CMSC 207- Lecture 28
CHAPTER 10: Graphs And Trees
(10.1 and 10.2 - Graphs)

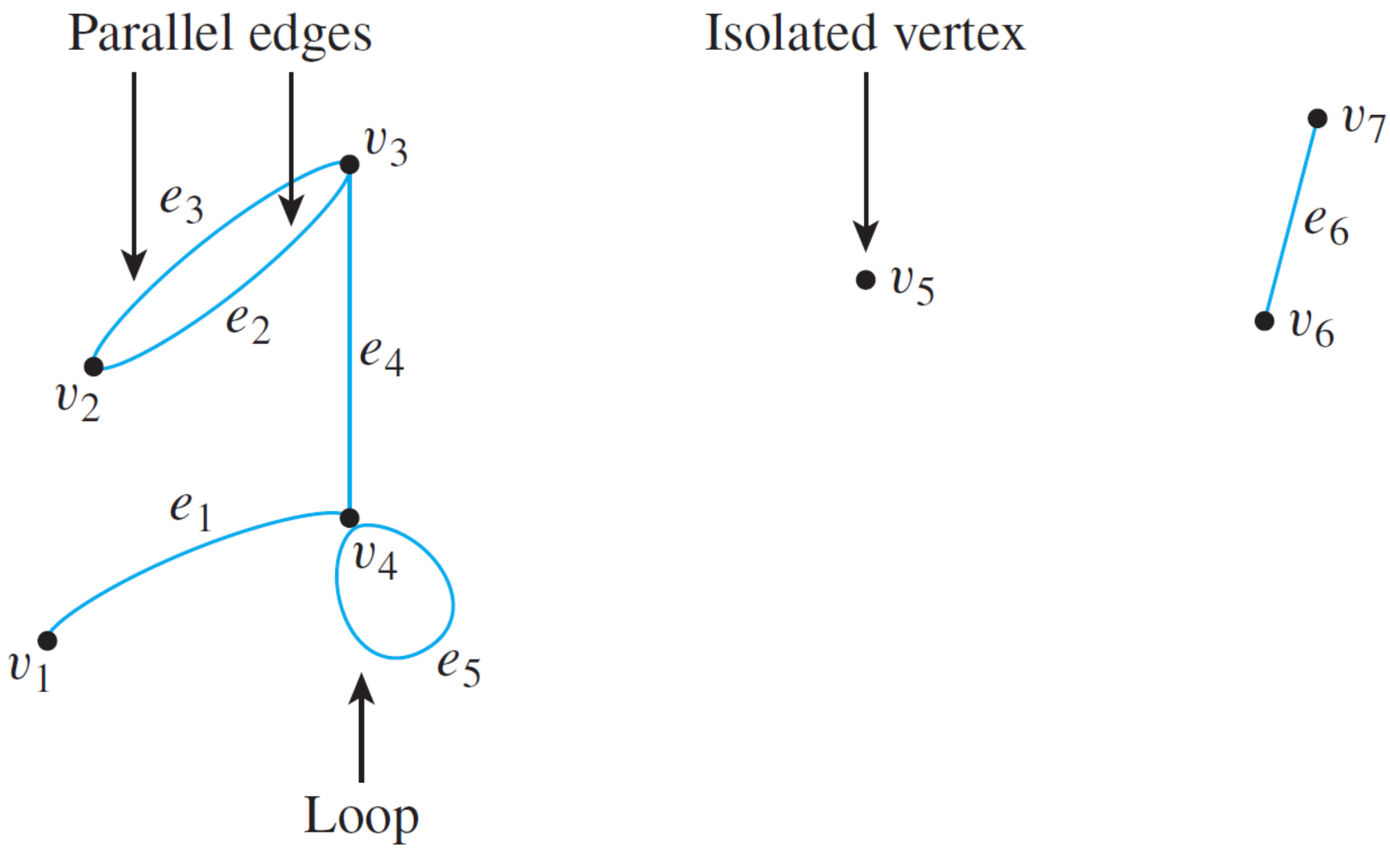
Dr. Ahmed Tarek

Graphs: Definitions and Basic Properties

In general, a graph consists of a set of vertices and a set of edges connecting various pairs of vertices.

Graphs: Definitions and Basic Properties

The edges may be straight or curved and should either connect one vertex to another or a vertex to itself, as shown below.



Graphs: Definitions and Basic Properties

In this drawing, the vertices have been labeled with ***v* ’s** and the edges with ***e*’s**.

When an edge connects a vertex to itself (as $\boldsymbol{e_5}$ does), it is called a ***loop***. When two edges connect the same pair of vertices (as e_2 and e_3 do), they are said to be ***parallel***.

If a vertex is unconnected by an edge to any other vertex in the graph (as $\boldsymbol{v_5}$ is), then the vertex is said to be ***isolated***.

Graphs: Definitions and Basic Properties

The formal definition of a graph follows.

• Definition

A **graph** G consists of two finite sets: a nonempty set $V(G)$ of **vertices** and a set $E(G)$ of **edges,** where each edge is associated with a set consisting of either one or two vertices called its **endpoints.** The correspondence from edges to endpoints is called the **edge-endpoint function.**

An edge with just one endpoint is called a **loop,** and two or more distinct edges with the same set of endpoints are said to be **parallel.** An edge is said to **connect** its endpoints; two vertices that are connected by an edge are called **adjacent;** and a vertex that is an endpoint of a loop is said to be **adjacent to itself.**

An edge is said to be **incident on** each of its endpoints, and two edges incident on the same endpoint are called **adjacent.** A vertex on which no edges are incident is called **isolated.**

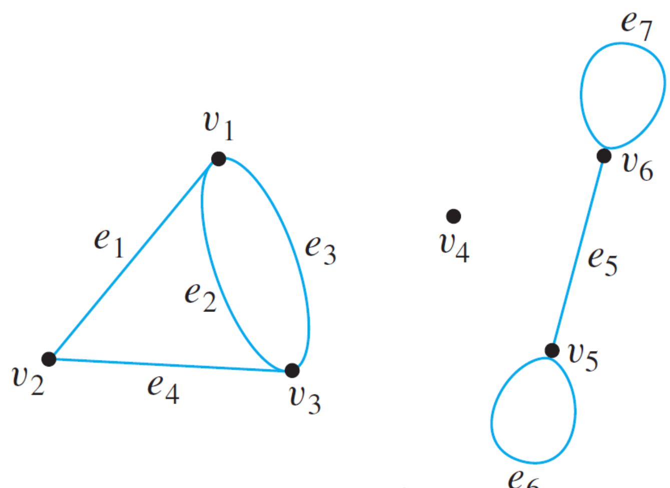
Graphs: Definitions and Basic Properties

Graphs have pictorial representations in which the vertices are represented by dots and the edges by line segments.

A given pictorial representation uniquely determines a graph.

Example 1 – *Terminology*

Consider the following graph:



a. Write the vertex set and the edge set, and give a table showing the edge-endpoint function.

b. Find all edges that are incident on v_1, all vertices that are adjacent to v_1, all edges that are adjacent to e_1, all loops, all parallel edges, all vertices that are adjacent to themselves, and all isolated vertices.

Example 1(a) – *Solution*

vertex set = $\{v_1, v_2, v_3, v_4, v_5, v_6\}$

edge set = $\{e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$

edge-endpoint function:

Edge	Endpoints
e_1	$\{v_1, v_2\}$
e_2	$\{v_1, v_3\}$
e_3	$\{v_1, v_3\}$
e_4	$\{v_2, v_3\}$
e_5	$\{v_5, v_6\}$
e_6	$\{v_5\}$
e_7	$\{v_6\}$

Example 1(a) – *Solution*

The isolated vertex v_4 does not appear in this table.

Although each edge must have either one or two endpoints, a vertex need not be an endpoint of an edge.

Example 1(b) – *Solution*

e_1, e_2, and e_3 are incident on v_1.

v_2 and v_3 are adjacent to v_1.

e_2, e_3, and e_4 are adjacent to e_1.

e_6 and e_7 are loops.

e_2 and e_3 are parallel.

v_5 and v_6 are adjacent to themselves.

v_4 is an isolated vertex.

Graphs: Definitions and Basic Properties

As noted earlier, a given pictorial representation uniquely determines a graph.

However, a given graph may have more than one pictorial representation.

Such things as the lengths or curvatures of the edges and the relative position of the vertices on the page may vary from one pictorial representation to another.

Graphs: Definitions and Basic Properties

Directed graph is similar to the definition of graph, except that one associates an ***ordered pair* of vertices** with each edge **instead of a *set* of vertices**.

Thus each edge of a directed graph can be drawn as an arrow going from the first vertex to the second vertex of the ordered pair.

Graphs: Definitions and Basic Properties

- **Definition**

A **directed graph,** or **digraph,** consists of two finite sets: a nonempty set $V(G)$ of vertices and a set $D(G)$ of directed edges, where each is associated with an ordered pair of vertices called its **endpoints.** If edge e is associated with the pair (v, w) of vertices, then e is said to be the (**directed**) **edge** from v to w.

Note that each directed graph has an **associated ordinary (undirected) graph**, which is obtained by ignoring the directions of the edges.

Special Graphs

One important class of graphs consists of those that do not have any **loops** or **parallel edges.** Such graphs are called ***simple*****.** In a simple graph, no two edges share the same set of endpoints, so specifying two endpoints is sufficient to determine an edge.

• Definition and Notation

A **simple graph** is a graph that does not have any loops or parallel edges. In a simple graph, an edge with endpoints v and w is denoted $\{v, w\}$.

Example 8 – *A Simple Graph*

Draw all simple graphs with the four vertices $\{u, v, w, x\}$ and two edges, one of which is $\{u, v\}$.

Solution:

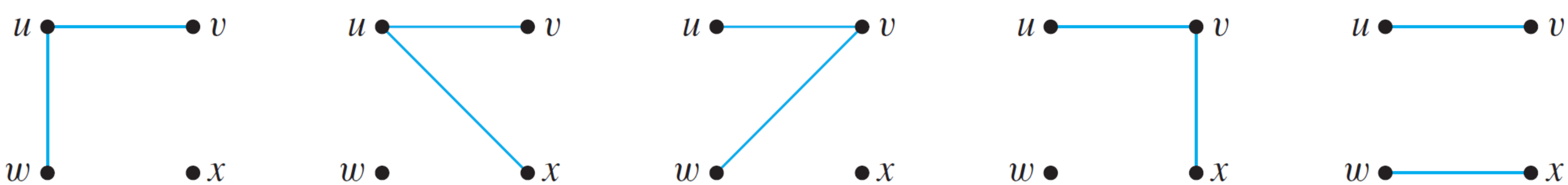
Each possible edge of a simple graph corresponds to a subset of two vertices.

Given four vertices, there are $\binom{4}{2} = 6$ such subsets in all:

$\{u, v\}$, $\{u, w\}$, $\{u, x\}$, $\{v, w\}$, $\{v, x\}$, and $\{w, x\}$.

Example 8 – *Solution*

Now one edge of the graph is specified to be {*u*, *v*}, so any of the remaining five from this list can be chosen to be the second edge. The possibilities are shown below.



Special Graphs

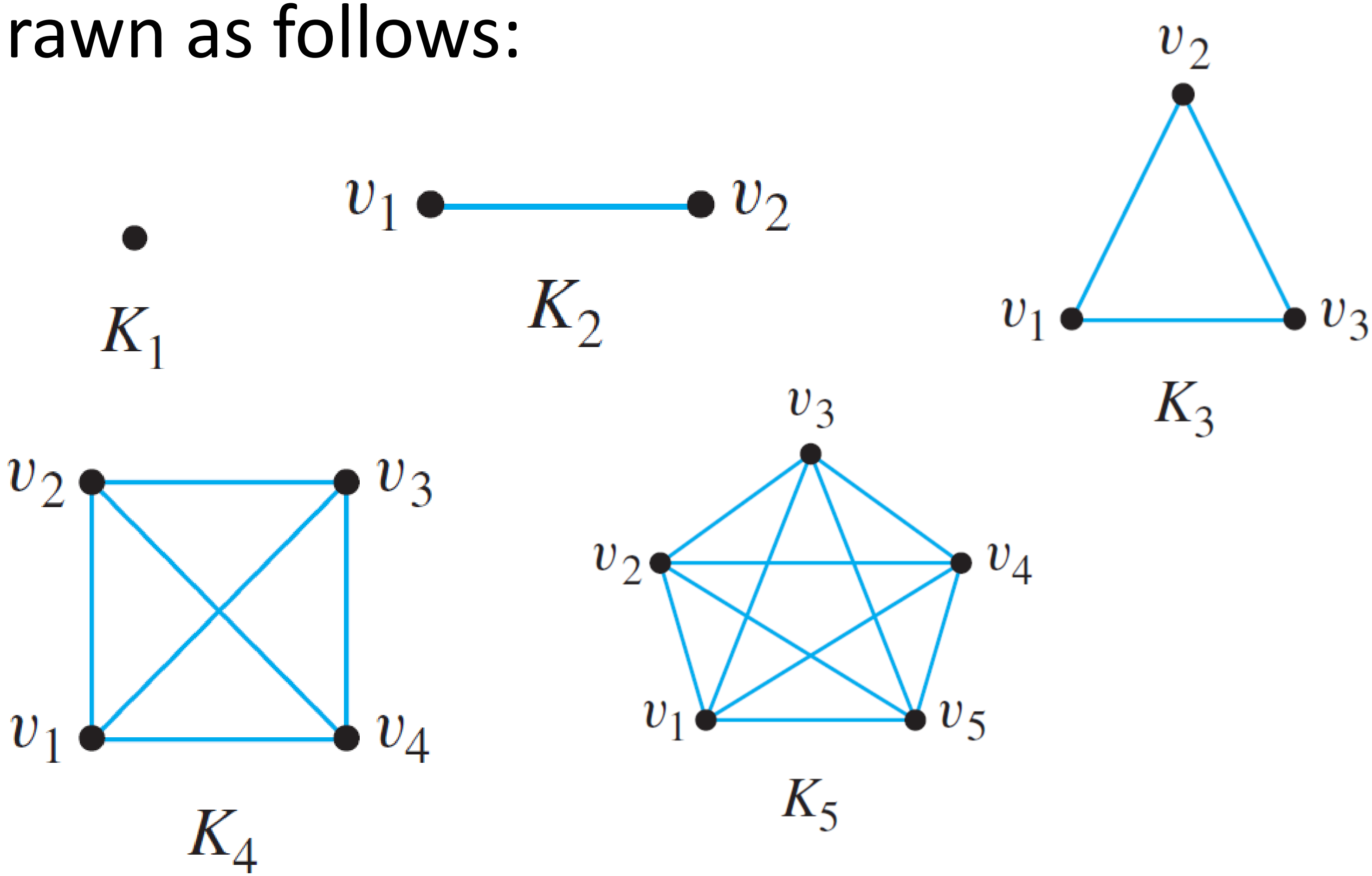
Another important class of graphs consists of those that are "**complete**" that has all pairs of vertices, which are connected by edges.

- **Definition**

Let n be a positive integer. A **complete graph on $\boldsymbol{n}$ vertices,** denoted K_n, is a simple graph with n vertices and exactly one edge connecting each pair of distinct vertices.

Example 9 – *Complete Graphs on n Vertices: K_1, K_2, K_3, K_4, K_5*

The complete graphs K_1, K_2, K_3, K_4, and K_5 can be drawn as follows:



Special Graphs

In yet another class of graphs, the vertex set can be separated into two subsets: Each vertex in one of the subsets is connected by exactly one edge to each vertex in the other subset, but not to any vertices in its own subset. Such a graph is called ***complete bipartite***.

• Definition

Let m and n be positive integers. A **complete bipartite graph on (m, n) vertices,** denoted $K_{m,n}$, is a simple graph with distinct vertices $v_1, v_2, \ldots, v_m$ and $w_1, w_2, \ldots, w_n$ that satisfies the following properties: For all $i, k = 1, 2, \ldots, m$ and for all $j, l = 1, 2, \ldots, n$,

1. There is an edge from each vertex v_i to each vertex w_j.
2. There is no edge from any vertex v_i to any other vertex v_k.
3. There is no edge from any vertex w_j to any other vertex w_l.

Special Graphs

• Definition

A graph H is said to be a **subgraph** of a graph G if, and only if, every vertex in H is also a vertex in G, every edge in H is also an edge in G, and every edge in H has the same endpoints as it has in G.

The Concept of Degree

The ***degree of a vertex*** is the number of end segments of edges that "**stick out of**" the vertex.

We will show that the sum of the degrees of all the vertices in a graph is twice the number of edges of the graph.

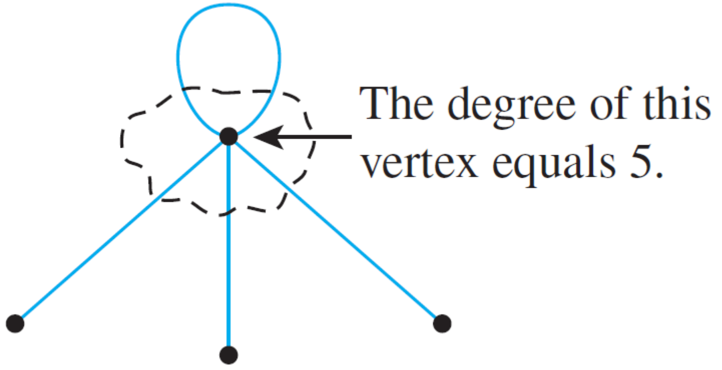
• Definition

Let G be a graph and v a vertex of G. The **degree of *v*,** denoted **deg(*v*),** equals the number of edges that are incident on v, with an edge that is a loop counted twice. The **total degree of *G*** is the sum of the degrees of all the vertices of G.

The Concept of Degree

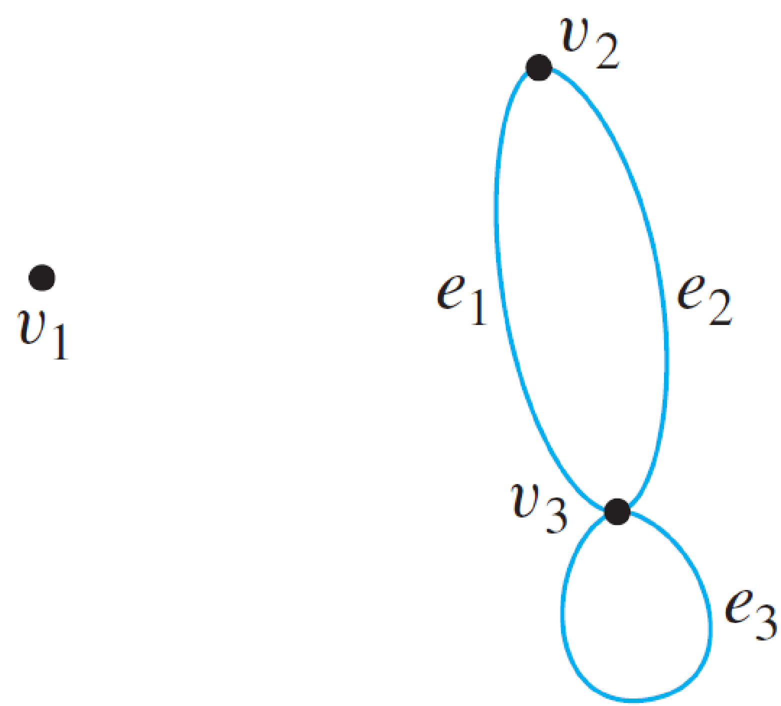
Since an edge that is a loop is counted twice, the degree of a vertex can be obtained from the drawing of a graph by counting how many end segments of edges are incident on the vertex.

This is illustrated below.



In-class Assignment #1

Find the degree of each vertex of the graph *G* shown below. Then find the total degree of *G*.



In-class Assignment #2:

- A graph has vertices of degrees 0, 2, 2, 3, and 9. How many edges does the graph have?
- A graph has vertices of degrees 1, 1, 4, 4, and 6. How many edges does the graph have?

The Concept of Degree

As the proof demonstrates, the conclusion is true even if the graph contains loops.

Theorem 10.1.1 The Handshake Theorem

If G is any graph, then the sum of the degrees of all the vertices of G equals twice the number of edges of G. Specifically, if the vertices of G are $v_1, v_2, \ldots, v_n$, where n is a nonnegative integer, then

$$\begin{aligned}\text{the total degree of } G &= \deg(v_1) + \deg(v_2) + \cdots + \deg(v_n)\\ &= 2 \cdot (\text{the number of edges of } G).\end{aligned}$$

Corollary 10.1.2

The total degree of a graph is even.

Definitions

• Definition

Let G be a graph, and let v and w be vertices in G.

A **walk from *v* to *w*** is a finite alternating sequence of adjacent vertices and edges of G. Thus a walk has the form

$$v_0 e_1 v_1 e_2 \cdots v_{n-1} e_n v_n,$$

where the v's represent vertices, the e's represent edges, $v_0 = v$, $v_n = w$, and for all $i = 1, 2, \ldots n$, v_{i-1} and v_i are the endpoints of e_i. The **trivial walk from *v* to *v*** consists of the single vertex v.

A **trail from *v* to *w*** is a walk from v to w that does not contain a repeated edge.

A **path from *v* to *w*** is a trail that does not contain a repeated vertex.

A **closed walk** is a walk that starts and ends at the same vertex.

A **circuit** is a closed walk that contains at least one edge and does not contain a repeated edge.

A **simple circuit** is a circuit that does not have any other repeated vertex except the first and last.

Definitions

For ease of reference, these definitions are summarized in the following table:

	Repeated Edge?	Repeated Vertex?	Starts and Ends at Same Point?	Must Contain at Least One Edge?
Walk	allowed	allowed	allowed	no
Trail	no	allowed	allowed	no
Path	no	no	no	no
Closed walk	allowed	allowed	yes	no
Circuit	no	allowed	yes	yes
Simple circuit	no	first and last only	yes	yes

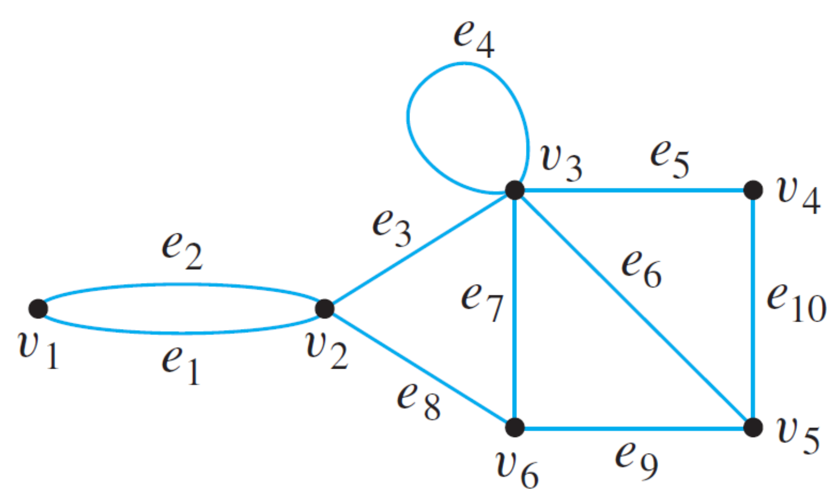
Often a walk can be specified by giving either a sequence of edges or a sequence of vertices.

Example 2 – *Walks, Trails Paths, and Circuits*

In the graph below, determine which of the following walks are trails, paths, circuits, or simple circuits.

a. $v_1 e_1 v_2 e_3 v_3 e_4 v_3 e_5 v_4$ **b.** $e_1 e_3 e_5 e_5 e_6$ **c.** $v_2 v_3 v_4 v_5 v_3 v_6 v_2$

d. $v_2 v_3 v_4 v_5 v_6 v_2$ **e.** $v_1 e_1 v_2 e_1 v_1$ **f.** v_1



Example 2 – *Solution*

a. This walk has a repeated vertex but does not have a repeated edge, so it is a **trail** from v_1 to v_4 but not a path.

b. This is **just a walk** from v_1 to v_5. It is not a trail because it has a repeated edge.

c. This walk starts and ends at $\boldsymbol{v_2}$, contains at least one edge, and does not have a repeated edge, so it is a **circuit**. Since **the vertex $\boldsymbol{v_3}$ is repeated** in the middle, it is not a simple circuit.

d. This walk starts and ends at $\boldsymbol{v_2}$, contains at least one edge, does not have a repeated edge, and does not have a repeated vertex. Thus it is a **simple circuit**.

Example 2 – *Solution*

e. This is just a **closed walk** starting and ending at v_1. It is **not a circuit** because **edge e_1** is repeated.

f. The first vertex of this walk is the same as its last vertex, but it **does not contain an edge**, and so it is not a circuit. It is a **closed walk** from v_1 to v_1. (It is also **a trail from v_1 to v_1**.)

Connectedness

The formal definition of connectedness is stated in terms of walks.

• Definition

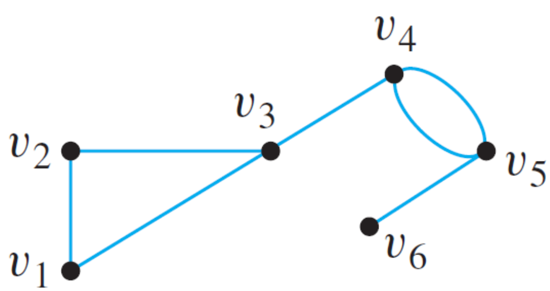
Let G be a graph. Two **vertices *v* and *w* of *G* are connected** if, and only if, there is a walk from v to w. The **graph *G* is connected** if, and only if, given *any* two vertices v and w in G, there is a walk from v to w. Symbolically,

$$G \text{ is connected} \quad \Leftrightarrow \quad \forall \text{ vertices } v, w \in V(G), \exists \text{ a walk from } v \text{ to } w.$$

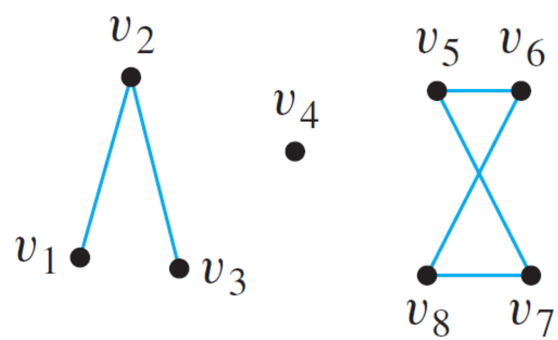
If you take the negation of this definition, you will see that a graph *G is not connected* if, and only if, there are two vertices of *G* that are not connected by any walk.

Example 3 – *Connected and Disconnected Graphs*

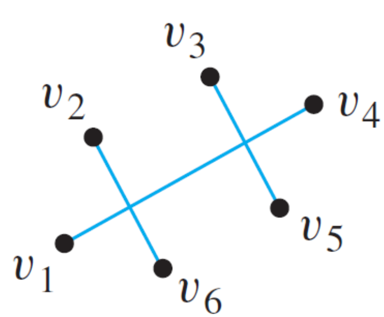
Which of the following graphs are connected?



(a)



(b)

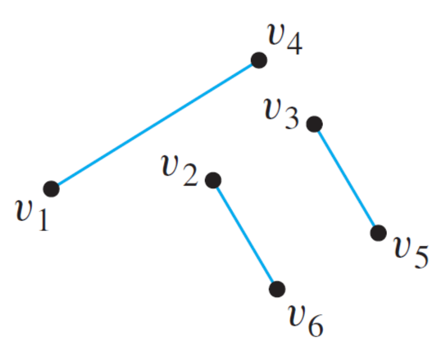


(c)

Example 3 – *Solution*

The graph represented in (*a*) is connected, whereas those of (*b*) and (*c*) are not. To understand why (*c*) is not connected, we know that in a drawing of a graph, two edges may cross at a point that is not a vertex.

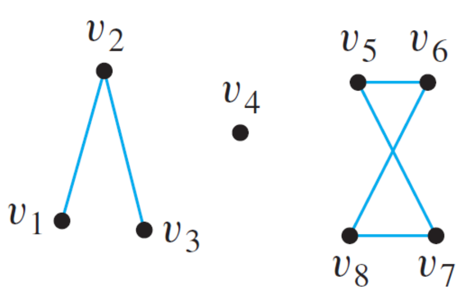
Thus the graph in **(*c*)** can be redrawn as follows:



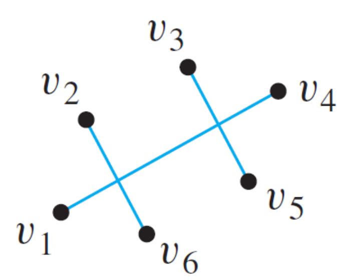
Connectedness

The graphs in (*b*) and (*c*) are both made up of three pieces, each of which is itself a connected graph.

A ***connected component*** of a graph is a connected subgraph of largest possible size.



(b)



(c)

Connectedness

• Definition

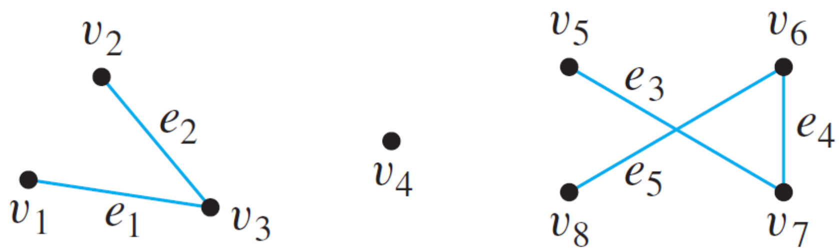
A graph H is a **connected component** of a graph G if, and only if,

1. H is subgraph of G;
2. H is connected; and
3. no connected subgraph of G has H as a subgraph and contains vertices or edges that are not in H.

Any graph is a kind of union of its connected components.

Example 4 – *Connected Components*

Find all connected components of the following graph G.



Solution:

G has three connected components: H_1, H_2, and H_3 with vertex sets V_1, V_2, and V_3 and edge sets E_1, E_2, and E_3, where

$$
\begin{aligned}
V_1 &= \{v_1, v_2, v_3\}, & E_1 &= \{e_1, e_2\}, \\
V_2 &= \{v_4\}, & E_2 &= \emptyset, \\
V_3 &= \{v_5, v_6, v_7, v_8\}, & E_3 &= \{e_3, e_4, e_5\}.
\end{aligned}
$$

In-class Practice Exercise #3

- Draw a graph with four vertices of degrees 1, 1, 1, and 4.
- Draw a graph with four vertices of degrees 1, 2, 3, and 4.