this class of operators possesses a distinct identity different from several known classes of operators, like the multiplication operators and the integral operators. No composition operator on ℓ^2 is compact, but every composition operator on ℓ^2 has a non-trivial invariant subspace. Similarly, the general invariant subspace problem of operators on a Hilbert space H is linked with the minimal invariant subspaces of an invertible composition operator C_T on $H^2(D)$ [257]. This has been a step forward in the direction of achieving the second goal and much more still is to be achieved.

Whenever a study is undertaken with some specific aims in mind, a lot of other things crop up as byproducts having connections with other branches of knowledge. This has happened with the study of the composition operators too. These operators are being used in statistical mechanics, distribution theory and topological dynamics besides their earlier applications in ergodic theory and classical mechanics.

If we look back at the works done on these operators, then it becomes evident that the most of the study has been concentrated on L^p-spaces, H^p-spaces or the locally convex spaces of continuous functions. An endeavour has been made in this monograph to present most of the results obtained so far. We have been a little prejudiced in presenting the materials in chapter II and chapter IV due to our special interest in the composition operators on L^p-spaces and on the weighted spaces of continuous functions. In order to contain the monograph within the limits some of the results are stated without proofs with appropriate references for the proofs. The background materials are not presented, and hence the readers are expected to have some knowledge of measure theory, analytic function theory and functional analysis to have a smooth sailing through the monograph. Besides being a reference book for researchers, this book may be used for a topic course to the advanced post-graduate students.

Chapter I starts with a broad and unified definition of the composition operator and the weighted composition operator and introduces the concrete spaces of functions on which the study of these operators is carried out in the subsequent chapters. A historical development of the theory of these operators is also presented in this chapter.

Chapter II deals with the composition operators on L^p-spaces. The composition operators on L^p-spaces are characterised and many examples are presented to illustrate the theory. Invertibility, compactness and different types of normality of these operators are studied in this chapter.

The composition operators and the weighted composition operators on functional Banach spaces are the subject matter of chapter III. H^p-spaces and ℓ^p-spaces are concrete examples of functional Banach spaces and results pertaining to the composition operators on these spaces are reported in this chapter. In chapter IV of this monograph we have studied the composition operators and the weighted composition operators on some locally convex spaces of continuous functions and cross sections. These spaces are broad enough to include many nice spaces which are used in analysis.

The composition operators and the weighted composition operators are employed in characterizations of isometries and homomorphisms on some spaces of functions. They have been utilised in the dynamical systems to study different types of motions. The ergodic theory and topological dynamics make use of the composition operators in developement of their theories. Some of these applications of these operators are presented in the last chapter of this book.

We have tried our best to collect and present many known results about these interesting operators, but still many results might not have found their place in this monograph. They can be found in the cited references which we have tried to keep as updated as possible. There will probably be some more references which are not included ; this is because we might not have seen them. We would like to apologise to those whose papers are not listed here. Most of the symbols used are standard ones, like H^p–spaces, L^p–spaces etc., for Hardy spaces, Lebesgue spaces etc., and they are introduced in chapter I. The number 2.4.5 indicates the fifth item of section 4 of chapter II and a reference such as [94] refers to the entry at No. 94 of the bibliography.

The work on this monograph began in the Fall of 1985 when I visited the University of Arkansas, U.S.A., as a Fulbright Faculty Fellow under Indo–U.S. fellowship programme. My several visits to the University of Massachusetts at Boston provided me opportunity to continue my work on this book. Thus five agencies have been involved in the support of the preparation of this monograph, they are : the University of Jammu, the University of Arkansas, the University of Massachusetts, the Council for International Exchange of Scholars and the University Grants Commission. I express my deep sense of thankfulness to each of them.

Normally, behind every work there is some one who gives inspiration, encouragement and motivation ; in this case there has been a man who is a mathematician, has been my teacher and is a trusted friend of mine. His name is V.L.N. Sarma. He introduced me to mathematics, taught me mathematics and inspired me to work in mathematics. I have no words to record my indebtedness and gratitude to Professor Sarma. I am thankful to many colleagues of mine and some of my Ph.D. students who contributed directly or indirectly towards the preparation of this book. They are : Dr. D.K. Gupta, Dr. Ashok Kumar, Dr. B.S. Komal, Dr. S.D. Sharma, Dr. T. Veluchamy, Dr. D. Chandra Kumar, Mr. Bhopinder Singh, Mr. Romesh Kumar. Dr. J. S. Manhas deserves a special thanks as he has collaborated with me in preparation of the last two chapters of the monograph.

Professor William Summers of the University of Arkansas worked as the faculty

associate during my visit under Indo–U.S. fellowship programme and collaborated with me in writing some research articles on composition operators. Professors Herbert Kamowitz and Dennis Wortman of the University of Massachusetts at Boston have read the manuscript of the monograph. They encouraged and inspired me to complete the work whenever I visited them in Boston. I intend to record my deep sense of gratitude and thankfulness to all these three mathematicians. I am indebted to my wife, Krishna whose patience, endurance and encouragement contributed a lot towards the completion of this work.

I am thankful to Professor L. Nachbin of the University of Rochester for inviting me to write this monograph for the series Mathematics Studies. Our thanks are due to the publisher for his co-operation in bringing out this book. The manuscript was typed by Mr. Ranjit Singh and Mr. Manjit Singh of our department and the Camera-ready manuscript was prepared by Mr. Dalip R. Mohun of M/s Shiroch Reprographics, Allahabad. We are thankful to each of them for their excellent job and sincere work pertaining to the manuscript. Errors and mistakes still remaining in the book would be due to our negligence.

May, 1993 R.K. Singh

CONTENTS

CHAPTER I

INTRODUCTION

1.1 DEFINITIONS AND HISTORICAL BACKGROUND

Let X be a non-empty set and let F_x be a vector space over the field $\mathbb{K}$ (where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) for every $x \in X$. Then the Cartesian product $\prod_{x \in X} F_x$ of the family $\{F_x : x \in X\}$ is a vector space under linear operations defined pointwise. Any element of $\prod_{x \in X} F_x$ is known as a cross-section over X, and the family $\{F_x : x \in X\}$ is known as a vector-fibration over X. By $L(X)$ we denote a topological vector space of the cross-sections over X. Let $T : X \to X$ be a mapping such that $f \circ T \in \prod_{x \in X} F_x$ whenever $f \in L(X)$. Then the correspondence $f \to f \circ T$ is a linear transformation from $L(X)$ to $\prod_{x \in X} F_x$ and it is called the composition transformation induced by T. This transformation is denoted by C_T. If π is a mapping defined on X such that $\pi.f \circ T \in \prod_{x \in X} F_x$ whenever $f \in L(X)$, then the correspondence $f \to \pi.f \circ T$ is a linear transformation from $L(X)$ to $\prod_{x \in X} F_x$. This transformation is called the weighted composition transformation induced by π and T, and we denote it by $W_{\pi,T}$. Our interest lies in the case in which C_T and $W_{\pi,T}$ take elements of $L(X)$ into elements of $L(X)$ and $C_T : L(X) \to L(X)$ and $W_{\pi,T} : L(X) \to L(X)$ are continuous. In this case we call C_T and $W_{\pi,T}$ the composition operator and the weighted composition operator on $L(X)$ induced by T and the pair (π, T) respectively. In case $F_x = \mathbb{K}$ for every $x \in X$, $\prod_{x \in X} F_x$ turns out to be the space of all scalar-valued functions on X and $L(X)$ is a topological vector space of complex-valued or real-valued functions. If $F_x = E$ for every $x \in X$, where E is a topological vector space over $\mathbb{K}$, then $\prod_{x \in X} F_x$ is the vector space of all vector-valued (E-valued) functions on X and $L(X)$ is a topological vector space of vector-valued (E-valued) functions. In this monograph an attempt is made to present a theory of these operators, (in particular of composition operators) on different topological vector spaces of functions based on the work done during the last two decades or so.

In order to have a mathematically fruitful and interesting theory of these operators it is desirable to have some structures (algebraic, topological, or a combination) on X and some conditions on the inducing mappings T and π. The following are three major situations occuring in the study of these operators :

(i) The underlying space X is a measure space and the inducing maps are measurable transformations,

(ii) The underlying space X is a region in $\mathbb{C}$ or $\mathbb{C}^n$ and the inducing maps are holomorphic functions,

(iii) The underlying space X is a topological space and the inducing maps are continuous functions.

In the first situation $L(X)$ is taken to be a topological vector space of measurable functions such as L^p–spaces; in the second case $L(X)$ is taken to be a topological vector space of analytic functions, like a Hardy space or a Bergman space or a Dirichlet space; and in the third case $L(X)$ is taken to be a topological vector space of continuous functions. There is quite a bit of overlap between these three cases and study becomes more interesting on the overlapping portion as it can be viewed from several angles. In the first situation in which $L(X)$ is an L^p–space the theory of the composition operators establishes a contact with ergodic theory, entropy theory, and classical mechanics; in the second case the theory touches differentiable dynamics, statistical mechanics, and the theory of distributions; and the third situation appears in topological dynamics, transformation groups, and in study of the continuous functions (see [136], [171], [237], [238], [270]).

So far as we know, the earliest appearance of a composition transformation was in 1871 in a paper of Schroeder [302], where it is asked to find a function f and a number α such that

$$(f \circ T)(z) = \alpha f(z)$$

for every z in an appropriate domain, whenever the function T is given. If z varies in the open unit disk and T is an analytic function, then a solution has been obtained by Koenigs [183] in 1884. In 1925 these operators were employed in Littlewood's subordination theory [221]. In the early 1930's the composition operators were used to study problems in mathematical physics and classical mechanics, especially worth mention is the work of Koopman ([199], [200]) on classical mechanics. In those days these operators were known as the substitution operators or translation operators. Banach used these operators to characterise isometries on Banach spaces of continuous functions [80]. In the 1940's and 1950's composition operators appeared in the work of Von Neumann and Halmos ([136], [140]) in the study of ergodic transformations. In 1966 Choksi [68] studied unitary composition operators and continued the study in later years.

The history of a systematic study of the composition operators is not that old. It was started in 1968 by Nordgren in his paper [253] though it had its appearance in 1966 in a paper of Ryff [297]. In his paper Nordgren studied composition operators on L^2 of the unit circle induced by inner functions. In 1969 Schwartz [303] and Ridge [278] wrote their Ph. D. theses on composition operators on H^p-spaces and composition operators on L^p-spaces, respectively. In 1972 Singh [324] completed his doctoral dissertation on composition operators under the supervision of Professor Nordgren. After this a group of mathematicians including Boyd [41], Caughran ([60], [61]), Cima ([73], [74]), Kamowitz ([160], [161]), Shapiro [309], Swanton ([384], [385]), and Wogen ([73], [74]) plunged into the study of composition operators on different function spaces. A series of lectures on composition operators by Nordgren at the Long Beach Conference on "Hilbert Space Operators" in 1977 ([254]) gave a further boost to the study of the composition operators and several more mathematicians such as Cowen ([83], [84]), Deddens [91], Iwanik [151], Roan [281], Whitley [399], etc., started their explorations of the properties of composition operators. In this way the 1970's had been a very fruitful decade so far as the study of composition operators on L^p-spaces and H^p-spaces is concerned. A group of research workers led by Singh had been engaged in studying composition operators at Jammu since 1973. The systematic study initiated in the 1970's has been continued and extended in several directions during the last decade. Worth mention are some new names such as Gupta [127], Komal [187], Kumar [202], Lambert [147], MacCluer ([228], [229], [230]), Mayer ([237], [238]), Sharma [311], Stanton [72], Summers [368], Takagi [387], Veluchamy [396], etc., joining the earlier group in exploration of the properties of the composition operators on different function spaces. In the later half of the last decade the study of these operators on spaces of continuous functions initiated by Kamowitz [164] picked up momentum. Feldman [109], Jamison and Rajagopalan [153], Singh and Summers [368], Singh and Manhas [352, 354], Takagi [388] made systematic studies of these operators on several spaces of continuous functions. Much has been known about this interesting, simple, but rich class of operators, but still there is much more to be explored.

This monograph aims at presenting a more or less consolidated and unified account of the systematic work done on the composition operators and related topics. Most of the results presented in this monograph are published in different mathematical journals; still there are many results which are either unpublished or in the process of publication. Particularly, Chapter IV contains many unpublished results which the authors have obtained recently.

Before proceeding to the next chapter, we would like to introduce the underlying spaces of functions on which the composition operators have been studied. We can divide these spaces into three broad categories :

(i) L^p–spaces.

(ii) Functional Banach spaces of functions.

(iii) Locally convex function spaces.

The next three sections of this chapter aim at introducing these spaces to the readers.

1.2 L^p–SPACES

Let $(X, \mathcal{S}, m)$ be a measure space and let p be a real number such that $1 \le p < \infty$. Let $\mathcal{L}^p(m)$ denote the set of all complex–valued measurable functions on X such that $|f|^p$ is m-integrable. Then $\mathcal{L}^p(m)$ is a complex linear space under the operations of pointwise addition and scalar multiplication. If $N^p(m)$ denotes the set of all null functions on X, then $N^p(m)$ is a subspace of $\mathcal{L}^p(m)$. Let $L^p(m)$ denote the quotient space $\mathcal{L}^p(m)/N^p(m)$. An element in $L^p(m)$ is a coset of the type $f+N^p(m)$, where f belongs to $\mathcal{L}^p(m)$. The coset $f+N^p(m)$ is denoted as $[f]$. Thus two functions g and h of $\mathcal{L}^p(m)$ belong to the same coset if and only if $g = h$ almost everywhere. On $L^p(m)$ we define a norm as

$$\|[f]\|_p = \left(\int |f|^p \, dm\right)^{\frac{1}{p}}.$$

Using the Minkowski inequality it can be shown that $L^p(m)$ is a normed linear space under the above norm. Under this norm $L^p(m)$ is complete. Thus $L^p(m)$ is a Banach space. The conjugate space of $L^p(m)$ is $L^q(m)$, where p and q are conjugate indices. For $p = 2$, $L^p(m)$ is a Hilbert space under the inner product defined as

$$\langle [f],\ [g] \rangle = \int f\bar{g}\ dm.$$

If X has a non-empty subset of measure zero, then it is evident that the elements of $L^p(m)$ are not functions on X but they are equivalence classes of functions on X. Two elements of $\mathcal{L}^p(m)$ are equivalent if they agree almost everywhere. Under this identification we regard $L^p(m)$ as a Banach space of functions. We shall take f as an element of $L^p(m)$ instead of taking $[f]$ as an element of $L^p(m)$. This function f represents all those functions of $\mathcal{L}^p(m)$ which are equal to f *a. e.*

If $X = \mathbb{N}$, the set of natural numbers, $\mathcal{S} = P(\mathbb{N})$, the power set of $\mathbb{N}$ and m the counting measure, then we denote the corresponding $L^p(m)$ by ℓ^p. This ℓ^p is the classical sequence space. A sequence $\{\alpha_n\}$ of complex numbers belongs to ℓ^p if $\sum_{n=1}^{\infty} |\alpha_n|^p < \infty$. The space ℓ^2 is the classical example of a Hilbert space given by Hilbert himself. If $w = \{w_n\}$ is a sequence of non-negative real numbers, and if the measure m on $P(\mathbb{N})$ is defined as

$$m(S) = \sum_{n \in S} w_n ,$$

then the corresponding $L^p(m)$ is denoted by $\ell^p(w)$. This is called the weighted sequence space with w as the sequence of weights.

A complex valued measurable function f on X is said to be essentially bounded if there exists an $M > 0$ such that the measure of the set

$$\{x \ :\ x \in X \ \text{ and } \ |f(x)| > M\}$$

is zero. By $\|f\|_\infty$ we denote the smallest such M, which is called the essential supremum of f. Let $\mathcal{L}^\infty(m)$ denote the set of all essentially bounded functions on X. Then $\mathcal{L}^\infty(m)$ is a linear space. By $L^\infty(m)$ we denote the quotient space $\mathcal{L}^\infty(m)/N^\infty$, where N^∞ is the subspace of null functions. With the essential supremum norm $L^\infty(m)$ becomes a Banach space. The symbol ℓ^∞ stands for the Banach space of all bounded sequences of complex numbers.

A measurable space $(X, \mathcal{B})$ is said to be a standard Borel space if X is a Borel subset of a complete separable metric space and $\mathcal{B}$ is the σ–algebra of Borel sets. Many measure theoretic pathologies in the study of composition operators are avoided if the underlying space is a standard Borel space. Most of the nice and useful examples of measurable spaces are standard Borel spaces.

1.3 FUNCTIONAL BANACH SPACES OF FUNCTIONS

Let X be a non-empty set and let $H(X)$ be a Banach space of complex–valued functions on X with pointwise addition and scalar multiplication. Let $x \in X$. Let δ_x be the mapping on $H(X)$ taking f into $f(x)$. Then it is obvious that δ_x is a linear functional on $H(X)$; it is called the evaluation functional induced by x. The space $H(X)$ is said to be a functional Banach space if each evaluation functional δ_x is continuous *i.e.*, if $\delta_x \in H^*(X)$ for every $x \in X$, where $H^*(X)$ denotes the dual space of $H(X)$. There are some Banach spaces of functions which are not functional Banach spaces, but there are quite a few which are functional Banach spaces. We shall introduce them later in this section.

In case $H(X)$ is a functional Hilbert space, using the Riesz-representation theorem for every $x \in X$ we can find a unique $f_x \in H(X)$ such that

$$g(x) = \delta_x(g) = \langle g, f_x \rangle$$

for every $g \in H(X)$. The function f_x is called the kernel function of X induced by x. Let $K(X) = \{f_x \ : \ x \in X\}$. Then $K(X)$ is a subset of $H(X)$. The complex valued function K defined on $X \times X$ as

$$K(x,y) = \langle f_y, f_x \rangle$$

is called the reproducing kernel of $H(X)$. It is clear that

$$K(x,y) = \delta_x(f_y) = f_y(x)$$

and

$$\bar{K}(x,y) = \delta_y(f_x) = f_x(y)$$

for every x and y in X. If $\{e_j \ : \ j \in J\}$ is an orthonormal basis for $H(X)$, then the reproducing kernel K of $H(X)$ is given by

$$K(x,y) = \sum_{j \in J}{}' e_j(x)\overline{e_j(y)} \qquad \text{[See [137] for details].}$$

Examples

The following are some of the familiar examples of functional Banach spaces.

(1.3.1) ℓ^p- SPACES

Let X be any (countable) set and let m be the counting measure defined on the power set of X. Then $L^p(m)$, denoted as $\ell^p(X)$ is a functional Banach space for $1 \le p \le \infty$. The continuity of the evaluation functionals follows from the fact that

$$|\delta_x(f)| = |f(x)| \le \| f \|.$$

In particular the unitary space $\mathbb{C}^n$ and the classical sequence space ℓ^p are functional Banach spaces. In case $p = 2$, $\ell^p(X)$ is a functional Hilbert space and the reproducing kernel of $\ell^2(X)$ is given by

$$K(x,y) = \begin{cases} 1, & x = y \\ 0, & x \ne y. \end{cases}$$

Thus the reproducing kernel of ℓ^2 is the characteristic function of the diagonal of $\mathbb{N} \times \mathbb{N}$.

(1.3.2) H^p-SPACES

Let $X = D$, the open unit disk in the complex plane and let $1 \le p < \infty$. Let $H^p(D)$ denote the vector space of holomorphic functions f on D such that

$$\sup_{0<r<1} \left\{ \frac{1}{2\pi} \int_0^{2\pi} \left| f\left(re^{i\theta}\right)\right|^p d\theta \right\} < \infty.$$

Then it is well-known that $H^p(D)$ is a Banach space under the norm defined as

$$\| f \| = \left\{ \sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} \left| f\left(re^{i\theta}\right)\right|^p d\theta \right\}^{\frac{1}{p}}.$$

If $f \in H^p(D)$ and $z \in D$, then we know from [99] that

$$|\delta_z(f)| = |f(z)| \le \frac{2^{1/p}\|f\|}{(1-|z|)^{1/p}}.$$

Hence δ_z is continuous for every $z \in D$. Thus $H^p(D)$ is a functional Banach space. In case $p = 2$, $H^p(D)$ is a functional Hilbert space and the reproducing kernel of $H^2(D)$, known as Szego kernel, is given by

$$K(x, y) = 1/(1 - x\bar{y}),$$

where x and y vary in D [137].

Let $n \in \mathbb{N}$ and let $X = D^n$, the Cartesian product of n copies of the disk D. Let $H^p(D^n)$ denote the vector space of all those holomorphic functions f on D^n such that

$$\|f\|^p = \left\{\sup_{0<r<1} \int_{(\partial D)^n} |f(rw)|^p \, dm_n(w)\right\} < \infty,$$

where m_n is the normalized Lebesgue measure on $(\partial D)^n$, ∂D denoting the boundary of D. Then $H^p(D^n)$ is a functional Banach space since

$$|f(z)| \le \|f\| \prod_{k=1}^{n} \left(1 - |z_k|^2\right)^{-\frac{1}{p}}$$

for $z = (z_1, z_2, \ldots, z_n) \in D^n$. For details see Rudin [287], and Singh and Sharma [359]. $H^2(D^n)$ is a functional Hilbert space. Similarly, if $X = D_n$, the unit ball of $\mathbb{C}^n$, then space $H^p(D_n)$ of all those holomorphic functions f on D_n, for which

$$\|f\|^p = \sup_{0<r<1} \left\{\int_{\partial D_n} |f(rw)|^p \, d\sigma(w)\right\} < \infty,$$

where σ is the rotation invariant Borel probability measure on the boundary ∂D_n of D_n, is a functional Banach space. We refer readers to Rudin [289] for further details about these spaces.

Let $X = P^+$, the upper-half plane and $H^p(P^+)$ denote the vector space of all those holomorphic functions f on P^+ for which

$$\| f \|^p = \sup_{y>0} \left\{ \int_{-\infty}^{\infty} | f(x+iy) |^p \, dx \right\} < \infty .$$

Then with this norm $H^p(P^+)$ is a functional Banach space since

$$|\delta_w(f)| = |f(w)| \le \frac{\|f\|}{(2\pi y)^{1/p}} ,$$

where $w = x+iy \in P^+$. For details see Duren [99] or Hoffman [146]. $H^2(P^+)$ is a functional Hilbert space and the reproducing kernel K is given by

$$K(w, u) = i / 2\pi(w - \bar{u}).$$

Note. All these spaces given in the set of examples are known as Hardy spaces and have very rich structures on them. More general Hardy spaces have been studied, but they are not presented here as we are not studying composition operators on them in this monograph.

(1.3.3) BERGMAN SPACES

Let $X = G$, a non–empty open connected subset of $\mathbb{C}$ and let m be the area Lebesgue measure on G. Let $A^p(G)$ be the linear space of all holomorphic functions f on G such that

$$\| f \|^p = \int_G | f |^p \, dm < \infty .$$

With the above norm $A^p(G)$ becomes a functional Banach space and for $p = 2$, $A^p(G)$ is a functional Hilbert space, where the inner product is given by

$$\langle f, g \rangle = \int_G f \bar{g} \, dm .$$

These spaces $A^p(G)$ for $1 \le p < \infty$ are known as Bergman spaces. In case $G = D$, the reproducing kernel of $A^2(D)$ is given by

$$K(x, y) = (1 / \pi) \left(1 / (1 - x\bar{y})^2\right)$$

for $x, y \in D$. This kernel is known as the Bergman kernel.

(1.3.4) SPACE OF BOUNDED FUNCTIONS

Let $H_b(X)$ denote the vector space of all complex-valued bounded functions on X. For $f \in H_b(X)$ we define $\| f \|$ as

$$\| f \| = \sup \left\{ | f(x) | \ : \ x \in X \right\}.$$

Then with this norm $H_b(X)$ is a functional Banach space since

$$| f(x) | \leq \| f \| .$$

1.4 LOCALLY CONVEX FUNCTION SPACES

Let X be a topological space, let E be a topological vector space, and let $A(X, E)$ be the vector space of all functions from X to E with linear operations defined pointwise. Then by a locally convex space of functions on X we mean a subspace $F(X, E)$ of $A(X, E)$ together with a family of seminorms making it into a seminormed linear space. If $E = \mathbb{K}$, then we write $F(X)$ for $F(X, \mathbb{K})$. Every Banach space of functions is a locally convex space of functions, but there are many locally convex spaces of functions which are not Banach spaces, nor even normed linear spaces; for example the space $C(X, E)$ of all continuous E-valued functions with compact-open topology, where X is non-compact and E is a locally convex space, is a locally convex space which is not normable.

If $F(X)$ is a locally convex space of functions on X, then functional δ_x defined on $F(X)$ as $\delta_x(f) = f(x)$, lies in the algebraic dual $F'(X)$ of $F(X)$ and the mapping taking x into δ_x is an embedding of X into $F'(X)$ and in this way X can be viewed as a subset of $F'(X)$. In case δ_x is continuous for every $x \in X$, it is clear that the topology of $F(X)$ is finer than the topology of pointwise convergence on $F(X)$. There could be many interesting locally convex topologies on $F(X)$ between these two topologies. In case X is a compact Hausdorff space and $F(X) = C(X)$, the space of continuous functions with supremum norm, we know that $C(X)$ is a Banach algebra and X turns out to be the maximal ideal space of $C(X)$. All examples given in Section 1.3 of this chapter are locally convex spaces of functions. In case of ℓ^p-spaces, we just endow $\mathbb{N}$ with discrete topology.

Now we shall define the weighted spaces of continuous functions and the weighted spaces of cross-sections which are also locally convex spaces of functions, and cover a wide range of function spaces.

A real–valued function f on a topological space X is said to be upper semicontinuous, abbreviated as *u.s.c.* if the inverse image of $(-\infty, a)$ is open for every real number a; equivalently, if the inverse image of $[a, \infty)$ is closed in X for every $a \in \mathbb{R}$. If the inverse image of (a, ∞) is open for every $a \in \mathbb{R}$, then the functiion f is said to be lower semicontinuous. Every continuous function is both *u.s.c.* and *l.s.c.*; it is clear that if a real-valued function f is both *u.s.c.* and *l.s.c.*, then it is continuous. If f is a characteristic function of a non-empty proper closed subset of X, then f is *u.s.c.* but not continuous.

A mapping $v : X \to \mathbb{R}^+$ is called a 'weight' on X if it is *u.s.c.*, where $\mathbb{R}^+$ is the set of all positive reals with usual topology. If V is a set of weights on X such that given any $x \in X$, there is some $v \in V$ for which $v(x) > 0$, we write $V > 0$. A set V of weights on X is said to be directed upward provided that for every pair $u_1, u_2 \in V$ and $\alpha > 0$, there exists $v \in V$ so that $\alpha u_i \leq v$ (pointwise on X) for $i = 1, 2$. By a system of weights we mean a set V of weights on X which additionally satisfies $V > 0$. Let $W(X)$ denote the family of all systems of weights on X. Let U and V be in $W(X)$. Then we say that $U \leq V$ if for every $u \in U$, there exists $v \in V$ such that $u \leq v$. This relation on $W(X)$ is reflexive and transitive. If $U \leq V$ and $V \leq U$, then we say that U and V are equivalent systems of weights on X and we write it as $U \approx V$. Let X be a completely regular Hausdorff space and let E be a Hausdorff locally convex topological vector space over $\mathbb{C}$. Let $cs(E)$ be the set of all continuous seminorms on E. By $C(X, E)$ we mean the collection of all continuous functions from X into E. If V is a system of weights on X, then the pair (X, V) is called a weighted topological system. Associated with each weighted topological system (X, V) we have the weighted spaces of continuous E–valued functions defined as:

$$CV_0(X, E) = \{f \in C(X, E) : v f \text{ vanishes at infinity on } X \text{ for each } v \in V\}.$$
$$CV_p(X, E) = \{f \in C(X, E) : vf(X) \text{ is precompact in } E \text{ for all } v \in V\},$$

and

$$CV_b(X, E) = \{f \in C(X, E) : vf(X) \text{ is bounded in } E \text{ for all } v \in V\}.$$

Obviously, $CV_0(X,E), CV_p(X,E)$ and $CV_b(X,E)$ are vector spaces and

$CV_p(X,E) \subset CV_b(X,E)$ while the upper semicontinuity of the weights yields that $CV_0(X,E) \subsetneq CV_p(X,E)$. Let $v \in V$, $q \in cs(E)$ and $f \in C(X,E)$. If we define

$$\| f \|_{v,q} = \sup \left\{ v(x)\, q(f(x)) \;:\; x \in X \right\},$$

then $\| \,.\, \|_{v,q}$ can be regarded as a seminorm on either $CV_b(X,E)$, $CV_p(X,E)$ or $CV_0(X,E)$ and the family $\left\{ \| \,.\, \|_{v,q} \;:\; v \in V,\, q \in cs(E) \right\}$ of seminorms defines a Hausdorff locally convex topology on each of these spaces. This topology will be denoted by ω_V, and the vector spaces $CV_0(X,E)$, $CV_p(X,E)$ and $CV_b(X,E)$ endowed with ω_V, are called the weighted locally convex spaces of vector-valued continuous functions. It has a basis of closed absolutely convex neighbourhoods of the origin of the form

$$B_{v,q} = \left\{ f \in CV_b(X,E) \;:\; \| f \|_{v,q} \leq 1 \right\}.$$

We also note that $CV_0(X,E)$ is a closed subspace of $CV_b(X,E)$. In case $E = \mathbb{C}$, we will omit E from our notation and write $CV_0(X)$, $CV_p(X)$ and $CV_b(X)$ in place of $CV_0(X,\mathbb{C})$, $CV_p(X,\mathbb{C})$ and $CV_b(X,\mathbb{C})$ respectively. We also write $\| \,.\, \|_v$ in place of $\| \,.\, \|_{v,q}$ for each $v \in V$, where $q(z) = |z|$, $z \in \mathbb{C}$. Moreover, if $E = (E,q)$ is any normed linear space and $v \in V$, we write $\| \,.\, \|_v$ instead of $\| \,.\, \|_{v,q}$. If U and V are two systems of weights on X such that $U \leq V$, then clearly $CV_0(X,E) \subset CU_0(X,E)$, $CV_p(X,E) \subset CU_p(X,E)$ and $CV_b(X,E) \subset CU_b(X,E)$, as well as the inclusion map is continuous in all cases.

The spaces $CV_0(X)$ and $CV_b(X)$ were first introduced by Nachbin [246], and the corresponding vector-valued analogues $CV_0(X,E), CV_p(X,E)$ and $CV_b(X,E)$ were subsequently studied in detail by Bierstedt [30] and Prolla [272]. For more details and examples of these spaces we refer to Bierstedt ([30], [31], [32]), Nachbin [246], Prolla ([272], [273], [274]) and Summers [380]. Besides examples, many standard spaces of continuous functions can be realized in this general setting and we shall list here certain instances for future reference. Let X be a completely regular Hausdorff space. We denote by χ_S, the characteristic function of a subset S of X. We have the following four systems of weights on X:

$V^1 = \left\{ \alpha\chi_K \;:\; \alpha \geq 0,\; K \subset X,\; K \text{ compact} \right\}$,

$V^2 = C_c^+(X)$, the set of all positive continuous functions on X with compact supports,

$V^3 = K^+(X)$, the set of all positive constant functions on X, and

$V^4 = C_0^+(X)$, the set of all positive continuous functions on X vanishing at infinity.

Further, if E is a locally convex space, then we define

$C_0(X,E) = \{f \in C(X,E) \ : \ f$ vanishes at infinity on $X\}$,

$C_p(X,E) = \{f \in C(X,E) \ : \ f(X)$ is precompact in $E\}$,

$C_b(X,E) = \{f \in C(X,E) \ : \ f(X)$ is bounded in $E\}$.

Example 1.4.1. Let X be a locally compact Hausdorff space, and let E be a locally convex space. Then

(i) $CV_0^1(X,E) = CV_p^1(X,E) = CV_b^1(X,E) = (C(X,E),\ k)$, where k denotes the compact -open topology ;

(ii) $CV_0^2(X,E) = CV_p^2(X,E) = CV_b^2(X,E) = (C(X,E),\ k)$;

(iii) $CV_0^3(X,E) = (C_0(X,E),u)$,

$CV_p^3(X,E) = (C_p(X,E),u)$, and

$CV_b^3(X,E) = (C_b(X,E),u)$, where u denotes the topology of uniform convergence on X, and

(iv) $CV_0^4(X,E) = CV_p^4(X,E) = CV_b^4(X,E) = (C_b(X,E),\beta)$, where β denotes the strict topology.

In order to introduce the weighted spaces of cross-sections, we need the following definitioins.

Let $\{F_x \ : \ x \in X\}$ be a vector fibration over X. Then by a weight on X we mean a function w defined on X such that $w(x)$ is a seminorm on F_x for each $x \in X$. For our convenience, we shall use the notation w_x for the seminorm $w(x)$ for each $x \in X$. By $w \leq w'$ we mean that $w_x \leq w'_x$, for every $x \in X$. Let W be a set of weights on X. Then W is said to be directed upward, if, for every pair $w, w' \in W$ and $\lambda > 0$, there exists $w'' \in W$ such that $\lambda\, w \leq w''$ and $\lambda\, w' \leq w''$. We hereafter assume that each set W of weights is directed upward. We write $W > 0$ if, for given $x \in X$, and $y \in F_x$, there is some $w \in W$ for which $w_x(y) > 0$. If a set W of weights on X additionally satisfies $W > 0$, then we shall refer W as a system of weights on X. If f is a cross-section over X and w is a weight on X, then we will denote by $w\,[f]$, the positive–valued function on X which takes x into

$w_x[\,f(x)\,]$. The weighted spaces of cross-sections over X with respect to the system of weights W are defined below :

$$LW_0(X) = \{f \in L(X) \;:\; w[f] \text{ is upper semicontinuous and vanishes at infinity on } X \text{ for each } w \in W\}$$

and

$$LW_b(X) = \{f \in L(X) \;:\; w[f] \text{ is a bounded function on } X \text{ for each } w \in W\}.$$

It is clear that $LW_0(X)$ and $LW_b(X)$ are vector spaces and $LW_0(X) \subset LW_b(X)$. Now, for $w \in W$ and $f \in L(X)$, if we define

$$\| f \|_w = \sup\{w_x[f(x)] \;:\; x \in X\},$$

then $\| . \|_w$ can be regarded as a seminorm on either $LW_b(X)$ or $LW_0(X)$ and the family $\{\| . \|_w \;:\; w \in W\}$ of seminorms defines Hausdorff locally convex topology on each of these spaces. We shall denote this topology by τ_W, and the vector–spaces $LW_0(X)$ and $LW_b(X)$ endowed with τ_W are called the weighted locally convex spaces of cross-sections. It also has a basis of closed absolutely convex neighbourhoods of the origin of the form

$$B_w = \{f \in LW_b(X) \;:\; \| f \|_w \leq 1\}.$$

For more details and examples of these weighted spaces of cross-sections we refer to ([32], [234], [247], [248], [274]). For the illustrations of these weighted spaces of cross-sections we are presenting the following examples :

Example 1.4.2. The weighted spaces $CV_0(X,E)$ and $CV_b(X,E)$ which have been defined earlier, can be written as weighted spaces of cross-sections, *e.g.*, $CV_0(X,E)$ is certainly a weighted space $L\tilde{W}_0(X)$ of cross-sections $f = (f(x))_{x \in X}$ with respect to the vector-fibration $\{F_x \;:\; x \in X\}$, where $F_x = E$ for each $x \in X$, and the set $\tilde{W} = \{\tilde{w}_{v,p} \;:\; v \in V,\; p \in cs(E)\}$ of weights on X defined by

$$\tilde{w}_{v,p}(x)\,[y] = v(x)p(y), \text{ for each } x \in X \text{ and } y \in E.$$

In the same way $CV_b(X,E)$ can be represented as a weighted space of cross-sections.

Example 1.4.3. Let $B(LW_0(X))$ denote the space of all continuous linear operators on $LW_0(X)$, the weighted space of cross-sections. Prolla [274] has represented the space $B(LW_0(X))$ as a weighted space of cross-sections over X with fibers $B(LW_0(X), F_x)$. The representation is given as follows :

For each $x \in X$, let $\delta_x : LW_0(X) \to F_x$ be the point evaluation at x, *i.e.*, $\delta_x(f) = f(x)$, for all $f \in LW_0(X)$. For $A \in B(LW_0(X))$, $\delta_x \circ A \in B(LW_0(X), F_x)$, and $R : A \to \hat{A} = (\delta_x \circ A)_{x \in X}$ represents $B(LW_0(X))$ as a weighted space of cross-sections over X with fibers $B(LW_0(X), F_x)$. A complete illustration of this example is given in [274]. For more examples of the weighted spaces of cross-sections we refer to Bierstedt [32].

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CHAPTER II

COMPOSITION OPERATORS ON L^p-SPACES

In this chapter we will concentrate on the study of the composition operators on L^p–spaces for $1 \leq p \leq \infty$. These spaces, known as Lebesgue spaces, often occur in several branches of mathematical physics. A characterization of measurable transformations inducing composition operators is given first and then the operators on L^p which are composition operators are characterized ; several examples are presented to illustrate the theory. The invertibility and compactness of these operators are studied in sections 2 and 3, respectively. The last section of this chapter studies different kinds of normality of the composition operators.

2.1 DEFINITIONS, CHARACTERIZATIONS AND EXAMPLES

Let $(X, \mathcal{S}, m)$ be a measure space. Then a mapping T from X into X is said to be a measurable transformation in case $T^{-1}(S) \in \mathcal{S}$ for every $S \in \mathcal{S}$. A measurable transformation T is said to be non-singular if $m(T^{-1}(S)) = 0$ whenever $m(S) = 0$. If T is non-singular, then the measure mT^{-1}, defined as $mT^{-1}(S) = m(T^{-1}(S))$ for every $S \in \mathcal{S}$, is an absolutely continuous measure on $\mathcal{S}$ with respect to m. If m is a σ–finite measure, then by the Radon-Nikodym theorem there exists a non-negative function f_T in $L^1(m)$ such that

$$mT^{-1}(S) = \int_S f_T dm$$

for every $S \in \mathcal{S}$. The function f_T is called the Radon-Nikodym derivative of mT^{-1} with respect to m.

Every non-singular transformation T from X into itself induces a linear transformation C_T on $L^p(m)$ into the linear space of all measurable functions on X, defined as

$$C_T f = f \circ T$$

for every $f \in L^p(m)$. In case C_T is continuous from $L^p(m)$ into itself, then it is called a composition operator on $L^p(m)$ induced by T.

Note. The non-singularity of T guarantees that C_T is well-defined, which may not be true in case T is not non-singular. For example, let X be the unit interval $[0,1]$ with Lebesgue measure and let T take x to $2x$ when $0 \le x \le \frac{1}{2}$ and to 1 whenever $\frac{1}{2} < x \le 1$. Then clearly T is not non-singular. Let f be the characteristic function of the half-open interval $[0,1)$ and g be the characteristic function of $[0,1]$. Then f and g represent the same element of $L^p(m)$ whereas $f \circ T$ and $g \circ T$ do not do so (*i.e.*, $f = g$ *a. e.*, but $C_T f \ne C_T g$ *a.e.*). Thus C_T is not well-defined. It has become clear now that the non-singularity of T is a necessary condition for it to induce a composition operator on $L^p(m)$.

In case of $L^\infty(m)$ it is not difficult to check that this condition of non-singularity is both necessary and sufficient for T to define a composition operator. If T is non-singular, then clearly $f \circ T \in L^\infty(m)$ whenever $f \in L^\infty(m)$ and

$$\| C_T f \|_\infty = \| f \circ T \|_\infty \le \| f \|_\infty .$$

Thus C_T is a contraction. It is an isometry on $L^\infty(m)$ if and only if m and mT^{-1} are equivalent, *i.e.* $m \ll mT^{-1}$ and $mT^{-1} \ll m$. ($\ll$ denotes the absolute continuity).

In case of $L^p(m)$, $1 \le p < \infty$, it turns out that the non-singularity of T alone is not a sufficient condition for it to induce a composition operator. For example, let X be the real line with the Lebesgue measure and let $T(x) = e^x$. Then clearly T is non-singular, but C_T is not even into map on $L^p(-\infty,\infty)$. This follows from the fact that $f = \chi_{[0,1]}$, the characteristic function of $[0,1]$ belongs to $L^p(-\infty,\infty)$ but $C_T f (= \chi_{[-\infty,0]})$ does not belong to $L^p(-\infty,\infty)$, for $1 \le p < \infty$. The following theorm characterizes the measurable transformations inducing composition operators on $L^p(m)$ for $1 \le p < \infty$.

Theorem 2.1.1. Let $(X, \mathcal{S}, m)$ be a σ-finite measure space and let $T : X \to X$ be a measurable transformation. Then T induces a composition operator C_T on $L^p(m)$ if and only if there exists a $b > 0$ such that

$$mT^{-1}(S) \le b\ m(S)$$

for every $S \in \mathcal{S}$.

Proof. Suppose C_T is the composition operator induced by T. If $S \in \mathcal{S}$ such that $m(S) < \infty$, then $\chi_S \in L^p(m)$ and

$$mT^{-1}(S) = \left\| C_T \chi_S \right\|^p \leq \| C_T \|^p \left\| \chi_S \right\|^p = \| C_T \|^p m(S).$$

Let $b = \left\| C_T \right\|^p$. Then

$$mT^{-1}(S) \leq b\ m(S).$$

If $m(S) = \infty$, then inequality is trivial.

Conversely, suppose that the condition is true. Then $mT^{-1} << m$; and hence the Radon-Nikodym derivative f_T of mT^{-1} with respect to m exists and

$$f_T \leq b \quad a.e.$$

Let $f \in L^p(m)$. Then

$$\| C_T f \|^p = \int | f \circ T |^p dm = \int | f |^p dmT^{-1} = \int | f |^p f_T dm \leq b \| f \|^p.$$

This shows that C_T is a bounded operator on $L^p(m)$. This completes the proof of the theorem.

Corollary 2.1.2. A measurable transformation $T : X \to X$ defines a composition operator on $L^p(m)$ $(1 \leq p < \infty)$ if and only if $mT^{-1} << m$ and $f_T \in L^\infty(m)$. In this case

$$\| C_T \| = \| f_T \|_\infty^{1/p} .$$

Proof. The proof follows from the above theorem.

Example 2.1.3. Let X be a locally compact abelian group and let m be the Haar measure on σ-algebra of Borel sets. Let $y \in X$. Then define $T_y : X \to X$ as

$$T_y(x) = yx$$

for every $x \in X$. Then C_{T_y} is a composition operator on $L^p(m)$ for $1 \leq p \leq \infty$. In particular let X be the real line with usual topology and with addition as the group operation. Then $T_y(x) = x + y$. The corresponding composition operators C_{T_y} are known as the translation operators and they have appeared in the work of Koopman [199] on classical mechanics.

Example 2.1.4. Let $(X, \mathcal{S}, m)$ be any σ-finite measure space and let T be a measure preserving transformation. Then C_T is an isometry on $L^p(m)$. In case T is invertible and $p = 2$, C_T is unitary operator. These operators have been used in ergodic theory and entropy theory [270].

Example 2.1.5. Let $X = \mathbb{Z}$, the set of all integers, and let $w = \{w_i : i \in \mathbb{Z}\}$ be a doubly infinite sequence of positive real numbers. Let $S \subset \mathbb{Z}$. Then define the measure m as

$$m(S) = \sum_{i \in S} w_i.$$

The L^p-space of this measure m will be denoted by $\ell^p(\mathbb{Z}_w)$, which is called a weighted doubly infinite sequence space. If the mapping $T : \mathbb{Z} \to \mathbb{Z}$ is defined by

$$T(i) = i - 1,$$

then the corresponding C_T is a weighted bilateral shift operator on $\ell^p(\mathbb{Z}_w)$. In case m is the counting measure and $p = 2$ the operator C_T is the classical bilateral shift operator on $\ell^2(\mathbb{Z})$. If $X = \mathbb{N}$ with the counting measure, then the mapping $T(i) = i + 1$ induces the adjoint of the unilateral shift on ℓ^2. For a detailed study of shift operators, see Shields [320].

Example 2.1.6. Let $T : \mathbb{Z} \to \mathbb{Z}$ be defined as $T(i) = -i$. Then C_T is the symmetry operator on $\ell^2(\mathbb{Z})$ which takes the i^{th} basis vector e_i into e_{-i} for every $i \in \mathbb{Z}$.

Example 2.1.7. If X is the real line with the Lebesgue measure and T is a monotonic function such that $1/T'$ is essentially bounded, then C_T is a continuous operator on $L^p(-\infty,\infty)$, where T' denotes the derivative of T. In particular, if $a > 0$ and $T(x) = ax$, then C_T is a composition operator on $L^p(-\infty,\infty)$ [329].

Example 2.1.8. Let X be the unit circle in the complex plane with the normalized Lebesgue measure and let T be a non-constant inner function. Then it has been shown in [253] that C_T is a composition operator on $L^2(m)$ and the norm of C_T is given by

$$\| C_T \|^2 = \frac{1+|T(0)|}{1-|T(0)|}.$$

Example 2.1.9. Let $X = \mathbb{N}$ with the counting measure. Then every injective map T from $\mathbb{N}$ to $\mathbb{N}$ induces a composition operator on ℓ^p and $\| C_T \| = 1$. The details can be obtained from [336].

Example 2.1.10. Let $(X, \mathcal{S}, m)$ be a measure space and let Y be a measurable subset of X, *i.e.*, $Y \in \mathcal{S}$. Let $T : Y \to X$ be measurable transformation. Let $f \in L^p(m)$. Then define $C_T f$ as

$$C_T f(x) = \begin{cases} f(T(x)) \ , & \text{if } \ x \in Y \\ 0 \ , & \text{if } \ x \in X \setminus Y \ . \end{cases}$$

If it is continuous on $L^p(m)$, then it is called the generalized composition operator induced by T and Y. It is clear that every composition operator is a generalized composition operator ; in this case $Y = X$. We can view the unilateral shift on ℓ^2 as a generalized composition operator. Let $X = \mathbb{Z}_+$ the set of all non-negative integers with the counting measure. Let $Y = \mathbb{N} \subset \mathbb{Z}_+$. Let $T : \mathbb{N} \to \mathbb{Z}_+$ be defined as $T(i) = i-1$. Then C_T is exactly the unilateral shift operator on ℓ^2.

Example 2.1.11. Let $X = \mathbb{R}$ with Lebesgue measure and let $T : \mathbb{R} \to \mathbb{R}$ be a rational function. Then C_T is a composition operator on $L^p(\infty, \infty)$ if and only if $\deg P > \deg Q$ and $T' \neq 0$ whenever it exists, where $T = P/Q$, P and Q being polynomials. In particular if T is a polynomial, then C_T is a composition operator on $L^p(\infty, \infty)$ if and only if $P' \neq 0$. For details see [329].

We have seen that Theorem 2.1.1 characterizes the mappings T which induce (generalized) composition operators on $L^p(m)$ in terms of the absolute continuity of the measure mT^{-1} and their Radon-Nikodym dervative f_T. Now we plan to characterize those operators (bounded) on $L^p(m)$ which are the composition operators. If an operator is a composition operator, it takes the characteristic functions of $L^p(m)$ into the characteristic functions, which are in a way building blocks for $L^p(m)$. This is clear because $C_T \chi_S = \chi_{T^{-1}(S)}$. This condition of taking the characteristic functions of $L^p(m)$ into the characteristic functions turns out to be a sufficient condition for a continuous operator to be a composition operator on L^p-spaces of certain nice measure spaces. These nice spaces are standard Borel spaces which are defined to be isomorphic to Borel subsets of complete separable metric spaces. Most of the measure spaces which we encounter in analysis are standard Borel spaces. We need to develop some machinery before we prove the theorem.

Let $(X, \mathcal{S}, m)$ be a σ-finite measure space. Then the σ-algebra $\mathcal{S}$ of measurable sets is a ring under the algebraic operations of symmetric difference and intersection as addition and multiplication respectively. Let $\mathfrak{Z}$ denote the collection of sets of measure zero, *i.e.*, $\mathfrak{Z} = \{S : S \in \mathcal{S} \text{ and } m(S) = 0\}$ Then $\mathfrak{Z}$ is an ideal of the ring $\mathcal{S}$. Let $\mathcal{S} / \mathfrak{Z}$ denote the quotient ring which is also a Boolean ring, since $\mathcal{S}$ is so. A mapping $h : \mathcal{S} \to \mathcal{S}$ is called a σ-endomorphism if it preserves the operations of symmetric difference and intersection, countable unions and the maximal element. Every σ-endomorphism h on $\mathcal{S}$ gives rise to a σ-endomorphism h' on $\mathcal{S} / \mathfrak{Z}$ defined by

$$h'(S \,\Delta\, \mathfrak{Z}) = h(S) \,\Delta\, \mathfrak{Z}.$$

If $T : X \to X$ is a measurable transformation , then the mapping $h_T : \mathcal{S} \to \mathcal{S}$ defined by

$$h_T(S) = T^{-1}(S)$$

is a σ-endomorphism on $\mathcal{S}$ and hence h'_T is a σ-endomorphism on $\mathcal{S} / \mathfrak{Z}$. Thus every measurable point transformation induces σ-endomorphisms on $\mathcal{S}$ and $\mathcal{S} / \mathfrak{Z}$. More generally, if $(X', \mathcal{S}', m')$ is a measure space and $T : X' \to X$ is a measurable transformation, then it induces σ-homomorphism $h_T : \mathcal{S} \to \mathcal{S}'$ and $h'_T : \mathcal{S} / \mathfrak{Z} \to \mathcal{S}' / \mathfrak{Z}'$, where $\mathfrak{Z}'$ denotes the ideal of sets of measure zero in $\mathcal{S}'$. The converse of the above statement is not true in general measure spaces. But if X is a standard Borel space, then it is shown by Sikorski in [323] that every σ-homomorphism from $\mathcal{S} / \mathfrak{Z}$ to $\mathcal{S}' / \mathfrak{Z}'$ is induced by a measurable transformation from X' to X. This result we shall record in the following theorem.

Theorem 2.1.12. Let $(X, \mathcal{S}, m)$ be a standard Borel space, let $(X', \mathcal{S}', m')$ be a measure space, and let ϕ be a σ-homomorphism from $\mathcal{S} / \mathfrak{Z} \to \mathcal{S}' / \mathfrak{Z}'$. Then there exists a measurable transformation $T : X' \to X$ such that

$$\phi = h'_T$$

Note. The proof of this theorem uses the rich structure of the Borel sets.

Let $\mathcal{K}$ denote the family of all characteristic functions of the members of $\mathcal{S}$, *i.e.*,

$$\mathcal{K} = \left\{\chi_S \ : \ S \in \mathcal{S}\right\}.$$

Let $\mathcal{K}^p = \mathcal{K} \cap L^p(m)$. Then the following theorem presents a characterization of (generalized) composition operators in terms of the invariance of $\mathcal{K}^p$.

Theorem 2.1.13. Let $(X, \mathcal{S}, m)$ be a standard Borel space and let A be an operator on $L^p(m)$. Then A is a (generalized) composition operator if and only if $\mathcal{K}^p$ is invariant under A, *i.e.*, $A\,\mathcal{K}^p \subset \mathcal{K}^p$.

Proof. Suppose A is a (generalized) composition operator on $L^p(m)$. Then there exists a measurable set $Y \in \mathcal{S}$ and a measurable transformation T from Y to X such that $A = C_T$. Let $\chi_S \in \mathcal{K}^p$, then $A\,\chi_S \in L^p\,(m)$. But

$$A\,\chi_S = C_T\,\chi_S = \chi_{T^{-1}(S)}\,.$$

Thus $A\chi_S \in \mathcal{K}^p$.

Conversely, suppose $A\,\mathcal{K}^p \subset \mathcal{K}^p$. Let $S \in \mathcal{S}$ be of finite measure. Then $\chi_S \in \mathcal{K}^p$. Hence $A\chi_S \in \mathcal{K}^p$. Thus there exists $W \in \mathcal{S}$ such that $A\chi_S = \chi_W$. Let us define $\phi_0\,(S) = W$. Thus ϕ_0 is defined on the collection of sets of finite measures. If S_1 and S_2 are disjoint measurable sets of finite measures, then

$$\begin{aligned} A\,(\chi_{S_1 \cup S_2}) &= A\,(\chi_{S_1} + \chi_{S_2}) \\ &= A\chi_{S_1} + A\chi_{S_2} \\ &= \chi_{W_1} + \chi_{W_2}. \end{aligned}$$

This shows that $m\,(W_1 \cap W_2) = 0$ and

$$\phi_0(S_1 \cup S_2) = \phi_0\,(S_1) \cup \phi_0\,(S_2).$$

It can be shown that ϕ_0 preserves intersection and difference. Since m is a σ-finite measure, there exists a sequence $\{S_i\}$ of pairwise disjoint measurable sets of finite measures such that

$$X = \bigcup_{i=1}^{\infty} S_i.$$

Let $X_i = \phi_0(S_i)$, *i.e.*, $A\chi_{S_i} = \chi_{X_i}$ for $i \in \mathbb{N}$, and let $X' = \bigcup_{i=1}^{\infty} X_i$. If S is any arbitrary member of $\mathcal{S}$, then we can write $\phi_0(S)$ as

$$\phi_0 (S) = \bigcup_{i=1}^{\infty} \phi_0 (S \cap S_i) \ .$$

It can be shown that $\phi_0 : \mathcal{S} \to \mathcal{S}'$ is a σ-homomorphism, where $\mathcal{S}'$ is a σ-algebra of measurable subsets of X'. This σ-homomorphism gives rise to a σ-homomorphism $\phi : \mathcal{S} / \mathfrak{X} \to \mathcal{S}' / \mathfrak{X}'$ defined as

$$\phi (S \,\Delta\, \mathfrak{X}) = \phi_0(S) \,\Delta\, \mathfrak{X}'.$$

Hence by Theorem 2.1.12 there exists a measurable transformation $T : X' \to X$ such that $\phi = h'_T$ If $m(S) < \infty$, then

$$A\chi_S = \chi_{T^{-1}(S)} = C_T\chi_S \ .$$

Thus A and C_T agree on $\mathcal{K}^p$ and hence on $L^p(m)$. This shows that $A = C_T$. This completes the proof of the theorem.

Corollary 2.1.14. Let $(X, \mathcal{S}, m)$ be a standard Borel space. Then an operator A on $L^p(m)$ is a generalized composition operator if and only if $A(f.g) = Af.Ag$, whenever f, g and $f.g$ belong to $L^p(m)$.

Proof. If $\chi_S \in \mathcal{K}^p$, then

$$A\chi_S = A\chi_S^2 = A\chi_S\, A\chi_S = (A\chi_S)^2.$$

This shows that $A\chi_S \in \mathcal{K}^p$. Hence by Theorem 2.1.13, A is a generalized composition operator.

Note. From the proof of Theorem 2.1.13 it is evident that if $A\,\mathcal{K}^p \subset \mathcal{K}^p$ and $X' = X$, then A is a composition operator on $L^p(m)$, rather than a generalized composition operator. If A is an injective operator with $A\,\mathcal{K}^p \subset \mathcal{K}^p$, then it can be deduced that $X' = X$ and hence A is a composition operator. This argument together with Theorem 2.1.13 gives the result of Von Neumann [251], which states that a unitary operator on $L^p(m)$ is a composition operator if and only if it is multiplicative. The invertible operators on ℓ^p taking basis vectors into basis vectors are composition operators. This we record in the following corollary.

Corollary 2.1.15. If A is an invertible operator on ℓ^p, then A is a composition operator if and only if the set $\{e_n \ : \ n \in \mathbb{N}\}$ is invariant under A, where e_n is the sequence defined as $e_n(k) = \delta_{nk}$, the Kronecker delta.

2.2 INVERTIBLE COMPOSITION OPERATORS

From now on in this chapter we shall restrict our study of the composition operators on $L^2(m)$ which has Hilbert space structure. Some of the results may be carried over to general L^p-spaces with minor changes in the proofs.

If π is an essentially bounded complex -valued measurable function on X, then the mapping M_π on $L^2(m)$, defined by $M_\pi f = \pi . f$, is a continuous operator with range in $L^2(m)$. This operator M_π is known as the multiplication operator induced by π. If C_T is a composition operator on $L^2(m)$, then it turns out that $C_T^* C_T$ is a multiplication operator and $C_T C_T^*$ is close to a multiplication operator. The following theorem presents these results.

Theorem 2.2.1. Let C_T be a composition operator on $L^2(m)$. Then

(i) $C_T^* C_T = M_{f_T}$.

(ii) $C_T C_T^* = M_{f_T \circ T} P$, where P is the projection of $L^2(m)$ on the closure of the range of C_T.

(iii) C_T has dense range if and only if $C_T C_T^* = M_{f_T \circ T}$.

Proof. (i) Let $f, g \in L^2(m)$. Then

$$\begin{aligned}\left\langle C_T^* C_T f, g\right\rangle = \langle C_T f, C_T g\rangle &= \int f \, \bar{g} \, dmT^{-1} \\ &= \int f_T \, f \, \bar{g} \, dm \\ &= \left\langle M_{f_T} f, g\right\rangle.\end{aligned}$$

Thus $C_T^* C_T = M_{f_T}$.

(ii) Let $f \in L^2(m)$. Then Pf belongs to the closure of the range of C_T. Hence there exists a sequence $\{C_T f_n\}$ in the range of C_T converging to Pf in norm. Thus

$$C_T\, C_T^*\, P\, f = \lim_n C_T\, C_T^*\, C_T\, f_n$$

$$= \lim_n C_T(f_T \;\; f_n)$$

$$= M_{f_T \circ T}\, P\, f.$$

Since $f - P f$ belongs to the orthogonal complement of the range of C_T which is equal to the kernel of C_T^*, we can conclude that $C_T\, C_T^*\, f = C_T\, C_T^*\, P\, f$. Thus

$$C_T C_T^* f = M_{f_T \circ T} P\, f \quad \text{for all} \quad f \in L^2(m).$$

Hence $C_T C_T^* = M_{f_T \circ T} P$.

(iii) If C_T has dense range, then by (ii) $P = I$, the identity operator. Hence $C_T C_T^* = M_{f_T \circ T} P$. Conversely, suppose $C_T C_T^* = M_{f_T \circ T} P$. Since $f_T \circ T \neq 0$ *a.e.*, $C_T C_T^*$ is an injection. Since C_T^* and $C_T\, C_T^*$ have the same kernel, we have the desired result. This completes the proof of the theorem.

Note. Although the results in the above theorem look very simple, they have many useful applications, as will be seen in this section and several other later sections. The results of this theorem were obtained within a time span of about ten years by Singh [325], Singh and Kumar [340], and Harrington and Whitley [144].

The following theorem characterizes injective composition operators in several ways.

Theorem 2.2.2. Let C_T be a composition operator on $L^2(m)$. Then the following are equivalent :

(i) C_T is an injection ;

(ii) f and $f \circ T$ have the same essential range for every $f \in L^2(m)$;

(iii) $m \ll m \circ T^{-1}$;

(iv) f_T is different from zero almost everywhere.

Proof. (i) $\Rightarrow$ (ii) : Assume C_T is an injection. The essential range of $f \circ T$ is always contained in the essential range of f for every f in $L^2(m)$. To prove the reverse inclusion let a belong to the essential range of f. Let G be a neighbourhood of a. Then $m(f^{-1}(G)) \neq 0$ by definition of essential range. Since C_T is an injection, we can conclude that $m(T^{-1}(f^{-1}(G))) \neq 0$. Thus a belongs to the essential range of $f \circ T$.

(ii) $\Rightarrow$ (iii) : Let $S \in \mathcal{S}$ such that $m T^{-1}(S) = 0$. Then the essential range of $C_T \chi_S$ is equal to the singleton set $\{0\}$. Hence by (ii) the essential range of χ_S is equal to $\{0\}$. This implies that $m(S) = 0$. Thus $m << m \circ T^{-1}$.

(iii) $\Rightarrow$ (iv) : This implication follows from the following equation :

$$m T^{-1}(S) = \int_S f_T \, dm.$$

(iv) $\Rightarrow$ (i) : Suppose f_T is different from zero almost everywhere. Then it is well-known that the corresponding multiplication operator M_{f_T} is an injection. Hence by part (i) of Theorem 2.2.1, $C_T^* C_T$ is an injection. Therefore, C_T is an injection. This completes the proof of the theorem.

Note. By the equality of two measurable sets S_1 and S_2 we mean $m(S_1 \Delta S_2) = 0$.

A measurable transformation T on a measure space $(X, \mathcal{S}, m)$ is said to be left invertible (or one-to-one) if there exists a measurable transformation T_1 such that $T_1 \circ T$ differs from the identity transformation atmost on a set of measure zero. Similarly, if $T \circ T_2$ differs from the identity transformation on a set of measure zero for some measurable transformation T_2 , then T is said to be right invertible (or onto). T is said to be invertible if

$$(T \circ U)(x) = (U \circ T)(x) \ \ a.e.,$$

for some measurable transformation U.

The following corollaries are consequences of Theorem 2.2.2.

Corollary 2.2.3. If C_T is an injection, then $C_T f$ is a characteristic function if and only if f is so.

Corollary 2.2.4. A composition operator C_T on ℓ^2 is an injection if and only if T is onto.

Corollary 2.2.5. Let C_T be a composition operator on $L^2(m)$. Let T be right invertible, and let a right inverse of T be non-singular. Then C_T is an injection.

Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be two σ-subalgebras of $\mathcal{S}$. Then $\mathcal{S}_1$ and $\mathcal{S}_2$ are said to be equivalent (written as $\mathcal{S}_1 = \mathcal{S}_2$) if for every $S_1 \in \mathcal{S}_1$ there exists $S_2 \in \mathcal{S}_2$ such that $S_1 = S_2$ and vice versa. If T is a measurable transformation, then $T^{-1}(\mathcal{S})$ is a σ-subalgebra of $\mathcal{S}$, where

$$T^{-1}(\mathcal{S}) = \left\{T^{-1}(S) \;:\; S \in \mathcal{S}\right\}.$$

The L^2-space with respect to the σ-subalgebra $T^{-1}(\mathcal{S})$, which we denote as $L^2(X, T^{-1}(\mathcal{S}), m)$, is a subspace of $L^2(m)$ and the range of every composition operator is a subspace (not necessarily closed) of this space. It turns out that the range of every composition operator is dense in $L^2(X, T^{-1}(\mathcal{S}), m)$. This we shall show in the following theorem.

Theorem 2.2.6. If C_T is a composition operator on $L^2(m)$, then the range of C_T is dense in $L^2(X, T^{-1}(\mathcal{S}), m)$.

Proof. In case X is of finite measure and $\chi_{S_1} \in L^2(X, T^{-1}(\mathcal{S}), m)$, we can obtain an $S_2 \in \mathcal{S}$ such that

$$\chi_{S_1} = \chi_{T^{-1}(S_2)} = C_T\, \chi_{S_2}.$$

Thus χ_{S_1} belongs to the range of C_T and hence all simple functions of $L^2(X, T^{-1}(\mathcal{S}), m)$ belong to the range of C_T. Thus the range of C_T is dense in $L^2(X, T^{-1}(\mathcal{S}), m)$. In case X is a σ-finite measure space the proof follows from the Lebesgue dominated convergence theorem.

In the following theorem we characterize surjective composition operators.

Theorem 2.2.7. A composition operator C_T on $L^2(m)$ is a surjection if and only if f_T is bounded away from zero on its support and $T^{-1}(\mathcal{S}) = \mathcal{S}$. (By the support of f_T we mean the set of all those points of X where f_T is non-zero).

Proof. By using the Hilbert space theory it can be proved that C_T has closed range if and only if $C_T^* C_T$ has closed range. But by part (i) of Theorem 2.2.1,

$C_T^* C_T = M_{f_T}$. Thus C_T has closed range if and only if M_{f_T} has closed range. But we know that a multiplication operator has closed range if and only if the inducing function is bounded away from zero on its support [340]. Thus C_T has closed range if and only if f_T is bounded away from zero on its support. Now to prove the theorem, suppose C_T is a surjection. Hence f_T is bounded away from zero on its support. Let $S \in \mathcal{S}$ be of finite measure. Then, since C_T is onto, there exists a function $f \in L^2(m)$ such that

$$C_T f = \chi_S .$$

Let $S_2 = \{x : x \in X \text{ and } f(x) = 1\}$. Then

$$C_T \chi_{S_2} = \chi_S$$

and hence $T^{-1}(S_2) = S$. Thus $S \in T^{-1}(\mathcal{S})$. This shows that $\mathcal{S} \subset T^{-1}(\mathcal{S})$. Since $T^{-1}(\mathcal{S})$ is a subalgebra of $\mathcal{S}$, we can conclude that $\mathcal{S} = T^{-1}(\mathcal{S})$.

On the other hand, suppose $T^{-1}(\mathcal{S}) = \mathcal{S}$ and f_T is bounded away from zero. Then by Theorem 2.2.6 and the argument given in the beginning of the proof we can conclude that the range of C_T is equal to L^2 (m). Thus C_T is a surjection. This completes the proof of the theorem.

Now the following results follow from the above theorem and some earlier results.

Corollary 2.2.8. A composition operator C_T on $L^2(m)$ has dense range if and only if $T^{-1}(\mathcal{S}) = \mathcal{S}$.

Corollary 2.2.9. If T is left invertible, then C_T has dense range.

Corollary 2.2.10. A composition operator C_T on ℓ^2 has dense range if and only if T is left invertible.

Now the following theorem gives a general characterization of invertibility of composition operators in terms of the properties of the Radon-Nikodym derivatives f_T and the subalgebra $T^{-1}(\mathcal{S})$.

Theorem 2.2.11. A composition operator C_T on L^2 (m) is invertible if and only if f_T is bounded away from zero (almost everywhere) on X and $T^{-1}(\mathcal{S}) = \mathcal{S}$.

Proof. The proof follows from Theorem 2.2.2. and Theorem 2.2.7.

Note. It is clear from the work presented in this section so far that the underlying σ-algebra $\mathcal{S}$ of the measurable sets plays a very important role in the invertibility of the composition operators. For example, if $\mathcal{S} = \{\phi, X\}$, then every composition operator on $L^2(m)$ is invertible. On the other hand, if we take the σ-algebra of all subsets of X, then the chances of invertibility of the composition operators are not that bright. The invertibility of C_T does not in general imply the invertibility of T, and vice versa. We shall quote the following examples to illustrate this observation.

Example 2.2.12. Let $X = [0, 1]$ with the Lebesgue measure m on the Borel subsets. Let $T(x) = \sqrt{x}$ for all $x \in X$. Then C_T is a composition operator on $L^2(m)$ by Example 2.1.7. It is clear that T is invertible ; $U(x) = x^2$ is an inverse of T. But since $\left\| C_T \, \chi_{[0, 1/n]} \right\|^2 / \left\| \chi_{[0, 1/n]} \right\|^2$ is equal to $1/n$ for every natural number n, we can conclude that C_T is not bounded away from zero, and hence it is not invertible.

Example 2.2.13. Let X be the same as in the above example, let C be the Cantor ternary set, and let h be the Cantor function. Let

$$T(x) = \frac{1}{2}\, x + \frac{1}{2}\, h(x) \quad \text{for} \quad x \in X.$$

Then T is a monotone function with $T'(x) = \frac{1}{2}$; and hence it defines a composition operator on $L^2(m)$. It can be proved that the characteristic function of $T(C)$ belongs to the kernel of C_T. Since $m(T(C)) = \frac{1}{2}$, the kernel of C_T is non-trivial. This shows that C_T is not invertible, although, T is so.

Now it is evident from the above theorem, note and examples that invertibility of a composition operator in the general case cannot be characterized in terms of invertibility of the inducing functions. However, if the underlying measure space is a standard Borel space, then this characterization is possible to some extent. This we shall prove in the following theorem.

Theorem 2.2.14. Let $(X, \mathcal{S}, m)$ be a standard Borel space and C_T be a composition operator on $L^2(m)$. Then C_T is invertible if and only if T is invertible with an inverse which induces a composition operator on $L^2(m)$.

Proof (outline). In case T is invertible and U is an inverse of T which induces a composition operator C_U on $L^2(m)$, it is clear that

$$C_T\, C_U = C_{UoT} = I = C_U\, C_T,$$

where I is the identity operator on $L^2(m)$. Thus C_T is invertible.

Conversely, suppose C_T is invertible. Then by Corollary 2.2.3, C_T^{-1} (the inverse of C_T) carries $\mathcal{H}^2$ into $\mathcal{H}^2$. Hence by Theorem 2.1.13 and the subsequent note, we conclude that C_T^{-1} is a composition operator on $L^2(m)$. Thus there exists a measurable transformation U such that $C_T^{-1} = C_U$. Since $C_T^{-1}C_T = C_U C_T = C_{ToU} = I$, we conclude that $f \circ T \circ U = f$ for every function f in $L^2(m)$. From this it can be deduced that $T \circ U(x) = x$ *a.e.* Similarly, $U \circ T(x) = x$ *a.e.* Thus U is an inverse of T defining the composition operator C_U. This takes care of the proof of the theorem.

Note. From the above theorem it is evident that if C_T is invertible, then C_T^{-1} is a composition operator. In general, this may not be true. We did not discuss the invertibility of generalized composition operators because they are not surjective in case $Y \neq X$. The above theorem holds form all measure spaces where σ-homomorphisms are induced by point mappings.

2.3 COMPACT COMPOSITION OPERATORS

An operator A on a Banach space is said to be compact if the image of the unit ball under A has compact closure. For an operator on a Hilbert space to be compact it should have compact image since the image of the unit ball is closed. In the case of a separable Hilbert space, compactness of an operator is equivalent to saying that it takes weakly convergent sequences into norm convergent sequences, *i.e.*, if $x_n \to x$ weakly, then $Ax_n \to Ax$ in the norm of the Hilbert space. In this section we shall study compact composition operators on $L^2(m)$. It turns out that there is a dearth of compact composition operators. Actually no composition operator on L^2 of a non-atomic measure space is compact. Even on ℓ^2, which is the L^2-space of an atomic measure space, no composition operator is comapct. But there are some weighted sequence spaces which do have compact composition operators.

Let $(X, \mathcal{S}, m)$ be a measure space, let $\varepsilon > 0$, and let π be a complex-valued measurable function on X. Then the set $\{x : x \in X \text{ and } |\pi(x)| > \varepsilon\}$ is denoted by X_ε^π. Let Z_ε^π be defined by

$$Z_\varepsilon^\pi = \left\{f\chi_{X_\varepsilon^\pi} : f \in L^2(m)\right\}.$$

Then Z_ε^π is a subspace of $L^2(m)$. An element of $L^2(m)$ belongs to Z_ε^π if it vanishes outside X_ε^π. The following theorem characterizes compact multiplication operators on $L^2(m)$ in terms of the dimension of Z_ε^π.

Theorem 2.3.1. Let M_π be a multiplication operator on $L^2(m)$. Then M_π is compact if and only if Z_ε^π is finite dimensional for every $\varepsilon > 0$.

Proof. Let M_π be compact. Then since Z_ε^π is invariant under M_π, we can conclude that the restriction of M_π to Z_ε^π is also compact. Since π is bounded away from zero on X_ε^π, we conclude that M_π is invertible on Z_ε^π [137]. Thus Z_ε^π is finite-dimensional.

Conversely, if Z_ε^π is finite-dimensional for every ε, then in particular $Z_{1/n}^\pi$ is finite-dimensional for every $n \in \mathbb{N}$. Let π_n be defined as

$$\pi_n = \pi\chi_{X_{1/n}^\pi}.$$

Then M_{π_n} is a finite rank operator and $M_{\pi_n} \to M_\pi$ in norm. Hence M_π is a compact operator. This completes the proof.

The following well-known result follows from the above theorem.

Corollary 2.3.2. If $(X, \mathcal{S}, m)$ is a non-atomic measure space, then no non-zero multiplication operator is compact.

Now the following theorem characterizes the compact composition operators on $L^2(m)$.

Theorem 2.3.3. Let C_T be a composition operator on $L^2(m)$. Then C_T is compact if and only if $Z_\varepsilon^{f_T}$ is finite-dimensional for every $\varepsilon > 0$.

Proof. Suppose C_T is compact. Then $C_T^* C_T$ is compact. Hence, by Theorem 2.1.15, M_{f_T} is compact. Thus by Theorem 2.3.1, $Z_\varepsilon^{f_T}$ is finite-dimensional. The converse follows from Theorem 2.3.1, and the fact that $C_T^* C_T$ is compact if and only if C_T is compact. This completes the proof of the theorem.

Corollary 2.3.4. [Ridge [279]]. If $(X, \mathcal{S}, m)$ is a non-atomic measure space, then no composition operator on $L^2(m)$ is compact.

Proof. If C_T is a compact composition operator on $L^2(m)$, then M_{f_T} is compact. Thus by Corollary 2.3.2, M_{f_T} is the zero operator. Hence C_T is the zero operator. This completes the proof, since no composition operator is the zero operator.

If $(X, \mathcal{S}, m)$ is a σ-finite measure space, then we can write $X = X_1 \cup X_2$, where X_1 and X_2 are non-atomic and atomic parts of X, respectively. Without loss of generality we can assume that atoms are points. In the following theorem we record necessary conditions for the compactness of the multiplication operators and the composition operators on $L^2(m)$. We assume that X has non-trivial non-atomic and atomic parts, *i.e.*, $m(X_1) \neq 0$ and $m(X_2) \neq 0$.

Theorem 2.3.5.

(i) A multiplication operator M_π on $L^2(m)$ is compact only if $\pi = 0$ *a.e.* on X_1.

(ii) If M_π is an injective multiplication operator on $L^2(m)$, then M_π is compact implies that $(X, \mathcal{S}, m)$ is an atomic measure space.

(iii) if C_T is a composition operator on $L^2(m)$, then $T^{-1}(X_2) = X$ whenever C_T is compact.

(iv) If $m(X) = \infty$, then the compactness of a composition operator C_T implies that $m(X_2) = \infty$.

Proof.

(i) Let m_2 be the restriction of the measure m on X_2 and let $m_1 = m - m_2$ [342]. It is clear that $L^2(m_1)$ is an invariant subspace of M_π; and hence $\pi = 0$ *a.e.* on X_1 by Corollary 2.3.2.

(ii) In case M_π is compact, the kernel of M_π includes $L^2(m_1)$. Since M_π is one–to–one, $L^2(m_1) = \{0\}$. Hence $m_1 = 0$. Thus $m = m_2$.

(iii) If C_T is compact, then M_{f_T} is compact. Hence by part (i), f_T is zero almost everywhere on X_1. Hence $mT^{-1}(X_1) = \int_{X_1} f_T \, dm = 0$. Thus

$$X = T^{-1}(X_2)\,.$$

(iv) In case C_T is compact and $m(X_2) < \infty$, by part (iii) and Theorem 2.1.1 we get a contradiction.

From the above theorems it is evident that the hope for existence of the compact composition operators lies in the case when the underlying measure space is atomic. The sequence spaces are examples of L^2–spaces of atomic measure spaces. In the rest of this section we shall study compact composition operators on Hilbert spaces of sequences.

Let $T : \mathbb{N} \to \mathbb{N}$ be a mapping and let $\varepsilon > 0$. Then define the set N_ε as

$$N_\varepsilon = \left\{ n : n \in \mathbb{N} \text{ and } mT^{-1}(\{n\}) > \varepsilon\, m(\{n\}) \right\}.$$

The following theorem characterizes compact composition operators on $\ell^2(w)$, which is defined in chapter I.

Theorem 2.3.6. Let C_T be a composition operator on $\ell^2(w)$. Then C_T is compact if and only if N_ε is a finite subset of $\mathbb{N}$ for every $\varepsilon > 0$.

Proof. Suppose $\{f_i\}$ is a sequence in $\ell^2(w)$ converging weakly to zero and let $\varepsilon > 0$. Suppose N_ε is finite with k elements. By Theorem 2.1.1, there exists $b > 0$ such that $mT^{-1}(\{n\}) \leq b(m(\{n\}))$ for every $n \in \mathbb{N}$. Thus

$$\begin{aligned}\|C_T\, f_i\|^2 = \int_{\mathbb{N}} |f_i|^2 dmT^{-1} &= \int_{N_\varepsilon} |f_i|^2 dmT^{-1} + \int_{\mathbb{N}\setminus N_\varepsilon} |f_i|^2 dmT^{-1} \\ &\leq bk\, |f_i(n_r)|^2 \; m(\{n_s\}) + \varepsilon \|f_i\|^2\,,\end{aligned}$$

where $|f_i(n_r)| = \max\left\{|f_i(n_t)| : n_t \in N_\varepsilon\right\}$ and $m(\{n_s\}) = \max\left\{m(\{n_t\}) : n_t \in N_\varepsilon\right\}$.

Since $\{f_i\}$ tends to zero pointwise, for $\varepsilon_1 > 0$ we can find $j \in \mathbb{N}$ such that for $i > j$ we have

$$\|C_T f_i\|^2 \leq \varepsilon_1\; bk\,.\, m(\{n_s\}) + \varepsilon \|f_i\|^2.$$

Since every weakly convergent sequence is norm bounded, we can conclude that the sequence $\{\|C_T f_i\|\}$ converges to 0. This shows that C_T is compact.

Conversely, suppose N_ε contains infinitely many elements for some $\varepsilon > 0$. Then C_T is bounded away from zero on the closure of the span of $\{e_j : j \in N_\varepsilon\}$. Thus the range of the restriction of C_T is a closed infinite dimensional subspace contained in the range of C_T. Thus by Problem 141 of [137], C_T is not compact. This completes the proof of the theorem.

The following corollaries are immediate consequences of the above theorem.

Corollary 2.3.7. A composition operator C_T on $\ell^2(w)$ is compact if and only if the sequence

$$\left\{\frac{mT^{-1}(\{i\})}{m(\{i\})}\right\} \text{ tends to zero as } i \text{ tends to } \infty.$$

Corollary 2.3.8. No composition operator on ℓ^2 is compact.

If $a > 0$ is a real number, and $w_i = a^i$ for $i \in \mathbb{N}$, then the corresponding $\ell^2(w)$ is denoted by ℓ_a^2. The following theorem characterizes composition operators on ℓ_a^2.

Theorem 2.3.9.

(i) Let $0 < a < 1$. Then a composition operator C_T on ℓ_a^2 is compact if and only if the sequence $\{i - T(i)\}$ tends to infinity as i tends to ∞.

(ii) Let $a > 1$. Then a composition operator C_T on ℓ_a^2 is compact if and only if the sequence $\{T(i) - i\}$ tends to infinity as i tends to ∞.

Proof. It is computational and follows from Corollary 2.3.7.

Examples 2.3.10.

(1) Let $0 < a < 1$ and let $T : \mathbb{N} \to \mathbb{N}$ be defined as

$$T(i) = \frac{j}{3} \quad \text{if} \quad j - 2 \leq i \leq j,$$

where j is a multiple of 3. Then T induces a composition operator on ℓ_a^2. Since

$$mT^{-1}(\{i\})/m(\{i\}) = a^{2i}\left(1 + a^{-1} + a^{-2}\right),$$

which tends to 0 as $i \to \infty$, we conclude that C_T is compact.

(2) Let $T : \mathbb{N} \to \mathbb{N}$ be defined as $T(i) = j^2$ for $j - 2 \leq i \leq j$, j being a multiple of 3. Then C_T is a compact composition operator on ℓ_a^2.

(3) Let $T : \mathbb{N} \to \mathbb{N}$ be defined as $T(i) = 2i$ for $i \in \mathbb{N}$. Then

$$\frac{mT^{-1}(\{i\})}{m(\{i\})} = \begin{cases} 0 & \text{if } i \text{ is odd} \\ \dfrac{1}{a^{i/2}} & \text{if } i \text{ is even.} \end{cases}$$

Thus C_T is compact on ℓ_a^2 for $a > 1$.

2.4 NORMALITY OF COMPOSITION OPERATORS

An operator A on a Hilbert space is said to be normal if it commutes with its adjoint A^*; it is said to be quasinormal if it commutes with A^*A. In case $A^*A - AA^*$ is a positive operator, A is said to be a hyponormal operator. If either A or A^* is a hyponormal, then A is said to be seminormal. A is said to be unitary if $A^*A = AA^* = I$. The main aim of this section is to throw some light on these types of composition operators on $L^2(m)$. Singh and Kumar [341] and Whitley [399] studied these types of composition operators simultaneously, but independently, and obtained almost identical results characterising normal, quasinormal, and unitary composition operators. These results are presented in the next few theorems of this section.

Theorem 2.4.1. Let C_T be a composition operator on $L^2(m)$. Then

(i) C_T is normal if and only if the range of C_T is dense in $L^2(m)$ and $f_T \circ T = f_T$ *a. e.*

(ii) C_T is quasinormal if and only if $f_T \circ T = f_T$ *a. e.*

Proof. (i) Suppose C_T is normal. Then $C_T^* C_T = C_T C_T^*$. Hence $\ker C_T = \ker C_T^* C_T = \ker M_{f_T} = \ker C_T C_T^* = \ker C_T^* = (\operatorname{ran} C_T)^{\perp}$. Let $E = \{x : f_T(x) = 0\}$. Then it can be proved that $m(E) = 0$. Hence $\ker M_{f_T} = \{0\}$. Thus $\ker C_T = \{0\}$. Thus $(\operatorname{ran} C_T)^{\perp} = \{0\}$. Hence the range of C_T is dense in $L^2(m)$. By Theorem 2.2.1 we have

$$M_{f_T} = M_{f_T \circ T} P,$$

where P is the projection of $L^2(m)$ on the closure of the range of C_T. If X is of finite measure, then $C_T \chi_X = \chi_{T^{-1}(X)} = \chi_X$. Hence χ_X belongs to the range of C_T. Thus

$$M_{f_T} \chi_X = M_{f_T \circ T} P \chi_X = M_{f_T \circ T} \chi_X .$$

Hence $f_T = f_T \circ T$ *a.e.* In case X is of infinite measure, it can be proved by writing X as a countable union of measurable sets of finite measures.

Conversely, suppose C_T has dense range and $f_T \circ T = f_T$ *a. e.* Then by part (iii) of Theorem 2.2.1, we have

$$C_T^* C_T = M_{f_T} = M_{f_T \circ T} = C_T C_T^* .$$

Hence C_T is normal.

(ii) Suppose C_T is quasinormal. Hence C_T commutes with $C_T^* C_T$. Thus by part (i) of Theorem 2.2.1, C_T commutes with M_{f_T}. Hence

$$C_T M_{f_T} = M_{f_T} C_T .$$

From this we can deduce that $f_T = f_T \circ T$.

Conversely, suppose $f_T = f_T \circ T$ *a. e.* Then for $f \in L^2(m)$,

$$C_T C_T^* C_T f = C_T(f_T f) = f_T \circ T \,.\, f \circ T$$

and

$$C_T^* C_T C_T f = C_T^* C_T(f \circ T) = f_T . f \circ T .$$

Thus $C_T C_T^* C_T f = f_T \circ T . f \circ T = f_T . f \circ T = C_T^* C_T C_T f$. This shows that C_T is quasinormal. This completes the proof of the theorem.

It turns out that in case of a finite measure space every normal composition operator is unitary and every quasinormal operator is an isometry. This we shall prove in the following theorem.

Theorem 2.4.2. Let $(X, \mathcal{S}, m)$ be a finite measure space and let C_T be a composition operator on $L^2(m)$. Then

(i) C_T is normal if and only if C_T is unitary.

(ii) C_T is quasinormal if and only if C_T is an isometry.

Proof.

(i) Suppose C_T is normal. Then by part (i) of Theorem 2.2.1, $C_T C_T^* = C_T^* C_T = M_{f_T}$. Thus to show that C_T is unitary it is enough to show that $f_T = 1$ *a. e.* By part (i) of Theorem 2.4.1, $f_T \text{o} T = f_T$ *a. e.* Let $S = \{x \ : \ x \in X \text{ and } f_T(x) \geq 1\}$. Then

$$S = \{x \ : \ x \in X \ \text{ and } \ (f_T \circ T)(x) \geq 1\} = T^{-1}(S) \, .$$

Since $m(X) < \infty$, $\chi_S \in L^2(m)$. Hence

$$\left\|C_T \chi_S\right\|^2 = \int \chi_S f_T dm = \int \chi_S f_T^2 \ dm \, .$$

Thus $f_T^2 = f_T$ on S. This shows that $f_T = 1$ *a.e.* on S. By a similar argument it can be shown that $f_T = 0$ *a.e.* on the complement of S. Now

$$m(X) = m\left(T^{-1}(X)\right) = \int f_T dm = \int_S f_T \, dm = m(S).$$

Thus the complemnt of S has measure zero. Hence $f_T = 1$ *a. e.* This proves that C_T is unitary. The converse is true in general.

(ii) Suppose C_T is quasinormal. Then $f_T \circ T = f_T$ *a.e.* by part (ii) of Theorem 2.4.1. Hence $f_T = 1$ *a. e.* as proved above. Since $C_T^* C_T = M_{f_T} = I$, we conclude that C_T is an isometry. The converse is true in general.

Note. From the proof of the above theorem it is apparent that in case of a finite measure space, normality or quasinormality of C_T implies that T is measure preserving which is equivalent to quasinormality of C_T. In case C_T has dense range, the normality of C_T is implied by the measure preserving property of T. If the underlying measure space is infinite, then normality does not imply that T is measure preserving.

Examples 2.4.3.

(1) Let $X = \mathbb{R}$ with Lebesgue measure and let $T(x) = ax + b$, $a \neq 1, 0$. Then C_T is a normal composition operator, but T is not measure preserving.

(2) For $i \in \mathbb{N}$, let $X_i = \{1_i, 2_i, 3_i, \ldots\}$ and let $X = \bigcup_{i=1}^{\infty} X_i$. Let the measure m be defined as

$$m(\{n_i\}) = \begin{cases} i^{(n-2)}/2 & \text{if } n \text{ is odd} \\ 1/i^{n/2} & \text{if } n \text{ is even.} \end{cases}$$

Let $T : X \to X$ be defined as

$$T(n_i) = \begin{cases} (n+2)_i\,, & \text{when } n \text{ is odd} \\ 1_i\,, & \text{when } n = 2 \\ (n-2)_i\,, & \text{when } n \text{ is even and greater than 2.} \end{cases}$$

Then it can be checked that $f_T(n_i) = 1/i$ for $n_i \in X_i$ and $f_T = f_T \circ T$. Since T is an injection, C_T has dense range, and hence we can conclude that C_T is normal. For a fixed n,

$$\left\| C_T \chi_{\{n_i\}} \right\|^2 \Big/ \left\| \chi_{\{n_i\}} \right\|^2 = 1/i \quad \text{for } i \in \mathbb{N}.$$

Hence C_T is not bounded away from zero. Thus it is not unitary.

The following corollary contains the results which can be obtained from Theorem 2.4.1 and some earlier results.

Corollary 2.4.4.

(i) A composition operator C_T on ℓ^2 is normal if and only if T is invertible.

(ii) If $w = \{w_1, w_2, w_3, \ldots\}$ is a strictly increasing (or strictly decreasing) sequence of positive numbers, then a composition operator C_T on $\ell^2(w)$ is normal if and only if T is the identity map.

There are not many multiplication operators which are composition operators. Actually there is only one and that is the identity operator. It can be proved that if $C_T = M_\pi$, then $\pi = 1$ *a. e.* We may need this result in the following theorem which characterizes unitary composition operators on L^2 of a standard Borel space.

Theorem 2.4.5. Let $(X, \mathcal{S}, m)$ be a standard Borel space and let C_T be a composition operator on $L^2(m)$. Then the following are equivalent :

(i) C_T is unitary,

(ii) T is an injection and $f_T = 1$ *a. e.*,

(iii) C_T is invertible and $f_T = 1$ *a. e.*,

(iv) C_T^* is a composition operator.

Proof. (i) $\Rightarrow$ (ii) If C_T is unitary, then

$$C_T^* C_T = C_T C_T^* = I .$$

Hence $M_{f_T} = I$. Thus $f_T = 1$ *a. e.* Since C_T is invertible by Theorem 2.2.14, the transformation T is an injection.

(ii) $\Rightarrow$ (i) If $f_T = 1$ *a. e.*, then C_T is an isometry and hence has closed range. If T is injection, then C_T has dense range by Corollary 2.2.9. Hence C_T is invertible.

(iii) $\Rightarrow$ (iv) Since C_T is invertible and $f_T = 1$ *a. e.*, we have

$$C_T^* = M_{f_T} C_T^{-1} = C_T^{-1} = C_{T^{-1}} .$$

(iv) $\Rightarrow$ (i) Suppose $C_T^* = C_U$ for some measurable transformation U. Then

$$M_{f_T} = C_T^* C_T = C_U C_T = C_{T \circ U}.$$

Thus by the argument given before the theorem we have

$$f_T = 1 \quad a.e.$$

This shows that C_T is an isometry. If C_T has dense range, then it would be unitary. To prove that C_T has dense range it is enough to prove that $T^{-1}(\mathcal{S}) = \mathcal{S}$. Let $S \in \mathcal{S}$ be of finite measure. If $\chi_S \in$ ran C_T, then there exists an $h \in L^2(m)$

such that

$$C_T h = \chi_S .$$

Since C_T is an injection, by Corollary 2.2.3, $h = \chi_{S_2}$ for some $S_2 \in \mathcal{S}$. Hence $S = T^{-1}(S_2) \in T^{-1}(\mathcal{S})$. If χ_S does not belong to the range of C_T, then we can write

$$\chi_S = C_T g + f,$$

where $f \in (\operatorname{ran} C_T)^{\perp}$. Now from this we have

$$\begin{aligned} C_U \chi_S &= C_T^* \, C_T \, g + C_T^* f \\ &= g \ \left(\because f \in \ker C_T^*\right). \end{aligned}$$

Thus $g = \chi_{U^{-1}(S)}$. It can be proved that $\chi_S = \chi_{T^{-1}(U^{-1}(S))}$. Hence $S \in T^{-1}(\mathcal{S})$. It can be proved that $S \in T^{-1}(\mathcal{S})$ for every $S \in \mathcal{S}$. Hence $\mathcal{S} \subset T^{-1}(\mathcal{S}) \subset \mathcal{S}$. Thus

$$\mathcal{S} = T^{-1}(\mathcal{S}) .$$

This proves the theorem.

If X is a locally compact Abelian group, then the mapping T_y taking x to yx defines unitary composition operators on $L^2(m)$. In particular, all translates of $\mathbb{R}$ define unitary composition operators on $L^2(-\infty, \infty)$. If the underlying measure space is an atomic measure space of finite measure, then it turns out that normal, unitary, and isometric composition operators coincide. We shall record this result in the following theorem.

Theorem 2.4.6. Let $(X, \mathcal{S}, m)$ be a finite atomic measure space and let C_T be a composition operator on $L^2(m)$. Then the following are equivalent :

(i) C_T is normal,

(ii) C_T is unitary,

(iii) C_T is an isometry,

(iv) C_T is quasinormal,

(v) C_T is a co-isometry.

Corollary 2.4.7. If $(X, \mathcal{S}, m)$ is a finite atomic measure space such that different atoms have different measures, then all five classes of composition operators on $L^2(m)$ given above coincide with the singleton set containing the identity operator.

Corollary 2.4.8. A composition operator on $\ell^2(w)$ is normal if and only if it is unitary if and only if it is an isometry if and only if it is quasinormal if and only if it is an identity operator, where $w_n = a^n$, $n \in \mathbb{N}$ for some $0 < a < 1$.

Let H be a Hilbert space, Let $B(H)$ denote the C^*–algebra of all bounded linear operators on H, let $C(H)$ denote the ideal of all compact operators, and let h be the canonical epimorphism from $B(H)$ to the Calkin algebra $B(H) / C(H)$. Then an operator A is said to be Fredholm if $h(A)$ is invertible in the Calkin algebra. A is said to be essentially normal, essentially unitary, or essentially isometric if $h(A)$ is a normal, unitary, or isometric element of $B(H) / C(H)$. A is said to be quasiunitary if $A^*A - I$ and $AA^* - I$ are finite rank operators.

In [202] Kumar has shown that the class of Fredholm composition operators on $L^2[0,1]$ coincides with the class of the invertible composition operators, and Singh and Veluchamy [369] has characterised Fredholm composition operators on L^2 of atomic measure spaces. We shall record a general characterization of Fredholm composition operators in the following theorem.

Theorem 2.4.9. A composition operator C_T on $L^2(m)$ is Fredholm if and only if $T^{-1}(X_1) = X_1$, $T^{-1}(X_2) = X_2$, $C_T / L^2(X_1)$ is invertible and $C_T / L^2(X_2)$ is Fredholm, where X_1 and X_2 are the non–atomic part and the atomic part of X respectively.

The class of Fredholm composition operators on ℓ^2 coincides with the class of essentially normal composition operators and with several other classes of the composition operators. This was proved by Singh and Veluchamy in [369]. We shall record the results in the following theorem.

Theorem 2.4.10. Let C_T be a composition operator on ℓ^2. Then the following are equivalent :

(i) C_T is Fredholm,

(ii) C_T is essentially normal,

(iii) C_T is quasi–unitary,

(iv) C_T is essentially unitary,

(v) C_T is essentially isometric.

If $(X, \mathcal{S}, m)$ is a σ–finite measure space and $T : X \to X$ is a measurable transformation, then $T \circ T$ is also a measurable transformation which we shall denote by T^2. If mT^{-1} is absolutely continuous with respect to m, then $m(T^2)^{-1}$ is absolutely continuous with respect to both mT^{-1} and m. Let g_T denote the Radon–Nikodym derivative of $m(T^2)^{-1}$ with respect to mT^{-1}. Then it is an easy exercise to show that

$$f_{T^2} = g_T f_T .$$

(Recall that f_T is the Radon–Nikodym derivative of mT^{-1} with respect to m).

In the following theorem we give results concerning the hyponormal composition operators. The first two parts of the theorem are due to Harrington and Whitley [144].

Theorem 2.4.11. Let $(X, \mathcal{S}, m)$ be a σ–finite measure space and let C_T be a composition operator on $L^2(m)$. Then

(i) C_T is hyponormal if and only if $\left\|\sqrt{f_T}\, f\right\| \geq \left\|\sqrt{f_T} \circ T\, Pf\right\|$ for every $f \in L^2(m)$.

(ii) C_T^* is hyponormal if and only if $f_T \circ T \geq f_T$ *a.e.* and the completion of the σ–algebra generated by set of type $S \cap X_0^{f_T}$ for $S \in \mathcal{S}$ is contained in $T^{-1}(\mathcal{S})$, where $X_0^{f_T} = \{x : f_T(x) > 0\}$.

(iii) C_T is quasi–hyponormal if and only if $f_{T^2} \leq g_T$ *a.e.*

Proof. (i) C_T is hyponormal if and only if $C_T^* C_T - C_T C_T^* \geq 0$. Thus C_T is hyponormal if and only if

$$\left\langle \left(C_T^* C_T - C_T C_T^* \right) f, f \right\rangle \geq 0 \quad \text{for all} \quad f \in L^2(m).$$

In light of Theorem 2.2.1, C_T is hyponormal if and only if

$$\left\langle M_{f_T} f, f \right\rangle \geq \left\langle M_{f_T \circ T} Pf, f \right\rangle \quad \text{for all} \quad f \in L^2(m).$$

From this we conclude that C_T is hyponormal if and only if

$$\left\| \sqrt{f_T}\, f \right\| \geq \left\| \sqrt{f_T \circ T}\, Pf \right\| \quad \text{for all} \quad f \in L^2(m).$$

This proves (i) (since P is the projection of $L^2(m)$ on $\overline{\text{ran } C_T}$, $P^2 = P$ and $P(f_T \circ T\, f) = f_T \circ T\, Pf$ [144]).

(ii) Suppose C_T^* is hyponormal. Then the kernel of C_T^* is contained in the kernel of C_T. Suppose S is a set of finite measure such that $S \subset X_0^{f_T}$, and S is not in $T^{-1}(\mathcal{S})$. Then χ_S does not belong to the closure of the range of C_T. Thus there exists a function f in the orthogonal complement of the closure of the range of C_T such that $\left\langle f, \chi_S \right\rangle \neq 0$. Since $f \in \ker C_T^* \subset \ker C_T = \ker M_{f_T}$, we have

$$f_T f = 0.$$

Thus we arrive at a contradiction. Hence $S \in T^{-1}(\mathcal{S})$. Let

$$S_1 = \left\{ x \ : \ \left(f_T \circ T \right)(x) < f_T\,(x) \right\}.$$

Then $S_1 \in T^{-1}(\mathcal{S})$. Using the hyponormality of C_T^*, it can be proved that $m(S_1) = 0$. Hence $f_T \circ T \geq f_T$ *a.e.* For the converse, suppose conditions are true. Let $f \in L^2(m)$. Then f can be written as

$$f = f_1 + f_2,$$

where f_1 belong to the closure of $\text{ran} C_T$ and f_2 to its orthogonal complement. It can be proved that

$$\left\| C_T^* f \right\|^2 - \left\| C_T f \right\|^2 = \int \left(f_T \circ T - f_T \right) |f_1|^2 dm.$$

Since $f_T \circ T \geq f_T$, we get the hyponormality of C_T^*.

(iii) Suppose C_T is quasi-hyponormal. Then

$$\left\| C_T^* C_T f \right\| \leq \left\| C_T C_T f \right\| \quad \text{for all} \ f \in L^2(m).$$

In particular,

$$\left\| M_{f_T} \chi_{S_1} \right\| \le \left\| C_{T^2} \chi_{S_1} \right\|,$$

for sets S_1 of finite measure. Thus

$$\int_X f_T^2 \chi_{S_1}\, dm \le \int_X \chi_{S_1}\, dm(T^2)^{-1} = \int_X f_{T^2} \chi_{S_1}\, dm.$$

Hence

$$\int_{S_1} \left(f_{T^2} - f_T^2 \right) dm \ge 0.$$

Hence $f_{T^2} \ge f_T^2$ *a.e.* on S_1, and hence *a. e.* on X. Thus $g_T \ge f_T$ *a.e.* The converse is easy.

The following corollary can be deduced from the above theorem.

Corollary 2.4.12.

(i) If C_T is hyponormal, then $g_T \ge f_T$ *a.e.*

(ii) If $(X, \mathcal{S}, m)$ is a finite measure space, then C_T is hyponormal if and only if $f_T = 1$ *a.e.*

(iii) If C_T is a composition operator on ℓ^2 induced by an injective map T, then C_T is normal if and only if C_T is hyponormal if and only if C_T is quasi-hyponormal.

Note. In case of ℓ^2 the classes of normal, quasinormal, hyponormal, and quasi-hyponormal operators are distinct and inclusions are proper. This can be seen in [334].

If $(X, \mathcal{S}, m)$ is a measure space and $T : X \to X$ is a measurable transformation, then $T^n : X \to X$ is also a measurable transformation for every $n \ge 1$, where T^n, as mentioned earlier is the composition of T with itself n times. If the measure mT^{-1} is absolutely continuous with respect to m, then $m(T^n)^{-1}$ is also absolutely continuous with respect to m ; by f_T^n we denote the Radon–Nikodym derivative of $m(T^n)^{-1}$

with respect to m. By $\mathcal{S}_\infty$ we denote the σ–algebra $\bigcap_{n=1}^{\infty}\left(T^n\right)^{-1}(\mathcal{S})$. If $\mathcal{A}$ is a σ–subalgebra of $\mathcal{S}$ such that $T^{-1}\mathcal{A} \subset \mathcal{A}$ and $(X, \mathcal{A}, m)$ is also σ–finite, then for $f \in L^2(X, \mathcal{S}, m)$, there exists a unique $\mathcal{A}$–measurable function g on X such that

$$\int_S g \, dm = \int_S f \, dm ,$$

for every $S \in \mathcal{A}$ of finite measure. This function g is called the conditional expectation of f with respect to $\mathcal{A}$ and we denote it by $E_{\mathcal{A}}(f)$ which is written as $E(f/\mathcal{A})$ in Statistics and Probability theory. Since each f_T^n is bounded, the conditional expectation $E_{\mathcal{A}}(f)$ is well defined, where $\mathcal{A} = (T^n)^{-1}(\mathcal{S})$, and in this case we denote it by $E_n(f)$. As an operator E_n is the orthogonal projection of $L^2(X, \mathcal{S}, m)$ onto $L^2(X, \mathcal{S}_n, m)$, where $\mathcal{S}_n = (T^n)^{-1}(\mathcal{S})$. An operator A on a Hilbert space is said to be subnormal if it has a normal extension on a larger Hilbert space. A sequence of the type $\left\{\int_0^a t^n d\mu(t)\right\}$ is called a moment sequence over the interval $[0,a]$, where μ is a Borel measure on $[0,a]$. It was proved by Lambert [206] that an Operator A on a Hilbert space is subnormal if and only if the sequence $\left\{\left\|A^n x\right\|^2\right\}$ is a moment sequence for every x in the Hilbert space. A sequence $\alpha = \{\alpha_n\}$ of positive real numbers is moment sequence if and only if for some interval I, the linear functional ϕ_α defined on the space $P(I)$ of polynomials by $\phi_\alpha\left(\sum a_n t^n\right) = \sum a_n \alpha_n$ is positive. This result is used by Lambert [207] to prove that C_T is subnormal on $L^2(m)$ if and only if the linear transformation $L : P(I) \to L^\infty(m)$ defined as $L\left(\sum a_n t^n\right) = \sum a_n f_T^n$ is positive. He has made use of this result to give several characterizations of subnormal composition operators in terms of moment sequence which we shall present in the following theorem.

Theorem 2.4.13. Let $(X, \mathcal{S}, m)$ be a measure space and let C_T be a composition operator on $L^2(m)$. Then the following are equivalent :

(i) C_T is subnormal operator.

(ii) For every $g \in L^2(m)$, there exists a Borel measure m_g on the interval $\left[0, \left\|f_T\right\|_\infty\right]$ such that

$$\int_0^{\|f_T\|_\infty} t^n \, dm_g(t) = \int_X f_T^n | g |^2 \, dm .$$

(iii) For every measurable set S of finite measure the sequence $\{m(T^n)^{-1}(S)\}$ is a moment sequence on $\left[0, \|f_T\|_\infty\right]$.

(iv) $\{f_T^n(x)\}$ is a moment sequence for almost every x in X.

Proof. (i) $\Rightarrow$ (ii) Let C_T be subnormal and let $g \in L^2(m)$. Then

$$\left\|C_T^n g\right\|^2 = \int \left|g \circ T^n\right|^2 \, dm = \int_X f_T^n |g|^2 dm \cdot$$

Since $\left\{\left\|C_T^n g\right\|^2\right\}$ is a moment sequence, there exists a measure m_g such that

$$\int_X t^n dm_g(t) = \int_X f_T^n |g|^2 dm .$$

(ii) $\Rightarrow$ (iii) Let $S \in \mathcal{S}$ be of finite measure and let $g = \chi_S \in L^2(m)$. Since

$$\int_X f_T^n \left| \chi_S \right|^2 dm = \int_S f_T^n \, dm = m(T^n)^{-1}(S),$$

we conclude by (ii) that $\{m(T^n)^{-1}(S)\}$ is a moment sequence.

(iii) $\Rightarrow$ (i) If $g = \chi_S$ is a characteristic function of a set of finite measure, then by (iii) $\left\{\left\|C_T^n g\right\|^2\right\}$ is a moment sequence. If g is a simple function, then $\left\{\left\|C_T^n g\right\|^2\right\}$ is a moment sequence. Since simple functions are dense, $\left\{\left\|C_T^n g\right\|^2\right\}$ is a moment sequence. Hence C_T is subnormal.

(i) $\Rightarrow$ (iv) Let $P_r(I)$ denote the set of all polynomials with rational complex coefficients and let $P_r^+(I)$ denote the set of all positive polynomials in $P_r(I)$, where $I = \left[0, \|f_T\|_\infty\right]$. If $p \in P_r^+(I)$, then $L(p) \geq 0$. Let $X_p = \left\{x \in X \ : \ L(p)\,(x) \geq 0\right\}$. Then $m(X \setminus X') = 0$, where $X' = \cup \{X_p \ : \ p \in P_r^+(I)\}$. Since $P_r^+(I)$ is dense in $P^+(I)$, it follows that $L(p)(x) \geq 0$ for $x \in X'$ and $p \in P^+(I)$. Thus by the statement preceding the theorem, we conclude that $\{f_T^n(x)\}$ is a moment sequence for every $x \in X'$.

(iv) $\Rightarrow$ (i) : Let $\{f_T^n(x)\}$ be a moment sequence for every $x \in Y$, where $m(X \setminus Y) =$

0. If $p \in P^+(I)$, then $L(p)(x) \geq 0$ on Y and hence L is a positive linear transformation. Thus it follows that C_T is subnormal. This completes the proof of the theorem.

Lambert has made a detailed study of the expectation operators and their relations with the composition operators. Even though T is not invertible, the functions

$$f_T\left(E_{T^{-1}(\mathcal{S})}g\right) \circ T^{-1} \quad \text{and} \quad \chi_F\left(E_{T^{-1}(\mathcal{S})}g\right) \circ T^{-1}$$

are well defined since $E_{T^{-1}(\mathcal{S})}\, g = h \circ T$ for some $\mathcal{S}$–measurable function h which is uniquely determined on F, the support of f_T. The adjoint C_T^* is given by

$$C_T^* g = f_T\left(E_{T^{-1}(\mathcal{S})}g\right) \circ T^{-1}.$$

For details we refer to [54]. The following theorem gives a relation between some reducing subspaces of C_T and the expectation operators induced by some σ–algebras.

Theorem 2.4.14. Let $(X, \mathcal{S}, m)$ be a σ–finite measure space and let C_T be a composition operator on $L^2(m)$. Let $S \in \mathcal{S}$ and let $\mathcal{A}$ be a σ–finite subalgebra of $\mathcal{S}$. Then

(i) $L^2(S)$ is a reducing subspace of C_T if and only if $T^{-1}(S) = S$.

(ii) $L^2(X, \mathcal{A}, m)$ is a reducing subspace of C_T if and only if f_T is $\mathcal{A}$–measurable and $E_{T^{-1}(\mathcal{S})}\, E_{\mathcal{A}} = E_{T^{-1}(\mathcal{A})}$.

(A subspace of a Hilbert space which is invariant under A and A^* is called a reducing subspace of A).

Proof. (i) If P is the orthogonal projection on $L^2(S)$, then $L^2(S)$ reduces C_T if and only if $PC_Tg = C_TPg$ for every $g \in L^2(m)$. From this we have

$$\chi_S \cdot g \circ T = \left(\chi_S \cdot g\right) \circ T = \chi_{T^{-1}(S)} \cdot g \circ T \qquad \text{for every } g \in L^2(m).$$

Thus P commutes with C_T if $\chi_S = \chi_{T^{-1}(S)}$. Let $\{X_n\}$ be a sequence of sets of finite measure such that $X = \bigcup_{n=1}^{\infty} X_n$. Let $g = C_T\chi_{X_n}$. Then from above we have

$$\chi_S = \chi_{T^{-1}(S)} \quad a.e. \text{ on } X_n.$$

This is true for all $n \in \mathbb{N}$. Hence $\chi_S = \chi_{T^{-1}(S)}$.

(ii) Suppose $L^2(X, \mathcal{A}, m)$ is a reducing subspace of C_T and let $g \in L^2(m)$. Then

$$E_{\mathcal{A}}(f_T g) = f_T E_{\mathcal{A}}(g).$$

Let $\{X_n\}$ be an increasing sequence of measurable sets and let $g_n = \chi_{X_n}$. Then $\{g_n\}$ converges to 1 pointwise and hence $\{E_{\mathcal{A}}(g_n)\}$ converges to 1 pointwise. Since $f_T.E_{\mathcal{A}}(g_n)$ is $\mathcal{A}$–measurable, we conclude that f_T is $\mathcal{A}$–measurable. By reducibility of C_T we have

$$E_{T^{-1}(\mathcal{S})}E_{\mathcal{A}} = E_{\mathcal{A}}\, E_{T^{-1}(\mathcal{S})},$$

and

$$E_{\mathcal{A}}\, E_{T^{-1}(\mathcal{S})} = E_{T^{-1}(\mathcal{S})\cap\mathcal{A}}.$$

Since $T^{-1}(\mathcal{A}) \subset \mathcal{A}$ and $T^{-1}(\mathcal{S})\cap\mathcal{A} \supset T^{-1}(\mathcal{A})$, we have

$$E_{T^{-1}(\mathcal{S})\cap\mathcal{A}} \geq E_{T^{-1}(\mathcal{A})}.$$

If $g \in L^2(X, T^{-1}(\mathcal{S})\cap\mathcal{A}, m)$, then we can write $g = h \circ T$, where h is a measurable function such that h is zero off the support of f_T. Now since $\sqrt{f_T}.h \in L^2(m)$, and $L^2(X, \mathcal{A}, m)$ reduces C_T, we conclude that

$$E_{\mathcal{A}}\left(\sqrt{f_T}.h\right) \circ T = E_{\mathcal{A}}\left(\sqrt{f_T} \circ T.\, h \circ T\right).$$

Thus

$$E_{\mathcal{A}}\left(\sqrt{f_T}\, h\right) \circ T = \sqrt{f_T} \circ T.\, h \circ T.$$

From this we conclude that $E_{\mathcal{A}}(f_T.g) = \sqrt{f_T}.g$ on the support of f_T, and hence h is $\mathcal{A}$–measurable and g is $T^{-1}(\mathcal{A})$–measurable. Thus

$$E_{T^{-1}(\mathcal{S}\cap\mathcal{A})} \leq E_{T^{-1}(\mathcal{A})},$$

and hence

$$E_{T^{-1}(\mathcal{S})}E_{\mathcal{A}} = E_{T^{-1}(\mathcal{A})}.$$

Conversely, if $\chi_S \in L^2(X, \mathscr{A}, m)$, then

$$C_T\chi_S = \chi_{T^{-1}(S)} \in L^2\left(X, T^{-1}(\mathscr{S}), m\right) \subset L^2(X, \mathscr{A}, m)$$

and

$$C_T^*\chi_S = f_T\left(E_{T^{-1}(\mathscr{S})}E_{\mathscr{A}}\chi_S\right)\circ T^{-1} = f_T\left(E_{T^{-1}(\mathscr{A})}\chi_S\right)\circ T^{-1}.$$

Since $E_{T^{-1}(\mathscr{A})}\chi_S$ is $T^{-1}(\mathscr{A})$–measurable, it can be written as $h \circ T$ for $\mathscr{A}$–measurable function h. Hence

$$C_T^*\chi_S = f_T.h \in L^2(X, \mathscr{A}, m) .$$

Since the span of the characteristic functions of the sets of finite measures is dense, we conclude that $L^2(X, \mathscr{A}, m)$ is invariant under C_T and C_T^*, and hence it is a reducing subspace of C_T. This completes the proof of the theorem.

An operator A on a Hilbert space is said to be centered if the set $\{A^{*n}A^n, A^kA^{*k};\ n, k \in \mathbb{Z}_+\}$ is commutative. In the following theorem we record a recent result of Lambert characterising the centered composition operators in terms of the measurability of Radon–Nikodym derivative with respect to a particular σ–subalgebra of $\mathscr{S}$.

Theorem 2.4.15. Let C_T be a composition operator on $L^2(X, \mathscr{S}, m)$ and let $\mathscr{S}_\infty = \bigcap_{n=1}^{\infty}\left(T^n\right)^{-1}\mathscr{S}$. Then C_T is centered if and only if the Radon–Nikodym derivative f_T of mT^{-1} is $\mathscr{S}_\infty$–measurable.

Outline of the proof. If C_T is a composition operator on $L^2(m)$, $Ug = C_Tg / \sqrt{f_T}\circ T$ and $Pg = \sqrt{f}.g$ for $g \in L^2(m)$, then U is a partial isometry, $C_T = UP$ is the polar decomposition of C_T and the polar decomposition of C_T^n is U_nP_n, where $P_ng = \sqrt{f_T^n}\,g$ and $U_ng = \dfrac{g}{f_T}\circ T^n$ for every $n \in \mathbb{Z}_+$. Thus C_T is centered if and only if $U_n = U^n$ for every $n \in \mathbb{Z}_+$. Now

$$U^n g = \left[\prod_{k=1}^{n} f_T \circ T^k\right]^{-1/2} . g \circ T^n.$$

Hence C_T is centered if and only if $f_T^n \circ T^n = \prod_{k=1}^{n} f_T \circ T^k$. If f_T is $\mathcal{S}_\infty$-measurable, then $E_n f_T = f_T$ for every $n \in \mathbb{Z}_+$. Using induction on n it can be proved that

$$\prod_{k=1}^{n} f_T \circ T^k = f_T^n \circ T^n.$$

If C_T is centered and $X = \bigcup_{k=1}^{\infty} X_k$, where $\{X_k\}$ is a pairwise disjoint sequence of sets of finite measure, then it can be shown that

$$\int_X (E_n f_T - f_T)dm = 0$$

and hence $E_n f_T = f_T$ *a.e.* for every $n \in \mathbb{Z}_+$. This will show that h is $\mathcal{S}_\infty$-measurable.

Note. If $\mathcal{S}_\infty = \{\phi, X\}$, then C_T is centered if and only if it is a constant multiple of an isometry; if $\mathcal{S}_\infty = \mathcal{S}$ then C_T is centered. Now $C_T / L^2(X, \mathcal{S}_\infty, m)$ is centered, and $C_T / L^2(X, \mathcal{S}', m)$ is centered if and only if $E_{\mathcal{S}'}(f_T)$ is $\mathcal{S}_\infty$-measurable.

2.5 WEIGHTED COMPOSITION OPERATORS

We know that a weighted composition operator $W_{\pi,T}$ on a function space $H(X)$ over a set X is a continuous linear transformation from $H(X)$ to itself defined by $W_{\pi,T}(f) = \pi . f \circ T$, where π is a function in X and T is a self map of X. If π induces the multiplication operator M_π on $H(X)$ defined as $M_\pi f = \pi . f$ and T induces the composition operator C_T on $H(X)$, then clearly $W_{\pi,T} = M_\pi C_T$. But it may happen that pair (π, T) induces the weighted composition operator $W_{\pi,T}$ while T may fail to do so. For example, if $\pi(x) = 0$ for every $x \in X$ and $T: X \to X$ is any map, then $W_{\pi,T}$ is a weighted composition operator whether T induces an operator or not. It is evident that the function $W_{\pi,T}(f)$ is obtained by multiplying by π the composite function $f \circ T$. If we multiply first by π and then compose the function $\pi . f$ with T, we get the operator $f \to (\pi . f) \circ T$, which we denote by $W_{T,\pi}$. It is clear that $W_{T,\pi} = W_{\pi\circ T,T}$. In this section we are interested in presenting some results on the weighted composition operators on L^2–spaces. A more general study of these operators with vector–valued or operator–valued weights on some spaces of continuous functions and cross sections is presented in section 4.4. The study of the weighted composition operators on L^2–spaces was initiated by Parrot in [261], where he studied the weighted translation operators. Peterson [269] explored the spectrum and commutant of the weighted translation operators, while Bastian [17] studied their decomposition. Embry and Lambert [103], and Hoover, Lambert and Quinn [147] further explored the properties and applications of these operators. Dharmadhikari [94] and Kumar [204] studied these operators in detail in their Ph.D. dissertations. The spectra and commutant of some weighted composition operators on $\ell^2(\mathbb{Z})$ were obtained by Carlson [57]. Takagi [389] made a study of compactness of the weighted composition operators on L^p–spaces, and Kamowitz and Wortman studied them on Sobolev type spaces. Some of the recent results we shall present in this section.

We assume that $\pi : X \to \mathbb{C}$ is an essentially bounded measurable function and $T : X \to X$ is a non–singular measurable transformation (to have more generalised weighted composition operators we can take the support of π as the domain of T). Now define the measure m_T^π on $\mathcal{S}$ as

$$m_T^\pi(S) = \int_{T^{-1}(S)} |\pi|^p dm .$$

Since $m(S) = 0$ implies $mT^{-1}(S) = 0$ and hence $m_T^\pi(S) = 0$, we conclude that $m_T^\pi << m$. Let f_T^π denote the Radon–Nikodym derivative of m_T^π with respect to m and let $\phi = (f_T^\pi)^{1/p}$. We know that if f_T is essentially bounded, then $W_{\pi,T}$ is a bounded operator on $L^p(m)$, but the converse is not true. The boundedness of $W_{\pi,T}$ is characterised in terms of boundedness of ϕ in the following theorem.

Theorem 2.5.1 $W_{\pi,T}$ is a bounded operator on $L^p(m)$ if and only if $\phi \in L^\infty(m)$.

Proof. Let $g \in L^2(m)$. Then

$$
\begin{aligned}
\| W_{\pi,T}\, g \|^p = \int_X | \pi . g \circ T |^p dm &= \int_X | g \circ T |^p dm_T^\pi \\
&= \int_X f_T^\pi \, | g |^p dm \\
&= \int_X \phi^p | g |^p dm \\
&= \int \left| M_\phi g \right|^p dm \\
&= \left\| M_\phi g \right\|^p .
\end{aligned}
$$

From this equality the proof of the theorem follows. This completes the proof of the theorem.

Theorem 2.5.2. Let $W_{\pi,T}$ be a weighted composition operator on $L^2(m)$. Then the following are equivalent :

(i) $W_{\pi,T}$ is compact,

(ii) M_ϕ is compact,

(iii) Z_ε^ϕ is finite dimensional for every $\varepsilon > 0$.

Proof. The equivalence of (i) and (ii) follows from the above theorem, and equivalence of (ii) and (iii) follows from Theorem 2.3.1.

Note. Theorem 2.3.1. is true on every L^p—space for $1 \le p < \infty$.

As before let $X = X_1 \cup X_2$ be the decomposition of X into non-atomic and atomic parts. Without loss of generality we can write $X_2 = \{x_i \ : \ i \in \mathbb{N}\}$, where each x_i is a point with non-zero finite point mass. Let $\alpha_i = m(\{x_i\})$. Then the following theorem of Tagaki [389] characterises the compactness of $W_{\pi,T}$, which is in a way generalization of Theorem 2.3.5.

Theorem 2.5.3. A weighted composition operator $W_{\pi,T}$ on $L^2(m)$ is compact if and only if $mT^{-1}(X_1) = 0$ and

$$\frac{m_T^\pi(\{x_i\})}{\alpha_i} \to 0 \quad \text{as } i \to \infty,$$

where this limit is taken to be zero if X_2 is finite.

Proof. Let $X_0 = \{x \in X_1 : \phi(x) > 0\}$.

Now,

$$\int_{T^{-1}(X_1)} |\pi|^p \, dm = m_T^{\pi}(X) = \int_{X_1} \phi^p \, dm,$$

from which it follows that $mT^{-1}(X_1) = 0$ if and only if $m(X_0) = 0$ *i.e.*, $\phi(x) = 0$ *a.e.* on X_1 if and only if $\pi(x) = 0$ *a.e.* on $T^{-1}(X_1)$. Now,

$$\alpha_i \phi^p(x_i) = \int_{\{x_i\}} \phi^p dm = m_T^{\pi}(\{x_i\}),$$

and hence

$$\phi(x_i) = \left\{\frac{m_T^{\pi}(\{x_i\})}{\alpha_i}\right\}^{1/p}.$$

Thus $W_{\pi,T}$ is compact if and only if $m(X_0) = 0$ and $\lim_{i\to\infty} \phi(x_i) = 0$. If $m(X_0) = 0$ and $\phi(x_i) \to 0$, then for each $\varepsilon > 0$ the set X_ε^ϕ is union of finitely many atoms and a set of measure zero. Thus Z_ε^ϕ is finite dimensional, and hence by Theorem 2.5.2 (iii), $W_{\pi,T}$ is compact.

On the other hand, if $m(X_0) \neq 0$ or $\phi(x_i) \not\to 0$, then we get $\varepsilon > 0$ such that $m(\{x \in X_1 : \phi(x) > \varepsilon\}) \neq 0$ or the set $\{x_i \in X_2 : \phi(x_i) > \varepsilon\}$ has infinitely many atoms. In each case Z_ε^ϕ is infinite dimensional and $W_{\pi,T}$ is not compact. This completes the proof of the theorem.

In the following theorem we record some results which are easy to prove.

Theorem 2.5.4. Let $(X, \mathcal{S}, m)$ be a σ-finite measure space. Then

(a) $W_{\pi,T} = 0$ if and only if π vanishes on $T^{-1}(S)$ *a.e.*, whenever $m(S) < \infty$.

(b) $W_{T,\pi}$ is an isometry on $L^2(m)$ if and only if $f_T|\pi|^2 = 1$ *a.e.*

(c) $W_{T,\pi}$ is unitary on $L^2(m)$ if and only if $T^{-1}(\mathcal{S}) = \mathcal{S}$ and $|\pi|^2 f_T = 1$ *a.e.*

(d) If T_A is a transitive algebra of bounded linear operators on ℓ^2 containing $W_{\pi,T}$ such that $\pi(n) \to 0$ as $n \to \infty$, then $T_A = B(\ell^2)$, the algebra of all bounded linear operators on ℓ^2.

Let $k \in \mathbb{Z}_+$ and $1 \le p \le \infty$. Then by S_k^p we denote the vector space of all functions f on the interval $[0, 1]$ such that $f, f', \ldots\ldots, f^{(k-1)}$ are absolutely continuous and $f^{(k)} \in L^p(0,1)$. If $f \in S_k^p$ then define

$$\| f \| = \left(\sum_{i=0}^{k} \| f^{(i)} \|_p^p \right)^{1/p}, \quad 1 \le p < \infty.$$

With this norm S_k^p is a Banach space.

Kamowitz and Wortman [169] have characterised some compact weighted composition operators on S_k^p and computed the spectrum of these operators. We shall present these results in the following theorem without proof.

Theorem 2.5.5.

(i) Let T be a self map of $[0,1]$ such that $T \in S_k^p \cap C^1[0,1]$ and T is of N-bounded variation for some $N \in \mathbb{N}$. Let $\pi \in S_k^\infty$. Then $W_{\pi,T}$ is a weighted composition operator on S_k^p.

(ii) Let $W_{\pi,T}$ be the weighted composition operator induced by π and T as given in (i). Then $W_{\pi,T}$ is compact if and only if $\pi.T' = 0$.

(iii) If the weighted composition operator $W_{\pi,T}$ is compact, then $\sigma(W_{\pi,T}) = \left\{ z \in \mathbb{C} \ : \ z^n = \pi(c) \ldots\ldots \pi\left(T^{n-1}(c)\right) \right.$ for some $n \in \mathbb{Z}_+$ and for some fixed point c of order $\left. n \right\} \cup \{0\}$.

($\sigma(W_{\pi,T})$ as usual denotes the spectrum of $W_{\pi,T}$).

The spectrum of a weighted composition operator on a general L^p—space has not been found so far. But the spectra in certain special cases have been computed by Carlson [57], Komal [189], Ridge [279], and recently by Takagi [389].

Let $Y_0 = X$ and let $Y_1 = \{x \in Y_0 \ : \ \pi(x) \ne 0\}$. Inductively, define

$$Y_k = \left\{x \in Y_{k-1} : \pi(T^{k-1}(x)) \neq 0\right\},$$

for every $k \in \mathbb{Z}_+$. Clearly $\{Y_k\}$ is a decreasing sequence of the measurable subsets of X. An atom c in X is called a fixed point of T of order k, $k \geq 1$, if $x \in Y_k$, $T^k(x) = x$ and $T^j(x) \neq x$ for $1 \leq j \leq k-1$. If $x \in X_2 \cap Y_k$, then $\{x, T(x), \cdots, T^k(x)\}$ is a subset of X_2. In the following theorem of Tagaki [389] the spectrum of a compact weighted composition operator is computed using the techniques of Kamowitz ([164], [169]).

Theorem 2.5.6. If $W_{\pi,T}$ is a compact weighted composition operator on $L^p(m)$, then

$$\sigma\left(W_{\pi,T}\right) \cup \{0\} = \Big\{\lambda \in \mathbb{C} : \lambda^k = \pi(c).\pi(T(c))\ldots\pi(T^{k-1}(c)) \text{ for some fixed atom } c \text{ of order } k \Big\} \cup \{0\}.$$

Proof. Suppose $\lambda \neq 0$ and for every $k \in \mathbb{Z}_+$

$$\lambda^k \neq \pi(c)\pi(T(c))\ldots\ldots\pi(T^{k-1}(c))$$

for every fixed atom c of order k. Let $g \in L^p(m)$ such that

$$W_{\pi,T}\ g = \lambda g\ . \tag{1}$$

Let $X = X_1 \cup X_2$ be the decomposition of X into non-atomic and atomic parts, and let $x \in X_2$. Then, since $X = \bigcup_{k=1}^{\infty} Y_k$, $x \in Y_k$ for some k. Using (1), by induction we have

$$\lambda^k g(x) = \pi(x)\pi(T(x))\ldots\ldots\pi\left(T^{k-1}(x)\right)g\left(T^k(x)\right)$$

for every $k \in \mathbb{Z}_+$. In case $x \in Y_k$ and $x \notin Y_{k+1}$, we have $\pi\left(T^k(x)\right) = 0$, and hence $\lambda g\left(T^k(x)\right) = 0$. From this it follows that

$$\lambda^{k+1} g(x) = \pi(x)\ldots\ldots\pi\left(T^{k-1}(x)\right)W_{\pi,T}\, g\left(T^k(x)\right) = 0.$$

Since $\lambda \neq 0$, we conclude that $g(x) = 0$. Let $x \in Y_k$ for every $k \in \mathbb{Z}_+$ and let $\text{orb}(x) = \{T^k(x) : k \in \mathbb{Z}_+\}$ be the positive orbit of x. In case $\text{orb}(x)$ is finite,

$T^j(x)$ is a fixed atom of T. If k is the order of $T^j(x)$, then

$$\lambda^k g\left(T^j(x)\right) = \pi\left(T^j(x)\right)\ldots\ldots\pi\left(T^{k-1}\left(T^j(x)\right)\right).g\left(T^j(x)\right).$$

From this we conclude that $g(T^j(x)) = 0$.

Since

$$\lambda^j g(x) = \pi(x)\ldots\ldots\pi\left(T^{j-1}(x)\right)\, g\left(T^j(x)\right) = 0\ ,$$

we get $g(x) = 0$. Now, suppose orb(x) is infinite. For $\varepsilon > 0$ consider the set

$$\left\{k\ :\ \left\{\frac{m\left(\left\{T^k(x)\right\}\right)}{m\left(\left\{T^{k+1}(x)\right\}\right)}\right\}^{1/p} \left|\pi\left(T^k(x)\right)\right| \geq \varepsilon\right\}.$$

If this set is infinite for some $\varepsilon > 0$, then since orb(x) is infinite, $\left(\frac{1}{\alpha_i}\, m_T^{\pi}\{x_i\}\right)^{1/p} \geq \varepsilon$ for infinitely many $i \in \mathbb{Z}_+$, which is impossible by Theorem 2.5.3. Thus for every $\varepsilon > 0$, there exists k_0 such that

$$\left(\frac{m\left(\left\{T^k(x)\right\}\right)}{m\left(\left\{T^{k+1}(x)\right\}\right)}\right)^{1/p} \left|\pi\left(T^k(x)\right)\right| < \varepsilon \quad \text{for every} \quad k \geq k_0\ .$$

If $k \geq k_0$, then by a simple computation it can be shown that

$$\left|\lambda^k g(x)\right| < \|\pi\|_\infty^{k_0}\left(\frac{1}{m\left(\left\{T^k(x)\right\}\right)}\right)^{1/p} \varepsilon^{k-k_0}\, \|g\|_p\ .$$

Hence

$$|g(x)| < \frac{\|\pi\|_\infty^{k_0}\ \|g\|_p}{\varepsilon^{k_0}\left(m\left(\left\{T^k(x)\right\}\right)\right)^{1/p}}\left(\frac{\varepsilon}{|\lambda|}\right)^k.$$

From this inequality taking $\varepsilon = \frac{|\lambda|}{2}$, we can conclude that $g(x) = 0$. This shows that $g(x) = 0$ for $x \in X_2$. Now we shall show that g is zero almost everywhere on

X_1. Let $x \in X_1$ such that $\pi(x) = 0$. Then

$$\lambda g(x) = \left(W_{\pi,T}g\right)(x) = \pi(x).(g \circ T)(x) = 0$$

and hence $g(x) = 0$. If $x \in X_1 \cap T^{-1}(X_2)$, since $T(x) \in X_2$, we conclude that $g(T(x)) = 0$, and hence $\lambda g(x) = 0$. Thus $g(x) = 0$ since $\lambda \neq 0$. Since $m\left(X_1 \cap T^{-1}(X_1)\right) = 0$, we conclude that $g(x) = 0$ *a. e.* on X_1. This shows that $g(x) = 0$ *a. e.* on X, and hence $\lambda \notin \sigma\left(W_{\pi,T}\right) \cup \{0\}$. This shows that

$$\sigma\left(W_{\pi,T}\right) \cup \{0\} \subset \left\{\lambda \ : \ \lambda^k = \pi(c)\pi(T(c)) \ldots\ldots \pi\left(T^{k-1}(c)\right), \text{ for some fixed atom } c \text{ of order } k \right\} \cup \{0\}.$$

The reverse inclusion can be obtained in the same way as done by Kamowitz [164, Proposition 3]. This completes the proof.

If $1 < p < \infty$, then $L^p(m)$ is reflexive and hence $W_{\pi,T}$ is weakly compact. In case $p = 1$, the class of the weakly compact weighted composition operators coincide with that of the compact weighted composition operators. This result we shall record in the following theorem.

Theorem 2.5.7 [389]. Let $W_{\pi,T}$ be a weighted composition operator on $L^1(m)$. Then the following are equivalent :

(i) $W_{\pi,T}$ is weakly compact,

(ii) $W_{\pi,T}$ is compact.

Note. Some results on the weighted composition operators on ℓ^2 are also given in section 3.4.

CHAPTER III

COMPOSITION OPERATORS ON FUNCTIONAL BANACH SPACES

This chapter is a study of the composition operators on the functional Banach spaces and functional Hilbert spaces which include among others the well known H^p–spaces of the unit disc, the unit polydisc, or the upper–half plane and ℓ^p–spaces. In the beginning of the chapter a characterization of the composition operators is presented in the general case and in the subsequent three sections composition operators on some special functional Banach spaces (Hilbert spaces) are studied.

Most of the work on the composition operators on Hardy spaces was done during the last fifteen years or so. Most of the mathematicians like Cima and Wogen [74], Cowen ([84], [85], [86]), Deddens [90], Kamowitz ([161], [162], [163], Nordgren ([253], [254], [257]), Roan ([282], [283]), Ryff [297], Schwartz [303], Shapiro and Taylor ([306], [309]) and Swanton [384] studied the properties of the composition operators on $H^p(D)$, where D is the unit disc of the complex plane. Singh and Sharma ([359], [360], [361]) studied these operators on $H^p(D^n)$ and $H^p(P^+)$, where D^n is the unit polydisc and P^+ is the upper–half–plane. During the middle of the last decade MacCluer ([228], [229], [230]) made an intensive study of the composition operators on $H^p(D_n)$, where D_n denotes the unit ball of $\mathbb{C}^n$. Recently Jafari [152] characterised those composition operators on $H^p(D^n)$ and weighted Hardy spaces, which are bounded or compact, and Zorboska ([411], [413]) studied these operators on weighted Hardy spaces, while Sharma and Kumar [314] explored some of their properties on Hardy Orlicz spaces. An extensive work on these operators on ℓ^p–spaces has been done by Singh and Gupta ([332], [334]), Singh and Komal ([336], [337], [338], [339]) and recently by Carlson ([57], [58]).

3.1 GENERAL CHARACTERIZATIONS

If X is a non–empty set and $H(X)$ is a functional Banach space of complex–valued functions, then we know that the evaluation functionals δ_x, defined as $\delta_x(f) = f(x)$, are continuous and hence belong to $H^*(X)$, the conjugate space of $H(X)$. If $T : X \to X$

is a mapping such that $f \circ T$ belongs to $H(X)$ whenever f belongs to $H(X)$, then the mapping taking f into $f \circ T$ is a linear transformation from $H(X)$ into itself. By the closed graph theorem this linear transformation is continuous (bounded). This transformation is denoted by C_T and is called the composition operator on $H(X)$ induced by T.

If A is an operator on $H(X)$, then A^*, the adjoint of A is an operator on $H^*(X)$ defined as

$$\left(A^* F\right)(f) = F(Af)$$

for every $F \in H^*(X)$ and $f \in H(X)$. It is evident that A^* is the composition operator on $H^*(X)$ induced by the operator A. If C_T is a composition operator on $H(X)$ and $\Delta = \left\{\delta_x \ : \ x \in X\right\}$, then clearly Δ is invariant under C_T^*; actually $C_T^* \ \delta_x = \delta_{T(x)}$. It turns out that this condition is also sufficient for an operator A on $H(X)$ to be a composition operator. This is evident from the following theorem.

Theorem 3.1.1. Let $H(X)$ be a functional Banach space over a non-empty set X and let A be an operator on $H(X)$. Then A is a composition operator if and only if Δ is invariant under A^* (*i.e.*, $A^* \ (\Delta) \subset \Delta$).

Proof. Suppose $A = C_T$ for some T. Let $\delta_x \in \Delta$ and $f \in H(X)$. Then $\left(C_T^* \ \delta_x \ \right)(f) = \ \delta_x(C_T f) = \delta_x(f \circ T) = f(T(x)) = \delta_{T(x)}(f)$. Thus $C_T^* \delta_x = \delta_{T(x)} \in \Delta$. Conversely, suppose Δ is invariant under A^*. Let $x \in X$. Then $\delta_x \in \Delta$ and hence $A^* \delta_x \in \Delta$. Thus there exists a $T(x) \in X$ such that $A^* \delta_x = \delta_{T(x)}$. The mapping T taking x to $T(x)$ is well defined. Now

$$(f \circ T)(x) = f(T(x)) = \delta_{T(x)}(f) = \left(A^* \delta_x\right)(f) = (Af)(x).$$

This shows that $C_T = A$. This completes the proof of the theorem.

If $H(X)$ is a functional Hilbert space and $x \in X$, then by Riesz–representation theorem there exists a unique $f_x \in H(X)$ such that

$$f(x) = \delta_x(f) = \langle f, f_x \rangle \quad \text{for all } f \in H(X) \, .$$

Let $K(X) = \left\{f_x \ : \ x \in X\right\}$. Then $K(X)$ is a subset of $H(X)$ and the invariance of

$K(X)$ under A^* is a necessary and sufficient condition for A to be a composition operator on $H(X)$. This we record in the following theorem the proof of which is similar to that of Theorem 3.1.1

Theorem 3.1.2. Let $H(X)$ be a functional Hilbert space and let A be an operator on $H(X)$. Then A is a composition operator on $H(X)$ if and only if $K(X)$ is invariant under A^*. In this case $A^* f_x = f_{T(x)}$.

Corollary 3.1.3. An operator A on $\ell^p, p \geq 1$ is a composition operator if and only if the set $\{ e_i : i \in \mathbb{N} \}$ is invariant under A^*.

Proof. Since ℓ^p is a functional Banach space and $\delta_i = e_i$ for all $i \in \mathbb{N}$, the proof follows from Theorem 3.1.1.

In case $H(X)$ is a functional Hilbert space, the kernel function K of $H(X)$ is defined as

$$K(x, y) = \langle f_y, f_x \rangle .$$

If $T : X \to X$ is a map and $\{x_1, x_2, \ldots, x_n\}$ is a finite subset of X, then by the symbol $\tilde{K}(x_1, x_2, \ldots, x_n)$ we denote the matrix whose (i, j)th entry is $K(x_i, x_j)$. Now the following theorem of Nordgren [254] characterises the mappings T which induce composition operators on $H(X)$.

Theorem 3.1.4. A mapping $T : X \to X$ induces a composition operator on $H(X)$ if and only if there exists an $M > 0$ such that

$$\tilde{K}(T(x_1), T(x_2), \ldots, T(x_n)) \leq M \, \tilde{K}(x_1, x_2, \ldots, x_n)$$

for every finite subset $\{x_1, x_2, \ldots, x_n\}$ of X.

Proof. Since $K(x_1, x_2, \ldots, x_n)$ is a Gram Matrix, it is a positive matrix. Suppose C_T is the composition operator induced by T. Then by Theorem 3.1.2 $C_T^* f_{x_i} = f_{T(x_i)}$ for $i = 1, 2, \ldots, n$. If $\alpha_1, \alpha_2, \ldots, \alpha_n$ are n complex numbers, then

$$\left\| \sum_{i=1}^{n} \alpha_i f_{T(x_i)} \right\|^2 = \left\| C_T^* \left(\sum_{i=1}^{n} \alpha_i f_{x_i} \right) \right\|^2 \leq \| C_T \|^2 \left\| \sum_{i=1}^{n} \alpha_i f_{x_i} \right\|^2 .$$

Computing the norm in $H(X)$ we see that the above inequality is equivalent to the following inequality

$$\sum_i \sum_j \overline{\alpha}_i \, \alpha_j \, K\big(T(x_i), \; T(x_j)\big) \le \|C_T\|^2 \sum_i \sum_j \overline{\alpha}_i \, \alpha_j \, K\big(x_i, x_j\big) ,$$

which is equivalent to the condition given in the theorem.

3.2 COMPOSITION OPERATORS ON SPACES $H^p(D)$, $H^p(D^n)$ AND $H^p(D_n)$

In this section we plan to present results on the composition operators on a very nice class of functional Banach spaces of analytic functions, namely, Hardy spaces, which have very rich algebraic and topological structures. The period of the last fifteen years or so had been very productive so far as the study of these operators on Hardy spaces is concerned, and due to some restrictions and necessity we would not be able to present all the results obtained so far. But a comprehensive bibliography is given and most of the omitted results can be found in the references cited there. Most of the works during the period concentrated on the study of the composition operators on $H^p(D)$. Singh and Sharma ([361], [317]) attempted to study these operators on $H^p(D^n)$. In her papers ([229], [230]) MacCluer studied composition operators on $H^p(D_n)$. The definitions of Hardy spaces $H^p(D)$, $H^p(D^n)$ and $H^p(D_n)$ are given in chapter–I. For further details about these spaces the readers should see Rudin ([287], [289]). In case $n = 1$, $H^p(D^n) = H^p(D_n) = H^p(D)$, and it is known that every holomorphic map $T : D \to D$ induces a composition operator on $H^p(D)$. This result does not carry over to the case of several complex variables. MacCluer [229], and Cima and Wogen [75] have produced examples of holomorphic mappings taking D_n into D_n and not inducing composition operators on $H^p(D_n)$ for $n > 1$; and Singh and Sharma [361] and Jafari [152] produced examples of unbounded composition operators on $H^p(D^n)$. In case of $H^p(D^n)$ if each component of the map $T : D^n \to D^n$ is a holomorphic map from D to D, then T induces a composition operator. Thus every automorphism induces a composition operator on $H^p(D^n)$. A characterisation of holomorphic mappings inducing composition operators on $H^p(D^n)$ has recently been given by Jafari [152] in terms of a measure associated with T. The following theorem contains some results on the composition operators on $H^p(D)$, most of which are due to Schwartz [303].

Theorem 3.2.1.

(i) Every holomorphic map T from D into itself induces a composition operator C_T on $H^p(D)$ for $1 \le p \le \infty$ and

$$\|C_T\|^p \le \frac{1+|T(0)|}{1-|T(0)|}, \quad (p \neq \infty).$$

(ii) If $f_n(z) = z^n$ for $n \in \mathbb{Z}_+$ and A is non-zero bounded operator on $H^p(D)$, then the following are equivalent :

(a) A is a composition operator,

(b) $A f_n = (Af_1)^n$ for every $n \in \mathbb{Z}_+$,

(c) $A(f.g) = Af \,.\, Ag$ for every bounded analytic functions f and g in $H^p(D)$.

(iii) A composition operator C_T on $H^p(D)$ is invertible if and only if T is a conformal automorphism of D.

Proof. (i) By a result of Rudin ([286], [287]) an analytic function f on D belongs to $H^p(D)$ if and only if $|f|^p$ has a harmonic majorant, and if ψ_p is the least harmonic majorant of f, then $\|f\|^p = \psi_f(0)$. Now if $T : D \to D$ is analytic and $f \in H^p(D)$, then $f \circ T$ is analytic on D and

$$|(f \circ T)(z)|^p = |f(T(z))|^p \le \psi_f(T(z)) \qquad \text{for all } z \in D,$$

Thus $\psi_f \circ T$ is a harmonic majorant for $f \circ T$ and hence $f \circ T \in H^p(D)$. This shows that T induces the composition operator C_T. In this case

$$\|C_T f\|^p = \|f \circ T\|^p = \psi_{f \circ T}(0) \le \psi_f(T(0)).$$

Hence by Harnack's inequality we have

$$\|C_T f\|^p \le \frac{1+|T(0)|}{1-|T(0)|}\|f\|^p.$$

(ii) Suppose $Af_n = (Af_1)^n$ for $n \in \mathbb{Z}_+$. If $T = Af_1$, then T belongs to $H^p(D)$. Now for every $n \in \mathbb{Z}_+$, we have

$$\left\| T^n \right\|^{1/n} = \left\| Af_n \right\|^{1/n} \leq \left\| A \right\|^{1/n} .$$

Taking limits of both sides as $n \to \infty$, it can be shown that

$$\left\| \tilde{T} \right\|_\infty \leq 1 ,$$

where $\tilde{T}$ denotes the radial limit of T which exists almost everywhere on the unit circle. Thus by the maximum modulus principle T maps D into D and it can not be a constant function of unit modulus. Thus T induces the composition operator C_T which agrees with A on f_n' s and hence $A = C_T$. Conversely, if A is a composition operator, then $Af_n = (Af_1)^n$ follows trivially. This proves the equivalence of (a) and (b). The equivalence of (a) and (c) follows easily from this.

(iii) Suppose A is the inverse of C_T and suppose f and g are bounded analytic functions in $H^p(D)$. Then

$$\begin{aligned} C_T A(fg) = A(fg)\circ T = f.g = (C_T Af)(C_T Ag) \\ = (Af.Ag)\circ T \end{aligned}$$

Thus $(A(fg) - Af.Ag)\circ T = 0$. Since T is non-constant as C_T is invertible, we conclude that the range of T is an open set. Hence

$$A(fg) = Af.Ag.$$

Thus by (ii) there exists an analytic function U from D into itself such that $A = C_U$. Since

$$(C_U C_T f_1)(z) = (T \circ U)(z) = z = (U \circ T)(z) \qquad \text{for all } z \in D,$$

we conclude that T is invertible with analytic inverse. Hence T is a conformal automorphism. The converse is easy to prove.

Note. In case of $H^\infty(D)$ every holomorphic T induces a contraction *i.e.*, $\left\| C_T \right\| \leq 1$. If T is an inner function, then it has been shown by Nordgren [253] that the inequality in part (i) of the theorem is equality and if 0 is a fixed point of T, then C_T is an isometry. In part (iii) invertibility of C_T implies that T is invertible in analytic sense which is the same as saying that T is a Mobius transformation.

Not every holomorphic map from D^n to D^n induces a composition operator on $H^p(D^n)$ for $n > 1$ as was shown by Singh and Sharma in [361]. For example,

$T(z_1,z_2)=(z_1,z_1)$ does not induce a composition operator on $H^2(D^2)$. But certain nice mappings always induce composition operators on $H^p(D^n)$. Such some mappings are given in the following theorem which is a partial generalization of part (i) of Theorem 3.2.1.

Theorem 3.2.2. Let $t_1,t_2,\ldots,t_n$ be n holomorphic functions from D into itself and let $T : D^n \to D^n$ be defined as

$$T(z_1,z_2,\ldots,z_n)=(t_1(z_1),t_2(z_2),\ldots,t_n(z_n)).$$

Then T induces a composition operator C_T on $H^p(D^n)$ for $1 \le p < \infty$ and in this case

$$\|C_T\|^p \le \prod_{k=1}^{n} \frac{1+|t_k(0)|}{1-|t_k(0)|}.$$

Proof. If $f \in H^p(D^n)$, then $|f|^p$ has an n-harmonic majorant (see Rudin [287]). Thus $\psi_f \circ T$ is an n-harmonic majorant of $|f \circ T|^p$ and hence $f \circ T \in H^p(D^n)$. Thus T induces a composition operator on $H^p(D^n)$. The inequality can be obtained after some computation (see [361] for details).

Corollary 3.2.3. (i) Every automorphism T of D^n induces a composition operator on $H^p(D^n)$.

(ii) Every proper holomorphic map from D^n to D^n induces a composition operator on $H^p(D^n)$.

Proof. The proof follows from the above theorem because every automorphism as well as every proper holomorhic map is of the type given in the statement of the theorem (see [250] for details).

Note. In the definition of T in the statement of Theorem 3.2.2 any permutation of $\{z_1,z_2,\ldots,z_n\}$ can be taken.

If $T=(T_1,T_2,\ldots,T_n)$ is a holomorphic map from D^n to D^n such that all but one T_i are constants, then T induces a composition operator on $H^p(D^n)$. For details we refer to [361].

The problem of characterization of holomorphic mappings inducing composition operators on $H^p(D^n)$ remained unsolved until the end of the last decade. In 1990 Jafari [152] presented a characterization in terms of the measures induced by the mappings. By ∂D^n we denote the distinguished boundary of D^n and m_n denotes the n-dimensional normalised Lebesgue area measure on ∂D^n. If R is a rectangle on ∂D^n, then $S(R)$ denotes the Corona associated to R. If $R = I_1 \times I_2 \times \ldots \times I_n \subset \partial D^n$, where I_j is the interval on ∂D of length σ_j and centre at $\exp\left(i\left(\theta_j^0 + \frac{\sigma_j}{2}\right)\right)$ for $j = 1, \ldots, n$, then

$$S(R) = S(I_1) \times S(I_2) \times \ldots \times S(I_n),$$

where $$S(I_j) = \left\{re^{i\theta} \in D : 1 - \sigma_j < r < 1 \ \text{ and } \ \theta_j^0 < \theta < \theta_j^0 + \sigma_j\right\}.$$

If G is any open subset in ∂D^n, then $S(G)$ is defined as

$$S(G) = \cup \ \{S(R) \ : \ R \ \text{ a rectangle in } \ G\}.$$

If $T : D^n \to D^n$ is a holomorphic map, then define

$$\tilde{T}(\xi) = \operatorname*{limit}_{r \to 1} T(r\,\xi) \quad \text{for} \quad \xi \in \partial D^n,$$

whenever the limit exists. Define the measure μ_T on $\overline{D^n}$ as

$$\mu_T(F) = m_n\left(\tilde{T}^{-1}(F)\right),$$

where F is a Borel subset of $\overline{D^n}$. Then it is clear that $\int_{\overline{D_n}} f\, d\mu_T = \int_{\partial D^n} f \circ \tilde{T}\, dm_n$ for $f \in C\left(\overline{D^n}\right)$. If μ is any finite, non-negative Borel measure on $\overline{D^n}$ and if A is the identity operator taking $H^p(D^n)$ into $L^p(\mu)$ for $1 < p < \infty$, then it has been recorded by Jafari [152] that A is bounded if and only if $\mu(S(G)) \le c\, m_n(G)$ for every open connected subset G of ∂D^n *i.e.* μ is a Carleson measure on $\overline{D^n}$. He further showed that the boundedness of C_T on a dense subset of $H^p(D^n)$ is equivalent to boundedness of C_T on $H^p(D^n)$ for $1 < p < \infty$. These results are used in the proof of the following theorem which characterises mappings inducing composition operators on $H^p(D^n)$ and gives a sufficient condition for unbounded composition operators.

Theorem 3.2.4. Let $T : D^n \to D^n$ be a holomorphic function and let $1 < p < \infty$. Then

(i) C_T is a composition operator on $H^p(D^n)$ if and only if $\mu_T(S(G)) < c\, m_n(G)$ for every open set G in ∂D^n and for some constant $c > 0$.

(ii) If $\sup_{z \in D^n} \prod_{i=1}^{n} \frac{(1 - |z_i|^2)}{1 - |T_i(z)|^2}$ is infinite, then T does not induce a composition operator on $H^p(D^n)$, where $T(z) = (T_1(z), T_2(z), \ldots, T_n(z))$.

Proof. (i) Since $\int_{\overline{D^n}} f\, d\mu_T = \int_{D^n} (f \circ \tilde{T}) dm_n$ gives rise to a continuous linear functional on $C\left(\overline{D^n}\right)$, we conclude that μ_T is a well-defined measure. If $f \in H^p\left(D^n\right) \cap C\left(\overline{D^n}\right)$, then

$$\begin{aligned} \|f \circ T\|_p^p &= \int_{\partial D^n} |f|^p \circ \tilde{T}\, dm_n \\ &= \int_{\overline{D^n}} |f|^p\, d\mu_T \ . \end{aligned}$$

If μ_T is a Carleson measure, then it follows from the statement preceding this theorem that C_T is continuous on $H^p\left(D^n\right) \cap C\left(\overline{D^n}\right)$ and hence on $H^p(D^n)$, since $H^p\left(D^n\right) \cap C\left(\overline{D^n}\right)$ is dense in $H^p(D^n)$. If C_T is bounded, then again by above argument μ is a Carleson measure. This outlines the proof of part (i).

(ii) It is sufficient to consider $p = 2$. Let $z \in D^n$. Define $g_z : D^n \to \mathbb{C}$ as

$$g_z(w) = \prod_{i=1}^{n} \frac{1}{1 - \overline{w}_i z_i}.$$

Then $g_z \in H^2(D^n)$, and hence

$$\|g_z\|^2 = g_z(z) = \prod_{i=1}^{n} \frac{1}{1 - |z_i|^2}.$$

We also have

$$\|g_{T(z)}\|^2 = g_{T(z)}(T(z)) = \prod_{i=1}^{n} \frac{1}{1-|T_i(z)|^2}.$$

Also

$$\|g_{T(z)}\|^2 = (g_{T(z)} \circ T)(z) = (C_T\, g_{T(z)})(z).$$

Hence

$$\|g_{T(z)}\|^2 \leq \|C_T \;\; g_{T(z)}\|_2 \left(\prod_{i=1}^{n} \frac{1}{1-|z_i|^2} \right)^{1/2} \qquad \text{(see [361]).}$$

From this it follows that

$$\|C_T\|^2 \geq \|g_{T(z)}\|^2 \prod_{i=1}^{n} (1-|z_i|^2) = \prod_{i=1}^{n} \frac{1-|z_i|^2}{1-|T_i(z)|^2}.$$

This is true for all $z \in D^n$; hence

$$\|C_T\|^2 \geq \sup_{z \in D^n} \left\{ \prod_{i=1}^{n} \frac{1-|z_i|^2}{1-|T_i(z)|^2} \right\}.$$

This takes care of the proof of part (ii).

Note. Using the theory of Carleson measures it follows that an analytic map $T : D^n \to D^n$ induces a composition operator if and only if μ_T is a Carleson measure on D^n. If $T(z_1, z_2) = (z_1, z_1)$, then T is an analytic self map of D^n and

$$\sup_{z \in D^2} \left\{ \frac{1-|z_2|^2}{1-|z_1|^2} \right\}$$

is infinite, and hence T does not induce a composition operator on $H^p(D^n)$ as reported earlier. For more such examples we refer to [111] and [152].

MacCluer produced several examples of analytic self-maps of D_n which do not induce composition operators on $H^p(D_n)$ for $n > 1$. Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{Z}_+^n$

such that $\alpha_i \neq 0$ for atleast two $i's$ and let $p(\alpha)$ be the product of the numbers of type $\alpha_i^{\alpha_i/2}$, where $\alpha_i \neq 0$. Let $q(\alpha) = |\alpha|^{|\alpha|/2}$. For $z = (z_1, z_2, \ldots, z_n) \in D_n$, define

$$T(z) = \left(\frac{q(\alpha)}{p(\alpha)} z^{\alpha}, 0, 0, \ldots, 0 \right).$$

Then it turns out that $T : D_n \to D_n$ is analytic. MacCluer [228] has produced a sequence $\{g_n\}$ in $H^p(D^n)$ such that $\|g_n \circ T\|$ tends to infinity as n does. Thus T does not induce a composition operator. If $T(z_1, z_2) = (z_1^2 + z_2^2, 0)$, then T is a self map of D_2 which does not induce a composition operator either. Similarly, if T_1 and T_2 are inner functions on D_n and $a, b > 0$, then the map

$$T(z_1, z_2) = \left(aT_1(z_1, z_2),\ bT_2(z_1, z_2)\right)$$

does not induce a composition operator on $H^p(D_2)$.

MacCluer has given a characterization of holomorphic self maps of D_n inducing composition operators on $H^p(D_n)$ in terms of the measures induced by self maps. The characterization is similar to one given recently by Jafari in [152] for $H^p(D^n)$ and recorded in the last theorem. We shall need some definition to present the characterization given by MacCluer.

Let m be the probability measure on the boundary ∂D_n of D_n, which is rotation invariant ([see Rudin [289]). If $T : D_n \to D_n$ is a holomorphic map and $\xi \in \partial D_n$, then define

$$\tilde{T}(\xi) = \lim_{r \to 1} T(r\xi),$$

which exists *a.e.* with respect to m. Thus T can be regarded as a map from $\overline{D_n}$ into itself. This map $\tilde{T}$ induces a Borel measure μ_T on $\overline{D_n}$ defined as

$$\mu_T(F) = m\left(\tilde{T}^{-1}(F)\right)$$

for every Borel subset F of $\overline{D_n}$. Let $\eta \in D_n$ and let $t > 0$. Then define the set S_η^t as

$$S_\eta^t = \left\{ z \in \overline{D} \ : \ \left|1 - \langle z, \eta \rangle\right| < t \right\}.$$

Let

$$h_\eta(t) = \mu_T\left(S_\eta^t\right).$$

The following theorem characterises the mappings inducing composition operators on $H^p(D_n)$.

Theorem 3.2.5. Let $T : D_n \to D_n$ be a holomorphic map. Then the following are equivalent :

(i) C_T is a composition operator on $H^p(D_n)$;

(ii) there exists a constant $b > 0$ such that $h_\eta(t) \leq b t^n$ for every $\eta \in D_n$ and $t > 0$;

(iii) μ_T is an m–Carleson measure on $\overline{D_n}$.

Proof. It is based on a characterization of m–Carleson measures on $\overline{D_n}$ given in [230].

One of the interesting classes of operators on any Banach space is the class of compact operators. We are interested here in reporting about the compact composition operators. In case of L^p–spaces there is scarcity of compact composition operators as we have seen in section three of chapter II. But in case of H^p–spaces there are many compact composition operators. For example, if T maps D into a polygon inscribed in the unit circle, then C_T is a compact composition operator on $H^p(D)$ as is shown by Shapiro and Taylor in [309]. A similar sufficient condition in terms of a geometric property of the range of the inducing function has been obtained by MacCluer in [230] in case of $H^p(D_n)$. Recently Jafari [152] presened a characterization of compact operators on $H^p(D^n)$, and Sharma and Singh [317] studied such operators on $H^2(D^n)$. While a suitable characterization of compact composition operators on functional Banach spaces of analytic functions is still awaitëd, it has been settled for most common spaces. In the following theorem we present some known results on compact composition operators on $H^p(D)$.

Theorem 3.2.6. Let C_T be a composition operator on $H^p(D)$. Then

(i) C_T is compact if and only if for every norm bounded sequence $\{f_n\}$ in $H^p(D)$ which converges to zero uniformly on compact subsets of the unit disc, the sequence $\{f_n \circ T\}$ converges to zero in norm.

(ii) C_T is compact implies that $\left| \tilde{T}(e^{i\theta}) \right| < 1$ *a.e.*

(iii) C_T is not compact if T possesses an angular derivative at some point of ∂D.

(iv) C_T is Hilbert–Schmidt if and only if $1/(1-|\tilde{T}|^2)$ is integrable with respect to Lebesgue measure on ∂D, $p = 2$.

(v) C_T is a trace class operator on $H^p(D)$ if T takes D into a polygon inscribed in the unit circle.

Proof. (i) This result is true in several Banach spaces of analytic functions and the proof is not difficult. It can be found in [303].

(ii) Since T maps D into D, it is obvious that $\left|\tilde{T}(e^{i\theta})\right| < 1$ *a.e.* Let $n \in \mathbb{N}$ and let f_n on D be defined as $f_n(z) = z^n$. Then $\{f_n\}$ is a norm bounded sequence converging to zero uniformly on compact subsets of D. If $\left|\tilde{T}(e^{i\theta})\right| = 1=1$ on a set of non–zero measure, then

$$\| C_T f_n \|^2 = \| f_n \circ T \|^2 = \int \left| T(e^{i\theta}) \right|^{2n} d\theta \not\to 0$$

as $n \to \infty$. Hence by (i), C_T is not compact.

(iii) Suppose T has angular derivative at $e^{i\theta}$ which can be assumed to be 1 without any loss of generality. By definition of the angular derivative it is clear that there exists a constant $k > 0$ such that

$$\frac{|1 - T(t)|}{|1 - t|} \le k \quad \text{for} \quad -1 < t < 1.$$

Let $$f_n(z) = \frac{1}{\sqrt{n(1-z)^{(n-1)/n}}} \quad \text{for} \quad z \in D.$$

Then it can be shown that $\{f_n\}$ is a weak null sequence in $H^2(D)$ and the sequence $\{f_n \circ T\}$ is bounded away from zero. Hence C_T is not compact (for details see [254]).

(iv) C_T is Hilbert–Schmidt if and only if $\sum\limits_{n=1}^{\infty} \|C_T f_n\|^2 < \infty$, where f_n is same as defined in (ii). Since

$$\sum_{n=1}^{\infty} \| C_T f_n \|^2 = \sum_{n=1}^{\infty} \int \left(\left| \tilde{T}\left(e^{i\theta}\right) \right|^{2n} \right) d\theta = \int \frac{1}{1 - \left| \tilde{T}\left(e^{i\theta}\right) \right|^2} \, d\theta \, ,$$

we conclude the result.

(v) See [309] for the proof.

Note. Results (i) and (ii) of the above theorem are due to Schwartz who has shown that the condition in (ii) is not sufficient. If T takes z to $\frac{1+z}{2}$ for $z \in D$, then T touches D at 1. But T does not induce a compact composition operator. This follows from part (iii) of the above theorem. Parts (iii) through (v) are due to Shapiro and Taylor [309] which are true for every $1 \le p \le \infty$, not necessarily $p = 2$ as taken in the theorem. On $H^{\infty}(D)$ a composition operator is compact if and only if $\| T \|_{\infty} < 1$. Cowen in his paper [84] has shown that compactness of C_T implies that $| a | < 1$, where a is the Denjoy–Wolff point of T, a non-elliptic Mobius transformation. The result of (iii) is extended to weighted Dirichlet spaces and $H^p(D_n)$ by MacCluer and Shapiro [233].

In the following theorem we record the characterizations of compact composition operators on $H^p(D^n)$ and $H^p(D_n)$ given by Jafari [152] and MacCluer [230] respectively. Proofs can be found in the cited reference.

Theorem 3.2.7. (i) A composition operator C_T on $H^p(D^n)$ is compact if and only if

$$\lim_{m_n(G)\to 0} \ \sup_{G \subset \partial D^n} \frac{\mu_T(S(G))}{m_n(G)} = 0.$$

(ii) A composition operator C_T on $H^p(D_n)$ is compact if and only if $h_\eta(t) = 0(t^n)$ uniformly in η as $t \to 0$.

Now we shall describe the spectra of composition operators on some Hardy spaces. From the known results so far, it is evident that in exploration of the spectra of composition operators the fixed points of the inducing functions play very important roles. The first result in this direction was obtained by Nordgren [253] who computed the spectra of composition operators induced by linear fractional transformations from D onto D. Then Kamowitz ([160], [161]) made a deep study of the spectra of composition operators induced by analytic mappings taking D into D with a condition that they are analytic in an open region containing D. Deddens [90] also obtained some results in special cases. Cowen [84] made a thorough study of the spectral properties of composition operators in terms of the behaviour of the inducing functions in vicinity of the Denjoy-Wolff points. The results are quite deep and computational; and hence we will content ourselves with their statements. For the proofs the readers can see the cited references. In several variables case Figura [112] and MacCluer ([228], [229]) reported the results about the spectra of composition operators. In this case too, the fixed points play very crucial roles.

If $T : D \to D$ is an inner function with a fixed point in D, then we can define a linear fractional transformation $L : D \to D$ such that L is onto and it takes the fixed point to 0. If $U = L \circ T \circ L^{-1}$, then U has a fixed point at 0, and hence by a result of [253], C_U is an isometry. Thus C_T is similar to C_U . If $T : D \to D$ is a linear fractional transformation, then the fixed points of T determine the behaviour of T. If T has one fixed point inside the unit circle and one outside, then it is called elliptic. In this case the function U has the form $U(z) = e^{i\theta} z$ for some θ. If two fixed points of T are on the unit circle ∂D, then T is called hyperbolic. It is known that the derivative T' of T takes a positive value $k > 1$ at one fixed point. If T has one fixed point on ∂D, then T is called parabolic. We shall record some known results about spectra in the following theorems. By $\sigma(C_T)$, we denote the spectrum of C_T.

Theorem 3.2.8 [Nordgren [254]]. Let C_T be a composition operator on $H^p(D)$. Then

(i) $\sigma(C_T) = \bar{D}$, if T is an inner function which is not a linear fractional transformation and has a fixed point in D.

(ii) $\sigma(C_T) =$ closure of the set $\left\{ e^{in\theta} \ : \ n \in \mathbb{Z}_+ \right\}$, if T is elliptic.

(iii) $\sigma(C_T) = \left\{ z \in \mathbb{C} \ : \ \frac{1}{\sqrt{k}} \le |z| \le \sqrt{k} \right\}$, if T is hyperbolic.

(iv) $\sigma(C_T) = \partial D$, if T is parabolic.

Theorem 3.2.9 [Kamowitz [161]]. Let $T : D \to D$ be an analytic map such that T is also analytic on $\bar{D}$ and C_T is a composition operator on $H^p(D)$ for $1 \le p < \infty$. Then

(i) $\sigma(C_T) = \left\{(T'(z_0))^n \ : \ n \in \mathbb{Z}_+\right\} \cup (0, 1)$, if C_T^n is compact for some $n \in \mathbb{Z}_+$ (z_0 is a fixed point of T in D) [61].

(ii) If C_T^n is not compact for every $n \in \mathbb{Z}_+$ and T is not surjective linear fractional transformation having no fixed point in D, then by Brouwer fixed point theorem, there exists a unique fixed point $z_0 \in \partial D$ such that $0 < T'(z_0) \leq 1$. Let $b = T'(z_0)$. Then if $b < 1$, $\sigma(C_T) = \left\{z \in \mathbb{C} \ : \ | z | \leq b^{-1/p}\right\}$ and if $b = 1$, $\sigma(C_T) \subset \bar{D}$.

(iii) Suppose T has a fixed point $z_0 \in D$ and T is not inner. Then $F = \bigcap\limits_{n=0}^{\infty} T^n(\partial D)$ is a finite subset of ∂D and $T : F \to F$ is a permutation.

Let k be the order of permutation, and let $c = \min\left\{\left(T^k\right)'(z) \ : \ z \in F\right\}$.

Then $c > 1$ and

$$\sigma\left(C_T\right) = \left\{z \in \mathbb{C} \ : \ | z | \leq 1 / c^{1/2k}\right\} \bigcup \left\{(T'(z_0))^n \ : \ n \in \mathbb{Z}_+\right\} \bigcup (1) .$$

Theorem 3.2.10 [Cowen [84]]. Let $T : D \to D$ be an analytic non-elliptic linear fractional transformation and let a be the Denjoy–Wolff point of T *i.e.* $T(a) = a$, $|T'(a)| \leq 1$ and $| a | \leq 1$. Then

(i) $\gamma(C_T) = 1$ if $|a| < 1$ and $\gamma(C_T) = 1/\sqrt{T'}(a)$ if $| a | = 1$ where $\gamma(C_T)$ denotes the spectral radius of C_T.

(ii) If T is an inner function which is not a linear fractional transformation, then

$$\sigma(C_T) = \left\{z \in \mathbb{C} \ : \ | z | \leq (T'(a))^{-1/2}\right\} \ \text{in case } | a | = 1.$$

Theorem 3.2.11 [Figura [112], MacCluer [229]]. Let $T : D_n \to D_n$ be a holomorphic map such that C_T is a composition operator on $H^p(D_n)$. Then

(i) If C_T is compact and z_0 is the fixed point of T in D_n, then $\sigma\left(C_T\right) = \left\{z \in \mathbb{C} \ : \ z = \lambda^{\alpha}(z_0) \ : \ \alpha \in \mathbb{Z}_+^n\right\} \cup (0)$, where $\lambda(z_0) = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ is the set of all eigen-values of the operator $dT(z_0) \ : \ \mathbb{C}^n \to \mathbb{C}^n$ defined by the derivative of T at z_0.

(ii) If T is an automorphism with a fixed point on ∂D_n, then $\sigma(C_T) = \partial D_n$.

(iii) If T is an automorphism fixing the point $(1,0,0,0\ldots,0)$ only, then $\gamma(C_T)=1$.

(iv) If $A : \mathbb{C}^n \to \mathbb{C}^n$ is a linear transformation with $\|A\|=1$ and $T : D_n \to D_n$ is defined as $T(z)=Az$.

Then

$$\sigma_p(C_T) = \left\{z \in \mathbb{C} \ : \ z = \lambda^\alpha, \ \alpha \in \mathbb{N}^n\right\},$$

where $\lambda=\{\lambda_1,\lambda_2,\ldots,\lambda_n\}=\sigma(A)$ and $\sigma_p(C_T)$ denotes the point spectrum of C_T.

(v) If $T(w)=Aw+b$ for $w\in D_n$ where $\|A\|+|b|<1$ then C_T is a trace class operator and

$$\sigma_p(C_T) = \left\{z \in \mathbb{C} \ : \ z = \lambda^\alpha, \ \alpha \in \mathbb{Z}_+^n\right\} \cup \{0\},$$

where $\lambda=\{\lambda_1,\lambda_2,\ldots,\lambda_n\}=\sigma(A)$.

The range of any operator is a linear manifold which is not necessarily closed or dense. But in some cases the range is closed or dense; for example if operator is an isometry, then the range is closed. In the following theorem, we give two results pertaining to the range of a composition operator; the first result is due to Cima, Thomson and Wogen [73] and the second due to Roan [283]. The first result has been generalised for the composition operators on L^p-spaces by Singh and Kumar [340].

Theorem 3.2.12. Let $T : D \to D$ be an analytic non-constant map, let $\tilde{T}$ be the boundary function of T, and let $f_{\tilde{T}}$ be the Radon–Nikodym derivative of the measure $m\tilde{T}^{-1}$ on ∂D with respect to the normalised Lebesgue measure m on ∂D. Then

(i) C_T has closed range in $H^2(D)$ if and only if $f_{\tilde{T}}$ is bounded away from 0 *a.e.* on ∂D.

(ii) If T is a weak* generator on $H^\infty(D)$, then the range of C_T is dense in H^p, $1<p<\infty$.

Proof. (i) Suppose $f_{\tilde{T}}$ is bounded away from zero and let $f \in H^p(D)$. Then

$$\begin{aligned} \| C_T f \|^p &= \int \left| \tilde{f} \circ \tilde{T} \right|^p dm = \int \left| \tilde{f} \right|^p dm\tilde{T}^{-1} \\ &= \int \left| \tilde{f} \right|^p f_{\tilde{T}}\, dm \\ &\geq \alpha \| f \|^p, \text{ for some } \alpha > 0. \end{aligned}$$

Hence C_T is bounded below and therefore has closed range. Conversely, if $f_{\tilde{T}}$ is not bounded away from zero, then for every $k \in \mathbb{N}$, the set

$$F_k = \left\{ e^{i\theta} : \left| f_{\tilde{T}}\left(e^{i\theta}\right) \right| < \frac{1}{k} \right\}$$

is of non-zero measure. Let $E_k = \tilde{T}^{-1}(F_k)$. Let $f_k \in H^\infty(D)$ such that $\left| \tilde{f}_k \right| = 1$ on F_k and $\left| \tilde{f}_k \right| = \frac{1}{2}$ of $\partial D - F_k$. Then it can be shown that

$$\frac{\left\| C_T f_k^{n_k} \right\|}{\left\| f_k^{n_k} \right\|} \to 0 \text{ as } k \to \infty.$$

Thus C_T is not bounded below and hence the range of C_T is not closed since T is non-constant.

(ii) For proof the readers are refered to [283].

Now we shall state the following theorem from [73] without proof which characterizes Fredholm composition operators on $H^p(D)$.

Theorem 3.2.13. Let $T : D \to D$ be an analytic map. Then C_T is Fredholm on $H^2(D)$ if and only if C_T is invertible if and only if T is a conformal automorphism of D.

The study of operator algebras is linked with the invariant subspace problem of operator theory. The reflexive algebra problem, the transitive algebra problem and the reductive algebra problem have connections with the invariant subspace problem. As mentioned in the introductory chapter, the second aim of the study of the composition operators has been concerned with establishing a link of composition operators with this outstanding problem. It has been shown that every composition operator on ℓ^2 has a non-trivial invariant subspace [336]. In 1987 Nordgren, Rosenthal and Wintrobe [257] showed that every operator on a Hilbert space has an invariant subspace if and only if the minimal invariant subspaces of a composition operator C_T on $H^2(D)$ induced by a hyperbolic surjective linear fractional transformation are one dimensional. Thus the study of invariant subspaces of a particular composition operator C_T, like $T(z) = (2z-1)/(2-z)$ may lead towards a solution of the invariant subspace problem. This follows from the study of algebras generated by composition operators.

Cima and Wogen [74] proved that the weakly closed algebra generated by the group of all invertible composition operators on $H^2(D)$ is reflexive. If f is a non-constant function in $H^p(D)$, then it turns out that the set $\{C_T f : T$ automorphism of D$\}$ has dense span. From this it follows that $\{0\}$, $\mathbb{C}$ and $H^2(D)$ are the only subspaces of $H^2(D)$ which are invariant under every invertible composition operator C_T. This result together with a result of Radjavi and Rosenthal [276] gives the reflexivity of the weakly closed algebra generated by all invertible composition operators. In [257] these results of Cima and Wogen are extended besides giving a nice structure theorem for invertible composition operators showing the reflexivity of such operators. Several consequences are derived including a link between general invariant subspace problem and minimal invariant subspaces of a hyperbolic composition operator on $H^2(D)$. In the following theorem we shall record the main results of [257].

Theorem 3.2.14. Let C_T be a composition operator on $H^2(D)$, $1 \leq p < \infty$. Then

(i) Every strongly closed algebra of operators on $H^2(D)$ containing C_T is reflexive if T is an elliptic automorphism of D of an infinite order.

(ii) If T is inner and $T(0) \neq 0$, then the common invariant subspace of C_T and the adjoint of the unilateral shift are the subspaces of the type $H^2(D) \ominus zgH^2(D)$, where g is inner and $g \circ T$ is a divisor of g.

(iii) Let $T : D \to D$ be an automorphism such that either T is parabolic with 1 as the fixed point or T is hyperbolic with 1 and -1 as the fixed points. Let $b(z)$ be the Blaschke product with zeros at $T^{(n)}(0)$, $n \in \mathbb{Z}$, and let K_0 be the subspace spanned by $\left\{T^{(n)} : n \in \mathbb{Z}\right\}$. Then $K_0 = (z.b(z)H^2(D))^{\perp} = \mathbb{C} \oplus \mathcal{L}$, and the compression of C_T to $\mathcal{L}$ is similar to a bilateral weighted shift.

(iv) If $T : D \to D$ is a parabolic or hyperbolic invertible linear fractional transformation, then C_T is super-reflexive.

(v) If $T : D \to D$ is a hyperbolic automorphism of D and λ is an interior point of $\sigma(C_T)$, then the operator $C_T - \lambda I$ is universal.

Note. There are some more results on composition operators on the Hardy spaces of the disc, the polydisc and the unit ball. Due to compactness of the monograph it would not be possible to list them all in this section of chapter III. They can be seen in the cited references. For example, Shapiro [306] computed the essential norm of composition operators using the Nevanlinna counting functions and Cowen [86] computed the adjoint of some composition operators and used it to characterise subnormal and hyponormal composition operators on $H^2(D)$. Some of the semi group properties of these operators studied by Siskakis ([373], [374]) are presented in the last chapter of this book.

The composition operators on several functional Banach spaces of analytic functions other than Hardy spaces have also been studied during the last two decades or so. Boyd [42] studied them on Bergman spaces; MacCluer and Shapiro [233] explored their properties on weighted Dirichlet spaces; Zorboska [414] studied them on S_a spaces; and Chan and Shapiro [65] on Hilbert spaces of entire functions. In [385] Swanton studied compact composition operators on Banach spaces of bounded analytic functions on a domain of $\mathbb{C}$, where as Mayer ([237], [238]) studied them on Banach spaces of analytic functions on a domain of $\mathbb{C}^n$ and used the results in the study of statistical mechanics.

Now in the next section we shall report the results obtained on the composition operators on the Hardy spaces of the upper half plane.

3.3 COMPOSITION OPERATORS ON $H^p(P^+)$

Let P^+ denote the upper half plane *i.e.*, $P^+ = \{w : w \in \mathbb{C} \text{ and } \operatorname{Im} w > 0\}$, where $\operatorname{Im} w$ denotes the imaginary part of w. Then the Hardy space $H^p(P^+)$ for $1 \le p < \infty$ is defined as

$$H^p\left(P^+\right) = \left\{f : f \text{ is analytic on } P^+ \text{ and } \sup_{y>0} \int_{-\infty}^{\infty} | f(x+iy) |^p \, dx < \infty\right\}.$$

Under the pointwise vector space operations and the norm defined as

$$\| f \|^p = \sup_{y>0} \int_{-\infty}^{\infty} | f(x+iy) |^p \, dx ,$$

$H^p(P^+)$ becomes a functional Banach space. The main aim of this section is to study composition operators on $H^p(P^+)$. In the case of several variables Figura, Jafari and MacCluer have given examples of the holomorphic mappings from D^n to D^n or from D_n to D_n which do not induce composition operators on Hardy spaces. Here too, though the case is of one variable, there are analytic functions mapping P^+ into P^+ without inducing composition operators on $H^p(P^+)$. A study of the composition operators on $H^p(P^+)$ was started by Singh and Sharma ([359], [360]). Actually, Sharma studied them in detail in his Ph.D. thesis [311]. Cowen [85] made a study of a particular type of the composition operators on $H^2(P^+)$ and employed them to explore some properties of Cesaro operators. We need some terminology to proceed further in this section.

Let $L(z)=\dfrac{i(1+z)}{1-z}$. Then L maps D onto P^+ and ∂D onto the real line, with L^{-1} given by

$$L^{-1}(w) = \frac{w-i}{w+i} .$$

Let Q be defined as

$$(Qf)(x)=\left(1/\sqrt{\pi}\right)\left(f\circ L^{-1}\right)(x)/(x+i).$$

Then Q is a well known isometric isomorphism of $L^p(m)$ onto $L^p(-\infty,\infty)$. Let $t : D \to D$ be an analytic map and let $T=L\circ t\circ L^{-1}$. Let $\beta(z)=\dfrac{1-t(z)}{1-z}$ for $z\in D$. Then using the Poisson integrals in the disc and in the upper half plane, it has been proved in [327] that C_T is a composition operator on $H^p(P^+)$ if and only if $M_\beta C_t$ is a bounded operator on $H^p(D)$. Let $A(P^+)$ denote the set of all analytic mappings T taking P^+ into itself such that the only singularity that T can have is a pole at ∞. In the following theorem we present some results on composition operators on $H^2(P^+)$.

Theorem 3.3.1.

(i) Let $T \in A(P^+)$. Then T induces a composition operator on $H^2(P^+)$ if and only if T has a pole at ∞.

(ii) If C_T is a composition operator on $H^2(P^+)$, then

$$\sup_{w\in P^+}\{(\operatorname{Im} w)/(\operatorname{Im} T(w))\} \leq \| C_T \|^2 .$$

(iii) If C_T is induced by some $T \in A(P^+)$, then C_T is invertible if and only if T is a conformal automorphism of P^+.

Proof. (i) Suppose T has a pole at ∞. Then T is analytic in a neighbourhood of ∞, and hence the function $t = L^{-1} \circ T \circ L$ is analytic in a neighbourhood of 1 and $\tilde{t}(1) = 1$. Thus T induces a composition operator C_T on $H^2(P^+)$ by an earlier statement. Conversely, if $f \circ T \in H^2(P^+)$ for every $f \in H^2(P^+)$, then $(f \circ T)(w)$ tends to zero as $w \to \infty$ within each half-plane $\operatorname{Im} w \geq \delta > 0$. Since the function taking w to $\frac{1}{i+w}$ belongs to $H^2(P^+)$, we conclude that $\frac{1}{(i+T(w))}$ tends to zero as $w \to \infty$. Thus T has a pole at ∞.

(ii) The reproducing kernel K for $H^2(P^+)$ is given by

$$K(w,u) = \frac{i}{2\pi(w-\bar{u})}.$$

Also, $\| f_u \|^2 = \langle f_u, f_u \rangle = K(u,u) = \frac{i}{4\pi \operatorname{Im} u}$.

We also know that $C_T^* f_w = f_{T(w)}$ for all $w \in P^+$.

Thus
$$\frac{\operatorname{Im} w}{\operatorname{Im} T(w)} = \frac{\| f_{T(w)} \|^2}{\| f_w \|^2} = \frac{\| C_T^* f_w \|^2}{\| f_w \|^2} \leq \| C_T \|^2 .$$

Hence
$$\sup\left\{\frac{\operatorname{Im} w}{\operatorname{Im} T(w)} : w \in P^+\right\} \leq \|C_T\|^2.$$

(iii) If T is a conformal automorphism, then T^{-1} is analytic and has a pole at ∞ since T has a pole at ∞. Thus C_T^{-1} is a composition operator on $H^2(P^+)$ and is the inverse of C_T. Conversely, suppose C_T is invertible. We know that

$$M_\beta\, C_t = PQ^{-1}\hat{P}^{-1}\, C_T \hat{P} Q P^{-1},$$

where $t = L^{-1} \circ T \circ L$, $\beta(z) = \dfrac{1 - t(z)}{1 - z}$, P, the Poisson integral in the disc and $\hat{P}$ is the Poisson integral in the upper half plane (see [327]). Thus $M_\beta C_t$ is invertible. Since M_β is subnormal and surjective we can conclude that it is invertible. Thus C_t is invertible, and hence t is a conformal automorphism. Thus T is a conformal automorphism. This completes the proof of the theorem.

Example 3.3.2. (i) Let $a > 0$ and $w_0 \in P^+$. Then define

$$T(w) = aw + w_0 \text{ for } w \in P^+.$$

Then T induces a composition operator on $H^2(P^+)$ which follows from part (i) of the above theorem. Cowen [85] later studied this composition operator and computed its norm which came out to be $\sqrt{1/a}$.

(ii) Let n be a positive integer and

$$T(w) = \frac{i\left((w+i)^{n+1} + w(w-1)^n\right)}{(w+i)^{n+1} - w(w-i)^n}$$

for $w \in P^+$. Then T maps P^+ into P^+ and induces a composition operator on $H^2(P^+)$.

(iii) Let T be a linear fractional transformation given by

$$T(w) = \frac{aw+b}{cw+d},$$

where a, b, c, d are real numbers such that $ad - bc > 0$ and $c \neq 0$. Then T maps P^+ into P^+ but does not induce a composition operator on $H^2(P^+)$ since the point at ∞ is not a pole of T.

Note. Example 3.3.2 (iii) shows that there are many analytic mappings taking P^+ into itself, but not inducing composition operators on $H^2(P^+)$. Theorem 3.3.1 part (i) characterises a particular type of mappings which induce composition operators. There are mappings which do not fall in this type and induce composition operators on $H^2(P^+)$. Thus a complete characterization of mappings inducing composition operators on $H^p(P^+)$ is an open problem.

In case of composition operators on Hardy spaces of the disc or the unit ball of $\mathbb{C}^n$ we have seen in section 2 of this chapter that compactness or non-compactness depends heavily on the boundary behaviour of the ranges of the inducing functions; for example, if the range of T touches ∂D too frequently, then chances for C_T to be compact become remote. A similar type of behaviour is visible in case of compact composition operators on $H^p(P^+)$. This we shall see in the following theorem.

Theorem 3.3.3. Let $T : P^+ \to P^+$ be a holomorphic mapping inducing the composition operator C_T on $H^2(P^+)$. Then

(i) C_T is not compact if $\lim\limits_{y \to 0} T(x+iy)$ exists *a.e.* and is a real number for every real number x.

(ii) C_T is not compact if there exists a $k > 0$ such that $\left|(i+nT(w))/(i+nw)\right| \leq k$ for every $w \in P^+$ and $n \in \mathbb{N}$.

(iii) Suppose $\lim\limits_{y \to 0} T(x+iy)$ exists *a.e.* and belongs to P^+. Let this limit be denoted by $T_*(x)$. Then C_T is a Hilbert–Schmidt composition operator if and only if

$$\int_{-\infty}^{\infty} \left[\operatorname{Im} T_*(x)\right]^{-1} dx < \infty .$$

Proof. (i) For $n \in \mathbb{Z}_+$, define the function f_n on P^+ as

$$f_n(w) = \left(1/\sqrt{\pi}\right)\left((w-i)^n/(w+i)^{n+1}\right).$$

Then f_n is a weak null sequence in $H^2(P^+)$. After a computation it can be shown that

$$\| C_T f_n \|^{2} = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{1}{1+(T_*(x))^2}\, dx .$$

Thus $\{C_T f_n\}$ does not converge to zero in norm. Hence C_T is not compact.

(ii) For $n \in \mathbb{N}$, let f_n on P^+ be defined as

$$f_n(w) = \frac{1}{\sqrt{n\left(\dfrac{i}{n}+w\right)}}.$$

Then $\{f_n\}$ is a pointwise null sequence which is norm bounded, and hence it is weak null. Now for $w \in P^+$, we have

$$\left|\left(C_T f_n\right)(w)\right|^2 = \left| n^{-1/2}\left(in^{-1}+T(w)\right)^{-1}\right|^2$$

$$= n^{-1}\left|\left(in^{-1}+T(w)\right)^{-1}\left(in^{-1}+w\right)\left(in^{-1}+w\right)^{-1}\right|^2$$

$$\geq k^{-2}\left| f_n(w)\right|^2 .$$

Hence $\| C_T f\|^2 \geq k^{-2}\pi$, since $\| f_n \| = \sqrt{\pi}$, for $n \in \mathbb{N}$. Thus C_T is notcompact.

(iii) Let n be a non-negative integer. Then define the function f_n on P^+ as

$$f_n(w) = \frac{(w+i)^n}{\sqrt{\pi}(w+i)^{n+1}} .$$

Then we know that the family $\{f_0, f_1, f_2, \ldots\}$ is an orthonormal basis for $H^2(P^+)$. The composition operator is Hilbert–Schmidt if and only if $\sum_{n=0}^{\infty} \| C_T f_n \|^2 < \infty$. But

$$\sum_{n=0}^{\infty}\| C_T f_n \|^2 = \sum_{n=0}^{\infty}\int_{-\infty}^{\infty}\left|(f_n \circ T)_*(x)\right|^2 dx = \sum_{n=0}^{\infty}\int_{-\infty}^{\infty}\left| f_n(T_*(x))\right|^2 dx .$$

A simple computation yields that C_T is Hilbert–Schmidt if and only if $\int_{-\infty}^{\infty}\left(\operatorname{Im} T_*(x)\right)^{-1} dx < \infty$. This completes the proof of the theorem.

Note. If C_T is a composition operator on $H^2(P^+)$ and $t = L^{-1} \circ T \circ L$, then it can be proved that C_t is not a compact composition operator [360]. Thus as an application of the results of this section we can conclude that there are many non–compact composition operators on $H^p(D)$. The analytic mappings inducing composition operators on $H^p(D)$ are not necessarily carried to the analytic mappings from P^+ to P^+ inducing composition operators on $H^p(P^+)$. For example, if $t(z) = \frac{i+z}{2i}$, then C_t is a non–compact composition operator on $H^p(D)$. If $T(w) = \frac{(i-3)w - 3i + 1}{(i-1)w + i - 1}$, then $T : P^+ \to P^+$ and $t = L^{-1} \circ T \circ L$. But T does not induce a composition operator on $H^2(P^+)$ since $T(w) \to \frac{i-3}{i-1}$ as $w \to \infty$. Thus the transforms under L of the mappings inducing composition operators on $H^2(P^+)$ induce non-compact composition operators on $H^p(D)$, but the converse is false as is evident from the above example. It would be worthwhile to find all those analytic mappings from D to D whose transforms induce composition operators on $H^2(P^+)$.

3.4. COMPOSITION OPERATORS ON ℓ^p-SPACES

Let $w = \{w_n\}$ be a sequence of non-negative real numbers. Then the weighted sequence space $\ell^p(w)$, for $1 \le p < \infty$ was defined in chapter I as the Banach space of all sequences $\{\alpha_n\}$ of complex numbers such that $\sum_{n=1}^{\infty} w_n |\alpha_n|^p < \infty$. In case the sequence $\{w_n\}$ of weights consists of non-zero terms, the space $\ell^p(w)$ is a functional Banach space. Thus such an $\ell^p(w)$ falls in the category of L^p-spaces as well as in the category of functional Banach spaces. If $p = 2$, then $\ell^p(w)$ is a Hilbert space. In case $w_n = 1$ for every $n \in \mathbb{N}$, $\ell^p(w)$ denoted as ℓ^p, which are classical examples of sequence spaces. ℓ^2 is a Hilbert space which was studied by Hilbert himself. In this section we shall present some results on composition operators on these sequence spaces. In case of abstract L^p-spaces as well as functional Banach spaces sometimes it becomes difficult to have a complete feeling of what is going on in the study of composition operators. In case of sequence spaces one can have a better feeling of the behaviour of composition operators and inducing functions which are sequences. This provides a concrete base for abstract thinking. In this section we shall assume that weights w_n' s are non-zero. The following theorem contains some preliminary results.

Theorem 3.4.1. Let $T : \mathbb{N} \to \mathbb{N}$ be a function. Then

(i) T induces a composition operator on $\ell^p(w)$ if and only if there exists $k > 0$ such that

$$\sum_{i \in T^{-1}(\{n\})} w_i \le k \; w_n ,$$

for every $n \in \mathbb{N}$. In this case $\| C_T \|^p = \inf$ of such k ' s.

(ii) T is an injection if and only if T induces an isometric composition operator on ℓ^p.

(iii) T induces invertible composition operator on ℓ^p if and only if it is invertible.

Proof. (i) Let S be a subset of $\mathbb{N}$. Then define

$$m(S)=\sum_{i\in S} w_i .$$

Then m becomes a measure on the σ–algebra of all subsets of $\mathbb{N}$ and $\ell^p(w)$ is just $L^p(m)$. Since each singleton set $\{n\}$ has non–zero measure, the proof follows from Theorem 2.1.1.

(ii) From (i) it is clear that T induces a composition operator on ℓ^p if and only if there exists $k>0$ such that $\# \; T^{-1}(\{n\}) \le k$ for every $n \in \mathbb{N}$. In case T is an injection, $\# \; T^{-1}(\{n\})$ is either 0 or 1; hence if we take $k=1$, the above inequality is satisfied. It is clear that 1 is the infimum of such k 's satisfying above inequality. Hence $\| C_T \| = 1$. The converse is obvious.

(iii) Suppose T is invertible. Then there exists $U:\mathbb{N}\to\mathbb{N}$ such that $(ToU)\,(n) = (UoT)\,(n) = n$ for every $n\in\mathbb{N}$. Thus U is an injection and hence C_U is an isometric composition operator on ℓ^p and $C_T C_U = C_U C_T = I$. Hence C_T is invertible. Conversely, suppose C_T is invertible. If T is not an injection, then $T(n)=T(m)$ for distinct n and m. Hence every sequence $\{\alpha_i\}$ in the range of C_T has $\alpha_n = \alpha_m$. Then C_T is not onto, which is a contradiction. Hence T is an injection. Similarly, if T is not a surjection, then the kernel of C_T is non–trivial, which is also a contradiction. Thus T is onto. This proves that T is invertible. This completes the proof.

The following theorem computes the adjoint of a composition operator on the Hilbert space $\ell^2(w)$ and presents some additional results.

Theorem 3.4.2. Let $T:\mathbb{N}\to\mathbb{N}$ be a map such that C_T is a composition operator on $\ell^2(w)$. Then

(i) The adjoint C_T^* is given by

$$\left(C_T^* x\right)(n) = \begin{cases} \dfrac{1}{w_n}\left(\displaystyle\sum_{i\in T^{-1}(\{n\})} w_i x_i\right), & \text{if } T^{-1}(\{n\}) \text{ is non – empty} \\ 0, & \text{if } T^{-1}(\{n\}) \text{ is empty} \end{cases}$$

for every $x\in\ell^2(w)$ and $n\in\mathbb{N}$.

(ii) C_T^* is a composition operator on ℓ^2 if and only if C_T is invertible if and only if C_T is unitary.

Proof. (i) Let x and y be in $\ell^2(w)$. Then

$$\begin{aligned}
\langle C_T\, x, y\rangle &= \sum_{n=1}^{\infty} w_n (x \circ T)(n).\overline{y(n)} \\
&= \sum_{n=1}^{\infty} \quad \sum_{i \in T^{-1}(\{n\})} w_i\, x(n)\overline{y(i)} \\
&= \sum_{n=1}^{\infty} x_n \quad \sum_{i \in T^{-1}(\{n\})} w_i \overline{y(i)} \\
&= \sum_{n=1}^{\infty} w_n x(n) \overline{(C_T^* y)(n)} \\
&= \langle x,\ C_T^*\ y\rangle
\end{aligned}$$

Thus C_T^* is the adjoint of C_T.

(ii) If $C_T = C_U$ for some U, then for every $n \in \mathbb{N}$

$$\chi_{\{T(n)\}} = C_T^* \chi_{\{n\}} = C_U\, \chi_{\{n\}} = \chi_{U^{-1}(\{n\})},$$

where χ_S denotes the sequence with value 1 on S and zero elsewhere. Thus $U^{-1}(\{n\}) = \{T(n)\}$. Hence U is invertible and therefore C_U is invertible. This shows that C_T is invertible. In case C_T is invertible, for $x \in \ell^2$ it is true that x and $x\, oT$ have the same ranges. Hence

$$\begin{aligned}
\| x \|^2 &= \sum_{n=1}^{\infty} | x_n |^2 \\
&= \sum_{n=1}^{\infty} | x_{T(n)} |^2 \\
&= \| C_T x \|^2 .
\end{aligned}$$

Thus C_T is unitary and hence $C_T^* = C_{T^{-1}}$. This completes the proof of the theorem.

In the following theorem we shall record the results characterising compact and

Hilbert–Schmidt composition operators on $\ell^2(w)$.

Theorem 3.4.3. Let $T : \mathbb{N} \to \mathbb{N}$ be a function such that C_T is a composition operator on $\ell^2(w)$. Then

(i) C_T is compact if and only if

$$\frac{1}{w_n} \sum_{i \in T^{-1}(\{n\})} w_i \to 0 \quad \text{as} \quad n \to \infty.$$

(ii) C_T is Hilbert–Schmidt if and only if the sequence $\left\{1/\sqrt{w_{T(n)}}\right\}$ belongs to $\ell^2(w)$.

Proof. The proof of (i) is obvious from Corollary 2.3.7. Actually (i) is a re-statement of the corollary in terms of the weights.

(ii) Since the functions $f_n = \frac{e_n}{\sqrt{w_n}}$, for $n \in \mathbb{Z}_+$ form an orthonormal basis for $\ell^2(w)$, the result can be obtained by using definition of the Hilbert–Schmidt operator to this basis.

One of the outstanding problems in operator theory is the invariant subspace problem: Does every bounded operator on an infinite dimensional separable Hilbert space have a non-trivial invariant closed subspace. During the last 30 years or so it has been proved that every well known operator like normal, compact or shifts etc. has an invariant subspace. An operator which commutes with a compact operator has an invariant subspace. In case of ℓ^2 we know that no composition operator is compact, but it turns out that every composition operator on ℓ^2 has a non-trivial invariant subspace. We shall prove this in the following theorem.

Theorem 3.4.4. Let C_T be a composition operator on ℓ^2. Then

(i) C_T has an invariant subspace.

(ii) C_T has a reducing subspace if there exists two distinct elements in $\mathbb{N}$ which are not in the same orbit of T. (two elements of $\mathbb{N}$ are said to be in the same orbit of T if each can be reached by composing T and T^{-1} sufficiently many times).

Proof. (i) If C_T is invertible, then by Theorem 3.4.2 (ii) it is unitary and hence normal. Thus it has an invariant subspace. If C_T is not invertible, then C_T is either not surjective or not injective. If C_T is not surjective, then the range of C_T being a closed subspace of ℓ^2 (see [340]) is an invariant subspace of C_T. If C_T is not injective, then the kernel of C_T is an invariant subspace of C_T. This completes the proof of part (i).

(ii) Suppose m_0 and n_0 are two distinct elements of $\mathbb{N}$ which are not in the same orbit of T. Let $F = \{n : n \in \mathbb{N}$, and n and n_0 are not in the same orbit of $T\}$. Let M be the span of e_n' s, where $n \in F$; e_n being the sequence whose n^{th} entry is 1 and rest 0. Then M is a proper (closed) subspace of ℓ^2 which is invariant under C_T and C_T^*. Thus M is a reducing subspace of C_T. This completes the proof of the theorem.

In the following theorem we record some more results without proofs which can be found in [337].

Theorem 3.4.5.

(i) Let $T : \mathbb{N} \to \mathbb{N}$ be a one-to-one map which is not onto. Then $\sigma(C_T) = \overline{D}$.

(ii) Let C_T be invertible on ℓ^2 and suppose for every $n \in \mathbb{N}$, there exists $m \in \mathbb{N}$ such that $T^m(n) = n$. Let $m_n = \inf\{m : T^m(n) = n\}$ and let $F = \{m_n : n \in \mathbb{N}\}$. Then $\sigma(C_T) = \bigcup_{q \in F} \{\lambda : \lambda^q = 1\}$.

(iii) If C_T is invertible and if for some $n \in \mathbb{N}$, there does not exist any $m \in \mathbb{N}$ such that $T^m(n) = n$, then $\sigma(C_T) = \partial D$.

(iv) A composition operator C_T on ℓ^2 is centered if and only if for every $k \in \mathbb{N}$, f_T^k is constant on $(T^p)^{-1}(\{n\})$, for every $n \in \mathbb{N}$ and $p \in \mathbb{N}$, where f_T^k is the function on $\mathbb{N}$ defined as $f_T^k(i) = \#(T^k)^{-1}(\{i\})$. [In terms of measure theoretic terminology f_T^k is the Radon–Nykodym derivative of $\mu(T^k)^{-1}$ with respect to the counting measure μ on $\mathbb{N}$].

Now we shall report some results on the weighted composition operators on ℓ^2.

Theorem 3.4.6. Let $W_{\pi,T}$ be a weighted composition operator on ℓ^2. Then

(i) $W_{\pi,T}$ is normal if and only if T/N_π is an injection and

$$| \pi(n) |^2 = \begin{cases} | \pi(n') |^2 , & \text{where } n' \in N_\pi \cap T^{-1}(\{n\}) \neq \varnothing \\ 0 , & \text{if } N_\pi \cap T^{-1}(\{n\}) = \varnothing, \end{cases}$$

where $N_\pi = \{n \in \mathbb{N} \ : \ \pi(n) \neq 0\}$.

(ii) $W_{\pi,T}$ is Hilbert–Schmidt if and only if $\pi \in \ell^2$.

(iii) $W_{\pi,T}$ is compact if and only if $\{\pi(n)\} \to 0$ as $n \to \infty$.

Proof. (i) Suppose $W_{\pi,T}$ is normal. Then by Theorem 3.4.2, we have

$$\pi \ C_T\left(\bar{\pi}\ (n)\ e_{T(n)}\right) = C_T^*\left(\sum_{i \in T^{-1}(\{n\})} | \pi(i) |^2 e_i \right).$$

From this, we have

$$\bar{\pi}(n) \sum_{j \in T^{-1}(\{T(n)\})} \pi(j) e_j = \sum_{i \in T^{-1}(\{n\})} | \pi(i) |^2 e_i \ .$$

If $T^{-1}(\{n\}) \cap N_\pi = \varnothing$, then $\bar{\pi}(n)\ \pi(n)\ e_n = 0$ and hence $\pi(n) = 0$. If $T^{-1}(\{T(n)\}) \cap N_\pi \neq \varnothing$, then it is a singleton set containing n and hence T/N_π is an injection. Thus

$$| \pi(n) |^2 = | \pi(n') |^2, \quad \text{for} \quad n' \in N_\pi \cap T^{-1}(\{n\}).$$

To prove converse, let $f \in \ell^2$. Then

$$f = \sum_{n=1}^{\infty} \alpha_n e_n.$$

Now

$$\begin{aligned}
W_{\pi,T}\, W_{\pi,T}^{*}\, f &= \pi \cdot \sum_{n=1}^{\infty} \alpha_n\, \overline{\pi}(n) \chi_{T^{-1}(\{T(n)\})} \\
&= \pi \cdot \sum_{n=1}^{\infty} \overline{\pi}(n) \alpha_n \sum_{i \in T^{-1}(\{T(n)\})} e_i \\
&= \sum_{n=1}^{\infty} \mid \pi(n) \mid^2 \alpha_n\, e_n \,.
\end{aligned}$$

Similarly,

$$\begin{aligned}
W_{\pi,T}^{*}\, W_{\pi,T}\, f &= C_T^{*}\left(\sum_{n=1}^{\infty} \alpha_n \sum_{i \in T^{-1}(\{n\})} \mid \pi(i) \mid^2 e_i \right) \\
&= \sum_{n=1}^{\infty} \alpha_n \left(\sum_{i \in T^{-1}(\{n\})} \mid \pi(i) \mid^2 \right) e_i \,.
\end{aligned}$$

Using the condition of the theorem, we have

$$W_{\pi,T}^{*}\, W_{\pi,T} = W_{\pi,T}\, W_{\pi,T}^{*} \,.$$

Hence $W_{\pi,T}$ is normal.

(ii) $W_{\pi,T}$ is Hilbert–Schmidt if and only if

$$\sum_{n=1}^{\infty} \| W_{\pi,T}\, e_n \|^2 < \infty \,.$$

Now

$$\begin{aligned}
\sum_{n=1}^{\infty} \| W_{\pi,T}\, e_n \|^2 &= \sum_{n=1}^{\infty} \sum_{i=1}^{\infty} \mid \pi(i) \mid^2 \mid C_T\, e_n(i) \mid^2 \\
&= \sum_{n=1}^{\infty} \sum_{i=1}^{\infty} \mid \pi(i) \mid^2 \mid e_{T(i)}\,(n) \mid^2 \\
&= \sum_{n=1}^{\infty} \mid \pi(n) \mid^2 \,.
\end{aligned}$$

From this the result follows.

(iii) Suppose $W_{\pi,T}$ is compact. Now $e_n \to 0$ weakly and thus

$$\left\| W^*_{\pi,T}\, e_n \right\| \to 0 \,.$$

Hence

$$|\,\pi(n)\,|\ \left\| C^*_T\, e_n \right\| \to 0 \,.$$

Since $\left\| C^*_T\, e_n \right\| = \left\| e_{T(n)} \right\| = 1$, we have $\pi(n) \to 0$. The converse is trivial.

The hyponormal and quasinormal weighted composition operators on $\ell^2(\mathbb{Z})$ have recently been studied by Carlson in [58].

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CHAPTER IV

COMPOSITION OPERATORS ON THE WEIGHTED LOCALLY CONVEX FUNCTION SPACES

In this chapter we shall undertake the study of the composition operators on the weighted locally convex spaces of continuous functions and cross-sections. The first section of this chapter is devoted to some classical results pertaining to these operators. In section 2 we shall concentrate on the study of composition operators on the weighted spaces of scalar-valued, vector-valued continuous functions and cross-sections, whereas the invertibility and compactness of these operators are presented in the third section. Finally, in the fourth section of this chapter we shall investigate some properties of the weighted composition operators on the weighted locally convex spaces.

4.1 INTRODUCTION, CHARACTERIZATION AND CLASSICAL RESULTS

In chapter III we have presented a study of the composition operators on some Banach spaces of functions equipped with the norm topologies. There are atleast two ways of generalising the theory of composition operators: one is to enlarge the domain of the composition operators, like taking spaces of continuous functions instead of just spaces of analytic functions; and other is to equip the domain spaces with different topologies, like on ℓ^{∞} one can have compact–open topology, strict topology besides the norm topology. This chapter is an attempt in the direction of generalizing the theory of the composition operators on broader and larger function spaces with locally convex topologies. We shall study composition operators on locally convex spaces of functions on topological spaces. This class of spaces certainly includes all Banach spaces of continuous functions and hence all Hardy spaces and ℓ^{p}–spaces that we discussed in the last chapter. One of the most important examples of the locally convex spaces of continuous functions is $C(X)$, the space of all continuous complex–valued functions on a compact topological space X with the supnorm topology; actually it is a C^{*}–algebra. If T is any continuous self map of X, then $f \circ T \in C(X)$ for every $f \in C(X)$, and hence the transformation C_T taking f to $f \circ T$ is a continuous operator on $C(X)$ with $\|C_T\| \leq 1$ and $C_T(fg) = C_Tf.C_Tg$ for every f and g in $C(X)$. Thus every continuous T induces a composition operator C_T on $C(X)$ which is multiplicative. In some cases the converse is also true. We shall record all these results together with some more results later in the section.

In order to have a rich supply of continuous functions, from now on in this chapter we shall assume that X is a completely regular Hausdorff space. By $C(X)$ we denote the vector space of all complex-valued continuous functions on X. The symbols $C_b(X)$, $C_0(X)$ and $C_c(X)$ are used for the vector spaces of bounded continuous functions, of continuous functions vanishing at infinity and of continuous functions with compact supports respectively. It is clear that $C_c(X) \subset C_0(X) \subset C_b(X) \subset C(X)$; in case X is compact, these spaces coincide with $C(X)$. By a locally convex space of continuous functions on X we mean a subspace $S(X)$ of $C(X)$ with a family of seminorms making $S(X)$ a topological vector space. The main aim of this chapter is to study composition operators on such spaces.

Let $S(X)$ be a locally convex space of continuous functions on X and let $x \in X$. Then the evaluation functional δ_x on $S(X)$ is defined as

$$\delta_x(f) = f(x), \quad \text{for every} \quad f \in S(X).$$

It is clear that δ_x is linear. But this may not be continuous for every $x \in X$. Actually, there are even Hilbert spaces of continuous functions where some evaluation functionals are discontinuous [137]. In case δ_x is continuous it lies in $S^*(X)$, the dual space of $S(X)$. If δ_x is continuous for every $x \in X$, then the set Δ defined as $\Delta = \{\delta_x : x \in X\}$ is a subset of $S^*(X)$. If A is any continuous operator on $S(X)$, then the adjoint A' of A, defined as $(A'F)(f) = F(Af)$, for $F \in S'(X)$ and $f \in S(X)$, where $S'(X)$ denotes the algebraic dual of $S(X)$, is a linear transformation from $S'(X)$ into itself. In case $S(X)$ is a normed linear space, the restriction of A' to $S^*(X)$ is a continuous operator with the same norm as that of A. Since every evaluation functional δ_x is linear, the set Δ of all evaluation functionals is contained in $S^*(X)$. The following theorem which is a generalization of Theorem 3.1.1 characterises the operators on $S(X)$ which are composition operators.

Theorem 4.1.1. Let $S(X)$ be a locally convex space of continuous functions on X and let A be a continuous operator on $S(X)$. Then A is a composition operator if and only if $A'(\Delta) \subset \Delta$, where $\Delta = \{\delta_x : x \in X\}$.

Proof. Suppose $A = C_T$ for some T. Let $\delta_x \in \Delta$. Then for $f \in S(X)$,

$$\begin{aligned}(A'\delta_x)(f) = (C_T'\delta_x)(f) = \delta_x(C_T f) &= (f \circ T)(x)\\ &= f(T(x))\\ &= \delta_{T(x)}(f).\end{aligned}$$

Thus $A'\delta_x = \delta_{T(x)} \in \Delta$. Conversely, suppose $A'(\Delta) \subset \Delta$. Let $\delta_x \in \Delta$. Then $A'\delta_x \in \Delta$. Hence there exists $y \in Y$ such that $A'\delta_x = \delta_y$. Let us define the map $T : X \to X$ as $T(x) = y$. Then T is well defined. Let $f \in S(X)$. Then for $x \in X$,

$$\begin{aligned}(f \circ T)(x) = f(T(x)) &= \delta_{T(x)}(f)\\ &= \delta_y(f)\\ &= (A'\delta_x)(f)\\ &= \delta_x(Af)\\ &= (Af)(x).\end{aligned}$$

Thus $f \circ T = Af$, for every $f \in S(X)$. Hence $C_T = A$. Thus A is a composition operator. This completes the proof of the theorem.

Note. In the proof of the above theorem we have not used the topology of the space $S(X)$. Thus this theorem is true for any topological vector space of functions on X.

After this general theorem let us consider some special cases, where function spaces have definite topologies. Some known results in the case of $C_b(X)$ with the supnorm topology arc given in the following two theorems.

Theorem 4.1.2. Let X be a completely regular Hausdorff space, and let $C_b(X)$ have the supnorm topology. Then

(i) Every continuous map $T : X \to X$ induces a composition operator C_T on $C_b(X)$ such that $\| C_T \| = 1$ and the set of all constant functions is invariant under C_T.

(ii) Let $T : X \to X$ be continuous. Then C_T is an injection if and only if

$\operatorname{ran} T$ is dense in X, where $\operatorname{ran} T$ denotes the range of T.

(iii) If T and S are continuous maps taking X into X such that $C_T = C_S$, then $T = S$.

Proof. (i) If $f \in C_b(X)$, then clearly $f \circ T \in C_b(X)$ and $\|C_T f\| \le \|f\|$. Hence $\|C_T\| < 1$. Since $\|C_T 1\| = 1$, we conclude that $\|C_T\| = 1$; (here 1 denotes the constant one function *i.e.*, $1(x) = 1$, for all $x \in X$). It is clear that each constant function goes to itself under C_T.

(ii) Suppose $\overline{\operatorname{ran} T} = X$. Let $f, g \in C_b(X)$ such that $C_T f = C_T g$. Thus $f \circ T = g \circ T$. This implies that f and g agree on the range of T which is dense in X; hence $f = g$. This proves that C_T is an injection. If $\overline{\operatorname{ran} T} \ne X$, then using the complete regularity of X we can find a non–zero function f in $C_b(X)$ such that $f(x) = 0$, for every $x \in \overline{\operatorname{ran} T}$. Thus $C_T f = 0$. Hence C_T is not an injection. This completes the proof.

(iii) Suppose $C_T = C_S$. Then $(f \circ T)(x) = (f \circ S)(x)$, for all $f \in C_b(X)$ and $x \in X$. Thus by the complete regularity $T(x) = S(x)$, for all $x \in X$. Hence $T = S$.

Note. (ii) and (iii) are true even if we replace $C_b(X)$ by $C(X)$ with just vector space structure. For details see [122].

Corollary 4.1.3. Every map $T : \mathbb{N} \to \mathbb{N}$ defines a composition operator on ℓ^∞. In general, if X is any discrete space, then every map from X into X defines a composition operator on $C_b(X)$, which we shall denote by $\ell^\infty(X)$.

Let $B \subset X$. Then by $B_Z(X)$ we denote the set of all those functions on X which vanish on B, *i.e.*,

$B_Z(X) = \{f \mid f : X \to \mathbb{C}$ such that $f(x) = 0$, for all $x \in B\}$. $B_Z^b(X)$ stands for $B_Z(X) \cap C_b(X)$. In the following theorem we record some more results on the composition operators on $C_b(X)$.

Theorem 4.1.4. Let C_T be a composition operator on $C_b(X)$ induced by a continuous map T. Then

(i) $\operatorname{ran} C_T$ is a closed self-adjoint subalgebra of $C_b(X)$.

(ii) $\ker C_T = (\operatorname{ran} T)_Z^b(X)$.

(iii) $\operatorname{ran} C_T$ is isometrically isomorphic to $C_b(X) / \ker C_T$.

(iv) $C_T^* \delta_x = \delta_{T(x)}$, for every $x \in X$.

(v) C_T is an isometry if $\operatorname{ran} T$ is dense in X.

Proof. (i) Clearly $\operatorname{ran} C_T$ is a subalgebra. If $g \in \operatorname{ran} C_T$, then $g = f \circ T$. Hence

$$\bar{g} = \overline{f \circ T} = \bar{f} \circ T = C_T \bar{f} \in \operatorname{ran} C_T .$$

Thus $\operatorname{ran} C_T$ is self-adjoint. The closedness of $\operatorname{ran} C_T$ will follow from (iii).

(ii) Let $f \in \ker C_T$. Then $f \circ T = 0$, *i.e.*, $f(T(x)) = 0$, for all $x \in X$. Thus $f \in (\operatorname{ran} T)_Z^b(X)$. Similarly the converse.

(iii) Let $\psi : C_b(X) / \ker C_T \to C_b(X)$ be defined as $\psi(g + \ker C_T) = C_T g$. This map ψ is well-defined and is linear, $C_b(X) / \ker C_T$ is a Banach space under the norm

$$\| g + \ker C_T \| = \inf\{\| g + h \| \ : \ h \in \ker C_T\}.$$

It can be shown that $\|\psi(g + \ker C_T)\| \leq \|g + \ker C_T\|$, for every $g \in C_b(X)$. If $h \in (g + \ker C_T)$, then by (ii) h coincides with g on $\operatorname{ran} T$. If $\overline{\operatorname{ran} T} = X$, then $\|C_T g\| = \|g \circ T\| = \|g\|$; and hence ψ is an isometry. If $\overline{\operatorname{ran} T} \neq X$, then by the complete regularity there exists $h \in C_b(X)$ such that $\|h\| = \|C_T g\|$ and $h(y) = g(y)$, for all $y \in \operatorname{ran} T$. Thus $h \in g + \ker C_T$, and hence

$$\|\psi(g + \ker C_T)\| = \|C_T g\| = \|h\| \geq \|g + \ker C_T\|.$$

Thus $\|\psi(g + \ker C_T)\| = \|g + \ker C_T\|$. Hence ψ is an isometry. This shows that $\operatorname{ran} C_T$ is a closed subalgebra of $C_b(X)$.

(iv) Let $f \in C_b(X)$. Then

$$\begin{aligned}\left(C_T^* \delta_x\right)(f) = \delta_x\left(C_T f\right) = \delta_x(f \circ T) \quad &= (f \circ T)(x) \\ &= f(T(x)) \\ &= \delta_{T(x)}(f).\end{aligned}$$

(v) It follows from the proof of part (iii).

Montador in [243] studied composition operators on C[0,1] and computed the spectrum in different cases.

4.2 COMPOSITION OPERATORS ON THE WEIGHTED LOCALLY CONVEX FUNCTION SPACES

The main aim of this section is to study composition operators on $CV_0(X)$, $CV_b(X)$, $CV_0(X,E)$, $CV_p(X,E)$, $LW_0(X)$, and $LW_b(X)$ which are defined earlier in the fourth section of chapter I. We shall complete this study of the composition operators in two phases. In the first phase we intend to initiate a study of these operators on the weighted spaces of scalar-valued and vector-valued continuous functions, and the second phase includes a study of the composition operators on the weighted spaces of cross-sections. Throughout the first phase we shall assume the following :

(4.2.a) X : a completely regular Hausdorff space;

(4.2.b) V : a system of weights on X;

(4.2.c) E : a non-zero locally convex Hausdorff space;

(4.2.d) for each $x \in X$, there exists $f_x \in CV_0(X)$ $(CV_0(X,E))$ such that $f_x(x) \neq 0$.

Let $T : X \to X$ be a continuous map, and let $V_T = \{v \circ T \ : \ v \in V\}$. Then it can be easily seen that V_T is also a system of weights on X. We note that the condition (4.2.d) is trivially satisfied if X is a locally compact space. The following theorem gives a necessary condition for T to induce a composition operator on $CV_b(X)$.

Theorem 4.2.1. Let $T : X \to X$ be a map such that $C_T : CV_0(X) \to CV_b(X)$ is a composition operator. Then T is continuous and $V \leq V_T$.

Proof. In view of (4.2.d), moreover, it readily follows from [246, Lemma 2, p. 69] that if $T : X \to X$ is any map for which the corresponding composition map C_T induced on $CV_0(X)$ has its range contained in $C(X)$, then T is necessarily continuous. We shall show that $V \leq V_T$. Let $v \in V$. Then, by continuity of C_T, there exists $u \in V$ such that $C_T(B_u) \subset B_v$. We claim that $v \leq 2u \circ T$. Fix $x_0 \in X$ and set $y = T(x_0)$. In case $u(y) > 0$, $G = \{x \in X : u(x) < 2u(y)\}$ is an open neighbourhood of y. Therefore, according to [246, Lemma 2, p. 69] there exists $g_y \in CV_0(X)$ such that $0 \leq g_y \leq 1$, $g_y(y) = 1$ and $g_y(X - G) = 0$. If we put $g = (2u(y))^{-1} g_y$, then it is clear that $g \in B_u$ and $C_T g \in B_v$. Thus $(g \circ T)(x)v(x) \leq 1$, for all $x \in X$. Hence $v(x_0) \leq 2(u \circ T)(x_0)$. Now, suppose $u(y) = 0$, and $v(x_0) > 0$. Let

$\varepsilon = v(x_0) / 2$. Then the set $G_1 = \{x \in X : u(x) < \varepsilon\}$ is an open neighbourhood of y and hence by [246, Lemma 2, p. 69] there exists $h_y \in CV_0(X)$ such that $0 \le h_y \le 1$, $h_y(y) = 1$ and $h_y(X - G_1) = 0$. Now, if we put $h = \varepsilon^{-1} h_y$, then clearly $h \in B_u$ and $C_T h \in B_v$. Thus, it follows that $v(x)(h \circ T)(x) \le 1$, for all $x \in X$. Hence $v(x_0) \le v(x_0) / 2$, which is a contradiction. Thus our claim is established. This completes the proof of the theorem.

In case of $CV_b(X)$ the condition in the above theorem turns out to be a sufficient condition also. This we prove in the following theorem.

Theorem 4.2.2 Let $T : X \to X$ be a map. Then C_T is a composition operator on $CV_b(X)$ if and only if $V \le V_T$ and T is continuous.

Proof. The necessity of the condition follows from Theorem 4.2.1. Now, we suppose that T is continuous and $V \le V_T$. Let $v \in V$ and $f \in CV_b(X)$. Then there exists $u \in V$ such that $v \le u \circ T$ and

$$\begin{aligned} \|C_T f\|_v &= \sup\{v(x)|f \circ T(x)| : x \in X\} \\ &\le \sup\{(u \circ T)(x)|(f \circ T)(x)| : x \in X\} \\ &= \sup\{(u \circ |f|)(T(x)) : x \in X\} \\ &\le \sup\{(u \circ |f|)(x) : x \in X\} = \|f\|_u. \end{aligned}$$

This proves that C_T is continuous and hence a composition operator on $CV_b(X)$.

Now, we shall present some examples of spaces and mappings which induce or do not induce composition operators.

Example 4.2.3. (i) Every mapping $T : \mathbb{N} \to \mathbb{N}$ induces a composition operator on $\ell^\infty (= CV_b(\mathbb{N})$, V being the system of positive constant sequences). In this case $V = V_T$.

(ii) Let $V = \{\lambda \chi_F : F$ a finite subset of $\mathbb{N}$ and $\lambda \ge 0\}$. Then every map $T : \mathbb{N} \to \mathbb{N}$ induces a composition operator on $CV_b(\mathbb{N})$. This follows from the fact that for every finite subset F of $\mathbb{N}$, there exists a finite subset K of $\mathbb{N}$ such that $F \subset T^{-1}(K)$, and hence $\lambda\chi_F \leq \lambda\chi_K \circ T$. This shows that $V \leq V_T$, and hence by Theorem 4.2.2, C_T is a composition operator on $CV_b(\mathbb{N})$.

(iii) Let $v : \mathbb{N} \to \mathbb{R}^+$ be defined as $v(n) = n$ for all $n \in \mathbb{N}$. Let $V = \{\lambda v \ : \ \lambda \geq 0\}$, and let $T : \mathbb{N} \to \mathbb{N}$ be a constant map. Then V_T is the set of all constant maps on $\mathbb{N}$ and clearly $V \not\leq V_T$. Thus T does not induce a composition operator on $CV_b(\mathbb{N})$.

(iv) Let $X = \mathbb{R}^+ \setminus \{0\}$ with the usual topology and let $v : X \to \mathbb{R}^+$ be defined as $v(x) = \frac{1}{x}$, for every $x \in X$. Let $V = \{\lambda v \ : \ \lambda \geq 0\}$, and let $T : X \to X$ be defined by $T(x) = x^2$, for all $x \in X$. Then $V \not\leq V_T$. Hence T does not induce a composition operator on $CV_b(X)$.

(v) Let $X = \mathbb{N}$ with discrete topology and let V be the system of constant weights. Then $CV_0(\mathbb{N}) = c_0$, the Banach space of null sequences of complex numbers. Let $T(n) = 2$, for all $n \in \mathbb{N}$. Then $V \leq V_T$ but C_T is not even an into map. If $f(n) = \frac{1}{n}$, then $f \in c_0$, but $C_T f \notin c_0$.

From Example 4.2.3 (v) it is clear that $V \leq V_T$ is not a sufficient condition for T to induce a composition operator on $CV_0(X)$; we need something more. In the following theorem we shall characterise the continuous mappings inducing composition operators on $CV_0(X)$. First of all we need a definition. For each $v \in V$ and $\delta > 0$, define the set $F(v, \delta)$ as $F(v, \delta) = \{x \in X \ : \ v(x) \geq \delta\}$. It is clear that $F(v, \delta)$ is a closed subset of X.

Theorem 4.2.4. If $T : X \to X$ is any function, then the following are equivalent :

(a) T induces a composition operator on $CV_0(X)$;

(b) T induces a composition operator C_T on $CV_b(X)$ and $CV_0(X)$ is invariant under C_T;

(c) (i) T is continuous, (ii) $V \leq V_T$, and (iii) for each $v \in V$, $\delta > 0$ and compact set $K \subset X$, $T^{-1}(K) \cap F(v, \delta)$ is compact;

(d) (i) T is continuous, (ii) $V \leq V_T$, and (iii) for each $v \in V$, $\delta > 0$ and $u \in V$ such that $v \leq u \circ T$, $T^{-1}(K) \cap F(v, \delta)$ is compact whenever K is a compact subset of $F(u, \delta)$.

Proof. (a) $\Rightarrow$ (c). If C_T is a composition operator on $CV_0(X)$, then by Theorem 4.2.1, T is continuous and $V \le V_T$. Now, let $v \in V$, $\delta > 0$ and K be a compact subset of X. Then by [246, Lemma 2, p. 69] there exists $h_K \in CV_0(X)$ such that $0 \le h_K \le 1$ and $h_K(x) = 1$ for all $x \in K$. Let $H = \{x \in X : v(x)(C_T h_K)(x) \ge \delta\}$. Then, since $C_T h_K \in CV_0(X)$, H is a compact subset of X. It is clear that $T^{-1}(K) \cap F(v,\delta) \subset H$. Since $T^{-1}(K) \cap F(v,\delta)$ is closed, we conclude that it is compact.

(c) $\Rightarrow$ (d). This follows easily.

(d) $\Rightarrow$ (b). If (d) holds, then by Theorem 4.2.2, C_T is a composition operator on $CV_b(X)$. Let $f \in CV_0(X)$. We would show that $C_T f \in CV_0(X)$. For this, let $v \in V$ and $\varepsilon > 0$. Let the set Y be defined as $Y = \{x \in X : v(x)\,|(C_T f)(x)| \ge \varepsilon\}$. If Y is empty, then Y is compact and hence we are done. Suppose Y is non–empty. Let us take $u \in V$ such that $v \le u \circ T$. Let $y \in Y$. Then

$$\varepsilon \le v(y)|(C_T f)(y)| \le (u \circ T)(y)|(C_T f)(y)|.$$

Let $K = \{x \in X : u(x)|f(x)| \ge \varepsilon\}$. Then K is compact, $K \subset F(u, \varepsilon/\|f\|_K)$ and $T(y) \in K$. Further $v(y) \ge \dfrac{\varepsilon}{|f(T(y))|} \ge \dfrac{\varepsilon}{\|f_K\|}$. This shows that $Y \subset F(v, \varepsilon/\|f\|_K)$.

Thus $Y \subset T^{-1}(K) \cap F(v, \varepsilon/\|f\|_K)$. This proves that Y is compact. Hence $C_T f \in CV_0(X)$. Thus $CV_0(X)$ is invariant under C_T..

(b) $\Rightarrow$ (a). This is evident. This completes the proof of the theorem.

The results given in the following corollary follow from the above theorem.

Corollary 4.2.5.

(i) If $T : X \to X$ is a homeomorphism, then C_T is a composition operator on $CV_0(X)$ if and only if $V \le V_T$.

(ii) Let $X = \mathbb{N}$, and $T : \mathbb{N} \to \mathbb{N}$ be a map. Then C_T is a composition operator on $CV_0(\mathbb{N})$ if and only if $V \le V_T$ and $T^{-1}(\{n\}) \cap F(v,\delta)$ is finite set for every $n \in \mathbb{N}$, $v \in V$ and $\delta > 0$.

(iii) C_T is a composition operator on c_0 if and only if $T^{-1}(\{n\})$ is a finite subset of $\mathbb{N}$ for every $n \in \mathbb{N}$. Thus every injective map induces a composition operator on c_0.

(iv) If $V = \{\lambda\chi_F : \lambda > 0$ and $F \subset \mathbb{N}$, finite$\}$, then every $T : \mathbb{N} \to \mathbb{N}$ induces a composition operator on $CV_0(\mathbb{N})$

Example 4.2.6. (i) Let $X = \mathbb{N}$, and $v(n) = n$, for $n \in \mathbb{N}$. Let $V = \{\lambda v : \lambda \geq 0\}$, and let $T : \mathbb{N} \to \mathbb{N}$ be defined as

$$T(n) = \begin{cases} \sqrt{n} \ , & \text{if } n \text{ is a perfect square} \\ n \ , & \text{otherwise.} \end{cases}$$

Then T is a surjection and $T^{-1}(\{n\})$ is finite for every $n \in \mathbb{N}$. Since $V \not\leq V_T$, T does not induce a composition operator on $CV_0(\mathbb{N})$.

(ii) Let $X = \mathbb{R}$, the set of real numbers with the usual topology. Let $v : \mathbb{R} \to \mathbb{R}^+$ be defined as $v(x) = |x|$, for all $x \in \mathbb{R}$, and let $V = \{\lambda v : \lambda \geq 0\}$. Let $T : \mathbb{R} \to \mathbb{R}$ be defined as $T(x) = x - 1$, for $x \in \mathbb{R}$. Then T is a homeomorphism, but does not induce a composition operator on $CV_0(\mathbb{R})$.

Remark 4.2.7. From the theorems and the examples given above it is clear that there are four variables playing important roles in determining the continuity of C_T; they are the underlying topological space X, the system of weights V, the space of continuous functions $CV_b(X)$ or $CV_0(X)$ and the continuous mappings $T : X \to X$. Mostly in our study of composition operators we fix the first three variables and characterise the mappings which induce composition operators. A study can be made by fixing the first and the last coordinates because we have seen the examples of mappings which do not define composition operators for one system of weights V but do define composition operators for another system of weights. This is in a way natural because the system of weights give the topology on the function spaces with respect to which we discuss the continuity of C_T. It would be worthwhile to characterise the system of weights for which every continuous mapping induces a composition operator on $CV_b(X)$ or $CV_0(X)$. We do not know if it may be easy or it may be difficult.

In the following theorem we shall give a description of the continuous composition maps from $CV_0(X)$ into $CV_b(X)$ which parallels a standard result for functional Hilbert spaces (see [254, page 46]). For each $x \in X$, the point evaluation δ_x defines a continuous linear functional on either $CV_0(X)$ or $CV_b(X)$. If we put

$\Delta(X) = \{\delta_x : x \in X\}$, then $\Delta(X)$ is a subset of either the continuous dual $CV_0(X)^*$ or $CV_b(X)^*$.

Theorem 4.2.8. Let $\Phi : CV_0(X) \to CV_b(X)$ be a linear transformation. Then there exists $T : X \to X$ such that $\Phi = C_T$ if and only if the transpose mapping Φ' from $CV_b(X)^*$ into the algebraic dual $CV_0(X)'$ leaves $\Delta(X)$ invariant. In case $\Phi'(\Delta(X)) \subset \Delta(X)$, moreover, T is necessarily continuous, and $\Phi = C_T$ is continuous if and only if $V \leq V_T$.

Proof. Suppose that $\Phi = C_T$ for some $T : X \to X$. Let $x \in X$ and $f \in CV_0(X)$. Then

$$\begin{aligned}(\Phi'\delta_x)(f) = (\delta_x \circ \Phi)(f) = \delta_x(\Phi(f)) &= \delta_x(C_T f)\\ &= f(T(x))\\ &= \delta_{T(x)}(f).\end{aligned}$$

This implies that $\Phi'\delta_x = \delta_{T(x)}$. Conversely, let us suppose that $\Phi'(\Delta(X)) \subset \Delta(X)$. For $x \in X$, if we define $T(x)$ to be the (unique) element of X such that $\Phi'(\delta_x) = \delta_{T(x)}$. Let $f \in CV_0(X)$. Then

$$\begin{aligned}\Phi(f)(x) = \delta_x(\Phi(f)) = (\delta_x \circ \Phi)(f) &= \Phi'(\delta_x)(f)\\ &= \delta_{T(x)}(f)\\ &= f(T(x))\\ &= C_T(f)(x).\end{aligned}$$

Thus $\Phi = C_T$. We also note that T is continuous since the range of C_T is contained in $C(X)$. In view of Theorem 4.2.1 and Theorem 4.2.2, $\Phi = C_T$ is continuous when $V \leq V_T$.

Remark 4.2.9. If $CV_0(X)$ is everywhere replaced by $CV_b(X)$ in the statement of Theorem 4.2.8, this yields a companion result which can be established by (essentially) the same argument.

One description of these continuous linear operators on $CV_0(X)$ which happen to be the composition operators follows immediately from Theorem 4.2.8. From several points of view, however, a characterisation entirely in terms of the operator itself would be preferable. We next give a deeper result which will serve to resolve the problem in a manner reflecting the classical work in this direction (for example, see [122, page 142]).

Theorem 4.2.10. Let $\Phi : CV_0(X) \to CV_b(X)$ be a continuous linear operator. Then there exists $T : X \to X$ such that $\Phi = C_T$ if and only if the following two conditions are satisfied :

(i) for each $x \in X$, there exists $g \in CV_0(X)$ such that $\Phi(g)(x) \neq 0$,

(ii) $\Phi(fg) = \Phi(f)\Phi(g)$, whenever $f, g \in CV_0(X) \cap C_b(X)$.

Proof. The conditions (i) and (ii) are obviously necessary and therefore we shall prove only the sufficient part. Assume that (i) and (ii) are satisfied. If we put $A = CV_0(X) \cap C_b(X)$, then clearly A is a self-adjoint subalgebra of $C_b(X)$ which separates the points of X. Now, we fix $x \in X$, and put

$$W_x = \{f \in A : \Phi(f)(x) = 0\}.$$

Then clearly W_x is a vector subspace of $CV_0(X)$, and the condition (ii) implies that W_x is a module over A. Further, the condition (i) implies that $\overline{W_x}$ is a proper subset of $CV_0(X)$. As we have an instance of the bounded case of the weighted approximation problem, it follows from [246, Theorem 1, p. 106] that there exists $y \in X$ such that $f(y) = 0$ for every $f \in W_x$. If we suppose that there exists $y_0 \in X$ with $y_0 \neq y$ such that $f(y_0) = 0$, for all $f \in W_x$, then taking $g \in A$ so that $g(y_0) = 1$, we also choose $h \in A$ such that $h(y_0) = 0$ and $h(y) = 1$. In this case since $\Phi(g)(x) \neq 0$, we put

$$f = h - \frac{\Phi(h)(x)}{\Phi(g)(x)}\, g.$$

From this it follows that $f \in W_x$ and $f(y_0) = 0$, since $f \in A$ and $\Phi(f)(x) = 0$. Further, it implies that $\Phi(h)(x) = 0$, which certainly does not hold, and so we see that there is a unique element $T(x) \in X$ such that $f(T(x)) = 0$ for every $f \in W_x$. Consequently, another application of [246, Theorem 1, page 106] even yields that

$$\overline{W_x} = \{f \in CV_0(X) : f(T(x)) = 0\}.$$

To show that $\Phi(f)(x) = f(T(x))$ for every $f \in CV_0(X)$, let us fix $g \in A$ such that

$g(T(x)) = 1$. If we put

$$h = [\Phi(g)(x)]^{-1} g^2 - g,$$

then $h \in W_x$ and the fact that $h(T(x)) = 0$ implies that $\Phi(g)(x) = 1$. Now, given any $f \in CV_0(X)$, let us put $h = f - f(T(x))g$. Then clearly $h \in \overline{W}_x$ since $h(T(x)) = 0$, and thus $\Phi(h)(x) = 0$. From this, it follows that $\Phi(f)(x) = f(T(x))$. This completes the proof of the theorem.

In particular, Theorem 4.2.10 serves to distinguish the composition operators among all continuous linear operators on $CV_0(X)$. Example 4.2.12 will demonstrate that a continuous linear operator $\Phi : CV_b(X) \to CV_b(X)$ which satisfies (i) and (ii) of Theorem 4.2.10 can fail to be a composition operator on $CV_b(X)$. Hence something more is needed in this case.

Theorem 4.2.11. Let $\Phi : CV_b(X) \to CV_b(X)$ be a continuous linear operator. Then there exists $T : X \to X$ such that $\Phi = C_T$ if and only if, the following two conditions are satisfied :

(i) for each $x \in X$, there exists $g \in CV_0(X)$ such that $\Phi(g)(x) \neq 0$,

(ii) $\Phi(fg) = \Phi(f)\Phi(g)$, whenever $f \in CV_b(X)$ and $g \in CV_0(X) \cap C_b(X)$.

Proof. The necessary part is obvious. Suppose that the conditions (i) and (ii) are true. Then by Theorem 4.2.10, there exists $T : X \to X$ such that $\Phi(f) = f \circ T$, for every $f \in CV_0(X)$. We fix $y \in X$ and consider any $f \in CV_b(X)$. Then $G = \{x \in X : |f(x)| < |f(T(y))| + 1\}$ is an open neighbourhood of $T(y)$ and so there exists $g \in CV_0(X) \cap C_b(X)$ such that $g(T(y)) = 1$ and $g(x) = 0$ for all $x \in X \backslash G$. Since $g^2 \in CV_0(X) \cap C_b(X)$, in view of (ii), it now follows that

$$\Phi\left(fg^2\right)(y) = \Phi(f)(y)\,\Phi\left(g^2\right)(y) = \Phi(f)(y).$$

Also, since $fg^2 \in CV_0(X)$, it implies that $\Phi(fg^2)(y) = (fg^2)(T(y)) = f(T(y))$.

With this the proof of the theorem is completed.

Now, we shall give two examples of which the first shows that condition (ii) of Theorem 4.2.11 cannot be replaced by the corresponding condition from Theorem 4.2.10, while the second demonstrates that (i) of Theorem 4.2.11 cannot be rephrased in terms of functions belonging to $CV_b(X)$.

Example 4.2.12. Let $X = \mathbb{R}^+$ with the usual topology induced by $\mathbb{R}$. Define $v : X \to \mathbb{R}^+$ as $v(x) = e^{-x}$ for all $x \in X$. Let $V = \{\lambda v : \lambda > 0\}$. For $y \in \beta X \setminus X$, we define an operator Φ on $CV_b(X)$ as

$$\Phi(f)(x) = f(x) + \beta(fv)(y),$$

where $f \in CV_b(X)$, $x \in X$ and $\beta(fv)$ denotes the Stone extension of fv to the Stone-Čech compactification βX of X. Then clearly Φ is a bounded linear operator on $CV_0(X)$. Moreover, given $x \in X$, there exists $g \in CV_0(X)$ such that $\Phi(g)(x) = g(x) \neq 0$. Since $C_b(X) \subset CV_0(X)$, it follows that

$$\Phi(fg)(x) = f(x)g(x) = \Phi(f)(x)\Phi(g)(x),$$

for every $x \in X$ whenever $f, g \in C_b(X) = C_b(X) \cap CV_0(X)$. However, Φ is not a composition operator on $CV_b(X)$. To see this, let us suppose that there does indeed exist some $T : X \to X$ such that $\Phi(f) = f \circ T$, for every $f \in CV_b(X)$. Setting $f(x) = e^x$, for all $x \in X$, since $f \in CV_b(X)$ and $f(T(0)) = \Phi(f)(0) = 2$ it would then follow that $T(0) = \log 2$. Thus, for any $g \in C_b(X)$, we would have that

$$g(\log 2) = \Phi(g)(0) = g(0),$$

and this contradiction serves to establish our claim.

Example 4.2.13. Let $X = \{0\} \cup [1, \infty)$ with the relative topology induced by $\mathbb{R}$. Define $v(x) = 1$, for every $x \in X$. Let $V = \{\lambda v : \lambda > 0\}$. Again for $y \in \beta X \setminus X$, we define

$$\Phi(f)(x) = \begin{cases} f(x), & x \in [1, \infty) \\ \beta(f)(y), & x = 0, \end{cases}$$

where $f \in CV_0(X) = C_b(X)$ and $\beta(f)$ the Stone extension of f to βX. Then Φ is obviously a bounded linear operator on $CV_b(X)$ such that $\Phi(fg) = \Phi(f)\Phi(g)$, for every $f, g \in CV_b(X)$. Moreover, $v \in CV_b(X)$ and $\Phi(v)(x) = 1$, for each $x \in X$, but $\Phi(g)(0) = 0$ for every $g \in CV_0(X) = C_0(X)$, whereby Φ is not a composition operator on $CV_b(X)$.

Now, we shall discuss some basic properties of the composition operators on the weighted spaces under consideration.

If $T : X \to X$ is a continuous map and $C_T : C(X) \to C(X)$ is a composition transformation, then it is obvious that

$$\ker C_T = \{f \in C(X) : N(f) \subset X \setminus T(X)\},$$

where $N(f) = \{x \in X : f(x) \neq 0\}$ and $T(X) = \operatorname{ran} T$. It is immediate that C_T is injective if $T(X)$ is dense in X since $N(f)$ is open and $f \in C(X)$. In view of condition (4.2.d), it follows that if the restriction $C_T\big|_{CV_0(X)}$ of C_T to some $CV_0(X)$ is injective, then $T(X)$ is dense in X. We shall record these observations in the following theorem.

Theorem 4.2.14. If a continuous function $T : X \to X$ induces a composition map $C_T : C(X) \to C(X)$, and if V is any system of weights on X for which (4.2.d) is satisfied, then the following are equivalent:

(a) $T(X)$ is dense in X;

(b) C_T is injective;

(c) $C_T\big|_{CV_b(X)}$ is injective;

(d) $C_T\big|_{CV_0(X)}$ is injective.

In the following theorem we study the range of a composition operator on $CV_0(X)$.

Theorem 4.2.15. Let V be a system of weights on X for which (4.2.d) is satisfied. Let U be a system of weights on X such that $U \leq V$, and assume that $T : X \to X$ induces a composition map $C_T : CV_0(X) \to CU_0(X)$. Then T is injective if and only if $C_T(CV_0(X))$ is dense in $CU_0(X)$.

Proof. We first suppose that $C_T(CV_0(X))$ is dense in $CU_0(X)$. Let $x, y \in X$ be such that $x \neq y$. Then there exists a function $h \in CV_0(X)$ such that $h(x) = 1$ and $h(y) = 0$. Choose $u \in V$ such that $u(x) \geq 1$ and $u(y) \geq 1$. Since $h \in CU_0(X)$ and $\operatorname{ran} C_T$ is dense in $CU_0(X)$, there exists $f \in CV_0(X)$ such that $\|f \circ T - h\|_u < \frac{1}{2}$. Thus $|f(T(x)) - 1| < \frac{1}{2}$ and $|f(T(y))| < \frac{1}{2}$. From this, it is clear that $T(x) \neq T(y)$. Thus T is an injection. Conversely, suppose that T is an injection. Let

$W = \operatorname{ran} C_T = \{f \circ T : f \in CV_0(X)\}$ and let $A = \{f \circ T : f \in C_b(X)\}$. Then W is a vector subspace of $CV_0(X)$ and A is a self-adjoint subalgebra of $C_b(X)$ which separates the points of X. Actually W is a submodule over A and W is everywhere different from zero. Since we again have an instance of the bounded case of the weighted approximation problem, the density of $C_T(CV_0(X)) = W$ in $CU_0(X)$ now follows as an immediate consequence of [246, Theorem 1, page 106]. This completes the proof of the theorem.

Given any closed subset F of X, the characteristic function of F is then a weight on X. If we put

$$\mathcal{F} = \mathcal{F}(X) = \left\{\lambda\chi_F : F \subset X,\ F \text{ finite},\ \lambda \geq 0\right\},$$

then $\mathcal{F}$ is a system of weights on X, and

$$C\mathcal{F}_0(X) = C\mathcal{F}_b(X) = \left(C(X), w(\mathcal{F})\right),$$

where $w(\mathcal{F})$ denotes the topology of pointwise convergence on X. Thus (4.2.d) is trivially satisfied, while $\mathcal{F}$ is the smallest system of weights on X in the sense that $\mathcal{F} \leq U$ for any system U of weights on X.

Theorem 4.2.16. Let $C_T : C(X) \to C(X)$ be the composition map induced by a continuous function $T : X \to X$, and let V be any system of weights on X for which (4.2.d) is satisfied. Then the following are equivalent :

(a) T is injective;

(b) $C_T(CV_0(X))$ is $w(\mathcal{F})$–dense in $C(X)$;

(c) $C_T(CV_b(X))$ is $w(\mathcal{F})$–dense in $C(X)$;

(d) $C_T(C(X))$ is $w(\mathcal{F})$–dense in $C(X)$.

Proof. Since $\mathcal{F} \leq V$, if T is injective, then (b) holds by Theorem 4.2.15. The implications (b) $\Rightarrow$ (c) and (c) $\Rightarrow$ (d) are obvious, while another application of Theorem 4.2.15, this time taking $U = V = \mathcal{F}$, shows that (d) $\Rightarrow$ (a).

Remark 4.2.17. One additional consequence of Theorem 4.2.15 is the fact that $CV_0(X)$ is always $w(\mathcal{F})$–dense in $C(X)$, which follows by setting $U = \mathcal{F}$ and $T(x) = x$, for all $x \in X$.

In case $T : X \to X$ happens to induce a composition map $C_T : CV_0(X) \to CV_0(X)$, then taking $U = V$ in Theorem 4.2.15 gives us that T is injective if and only if $C_T\left(CV_0(X)\right)$ is dense in $CV_0(X)$. Furthermore, even if T only induces a composition map $C_T : CV_b(X) \to CV_b(X)$, it is still the case that T will be injective when $C_T(CV_b(X))$ is dense in $CV_b(X)$, indeed; (c) of Theorem 4.2.16 would then be satisfied in view of the preceding remark. However, as we next demonstrate, the converse assertion can fail to hold in this case.

Example 4.2.18. Let $X = \mathbb{R}$ with the usual topology. Define $v : X \to \mathbb{R}^+$ as $v(x) = 1$, for all $x \in X$. Let $V = \{\lambda v : \lambda > 0\}$. Then $CV_b(X) = C_b(X)$. Now, if we define $T : X \to X$ as $T(x) = \tan^{-1} x$, $x \in X$, then $V \leq V_T$, T is injective and certainly induces a composition map $C_T : CV_b(X) \to CV_b(X)$. Let $f(x) = \sin x$, $x \in X$. Then $f \in C_b(X)$. Suppose there exists $g \in C_b(X)$ such that $\|g \circ T - f\|_v < \frac{1}{3}$. Since $|g(T(k\pi))| < \frac{1}{3}$ for each $k \in \mathbb{N}$, we have that $|g(\pi / 2)| \leq \frac{1}{3}$. Similarly, however, we would also have that $|g(\pi / 2) - 1| \leq \frac{1}{3}$, which is a contradiction. Thus $C_T(CV_b(X))$ is not dense in $CV_b(X)$.

Turning to the question as to when a composition map $C_T : CV_b(X) \to CV_b(X)$, say, will actually be surjective, we begin with the following proposition.

Proposition 4.2.19. Let $C_T : C(X) \to C(X)$ be the composition map induced by a continuous function $T : X \to X$, and let V be any system of weights on X for which (4.2.d) is satisfied. If $CV_0(X) \subset C_T(C(X))$, then $T : X \to T(X)$ is a homeomorphism.

Proof. From Theorem 4.2.16, it follows that T is injective. Now, given any $g \in CV_0(X)$, since there exists $f \in C(X)$ such that $g = f \circ T$, $g \circ T^{-1} = f\big|_{T(X)}$ is continuous on $T(X)$. Again, in view of condition (4.2.d), it follows that $T^{-1} : T(X) \to X$ is continuous. This completes the proof.

Let $T : X \to T(X) \subset X$ be a homeomorphism and let V be a system of weights on X. Let $V_{T^{-1}} = \{v \circ T^{-1} : v \in V\}$. Then it can be easily proved that $V_{T^{-1}}$ is a system of weights on $T(X)$ for which (4.2.d) is satisfied. Now the following theorem characterises surjective composition transformations which is a generalisation of [122, Theorem 10.3(b), page 141].

Theorem 4.2.20. Let $C_T : CV_b(X) - CV_b(X)$ be a composition map induced by the function $T : X \to X$. Then C_T is surjective if and only if

(i) $T : X \to T(X)$ is a homeomorphism, and

(ii) given $g \in C(V_{T^{-1}})_b(T(X))$, there exists $f \in CV_b(X)$ such that $f|_{T(X)} = g$.

Proof. Suppose that C_T is surjective. Then (i) follows from Proposition 4.2.19. Further, given $g \in C(V_{T^{-1}})_b(T(X))$, $v \in V$, and any $x \in X$, $g \circ T \in C(X)$, and

$$|g \circ T(x)|v(x) = |g(T(x))|v \circ T^{-1}(T(x)) \leq \|g\|_{v \circ T^{-1}}.$$

This implies that $g \circ T \in CV_b(X)$ and so there exists $f \in CV_b(X)$ such that $f \mathrm{o} T = g \mathrm{o} T$, which is to say that $f|_{T(X)} = g$. Conversely, let us assume that (i) and (ii) are both satisfied. If we fix $g \in CV_b(X)$, then $g \circ T^{-1} \in C(T(X))$. Now, for given $v \in V$ and $x \in X$, we have that

$$\left|g \circ T^{-1}(T(x))\right|v \circ T^{-1}(T(x)) = |g(x)|v(x) \leq \|g\|_v,$$

which implies that $g \circ T^{-1} \in C(V_{T^{-1}})_b(T(X))$. Thus, according to (ii), there exists $f \in CV_b(X)$ such that $f|_{T(X)} = g \circ T^{-1}$, and hence $f \mathrm{o}\, T = g$. The proof of the theorem is completed.

Remark 4.2.21. If $T : X \to T(X) \subset X$ is a homeomorphism, then the preceding argument also shows that $\Phi(f) = f \circ T^{-1}$ for each $f \in CV_b(X)$ actually defines a topological isomorphism between $CV_b(X)$ and $C(V_{T^{-1}})_b(T(X))$ and it readily follows that $\Phi(CV_0(X)) = C(V_{T^{-1}})_0(T(X))$.

There is a functional analytic approach to surjectivity, however, which can sometimes offer a way around this problem while simultaneously providing additional information. According to Ptak's open mapping theorem (see [301, page 163]), if

$CV_0(X)$ say, happens to be B–complete (or fully complete), and if $T : X \to X$ induces a nearly open composition operator $C_T : CV_0(X) \to CV_0(X)$ such that $C_T(CV_0(X))$ is dense in $CV_0(X)$, then C_T is necessarily an open surjection. Consequently, when V is a system of weights generated by a single continuous weight on X, for example, C_T will be (open and) surjective as soon as $C_T : CV_0(X) \to CV_0(X)$ is a continuous nearly open operator with dense range. In the following theorem we shall characterise those composition maps $C_T : CV_0(X) \to CV_0(X)$ which are nearly open in the following sense : given any $v \in V$, there exists $u \in V$, such that $B_u \subset \overline{C_T(B_v)}$.

Theorem 4.2.22. If $T : X \to X$ induces a composition map $C_T : CV_0(X) \to CV_0(X)$, then C_T is nearly open if and only if T is injective and $V_T \leq V$.

Proof. Suppose that C_T is nearly open. Let $v \in V$ and $f \in CV_0(X)$. Then there exists $u \in V$ such that $B_u \subset \overline{C_T(B_v)}$. If we put $\lambda = \|f\|_u + 1$ then there exists $g \in B_v$ such that $g \circ T \in \lambda^{-1}(f + B_v)$. Further, it implies that $C_T(\lambda g) = \lambda g \circ T \in f + B_v$. This proves that $C_T(CV_0(X))$ is dense in $CV_0(X)$. From Theorem 4.2.15, it follows that T is injective. In order to show that $V_T \leq V$, let us fix $v \in V$, $x \in X$. and $\varepsilon > 0$. Since C_T is nearly open, there exists $u \in V$ such that $B_u \subset \overline{C_T(B_v)}$. Let $G = \{y \in X : u(y) < u(x) + \varepsilon / 2\}$. Then G is an open neighbourhood of x, and therefore, by [246, Lemma 2, p. 69] there exists $g \in CV_0(X)$ such that $0 \leq g \leq 1$, $g(x) = 1$ and $g(X \backslash G) = 0$. Let $\alpha = (u(x) + \varepsilon / 2)^{-1}$ and $y \in X$. Then

$$\alpha g(y) u(y) < 1.$$

Hence $\alpha g \in B_u$. Choose $w \in V$ such that $w(x) \geq 1$ and let $f \in B_v$ such that $\|f \circ T - \alpha g\|_w \leq \alpha\varepsilon[2(v(T(x)) + 1]^{-1}$. From this, we get

$$\left| f(T(x)) - \alpha \right| \leq \left| f(T(x)) - \alpha g(x) \right| w(x) \leq \alpha\varepsilon\left[2\left(v(t(x)) + 1\right)\right]^{-1}.$$

Further, it implies that

$$1 \geq \left| f(T(x)) \right| v(T(x)) \geq \alpha v(T(x)) = \left| f(T(x)) - \alpha \right| v(T(x)) \geq \alpha\left(v(T(x)) - \varepsilon / 2\right).$$

Since this gives us that $v(T(x)) \leq \alpha^{-1} + \varepsilon / 2 = u(x) + \varepsilon$, we see that $v(T(x)) \leq u(x)$, for all $x \in X$. Thus $V_T \leq V$.

Conversely, suppose that T is injective and $V_T \leq V$. Let $v \in V$. Then there exists $u \in V$ such that $v \circ T \leq u$. Choose $f \in CV_0(X)$ such that $\|f \circ T\|_u < 1$. Since T is an injection, from an application of Theorem 4.2.15, it follows that $C_T(CV_0(X))$ is dense in $CV_0(X)$. In order to show that $B_u \subset \overline{C_T(B_v)}$, it is enough to show that $f \circ T \in \overline{C_T(B_v)}$. To this end, given $w \in V$, we put $K_1 = \{x \in X : |f(T(x))|w(x) \geq 1\}$ and $K_2 = \{x \in X : |f(x)|v(x) \geq 1\}$. Then K_1 and K_2 are both compact. Since $|f(T(x))|v(T(x)) \leq |f(T(x))|u(x) < 1$, for every $x \in X$, we have that $T(x) \notin K_2$. Also, since T is continuous, $T(K_1)$ is compact and $T(K_1) \cap K_2 = \varnothing$. We choose $g \in C_b(X)$ such that $0 \leq g \leq 1$, $g(K_2) = 1$ and $g(T(K_1)) = 0$. If we put $h = f - gf$, then $h \in CV_0(X)$ and

$$|h(x)|v(x) = (1 - g(x))|f(x)|v(x) < 1,$$

for all $x \in X$. This implies that $h \in B_v$. Also, for any $x \in X$,

$$|f(T(x)) - h(T(x))|w(x) = g(T(x))|f(T(x))|w(x) < 1,$$

and this yields the desired argument.

The following corollary is an immediate consequence of Theorem 4.2.22 taken together with Theorem 4.2.4.

Corollary 4.2.23. Let $T : X \to X$ be a map. Then C_T is a nearly open composition operator on $CV_0(X)$ if and only if (i) T is a continuous injection, (ii) $V_T \approx V$, and (iii) for each $v \in V$, $\delta > 0$, and compact set $K \subset X$, $T^{-1}(K) \cap F(v, \delta)$ is compact.

Remark 4.2.24. If $C_T : CV_b(X) \to CV_b(X)$ is a nearly open composition map induced by the function $T : X \to X$, then the necessity argument from the proof of Theorem 4.2.22 can be adapted (by using Theorem 4.2.16 in place of Theorem 4.2.15) to as well show that T must be injective and $V_T \leq V$. However, Example 4.2.18 makes clear that the converse assertion is false. Example 4.2.18 also shows that condition (iii) cannot be omitted from the statement of Corollary 4.2.23.

Now our efforts are to extend the theory of composition operators obtained so far, to the weighted spaces of vector-valued continuous functions, especially on $CV_0(X,E)$ and $CV_p(X, E)$. We begin with the following theorem.

Theorem 4.2.25. If $T : X \to X$ induces a composition operator $C_T : CV_0(X,E) \to CV_p(X,E)$, then T is continuous, V_T is a system of weights on X, and for every $v \in V$ and $p \in cs(E)$, there exist $u \in V$ and $q \in cs(E)$ such that

$$v(x)\ p(y) \le u(T(x))\ q(y)\ , \quad \text{for every } x \in X \text{ and } y \in E\ .$$

Proof. The continuity of T follows from the same remark as given in Theorem 4.2.1. It is clear that V_T is a system of weights on X. Now, it remains to show that for every $v \in V$ and $p \in cs(E)$, there exist $u \in V$ and $q \in cs(E)$ such that $v(x)p(y) \le u(T(x))q(y)$, for all $x \in X$ and $y \in E$. Now, fix $v \in V$ and $p \in cs(E)$. Then there exist $u \in V$ and $q \in cs(E)$ such that $C_T(B_{u,q}) \subset B_{v,p}$. We claim that $v(x)p(y) \le 2u(T(x))q(y)$, for all $x \in X$ and $y \in E$. Take $x_0 \in X$ and $y_0 \in E$. Let $T(x_0) = s$ and $u(s)q(y_0) = \varepsilon$. If $\varepsilon > 0$, then the set $G = \{x \in X : u(x) < 2\varepsilon / q(y_0)\}$ is an open neighbourhood of s. It follows from [246, Lemma 2, p. 69] that there exists $f \in CV_0(X)$ such that $0 \le f \le 1$, $f(s) = 1$, and $f(X \setminus G) = 0$. If we define $g(x) = f(x)\, y_0$ for all $x \in X$, then $g \in CV_0(X,E)$. Let $h = (2\varepsilon)^{-1} g$. Then clearly $h \in B_{u,q}$ and $h \circ T \in B_{v,p}$. From this, it follows that

$$v(x_0)p(y_0) \le 2u\big(T(x_0)\big)q(y_0).$$

On the other hand, if $\varepsilon = 0$, then we can establish the inequality by proceeding analogously. For detailed proof see [352, Theorem 2.1].

Theorem 4.2.26. A map $T : X \to X$ induces a composition operator $C_T : CV_p(X,E) \to CV_p(X,E)$ if and only if

(i) T is continuous,

(ii) for every $v \in V$ and $p \in cs(E)$, there exist $u \in V$ and $q \in cs(E)$ such that $v(x)p(y) \le u(T(x))q(y)$, for all $x \in X$ and $y \in E$.

Proof. The necessary part follows from Theorem 4.2.25. To establish the sufficient part, assume that both (i) and (ii) hold. Let $\{f_\alpha\}$ be a net in $CV_p(X,E)$ such that for every $v \in V$ and $p \in cs(E)$, $\|f_\alpha\|_{v,p} \to 0$. Then

$$\begin{aligned}\|f_\alpha \circ T\|_{v,p} &= \sup\{v(x)p(f_\alpha(T(x))) : x \in X\}\\ &\leq \sup\{u(T(x))q(f_\alpha(T(x))) : x \in X\}\\ &\leq \sup\{u((x)q(f_\alpha(x))) : x \in X\}\\ &= \|f_\alpha\|_{u,q} \to 0.\end{aligned}$$

This completes the proof.

Remark 4.2.27. If we take $E = \mathbb{C}$ in the above theorem, then this reduces to Theorem 4.2.2. Again, if both conditions (i) and (ii) of Theorem 4.2.26 hold, then T can induce a composition operator C_T on $CV_0(X, E)$ if only $CV_0(X, E)$ were to be invariant under the composition operator $C_T : CV_p(X, E) \to CV_p(X, E)$. A generalised version of Theorem 4.2.4 is presented in the following theorem.

Theorem 4.2.28. Let $T : X \to X$ be a map. Then the following are equivalent:

(a) T induces a composition operator C_T on $CV_0(X, E)$;

(b) T induces a composition operator C_T on $CV_p(X, E)$ and $CV_0(X, E)$ is invariant under C_T ;

(c) (i) T is continuous, (ii) for every $v \in V$ and $p \in cs(E)$, there exist $u \in V$ and $q \in cs(E)$ such that $v(x)p(y) \leq u(T(x))q(y)$, for all $x \in X$ and $y \in E$, (iii) for each $v \in V$, $\delta > 0$ and compact set $K \subset X$, $T^{-1}(K) \cap F(v, \delta)$ is compact;

(d) (i) T is continuous, (ii) for evey $v \in V$ and $p \in cs(E)$, there exist $u \in V$ and $q \in cs(E)$ such that $v(x)p(y) \leq u(T(x))q(y)$, for all $x \in X$ and $y \in E$, (iii) for each $v \in V$, $p \in cs(E)$, $\delta > 0$ and $u \in V$, $q \in cs(E)$ such that $v(x)p(y) \leq u(T(x))q(y)$, for every $x \in X$ and $y \in E$, $T^{-1}(K) \cap F(v, \delta)$ is compact whenever K is a compact subset of $F(u, \delta)$.

Proof (a) $\Rightarrow$ (c). From Theorem 4.2.25, it follows that the conditions c(i) and c(ii) are true whenever (a) holds. To establish c(iii), let $v \in V$, $\delta > 0$ and $K \subset X$ be a compact subset. In light of [246, Lemma 2, p. 69] there exists $f \in CV_0(X)$ such that $0 \leq f \leq 1$, $f(K) = 1$. Since E is a non-zero locally convex Hausdorff space, there exists a non-zero vector $y_0 \in E$ and $p' \in cs(E)$ such that $p'(y_0) \neq 0$. If we put $g(x) = f(x)y_0 / p'(y_0)$ for every $x \in X$, then $g \in CV_0(X, E)$ and $g \circ T \in CV_0(X, E)$. Let $H = \{x \in X : v(x)p'(g(T(x))) \geq \delta\}$. Then H is compact and it readily follows that $T^{-1}(K) \cap F(v,\delta) \subset H$, and hence it is compact.

(c) $\Rightarrow$ (d). In this case (d) is an obvious consequence of (c).

(d) $\Rightarrow$ (b). In this case, it follows from Theorem 4.2.26 that T induces the composition operator C_T on $CV_p(X,E)$. To complete the proof of (b), it remains to prove that $CV_0(X,E)$ is invariant under C_T. Let $f \in CV_0(X,E)$, $v \in V$, $p \in cs(E)$ and $\delta > 0$. Consider the set $S = \{x \in X : v(x)p(f(T(x))) \geq \delta\}$. In case S is empty, compactness follows obviously. Assume that S is non-empty. Let us take $u \in V$ and $q \in cs(E)$ such that $v(x)p(y) \leq u(T(x))q(y)$, for every $x \in X$ and $y \in E$. Now, if we set $H = \{x \in X : u(x)q(f(x)) \geq \delta\}$, then H is a compact subset of X and $T(S) \subset H$. Let $r = p + q$ and let $M_r = \sup\{r(f(x)) : x \in H\}$. Then $M_r > 0$ and $H \subset F(u, \delta / M_r)$. Thus from d(iii), it follows that $T^{-1}(H) \cap F(v, \delta / M_r)$ is compact. Since $S \subset T^{-1}(H)$ and $S \subset F(v, \delta / M_r)$, it follows that S is a compact subset of X. This proves that $C_T f \in CV_0(X,E)$. Finally, (a) clearly follows from (b). This completes the proof of the theorem.

Now, we plan to characterise the linear transformations which are composition operators from $CV_0(X,E)$ into $CV_p(X,E)$ and this generalises our Theorem 4.2.8. We begin with the following notations and definitions.

Let $F(CV_0(X,E),E)$ be the vector space of all linear mappings from $CV_0(X,E)$ into E and let $L(CV_p(X,E),E)$ be a vector subspace of $F(CV_0(X,E),E)$ consisting of all continuous linear mapping from $CV_p(X,E)$ into E.

For each $x \in X$, we define a mapping $\delta_x : CV_p(X,E) \to E$ as $\delta_x(f) = f(x)$, for all $f \in CV_p(X,E)$. Clearly, it is linear, we claim that δ_x is continuous. Let $p \in cs(E)$. Since for $x \in X$, there exists $v \in V$ such that $v(x) > 0$, we have that $p(\delta_x(f)) = p(f(x)) \leq \left[\frac{1}{v(x)}\right]\|f\|_{v,p}$, for all $f \in CV_p(X,E)$. This proves that δ_x is a continuous transformation. Let $\Delta(X) = \{\delta_x : x \in X\}$. Then $\Delta(X)$ is a subset of $L(CV_p(X,E),E)$.

Theorem 4.2.29. Let $\Phi : CV_0(X,E) \to CV_p(X,E)$ be a linear transformation. Then there exists $T : X \to X$ such that $\Phi = C_T$ if and only if the composition transformation $C_\Phi : L(CV_p(X,E),E) \to F(CV_0(X,E),E)$ induced by the linear transformation Φ, leaves $\Delta(X)$ invariant. In case $C_\Phi(\Delta(X)) \subset \Delta(X)$, and T is continuous, $\Phi = C_T$ is continuous if and only if for every $v \in V$ and $p \in cs(E)$, there exist $u \in V$ and $q \in cs(E)$ such that $v(x)p(y) \leq u(T(x))q(y)$, for all $x \in X$ and $y \in E$.

Proof. Suppose $\Phi = C_T$, for some $T : X \to X$. If $x \in X$, and $f \in CV_0(X, E)$. Then

$$\begin{aligned}(C_\Phi \delta_x)(f) &= (\delta_x \circ \Phi)(f) \\ &= \delta_x(\Phi(f)) = \delta_x(C_T(f)) = \delta_x(f \circ T) \\ &= f(T(x)) = \delta_{T(x)}(f).\end{aligned}$$

This proves that $C_\Phi \delta_x = \delta_{T(x)}$. Conversely, let us suppose that $C_\Phi(\Delta(X)) \subset \Delta(X)$. For $x \in X$, if we take $T(x)$ to be the unique element of X such that $C_\Phi(\delta_x) = \delta_{T(x)}$, and consider any function $f \in CV_0(X, E)$, then

$$\begin{aligned}\Phi(f)(x) &= \delta_x(\Phi(f)) \\ &= (\delta_x \circ \Phi)(f) \\ &= C_\Phi(\delta_x)(f) \\ &= \delta_{T(x)}(f) \\ &= f(T(x)) \\ &= C_T(f)(x).\end{aligned}$$

This shows that $\Phi = C_T$. As noted earlier, T is continuous, while C_T is continuous exactly when for every $v \in V$ and $p \in cs(E)$, there exist $u \in V$ and $q \in cs(E)$ such that $v(x)p(y) \leq u(T(x))q(y)$, for all $x \in X$ and $y \in E$, in view of Theorem 4.2.25 and Theorem 4.2.26 proved earlier.

In the following results we shall carry the basic properties of composition operators discussed so far to the weighted spaces of vector-valued continuous functions. The following theorem is a straight forward extension of Theorem 4.2.14.

Theorem 4.2.30. Let $C_T : C(X, E) \to C(X, E)$ be a composition map induced by a continuous function $T : X \to X$, and let V be any system of weights on X for which (4.2.d) is satisfied. Then the following are equivalent :

(i) $T(X)$ is dense in X;

(ii) C_T is injective;

(iii) $C_T \big| CV_p(X, E)$ is injective;

(iv) $C_T \big| CV_0(X, E)$ is injective.

Now, we shall present the vector-valued analogue of Theorem 4.2.15 in which we discuss the range of a composition operator.

Theorem 4.2.31. Let V be a system of weights on X for which (4.2.d) is satisfied, let U be a system of weights on X such that $U \leq V$ and assume that $T : X \to X$ induces a composition map $C_T : CV_0(X, E) \to CU_0(X, E)$. Then T is injective if and only if $C_T(CV_0(X, E))$ is dense in $CU_0(X, E)$.

Proof. Suppose that $C_T(CV_0(X, E))$ is dense in $CU_0(X, E)$, and let $x_1, x_2 \in X$ be such that $x_1 \neq x_2$. Choose $g \in CV_0(X, E)$, $p \in cs(E)$ and $u \in U$ such that $g(x_1) = 0$, $p(g(x_2)) = 1$ and $u(x_1), u(x_2) \geq 1$. Since $g \in CU_0(X, E)$, there exists $f \in CV_0(X, E)$ such that

$$\|f \circ T - g\|_{u,p} < \tfrac{1}{2}\,.$$

From this it follows that $p(f(T(x_1))) < \frac{1}{2}$ and $|\, p(f(T(x_2))) - 1 \,| < \frac{1}{2}$ and this yields that $T(x_1) \neq T(x_2)$. Conversely, if T is injective and $A = \{f \circ T \ : \ f \in C_b(X)\}$ is a self-adjoint subalgebra of $C_b(X)$ which separates the points of X, then the vector subspace $Y = C_T\big(CV_0(X, E)\big)$ is a module over A and the density of Y in $CU_0(X, E)$ now follows from the vector-valued analogue of the bounded case of the weighted approximation problem [383, Theorem 4.1, page 181]. This completes the proof.

Let K be a closed subset of X, and χ_K be the characteristic function of K. Then we define $\mathcal{U} = \Big\{\lambda\chi_K \ : \ \lambda > 0,\ K \subset X,\ K \text{ finite}\Big\}$. Then clearly $\mathcal{U}$ is a system of weights on X, and

$$C\mathcal{U}_0(X, E) = C\mathcal{U}_p(X, E) = \big(C(X, E), w_{\mathcal{U}}\big),$$

where $w_{\mathcal{U}}$ denotes the topology of pointwise convergence on X.

Now we shall present the following theorems which are the vector-valued analogue of Theorem 4.2.16, Theorem 4.2.19 and Theorem 4.2.20. For the proof of these results we refer to [352].

Theorem 4.2.32. Let $C_T : C(X, E) \to C(X, E)$ be a composition map induced by a continuous function $T : X \to X$, and let V be any system of weights on X for which (4.2.d) is satisfied. Then the following are equivalent :

(i) T is injective;

(ii) $C_T(CV_0(X, E))$ is $w\mathcal{U}$-dense in $C(X, E)$;

(iii) $C_T(CV_p(X, E)$ is $w\mathcal{U}$-dense in $C(X, E)$;

(iv) $C_T(C(X, E)$ is $w\mathcal{U}$-dense in $C(X, E)$.

Theorem 4.2.33. Let $C_T : C(X, E) \to C(X, E)$ be a composition map induced by a continuous function $T : X \to X$, and let V be any system of weights on X for which (4.2.d) is satisfied. If $CV_0(X, E) \subset C_T(C(X, E))$, then $T : X \to T(X)$ is a homeomorphism.

Theorem 4.2.34. Let $T : X \to X$ be a function such that $C_T : CV_b(X, E) \to CV_b(X, E)$ is a composition map. Then C_T is surjective if and only if

(i) $T : X \to T(X)$ is a homeomorphism, and

(ii) given $g \in CV(T^{-1})_b(T(X), E)$, there exists $f \in CV_b(X, E)$ such that $f|_{T(X)} = g$.

In the following theorem, we shall give a generalisation of Theorem 4.2.22 which characterises those composition maps $C_T : CV_0(X, E) \to CV_0(X, E)$ which are nearly open in the sense that given any $v \in V$ and $p \in cs(E)$, there exist $u \in V$ and $q \in cs(E)$ such that $B_{u,q} \subset \overline{C_T(B_{v,p})}$.

Theorem 4.2.35. Let $T : X \to X$ be a function such that $C_T : CV_0(X, E) \to CV_0(X, E)$ is a composition map. Then C_T is nearly open if and only if

(i) T is injective,

(ii) for every $v \in V$ and $p \in cs(E)$, there exist $u \in V$ and $q \in cs(E)$ such that $v(T(x))p(y) \leq u(x)q(y)$, for all $x \in X$ and $y \in E$.

Proof. Suppose that C_T is nearly open. Let $v \in V$ and $p \in cs(E)$. Then there exist $u \in V$ and $q \in cs(E)$ such that $B_{u,q} \subset \overline{C_T(B_{v,p})}$. To show that T is injective, in view of Theorem 4.2.31, it is enough to show that $C_T(CV_0(X,E))$ is dense in $CV_0(X,E)$. Let $f \in CV_0(X,E)$ and $\alpha = \|f\|_{u,q} + 1$. Then $\alpha^{-1}f \in B_{u,q}$ and there exists $g \in B_{v,p}$ such that $g \circ T \in \alpha^{-1}(f + B_{v,p})$. Further, it implies that $\alpha g \circ T \in f + B_{v,p}$. This proves that $C_T(CV_0(X,E))$ is dense in $CV_0(X,E)$. Now, we establish the condition (ii). Fix $x_0 \in X$ and $y_0 \in E$. We claim that

$$v(T(x_0))\, p\,(y_0) \leq 2u\,(x_0)\, q(y_0)\,.$$

Set $u(x_0)q(y_0) = \varepsilon$. In case $\varepsilon > 0$, the set $G = \{x \in X : u(x) < \frac{3\varepsilon}{2q(y_0)}\}$ is an open neighbourhood of x_0. Therefore, according to [246, Lemma 2, p. 69], there exists $g \in CV_0(X)$ such that $0 \leq g \leq 1$, $g(x_0) = 1$ and $g(X \setminus G) = 0$. We define $h_1(x) = g(x)y_0$, for all $x \in X$. Then $h_1 \in CV_0(X,E)$. Let $\beta = (3\varepsilon/2)^{-1}$ and $h = \beta h_1$. Then $h \in B_{u,q}$. Let $w \in V$ be such that $w(x_0) \geq 1$ and $p' \in cs(E)$ such that $p' = 2p$ with $p'(y_0) \geq 1$. That is, $p(y_0) \geq \frac{1}{2}$. Therefore, we can find $f \in B_{v,p}$ such that

$$\|f \circ T - h\|_{w,p'} \leq \beta\varepsilon\left[2\big(v(T(x_0))p(y_0) + 1\big)\right]^{-1}.$$

Further, it implies that

$$p\big(f(T(x_0)) - \beta y_0\big) \leq \alpha\varepsilon\left[2\big(v(T(x_0))p(y_0) + 1\big)\right]^{-1}. \tag{1}$$

Consequently, we have that

$$1 \geq v(T(x_0))p\big(f(T(x_0))\big) \geq v(T(x_0))\beta p(y_0) - \frac{1}{2}\, v(T(x_0))p'\big(f(T(x_0)) - \beta y_0\big). \tag{2}$$

From (1), we have

$$-\frac{1}{2}\, v\,(T(x_0))\, p'\,\big(f(T(x_0)) - \beta y_0\big) \geq -v(T(x_0))\, p\,(y_0)\, \beta\varepsilon\left[2\big(v(T(x_0))\, p\,(y_0) + 1\big)\right]^{-1}. \tag{3}$$

Further, using (3) in (2), it follows that

$$1 \geq \beta\,[v(T(x_0))p(y_0) - \varepsilon / 2].$$

Thus we have

$$v\big(T(x_0)\big)p(y_0) \leq 2u(x_0)q(y_0).$$

Hence in this case our claim is established. Similarly, we can establish our claim in case $u(x_0)q\ (y_0) = 0$. For details of proof see [352]. With this the proof of the theorem is completed.

Now, we define some notations which are needed to state the following corollary which is an immediate consequence of Theorem 4.2.35 taken together with Theorem 4.2.28.

Let $V \times cs(E) = \{(v, p) : v \in V, p \in cs(E)\}$, where $(v, p) : X \times E \to \mathbb{R}^+$ is defined as $(v, p)(x, y) = v(x)p(y)$, for all $x \in X$ and $y \in E$. We write $V \times cs(E) \leq U \times cs(E)$, whenever given $v \in V$ and $p \in cs(E)$, there exist $u \in U$ and $q \in cs(E)$ such that $(v, p) \leq (u, q)$ *i.e.*, $v(x)p(y) \leq u(x)q(y)$, for every $x \in X$ and $y \in E$. If $V \times cs(E) \leq U \times cs(E)$ and $U \times cs(E) \leq V \times cs(E)$, then they are equivalent and we denote it by $V \times cs(E) \approx U \times cs(E)$.

Corollary 4.2.36. Let $T : X \to X$ be a function. Then $C_T : CV_0(X, E) \to CV_0(X, E)$ is a nearly open composition operator if and only if

(i) T is a continuous injection,

(ii) $V_T \times cs(E) \approx V \times cs(E)$,

(iii) for every $v \in V, \varepsilon > 0$ and compact set $K \subset X$, $T^{-1}(K) \cap F(v, \varepsilon)$, is compact.

Finally, we begin with a study of composition operators on the weighted spaces of cross-sections which is the second phase of our study in this section. In this phase of study we shall characterise the self-maps $T : X \to X$ which induce the composition operators on the weighted spaces $LW_b(X)$ and $LW_0(X)$ defined in section 4 of chapter I. Besides, an algebraic criterion for distinguishing the composition operators among all continuous linear operators on these weighted spaces, we give some basic properties of these operators. Most precisely, this study presents a unification and a very broad generalisation of the theory of these operators on the weighted spaces of functions. We begin with the following characterization of composition operators.

Theorem 4.2.37. Let $T : X \to X$ be a map such that for every $x \in X$, $F_{T(x)} \subset F_x$. Then $C_T : LW_b(X) \to LW_b(X)$ is a composition operator if and only if $W \leq W \circ T$, where $W \circ T = \{w \circ T : w \in W\}$.

Proof. Let $W \leq W \circ T$. Then for given $w \in W$, there exists $w' \in W$ such that $w \leq w' \circ T$. Thus $w_x(y) \leq w'_{T(x)}(y)$, for every $x \in X$ and $y \in F_{T(x)}$. Let $\{f_\alpha\}$ be a net in $LW_b(X)$ such that $\|f_\alpha\|_w \to 0$, for every $w \in W$. Then

$$\begin{aligned}\|f_\alpha \circ T\|_w &= \sup\{w_x[f_\alpha(T(x))] : x \in X\}\\ &\leq \sup\{w'_x[f_\alpha(x)] : x \in X\} = \|f_\alpha\|_{w'} \to 0.\end{aligned}$$

This shows that C_T is a composition operator on $LW_b(X)$.

Conversely, suppose C_T is a composition operator and let $w \in W$. Then there exists $w' \in W$ such that $C_T(B_{w'}) \subset B_w$. Now, we claim that $w \leq 2\ w' \circ T$. Fix $x_0 \in X$ and $y_0 \in F_{T(x_0)}$. Set $w'_{T(x_0)}(y_0) = \varepsilon$. In case $\varepsilon > 0$, we define the function $g : X \to \bigcup_{x \in X} F_x$ as

$$g(x) = \begin{cases}(2\varepsilon)^{-1} y_0\ , & x = T(x_0)\\ 0\ , & \text{otherwise.}\end{cases}$$

Then $g \in B_{w'}$ and $g \circ T \in B_w$. This gives us that

$$w_{x_0}(y_0) \leq 2\ w'_{T(x_0)}(y_0).$$

On the other hand, suppose that $\varepsilon = 0$ and $w_{x_0}(y_0) > 0$. Let $\beta = \frac{w_{x_0}(y_0)}{2}$. Then we define the function $h : X \to \bigcup_{x \in X} F_x$ as

$$h(x) = \begin{cases}\beta^{-1} y_0\ , & x = T(x_0),\\ 0\ , & \text{otherwise.}\end{cases}$$

Clearly $h \in B_{w'}$ and $h \circ T \in B_w$. This implies that $w_{x_0}(y_0) \leq \frac{w_{x_0}(y_0)}{2}$, which is an impossibility. With this the proof of the theorem is completed.

Now, we shall see that the map $T : X \to X$ with the condition that $W \leq W \circ T$ does not induce a composition operator on $LW_0(X)$. To see this, let $\{F_n : n \in \mathbb{N}\}$ be the vector–fibration over $\mathbb{N}$, where for every $n \in \mathbb{N}$, $F_n = \mathbb{R}^n$ with the usual norm. Let v be the function on $\mathbb{N}$ defined as $v(n) = \| \, . \, \|_n$, for every $n \in \mathbb{N}$, and let $W = \{\lambda v : \lambda > 0\}$. Then W is a system of weights on $\mathbb{N}$. We define the map $T : \mathbb{N} \to \mathbb{N}$ as $T(n) = 1$, for every $n \in \mathbb{N}$. Then clearly for every $n \in \mathbb{N}$, $F_{T(n)} \subset F_n$ and $W \leq W \circ T$. If we define the function $f : \mathbb{N} \to \bigcup_{n \in \mathbb{N}} F_n$ as $f(n) = (0, 0, \cdots, 1/n)$, then clearly $f \in LW_0(\mathbb{N})$. But $f \circ T \notin LW_0(\mathbb{N})$. So, this motivates us to look for some more conditions on the map T to induce a composition operator on $LW_0(X)$. This we shall establish in the following theorem.

Theorem 4.2.38. Let $T : X \to X$ be a continuous function such that for every $x \in X$, $F_{T(x)} \subset F_x$. Then the following are equivalent :

(a) T induces a composition operator C_T on $LW_0(X)$;

(b) (i) $W \leq W \circ T$, (ii) for each $f \in L(X)$ such that $w[f]$ is upper semicontinuous for every $w \in W$ and given $w \in W$, $\varepsilon > 0$ and compact set $K \subset X$, $T^{-1}(K) \cap \{x \in X : w_x[f(T(x))] \geq \varepsilon\}$ is compact,

(c) (i) $W \leq W \circ T$, (ii) for each $f \in L(X)$ such that $w[f]$ is upper semicontinuous for every $w \in W$, and given $w \in W$, $\varepsilon > 0$ and $w' \in W$ such that $w \leq w' \circ T$, $T^{-1}(K) \cap \{x \in X : w_x[f(T(x))] \geq \varepsilon\}$ is compact, whenever K is a compact subset of $\{x \in X : w'_x[f(x)] \geq \varepsilon\}$,

(d) T induces a composition operator C_T on $LW_b(X)$ and $LW_0(X)$ is invariant under C_T.

Proof. If condition (a) holds, then (b) (i) follows from Theorem 4.2.37. Now, let $w \in W$ and $f \in L(X)$. If $\varepsilon > 0$ and $K \subset X$ is a compact subset, then we define the function $g : X \to \bigcup_{x \in X} F_x$ as

$$g(x) = \begin{cases} f(x) & , \quad x \in K \\ 0 & , \quad \text{otherwise.} \end{cases}$$

Clearly $g \in LW_0(X)$ and $g \circ T \in LW_0(X)$. This implies that the set $F = \{x \in X : w_x[g(T(x))] \geq \varepsilon\}$ is compact. Thus $T^{-1}(K) \bigcap \{x \in X : w_x[f(T(x))] \geq \varepsilon\}$ is compact as it is a closed subset of F. (c) is an immediate consequence of (b). Now, if condition (c) holds, then from Theorem 4.2.37 it follows that C_T is a composition operator on $LW_b(X)$. To show that $LW_0(X)$ is invariant under C_T, let $g \in LW_0(X)$, $w \in W$ and $\varepsilon > 0$. Let $S = \{x \in X : w_x[g(T(x))] \geq \varepsilon\}$. Now, we choose $w' \in W$ such that $w \leq w' \circ T$. Then clearly the set $G = \{x \in X : w'_x[g(x)] \geq \varepsilon\}$ is compact and $T(S) \subset G$. Since $S \subset T^{-1}(G) \bigcap \{x \in X : w_x[g(T(x))] \geq \varepsilon\}$ and in view of (c) (ii), it follows that the set $T^{-1}(G) \bigcap \{x \in X : w_x[g(T(x))] \geq \varepsilon\}$ is compact. Thus $C_T g \in LW_0(X)$. Finally, (a) is an immediate consequence of (d). This completes the proof of the theorem.

Example 4.2.39. Let $X = (0, \infty) = \mathbb{R}^+$, the set of positive reals with the usual topology, and let $L^p(X, \mu)$ $(1 \leq p < \infty)$ be the Banach space of all complex-valued measurable functions whose p^{th}-power is integrable on X with respect to the measure μ. Let $H^p(D)$ $(1 \leq p < \infty)$ be the Hardy space of analytic functions on the open unit disc D. Let $\{F_s : s \in \mathbb{R}^+\}$ be the vector-fibration over $\mathbb{R}^+$, where

$$F_s = \begin{cases} L^s(X, \mu), & 1 \leq s < \infty \\ H^{1/s}(D), & 0 < s < 1. \end{cases}$$

Let v be the function defined on $\mathbb{R}^+$ such that

$$v(s) = \begin{cases} \|\cdot\|_s, & 1 \leq s < \infty, \quad \text{where } \|.\|_s \text{ is the usual norm of } L^s(X, \mu), \\ \|\cdot\|_{1/s}, & 0 < s < 1, \quad \text{where } \|.\|_{1/s} \text{ is the usual norm of } H^{1/s}(D). \end{cases}$$

Let $W = \{\lambda v : \lambda \geq 0\}$. Then W is a system of weights on $\mathbb{R}^+$. Now, for each $t \in [1, \infty)$, we define the mapping $T_t : \mathbb{R}^+ \to \mathbb{R}^+$ as

$$T_t(s) = \begin{cases} s + t, & 1 \leq s < \infty \\ st^{-1}, & 0 < s < 1. \end{cases}$$

We note that as the positive reals increase the corresponding fibre spaces defined by these reals form a nested sequence and therefore the map $T_t : \mathbb{R}^+ \to \mathbb{R}^+$ satisfies the requirement that for every $s \in \mathbb{R}^+, F_{T_t(s)} \subset F_s$. In view of Theorem 4.2.37, it follows that C_{T_t} is a composition operator on $LW_b(\mathbb{R}^+)$.

Example 4.2.40. Let $X = \mathbb{Z}^+$, the set of positive integers with the discrete topology, and let $\{F_n : n \in \mathbb{Z}^+\}$ be the vector–fibration over $\mathbb{Z}^+$, where for every $n \in \mathbb{Z}^+, F_n = \mathbb{C}^n$. Let v be the mapping defined on $\mathbb{Z}^+$ as $v(n) = \|.\|_n$, for all $n \in \mathbb{Z}^+$ where $\|.\|_n$ is the norm of $\mathbb{C}^n$. Let $W = \{\lambda v : \lambda \geq 0\}$. Then W is a system of weights on $\mathbb{Z}^+$. Define the mapping $T : \mathbb{Z}^+ \to \mathbb{Z}^+$ as $T(n) = n-1$, for all $n \in \mathbb{Z}^+ \setminus \{1\}$ and $T(1) = 1$. Then clearly for every $n \in \mathbb{Z}^+$, $F_{T(n)} \subset F_n$ and $W \leq W \circ T$. Thus by Theorem 4.2.37, T induces a composition operator on $LW_b(X)$.

In the following theorem we shall give a description of the continuous composition operators from $LW_b(X)$ into itself which parallels a standard result for functional Hilbert spaces proved by Caughran and Schwartz [61].

For each $x \in X$, we define the mapping $\delta_x : LW_b(X) \to F_x$ as $\delta_x(f) = f(x)$, for every $f \in LW_b(X)$. Clearly δ_x is linear. For each $x \in X$, we denote by $F(LW_b(X), F_x)$ the vector space of all linear mappings from $LW_b(X)$ to F_x, whereas $S(LW_b(X), F_x)$ denotes a vector subspace of $F(LW_b(X), F_x)$ containing δ_x.

Theorem 4.2.41. Let $\Phi : LW_b(X) \to LW_b(X)$ be a linear transformation. Then there is a map $T : X \to X$ such that $\Phi = C_T$ if and only if for every $x \in X$, the composition transformation $C_\Phi : S(LW_b(X), F_x) \to F(LW_b(X), F_x)$ induced by Φ satisfies $C_\Phi(\delta_x) = \delta_{T(x)}$. In case $C_\Phi(\delta_x) = \delta_{T(x)}$, $\Phi = C_T$ is continuous if and only if $W \leq W \circ T$.

Proof. If $\Phi = C_T$, for some $T : X \to X$, then for $x \in X$ and $f \in LW_b(X)$, we have

$$\begin{aligned} C_\Phi(\delta_x)(f) = (\delta_x \circ \Phi)(f) = \delta_x(\Phi f) &= \delta_x(C_T f) \\ &= f(T(x)) \\ &= \delta_{T(x)}(f) \cdot \end{aligned}$$

Thus $C_\Phi(\delta_x) = \delta_{T(x)}$. Conversely, if $x \in X$, then let $T(x)$ be the unique element of X such that $C_\Phi(\delta_x) = \delta_{T(x)}$. To show that $\Phi = C_T$, let $f \in LW_b(X)$. Then

$$\begin{aligned}\Phi(f)(x) = \delta_x(\Phi f) = (\delta_x \circ \Phi)(f) &= C_\Phi(\delta_x)(f)\\ &= \delta_{T(x)}(f)\\ &= f(T(x))\\ &= (f \circ T)(x).\end{aligned}$$

Thus $\Phi = C_T$. From Theorem 4.2.37, it follows that $\Phi = C_T$ is continuous if and only if $W \leq W \circ T$. This completes the proof of the theorem.

Finally, we wind up this section by characterising those composition maps $C_T : LW_0(X) \to LW_0(X)$ which are nearly open in the sense that given any $w \in W$, there exists $w' \in W$ such that $B_{w'} \subset \overline{C_T(B_w)}$. Also, the open mapping theorem of Ptak's [301] says that $LW_0(X)$ is B–complete (or fully complete), and if $T : X \to X$ induces a nearly open composition operator $C_T : LW_0(X) \to LW_0(X)$ such that $C_T(LW_0(X))$ is dense in $LW_0(X)$, then C_T is necessarily an open surjection. Now, if the vector fibration $\{F_x : x \in X\}$ is such that each fibre space F_x is a Banach space, and W is a system of weights on X generated by a single weight v on X defined by $v(x) = \| \, . \, \|_x$, for all $x \in X$, where $\| \, . \, \|_x$ is the norm of the Banach space F_x, then C_T will be open surjection if $C_T : LW_0(X) \to LW_0(X)$ is continuous nearly open operator with dense range. First of all, we shall establish the following proposition which characterises the self maps for which the corresponding composition maps $C_T : LW_0(X) \to LW_0(X)$ have dense range. We shall work with the following requirements while establishing remaining results of this section.

(4.2.41a) For each $x \in X$, there exists $f_x \in LW_0(X)$ such that $f_x(x) \neq 0$,

(4.2.41b) If given any vector subspace Z of $LW_0(X)$, then for all $f \in L(X)$ and $x_0 \in X$, given $w \in W$ and $\varepsilon > 0$, there exist $g \in Z$ and neighbourhood N of x_0 in X, such that $w_x[f(x) - g(x)] < \varepsilon$, for all $x \in N$ [247, page 307(b)].

Proposition 4.2.42. Let W' be a system of weights on X such that $LW_0'(X)$ satisfies the condition (4.2.41b), and let W be a system of weights on X such that $W' \leq W$. Let $T : X \to X$ be a continuous map such that for every $x \in X$, $F_{T(x)} \subset F_x$ and for given $x_1, x_2 \in X$, there is $w \in W$ such that $w_{T(x_1)} \leq w_{x_1}$ on $F_{T(x_1)}$ and $w_{T(x_2)} \leq w_{x_2}$ on $F_{T(x_2)}$, and T induces the composition map $C_T : LW_0(X) \to LW_0'(X)$. Then $C_T(LW_0(X))$ is dense in

$LW_0'(X)$ if and only if T is injective.

Proof. Suppose that $C_T(LW_0(X))$ is dense in $LW_0'(X)$. Let $x_1, x_2 \in X$ be such that $x_1 \neq x_2$. Choose $h \in LW_0'(X)$ and $w \in W'$ such that $h(x_1) = 0$ and $h(x_2) \neq 0$, $w_{x_2}[h(x_2)] \geq 1$, $w_{T(x_1)} \leq w_{x_1}$ on $F_{T(x_1)}$ and $w_{T(x_2)} \leq w_{x_2}$ on $F_{T(x_2)}$. Since $h \in LW_0'(X)$, there exists $f \in LW_0(X)$ such that $\|f \circ T - h\|_w < \frac{1}{2}$.

From this, it follows that

$$w_{T(x_1)}\left[f\left(T(x_1)\right)\right] < \tfrac{1}{2}, \tag{1}$$

and

$$\left|w_{T(x_2)}\left[f\left(T(x_2)\right)\right] - 1\right| < \tfrac{1}{2}. \tag{2}$$

Now, if $T(x_1) = T(x_2)$, then it contradicts (1) and (2), and therefore $T(x_1) \neq T(x_2)$.

Conversely, if T is injective, then $A = \{f \circ T : f \in C_b(X)\}$ is a self-adjoint subalgebra of $C_b(X)$ which separates the points of X. Since the vector subspace $Z = \{f \circ T : f \in LW_0(X)\}$ is a module over A and $LW_0'(X)$ satisfies the condition (4.2.41b), the density of $C_T(LW_0(X)) = Z$ in $LW_0'(X)$ now follows from [247, Theorem 11] as it is the bounded case of the strict weighted approximation problem. This completes the proof.

Theorem 4.2.43. Let W be a system of weights on X such that $LW_0(X)$ satisfies the condition (4.2.41b). Let $T : X \to X$ be a continuous map such that for every $x \in X$, $F_{T(x)} \subset F_x$ and for given $x_1, x_2 \in X$, there is $w \in W$ such that $w_{T(x_1)} \leq w_{x_1}$ on $F_{T(x_1)}$ and $w_{T(x_2)} \leq w_{x_2}$ on $F_{T(x_2)}$, and T induces a composition map $C_T : LW_0(X) \to LW_0(X)$. Then C_T is nearly open if and only if

(i) $W o T \leq W$;

(ii) T is injective.

Proof. Suppose that C_T is a nearly open composition operator, and let $w \in W$. Choose $w' \in W$ such that $B_{w'} \subset \overline{C_T(B_w)}$. To show that T is injective, in view of Theorem 4.2.42, it is enough to establish that $C_T(LW_0(X))$

is dense in $LW_0(X)$. Let $f \in LW_0(X)$ and let $\lambda = \| f \|_{w'} + 1$. Then $\lambda^{-1} f \in B_{w'}$. Further, there exists $g \in B_w$ such that $\lambda^{-1} f = g \circ T$. This implies that $C_T(\lambda g) \in f + B_w$. Thus $C_T(LW_0(X))$ is dense in $LW_0(X)$. Now, we shall show that $W \circ T \leq W$. Let $w \in W$. Choose $w' \in W$ such that $B_{w'} \subset \overline{C_T(B_w)}$. Claim that $w \circ T \leq 2w'$, *i.e.*, $w_{T(x)}(y) \leq 2w'_x(y)$, for every $x \in X$ and $y \in F_{T(x)}$. Fix $x_0 \in X$ and $y_0 \in F_{T(x_0)}$. Put $w'_{x_0}(y_0) = \varepsilon$. In case $\varepsilon > 0$, we define the function $f : X \to \bigcup_{x \in X} F_x$ as

$$f(x) = \begin{cases} \varepsilon^{-1} y_0, & x = x_0 \\ 0 \quad , & \text{otherwise.} \end{cases}$$

Obviously, $f \in B_{w'}$. Choose $w'' \in W$ such that $w \leq w''$ and $w''_{T(x_0)} \leq w_{x_0}$ on $F_{T(x_0)}$. We can find $g \in B_w$ so that

$$\| f - g \circ T \|_{w''} \leq \frac{\varepsilon^{-1}}{3} \left(w_{T(x_0)}(y_0) + \varepsilon \right).$$

Further, it implies that

$$w''_{x_0}\left[\varepsilon^{-1} y_0 - g\big(T(x_0)\big)\right] \leq \frac{\varepsilon^{-1}}{3} \left(w_{T(x_0)}(y_0) + \varepsilon \right).$$

Consequently with the choice of w'', we have

$$w_{T(x_0)}\left[\varepsilon^{-1} y_0 - g\big(T(x_0)\big)\right] \leq \frac{\varepsilon^{-1}}{3} \left(w_{T(x_0)}(y_0) + \varepsilon \right).$$

Since $g \in B_w$, we have

$$1 \geq w_{T(x_0)}\left[g\big(T(x_0)\big)\right] \geq \varepsilon^{-1} w_{T(x_0)}(y_0) - \frac{\varepsilon^{-1}}{3} \left(w_{T(x_0)}(y_0) + \varepsilon \right).$$

Further, it implies that $w_{T(x_0)}(y_0) \leq 2w'_{x_0}(y_0)$. On the other hand, suppose $\varepsilon = 0$ and $w_{T(x_0)}(y_0) > 0$. Put $\beta = w_{T(x_0)}(y_0) / 4$ and define the function $f : X \to \bigcup_{x \in X} F_x$ as

$$f(x) = \begin{cases} \beta^{-1} y_0, & x = x_0 \\ 0 \quad , & \text{otherwise.} \end{cases}$$

Clearly $f \in B_w$. Again, we can choose $w'' \in W$ in the same way so that we can find $g \in B_w$ such that $\|f - g \circ T\|_{w''} \leq 2$. From this, it follows that $w''_{x_0}[\beta^{-1} y_0 - g(T(x_0))] \leq 2$. Further it implies that $w_{T(x_0)}[\beta^{-1} y_0 - g(T(x_0))] \leq 2$. Also, since $g \in B_w$, we have

$$1 \geq w_{T(x_0)}\left[g\left(T(x_0)\right)\right] \geq w_{T(x_0)}\left[\beta^{-1} y_0\right] - w_{T(x_0)}\left[\beta^{-1} y_0 - g\left(T(x_0)\right)\right] \geq 2,$$

which is a contradiction. Thus our claim holds in both the cases.

Conversely, suppose T is injective and $W \circ T \leq W$. Let $w \in W$. Then choose $w' \in W$ such that $w \circ T \leq w'$. Let $f \in LW_0(X)$ be such that $\|f \circ T\|_{w'} < 1$. Since $C_T(LW_0(X))$ is dense in $LW_0(X)$, it is enough to show that $f \circ T \in \overline{C_T(B_w)}$. For any $w'' \in W$, if we put $K_1 = \{x \in X : w''_x[f(T(x))] \geq 1\}$ and $K_2 = \{x \in X : w_x[f(x)] \geq 1\}$, then K_1 and K_2 are compact sets. Since for every $x \in X, w_{T(x)}[f(T(x))] \leq w'_x[f(T(x))] < 1$, $T(x) \notin K_2$. Since T is continuous, $T(K_1)$ is compact and $T(K_1) \cap K_2 = \varnothing$. Choose $g \in C_b(X)$ such that $0 \leq g \leq 1$, $g(K_2) = 1$, $g(T(K_1)) = 0$. Let $h = f - gf$. Then $h \in LW_0(X)$. Now, for every $x \in X$, we have $w_x[h(x)] = |1 - g(x)| w_x[f(x)] < 1$. This implies that $h \in B_w$. Then for $w'' \in W$, we have $w''_x[f(T(x)) - h(T(x))] = |g(T(x))| w''_x[f(T(x))] < 1$, for all $x \in X$. This implies that $\|f \circ T - h \circ T\|_{w''} \leq 1$. Thus $f \circ T \in \overline{C_T(B_w)}$. Hence the proof of the theorem is completed.

The next theorem is an immediate consequence of Theorem 4.2.38 and Theorem 4.2.43.

Theorem 4.2.44. Let W and T be as in the hypothesis of Theorem 4.2.43. Then $C_T : LW_0(X) \to LW_0(X)$ is a nearly open composition operator if and only if

(i) T is an injection;

(ii) $W \approx W \circ T$;

(iii) for each $f \in L(X)$ such that $w[f]$ is upper semicontinuous for every $w \in W$, and given $w \in W$, $\varepsilon > 0$ and compact set $K \subset X$, $T^{-1}(K) \bigcap \{x \in X : w_x[f(T(x))] \geq \varepsilon\}$ is compact.

4.3 INVERTIBLE AND COMPACT COMPOSITION OPERATORS ON WEIGHTED FUNCTION SPACES

In this section we plan to characterise invertible and compact composition operators on the weighted locally convex spaces of functions. We shall complete the study of this section in two phases. In the first phase we endeavour to obtain invertible composition operators on $CV_0(X)$, $CV_0(X, E)$ and $LW_0(X)$. Compact composition operators on $C(X, E)$ and $CV_0(X, E)$ are studied in the second phase. For more details of these results we refer to ([356], [358], [367]). The characterizations of invertible composition operators are based on the invertibility criterion established in [356] and which is based on the new concept of a bounded below operator on a Hausdorff topological vector space. For our convenience, we shall present this new invertibility criterion and the definition of a bounded below operator.

Defintion 4.3.1. Let E be a Hausdorff topological vector space, and let $A : E \to E$ be a continuous linear operator. Then A is said to be bounded below if for every neighbourhood N of the origin, there exists a neighbourhood M of the origin such that $A(N^c) \subset M^c$, where the symbol 'c' stands for the complement of a neighbourhood.

The following elementary properties of a bounded below operator are proved in [356, Theorem 3.3].

(4.3.1a) A is bounded below $\Rightarrow A$ is one–one.

(4.3.1b) A, B are bounded below $\Rightarrow$ so is their composition $A \circ B$,

(4.3.1c) A, B are such that $A \circ B$ is bounded below $\Rightarrow B$ is bounded below.

Theorem 4.3.2. Let E be a complete Hausdorff topological vector space and let $A : E \to E$ be a continuous linear operator. Then A is invertible if and only if A is bounded below and has dense range.

Proof. If A is invertible, then ran $A = E$ and hence A has dense range. Let N

be a neighbourhood of the origin. Then there exists a neighbourhood M of the origin such that $A^{-1}(M) \subset N$. Claim that $A(N^c) \subset M^c$. Let $y \in A(N^c)$. Then $A^{-1}y \in N^c$. This implies that $A^{-1}y \notin N$. Thus it follows that $y \in M^c$.

Conversely, suppose that A is bounded below and has dense range. First we claim that ran A is a closed subspace of E. Let $\{Ax_\alpha\}$ be a Cauchy net, and let N be a neighbourhood of the origin. Then there exists a neighbourhood M of the origin such that $A(N^c) \subset M^c$. Since $\{Ax_\alpha\}$ is a Cauchy net, for given neighbourhood M of the origin, there exists α_0 such that $Ax_\alpha - Ax_\beta \in M$ for every $\alpha, \beta \geq \alpha_0$. Further, it implies that $A(x_\alpha - x_\beta) \notin M^c$. Thus it follows that $x_\alpha - x_\beta \in N$. This proves that $\{x_\alpha\}$ is a Cauchy net in E and hence it converges. Let $x = \lim_\alpha x_\alpha$. Then $\lim_\alpha Ax_\alpha = Ax$ is in the range of A. This proves that ran A is a closed subspace of E, and hence ran $A = E$. Thus A^{-1} is well defined. Finally, we shall show that A^{-1} is continuous. To this end, let N be a neighbourhood of the origin. Then there is a neighbourhood M of the origin such that $A(N^c) \subset M^c$. Claim that $A^{-1}(M) \subset N$. Let $y \in A^{-1}(M)$. Then $Ay \in M$. This means that $Ay \notin M^c$. Thus it follows that $y \notin N^c$ and consequently $y \in N$. This completes the proof of the theorem.

In order to have the characterization of invertible composition operators, the completeness of the underlying weighted spaces is an essential requirement as we are using the invertibility criterion proved above. The completeness of the weighted spaces has been studied in detail by Summers [381], Prolla [272] and Bierstedt [31]. Some of the results regarding completeness we shall state below. The following result is a comparison test for completeness in weighted spaces proved by Summers [381, Theorem 3.6].

Theorem 4.3.3. Let U be a system of weights on X such that $CU_0(X)$ is complete, and let V be a system of weights on X such that $U \leq V$. Then $CV_0(X)$ is complete.

If E is a complete locally convex Hausdorff space, then the vector-valued analogue of [381, Theorem 3.6] is proved by Prolla [272, Theorem 3]. Further, Bierstedt [31, Proposition 22] has obtained a sufficient condition for completeness in weighted spaces which is presented below.

Proposition 4.3.4. Let X be a $V_{\mathbb{R}}$-Space. Then $CV_0(X, E)$ and $CV_b(X, E)$

are complete for each complete locally convex space E.

In the following theorem we shall establish the characterisation of invertible composition operators on the weighted space $CV_0(X)$.

Theorem 4.3.5. Let V be a system of weights on X such that $CV_0(X)$ is complete, and let $T : X \to X$ be a map such that C_T is a composition operator on $CV_0(X)$. Then C_T is invertible if and only if

(i) $T(X)$ is dense in X;

(ii) T is injective;

(iii) $V_T \leq V$.

Proof. From Theorem 4.3.2, it follows that C_T is invertible if and only if C_T is bounded below and has dense range. Suppose that the conditions (i) to (iii) hold. Let $v \in V$. By (iii), there exists $u \in V$ such that $v \circ T \leq u$. We claim that $C_T(B_v^c) \subset B_u^c$. Let $f \in CV_0(X)$ be such that $f \in B_v^c$. Then we have

$$\begin{aligned} 1 < \|f\|_v &= \sup\{v(x)|f(x)| : x \in X\} \\ &\leq \sup\{u(x)|f \circ T(x)| : x \in X\} = \|C_T f\|_u. \end{aligned}$$

Thus $C_T f \in B_u^c$. This proves that C_T is bounded below. Also, since T is injective, it follows from Theorem 4.2.15 that C_T has dense range. Conversely, suppose that C_T is invertible. Then C_T is bounded below and has dense range. Since C_T is one–one, it follows from Theorem 4.2.14 that $T(X)$ is dense in X. The injectivity of T follows from Theorem 4.2.15 as C_T has dense range. To establish the condition (iii), let $v \in V$. There there is $u \in V$ such that $C_T(B_v^c) \subset B_u^c$. We claim that $v \circ T \leq 2u$. Fix $x_0 \in X$ and let $u(x_0) = \varepsilon$. If $\varepsilon > 0$, then the set $G = \{x \in X : u(x) < 2\varepsilon\}$ is an open neighbourhood of x_0. Therefore, according to [246, Lemma 2, p. 69], there exists $f \in CV_0(X)$ such that $0 \leq f \leq 1$, $f(x_0) = 1$ and $f(X \setminus G) = 0$. Let $g = (2\varepsilon)^{-1} f$. Then $g \in B_u$. Again, there exists $h \in CV_0(X)$ such that $h \circ T = g$ because C_T is onto. This implies that $h \circ T \notin B_u^c$. Further, we have $h \notin B_v^c$. Thus $v(x)|h(x)| \leq 1$, for all $x \in X$. This yields that $v(T(x_0)) \leq 2u(x_0)$. On the other hand, if $\varepsilon = 0$ and $v(T(x_0)) > 0$, then we put $\beta = v(T(x_0)) / 2$ and $G_1 = \{x \in X : u(x) < \beta\}$. Clearly G_1 is an open neighbourhood of x_0. Thus according to [246, Lemma 2, p. 69], there exists $f \in CV_0(X)$ such that $0 \leq f \leq 1$, $f(x_0) = 1$ and $f(X \setminus G_1) = 0$. Let

$g = \beta^{-1}f$. Then $g \in B_u$, and there exists $h \in CV_0(X)$ such that $h \circ T = g$. It now follows that $h \circ T \notin B_u^c$ and $h \notin B_v^c$. Further, it implies that $v(T(x_0)) \leq v(T(x_0))/2$, a contradiction. Thus our claim is established. This completes the proof of the theorem.

Corollary 4.3.6. Let V be a system of weights on X such that $CV_0(X)$ is complete, and let $T : X \to X$ be a map such that C_T is a composition operator on $CV_0(X)$. Then C_T is invertible if and only if C_T is nearly open and $T(X)$ is dense in X.

Proof. It follows from Theorem 4.3.5 and Theorem 4.2.22.

In the following theorem we present a characterization of invertible composition operators on the weighted spaces of vector-valued continuous functions.

Theorem 4.3.7. Let E be a complete locally convex Hausdorff space, and let V ba a system of weights on X such that $CV_0(X, E)$ is complete. Let $T : X \to X$ be a map such that C_T is a composition operator on $CV_0(X, E)$. Then C_T is invertible if and only if

(i) $T(X)$ is dense in X;

(ii) T is injective;

(iii) for every $v \in V$ and $p \in cs(E)$, there exist $u \in V$ and $q \in cs(E)$ such that $v(T(x))p(y) \leq u(x)q(y)$, for every $x \in X$ and $y \in E$.

Proof. Suppose that the conditions (i) through (iii) are true. Let $v \in V$ and $p \in cs(E)$. Then there is $u \in V$ and $q \in cs(E)$ such that $v(T(x)p(y) \leq u(x)q(y)$, for every $x \in X$ and $y \in E$. Claim that $C_T(B^c_{v,p}) \subset B^c_{u,q}$. Let $f \in CV_0(X, E)$ be such that $f \in B^c_{v,p}$. Then we have

$$\begin{aligned} 1 < \|f\|_{v,p} &= \sup\{v(x)p(f(x)) : x \in X\} \\ &\leq \sup\{u(x)q(f \circ T(x)) : x \in X\} = \|C_T f\|_{u,q}. \end{aligned}$$

This proves that $f \circ T \in B_{u,q}$. Thus C_T is bounded below. Also, C_T has dense range, which follows from Theorem 4.2.31. From Theorem 4.3.2 it follows that C_T is invertible. Conversely, if C_T is invertible, then we conclude that C_T is bounded below and has dense range. Again from Theorem 4.2.30 and Theorem 4.2.31, it follows that $T(X)$ is dense in X and T is one–one. To establish the condition (iii), let $v \in V$ and $p \in cs(E)$. Then there are $u \in V$ and $q \in cs(E)$ such that $C_T(B^c_{v,p}) \subset B^c_{u,q}$. Claim that $v(T(x))p(y) \le 2u(x)q(y)$ for every $x \in X$ and $y \in E$. Fix $x_0 \in X$ and $y_0 \in E$. Let $u(x_0)q(y_0) = \varepsilon$. If $\varepsilon > 0$, then the set $G = \{x \in X : u(x) < 2\varepsilon / q(y_0)\}$ is an open neighbourhood of x_0. According to [246, Lemma 2, p. 69] there exists $f \in CV_0(X)$ such that $0 \le f \le 1$, $f(x_0) = 1$ and $f(X \setminus G) = 0$. Define the function $g : X \to E$ as $g(x) = f(x)y_0$, for every $x \in X$. Then $g \in CV_0(X, E)$. Let $h = (2\varepsilon)^{-1}g$. Then $h \in B_{u,q}$. Now, there exists a function $k \in CV_0(X, E)$ such that $k \circ T = h$. Then $k \circ T \notin B^c_{u,q}$ and $k \notin B^c_{v,p}$. From this, we conclude that

$$v(T(x_0))\, p\,(y_0) \le 2u(x_0)\, q(y_0)\,.$$

On the other hand, if $\varepsilon = 0$, then the following three cases arise :

(a) $u(x_0) = 0, \quad q(y_0) \ne 0,$

(b) $u(x_0) \ne 0, \quad q(y_0) = 0,$

(c) $u(x_0) = 0, \quad q(y_0) = 0.$

Assume the case (a) and suppose that $v(T(x_0))p(y_0) > 0$. Let $\beta = v(T(x_0))p(y_0)/2$. Then the set $G = \{x \in X : u(x) < \frac{\beta}{q(y_0)}\}$ is an open neighbourhood of x_0. Therefore, according to [246, Lemma 2, p. 69] there exists $f \in CV_0(X)$ such that $0 \le f \le 1$, $f(x_0) = 1$ and $f(X \setminus G) = 0$. Define the function $g : X \to E$ as $g(x) = f(x)y_0$, for every $x \in X$. Then $g \in CV_0(X, E)$. Let $h = \beta^{-1}g$. Then $h \in B_{u,q}$ and there exists $h_1 \in CV_0(X, E)$ such that $h_1 \circ T = h$. Thus $h_1 \circ T \notin B^c_{u,q}$ and $h_1 \notin B^c_{v,p}$. Further it implies that $v(T(x_0))p(y_0) \le v(T(x_0))p(y_0)/2$, which is a contradiction. Hence in this case our claim is established. Similarly, we can establish our claim in both cases (b) and (c). For detailed proof see [356].

The following corollary is an immediate consequence of Theorem 4.3.7 and Theorem 4.2.35.

Corollary 4.3.8. Let E be a complete locally convex Hausdorff space, and let V be a system of weights on X such that $CV_0(X,E)$ is complete. Let $T : X \to X$ be a map such that C_T is a composition operator on $CV_0(X,E)$. Then C_T is invertible if and only if C_T is nearly open and $T(X)$ is dense in X.

Finally in the first phase of our study we endeavour to extend the theory of invertible composition operators to the weighted spaces of cross-sections. The self-maps inducing composition operators on $LW_0(X)$ and $LW_b(X)$ have been characterised in section 2 of this chapter. From now onwards we assume the following condition .

(4.3.8a) For each $x \in X$, there exists $f_x \in LW_0(X)$ such that $f_x(x) \neq 0$.

Theorem 4.3.9. Let $T : X \to X$ be a continuous map such that for every $x \in X$, $F_{T(x)} \subset F_x$ and T induces a composition operator C_T on $LW_0(X)$. Then C_T is bounded below if and only if T is onto and $W \circ T \leq W$. , *i.e.*, for every $w \in W$, there exists $w' \in W$ such that $w_{T(x)}(y) \leq w'_x(y)$, for every $x \in X$ and $y \in F_{T(x)}$.

Proof. Suppose that the conditions hold. Let $w \in W$. Then there is $w' \in W$ such that $w_{T(x)}(y) \leq w'_x(y)$, for every $x \in X$, and $y \in F_{T(x)}$. Claim that $C_T(B^c_w) \subset B^c_{w'}$. Let $f \in LW_0(X)$ be such that $f \in B^c_w$. Then we have

$$\begin{aligned} 1 < \|f\|_w &= \sup\{w_x[f(x)] : x \in X\} \\ &\leq \sup\{w'_x[f(T(x))] : x \in X\} = \|C_T f\|_{w'}. \end{aligned}$$

This proves that $C_T f \in B^c_{w'}$. Conversely, suppose that C_T is bounded below. Assume on the contrary that T is not onto. Then there is $y \in X$ such that for every $x \in X$, $T(x) \neq y$. Choose $f \in LW_0(X)$ such that $f(y) \neq 0$. Define the function $g : X \to \bigcup\limits_{x \in X} F_x$ as

$$g(x) = \begin{cases} f(x) , & x = y \\ 0 , & \text{otherwise.} \end{cases}$$

Then $g \in LW_0(X)$ and $g \circ T = 0$, which contradicts the fact that C_T is one–one. To show that $W \circ T \leq W$, let $w \in W$. Then there exists $w' \in W$ such that $C_T(B^c_w) \subset B^c_{w'}$. Claim that $w \circ T \leq 2w'$. Fix $x_0 \in X$ and $y_0 \in F_{T(x_0)}$. Let

$w'_{x_0}(y_0) = \varepsilon$. In case $\varepsilon > 0$, we define the function $g : X \to \bigcup_{x \in X} F_x$ as

$$g(x) = \begin{cases} (2\varepsilon)^{-1} y_0 \ , & x = T(x_0) \\ 0 \ , & \text{otherwise.} \end{cases}$$

Then $g \circ T \notin B^c_{w'}$. This implies that $g \notin B^c_w$. Thus, it follows that $w_{T(x_0)}(y_0) \leq 2w'_{x_0}(y_0)$. On the other hand, suppose $\varepsilon = 0$ and $w_{T(x_0)}(y_0) > 0$. Let $\beta = w_{T(x_0)}(y_0) / 2$. Then we define the function $h : X \to \bigcup_{x \in X} F_x$ as

$$h(x) = \begin{cases} \beta^{-1} y_0 \ , & x = T(x_0) \\ 0 \ , & \text{otherwise.} \end{cases}$$

Obviously $h \circ T \notin B^c_{w'}$ and $h \notin B^c_w$. Further, it implies that $w_{T(x_0)}(y_0) \leq w_{T(x_0)}(y_0) / 2$, a contradiction. This completes the proof of the theorem.

Theorem 4.3.10. Let W be a system of weights on X such that $LW_0(X)$ satisfies the condition (4.2.41b). Let $T : X \to X$ be a continuous map such that for every $x \in X$, $F_{T(x)} \subset F_x$ and for given $x_1, x_2 \in X$, there is $w \in W$ such that $w_{T(x_1)} \leq w_{x_1}$ on $F_{T(x_1)}$ and $w_{T(x_2)} \leq w_{x_2}$ on $F_{T(x_2)}$, and T induces the composition operator C_T on $LW_0(X)$. Then C_T has dense range if and only if T is injective.

Proof. It follows from Proposition 4.2.42.

The following result is an immediate consequence of Theorem 4.3.9, Theorem 4.3.10 and Theorem 4.3.2.

Theorem 4.3.11. Let W be a system of weights on X such that $LW_0(X)$ is complete and satisfies the condition (4.2.41b). Let $T : X \to X$ be a continuous map such that for every $x \in X$, $F_{T(x)} \subset F_x$ and for given $x_1, x_2 \in X$ there is $w \in W$ such that $w_{T(x_1)} \leq w_{x_1}$ on $F_{T(x_1)}$ and $w_{T(x_2)} \leq w_{x_2}$ on $F_{T(x_2)}$, and T induces a composition operator C_T on $LW_0(X)$. Then C_T is invertible if and only if T is bijective and $W \circ T \leq W$.

Corollary 4.3.12. Let W and T be as in the hypothesis of Theorem 4.3.11. Then C_T is invertible if and only if C_T is nearly open and T is onto.

Proof. It follows from Theorem 4.3.11 and Theorem 4.2.43.

Under the second phase of the study in this section we shall characterise compact composition operators on the weighted spaces of vector–valued continuous functions. By a compact operator we mean a linear transformation A from a topological vector space E into itself such that the image of every bounded subset of E under A is relatively compact in E. Let B be a subset of $CV_b(X, E)$ (or $CV_0(X, E)$). Then B is bounded if for each $v \in V$ and $p \in cs(E)$, there exists a constant $m_{v,p} > 0$ such that $\|f\|_{v,p} \leq m_{v,p}$, for every $f \in B$. If A is a compact operator on $CV_b(X, E)$ (or $CV_0(X, E)$) and $\{f_n\}$ is a bounded sequence in $CV_b(X, E)$, then there exists a subsequence $\{f_{n_k}\}$ and $g \in CV_b(X, E)$ such that $\{Af_{n_k}\}$ converges to g. If X is a compact Hausdorff space, V consists of constant weights and $E = \mathbb{K}$, then $CV_b(X) = CV_0(X) = C(X)$ with the topology of uniform convergence on X. In this case compact composition operators have been characterised by Kamowitz [164]. Singh and Summers [367] generalised the result of Kamowitz to the spaces of vector–valued continuous functions on a completely regular Hausdorff space. The main theorem which characterises compact composition operators on the weighted spaces $CV_b(X, E)$ (or $CV_0(X, E)$) is a generalisation of a result presented in [168]. First we present the following propositions without proofs which are used in the proof of the main theorem. For details of proof of these propositions we refer to [358].

Proposition 4.3.13. For any system of weights V on X, the following statements are equivalent :

(a) Every $v \in V$ is bounded on X;

(b) For every $y \in E$, $1_y \in CV_b(X, E)$, where $1_y : X \to E$ denotes the constant map given by $1_y(x) = y$, for every $x \in X$;

(c) Every constant function on X induces a composition operator on $CV_b(X, E)$.

Proposition 4.3.14. For any system V of weights on X, the following statements are equivalent :

(a) Every $v \in V$ vanishes at infinity on X (*i.e.*, for given $v \in V$ and $\alpha > 0$, the set $N(v, \alpha) = \{x \in X : v(x) \geq \alpha\}$ is compact in X) ;

(b) For every $y \in E$, $1_y \in CV_0(X, E)$;

(c) Every constant function on X induces a composition operator on $CV_0(X, E)$.

Proposition 4.3.15. Let V be a system of weights on X and let $x \in X$. Then there exists an open set containing x on which each $v \in V$ is bounded.

Now, we shall prove the main theorem which shows that the collection of compact composition operators on the weighted spaces is not too large if the underlying topological space is connected and $\dim E < \infty$.

Theorem 4.3.16. Let E be a non-zero finite dimensional locally convex Hausdorff space and let V be a system of weights on a connected completely regular Hausdorff space X such that every constant function on X induces a composition operator on $CV_b(X, E)$ (or $CV_0(X, E)$). Let $T : X \to X$ be a map such that C_T is a composition operator on $CV_b(X, E)$. Then C_T is compact if and only if the inducing function is constant.

Proof. First suppose that C_T is compact on $CV_b(X, E)$. Let $x_1, x_2 \in X$ and let $y_1 = T(x_1) \neq y_2 = T(x_2)$. Then by Proposition 4.3.15, there exists an open set G containing y_1 on which every $v \in V$ is bounded and $y_2 \notin G$. Since X is completely regular, there exists a continuous function $f : X \to [0, 1]$ such that $f(y_1) = 1$ and $f(x) = 0$, for every $x \notin G$. Let $z \in E$ and $q \in cs(E)$ be such that $q(z) \neq 0$. Now, we define $f_z : X \to E$ as $f_z(x) = f(x)z / q(z)$, for every $x \in X$. Then $f_z \in CV_b(X, E)$. Let $g_n(x) = f^{n-1}(x)f_z(x)$, for every $n \in \mathbb{N}$, where f^n is the product of f with itself n-times. Let $F = \{g_n : n \in \mathbb{N}\}$. Then clearly F is a bounded subset of $CV_b(X, E)$. Since C_T is compact, there exists a subsequence $\{g_{n_k}\}$ of $\{g_n\}$ and $g \in CV_b(X, E)$ such that $\{C_T g_{n_k}\}$ converges to g. Thus it follows that for every $v \in V$ and $p \in cs(\ddot{E})$, $\| g_{n_k} \circ T - g \|_{v,p} \to 0$. This implies that $v(x)p(g_{n_k}(T(x)) - g(x)) \to 0$, for every $x \in X$. From this, it follows that $q \circ g_{n_k}(T(x)) \to q \circ g(x)$, for each $x \in X$. Further, it implies that $q \circ g$ is a characteristic function with $q \circ g\,(x_1) = 1$ and $q \circ g\,(x_2) = 0$, a contradiction since $q \circ g$ is continuous and X is connected. Thus $T(x_1) = T(x_2)$, for every $x_1, x_2 \in X$. This proves that T is a constant function. The converse is obvious. This completes the proof of the theorem.

Remark 4.3.17. If E is infinite dimensional, then the constant functions do not always induce compact composition operators.

4.4 WEIGHTED COMPOSITION OPERATORS ON WEIGHTED FUNCTION SPACES

The main objective of this section is to characterise weighted composition operators on the weighted spaces of continuous functions and the weighted spaces of cross-sections, and then obtain some results pertaining to invertibility and compactness of these operators.

Let X be a compact Hausdorff space, and E be a Banach space. Then clearly $C(X, E)$ is a Banach space under the supremum norm defined as

$$\|f\| = \sup\{\|f(x)\| : x \in X\}.$$

Now, we say that an operator S on $C(X, E)$ has the disjoint support property if $\|f(x)\|\,\|g(x)\| = 0$, for every $x \in X$ implies that $\|Sf(x)\|\,\|Sg(x)\| = 0$, for every $x \in X$. The weighted composition operators $W_{\pi,T}$ on $C(X, E)$ defined by $(W_{\pi,T}f)(x) = \pi(x)f(T(x))$, where T is a self-map of X and for each x, $\pi(x)$ is a bounded linear operator on E, are characterised in term of the disjoint support property. Kamowitz [164] has studied weighted composition operators in the $C(X)$ setting. Jamison and Rajagopalan [153] has extended the results of Kamowitz [164] in two ways. Firstly, by weakening his hypothesis on the self-map T and secondly, by extending his results to the setting of vector-valued functions. In particular, Jamison and Rajagopalan do not require that the map T be continuous everywhere. This is stated as a hypothesis in the paper of Kamowitz [164] and Uhlig [394]. Examples are given in [153], illustrating as to why this continuity is not necessary. One of the examples is as follows : Take $X = [0, 1]$, $E = \mathbb{C}$, and $\pi(x) = 0$, for every $x \in X$, and let $T(x) = 1$ on the rationals and 0 on the irrationals. The resulting operator $W_{\pi,T}$ is continuous and even compact but T is discontinuous at every point. In the lattice setting such operators have been studied by Feldman and Porter [110] as well as by Arendt and Hart [12]. However, in [153] the authors do not assume any lattice structure. In the scalar case, characterisation of such operators do need lattice structure as is shown in [11] that if S is a bounded linear operator on $C(X)$ with the disjoint support property, then $Sf = h.f(T)$, where $h = S(1)$ and T is a self map of X which is continuous on $\{x \in X : h(x) \neq 0\}$.

Now, we shall present a characterisation of weighted composition operators obtained by Jamison and Rajagopalan [153], which generalises the results of Kamowitz [164] and Arendt [11].

Theorem 4.4.1. Let $S : C(X, E) \to C(X, E)$ be a bounded linear operator. Then S has the disjoint support property if and only if there is a self-map T of X and a strongly continuous operator-valued function π defined on X with values in $B(E)$ such that T is continuous on the set $X \setminus W$, where $N = \{x \in X : \pi(x) = 0\}$ and $Sf(x) = \pi(x)f(T(x))$, for every $f \in C(X, E)$.

Proof. If S is a bounded linear operator on $C(X, E)$ with the given conditions, then clearly it satisfies the disjoint support property. On the other hand, suppose that S is a bounded linear operator on $C(X, E)$ that satisfies the disjoint support property. Let $y \in E$ and $f \in C(X, \mathbb{R})$. Define the function $f_y(x) = f(x)y$, for every $x \in X$. Then $f_y \in C(X, E)$. By 1_y we mean the function defined by $1_y(x) = y$, for every $x \in X$. Now, we fix $y \in E$ and $y^* \in E^*$. We define the map S' on $C(X, \mathbb{R})$ as $S'f(x) = y^*((Sf_y)(x))$, for every $x \in X$. Clearly S' is a bounded linear operator on $C(X, \mathbb{R})$ such that it satisfies the disjoint support property. From the scalar case [11], it now follows that there exist a self map T_{y,y^*} and a scalar-valued function h_{y,y^*} such that for each $f \in C(X, \mathbb{R})$,

$$S'f(x) = h_{y,y^*}(x)\, f\left(T_{y,y^*}(x)\right).$$

If we put $f = 1$ (the constant one function), then we see that for every x, $h_{y,y^*}(x) = y^*((S1_y)(x))$. Thus h_{y,y^*} is weak*-continuous for fixed x and y. Let $y^* \in E^*$ be fixed and for fixed $x \in X$ and $y \in E$, we define $\pi(x)y = (S1_y)(x)$. Thus for each x, $\pi(x)$ is a linear operator such that $\| \pi(x) \| \leq \| S \|$ and the map $x \to \pi(x)$ is continuous in the strong operator topology. The proof can be completed by showing that T_{y,y^*} is independent of y and y^*, and is continuous on the set

$\{x \in X : \pi(x) \neq 0\}$. For details of proof we refer to [153].

In the following example it is made clear that the map $x \to \pi(x)$ is only required to be continuous in the strong operator topology.

Example 4.4.2. Let E be a separable infinite dimensional Hilbert space and let $X = \{0, 1, \frac{1}{2}, \frac{1}{3}, \ldots\}$ with subspace topology. Let $\{e_i\}$ denote an orthonormal basis for the Hilbert space E. For $x \in X$, we define the linear transformation $\pi(x)$ as follows :

$$\pi(0) = 0 \quad \text{and} \quad \pi(1/k)y = \langle y, e_k \rangle e_1.$$

Take T as the identity map *i.e.*, $T(x) = x$, for every $x \in X$. Now, if $f \in C(X, E)$ then the weighted composition operator is given by

$$Sf(x) = \begin{cases} 0, & x = 0 \\ \pi(1/k)f(1/k) = \langle f(1/k), e_k \rangle e_1, & \text{for } x = 1/k. \end{cases}$$

It is clear that the map $x \to \pi(x)$ is continuous in the strong operator topology. But $\|\pi(1/n)\| = 1$ for all n and $\pi(0) = 0$ implies that the mapping $x \to \pi(x)$ is not continuous in the uniform operator topology.

Now, we shall present a generalization of Theorem 4.4.1 which is obtained by Singh and Singh [365] by taking E to be a locally convex Hausdorff space. In this case $C(X, E)$ becomes a locally convex Hausdorff space with respect to the topology induced by a family $\{\|\cdot\|_p : p \in cs(E)\}$ of seminorms on $C(X, E)$, where $\|f\|_p = \sup\{p(f(x)) : x \in X\}$. By $B(E)$ we mean the locally convex space of all continuous linear operators on E equipped with the strong operator topology. Let f and $g \in C(X, E)$. The we say that f and g have the disjoint support property if $p(f(x))p(g(x)) = 0$, for each $x \in X$ and $p \in cs(E)$ and we write $f \perp g$. We say that a linear transformation $S : C(X, E) \to C(X, E)$ has the disjoint support property if $f \perp g$ implies that $Sf \perp Sg$.

Example 4.4.3. For any $k \in \mathbb{N}$, define

$$f_k(n) = \begin{cases} e_n, & \text{if } n \leq k \\ 0, & \text{otherwise} \end{cases}$$

and

$$g_k(n) = \begin{cases} e_n, & \text{if } \ n > k \\ 0 \ , & \text{otherwise.} \end{cases}$$

where $e_n = \{0, 0, \ldots, \hat{1}, 0, \ldots\}$, cap denotes the n^{th} place. Then $f_k, g_k \in C(\mathbb{N}, \ell^2)$ and $f_k \perp g_k$ for each $k \in \mathbb{N}$. If we define $S : C(\mathbb{N}, \ell^2) \to C(\mathbb{N}, \ell^2)$ as $Sf(n) = f(n+1)$ for each $f \in C(\mathbb{N}, \ell^2)$ and $n \in \mathbb{N}$, then S has the disjoint support property.

Theorem 4.4.4. Let $S : C(X, E) \to C(X, E)$ be a continuous linear operator. Then S has the disjoint support property if and only if $S = W_{\pi,T}$, for some self map T on X and $\pi \in C(X, B(E))$ such that T is continuous on the set $N^\pi = \{x \in X \ : \ \pi(x) \neq 0\}$.

Proof. Suppose that $S = W_{\pi,T}$, for some π and T. Let $f, g \in C(X, E)$ be such that $f \perp g$. Then for every $x \in X$ and $p \in cs(E)$, we have

$$\begin{aligned} p\big((Sf)(x)\big)p\big((Sg)(x)\big) &= p\big(\pi(x)f(T(x))\big)p\big(\pi(x)g(T(x))\big) \\ &= q\big(f(y)\big)q\big(g(y)\big), \\ &= 0, \quad \text{since } f \perp g. \end{aligned}$$

Thus $Sf \perp Sg$ whenever $f \perp g$. This shows that S has the disjoint support property. Conversely, if S is so, then for $z \in E$ and $z^* \in E^*$, we define the operator A_{z,z^*} on $C(X)$ as $(A_{z,z^*}f)(x) = z^*((Sf_z)(x))$, where f_z is defined by $f_z(x) = z$, for every $x \in X$. From [365, Proposition 3.2], it follows that A_{z,z^*} has the disjoint support property and further Proposition 3.3 of [365] gives the existence of $\pi \in C(X, B(E))$. Again, from Proposition 3.1 of [365] we have a self map T_{z,z^*} on X and $h_{z,z^*} \in C(X)$ such that

$$(A_{z,z^*}f)(x) = h_{z,z^*}(x)f\big(T_{z,z^*}(x)\big).$$

Now, if we take $f = 1$ (the constant one function), then

$$(A_{z,z^*}1)\ (x) = h_{z,z^*}(x), \quad \text{for every } x \in X.$$

Further, it implies that

$$z^*(\pi(x)z) = z^*\big((S1_z)(x)\big) = h_{z,z^*}(x), \quad \text{for every } x \in X.$$

From Proposition 3.4 of [365] and the last equation, it follows that $(A_{z,z^*}f)(x) = h_{z,z^*}(x)f(T(x))$, where $T(x)$ is the value of $T_{z,z^*}(x)$ for all $x \in X$ as T_{z,z^*} is independent of z and z^*. Thus for each $f \in C(X)$, $z \in E$, $z^* \in E^*$ and $x \in N^c$, we have

$$z^*\big((S1_z)\,(x)\big) = z^*\big((\pi(x)z)\;f(T(x))\big) = z^*\big(\pi(x)f_z(T(x))\big).$$

Thus it implies that

$$Sf_z(x) = \begin{cases} \pi(x)(f_z(T(x))) & , \quad x \in N^{\pi} \\ 0 & , \quad x \in N, \end{cases}$$

where $\pi \in C(X, B(E))$ and the self map T is continuous on the set N^{π}. Also, the above equality is valid for every $f \in C(X, E)$ because the closed linear span of the set $\{f_z : z \in E,\ f \in C(X)\}$ is dense in $C(X, E)$. This completes the proof of the theorem.

From the results proved so far in this section it is clear that the characterisations of continuous linear operators which are weighted composition operators on $C(X)$ and $C(X, E)$ have been presented by different authors. But in case of the weighted spaces of continuous functions characterisation of such operators has not been obtained so far. We are continuing our efforts in this direction. However, we have characterised weighted composition operators on the weighted spaces of continuous functions and the weighted spaces of cross-sections in terms of the inducing functions.

Before establishing the desired characterisations, we shall present the following theorem of [353].

Theorem 4.4.5. Let E be a locally convex algebra and let $\pi : X \to E$ a continuous function. Then $M_{\pi} : CV_0(X, E) \to CV_0(X, E)$ is a multiplication operator if and only if for every $v \in V$ and $p \in cs(E)$, there exist $u \in V$ and $q \in cs(E)$ such that $v(x)p(\pi(x)y) \le u(x)q(y)$ for every $x \in X$ and $y \in E$.

Proof. Let $v \in V$ and $p \in cs(E)$. Then there exist $u \in V$ and $q \in cs(E)$ such that $v(x)p(\pi(x)y) \le u(x)q(y)$, for every $x \in X$ and $y \in E$. Let $\{f_{\alpha}\}$ be a net in $CV_0(X, E)$ such that for every $v \in V$ and $p \in cs(E)$, $\|f_{\alpha}\|_{v,p} \to 0$. Then

$$\begin{aligned}\|\pi f_\alpha\|_{v,p} &= \sup\{v(x)p(\pi(x)f_\alpha(x)) : x \in X\} \\ &\le \sup\{u(x)q(f_\alpha(x)) : x \in X\} = \|f_\alpha\|_{u,q} \to 0.\end{aligned}$$

This proves that M_π is continuous at the origin. Hence M_π is a multiplication operator on $CV_0(X,E)$. Conversely, suppose that M_π is a multiplication operator on $CV_0(X,E)$. Let $v \in V$ and $p \in cs(E)$. Then there exist $u \in V$ and $q \in cs(E)$ such that $M_\pi(B_{u,q}) \subset B_{v,p}$. Now, we claim that $v(x)p(\pi(x)y) \le 2u(x)q(y)$, for every $x \in X$ and $y \in E$. Fix $x_0 \in X$ and $y_0 \in E$. Set $u(x_0)q(y_0) = \varepsilon$. In case $\varepsilon > 0$, the set $G = \{x \in X : u(x)q(y_0) < 2\varepsilon\}$ is an open neighbourhood of x_0. According to [246, Lemma 2, p. 69] there exists $f \in CV_0(X)$ such that $0 \le f \le 1$, $f(x_0) = 1$ and $f(X \setminus G) = 0$. Now, we define the function $g : X \to E$ as $g(x) = f(x)y_0$, for every $x \in X$. Then clearly $g \in CV_0(X,E)$. If we put $h = (2\varepsilon)^{-1}g$, then $h \in B_{u,q}$ and $\pi h \in B_{v,p}$. Thus it follows that

$$v(x)p(\pi(x)h(x)) \le 1, \text{ for every } x \in X.$$

Further, it implies that

$$v(x_0)p(\pi(x_0)y_0) \le 2u(x_0)q(y_0).$$

On the other hand, suppose $u(x_0)q(y_0) = 0$. Then the following three cases arise :

(i) $u(x_0) = 0,\ q(y_0) \ne 0,$

(ii) $u(x_0) \ne 0,\ q(y_0) = 0,$

(iii) $u(x_0) = 0,\ q(y_0) = 0$.

Assume that case (i) holds. On the contrary suppose that $v(x_0)p(\pi(x_0)y_0) > 0$. Put $\varepsilon = v(x_0)p(\pi(x_0)y_0)/2$. Consider the set $G = \{x \in X : u(x)q(y_0) < \varepsilon\}$. Then G is an open neighbourhood of x_0. Therefore according to [246, Lemma 2, p. 69] there exists $f \in CV_0(X)$ such that $0 \le f \le 1$, $f(x_0) = 1$ and $f(X \setminus G) = 0$. Define the function $g : X \to E$, as $g(x) = f(x)y_0$ for every $x \in X$. Then $g \in CV_0(X,E)$. Let $h = \varepsilon^{-1}g$. Then $h \in B_{u,q}$ and $\pi h \in B_{v,p}$. From this, it follows that

$$v(x_0)p(\pi(x_0)y_0) \le \frac{v(x_0)p(\pi(x_0)y_0)}{2},$$

which is impossible. Thus in this case our claim is established. We can easily settle our claim by putting $N_1 = \{x \in X : u(x) < \varepsilon + u(x_0)\}$ and $N_2 = \{x \in X : u(x) < \varepsilon\}$

as the neighbourhoods of x_0 in case of (ii) and (iii) respectively. The proof can be completed in the same way as we did in case (i).

Remark 4.4.6. In Theorem 4.4.5, if we replace the weighted space $CV_0(X, E)$ by $CV_b(X, E)$ even then the result is true. In case $E = \mathbb{C}$, Theorem 4.4.5 reduces to Theorem 2.1 of [348].

Now, we shall characterise those mappings $T : X \to X$ and $\pi : X \to E$, which induce weighted composition operators on $CV_b(X, E)$.

Theorem 4.4.7. Let E be a locally convex algebra with identity e, let $T : X \to X$ and $\pi : X \to E$ be continuous functions. Then $W_{\pi,T} : CV_b(X, E) \to CV_b(X, E)$ is a weighted composition operator if and only if for every $v \in V$ and $p \in cs(E)$, there exist $u \in V$ and $q \in cs(E)$ such that $v(x)p(\pi(x)y) \leq u(T(x))q(y)$, for every $x \in X$ and $y \in E$.

Proof. The proof is analogous to the proof of Theorem 4.4.5.

Remark 4.4.8. In Theorem 4.4.7 if we take $\pi : X \to E$ as the constant function defined by $\pi(x) = e$, for every $x \in X$, then it reduces to Theorem 4.2.26 which characterises composition operators on $CV_b(X, E)$ induced by T. Again, in the above theorem if we take $T : X \to X$ as the identity map, then Thorem 4.4.7 reduces to Theorem 4.4.5 which characterises multiplication operators on $CV_b(X, E)$ induced by π. In case $E = \mathbb{C}$, Theorem 4.4.7 characterises weighted composition operators on $CV_b(X)$ [364].

Remark 4.4.9. Let $T : X \to X$ and $\pi : X \to \mathbb{C}$ be continuous mappings. Then the pair (π, T) induces a weighted composition operator $W_{\pi,T}$ on $CV_0(X, E)$ only if the following conditions are satisfied :

(i) for each $v \in V$ and $p \in cs(E)$, there exist $u \in V$ and $q \in cs(E)$ such that $v(x)|\pi(x)|p(y) \leq u(T(x))q(y)$, for all $x \in X$ and $y \in E$;

(ii) for each $v \in V$, $\varepsilon > 0$ and compact set $K \subset X$, $T^{-1}(K) \cap N(v|\pi|, \varepsilon)$ is compact, where $N(v|\pi|, \varepsilon) = \{x \in X \ : \ v(x)|\pi(x)| \geq \varepsilon\}$.

Also, note that we are still investigating the necessary and sufficient conditions for the continuous maps $T : X \to X$ and $\pi : X \to E$ to induce weighted composition operators on $CV_0(X, E)$. In case $E = \mathbb{C}$, we have the solution in the following theorem which we shall record without proof.

Theorem 4.4.10 [364]. Let $T : X \to X$ and $\pi : X \to \mathbb{C}$ be continuous mappings. Then the following are equivalent :

(a) π and T induces a weighted composition operator $W_{\pi,T}$ on $CV_0(X)$;

(b) π and T induces a weighted composition operator $W_{\pi,T}$ on $CV_b(X)$ and $CV_0(X)$ is invariant under $W_{\pi,T}$;

(c) (i) $V|\pi| \le V_T$, and (ii) for each $v \in V$, $\varepsilon > 0$ and compact subset K of X, the set $T^{-1}(K) \cap N(v|\pi|, \varepsilon)$ is compact;

(d) (i) $V|\pi| \le V_T$ and (ii) for each $v \in V$, $\varepsilon > 0$ and $u \in V$ such that $v|\pi| \le u \circ T$, the set $T^{-1}(K) \cap N(v|\pi|, \varepsilon)$ is compact whenever K is a compact subset of $N(u, \varepsilon)$.

Now, we shall present the characterisation of weighted composition operators on $CV_b(X, E)$ induced by operator-valued functions π on X and self-maps T on X.

Theorem 4.4.11. Let E be a locally convex Hausdorff space such that each convergent net in E is bounded. Let $\pi \in C(X, B(E))$ and $T \in C(X, X)$. Then $W_{\pi,T}$ is a weighted composition operator on $CV_b(X, E)$ if and only if for every $v \in V$ and $p \in cs(E)$, there exist $u \in V$ and $q \in cs(E)$ such that $v(x)p(\pi_x(y)) \le u(T(x))q(y)$, for every $x \in X$ and $y \in E$.

Proof. It follows from Theorem 3.2 of [363].

Remark 4.4.12. If $\pi \in C(X, B(E)$ and $T \in C(X, X)$ satisfy the (sufficient) condition of Theorem 4.4.11, then $W_{\pi,T}$ is a weighted composition operator on $CV_0(X, E)$ if only $CV_0(X, E)$ happens to be invariant under $W_{\pi,T}$. But this is not the case as can be seen from the example preceding Theorem 2.3 of [368]. So, we need to have some more conditions on π and T so that $W_{\pi,T}$ is a weighted composition operator on $CV_0(X, E)$. These conditions are still under investigation.

In the following theorems we shall present very broad generalisations of these operators on the weighted spaces of functions. In particular, we shall characterise the self maps $T : X \to X$ and the operator–valued maps $\pi : X \to \bigcup_{x \in X} (L(F_x))$ which induce the weighted composition operators on the weighted spaces of cross–sections $LW_0(X)$ and $LW_b(X)$. Here $\pi : X \to \bigcup_{x \in X} (L(F_x))$ is a map such that $\pi(x) \in L(F_x)$, for every $x \in X$, where $L(F_x)$ denotes the vector space of all linear transformations from F_x to itself.

Theorem 4.4.13. Let $T : X \to X$ be a map such that for every $x \in X$, $F_{T(x)} \subset F_x$ and $\pi : x \to \bigcup_{x \in X} (L(F_x))$ let $\pi : X \to \bigcup_{x \in X} (L(F_x))$ be a mapping such that for every $x \in X$, $\pi(x) \in L(F_x)$. Then $W_{\pi,T} : LW_b(X) \to LW_b(X)$ is a weighted composition operator if and only if $W \pi \leq W \circ T$, *i. e.*, for every $w \in W$, there exists $w' \in W$ such that $w_x(\pi_x(y)) \leq w'_{T(x)}(y)$, for every $x \in X$ and $y \in F_{T(x)}$.

Proof. It is analogous to the proof of Theorem 4.2.37.

Theorem 4.4.14. Let $T : X \to X$ be a continuous map such that for every $x \in X$, $F_{T(x)} \subset F_x$, and let π be the same map as in Theorem 4.4.13. Then the following are equivalent:

(a) π and T induces a weighted composition operator $W_{\pi,T}$ on $LW_0(X)$;

(b) (i) $W.\pi \leq W \circ T$, (ii) for each $f \in L(X)$ such that $w[f]$ is upper semicontinuous for every $w \in W$ and given $w \in W$, $\varepsilon > 0$, and compact set $K \subset X$, $T^{-1}(K) \bigcap \{x \in X : w_x[\pi_x(f(T(x))] > \varepsilon\}$ is compact;

(c) (i) $W.\pi \leq W \circ T$, (ii) for each $f \in L(X)$ such that $w[f]$ is upper semicontinuous for every $w \in W$ and given $w \in W$, $\varepsilon > 0$, and $w' \in W$ such that $w.\pi \leq w' \circ T$, $T^{-1}(K) \bigcap \{x \in X : w_x[\pi_x(f(T(x))] \geq \varepsilon\}$ is compact whenever K is a compact subset of $\{x \in X : w'_x[f(x)] \geq \varepsilon\}$;

(d) π and T induces a weighted composition operator $W_{\pi,T}$ on $LW_b(X)$ and $LW_0(X)$ is invariant under $W_{\pi,T}$.

Proof. The proof is analogous to the proof of Theorem 4.2.38.

Remark 4.4.15. If we define the function $\pi : X \to \bigcup_{x \in X} (L(F_x))$ as $\pi(x) = I$, the identity transformation, for every $x \in X$, then Theorem 4.4.13 and Theorem 4.4.14

reduce to Theorem 4.2.37 and Theorem 4.2.38 respectively. In case we define $T : X \to X$ as the identity map, then both Theorem 4.4.13 and Theorem 4.4.14 reduce to the following result which characterises the multiplication operators on $LW_0(X)$.

Theorem 4.4.16 [350, Theorem 3.4]. Let $\pi : X \to \bigcup_{x \in X}(L(F_x))$ be a map such that for every $x \in X$, $\pi(x) \in L(F_x)$. Then $M_\pi : LW_0(X) \to LW_0(X)$ is a multiplication operator if and only if $W\pi \leq W$.

Remark 4.4.17. We note that if $\pi \in C(X, B(E))$ induces a multiplication operator on $CV_i(X, E)$ and $T \in C(X, X)$ induces a composition operator C_T on $CV_i(X, E)$, then the pair (π, T) induces a weighted composition operator $W_{\pi,T}$ on $CV_i(X, E)$, where $i \in \{0, b\}$. But the converse argument may not be true. For example, take $\pi(x) = 0$, for each $x \in X$ and T to be any self-map on X which does not induce composition operator on $CV_b(X, E)$ (or $CV_0(X, E)$), then obviously $W_{\pi,T}$ is a weighted composition operator on $CV_b(X, E)$ as well as on $CV_0(X, E)$. So, it is remarkable to observe that even if one of π or T does not induce the corresponding operator, π and T taken together may still induce a weighted composition operator. This we shall further illustrate in the following examples :

Example 4.4.18. Let $X = \mathbb{N}$ with the discrete topology, and let $V = \{\lambda v : \lambda \geq 0\}$, where $v(n) = n$, for every $n \in \mathbb{N}$. Let $E = \ell^2$, the Hilbert space of all square summable sequences of complex numbers. Define $\pi : \mathbb{N} \to B(\ell^2)$ as $\pi(n) = \frac{1}{n} U^n$, for every $n \in \mathbb{N}$, where U denotes the unilateral shift operator on ℓ^2. Then π is a continuous bounded operator–valued function and hence it induces a multiplication operator M_π on $CV_b(X, E)$. Let $T : \mathbb{N} \to \mathbb{N}$ be defined as

$$T(n) = \begin{cases} \sqrt{n}, & \text{if } n \text{ is a perfect square} \\ n\ , & \text{otherwise.} \end{cases}$$

Obviously T does not induce a composition operator on $CV_b(X, E)$. Since for every $v \in V$, there is $u \in V$ such that $v(n) \| \pi_n(y) \| \leq u(T(n)) \| y \|$, for every $n \in \mathbb{N}$ and $y \in \ell^2$. Thus in view of Theorem 4.4.7, it follows that $W_{\pi,T}$ is a weighted composition operator on $CV_b(X, E)$.

Example 4.4.19. Let X, V and E be the same as defined in Example 4.4.18. Let $\pi : \mathbb{N} \to B(\ell^2)$ be defined as $\pi(n) = nU^n$, for every $n \in \mathbb{N}$. Let $T : \mathbb{N} \to \mathbb{N}$ be defined as $T(n) = n^2$, for every $n \in \mathbb{N}$. Then π does not induce a multiplication operator M_π on $CV_b(X, E)$ and T defines a composition operator C_T on $CV_b(X, E)$. But the pair (π, T) induces a weighted composition operator $W_{\pi,T}$ on $CV_b(X, E)$.

Now our efforts are to present the characterisations of invertible weighted composition operators on the weighted spaces of continuous functions and the weighted spaces of cross-sections. The following theorem characterises the invertible weighted composition operators induced by the vector-valued mappings and the self-maps on the weighted spaces of vector-valued continuous functions.

Theorem 4.4.20. Let E be a complete locally multiplicatively convex commutative algebra, let $csm(E)$ be the set of all continuous submultiplicative seminorms on E, and let V be a system of weights on X such that $CV_0(X, E)$ is complete. Let $\pi : X \to E$ and $T : X \to X$ be continuous mappings such that $W_{\pi,T} \in B(CV_0(X, E))$. Then $W_{\pi,T}$ is invertible if and only if

(i) $\pi(x) \neq 0$, for every $x \in X$;

(ii) T is injective;

(iii) $T(X)$ is dense in X;

(iv) for every $v \in V$ and $p \in csm(E)$, there exist $u \in V$ and $q \in csm(E)$ such that $v(T(x))p(y) \le u(x)q(\pi(x)y)$, for every $x \in X$ and $y \in E$.

Proof. Suppose that the conditions (i) to (iv) are true. In view of Theorem 4.3.2, it is enough to show that the weighted composition operator $W_{\pi,T}$ is bounded below and has dense range. Since $W_{\pi,T}(CV_0(X, E))$ is a vector subspace of $CV_0(X, E)$ which is a module over the self-adjoint algebra $C_b(X)$ which separates the points of X and in view of (4.2.d), $W_{\pi,T}(CV_0(X, E))$ is everywhere different from zero, the density of $W_{\pi,T}(CV_0(X, E))$ in $CV_0(X, E)$ now follows from the vector-valued analogue of the instance of the bounded case of the strict weighted approximation problem [383]. Let $v \in V$ and $p \in csm(E)$. Then by (iv), there exist $u \in V$ and $q \in csm(E)$ such that $v(T(x))p(y) \le u(x)q(\pi(x)y)$, for every $x \in X$ and $y \in E$. Claim that $W_{\pi,T}(B^c_{v,p}) \subset B^c_{u,q}$. Let $f \in CV_0(X, E)$ be such that $f \in B^c_{v,p}$. Then we have

$$
\begin{aligned}
1 < \|f\|_{v,p} &= \sup\{v(x)p(f(x)) : x \in X\} \\
&\le \sup \overline{\{u(x)q(\pi(x)f(T(x))) : x \in X\}} \\
&= \sup\{u(x)q(\pi(x)(f \circ T)(x)) : x \in X\} \\
&= \|W_{\pi,T}\, f\|_{u,q}.
\end{aligned}
$$

This shows that $W_{\pi,T} f \in B^c_{u,q}$. Thus $W_{\pi,T}$ is bounded below. Conversely, if $W_{\pi,T}$ is invertible, then we conclude that $W_{\pi,T}$ is bounded below and has dense range. To establish the condition (i), assume on the contrary that for some $x_0 \in X$, $\pi(x_0) = 0$. Choose $g \in CV_0(X, E)$ such that $g(x_0) \neq 0$. Again, we choose $v \in V$ and $p \in csm(E)$ such that $v(x_0) \ge 1$ and $p(g(x_0)) = 1$. Since $g \in CV_0(X, E)$, there exists $f \in CV_0(X, E)$ such that $\|\pi.f \circ T - g\|_{v,p} < \frac{1}{2}$. This further implies that $v(x)p(\pi(x)f(T(x)) - g(x)) < \frac{1}{2}$, for every $x \in X$. Thus we have $v(x_0)p(\pi(x_0)f(T(x_0)) - g(x_0)) < \frac{1}{2}$, which is impossible. Hence $\pi(x) \neq 0$, for every $x \in X$. To show that T is injective, let $x_1, x_2 \in X$ be such that $x_1 \neq x_2$. Then $\pi(x_1) \neq 0$ and $\pi(x_2) \neq 0$. Choose $g \in CV_0(X, E)$ such that $g(x_1) = 0$ and $g(x_2) \neq 0$. Further we choose $v \in V$ and $p \in csm(E)$ such that $v(x_1) \ge 1$, $v(x_2) \ge 1$ and $p(\pi(x_1)) \neq 0$, $p(\pi(x_2)) \neq 0$, and $p(g(x_2)) \neq 0$. Let $\varepsilon = p(\pi(x_1)) + p(\pi(x_2))$, and let $h = \dfrac{1}{p(\pi(x_1))p(g(x_2))}\, g$. Then $h \in CV_0(X, E)$. Now, there exists $f \in CV_0(X, E)$ such that $\|W_{\pi,T}\, f - h\|_{v,p} < \frac{1}{2\varepsilon}$. Further, it implies that $v(x)p(\pi(x)f(T(x)) - h(x)) < \frac{1}{2\varepsilon}$, for every $x \in X$. From this it follows that $p(\pi(x_1)f(T(x_1))) < \frac{1}{2\varepsilon}$ and $\left|p(\pi(x_2)f(T(x_2))) - \dfrac{1}{p(\pi(x_1))}\right| < \frac{1}{2\varepsilon}$. This further yields that $p(\pi(x_2)\pi(x_1)f(T(x_1))) < \frac{1}{2}$ and $|p(\pi(x_1)\pi(x_2)f(T(x_2))) - 1| < \frac{1}{2}$. Thus we conclude that $T(x_1) \neq T(x_2)$. Since $W_{\pi,T}$ is one–one, therefore if $f \in CV_0(X, E)$ is such that $W_{\pi,T} f = 0$, then $f = 0$. Thus $f \circ T = 0$ implies that $f = 0$. From Theorem 4.2.30, it follows that $T(X)$ is dense in X. Now, we shall establish the condition (iv). Let $v \in V$ and $p \in csm(E)$. Then there exist $u \in V$ and $q \in csm(E)$ such that $W_{\pi,T}(B^c_{v,p}) \subset B^c_{u,q}$. Now, we claim that $v(T(x))p(y) \le 2u(x)q(\pi(x)y)$, for every $x \in X$ and $y \in E$. Fix $x_0 \in X$ and $y_0 \in E$. Let $\varepsilon = u(x_0)q(\pi(x_0)y_0)$. If $\varepsilon > 0$, then the set $G = \{x \in X \ : \ u(x) < 2\varepsilon / q(\pi(x_0)y_0)\}$ is an open neighbourhood of x_0. Therefore according to [246, Lemma 2, p. 69], there exists $f \in CV_0(X)$

such that $0 \le f \le 1$, $f(x_0) = 1$ and $f(X \setminus G) = 0$. Define the function $g : X \to E$ as $g(x) = f(x)y_0$, for every $x \in X$. Then $g \in CV_0(X, E)$. Let $g_1(x) = \pi(x_0)\,(2\varepsilon)^{-1} g(x)$, for every $x \in X$. Then obviously $g_1 \in B_{u,q}$. Since $W_{\pi,T}$ is onto, there exists $h \in CV_0(X, E)$ such that $\pi . h \circ T = g_1$. This yields that $\pi . h \circ T \notin B^c_{u,q}$. From this it follows that $h \notin B^c_{v,p}$. Thus we have $v(x)p(h(x)) \le 1$, for every $x \in X$. Now, it readily follows that

$$v\big(T(x_0)\big)p(y_0) \le 2u(x_0)q\big(\pi(x_0)y_0\big).$$

On the other hand, if $\varepsilon = 0$, then the following three cases arise :

(a) $u(x_0) = 0, \qquad q(\pi(x_0)y_0) \neq 0,$

(b) $u(x_0) \neq 0, \qquad q(\pi(x_0)y_0) = 0,$

(c) $u(x_0) = 0, \qquad q(\pi(x_0)y_0) = 0.$

Suppose that case (a) holds. On the contrary assume that $v(T(x_0))p(y_0) > 0$. Let $\beta = v(T(x_0)p(y_0)/2$. Then the set $G = \left\{x \in X \ : \ u(x) < \beta/q\big(\pi(x_0)y_0\big)\right\}$ is an open neighbourhood of x_0. According to [246, Lemma 2, p. 69], there exists $f \in CV_0(X)$ such that $0 \le f \le 1$, $f(x_0) = 1$ and $f(X \setminus G) = 0$. Again, define the function $g : X \to E$ as $g(x) = f(x)y_0$, for every $x \in X$. Then $g \in CV_0(X, E)$. If we put $g_1 = \beta^{-1}\pi(x_0)g$, then obviously $g_1 \in B_{u,q}$. Also there exists $h \in CV_0(X, E)$ such that $W_{\pi,T}h = g_1$. Thus we have $\pi . h \circ T \notin B^c_{u,q}$ and $h \notin B^c_{v,p}$. From this we conclude that $v(T(x_0))p(y_0) \le \dfrac{v(T(x_0))p(y_0)}{2}$, which is a contradiction. Thus our claim is established in this case as well. In the same way we can establish our claim for other cases. With this the proof of the theorem is completed.

Remark 4.4.21. In the above theorem if we take $T : X \to X$ as the identity map, then it reduces to Theorem 4.9 of [355] which characterises invertible multiplication operators on $CV_0(X, E)$. In case $E = \mathbb{C}$, the above theorem reduces to [355, Theorem 3.5] which characterises invertible weighted composition operators on $CV_0(X)$.

Now, our efforts are to generalize the theory of invertible weighted composition operators on the weighted spaces $LW_0(X)$ and $LW_b(X)$ of cross-sections. For the following results we assume the requirement that for each $x \in X$ there exists

$f_x \in LW_0(X)$ such that $f_x(x) \neq 0$.

Theorem 4.4.22. Let $T : X \to X$ be a continuous map such that for every $x \in X$, $F_{T(x)} \subset F_x$ and let $\pi : X \to \mathbb{C}$ be a map such that $W_{\pi,T} \in B(LW_0(X))$. Then $W_{\pi,T}$ is bounded below if and only if T is onto and $W \circ T \leq W.\pi$, *i.e.*, for every $w \in W$, there exists $w' \in W$ such that $w_{T(x)}(y) \leq w'_x(\pi(x)y)$, for every $x \in X$ and $y \in F_{T(x)}$.

Proof. Suppose that the conditions hold, and let $w \in W$. Then there exists $w' \in W$ such that $W_{T(x)}(y) \leq w'_x(\pi(x)y)$, for every $x \in X$ and $y \in F_{T(x)}$. We claim that $W_{\pi,T}(B^c_w) \subset B^c_{w'}$. Let $f \in LW_0(X)$ be such that $f \in B^c_w$. Then we have

$$\begin{aligned} 1 < \|f\|_w &= \sup\{w_x[f(x)] : x \in X\} \\ &\leq \sup\{w'_x[\pi(x)f(T(x))] : x \in X\} \\ &= \|\pi . f \circ T\|_{w'}. \end{aligned}$$

Thus $W_{\pi,T}f \in B^c_{w'}$. This proves that $W_{\pi,T}$ is bounded below. Conversely, suppose that the weighted composition operator $W_{\pi,T}$ is bounded below. Let $w \in W$. Then there is $w' \in W$ such that $W_{\pi,T}(B^c_w) \subset B^c_{w'}$. We claim that $w_{T(x)}(y) \leq 2w'_x(\pi(x)y)$, for every $x \in X$ and $y \in F_{T(x)}$. Fix $x_0 \in X$ and $y_0 \in F_{T(x_0)}$. Let $\varepsilon = w'_{x_0}(\pi(x_0)y_0)$. In case $\varepsilon > 0$, we define the function $g : X \to \bigcup_{x \in X} F_x$ as

$$g(x) = \begin{cases} (2\varepsilon)^{-1} y_0, & x = T(x_0) \\ 0 \quad , & \text{otherwise.} \end{cases}$$

Obviously $W_{\pi,T}g \notin B^c_{w'}$ and $g \notin B^c_w$. From this, we conclude that $w_{T(x_0)}(y_0) \leq 2w'_{x_0}(\pi(x_0)y_0)$. On the other hand, if $\varepsilon = 0$ and $w_{T(x_0)}(y_0) > 0$, then we set $\beta = w_{T(x_0)}(y_0) / 2$. Define the function $h : X \to \bigcup_{x \in X} F_x$ as

$$h(x) = \begin{cases} \beta^{-1} y_0, & x = T(x_0) \\ 0 \quad , & \text{otherwise.} \end{cases}$$

Clearly $W_{\pi,T}h \notin B^c_{w'}$ and $h \notin B^c_w$. Thus it follows that $w_{T(x_0)}(y_0) \leq w_{T(x_0)}(y_0) / 2$.

This is a contradiction, and hence our claim is established. Assume on the contrary that T is not onto. Then there exists $y \in X$ such that for every $x \in X$, $T(x) \neq y$. Now, we choose $f \in LW_0(X)$ such that $f(y) \neq 0$. Define the function $g : X \to \bigcup_{x \in X} F_x$ as

$$g(x) = \begin{cases} f(x), & x = y \\ 0 \quad , & \text{otherwise.} \end{cases}$$

Then clearly $0 \neq g \in LW_0(X)$. Since $g \circ T = 0$ implies that $\pi . g \circ T = 0$, which contradicts the fact that $W_{\pi,T}$ is one–one. Thus T is onto. This completes the proof.

Theorem 4.4.23. Let W be a system of weights on X such that $LW_0(X)$ satisfies the condition (4.2.41b). Let $T : X \to X$ be a continuous map such that for every $x \in X$, $F_{T(x)} \subset F_x$ and for given $x_1, x_2 \in X$, there is $w \in W$ such that $w_{T(x_1)} \leq w_{x_1}$ on $F_{T(x_1)}$ and $w_{T(x_2)} \leq w_{x_2}$ on $F_{T(x_2)}$, and $\pi : X \to \mathbb{C}$ be such that $W_{\pi,T} \in B(LW_0(X))$. Then $W_{\pi,T}$ has dense range if and only if $\pi(x) \neq 0$, for every $x \in X$ and T is injective.

Proof. Suppose that the weighted composition operator $W_{\pi,T}$ has dense range. On the contrary we assume that for some $x_0 \in X$, $\pi(x_0) = 0$. Then there is $f \in LW_0(X)$ such that $f(x_0) \neq 0$. Choose $w \in W$ such that $w_{x_0}[f(x_0)] \neq 0$. Let $g = \dfrac{f}{w_{x_0}[f(x_0)]}$. Then $g \in LW_0(X)$. Now, there exists $h \in LW_0(X)$ such that $\left\| W_{\pi,T}h - g \right\|_w < \frac{1}{2}$. From this, we conclude that

$$\frac{w_{x_0}\left[f(x_0)\right]}{w_{x_0}\left[f(x_0)\right]} < \frac{1}{2},$$

which is a contradiction. To show that T is one–one, let $x_1, x_2 \in X$ be such that $x_1 \neq x_2$. Then $\pi(x_1) \neq 0$ and $\pi(x_2) \neq 0$. Let $\beta = |\pi(x_1)|\,|\pi(x_2)|$ and $\varepsilon = |\pi(x_1)| + |\pi(x_2)|$. Then choose $f \in LW_0(X)$ such that $f(x_1) = 0$ and $f(x_2) \neq 0$. Again, we choose $w \in W$ such that $w_{x_2}[f(x_2)] \neq 0$, $w_{T(x_1)} \leq w_{x_1}$ on $F_{T(x_1)}$ and $w_{T(x_2)} \leq w_{x_2}$ on $F_{T(x_2)}$. Let $g = \dfrac{\pi(x_2)}{w_{x_2}[f(x_2)]} f$. Then $g \in LW_0(X)$

and therefore, there exists $h \in LW_0(X)$ such that $\left\|W_{\pi,T}h - g\right\|_w < \frac{\beta}{2\varepsilon}$. From this, it follows that $w_{x_1}[\pi(x_1)h(T(x_1))] < \frac{\beta}{2\varepsilon}$ and $w_{x_2}[\pi(x_2)h(T(x_2)) - g(x_2)] < \frac{\beta}{2\varepsilon}$. Further, it implies that $w_{T(x_1)}[h(T(x_1))] < \frac{1}{2}$ and $\left|w_{T(x_2)}[h(T(x_2))] - 1\right| < \frac{1}{2}$. In case $T(x_1) = T(x_2)$, it is a contradiction. Thus T is an injective map. Conversely, suppose that the conditions hold. Then the range of $W_{\pi,T}$ is a vector subspace of $LW_0(X)$ such that it is a module over the self-adjoint algebra $C_b(X)$ which separates the points of X and also under the condition (4.2.41b), the density of the range of $W_{\pi,T}$ in $LW_0(X)$ now follows from the bounded case of the strict weighted approximation problem [247, Proposition 2 and Theorem 10].

The following theorem immediately follows from Theorem 4.3.2, Theorem 4.4.22 and Theorem 4.4.23.

Theorem 4.4.24. Let W be a system of weights on X such that $LW_0(X)$ is complete and satisfies the condition (4.2.41b). Let $\pi : X \to \mathbb{C}$ and $T : X \to X$ be the same as in Theorem 4.4.23. Then the weighted composition operator $W_{\pi,T}$ is invertible if and only if

(i) $\pi(x) \neq 0$, for every $x \in X$;

(ii) T is bijective ;

(iii) $W \circ T \leq W\pi$.

Finally, we shall discuss compact weighted composition operators on certain spaces of continuous functions. From the known information, we observe that a little development has been made in this direction. Feldman [109], Jamison and Rajagopalan [153], Kamowitz [164], Singh and Summers [367], Takagi ([388], [390]) are some of mathematicians who have characterised compact weighted composition operators on $C(X)$, $C(X, E)$ and certain subspaces of $C(X, E)$. But their generalisations to the weighted spaces of functions are not obtained so far. Here our efforts are to present the results obtained so far by different researchers with brief descriptions.

In [164], Kamowitz characterized compact weighted endomorphisms of $C(X)$, the

Banach algebra of continuous functions on a compact Hausdorff space X. It is easy to see that every non-zero endomorphism H of $C(X)$ has the form $Hf = f \circ T$ for some continuous function $T : X \to X$. Consequently, the weighted endomorphisms of $C(X)$ have the form $f(x) \to \pi(x)f(T(x))$ for some $\pi \in C(X)$ and continuous function $T : X \to X$. We have denoted this map by $W_{\pi,T}$. We say that a linear operator S on a Banach space B is compact if $S((B)_1)$ is relatively compact in B, where $(B)_1$ is the unit ball in B. Now, we shall record without proof in the following theorem the main result of [164].

Theorem 4.4.25 : Suppose X is a compact Hausdorff space, $\pi \in C(X)$, and T is a continuous function from X into X. Then the map $W_{\pi,T} : C(X) \to C(X)$ is compact if and only if for each connected component K of $\{x : \pi(x) \neq 0\}$, there exists an open set $G \supset K$ such that T is constant on G.

In [387], Takagi defines a weighted composition operator on a function algebra as a generalisation of a weighted endomorphism, and characterise compact weighted composition operators on a function algebra satisfying a certain condition. This generalisation includes Kamowitz's results as corollaries and have applications to compact weighted composition operators on the Hardy class $H^\infty(D)$. We begin with some definitions. Let F be a function algebra on a compact Hausdorff space X, that is, a uniformly closed subalgebra of $C(X)$ which contains the constants and separates the points of X. Let M_F be the maximal ideal space of F, and let M_F^∞ be the union of M_F and the zero functional on F. Since M_F^∞ is a subset of the dual of F, it is equipped with the relative w^*-topology and norm topology respectively. For each $f \in F$, we define $\hat{f}(\phi) = \phi(f)$, for every $\phi \in M_F^\infty$, and supt $f = \{x \in X : f(x) \neq 0\}$. Clearly supt f is open. If H is an endomorphism of a function algebra F, then H has the representation $Hf(x) = \hat{f}(\phi(x))$, $x \in X$, $f \in F$, for some continuous map ϕ from X into M_F^∞, which is given by

$$\phi(x) = H^*(\hat{x}), \quad x \in X,$$

where H^* is the adjoint of H and $\hat{x}$ is the evaluation functional at x, *i.e.*, $\hat{x}(f) = f(x)$ for each $f \in F$. Note that when $H(1) = 1$, ϕ maps X into M_F. Thus a weighted endomorphism of F has the form

$$f(x) \to \pi(x)\hat{f}(\phi(x)), \quad x \in X, \quad f \in F,$$

for some $\pi \in F$ and some continuous map ϕ from X into M_F^∞. In [387], Takagi defined the weighted composition operators, which involve weighted endomorphisms and this we give in the following definition.

Definition 4.4.26. Let A be a linear operator from F to F. Then A is a weighted composition operator on F if there exist an element π in F and a continuous map T from supt π into M_F^∞ such that

$$Af(x) = \begin{cases} \pi(x)\hat{f}(T(x)) & , \quad x \in \text{supt } \pi \\ 0 & , \quad x \in X \setminus \text{supt } \pi, \end{cases}$$

for each $f \in F$. We write $W_{\pi,T}$ for A.

Theorem 4.4.27. Let $W_{\pi,T}$ be a weighted composition operator on F. Then $W_{\pi,T}$ is compact if and only if T is a continuous map from supt π into M_F^∞ with respect to the norm topology.

Proof. Suppose that $W_{\pi,T}$ is a compact weighted composition operator on F, and let $\{x_\alpha\}$ be a net in supt π converging to x. We note that if $K = \{f \in F : \|f\| \le 1\}$, then the compactness of $W_{\pi,T}$ implies that $W_{\pi,T}(K)$ is relatively compact in F, and so is in $C(X)$. According to the Ascoli–Arzela theorem, it is equivalent that $W_{\pi,T}(K)$ is equicontinuous, *i.e.*,

$$\sup_{f \in K}\left\{\left|(W_{\pi,T}f)\,(x_\alpha) - (W_{\pi,T}f)\,(x)\right|\right\} \to 0, \tag{1}$$

as $x_\alpha \to x$ in X. Now, for any $x, x_\alpha \in$ supt π, it follows that

$$\left\|T(x_\alpha) - T(x)\right\| \le \frac{1}{|\pi(x)|}\left(\left|\pi(x) - \pi(x_\alpha)\right|\right) + \sup_{f \in K}\left\{\left|(W_{\pi,T}f)(x_\alpha) - (W_{\pi,T}f)(x)\right|\right\}.$$

Using the continuity of π and (1), we have

$$\left\|T\left(x_\alpha\right) - T(x)\right\| \to 0 \quad \text{as} \quad x_\alpha \to x.$$

Conversely, assume that T is a continuous map. To show that $W_{\pi,T}$ is a compact operator, it is enough to establish (1). Let $\{x_\alpha\}$ be a net in X converging to x. If $x \in$ supt π, then we can assume that $\{x_\alpha\} \subset$ supt π, since supt π is open. Further, it implies that

$$\sup_{f\in K}\left\{\left|(W_{\pi,T}f)\,(x_\alpha)-(W_{\pi,T}f)\,(x)\right|\right\}\le\|\pi\|\,\left\|T(x_\alpha)-T(x)\right\|+\left|\pi(x_\alpha)-\pi(x)\right|\to 0$$

as $x_\alpha \to x$. If $x \notin \operatorname{supt}\pi$, then we have

$$\sup_{f\in K}\left\{\left|(W_{\pi,T}f)\,(x_\alpha)-(W_{\pi,T}f)(x)\right|\right\}=\sup_{f\in K}\left\{\left|(W_{\pi,T}f)\,(x_\alpha)\right|\right\}$$

$$=\begin{cases}\left|\pi(x_\alpha)\right|\,\left\|T(x_\alpha)\right\|\le\left|\pi(x_\alpha)\right|\ , & x_\alpha\in\operatorname{supt}\pi\\ 0\ , & x_\alpha\notin\operatorname{supt}\pi.\end{cases}$$

Thus $\sup_{f\in K}\left\{\left|(W_{\pi,T}f)\,(x_\alpha)-(W_{\pi,T}f)(x)\right|\right\}\to 0$ as $x_\alpha \to x$. This completes the proof of the theorem.

Now, we shall present relations between compact weighted composition operators and Gleason parts established by Takagi [387]. For this, we need some definitions.

Note that M_F is divided into Gleason parts $\{P_\alpha\}$ for F, as follows :

$$M_F=\bigcup_\alpha P_\alpha,\ P_\alpha\cap P_\beta=\varnothing\ \ (\alpha\ne\beta)\ .$$

The part P containing $m_0 \in M_F$ is defined by

$$P=\left\{m\in M_F:\left\|m-m_0\right\|<2\right\}.$$

Obviously, each part is open in M_F with the norm topology and is therefore open in M_F^∞ with the norm topology. Since $\{0\}$ is so, we consider $\{0\}$ as a part of F. Thus M_F^∞ is divided into parts, and each part is open and closed in M_F^∞ with the norm topology. Now, we have the following results proved by Takagi [387].

Theorem 4.4.28. Let $W_{\pi,T}$ be a weighted composition operator on F. If $W_{\pi,T}$ is compact, then for each connected component K of $\operatorname{supt}\pi$, there exist an open set $G \subset \operatorname{supt}\pi$ and a part P for F such that

$$K\subset G,\ \ T(G)\subset P.$$

Proof. Let K be a connected component of $\operatorname{supt}\pi$, and let $x_0 \in K$. Then

$T(x_0)$ is in some part P for F. If we set $G = \{x \in \operatorname{supt} \pi : T(x) \in P\}$, then clearly G is open and closed in $\operatorname{supt} \pi$ because T is a continuous map from $\operatorname{supt} \pi$ into M_F^∞ with the norm topology, and P is open and closed in M_F^∞ with the norm topology. Also $K \subset G$, if not, then the disconnection $K = (K \cap G) \cup (K \cap (\operatorname{supt} \pi \setminus G))$ induces a contradiction.

The following theorem is the converse of the above theorem.

Theorem 4.4.29. Let $W_{\pi,T}$ be a weighted composition operator on F. Suppose that for each connected component K of $\operatorname{supt} \pi$, there exist an open set $G \subset \operatorname{supt} \pi$ and an element $m \in M_F^\infty$ such that

$$K \subset G, \quad T|_G = m. \quad \text{Then } W_{\pi,T} \text{ is compact .} \tag{2}$$

Proof. Let $x_0 \in \operatorname{supt} \pi$, and let K be a connected component containing x_0. Now, choose an open set G satisfying (2). Then $x_0 \in G$ and $\|T(x) - T(x_0)\| = \|m - m\| = 0$ and, for every $x \in G$. Thus T is a continuous map from $\operatorname{supt} \pi$ into M_F^∞ with the norm topology. The proof follows from Theorem 4.4.27.

According to Theorem 4.4.29, when each part for F is a one–point part for example, when $F = C(X)$, the converse to Theorem 4.4.28 is true. If there exists a non–trivial part, does the converse to Theorem 4.4.28 hold ?

Let P be a non–trivial part. Then P satisfies the condition (α) if P has the following property :

(4.4.29α) For any $m \in P$, there are some open neighbourhood N_m of m in P and a homeomorphism ρ from a polydisc D^n (a disc if $n = 1$, n depends on N_m) onto N_m such that $\hat{f} \circ \rho$ is an analytic function on D^n for all $f \in F$.

This condition was introduced by Ohno and Wada [260].

Now, we have the following main result which characterises compact weighted composition operators on F [387, Theorem 2].

Theorem 4.4.30. Suppose that every non–trivial part for F satisfies (4.4.29α). Let $W_{\pi,T}$ be a weighted composition operator on F. Then $W_{\pi,T}$ is compact if and only if for each connected component K of $\operatorname{supt} \pi$, there exist an open set $G \subset \operatorname{supt} \pi$ and a part P for F such that

$$K \subset G, \qquad T(G) \subset P.$$

Proof. If $W_{\pi,T}$ is a compact weighted composition operator on F, then the conditions follows from Theorem 4.4.28. On the other hand, to prove that $W_{\pi,T}$ is compact, it is enough to show that T is a continuous map from supt π into M_F^∞ with the norm topology. Fix $x_0 \in$ supt π. Then there is an open $G \subset$ supt π such that $x_0 \in G$ and $T(G) \subset P$, where P is a part for F. Now, if P is a one-point part, then it is proved in Theorem 4.4.29 that T is continuous at x_0 with respect to the norm topology. So, we assume that P is non-trivial. From the definition of weighted composition operators, it follows that T is a continuous map from supt π into M_F^∞ with the weak*-topology. The proof can be completed by showing that the identity map I from P with the weak*-topology on P with the norm topology is continuous at $T(x_0)$. For detailed proof we refer to [387].

Kamowitz [164] has proved a theorem which characterises compact weighted endomorphisms of $C(X)$. This theorem immediately follows from Theorem 4.4.30 as each part of $M_{C(X)}(= X)$ is a one-point part. This we present in the following corollary.

Corollary 4.4.31 [164, Theorem A]. Let $W_{\pi,T}$ be a weighted endomorphism of $C(X)$. Then $W_{\pi,T}$ is compact if and only if for each connected component K of supt π, there exists an open set $G \supset K$ such that T is constant on G.

Remark 4.4.32. In [387], Takagi has given a counter example to the question : does the converse to theorem 4.4.28 hold? If every part for F satisfies (α), Theorem 4.4.30 answered "yes". But for the general case, the answer is "no". Indeed, there exist a function algebra F and a weighted composition operator $W_{\pi,T}$ on F such that

(i) for each connected component K of supt π, there are open set $G \subset$ supt π and a part P for F such that $K \subset G$, $T(G) \subset P$;

(ii) $W_{\pi,T}$ is not compact.

In [153], Jamison and Rajagopalan generalised some of the results pertaining to compact weighted composition operators of $C(X)$ to vector setting of $C(X,E)$, where

E is a Banach space. Here we shall give a brief description of these operators on $C(X,E)$. Also, note that the weighted composition operators $W_{\pi,T}$ on $C(X,E)$ take the following form :

$$\left(W_{\pi,T}f\right)(x) = \pi(x)f(T(x)),$$

where T is a self map of X and for each x, $\pi(x)$ is a bounded linear operator on E. To obtain the main result of compact weighted composition operators on $C(X,E)$, we need the following definitions. We denote by $N = \{x \in X : \pi(x) = 0$, the zero operator on $E\}$. By $|G|$, we denote the cardinality of a set G. A sequence $\{f_n\}$ is said to be ε–uniform Cauchy on $G \subset X$ if given $\varepsilon > 0$, there exists N_ε such that $\left\|f_n(x) - f_m(x)\right\| < \varepsilon$, for all $m, n > N_\varepsilon$ and $x \in G$. It is well–known that a sequence $\{f_n\}$ converges in $C(X,E)$ if and only if it is ε–uniform Cauchy on X for every $\varepsilon > 0$.

Theorem 4.4.33. Let $W_{\pi,T}$ be a weighted composition operator on $C(X,E)$. Then the following conditions are necessary and sufficient for compactness of $W_{\pi,T}$.

(a) T is continuous in $X \setminus N$,

(b) $x \to \pi(x)$ is continuous in the uniform operator topology,

(c) If K is a compact subset of $X \setminus N$, then $|T(K)| < \infty$,

(c′) If K is a connected component of $X \setminus N$, there exists an open subset G of X, such that $G \subset X \setminus N$, $K \subset G$ and $|T(G)| < \infty$,

(d) If $\{e_n\}$ is a sequence in E, $\varepsilon > 0$ and K is a compact subset of $X \setminus N$, then there exists a subsequence $\{e_{n_k}\}$ such that $\{\pi(x)e_{n_k}\}$ is ε–uniformly Cauchy on K,

(e) If $\{f_n\}$ is a bounded sequence in $C(X, E)$ and $\varepsilon > 0$, then there exists a subsequence $\{f_{n_k}\}$ and a neighbourhood $N_\varepsilon \supset Z$ such that $\left\|W_{\pi,T}f_{n_k}(x)\right\| < \varepsilon$ for every $x \in N_\varepsilon$, where $Z = \{x \in X : W_{\pi,T}(f_n)(x) = 0$ for every $n\}$.

Proof. Assume that the conditions from (a) to (e) are true. The continuity of $W_{\pi,T}$ follows from condition (a) and (b). Now, it remains to show that $W_{\pi,T}$ is compact. Let $\varepsilon > 0$ and let $\{f_n\}$ be a sequence in $C(X, E)$ such that $\left\|f_n\right\| \le 1$. Then by condition (e) there exists a subsequence $\{f_n'\}$ and a neighbourhood $N_\varepsilon \supset N$ such that $\|Tf_n'(x)\| < \varepsilon$, for every $x \in N_\varepsilon$. Let $K_\varepsilon = X \setminus N_\varepsilon$. Then K_ε is compact and by condition (c), $|T(K_\varepsilon)| < \infty$. Now, there exists $x_1, x_2, \dots, x_p$ such that $T(K_\varepsilon) = \{x_1, x_2, \dots, x_p\}$. If we put $F_i = T^{-1}(x_i) \cap K_\varepsilon$, then clearly the set F_i is compact and $|T(F_i)| = 1$. Without loss of generality, the original subsequence can be relabeled as $\{f_n\}$. Now, if $x \in F_1$, then $\{f_j(T)\}$ is a sequence of constant vectors in E. According to condition (d), there exists a subsequence $\{f_{1k}\}$ of $\{f_n\}$ such that $\{\pi(x)f_{1k}(T)\}$ is ε–uniform Cauchy on F_1. Since the sequence $\{f_{1k}(T)\}$ is again a sequence of constant vectors in E, condition (d) implies that there is a subsequence $\{f_{2k}\}$ of $\{f_{1k}\}$ such that $\{\pi(x)f_{2k}(T)\}$ is ε–uniform Cauchy on $F_1 \cup F_2$. This process gives a sequence $\{f_{pk}\}$ such that $\{\pi(x)f_{pk}(T)\}$ is ε–uniform Cauchy on K_ε. Further, it follows that $\{Tf_{pk}\}$ is ε–uniform Cauchy on all of X because $\|\pi(x)f_{pk}(T)\| < \varepsilon$, for $x \in N_\varepsilon$. Since $C(X, E)$ is complete, there exists $g \in C(X, E)$ such that $Tf_{pk} \to g$. This proves that $W_{\pi,T}$ is a compact weighted composition operator.

Conversely, we show that these conditions are necessary. Condition (a) follows from the continuity of $W_{\pi,T}$. To establish (b), suppose that the map $x \to \pi(x)$ is not continuous in the uniform norm at some $x_0 \in X$. Then there exists a $\delta > 0$ such that for each open set G containing x_0, there exists an x_G such that $\|\pi(x_G) - \pi(x_0)\| > \delta$. Further, it implies that there is a net $\{e_G\}$ such that $\|e_G\| \le 1$ and $\|\pi(x_G)e_G - \pi(x_0)e_G\| > \delta$ for all G in Ω, where Ω denotes the class of all neighbourhoods of x_0. For each compact neighbourhood G of x_0, let g_G be the constant function defined by $g_G(x) = e_G$. Then $(W_{\pi,T}g_G)(x) = \pi(x)e_G$, for each $x \in X$. Since $W_{\pi,T}$ is compact, there is a subnet of $\{g_G\}$ which can be labelled (without loss of generality) as $\{g_G\}$, and a function $h \in C(X, E)$ such that $W_{\pi,T}g_G \to h$ along Ω. Again, we have $\|W_{\pi,T}g_G(x_G) - h(x_G)\| \to 0$ and $\|W_{\pi,T}g_G(x_0) - h(x_0)\| \to 0$ along Ω. Also, we have $\|W_{\pi,T}g_G(x_0) - W_{\pi,T}g_G(x_G)\| = \|\pi(x_G)e_G - \pi(x_0)e_G\| > \delta$, which is a contradiction. Thus it follows that the map $x \to \pi(x)$ is continuous in the uniform operator topology. Now, for establishing condition (c), suppose that K is a compact subset of $X \setminus N$ and that $x_0 \in K$. We claim that there exists a neighbourhood G of x_0 such that $|T(G)| < \infty$. Since $x_0 \in X \setminus N$, $\pi(x_0) \ne 0$. Assume that $z \in E$ such that $\|z\| = 1$ and $\|\pi(x_0)z\| > 0$.

Also, there exists an $\varepsilon > 0$ and a compact neighbourhood G of x_0 such that $G \subset X \setminus N$ and $\|\pi(y)z\| > 0$, for all $y \in G$ because the map $x \to \pi(x)$ is continuous in the strong operator topology. If $|T(G)|$ were not finite, then there would exist a sequence $\{f_n\}$ containing no convergent subsequence. For $x \in G$, define $h_n(x) = f_n(T(x))$. Let $\{g_n\}$ be a (norm preserving) Tietze extension of the $h_n's$ to all of X. The functions $g_n(x)z \in C(X, E)$ and clearly $\{W_{\pi,T}(g_n z)\}$ contains no convergent subsequence. This contradicts the compactness of $W_{\pi,T}$. To show that condition (d) is also necessary, let K be a compact subset of $X \setminus N$ and let $\{e_n\}$ be a sequence in E. Then there is a function $g \in C(X)$ such that $g(x) = 1$ on K and $0 < g < 1$. Define the function $f_n(x) = g(x)e_n$. Then there exists a subsequence $\{f_{n(k)}\}$ such that $\{W_{\pi,T} f_{n(k)}\}$ converges in $C(X, E)$ because $\|f_n\| < \infty$. Thus the subsequence $W_{\pi,T} f_{n(k)}(x) = \pi(x)e_{n(k)}$ is ε-uniformly Cauchy on K. Finally, for proving (e) let $\{f_n\}$ be a bounded sequence. Then by the compactness of $W_{\pi,T}$, there exists a subsequence $\{f_{n_k}\}$ such that $\{W_{\pi,T} f_{n_k}\}$ converges to some $g \in C(X, E)$. Also $g(x) = 0$ for $x \in Z$. Thus, if $\varepsilon > 0$, then there exists an open set $N_\varepsilon \supset Z$ such that $\varepsilon > \|g(y) - g(x)\| = \|g(y)\|$, for every $y \in N_\varepsilon$. This completes the proof of the theorem.

At the end of this chapter, we shall outline the generalisation of compact weighted composition operators on certain subspaces of $C(X, E)$ obtained by Takagi [388] in the function algebra setting using Gleason parts technique. These spaces are defined as follows.

Let F be a closed subalgebra of $C(X)$ which contains the constants and separates points of X. Let

$$F(X, E) = \left\{f \in C(X, E) : e^* \circ f \in F, \text{ for all } e^* \in E^*\right\},$$

where E^* is the dual space of E. Then clearly $F(X, E)$ is a Banach space relative to the same norm. For example, as a generalization of the disc algebra $A(\overline{D})$ on the closed unit disc $\overline{D}$, consider the space $\{f \in C(\overline{D}, E) : f$ is an analytic E-valued function on the open unit disc $D\}$. Here f is said to be analytic on D when it is differentiable at each point of D, in the sense that the limit of the usual difference quotient exists in the norm topology. It is known that this space coincides the following space,

$$\left\{f \in C(\overline{D}, E) : e^* \circ f \in A(\overline{D}) \text{ for all } e^* \in E^*\right\}$$

(see [63, p. 126]). We shall record the following result without proof.

Theorem 4.4.34 [388]. (A) Let $W_{\pi,T}$ be a weighted composition operator on $F(X,E)$. If $W_{\pi,T}$ is compact, then

(a) for each connected component K of $S(\pi) = \{x \in X : \pi(x) \neq 0\}$, there exists an open set G containing K and a part P for F such that $T(G) \subset P$;

(b) the map $\pi : X \to B(E)$ is continuous in the uniform operator topology, *i.e.*, $\|\pi(x_\alpha) - \pi(x)\|_{B(E)} \to 0$ as $x_\alpha \to x$;

(c) for any $x \in S(\pi)$, $\pi(x)$ is a compact operator on E;

(B) In addition, we assume that every non–trivial part for F has the property (4.4.29 α). If a weighted composition operator $W_{\pi,T}$ on $F(X,E)$ satisfies the above conditions (a) to (c), then $W_{\pi,T}$ is compact.

Remark 4.4.35 From the above theorem we observe that if $F = C(X)$ then $F(X,E) = C(X,E)$. Note that every part for $C(X)$ is one point. Theorem 4.4.34 yields the following corollary, which says that the condition (e) in Theorem 4.4.33 is removable.

Corollary 4.4.36. Let $W_{\pi,T}$ be a weighted composition operator on $C(X,E)$. Then $W_{\pi,T}$ is compact if and only if

(i) for each connected compact K of $S(\pi) = \{x \in X : \pi(x) \neq 0\}$, there exists an open set G containing K such that T is constant on G.

(ii) the map π is continuous in the uniform operator topology, and

(iii) for each $x \in S(\pi)$, $\pi(x)$ is a compact operator on E.

In the above corollary, if we consider $E = \mathbb{C}$, then the space $F(X, \mathbb{C})$ is a function algebra F on X, and we obtain results of [387] as the conditions (b) and (c) in Theorem 4.4.34 are trivially satisfied. Let I_E be the identity operator on E, and let $\pi(x) = I_E$ for every $x \in X$. Then the weighted composition operator $W_{\pi,T}$ on $F(X,E)$ induced by this map π turns out to be a composition operator. If E is an infinite dimensional Banach space, then I_E is not compact, and so the above map π does not satisfy the condition (c) in Theorem 4.4.34. Thus the following corollary immediately follows from the part (A) of Theorem 4.4.34 [367].

Corollary 4.4.37 [388]. If E is infinite dimensional, then there is no compact composition operator on $F(X, E)$.

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CHAPTER V

SOME APPLICATIONS OF COMPOSITION OPERATORS

In the chapters II through IV we have presented a study of the composition operators on different function spaces. These natural type of operators have been appearing in explicit or implicit form in different areas of mathematical sciences, like classical mechanics, ergodic theory, dynamical systems, Markov processes, theory of semigroups, isometries and homomorphisms etc. Some of these interactions and applications we shall present in this chapter. Isometries play very important roles in the study of some mathematical structures. Most of the isometries on suitable function spaces are related to the composition operators. We have presented characterizations of isometries on some function spaces of different categories in the first section of this chapter. The second section concentrates on the applications of the composition operators in ergodic theory and topological dynamics. The translations-induced composition operators are very important for the study of different types of motions. Applications of the composition operators in dynamical systems are illustrated in the third section. The last section of this chapter is devoted to show the involvement of the composition operators in the study of the algebra homomorphisms on some function algebras.

5.1 ISOMETRIES AND COMPOSITION OPERATORS

In the present section we have made an effort to report some known results on the isometries between some function spaces, which illustrates the employment of the composition operators in the study of the isometries. For our convenience, we shall present these results in this section in two phases. In the first phase we shall be dealing with the isometries on the spaces of continuous functions and L^p-spaces and in the second phase H^p-spaces and Bergman spaces are taken for the study of the isometries in terms of composition operators.

We begin with the classical Banach–Stone theorem. If X and Y are compact Hausdorff spaces and $T : X \to Y$ is a continuous map, then it is well-known that the composition operator $C_T : C(Y) \to C(X)$ is an isometry if and only if T is surjective, and C_T is a surjective isometry if and only if T is a homeomorphism. Further, if $T : X \to Y$ is a homeomorphism and $\pi : X \to \mathbb{C}$ is a continuous function with $|\pi(x)| = 1$ for all $x \in X$, then the weighted composition operator $W_{\pi,T} : C(Y) \to C(X)$ is a surjective isometry. The converse of the foregoing remark is the Banach–Stone theorem which says that every surjective isometry from $C(Y)$ to $C(X)$ is a weighted composition operator. To establish the Banach–Stone theorem we need to give some definitions.

Definitions 5.1.1. If E is a vector space and $x_1, x_2, \in E$, then the set $\{tx_1 + (1-t)x_2 : 0 < t < 1\}$ is called the open line segment between x_1 and x_2. In case $x_1 \neq x_2$ this line segment is called proper. If K is a convex subset of E, then a point $y \in K$ is said to be an extreme point of K if there is no proper open line segment that contains y and lies entirely in K. By the symbol $ext\,K$ we denote the set of all extreme points of K. If E is a normed linear space, then $(E)_1$ denotes the closed unit ball of E.

If X is a topological space, then by $M(X)$ we denote the normed linear space of all complex-valued regular Borel measures on X with the total variation norm. If X is a compact Hausdorff space, then one can prove that the set of all extreme points of $(M(X))_1$ is $\{\alpha\, \delta_x : |\alpha| = 1 \text{ and } x \in X\}$ and the set of all extreme points of $P(X)$, the set of probability measures on X, is $\{\delta_x : x \in X\}$ (For details see [80]).

Theorem 5.1.2. [Banach–Stone Theorem]. Let X and Y be compact Hausdorff spaces, and let $A : C(Y) \to C(X)$ be a surjective isometry. Then there is a homeomorphism $T : X \to Y$ and a function π in $C(X)$ such that $|\pi(x)| = 1$ for all $x \in X$ and

$$(Af)(x) = \pi(x) f(T(x)), \text{ for all } f \in C(Y)$$

and every $x \in X$ (*i.e.*, $A = W_{\pi,T}$).

Proof. Suppose $A : C(Y) \to C(X)$ is a surjective isometry. Then the adjoint operator $A^* : M(X) \to M(Y)$ is also a surjective isometry. Now, it is straightforward to check that A^* is a weak*-homeomorphism of $(M(X))_1$ onto $(M(Y))_1$ that distributes over convex combinations. Further, it implies that

$$A^*\left(ext(M(X))_1\right) = ext(M(Y))_1. \tag{1}$$

From the remarks preceding this theorem, it follows that for every $x \in X$ there is a uniue $T(x)$ in Y and a unique scalar $\pi(x)$ such that $|\pi(x)| = 1$ and $A^*(\delta_x) = \pi(x)\delta_{T(x)}$. Thus by the uniqueness, $\pi : X \to \mathbb{C}$ and $T : X \to Y$ are well-defined functions. First of all we show that the function π is continuous. Let $\{x_\alpha\}$ be a net in X such that $x_\alpha \to x$. Then clearly $\delta_{x_\alpha} \to \delta_x$ (weak*) in $M(X)$. Further, we have $A^*(\delta_{x_\alpha}) \to A^*(\delta_x)$ (weak*) in $M(Y)$. That is, $\pi(x_\alpha)\delta_{T(x_\alpha)} \to \pi(x)\delta_{T(x)}$. In particular, $\pi(x_\alpha) = A^*(\delta_{x_\alpha})(1) \to A^*(\delta_x)(1) = \pi(x)$. This proves that π is a continuous map. Now, we shall show that the map $T : X \to Y$ is a homeomorphism. To this end, let $\{x_\alpha\}$ be a net in X such that $x_\alpha \to x$. Then $\pi(x_\alpha)\delta_{T(x_\alpha)} \to \pi(x)\delta_{T(x)}$ (weak*) in $M(Y)$. Since π is continuous, $\pi(x_\alpha) \to \pi(x)$ in $\mathbb{C}$. Thus it follows that $\delta_{T(x_\alpha)} \to \delta_{T(x)}$. Since the map $x \to \delta_x$ is a homeomorphism from X into $(\Delta(X), wk^*)$, we conclude that $T(x_\alpha) \to T(x)$. This proves that T is continuous. To show that T is an injection, let $x_1, x_2 \in X$ be such that $x_1 \neq x_2$. Then $\overline{\pi(x_1)}\,\delta_{x_1} \neq \overline{\pi(x_2)}\,\delta_{x_2}$ and hence $T(x_1) \neq T(x_2)$. Now, we fix $y \in Y$. Then by the surjectivity of A^* there exists $\mu \in M(X)$ such that $A^*(\mu) = \delta_y$. In view of (1), it follows that $\mu \in ext(M(X))_1$. Thus $\mu = \beta\,\delta_x$, for some $x \in X$ and $\beta \in \mathbb{C}$ such that $|\beta| = 1$. This implies that $\delta_y = A^*(\beta\,\delta_x) = \beta\,\pi(x)\,\delta_{T(x)}$. Further, it implies that $\beta = \overline{\pi(x)}$ and $T(x) = y$. Thus $T : X \to Y$ is a continuous bijection and hence must be a homeomorphism. Let $f \in C(Y)$ and $x \in X$. Then

$$\delta_x(Af) = A^*\left(\delta_x\right)(f) = \pi(x)\delta_{T(x)}(f) = \pi(x)f\left(T(x)\right).$$

Thus $(Af)(x) = \pi(x)f\left(T(x)\right)$. This completes the proof of the theorem.

The classical Banach-Stone theorem has been extended in various directions; for example, Nagasawa [249] extended it to function algebras; Amir [8], Cambern [47] and

Cengiz [62] to regular subspaces; Cambern and Pathak [53], Pathak [263] and Pathak and Vasavada [265] to spaces of differentiable functions; Pathak [264], Vasavada [395] and Rao and Roy [277] to the spaces of absolutely continuous functions. In most of the above-mentioned papers the following situation was considered.

Let S be a subspace of the Banach space $C(X)$, which separates points of X, and let L_S be a linear map from S into a Banach space E. Assume that the norm on S is given by one of the following formulas :

(F_1) $\| f \| = \max\{\| f \|_\infty, \| L_S f \|\}$, for $f \in S$, where $\| . \|_\infty$ is the usual supremum norm on $C(X)$,

(F_2) $\| f \| = \| f \|_\infty + \| L_S f \|$, for $f \in S$,

(F_3) $\| f \| = \sup\{| f(x) | + | L_S f(x) | \; : \; x \in X\}$, $f \in S$, where in this case we assume that $E = C(X)$.

For example, $L_S : C^1[0,1] \to C[0,1]$ can be defined as $L_S(f) = f'$ or $L_S : AC[0,1] \to L^1[0,1]$ can be defined as $L_S(f) = f'$, where $AC[0,1]$ denotes the space of all absolutely continuous functions on $[0, 1]$.

Next assume that M is a subspace of $C(Y)$, which separates points of Y, and that the norm on M is given by a map $L_M : M \to E$, via the same formulas as the norm on S. Then the question arises whether any isometry A from S onto M is of the canonical form

$$A(f)(y) = \pi(y) f \circ T(y), \quad f \in S \text{ and } y \in Y, \tag{2}$$

where T is a homeomorphism from Y onto X and π is a scalar-valued function defined on Y such that $|\pi(y)| = 1$, for all $y \in Y$. A simple technical scheme to verify the above mentioned problem has been given by Jarosz and Pathak [156]. This scheme covers all the results of the references mentioned after the proof of the Banach-Stone theorem. Now, we shall highlight the results of this scheme. In this effort, we need some definitions and notations.

Let $Z = X \cup (E^*)_1$ and let $\tilde{S}$ be a normed subspace of the Banach space $C(Z)$ with the usual supremum norm. Then it is clear that every extreme point of $(\tilde{S}^*)_1$ is of the form $\lambda\delta_y$, where $y \in Z$ and $|\lambda| = 1$. Further, we note that if the norm on S is defined by the formula (F_1), then there is an isometric embedding $\psi : S \to \tilde{S}$ given by $\psi(f) = \tilde{f}$, for every $f \in S$, where $\tilde{f}$ is defined by $\tilde{f}(x) = f(x)$, $x \in X$, and $\tilde{f}(e^*) = e^*(L_S f)$, $e^* \in (E^*)_1$. Hence any extreme point of

$(S^*)_1$ is of the form $f \to \alpha f(x)$, where $x \in X$ and $\alpha \in \partial D$, or of the form $f \to e^* \circ L_S(f)$, where $e^* \in ext(E^*)_1$. Again, if $\tilde{S}$ is a normed subspace of $C(X \times (E^*)_1)$, and if the norm on S is given by the formula (F_2), then there is an isometry $A : S \to \tilde{S}$ defined as $A(f) = \tilde{f}$, for every $f \in S$, where $\tilde{f}$ is given by $\tilde{f}(x, e^*) = f(x) + e^*(L_S f)$, $(x, e^*) \in X \times (E^*)_1$. In this case any extreme point of $(S^*)_1$ is of the form $f \to \alpha f(x) + e^* \circ L_S(f)$, where $x \in X, e^* \in ext(E^*)_1$ and $\alpha \in \partial D$. Also, if the norm on S is given by the formula (F_3), then a suitable choice is $Z = X \times \partial D$ and $\tilde{f}(x, \lambda) = f(x) + \lambda(L_S f)(x)$, $x \in X$, $\lambda \in \partial D$. Thus in this case any extreme point of $(S^*)_1$ is given by $f \to \alpha f(x) + \beta(L_S f)(x)$, where $x \in X$ and $\alpha, \beta \in \partial D$.

Let S be a subspace of $C(X)$, where X is a compact Hausdorff space. We say that S is F_1–subspace of X, F_2–subspace of X or F_3–subspace of X if there is a Banach space E and a linear map $L_S : S \to E$ such that

(a) the norm on S is given by the formula (F_1), (F_2) or (F_3) respectively.

(b) for any $x_1, x_2 \in X$, the evaluation functionals δ_{x_1} and δ_{x_2} are linearly independent, and if the corresponding assumptions listed below are satisfied :

(cF_1) $X_0 = \{x \in X : \delta_x \in ext(S^*)_1\}$ is a dense subset of X,

(cF_2) there is an $e_0^* \in ext(E^*)_1$ such that

$$X_0 = \{x \in X : \text{for all } \alpha \in \partial D,\ \delta_x + \alpha e_0^* \circ L_S \in ext(S^*)_1\}$$

is a dense subset of X,

(cF_3) $X_0 = \{x \in X :$ for all $\alpha, \beta \in \partial D, \alpha\delta_x + \beta\delta_x \circ L_S \in ext\ (S^*)_1\}$ is a dense subset of X,

(dF_2) if $\alpha\delta_x + e^* \circ L = \alpha_0\delta_{x_0} + \beta_0 e_0^* \circ L_S$, where $e^*, e_0^* \in ext(E^*)_1$, $x, x_0 \in X$, $\alpha\delta_x + e^* \circ L_S \in ext(S^*)_1$ and α_0, β_0 are scalars, then $x = x_0$ and $\alpha = \alpha_0$,

(dF_3) if $\alpha\,\delta_x + \delta_x \circ L_S = \alpha_0\delta_{x_0} + \beta_0\delta_{x_0} \circ L_S$, where $x, x_0 \in X$, $\alpha\,\delta_x + \delta_x \circ L_S \in ext(S^*)_1$ and α_0, β_0 are scalars, then $x = x_0$ and $\alpha = \alpha_0$.

In the following theorem we shall record the results without proofs which are due to Jarosz and Pathak [156]. Moreover, these results provide a scheme which is under investigation.

Theorem 5.1.3.

(i) Let S and M be F_1–subspaces of $C(X)$ and $C(Y)$, respectively. Let $X_0 = \{x \in X : \delta_x \in ext(S^*)_1\}$, $\tilde{X} = \{\alpha\delta_x : x \in X_0, \alpha \in \partial D\}$, $Y_0 = \{y \in Y : \delta_y \in ext(M^*)_1\}$ and $\tilde{Y} = \{\alpha\delta_y : y \in Y_0, \alpha \in \partial D\}$. Then an isometry A from S onto M is of the canonical form (2) if and only if $A^*(\tilde{Y}) = \tilde{X}$.

(ii) Let S and M be F_2–subspaces of $C(X)$ and $C(Y)$, respectively. Then an isometry A from S onto M is of the canonical form (2) if and only if for any $\alpha_i\delta_{y_i} + e_i^* \circ L_M \in ext(M^*)_1$, $i = 1,2$, the following two implications hold :

(a) $y_1 = y_2$ if and only if $x_1 = x_2$ and

(b) $e_1^* \circ L_M$ and $e_2^* \circ L_M$ are proportional if and only if $G_1 \circ L_S$ and $G_2 \circ L_S$ are proportional, here by x_i and G_i we denote the elements of X and $ext(E^*)_1$, respectively, such that $A^*(\alpha_i\delta_{y_i} + e_i^* \circ L_M) = \beta_i\delta_{x_i} + G_i \circ L_S$, for some scalars β_i, $i = 1, 2$.

(iii) Let S and M be F_3–subspaces of $C(X)$ and $C(Y)$, respectively. Then an isometry A from S onto M is of the canonical form (2) if and only if the following two conditions hold :

(a) for any $\alpha_i\delta_{y_i} + \beta_i\delta_{y_i} \circ L_M \in ext(M^*)_1, i = 1, 2$, we have $y_1 = y_2$ if and only if $x_1 = x_2$, where by x_i we denote an element of X such that $A^*\left(\alpha_i\delta_{y_i} + \beta_i\delta_{y_i} \circ L_M\right) = \alpha_i'\delta_{x_i} + \beta_i'\delta_{x_i} \circ L_S$, for some scalars α_i', β_i', $i = 1, 2$;

(b) $A^*(\{\alpha\,\delta_y : y \in Y, \alpha \in \partial D\}) \cap \{\alpha\delta_x \circ L_S : x \in X, \alpha \in \partial D\} = \varnothing$.

For the verification of the assumptions of the scheme, it is necessary to have at least partial description of the extreme functionals in the unit ball of the dual spaces. Hence in some cases it is easier to apply the following theorem, which is an immediate consequence of a theorem of [155].

Theorem 5.1.4. Let S be a subspace of $C(X)$ such that

(i) S is supnorm dense in $C(X)$,

(ii) the norm on S is given by a map $L_S : S \to E$, via the formula (F_1) or (F_2),

(iii) S contains the constant function 1 and $L_S(1) = 0$.

Assume next that M is a subspace of $C(Y)$ which satisfies the analogous assumptions (i) – (iii). Then any surjective isometry A from S to M, such that $A(1) = 1$ is of the form

$$A(f) = f \circ T, \text{ for } f \in S,$$

where T is a homeomorphism from Y onto X.

Jarosz and Pathak [156] have given various examples to illustrate the above scheme.

Example 5.1.5. Let $S = C^1(X)$ and $M = C^1(Y)$ be the algebras of continuously differentiable functions defined on the compact subsets X and Y of the real line $\mathbb{R}$, respectively. Here it is not assumed that the sets X and Y do not contain isolated points but the derivative of a function $f \in C^1(X)$ is defined only on the set of non-isolated points of X. Suppose that the norms on S and M are given by

$$\| f \| = \max\left(\| f \|_\infty, \| f' \|_\infty\right), \quad f \text{ belongs to } S \text{ or } M.$$

Then it is shown [156, Example 1] with the help of Theorem 5.1.3 that any isometry A from $C^1(X)$ onto $C^1(Y)$ is of the form

$$A(f)(y) = \pi(y) f(T(y)), \quad \text{for } f \in S,\ y \in Y,$$

where $\pi \in C^1(Y)$, $|\pi(y)| = 1$, for all $y \in Y$ and T is a homeomorphism from Y onto X. Moreover, this example holds for arbitrary subsets of the real line not necessarily bounded and closed. In that case S is considered as a subspace of $C(\beta X)$ and the proof is slightly more technical. The general form of the isometries of $C^1(X)$ spaces, defined on a compact subset X of the real line, without isolated points, was investigated by Pathak and Vasavada [265]. If the spaces $S = C^1(X)$ and $M = C^1(Y)$ of this example are provided with the norm given by the formula (F_2), *i.e.*, the norm is given by

$$\| f \| = \| f \|_\infty + \| f' \|_\infty, \quad f \text{ belongs to } S \text{ or } M,$$

then it is proved in [156] that any isometry from S onto M is of the canonical form (2). The isometries of $C^1[0, 1]$ with the above (F_2)–norm were characterised by Rao and Roy [277].

Example 5.1.6. Let $S = AC(X)$ be the space of all absolutely continuous scalar valued functions defined on a compact subset X of the real line, such that $X = \overline{\text{int } X}$. Define the norm on S by

$$\| f \| = \max\left(\| f \|_\infty, \| f' \|_1\right), \qquad \text{for } f \in S,$$

where $\| f' \|_1 = \int_X | f' | dm$ and m is the Lebesgue measure. Then Jarosz and Pathak [156, Example 2] have shown that any isometry from $AC(X)$ onto $AC(Y)$ is canonical. The isometries of the space $AC[0,1]$ have been described by Cambern [47] and by Rao and Roy [277], where as the isometries of $C^1(X)$ spaces with the (F_3)–norm have first been considered by Cambern [47] for $X = [0,1]$ and then by Cambern and Pathak [53] for any compact subset X of the real line without isolated points.

Example 5.1.7. Let X be a compact subset of the real line. For $1 \le p \le \infty$, we define

$$AC^p(X) = \left\{ f \in C(X) \ : \ f' \text{ exists } a.e., \ f' \in L^p(X) \right\}$$

and define the norm on $AC^p(X)$ by

$$\| f \| = \| f \|_\infty + \| f' \|_p.$$

Rao and Roy [277] proved that any isometry from $AC^p([0, 1])$ onto itself is canonical in case $p = 1$ or $p = \infty$ and asked whether the same holds for $1 < p < \infty$. The answer is positive and it is an immediate consequence of the following theorem which is due to Jarosz and Pathak [156].

Theorem 5.1.8. Let X be a compact Hausdorff space, and let S be a subspace of $C(X)$ such that

(i) S is supnorm dense in $C(X)$,

(ii) the norm on S is given by a map $L_S : S \to E$ via the formula (F_3),

(iii) S contains the constant function 1 and $L_S(1) = 0$,

(iv) $\dim(L_S(S)) \geq 2$,

(v) E is strictly convex,

(vi) for any unimodular function $\theta \in S$ such that $L_S(\theta) = 0$, the map $f \to f / \theta$ is a well-defined isometry of S onto itself.

Assume next that M is a subspace of $C(Y)$, Y a compact Hausdorff space, which also satisfies assumptions (i) through (vi). Then any isometry A from S onto M is of the form

$$A(f) = \theta.\, f \circ T, \quad f \in S,$$

where T is a homeomorphism from Y onto X and $\theta \in M$ is a unimodular function such that $L_M(\theta) = 0$.

Proof. In view of Theorem 5.1.4 and by assumption (vi) it is enough to show that $A(1)$ is a unimodular function on Y such that $L_M(A(1)) = 0$. We say that an element $g \in S$ has the P-property if $\| g \| = 1$ and if for any $f \in S$ there is a $\beta \in \partial D$ such that

$$\| g + \beta f \| = \| g \| + \| f \| .$$

It is clear that this property is preserved by the isometry A. Let $f \in S$ and $\beta \in \partial D$ be such that $\| f \|_\infty = \sup_{x \in X} \operatorname{Re}(\beta f(x))$. Then by the definition of the norm on S we have

$$\| 1 + \beta f \| = \| 1 + \beta f \|_\infty + \| L_S f \| = 1 + \| f \|_\infty + \| L_S(f) \| = 1 + \| f \| .$$

To complete the proof we have to prove that if $g \in S$ has the P-property, then g is a unimodular function on X and $L_S(g) = 0$. First of all we prove that $| g | = c$ on X for some constant c and then from the definition of the norm on S we get $c = \| g \|_\infty = \| g \| = 1$. Suppose that there exists an $x_0 \in X$ such that

$|g(x_0)| < \|g\|_\infty$. Then by assumption (i) there is an $f \in S$ such that $\|f\|_\infty = \|g\|_\infty - |g(x_0)|$ and $|f(x)| \le \|g\|_\infty - |g(x)| + \frac{1}{2}(\|g\|_\infty - |g(x_0)|)$, for $x \in X$. For any $\beta \in \partial D$, it follows that $\|g + \beta f\|_\infty \le \|g\|_\infty + \frac{1}{2}\|f\|_\infty$. Thus

$$\begin{aligned}\|g + \beta f\| &= \|g + \beta f\|_\infty + \|L_S(g + \beta f)\| \\ &\le \|g\|_\infty + \frac{1}{2}\|f\|_\infty + \|L_S(g)\| + \|L_S(f)\| \\ &= \|g\| + \|f\| - \frac{1}{2}\|f\|_\infty < 1 + \|f\|.\end{aligned}$$

Hence $|g| =$ constant. If $L_S(g) \neq 0$, then by assumption (iv) there is an $f \in S$ such that $L_S(f)$ and $L_S(g)$ are not proportional. Since E is strictly convex, for any $h, h' \in E$ we have $\|h + h'\| = \|h\| + \|h'\|$ if and only if h and h' are proportional and hence for any $\beta \in \partial D$, we have

$$\begin{aligned}\|g + \beta f\| &= \|g + \beta f\|_\infty + \|L_S(g + \beta f)\| \\ &< \|g\|_\infty + \|f\|_\infty + \|L_S(g)\| + \|L_S(f)\| \\ &= 1 + \|f\|.\end{aligned}$$

This proves that $L_S(g) = 0$. Hence the proof of the theorem is completed.

Besides the above general scheme, the classical Banach–Stone theorem has been generalised in another direction to the spaces of the form $C(X, E)$ consisting of continuous vector-valued functions from the compact Hausdorff space X to a Banach space E. In 1950 Jerison [157] started investigation of the problem of determining geometric conditions on a Banach space E which would allow generalizations of the classical Banach–Stone theorem to the spaces of the type $C(X, E)$. In particular, Jerison showed, among other things, that if E is strictly convex, then for any isometry A from $C(X, E)$ onto $C(Y, E)$ there exists a homeomorphism T from the space Y onto X, a continuous function $y \to \theta(y)$ from Y into the space of bounded linear operators on E (given its strong operator topology) such that for each $y \in Y$, $\theta(y)$ is an isometry on E and

$$(Af)(y) = \theta(y)f(T(y)), \quad \text{for } f \in C(X, E), \; y \in Y. \tag{3}$$

Further, in 1989 Fleming and Jamison [116] presented another generalised version of the classical Banach-Stone theorem using the theory of hermitian operators. We recall that a bounded linear operator A on a Banach space E is norm hermitian if $[Ax, x]$ is real for every $x \in E$, where $[\ .\ ]$ is a semi-inner product on E with the property that $[x, x] = \| x \|^2$. There are other equivalent definitions of hermitian operators but here we are not in need of them and for more details of these things the reader is referred to [226]. A Banach space E is said to have trivial hermitians if the only hermitian operators are real multiples of the identity operator. Some well-known Banach spaces have this property; for example, $H^p(D)$ for $1 \leq p < \infty$, the space $C^1[0,1]$ of continuously differentiable functions, and $AC[0,1]$, the space of absolutely continuous functions [26, 28]. Fleming and Jamison [116, Proposition 6.1] have shown that if X and Y are compact Hausdorff spaces and E and F are Banach spaces with trivial hermitians, then an operator A from $C(X, E)$ into $C(Y, F)$ is a surjective isometry if and only if there exists a homeomorphism T from Y onto X, a continuous function $y \to \theta(y)$ from Y into the space of bounded linear operators from E to F (given its strong operator topology) such that for each y, $\theta(y) : E \to F$ is a surjective isometry and

$$(Af)(y) = \theta(y) f(T(y)), \quad \text{for every } f \in C(X, E), \text{ and } y \in Y.$$

A more detailed account on the Banach–Stone theorem can be found in the monograph of Behrends [19] which contains Behrends's own results. The most recent results known to the authors are due to Cambern and Jarosz [50] who described the isometries of the spaces $C(X, E^*)$, where X is a compact Hausdorff space and E^* is equipped with weak*–topology. In fact, Cambern and Jarosz [50, Theorem 2,3,4] have shown that if E^* and F^* have trivial centralizers (for the definition and properties of the centralizer of a Banach space the reader is referred to [19]) and satisfy a topological condition, namely that the weak* and norm topologies coincide on the surface of the unit ball, then for every surjective isometry $A : C(X, E^*) \to C(Y, F^*)$, there exists a homeomorphism T from Y onto X, an operator-valued mapping $y \to \theta(y)$, and a dense set G in Y on which

$$(Af)(y) = \theta(y) f(T(y)), \quad f \in C\left(X, E^*\right). \tag{4}$$

In addition, if the spaces E^* and F^* are separable, then the operators $\theta(y)$ are surjective isometries and the map $y \to \theta(y)$ is continuous. Moreover, when X and Y are metric spaces then the representation (4) holds on whole of Y.

Now, we shall present the characterizations of isometries of L^p-spaces. In the beginning, the isometries of $L^p[0,1]$, $1 \leq p < \infty$, $p \neq 2$ were described by Banach [16], and then Lamperti [214] carried these results to the spaces $L^p(X)$ for any σ-finite measure space $(X, \mathcal{S}, \mu)$. Further, Cambern [49] and Sourour [378] have investigated the surjective isometries of $L^p(X, H)$ and $L^p(X, E)$, $1 \leq p < \infty$, $p \neq 2$, for a separable Hilbert space H and a separable Banach space E, respectively. Here we are presenting some of the results on the isometries of these spaces.

If $(X, \mathcal{S}, \mu)$ is a measure space, then we recall that a set isomorphism is a map ϕ of $\mathcal{S}$ into itself, defined modulo null sets, such that $\phi(X \setminus S) = \phi(X) \setminus \phi(S)$, $\phi\left(\bigcup_{n=1}^{\infty} S_n\right) = \bigcup_{n=1}^{\infty} \phi(S_n)$, and $\mu(\phi(S)) = 0$ if and only if $\mu(S) = 0$, for all sets S and sequence $\{S_n\}$ in $\mathcal{S}$. A set isomorphism induces a linear transformation, also denoted by ϕ, on the space of measurable functions, which is characterised by $\phi\chi_S = \chi_{\phi(S)}$. Also, we note that if $1 \le p < \infty$, $p \neq 2$ and f, g are in $L^p(\mu)$, then

$$\| f + g \|^p + \| f - g \|^p = 2\left(\| f \|^p + \| g \|^p\right) \tag{5}$$

if and only if $f.\, g = 0$, *a. e.* The proof of the foregoing remark can be established by using Lemma 14 of [285, p. 331].

Theorem 5.1.9. Let $1 \le p < \infty$, $p \neq 2$, and let $A : L^p[0,1] \to L^p[0,1]$ be an isometry. Then there is a Borel measurable mapping T from $[0,1]$ onto itself and a function $\theta \in L^p[0,1]$ such that

$$Af = W_{\theta,T} f, \qquad \text{for every } f \in L^p[0,1].$$

The function θ is uniquely determined (within *a.e.* equivalence) and T is uniquely determined (within *a.e.* equivalence) on the set where $\theta \neq 0$. For any Borel set S, we have

$$\int_{T^{-1}(S)} | \theta |^p dt = \int_S dt.$$

Proof. Let $f \in L^p[0,1]$. Then supt f, the support of f, is an element in $\mathcal{S}/\mathfrak{X}$, the σ-algebra of Borel sets modulo sets of measure zero. Now, we define a map $\phi : \mathcal{S} \to \mathcal{S}/\mathfrak{X}$ by

$$\phi(S) = \text{supt } A\chi_S, \quad \text{for } S \in \mathcal{S}.$$

Also, if S_1 and S_2 are disjoint sets, then by the foregoing argument (5), it follows that

$$\left\| \chi_{S_1} + \chi_{S_2} \right\|^p + \left\| \chi_{S_1} - \chi_{S_2} \right\|^p = 2\left(\left\| \chi_{S_1} \right\|^p + \left\| \chi_{S_2} \right\|^p \right),$$

and

$$\left\|A\,\chi_{S_1}+A\,\chi_{S_2}\right\|^p+\left\|A\,\chi_{S_1}-A\,\chi_{S_2}\right\|^p=2\left(\left\|A\,\chi_{S_1}\right\|^p+\left\|A\,\chi_{S_2}\right\|^p\right),$$

since A is norm preserving. From this we conclude that $\phi(S_1).\phi(S_2)=0$ *a.e.*. This shows that ϕ takes disjoint sets to disjoint sets. Again, we note that if S_1 and S_2 are disjoint then

$$\chi_{S_1\cup S_2}=\chi_{S_1}+\chi_{S_2}$$

and

$$A\,\chi_{S_1\cup S_2}=A\,\chi_{S_1}+A\,\chi_{S_2}.$$

Thus $\phi(S_1\cup S_2)=\phi(S_1)+\phi(S_2)$, for any pair of Borel sets. Also, for $\phi([0,1])=Y$, we have

$$\phi([0.1]\setminus S]=Y\setminus\phi(S).$$

This proves that ϕ is a homomorphism of $\mathcal{S}$ into the algebra of Borel subsets of Y. If $S=\bigcup_{n=1}^{\infty}S_n$ where $\{S_n\}$ is a sequence of disjoint sets, then

$$\chi_S=\lim_m\sum_{n=1}^{m}\chi_{S_n}\quad\text{is in}\quad L^p[0,1],$$

and by the continuity of A, $A\,\chi_S=\lim_m\sum_{n=1}^{m}A\,\chi_{S_n}$. Thus we have $\phi(S)=\bigcup_{n=1}^{\infty}\phi(S_n)$.

Hence ϕ is a σ-homomorphism. Therefore, there exists a Borel measurable mapping T of Y into [0,1] such that $\phi(S)=T^{-1}(S)$. The map T must be onto all of [0,1] except for a set of measure zero because ϕ can only take sets of measure zero into sets of measure zero. Now extend the map T on all of [0,1] by setting $T(t)=0$ for $t\in Y$. Consider the function $1=\chi_{[0,1]}$ and set $\theta=A(1)$. Then θ belongs to $L^p[0,1]$ and supt $\theta=Y$. Let $S\in\mathcal{S}$. Then we have $1=\chi_S+\chi_{[0,1]\setminus S}$ and $\theta=A\,\chi_S+A\,\chi_{[0,1]\setminus S}$. Since the functions on the right side of the equality have disjoint supports, we have

$$A\,\chi_S=\theta.\chi_{\phi(S)}=\theta.\left(\chi_S\circ T\right).$$

This representation is also true for simple functions. Since the set of simple functions is dense in $L^p[0,1]$, it follows that

$$Af = \theta . f \circ T, \qquad \text{for all} \quad f \in L^p[0,1].$$

Thus $Af = W_{\theta,T} f$. Finally, for $S \in \mathcal{S}$, we have

$$\begin{aligned} \int_{T^{-1}(S)} |\theta|^p dt = \int \chi_{T^{-1}(S)} |\theta|^p dt = \int \left| \theta . \chi_S \circ T \right|^p dt \\ = \int \left| A \chi_S \right|^p dt \\ = \int \chi_S \, dt \\ = \int_S dt . \end{aligned}$$

This completes the proof of the thoerem.

Lamperti [214] has generalised the above theorem and determined the isometries of $L^p(X)$ for any σ-finite measure space $(X, \mathcal{S}, \mu)$. He has shown that an operator $A : L^p(X) \to L^p(X)$, $p \neq 2$ is an isometry if and only if there exists a set isomorphism ϕ and a function $\theta : X \to \mathbb{C}$ such that

$$(Af)(x) = \theta(x)(\phi(f))(x),$$

and

$$|\theta|^p = \frac{d\mu\phi^{-1}}{d\mu} \quad a.e. \quad \text{on} \quad \phi(X).$$

In [49] Cambern has further generalised the above result to the L^p-spaces of vector-valued functions. Moreover, if $(X, \mathcal{S}, \mu)$ is a finite measure space and $L^p(X, H)$ is the Banach space of all measurable functions f defined on X and taking values in a separable Hilbert space H, such that $\| f(x) \|^p$ is integrable, then Cambern has shown that an operator $A : L^p(X, H) \to L^p(X, H)$, $1 \leq p < \infty$, $p \neq 2$, is a surjective isometry if and only if A is of the form $(A(f))(x) = \psi(x)\theta(x)(\phi(f))(x)$, where ϕ is a set isomorphism of $\mathcal{S}$ onto itself, ψ is a weakly measurable operator-valued function such that $\psi(x)$ is $(a.e.)$ an isometry of H onto itself, and θ is a scalar function which is defined by ϕ through a formula involving Radon–Nikodym derivatives. In order to have a touch of the proof of the above statement, it will be more suitable to present the proof of the results on the surjective isometries of $L^p(X,E)$ spaces, where E is a separable Banach space. This result has been obtained by Sourour [378] using the properties of hermitian operators. We shall present the characterization in the following theorem.

Theorem 5.1.10. Let $(X, \mathcal{S}, \mu)$ be a finite measure space, let A be an operator on $L^p(X,E)$, $1 \le p < \infty$, $p \ne 2$ and assume that E is not the ℓ^p–direct sum of two nonzero Banach spaces (for the same p). Then A is a surjective isometry if and only if

$$(Af)(.) = \psi(.)\,\theta(.)\,\big(\phi(f)\big)(.), \qquad \text{for} \quad f \in L^p(X,E),$$

where ϕ is a set isomorphism of the measure space onto itself, ψ is a strongly measurable map of X into $B(E)$ with $\psi(x)$ a surjective isometry of E for almost all $x \in X$, and $\theta = (d\upsilon/d\mu)^{1/p}$ where $\upsilon = \mu \circ \phi^{-1}$.

In order to prove this theorem we shall present some propositions and preliminary results. If ϕ is a set homomorphism of the measure space $(X, \mathcal{S}, \mu)$, then there is a maximal element $X_0 \in \mathcal{S}$ for which $\phi(X_0)$ is a null set. The set X_0 is maximal in the sense that if $\phi(S)$ is a null set, then $S \setminus X_0$ is a null set. The existence of X_0 follows from Zorn's lemma. If $\{S_\alpha\}$ is an increasing chain of sets for which $\mu(\phi(S_\alpha)) = 0$, then there are only countably many essentially different $S_\alpha' s$ and hence $\mu(\phi(\bigcup_\alpha S_\alpha)) = 0$. This set X_0 is called the kernel of ϕ. Also, note that ϕ restricted to subsets of $X \setminus X_0$ is a set isomorphism. If ϕ is a set homomorphism and if X_0 is its kernel, then define the measure υ by

$$\upsilon(R) = \mu\big(\phi^{-1}(R) \setminus X_0\big), \quad R \in \mathcal{S}. \tag{6}$$

Note that $\phi^{-1}(S)$ is not unique, but $\phi^{-1}(R) \setminus X_0$ is unique upto a null set. The measure υ is absolutely continuous with respect to μ and let

$$\theta = \big(d\upsilon/d\mu\big)^{1/p}. \tag{7}$$

Also, we say that E is the ℓ^p–direct sum of two Banach spaces E_1 and E_2 if E is isometrically isomorphic to $E_1 \oplus_p E_2$, where the norm on the direct sum is given by

$$\|y_1 + y_2\| = \left\{\|y_1\|^p + \|y_2\|^p\right\}^{1/p}.$$

In what follows $(X, \mathcal{S}, \mu)$ will be a finite measure space.

Proposition 5.1.11.

(i) Let E be a Banach space, and let ϕ be set homomorphism on $(X, \mathcal{S}, \mu)$. Let f be a E-valued measurable function and let $\{f_n\}$ be a sequence of simple functions converging to f in measure. Then $\{\phi(f_n)\}$ is a Cauchy sequence in measure and hence converges in measure to a measurable function $\phi(f)$.. Furthermore, $\phi(f)$ depends only on f and not on the particular sequence $\{f_n\}$.

(ii) $\phi\big(\| f(\,.\,) \|\big) = \| \phi(f)(\,.\,) \|$ for every measurable function f.

(iii) Let $f \in L^p(X,E)$. Then $\theta\,\phi(f) \in L^p(X,E)$ and $\| \theta\, \phi(f) \| \le \| f \|$. Equality holds for all f if and only if ϕ is a set isomorphism.

Proof. (i) We observe that

$$\left\{x \ : \ \| \phi(f_n)(x) - \phi(f_m)(x) \| > \alpha\right\} = \phi\left\{x \ : \ \| f_n(x) - f_m(x) \| > \alpha\right\}.$$

Also, the measure $\mu(\phi(.))$ is absolutely continuous with respect to μ and therefore is μ-continuous [97, p. 113]. That is, given $\varepsilon > 0$, there is $\delta > 0$ such that $\mu(\phi(S)) < \varepsilon$ whenever $\mu(S) < \delta$. Let $\alpha > 0$. Then choose n_0 such that

$$\mu\left\{x \ : \| f_n(x) - f_m(x) \| > \alpha\right\} < \delta \qquad \text{for } n, m \ge n_0 .$$

This implies that $\mu\left\{x \ : \ \left\|(\phi(f_n))(x) - (\phi(f_m))(x)\right\| > \alpha\right\} < \varepsilon$, for $n, m \ge n_0$. Thus $\{\phi(f_n)\}$ is a Cauchy sequence in measure and hence it converges in measure to a function which will be denoted by $\phi(f)$ [97, p. 145]. Now, we shall show that $\phi(f)$ is well-defined. For this, it is sufficient to prove that if $\{f_n\}$ is a sequence of simple functions such that $f_n \to 0$ in measure and $\phi(f_n) \to g$ in measure, then $g = 0$. Again in light of [97, p. 145], we may assume, by taking a subsequence if necessary, that $\{f_n\}$ and $\{\phi(f_n)\}$ converge almost uniformly. Fix $\varepsilon > 0$ and select δ as above. Then choose a set $S_0 \in \mathcal{S}$ such that $\mu(S_0) < \delta$ and $f_n \to 0$ uniformly on $X \setminus S_0$. Now it is easy to see that $\phi(f_n) \to 0$ uniformly on $\phi(X) \setminus \phi(S_0)$

and $\mu(\phi(S_0)) < \varepsilon$. This implies that $\phi(f_n) \to 0$ almost uniformly on $\phi(X)$ and hence $g = 0$ *a.e.* on $\phi(X)$. Since $\phi(f_n) = 0$ on $X \setminus \phi(X)$, therefore we have $g = 0$ *a. e.* This completes the proof of (i).

(ii) Let g be the function defined as

$$g(x) = \| f(x) \| , \quad \text{for all} \quad x \in X.$$

Then $\| f(\,.\,) \|$ is the scalar function g. We observe that the result follows easily for simple functions and hence holds for all measurable functions because if $f_n \to f$ in measure then $\| f_n(.) \| \to \| f(.) \|$ in measure.

(iii) Let X_0 be the kernel of ϕ and let $f \in L^p(X, E)$. We set $f_0 = \chi_{X_0} f$, $f_1 = f - f_0$, $g(x) = \| f(x) \|$ and $g_1(x) = \| f_1(x) \|$, for all $x \in X$. Then $g, g_1 \in L^p(X)$. Since ϕ restricted to $X \setminus X_0$ is a set isomorphism, in view of Lamperti's result [214] it follows that $\|\theta\ \phi(g_1)\| = \|g_1\|$. Further, from the above part (ii), we get $\|\theta\ \phi(f_1)\| = \|f_1\|$. Since $\phi(f_0) = 0$, we have the desired result. With this the proof of the proposition is completed.

Proposition 5.1.12. Let A be a bounded linear operator on $L^p(X, E)$, $1 \le p < \infty$ such that A maps functions of (almost) disjoint supports into functions of (almost) disjoint supports. Then there is a set homomorphism ϕ of $\mathcal{S}$ and a strongly measurable map ψ from X into $B(E)$ such that

$$(Af)(.) = \psi(.)\left(\phi(f)\right)(.), \quad \text{for every } f \in L^p(X, E) .$$

Proof. Let $\{e_n\}$ be a countable linearly independent subset of E such that linear span of $\{e_n\}$, denoted by K, is dense in E. Let K_0 be the set of all linear combinations of $\{e_n\}$ with complex rational coefficients and suppose that A satisfy the hypothesis of the proposition. Now, we fix $S \in \mathcal{S}$. Then define the set function $\phi : \mathcal{S} \to \mathcal{S}$ as

$$\phi(S) = \bigcup_n \operatorname{supt}\left(A\left(\chi_S e_n\right)\right).$$

It is now straightforward that $A\left(\chi_{S_1} e_n\right)$ and $A\left(\chi_{S_2} e_m\right)$ are almost disjoint for every n and m in case S_1 and S_2 are disjoint sets. Thus $\phi(S_1)$ and $\phi(S_2)$ are almost disjoint and the equation $A\left(\chi_{S_1} e_n\right) + A\left(\chi_{S_2} e_n\right) = A\left(\chi_{S_1 \cup S_2} e_n\right)$ implies that

$\phi(S_1 \cup S_2) = \phi(S_1) \cup \phi(S_2)$ within a null set. Further it can be extended to countable unions of disjoint sets because A is continuous. Also the extension to countable unions of any sets easily follows. This shows that ϕ is a set homomorphism. Let X_0 be the kernel of ϕ. Since the null space of A is the space of functions vanishing (*a.e.*) on $X \setminus X_0$, therefore ϕ is a set–isomorphism if and only if A is one–to–one. Assume that υ and θ are as in (6) and (7). We can choose $\theta(x) > 0$ for every $x \in \phi(X)$ and $\theta(x) = 0$ for every $x \notin \phi(X)$. Set $f_n = A(1_{e_n})$, where 1_{e_n} is the constant function on X taking values e_n. We may assume that $f_n(x) = 0$ for all n and $x \notin \phi(X)$. For every $x \in X$, we define

$$\psi(x)e_n = f_n(x), \quad \text{for every } n \in \mathbb{N}.$$

After extending $\psi(x)$ linearly to K, we get

$$\psi(x)\left(\sum_{i=1}^{k} \lambda_i e_i\right) = \sum_{i=1}^{k} \lambda_i f_i(x).$$

Thus it implies that for every $y \in K$, $\psi(.)\,(y) = A(1_y)$ *a.e.* Now we shall show that $\psi(x)$ is a bounded operator on E. Let $S \in \mathcal{S}$, $S_1 = S \setminus X_0$, and $y \in K_0$. Then

$$\begin{aligned}
\int_{\phi(S)} \| \psi(x)y \|^p d\mu(x) &= \int_{\phi(X)} \left\| \left(A(1_y)\right)(x) \right\|^p d\mu(x) \\
&= \int \left\| A\left(\chi_{\phi(S)}y\right)(x) \right\|^p d\mu(x) \\
&= \left\| A\left(\chi_{S_1}y\right) \right\|^p \\
&\le \| A \|^p \mu(S_1) \| y \|^p \\
&= \| A \|^p \| y \|^p \int_{\phi(S)} (\theta(x))^p d\mu(x) .
\end{aligned}$$

Thus $\| \psi(x)y \| \le \| A \| \| y \| \, | \theta(x) |$, for almost all $x \in \phi(X)$. For $x \notin \phi(X)$, the same inequality is trivial. Hence there is a null set S_0 such that

$$\| \psi(x)y \| \le \| A \| \| y \| \theta(x), \quad \text{for } y \in K_0,\ x \notin S_0. \tag{8}$$

Further, if $y = \sum_{i=1}^{n} \lambda_i e_i$ and E_n is the linear span of $e_1, \ldots, e_n$, then the restriction of $\psi(x)$ to E_n is a linear map between two finite dimensional spaces and hence is bounded. Also the norm $\| \psi(x) \| \le \| A \| \theta(x)$ since $K_0 \cap E_n$ is dense in E_m. This proves that (8) holds for all $y \in K$ and thus $\psi(x)$ can be extended to a bounded linear operator on E and $\| \psi(x) \| \le \| A \| \theta(x)$ for $x \notin S_0$. Finally, we show that $\psi : X \to B(E)$ is strongly measurable. To this end, let $y \in E$ and let $\{y_n\} \subset K$ be such that $y_n \to y$. Since $\psi(\,.\,)y_n = A\left(1_{y_n}\right)$ *a.e.*, we conclude from the continuity of almost all $\psi(x)$ and of A that $\psi(\,.\,)y = A\left(1_y\right)$ and hence ψ is measurable. Now, if we set $(A_1 f)(.) = \psi(.)(\phi(f))(.)$, then

$$\begin{aligned}\int \| (A_1 f)(x) \|^p d\mu(x) &\le \int \| \psi(x) \|^p \| (\phi(f))(x) \|^p d\mu(x) \\ &\le \| A \|^p \int (\theta(x))^p \| (\phi(f))(x) \|^p d\mu(x) \\ &\le \| A \|^p \| f \|^p \text{ by Proposition 5.1.11 (iii)}.\end{aligned}$$

This shows that A_1 is a bounded linear operator on $L^p(X, E)$. It is already proved that A_1 agrees with A on constant functions. Also, since

$$A\left(\chi_S y\right) = \chi_{\phi(S)} A\left(1_y\right) = \psi(\,.\,)\chi_{\phi(S)} y = A_1\left(\chi_S y\right),$$

we conclude that A agrees with A_1 on simple functions and this yields that $A = A_1$. This completes the proof of the proposition.

Proposition 5.1.13. Let A, ϕ, ψ and θ be as in Proposition 5.1.12. Then A is an isometry if and only if

(i) ϕ is a set isomorphism from $\mathcal{S}$ into itself,

(ii) $\psi(x) = \pi(x)\theta(x)$, where $\pi(x)$ is an isometry from E into itself for almost all x. Furthermore, A is onto if and only if ϕ and almost all $\pi(x)$ are onto.

Proof. Suppose (i) and (ii) hold. Then Proposition 5.1.11 (iii) gives that A is an isometry. Conversely, we assume that A is an isometry. Then A is one–one and this further implies that the set X_0, the kernel of ϕ, is a μ–null set. This proves (i). Now, define $\pi(x) = \psi(x) / \theta(x)$, for $x \in \phi(X)$ and $\pi(x) = I$, the identity operator, otherwise. Let $S \in \mathcal{S}$ and $f = \chi_S y$. Then it follows that

$$\begin{aligned}\int_{\phi(S)} (\theta(x))^p \| \pi(x) y \|^p d\mu(x) &= \mu(S) \| y \|^p \\ &= \int_{\phi(S)} (\theta(x))^p \| y \|^p d\mu(x).\end{aligned}$$

Further, it implies that $\| \pi(x)y \| = \| y \|$ for almost all $x \in \phi(X)$ as the above equality holds for all S. Note that the same equality is also true for $x \notin \phi(X)$. Let $\{y_n\}$ be a countable dense set in E and let S_0 be a null set such that $\| \pi(x)y_n \| = \| y_n \|$ for all n and all $x \notin S_0$. From the boundedness of $\pi(x)$, it follows that $\| \pi(x)y \| = \| y \|$ for all y and almost all x. This establishes (ii). Now, if we suppose that ϕ and almost all $\pi(x)$ are onto, and if we set $A_1 f = \theta(.)(\phi(f))(.)$ and $A_2 f = \pi(.)f(.)$, then it is easy to show that both A_1 and A_2 and hence A are surjective. On the other hand if A is onto, then ϕ is clearly onto and thus A_2 must be surjective. In particular, the functions $1_{y_n} \in A_2(L^p(X,E))$. Thus there exist functions $f_n \in L^p(X,E)$ and a null set S_0 such that $\pi(x)f_n(x) = y_n$ for all $x \notin S_0$. This implies that $K \subset \pi(x)(E)$ for $x \notin S_0$. Since $\pi(x)$ is an isometry, therefore it has a closed range. That is $\pi(x)(E) = E$, for almost all $x \in X$. This completes the proof of the proposition.

Proposition 5.1.14. Let $A : L^p(X,E) \to L^p(X,E)$ be an operator $1 \le p < \infty$, $p \ne 2$. Then A is hermitian if and only if $(Af)(.) = \psi(.)f(.)$ for a hermitian valued strongly measurable map ψ from X into $B(E)$.

Proof. For the proof the reader is referred to [378].

Proof of Theorem 5.1.10. First of all we show that A is an isometry whenever A is of the given form. We set $(A_1 f)(x) = \theta(x)(\phi(f))(x)$ and $(A_2 f)(x) = \psi(x)(f(x))$. Then $A = A_2 A_1$ and it is immediate that A_2 is an isometry from $L^p(X,E)$ onto itself. Also Proposition 5.1.11 (iii) gives us that A_1 is norm preserving. To show that A_1 is onto, it suffices to do that for the scalar-valued case since this implies that the range of A_1 contains $\{g(.)y : g \in L^p(X), y \in E\}$ which is dense in $L^p(X,E)$. The scalar-valued case can be proved by showing that 0 is the only function in $L^q(X)$, $1/p + 1/q = 1$ which annihilates $\{\theta(.)\phi(f(.)) : f \in L^p(X)\}$. Now suppose that A is an isometry of $L^p(X,E)$ onto itself. We note that if J is a hermitian operator on $L^p(X,E)$ then so is AJA^{-1}. Let $S \in \mathcal{S}$. Then define J_1 by $(J_1 f)(.) = \chi_S f$. Thus AJ_1A^{-1} is a hermitian projection. In view of Proposition 5.1.14, it follows that

$$\left(AJ_1A^{-1}f\right)(x) = P(x)f(x), \quad f \in L^p(X,E),$$

where $P(x)$ is a hermitian projection for almost all $x \in X$. Thus

$$\left(A\left(\chi_S g\right)\right)(.) = P(.)(Ag)(.)\,, \quad g \in L^p(X,E)$$

and

$$\left(A\left(\chi_{X\setminus S}g\right)\right)(.) = Q(.)(Ag)(.), \quad g \in L^p(X,E)$$

where $Q(x) = I - P(x)$. Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be the subspaces of functions in $L^p(X,E)$ vanishing on $X \setminus S$, and S respectively. Also, set $\mathcal{N}_1 = \{P(.)f(.) : f \in L^p(X,E)\}$ and $\mathcal{N}_2 = \{Q(.)f(.) : f \in L^p(X,E)\}$. Further, it implies that $A\mathcal{M}_i = \mathcal{N}_i$ for $i = 1, 2$. Now it is immediate that $\| f_1 + f_2 \|^p = \| f_1 \|^p + \| f_2 \|^p$ for any $f_1 \in \mathcal{N}_1$ and $f_2 \in \mathcal{N}_2$ since the same holds for any $f_1 \in \mathcal{M}_1$ and $f_2 \in \mathcal{M}_2$. Fix $y \in E$ and $S \in \mathcal{S}$. Let $f_1(.) = P(.)\chi_S(.)y$ and $f_2(.) = Q(.)\,\chi_S(.)y$. Then we have $f_i \in \mathcal{N}_i$ and

$$\int_S \| P(x)y \|^p d\mu + \int_S \| Q(x)y \|^p d\mu(x) = \int_S \| y \|^p d\mu(x)\,.$$

Since the above equality holds for any $S \in \mathcal{S}$, it follows that

$$\| y \|^p = \| P(x)y \|^p + \| Q(x)y \|^p \ \ a.e.$$

Therefore, there exists a null set S_0 such that the above equality holds for all $x \notin S_0$ and all y in a countable dense set; and hence the same is true for $x \notin S_0$ and all $y \in E$. Thus E is the ℓ^p-direct sum of $P(x)(E)$ and $Q(x)(E)$, and hence for almost all $x, P(x) = 0$ or I. That is, $P(.) = \chi_S(.)I$ *a. e.*, for some $S \in \mathcal{S}$. Thus it follows that A maps $\mathcal{M}_1$ onto the space of functions which (almost) vanish on $X \setminus S$ and maps $\mathcal{M}_2$ onto the space of functions which (almost) vanish on S. Now Proposition 5.1.12 and Proposition 5.1.13 implies that A is of the desired form. This completes the proof of the theorem.

Remark. In a recent paper [218] Lin has considered the isometries of $L^2(X,E)$ under certain conditions on E. In fact, he proved that if E is not one dimensional and if E is separable with trivial L^2-structure and $(X, \mathcal{S}, \mu)$ is σ-finite, then the hermitian operators and isometries on $L^2(X,E)$ have forms like the hermitian operators and isometries on $L^p(X,E)$.

Now, we shall present some results on the isometries of some H^p-spaces which falls in the second phase of study presented in this section. The theory goes back to the papers of Deleeuw, Rudin and Wermer [92], and Nagasawa [249] in which they have described the isometries of $H^1(D)$ and $H^\infty(D)$ whereas the isometries of $H^p(\partial D)$ for $1 < p < \infty, p \neq 2$, have been investigated by Forelli [118]. Since then the efforts have been made by many mathematicians to generalise the obtained results to the spaces of vector–valued functions. In [48] the isometries of $H^\infty(D)$ were generalised to the context of $H^\infty(D, H)$ for a finite dimensioinal Hilbert space H and very recently in 1990 Lin [219] has extended the results on isometries of $H^\infty(D, H)$ to the spaces $H^\infty(D, E)$ for certain infinite dimensional Banach spaces E. On the other hand Cambern and Jarosz [52] have been able to carry the results of $H^1(D)$ to the spaces $H^1(D, H)$ for a finite dimensional Hilbert space H, and in the same paper an open question has been raised for infinite dimensional range spaces. In case of $H^p(\partial D)$, $1 < p < \infty$, $p \neq 2$, Forelli's result is still in the scalar case and there is no generalization to the spaces of vector–valued functions as far as we know.

We note that in each of the cases mentioned above the composition operators are employed to characterise the isometries. In particular these isometries are like the weighted composition operators. For a comprehensive account of these results on $H^1(D)$ and $H^\infty(D)$ we refer the book [146] by Hoffman. Here we are demonstrating the application of composition operators in characterizing the isometries of H^p, $1 < p < \infty$, $p \neq 2$. In this direction, we need to write some notations and results which are used in characterising the isometries of H^p–spaces.

Let μ be the Lebesgue measure on the unit circle ∂D with $\int d\mu = 1$ and consider the space L^p of complex–valued μ–measurable functions f such that $\int |f|^p d\mu$ is finite. Also, note that H^p is the closure in L^p of the algebra of analytic polynomials

$$\sum_{n=0}^{\infty} c_n z^n,\ z \in \partial D.$$

For $p \geq 1$, H^p has a clear alternative description as the subspace in L^p of functions whose Fourier Coefficients vanish for negative indices. If T is a non–constant inner function and if υ is the Borel measure on the unit circle induced by T *i.e.*, $\upsilon(S) = \mu(T^{-1}(S))$, for Borel subsets S of the unit circle, then

$$\int f d\upsilon = \int f \circ T\ d\mu,$$

where f is a Borel–measurable function, and the Fourier–Stieltjes coefficient of υ at n is

$$\int \bar{T}^n d\mu.$$

As averaging with μ is a multiplicative linear functional on H^∞,

$$\int T^n d\mu = \left(\int T \, d\mu\right)^n,$$

when $n > 0$, and thus the Fourier–Stieltjes coefficient of υ at $n > 0$ is

$$\left(\int \bar{T} d\mu\right)^n.$$

Therefore

$$\upsilon = P\,\mu, \tag{9}$$

where P is the Poisson kernel given by

$$P(z) = \left(1 - |a|^2\right) \Big/ |1 - az|^2,$$

where $a = \int \bar{T}\, d\mu$. We say that P is the Poisson kernel induced by T. Now, since υ is absolutely continuous with respect to μ, in view of the definition of υ it follows that T^{-1} takes subsets of the unit circle with μ measure zero into sets of the same kind. Here by $\mathcal{S}$ we denote the collection of μ–measurable subsets of the unit circle, and by $\mathcal{S}_T$, the collection of sets $S_1 \,\Delta\, S_2$, where $S_1 = T^{-1}(S)$ with $S \in \mathcal{S}$ and $S_2 \in \mathcal{S}$ with $\mu(S_2) = 0$. Thus $\mathcal{S}_T \subset \mathcal{S}$. Also, we say that an inner function T is a conformal map of the unit disk onto itself if T is (*a.e.*) the restriction of such a function to the unit circle. Furthermore, we shall record some results without proofs in the following proposition which are due to Forelli [118].

Proposition 5.1.15.

(i) Assume that $f_i \in L^p(\mu_i)$, $i = 1, 2$ and that for all z

$$\int |1 + zf_1|^p d\mu_1 = \int |1 + zf_2|^p d\mu_2.$$

Then

$$\int |f_1|^2 d\mu_1 = \int |f_2|^2 d\mu_2 .$$

(ii) Let $\mathcal{M}$ be a subalgebra of $L^\infty(\mu_1)$ that contains constants, and let A be a linear transformation of $\mathcal{M}$ into $L^\infty(\mu_2)$ with $A(1) = 1$. Suppose that $p \neq 2$ and

$$\int | Af |^p d\mu_2 = \int | f |^p d\mu_1,$$

for all $f \in \mathcal{M}$. Then A is multiplicative.

Theorem 5.1.16.

(i) Let $p \neq 2$ and let A be an isometry of H^p into H^p. Then there is a non-constant inner function T and a function ψ in H^p such that

$$Af = \psi\, C_T f, \qquad \text{for } f \in H^p. \tag{10}$$

T and ψ are related by

$$\int_S | \psi |^p d\mu = \int_S 1 / P(T) d\mu, \quad S \in \mathcal{S}_T, \tag{11}$$

where P is the Poisson kernel induced by T. Conversely, when a non-constant inner function T and a function ψ in H^p are related by (11), (10) defines an isometry of H^p into H^p.

(ii) Let $p \neq 2$ and let A be an isometry of H^p onto H^p. Then

$$Af = b(dT / dz)^{1/p} C_T f, \tag{12}$$

where T is a conformal map of the unit disk onto itself and b is a unimodular complex number. Conversely, (12) defines an isometry of H^p onto H^p.

Proof. (i) Assume that A is an isometry of H^p into H^p and let $\psi = A(1)$. Then $\psi \in H^p$ such that $\psi \neq 0$ and ψ cannot vanish on any set of positive μ–measure. Let υ be the measure such that $d\upsilon = | \psi |^p d\mu$. Then υ and μ are mutually absolutely continuous. If a linear transformation $L : H^p \to L^p(\upsilon)$ is defined by $Lf = Af / \psi$, then L is an isometry of H^p into $L^p(\upsilon)$ with $L(1) = 1$. Let f_0 be the inner function defined as $f_0(z) = z$. Then

$$\int \left| L\left(f_0^n\right) \right|^p d\upsilon = 1.$$

Further, in view of Proposition 5.1.15 (i), we get

$$\int \left| L\left(f_0^n\right) \right|^2 d\upsilon = 1.$$

This implies that $\left| L\left(f_0^n\right) \right| = 1$ as $p \neq 2$. Thus L takes the algebra generated by f_0 into $L^\infty(\upsilon)$. From Proposition 5.1.15 (ii) we know that L is multiplicative on this algebra, therefore for a polynomial g we have $L(g(f_0)) = g(Lf_0)$ and

$$A\left(g\left(f_0\right)\right) = \psi C_T g, \tag{13}$$

where $T = Lf_0$. Since $\psi \in H^p$, we have $\psi = F.G$, where F is an inner function and G is an outer function in H^p. Further (13) implies that $F.T^n$ is an inner function for $n \geq 0$. Now, we shall show that T is an inner function. Let $\mathcal{M}$ be the closed subspace of L^2 spanned by $f_0^j T^k (j, k \geq 0)$. Then $\mathcal{M}$ is invariant under multiplication by f_0, but not by $\bar{f}_0$ as $F(\mathcal{M})$ is contained in H^2. Thus $\mathcal{M} = \theta(H^2)$, where $|\theta| = 1$. Now $f_0^j T^k \theta$ is in $\mathcal{M}$ for $j, k \geq 0$, and θ can be approximated by polynomials in f_0 and T. Thus $\theta^2 \in \mathcal{M}$ and $\theta^2 = \theta g$, where $g \in H^2$. This further implies that $\theta \in H^2$ and $\mathcal{M} = H^2$. Thus T is an inner function because $T \in \mathcal{M}$. Since the algebra generated by f_0 is dense in H^p and A is bounded, it follows that

$$Af = \psi f \circ T, \quad \text{for all } f \in H^p,$$

where $\psi \in H^p$ and T is a non-constant inner function. Also, for $f \in H^p$, we have

$$\int |\psi|^p |f \circ T|^p d\mu = \int |f|^p d\mu. \tag{14}$$

Let $S = T^{-1}(S_1)$, where $S_1 \in \mathcal{S}$. From (14) it follows that

$$\int_S |\psi|^p d\mu = \int_{S_1} d\mu \tag{15}$$

because the characteristic function of S_1 can be approximated by the moduli of functions in H^p, and we get from (9) that

$$\int_{S_1} d\mu = \int_{S_1} 1/P \, d\upsilon = \int_S 1/P(T) d\mu \,. \tag{16}$$

Thus (15) and (16) imply the desired form (11). Conversely, if T is a non-constant inner function and $\psi \in H^p$ and they are related by (11), then (15) is an outcome of (11) and (16). Further, if A is restricted to simple functions, then (15) shows that A, defined by (10), is an isometry of L^p into itself. Thus A is an isometry of L^p. Since A takes the algebra generated by f_0 into H^p, it is immediate that A takes H^p

into H^p. This completes the proof of (i).

(ii) Since A and A^{-1} are isometries, therefore we have $Af = \psi f \circ T$ and

$$A^{-1}f = \eta\, f \circ \phi.$$

Thus from the relation $AA^{-1}f = A^{-1}Af = f$, it follows that

$$\eta(T)f(\phi(T)) = \eta\psi(\phi)f(T(\phi)) = f.$$

Now, taking $f = 1$, we get

$$\psi\eta(T) = \eta\psi(\phi) = 1.$$

Further, it implies that

$$f(\phi(T)) = f(T(\phi)) = f.$$

This proves that T is a conformal map of the unit disk onto itself. Now, since T is a conformal map of the unit disk onto itself, we have $\mathcal{S}_T = \mathcal{S}$ and $|dT / dz| = 1 / P(T)$, where P is the Poisson kernel induced by T. Thus (11) reduces to

$$\int_S |\psi|^p d\mu = \int_S |dT / dz| d\mu ,$$

where $S \in \mathcal{S}$. This shows that ψ and $(dT / dz)^{1/p}$ have the same modulus. ψ is an outer function, for otherwise the non-constant inner factor of ψ would divide every function in H^p, and as $(dT / dz)^{1/p}$ is also an outer function and has the same modulus as ψ, $\psi = b(dT / dz)^{1/p}$, where b is a unimodular complex number. With this the proof of the theorem is completed.

Remark. As we have already pointed out that the above result is not obtained in the setting of vector-valued functions whereas the isometries of $H^1(D)$ and $H^\infty(D)$ are settled for the vector-valued case. If E is a uniformly convex and uniformly smooth Banach space, then Lin [219] has shown that every isometry A of $H^\infty(D, E)$ onto $H^\infty(D, E)$ is of the form

$$Af = \psi(C_T f), \qquad f \in H^\infty(D, E)$$

where ψ is an isometry from E onto E and T is a conformal map of D onto itself. In case E is a finite dimensional Hilbert space, Cambern and Jarosz have investigated the isometries of the Banach space $H^1(D, E)$ onto itself and proved that every such an isometry A is of the form

$$(Af)(z) = \psi f(T(z))T'(z), \qquad \text{for } f \in H^1(D, E),$$

where T is a conformal map of D onto itself and ψ is a unitary operator on E. For the details of these results the reader is referred to [219] and [52]. Also note that the composition operators are employed in characterizing the surjective isometries of the Bergman spaces and for this detail we refer to [185] and [240].

5.2 ERGODIC THEORY AND COMPOSITION OPERATORS

The ergodic theory interacts with the problems arising in the physical sciences and the theory of the composition operators interacts with the ergodic theory. Some of these interactions are highlighted in this section.

Let G be a group with the identity e and let X be a non-empty set. Let $u : G \times X \to X$ be a mapping such that $u(e, x) = x$ and $u(st, x) = u(s, u(t, x))$ for every $x \in X$ and for every $s, t \in G$. Then u is called an action of G on X or a motion on X induced by G. If $x \in X$, then the function $u^x : G \to X$ defined as $u^x(t) = u(t, x)$ is called a motion through the point x and the range of this function is called the orbit of x which we denote by the symbol orb(x). If $t \in G$, then the function $u_t : X \to X$ defined as $u_t(x) = u(t, x)$ is a bijection and $(u_t)^{-1} = u_{t^{-1}}$. If G is a topological group, X is a topological space and the mapping $u : G \times X \to X$ satisfies an additional condition of continuity, then the triple (G, X, u) is called a transformation group. The transformation group $(\mathbb{Z}, X, u)$ is known as a discrete dynamical system and $(\mathbb{R}, X, u)$ is called a dynamical system, where $\mathbb{Z}$ is the group of integers under addition with the discrete topology and $\mathbb{R}$ is the group of real numbers under addition with the usual topology. If we replace $\mathbb{Z}$ and $\mathbb{R}$ by $\mathbb{Z}^+$ and $\mathbb{R}^+$ respectively, then we have semidynamical systems. For a detailed study of the transformation groups and the dynamical systems we refer to [102] and [397].

If $(X, \mathcal{S}, \mu)$ is a σ-finite measure space and $T : X \to X$ is a non-singular measurable transformation such that the Radon-Nikodym derivative of μT^{-1} with respect to μ is essentially bounded, then by Theorem 2.1.1 we know that T induces the composition operator C_T on $L^p(\mu)$ for $p \geq 1$. If $u_T : \mathbb{Z}^+ \times X \to X$ and $u'_T : \mathbb{Z}^+ \times L^p(\mu) \to L^p(\mu)$ are defined as $u_T(n, x) = T^n(x)$ and $u'_T\ (n, f) = C_T^n f$, then these are discrete actions induced by T on X and $L^p(\mu)$ respectively. If T is a measure preserving transformation, then we know from Example 2.1.4 that C_T is an isometry on $L^p(\mu)$, and if T is also invertible, then C_T is a unitary operator on the

Hilbert Spaces $L^2(\mu)$. Thus every measure preserving transformation gives rise to discrete motions on X and $L^p(\mu)$. More, generally, if $\mathcal{F} = \{T_t : t \in \mathbb{R}\}$ is a group (or semigroup) of measure preserving transformations X under the operation of composition of mappings such that $T_0 = I$ and $T_{s+t} = T_s \circ T_t$ and for every measurable function $f : X \to \mathbb{R}$ the function $\phi : \mathbb{R} \times X \to \mathbb{R}$ defined by $\phi(t, x) = f(T_t(x))$ is measurable, then the family $\mathcal{F}$ induces motions u and u' of $\mathbb{R}$ on X and $L^p(\mu)$ respectively, given by $u(t, x) = T_t(x)$ and $u'(t, f) = C_{T_t} f$ for $x \in X$, $f \in L^p(\mu)$ and $t \in \mathbb{R}$. If $T : X \to X$ is an invertible measure preserving transformation, then the discrete actions u_T and u'_T come from the families $\mathcal{F} = \{T^n : n \in \mathbb{Z}\}$ and $\mathcal{F}' = \{C_T^n : n \in \mathbb{Z}\}$. The motions induced by such families of measure preserving transformations and the corresponding composition operators are known as the measurable dynamical systems and play very important roles in ergodic theory, Markov processes and measurable and topological dynamics. The theory of the composition operators induced by the measure preserving transformations interacts with ergodic theory. Some of the results of this interaction are presented in this section to have some flavour of the applications of the composition operators.

Definition 5.2.1. Let $(X, \mathcal{S}, \mu)$ be a probability measure space. Then an operator A on $L^1(\mu)$ is said to be doubly stochastic if

(i) $Af \geq 0$, when $f \geq 0$,

(ii) $\int_X Af \, d\mu = \int_X f d\mu$,

(iii) $Af = f$, when f is a constant function.

(equality and inequality here are taken to be almost everywhere).

It is clear that if $T : X \to X$ is a measure preserving transformation, then $\int_X C_T f \, d\mu = \int_X f d\mu T^{-1} = \int_X f d\mu$, and hence C_T is doubly stochastic. It turns out that in case of the standard Borel spaces the composition operators are the only doubly stochastic isometric operators in some cases. This we shall present in the following theorem.

Theorem 5.2.2. Let $(X, \mathcal{S}, \mu)$ be a standard Borel probability measure space, and let A be a doubly stochastic operator on $L^1(\mu)$ which is an isometry on $L^2(\mu)$. Then there exists a measure preserving transformation $T : X \to X$ such that $A = C_T$.

Proof. Let $S \in \mathcal{S}$. Then $A\chi_S \geq 0$ and $\int_X A\chi_S d\mu = \int_X \chi_S d\mu = \mu(S) \leq 1$. Hence $0 \leq A\chi_S \leq 1$. Since A is an isometry on $L^2(\mu)$, we have

$$\int_X \left(A\chi_S\right)^2 d\mu = \left\langle A\chi_S, A\chi_S\right\rangle = \left\langle \chi_S, \chi_S\right\rangle$$
$$= \int_X \chi_S d\mu = \int_X A\chi_S d\mu.$$

From this we conclude that

$$\left(A\chi_S\right)^2 = A\chi_S$$

and hence $A\chi_S$ is a characteristic function. Thus the set K of all characteristic functions of $L^1(\mu)$ is invariant under A. Hence by Theorem 2.1.13, $A = C_T$ for some measurable transformation T. Since

$$\mu(S) = \int_X \chi_S d\mu = \int_X C_T \chi_S d\mu = \mu T^{-1}(S)$$

for every $S \in \mathcal{S}$, we conclude that T is measure–preserving. This completes the proof of the theorem.

Note. If C_T is unitary on $L^2(\mu)$, then the doubly stochastic operator A gives rise to a discrete measurable dynamical systems on X and on $L^p(\mu)$, $p \geq 1$.

If $T : X \to X$ is a measure preserving transformation, then the family $\{T^n : n \in \mathbb{Z}^+\}$ gives rise to a discrete measurable semidynamical system. It turns out that the orbit of almost every point of a measurable subset S of X has non–empty intersection with S. This is shown in the following classical theorem of Poincare.

Theorem 5.2.3. [Poincare Recurrence Theorem]. Let T be a measure preserving transformation on a finite measure space $(X, \mathcal{S}, \mu)$ and let $S \in \mathcal{S}$. Then for almost every $s \in S$, there exists $n \in \mathbb{Z}^+$ such that $T^n(s) \in S$ (*i.e.* $\text{orb}(s) \cap S \neq \varnothing$).

Proof. Suppose the conclusion of the theorem is not true. Then the set

$$F = \left\{ s \in S \ : \ T^n(s) \notin S, \quad \text{for every } n \in \mathbb{Z}^+ \right\}$$

has non-zero measure and

$$F = S \cap T^{-1}(X \setminus S) \cap T^{-2}(X \setminus S) \cap \ldots\ldots .$$

If $x \in F$, then $T^n(x) \notin F$ for every $n \in \mathbb{N}$. Hence $F \cap T^{-1}(F) = \varnothing$, for every $n \in \mathbb{N}$. Since T is measure preserving and $\mu(X) < \infty$, we get a contradiction. With this contradiction the proof of the theorem is completed.

Note. If T is measure preserving, then T^n is also measure preserving for every $n \in \mathbb{N}$. If F_n is the set of points of S which never return to S under T^n, then by the above theorem $\mu(F_n) = 0$. If $x \in S \setminus (\bigcup_{n \in \mathbb{N}} F_n)$ then $T^n(x) \in S$ for infinitely many n's. Thus almost every point of S returns to S under T infinitely many times. This is a stronger version of the above theorem.

Examples of Measure Preserving Transformations :

(i) Let $X = \mathbb{R}$, μ be the Lebesgue measure, and for every $t \in \mathbb{R}$, let $T_t : \mathbb{R} \to \mathbb{R}$ be defined as $T_t(x) = x + t$, $x \in \mathbb{R}$. Then the family $\{T_t : t \in \mathbb{R}\}$ is a group of measure preserving transformations and gives rise to measurable dynamical systems on $\mathbb{R}$ and $L^p(\mu)$ for $p \geq 1$.

(ii) Let $X = [0, 1]$, $\mathcal{S}$ be the σ-algebra of all Borel sets. Let $0 < a < 1$ and $T_a(x)$ be the fractional part of $x + a$. Then $T_a : X \to X$ is a measure-preserving transformation.

(iii) Let $X = \{z \in \mathbb{C} : |z| = 1\} = \partial D$ with the normalised Lebesgue measure and let $0 < a < 1$. Then the mapping $T_a : X \to X$ defined as $T_a(z) = e^{2\pi i a} z$ is a measure preserving transformation on the unit circle ∂D.

(iv) Let $X = [0, 1]$. Define $T : X \to X$ as

$$T(x) = \begin{cases} 2x \quad , & \text{for } \ 0 \leq x < \frac{1}{2} \\ 2x - 1, & \text{for } \ \frac{1}{2} \leq x < 1. \end{cases}$$

Then T is a non-invertible measure-preserving transformation.

(v) If X is a compact group with the Haar measure, then every automorphism of X is a measure preserving transformation.

(vi) If $X = [0, 1] \times [0, 1]$ with the area measure and $T(x, y) = (T_a(x), y)$, T_a given by (i), then T is a measure–preserving transformation.

(vii) If $T : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ is defined by $T(x, y) = (2x, \frac{1}{2} y)$, then T preserves area measure on $\mathbb{R} \times \mathbb{R}$.

(viii) Let X be a locally compact group with a left invariant Haar measure and $c \in X$. Then the mapping $T_c : X \to X$ defined as $T_c(x) = cx$ is measure–preserving.

(ix) If X is the torus and $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is a unimodular matrix of integers, then the mapping $T : X \to X$ defined as $T(x, y) = (x^a y^b, x^c y^d)$ is measure–preserving.

There have been implicit applications of the theory of the composition operators in several types of ergodic results. We shall give only representative samples to illustrate the involvement of these operators in earlier results in ergodic theory.

If T is a measure preserving transformation on $(X, \mathcal{S}, \mu)$, then we know that the composition operator C_T is an isometry on $L^p(\mu)$, $p \geq 1$, and hence $u : \mathbb{Z}^+ \times L^p(\mu) \to L^p(\mu)$, defined as

$$u(n, f) = C_T^n f = f \circ T^n,$$

is a semidynamical (discrete) system on $L^p(\mu)$. If $f \in L^p(\mu)$, then the orbit of f consists of the functions $f, C_T f, C_T^2 f, \ldots, C_T^n f, \ldots$. These elements of the orbit of f give rise to the sequence $\{g_n\}$ in $L^p(\mu)$ given by

$$g_n = \frac{1}{n} \sum_{k=0}^{n-1} C_T^k f.$$

The study of the convergence of the sequence $\{g_n\}$ comes under the realm of the ergodic theory. A limit of this sequence is called a time mean of f and $\int_X f \, d\mu$ is called the space mean of f. For some suitable transformations the time mean and the space mean are equal (*a.e*). The convergence of the sequence $\{g_n\}$ in almost everywhere sense was proved by Birkhoff, which we shall present in the following theorem.

Theorem 5.2.4 [Birkhoff Ergodic Theorem]. Let $(X, \mathcal{S}, \mu)$ be a finite measure space and let $T : X \to X$ be a measure preserving transformation. Then for every $f \in L^1(\mu)$,

(i) the sequence $\left\{\frac{1}{n}\sum_{k=0}^{n-1} C_T^k f\right\}$ converges *a.e.* to a function $g \in L^1(\mu)$.

(ii) $C_T g = g$.

(iii) $\int_X g\,d\mu = \int_X f\,d\mu$.

In order to prove this theorem we need the following results which we shall present without proof. We refer to [44] for details.

Theorem 5.2.5. Let A be a contraction on $L^1(\mu)$ taken as real Banach space. For $f \in L^1(\mu)$, let $A_n f(x) = \sum_{k=0}^{n} A^k f(x)$ and for $\alpha \in \mathbb{R}$, let

$$S(f) = \left\{x \in X \ : \ \sup_n A_n f(x) > 0\right\},$$

$$S^*(f, \alpha) = \left\{x \in X \ : \ \sup_n \frac{1}{n} A_n f(x) > \alpha\right\},$$

and

$$S_*(f, \alpha) = \left\{x \in X \ : \ \inf_n \frac{1}{n} A_n f(x) < \alpha\right\}.$$

Then

(i) $\int_{S(f)} f\,d\mu \geq 0$ (Maximal Ergodic Theorem).

(ii) For every $\alpha, \beta \in \mathbb{R}$,

$$\alpha\mu\left(S^*(f, \alpha)\right) \leq \int_{S^*(f,\alpha)} f\,d\mu,$$

and

$$\beta\mu(S_*(f, \beta)) \geq \int_{S_*(f,\beta)} f\,d\mu\,.$$

Proof of Theorem 5.2.4. Let $f \in L^1(\mu)$ such that f is real valued and let

$$g_n = \frac{1}{n}\sum_{k=0}^{n-1} C_T^k f.$$

Let $\bar{f}(x) = \limsup_n g_n(x)$, $\underline{f}(x) = \liminf_n g_n(x)$, $f^*(x) = \sup_n g_n(x)$ and $f_*(x) = \inf_n g_n(x)$.

For $\beta < \alpha$, let

$$S_\alpha^\beta = \left\{x \in X : \underline{f}(x) < \beta < \alpha < \bar{f}(x)\right\}.$$

Since $\bar{f}(T(x)) = \bar{f}(x)$ and $\underline{f}(T(x)) = \underline{f}(x)$, T maps S_α^β into itself. If $\mu(S_\alpha^\beta) = \lambda \neq 0$, then T preserves the measure μ/λ on S_α^β with relative σ-algebra. Applying result (ii) of the last theorem and using the fact that

$$S^*(f,\alpha) = S_*(f,\beta) = S_\alpha^\beta,$$

we get

$$\alpha \leq \frac{1}{\lambda}\int_{S_\alpha^\beta} f\,d\mu \leq \beta,$$

which is a contradiction. Thus $\mu(S_\alpha^\beta) = 0$. If $S = \{x \in X : \underline{f}(x) < \bar{f}(x)\}$, then $S = \bigcup_{\beta<\alpha} S_\alpha^\beta$, α and β being rational numbers. Thus $\mu(S) = 0$. Hence $\bar{f}(x) = \underline{f}(x)$ *a.e.* This shows that the sequence $\{g_n\}$ converges *a.e.* Since

$$\begin{aligned}\int_X |\, g_n(x)\,|\,d\mu &\leq \frac{1}{n}\int\sum_{k=0}^{n-1}\left|\left(C_T^k f\right)\right|d\mu \\ &= \|f\|_1,\end{aligned}$$

we can conclude by Fatou's lemma that

$$\int_X |\bar{f}|\,d\mu \leq \int_X |f|\,d\mu.$$

Hence $\bar{f} \in L^1(\mu)$. Let $g = \bar{f}$. Then $\lim_n g_n(x) = g(x)$ *a.e.* If f is a complex-valued L^1-function, then we can prove the above result for real and imaginary parts of f.

Hence it is true for every $f \in L^1(\mu)$. Since $\bar{f} \circ T = \bar{f}$, we have $C_T g = g$. This proves (i) and (ii). To prove (iii), let $X_k^n = \left\{x \in X \ : \ \frac{k}{n} \le \bar{f}(x) < \frac{k+1}{n}\right\}$, where $k \in \mathbb{Z}$ and $n \in \mathbb{N}$. Let $\varepsilon > 0$ be an arbitrarily small number. Then it can be seen that

$$\int_{X_k^n} f d\mu \ge \left(\frac{k}{n} - \varepsilon\right)\mu\left(X_k^n\right),$$

and hence

$$\int_{X_k^n} f d\mu \ge \frac{k}{n}\mu\left(X_k^n\right).$$

Thus

$$\int_{X_k^n} \bar{f} d\mu \le \frac{k+1}{n}\mu\left(X_k^n\right) \le \frac{1}{n}\mu\left(X_k^n\right) + \int_{X_k^n} f d\mu.$$

From this we get

$$\int_X \bar{f} d\mu \le \frac{\mu(X)}{n} + \int_X f d\mu.$$

This is true for every n, and hence

$$\int_X \bar{f} d\mu \le \int_X f d\mu.$$

Since $\bar{f} = \underline{f}$ *a.e.*, taking $-f$ at the place of f we have

$$\int_X \bar{f} d\mu \ge \int_X f d\mu.$$

Thus

$$\int_X g d\mu = \int_X f d\mu.$$

This completes the proof of the theorem.

In the following corollary we present the L^p–ergodic theorem proved by Von Neumann.

Corollary 5.2.6. Let T be a measure preserving transformation on a probability measure space $(X, \mathcal{S}, \mu)$ and let $f \in L^p(\mu)$. Then there exists a $g \in L^p(\mu)$ such that the sequence $\{g_n\}$ converges to a g in L^p–norm and g is a fixed point of C_T, where

$$g_n = \frac{1}{n}\sum_{k=0}^{n-1} C_T^k f \quad \text{for every } n \in \mathbb{N}.$$

Outline of the proof. Let $\varepsilon > 0$. Then there exists an $f' \in L^\infty(\mu)$ such that $\|f - f'\| < \varepsilon / 4$. Let $g_n' = \frac{1}{n}\sum_{k=0}^{n-1} C_T^k f'$. By Theorem 5.2.5 and the bounded convergence theorem the sequence $\{g_n'\}$ converges to $\bar{f}'$ in the L^p–norm, where $\bar{f}'(x) = \lim_n \sup g_n' (x)$. Now

$$\begin{aligned}\|g_n - g_{n+m}\|_p &\leq \|g_n - g_n'\|_p + \|g_n' - g_{n+m}'\|_p + \|g_{n+m}' - g_{n+m}\|_p \\ &< \varepsilon / 4 + \varepsilon / 2 + \varepsilon / 4 = \varepsilon,\end{aligned}$$

for suitable choice of n. Thus $\{g_n\}$ is a Cauchy sequence, and hence there exists a $g \in L^p(\mu)$ such that $g_n \to g$ in L^p–norm. It can be proved that $C_T g = g$.

There is a relation between a minimal measurable dynamical system on $(X, \mathcal{S}, \mu)$ and the set of fixed points of the composition operator. We need the following definition to present this relation.

Definition 5.2.7. Let T be a measure preserving transformation on a measure space $(X, \mathcal{S}, \mu)$. Then T is said to be ergodic if $T^{-1}(S) = S$ implies that either $\mu(S) = 0$ or $\mu(X \setminus S) = 0$. A doubly stochastic operator A on $L^1(\mu)$ is said to be ergodic if the constant functions are the only fixed points of A (*i.e.*, $Af = f$, $f \in L^1(\mu)$ implies that f is a constant function *a.e.*).

Note. If $T^{-1}(S) = S$, then $x \in S$ if and only if $T^n(x) \in S$ for every $n \in \mathbb{N}$. This means that S is an invariant subsystem under the motion induced by T. In case of ergodic transformation X is the only non–trivial invariant system.

The following theorem presents a connection between the ergodicity of T and the ergodicity of C_T.

Theorem 5.2.8. Let T be a measure preserving transformation on a probability measure space $(X, \mathcal{S}, \mu)$. Then T is ergodic if and only if the composition operator C_T is ergodic.

Proof. Suppose that the composition operator C_T is ergodic and suppose that $T^{-1}(S) = S$, where $S \in \mathcal{S}$. Then $\chi_{T^{-1}(S)} = \chi_S$, and hence $C_T\chi_S = \chi_S$. Thus $\chi_S = c$ *a.e.*, for some constant c. From this it follows that $\mu(S) = 0$ or $\mu(X \setminus S) = 0$. This shows that T is ergodic. Conversely, suppose T is ergodic. Assume $C_T f = f$ for $f \in L^1(\mu)$. Let $k \in \mathbb{Z}$ and $n \in \mathbb{Z}^+$. Let

$$X_n^k = \left\{x \in X \ : \ \frac{k}{2^n} \le f(x) < \frac{k+1}{2^n}\right\}.$$

Then $T^{-1}(X_n^k) = X_n^k$. Hence either $\mu(X_n^k) = 0$ or $\mu(X \setminus X_n^k) = 0$. Since $X = \bigcup_{k \in \mathbb{Z}} X_n^k$, for each n, there exists a unique k_n such that $\mu\left(X_n^{k_n}\right) = 1$. Let $X' = \bigcap_{n=0}^{\infty} X_n^{k_n}$. Then $\mu(X') = 1$ and f is constant on X'. This completes the proof of the theorem.

Notes. (i) If T is ergodic, then for every measurable set S of non-zero measure, $\mu\left(\bigcup_{n=0}^{\infty} T^{-n}(S)\right) = 1$, and for every pair of measurable sets S_1 and S_2 of non-zero measures, there exists $n \in \mathbb{N}$ such that $\mu(T^{-n}(S_1) \cap S_2) \neq 0$.

(ii) If T is an invertible measure preserving transformation, then T is ergodic if and only if 1 is a simple eigen value of C_T. If T is ergodic, then the absolute value of every proper function of C_T is constant and the set of all proper values of C_T is a subgroup of the unit circle.

(iii) If T is ergodic, then the sequence $\{g_n\}$ converges to the constant c *a.e.*, where $c = \int_X f d\mu$ and $g_n = \frac{1}{n}\sum_{k=0}^{n-1} C_T^k f$. Also,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \mu\left(S_1 \cap T^{-1}(S_2)\right) = \mu(S_1)\, \mu(S_2), \text{ for } S_1, S_2 \in \mathcal{S}.$$

Definitiion 5.2.9. Let T be a measure preserving transformation on $(X, \mathcal{S}, \mu)$. Then

(i) T is said to be weak-mixing if $\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \left| \mu\left(T^{-k}(S) \cap W\right) - \mu(S)\mu(W) \right| = 0$, for every $S, W \in \mathcal{S}$.

(ii) T is said to be strong-mixing if $\lim_{k\to\infty} \mu\left(T^{-k}(S) \cap W\right) = \mu(S)\, \mu(W)$ for every $S, W \in \mathcal{S}$.

In the following theorem we shall present the characterizations of ergodicity, weak-mixing and strong-mixing in terms of the behaviour of the composition operator on the Hilbert space $L^2(\mu)$.

Theorem 5.2.10. Let T be a measure preserving transformation on a probability measure space $(X, \mathcal{S}, \mu)$ and let C_T be the composition operator induced by T on $L^2(\mu)$. Then

(i) T is ergodic if and only if $\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \left\langle C_T^k f, f \right\rangle = \langle 1, f \rangle \langle f, 1 \rangle$, for every $f \in L^2(\mu)$.

(ii) T is weak-mixing if and only if $\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \left| \left\langle C_T^k f, f \right\rangle - \langle f, 1 \rangle \langle f, 1 \rangle \right| = 0$, for every $f \in L^2(\mu)$.

(iii) T is strong-mixing if and only if $\lim_n \left\langle C_T^n f, f \right\rangle = \langle f, 1 \rangle \langle 1, f \rangle$, for every $f \in L^2(\mu)$.

Outline of the proof. We shall prove only (iii) to give the ideas about the proofs of (i) and (ii). Suppose T is strong-mixing and suppose S and W are measurable sets. Then χ_S and χ_W are in $L^2(\mu)$, and

$$\left\langle C_T^n \chi_S, \chi_W \right\rangle = \mu\left(T^{-k}(S) \cap W\right),$$

and

$$\langle \chi_S, 1\rangle \langle 1, \chi_W\rangle = \mu(S)\,\mu(W)\,.$$

Hence
$$\langle C_T^n\, \chi_S, \chi_W\rangle \to \langle \chi_S, 1\rangle \langle 1, \chi_W\rangle.$$

From this we can conclude that $\langle C_T^n h, h\rangle \to \langle 1, h\rangle\langle h, 1\rangle$ for every simple functions. Since the set of all simple functions is dense in $L^2(\mu)$, using Schwartz inequality and the fact that C_T^n is an isometry on $L^2(\mu)$ for every $n \in \mathbb{N}$ we conclude that

$$\langle C_T^n f, f\rangle \to \langle 1, f\rangle \langle f, 1\rangle, \text{ for every } f \in L^2(\mu).$$

Conversely, suppose $\lim\limits_{n\to\infty} \langle C_T^n f, f\rangle = \langle f, 1\rangle \langle 1, f\rangle$, for every $f \in L^2(\mu)$. Let H_f be the smallest invariant subspace of C_T containing f and all constant functions. Let

$$H_f^1 = \left\{ g \in L^2(\mu) \ : \ \lim_n \langle C_T^n f, g\rangle = \langle f, 1\rangle \langle 1, g\rangle \right\}.$$

Then H_f^1 is an invariant subspace of $L^2(\mu)$ containing f and all constant functions, and hence $H_f \subset H_f^1$. Let $g \in H_f^{1}$. Then $\langle C_T^n\, f, g\rangle = 0$ and $\langle 1, g\rangle = 0$. Hence $H_f^1 = L^2(\mu)$. Thus

$$\lim_n \langle C_T^n\, f, g\rangle = \langle f, 1\rangle \langle 1, g\rangle, \text{ for every } f, g \in L^2(\mu).$$

Let S and W be measurable sets and let $f = \chi_S$ and $g = \chi_W$. Then we have

$$\lim_n \mu\left(T^{-n}(S) \cap W\right) = \mu(S)\,\mu(W)\,.$$

This shows that T is strong-mixing.

Using the spectral theorem we get the following theorem which we shall present without proof.

Theorem 5.2.11. Let T be an invertible measure preserving transformation on $(X, \mathcal{S}, \mu)$. Then T is weak-mixing if and only if 1 is the only eigen value of C_T and the only eigen functions are the constant functions in $L^2(\mu)$.

Let T be a measure preserving transformation on a measure space $(X, \mathcal{S}, \mu)$. Then we know that C_T is an isometry on $L^2(\mu)$, and hence it is injective. In case $(X, \mathcal{S}, \mu)$

is a finite measure space, then it follows from Theorem 2.4.2 that C_T is unitary if and only if T induces an automorphism of $\mathcal{S}/\mathcal{Z}$

Definition 5.2.12. Let T_1 and T_2 be two measure preserving transformations on a measure space $(X, \mathcal{S}, \mu)$. Then T_1 and T_2 are said to spectrally isomorphic if there exists a unitary operator $A : L^2(\mu) \to L^2(\mu)$ such that

$$C_{T_1} A = A\, C_{T_2}.$$

Let T be an invertible measure preserving transformation on a probability measure space $(X, \mathcal{S}, \mu)$. Then we say that T has countable Lebesgue spectrum if there exists a sequence $\{f_k\}_{k=0}^{\infty}$ in $L^2(\mu)$ such that

(i) $f_0 = 1$,

(ii) $\bigcup_{n \in \mathbb{Z}} \left\{ C_T^n f_k \ : \ k \in \mathbb{Z}^+ \right\}$ is an orthonormal basis for $L^2(\mu)$.

The following theorem is easy to prove.

Theorem 5.2.13. If T_1 and T_2 are two measure preserving transformations with countable Lebesgue spectrum, then they are spectrally isomorphic.

Outline of the Proof. Let $\mathcal{B}_1 = \bigcup_{n \in \mathbb{Z}} \left\{ C_{T_1}^n f_k \ : \ k \in \mathbb{Z}^+ \right\}$ and $\mathcal{B}_2 = \bigcup_{n \in \mathbb{Z}} \left\{ C_{T_2}^n g_k \ : \ k \in \mathbb{Z}^+ \right\}$ be two orthonormal basis of $L^2(\mu)$. Let us define $A : \mathcal{B}_1 \to \mathcal{B}_2$ as $A(f_0) = g_0$ and $A(C_{T_1}^n f_k) = C_{T_2}^n g_k$. We can extend it linearly on $L^2(\mu)$ as a unitary operator. It is clear that $A\, C_{T_1} = C_{T_2}\, A$. Hence T_1 and T_2 are spectrally isomorphic.

Theorem 5.2.14. If T_1 and T_2 are spectrally isomorphic measure preserving transformations on a probability space $(X, \mathcal{S}, \mu)$, then

(i) T_1 is ergodic if and only if T_2 is ergodic.

(ii) T_1 is weak–mixing if and only if T_2 is weak–mixing.

(iii) T_1 is strong–mixing if and only if T_2 is strong–mixing.

Proof. Just to illustrate we shall prove (i). Suppose T_1 is ergodic. Then by Theorem 5.2.8 constant functions are the only fixed points of C_{T_1}. Suppose $f \in L^2(\mu)$ and $C_{T_2} f = f$. Since T_1 and T_2 are spectrally isomorphic, there exists a unitary operator A such that

$$C_{T_2} = A^{-1} C_{T_1} A.$$

Hence

$$A^{-1} C_{T_1} A f = f.$$

Thus

$$C_{T_1} Af = Af.$$

Hence Af is a constant function in $L^2(\mu)$. Thus f is a constant function since the closed subspace of all constant functions has dimension 1. Thus T_2 is ergodic. Similarly, we can show that the ergodicity of T_2 implies the ergodicity of T_1. Similarly (ii) and (iii) can be proved.

Definition 5.2.15. A measure preserving ergodic transformation T is said to have discrete spectrum if there exists an orthonormal basis for $L^2(\mu)$ consisting of eigen functions of C_T. Two measure preserving transformations T_1 and T_2 are said to be conjugate if there exists a σ-algebra automorphism Φ on $\mathcal{S}/\mathfrak{X}$ such that $\Phi\, h'_{T_1} = h'_{T_2}\, \Phi$, where h'_T is σ-homomorphism induced by T (see section 2.1).

We shall record the following theorem due to Halmos and Von Neumann [398]. The properties of the composition operators are employed to prove it.

Theorem 5.2.16. Let T_1 and T_2 be two measure preserving ergodic transformations on a probability measure space having discrete spectrum. Then the following statements are equivalent :

(i) T_1 and T_2 are spectrally isomorphic,

(ii) T_1 and T_2 are conjugate transformations ,

(iii) The point spectrum of C_{T_1} is same as the point spectrum of C_{T_2}.

Note. Ergodic properties of non-measure preserving transformations and Markov processes induced by the composition operators and the weighted composition operators have been studied by Hoover, Lambert and Quinn [147], Lambert [209] and Quinn [275]. A relation between composition operators and expectation operators is presented in section 2.4. Some spectral properties of the composition operators are used in statistical mechanics [237].

The composition operators also play an important role in establishing a relation between measurable dynamics and topological dynamics. We shall present this relation in the following few pages.

Let X be a compact Hausdorff space and let $T : X \to X$ be a continuous map. Then we know from Theorem 4.1.2 that $C_T : C(X) \to C(X)$ is a continuous algebra homomorphism and $\|C_T\| = 1$. The mappings T and C_T give rise to a discrete semidynamical systems on X and $C(X)$ respectively. If T is a homeomorphism, then the discrete dynamical systems are induced by T and C_T. A study of the actions induced by T and C_T comes under the realm of topological dynamics. As earlier $M(X)$ denotes the normed linear space of all (signed) Borel measures. If $\mu \in M(X)$, then $\|\mu\| = |\mu|(X)$, where $|\mu|$ denotes the total variation of μ. By the Riesz-representation theorem we know that the dual space of the Banach space $C(X)$ is isometrically isomorphic to $M(X)$ and the representation is given by

$$F_\mu(f) = \int_X f d\mu, \quad \text{where} \quad \mu \in M(X) \quad \text{and} \quad f \in C(X).$$

The correspondence $\mu \to F_\mu$ is an isometric isomorphism.

If $T : X \to X$ is a continuous map, then $C_T : C(X) \to C(X)$ and $C_T^* : M(X) \to M(X)$ are bounded linear operators. The adjoint C_T^* of C_T is given by

$$\left(C_T^*\mu\right)(f) = F_\mu\left(C_T f\right) = \int_X f \circ T \, d\mu .$$

It can be easily checked that

(i) $C_T^*\mu \geq 0$ if $\mu \geq 0$.

(ii) $\| C_T^*\mu \| \leq \| \mu \|$.

(iii) $(C_T^*\mu)(X) = \mu(X)$.

(iv) If T is an injection, then $\left\| C_T^*\mu \right\| = \| \mu \|$.

(v) If T is a homeomorphism, then C_T and C_T^* are invertible operators and $(C_T^*)^{-1} = C_{T^{-1}}^* = (C_T^{-1})^*$.

The following theorem shows that every topological semidynamical system is a semi measurable dynamical system.

Theorem 5.2.17. Let X be a compact Hausdorff space and $T : X \to X$ be a continuous map. Then there exists a probability measure μ on the Borel subsets of X such that T is a measure preserving transformation with respect to μ.

Proof. (outline) Let $P = \{\mu \in M(X) : \mu \geq 0 \text{ and } \| \mu \| = 1\}$. Then by the Banach–Alaoglu theorem, P is w^*–compact. It can be shown that P is a C_T^* –invariant non–empty convex subset of $M(X)$. By Kakutani–Markov fixed point theorem [97] there exists a $\mu \in P$ such that $C_T^*\mu = \mu$. Now, for $f \in C(X)$

$$\begin{aligned}\int_X C_T f \, d\mu = F_\mu(C_T f) &= (C_T^*\mu)(f)\\ &= F_\mu(f)\\ &= \int_X f d\mu.\end{aligned}$$

From this it follows that

$$\mu\left(T^{-1}(S)\right) = \mu(S)$$

for every Borel set S. This shows that T is a measure preserving transformation with respect to measure μ. With this the outline of the proof is completed.

Note. A measure preserving transformation is essentially a surjection. Thus, if T in the above theorem is a constant map, then the fixed points of C_T^* should concentrate at a single point like Dirac delta measures.

Definitions 5.2.18. Let X be a compact Hausdorff space and $T : X \to X$ be a continuous map. Let $(\mathbb{Z}^+, X, u_T)$ denotes the (discrete) semidynamical system induced by T, where $u_T(n, x) = T^n(x)$ for $n \in \mathbb{Z}^+$ and $x \in X$. Then

(i) $(\mathbb{Z}^+, X, u_T)$ is said to be minimal if X is the only non-empty, closed invariant subset of itself.

(ii) $(\mathbb{Z}^+, X, u_T)$ is recurrent if for every open subset U of X and for every $x \in U$ there exists $n \in \mathbb{Z}^+$ such that $T^n(x) \in U$.

(iii) $(\mathbb{Z}^+, X, u_T)$ is uniquely ergodic if there exists a unique $\mu \in M(X)$ such that $C_T^*\mu = \mu$.

(iv) $(\mathbb{Z}^+, X, u_T)$ is strictly ergodic if it is uniquely ergodic and minimal.

(v) If T is a homeomorphism, then the discrete dynamical system $(\mathbb{Z}, X, u_T)$ is said to be equicontinuous if the family $\{T^n : n \in \mathbb{Z}\}$ is equicontinuous.

In the following theorems we present some results in which composition operators have been used implicitcly.

Theorem 5.2.19. Let $(\mathbb{Z}^+, X, u_T)$ be a semidynamical system induced by a continuous map T on a compact Hausdorff space X. Then

(i) The system $(\mathbb{Z}^+, X, u_T)$ is minimal if and only if $\overline{\text{orb}(x)} = X$, for every $x \in X$.

(ii) If $(\mathbb{Z}^+, X, u_T)$ is minimal, $f \in C(X)$ and $C_T f = f$, then f is constant.

(iii) $(\mathbb{Z}^+, X, u_T)$ has a minimal subsystem.

Proof. (i) If $x \in X$, then $\overline{\text{orb}(x)}$ is non-empty, closed invariant subset of X and $\overline{\text{orb}(x)} = X$ in case X is minimal. If X is not minimal, then it has a non-empty, closed, proper invariant set X_0. If $x \in X_0$, then $\overline{\text{orb}(x)} \subset X_0$. Thus $\overline{\text{orb}(x)} \neq X$. This proves (i)

(ii) Proof is similar to that of Theorem 5.2.8.

(iii) Let $\mathscr{C}$ be the collection of all non-empty closed, invariant subsets of X. If S, $W \in \mathscr{C}$, then we say that $S \leq W$ if $S \supset W$. This relation turns $\mathscr{C}$ into a partially ordered set. Use Zorn's lemma to get a maximal element $X_0 \in \mathscr{C}$. Then $(\mathbb{Z}^+, X_0, u_T)$ is a minimal subsystem.

Theorem 5.2.20. Let $T : X \to X$ be a homeomorphism inducing the dynamical system $(\mathbb{Z}, X, u_T)$. Then we have the following :

(i) If $(\mathbb{Z}, X, u_T)$ is minimal and equicontinuous, then it is strictly ergodic.

(ii) $(\mathbb{Z}, X, u_T)$ is uniquely ergodic if and only if the sequence $\left\{\frac{1}{n}\sum_{k=0}^{n-1} C_T^k f\right\}$ converges uniformly to a constant function for every $f \in C(X)$.

Proof. We shall prove just part (i). Let $f \in C(X)$. Then $\{T^n : n \in \mathbb{Z}\}$ is equicontinuous and hence $\{C_T^n \circ f : n \in \mathbb{Z}\}$ is equicontinuous. By the Arzela–Ascoli theorem there exists a subsequence of $\{f \circ T^n\}$ which converges uniformly to a function $g \in C(X)$. Now

$$\begin{aligned} C_T g &= C_T\left(\lim_{n_k} f \circ T^{n_k}\right) \\ &= \lim_{n_k} f \circ T^{n_k+1} \\ &= g. \end{aligned}$$

Thus g is a fixed point of C_T and hence by Theorem 5.2.19 (ii), g is a constant function, say $\lambda(f)$. If μ is any normalised Borel measure on X such that $C_T^*\mu = \mu$, then

$$\mu\left(f \circ T^{n_k}\right) = \int_X f \circ T^{n_k} d\mu = \mu(f).$$

Since $\mu(f \circ T^{n_k}) \to \mu(g)$, we conclude that $\mu(f) = \lambda(f)$ for every $f \in C(X)$. Thus $\mu = \lambda$. This shows that the system is strictly ergodic.

(ii) (Outline of the proof). Suppose $(\mathbb{Z}, X, u_T)$ is uniquely ergodic with invariant measure μ. Let $Y = \{g - C_T g : g \in C(X)\}$. Then it follows that the subspace of measures in $M(X)$ which vanish on Y is one dimensional. This means $\overline{Y} = \{f - \mu(f) : f \in C(X)\}$. Thus for any $f \in C(X)$ and any $\varepsilon > 0$, there is $g \in C(X)$ such that

$$\| f - \mu(f) - (g - C_T g) \| < \varepsilon .$$

From this it follows that

$$\left\|\frac{1}{n}\sum_{k=0}^{n-1} C_T^k f - \mu(f) - \frac{1}{n}\left(g - C_T^n g\right)\right\| \le \frac{1}{n}\sum \left\| C_T^n\left(f - \mu(f) - \left(g - C_T g\right)\right)\right\| < \varepsilon .$$

It follows from this that $\frac{1}{n}\sum_{k=0}^{n-1} C_T^k f \to \mu(f)$ uniformly. Conversely, suppose $\frac{1}{n}\sum_{k=0}^{n-1} C_T^k f \to \lambda(f)$ uniformly, where $\lambda(f)$ is a constant depending on f. Let μ be any measure such that $C_T^* \mu = \mu$. Since $\left\{\frac{1}{n}\sum C_T^k f\right\}$ is uniformly bounded sequence, using bounded convergence theorem it can be concluded that

$$\mu(f) = \lambda(f).$$

This shows that

$$\mu = \lambda.$$

This completes the proof of the theorem.

The discrete semidynamical systems or the discrete dynamical systems on a compact Hausdorff space X are actually special cases of a more generalised system on X which is called a flow. A flow on X is a triple (S, X, u), where S is a semigroup and $u : S \times X \to X$ such that $u_s : X \to X$ defined as $u_s(x) = u(s, x)$ is continuous for every $s \in S$ and $u(st, x) = u(s, u(t, x))$ for every $s, t \in S$ and for every $x \in X$. In case S is a topological group and u is jointly continuous we say that flow (S, X, u) is a transformation group in case $u(e, x) = x$ for every $x \in X$. Transformation group we have defined in the beginning of this section. In case S is a semitopological group, we have a semitransformation group. If (S, X, u) is a flow on X, then the enveloping semigroup of this flow is by definition the closure in X^X of the semigroup $\{u_s : s \in S\}$, where X^X denotes the set of all functions from X to X with product topology. This semigroup is denoted by $\Sigma(S, X)$. Let (S, X, u) and (S, X', u') be two flows with the same phase semigroup S. Then a continuous mapping $\phi : X \to X'$ is said to be a (flow) homomorphism from flow (S, X, u) into flow (S, X', u') if

$$\phi(u(s, x)) = u'(s, \phi(x)),$$

for every $s \in S$ and $x \in X$. If the homomorphism ϕ is a bijectiion, then it is called an isomorphism. If (S, X, u) is a flow, then $C_{u_s} : C(X) \to C(X)$ defined as $C_{u_s} f = f \circ u_s$ is a bounded linear operator on $C(X)$. In the following theorem we record some results which employ the theory of composition operators.

Theorem 5.2.21. Let (S, X, u) and (S, X', u') be two flows and let $\phi : X \to X'$ be a surjective flow homomorphism.

(i) Then there exists a unique continuous homomorphism $\theta : \sum(S, X) \to \sum(S, X')$ such that

$$\big(\theta(f)\big)\big(\phi(x)\big) = \phi\big(f(x)\big) ,$$

for every $x \in X$ and for every $f \in \Sigma(S, X)$.

(ii) The mapping $h : S \to B(C(X))$ defined as $h(s) = C_{u_s}$ is an antirepresentation of the semigroup by the bounded linear operators on $C(X)$, where $B(C(X))$ denotes the Banach algebra of all bounded linear operators on $C(X)$.

(iii) If $h : S \to B(C(X))$ is an anti representation of S by operators on $C(X)$ such that under $h(s)$ the constant functions are fixed and $h(s)$ is multiplicative for every $s \in S$, then there exists an action $u : S \times X \to X$ such that

$$h(s) = C_{u_s} \quad \text{for every } s \in S.$$

Some of the theory of semidynamical systems can be generalised to the flows. We refer to [22] for details.

The semigroup of operators (linear or non–linear) have an intimate relation with the dynamical systems and with the solutions of different types of evolution equations arising out of physical situations. A detailed theory of the semigroups of operators has been given in [267] and [397]. In recent years the ergodic theorems have been extended in several directions including the semigroup of operators. The theory of the composition operators contributes directly or indirectly in these generalizations and extensions.

Definition 5.2.22. Let E be a metric space. Then by a C_0–semigroup of operators on E we mean a family $\{A_t : t \in \mathbb{R}^+\}$ of continuous mappings $A_t : E \to E$ such that

(i) $A_0 = I$, the identity mapping,

(ii) $A_{t+s} = A_t \circ A_s$ for $t, s \in \mathbb{R}^+$,

(iii) for every $x \in E$, the mapping $u^x : \mathbb{R}^+ \to E$ defined as $u^x(t) = A_t(x)$, is continuous.

If $\{A_t : t \in \mathbb{R}^+\}$ is a C_0–semigroup of operators on E, then the mapping $u : \mathbb{R}^+ \times E \to E$ defined as

$$u(t, x) = A_t(x),\ t \in \mathbb{R}^+ \text{ and } x \in X,$$

is a motion and hence gives rise to a semidynamical system $(\mathbb{R}^+, X, u)$. The converse is also true. That is every motion of $\mathbb{R}^+$ on E gives rise to a C_0–semigroup of operators on E. If E is a topological vector space and A_t is linear for every $t \in \mathbb{R}^+$, then C_0–semigroup $\{A_t : t \in \mathbb{R}^+\}$ gives rise to a linear semidynamical system which has been the subject matter of study for the last several decades. If $\{A_t : t \in \mathbb{R}^+\}$ is a C_0–semigroup of (linear) operators on a Banach space, then the real valued function $t \to \|A_t\|$ is dominated by the function $t \to Me^{\omega t}$ for some $\omega \in \mathbb{R}$ *i.e.*,

$$\| A_t \| \le Me^{\omega t} \quad \text{for every} \quad t \in \mathbb{R}^+ .$$

For details see [397]. The group (semigroup) of translations–induced composition operators on nice function spaces on $\mathbb{R}$ (or $\mathbb{R}^+$) is a very useful group (semigroup) of linear operators and it is employed in the study of different types of periodic motions in some state spaces.

If E is a Banach space, then for every $t \in \mathbb{R}$ the translation $T_t : \mathbb{R} \to \mathbb{R}$, defined as $T_t(s) = s + t$ for $s \in \mathbb{R}$ gives rise to the composition operator $C_{T_t} : C_b(\mathbb{R}, E) \to C_b(\mathbb{R}, E)$, where $C_b(\mathbb{R}, E)$ denotes the Banach space of all bounded continuous functions from $\mathbb{R}$ to E with supnorm. The family $\{C_{T_t} : t \in \mathbb{R}\}$ is a group of bounded linear operators, and gives rise to a dynamical system on $C_b(\mathbb{R}, E)$. We shall denote C_{T_t} by C_t. If $\mathbb{R}$ is replaced by $\mathbb{R}^+$, then the family $\{C_t : t \in \mathbb{R}^+\}$ gives rise to a C_0–continuous semigroup of operators on $C_b(\mathbb{R}^+, E)$. A function $f \in C_b(\mathbb{R}, E)$ is said to be almost periodic (*a.p*) if the set $\{C_t f : t \in \mathbb{R}\}$ is relatively compact in $C_b(\mathbb{R}, E)$ *i.e.*, the orbit of f under the motion induced by $\{C_t : t \in \mathbb{R}\}$ has compact closure in the supnorm topology of $C_b(\mathbb{R}, E)$. A function $f \in C_b(\mathbb{R}^+, E)$ is asymptotically almost periodic (*a.a.p*) if the set $\{C_t f : t \in \mathbb{R}^+\}$ is relatively compact in the space $C_b(\mathbb{R}^+, E)$. A function $f \in C_b(\mathbb{R}^+, E)$ is Eberlein weakly almost periodic (*E–w.a.p*) if the set $\{C_t f : t \in \mathbb{R}^+\}$ is relatively compact in the weak topology of $C_b(\mathbb{R}^+, E)$. The spaces of all almost periodic functions, all asymptotically almost periodic functions and Eberlein–weakly almost periodic functions will be denoted by $AP(\mathbb{R}, E)$, $AAP(\mathbb{R}^+, E)$ and $W(\mathbb{R}^+, E)$ respectively. The symbol $W_0(\mathbb{R}^+, E)$ stands for the subspace of $W(\mathbb{R}^+, E)$ consisting of the functions f such that the zero function belongs to the weak closure of $\{C_t f : t \geq 0\}$. The space $AP(\mathbb{R}, E)$ is a C^*–algebra and hence it is isometrically isomorphic to $C(Y)$ for some compact Hausdorff space. For details we refer to [97]. These functions are further introduced in the next section and their generalizations to topological semigroups is reported.

If K is a (closed) subset of a Banach space E, $T : \mathbb{R}^+ \to K$ is a function and Y is a Banach space, then for every function $f : K \to Y$, $f \circ T$ is a function from $\mathbb{R}^+$ to Y and the mapping $f \to f \circ T$ is a linear transformation from $F(K, Y)$ to $F(\mathbb{R}^+, Y)$, where $F(K, Y)$ as mentioned earlier denotes the vector space of all functions from K to Y with pointwise linear operations. If $T : \mathbb{R}^+ \to K$ is continuous and $f : K \to Y$ is continuous, then $f \circ T \in C(\mathbb{R}^+, Y)$. If T and f are suitably chosen such that the vector–valued function $f \circ T : \mathbb{R}^+ \to Y$ is integrable on $[o, \alpha]$ for every $\alpha \in \mathbb{R}^+$, then the study of convergence of

$$\lim_{\alpha \to \infty} \frac{1}{\alpha} \int_0^\alpha (f \circ T)(t) dt$$

in different topologies of Y or $B(Y)$ comes under the realm of the generalised ergodic theory of the flows. In the recent years many results have been obtained in this direction. We refer to [97, 292, 295, 296] for further details.

If $\{A_t : t \in \mathbb{R}^+\}$ is a C_0–semigroup of operators on a closed subset K of a Banach space E, then for every $x \in K$ the mapping $T_x : \mathbb{R}^+ \to K$ defined as $T_x(t) = A_t(x)$ is continuous and it is known as the motion through x. If $E = Y$, and $f : K \to E$ is the inclusion map, then

$$\frac{1}{\alpha} \int_0^\alpha (f \circ T_x)(t) dt = \frac{1}{\alpha} \int_0^\alpha A_t(x) dt.$$

The study of the convergence of this type of time averages was started by Baillon [15] in 1975 where he proved that

$$\lim_{\alpha \to \infty} \frac{1}{\alpha} \int_0^\alpha A_t(x) dt$$

exists for an odd contraction semigroup $\{A_t : t \in \mathbb{R}^+\}$ of operators on real Hilbert spaces. Many Mathematical Scientists extended the results of Baillon to different types of flows on some special Banach spaces. Some of these results we shall present here without proof. We shall need some definitions.

Definitions 5.2.23. Let $\{A_t : t \in \mathbb{R}^+\}$ be a semigroup of operators on a closed subset K of E. Then a function $f \in C(\mathbb{R}^+, K)$ is said to be an almost orbit of $\{A_t : t \in \mathbb{R}^+\}$ if

$$\lim_{t \to \infty} \sup_{s \in \mathbb{R}^+} \left\| (C_s f)(t) - (A_s \circ f)(t) \right\| = 0.$$

If $x \in E$, then the Ω-limit set of the function $f : \mathbb{R}^+ \to K$ is the set $\{ y \in K :$ there is a sequence $0 < t_n \to \infty$ such that $f(t_n) \to y$ in norm$\}$. The Ω-limit set of f is denoted by $\omega(f)$. The Ω-limit set of u^x is denoted by $\omega(x)$. Similarly, taking convergence of $\{f(t_n)\}$ in weak topology we can define weak Ω-limit set of f, which we denote by $\omega_w(f)$. A function $F : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ is called strongly regular if

(i) $\{F(s,.) : s \in \mathbb{R}^+\}$ is a bounded subset of $L^1(\mathbb{R}^+)$,

(ii) $\lim_{s \to \infty} \int_0^\infty F(s,t)dt = 1$,

(iii) $\lim_{s \to \infty} \int_0^\alpha |F(s,t)|dt = 0$, for every $\alpha > 0$,

(iv) $\lim_{s \to \infty} \int_0^\alpha |F(s,t+h) - F(s,t)|dt = 0$, for every $h \in \mathbb{R}^+$.

We shall conclude this section with the following theorem in which we record some recent ergodic results of [295] pertaining to flows induced by the semigroup of operators.

Theorem 5.2.24.

(i) If $\{A_t : t \geq 0\}$ is a contraction semigroup of operators on a closed convex subset K of a uniformly convex Banach space E, and if $f : \mathbb{R}^+ \to K$ is a bounded almost orbit of $\{A_t : t \geq 0\}$ such that $\lim_{t \to 0} \|f(t+h) - f(t)\|$ exists uniformly in $h \in \mathbb{R}^+$, then there exist a unique z in the closed convex hull of $\omega_w(f)$ such that $\lim_{s \to \infty} \int_0^\infty F(s,t)f(t+h)dt = z$ in norm topology and convergence is uniform over $h \in \mathbb{R}^+$, where $F : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ is a strongly regular kernel function. The point z is a fixed point of the motion induced by $\{A_t : t \geq 0\}$.

(ii) Let K be a weakly closed subset of a Banach space E, let $\{A_t : t \geq 0\}$ be a

C_0–semigroup of operators on K and let $f \in W(\mathbb{R}^+, K)$ such that f is an almost orbit of $\{A_t : t \geq 0\}$ and restriction of A_t to the weak closure of the range of f is weak to weak continuous for $t \in \mathbb{R}^+$. Then

(a) $f = u^y + \phi$, for unique $y \in \omega_w(f)$ and unique $\phi \in W_0(\mathbb{R}^+, E)$.

(b) u^y is the restriction of a function $g \in AP(\mathbb{R}, E)$.

(c) The Ω–limit set $\omega(y)$ of u^y has the structure of a compact abelian group. If μ is the normalised Haar–measure on $\omega(y)$, K' is the w–closure of the range of f with relative weak topology and Y is any Banach space, then $\frac{1}{\alpha}\int_0^\alpha g\big(f(t+h)\big)dt$ converges to $\int_{\omega(y)} g d\mu$

in weak topology as $\alpha \to \infty$, for every weak to weak continuous function $g : K' \to Y$. Also

$$\lim_{\alpha\to\infty} \frac{1}{\alpha}\int_0^\alpha g\big(f(t+h)\big)dt = \int_{\omega(y)} g d\mu$$

in norm topology for every $g \in C(K', Y)$.

(both limits exists uniformly over $h \geq 0$).

5.3 DYNAMICAL SYSTEMS AND COMPOSITION OPERATORS

In the beginning of this section we obtain (linear) dynamical systems induced by some composition operators and multiplication operators on the weighted spaces of functions and then give an asymptotic behaviour of motions of dynamical systems on these spaces. A study of semigroups of composition operators on Hardy spaces as well as on Bergman spaces is presented in this section. Finally, we wind up this section by giving some details of weighted composition semigroups and weighted translation semigroups on Hardy spaces and $L^2\big(\mathbb{R}^+\big)$ respectively.

Let $\omega \in \mathbb{R}$, let $T_\omega : \mathbb{R} \to \mathbb{R}$ be the translation defined as $T_\omega(t) = t + \omega$ for $t \in \mathbb{R}$, and let $v(t) = e^{-|t|}$, $t \in \mathbb{R}$. Let $\nabla : \mathbb{R} \times CV_0\big(\mathbb{R}\big) \to CV_0\big(\mathbb{R}\big)$ be defined as $\nabla(\omega, f) = f \circ T_\omega$, where $V = \{\lambda v : \lambda > 0\}$. Then by Example 3.2 of [293], $\big(\mathbb{R},\ CV_0\big(\mathbb{R}\big),\ \nabla\big)$ is a linear dynamical system. For the sake of simplicity we say that ∇ is a dynamical system on $CV_0\big(\mathbb{R}\big)$. There are systems of weights for which $CV_0\big(\mathbb{R}\big)$ is not translation–invariant and hence in such cases we do not get translation-induced dynamical systems; for examples see [362]. Recently Singh and Summers in [368] characterised some systems for which the translations T_ω induce dynamical systems on $CV_0(\mathbb{R})$. Let $v : \mathbb{R} \to \mathbb{R}$ be any continuous strictly positive weight function let $V = \{\lambda v : \lambda > 0\}$. In view of Corollary 4.2.5, it follows that each translation $T_\omega : \mathbb{R} \to \mathbb{R}$, induces a composition operator on $CV_0(\mathbb{R})$ if $V \leq V \circ T_\omega$. Thus $V \leq V \circ T_\omega$ is a necessary condition for ∇ to be a dynamical system on $CV_0(\mathbb{R})$. Now, we show that this condition is also sufficient. We shall present this result in the following theorem.

Theorem 5.3.1. Let $v \in C(\mathbb{R})$ be such that $v(t) > 0$, for every $t \in \mathbb{R}$, and let $V = \{\lambda v : \lambda > 0\}$. Consider the function $\nabla : \mathbb{R} \times CV_0(\mathbb{R}) \to C(\mathbb{R})$ defined by $\nabla(\omega, f) = f \circ T_\omega$, for $\omega \in \mathbb{R}$ and $f \in CV_0(\mathbb{R})$. Then the following are equivalent:

(a) ∇ is a (linear) dynamical system on $CV_0(\mathbb{R})$,

(b) $V \leq V \circ T_\omega$, for every $\omega \in \mathbb{R}$,

(c) $C_{T_\omega}(CV_0(\mathbb{R})) \subset CV_b(\mathbb{R})$, for each $\omega \in \mathbb{R}$.

In order to give the proof of the above theorem, we need to prove the following two lemmas. Let $V \leq V \circ T_\omega$, for some $\omega \in \mathbb{R}$. Then define

$$\lambda(\omega) = \inf\{\lambda \geq 0 \ : \ v \leq \lambda v \circ T_\omega\},$$

which is the same as setting

$$\lambda(\omega) = \sup\left\{\frac{v(t)}{v(t+\omega)} \ : \ t \in \mathbb{R}\right\}.$$

Lemma 5.3.2. Let $V \leq V \circ T_\omega$, for every $\omega \in \mathbb{R}$. Then there exists $\varepsilon > 0$ and a function $F : \mathbb{R} \to \mathbb{R}^+$ such that, given $\omega \in \mathbb{R}$, $\lambda(\tau) \leq F(\omega)$ whenever $\tau \in \mathbb{R}$ with $|\tau - \omega| < \varepsilon$.

Proof. For each $n \in \mathbb{N}$, we set $K_n = \{\tau \in \mathbb{R} : \lambda(\tau) \leq n\}$. Since K_n is closed in $\mathbb{R}$ for every $n \in \mathbb{N}$ and $\mathbb{R} = \bigcup_{n=1}^{\infty} K_n$, by the Baire category theorem there exists $m \in \mathbb{N}$ such that $\operatorname{int}(K_m) \neq \emptyset$. Now, we choose $\sigma \in K_m$ and $\varepsilon > 0$ such that $(\sigma - \varepsilon, \sigma + \varepsilon) \subset K_m$. Fix $\omega \in \mathbb{R}$ and set $F(\omega) = m\,\lambda(\omega - \sigma)$. For $\tau \in \mathbb{R}$ such that

$|\omega-\tau|<\varepsilon$ and $t\in\mathbb{R}$, we then have that

$$\frac{v(t)}{v(t+\tau)}=\frac{v(t)}{v(t+\tau-\omega+\sigma)}\cdot\frac{v(t+\tau-\omega+\sigma)}{v(t+\tau)}$$
$$\leq \lambda(\tau-\omega+\sigma)\,\lambda\,(\omega-\sigma)\leq F(\omega).$$

Thus it follows that $\lambda(\tau)\leq F(\omega)$.

Lemma 5.3.3. Let $V\leq V\circ T_\omega$ for every $\omega\in\mathbb{R}$, and let $\nabla\ :\ \mathbb{R}\times CV_0(\mathbb{R})\to C(\mathbb{R})$ be the function defined by $\nabla(\omega,f)=f\circ T_\omega$, for $\omega\in\mathbb{R}$ and $f\in CV_0(\mathbb{R})$. Then ∇ is a (linear) dynamical system on $CV_0(\mathbb{R})$.

Proof. We note that if $V\leq V\circ T_\omega$ for each $\omega\in\mathbb{R}$, then $\nabla(\omega,f)=f\circ T_\omega\in CV_0(\mathbb{R})$ for every $\omega\in\mathbb{R}$ and all $f\in CV_0(\mathbb{R})$. Also, the required group properties are satisfied by ∇. In order to show that ∇ is a (linear) dynamical system on $CV_0(\mathbb{R})$, in view of Theorem 1 of [66], it is enough to show that ∇ is a separately continuous map. Fix $\omega\in\mathbb{R}$, and let $\{f_n\}$ be a sequence in $CV_0(\mathbb{R})$ such that $f_n\to f$ in $CV_0(\mathbb{R})$. Then for $v\in V$ we have

$$\begin{aligned}\|\nabla(\omega,f_n)-\nabla(\omega,f)\|_v&=\|f_n\circ T_\omega-f\circ T_\omega\|_v\\&=\sup\{v(t)\,|\,f_n(t+\omega)-f(t+\omega)\,|\ :\ t\in\mathbb{R}\}\\&\leq\sup\{v(t)\,|\,f_n(t)-f(t)\,|\ :\ t\in\mathbb{R}\}\\&=\|f_n-f\|_v\to 0\ \text{ as }\ n\to\infty.\end{aligned}$$

This proves the continuity of ∇ in the first argument. Now we fix $f\in CV_0(\mathbb{R})$ and show that the function $\nabla(.,f)\ :\ \mathbb{R}\to CV_0(\mathbb{R})$ is continuous. For this, we fix $\omega\in\mathbb{R}$ and $\varepsilon>0$. Then by Lemma 5.3.2 there exist $\delta_1>0$ and $F(\omega)>0$ such that $\lambda(\tau)\leq F(\omega)$ for all $\tau\in\mathbb{R}$ with $|\tau-\omega|<\delta_1$. Since vf is a continuous function vanishing at infinity on $\mathbb{R}$, there exists $\delta_2>0$ such that

$$|v(s)f(s)-v(t)f(t)|<\varepsilon/2\,F(\omega),\ \text{for}\ s,t\in\mathbb{R}$$

such that $|s-t|<\delta_2$, and there also exists $t_0\in\mathbb{R}^+$ so that

$$v(t)\,|f(t)|<\varepsilon/4\,F(\omega)\tag{1}$$

when $t\in\mathbb{R}$ and $|t|>t_0$. Now we put $\eta=\inf\{v(t)\ :\ |t|\leq t_0\}$. There exists

$\delta_3 > 0$ such that if $s, t \in \mathbb{R}$ with $|s|, |t| \le t_0 + 1$ and $|s - t| < \delta_3$, then

$$| v(s) - v(t) | < \eta \, \varepsilon \Big/ \Big[2\big(\| f \|_v + 1 \big) F(\omega) \Big] . \tag{2}$$

Take $\delta = \min\{\delta_1, \delta_2, \delta_3, 1\}$. Let $\tau \in \mathbb{R}$ be such that $|\tau - \omega| < \delta$. Then for any $t \in \mathbb{R}$, it follows from either (1) or (2) that

$$v(t+\omega) | f(t+\omega) | \left| \frac{v(t)}{v(t+\tau)} - \frac{v(t)}{v(t+\omega)} \right| < \varepsilon / 2.$$

Further, it implies that

$$v(t) | f(t+\tau) - f(t+\omega) | \le \frac{v(t)}{v(t+\tau)} |v(t+\tau) f(t+\tau) - v(t+\omega) f(t+\omega)| + \varepsilon / 2.$$

Thus it follows that ∇ is continuous in the second argument. This completes the proof of the lemma.

Proof of Theorem 5.3.1. The implication (b) $\Rightarrow$ (a) is an immediate consequence of Lemma 5.3.3 whereas assertion (c) obviously follows from (a). To complete the proof we need to prove the implication (c) $\Rightarrow$ (b). Suppose on the contrary that (b) does not holds. Then there exist $\omega \in \mathbb{R}$ and a sequence $\{t_n\}$ in $\mathbb{R}$ such that $v(t_n) \ge n^2 v(t_n + \omega)$ for each $n \in \mathbb{N}$. Further, we may suppose that there exists $g \in C_0(\mathbb{R})$ for which $g(t_n + \omega) = 1/n$, $n \in \mathbb{N}$. Now, if we put $f = g/v$, then $f \in CV_0(\mathbb{R})$. Since

$$v(t_n) \left| f(t_n + \omega) \right| \ge n^2 g(t_n + \omega) = n, \text{ for each } n \in \mathbb{N},$$

it follows that $f \circ T_\omega \in CV_b(\mathbb{R})$, which is a contradiction. This completes the proof of the theorem.

Now, we give two examples which help to illustrate the above situation.

Example 5.3.4. Let $v : \mathbb{R} \to \mathbb{R}^+$ be defined as $v(t) = e^{-|t|}$, for $t \in \mathbb{R}$. If we take any $\omega \in \mathbb{R}$, then $V \le V \circ T_\omega$ since $v(t) \le e^{|\omega|}. v(t+\omega)$ for all $t \in \mathbb{R}$ (see [293], Example 3.2).

Example 5.3.5. If $v(t) = e^{-t^2}$, for $t \in \mathbb{R}$, then it is obvious that $V \le V \circ T_\omega$ only if $\omega = 0$.

Now our efforts are to present some results on dynamical systems induced by the multiplication operators on the weighted spaces of continuous functions and the weighted spaces of cross–sections. Let E be a commutative Banach algebra with unit element, and let $C_b(\mathbb{R},E)$ be the Banach algebra of all bounded continuous functions from $\mathbb{R}$ to E, and let V be a system of weights on $\mathbb{R}$. Then clearly $CV_b(\mathbb{R},E)$ is a locally convex Hausdorff space with the weighted topology ω_V. For $g \in C_b(\mathbb{R},E)$ and for each $t \in \mathbb{R}$, we define the map $\pi_t : \mathbb{R} \to E$ as $\pi_t(\omega) = e^{t\, g(\omega)}$, for every $\omega \in \mathbb{R}$. It is easy to see that π_t is a continuous bounded map from $\mathbb{R}$ to E, and hence by Corollary 3.8 of [353], π_t induces a multiplication operator M_{π_t} on the weighted locally convex space $CV_b(\mathbb{R},E)$ for any system of weights V on $\mathbb{R}$.

Theorem 5.3.6. Let $g \in C_b(\mathbb{R},E)$, and let $\nabla_g : \mathbb{R} \times CV_b(\mathbb{R},E) \to C(\mathbb{R},E)$ be the function defined by $\nabla_g(t,f) = M_{\pi_t} f$, for $t \in \mathbb{R}$ and $f \in CV_b(\mathbb{R},E)$. Then ∇_g is a dynamical system on $CV_b(\mathbb{R},E)$.

Proof. Since for every $t \in \mathbb{R}$, M_{π_t} is a multiplication operator on $CV_b(\mathbb{R},E)$, we can conclude that $\nabla_g(t,f) = M_{\pi_t} f$ belongs to $CV_b(\mathbb{R},E)$ whenever $t \in \mathbb{R}$ and $f \in CV_b(\mathbb{R},E)$. Also, we can easily see that $\nabla_g(0\ ,f) = f$ and $\nabla_g(t+s,f) = \nabla_g(t,\nabla_g(s,f))$, for every $t,s \in \mathbb{R}$ and $f \in CV_b(\mathbb{R},E)$. In order to show that ∇_g is a dynamical system on $CV_b(\mathbb{R},E)$, it is enough to establish that ∇_g is continuous. To this end, let $\{(t_\alpha,f_\alpha)\}$ be a net in $\mathbb{R} \times CV_b(\mathbb{R},E)$ such that $(t_\alpha,f_\alpha) \to (t,f)$ in $\mathbb{R} \times CV_b(\mathbb{R},E)$. We shall show that $\nabla_g(t_\alpha,f_\alpha) \to \nabla(t,f)$ in $CV_b(\mathbb{R},E)$. Let $v \in V$. Then

$$
\begin{aligned}
\left\| \nabla_g(t_\alpha,f_\alpha) - \nabla_g(t,f) \right\|_v &= \left\| M_{\pi_{t_\alpha}} f_\alpha - M_{\pi_t} f \right\|_v = \left\| \pi_{t_\alpha} f_\alpha - \pi_t f \right\|_v \\
&= \sup\left\{ v(\omega) \left\| \pi_{t_\alpha}(\omega) f_\alpha(\omega) - \pi_t(\omega) f_\alpha(\omega) + \pi_t(\omega) f_\alpha(\omega) - \pi_t(\omega) f(\omega) \right\| \ :\ \omega \in \mathbb{R} \right\} \\
&\le \sup\left\{ v(\omega) \left\| \pi_{t_\alpha}(\omega) - \pi_t(\omega) \right\| \left\| f_\alpha(\omega) \right\| \ :\ \omega \in \mathbb{R} \right\} \\
&\qquad + \sup\left\{ v(\omega) \left\| \pi_t(\omega) \right\| \left\| f_\alpha(\omega) - f(\omega) \right\| \ :\ \omega \in \mathbb{R} \right\} \\
&= \sup\left\{ v(\omega) \left\| f_\alpha(\omega) \right\| \left\| e^{tg(\omega)} \right\| \left\| \left(e^{t_\alpha g(\omega) - tg(\omega)} - 1 \right) \right\| \ :\ \omega \in \mathbb{R} \right\} \\
&\qquad + \sup\left\{ v(\omega) \left\| f_\alpha(\omega) - f(\omega) \right\| \left\| e^{tg(\omega)} \right\| \ :\ \omega \in \mathbb{R} \right\}
\end{aligned}
$$

$$\leq \sup\left\{ v(\omega) \parallel f_\alpha(\omega) \parallel e^{|t| \| g \|_\infty} \left(e^{|t_\alpha - t| \| g \|_\infty} - 1 \right) \ : \ \omega \in \mathbb{R} \right\}$$

$$+ \sup\left\{ v(\omega) \parallel f_\alpha(\omega) - f(\omega) \parallel e^{|t| \| g \|_\infty} \ : \ \omega \in \mathbb{R} \right\}$$

$$= e^{|t| \| g \|_\infty} \left(e^{|t_\alpha - t| \| g \|_\infty} - 1 \right) \parallel f_\alpha \parallel_v + e^{|t| \| g \|_\infty} \parallel f_\alpha - f \parallel_v$$

$$\longrightarrow 0 \ \text{ as } \ | t_\alpha - t | \to 0 \ \text{ and } \ \parallel f_\alpha - f \parallel_v \to 0.$$

This proves that ∇_g is continuous and hence ∇_g is a (linear) dynamical system on $CV_b(\mathbb{R},E)$.

Remark 5.3.7. If $\mathcal{F} = \{M_{\pi_t} \ : \ t \in \mathbb{R}\}$ is the family of multiplication operators on $CV_b(\mathbb{R},E)$ defined above, then it satisfies the following :

(i) $M_{\pi_{t+s}} f = M_{\pi_t}(M_{\pi_s} f)$, for all $t, s \in \mathbb{R}$ and $f \in CV_b(\mathbb{R},E)$.

(ii) $M_{\pi_0} f = f$, for all $f \in CV_b(\mathbb{R},E)$.

(iii) $\lim_{t \to 0} M_{\pi_t} f = f$, for all $f \in CV_b(\mathbb{R},E)$.

Thus the family $\mathcal{F}$ is a C_0–group of multiplication operators on the locally convex space $CV_b(\mathbb{R},E)$ which turns out to be locally equicontinuous under certain conditions on the system of weights V on $\mathbb{R}$. To show that the family $\mathcal{F}$ is locally equicontinuous, it is to be established that for any fixed $s \in \mathbb{R}$, the subfamily $\mathcal{F}_s = \{M_{\pi_t} \ : \ -s \leq t \leq s\}$ is equicontinuous on $CV_b(\mathbb{R},E)$. Let V be a system of weights on $\mathbb{R}$ such that for each compact subset K of $\mathbb{R}$, there exists a weight $v \in V$ such that $\inf_{t \in K} v(t) > 0$. Let U be a countable set of non-negative upper semicontinuous functions on $\mathbb{R}$ such that $W = \{\alpha u \ : \ \alpha \geq 0, \ u \in U\}$ is a system of weights on $\mathbb{R}$ with $W \approx V$. Then clearly the weighted space $CV_b(\mathbb{R},E)$ is metrizable. In case $E = \mathbb{C}$, the metrizable weighted space $CV_b(\mathbb{R})$ is a special case of the result proved by Summers [380, Theorem 2.10]. Completeness of the weighted space $CV_b(\mathbb{R},E)$ follows from Proposition 4.3.4 and a remark following Proposition 7 in [32]. Thus the weighted space $CV_b(\mathbb{R},E)$ is completely metrizable. For each $s \in \mathbb{R}$, the family $\mathcal{F}_s = \{M_{\pi_t} \ : \ -s \leq t \leq s\}$ is a bounded set in $B(CV_b(\mathbb{R},E))$ since the map $t \to M_{\pi_t}$ is continuous, where $B(CV_b(\mathbb{R},E))$ is the locally convex space of all continuous linear operators on $CV_b(\mathbb{R},E)$ with strong operator topology. Also, for each $f \in CV_b(\mathbb{R},E)$, the set $\mathcal{F}_s = \{M_{\pi_t} f \ : \ -s \leq t \leq s\}$ is bounded in $CV_b(\mathbb{R},E)$. From a corollary of the Banach–Steinhaus Theorem [288, Theorem 2.6] it follows that the family $\mathcal{F}$ is locally equicontinuous.

Let $B(E)$ be the Banach algebra of all bounded linear operators on a Banach Space E and let V be a system of weights on $\mathbb{R}$ such that the weighted space $CV_b(\mathbb{R}, E)$ is metrizable. Fix $g \in C_b(\mathbb{R})$ and $A \in B(E)$. Then for each $t \in \mathbb{R}$, we define the mapping $\pi_t : \mathbb{R} \to B(E)$ as

$$\pi_t(\omega) = e^{tg(\omega)A}, \text{ for every } \omega \in \mathbb{R}.$$

Thus π_t is a continuous bounded operator–valued mapping and hence by Remark 2.2 (ii) of [349] it follows that M_{π_t} is a multiplication operator on $CV_b(\mathbb{R}, E)$. Now, for $g \in C_b(\mathbb{R})$ and $A \in B(E)$ if we define the function $\nabla_{A,g} : \mathbb{R} \times CV_b(\mathbb{R}, E) \to C(\mathbb{R}, E)$ as $\nabla_{A,g}(t, f) = M_{\pi_t} f$, for $t \in \mathbb{R}$ and $f \in CV_b(\mathbb{R}, E)$, then $\nabla_{A,g}$ is a dynamical system on $CV_b(\mathbb{R}, E)$ (see [349, Theorem 3.1]).

Let $\{F_t : t \in \mathbb{R}\}$ be the vector–fibration over $\mathbb{R}$, where each F_t is a Banach space over the field $\mathbb{C}$ with the norm denoted by $\| . \|_t$ and by $B(F_t)$ we mean the Banach algebra of all bounded linear operators on F_t with the operator norm $||| \, . \, |||_t$. By $F_b(\mathbb{R})$ we denote the normed linear space of all bounded functions on $\mathbb{R}$ with supnorm. Fix $g \in F_b(\mathbb{R})$. For each $t \in \mathbb{R}$, if we define the linear transformation $A_t : F_t \to F_t$ as

$$A_t(y) = g(t)y, \quad \text{for every } y \in F_t,$$

then $\| A_t(y) \|_t = \| g(t)y \|_t = | g(t) | \| y \|_t \le M \| y \|_t$ for every $y \in F_t$, where $M = \| g \|_\infty$. Thus it follows that each A_t is a bounded linear operator on F_t and hence $A_t \in B(F_t)$. Also, note that $||| A_t |||_t \le M$, for every $t \in \mathbb{R}$. If w is the function defined on $\mathbb{R}$ such that $w(t) = \| . \|_t$, for every $t \in \mathbb{R}$, then w is a weight on $\mathbb{R}$ and the set $W = \{\lambda w : \lambda > 0\}$ is a system of weights on $\mathbb{R}$. Let $L(\mathbb{R})$ be a vector space of cross-sections over $\mathbb{R}$. Then it readily follows that the weighted space $LW_0(\mathbb{R})$ is a normed linear space with respect to the system of weights W on $\mathbb{R}$.

For $s \in \mathbb{R}$, define the operator–valued mapping $\pi_s : \mathbb{R} \to \bigcup_{t \in \mathbb{R}} \left(B(F_t)\right)$ as

$$\pi_s(t) = e^{sA_t}, \quad \text{for every } t \in \mathbb{R}.$$

Now, it can be easily seen that each π_s induces a multiplication operator M_{π_s} on $LW_0(\mathbb{R})$.

In the following theorem we shall obtain a (linear) dynamical system on the weighted spaces of cross-sections.

Theorem 5.3.8. Let $g \in F_b(\mathbb{R})$ and let $\nabla_g : \mathbb{R} \times LW_0(\mathbb{R}) \to L(\mathbb{R})$ be the function defined by $\nabla_g(s,f) = M_{\pi_s} f$, for $s \in \mathbb{R}$ and $f \in LW_0(\mathbb{R})$. Then ∇_g is a dynamical system on $LW_0(\mathbb{R})$.

Proof. Since M_{π_s} is a multiplication operator on $LW_0(\mathbb{R})$ for each $s \in \mathbb{R}$, we can conclude that $\nabla_g(s,f)$ belongs to $LW_0(\mathbb{R})$, whenever $s \in \mathbb{R}$ and $f \in LW_0(\mathbb{R})$. Thus ∇_g is a function from $\mathbb{R} \times LW_0(\mathbb{R}) \to LW_0(\mathbb{R})$. One can easily show that $\nabla_g(0,f) = f$, for every $f \in LW_0(\mathbb{R})$ and $\nabla_g(s+t,f) = \nabla_g(s,\nabla_g(t,f))$, for every $s,t \in \mathbb{R}$ and $f \in LW_0(\mathbb{R})$. Now, we shall show that ∇_g is continuous. Let $\{(s_n,f_n)\}$ be a sequence in $\mathbb{R} \times LW_0(\mathbb{R})$ such that $(s_n,f_n) \to (s,f)$ in $\mathbb{R} \times LW_0(\mathbb{R})$. Let $w' \in W$. Then $w' = \lambda w$, $\lambda > 0$. Now

$$
\begin{aligned}
\left\| \nabla_g(s_n,f_n) - \nabla_g(s,f) \right\|_{w'} &= \sup\Big\{ \left\| \pi_{s_n}(t)f_n(t) - \pi_s(t)f_n(t) + \pi_s(t)f_n(t) \right. \\
&\qquad\qquad \left. -\pi_s(t)f(t) \right\|_t \; : \; t \in \mathbb{R} \Big\} \\
&\le \sup\left\{ e^{|s|M}\left(e^{|s_n-s|M} - 1\right) w_t'[f_n(t)] \; : \; t \in \mathbb{R} \right\} \\
&\qquad + \sup\left\{ e^{|s|M} w_t'[f_n(t) - f(t)] \; : \; t \in \mathbb{R} \right\} \\
&= e^{|s|M}\left(e^{|s_n-s|M} - 1\right) \| f_n \|_{w'} + e^{|s|M} \| f_n - f \|_{w'} \\
&\to 0 \quad \text{as} \quad |s_n - s| \to 0 \quad \text{and} \quad \| f_n - f \|_{w'} \to 0.
\end{aligned}
$$

This shows that the map ∇_g is continuous and hence ∇_g is a (linear) dynamical system on $LW_0(\mathbb{R})$. This completes the proof of the theorem.

Remark 5.3.9. Let $\mathcal{F} = \{M_{\pi_s} : s \in \mathbb{R}\}$ be the family of multiplication operators on the weighted space $LW_0(\mathbb{R})$ as defined earlier. Then we have the following straightforward observations :

(i) $M_{\pi_{s+t}} f = M_{\pi_s}\left(M_{\pi_t} f\right)$, for every $f \in LW_0(\mathbb{R})$.

(ii) $M_{\pi_0} f = f$, for every $f \in LW_0(\mathbb{R})$.

(iii) $\lim_{t \to 0} M_{\pi_t} f = f$, for all $f \in LW_0(\mathbb{R})$

Also, by the same argument as used in Remark 5.3.7 we can conclude that the C_0–group $\mathcal{F}$ is locally equicontinuous.

Ruess and Summers [290, 294] have studied asymptotic behaviour of motions of dynamical systems that occur in the context of the abstract Cauchy problem. While making this study one cannot ignore the significance of the composition operators, which we shall outline here as under.

Recall that for a fixed $\omega \in \mathbb{R}$, the mapping $T_\omega : \mathbb{R} \to \mathbb{R}$ is defined as $T_\omega(t) = t + \omega$, for every $t \in \mathbb{R}$. Let $V = \{\lambda \chi_K \ : \ \lambda \geq 0, \ K \subset \mathbb{R}, \ K$ compact$\}$ and $U = K^+(\mathbb{R})$ be the system of constant weights on $\mathbb{R}$. Then from Example 1.4.1 we observe that

$$CV_0(\mathbb{R}, E) = CV_p(\mathbb{R}, E) = CV_b(\mathbb{R}, E) = (C(\mathbb{R}, E), k)$$

and

$$CU_0(\mathbb{R}, E) = (C_0(\mathbb{R}, E), \ u), CU_p(\mathbb{R}, \ E) = (C_p(\mathbb{R}, \ E), \ u),$$
$$CU_b(\mathbb{R}, E) = (C_b(\mathbb{R}, E), \ u),$$

where k and u denote the compact–open topology and the topology of uniform convergence on $\mathbb{R}$ respectively. From Theorem 4.2.26 and Theorem 4.2.28, it follows that each translation $T_\omega : \mathbb{R} \to \mathbb{R}$ induces a composition operator C_{T_ω} on either $(C(\mathbb{R}, E), k)$, $(C_p(\mathbb{R}, E), u)$ or $(C_b(\mathbb{R}, E), u)$. In [290], Ruess and Summers derive a complete solution of the following problem :

Problem 5.3.10. Characterise those $f \in C_b(\mathbb{R}^+, E)$ for which the family of functions $\mathcal{F}^+ = \{C_{T_\omega} f \ : \ \omega \in \mathbb{R}^+\}$ is relatively compact in $(C_b(\mathbb{R}^+, E), u)$ where C_{T_ω} is the composition operator on $(C_b(\mathbb{R}^+, E), u)$ induced by T_ω.

In this problem if we take the range space E as a Banach space and $\mathbb{R}^+$ is replaced by $\mathbb{R}$, then the solution is obtained by Bochner [36]. That is, if $f \in C(\mathbb{R}, E)$, then the family of mappings $\mathcal{F} = \{C_{T_\omega} f \ : \ \omega \in \mathbb{R}\}$, is relatively compact in $(C_b(\mathbb{R}, E), u)$ if and only if f is almost periodic. In order to present Bochner's result we shall need the following definitions.

Definition 5.3.11. Let $G \subset \mathbb{R}$ be a set of real numbers. Then G is said to be relatively dense if there exists a number $m > 0$ such that any interval $(\alpha, \alpha + m) \subset \mathbb{R}$ of length m contains at least one member from G. Let E be a Banach space. Then a number τ is called an ε–almost period of $f : \mathbb{R} \to E$ if

$$\sup_{t \in \mathbb{R}} \| f(t + \tau) - f(t) \| \leq \varepsilon . \tag{1}$$

Definition 5.3.12. Let $f : \mathbb{R} \to E$ be a continuous function. Then f is said to be almost periodic if it has a relatively dense set of ε–almost periods for each $\varepsilon > 0$, *i.e.*, if there is a number $\rho = \rho(\varepsilon)$ such that each interval $(\alpha, \alpha + \rho) \subset \mathbb{R}$ contains at least one number $\tau = \tau_\varepsilon$ satisfying (1).

Theorem 5.3.13 [Bochner's Theorem]. Let E be a Banach space and let $f : \mathbb{R} \to E$ be a continuous function. Then f is almost periodic if and only if the family of functions $\mathcal{F} = \{C_{T_\omega} f \ : \ \omega \in \mathbb{R}\}$ is compact in $(C_b(\mathbb{R}, E), u)$, where $C_{T_\omega} : C_b(\mathbb{R}, E) \to C_b(\mathbb{R}, E)$ is the composition operator induced by T_ω.

Proof. Suppose that f is an almost periodic function. We denote by $\mathbb{Q}$, the set of all rational points on $\mathbb{R}$. Let $\{C_{T_{\omega_n}} f\}$ be an arbitrary sequence in $\mathcal{F}$. By applying the diagonal process, we can choose a subsequence of $\{C_{T_{\omega_n}} f\}$ and label it again by $\{C_{T_{\omega_n}} f\}$ which converges for any $r \in \mathbb{Q}$ since the closure of the range of f is compact. We shall show that the sequence $\{C_{T_{\omega_n}} f\}$ converges in $(C_b(\mathbb{R}, E), u)$. Fix $\varepsilon > 0$ and let $\rho = \rho_\varepsilon$ be the corresponding length. Since f is uniformly continuous, we choose $\delta = \delta_\varepsilon > 0$ and subdivide the segment $[0, \rho]$ into p–segments $\Delta_k (k = 1, 2, \ldots, p)$ of length not greater than δ; and further we choose rational point r_k from each Δ_k. Suppose that $n = n_\varepsilon$ is choosen so that

$$\left\| C_{T_{\omega_n}} f(r_k) - C_{T_{\omega_n}} f(r_k) \right\| < \varepsilon, \tag{2}$$

for $n, m \geq n_\varepsilon$ and $k = 1, \ldots, p$. For every $t_0 \in \mathbb{R}$, we find a $\tau = \tau_0$ such that

$$0 \leq t_0 + \tau \leq \rho \quad \text{if and only if} \quad -t_0 \leq \tau \leq -t_0 + \rho.$$

Suppose that the number $t_0' = t_0 + \tau$ falls in the interval Δ_{k_0} and that $r_{k_0} \in \Delta_{k_0}$ is the rational point choosen earlier. Then by the choice of δ we have

$$\left\| C_{T_{\omega_n}} f(t_0') - C_{T_{\omega_n}} f(r_{k_0}) \right\| < \varepsilon, \tag{3}$$

and

$$\left\| C_{T_{\omega_m}} f(t_0') - C_{T_{\omega_m}} f(r_{k_0}) \right\| < \varepsilon. \tag{4}$$

From (2), (3) and (4), it follows that $\left\| C_{T_{\omega_n}} f(t_0) - C_{T_{\omega_m}} f(t_0) \right\| < 5\varepsilon$. Since $t_0 \in \mathbb{R}$ was arbitrary, the inequality implies that the sequence $\{C_{T_{\omega_n}} f\}$ converges in $(C_b(\mathbb{R}, E), u)$. Thus the family $\mathcal{F}$ is compact in $(C_b(\mathbb{R}, E), u)$.

Conversely, we suppose that the family $\mathcal{F} = \{C_{T_\omega} f \ : \ \omega \in \mathbb{R}\}$ is compact in $(C_b(\mathbb{R}, E), u)$. Now, it is clear that the function f is bounded. From the boundedness of f we conclude that the family is contained in $C_b(\mathbb{R}, E)$. Further, by a criterion of Hausdorff, it follows that for every $\varepsilon > 0$, there are numbers $s_1, s_2, \ldots, s_p$ such that for all $s \in \mathbb{R}$ there is $k = k(s)$ such that

$$\sup_{t \in \mathbb{R}} \left\{ \left\| C_{T_s} f(t) - C_{T_{s_k}} f(t) \right\| \right\} < \varepsilon . \tag{5}$$

Further, it implies that

$$\sup_{t \in \mathbb{R}} \left\{ \left\| C_{T_{s-s_k}} f(t) - C_{T_0} f(t) \right\| \right\} < \varepsilon .$$

Thus the numbers $s - s_k (k = 1, \ldots, p)$ are ε–almost periods for f. To complete the proof, we need to prove that the set of numbers $s - s_k$ is relatively dense. Let $M = \max_{1 \le k \le p} | s_k |$. Then $s - M \le s - s_k \le s + M$. Since s is arbitrary, the last inequality implies that every interval of length $2M$ contains an ε–almost period for f. This completes the proof of the theorem.

Using the composition operators induced by translations the theory of different types of periodic functions has been extended to functions on semitopological semigroups. If S is a semitopological semigroup, then by definition left translation $L_s : S \to S$ and the right translation $R_s : S \to S$ are continuous. A function $f \in C_b(S)$ is said to be almost periodic if the closure of the set $\{C_{R_s} f \ : \ s \in S\}$ is compact in the norm topology of $C_b(S)$, where $C_{R_s} : C_b(S) \to C_b(S)$ is the composition operator induced by R_s. If S is a compact topological semigroup (*i.e.*, multiplication is jointly continuous), then $C_b(S) = AP(S)$. Thus every function on the unit circle is almost periodic. It is well known that the space $AP(S)$ of all almost periodic functions on S is a translation invariant C^*-subalgebra of $C_b(S)$ containing all constant functions. The following theorem shows that $AP(S)$ is invariant under the composition operators induced by continuous homomorphisms.

Theorem 5.3.14. Let S_1 and S_2 be two semitopological semigroups and let $T : S_1 \to S_2$ be a continuous homomorphism. Then

$$C_T\big(AP(S_2)\big) \subset AP(S_1)\ .$$

Proof. Let $f \in AP(S_2)$ and $s \in S_1$. Then

$$\begin{aligned}
\left(C_{R_s} C_T f\right)(x) &= \left(f \circ T \circ R_s\right)(s) \\
&= \left(f \circ T\right)(xs) \\
&= f\big(T(x)T(s)\big) \\
&= \left(C_{R_{T(s)}} f\right)\!\big(T(x)\big) \\
&= \left(C_T C_{R_{T(s)}}\, f\right)(x), \quad x \in S_1.
\end{aligned}$$

Thus

$$\left\{C_{R_s} C_T f \ :\ s \in S_1\right\} = \left\{C_T C_{R_{T(s)}} \ :\ s \in S_1\right\}.$$

Since C_T is a bounded operator, we conclude that the set $\{C_{R_s}\, C_T f \ :\ s \in S_1\}$ has compact closure in $C_b(S_1)$. Hence $C_T f \in AP(S_1)$. This shows that $C_T(AP(S_2)) \subset AP(S_1)$.

Note. (i) If S_1 is a subsemigroup of S_2, then taking T as the inclusion map it can be seen that if $f \in AP(S_2)$, then $f / S_1 \in AP(S_1)$.

(ii) Taking weak topology on $C_b(S)$ we can define weakly almost periodic functions on S. The space of all weakly almost periodic functions on S is denoted by

$WAP(S)$. With the help of finite dimensional unitary representation of S the strongly almost periodic function has been defined and studied (see [22] for details).

(iii) Vector-valued almost periodic functions and weakly almost periodic functions on S can be defined taking the space $C_b(S,E)$, where E is a Banach space or a locally convex space. We have introduced these functions on $\mathbb{R}^+$ (or $\mathbb{R}$) in the last section. If S is a compact semitopological semigroup and E is a Banach space, then it is not difficult to see that

$$C_b(S,E) = WAP(S,E).$$

If $T: S_1 \to S_2$ is a continuous homomorphism, then

$$C_T\big(WAP(S_2,E)\big) \subset WAP(S_1,E)\ .$$

After some years of the Bochner's result, Frechet [119, 120] obtained an analogous result for subintervals of $\mathbb{R}$ of the form $[a, \infty)$, $a \in \mathbb{R}$ and the range space E restricted to finite dimensional space. Here we shall present an account of some of these results and related ideas obtained by Ruess and Summers [290] in the general setting of an arbitrary locally convex range space. In order to proceed further, we need the following definitions.

Definition 5.3.15. Let $a \in \mathbb{R}$, and let $I_a = \{t \in \mathbb{R} : t \geq a\}$. Then a subset G of I_a is said to be relatively dense in I_a if there exists $m > 0$ such that for each $t \in I_a$, the closed interval $[t,\ t+m]$ contains at least one member of G.

Definition 5.3.16 [119, 120]. Let $a \in \mathbb{R}$ and let E be a locally convex space. Then a continuous function $f : I_a \to E$ is said to be asymptotically almost periodic $(a.a.p)$ if for any $\varepsilon > 0$ and $q \in cs(E)$, there exists $\rho = \rho(\varepsilon, q) \geq a$ and a relatively dense set $G = G(\varepsilon,\ q)$ in I_ρ such that $q(f(t+\tau) - f(t)) \leq \varepsilon$, for each $\tau \in G$ and each $t \in I_\rho$ with $t + \tau \geq \rho$.

Definition 5.3.17. Let $a \in \mathbb{R}$ and let E be a locally convex space. Then a subset H of $C(I_a, E)$ is said to be equi-asymptotically almost periodic if for $\varepsilon > 0$ and $q \in cs(E)$, there exist $\rho \geq a$ and a relatively dense set G in I_ρ such that $q(h(t+\tau) - h(t)) \leq \varepsilon$ for each $\tau \in G$ and every $t \in I_\rho$ with $t + \tau \geq \rho$; and for all $h \in H$.

Now we state the following theorem which is the extension of Frechet's theorem and serves to resolve Problem 5.3.10 completely.

Theorem 5.3.18 [290, Theorem 3.1]. Let $a \in \mathbb{R}$ and let $f \in C(I_a, E)$, where E is a locally convex space. Then the family $\mathcal{F}^+ = \{C_{T_\omega} f : \omega \geq 0\}$ of mappings is a precompact subset of $(C_b(I_a, E), u)$ if and only if f is asymptotically almost periodic.

Indeed, we shall present here the proof of a stronger result which is stated below.

Theorem 5.3.19. Let $X = \mathbb{R}$ (respectively, $X = I_a$, where $a \in \mathbb{R}$), and let E be a locally convex space. Then for a subset H of $C(X, E)$, the following are equivalent :

(a) (i) H is a precompact in $(C(X, E), k)$, and

(ii) H is equi–almost periodic (respectively, equi–asymptotically almost periodic);

(b) the family $\mathcal{F}(H) = \{C_{T_\omega} h : h \in H, \omega \in \mathbb{R}\}$ (respectively, $\mathcal{F}^+(H) = \{C_{T_\omega} h : h \in H, \omega \in \mathbb{R}^+\}$) of mappings is a precompact subset of $(C_b(X, E), u)$.

Before attending the proof of the above theorem we shall outline the proof of the following proposition which is to be used in establishing Theorem 5.3.19.

Proposition 5.3.20. Let $X = I_a$, $a \in \mathbb{R}$ (respectively, $X = \mathbb{R}$) and let E be a locally convex space. Assume that H is a precompact subset of $(C(X, E), k)$. If H is equi–asymptotically almost periodic (respectively, equi–almost periodic), then H is uniformly equicontinuous on X and $H(X)$ is precompact in E.

Proof. (outline) We shall outline the proof for the case when $X = I_a$ and H is equi-asymptotically almost periodic. For the almost periodic case the proof follows exactly on the same line with slight changes. To show that H is uniformly equicontinuous, we fix $\varepsilon > 0$ and $p \in cs(E)$. Then by Definition 5.3.15 and Definition 5.3.17 there exist $\rho \geq a$, $m > 0$, and a relatively dense set G in I_ρ

such that, for each $\tau \in G$ and every $t \in I_\rho$ with $t + \tau \geq \rho$, $p(h(t+\tau) - h(t)) < \varepsilon/3$ for all $h \in H$, while $[t, t+m] \cap G \neq \varnothing$ for any $t \in I_\rho$. Set $N = \max\{\rho, m\}$. Choose $\tau_k \in [kN, (k+1)N] \cap G$, $k = 1,2,\ldots,$. From Theorem 2.1 of [290], it follows that H is equicontinuous on I_a and hence H is uniformly equicontinuous on the closed interval $[a, 5N]$. Thus there exists $\delta \in (0, N/2)$ such that $p(h(t_1) - h(t_2)) < \varepsilon/3$ whenever $h \in H$ and $t_1, t_2 \in [a, 5N]$ with $|t_1 - t_2| < \delta$. On the other hand, suppoe that $t_1, t_2 > 4N$ with $|t_1 - t_2| < \delta$. Then, we choose $k \in \mathbb{N}$ such that $t_1, t_2 \in [kN, (k+2)N]$. Let $s_i = t_i - \tau_{k-2}, i = 1, 2$. Now, since $s_1, s_2 \in [N, 4N]$ and $|s_1 - s_2| < \delta$, it follows that $p(h(t_1) - h(t_2)) < \varepsilon$ for all $h \in H$. This implies that H is uniformly equicontinuous on all of I_a. Now, we shall show that $H(I_a)$ is precompact in E. We start from the equicontinuity of H to obtain a finite (open) cover $\{G_i\}_{i=1}^n$ of $[a, 3N]$ and $t_i \in G_i, i = 1, \ldots, n,$ such that, for every $h \in H, p(h(t) - h(t_i)) < \varepsilon/2$ for $t \in G_i, i \in \{1, \ldots, n\}$. On the other hand, if $t > 3N$, then after some computation we have that $p(h(t) - h(t_i)) < \varepsilon$, for any $h \in H$.. Thus we have proved that $H(I_a) \subset \bigcup_{i=1}^{n} \{H(t_i) + \varepsilon B_q\}$. This shows that $H(I_a)$ is precompact in E since for each $i \in \{1, \ldots, n\}, H(t_i)$ is precompact in E in view of Theorem 2.1 of [290].

Proof of Theorem 5.3.19. (a) $\Rightarrow$ (b). Assume the case $X = I_a$, where $a \in \mathbb{R}$. By Proposition 5.3.20, it follows that $H(I_a)$ is precompact in E and also we have the inclusion $\mathcal{F}^+(H) \subset (C_p(I_a, E), u)$. For each $t \in I_a$, the set $\mathcal{F}^+(H)(t)$ is precompact in E because $H(I_a)$ is precompact in E. To show that the set $\mathcal{F}^+(H)$ is precompact in $(C_b(I_a, E), u)$, it suffices to establish the precompactness of this set in $(C_p(I_a, E), u)$. For this, it is enough to settle the finite covering condition 3(ii) of Theorem 2.2 of [290]. Fix $p \in cs(E)$ and $\varepsilon > 0$. As in the proof of Proposition 5.3.20, we choose $\rho \geq a$, $m > 0$ and a relatively dense set G in I_ρ such that, for each $\tau \in G$ and each $t \in I_\rho$ with $t + \tau \geq \rho$, $p(h(t+\tau) - h(t)) < \varepsilon/2$ for all $h \in H$, while $[t, t+m] \cap G \neq \varnothing$, for any $t \in I_a$. Further, we set $N = \max\{\rho, m\}$, $\tau_0 = 0$ and also fix $\tau_k \in [kN, (k+1)N] \cap G$, $k = 1, 2, \ldots$. Since by Proposition 5.3.20, H is uniformly equicontinuous on I_a, we conclude that the set $\mathcal{F}^+(H)$ is also uniformly equicontinuous. Thus we obtain a finite cover $\{F_i\}_{i=1}^n$ of $[N, 3N]$ by (relatively open) subsets of I_ρ and $s_i \in F_i$, $i, \ldots, n$, such that, for every $h \in H$ and $\omega \in \mathbb{R}^+$, $p(h \circ T_\omega(s) - h \circ T_\omega(s_i)) < \varepsilon/2$ whenever $s \in F_i$, $i \in \{1, \ldots, n\}$. Let us fix $G_i = \bigcup_{k=0}^{\infty} \{F_i + \tau_k\}$, $i = 1, \ldots, n$. Then by taking $t \in G_i$ for $i \in \{1, \ldots, n\}$, we choose $s \in F_i$ and $k \in \mathbb{N} \cup \{0\}$ such that $t = s + \tau_k$. Thus from the above estimates, it follows that $p(h \circ T_\omega(t) - h \circ T_\omega(s_i)) < \varepsilon$, for every $h \in H$ and for all $\omega \in \mathbb{R}^+$. Also, an obvious computation shows that $I_N \subset \bigcup_{i=1}^{n} G_i$. This further implies that $\left\{X_{s_i}\left(\mathcal{F}^+(H), v, p, \varepsilon\right) \ : \ i = 1, \ldots, n\right\}$ covers I_N, where for each $i \in \{1, \ldots, n\}$,

$$X_{s_i}\left(F^+(H), v, p, \varepsilon\right) = \left\{t \in X : \sup\left\{p\left(v(t)h(t) - v(s_i)h(s_i)\right) \ : \ h \in \mathcal{F}^+(H)\right\} \le \varepsilon\right\}$$

and of course here $v(t) = 1$ for each $t \in I_a$. Since the equicontinuity of $\mathcal{F}^+(H)$ makes it possible to trivially cover $[a, N]$ by finitely many sets of the same prescribed form, we conclude that the finite covering condition 3(ii) of Theorem 2.2 of [290] is saisfied. Thus the family $\mathcal{F}^+(H) = \{C_{T_\omega} h \ : \ h \in H, \ \omega \in \mathbb{R}^+\}$ of mappings is precompact in $(C_b(I_a, E), u)$. For the case $X = \mathbb{R}$ with H as precompact and equi-almost periodic, the precompactness of the family $\mathcal{F}(H) = \{C_{T_\omega} h \ : \ h \in H, \ \omega \in \mathbb{R}\}$ in $(C_b(X, E), u)$ can be obtained by the same argument as adapted in the first case.

(b) $\Rightarrow$ (a) If (b) holds, then (a) (i) follows easily. Further, according to [290, Theorem 2.1] it follows that the family $\mathcal{F}(H)$ (respectively, $\mathcal{F}^+(H)$) is a precompact subset of $(C_p(\mathbb{R}, E), u)$ (respectively, $(C_p(I_a, E), u)$). Now, we can use Theorem 2.2 of [290] to show that H is equi-almost periodic (respectively, equi-asymptotically almost periodic). Let $p \in cs(E)$ and $\varepsilon > 0$. Then by 3(ii) of Theorem 2.2. of [290], there exists a finite cover $\{F_i\}_{i=1}^{n}$ of $\mathbb{R}$ (respectively, I_b, where $b = \max\{a, 1\}$ and $F_i \subset I_b$, $i = 1, \ldots, n$) and $t_i \in F_i$, $i = 1, \ldots, n$, such that for all $h \in H$ and $\omega \in \mathbb{R}$ (respectively, all $\omega \in \mathbb{R}^+$), $p(h \circ T_\omega(t) - h \circ T_\omega(t_i)) < \varepsilon$ whenever $t \in F_i$, $i \in \{1, \ldots, n\}$. At this moment, let us consider the case $X = I_a$. Then setting $\rho = m = \max\{t_1, t_2, \ldots, t_n\}$, we note that $\rho \ge a$ and $m > 0$. Consider the set $G = \left\{\bigcup_{i=1}^{n} (F_i - t_i)\right\} \cap I_\rho$. For any $t \in I_\rho$, $t + m \ge b$, such that $t + m \in F_i$ for some $i \in \{1, \ldots, n\}$, while $t \le (t + m) - t_i \le t + m$, and hence G is relatively dense in I_ρ. For $t \in I_\rho$ and $\tau \in G$, we choose $i \in \{1, \ldots, n\}$ and $s \in F_i$ such that $\tau = s - t_i$. Since $t - t_i \ge 0$, it follows that

$$p\left(h(t + \tau) - h(t)\right) = p\left(h \circ T_{t - t_i}(s) - h \circ T_{t - t_i}(t_i)\right) < \varepsilon,$$

for all $h \in H$. This proves that H is equi-asymptotically almost periodic. For the case

$X = \mathbb{R}$, we fix $m > 2\max\{|t_1|,\ldots,|t_n|\}$ and set $G = \bigcup_{i=1}^{n}(F_i - t_i)$. Then G is relatively dense in $\mathbb{R}$. Now, for given $t \in \mathbb{R}$ and any $\tau \in G$, it readily follows by the same argument used in the preceding case that $p(h(t+\tau) - h(t)) < \varepsilon$, for all $h \in H$. Thus we conclude that H is equi–almost periodic. This completes the proof of the theorem.

Now, we shall present the following theorem without proof which states that almost periodic and asymptotically almost periodic functions can be factored through a reflexive Banach space so as to maintain the respective periodicity properties.

Theorem 5.3.21 [290, Theorem 3.5]. Let $X = \mathbb{R}$ (respectively, $X = I_a$, where $a \in \mathbb{R}$), and let E be a Frechet space. Then the following are equivalent for a subset H of $C(X,E)$.

(a) (i) H is relatively compact in $(C(X,E),k)$, and

(ii) H is equi–almost periodic (respectively, equi–asymptotically almost periodic);

(b) there is a compact disk K in E such that

(i) the Banach space E_K is reflexive, where E_K denotes the linear span of K in E under the norm defined by the guage of K,

(ii) $H(X) \subset E_K$, and

(iii) the family $\mathcal{F}(H) = \{C_{T_\omega} h : h \in H, \ \omega \in \mathbb{R}\}$ (respectively, $\mathcal{F}^+(H) = \{C_{T_\omega} h : h \in H, \ \omega \in \mathbb{R}^+\}$) of mappings is relatively compact as a subset of $(C_p(X, E_K), u)$;

(c) there is a compact disk K in E such that

(i) E_K is a reflexive Banach space,

(ii) $H(X) \subset E_K$,

(iii) H is relatively compact as a subset of $C(X, E_K), k)$, and

(iv) H is equi–almost periodic (respectively, equi–asymptotically almost periodic) as a subset of $C(X, E_K)$.

Let E be a locally convex space. Then E under its associated weak topology $\sigma(E,E^*)$ is denoted by E_w. Let $a \in \mathbb{R}$. Then a function $f : I_a \to E$ is called weakly asymptotically almost periodic (*w.a.a.p*) if $f : I_a \to E_w$ is asymptotically almost periodic. If E is a Banach space, then it has been shown in [7, p. 45] that a weakly almost periodic (*w.a.p*) function $f : \mathbb{R} \to E$ is almost periodic if and only if the range of f is relatively compact in E. This version holds in general as is shown by Ruess and Summers [290, Thorem 3.6] and this we record here without proof.

Theorem 5.3.22. Let $X = \mathbb{R}$ (respectively, $X = I_a$, where $a \in \mathbb{R}$) and let E be a locally convex space. Then a function $f : X \to E$ is almost periodic (respectively, asymptotically almost periodic) if and only if f is weakly almost periodic (respectively, weakly asymptotically almost periodic) and $f(X)$ is precompact in E.

It is remarked in [290] that a function $f : \mathbb{R} \to E$ is weakly almost periodic if $f : \mathbb{R} \to E_w$ is almost periodic. According to Theorem 5.3.19, it is equivalent to requiring that the family $\mathcal{F} = \{C_{T_\omega} f : \omega \in \mathbb{R}\}$ of mappings be precompact in $(C_b(\mathbb{R}, E_w), u)$. So if $f : \mathbb{R} \to E$ is weakly almost periodic, then under what conditions the family $\mathcal{F}$ will be relatively compact in $(C_b(\mathbb{R}, E_w), u)$. For favourable situation, the quasi completeness of E_w is to be taken as hypothesis. In fact, E_w is quasicomplete if and only if E is semireflexive [186, p. 299]. However, it is clear that if the family $\mathcal{F}$ is relatively compact subset of $(C_b(\mathbb{R}, E_w), u)$, then $f : \mathbb{R} \to E$ is a weakly almost periodic function and the range $f(\mathbb{R})$ is weakly relatively compact in E. The converse assertion is also true and this we shall record in the following theorem.

Theorem 5.3.23. Let $X = \mathbb{R}$ (respectively, $X = I_a$, where $a \in \mathbb{R}$), and let E be a locally convex space. For a function $f : X \to E$, the following are equivalent :

(a) (i) f is weakly almost periodic (respectively, weakly asymptotically almost periodic), and

(ii) $f(X)$ is weakly relatively compact in E;

(b) the family $\mathcal{F} = \{C_{T_\omega} f : \omega \in \mathbb{R}\}$ (respectively, $\mathcal{F}^+ = \{C_{T_\omega} f : \omega \in \mathbb{R}^+\}$ of mappings is relatively compact as a subset of $(C_b(X, E_w), u)$.

Proof. (b) follows from (a) as it is an immediate consequence of Theorem 5.3.19 and the fact that for each $t \in X$, the point evaluation δ_t is a continuous map from $(C_b(X, E_w), u)$ into E_w. For the converse assertion, we shall deal with the case when

$X = \mathbb{R}$ and $f : \mathbb{R} \to E$ is weakly almost periodic because the argument is same for the other case. From Theorem 5.3.19 it is known that the family $\mathcal{F}$ is precompact in $(C_b(\mathbb{R}, E_w), u)$. Therefore, for a net $\{f \circ T_{\omega_\alpha}\}$ in $\mathcal{F}$, there is a subnet $\{f \circ T_{\omega_\lambda}\}$, say, which is a Cauchy net in $(C_b(\mathbb{R}, E_w), u)$. Let us put $F = ((E^*)', \sigma((E^*)', E^*))$, where $(E^*)'$ is the algebraic dual of E^*. Therefore $(C_b(\mathbb{R}, F), u)$ is complete since F is complete. Thus the net $\{f \circ T_{\omega_\lambda}\}$ converges in $(C_b(\mathbb{R}, F), u)$ to some $g \in (C_b(\mathbb{R}, F), u)$. From (a) (ii), it follows that $f(\mathbb{R})$ is relatively compact in E_w with respect to $\sigma(E, E^*)$. Let $K = \overline{f(\mathbb{R})}$. Then it is also $\sigma((E^*)', E^*)$ compact and this yields that $g(\mathbb{R}) \subset K$. This proves that $g \in (C_b(\mathbb{R}, E_w), u)$. With this the proof of the theorem is completed.

We know that the set of semigroups of bounded linear operators on a Banach space E is in one–to–one correspondence with the set of (linear) semidynamical systems on E. So, to know the behaviour of the (linear) semidynamical systems induced by the composition operators, it is desirable to make a study of the semigroups of composition operators and weighted composition operators. The theory of semigroups of composition operators and weighted composition semigroups on Hardy spaces and Bergman spaces has been studied by Aleman [5], Berkson [24], Berkson and Porta [28], Cowen [85], Embry and Lambert [103, 104, 105], Evard and Jafari [107], Konig [198] and Siskakis [372, 373, 374, 375, 376]. We are presenting here some of the significant results obtained by them. Before proceeding in this direction, we shall recall some basic definitions and examples.

Definition 5.3.24. Let E be a Banach space. Then

(i) a one–parameter family $\{S_t : 0 \le t < \infty\}$ of bounded linear operators from E into itself is said to be a semigroup of bounded linear operators on E if

(a) $S_0 = I$, the identity operator on E,

(b) $S_{t+s} = S_t \circ S_s$, for every $t, s \ge 0$.

The axioms (a) and (b) are called the identity axiom and the semigroup property respectively.

(ii) a semigroup of bounded linear operators $\{S_t : t \ge 0\}$ is said to be uniformly continuous if

(c) $\lim_{t\to 0^+} \| S_t - I \| = 0.$

(iii) a semigroup of bounded linear operators $\{S_t \ : \ t \geq 0\}$ on E is said to be strongly continuous semigroup of bounded linear operators if

(d) $\lim_{t\to 0^+} S_t(y) = y,$ for each $y \in E.$

If $\{S_t \ : \ t \geq 0\}$ is a strongly continuous semigroup of bounded linear operators on E, then by [407, p. 233], axiom (d) is equivalent to the condition

(e) $\lim_{t\to s} S_t(y) = S_s(y),$ for each $s \geq 0$ and $y \in E.$

A strongly continuous semigroup of bounded linear operators on E will be called a semigroup of class C_0 or simply a C_0–semigroup.

Some examples of strongly continuous semigroups of bounded linear operators are given below.

Example 5.3.25. Let $C_b^u(\mathbb{R})$ denote the Banach space of all bounded uniformly continuous complex–valued functions on $\mathbb{R}$ with the supremum norm. For $t > 0$, we define

$$N_t(\omega) = \frac{1}{\sqrt{2\pi t}} e^{-\omega^2/2t}, \omega \in \mathbb{R},$$

which is the Gaussian probability density. For any $t \geq 0$, define $S_t : C_b^u(\mathbb{R}) \to C_b^u(\mathbb{R})$ as

$$\begin{aligned} (S_t f)(x) &= \int_{-\infty}^{\infty} N_t(x-\omega) f(\omega) d\omega, && \text{for } t > 0, \\ &= f(x) && \text{for } t = 0. \end{aligned}$$

Since $\int_{-\infty}^{\infty} N_t(x-\omega) d\omega = 1,$ it follows that each S_t is continuous because $\| S_t f \| \leq \| f \| \int_{-\infty}^{\infty} N_t(x-\omega) d\omega = \| f \|$. From the definition of S_t, $S_0 = I$, the identity operator. The semigroup property $S_{t+s} = S_t \circ S_s$, for any $t, s \geq 0$, follows from the well known formula concerning the Gaussian probability distribution :

$$\frac{1}{\sqrt{2\pi(t+s)}} e^{-\omega^2/2(t+s)} = \frac{1}{\sqrt{2\pi(t)}} \cdot \frac{1}{\sqrt{2\pi s}} \int_{-\infty}^{\infty} e^{-(\omega-\rho)^2/2t} . e^{-\rho^2/2s} \, d\rho.$$

Now, we shall show that the semigroup $\{S_t \ : \ t \geq 0\}$ is strongly continuous. Note that

$$f(x) = \int_{-\infty}^{\infty} N_t(x-\omega) f(x) d\,\omega.$$

Thus $(S_t f)(x) - f(x) = \int_{-\infty}^{\infty} N_t(x-\omega)\,[f(\omega) - f(x)]d\omega,$ which is, by the change of variable $\dfrac{x-\omega}{\sqrt{t}} = z,$ equal to

$$\begin{aligned}
&\int_{-\infty}^{\infty} N_t(\sqrt{t}\ z)\big[f(x-\sqrt{t}\ z) - f(x)\big]\sqrt{t}\ dz \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi t}}\, e^{-\frac{(tz^2)}{2t}}\big[f(x-\sqrt{t}\ z) - f(x)\big]\sqrt{t}\ dz \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}}\, e^{-z^2/2}\,\big[f(x-\sqrt{t}\ z) - f(x)\big]\, dz \\
&= \int_{-\infty}^{\infty} N_1(z)\big[f(x-\sqrt{t}\ z) - f(x)\big]\, dz.
\end{aligned}$$

Spliting the integral of the right hand side, we get

$$\begin{aligned}
\big|\, (S_t f)\,(x) - f(x)\,\big| \leq &\int_{|\sqrt{t}\ z| \leq \delta} N_1(z)\,\big|\, f(x-\sqrt{t}\ z) - f(x)\,\big|\, dz \\
&+ \int_{|\sqrt{t}\ z| > \delta} N_1(z)\,\big|\, f(x-\sqrt{t}\ z) - f(x)\,\big|\, dz.
\end{aligned}$$

By the uniform continuity of f, for any $\varepsilon > 0$, there exists a number $\delta = \delta(\varepsilon) > 0$ such that $|f(x_1) - f(x_2)| \leq \varepsilon$ whenever $|x_1 - x_2| \leq \delta$. Thus, we have

$$\begin{aligned}
\big|\, (S_t f)\,(x) - f(x)\,\big| &\leq \varepsilon \int_{|\sqrt{t}\ z| \leq \delta} N_1(z)dz + 2\|\, f\,\| \int_{|\sqrt{t}\ z| > \delta} N_1(z)dz \\
&\leq \varepsilon + 2\,\|\, f\,\| \int_{|\sqrt{t}\ z| > \delta} N_1(z)dz.
\end{aligned}$$

The second term of the right hand side tends to zero at $t \to 0$ since the integral $\int_{-\infty}^{\infty} N_1(z)dz$ converges. This shows that $\lim\limits_{t\to 0^+} \sup\limits_{x\in X} |(S_t f)(x) - f(x)| = 0.$ Thus $\lim\limits_{t\to 0^+} S_t f = f$. Also $\lim\limits_{t\to s} S_t f = S_s f$, for each $s \geq 0$ and each $f \in C_b^u(\mathbb{R})$. This proves that the family $\{S_t : t \geq 0\}$ is a C_0–contraction semigroup of bounded linear operator on $C_b^u(\mathbb{R})$.

Example 5.3.26. Let $\lambda > 0$ and $\mu > 0$. For any $t \geq 0$, we define $S_t : C_b^u(\mathbb{R}) \to C_b^u(\mathbb{R})$, as

$$(S_t f)(x) = e^{-\lambda t} \sum_{k=0}^{\infty} \frac{(\lambda t)^k}{k!} f(x - k\mu), \text{ for each } f \in C_b^u(\mathbb{R})$$

and each $x \in \mathbb{R}$. From the definition, it is clear that $S_0 = I$, the identity operator. For the semigroup property and the strong continuity of $\{S_t : t \geq 0\}$ we refer to [299, p. 5]. Thus $\{S_t : t \geq 0\}$ is a contraction semigroup of bounded linear operators on $C_b^u(\mathbb{R})$.

Example 5.3.27. For $t \geq 0$, let $T_t : D \to D$ be defined as

$$T_t(z) = \frac{e^{-t} z}{\left(e^{-t} - 1\right)z + 1},$$

for each $z \in D$. Then T_t satisfies the following properties.

(i) $T_0 = I$, the identity map on D.

(ii) $T_{t+s} = T_t \circ T_s$, for $t, s \geq 0$.

(iii) $\lim_{t \to s} T_t(z) = T_s(z)$, for each $z \in D$ and each $s \geq 0$.

Thus the family $\{T_t : t \geq 0\}$ is a semigroup of analytic functions of D. Further, in view of Littlewood's subordination theorem [99, Theorem 1.7], each T_t induces a composition operator C_{T_t} on $H^p(D)$ $(1 \leq p \leq \infty)$. Further, let $h_t(z) = T_t(z) \,/\, z$, for $z \in D$. Then h_t is a bounded analytic map for each $t \geq 0$. For any $t \geq 0$, we define $S_t : H^p(D) \to H^p(D)$ as

$$S_t(f) = h_t . C_{T_t}(f), \text{ for each } f \in H^p(D).$$

Thus the family $\{S_t : t \geq 0\}$ is a C_0–semigroup of bounded linear operators on $H^p(D)$.

In the following examples we present C_0–semigroups of composition operators.

Example 5.3.28. For each $t \in \mathbb{R}^+$, we define $T_t : \mathbb{R}^+ \to \mathbb{R}^+$ as $T_t(s) = s + t$, for every $s \in \mathbb{R}^+$. Each T_t induces a composition operator C_{T_t} on $C_b^u(\mathbb{R}^+)$. Further, for each $t \in \mathbb{R}^+$, define $S_t : C_b^u(\mathbb{R}^+) \to C_b^u(\mathbb{R}^+)$ as $S_t f = C_{T_t} f$, for each $f \in C_b^u(\mathbb{R}^+)$. Then clearly the family $\{S_t : t \geq 0\}$ is a C_0-contraction semigroup of composition operators. In this example if we replace $\mathbb{R}^+$ by $\mathbb{R}$, then for each $t \geq 0$, $T_t : \mathbb{R} \to \mathbb{R}$, defined by $T_t(s) = s + t$, for every $s \in \mathbb{R}$, still induces a composition operator C_{T_t} on $C_b^u(\mathbb{R})$. Therefore the family $\{S_t : t \geq 0\}$ as taken above is a C_0–contraction semigroup of composition operators.

Example 5.3.29. Let $H^2(P^+)$ be the Hardy space of the upper half plane. For $t \geq 0$, we define $T_t : P^+ \to P^+$ as $T_t(w) = w + tw_0$, where $w_0 \in P^+$. Then each T_t induces a composition operator C_{T_t} on $H^2(P^+)$ [359, Example 1]. It is easy to see that the family $\{C_{T_t} : t \geq 0\}$ is a C_0-semigroup of composition operators on $H^2(P^+)$. In this example, if we define for each $t \geq 0$, $T_t : P^+ \to P^+$ as $T_t(w) = e^{-t}(w - i) + i$, for each $w \in P^+$, then again each T_t induces the composition operator C_{T_t} on $H^2(P^+)$, and therefore the family $\{C_{T_t} : t \geq 0\}$ is a C_0–semigroup of composition operators on $H^2(P^+)$.

Example 5.3.30. Consider the Lebesgue space $L^p(\mu)$, $1 \leq p < \infty$, where μ is the Lebesgue measure of $\mathbb{R}$. For each $t \geq 0$, we define $T_t : \mathbb{R} \to \mathbb{R}$ as $T_t(s) = s + t$, for every $s \in \mathbb{R}$. Then each translation T_t induces the composition operator C_{T_t} on $L^p(\mu)$ and hence the family $\{C_{T_t} : t \geq 0\}$ is a C_0–semigroup of composition operators on $L^p(\mu)$.

Now, we are presenting a one–to–one correspondence between (linear) semi-dynamical systems on a Banach space E and C_0–semigroups of bounded linear operators on E.

Let $\{S_t : t \geq 0\}$ be a C_0–semigroup of bounded linear operators on a Banach space E. Then we define a map $\nabla : \mathbb{R}^+ \times E \to E$ as $\nabla(t, y) = S_t(y)$, for each $t \in \mathbb{R}^+$ and $y \in E$. Clearly ∇ is continuous in each variable and hence by [66, Theorem 1], ∇ is jointly continuous. Also, $\nabla(0, y) = S_0(y) = I(y) = y$, for each $y \in E$, and

$$\begin{aligned}\nabla(t + s, y) = S_{t+s}(y) = S_t(S_s(y)) &= S_t(\nabla(s, y)) \\ &= \nabla(t, \nabla(s, y)),\end{aligned}$$

for each $s, t \in \mathbb{R}^+$ and every $y \in E$. This proves that $(\mathbb{R}^+, E, \nabla)$ is a semi–

dynamical system. To show that ∇ is linear, let $\alpha, \beta \in \mathbb{C}$, $x, y \in E$ and $t \in \mathbb{R}^+$. Then

$$\begin{aligned}\nabla(t, \alpha x + \beta y) = S_t(\alpha x + \beta y) &= \alpha S_t(x) + \beta S_t(y), \\ &= \alpha\nabla(t, x) + \beta\nabla(t, y).\end{aligned}$$

Conversely, suppose that $(\mathbb{R}^+, E, \nabla)$ is a linear semidynamical system. For any $t \in \mathbb{R}^+$, we define $S_t : E \to E$ as $S_t(y) = \nabla(t, y)$, for every $y \in E$. Then $S_0(y) = \nabla(0, y) = y = I(y)$, for each $y \in E$. The linearity of S_t is obvious. Now, for each $t, s \in \mathbb{R}^+$, we have $S_{t+s}(y) = \nabla(t+s, y) = \nabla(t, \nabla(s, y)) = \nabla(t, S_s(y)) = S_t(S_s(y))$, for each $y \in E$. Thus $S_{t+s} = S_t \circ S_s$, for each $t, s \geq 0$. Also, the strong continuity of S_t, for $t \geq 0$, follows from the continuity of the map ∇. Thus we have shown that $\{S_t : t \geq 0\}$ is a C_0–semigroup of bounded linear operators on E.

Now, our aim is to present a one–to–one correspondence between the set of one–parameter semigroups of composition operators on Hardy spaces of the unit disc and set of one–parameter semigroups of holomorphic mappings of the unit disc into itself; and also to characterise the infinitesimal generators of these semigroups. For the details of this study we refer to the recent works of Berkson and Porta [28]. In order to write these results, we need to add some more definitions and notations.

Let G be an open set in the complex plane $\mathbb{C}$. Then by a one–parameter semigroup $\{T_t : t \in \mathbb{R}^+\}$ of holomorphic mappings of G into itself we mean a homomorphism $t \to T_t$ of the additive semigroup of positive real numbers $\mathbb{R}^+$ into the semigroup (under composition) of all analytic mappings of G into G such that T_0 is the identity map of G and $T_t(z)$ is continuous in (t, z) on $\mathbb{R}^+ \times G$. Also, write $T(t, z)$ for $T_t(z)$ and denote $\partial T(t,z)/\partial t$ by $T_1(t, z)$. By $\mathcal{F}(G)$ we denote the family of all such one–parameter semigroups on G.

Theorem 5.3.31. Let G be an open set in $\mathbb{C}$, and let $\{T_t : t \in \mathbb{R}^+\}$ be a one–parameter semigroup of holomorphic mappings of G into G. Then there is a holomorphic mapping $H : G \to \mathbb{C}$ such that

$$\frac{\partial T(t, z)}{\partial t} = H(T(t, z)), \quad \text{for } t \in \mathbb{R}^+,\ z \in G. \tag{1}$$

Proof. Let K be a compact convex subset of G. For a suitable $\alpha \in (0,1)$, the compact set $\bigcup\{T_t(K) : 0 \le t \le \alpha\}$ has its convex hull, which is compact and contained in G. Thus, there exists $\eta \in (0,\alpha]$ such that

$$| T_{2t}(z) - 2T_t(z) + z | \le (1/10) | T_t(z) - z |, \ 0 \le t \le \eta, \ \ z \in K. \tag{2}$$

Indeed, the minorant in (2) is the absolute value of the integral of $\frac{d}{d\xi}[\, T_t(\xi) - \xi \,]$ along the line segment from z to $T_t(z)$, and (by virtue of Cauchy's integral formula) this integrand has modulus less than 1/10 for sufficiently small t. In view of (2), we find that $|T_t(z) - z| \le (10/19)|T_{2t}(z) - z|$, for $z \in K$, $0 \le t \le \eta$. For convenience, replace 10/19 by $2^{-2/3}$ in this last inequality. Then a straight forward argument shows that there exists a constant c (depending on K) such that

$$|T_t(z) - z| \le c\,t^{2/3}, \text{ for } z \in K, \ 0 \le t \le 1.$$

Cover the convex hull $\tilde{K}$ of $\bigcup\{T_t(K) : 0 \le t \le \alpha\}$ by a finite collection of closed discs, each of which is contained in a closed disc contained in G having the same center and a strictly larger radius. Now, by applying the last inequality to each of the larger discs (with a suitable constant c in each case), and using Cauchy's integral formula for $\frac{d}{d\xi}[\, T_t(\xi) - \xi \,]$, we get a constant $\tilde{c}$ such that for $0 \le t \le 1$, the modulus of this derivative does not exceed $\tilde{c}\, t^{2/3}$ on $\tilde{K}$. Again by the same argument as employed in establishing (2), it follows that for $0 \le t \le \alpha$, and $z \in K$,

$$|T_{2t}(z) - 2T_t(z) + z| \le \tilde{c}\, t^{2/3} |T_t(z) - z| < \tilde{c}\, c t^{4/3}.$$

Further, we have

$$\left| [T_{2t}(z) - z](2t)^{-1} - [T_t(z) - z]t^{-1} \right| \le \frac{\tilde{c}c}{2} t^{1/3}, \text{ for } z \in K, \ 0 < t \le \alpha.$$

From this, it follows that $\lim\limits_n 2^n(T(2^{-n}, z) - z) = H(z)$ exists uniformly on compact subsets of G. In particular, H is analytic on G. For $z_0 \in G$ and $t > 0$, $\{T_s(z_0) : 0 \le s \le t\}$ is a compact subset of G. Thus $2^n[T(s + 2^{-n}, z_0) - T(s, z_0)]$ tends uniformly to $H(T(s,z_0))$ for $s \in [0,t]$. From calculus, we deduce that

$$T_t(z) = z + \int_0^t H(T_s(z))\, ds, \text{ for } z \in G, \ t \in \mathbb{R}^+.$$

This completes the proof of the theorem.

Definition 5.3.32. Let G and $\{T_t : t \in \mathbb{R}^+\}$ be as in Theorem 5.3.31. Then the function $H : G \to \mathbb{C}$ satisfying (1) is uniquely determined as $T_t(0,.)$ and is called the infinitesimal generator of $\{T_t : t \in \mathbb{R}^+\}$.

The *infinitesimal* generators of semigroups of holomorphic mappings of the right half plane into itself are described as follows.

Let $P_r^+ = \{z \in \mathbb{C} : \operatorname{Re} z > 0\}$. If $H : P_r^+ \to \mathbb{C}$ is analytic and if the initial value problem

$$\frac{\partial T(t,z)}{\partial t} = H(T(t,z)),\ T(0,z) = z \tag{3}$$

has a solution on $\mathbb{R}^+ \times P_r^+$, then it follows from the analyticity of H and the convexity of P_r^+, that for every $z_0 \in P_r^+$ and every $\tau > 0$, the initial value problem $\frac{dw}{dt} = H(w)$, $w(0) = z_0$ has a unique solution on the interval $0 \le t \le \tau$. In particular, (3) has a unique solution T on $\mathbb{R}^+ \times P_r^+$, and (since the initial value problem is autonomous) $T_{t+s} = T_t \circ T_s$ on P_r^+, for $t, s \in \mathbb{R}^+$. Now, it is clear that for $z_0 \in P_r^+$ and $s > 0$, the problem $\frac{dw}{dt} = H(w)$, $w(s) = z_0$ has at most one solution on the interval $0 \le t \le s$. Thus it follows that for each $t \in \mathbb{R}^+$, T_t is one–to–one. Standard techniques (such as the method of proof of [76, Theorem 4.1] with suitable modifications) show that $T(.,.)$ is continuous on $\mathbb{R}^+ \times P_r^+$. Further, the method of proof of [150, Theorem 9], with obvious modifications, now shows that for $t \ge 0$, $T_t(\,.\,)$ is analytic on P_r^+, and its derivative with respect to the complex variable is given by

$$T_t'(z) = \exp\left[\int_0^t H'(T(w,z))dw\right], \quad \text{for} \quad z \in P_r^+.$$

The foregoing remarks are summarized in the following theorem.

Theorem 5.3.33 [28, Proposition 2.2] Let $\mathcal{G}(P_r^+)$ be the set of all infinitesimal generators of one–parameter semigroups of holomorphic mappings of P_r^+ into itself, and let $\mathcal{F}(P_r^+)$ be the set of all such one–parameter semigroups. Then $\mathcal{G}(P_r^+)$ consists of all analytic functions H on P_r^+ such that the initial value problem (3) has a global solution T on $\mathbb{R}^+ \times P_r^+$. The correspondence which assigns to each member of $\mathcal{F}(P_r^+)$ its infinitesimal generator is one–to–one, and for each $H \in \mathcal{G}(P_r^+)$, the corresponding semigroup is the unique solution on $\mathbb{R}^+ \times P_r^+$ of the initial value problem (3). If $\{T_t : t \in \mathbb{R}^+\} \in \mathcal{F}(P_r^+)$, then T_t is univalent for all $t \in \mathbb{R}^+$.

Now, we shall record some more results of the non–trivial generators (without proofs) with additional notations.

Let $\mathcal{G}_1(P_r^+)$ be the class of all analytic functions on P_r^+, not identically zero, which map P_r^+ into its closure in $\mathbb{C}$, and let $\mathcal{G}_2(P_r^+)$ be the class of all analytic functions H on P_r^+ of the form

$$H(z) = -F(z)\,(z - ib)^2, \quad \text{for} \quad z \in P_r^+, \tag{4}$$

where $F \in \mathcal{G}_1(P_r^+)$, and b is a real constant. Let $\mathcal{G}_3(P_r^+)$ be the class of all analytic functions H on P_r^+ of the form

$$H(z) = F(z)\left(\overline{\xi} + z\right)(\xi - z), \quad \text{for} \quad z \in P_r^+, \tag{5}$$

where $F \in \mathcal{G}_1(P_r^+)$, and ξ is a constant belonging to P_r^+.

Theorem 5.3.34. $\mathcal{G}(P_r^+) \setminus \{0\}$ is the disjoint union of $\mathcal{G}_1(P_r^+), \mathcal{G}_2(P_r^+)$ and $\mathcal{G}_3(P_r^+)$. If $H \in \mathcal{G}_1(P_r^+)$ (respectively, H is of the form (4)), and if $\{T_t : t \in \mathbb{R}^+\}$ is the semigroup corresponding to H, then for each $z \in P_r^+, T_t(z) \to \infty$ (respectively, $T_t(z) \to ib$) as $t \to +\infty$. If H is of the form (5) with corresponding semigroup $\{T_t : t \in \mathbb{R}^+\}$, then : (i) $T_t(z) \to \xi$ as $t \to +\infty$ for each $z \in P_r^+$ if and only if $F(P_r^+) \subset P_r^+$, and (ii) $T_t(\xi) = \xi$, for $t \in \mathbb{R}^+$.

Remark. It is clear from the limiting behaviour of semigroups (as $t \to +\infty$) that if H has a representation in either of the forms (4), (5), then such a representation is unique.

Let $\mathcal{P}$ be the class of all analytic functions F on the open unit disc D such that Re $F \geq 0$, and F is not the zero function. Let $\mathcal{A}$ be the class of all functions H on D of the form

$$H(z) = \overline{\alpha} \quad F(z)\,(z-\alpha)^2, \tag{6}$$

where $|\alpha| = 1$, $F \in \mathcal{P}$. Let $\mathcal{B}$ be the class of all functions H on D of the form

$$H(z) = \ F(z)\left(\overline{\beta} z - 1\right)(z-\beta), \tag{7}$$

where $\beta \in D$ and $F \in \mathcal{P}$.

In the preceding notation, Theorem 5.3.34 and the remark immediately following it can be rephrased as follows.

Theorem 5.3.35. $\mathcal{G}(D)\setminus\{0\}$ is the disjoint union of $\mathcal{A}$ and $\mathcal{B}$. If H has the form (6), then its corresponding semigroup $\{T_t \ : \ t \in \mathbb{R}^+\}$ satisfies $T_t(z) \to \alpha$ as $t \to +\infty$ for each $z \in D$. If H has the form (7), then for its corresponding semigroup $\{T_t \ : \ t \in \mathbb{R}^+\}$ we have : (i) $T_t(z) \to \beta$ as $t \to +\infty$ for each $z \in D$ if and only if Re $F > 0$ on D; and (ii) $T_t(\beta) \to \beta$ for all $t \in \mathbb{R}^+$. A representation in either of the forms (6), (7) is unique.

Let $T : D \to D$ be an analytic map. Then it is well known that T induces the composition operator C_T on $H^p(D)$, $0 < p \leq \infty$ [99, p. 29]. Since $T = C_T f_1$, where the function f_n is defined by $f_n(z) = z^n, n = 0,1,2,\ldots,$ we see that for fixed p, the correspondence between analytic maps of D into itself and the composition operators they induce is one–to–one.

Theorem 5.3.36. Suppose $1 \leq p < \infty$. There is a one–to–one correspondence between $\mathcal{F}(D)$ and the strongly continuous one–parameter semigroups of composition operators on $H^p(D)$. For $\{T_t \ : \ t \in \mathbb{R}^+\}$ belongs to $\mathcal{F}(D)$, the corresponding semigroup of composition operators is given by

$$C_{T_t} f = f \circ T_t, \quad \text{for} \quad t \in \mathbb{R}^+ \quad \text{and} \quad f \in H^p(D)\,. \tag{8}$$

Proof. Let $\{T_t : t \in \mathbb{R}^+\} \in \mathcal{F}(D)$. Then we shall show that the semigroup $\{C_{T_t} : t \in \mathbb{R}^+\}$ defined by (8) is strongly continuous on $H^p(D)$. For each $t \in \mathbb{R}^+$, let $\sum_{n=0}^{\infty} a_{n,t} z^n$ be the Maclaurin series expansion for T_t. By applying Theorem 5.3.33 to D because each T_t is univalent, we conclude from the area theorem [145, Lemma 1.1] that

$$\sum_{n=1}^{\infty} n \left| a_{n,t} \right|^2 \leq 1, \quad \text{for } t \in \mathbb{R}^+. \tag{9}$$

Further, it implies that the family $\{T_t : t \in \mathbb{R}^+\}$ is a totally bounded subset of $H^2(D)$. Let $\{t_n\}$ be a sequence in $\mathbb{R}^+$ such that $t_n \to s$. Then there exists a subsequence $\{t_{n_k}\}$ and an $f \in H^2(D)$ such that $\left\| T_{t_{n(k)}} - f \right\|_2 \to 0$. This implies that for each $z \in D$, $T_{t_{n(k)}}(z) \to f(z)$. Since $T_t(z)$ is a continuous function of t, we therefore have $\left\| T_{t_{n(k)}} - T_s \right\|_2 \to 0$. This proves that the map $t \to T_t$ is continuous from $\mathbb{R}^+$ into $H^2(D)$. Also, the map $t \to T_t$ is continuous from $\mathbb{R}^+$ into $H^p(D)$ as we can use the Lebesgue bounded convergence theorem whenever we go out to the boundary $|z| = 1$. Moreover, for each polynomial P, the map $t \to C_{T_t}P$ is continuous form $\mathbb{R}^+$ into $H^p(D)$. Since for each $t \in \mathbb{R}^+$, we have in view of Theorem 3.2.1 (i) that

$$\left\| C_{T_t} \right\| \leq \left[\frac{1 + |T_t(0)|}{1 - |T_t(0)|} \right]^{1/p}.$$

Thus we conclude that $\{C_{T_t} : t \in \mathbb{R}^+\}$ is uniformly bounded on compact subsets. Since the polynomials are dense in $H^p(D)$, it follows that $\{C_{T_t} : t \in \mathbb{R}^+\}$ is strongly continuous.

Conversely, suppose $\{C_{T_t} : t \in \mathbb{R}^+\}$ is a strongly continuous semigroup of composition operators on $H^p(D)$, and let $\{T_t : t \in \mathbb{R}^+\}$ be the family of holomorphic mappings satisfying (8). From the strong continuity, it follows that the map $t \to T_t$ is continuous from $\mathbb{R}^+$ into $H^p(D)$. Also, Cauchy's integral formula implies the continuity of $T(.,.)$ on $\mathbb{R}^+ \times D$. This completes the proof.

Theorem 5.3.37. Let $\{T_t : t \in \mathbb{R}^+\}$ belong to $\mathcal{F}(D)$ and let $1 \leq p < \infty$. Let H be the infinitesimal generator of $\{T_t : t \in \mathbb{R}^+\}$, $\{C_{T_t} : t \in \mathbb{R}^+\}$ be the semigroup of composition operators as in (8), and Γ, the infinitesimal generator of $\{C_{T_t} : t \in \mathbb{R}^+\}$. Then the domain $\mathfrak{D}(\Gamma)$ of Γ, consists of all $f \in H^p(D)$ such that $Hf' \in H^p(D)$, and $\Gamma(f) = Hf'$, for $f \in \mathfrak{D}(\Gamma)$.

Proof. If $T_t = f_1$, for $t \in \mathbb{R}^+$, where f_1 is given in the remark preceding Theorem 5.3.36, then the assertion is trivial. Thus, we assume that $H \in \mathcal{G}(D) \setminus \{0\}$. Also, since $C_{T_t} f_0 = f_0$, $\| C_{T_t} \| > 1$, for $t \in \mathbb{R}^+$. Further from [97, Lemma VIII 1.4 and Theorem VIII 1.11] we see that $\omega_0 = \lim_{t \to +\infty} t^{-1} \log \| C_{T_t} \|$ exists, $0 < \omega_0 < \infty$. Also, if $\operatorname{Re} \lambda > \omega_0$, then λ is in the resolvent set of Γ. The description of $\mathfrak{D}(\Gamma)$ in the statement of the present theorem certainly defines a linear manifold M in $H^p(D)$, and hence the linear transformation $L : M \to H^p(D)$ defined by $Lf = Hf'$, clearly extends Γ. Let us first suppose that H has the form (6). In particular, on $| H | > 0$ on D. Let h be a primitive on D of $(1/H)$. Then for $t \in \mathbb{R}^+$, $z \in D$,

$$h(T(t, z)) = t + h(z) . \tag{10}$$

For $r > \max\left\{\omega_0, \log\left[\left(1 + | T(1,0) |\right)\left(1 - | T(1,0) |\right)^{-1}\right]\right\}$, we show that $(L - r)$ is one–to–one. Let $f \in H^p(D)$, and let $Hf' = rf$ on D. Then there exists a complex constant k such that $f = k \exp(rh)$. From this and (10) it follows that $f(T_t) = e^{rt} f$, for $t \in \mathbb{R}^+$. Thus we have

$$e^{rt} \| f \|_1 \leq \frac{1 + | T_t(0) |}{1 - | T_t(0) |} \| f \|_1 .$$

If f is not the zero function, then by taking $t = 1$ in this last inequality, we get a contradiction to the choice of r. Since $(L - r)$ is one–to–one and extends $(\Gamma - r)$, while the range of the latter is $H^p(D)$, it follows that $L = \Gamma$. Finally, suppose that H has the form (7). If f is an analytic function on D such that f is not the zero function and λ is a complex number such that $Hf' = \lambda f$ on D, then select r such that $| \beta | < r < 1$, and f has no zeros on $| z | = r$. Now, we have

$$\begin{aligned} (2\pi i)^{-1} \int_{| z | = r} [f'(z) / f(z)] dz &= (2\pi i)^{-1} \int_{| z | = r} [\lambda / H(z)]\, dz \\ &= \lambda\left[F(\beta)\left(| \beta |^2 - 1\right)\right]^{-1}. \end{aligned} \tag{11}$$

In view of the argument principle, we conclude that the set of eigenvalues of L is countable in this case. Moreover, we can choose a real number $\lambda < \omega_0$ such that $(L - \lambda)$ is one–to–one. Thus $L = \Gamma$ as in the previous case. This completes the proof.

The following corollary is an immediate consequence of Theorem 5.3.37.

Corollary 5.3.38. If $1 \leq p < \infty$ and $\{C_{T_t} : t \in \mathbb{R}^+\}$ is a strongly continuous one–parameter semigroup of composition operators on $H^p(D)$, then $\{C_{T_t} : t \in \mathbb{R}^+\}$ has a bounded infinitesimal generator if and only if C_{T_t} is the identity operator for every $t \in \mathbb{R}^+$.

Further Siskakis [373] has discussed the strong continuity and the infinitesimal generator of the weighted composition semigroups on Hardy spaces. In this setting, the study of the weighted composition semigroups includes the study of semigroups of composition operators. So, we shall write here some of the results of [373] which generalise the previous results.

We have already observed in Theorem 5.3.36 that a one–parameter semigroup $\{T_t : t \in \mathbb{R}^+\}$ of holomorphic mappings of D into itself, induces a strongly continuous semigroup $\{C_{T_t} : t \in \mathbb{R}^+\}$ of composition operators on $H^p(D)$, $1 \leq p < \infty$. Let $\theta : D \to \mathbb{C}$ be an analytic function and for $t \in \mathbb{R}^+$, let W_{θ,T_t} be the map defined by

$$W_{\theta,T_t}(f) = \frac{\theta \circ T_t}{\theta}\, f \circ T_t, \qquad \text{for } f \in H^p(D). \tag{12}$$

Then after imposing certain conditions on θ, we shall see that $\left\{W_{\theta,T_t} : t \in \mathbb{R}^+\right\}$ is a semigroup of weighted composition operators on $H^p(D)$. Further, we shall show that under certain conditions on the weight function θ, the weighted composition semigroup $\left\{W_{\theta,T_t} : t \in \mathbb{R}^+\right\}$ is strongly continuous. In case $\theta = 1$, this study reduces to [28]. Also it should be noted here that weighted composition operators defined by (12) include the one–parameter groups of isometries of $H^p(D)$ studied in [26].

In Theorem 5.3.35, it is shown that the infinitesimal generator H has the unique representation

$$H(z) = F(z)\left(\overline{\beta}\, z - 1\right)(z - \beta), \tag{13}$$

where $|\beta| \leq 1$ and F is analytic with $\operatorname{Re} F \geq 0$ on D. The distinguished point β in (13) is called the Denjoy–Wolff (DW) point of $\{T_t : t \in \mathbb{R}^+\}$. Further, if one-parameter semigroup $\{T_t : t \in \mathbb{R}^+\}$ is with DW point β, and $\theta : D \to \mathbb{C}$ is analytic with no zeros in $D - \{\beta\}$, then for analytic function f on D, the function $W_{\theta,T_t}(f)$ defined by (12) is analytic on D except possibly when $\beta \in D$ and θ has a zero at β. In this case $\theta(T_t(z))/\theta(z)$ becomes analytic on D if we assign it the value $\left[T_t'(\beta)\right]^n$ at β, where n is the order of the zero.

Theorem 5.3.39. Suppose $0 < p < \infty$, $\{T_t : t \in \mathbb{R}^+\}$ is a semigroup of analytic functions with DW point β, and $\theta : D \to \mathbb{C}$ is analytic with no zeros in $D - \{\beta\}$. Then $\lim\limits_{t \to 0} \| W_{\theta,T_t}(f) - f \|_p = 0$ for each $f \in H^p(D)$ if either of the following two conditions is satisfied :

(a) $\limsup\limits_{t \to 0} \left\| \dfrac{\theta \circ T_t}{\theta} \right\|_\infty \leq 1$;

(b) $\theta \in H^q(D)$ for some $q > 0$ and $\limsup\limits_{t \to 0} \left\| \dfrac{\theta \circ T_t}{\theta} \right\|_\infty < \infty$.

Proof. In view of conditions (a) and (b), $\theta \circ T_t / \theta \in H^\infty(D)$ for sufficiently small t. Let $t > 0$. Then there exists a small δ and an integer n such that $t = n\,\delta$. Thus the equation

$$\frac{\theta \circ T_t}{\theta} = \prod_{k=1}^{n} \cdot \frac{\theta \circ T_{k\delta}}{\theta \circ T_{(k-1)\delta}}$$

shows that $\dfrac{\theta \circ T_t}{\theta} \in H^\infty(D)$. Hence for each $t \in \mathbb{R}^+$, W_{θ,T_t} is a bounded linear operator on $H^p(D)$. Also, for $f \in H^p(D)$, we have

$$\| W_{\theta,T_t}(f) \|_p^p \leq \left\| \frac{\theta \circ T_t}{\theta} \right\|_\infty^p \| f \circ T_t \|_p^p \leq \left\| \frac{\theta \circ T_t}{\theta} \right\|_\infty^p \left[\frac{1 + |T_t(0)|}{1 - |T_t(0)|}\right] \| f \|_p^p \,. \tag{14}$$

Thus the set $\left\{ \| W_{\theta,T_t} \| \; : \; t \in [0,1] \right\}$ is bounded. Suppose condition (a) holds and consider the case $1 < p < \infty$. Fix $p > 1, f \in H^p(D)$ and a sequence $t_n \to 0$. Since $H^p(D)$ is a reflexive Banach space, the inequality (14) and the fact that $W_{\theta,T_{t_n}}(f)(z) \to f(z)$ for each $z \in D$ show that $W_{\theta,T_{t_n}}(f)$ converges weakly to f in $H^p(D)$. From this and [97, Lemma II.3.27], it follows that

$$\| f \|_p \leq \lim_{n\to\infty} \inf \left\| W_{\theta,T_{t_n}}(f) \right\|_p .$$

Also,

$$\lim_{n\to\infty} \sup \left\| W_{\theta,T_{t_n}}(f) \right\|_p \leq \lim_{n\to\infty} \sup \left\| \frac{\theta \circ T_{t_n}}{\theta} \right\|_\infty \lim_{n\to\infty} \left[\frac{1 + |T_{t_n}(0)|}{1 - |T_{t_n}(0)|} \right]^{1/p} \| f \|_p$$
$$= \| f \|_p .$$

This implies that $\left\| W_{\theta,T_{t_n}}(f) \right\|_p \to \| f \|_p$ as $n \to \infty$. This together with the weak convergence of $W_{\theta,T_{t_n}}(f)$ to f and the uniform convexity of $H^p(D)$ for $1 < p < \infty$ implies that $\lim_{n\to\infty} \left\| W_{\theta,T_{t_n}}(f) - f \right\|_p = 0$. Now, consider the case $0 < p \leq 1$. In this case the metric on $H^p(D)$ is $\| f - g \|_p^p$. Fix q, $1 < q < \infty$, and $g \in H^q(D)$, and consider the triangle inequality.

$$\left\| W_{\theta,T_t}(f) - f \right\|_p^p \leq \left\| W_{\theta,T_t}(f) - W_{\theta,T_t}(g) \right\|_p^p + \left\| W_{\theta,T_t}(g) - g \right\|_p^p + \| g - f \|_p^p .$$

We have

$$\left\| W_{\theta,T_t}(f) - W_{\theta,T_t}(g) \right\|_p^p \leq \left\| W_{\theta,T_t} \right\|^p \| f - g \|_p^p$$

and

$$\left\| W_{\theta,T_t}(g) - g \right\|_p^p \leq \left\| W_{\theta,T_t}(g) - g \right\|_q^p .$$

Thus for $t \in [0, 1]$ we have

$$\left\| W_{\theta,T_t}(f) - f \right\|_p^p \leq (M + 1) \| f - g \|_p^p + \left\| W_{\theta,T_t}(g) - g \right\|_q^p ,$$

where M is a common bound of $\| W_{\theta,T_t} \|^p$ for $t \in [0, 1]$. From this inequality and the first part of the proof, together with the fact that $H^q(D)$ is dense in $H^p(D)$, we get the desired conclusion.

Now suppose that condition (b) holds. Since $\{\| W_{\theta,T_t} \| : 0 \le t \le 1\}$ is bounded, and the polynomials are dense in $H^p(D)$, it is enough to show that for each polynomial P, $\lim_{t\to 0} \| W_{\theta,T_t}(P) - P \|_p = 0$. Further, this is equivalent to showing that $\lim_{t\to 0} \| W_{\theta,T_t}(f_n) - f_n \|_p = 0$, for each $n \ge 0$, where $f_n(z) = z^n$. We shall consider the case $n = 1$ (the proof for other values of n is similar). On the contrary we suppose that there is a sequence $\{t_k\}$ tending to zero such that

$$\left\| \frac{\theta \circ T_{t_k}}{\theta} T_{t_k} - f_1 \right\|_p \ge \alpha > 0, \quad \text{for} \quad k = 1, 2, 3 \dots .$$

Choosing $\theta = 1$ in the proof under condition (a), we have $\lim_{t\to 0} \| f \circ T_t - f \|_p = 0$, for each $f \in H^p(D)$. In particular,

$$\| T_{t_k} - f_1 \|_p \to 0 \quad \text{as} \quad k \to \infty .$$

Thus there is a subsequence $\{t_{k_n}\}$ such that

$$T_{t_{k_n}}\left(e^{i\theta}\right) \to e^{i\theta} \quad a.e \quad \text{on} \quad \partial D ,$$

where $T_t(e^{i\theta})$ denotes the boundary function of T_t. Now $\theta \in H^q(D)$ with $q > 0$, therefore

$$\lim_{n\to\infty} \| \theta \circ T_{t_{k_n}} - \theta \|_q = 0 .$$

Hence there is a further subsequence, which we may label as $\{t_m\}$, such that

$$\left(\theta \circ T_{t_m}\right)\left(e^{i\theta}\right) \to \theta\left(e^{i\theta}\right) \quad a.e. \quad \text{on} \quad \partial D.$$

It follows that the boundary function $\beta_{t_m}\left(e^{i\theta}\right)$ of $\theta(T_t(z)) / \theta(z)$ satisfies

$\beta_{t_m}\left(e^{i\theta}\right) \to 1$ *a.e.* on ∂D. Further, it implies that

$$\beta_{t_m}\left(e^{i\theta}\right)T_{t_m}\left(e^{i\theta}\right) - e^{i\theta} \to 0 \qquad a.e. \text{ on } \quad \partial D.$$

Since $\left| \beta_{t_m}\left(e^{i\theta}\right)T_{t_m}\left(e^{i\theta}\right) - e^{i\theta} \right| \leq \| \theta \circ T_t / \theta \| + 1$, by applying the bounded convergence theorem we have

$$\int_0^{2\pi} \left| \beta_{t_m}\left(e^{i\theta}\right) T_{t_m}\left(e^{i\theta}\right) - e^{i\theta} \right|^p d\theta \to 0 \text{ as } m \to \infty.$$

From this, it follows that

$$\left\| \frac{\theta \circ T_{t_m}}{\theta} T_{t_m} - f_1 \right\|_p \to 0 \text{ as } m \to \infty,$$

which is a contradiction to the original choice of $\{t_k\}$. This completes the proof.

Now, we shall record the following theorem without proof which gives the infinitesimal generator of $\{W_{\theta,T_t} : t \in \mathbb{R}^+\}$.

Theorem 5.3.40 [373, Theorem 2]. Suppose $1 \leq p < \infty$ and $\{T_t : t \in \mathbb{R}^+\}$ is a semigroup of analytic functions with infinitestimal generator H and DW point β. Also, let $\theta : D \to \mathbb{C}$ be analytic with no zeros in $D - \{\beta\}$, and suppose that condition (a) or (b) of Theorem 5. 3. 39 is satisfied. Then the infinitesimal generator Δ_p of $\{W_{\theta,T_t} : t \in \mathbb{R}^+\}$ on $H^p(D)$ has domain

$$\mathfrak{D}\left(\Delta_p\right) = \left\{ f \in H^p(D) \ : \ Hf' + \theta' \frac{H}{\theta} f \in H^p(D) \right\},$$

and

$$\Delta_p(f) = Hf' + \theta' \frac{H}{\theta} f, \text{ for each } f \in \mathfrak{D}\left(\Delta_p\right).$$

In [375] Siskakis has continued the above study and presented the results in the setting of Bergman spaces. In particular, he has proved that semigroups of composition operators on weighted Bergman spaces are continuous in the strong operator topology but not in the uniform operator topology, and also identified the infinitesimal generators of such semigroups. For details of these results we refer to [375].

Evard and Jafari [107] has investigated the properties of semigroups of weighted composition operators on Hardy spaces which are related to the properties of the weight functions, called multipliers. They gave a necessary and sufficient condition for a one-parameter semigroup of holomorphic mappings of a region G in the complex plane into itself which generate a multiplicative semigroup of operators when it is applied to a one-parameter semigroup of composition operators. Under an additional hypothesis on the family of multipliers they prove that the semigroups of weighted composition operators generated from the semigroups of composition operators by these multipliers is strongly continuous on Hardy spaces. Now, we shall need additional notations to present the proofs of some of the results of [107].

Let G be a region in $\mathbb{C}$, let $A(G)$ be a Banach space of analytic functions on G. Let $\{\theta_t : t \in \mathbb{R}^+\}$ be a family of analytic functions in $A(G)$, and let M_{θ_t} be the operator of pointwise multiplication by θ_t on the Banach space $A(G)$. Consider a one-parameter semigroup $\{T_t : t \in \mathbb{R}^+\}$ of holomorphic mappings of the region G into itself and let $\{C_{T_t} : t \in \mathbb{R}^+\}$ be the corresponding semigroup of composition operators on $H^p(G)$, $1 \leq p < \infty$ as given in Theorem 5.3.36. Now, we set $W_{\theta_t,T_t} = M_{\theta_t}C_{T_t}$ and $\tilde{W}_{\theta_t,T_t} = C_{T_t}M_{\theta_t}$. Then in the following theorems we shall characterise multipliers that generate semigroups.

Theorem 5.3.41. The family $\left\{W_{\theta_t,T_t} : t \in \mathbb{R}^+\right\}$ is a semigroup of $A(G)$ if and only if the family $\left\{\theta_t : t \in \mathbb{R}^+\right\}$ satisfy the condition

$$\theta_{s+t} = \theta_s.\theta_t \circ T_s. \tag{15}$$

Proof. In case $\left\{W_{\theta_t,T_t} : t \in \mathbb{R}^+\right\} = \{0\}$, then $\theta_t = 0$, for every $t \in \mathbb{R}^+$. Thus condition (15) is vacuously satisfied. Without loss of generality we assume that $\left\{W_{\theta_t,T_t} : t \in \mathbb{R}^+\right\}$ is a non-zero semigroup. Fix $f \in A(G)$. By definition of W_{θ_t,T_t}, we have

$$\begin{aligned} W_{\theta_s,T_s}W_{\theta_t,T_t}f(z) &= W_{\theta_s,T_s}M_{\theta_t}C_{T_t}f(z) \\ &= W_{\theta_s,T_s}\big(\theta_t(z)f(T_t(z))\big) \\ &= \theta_s(z)\theta_t\big(T_s(z)\big)f\big(T_t\big(T_s(z)\big)\big) \,. \end{aligned}$$

Since the family $\left\{T_t \;:\; t \in \mathbb{R}^+\right\}$ is a semigroup,

$$W_{\theta_s,T_s}W_{\theta_t,T_t}f(z) = \theta_s(z)\theta_t\big(T_s(z)\big)f\big(T_{t+s}(z)\big).$$

Since

$$W_{\theta_{t+s},T_{t+s}}f(z) = \theta_{t+s}(z)f\big(T_{t+s}(z)\big) \,,$$

it follows that the family $\left\{W_{\theta_t,T_t} \;:\; t \in \mathbb{R}^+\right\}$ forms a semigroup if $\theta_s(z)\theta_t\big(T_s(z)\big) = \theta_{s+t}(z)$. This proves the sufficient part. On the other hand suppose that $\left\{W_{\theta_t,T_t} \;:\; t \in \mathbb{R}^+\right\}$ is a semigroup. Then

$$W_{\theta_{t+s},T_{t+s}}f(z) = W_{\theta_t,T_t}W_{\theta_s,T_s}f(z) \quad \text{for} \quad \text{every} \quad f \in A(G).$$

By the definition of W_{θ_t,T_t}, we have

$$\theta_{s+t}(z)f\big(T_{s+t}(z)\big) = \theta_s(z)\theta_t\big(T_s(z)\big)f\big(T_{t+s}(z)\big) \quad \text{for every} \quad f \in A(G).$$

Since $\left\{T_t \;:\; t \in \mathbb{R}^+\right\}$ is a semigroup and $T_0 = I$, the identity mapping, T_t are non-constant for every $t \in \mathbb{R}^+$. By the open mapping theorem, since $T_t(G)$ is open for every $t \in \mathbb{R}^+$, $f \circ T_{s+t}$ is constant for nonconstant $f \in A(G)$. Since the zeros of a non-zero holomorphic function are isolated, we have

$$\theta_{s+t}(z) = \theta_s(z)\theta_t\big(T_s(z)\big), \quad \textit{a.e.}$$

Since θ_t are pointwise multipliers of a holomorphic function space, θ_t are holomorphic for every $t \in \mathbb{R}^+$. Thus condition (15) holds.

Theorem 5.3.42. The family $\left\{\tilde{W}_{\theta_t,T_t} \;:\; t \in \mathbb{R}^+\right\}$ is a semigroup if and only if the family of multipliers $\left\{\theta_t \;:\; t \in \mathbb{R}^+\right\}$ satisfy the condition

$$\theta_{s+t}\big(T_{s+t}(z)\big) = \theta_s\big(T_s(z)\big)\theta_t\big(T_{t+s}(z)\big) \,. \tag{16}$$

Proof. Without loss of generality suppose that the family $\left\{\tilde{W}_{\theta_t, T_t} \ : \ t \in \mathbb{R}^+\right\}$ is a non-trivial semigroup. Fix $t, s \in \mathbb{R}^+$ and $f \in A(G)$. Then

$$\tilde{W}_{\theta_t, T_t}(f)(z) = C_{T_t} M_{\theta_t} f(z) = C_{T_t} \theta_t f(z) = \theta_t(T_t(z)) f(T_t(z)) .$$

Also

$$\tilde{W}_{\theta_s, T_s} \ \tilde{W}_{\theta_t, T_t} \ f(z) = \theta_s(T_s(z))\theta_t(T_{s+t}(z)) f(T_{s+t}(z)) .$$

On the other hand,

$$\tilde{W}_{\theta_{s+t}, T_{s+t}} \ f(z) = C_{T_{t+s}} \ M_{\theta_{t+s}} \ f(z) .$$

Thus if condition (16) holds, then the family $\left\{\tilde{W}_{\theta_t, T_t} \ : \ t \in \mathbb{R}^+\right\}$ of operators forms a semigroup. The converse can be established by the same argument as we gave in Theorem 5.3.41. This completes the proof.

Note. Many examples of multipliers satisfying the necessary and sufficient conditions of the above theorems are given in [107] to illustrate the theory. These examples give the general form of the multipliers $\left\{\theta_t \ : \ t \in \mathbb{R}^+\right\}$ that generate the semigroups $\left\{W_{\theta_t, T_t} \ : \ t \in \mathbb{R}^+\right\}$ and $\left\{\tilde{W}_{\theta_t, T_t} \ : \ t \in \mathbb{R}^+\right\}$. Further, an interesting relationship between the one-parameter groups of operators $\left\{W_{\theta_t, T_t} \ : \ t \in \mathbb{R}^+\right\}$ and $\left\{\tilde{W}_{\theta_t, T_t} \ : \ t \in \mathbb{R}^+\right\}$ has been obtained.

Let G be a simply connected region in $\mathbb{C}$, and let μ denote the Lebesgue measure on the boundary of G. Let θ^* denote the non-tangential limits of θ at the boundary of G. In [356, 360] Singh and Sharma demonstrate that if G is the upper half plane in $\mathbb{C}$, then for C_T to be a composition operator on $H^2(G)$, the inducing self map T must have a pole at infinity. Of course, the Hardy spaces of this domain for $p < \infty$ do not contain the constant functions. In addition to the other results here we shall record a sufficient condition for a simply connected region G whose Hardy spaces contain the constant functions to be mapped into themselves under composition with all holomorphic self maps of the region G into itself.

Theorem 5.3.43 [107]. Let $1 \leq p \leq \infty$ and suppose G is a simply connected proper subset of the complex plane $\mathbb{C}$ such that $H^p(G)$ contains the constant functions. Then

(i) M_θ is a multiplication operator on $H^p(G)$ if and only if θ^* is essentially bounded *i.e.*, $H^p(G)$ remains invariant under multiplication by θ if and only if $\theta \in H^\infty(G)$.

(ii) Let T be a biholomorphic map of a region G onto the unit disk D. A necessary and sufficient condition for C_T to be a composition operator from $H^p(D)$ into $H^p(G)$ is that $(T^{-1})' \in H^\infty(D)$.

(iii) A sufficient condition for $H^p(G)$ to remain invariant under all holomorphic maps of the region G into itself is that there exists a biholomorphic map T mapping the region G onto the unit disk D such that $T' \in H^\infty(G)$ and $(T^{-1})' \in H^\infty(D)$.

(iv) For $t \geq 0$, M_{θ_t} is a multiplication operator on $H^p(G)$ if and only if the inducing function $\theta_t \in H^\infty(G)$.

(v) Let $\theta_t \in H^\infty(G)$ for every $t \in \mathbb{R}^+$. Then the one-parameter semigroup $\left\{W_{\theta_t,T_t} : t \in \mathbb{R}^+\right\}$ of weighted composition operators is uniformly bounded on $H^p(G)$, for each compact subset of $\mathbb{R}^+$.

In the following theorem we show that under an additional necessary boundedness hypothesis on each of the multipliers θ_t, the semigroup $\left\{W_{\theta_t,T_t} : t \in \mathbb{R}^+\right\}$ of weighted composition operators is strongly continuous on $H^p(G)$ for $1 \leq p < \infty$.

Theorem 5.3.44. If $\left\{\theta_t : t \in \mathbb{R}^+\right\}$ is a one-parameter family of multipliers of $H^p(G)$ generating a semigroup $\left\{W_{\theta_t,T_t} : t \in \mathbb{R}^+\right\}$ and $\theta_t \in H^\infty(G)$, then the family of operators $\left\{W_{\theta_t,T_t} : t \in \mathbb{R}^+\right\}$ is a strongly continuous semigroup of operators on $H^p(G)$ for $1 \leq p < \infty$.

Proof. Fix $1 \leq p < \infty$. To show that $\left\{W_{\theta_t,T_t} : t \in \mathbb{R}^+\right\}$ is strongly continuous, we need to show that if $\{t_n\}$ is a sequence in $\mathbb{R}^+$ tending to t, then

$\left\| W_{\theta_{t_n}, T_{t_n}} f - W_{\theta_t, T_t} f \right\| \to 0$, for every $f \in H^p(G)$. In view of Theorem 5.3.43 (iv), since the biholomorphic maps from G onto D having bounded derivatives map $H^p(D)$ onto $H^p(G)$ boundedly, the strong continuity of the one-parameter semigroup of composition operators on the unit disk (see Theorem 5.3.36) implies the strong continuity of the one-parameter semigroup of composition operators on $H^p(G)$ for $1 \le p < \infty$. In particular, if $\{t_n\}$ is a sequence in $\mathbb{R}^+$ such that $t_n \to 0$, since $C_{T_0} f = f$, for every $f \in H^p(G)$,

$$\lim_{n \to \infty} \left\| C_{T_{t_n}} f - f \right\|_p = 0.$$

In view of triangle inequality, we have

$$\begin{aligned} \left\| W_{\theta_{t_n}, T_{t_n}} f - f \right\|_p &= \left\| M_{\theta_{t_n}} C_{T_{t_n}} f - f \right\|_p \\ &= \left\| M_{\theta_{t_n}} \left(C_{T_{t_n}} f - f \right) + M_{\theta_{t_n}} f - f \right\|_p \\ &\le \left\| M_{\theta_{t_n}} \right\|_p \left\| C_{T_{t_n}} f - f \right\|_p + \left\| M_{\theta_{t_n}} f - f \right\|_p . \end{aligned}$$

Since for each $t \in \mathbb{R}^+$, M_{θ_t} is a multiplication operator on $H^p(G)$, by the Banach–Steinhaus theorem and an argument similar to that of Theorem 5.3.43 (vi), it follows that the family $\left\{ M_{\theta_t} : t \in \mathbb{R}^+ \right\}$ is uniformly bounded on compact subsets of $\mathbb{R}^+$. Therefore, it is enough to establish that $\left\| M_{\theta_{t_n}} f - f \right\|_p \to 0$ as $t_n \to 0$. Also, for every $f \in H^p(G)$,

$$\lim_{n \to \infty} M_{\theta_{t_n}} f(z) = f(z), \quad \text{for every} \quad z \in G, \tag{17}$$

since θ_{t_n} tends to the constant one function pointwise as $t_n \to 0$. Since $\left\{ M_{\theta_{t_n}} : n \in \mathbb{N} \right\}$ is uniformly bounded, Theorem 5.3.43 (v) implies that the multipliers θ_{t_n} are uniformly bounded. Then by the dominated convergence theorem we conclude that

$$\lim_{n \to \infty} \left\| M_{\theta_{t_n}} f \right\|_p \to \| f \|_p . \tag{18}$$

From statements (17) and (18) we shall deduce that

$$\lim_{n\to\infty} \left\| M_{\theta_{t_n}} f - f \right\|_p = 0 . \tag{19}$$

To show (19), let $f \in H^p(G)$, $\| f \|_p = 1$ and $\varepsilon > 0$ be fixed. The boundary ∂G of the region G has finite measure since the constant functions belong to $H^p(G)$. Thus in view of Egoroff 's theorem there exists a set $K \subset \partial G$ such that $\int_K | f |^p < \varepsilon$, and on the complement of K, $M_{\theta_{t_n}} f$ converge uniformly to f. Further, by Fatou's theorem, since

$$\liminf_n \int_{K^c} \left| M_{\theta_{t_n}} f \right|^p dm \geq \int_{K^c} | f |^p \, dm,$$

we have $\int_K \left| M_{\theta_{t_n}} f \right|^p dm < \varepsilon$ for large n. Hence

$$\begin{aligned} \int_{\partial G} \left| M_{\theta_{t_n}} f - f \right|^p dm &= \int_K \left| M_{\theta_{t_n}} f - f \right|^p dm + \int_{K^c} \left| M_{\theta_{t_n}} f - f \right|^p dm \\ &\leq \alpha \int_K \left| M_{\theta_{t_n}} f \right|^p dm + \int_K | f |^p dm \leq (\alpha + 1)\, \varepsilon, \end{aligned}$$

for some constant α. Thus $\left\| M_{\theta_{t_n}} f - f \right\|_p \to 0$ as $t_n \to 0$. This proves that $\left\| W_{\theta_{t_n}, T_{t_n}} f - f \right\|_p \to 0$ as $t_n \to 0$. Let $H(t) = W_{\theta_t, T_t} f$. Now, since $\left\{ W_{\theta_t, T_t} : t \in \mathbb{R}^+ \right\}$ is a semigroup, if $t_n \to t$ and $\varepsilon_n = t_n - t$, then

$$\lim_{t_n \to t} H(t_n) = \lim_{\varepsilon_n \to 0} W_{\theta_{\varepsilon_n}, T_{\varepsilon_n}} W_{\theta_t, T_t} f = W_{\theta_t, T_t} f = H(t) .$$

This shows that the function $H(t)$ is strongly right continuous. It can be shown that $H(t)$ is strongly continuous for all $t \in \mathbb{R}^+$. For details we refer to [107]. This completes the proof of the theorem.

Finally, we wind up this section by giving some details of weighted translation semigroups which are continuous analogues of the weighted unilateral shifts where as it is well known that the semigroup of translations on $L^2(\mathbb{R}^+)$ is in many ways a continuous analogue of the unilateral shift. The semigroups under investigation in some sense are special cases of the semigroups of weighted composition operators. The weighted translation semigroups and some of their properties have been dealt with in detail by Embry and Lambert [103, 104, 105]. Here we shall present some of the basic structure of the semigroups under investigation.

Let $L^2(\mathbb{R}^+)$ be the Hilbert space of square integrable measurable functions from $\mathbb{R}^+$ into $\mathbb{C}$. For $t \in \mathbb{R}^+$, let $T_t : \mathbb{R}^+ \to \mathbb{R}^+$ be defined as $T_t(x) = x - t$ for $x \geq t$ and zero otherwise, and let C_{T_t} be the corresponding composition operator on $L^2(\mathbb{R}^+)$. Clearly the family $\left\{C_{T_t} : t \in \mathbb{R}^+\right\}$ is a semigroup of composition operators on $L^2(\mathbb{R}^+)$. Let $\theta : \mathbb{R}^+ \to \mathbb{C}$ be a measurable, almost everywhere non-zero function such that for each fixed t, the function θ_t, defined by

$$\theta_t(x) = \begin{cases} \dfrac{\theta(x)}{\theta(x-t)} & , \quad x \geq t \\ 0 & , \quad x < t \end{cases}$$

is essentially bounded. This defines the multiplication operator M_{θ_t} on $L^2(\mathbb{R}^+)$. Now, for $t \in \mathbb{R}^+$, we define $W_{\theta_t, T_t} f(x) = \theta_t(x)\,(C_{T_t} f(x))$. It is easy to see that the family $\left\{W_{\theta_t, T_t} : t \in \mathbb{R}^+\right\}$ of weighted composition operators is a semigroup in $B(L^2(\mathbb{R}^+))$. From now onwards we shall assume that θ is continuous on $\mathbb{R}^+$ and all expressions involving a symbol such as $g(x-t)$ are zero fox $x < t$.

Theorem 5.3.45. The semigroup $\left\{W_{\theta_t, T_t} : t \in \mathbb{R}^+\right\}$ is strongly continuous if and only if there exist numbers M and m in $\mathbb{R}$ such that for each $t \in \mathbb{R}^+$,

$$\operatorname*{ess\,sup}_{x \in \mathbb{R}^+} \left| \frac{\theta(x+t)}{\theta(x)} \right| \leq M e^{mt}. \tag{20}$$

Proof. From [97, page 619], it follows that if $\left\{W_{\theta_t, T_t} : t \in \mathbb{R}^+\right\}$ is strongly continuous, then there exists M and m such that (20) holds. Conversely, we first suppose that (20) holds with $m = 0$. That is, the family $\left\{\theta_t : t \in \mathbb{R}^+\right\}$ is uniformly bounded in $L^\infty(\mathbb{R}^+)$. Simple computations show that $\left\| W_{\theta_t, T_t} \right\| = \| \theta \|_\infty$, therefore the family $\left\{W_{\theta_t, T_t} : t \in \mathbb{R}^+\right\}$ is uniformly bounded. So, in order to show that the semigroup is strongly continuous it suffices to show that $\lim_{t \to 0} W_{\theta_t, T_t} f = f$, for all f

in a dense subset of $L^2(\mathbb{R}^+)$. Let f be a continuous function with compact support. Then

$$\begin{aligned}\| W_{\theta_t,T_t} f - f \|^2 &= \int_0^\infty | \theta_t(x) f(T_t(x)) - f(x) |^2 \, dx \\ &= \int_0^\infty \left| \frac{\theta(x)}{\theta(x-t)} f(x-t) - f(x) \right|^2 dx.\end{aligned}$$

For any t and x, we have

$$\left| \frac{\theta(x)}{\theta(x-t)} f(x-t) - f(x) \right|^2 \leq (M+1)^2 \| f \|_\infty^2.$$

Also, for each fixed x,

$$\lim_{t \to 0} \left| \frac{\theta(x)}{\theta(x-t)} f(x-t) - f(x) \right| = 0.$$

Thus by the bounded convergence theorem,

$$\lim_{t \to 0} \| W_{\theta_t, T_t} f - f \| = 0.$$

Now, we suppose that (20) holds for M and m in $\mathbb{R}$. If we define ψ on $\mathbb{R}^+$ as $\psi(x) = e^{-mx}\theta(x)$, then for all t and $x \geq t$, we have

$$\begin{aligned}\left| \frac{\psi(x)}{\psi(x-t)} \right| &= \left| \frac{e^{-mx}\theta(x)}{e^{-m(x-t)}\theta(x-t)} \right| \\ &= e^{-mt} \left| \frac{\theta(x)}{\theta(x-t)} \right| \leq M.\end{aligned}$$

Further, if we set $S_t = e^{-mt} W_{\theta_t,T_t}$, then

$$\begin{aligned}(S_t f)(x) &= e^{-mt}\theta_t(x) f(T_t(x)) \\ &= e^{-mt} \frac{\theta(x)}{\theta(x-t)} f(x-t) \\ &= \frac{\psi(x)}{\psi(x-t)} f(x-t).\end{aligned}$$

Hence the semigroups $\left\{S_t \;:\; t \in \mathbb{R}^+\right\}$ and $\left\{W_{\theta_t,T_t} = e^{mt} S_t \;:\; t \in \mathbb{R}^+\right\}$ are strongly continuous. This completes the proof of the theorem.

Note. The semigroups $\left\{W_{\theta_t,T_t} \;:\; t \in \mathbb{R}^+\right\}$ defined above can be seen as a continuous analogue of unilateral weighted shifts, and for general information about weighted shifts the reader is referred to [137, 173]. The semigroups appearing in Theorem 5.3.45 are referred as weighted translation semigroups with symbol θ.

Theorem 5.3.46. Let $\left\{W_{\theta_t,T_t} \;:\; t \in \mathbb{R}^+\right\}$ and $\left\{W_{\psi_t,T_t} \;:\; t \in \mathbb{R}^+\right\}$ be weighted translation semigroups with symbols θ and ψ respectively. Then $\left\{W_{\theta_t,T_t} \;:\; t \in \mathbb{R}^+\right\}$ is unitary equivalent to $\left\{W_{\psi_t,T_t} \;:\; t \in \mathbb{R}^+\right\}$ if and only if the function $|\theta/\psi|$ is constant on $\mathbb{R}^+$.

Proof. First suppose that $|\theta/\psi| = c$ for some constant c. Now, if we define an operator A on $L^2(\mathbb{R}^+)$ by $(Af)(x) = c(\psi(x)/\theta(x))f(x)$, then clearly A is a unitary operator. Moreover

$$\begin{aligned}\left(AW_{\theta_t,T_t}\, f\right)(x) &= c.\frac{\psi(x)}{\theta(x)}\,\frac{\theta(x)}{\theta(x-t)}\, f(x-t)\\ &= c.\frac{\psi(x)}{\theta(x-t)}\, f(x-t),\end{aligned}$$

and

$$\begin{aligned}\left(W_{\psi_t,T_t}\, Af\right)(x) &= \frac{\psi(x)}{\psi(x-t)}\,.\,c\,\frac{\psi(x-t)}{\theta(x-t)}\, f(x-t)\\ &= c\,\frac{\psi(x)}{\theta(x-t)}\, f(x-t).\end{aligned}$$

Thus for every $t \in \mathbb{R}^+$,

$$A\, W_{\theta_t,\,T_t} = W_{\psi_t,T_t}\, A\,.$$

For the converse part we refer to [104].

Let $\left\{W_{\theta_t,T_t} \;:\; t \in \mathbb{R}^+\right\}$ be a semigroup with continuous symbol θ. Then we say that this semigroup is hyponormal, subnormal, quasinormal or normal if each W_{θ_t,T_t} has the given property. Each of these properties is preserved under unitary equivalence and hence by Theorem 5.3.46 we may assume θ to be positive. Let M_t be the closed linear span of the functions $\theta\chi_n^{(t)}$, where $\chi_n^{(t)}$ is the characteristic function of the interval $[nt, (n+1)t]$. In the following theorem, we shall record the results without proofs which are due to Embry and Lambert [104].

Theorem 5.3.47. Let $\left\{W_{\theta_t,T_t} \;:\; t \in \mathbb{R}^+\right\}$ be a semigroup with continuous symbol θ. Then

(i) $\left\{W_{\theta_t,T_t} \;:\; t \in \mathbb{R}^+\right\}$ is subnormal if and only if $\left\{W_{\theta_t,T_t}\big|_{M_t} \;:\; t \in \mathbb{R}^+\right\}$ is subnormal.

(ii) $\left\{W_{\theta_t,T_t} \;:\; t \in \mathbb{R}^+\right\}$ is hyponormal if and only if $\left\{W_{\theta_t,T_t}\big|_{M_t} \;:\; t \in \mathbb{R}^+\right\}$ is hyponormal.

(iii) $\left\{W_{\theta_t,T_t} \;:\; t \in \mathbb{R}^+\right\}$ is hyponormal if and only if $\log\theta$ is convex.

(iv) Assume that θ is a C^1–function on $(0, \infty)$ and $\theta'/\theta \in L^\infty(\mathbb{R}^+)$. Then the infinitesimal generators H of $\left\{W_{\theta_t,T_t} \;:\; t \in \mathbb{R}^+\right\}$ and H^* of $\left\{W^*_{\theta_t,T_t} \;:\; t \in \mathbb{R}^+\right\}$ are given by

(a) $$\mathfrak{D}(H) = \left\{f \;:\; f \in L^2\left(\mathbb{R}^+\right),\ f \text{ absolutely continuous},\ f' \in L^2\left(\mathbb{R}^+\right),\ f(0) = 0\right\}$$

and

$$H(f) = -f' + (\theta'/\theta)f;$$

(b) $$\mathfrak{D}\left(H^*\right) = \left\{f \;:\; f \in L^2\left(\mathbb{R}^+\right),\ f \text{ absolutely continuous},\ f' \in L^2\left(\mathbb{R}^+\right)\right\}$$

and

$$H^*(f) = f' + (\theta'/\theta)f.$$

(v) Assume that θ is a C^1-function and $\theta'/\theta \in L^\infty(\mathbb{R}^+)$. Then $\left\{W_{\theta_t,T_t} : t \in \mathbb{R}^+\right\}$ is hyponormal if and only if its infinitesimal generator H is hyponormal.

Note. For a more general characterization of subnormal weighted translation semigroups we refer to Embry and Lambert [103]. In [103], it is shown that a weighted translation semigroup $\left\{W_{\theta_t,T_t} : t \in \mathbb{R}^+\right\}$ on $L^2(\mathbb{R}^+)$ with symbol θ is subnormal if and only if θ^2 is a product of an exponential function and the Laplace–Stieljes transform of an increasing function of total variation one.

5.4 HOMOMORPHISMS AND COMPOSITION OPERATORS

If we have an algebra of functions on a set X with pointwise operations, then we know that every composition transformation is an algebra homomorphism. In many function algebras the converse is also true under suitable conditions. In this section we shall present some results which will illustrate the application of the composition operators in the characterization of algebra homomorphisms. Some partial results we have already given in chapters III and IV. For example, see Theorem 3.2.1 and Theorem 4.2.10.

If X and Y are two compact Hausdorff spaces, then we know that $C(X)$ and $C(Y)$ are Banach algebras of the continuous complex–valued functions on X and Y respectively with supremum norm topology. They are also C^*–algebras and the maximal ideal spaces of $C(X)$ and $C(Y)$ are homeomorphic to X and Y respectively. We refer to [96] for details. If $T : Y \to X$ is a continuous map, then we know that it induces the composition operator $C_T : C(X) \to C(Y)$ which is a $*$–homomorphism. It turns out that every non–zero $*$–homomorphism from $C(X)$ to $C(Y)$ is a composition operator. This result we shall present in the following theorem.

Theorem 5.4.1. Let X and Y be two compact Hausdorff spaces and let $A : C(X) \to C(Y)$ be a non–zero algebra homomorphism such that $A(\overline{f}) = \overline{A(f)}$, for every $f \in C(X)$. Then there exists $T \in C(Y, X)$ such that $A = C_T$. If A is a bijection, then T is a homomorphism, (consequently A is continuous).

Proof. Let $y \in Y$ and let $\phi_y : C(Y) \to \mathbb{C}$ be defined as $\phi_y(f) = f(y)$, $f \in C(Y)$. Then ϕ_y is a bounded multiplicative linear functional on $C(Y)$. Let $\psi_y : C(X) \to \mathbb{C}$ be defined as

$$\psi_y(f) = \phi_y(Af), \quad f \in C(X).$$

Then ψ_y is a bounded multiplicative linear functional on $C(X)$. Since the maximal ideal space of $C(X)$ is homeomorphic to X under the homeomorphism defined by the point evaluations [96], there exists a unique $x \in X$ such that

$$\psi_y(f) = \delta_x(f) = f(x), \; f \in C(X).$$

Let $T : Y \to X$ be defined as $T(y) = x$, where $x \in X$ such that

$$\psi_y(f) = f(x), \text{ for every } f \in C(X).$$

Then it is clear that

$$(Af)(y) = f(T(y)), \; y \in Y \text{ and } f \in C(Y)$$

and

$$\| Af \| \le \| f \|, \; f \in C(Y).$$

This shows that A is a bounded operator. Let $y_0 \in Y$, let $x_0 = T(y_0)$ and let U be an open set containing x_0. Since X is completely regular, there exists a function $f \in C(X)$ such that $f(x_0) = 1$ and $f(x) = 0$ for every $x \in X \setminus U$. Since $Af \in C(Y)$, the set $W = \{y \in Y : (Af)(y) \neq 0\}$ is an open set containing y_0. Let $y \in W$. Then $(Af)(y) = f(T(y)) \neq 0$. Hence $T(y) \in U$. Thus $T(W) \subset U$. This proves the continuity of T at y_0 and hence everywhere in Y. Thus $T \in C(Y, X)$. Since $(Af)(y) = f(T(y))$ for every $y \in Y$ and $f \in C(X)$, we conclude that $A = C_T$. In case A is a bijection, $A^{-1} : C(Y) \to C(X)$ is a $*$–homomorphism and hence there is a $T_1 \in C(X, Y)$ such that $A^{-1} = C_{T_1}$ *i.e.*, $(A^{-1} f)(x) = f(T_1(x))$, $f \in C(Y)$ and $x \in X$. It is clear from this that

$$f(x) = f(T(T_1(x))), \quad x \in X, \; f \in C(X).$$

From this we conclude that $T \circ T_1$ is the identity function on X and similarly, $T_1 \circ T$ is the identity function on Y. Hence T is a homeomorphism. This completes the proof of the theorem.

Notes. (i) From the above proof it is evident that the evaluation functionals play an important role in creation of the mapping $T : Y \to X$. If a self-adjoint Banach algebra $B(X)$ of complex-valued functions on X is such that every non-zero multiplicative linear functional is a point-evaluation, then every non-zero homomorphism on $B(X)$ preserving the complex conjugation will be a composition operator.

(ii) If X and Y are completely regular spaces, then every non-zero ring homomorphism $A : C(X,\mathbb{R}) \to C(Y,\mathbb{R})$ is C_T for some $T \in C(Y,X)$ provided A takes the constant one function on X to the constant one function on Y and Y is real compact (see [122] for further details).

(iii) If X and Y are completely regular Hausdorff spaces, $T \in C(Y,X)$ and $\hat{T} : \beta Y \to \beta X$ is the Stone extension of T, then $C_T : C_b(X) \to C_b(Y)$ is an algebra homomorphism and $C_{\hat{T}} : C(\beta Y) \to C(\beta X)$ is an algebra homomorphism and they have several common properties.

(iv) Let E be any commutative Banach algebra and let X be the maximal ideal space of E. Then for every non-zero homomorphism $A : E \to E$ there exists a mapping $T : X \to X$ such that

$$\widehat{Af}\,(x) = \hat{f}(T(x)), \quad \text{for every} \quad x \in X \quad \text{and} \quad f \in E,$$

where $\hat{f}$ denotes the Gelfand transform of f. We refer to [167] for details.

Definition 5.4.2. Let $M(\mathbb{C})$ denote the set of all finite regular (signed) Borel measures on the complex plane $\mathbb{C}$ and let $M_e(\mathbb{C}) = \{\mu \in M(\mathbb{C}) : \int_{\mathbb{C}} e^{|\mathrm{Re}\,\lambda|} d|\mu| < \infty\}$. Let $EA[-1,1] = \{f : f(x) = \int_{\mathbb{C}} e^{x\lambda}\, d\mu(\lambda),$ for some $\mu \in M_e(\mathbb{C}), x \in [-1,1]\}$. If $f \in EA[-1,1]$, then define

$$\| f \| = \inf\left\{\int_{\mathbb{C}} e^{|\mathrm{Re}\,\lambda|} d|\mu|(\lambda) \;:\; \mu \in M_e(\mathbb{C}) \text{ and } f(x) = \int_{\mathbb{C}} e^{x\lambda} d\mu(\lambda)\right\}.$$

This norm makes $EA[-1,1]$ into a semi-simple Banach algebra with the maximal ideal space $[-1, 1]$. The dual space of $EA[-1,1]$ is isometrically isomorphic to $D[-1, 1]$, the Banach algebra of entire functions ϕ such that $\| \phi \| = \sup_{\lambda \in \mathbb{C}}\left\{ |\phi(\lambda)| e^{-|\mathrm{Re}\,\lambda|}\right\} < \infty$. If $\phi \in D[-1,1]$, then the functional $\psi_\phi : EA[-1,1] \to \mathbb{C}$ is given by $\psi_\phi(f) = \int_{\mathbb{C}} \phi(\lambda) d\mu(\lambda)$, where $\mu \in M_e(\mathbb{C})$ such that $f(x) = \int_{\mathbb{C}} e^{x\lambda}\, d\mu(\lambda)$. For complete details we refer to [37].

In the following theorem we shall show that if $T : [-1,1] \to [-1,1]$ is a non–linear polynomial, then it does not induce a composition operator on $EA[-1,1]$.

Theorem 5.4.3. Let $T : [-1,1] \to [-1,1]$ be a non–linear polynomial. Then C_T is not an operator on $EA[-1,1]$.

Proof. Suppose C_T is an operator on $EA[-1,1]$. Let $u(x) = x$, for every $x \in [-1,1]$. Then $u \in EA[-1,1]$ and $C_T u = T$. Let $n > 0$ and let $\phi_n(z) = \int_{-1}^{1} e^{zx} e^{-inT(x)} dx$. Then ϕ_n is an entire function such that

$$\sup\left\{|\phi_n(z)|\, e^{-|\operatorname{Re} z|} \;:\; z \in \mathbb{C}\right\} \le 2.$$

Thus $\phi_n \in D[-1,1]$ for every $n > 0$. It follows from a result of [37] that

$$\psi_{\phi_n}\left(e^{zu}\right) = \phi_n(z) = \int_{-1}^{1} e^{zx}\, e^{-inT(x)}\, dx.$$

Since the set $\{e^{zu} : z \in \mathbb{C}\}$ generates $EA[-1,1]$, we have $\psi_{\phi_n}(f) = \int_{-1}^{1} f(x).\, e^{-inT(x)} dx$, for every $f \in EA[-1,1]$. Thus

$$\begin{aligned} \|\psi_{\phi_n}\| &= \sup\left\{|\phi_n(z)|\, e^{-|\operatorname{Re} z|} \;:\; z \in \mathbb{C}\right\} \\ &= \sup\left\{|\phi_n(it)| \;:\; t \in \mathbb{R}\right\} \quad \text{[by Phragmen – Lindelöff theorem]} \\ &= \sup_{t \in \mathbb{R}} \left\{ \left| \int_{-1}^{1} e^{i(tx - nT(x))}\, dx \right| \right\}. \end{aligned}$$

Let $f_n = C_T(e^{inu}) = e^{inT(x)}$. Then $f \in EA[-1,1]$ and $\psi_{\phi_n}(f_n) = 2$. Now, since C_T is a bounded operator on $EA[-1,1]$, we have

$$\begin{aligned} 2 &= \left\| \psi_{\phi_n}\left(C_T\left(e^{inu}\right)\right) \right\| \\ &\le \|\psi_{\phi_n}\|\, \|C_T\| \\ &= \|C_T\| \sup_{t \in \mathbb{R}} \left| \int_{-1}^{1} e^{i(tx - nT(x))}\, dx \right|. \end{aligned}$$

But if T is non–linear polynomial, then the right hand side approaches to zero as n approaches to ∞. This is a contradiction. Hence T does not induce a composition operator on $EA[-1,1]$. This completes the proof of the theorem.

We shall present the following theorem without proof. This theorem resembles a theorem of Beurling–Helson given in [172].

Theorem 5.4.4. If $T : [-1,1] \to [-1,1]$ induces a composition operator on $EA[-1,1]$, then there exists an interval $I \subset [-1,1]$ such that T is a linear function on I. (See [170] for details of the proof).

Let $\alpha, \beta \in \mathbb{R}$ with the condition that $|\alpha| + |\beta| \leq 1$ and let $T_\alpha^\beta : [-1,1] \to [-1,1]$ be defined as $T_\alpha^\beta(x) = \alpha x + \beta$, $x \in [-1,1]$. Then $C_{T_\alpha^\beta}$ is an operator on $EA[-1,1]$ and we shall denote it simply by C_α^β. The following theorem characterizes the non-zero algebra endomorphisms of $EA[-1,1]$ in terms of the composition operators induced by the linear functions.

Theorem 5.4.5. Let A be a non-zero endomorphism of $EA[-1,1]$. Then $A = C_\alpha^\beta$ for some $\alpha, \beta \in \mathbb{R}$ such that $|\alpha| + |\beta| \leq 1$.

Proof. Let A be a non-zero endomorphism. Then $A = C_T$ for some $T : [-1,1] \to [-1,1]$ as the maximal ideal space of $EA[-1,1]$ is homeomorphic to $[-1,1]$ and T can be produced. We shall prove that $T = T_\alpha^\beta$. By the last theorem there exists $\alpha, \beta \in \mathbb{R}$ and an interval I such that $T = T_\alpha^\beta$ on I. That is,

$$T(y) = \alpha y + \beta \ \text{ for } \ y \in I.$$

Let k be a natural number and let $\mu_k' \in M_e(\mathbb{C})$ such that

$$e^{ikT(y)} = \int_{\mathbb{C}} e^{y\lambda} d\mu_k'(\lambda) .$$

Since $\left\| e^{ikT(u)} \right\| \leq \| C_T \|$, we can conclude that

$$\int_{\mathbb{C}} e^{|RE \lambda|} d|\mu_k'|(\lambda) \leq 2 \| C_T \| .$$

Let $\mu_k(\lambda) = e^{-ik\beta} \mu_k'(\lambda + ik\alpha)$. Then the sequences $\left\{e^{|\operatorname{Re}\lambda|} d\mu_k\right\}$ and $\{d\mu_k\}$ are uniformly bounded. Also

$$\begin{aligned} \int_{\mathbb{C}} e^{y\lambda} d\mu_k(\lambda) &= e^{-i(\alpha ky + \beta k)} \int_{\mathbb{C}} e^{y\lambda} d\mu_k'(\lambda) \\ &= e^{i[kT(y) - k(\alpha y + \beta)]} . \end{aligned}$$

Let $T_1(y) = T(y) - \alpha y - \beta$. Then $T_1(y) = 0$ if $y \in I$ and

$$\int_{\mathbb{C}} e^{y\lambda}\, d\mu_k(\lambda) = e^{ikT_1(y)}.$$

Let $\sigma_n = \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}\mu_k$. Then σ_n is a Borel measure on $\mathbb{C}$ and

$$\int e^{|\operatorname{Re}\lambda|}\, d|\sigma_n|(\lambda) \le \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}\int_{\mathbb{C}} e^{|\operatorname{Re}\lambda|} d|\mu_k| \le 2\,\|C_T\|.$$

Hence as a Gelfand transform

$$\hat{\sigma}_n(y) = \int_{\mathbb{C}} e^{y\lambda}\, d\sigma_n(\lambda)$$

and $\{\hat{\sigma}_n : n \ge 0\}$ is a bounded subset of $C[-1,1]$ with supremum norm. Since $M(\mathbb{C})$ is the conjugate space of $C_0(\mathbb{C})$, there exists a Borel measure σ on $\mathbb{C}$ such that for $f \in C_0(\mathbb{C})$,

$$\int_{\mathbb{C}} f(\lambda)\, e^{|\operatorname{Re}\lambda|}\, d\sigma_n(\lambda) \quad \text{tends to} \quad \int_{\mathbb{C}} f(\lambda) e^{|\operatorname{Re}\lambda|}\, d\sigma(\lambda)$$

and

$$\int_{\mathbb{C}} f(\lambda)\, d\sigma_n(\lambda) \quad \text{tends to} \quad \int_{\mathbb{C}} f(\lambda)\, d\sigma(\lambda).$$

Let

$$f(\lambda) = \frac{e^{b\lambda} - e^{a\lambda}}{\lambda}\, e^{-|\operatorname{Re}\lambda|}, \quad \lambda \in \mathbb{C},$$

where $a, b \in [-1, 1]$ such that $a < b$. Then $f \in C_0(\mathbb{C})$. Also

$$\hat{\sigma}_n(y) = \int_{\mathbb{C}} e^{y\lambda}\, d\sigma_n(\lambda) = \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}\int_{\mathbb{C}} e^{y\lambda}\, d\mu_k(\lambda)$$
$$= \left(\frac{1 + e^{iT_1(y)}}{2}\right)^n.$$

Let $J = \{y : T_1(y) = 0\}$. Then $\hat{\sigma}_n(y) \to \chi_J(y)$, for every $y \in [-1, 1]$. Using the Lebesgue Dominated Convergence Theorem we can conclude that

$$\int_{-1}^{1} \chi_{[a,b]}(y)\,\hat{\sigma}_n(y)\,dy \to \int_{-1}^{1} \chi_{[a,b]}(y)\,\chi_J(y)dy,$$

for every $a, b \in [-1, 1]$ such that $a < b$. Thus

$$\int_{-1}^{1} \chi_{[a,b]}(y)\,\hat{\sigma}_n(y)\,dy = \int_{-1}^{1} \chi_{[a,b]}(y)\,\chi_J(y)dy.$$

From this we can conclude that

$$\hat{\sigma}(y) = \chi_J(y) \quad a.e.$$

Since the interval $I \subset J$ and $\hat{\sigma}$ is continuous, we conclude that $\hat{\sigma}(y) = 1$ for every $y \in [-1, 1]$ and thus $J = [-1, 1]$. Hence $T(y) = \alpha\, y + \beta$ for every $y \in [-1, 1]$. Since T is a self map on $[-1, 1]$, we conclude that $|\alpha| + |\beta| \leq 1$. This completes the proof of the theorem.

Note. C_α^β is an automorphism on $E\,A\,[-1, 1]$ if and only if $\alpha = 1$ or -1 and $\beta = 0$.

Let X be a compact metric space with a metric d and let $Lip\,(X, d)$ denote the Banach algebra of all complex-valued functions f on X such that

$$\| f \|_\infty + \sup_{x \neq y} \frac{|f(x) - f(y)|}{d(x,y)} < \infty.$$

where $\| f \|_\infty$, as usual denotes the sup norm of f. If $f \in Lip\,(X, d)$, then

$$\| f \| = \| f \|_\infty + \sup_{x \neq y} \frac{|f(x) - f(y)|}{d(x,y)}.$$

$Lip\,(X, d)$ is a commutative semi-simple Banach algebra and it is called a Lipschitz algebra. It has been shown in [318] that the maximal ideal space of $Lip\,(X, d)$ is homeomorphic to X and every non-zero endomorphism of $Lip\,(X, d)$ is a composition operator C_T induced by a map $T : X \to X$ such that

$d(T(x), T(y)) \le M\, d(x.y)$ for some $M > 0$ and every $x, y \in X$. A mapping $T : X \to X$ is called a super contraction if

$$\lim_{d(x, y) \to 0} \frac{d(T(x), T(y))}{d(x, y)} = 0 .$$

Clearly every constant map is a super contraction. There are compact metric spaces which have non–constant super contraction. For details we refer to [166]. It turns out that every non–zero compact endomorphism on $Lip\,(X, d)$ is induced by a super contraction. This we shall present in the following theorem.

Theorem 5.4.6. Let A be a non–zero endomorphism of $Lip\,(X, d)$ induced by the map $T : X \to X$. Then A is compact if and only if T is a super contraction.

Proof. (outline) Suppose $T : X \to X$ is not a super contraction. Then there exist $\varepsilon > 0$ and $x_n, y_n \in X$ such that

$$d(x_n, y_n) < \frac{1}{n^2} \quad \text{and} \quad \frac{d(T(x_n),\ T(y_n))}{d(x_n, y_n)} \ge \varepsilon > 0 .$$

Let $n \in \mathbb{N}$ and let

$$f_n(x) = \frac{1 - e^{-nd(x, T(y_n))}}{n} .$$

Then $\| f_n \|_\infty < \frac{1}{n}$, $\sup\limits_{x \neq y} \frac{| f_n(x) - f_n(y) |}{d(x, y)} < 1$ and $\| f_n \| < \frac{1}{n} + 1$. If C_T is compact, then there exists a subsequence $\{f_{n_k}\}$ and $g \in Lip(X, d)$ such that

$$C_T f_{n_k} \to g \quad \text{in the norm of} \quad Lip(X,\ d) .$$

Since $f_{n_k} \to 0$ (uniformly), we conclude that $g = 0$. It follows that

$$\frac{\left| f_{n_k}(T(x)) - f(T(y)) \right|}{d(x, y)} < \varepsilon / 2$$

for large k and $x, y \in X$, $x \neq y$. From this we get

$$\frac{\varepsilon}{2} > \frac{1 - e^{-n_k d\left(T\left(x_{n_k}\right),\, T\left(y_{n_k}\right)\right)}}{n_k d\left(x_{n_k}, y_{n_k}\right)} = \frac{e^{-n_k t} d\left(T\left(x_{n_k}\right), T\left(y_{n_k}\right)\right)}{d\left(x_{n_k}, y_{n_k}\right)}$$

for some t depending on k such that $0 < t < d\left(T\left(x_{n_k}\right), T\left(y_{n_k}\right)\right)$. Since $T : X \to X$ induces the composition operator C_T, there exists $M > 0$ such that $d(T(x), T(y) \leq M\ d(x, y)$ for $x, y \in X$. Thus

$$0 < \varepsilon \leq \frac{d\left(T\left(x_{n_k}\right), T\left(y_{n_k}\right)\right)}{d\left(x_{n_k}, y_{n_k}\right)} < \frac{\varepsilon}{2}\ e^{n_k t} < \frac{\varepsilon}{2}\, e^{\frac{M}{n_k}}$$

for $0 < t \leq M / n_k^2$. If k tends to infinity, we get $0 < \varepsilon \leq \varepsilon/2$, which is a contradiction. Thus T is a super contraction.

Conversely, if $T : X \to X$ is super contraction and $\{f_n\}$ is a bounded sequence in $Lip\ (X, d)$, then $\{f_n\}$ is equicontinuous, and by Arzela–Ascoli theorem $f_{n_k} \to g$ for some $g \in C(X)$. It can be shown that the sequence $\{C_T\ f_{n_k}\}$ is a Cauchy sequence in $Lip\ (X, d)$. Hence there exists an $f \in Lip\ (X, d)$ such that $C_T f_{n_k} \to f$. This shows that C_T is compact and $f = C_T g$. This completes the proof of the theorem.

Notes.

(i) If C_T is a non-zero compact endomorphism of $Lip\ (X, d)$, then the spectrum of C_T is the set $\{0, 1\}$.

(ii) If $0 < \alpha \leq 1$ and (X, d) is a compact metric space, then $d^{\alpha} : X \times X \to \mathbb{R}^+$ defined as $d^{\alpha}(x, y) = (d(x, y))^{\alpha}$ is a metric on X. Let $Lip_{\alpha}(X, d) = Lip(X, d^{\alpha})$. Let

$$lip_{\alpha}(X, d) = \left\{ f \in Lip_{\alpha}(X, d)\ :\ \lim_{d(x,y) \to 0} \frac{|f(x) - f(y)|}{(d(x, y))^{\alpha}} = 0 \right\},\quad 0 < \alpha < 1.$$

Then $lip_{\alpha}(X, d)$ is a closed subalgebra of $Lip_{\alpha}(X, d)$ and the second dual of $lip_{\alpha}(X, d)$ is isometrically isomorphic to $Lip_{\alpha}(X, d)$. Every non-zero compact endomorphism of $lip_{\alpha}(X, d)$ is also induced by a super contractive mapping and spectrum in this case is also $\{0, 1\}$.

(iii) If (X_1, d_1) and (X_2, d_2) are compact metric spaces, and $A\ :\ Lip(X_1, d_1) \to Lip(X_2, d_2)$ is a non-zero algebra homomorphism, then there exists a map $T : X_2 \to X_1$ such that $d_1(T(x), T(y)) \leq M\, d_2(x,y)$, for some $M > 0$ and $A = C_T$. See [318] for details.

(iv) Let

$$E = \left\{ f \in C^{\infty}[0,1] \ : \ \| f \| = \sum_{n=0}^{\infty} \frac{\| f^{(n)} \|_{\infty}}{(n\,!)^2} < \infty \right\}.$$

Then E is a semi-simple commutative Banach algebra with $[0, 1]$ as the maximal ideal space and E admits a non-trivial compact endomorphism. If $T : [0,1] \to [0,1]$ is defined as $T(x) = x/2$, then the composition operator C_T induced by T is a non-trivial compact endomorphism of E. Details can be looked into [170].

In the recent years the lattice homomorphisms and the disjointness preserving operators have been characterised in terms of the weighted composition operators. Feldman and Porter in [110] characterized lattice homomorphisms on real Banach lattices having locally compact representation spaces. Jamison and Rajagopalan in [153] have characterised operators on $C(X, E)$ with the disjoint support property. Recently Chan [64] generalised this result to operators on spaces of cross-sections over X. We shall present the result in the general setting.

Let X be a compact Hausdorff space, let $\{E_x : x \in X\}$ be a family of Banach spaces and let $L(X)$ be a closed subspace of the cross-sections over X *i.e.*, $L(X)$ is a closed subspace of the product space $\prod_{x \in X} E_x$. The space $L(X)$ is said to be a function module if

(a) $h.f \in L(X)$ for every $f \in L(X)$ and $h \in C(X)$;

(b) the function $u_f : X \to \mathbb{R}^+$ defined as $u_f(x) = \| f(x) \|$ is upper semicontinuous for every $f \in L(X)$;

(c) $E_x = \{f(x) \ : \ f \in L(X)\}$ for every $x \in X$;

(d) $\{x \ : \ E_x \neq \{0\}\}$ is dense in X.

An operator $A : L(X) \to L(X)$ is said to have disjoint support property if $\| Af(x) \| \| Ag(x) \| = 0$, for every $x \in X$ whenever $\| f(x) \| \| g(x) \| = 0$, for every

$x \in X$, for $f, g \in L(X)$. In the following theorem it is shown that every continuous operator on a function module $L(X)$ having disjoint support property is a weighted composition operator $W_{\pi,T}$ induced by a pair (π, T) where $\pi : X \to \bigcup_{x \in X} B(E_x)$ is an operator-valued map such that $\pi(x) \in B(E_x)$, for every $x \in X$ and T is a self map of X.

Theorem 5.4.7. Let $L(X)$ be a function module in $\prod_{x \in X} E_x$ and let $A : L(X) \to L(X)$ be a continuous operator. Then A has the disjoint support property if and only if $A = W_{\pi,T}$ for some $\pi : X \to \bigcup_{x \in X} B(E_x)$ such that $\pi(x) \in B(E_x)$ for every $x \in X$, and for some $T : X \to X$.

Proof. If $A = W_{\pi,T}$ and $f, g \in L(X)$ such that $\| f(x) \| \| g(x) \| = 0$ for every $x \in X$, then

$$\begin{aligned} \| A f(x) \| \| A g(x) \| &= \| \pi(x) f(T(x)) \| \| \pi(x) g(T(x)) \| \\ &\leq \| \pi(x) \|^2 \| f(T(x)) \| \| g(T(x)) \| \\ &= 0, \text{ for every } x \in X. \end{aligned}$$

This shows that A has the disjoint support property. Let $x \in X$ such that $Ag(x) \neq 0$ for some $g \in L(X)$. Let $\mathcal{F}_x = \{K \subset X : K$ is non-empty and compact, $f \in L(X)$, $f(k) = 0$, for every $k \in K$ implies $A f(x) = 0\}$. Under set inclusion $\mathcal{F}_x$ is a partially ordered set. Using property (b) of the function module $L(X)$ it can be shown that $\mathcal{F}_x$ has a minimal element. If $K_0 \in \mathcal{F}_x$ is a minimal element, then using Riesz-representation theorem it can be concluded that there exists unique $y \in X$ such that $K_0 = \{y\}$. Now define $T(x) = y$ and $\pi(x)(z) = (A f)(x)$, where $z \in E_x$ and f is any element of $L(X)$ such that $f(y) = z$. For other $x \in X$, let $\pi(x) = 0$ and $T(x)$ be any arbitrary element. Thus we have

$$(A f)(x) = \pi(x)(z) = \pi(x)\big(f(T(x))\big)$$

for every $f \in L(X)$ and $x \in X$. This shows that

$$A = W_{\pi,T}.$$

This completes the outline of the proof. For detailed proof we refer to [64].

Remark 5.4.8. If $E_x = E$ for every $x \in X$ and $L(X) = C(X,E)$, then $L(X)$ is a function module and the characterisation given by Jamison and Rajagopalan follows from the above theorem (see Theorem 4.4.1).

The following theorem of Feldman and Porter [110] presents a representation of the lattice homomorphisms between Banach lattices using the theory of the weighted composition operators. We shall just state this theorem without proof.

Theorem 5.4.9. Let E_1 and E_2 be Banach lattices having locally compact representative spaces X_1 and X_2 respectively, and let $A : E_1 \to E_2$ be a lattice homomorphism. Then there exists a function $\pi : X_2 \to \mathbb{R}^+$ and a continuous function $T : P_\pi \to X_1$ such that

$$(A\ f)\ (y) = \begin{cases} \pi(y)\ f(T(y)), & \text{if } y \in P_\pi \\ 0 & , \text{if } y \text{ is an interior point of } Z_\pi; \end{cases}$$

where $P_\pi = \{y \in X_2 : \pi(y) > 0\}$ and $Z_\pi = \{y \in X_2 : \pi(y) = 0\}$.

Let X be an open subset of the complex plane $\mathbb{C}$, and let $H(X)$ denote the algebra of all holomorphic functions on X. Then it turns out that the evaluation functionals are the only non-zero multiplicative linear functionals on $H(X)$. This we shall prove in the following theorem.

Theorem 5.4.10. If $\Phi : H(X) \to \mathbb{C}$ is a non-zero multiplicative linear functional, then there exists a $z_0 \in X$ such that $\Phi = \delta_{z_0}$.

Proof. Let $\alpha \in \mathbb{C}$ and let α_c denote the constant function $\alpha_c(z) = \alpha$. If $\Phi : H(X) \to \mathbb{C}$ is a non-zero multiplicative linear functional, then $\Phi(1_c) = \Phi(1_c.1_c) = \Phi(1_c)\,\Phi(1_c)$. Thus $\Phi(1_c)$ is equal to zero or one. In case $\Phi(1_c) = 0$, we have $\Phi(f) = \Phi(f.1_c) = 0$, for every $f \in H(X)$. Hence $\Phi = 0$. Thus $\Phi(1_c) = 1$. Now $\Phi(\alpha_c) = \Phi(\alpha_c.1_c) = \alpha\Phi(1_c) = \alpha$. Let $u(z) = z$ be the identity function. Let $z_0 = \Phi(u)$. Then $z_0 \in X$. For, if $z_0 \notin X$, then the function $f_{z_0} : X \to \mathbb{C}$ defined as $f_{z_0}(z) = \dfrac{1}{z - z_0}$ belongs to $H(X)$. Thus

$$\left(u - z_{0_c}\right) g = 1_c, \quad \text{for some} \quad g \in H(X).$$

Hence $\Phi(1_c) = \Phi(u - z_{0_c}) \,.\, \Phi(g) = (z_0 - z_0)\, \Phi(g) = 0$, which is a contradiction to the fact that $\Phi(1_c) = 1$. Thus $z_0 \in X$. Let $f \in H(X)$ and let $g : X \to \mathbb{C}$ be defined as

$$g(z) = \begin{cases} \dfrac{f(z) - f(z_0)}{z - z_0}, & z \neq z_0 \\ f'(z_0) & , z = z_0 . \end{cases}$$

Then $g \in H(X)$. Thus

$$\left(u - z_{0_c}\right) g = f - f(z_0)_c \,.$$

Hence $\Phi((u - z_{0_c}) \,.\, g) = \Phi\,(f - f(z_0)_c)$. Thus $\Phi(f) = \delta_{z_0}(f)$. This shows that $\Phi = \delta_{z_0}$.

If X and Y are open subsets of $\mathbb{C}$ and $T : Y \to X$ is a holomorphic map, then $C_T : H(X) \to H(Y)$ is a non-zero algebra homomorphism. It turns out that every non-zero algebra homomorphism from $H(X)$ to $H(Y)$ is a composition transformation if Y is connected. We shall prove this result in the following theorem.

Theorem 5.4.11. Let X and Y be two open subsets of the complex plane and suppose that Y is connected. Let $A : H(X) \to H(Y)$ be a non-zero algebra homomorphism. Then there exists a holomorphic map $T : Y \to X$ such that $A = C_T$.

Proof. Since $1_c \in H(X)$ and $A(1_c) = A(1_c) \,.\, A(1_c)$, using connectedness of Y we can conclude that $A(1_c) = 1_c$. From this we get $A(\alpha_c) = \alpha_c$ for every $\alpha \in \mathbb{C}$. Now, if $y \in Y$, then define $\Phi_y(f) = (A\,f)\,(y)$. Then Φ_y is a multiplicative linear functional on $H(X)$, and hence by the last theorem, there exists a $z \in X$ such that

$$\Phi_y(f) = f(z) \,.$$

If $u(z) = z$, for $z \in X$, then $\Phi_y(u) = u(z)$. Hence $(Au)\,(y) = z$. Thus $(A\,f)\,(y) = f(z) = f(A\,u)\,(y)$, $y \in Y$. Let $T = A\,u$. Then clearly $A\,f = C_T f$, for every $f \in H(X)$. This completes the proof of the theorem.

Notes.

(i) The above theorem need not be true in case of holomorphic functions of several variables; see [225] for details.

(ii) An algebra homomorphism from $H(X)$ to $H(Y)$ is continuous with respect to compact–open topology on $H(X)$ and $H(Y)$.

(iii) $H(X)$ and $H(Y)$ are isomorphic as algebras if and only if X and Y are conformally equivalent.

If X is an open subset of $\mathbb{C}$, then a function $T : X \to \mathbb{C}$ is said to be anti–conformal if $\overline{T}$ is conformal. Two regions X and Y are said to be anticonformally equivalent if there exists an anticonformal map $T : X \to Y$.

If $T : X \to Y$ is an anticonformal onto map, then $f \circ T \in H(X)$ for every $f \in H(Y)$ and the mapping $f \to \overline{f \circ T}$ is a ring–isomorphism from $H(Y)$ to $H(X)$. In the following theorem of Bers [29] we record the representation of a ring isomorphism from $H(X)$ to $H(Y)$.

Theorem 5.4.12. Let X and Y be two regions in $\mathbb{C}$. Then $H(X)$ and $H(Y)$ are ring–isomorphic if and only if X and Y are either conformally or anticonformally equivalent. Every ring–isomorphism A from $H(X)$ to $H(Y)$ is induced by a conformal mapping or an anticonformal mapping *i.e.*, $A\,f = C_T f$ or $A\,f = \overline{C_{T'} f}$ for some conformal map T or some anticonformal map T'.

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