CMSC 207- Lecture 21
CHAPTER 8: Relations (8.2)

Dr. Ahmed Tarek

Reflexivity, Symmetry, and Transitivity

Let A = {2, 3, 4, 6, 7, 9} and define a relation R on A as follows: For all x, $y \in A$, $x \; R \; y \quad \Leftrightarrow \quad 3 \mid (x - y)$.

Then 2 R 2 because 2 – 2 = 0, and 3 | 0.

Similarly, 3 R 3, 4 R 4, 6 R 6, 7 R 7, and 9 R 9.

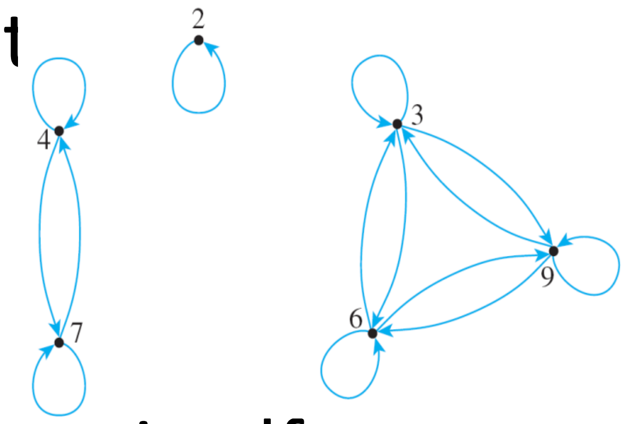
Also 6 R 3 because 6 – 3 = 3, and 3 | 3.

And 3 R 6 because 3 – 6 = –(6 – 3) = –3, and 3 | (–3).

Similarly, 3 R 9, 9 R 3, 6 R 9, 9 R 6, 4 R 7, and 7 R 4.

Reflexivity, Symmetry, and Transitivity

Thus the directed graph for *R* has the appearance shown at the right. This graph has three important properties:



1. Each point of the graph has an arrow looping around from it back to itself.

2. In each case where there is an arrow going from one point to a second, there is an arrow going from the second point back to the first.

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3. In each case where there is an arrow going from one point to a second and from the second point to a third, there is an arrow going from the first point to the third. That is, there are no “incomplete directed triangles” in the graph.

Properties (1), (2), and (3) correspond to properties of relations called *reflexivity*, *symmetry*, and *transitivity*.

• Definition

Let R be a relation on a set A.

1. R is **reflexive** if, and only if, for all $x \in A$, $x\ R\ x$.
2. R is **symmetric** if, and only if, for all $x, y \in A$, ***if*** $x\ R\ y$ then $y\ R\ x$.
3. R is **transitive** if, and only if, for all $x, y, z \in A$, ***if*** $x\ R\ y$ and $y\ R\ z$ then $x\ R\ z$.

Reflexivity, Symmetry, and Transitivity

Because of the equivalence of the expressions $x\ R\ y$ and $(x, y) \in R$ for all x and y in A, the reflexive, symmetric, and transitive properties can also be written as follows:

1. R is reflexive $\Leftrightarrow$ for all x in A, $(x, x) \in R$.
2. R is symmetric $\Leftrightarrow$ for all x and y in A, ***if*** $(x, y) \in R$ then $(y, x) \in R$.
3. R is transitive $\Leftrightarrow$ for all x, y and z in A, ***if*** $(x, y) \in R$ and $(y, z) \in R$ then $(x, z) \in R$.

Reflexivity, Symmetry, and Transitivity

In informal terms, properties (1)–(3) say the following:

1. **Reflexive:** Each element is related to itself.
2. **Symmetric:** If any one element is related to any other element, then the second element is related to the first.
3. **Transitive:** If any one element is related to a second and that second element is related to a third, then the first element is related to the third. The definitions of reflexivity, symmetry, and transitivity are universal statements.

Reflexivity, Symmetry, and Transitivity

This means that to prove a relation has one of these properties, you use either the method of exhaustion or the method of generalizing from the generic particular.

Now consider what it means for a relation *not* to have one of the properties defined previously. We have known that the negation of a universal statement is existential.

Hence if *R* is a relation on a set *A*, then

1. *R* is **not reflexive** $\Leftrightarrow$ $x \not R x$ there is an element *x* in *A* such that $(x, x) \notin R$.

Reflexivity, Symmetry, and Transitivity

2. *R* is **not symmetric** $\Leftrightarrow$ there are elements *x* and *y* in *A* such that $x\ R\ y$ but $y \not R\ x$ *[that is, such that* $(x, y) \in R$ *but* $(y, x) \notin R$*]*.

3. *R* is **not transitive** $\Leftrightarrow$ there are elements *x*, *y* and *z* in *A* such that $x\ R\ y$ and $y\ R\ z$ but $x \not R\ z$ *[that is, such that* $(x, y) \in R$ *and* $(y, z) \in R$ *but* $(x, z) \notin R$*]*.

You can show that a relation does *not* have one of the properties by finding a counterexample.

Example 1 – *Properties of Relations on Finite Sets*

Let $A = \{0, 1, 2, 3\}$, and define relations R, S, and T on A as follows:

$R = \{(0, 0), (0, 1), (0, 3), (1, 0), (1, 1), (2, 2), (3, 0), (3, 3)\}$, $S = \{(0, 0), (0, 2), (0, 3), (2, 3)\}$, $T = \{(0, 1), (2, 3)\}$.

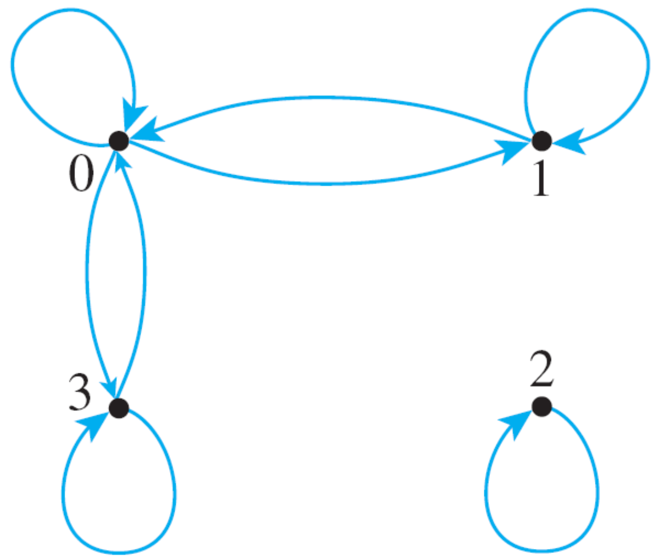
a. Is R reflexive? symmetric? transitive?

b. Is S reflexive? symmetric? transitive?

c. Is T reflexive? symmetric? transitive?

Example 1(a) – *Solution*

The directed graph of *R* has the appearance shown below.



***R* is reflexive:** There is a loop at each point of the directed graph. This means that each element of *A* is related to itself, so *R* is reflexive.

Example 1(a) – *Solution*

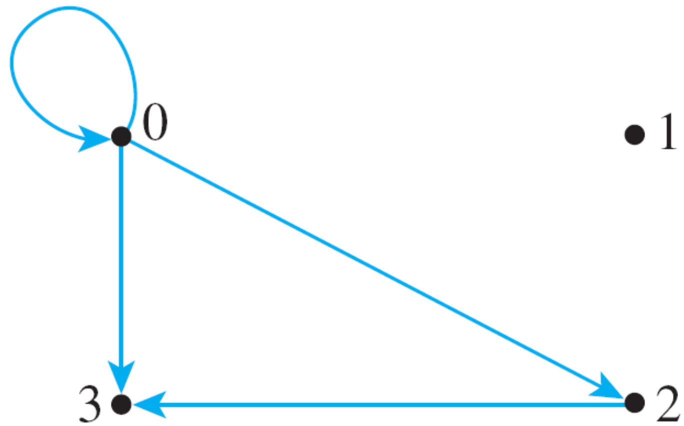
***R* is symmetric:** In each case where there is an arrow going from one point of the graph to a second, there is an arrow coming from the second point back to the first.

This means that whenever one element of A is related by R to a second, then the second is related to the first. Hence R is symmetric.

***R* is not transitive:** There is an arrow going from 1 to 0 and an arrow going from 0 to 3, but there is no arrow going from 1 to 3. This means that there are elements of A—0, 1, and 3—such that $1\ R\ 0$ and $0\ R\ 3$ but $1 \not R\ 3$. Hence R is **not transitive**.

Example 1(b) – *Solution*

The directed graph of *S* has the appearance shown below.



S is not reflexive: There is no loop at 1. Thus (1, 1) $\notin$ *S*, and so *S* is not reflexive.

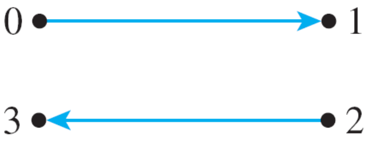
S is not symmetric: There is an arrow from 0 to 2 but not from 2 to 0. Hence (0, 2) $\in$ *S* but (2, 0) $\notin$ *S*, and so *S* is not symmetric.

Example 1(b) – *Solution*

***S* is transitive:** There are three cases for which there is an arrow going from one point of the graph to a second and from the second point to a third:
Namely, there are arrows going from 0 to 2 and from 2 to 3; there are arrows going from 0 to 0 and from 0 to 2; and there are arrows going from 0 to 0 and from 0 to 3. In each case there is an arrow going from the first point to the third. (Note again that the "first," "second," and "third" points need not be distinct.)
This means that whenever $(x, y) \in S$ and $(y, z) \in S$, then $(x, z) \in S$, for all $x, y, z \in \{0, 1, 2, 3\}$, and so S is transitive.

Example 1(c) – *Solution*

The directed graph of T has the appearance shown at right.



***T* is not reflexive:** There is no loop at 0, for example. Thus $(0, 0) \notin T$, so T is not reflexive.

***T* is not symmetric:** There is an arrow from 0 to 1 but not from 1 to 0. Thus $(0, 1) \in T$ but $(1, 0) \notin T$, and so T is not symmetric.

***T* is transitive:** The transitivity condition is **vacuously true for *T*.** To see this, observe that the transitivity condition says that For all $x, y, z \in A$, if $(x, y) \in T$ and $(y, z) \in T$ then $(x, z) \in T$.

Example 1(c) – *Solution*

The only way for this to be false would be for there to exist elements of A that make the hypothesis true and the conclusion false.
That is, there would have to be elements x, y, and z in A such that $(x, y) \in T$ and $(y, z) \in T$ and $(x, z) \notin T$.
Only elements in T are (0, 1) and (2, 3), and these do not have the potential to link up. Hence the hypothesis is never true. It follows that it is impossible for T *not* to be transitive, and thus T is transitive.

- **In-class Assignments #1 and #2**

In-class Assignment #1

- Let **A = {3, 4, 5, 6, 7, 8, 9}** and define a binary relation $\mathbf{R_3}$ on **A** as follows:
- For all **x, y** $\in$ **A**, **xRy** if and only if $\mathbf{x} \mid \mathbf{y}$. Which of the properties does $\mathbf{R_3}$ possess?
- List the pairs in the relation.

In-class Assignment #2

- Let S = {(0, 0), (0, 3), (1, 0), (1, 2), (2, 0), (3, 2)}.
- Draw the directed graph.
- Determine whether the relation is reflexive, symmetric, and transitive.
- Give a counterexample in each case in which the relation does not satisfy one of the properties.

Example 2 – *Properties of Equality*

Define a relation R on **R** (the set of all real numbers) as follows: For all real numbers x and y.

$$\boxed{x \; R \; y \quad \Leftrightarrow \quad x = y.}$$

a. Is R reflexive?

b. Is R symmetric?

c. Is R transitive?

Example 2(a) – *Solution*

***R* is reflexive:** *R* is reflexive if, and only if, the following statement is true: **For all *x* ∈ R, *x R x*.**

Since ***x R x*** just means that ***x* = *x*,** this is the same as saying: **For all *x* ∈ R, *x* = *x*.**

But this statement is certainly true; every real number is equal to itself.

Example 2(b) – *Solution*

***R* is symmetric:** *R* is symmetric if, and only if, the following statement is true:

For all x, $y \in$ R, if $x\ R\ y$ then $y\ R\ x$.

By definition of R, $x\ R\ y$ means that $x = y$ and $y\ R\ x$ means that $y = x$. Hence *R* is symmetric if, and only if,

For all x, $y \in$ R, if $x = y$ then $y = x$.

But this statement is certainly true; if one number is equal to a second, then the second is equal to the first.

Example 2(c) – *Solution*

***R* is transitive:** *R* is transitive if, and only if, the following statement is true:

For all $x, y, z \in$ R, if $x\ R\ y$ and $y\ R\ z$ then $x\ R\ z$.

By definition of *R*, $x\ R\ y$ means that $x = y$, $y\ R\ z$ means that $y = z$, and $x\ R\ z$ means that $x = z$. Hence ***R*** is transitive if, and only if, the following statement is true:

For all $x, y, z \in$ R, if $x = y$ and $y = z$ then $x = z$.

But this statement is certainly true: If one real number equals a second, and the second equals a third, then the first equals the third.

Example 4 – *Properties of Congruence Modulo 3*

Define a relation ***T*** on **Z** (the set of all integers) as follows: **For all integers *m* and *n*,**

$$\boxed{m\ T\ n \quad \Leftrightarrow \quad 3 \mid (m-n).}$$

This relation is called **congruence modulo 3.**

a. Is *T* reflexive?

b. Is *T* symmetric?

c. Is *T* transitive?

Example 4(a) – *Solution*

***T* is reflexive:** To show that T is reflexive, it is necessary to show that: **For all $m \in$ Z, $m\ T\ m$.**

By definition of T, this means that

$$\textbf{For all } m \in \text{Z},\ 3 \mid (m - m).$$

Or, since $m - m = 0$, **For all $m \in$ Z, $3 \mid 0$.**

But this is true: **$3 \mid 0$ since $0 = 3 \cdot 0$.** Hence ***T* is reflexive**. This reasoning is formalized in the following proof.

Example 4(b) – *Solution*

***T* is symmetric:** To show that *T* is symmetric, it is necessary to show that

For all $m, n \in Z$, if $m \, T \, n$ then $n \, T \, m$.

By definition of *T* this means that

For all $m, n \in Z$, if $3 \mid (m - n)$ then $3 \mid (n - m)$. Suppose *m* and *n* are particular but arbitrarily chosen integers such that $3 \mid (m - n)$. since $3 \mid (m - n)$, then $m - n = 3k$ for some integer *k*. Now $(n - m) = -(m - n)$. Hence, you can multiply both sides of this equation by –1 to obtain $-(m - n) = -3k$, which is equivalent to $n - m = 3(-k)$.

Example 4(b) – *Solution*

Since $-k$ is an integer, this equation shows that

$$\mathbf{3 \mid (\boldsymbol{n} - \boldsymbol{m}).}$$

It follows that *T* is symmetric.

Example 4(c) – *Solution*

***T* is transitive:** To show that T is transitive, it is necessary to show that

For all $m, n, p \in$ Z, if m T n and n T p then m T p.

By definition of T this means that

For all $m, n \in$ Z,

if 3 | ($m - n$) and 3 | ($n - p$) then 3 | ($m - p$).

Suppose m, n, and p are particular but arbitrarily chosen integers such that 3 | ($m - n$) and 3 | ($n - p$).

Example 4(c) – *Solution*

By definition of "divides," since

$$\mathbf{3 \mid (\mathit{m} - \mathit{n}) \text{ and } 3 \mid (\mathit{n} - \mathit{p}),}$$

then $m - n = 3r$ for some integer r,

and $n - p = 3s$ for some integer s.

Now $\mathbf{(\mathit{m} - \mathit{n}) + (\mathit{n} - \mathit{p}) = \mathit{m} - \mathit{p}.}$

Add these two equations together to obtain

$$\mathbf{(\mathit{m} - \mathit{n}) + (\mathit{n} - \mathit{p}) = 3\mathit{r} + 3\mathit{s},}$$

which is equivalent to $\mathbf{\mathit{m} - \mathit{p} = 3(\mathit{r} + \mathit{s}).}$

*[Thus we have found **an integer** so that* $\mathbf{m - p = 3 \cdot}$ **(*that integer*)**.*]*

Example 4(c) – *Solution*

Since r and s are integers, $r + s$ is an integer. So this equation shows that **3 | (m – p).**

It follows that T is transitive.