CMSC 207- Lecture 4 CHAPTER 2: Logical Form & Logical Equivalence 2.1

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Logical Form and Logical Equivalence

- The central concept of Deductive Logic , where there are rules that help to deduce the conclusion is the concept of Argument Form. An argument is a sequence of statements aimed at demonstrating the truth of an assertion.
- The assertion at the end of the sequence is called the *conclusion*, and the preceding statements are called *premises*.
- It is not always possible to verify the truth of the premises involved in an argument.

Logical Form and Logical Equivalence

- It is possible to use logic to determine whether the truth of the conclusion follows necessarily from the truth of the premises – that is, whenever the premises are all true, whether the conclusion is also true.
- If so, the Argument Form is a Valid Argument Form.

Example 1 – *Identifying Logical Form*

- Fill in the blanks below so that argument (b) has the same form as argument (a). Then represent the common form of the arguments using letters to stand for component sentences.

- **a.** If Jane is a math major or Jane is a computer science major, then Jane will take Math 150. Jane is a computer science major. Therefore, Jane will take Math 150.

b. If logic is easy or , $\underset{\text{(1)}}{\rule{2cm}{0.4pt}}$ then $\underset{\text{(2)}}{\rule{2cm}{0.4pt}}$.

I will study hard.

Therefore, I will get an A in this course.

Example 1 – *Solution*

1. I (will) study hard.

2. I will get an A in this course.

Common Argument Form: If *p* or *q*, then *r*.

q.

- Therefore, *r*.

Statements

- Definitions of Formal Logic have been developed so that they agree, and align with the Natural or Intuitive Logic used by people.
- The differences that exist between formal and intuitive logic are necessary to avoid ambiguity, and obtain consistency.
- In mathematical theory, new terms are defined by using the previously defined terms.

Statements

• Definition

A **statement** (or **proposition**) is a sentence that is true or false but not both.

Compound Statements

- We now introduce three symbols that are used to build more complicated logical expressions out of simpler ones.

The symbol ~ denotes *not*, $\wedge$ denotes *and*, and $\vee$ denotes *or*.

- Given a statement *p*, the sentence "~*p*" is "not *p*" or "It is not the case that *p*", and is called the **negation of *p***. In some computer languages the symbol • is used in place of ~.

Compound Statements

- Given another statement q, the sentence "$p \wedge q$" is read "p and q" and is called the **conjunction of *p* and *q***.

The sentence "$p \vee q$" is read "p or q" and is called the **disjunction of *p* and *q***.

- In expressions that include the symbol ~ as well as $\wedge$ or $\vee$, the **order of operations** specifies that ~ is performed first.

For instance, $\boldsymbol{\sim p \wedge q = (\sim p) \wedge q}$**.**

- In logical expressions, as in ordinary algebraic expressions, the order of operations can be overridden through the use of parentheses. Thus $\sim(p \wedge q)$ represents the negation of the conjunction of p and q.

- In this, as in most treatments of logic, the symbols $\wedge$ and $\vee$ are considered coequal in order of operation, and an expression such as $p \wedge q \vee r$ is considered ambiguous – no order. This expression must be written as either $(p \wedge q) \vee r$ or $p \wedge (q \vee r)$ to have meaning.

Example 2 – *Translating from English to Symbols: But and Neither-Nor (Contd.)*

- Write each of the following sentences symbolically, letting
h = "It is summer" and s = "It is sunny."
- **a.** It is not summer but it is sunny.
- **b.** It is neither summer nor sunny.

Example 2 – *Translating from English to Symbols: But and Neither-Nor*

- Solution:
- **a.** The given sentence is equivalent to "It is not summer and it is sunny," which can be written symbolically as $\sim h \wedge s$.
- **b.** To say it is neither summer nor sunny means that it is not summer and it is not sunny. Therefore, the given sentence can be written symbolically as $\sim h \wedge \sim s$.

Truth Values

- In Example 2, we built compound sentences out of component statements and the terms *not*, *and*, and *or*.
- If such sentences are to be statements, they must have well-defined **truth values**—they must be either true or false but not both.
- We now define such compound sentences as statements by specifying their truth values in terms of the statements that compose them.

Truth Values

- *The negation of a statement is a statement that exactly expresses what it would mean for the statement to have the opposite truth value.*

• Definition

If p is a statement variable, the **negation** of p is "not p" or "It is not the case that p" and is denoted $\sim p$. It has opposite truth value from p: if p is true, $\sim p$ is false; if p is false, $\sim p$ is true.

p	$\sim p$
T	F
F	T

Truth Table for ~*p*

- The truth values for negation are summarized in a *truth table*.

Truth Values

• Definition

If p and q are statement variables, the **conjunction** of p and q is "p and q," denoted $p \wedge q$. It is true when, and only when, both p and q are true. If either p or q is false, or if both are false, $p \wedge q$ is false.

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Truth Table for $p \wedge q$

- The truth values for conjunction can also be summarized in a truth table.

Truth Values

• Definition

If p and q are statement variables, the **disjunction** of p and q is "p or q," denoted $p \vee q$. It is true when either p is true, or q is true, or both p and q are true; it is false only when both p and q are false.

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Truth Table for $p \vee q$

- Above is the truth table for disjunction:

Evaluating the Truth of More General Compound Statements

- Now that truth values have been assigned to $\sim p$, $p \wedge q$, and $p \vee q$, consider the question of assigning truth values to more complicated expressions such as $\sim p \vee q$, $(p \vee q) \wedge \sim(p \wedge q)$, and $(p \wedge q) \vee r$. Such expressions are called *statement forms* (or *propositional forms*).

• Definition

A **statement form** (or **propositional form**) is an expression made up of statement variables (such as p, q, and r) and logical connectives (such as $\sim$, $\wedge$, and $\vee$) that becomes a statement when actual statements are substituted for the component statement variables. The **truth table** for a given statement form displays the truth values that correspond to all possible combinations of truth values for its component statement variables.

Evaluating the Truth Values of More General Compound Statements

- To compute the truth values for a statement form, follow rules similar to those used to evaluate algebraic expressions.
- For each combination of truth values for the statement variables, first evaluate the expressions within the innermost parentheses, then evaluate the expressions within the next innermost set of parentheses, and so forth until you have the truth values for the complete expression.

Example 4 – ***Truth Table for Exclusive Or***

- Construct the truth table for the statement form $\boldsymbol{(p \vee q) \wedge \sim(p \wedge q)}$.
- When *or* is used in its exclusive sense, the statement "*p* or *q*" means "*p* or *q* but not both" or "*p* or *q* and not both *p* and *q*," which translates into symbols as $(p \vee q) \wedge \sim(p \wedge q)$.
- This is sometimes abbreviated:

$$p \oplus q \text{ or } p \text{ XOR } q.$$

Example 4 – *Solution*

- Set up columns labeled p, q, $p \vee q$, $p \wedge q$, $\sim(p \wedge q)$, and $(p \vee q) \wedge \sim(p \wedge q)$.

p	q	$p \vee q$	$p \wedge q$	$\sim(p \wedge q)$	$(p \vee q) \wedge \sim(p \wedge q)$
T	T	T	T	F	F
T	F	T	F	T	T
F	T	T	F	T	T
F	F	F	F	T	F

Truth Table for *Exclusive Or*: $(p \vee q) \wedge \sim(p \wedge q)$

- Fill in the p and q columns with all the logically possible combinations of T's and F's. Then use the truth tables for $\vee$ and $\wedge$ to fill in the $p \vee q$ and $p \wedge q$ columns with the appropriate truth values.

- Next fill in the $\sim(p \wedge q)$ column by taking the opposites of the truth values for $p \wedge q$.

- For example, the entry for $\sim(p \wedge q)$ in the first row is F because in the first row the truth value of $p \wedge q$ is T.

- Finally, fill in the $(p \vee q) \wedge \sim(p \wedge q)$ column by considering the truth table for an *and* statement together with the computed truth values for $p \vee q$ and $\sim(p \wedge q)$.

Example 4 – *Solution*

- The entry in the second row is T because both components are true in this row.

p	q	$p \vee q$	$p \wedge q$	$\sim(p \wedge q)$	$(p \vee q) \wedge \sim(p \wedge q)$
T	T	T	T	F	F
T	F	T	F	T	T
F	T	T	F	T	T
F	F	F	F	T	F

Truth Table for *Exclusive Or*: $(p \vee q) \wedge \sim(p \wedge q)$

Logical Equivalence

- The table shows that for each combination of truth values for p and q, $p \wedge q$ is true when, and only when, $q \wedge p$ is true.

- In such a case, the statement forms are called *logically equivalent*, and we say

that (1) and (2) are *logically equivalent statements*.

p	q	$p \wedge q$	$q \wedge p$
T	T	T	T
T	F	F	F
F	T	F	F
F	F	F	F

$p \wedge q$ and $q \wedge p$ always have the same truth values, so they are logically equivalent

Logical Equivalence

• Definition

Two *statement forms* are called **logically equivalent** if, and only if, they have identical truth values for each possible substitution of statements for their statement variables. The logical equivalence of statement forms P and Q is denoted by writing $P \equiv Q$.

Two *statements* are called **logically equivalent** if, and only if, they have logically equivalent forms when identical component statement variables are used to replace identical component statements.

Testing Whether Two Statement Forms *P* and *Q* Are Logically Equivalent

1. Construct a truth table with one column for the truth values of P and another column for the truth values of Q.

Logical Equivalence

2. Check each combination of truth values of the statement variables to see whether the truth value of P is the same as the truth value of Q.

a. If in each row the truth value of P is the same as the truth value of Q, then P and Q are logically equivalent.

b. If in some row P has a different truth value from Q, then P and Q are not logically equivalent.

Example 6 – *Double Negative Property:* $\sim(\sim p) \equiv p$

- Construct a truth table to show that the negation of the negation of a statement is logically equivalent to the statement, annotating the table with a sentence of explanation.

- **Solution:**

p	$\sim p$	$\sim(\sim p)$
T	F	T
F	T	F

p and $\sim(\sim p)$ always have the same truth values, so they are logically equivalent

Logical Equivalence

- There are two ways to show that statement forms *P* and *Q* are *not* logically equivalent. As indicated previously, one is to use a truth table to find rows for which their truth values differ.
- The other way is to find concrete statements for each of the two forms, one of which is true and the other of which is false.
- The next example illustrates both of these ways.

Example 7 – *Showing Nonequivalence*

- Show that the statement forms $\sim(p \wedge q)$ and $\sim p \wedge \sim q$ are not logically equivalent.

- **Solution:**

- **a.** This method uses a truth table annotated with a sentence of explanation.

p	q	$\sim p$	$\sim q$	$p \wedge q$	$\sim(p \wedge q)$		$\sim p \wedge \sim q$
T	T	F	F	T	F		F
T	F	F	T	F	T	$\neq$	F
F	T	T	F	F	T	$\neq$	F
F	F	T	T	F	T		T

$\sim(p \wedge q)$ and $\sim p \wedge \sim q$ have different truth values in rows 2 and 3, so they are not logically equivalent

Example 7 – *Solution*

- **b.** This method uses an example to show that $\sim(p \wedge q)$ and $\sim p \wedge \sim q$ are not logically equivalent. Let p be the statement "$0 < 1$" and let q be the statement "$1 < 0$."
- Then

$\sim(p \wedge q)$ is "It is not the case that both $0 < 1$ and $1 < 0$,"

which is true.

- On the other hand,

$$\sim p \wedge \sim q \quad \text{is} \quad \text{“}0 \not< 1 \quad \text{and} \quad 1 \not< 0\text{,”}$$

which is false.

Example 7 – *Solution*

- This example shows that there are concrete statements you can substitute for p and q to make one of the statement forms true, and the other one false. Therefore, the statement forms are not logically equivalent.

Logical Equivalence

- The two logical equivalences are known as **De Morgan's laws** of logic in honor of Augustus De Morgan, who was the first to state them in formal mathematical terms.

De Morgan's Laws

The negation of an *and* statement is logically equivalent to the *or* statement in which each component is negated.

The negation of an *or* statement is logically equivalent to the *and* statement in which each component is negated.

- Symbolically we can represent the two logic equivalences as:

$$\boxed{\sim(p \wedge q) \equiv \sim p \vee \sim q}$$

and

$$\boxed{\sim(p \vee q) \equiv \sim p \wedge \sim q.}$$

Example 9 – *Applying De Morgan's Laws*

- **Write negations for each of the following statements:**
- **a.** John is 6 feet tall and he weighs at least 200 pounds.
- **b.** The bus was late or Tom's watch was slow.
- **Solution:**
- **a.** John is not 6 feet tall or he weighs less than 200 pounds.
- **b.** The bus was not late and Tom's watch was not slow.

Tautologies and Contradictions

• Definition

A **tautology** is a statement form that is always true regardless of the truth values of the individual statements substituted for its statement variables. A statement whose form is a tautology is a **tautological statement.**

A **contradication** is a statement form that is always false regardless of the truth values of the individual statements substituted for its statement variables. A statement whose form is a contradication is a **contradictory statement.**

- According to this definition, the truth of a tautological statement and the falsity of a contradictory statement are due to the logical structure of the statements themselves and are independent of the meanings of the statements.

Example 13 – *Logical Equivalence Involving Tautologies and Contradictions*

- If **t** is a tautology and **c** is a contradiction, show that:

$p \wedge \mathbf{t} \equiv p$ and $p \wedge \mathbf{c} \equiv \mathbf{c}$.

- Solution:

p	$\mathbf{t}$	$p \wedge \mathbf{t}$	p	$\mathbf{c}$	$p \wedge \mathbf{c}$
T	T	T	T	F	F
F	T	F	F	F	F

same truth values, so $p \wedge \mathbf{t} \equiv p$

same truth values, so $p \wedge \mathbf{c} \equiv \mathbf{c}$

Example 14 – *Simplifying Statement Forms*

- Use Theorem 2.1.1 to verify the logical equivalence

$$\sim(\sim p \wedge q) \wedge (p \vee q) \equiv p.$$

- **Solution:**
- Use the laws of Theorem 2.1.1 to replace sections of the statement form on the left by logically equivalent expressions.
- Each time you do this, you obtain a logically equivalent statement form.

Example 14 – *Solution*

Continue making replacements until you obtain the statement form on the right.

$$\sim(\sim p \wedge q) \wedge (p \vee q) \equiv (\sim(\sim p) \vee \sim q) \wedge (p \vee q) \quad \text{by De Morgan's laws}$$

$$\equiv (p \vee \sim q) \wedge (p \vee q) \quad \text{by the double negative law}$$

$$\equiv p \vee (\sim q \wedge q) \quad \text{by the distributive law}$$

$$\equiv p \vee (q \wedge \sim q) \quad \text{by the commutative law for } \wedge$$

$$\equiv p \vee \mathbf{c} \quad \text{by the negation law}$$

$$\equiv p \quad \text{by the identity law.}$$