CMSC 207- Lecture 8
CHAPTER 3: The Logic of Quantified Statements
Sections 3.1 & 3.2

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Predicates and Quantified Statements I

In logic, predicates are obtained by removing some or all of the nouns from a statement. For example, let P stand for "is a student at Bedford College" and let Q stand for "is a student at." Then both P and Q are *predicate symbols*.

The sentences "x is a student at Bedford College" and "x is a student at y" are symbolized as $P(x)$ and as $Q(x, y)$ respectively, where x and y are *predicate variables* that take values in appropriate sets.

Predicates and Quantified Statements I

For simplicity, *predicate* are defined to be a predicate symbol together with suitable predicate variables. In some other treatments of logic, such objects are referred to as **propositional functions** or **open sentences**.

> **• Definition**
>
> A **predicate** is a sentence that contains a finite number of variables and becomes a statement when specific values are substituted for the variables. The **domain** of a predicate variable is the set of all values that may be substituted in place of the variable.

Predicates and Quantified Statements I

When an element in the domain of the variable of a
one-variable predicate is substituted for the variable, the resulting statement is either true or false. The set of all such elements that make the predicate true is called the *truth set* of the predicate.

• Definition

If $P(x)$ is a predicate and x has domain D, the **truth set** of $P(x)$ is the set of all elements of D that make $P(x)$ true when they are substituted for x. The truth set of $P(x)$ is denoted

$$\{x \in D \mid P(x)\}.$$

Example 2 – *Finding the Truth Set of a Predicate*

Let $Q(n)$ be the predicate "n is a factor of 8." Find the truth set of $Q(n)$ if

a. the domain of n is the set $\mathbf{Z}^+$ of all positive integers

b. the domain of n is the set $\mathbf{Z}$ of all integers.

Solution:

a. The truth set is $\{1, 2, 4, 8\}$ because these are exactly the positive integers that divide 8 evenly.

b. The truth set is $\{1, 2, 4, 8, -1, -2, -4, -8\}$ because the negative integers $-1, -2, -4$, and -8 also divide into 8 without leaving a remainder.

The Universal Quantifier: ∀

One sure way to change predicates into statements is to assign specific values to all their variables.

For example, if *x* represents the number 35, the sentence "*x* is (evenly) divisible by 5" is a true statement since $35 = 5 \cdot 7$. Another way to obtain statements from predicates is to add **quantifiers**.

Quantifiers are words that refer to quantities such as "some" or "all" and tell for how many elements a given predicate is true.

The Universal Quantifier: $\forall$

The symbol $\forall$ denotes "for all" and is called the **universal quantifier.**

The domain of the predicate variable is generally indicated between the $\forall$ symbol and the variable name or immediately following the variable name. Some other expressions that can be used instead of *for all are for every*, *for arbitrary*, *for any*, *for each*, and *given any*.

The Universal Quantifier: ∀

Sentences that are quantified universally are defined as statements by giving them the truth values specified in the following definition:

• Definition

Let $Q(x)$ be a predicate and D the domain of x. A **universal statement** is a statement of the form "$\forall x \in D, Q(x)$." It is defined to be true if, and only if, $Q(x)$ is true for every x in D. It is defined to be false if, and only if, $Q(x)$ is false for at least one x in D. A value for x for which $Q(x)$ is false is called a **counterexample** to the universal statement.

Example 3 – *Truth and Falsity of Universal Statements*

a. Let D = {1, 2, 3, 4, 5}, and consider the statement $\forall x \in D, x^2 \geq x.$

Show that this statement is true.

$$\forall x \in \mathbf{R}, x^2 \geq x.$$

b. Consider the statement: Find a counterexample to show that this statement is false.

Example 3 – *Solution*

a. Check that "$x^2 \geq x$" is true for each individual x in D.

$$1^2 \geq 1, \qquad 2^2 \geq 2, \qquad 3^2 \geq 3, \qquad 4^2 \geq 4, \qquad 5^2 \geq 5.$$

Hence "$\forall x \in D, x^2 \geq x$" is true.

b. *Counterexample:* Take $x = \frac{1}{2}$. Then x is in $\mathbf{R}$ (since $\frac{1}{2}$ is a real number) and

$$\left(\frac{1}{2}\right)^2 = \frac{1}{4} \not\geq \frac{1}{2}.$$

Hence "$\forall x \in \mathbf{R}, x^2 \geq x$" is false.

The Universal Quantifier: ∀

The technique used to show the truth of the universal statement in Example 3(a) is called the **method of exhaustion.**

It consists of showing the truth of the predicate separately for each individual element of the domain.

This method can, in theory, be used whenever the domain of the predicate variable is finite.

The Existential Quantifier: ∃

The symbol ∃ denotes "there exists" and is called the **existential quantifier.** For example, the sentence "There is a student in Math 140" can be written as ∃ a person *p* such that *p* is a student in Math 140, or, more formally,

$\exists p \in P$ such that p is a student in Math 140,

where *P* is the set of all people. The domain of the predicate variable is generally indicated either between the ∃ symbol and the variable name or immediately following the variable name.

The Existential Quantifier: ∃

Sentences that are quantified existentially are defined as statements by giving them the truth values specified in the following definition.

• Definition

Let $Q(x)$ be a predicate and D the domain of x. An **existential statement** is a statement of the form "$\exists x \in D$ such that $Q(x)$." It is defined to be true if, and only if, $Q(x)$ is true for at least one x in D. It is false if, and only if, $Q(x)$ is false for all x in D.

Example 4 – *Truth and Falsity of Existential Statements*

a. Consider the statement

$$\exists m \in \mathbf{Z}^{+} \text{ such that } m^2 = m.$$

Show that this statement is true.

b. Let $E = \{5, 6, 7, 8\}$ and consider the statement

$$\exists m \in E \text{ such that } m^2 = m.$$

Show that this statement is false.

Example 4 – *Solution*

a. Observe that $1^2 = 1$. Thus "$m^2 = m$" is true for at least one integer m. Hence "$\exists m \in \mathbf{Z}$ such that $m^2 = m$" is true.

b. Note that $m^2 = m$ is not true for any integers m from 5 through 8:

$$5^2 = 25 \neq 5, \qquad 6^2 = 36 \neq 6, \qquad 7^2 = 49 \neq 7, \qquad 8^2 = 64 \neq 8.$$

Thus "$\exists m \in E$ such that $m^2 = m$" is false.

Example 5 – *Translating from Formal to Informal Language*

Rewrite the following formal statements in a variety of equivalent but more informal ways. Do not use the symbol ∀ or ∃.

a. $\forall x \in \mathbf{R}, x^2 \geq 0.$

b. $\forall x \in \mathbf{R}, x^2 \neq -1.$

c. $\exists m \in \mathbf{Z}^+ \text{such that } m^2 = m.$

Example 5 – *Solution*

a. All real numbers have nonnegative squares.

Or: Every real number has a nonnegative square.

Or: Any real number has a nonnegative square.

Or: The square of each real number is nonnegative.

b. All real numbers have squares that are not equal to −1.

Or: No real numbers have squares equal to −1. (The words *none are* or *no . . . are* are equivalent to the words *all are not*.)

Example 5 – *Solution*

c. There is a positive integer whose square is equal to itself.

Or: We can find at least one positive integer equal to its own square.

Or: Some positive integer equals its own square.

Or: Some positive integers equal their own squares.

Universal Conditional Statements

The most important form of statement in mathematics is the **universal conditional statement:**

$$\forall x, \text{ if } P(x) \text{ then } Q(x).$$

Example 8 – *Writing Universal Conditional Statements Informally*

Rewrite the following statement informally, without quantifiers or variables.

$$\forall x \in \mathbf{R}, \text{ if } x > 2 \text{ then } x^2 > 4.$$

Solution:

If a real number is greater than 2 then its square is greater than 4.

Or: Whenever a real number is greater than 2, its square is greater than 4.

Example 8 – *Solution*

Or: The square of any real number greater than 2 is greater than 4.

Or: The squares of all real numbers greater than 2 are greater than 4.

Equivalent Forms of Universal and Existential Statements

Observe that the two statements "∀ real numbers *x*, if *x* is an integer then *x* is rational" and "∀ integers *x*, *x* is rational" mean the same thing.

$$\forall x \in U, \text{ if } P(x) \text{ then } Q(x)$$

Both have informal translations "All integers are rational." In fact, a statement of the form

can always be rewritten in the form

$$\forall x \in D, Q(x)$$

by narrowing *U* to be the domain *D* consisting of all values of the variable *x* that make *P*(*x*) true.

Equivalent Forms of Universal and Existential Statements

Conversely, a statement of the form

$$\forall x \in D,\ Q(x)$$

can be rewritten as

$$\forall x, \text{ if } x \text{ is in } D \text{ then } Q(x).$$

Example 10 – *Equivalent Forms for Universal Statements*

Rewrite the following statement in the two forms "∀x, if _______ then _______" and "∀ _______ x, ________": All squares are rectangles.

Solution:

∀x, if x is a square then x is a rectangle.

∀ squares x, x is a rectangle.

Equivalent Forms of Universal and Existential Statements

Similarly, a statement of the form

“$\exists x$ such that $p(x)$ and $Q(x)$”

can be rewritten as

“$\exists x\ \varepsilon D$ such that $Q(x)$,”

where D is the set of all x for which $P(x)$ is true.

Implicit Quantification

Mathematical writing contains many examples of implicitly quantified statements. Some occur, through the presence of the word *a* or *an*. Others occur in cases where the general context of a sentence supplies part of its meaning.

For example, in an algebra course in which the letter x is always used to indicate a real number, the predicate: If $x > 2$ then $x^2 > 4$

is interpreted to mean the same as the statement $\forall$ real numbers x, if $x > 2$ then $x^2 > 4$.

Implicit Quantification

Mathematicians often use a double arrow to indicate implicit quantification symbolically.

For instance, they might express the above statement as

$$x > 2 \Rightarrow x^2 > 4.$$

• Notation

Let $P(x)$ and $Q(x)$ be predicates and suppose the common domain of x is D.

- The notation $\boldsymbol{P(x) \Rightarrow Q(x)}$ means that every element in the truth set of $P(x)$ is in the truth set of $Q(x)$, or, equivalently, $\forall x,\ P(x) \to Q(x)$.
- The notation $\boldsymbol{P(x) \Leftrightarrow Q(x)}$ means that $P(x)$ and $Q(x)$ have identical truth sets, or, equivalently, $\forall x,\ P(x) \leftrightarrow Q(x)$.

Negations of Quantified Statements

The general form of the negation of a universal statement follows immediately from the definitions of negation and of the truth values for universal and existential statements.

Theorem 3.2.1 Negation of a Universal Statement

The negation of a statement of the form

$$\forall x \text{ in } D,\ Q(x)$$

is logically equivalent to a statement of the form

$$\exists x \text{ in } D \text{ such that } \sim Q(x).$$

Symbolically, $\sim(\forall x \in D,\ Q(x)) \equiv \exists x \in D \text{ such that } \sim Q(x)$.

The general form for the negation of an existential statement follows immediately from the definitions of negation and of the truth values for existential and universal statements.

Theorem 3.2.2 Negation of an Existential Statement

The negation of a statement of the form

$$\exists x \text{ in } D \text{ such that } Q(x)$$

is logically equivalent to a statement of the form

$$\forall x \text{ in } D, \sim Q(x).$$

Symbolically, $\sim(\exists x \in D \text{ such that } Q(x)) \equiv \forall x \in D, \sim Q(x).$

Write formal negations for the following statements:

a. $\forall$ primes p, p is odd.
b. $\exists$ a triangle T such that the sum of the angles of T equals 200°.

Solution:

a. By applying the rule for the negation of a $\forall$ statement, you can see that the answer is

$$\exists \text{a prime } p \text{ such that } p \text{ is not odd.}$$

b. By applying the rule for the negation of a ∃ statement,
you can see that the answer is

∀ triangles T, the sum of the angles of T does not equal 200°.

Negations of universal conditional statements are of special importance in mathematics.

The form of such negations can be derived from facts that have already been established.

$$\sim(\forall x, P(x) \to Q(x)) \equiv \exists x \text{ such that } \sim(P(x) \to Q(x)). \qquad 3.2.1$$

By definition of the negation of a *for all* statement,

But the negation of an if-then statement is logically equivalent to an *and* statement. More precisely,

$$\sim(P(x) \to Q(x)) \equiv P(x) \wedge \sim Q(x). \qquad 3.2.2$$

Substituting (3.2.2) into (3.2.1) gives

$$\sim(\forall x,\ P(x) \to Q(x)) \equiv \exists x \text{ such that } (P(x) \wedge \sim Q(x)).$$

Written less symbolically, this becomes

Negation of a Universal Conditional Statement

$$\sim(\forall x, \text{if } P(x) \text{ then } Q(x)) \equiv \exists x \text{ such that } P(x) \text{ and } \sim Q(x).$$

Write a formal negation for statement (a) and an informal negation for statement (b).

a. $\forall$ people *p*, if *p* is blond then *p* has blue eyes.
b. If a computer program has more than 100,000 lines, then it contains a bug.

Solution:

a. $\exists$ a person *p* such that *p* is blond and *p* does not have blue eyes.

b. There is at least one computer program that has more than 100,000 lines and does not contain a bug.

The negation of a *for all* statement is a *there exists* statement, and the negation of a *there exists* statement is a *for all* statement.

These facts are analogous to De Morgan's laws, which state that the negation of an *and* statement is an *or* statement and that the negation of an *or* statement is an *and* statement.

If $Q(x)$ is a predicate and the domain D of x is the set $\{x_1, x_2, \ldots, x_n\}$, then the statements

$$\forall x \in D,\ Q(x)$$

and

$$Q(x_1) \wedge Q(x_2) \wedge \cdots \wedge Q(x_n)$$

are logically equivalent.

Similarly, if $Q(x)$ is a predicate and $D = \{x_1, x_2, . . . , x_n\}$, then the statements

$$\exists x \in D \text{ such that } Q(x)$$

and $$Q(x_1) \vee Q(x_2) \vee \cdots \vee Q(x_n)$$

are logically equivalent.

In general, a statement of the form

$$\forall x \text{ in } D, \text{ if } P(x) \text{ then } Q(x)$$

is called **vacuously true** or **true by default** if, and only if, $P(x)$ is false for every x in D.

We have known that a conditional statement has a contrapositive, a converse, and an inverse.

The definitions of these terms can be extended to universal conditional statements.

• Definition

Consider a statement of the form: $\forall x \in D$, if $P(x)$ then $Q(x)$.

1. Its **contrapositive** is the statement: $\forall x \in D$, if $\sim Q(x)$ then $\sim P(x)$.
2. Its **converse** is the statement: $\forall x \in D$, if $Q(x)$ then $P(x)$.
3. Its **inverse** is the statement: $\forall x \in D$, if $\sim P(x)$ then $\sim Q(x)$.

Write a formal and an informal contrapositive, converse, and inverse for the following statement:

If a real number is greater than 2, then its square is greater than 4.

Solution:

The formal version of this statement is

$$\forall x \in \mathbf{R}, \text{ if } x > 2 \text{ then } x^2 > 4.$$

Contrapositive: $\forall x \in \mathbf{R}$, if $x^2 \leq 4$ then $x \leq 2$.

Or: If the square of a real number is less than or equal to 4, then the number is less than or equal to 2.

Converse: $\forall x \in \mathbf{R}$, if $x^2 > 4$ then $x > 2$.

Or: If the square of a real number is greater than 4, then the number is greater than 2.

Inverse: $\forall x \in \mathbf{R}$, if $x \leq 2$ then $x^2 \leq 4$.

Or: If a real number is less than or equal to 2, then the square of the number is less than or equal to 4.

Let $P(x)$ and $Q(x)$ be any predicates, let D be the domain of x, and consider the statement $\forall x \in D, \text{if } P(x) \text{ then } Q(x)$

and its contrapositive

Any particular x in D that makes "if $P(x)$ then $Q(x)$" true also makes "if $\sim Q(x)$ then $\sim P(x)$" true (by the logical equivalence between $p \to q$ and $\sim q \to \sim p$).

$$\forall x \in D, \text{if } \sim Q(x) \text{ then } \sim P(x).$$

It follows that the sentence “If $P(x)$ then $Q(x)$” is true for all x in D if, and only if, the sentence “If $\sim Q(x)$ then $\sim P(x)$” is true for all x in D.

Thus we write the following and say that a universal conditional statement is logically equivalent to its contrapositive:

$$\boxed{\forall x \in D, \text{if } P(x) \text{ then } Q(x) \equiv \forall x \in D, \text{if } \sim Q(x) \text{ then } \sim P(x)}$$

The definitions of *necessary, sufficient,* and *only if* can also be extended to apply to universal conditional statements.

• Definition

- "$\forall x$, $r(x)$ is a **sufficient condition** for $s(x)$" means "$\forall x$, if $r(x)$ then $s(x)$."
- "$\forall x$, $r(x)$ is a **necessary condition** for $s(x)$" means "$\forall x$, if $\sim r(x)$ then $\sim s(x)$" or, equivalently, "$\forall x$, if $s(x)$ then $r(x)$."
- "$\forall x$, $r(x)$ **only if** $s(x)$" means "$\forall x$, if $\sim s(x)$ then $\sim r(x)$" or, equivalently, "$\forall x$, if $r(x)$ then $s(x)$."