CMSC 207- Lecture 26
CHAPTER 9: Counting and Probability (9.5 & 9.6)

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Counting Subsets of a Set: Combinations

- Given a set S with n elements, how many subsets of size r can be chosen from S?
- The number of subsets of size r that can be chosen from S equals the number of subsets of size r that S has.
- Each individual subset of size r is called an ***r*-*combination*** of the set.

Counting Subsets of a Set: Combinations

• Definition

Let n and r be nonnegative integers with $r \leq n$. An ***r*-combination** of a set of n elements is a subset of r of the n elements. As indicated in Section 5.1, the symbol

$$\binom{n}{r},$$

which is read "n choose r," denotes the number of subsets of size r (r-combinations) that can be chosen from a set of n elements.

- We have known that calculators generally use symbols like $C(n, r)$, ${}_nC_r$, $C_{n,r}$, or ${}^nC_r$ instead of $\binom{n}{r}$.

Example 1 – 3-*Combinations*

- Let S = {Ann, Bob, Cyd, Dan}. Each committee consisting of three of the four people in S is a 3-combination of S.
- **a.** List all such 3-combinations of S.
- **b.** What is $\binom{4}{3}$?
- Solution:
- **a.** Each 3-combination of S is a subset of S of size 3. But each subset of size 3 can be obtained by leaving out one of the elements of S. The 3-combinations are {Bob, Cyd, Dan} leave out Ann

Example 1 – *Solution*

- {Ann, Cyd, Dan} leave out Bob
- {Ann, Bob, Dan} leave out Cyd
- {Ann, Bob, Cyd} leave out Dan.
- **b.** Because $\binom{4}{3}$ is the number of 3-combinations of a set with four elements, by part (**a**), $\binom{4}{3} = 4$.

Counting Subsets of a Set: Combinations

- There are two distinct methods that can be used to select *r* objects from a set of *n* elements. In an **ordered selection**, it is not only what elements are chosen but also the order in which they are chosen that matters.
- Two ordered selections are said to be the same if the elements chosen are the same and also if the elements are chosen in the same order. **An ordered selection of *r* elements from a set of *n* elements is an *r*-permutation of the set**.

Counting Subsets of a Set: Combinations

- In an **unordered selection**, on the other hand, it is only the identity of the chosen elements that matters. Two unordered selections are said to be the same if they consist of the same elements, regardless of the order in which the elements are chosen.
- An unordered selection of r elements from a set of n elements is the same as an ***r*-combination of the set**.

Counting Subsets of a Set: Combinations

Theorem 9.5.1

The number of subsets of size r (or r-combinations) that can be chosen from a set of n elements, $\binom{n}{r}$, is given by the formula

$$\binom{n}{r} = \frac{P(n, r)}{r!} \qquad \text{first version}$$

or, equivalently,

$$\binom{n}{r} = \frac{n!}{r!(n-r)!} \qquad \text{second version}$$

where n and r are nonnegative integers with $r \leq n$.

Example 4 – *Calculating the Number of Teams*

- Consider again the problem of choosing five members from a group of twelve to work as a team on a special project. How many distinct five-person teams can be chosen?

- Solution:

The number of distinct five-person teams is the same as the number of subsets of size 5 (or 5-combinations) that can be chosen from the set of twelve. This number is $\binom{12}{5}$.

$$\binom{12}{5} = \frac{12!}{5!(12-5)} = \frac{1\cancel{2}\cdot 11\cdot 1\cancel{0}\cdot 9\cdot 8\cdot \cancel{7!}}{(\cancel{5}\cdot\cancel{4}\cdot\cancel{3}\cdot\cancel{2}\cdot 1)\cdot\cancel{7!}} = 11\cdot 9\cdot 8 = 792.$$

Example 7 – *Teams with Members of Two Types*

- Suppose the group of twelve consists of five men and seven women.
- **a.** How many five-person teams can be chosen that consist of three men and two women?
- **b.** How many five-person teams contain at least one man?
- **c.** How many five-person teams contain at most one man?
- Solution: **a.** To answer this question, think of forming a team as a two-step process: **Step 1:** Choose the men.

Example 7 – *Solution*

- **Step 2:** Choose the women.
- There are $\binom{5}{3}$ ways to choose the three men out of the five and $\binom{7}{2}$ ways to choose the two women out of the seven.
- Hence, by the product rule,

$$\begin{bmatrix}\text{number of teams of five that}\\ \text{contain three men and two women}\end{bmatrix} = \binom{5}{3}\binom{7}{2} = \frac{5!}{3!2!}\cdot\frac{7!}{2!5!}$$

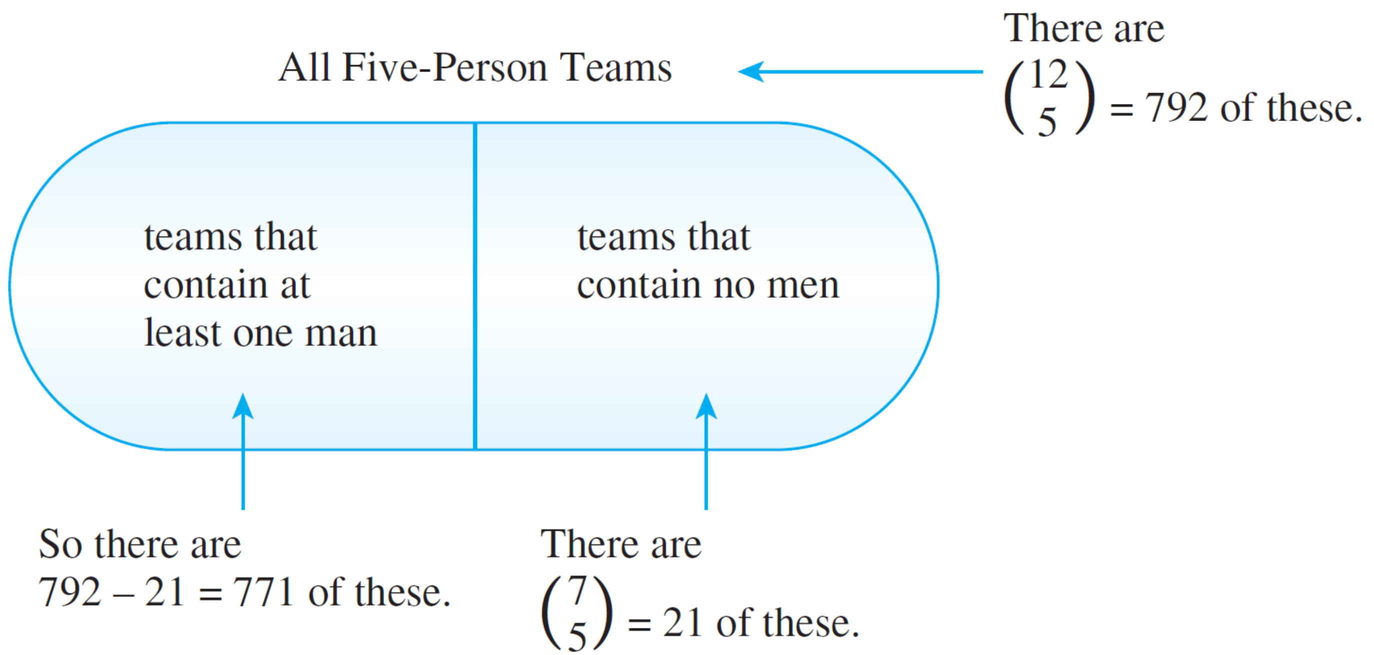
$$= \frac{7\cdot 6\cdot 5\cdot \cancel{4}\cdot \cancel{3}\cdot \cancel{2}\cdot 1}{\cancel{3}\cdot \cancel{2}\cdot 1\cdot \cancel{2}\cdot 1\cdot \cancel{2}\cdot 1}$$

$$= 210.$$

Example 7 – *Solution*

- **b.** Use either the addition rule or the difference rule. The solution by the difference rule is shorter. Observe that the set of five-person teams containing at least one man equals the set difference between the set of all five-person teams and the set of five-person teams that do not contain any men.

- See Figure.


All Five-Person Teams
There are
$\binom{12}{5} = 792$ of these.
teams that contain at least one man
teams that contain no men
So there are
792 – 21 = 771 of these.
There are
$\binom{7}{5} = 21$ of these.

Example 7 – *Solution*

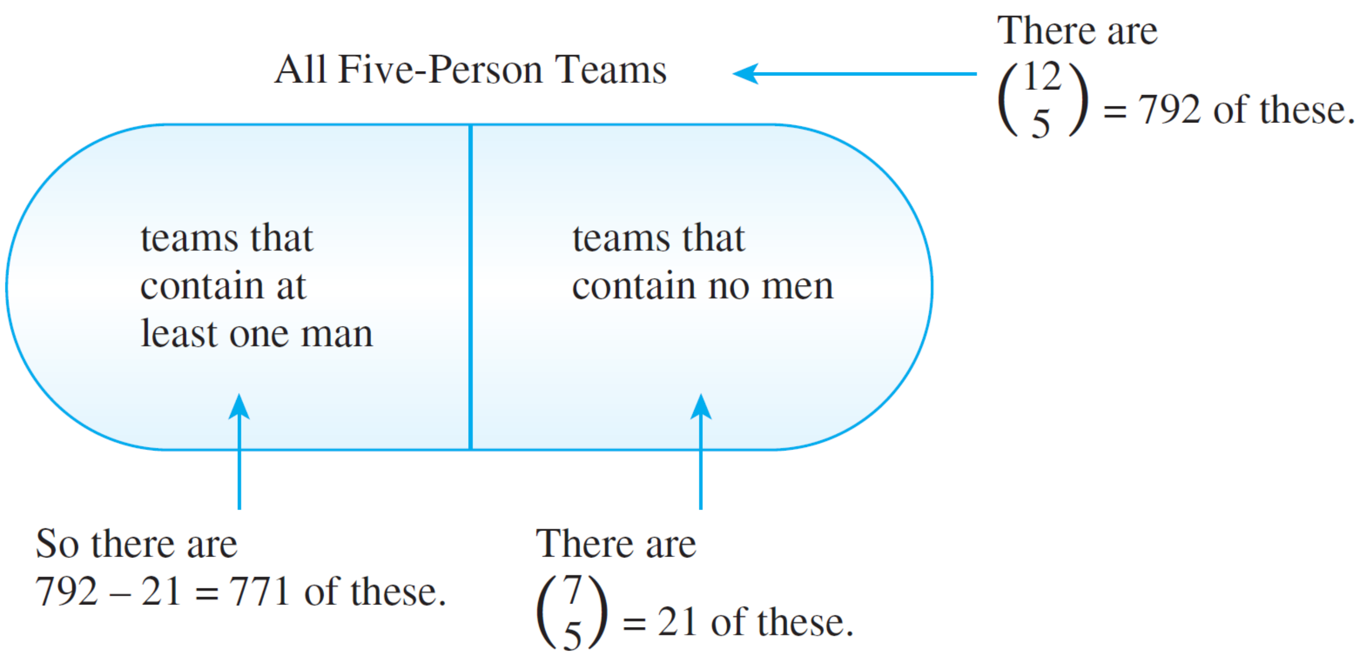
- Now a team with no men consists entirely of five women chosen from the seven women in the group, so there are $\binom{7}{5}$ such teams. The total number of five-person teams is $\binom{12}{5}$ = 792.
- Hence, by the difference rule,

$$\begin{bmatrix}\text{number of teams}\\ \text{with at least}\\ \text{one man}\end{bmatrix} = \begin{bmatrix}\text{total number}\\ \text{of teams}\\ \text{of five}\end{bmatrix} - \begin{bmatrix}\text{number of teams}\\ \text{of five that do not}\\ \text{contain any men}\end{bmatrix}$$

$$= \binom{12}{5} - \binom{7}{5} = 792 - \frac{7!}{5!\cdot 2!}$$

Example 7 – *Solution*

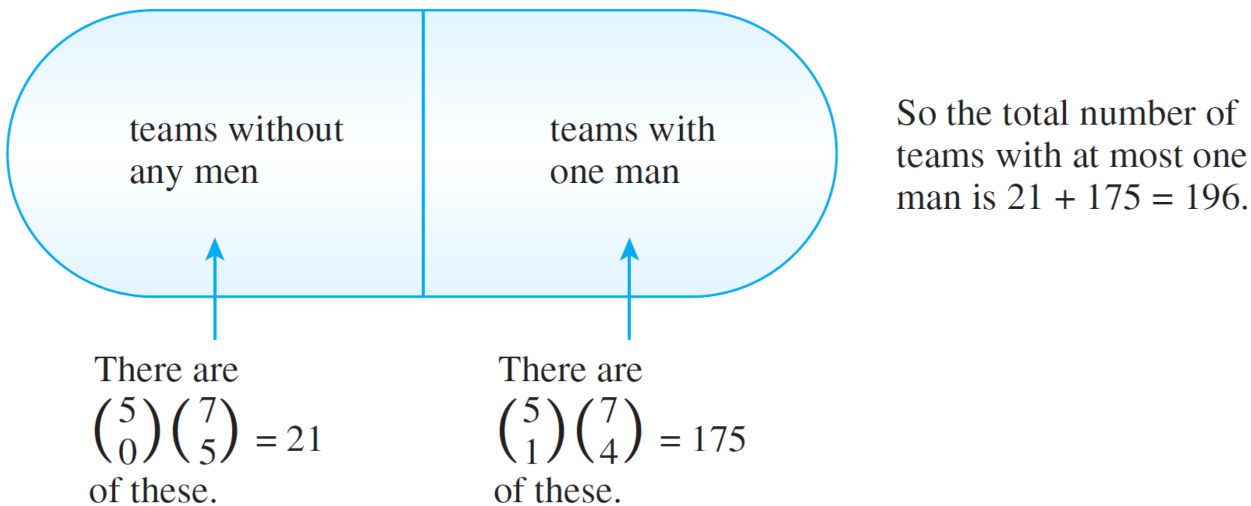
$$= 792 - \frac{7 \cdot \overset{3}{\cancel{6}} \cdot \cancel{5!}}{\cancel{5!} \cdot \cancel{2} \cdot 1} = 792 - 21 = 771.$$



Example 7 – *Solution*

- **c. T**he set of teams containing at most one man can be partitioned into the set that does not contain any men and the set that contains exactly one man.

- Hence, by the addition rule,



Teams with At Most One Man

Example 7 – *Solution*

$$\begin{bmatrix}\text{number of teams}\\ \text{with at}\\ \text{most one man}\end{bmatrix} = \begin{bmatrix}\text{number of}\\ \text{teams without}\\ \text{any men}\end{bmatrix} + \begin{bmatrix}\text{number of}\\ \text{teams with}\\ \text{one man}\end{bmatrix}$$

$$= \binom{5}{0}\binom{7}{5} + \binom{5}{1}\binom{7}{4}$$

$$= 21 + 175$$

$$= 196.$$

Counting Subsets of a Set: Combinations

- **Following is the generalized theorem.**

Theorem 9.5.2 Permutations with sets of Indistinguishable Objects

Suppose a collection consists of n objects of which

n_1 are of type 1 and are indistinguishable from each other

n_2 are of type 2 and are indistinguishable from each other

$\vdots$

n_k are of type k and are indistinguishable from each other,

and suppose that $n_1 + n_2 + \cdots + n_k = n$. Then the number of distinguishable permutations of the n objects is

$$\binom{n}{n_1}\binom{n-n_1}{n_2}\binom{n-n_1-n_2}{n_3}\cdots\binom{n-n_1-n_2-\cdots-n_{k-1}}{n_k}$$

$$= \frac{n!}{n_1!\,n_2!\,n_3!\cdots n_k!}.$$

In-class Assignment #1

- A computer programming team has 13 members. How many ways a group of seven be chosen to work on a project?
- Suppose seven team members are women and six are men. How many groups of seven can be chosen that contain four women and three men?

r-Combinations with Repetition Allowed

- How many ways are there to choose *r* elements without regard to order from a set of *n* elements if *repetition is allowed*?

• Definition

An ***r*-combination with repetition allowed**, or **multiset of size *r***, chosen from a set X of n elements is an unordered selection of elements taken from X with repetition allowed. If $X = \{x_1, x_2, \ldots, x_n\}$, we write an r-combination with repetition allowed, or multiset of size r, as $[\boldsymbol{x_{i_1}}, \boldsymbol{x_{i_2}}, \ldots, \boldsymbol{x_{i_r}}]$ where each x_{i_j} is in X and some of the x_{i_j} may equal each other.

Example 1 – *r-Combinations with Repetition Allowed*

- Write a complete list to find the number of 3-combinations with repetition allowed, or multisets of size 3, that can be selected from {1, 2, 3, 4}. Observe that because the order in which the elements are chosen does not matter, the elements of each selection may be written in increasing order, and writing the elements in increasing order will ensure that no combinations are overlooked.
- Solution: [1, 1, 1] ; [1, 1, 2]; [1, 1, 3]; [1, 1, 4] all combinations with 1, 1

  [1, 2, 2] ; [1, 2, 3]; [1, 2, 4]; all additional combinations with 1, 2

Example 1 – *Solution*

[1, 3, 3] ; [1, 3, 4]; [1, 4, 4];	all additional combinations with 1, 3 or 1, 4
[2, 2, 2] ; [2, 2, 3]; [2, 2, 4];	all additional combinations with 2, 2
[2, 3, 3] ; [2, 3, 4]; [2, 4, 4];	all additional combinations with 2, 3 or 2, 4
[3, 3, 3] ; [3, 3, 4]; [3, 4, 4];	all additional combinations with 3, 3 or 3, 4
[4, 4, 4]	the only additional combination with 4, 4

- Thus there are twenty 3-combinations with repetition allowed.

r-Combinations with Repetition Allowed

- By Theorem 9.5.1, this number is $\binom{r+n-1}{r}$.

Theorem 9.5.1

The number of subsets of size r (or r-combinations) that can be chosen from a set of n elements, $\binom{n}{r}$, is given by the formula

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or, equivalently,

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where n and r are nonnegative integers with $r \leq n$.

r-Combinations with Repetition Allowed

Theorem 9.6.1

The number of r-combinations with repetition allowed (multisets of size r) that can be selected from a set of n elements is

$$\binom{r+n-1}{r}.$$

This equals the number of ways r objects can be selected from n categories of objects with repetition allowed.

In-class Practice Exercise #2

- With repetition allowed, how many multisets of size four can be chosen from a set of three elements?
- How many multisets of size five can be chosen from a set of three elements? (**Hints:** use **Theorem 9.6.1**)