CMSC 207- Lecture 18
CHAPTER 7: Functions (7.1 & 7.2)

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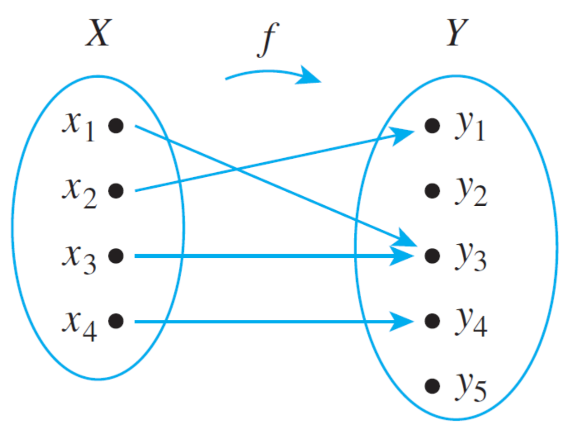
Arrow Diagrams

If X and Y are finite sets, you can define a function f from X to Y by drawing an arrow diagram.

You make a list of elements in X and a list of elements in Y, and draw an arrow from each element in X to the corresponding element in Y, as shown



Arrow Diagrams

This arrow diagram defines a function, because:

1. Every element of X has an arrow coming out of it.
2. No element of X has two arrows coming out of it that point to two different elements of Y.

Example 2 – *A Function Defined by an Arrow Diagram*

Let $X = \{a, b, c\}$ and $Y = \{1, 2, 3, 4\}$. Define a function f from X to Y by the arrow diagram in Figure 7.1.3.

a. Write the domain and co-domain of f.

b. Find $f(a)$, $f(b)$, *and* $f(c)$.

c. What is the range of f?

d. Is c an inverse image of 2?
Is b an inverse image of 3?

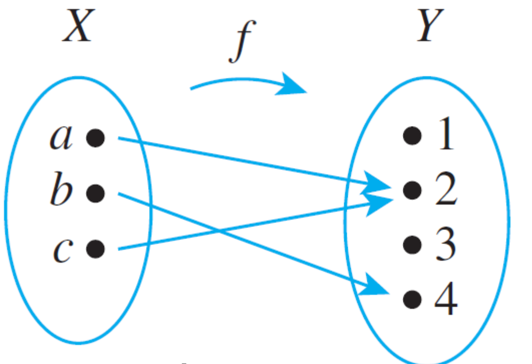


Figure 7.1.3

e. Find the inverse images of 2, 4, and 1.

f. Represent f as a set of ordered pairs.

Example 2 – *Solution*

a. domain of $f = \{a, b, c\}$, co-domain of $f = \{1, 2, 3, 4\}$

b. $f(a) = 2,\ f(b) = 4,\ f(c) = 2$

c. range of $f = \{2, 4\}$

d. Yes, No

e. inverse image $2 = \{a, c\}$

inverse image of $4 = \{b\}$

inverse image of $1 = \emptyset$ (*since no arrows point to* 1)

f. $\{(a, 2), (b, 4), (c, 2)\}$

Arrow Diagrams

Theorem 7.1.1 A Test for Function Equality

If $F: X \to Y$ and $G: X \to Y$ are functions, then $F = G$ if, and only if, $F(x) = G(x)$ for all $x \in X$.

Example 3 – *Equality of Functions*

a. Let $J_3 = \{0, 1, 2\}$, and define functions f and g from J_3 to J_3 as follows: For all x in J_3,

$$f(x) = (x^2 + x + 1)\ mod\ 3 \quad \text{and} \quad g(x) = (x+2)^2\ mod\ 3.$$

Does $f = g$?

b. Let F: **R** → **R** and G: **R** → **R** be functions. Define new functions $F + G$: **R** → **R** and $G + F$: **R** → **R** as follows: For all $x \in$ **R**,

$$(F+G)(x) = F(x) + G(x) \quad \text{and} \quad (G+F)(x) = G(x) + F(x).$$

Does $F + G = G + F$?

Example 3 – *Solution*

a. Yes, the table of values shows that $f(x) = g(x)$ for all x in J_3.

x	$x^2 + x + 1$	$f(x) = (x^2 + x + 1)$ *mod* 3	$(x + 2)^2$	$g(x) = (x + 2)^2$ *mod* 3
0	1	$1 \ mod \ 3 = 1$	4	$4 \ mod \ 3 = 1$
1	3	$3 \ mod \ 3 = 0$	9	$9 \ mod \ 3 = 0$
2	7	$7 \ mod \ 3 = 1$	16	$16 \ mod \ 3 = 1$

b. Again the answer is yes. For all real numbers x,

$$(F + G)(x) = F(x) + G(x)$$ by definition of $F + G$

$$= G(x) + F(x)$$ by the commutative law for addition of real numbers

$$= (G + F)(x)$$ by definition of $G + F$

Hence $F + G = G + F$.

Example 4 – *The Identity Function on a Set*

Given a set X, define a function I_X from X to X by

$$I_X(x) = x$$

for all x in X.

The function I_X is called the **identity function on *X*** because it sends each element of X to the element that is identical to it. Thus the identity function can be pictured as a machine that sends each piece of input directly to the output chute without changing it in any way.

Examples of Functions

• Definition Logarithms and Logarithmic Functions

Let b be a positive real number with $b \neq 1$. For each positive real number x, the **logarithm with base b of x,** written $\log_b x$, is the exponent to which b must be raised to obtain x. Symbolically,

$$\log_b x = y \quad \Leftrightarrow \quad b^y = x.$$

The **logarithmic function with base b** is the function from $\mathbf{R}^+$ to $\mathbf{R}$ that takes each positive real number x to $\log_b x$.

Examples of Functions

We have known that if *S* is a nonempty, finite set of characters, then a **string over *S*** is a finite sequence of elements of *S.*

The number of characters in a string is called the **length** of the string. The **null string over *S*** is the "string" with no characters.

It is usually denoted $\in$ and is said to have length 0.

Boolean Functions

• Definition

An **(*n*-place) Boolean function** f is a function whose domain is the set of all ordered n-tuples of 0's and 1's and whose co-domain is the set $\{0, 1\}$. More formally, the domain of a Boolean function can be described as the Cartesian product of n copies of the set $\{0, 1\}$, which is denoted $\{0, 1\}^n$. Thus $f: \{0, 1\}^n \to \{0, 1\}$.

In-class Assignment #1

Functions Acting on Sets

Given a function from a set *X* to a set *Y*, you can consider the set of images in *Y* of all the elements in a subset of *X* and the set of inverse images in *X* of all the elements in a subset of *Y*.

• Definition

If $f: X \to Y$ is a function and $A \subseteq X$ and $C \subseteq Y$, then

$$f(A) = \{y \in Y \mid y = f(x) \text{ for some } x \text{ in } A\}$$

and

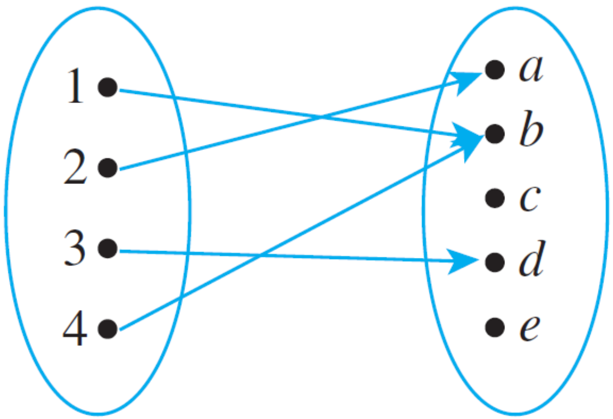
$$f^{-1}(C) = \{x \in X \mid f(x) \in C\}.$$

$f(A)$ is called the **image of A**, and $f^{-1}(C)$ is called the **inverse image of C**.

Example 13 – *The Action of a Function on Subsets of a Set*

Let $X = \{1, 2, 3, 4\}$ and $Y = \{a, b, c, d, e\}$, and define:

$F : X \to Y$ by the following arrow diagram:



Let $A = \{1, 4\}$, $C = \{a, b\}$, and $D = \{c, e\}$. Find $F(A)$, $F(X)$, $F^{-1}(C)$, and $F^{-1}(D)$.

Example 13 – *Solution*

$$F(A) = \{b\}$$

$$F(X) = \{a, b, d\}$$

$$F^{-1}(C) = \{1, 2, 4\}$$

$$F^{-1}(D) = \emptyset$$

One-to-One and Onto, Inverse Functions

Two important properties that functions may satisfy are: the property of being ***one-to-one*** and the property of being ***onto***.

Functions that satisfy both properties are called ***one-to-one correspondences*** or ***one-to-one onto*** *functions*.

When a function is a one-to-one correspondence, the elements of its domain and co-domain match up perfectly, and we can define an *inverse function* from the co-domain to the domain that "undoes" the action of the function.

One-to-One Functions

A function may send several elements of its domain to the same element of its co-domain.

In terms of arrow diagrams, this means that two or more arrows that start in the domain can point to the same element in the co-domain.

On the other hand, if no two arrows that start in the domain point to the same element of the co-domain then the function is called ***one-to-one*** or ***injective***.

One-to-One Functions

For a one-to-one function, each element of the range is the image of at most one element of the domain.

• **Definition**

Let F be a function from a set X to a set Y. F is **one-to-one** (or **injective**) if, and only if, for all elements x_1 and x_2 in X,

$$\text{if } F(x_1) = F(x_2), \text{ then } x_1 = x_2,$$

or, equivalently,

$$\text{if } x_1 \neq x_2, \text{ then } F(x_1) \neq F(x_2).$$

Symbolically,

$$F\colon X \to Y \text{ is one-to-one} \quad \Leftrightarrow \quad \forall x_1, x_2 \in X, \text{if } F(x_1) = F(x_2) \text{ then } x_1 = x_2.$$

To obtain a precise statement of what it means for a function *not* to be one-to-one, take the negation of one of the equivalent versions of the definition above.

One-to-One Functions

Thus:

A function $F: X \to Y$ is *not* one-to-one $\Leftrightarrow$ $\exists$ elements x_1 and x_2 in X with $F(x_1) = F(x_2)$ and $x_1 \neq x_2$.

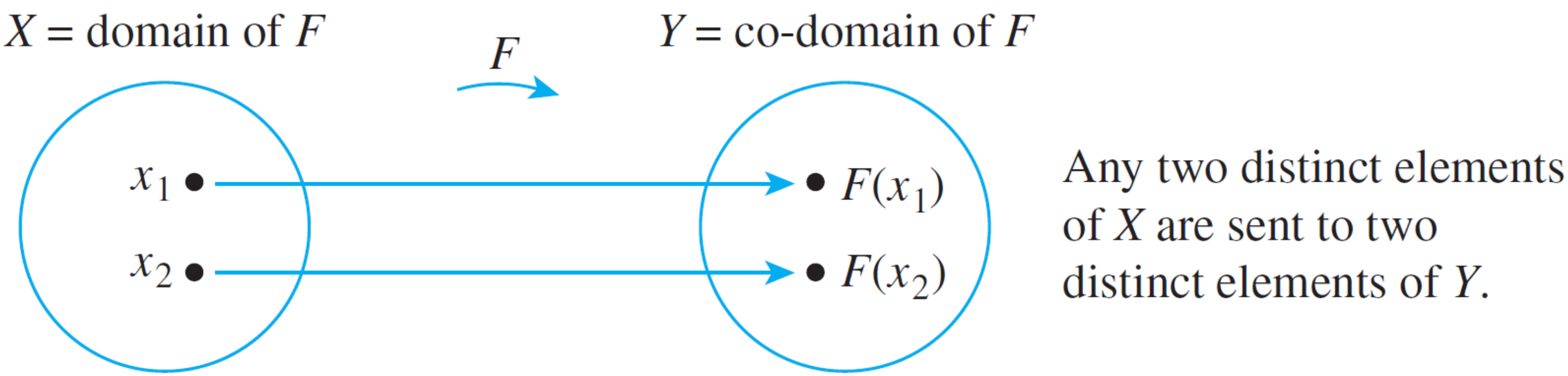
That is, if elements x_1 and x_2 can be found that have the same function value but are not equal, then F is not one-to-one.

One-to-One Functions

In terms of arrow diagrams, a one-to-one function can be thought of as a function that separates points. That is, it takes distinct points of the domain to distinct points of the co-domain.

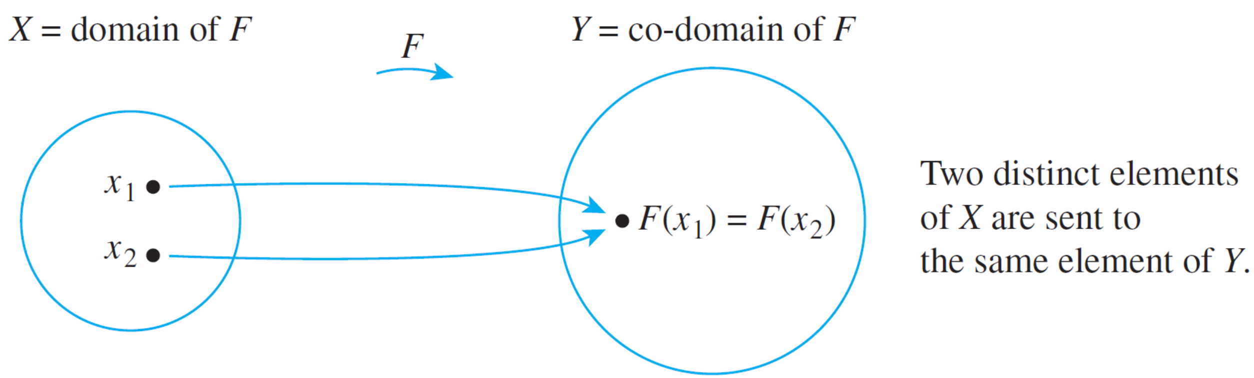
A function that is not one-to-one fails to separate points. That is, at least two points of the domain are taken to the same point of the co-domain.

One-to-One Functions



A One-to-One Function Separates Points

Figure 7.2.1 (a)



A Function That Is Not One-to-One Collapses Points Together

Figure 7.2.1 (b)

One-to-One Functions on Infinite Sets

Now suppose f is a function defined on an infinite set X. By definition, f is one-to-one if, and only if, the following universal statement is true:

$$\forall x_1, x_2 \in X, \text{ if } f(x_1) = f(x_2) \text{ then } x_1 = x_2.$$

Thus, to prove f is one-to-one, you will generally use the method of direct proof: **suppose** x_1 and x_2 are elements of X such that

$f(x_1) = f(x_2)$ and **show** that $x_1 = x_2$.

One-to-One Functions on Infinite Sets

To show that f is *not* one-to-one, you will ordinarily **find** elements x_1 and x_2 in X so that $f(x_1) = f(x_2)$ but $x_1 \neq x_2$.

In-class Assignment #2

Onto Functions

We have noted that there may be an element of the co-domain of a function that is not the image of any element in the domain.

On the other hand, *every* element of a function's co-domain may be the image of some element of its domain. Such a function is called ***onto*** or ***surjective***. When a function is onto, its range is equal to its co-domain.

- **Definition**

Let F be a function from a set X to a set Y. F is **onto** (or **surjective**) if, and only if, given any element y in Y, it is possible to find an element x in X with the property that $y = F(x)$.

Symbolically:

$$F\colon X \to Y \text{ is onto} \quad \Leftrightarrow \quad \forall y \in Y, \exists x \in X \text{ such that } F(x) = y.$$

Onto Functions

To obtain a precise statement of what it means for a function ***not*** to be onto, take the negation of the definition of onto:

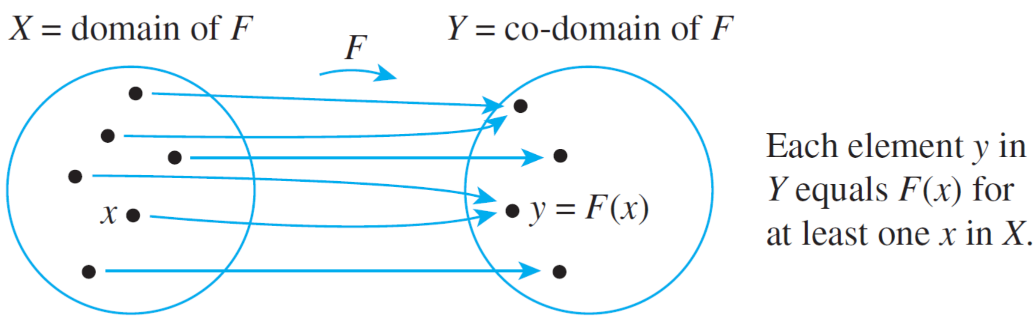
$$F: X \to Y \text{ is } \textit{not} \text{ onto} \quad \Leftrightarrow \quad \exists y \text{ in } Y \text{ such that } \forall x \in X, F(x) \neq y.$$

There is some element in *Y* that is *not* the image of *any* element in *X*. In terms of arrow diagrams, a function is onto if each element of the co-domain has an arrow pointing to it from some element of the domain.

A function is not onto if at least one element in its co-domain does not have an arrow pointing to it.

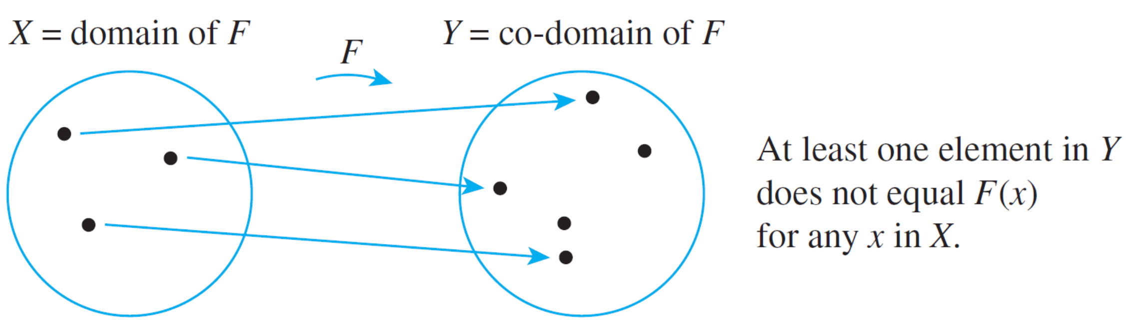
Onto Functions

This is illustrated in Figure 7.2.3.



A Function That Is Onto

Figure 7.2.3 (a)



A Function That Is Not Onto **Figure 7.2.3 (b)**

Equivalently, for each positive real number *x* and real number *y*, $\log_b x = y \quad \Leftrightarrow \quad b^y = x.$

It can be shown using calculus that both the exponential and logarithmic functions are one-to-one and onto.

Therefore, by definition of one-to-one, the following properties hold true:

For any positive real number b with $b \neq 1$,

$$\text{if } b^u = b^v \text{ then } u = v \quad \text{for all real numbers } u \text{ and } v, \tag{7.2.5}$$

and

$$\text{if } \log_b u = \log_b v \text{ then } u = v \quad \text{for all positive real numbers } u \text{ and } v. \tag{7.2.6}$$

One-to-One Correspondences

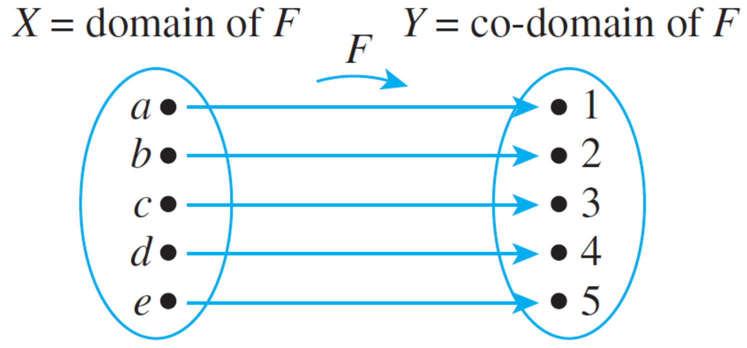
Consider a function $F: X \rightarrow Y$ that is both one-to-one and onto. Given any element x in X, there is a unique corresponding element $y = F(x)$ in Y (since F is a function).

Also given any element y in Y, there is an element x in X such that $F(x) = y$ (since F is onto) and there is only one such x (since F is one-to-one).

One-to-One Correspondences

Thus, a function that is one-to-one and onto sets up a pairing between the elements of *X* and the elements of *Y* that matches each element of *X* with exactly one element of *Y* and each element of *Y* with exactly one element of *X*.

Such a pairing is called a *one-to-one correspondence* or *bijection* and is illustrated by the arrow diagram in Figure 7.2.5.



An Arrow Diagram for a One-to-One Correspondence

Figure 7.2.5

One-to-One Correspondences

One-to-one correspondences are often used as aids to counting.

• **Definition**

A **one-to-one correspondence** (or **bijection**) from a set X to a set Y is a function $F: X \to Y$ that is both one-to-one and onto.

Example 10 – *A Function of Two Variables*

Define a function:

F: $\mathbf{R} \times \mathbf{R} \to \mathbf{R} \times \mathbf{R}$ as follows: For all $(x, y) \in \mathbf{R} \times \mathbf{R}$,

$$F(x, y) = (x + y, x - y).$$

Is *F* a one-to-one correspondence from $\mathbf{R} \times \mathbf{R}$ to itself?

Solution:

The answer is yes. To show that *F* is a one-to-one correspondence, you need to show both that *F* is one-to-one and that *F* is onto.

Example 10 – *Solution*

Proof that *F* is one-to-one:

Suppose that (x_1, y_1) and (x_2, y_2) are any ordered pairs in $\mathbf{R} \times \mathbf{R}$ such that

$$(x_1 + y_1, x_1 - y_1) = (x_2 + y_2, x_2 - y_2).$$

[We must show that $(x_1, y_1) = (x_2, y_2)$*.]* By definition of *F*, $F(x_1, y_1) = F(x_2, y_2).$

For two ordered pairs to be equal, both the first and second components must be equal. Thus x_1, y_1, x_2, and y_2 satisfy the following system of equations:

$$x_1 + y_1 = x_2 + y_2 \qquad (1)$$

$$x_1 - y_1 = x_2 - y_2 \qquad (2)$$

Example 10 – *Solution*

Adding equations (1) and (2) gives that

$$2x_1 = 2x_2, \quad \text{and so} \quad x_1 = x_2.$$

Substituting $x_1 = x_2$ into equation (1) yields

$$x_1 + y_1 = x_1 + y_2, \quad \text{and so} \quad y_1 = y_2.$$

Thus, by definition of equality of ordered pairs, $(x_1, y_1) = (x_2, y_2)$. *[as was to be shown].*

Inverse Functions

If *F* is a one-to-one correspondence from a set *X* to a set *Y*, then there is a function from *Y* to *X* that "undoes" the action of *F*; that is, it sends each element of *Y* back to the element of *X* that it came from. This function is called the *inverse function* for *F*.

Theorem 7.2.2

Suppose $F: X \to Y$ is a one-to-one correspondence; that is, suppose F is one-to-one and onto. Then there is a function $\boldsymbol{F^{-1}: Y \to X}$ that is defined as follows:

Given any element y in Y,

$$\boldsymbol{F^{-1}(y)} = \text{that unique element } x \text{ in } X \text{ such that } F(x) \text{ equals } y.$$

In other words,

$$F^{-1}(y) = x \quad \Leftrightarrow \quad y = F(x).$$

Inverse Functions

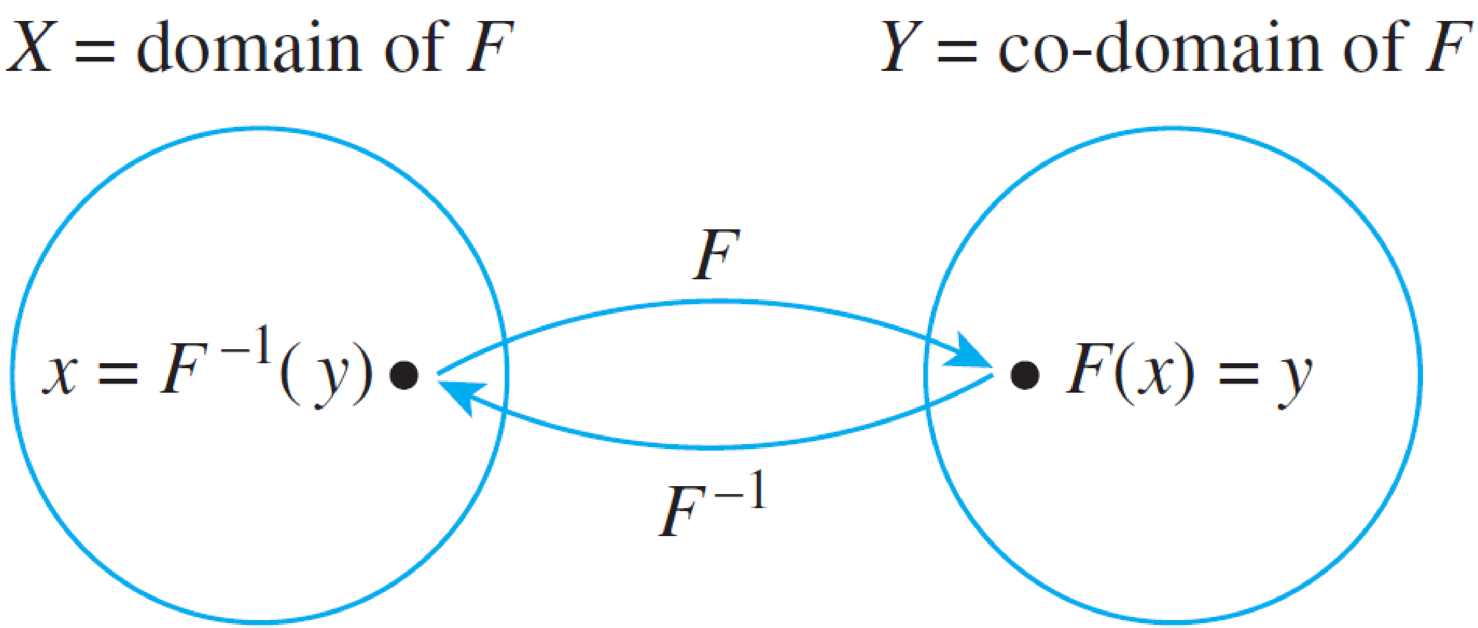
Given an element y in Y, there is an element x in X with $F(x) = y$ because F is onto; x is unique because F is one-to-one.

• Definition

The function F^{-1} of Theorem 7.2.2 is called the **inverse function** for F.

Inverse Functions

The diagram that follows illustrates the fact that an inverse function sends each element back to where it came from.



Example 13 – *Finding an Inverse Function for a Function Given by a Formula*

The function f: **R** → **R** defined by the formula

$$f(x) = 4x - 1 \quad \text{for all real numbers } x$$

Find its inverse function.

Solution:

For any *[particular but arbitrarily chosen] y* in **R**, by definition of f^{-1}, $f^{-1}(y)$ = that unique real number x such that $f(x) = y$.

Example 13 – *Solution*

But

$$
\begin{aligned}
& f(x) = y \\
\Leftrightarrow \quad & 4x - 1 = y && \text{by definition of } f \\
\Leftrightarrow \quad & x = \frac{y+1}{4} && \text{by algebra.}
\end{aligned}
$$

Hence $$f^{-1}(y) = \frac{y+1}{4}.$$

Inverse Functions

The following theorem follows easily from the definitions.

Theorem 7.2.3

If X and Y are sets and $F\colon X \to Y$ is one-to-one and onto, then $F^{-1}\colon Y \to X$ is also one-to-one and onto.