CMSC 207- Lecture 14
CHAPTER 5: Sequences, Mathematical Induction, and Recursion (5.3)

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Mathematical Induction II

Mathematical Induction II is also known as the Strong Induction.

Strong Induction or Mathematical Induction II is used to Prove the Properties of Division, Recurrence Relations with multiple initial conditions as well as Inequalities.

Just like Mathematical Induction I, or the Weak Induction, Strong Induction is a powerful proof technique in discrete computational structures.

Mathematical Induction II – Proof Process

- **3 Parts in the Proof Process:**
- **Base:** Where we show P(n) is true for the base case, **n = a** or, P(a) is true.
- **Induction:** Assuming P(k) true for all values k = a through k=k (Inductive Hypothesis), show that P(k+1) is also true based on P(k) is true.
- **Conclusion:** P(a) is true. If P(n) is true, P(n+1) is also true. As P(a) is true, so P(a+1) is true. As P(a+1) is true, so P(a+2) is also true. As P(a+2) is true, so P(a+3) is true. Proceeding in this way, P(n) is true for all $n \geq a$. (**Hence, Proved**)

Example 1 – *Proving a Divisibility Property*

Use mathematical induction II (Strong Induction) for the following problem.

Given the following sequence: $a_1=1$ $a_2=3$ $a_3=8$
$a_i = a_{i-1} + 4a_{i-2}$ for $i>=4$
Prove or disprove that for all a_n for $n>=3$
that $4|a_n$ using strong induction.

Solution: Proof:

P(n): $4|a_n$
for $n>=3$

Basis or the Base Case: P(3): $4|a_3$ or, $4|8$, which is true. So, P(3) is true for $n = 3$.

Example 1 – *Solution*

Induction: For Induction Hypothesis, assume that P(k) is true for k>=3. So, we will need to prove P(k+1). As P(k) is true, so

$a_k = a_{k-1} + 4a_{k-2}$ for k>=4

Now, $4|a_k$ or, $4|(a_{k-1} + 4a_{k-2})$ (by Induction hypothesis)

Prove that $4|a_{k+1}$

$a_{k+1} = a_k + 4a_{k-1} = (a_{k-1} + 4a_{k-2}) + 4a_{k-1}$

Example 1 – *Solution* cont'd

However, $4|a_k$ or, $4|(a_{k-1} + 4a_{k-2})$ is true (by the Inductive hypothesis), and by the property of divisibility, $\mathbf{4|4a_{k-1}}$ (due to the multiplication factor of 4).

Hence, $4|((a_{k-1} + 4a_{k-2}) + 4a_{k-1})$ or $4 \mid a_{k+1}$

Q.E.D

Additional Example & *Solution*

Following Proposition has limited implication. The Proposition is proved using Mathematical Induction II:

Proposition 5.3.1

For all integers $n \geq 0$, $2^{2n} - 1$ is divisible by 3.

Proof (by mathematical induction):

Let the property $P(n)$ be the sentence "$2^{2n} - 1$ is divisible by 3."

$$2^{2n} - 1 \text{ is divisible by } 3. \qquad \leftarrow P(n)$$

Additional Example & *Solution*

Show that *P*(0) is true:

To establish *P*(0), we must show that

$$2^{2 \cdot 0} - 1 \text{ is divisible by } 3. \qquad \leftarrow P(0)$$

But

$$2^{2 \cdot 0} - 1 = 2^0 - 1 = 1 - 1 = 0$$

and 0 is divisible by 3 because $0 = 3 \cdot 0$.

Hence *P*(0) is true.

Additional Example & *Solution*

Show that for all integers $k \geq 0$, if $P(k)$ is true then $P(k + 1)$ is also true:

[Suppose that P(k) is true for a particular but arbitrarily chosen integer k ≥ 0. That is:]

Let k be any integer with $k \geq 0$, and suppose that

$$2^{2k} - 1 \text{ is divisible by } 3. \qquad \leftarrow P(k)$$

inductive hypothesis

By definition of divisibility, this means that

$$2^{2k} - 1 = 3r \quad \text{for some integer } r.$$

Additional Example & *Solution*

[We must show that P(*k* + 1) *is true. That is:]* We must show that

$$2^{2(k+1)} - 1 \text{ is divisible by } 3. \qquad \leftarrow \; P(k+1)$$

But

$$\begin{aligned} 2^{2(k+1)} - 1 &= 2^{2k+2} - 1 \\ &= 2^{2k} \cdot 2^2 - 1 && \text{by the laws of exponents} \\ &= 2^{2k} \cdot 4 - 1 \\ &= 2^{2k}(3+1) - 1 \end{aligned}$$

Additional Example & *Solution*

$$= 2^{2k} \cdot 3 + (2^{2k} - 1)$$ by the laws of algebra

$$= 2^{2k} \cdot 3 + 3r$$ by inductive hypothesis

$$= 3(2^{2k} + r)$$ by factoring out the 3.

But $(2^{2k} + r)$ is an integer because it is a sum of products of integers, and so, by definition of divisibility, $2^{2(k+1)} - 1$ is divisible by 3 *[as was to be shown]*. *[Since we have proved the basis step and the inductive step, we conclude that the proposition is true.]*

In-class Practice Exercises

- Show by Induction that $(7^n – 1)$ is divisible by 6, for each integer $n \geq 0$.

- Show by Mathematical Induction that $(n^3 – 7n + 3)$ is divisible by 3, for each integer $n \geq 0$.

Mathematical Induction II

The next example illustrates the special usage of Mathematical Induction II or the Strong Induction to prove an inequality.

Example 2 – *Proving an Inequality*

Use mathematical induction to prove that for all integers $n \geq 3$, $2n + 1 < 2^n$.

Solution:

In this example the property $P(n)$ is the inequality

$$\boxed{2n + 1 < 2^n.} \quad \leftarrow \text{the property } (P(n))$$

By substitution, the statement for the basis step, $P(3)$, is

$$\boxed{2 \cdot 3 + 1 < 2^3.} \quad \leftarrow \text{basis } (P(3))$$

Example 2 – *Solution*

The supposition for the inductive step, *P*(*k*), is

$$\boxed{2k + 1 < 2^k,} \qquad \leftarrow \text{inductive hypothesis } (P(k))$$

and the conclusion to be shown is

$$\boxed{2(k + 1) + 1 < 2^{k+1}.} \qquad \leftarrow \text{to show } (P(k + 1))$$

To prove the basis step, observe that the statement *P*(3) is true because $2 \cdot 3 + 1 = 7$, $2^3 = 8$, and $7 < 8$.

Example 2 – *Solution*

To prove the inductive step, suppose the inductive hypothesis, that $P(k)$ is true for an integer $k \geq 3$.

This means that $2k + 1 < 2^k$ is assumed to be true for a particular but arbitrarily chosen integer $k \geq 3$. Then derive the truth of $P(k + 1)$. Or, in other words, show that the inequality is true. But by multiplying out and regrouping,

$$2(k+1)+1 < 2^{k+1}$$

$$2(k+1)+1 = 2k+3 = (2k+1)+2, \qquad 5.3.1$$

$$(2k+1)+2 < 2^k + 2. \qquad 5.3.2$$

and by substitution from the inductive hypothesis,

Example 2 – *Solution*

Hence

$$(2k + 1) + 2 < 2^k + 2.$$

The left-most part of equation (5.3.1) is less than the right-most part of inequality (5.3.2).

If it can be shown that $2^k + 2$ is less than 2^{k+1}, then the desired inequality will have been proved. But since the quantity 2^k can be added to or subtracted from an inequality without changing its direction,

$$2^k + 2 < 2^{k+1} \quad \Leftrightarrow \quad 2 < 2^{k+1} - 2^k = 2^k(2 - 1) = 2^k.$$

And since multiplying or dividing an inequality by 2 does not change its direction,

$$2 < 2^k \quad \Leftrightarrow \quad 1 = \frac{2}{2} < \frac{2^k}{2} = 2^{k-1}$$

by the laws of exponents.

Example 2 – *Solution*

This last inequality is clearly true for all $k \geq 2$. Hence it is true that $2(k+1)+1 < 2^{k+1}$.

This discussion is made more flowing (but less intuitive) in the following formal proof:

Proposition 5.3.2

For all integers $n \geq 3$, $2n + 1 < 2^n$.

Proof (by Strong Induction):

Let the property *P*(*n*) be the inequality

$$2n + 1 < 2^n. \qquad \leftarrow P(n)$$

Example 2 – *Solution*

Show that *P*(3) is true:

To establish *P*(3), we must show that

$$2 \cdot 3 + 1 < 2^3. \qquad \leftarrow P(3)$$

But

$$2 \cdot 3 + 1 = 7 \quad \text{and} \quad 2^3 = 8 \quad \text{and} \quad 7 < 8.$$

Hence, *P*(3) is true.

Example 2 – *Solution*

Show that for all integers *k* ≥ 3, if *P*(*k*) is true then *P*(*k* + 1) is also true:

[Suppose that P(*k*) *is true for a particular but arbitrarily chosen integer k* ≥ 3. *That is:]*

Suppose that *k* is any integer with *k* ≥ 3 such that $2k + 1 < 2^k$. $\leftarrow P(k)$ inductive hypothesis

[We must show that P(*k* + 1) *is true. That is:]* We must show that $2(k + 1) + 1 < 2^{(k+1)}$.

Example 2 – *Solution*

Or, equivalently,

$$2k + 3 < 2^{(k+1)}. \qquad \leftarrow P(k+1)$$

But

$$2k + 3 = (2k + 1) + 2 \qquad \text{by algebra}$$

$$< 2^k + 2^k \qquad \text{because } 2k + 1 < 2^k \text{ by the inductive hypothesis and because } 2 < 2^k \text{ for all integers } k \geq 2$$

- $2k + 3 < 2 \cdot 2^k = 2^{k+1}$ by the laws of exponents.

[Since we have proved the basis step and the inductive step, we conclude that the proposition is true.]

Another Example of Strong Induction

Given the following sequence:

$a_1=1$ $a_2=7$ $a_i = 2a_{i-1} + 5a_{i-2}$ for $i>=3$

Prove or disprove that for all $n>=1$,

a_n is always odd using strong induction.

Proof:

P(n) : a_n is always odd for $n>=1$

Basis: P(1): a_1 is always odd. Since, $a_1 = 1$, therefore, P(1) is true.

Induction: Induction Hypothesis, P(k) is true for all $k>=1$. Show that P(k+1) is also true, i.e.,

a_{k+1} is always odd for $k>=1$

Example on Strong Induction Contd.

According to the Induction Hypothesis, P(k) is true. So, $a_k = 2a_{k-1} + 5a_{k-2}$ is always odd for k>=3. Now, for P(k+1), $a_{k+1} = 2a_k + 5a_{k-1}$

$= 2*(2a_{k-1} + 5a_{k-2}) + 5a_{k-1}$

$= 4a_{k-1} + 10a_{k-2} + 5a_{k-1}$

$= 9a_{k-1} + 10a_{k-2}$

By Strong Induction, a_{k-1} and a_{k-2} are both odd. Now, An **Odd Number × An Even Number is even** . So $10a_{k-2}$ is even. Also, An Odd Number × An Odd Number is Odd. Hence, $9a_{k-1}$ is Odd. Also, An Odd Number + An Even Number provides with an odd number (Example: 5 + 4 = 9). **So, $9a_{k-1} + 10a_{k-2}$ will always provide with an odd number.** This proves P(k+1) or, $a_{k+1} = 2a_k + 5a_{k-1}$ is Odd. **Q.E.D.**

In-class Practice Exercise

- Show by strong induction that $n^2 < 2^n$, for all integers $n \geq 5$.

An Interesting Example on Strong Induction

- A chocolate bar consists of unit squares arranged in an **n×m rectangular grid**. It is possible to split the bar into individual unit squares, by breaking along the lines. Show that the number of breaks needed is **nm−1**.
- **Proof: Basis-**For a 1×1 square, we are already done, so no break is needed. Therefore, no steps are needed.
  $1\times1-1=0$, so the base case is true.

An Interesting Example on Strong Induction-Contd.

- **Induction Step:** Let P(n,m) denote the number of breaks needed to split up an n×m square.
- With Induction Hypothesis, we may assume that the first break is along a row, and we get an **n1×m** and an **n2×m** bar, where **n1+n2=n.** By the induction hypothesis, the number of further breaks that we need is **n1×m−1** and **n2×m−1** (for instance, with 2 rows, and 2 columns, we will need 1 break for the 2 rows, and 1 break each for the 2 columns in a broken piece of a chocolate totaling (1+1+1) = 3 = (2x2 -1)). Hence, the total number of breaks that we need is:

1+(n1×m−1)+(n2×m−1)=(n1+n2)×m−1=n×m−1. Q.E.D.