CMSC 207- Lecture 3
CHAPTER 1: SPEAKING MATHEMATICALLY – Contd.

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Relations and Functions

Definition

Let A and B be sets. A **relation R from A to B** is a subset of $A \times B$. Given an ordered pair (x, y) in $A \times B$, **x is related to y by R**, written $x\ R\ y$, if, and only if, (x, y) is in R. The set A is called the **domain** of R and the set B is called its **co-domain**.

The notation for a relation R may be written symbolically as follows:

$$\boldsymbol{x\ R\ y} \quad \text{means that} \quad \boldsymbol{(x, y) \in R.}$$

The notation $x \not R\ y$ means that x is not related to y by R:

$$\boldsymbol{x \not R\ y} \quad \text{means that} \quad \boldsymbol{(x, y) \notin R.}$$

Example – *A Relation as a Subset of Cartesian Product*

- Let $A = \{3, 4\}$ and $B = \{6, 7, 8\}$ and define a relation R from A to B as follows: Given any $(x, y) \in A \times B$, $(x, y) \in R$ means that $\frac{x - y}{2}$ is an integer.
- a. State explicitly which ordered pairs are in $A \times B$ and which are in R.
- b. Is $3\ R\ 7$? Is $4\ R\ 6$? Is $3\ R\ 6$?
- c. What are the domain and co-domain of R?

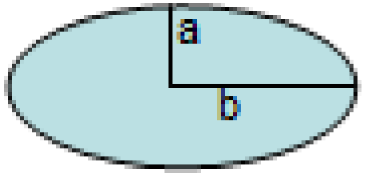
Example – *Solution*

- a. $A \times B = \{(3, 6), (3, 7), (3, 8), (4, 6), (4, 7), (4, 8)\}$. To determine explicitly the composition of *R*, examine each ordered pair in $A \times B$ to see whether its elements satisfy the defining condition for *R*.
- So, R ={(3, 7), (4, 6), (4, 8)}
  - (3, 7) $\in R$ because (3-7)/2 = 2, an integer
  - (4, 6) $\in R$ because (4-6)/2 = -1, an integer
  - (3, 6) $\notin R$ because (3-6)/2 = -1.5, NOT an integer

For R, Domain = {3, 4}, & Co-domain = {6, 7, 8}

Example – *The Ellipse Relation*

- Define a relation ***E*** from **R** to **R** as follows: For any $(x, y) \in \mathbf{R} \times \mathbf{R}$, $(x, y) \in E$ means that $(x^2/9)+(y^2/4)=1$



a. Is $(3, 0) \in$ E? Is $(0, 0) \in$ E?
Is $(0, 2) \in$ E? Is $(3, 2) \in$ E?

b. What are the domain and co-domain of *E*?

Here, for the Ellipse represented by the *Relation E*, half of the major axis, b = 3, and half of the minor axis, a = 2

Example: The Ellipse Relation – *Solution*

a. YES. $(3, 0) \in \mathbf{E}$. As $(3^2/9)+(0^2/4)=1$, so it is true.

NO. $(0, 0) \notin \mathbf{E}$. As $(0^2/9)+(0^2/4) = 0 \neq 1$, so it is NOT True.

YES. $(0, 2) \in \mathbf{E}$. As $(0^2/9)+(2^2/4)=1$, so it is true.

NO. $(3, 2) \notin \mathbf{E}$. As $(3^2/9)+(2^2/4) = 1 + 1 = 2 \neq 1$, so it is NOT True.

b. The domain, and co-domain of ***E*** are both **R**, the set of all real numbers.

Arrow Diagram of a Relation

Arrow Diagram of a Relation

- Suppose R is a relation from a set A to a set B.

The **arrow diagram for Relation, *R*** is obtained as follows:

- **1.** Represent the elements of A as points in one elliptical region and the elements of B as points in another elliptical region.
- **2.** For each x in A and y in B, draw an arrow from x to y if, and only if, x is related to y by R. Symbolically:
- **Draw an arrow from *x* to *y***
- **if, and only if, $x \, R \, y$**

OR, **if, and only if, $(x, y) \in R$.**

Example – *Arrow Diagrams of Relations*

- Let $A = \{1, 2, 3\}$ and $B = \{1, 5, 3\}$ and define relations S and T from A to B as follows:

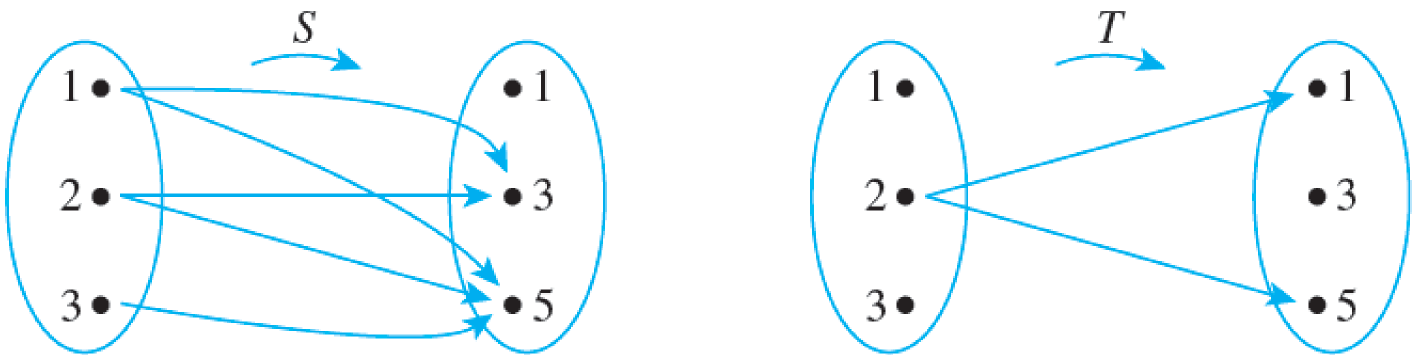
- For every $(x, y) \in A \times B$,

- $(x, y) \in S$ means that $x < y$ *S* is a "less than" relation.

- $T = \{(2, 1), (2, 5)\}$.

- Draw arrow diagrams for S and T.

Example – *Solution*



•These example relations illustrate that it is possible to have several arrows coming out of the same element of *A* pointing to different elements in B. Also, it is quite possible to have an element of *A* that does not have an arrow coming out of it.

Functions

Definition

A function F from a set A to a set B is a relation with domain A and co-domain B that satisfies the following two properties:

1. For every element x in A, there is an element y in B such that $(x, y) \in F$.
2. For all elements x in A and y and z in B,

$$\text{if} \quad (x, y) \in F \text{ and } (x, z) \in F, \quad \text{then} \quad y = z.$$

Functions

- The two Properties (1) and (2) of a Function can be stated less formally as follows: A relation *F* from *A* to *B* is a function if, and only if:
- 1. Every element of *A* is the first element of an ordered pair of *F*.
- 2. No two distinct ordered pairs in *F* have the same first element.

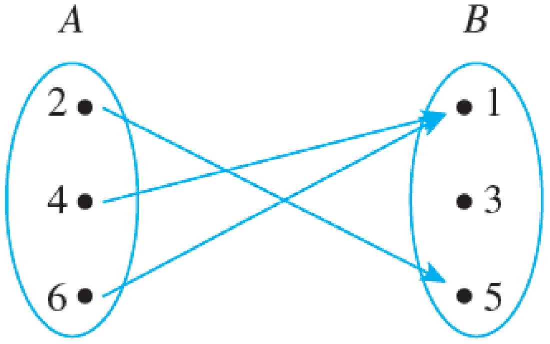
Therefore, if x is an arbitrary integer, following is a function of x: F(x) = 2x + 7

Function Notation

If A and B are sets and F is a function from A to B, then given any element x in A, the unique element in B that is related to x by F is denoted $\boldsymbol{F(x)}$, which is read "$\boldsymbol{F}$ **of** $\boldsymbol{x}$."

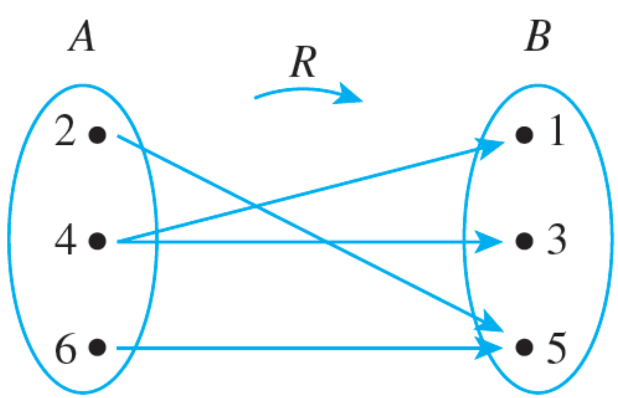
Example – *Functions and Relations on Finite Sets*

- Let $A = \{2, 4, 6\}$ and $B = \{1, 3, 5\}$. Which of the relations R, S, and T defined below are functions from A to B?
- a. $R = \{(2, 5), (4, 1), (4, 3), (6, 5)\}$
- b. For every $(x, y) \in A \times B$, $(x, y) \in S$ means that $y = x + 1$.
- c. T is defined by the arrow diagram



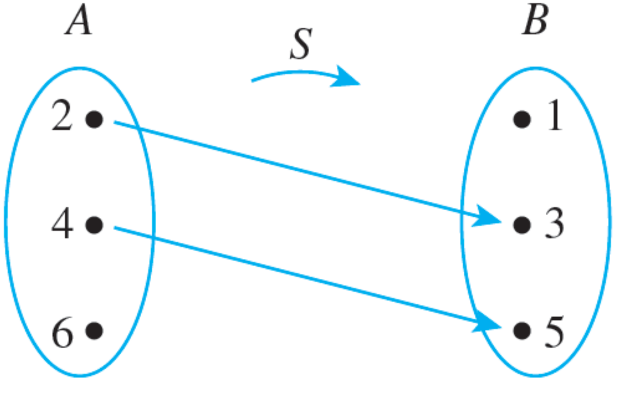
Example – *Solution*

- a. *R* is not a function because it does not satisfy property (2) for being a function. The ordered pairs (4, 1) and (4, 3) have the same first element but different second elements. You can see this graphically if you draw the arrow diagram for *R*. There are two arrows coming out of 4: One points to 1 and the other points to 3.



Example – *Solution* Continued.

- b. S is not a function because it does not satisfy property (1). It is not true that every element of A is the first element of an ordered pair in S. For example, $6 \in A$ but there is no y in B such that $y = 6 + 1 = 7$. You can also see this graphically by drawing the arrow diagram for S.



Example – *Solution* continued

- c. T is a function: Each element in {2, 4, 6} is related to some element in {1, 3, 5}, and no element in {2, 4, 6} is related to more than one element in {1, 3, 5}. When these properties are stated in terms of the arrow diagram, they become (1) there is an arrow coming out of each element of the domain, and (2) no element of the domain has more than one arrow coming out of it.
- Therefore, $\boldsymbol{T(2) = 5}$**,** $\boldsymbol{T(4) = 1}$**,** and $\boldsymbol{T(6) = 1}$.

Example– *Functions and Relations on Sets of Strings*

- Let $A = \{a, b\}$ be an Alphabet, and let S be the set of all strings over A.

- a. Define a relation L from S to $\mathbf{Z}^{nonneg}$ as follows:

  For every string s in S and for every nonnegative integer n, $(s, n) \in L$ means that the length of s is n.

  Here, L is a function because every string in S has one and only one length.

  Find ***L*(*abababab*)** and ***L*(*bbbaaa*)**.

Example – *Functions and Relations on Sets of Strings*

continued.

b. Define a relation C from S to S as follows: For all strings s and t in S,

- $(s, t) \in C$ means that $t = as$,
- where as is the string obtained by appending the character a on the left of the characters in s. (C is called **concatenation** by character a on the left.)

Example – *Functions and Relations on Sets of Strings* continued

- Also, C is a function because every string in S consists entirely of a's and b's and adding an additional a on the left creates a new string that also consists of a's and b's, and therefore, is also in set, S.
- Find ***C*(*abababab*)** and ***C*(*bbbaaa*).**

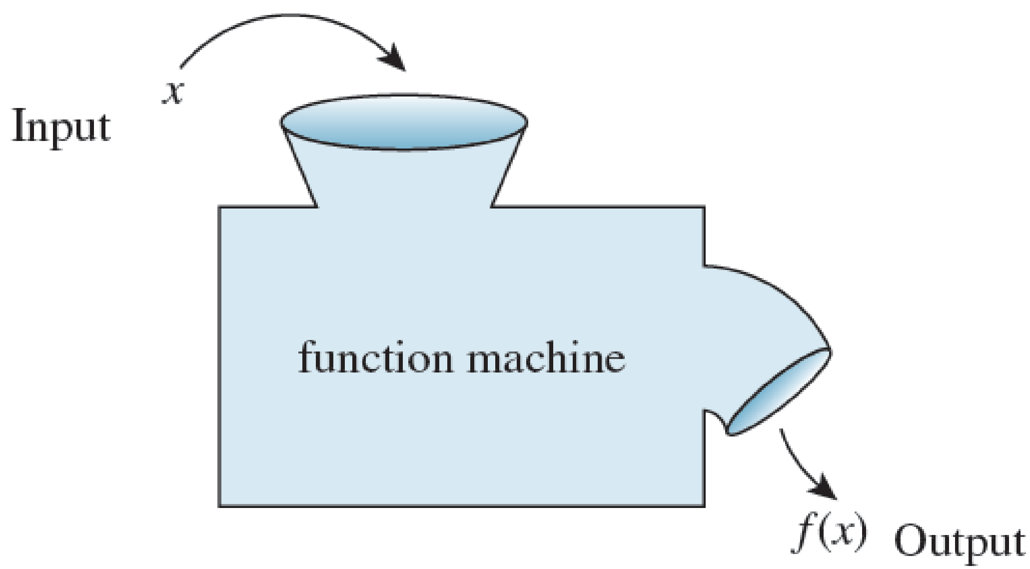
Example – *Solution*

- **a.** $L(ababab ab) = 8$ and $L(bbbaaa) = 6$

- **b.** $C(abababab) = aabababab$ and $C(bbbaaa) = abbbaaa$

Function Machines

- A useful way to think of a function is as a machine. Suppose f is a function from X to Y and an input x of X is given. Imagine f to be a machine that processes x in a certain way to produce the output $f(x)$.



Example – *Functions Defined by Formulas*

- The **squaring function** *f* from **R** to **R** is defined by the formula $f(x) = x^2$
- For every real number *x*. This means that no matter what real number input is substituted for *x*, the output of *f* will be the square of that number.
- In other words, *f* sends each real number *x* to x^2, or, symbolically, $f\colon x \to x^2$.

Example – *Functions Defined by Formulas*

continued.

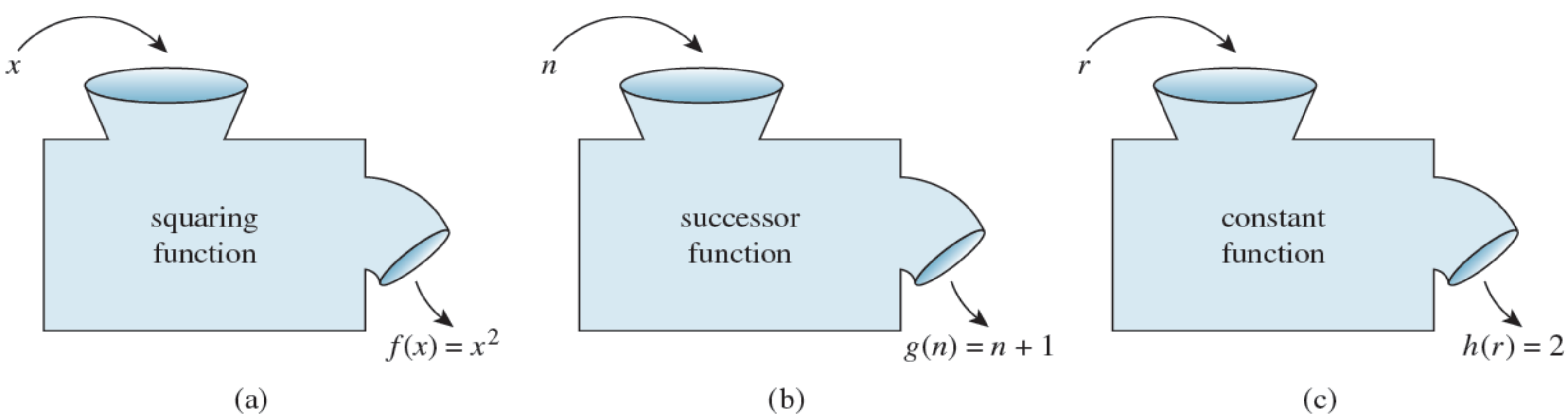
- The **successor function** g from **Z** to **Z** is defined by the formula $g(n) = n + 1$.
- Thus, no matter what integer is substituted for n, the output of g will be that number plus 1:
- In other words, g sends each integer n to $n + 1$, or, symbolically, $g: n \to n + 1$.

Example – *Functions Defined by Formulas*

continued

- An example of a **constant function** is the function h from **Q** to **Z** defined by the formula $h(r) = 2$ for all rational numbers r. This function sends each rational number r to 2. Hence, no matter what the input, the output is always 2: $h(\square) = 2$ or $h: r \to 2$.

The functions f, g, and h are represented by the function machines



Functions and Relations

- A relation is a subset of a Cartesian product and a function is a special kind of relation. If *f* and *g* are functions from a set *A* to a set *B*, then

$$f = \{(x, y) \in A \times B \mid y = f(x)\} \quad \text{and} \quad g = \{(x, y) \in A \times B \mid y = g(x)\}.$$

- It follows that

$$\boxed{\boldsymbol{f} \ \textbf{equals} \ \boldsymbol{g}, \quad \text{written } \boldsymbol{f} = \boldsymbol{g}, \quad \text{if, and only if,} \quad \boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{g}(\boldsymbol{x}) \text{ for all } x \text{ in } A.}$$

Example – *Equality of Functions*

- Define functions f and g from **R** to **R** by the following formulas:

$$f(x) = |x| \quad \text{for every } x \in \mathbf{R}.$$
$$g(x) = \sqrt{x^2} \quad \text{for every } x \in \mathbf{R}.$$

- Does $f = g$?

- Yes. Because the absolute value of any real number equals the square root of its square,
- $|x| = \sqrt{x^2}$ for all $x \in$ **R**. Hence $f = g$.