1) $\cdot\ B = S^{-1} A S$

$\cdot\ B\underline{x} = \lambda \underline{x}$

$\Leftrightarrow S^{-1} A S \underline{x} = \lambda \underline{x}$

$\Leftrightarrow A S \underline{x} = S \lambda \underline{x}$

$\Leftrightarrow A(S\underline{x}) = \lambda (S\underline{x})$

2) $\langle \sin x, \cos x \rangle$

$= \int_0^{\pi} \sin(x) \cos(x)\, dx$

let $u = \sin(x)$

$du = \cos(x)\, dx$

$= \int u\, du$

$= \frac{1}{2} [u^2]$

$= \frac{1}{2} [\sin^2(x)]_0^{\pi}$

$= \frac{1}{2} (\sin^2(\pi) - \sin^2(0))$

$= 0$ (orthogonal)

3) $\|a_4\| = \sqrt{a^2+b^2+c^2+d^2} = 1$

④ $\Leftrightarrow a^2+b^2+c^2+d^2 = 1$

$\cdot\ a_1 \cdot a_4 = 0 = \frac{1}{2}a + \frac{1}{2}b + \frac{1}{2}c + \frac{1}{2}d$

① $\Leftrightarrow 0 = a+b+c+d$

$\cdot\ a_2 \cdot a_4 = 0 = \frac{1}{2}a - \frac{1}{2}b + \frac{1}{2}c - \frac{1}{2}d$

② $\Leftrightarrow 0 = a-b+c-d$

$\cdot\ a_3 \cdot a_4 = 0 = \frac{1}{\sqrt{2}}a - \frac{1}{\sqrt{2}}c$

③ $0 = a-c$

from ① & ② $0 = 2a+2c$

$\Leftrightarrow \boxed{0 = a+c}$

from ③ $0 = a-c$ +

$0 = 2a$

$\boxed{\begin{matrix} a=0 \\ c=0 \end{matrix}}$

from ④ $a^2+b^2+c^2+d^2=1$

$\Leftrightarrow \boxed{b^2+d^2=1}$

from ① & ② $b+d=0$

$\Leftrightarrow b=-d$

$2d^2 = 1$

$d^2 = \frac{1}{\sqrt{2}}$

$d = \pm \frac{1}{\sqrt{2}}$

$\therefore (a,b,c,d) = (0, \frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}})$

or $(0, -\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}})$

4) if $\theta \neq k\pi$, $k \in \mathbb{Z}$

$\Rightarrow \cos\theta \neq 1$, $\cos\theta \neq -1$

$\Rightarrow \sin\theta \neq 0$

$\det(A - \lambda I) = 0$

$\hookrightarrow \left| \begin{bmatrix} \cos\theta - \lambda & -\sin\theta \\ \sin\theta & \cos\theta - \lambda \end{bmatrix} \right| = 0$

$\Leftrightarrow (\cos\theta - \lambda)^2 + \sin^2\theta = 0$

$\Leftrightarrow \cos^2\theta + \lambda^2 - 2\lambda\cos\theta + \sin^2\theta = 0$

$\Leftrightarrow \lambda^2 - (2\cos\theta)\lambda + 1 = 0$

$$\lambda_{1,2} = \frac{2\cos\theta \pm \sqrt{4\cos^2\theta - 4}}{2}$$

$$= \frac{2\cos\theta \pm \sqrt{-4\sin^2\theta}}{2}$$

gonna be on imaginary part since $\sin\theta \neq 0$ and $\sin^2\theta \geq 0$

$$\cancel{= \cos\theta \pm \sin\theta}$$

$$= \cos\theta \pm i\sin\theta$$

$A\underline{x} = \lambda\underline{x}$

6). $Q^T Q = I$

$\cdot\ Q^T = Q^{-1}$

$$(Q^m)^{-1} = (Q^m)^T$$

$$= (\underbrace{Q Q \dots Q}_{m \text{ times}})^T = (Q^T)^m$$

7) $\overline{\alpha+\beta} = \overline{(re(\alpha)+im(\alpha)+re(\beta)+im(\beta))}$

$= \overline{(re(\alpha+\beta)+im(\alpha+\beta))}$

$= re(\alpha+\beta) - im(\alpha+\beta)$

$= [re(\alpha) - im(\alpha)] + [re(\beta) - im(\beta)]$

$= \bar{\alpha} + \bar{\beta}$

$\overline{\alpha\beta} = \overline{(re(\alpha)+im(\alpha))+(re(\beta)+im(\beta))}$

$= \overline{re(\alpha\beta) \cancel{+} \overset{rel}{\cancel{im}}(\alpha\beta) + im(\alpha\beta) + im(\alpha\beta)}$

$= -2\, im(\alpha\beta)$

Claim: $\overline{\alpha\beta} = \bar{\alpha}\bar{\beta}$

LHS $= \overline{(re(\alpha)+im(\alpha))+(re(\beta)+im(\beta))}$

$= \overline{(re(\alpha\beta) - re(\alpha\beta) + 2im(\alpha\beta))}$

$= -2\, im(\alpha\beta)$

RHS $= (re(\alpha) - im(\alpha))(re(\beta) - im(\beta))$

$= re(\alpha\beta) - re(\alpha\beta) - 2im(\alpha\beta)$

$= -2\, im(\alpha\beta)$

claim $\overline{AB} = \bar{A}\ \bar{B}$

~~LHS = $(\overline{ab}_{ij})$~~

$$\overline{AB}\ \cancel{\text{LHS}} = \overline{\left(\sum_{k=1}^{n} a_{ik} b_{kj}\right)}$$

$$= \left(\sum_{k=1}^{n} \overline{a_{ik} b_{kj}}\right)$$

$$= \sum_{k=1}^{n} \overline{a_{ik}}\ \overline{b_{kj}} = \bar{A}\ \bar{B}$$

5) a) $Qx = \lambda x$

$\Leftrightarrow \|Qx\|_2 = \|x\|_2$

$\Leftrightarrow \|\lambda x\| = \|x\|_2$

$\Leftrightarrow |\lambda| = 1$

b) $Q^TQ = 1$

$\Leftrightarrow \det(Q^TQ) = I$

$\Leftrightarrow \det(Q^T)\det(Q) = 1$

$\Leftrightarrow |\det Q| = 1$

8) P_A & P_B as polynomial of A and B

$P_B(t) = \det(B - tI) = \det(S^{-1}AS - tI)$

$= \det(S^{-1}(A - tI)S) = \det(S^{-1})$

$= \det(A - tI) = PA(t)$

Since both are equal, they will have the same eigen value and characteristic polynomial

9) a) $Col(H) = span\left(\begin{bmatrix} 0.6 \\ 0.3 \\ 0.6 \end{bmatrix}, \begin{bmatrix} -0.033 \\ 0.133 \\ -0.033 \end{bmatrix}\right)$

b) $\begin{bmatrix} 0.6 & 0.3 \\ 0.3 & 0.3 \\ 0.6 & 0.3 \end{bmatrix} = \underbrace{\begin{bmatrix} -0.033 \\ 0.133 \\ -0.033 \end{bmatrix}}_{Q} \underbrace{\begin{bmatrix} 0.6 & 0.3 \\ 0 & 0.15 \\ 0 & 0 \end{bmatrix}}_{R}$

$\downarrow$

$\begin{bmatrix} 0.6 & 0.3 \\ 0 & 0.15 \\ 0 & 0 \end{bmatrix}$ upper triangular

c) $y = Hx$

$\begin{pmatrix} -0.4 \\ -0.3 \\ -0.9 \end{pmatrix} = \begin{pmatrix} 0.6 & 0.3 \\ 0.3 & 0.3 \\ 0.6 & 0.3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$x = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ //