LITTLEWOOD'S PROBLEM FOR SETS WITH MULTIDIMENSIONAL STRUCTURE

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Abstract. We give L^1-norm estimates for exponential sums of a finite sets A consisting of integers or lattice points. Under the assumption that A possesses sufficient multidimensional structure, our estimates are stronger than those of McGehee-Pigno-Smith and Konyagin. These theorems improve upon past work of Petridis.

1. Introduction

Let A be a finite set of integers. The relationship between the additive structure of A and the exponential sum[1]

$$F(t) = \sum_{a \in A} e(at)$$

is well-documented (see [TV], Chapter 4). In particular, we have

$$\int_0^1 |F(t)|^{2k}\, dt = |\{(a_1, \ldots, a_2k) \in A^{2k} : a_1 + \cdots + a_k = a_{k+1} + \cdots + a_{2k}\}|,$$

so that the larger the $2k$'th moment of F, the more additive structure A possesses. For $1 \leq p < 2$, we expect the L^p-norms of F to be smaller for additively structured sets. This led Littlewood to conjecture (see, for instance [HL]) that

$$\inf_{\substack{A \subseteq \mathbb{Z} \\ |A|=N}} \int_0^1 \left| \sum_{a \in A} e(at) \right| dt = \int_0^1 \left| \sum_{n=1}^{N} e(nt) \right| dt. \tag{1}$$

It was a great feat when the estimate

$$\inf_{\substack{A \subseteq \mathbb{Z} \\ |A|=N}} \int_0^1 \left| \sum_{a \in A} e(at) \right| dt \geq C \int_0^1 \left| \sum_{n=1}^{N} e(nt) \right| dt \tag{2}$$

was established by McGehee, Pigno and Smith in [MPS] and independently by Konyagin in [Ko] for some absolute constant $C > 0$. Here we will record the theorem of McGehee, Pigno and Smith as it will be used repeatedly in this article.

Theorem 1.1. *Let $a_1 < \ldots < a_n$ be a sequence of integers and let $u_1, \ldots, u_n$ be complex numbers. Then*

$$\int_0^1 \left| \sum_{j=1}^{n} u_j e(a_j t) \right| dt \geq C_{MPS} \sum_{j=1}^{n} \frac{|u_j|}{j},$$

[1]Throughout, we will use the notation $e(z) = e^{2\pi i z}$.

where $C_{MPS} > 0$ is an absolute constant.

The estimate (2) leaves open a few questions. First, it remains to establish the sharp constant, i.e. to prove (1). Second, one would like to characterize the sets A for which (2) holds, a question sometimes referred to as the *Inverse Littlewood Problem*, see [G]. This article concerns the latter problem and we interpret it as follows: if A possesses some structure which is decidedly unlike an arithmetic progression, can the estimate (2) be improved? Specifically, we will explore how the notion of dimension can be leveraged. Such questions have already by investigated by Petridis [P], and where appropriate we will compare results.

The first notion of dimension we will explore is quite literal - we consider A a subset of the lattice $\mathbb{Z}^r$. Since $\mathbb{Z}^r$ contains one-dimensional sets, one must take steps to ensure A is truly multidimensional, which we now do.

For $i = 1, \ldots, r$, let $\pi_i : \mathbb{Z}^r \to \mathbb{Z}$ denote the i'th coordinate projection and let

$$A_i = \pi_i(A)$$

denote the image of A under this projection; for $a_i \in \pi_i(A)$ let

$$A_i^*(a_i) = \pi_i^{-1}(a_i) \cap A$$

denote the fibre of A above a_i. Our first estimate extends Theorem 1.1 to higher dimensional sets.

Theorem 1.2. *Suppose $A \subseteq \mathbb{Z}^r$ and A_1 is ordered as*

$$A_1 = \{a_{1,1} < \ldots < a_{1,n}\}.$$

Then we have the estimate

$$\int_{[0,1]^r} \left| \sum_{\boldsymbol{a} \in A} e(\boldsymbol{a} \cdot \boldsymbol{t}) \right| d\boldsymbol{t} \geq C_{MPS} \sum_{j=1}^{n} \frac{1}{j} \int_{[0,1]^{r-1}} \left| \sum_{\boldsymbol{a}^* \in A_1^*(a_{1,j})} e(\boldsymbol{a}^* \cdot \boldsymbol{t}) \right| d\boldsymbol{t}.$$

We say $A \subseteq \mathbb{Z}$ is n-strongly 1-dimensional if $|A| \geq n$. Inductively, if $(n_1, \ldots, n_r)$ is a r-tuple of natural numbers then we say a set $A \subseteq \mathbb{Z}^r$ is $(n_1, \ldots, n_r)$-strongly r-dimensional if $|A_1| \geq n_1$ and $|A_1^*(a_1)|$ is $(n_2, \ldots, n_r)$-strongly $(r-1)$-dimensional for each $a_1 \in A_1$.

Theorem 1.3. *Suppose $A \subseteq \mathbb{Z}^r$ is a $(n_1, \ldots, n_r)$-strongly r-dimensional subset of $\mathbb{Z}^r$. Then*

$$\int_{[0,1]^r} \left| \sum_{\boldsymbol{a} \in A} e(\boldsymbol{a} \cdot \boldsymbol{t}) \right| d\boldsymbol{t} \geq C_{MPS}^r \log(n_1) \cdots \log(n_r).$$

Here we have strived to make the dependence on the implicit constant from Theorem 1.1 explicit. This estimate is an improvement on Theorem 1.2 in [P] and is best-possible up to the constant C_{MPS}^r.

We now move to the case of subsets A of $\mathbb{Z}$, which are one dimensional but have a structure if higher dimensional sets. As motivation, recall that a *generalized arithmetic progression of rank* 2 is a set of the form

$$G = \{am + bn : 1 \leq m \leq M,\ 1 \leq n \leq N\}.$$

These sets arise as projections of boxes in $\mathbb{Z}^2$, hence we think of them as possessing multidimensional structure. To guarantee that the elements $am + bn$ are distinct, it is sufficient to impose the condition $aM < b$. It is this sort of condition, which can be viewed as a multiscale condition, that motivates the last theorem of this article. To state it, we begin with appropriate notions of a multidimensional subset of $\mathbb{Z}$. We say $A \subseteq \mathbb{Z}$ is n-strongly 1-dimensional if $|A| \geq n$. For $\delta_1, \ldots, \delta_{r-1} > 0$, we define inductively that a finite set $A \subseteq \mathbb{Z}$ is $(\delta_1, \ldots, \delta_{r-1}; n_1, \ldots, n_r)$-strongly r-dimensional if there are numbers d_1 and d_2 with $d_2 > (2 + \delta_1)d_1$ and such that

$$A = \bigcup_{k \in I} A_k + kd_2$$

for some set I of consisting of at least n_1 integers and subsets $A_k \subseteq \{-d_1, \ldots, d_1\}$ which are each $(\delta_2, \ldots, \delta_{r-1}; n_2, \ldots, n_r)$-strongly $(r-1)$-dimensional.

Theorem 1.4. *Let $\delta, \ldots, \delta_{r-1} > 0$ and $n_1, \ldots, n_r$ be positive integers satisfying*

$$n_i \geq 2^{21} C_{MPS}^3 \prod_{j=i}^{r} (\log(n_j))^3$$

for each i. Suppose A is a $(\delta_1 \ldots, \delta_{r-1}; n_1, \ldots, n_r)$-strongly r-dimensional subset of $\mathbb{Z}$. Then

$$\int_0^1 \left| \sum_{a \in A} e(at) \right| dt \geq C_{\delta_1 \ldots, \delta_{r-1}} \log(n_1) \cdots \log(n_r),$$

where

$$C_{\delta_1 \ldots, \delta_{r-1}} = C_{MPS}^r 2^{-8r+8} \prod_{j=1}^{r-1} (1 + \log(1 + 2/\delta_j))^{-1}.$$

This theorem is also best-possible up to the constant, as the lower bound is realized by an appropriately chosen r-dimensional arithmetic progression, see Theorem 3.3 in [S]. Estimates for multidimensional subsets of $\mathbb{Z}$ were established in Theorem 1.3 of [P] as well. There, the bounds are likely not as sharp as in Theorem 1.4, but the hypotheses are somewhat different, relying on the notion of a *Freiman isomorphism*. It might also be noted that Theorem 1.4 holds for two-dimensional sets, which was not established in [P].

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2. Strongly multidimensional sets in $\mathbb{Z}^r$

The basic argument relies on the solution to Littlewood's problem by McGehee-Pigno-Smith, and in particular their generalized Hardy inequality.

Proof of Theorem 1.2. Let $(t_1, t_2, \dots, t_r) \in [0,1]^r$ and write

$$\sum_{\boldsymbol{a}\in A} e(\boldsymbol{a}\cdot\boldsymbol{t}) = \sum_{j=1}^{n} e(a_{1,j}t_1)\left(\sum_{\boldsymbol{a}^*\in A_1^*(a_1)} e(\boldsymbol{a}^*\cdot(t_2,\dots,t_r))\right).$$

We interpret this as a trigonometric polynomial in the variable t_1 with complex coefficients. By Theorem 1.1 we have

$$\int_0^1 \left|\sum_{\boldsymbol{a}\in A} e(\boldsymbol{a}\cdot\boldsymbol{t})\right| dt_1 \geq C_{MPS}\sum_{j=1}^{n}\frac{1}{j}\left|\sum_{\boldsymbol{a}^*\in A_1^*(a_1)} e(\boldsymbol{a}^*\cdot(t_2,\dots,t_r))\right|.$$

Integrating over $t_2, \dots, t_r$ completes the proof. □

Iterated application of Theorem 1.2 leads to the proof of Theorem 1.3.

Proof of Theorem 1.3. We proceed by induction on r, and when $r = 1$, this follows immediately from Theorem 1.1. By the preceding proposition, we have

$$\int_{[0,1]^d}\left|\sum_{\boldsymbol{a}\in A} e(\boldsymbol{a}\cdot\boldsymbol{t})\right| d\boldsymbol{t} \geq C_{MPS}\sum_{j=1}^{n}\frac{1}{j}\int_{[0,1]^{r-1}}\left|\sum_{\boldsymbol{a}^*\in A_1^*(a_1)} e(\boldsymbol{a}^*\cdot\boldsymbol{t})\right| d\boldsymbol{t},$$

and each of the sets $A_1^*(a_1)$ is $(n_2, \dots, n_r)$-strongly $(r-1)$-dimensional. By induction,

$$\int_{[0,1]^{r-1}}\left|\sum_{\boldsymbol{a}^*\in A_1^*(a_1)} e(\boldsymbol{a}^*\cdot\boldsymbol{t})\right| d\boldsymbol{t} \geq C_{MPS}^{r-1}\log(n_2)\cdots\log(n_r)$$

and since $n \geq r_1$ the theorem is proved. □

3. Lemmata

Throughout, we will say f is a trigonometric polynomial of degree d if

$$f(t) = \sum_{|n|\leq d} a_n e(nt).$$

Notice L^1-norms are preserved if we translate the support of $\widehat{f}$ by d. We need the following.

Lemma 3.1 (Bernstein's inequality)**.** *Let $f : [0,1] \to \mathbb{C}$ be a trigonometic polynomial of degree d. Then*

$$\|f'\|_{L^1([0,1])} \leq 2d\|f\|_{L^1([0,1])}.$$

Proof. See [Ka, Chapter 1, Excercise 7.16]. □

Lemma 3.2. *Let n be a positive integer. Then for any trigonometric polynomial f of degree d we have*

$$\left| \|f\|_{L^1([0,1])} - \frac{1}{N}\sum_{j=1}^{N} |f(j/N)| \right| \leq \frac{d}{N}\|f\|_{L^1([0,1])}$$

for any trigonometric polynomial f of degree d.

Proof. We begin with the classical estimate

$$\left| \int_0^1 |f(t)|dt - \frac{1}{N}\sum_{j=1}^{N} |f(j/N)| \right| \leq \frac{1}{N}\mathrm{Var}(|f|)$$

where

$$\mathrm{Var}(g) = \sup_{0=x_0<x_1<\ldots<x_M=1} \sum_{j=1}^{M} |g(x_j) - g(x_{j-1})|$$

is the total variation of g. Since

$$\mathrm{Var}(|f|) \leq \mathrm{Var}(f) = \int_0^1 |f'(t)|dt$$

the result now follows from Bernstein's inequality. □

One of the key ideas that goes into the proof of Theorem 1.4 is to amplify the gaps between the different pieces of A. To do so, we need a function that can isolate the various subsets A_k.

Lemma 3.3. *Let M and N be integers with $2 \leq M < N$ and let $R \geq 2N + 4M + 1$. Then there is a function $K_{M,N}$ with the following properties:*

(1) $K_{M,N}(k) = 1$ *for* $|k| \leq N$,
(2) $K_{M,N}(k) = 0$ *for* $|k| \geq N + 2M$, *and*
(3)

$$\frac{1}{R}\sum_{j=1}^{R} |\widehat{K_{M,N}}(j/R)| \leq (16 + 8\log(1 + N/M)).$$

Proof. Recall that the Dirichlet kernel of order N is

$$D_N(t) = \sum_{|n|\leq N} e(nt) = \frac{\sin(\pi(2N+1)t)}{\sin(\pi t)},$$

and the Fejer kernel or order N is

$$F_N(t) = \sum_{|n|\leq N} \left(1 - \frac{|n|}{N+1}\right) e(nt) = \frac{1}{N+1}\frac{(\sin(\pi(N+1)t))^2}{(\sin(\pi t))^2}.$$

We have

$$|D_N(t)| \leq 2N+1,\ 0 \leq F_N(t) \leq N+1,\ \int_0^1 F_N(t)dt = 1.$$

Now let M and N be integers with $M < N$ and define

$$K_{M,N}(k) = \frac{1}{M} \sum_{\substack{|n| \le M-1 \\ |n-k| \le N+M}} \left(1 - \frac{|n|}{M}\right).$$

Then

$$\widehat{K_{M,N}}(t) = \frac{1}{M} D_{N+M}(t) F_{M-1}(t).$$

First, observe that if $|k| \le N$ then

$$K_{M,N}(k) = \frac{1}{M} \sum_{|n| \le M-1} \left(1 - \frac{|n|}{M}\right) = 1.$$

Meanwhile if $|k| \ge N + 2M$ then $K_{M,N}(k) = 0$ since the defining sum is empty. Thus we have (1) and (2).

For (3), we have

$$\int_0^1 |\widehat{K_{M,N}}(t)| dt = 2I_1 + 2I_2 + 2I_3$$

where

$$I_1 = \frac{1}{M} \int_0^{\frac{1}{N+M}} |D_{M+N}(t)| F_{M-1}(t) dt \le 3,$$

$$I_2 = \frac{1}{M} \int_{\frac{1}{N+M}}^{\frac{1}{M}} |D_{M+N}(t)| F_{M-1}(t) dt,$$

and

$$I_3 = \frac{1}{M} \int_{\frac{1}{M}}^{\frac{1}{2}} |D_{M+N}(t)| F_{M-1}(t) dt.$$

Using $2t \le |\sin(\pi t)| \le \pi t$ for $|t| \le \frac{1}{2}$, we have

$$\begin{aligned} I_2 &\le \frac{1}{M^2} \int_{\frac{1}{N+M}}^{\frac{1}{M}} \frac{|\sin(\pi M t)|^2}{|\sin(\pi t)|^3} dt \\ &\le \frac{\pi^2}{8} \int_{\frac{1}{N+M}}^{\frac{1}{M}} \frac{dt}{t} \\ &\le 2 \log(1 + N/M). \end{aligned}$$

Finally,

$$I_3 \le \frac{1}{M^2} \int_{\frac{1}{M}}^{\frac{1}{2}} \frac{1}{t^3} dt \le 1.$$

By Lemma 3.2,

$$\frac{1}{R} \sum_{j=1}^{R} |\widehat{K_{M,N}}(j/R)| \le 2 \|\widehat{K_{M,N}}\|_{L^1([0,1])} \le 4(4 + 2\log(1 + N/M)).$$

$\square$

Lemma 3.4. *Let R a positive integer and $K : \mathbb{Z} \to \mathbb{C}$ be a periodic function with period R. Then*

$$\int_0^1 \left| \sum_m a_m K(m) e(mt) \right| dt \leq \frac{1}{R} \sum_{j=1}^{R} |\widehat{K}(j/R)| \int_0^1 \left| \sum_m a_m e(mt) \right| dt.$$

Proof. By orthogonality of characters modulo R

$$\sum_m a_m K(m) e(mt) = \frac{1}{R} \sum_{j=1}^{R} \widehat{K}(j/R) \sum_m a_m e(m(t + y/R)).$$

So by the triangle inequality,

$$\int_0^1 \left| \sum_m a_m K(m) e(mt) \right| dt \leq \frac{1}{R} \sum_{j=1}^{R} |\widehat{K}(j/R)| \int_0^1 \left| \sum_m a_m e(m(t + y/R)) \right| dt.$$

□

Given a set $I \subseteq \mathbb{Z}$, a positive integer q, and an arbitrary integer s, we define

$$I(q; s) = \{k \in I : k = s \pmod{q}\}.$$

The following lemma is used to amplify the space between the sets A_k.

Lemma 3.5. *Let d_1, d_2 and q be positive integers with $(2 + 2\delta)d_1 + 4 \leq d_2$ for some $\delta > 0$ and $q \geq 4$. Suppose I is a finite set of integers, and let*

$$F(t) = \sum_{k \in I} f_k(t) e(d_2 k t)$$

where each f_k is a trigonometric polynomial of degree at most d_1. Then for any integer s, we have

$$\int_0^1 \left| \sum_{k \in I(q;s)} f_k(t) e(d_2 k t) \right| dt \leq (16 + 8 \log(1 + 2/\delta)) \|F\|_{L^1([0,1])}.$$

Proof. If necessary we may replace I with $I - s$ while preserving the L^1-norm, and so there is no loss of generality in assuming $s = 0$. By definition, we can write

$$F(t) = \sum_m a_m e(mt)$$

where the coefficients a_m are supported on numbers of the form

$$m = d_2 k + l, \ |l| \leq d_1. \tag{3}$$

Let $M = \lceil \delta d_1/2 \rceil$ and $N = d_1$, and let $K_{M,N}$ be the function from Lemma 3.3. The support of $K_{M,N}$ is contained in the interval

$$[-N - 2M, N + 2M] \subseteq [-d_2/2, d_2/2]$$

and $K_{M,N}$ is identically 1 on $\{-d_1, \ldots, d_1\}$. Extend $K_{M,N}$ periodically with period qd_2. Then by Lemma 3.4,

$$\int_0^1 \left| \sum_m a_m K_{M,N}(m) e(mt) \right| dt \leq (16 + 8 \log(1 + N/M)) \|F\|_{L^1([0,1])}.$$

Now $a_m K_{M,N}(m)$ is only non-zero if $m = jqd_2 + l'$ with $-d_2/2 \leq l' \leq d_2/2$. By (3), we have

$$l - l' = d_2(jq - k),$$

and since $|l - l'| \leq d_1 + d_2/2 < d_2$, this can only happen if $k = jq$ and $l = l'$. So we are left with coefficients supported on integers of the form $jqd_2 + l$ with $-d_1 \leq l \leq d_1$. However, $K_{M,N}$ is identically 1 on numbers of the form $jqd_2 + l$ with $-d_1 \leq l \leq d_1$. In summary,

$$\sum_m a_m K_{M,N}(m) e(mt) = \sum_{\substack{k \in I \\ k = 0 \pmod q}} e(kd_2 t) \sum_{|l| \leq d_1} a_{kd_2 + l} e(lt) = \sum_{k \in I(q;0)} f_k(t) e(kd_2 t).$$

□

Finally, in order to apply Lemma 3.5 effectively, we need a good modulus q. Such a modulus is guaranteed by the following lemma.

Lemma 3.6. *Let I be a set integers with $|I| \geq 8$. Then there are positive integers q and s such that*

$$|I|^{1/3}/8 \leq |I(q;s)| \leq q^{1/2}$$

Proof. For each $j \geq 1$, choose any s_j so that $|I(4^j; s_j)|$ is maximal. Then $|I(4^j; s_j)| \geq 4^{-j}|I|$ by the pigeonhole principle. We have

$$|I(4; s_1)| \geq |I|/4 \geq 2$$

while for sufficiently large j we have $|I(4^j; s_j)| = 1 \leq 2^j$, there is a minimal j_0 so that $|I(4^{j_0}; s_{j_0})| \leq 2^{j_0}$. We let $q = 4^{j_0}$, and $s = s_{j_0}$. Then

$$\frac{|I|}{q} \leq |I(q,s)| \leq q^{1/2},$$

so that in particular $|I|^{1/3} \leq q^{1/2}$. By minimality of j_0

$$q^{1/2}/2 = 2^{j_0 - 1} \leq |I(q/4; s_{j_0 - 1})| \leq 4I(q;s),$$

so that $|I(q;s)| \geq |I|^{1/3}/8$. □

4. Multidimensional subsets of $\mathbb{Z}$

The following proposition can be viewed as a sort of analog to Theorem 1.2.

Proposition 4.1. *Let d_1, d_2 positive integers with $(2+\delta)d_1 < d_2$. Suppose I is a finite set of integers, and let*

$$F(t) = \sum_{k\in I} f_k(t)e(d_2kt)$$

where

$$f_k(t) = \sum_{|n|\leq d_1} a_{n,k}e(nt).$$

Let q and s an integers with $q \geq 2$ and suppose

$$I(q;s) = \{k_1 < \ldots < k_J\}.$$

Then we have have

$$\|F\|_{L^1([0,1])} \geq \frac{1}{16+8\log(1+2/\delta)}\sum_{j=1}^{J}\|f_{k_j}\|_{L^1([0,1])}\left(\frac{C_{MPS}}{2j} - \frac{2d_1}{qd_2}\right).$$

Proof. By the Lemma 3.5, we have

$$\|F\|_{L^1([0,1])} \geq \frac{1}{16+8\log(1+2/\delta)}\int_0^1\left|\sum_{j=1}^{J} f_{k_j}(t)e(k_jd_2t)\right| dt.$$

Next we write each $k_j \in I(q;s)$ as $k_j = b_{k_j}q + s$, so the above integral becomes

$$\begin{aligned}\int_0^1\left|\sum_{j=1}^{J} f_{k_j}(t)e((b_{k_j}q+s)d_2t)\right| dt &= \int_0^1\left|\sum_{j=1}^{J} f_{k_j}(t)e(b_{k_j}qd_2t)\right| dt\\ &= \frac{1}{qd_2}\int_0^{qd_2}\left|\sum_{j=1}^{J} f_{k_j}(u/qd_2)e(b_{k_j}u)\right| du.\end{aligned}$$

after the change of variables $u = qd_2t$. Next, by breaking the integral into intervals of unit length, we get

$$\frac{1}{qd_2}\sum_{m=1}^{qd_2}\int_{m-1}^{m}\left|\sum_{j=1}^{J} f_{k_j}(u/qd_2)e(b_{k_j}u)\right| du \geq T_1 - T_2$$

where

$$T_1 = \frac{1}{qd_2}\sum_{m=1}^{qd_2}\int_{m-1}^{m}\left|\sum_{j=1}^{J} f_{k_j}((m-1)/qd_2)e(b_{k_j}u)\right| du$$

and

$$T_2 = \frac{1}{qd_2}\sum_{m=1}^{qd_2}\int_{m-1}^{m}\left|\sum_{j=1}^{J}(f_{k_j}(u/qd_2) - f_{k_j}((m-1)/qd_2))e(b_{k_j}u)\right| du.$$

By periodicity followed by Theorem 1.1, we have the estimate

$$T_1 = \frac{1}{qd_2}\sum_{m=1}^{qd_2}\int_0^1 \left|\sum_{j=1}^{J} f_{k_j}((m-1)/qd_2)e(b_{k_j}u)\right| du \geq \frac{C_{MPS}}{qd_2}\sum_{m=1}^{qd_2}\sum_{j=1}^{J}\frac{|f_{k_j}((m-1)/qd_2)|}{j}.$$

An application of Lemma 3.2, bearing in mind $qd_2 > 2d_1$, yields

$$\frac{1}{qd_2}\sum_{m=1}^{qd_2}|f_{k_j}((m-1)/qd_2)| \geq \frac{1}{2}\|f_{k_j}\|_{L^1([0,1])}.$$

To complete the proof of the proposition it suffices to estimate T_2 appropriately. For u in the interval $m-1 \leq u \leq m$ we have

$$\begin{aligned}\left|\sum_{j=1}^{J}(f_{k_j}(u/qd_2) - f_{k_j}((m-1)/qd_2))e(b_{k_j}u)\right| &= \left|\sum_{j=1}^{J}\int_{\frac{m-1}{qd_2}}^{\frac{u}{qd_2}} f'_{k_j}(v)e(b_{k_j}u)dv\right| \\ &\leq \int_{\frac{m-1}{qd_2}}^{\frac{m}{qd_2}}\left|\sum_{j=1}^{J} f'_{k_j}(v)e(b_{k_j}u)\right| dv\end{aligned}$$

by the triangle inequality and positivity. Thus we get the upper bound

$$\begin{aligned}T_2 &\leq \frac{1}{qd_2}\sum_{m=1}^{qd_2}\int_{\frac{m-1}{qd_2}}^{\frac{m}{qd_2}}\int_{m-1}^{m}\left|\sum_{j=1}^{J} e(b_{k_j}u)f'_{k_j}(v)\right| dudv \\ &= \frac{1}{qd_2}\int_0^1\int_0^1\left|\sum_{j=1}^{J} e(b_{k_j}u)f'_{k_j}(v)\right| dudv \\ &\leq \frac{1}{qd_2}\sum_{j=1}^{J}\int_0^1\left|f'_{k_j}(v)\right| dv.\end{aligned}$$

By Lemma 3.1, we obtain

$$T_2 \leq \frac{2d_1}{qd_2}\sum_{j=1}^{J}\|f_{k_j}\|_{L^1([0,1])}.$$

□

Proof of Theorem 1.4. As in the proof of 1.3, we proceed by induction on r. When $r = 1$, this follows immediately from Theorem 1.1.

For the inductive step, we begin by writing A as

$$A = \bigcup_{k\in I} A_k + kd_2$$

for some $(n_2, \ldots, n_r)$-strongly regular sets $A_k \subseteq \{-d_1, \ldots, d_1\}$, where d_1 and d_2 with $16 < 4d_1 < d_2$. If we let

$$f_k(t) = \sum_{a\in A_k} e(at)$$

then

$$\sum_{a\in A} e(at) = \sum_{k\in I} f_k(t)e(d_2kt)$$

is of the necessary form to apply Proposition 4.1. If it is the case that

$$\|f_k\|_{L^1([0,1])} \geq C_{\delta_2,\ldots,\delta_{r-1}} \log(n_1)\cdots\log(n_r)$$

for some k, then we can choose q so large and s in such a way that $I(q;s) = \{k\}$. This gives that

$$\|F\|_{L^1([0,1])} \geq \frac{C_{MPS}}{64 + 32\log(1 + 2/\delta_1))}\|f_k\|_{L^1([0,1])},$$

yielding the theorem immediately. Thus there is no loss of generality in assuming

$$\|f_k\|_{L^1([0,1])} \leq C_{\delta_2,\ldots,\delta_{r-1}} \log(n_1)\cdots\log(n_r) \tag{4}$$

for each $k \in I$.

Now we choose q and s as in Lemma 3.6 applied to I to get

$$I(q;s) = \{k_1 < \ldots < k_J\}$$

satisfying

$$\frac{n_1^{1/3}}{8} \leq J \leq q^{1/2}.$$

Then

$$\int_0^1 \left|\sum_{a\in A} e(at)\right| dt \geq \frac{1}{2^4(1 + \log(1 + 2/\delta))} \sum_{j=1}^{J} \|f_{k_j}\|_{L^1([0,1])} \left(\frac{C_{MPS}}{2j} - \frac{2d_1}{qd_2}\right). \tag{5}$$

By induction,

$$\sum_{j=1}^{J} \frac{\|f_{k_j}\|_{L^1([0,1])}}{j} \geq C_{\delta_2,\ldots,\delta_{r-1}} \log(J)\log(n_2)\cdots\log(n_r) \geq \frac{1}{4}C_{\delta_2,\ldots,\delta_{r-1}} \log(n_1)\cdots\log(n_r).$$

By (4), the error term in (5) is at most

$$\frac{2Jd_1}{qd_2}C_{\delta_2,\ldots,\delta_{r-1}} \log(n_1)\cdots\log(n_r) \leq C_{\delta_2,\ldots,\delta_{r-1}} \frac{\log(n_1)\cdots\log(n_r)}{q^{1/2}}.$$

As guaranteed by Lemma 3.6 and the hypotheses of the theorem,

$$q^{1/2} \geq n_1^{1/3}/8 \geq 16C_{MPS}\log(n_1)\cdots\log(n_r).$$

So, we have shown

$$\begin{aligned}\int_0^1 \left|\sum_{a\in A} e(at)\right| dt &\geq \frac{1}{2^4(1 + \log(1 + 2/\delta))}\frac{C_{\delta_2,\ldots,\delta_{r-1}}}{2^4} \log(n_1)\cdots\log(n_r) \\ &= C_{\delta_1,\ldots,\delta_{r-1}} \log(n_1)\cdots\log(n_r).\end{aligned}$$

□

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