CMSC 207- Lecture 22
CHAPTER 8: Relations (8.3)

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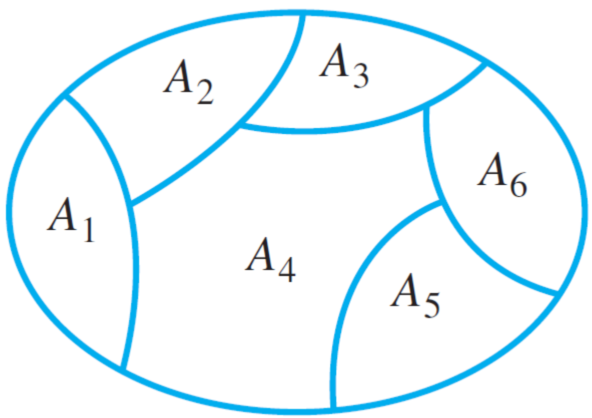
The Relation Induced by a Partition

A **partition** of a set *A* is a finite or infinite collection of nonempty, mutually disjoint subsets whose union is *A*. The diagram of Figure 8.3.1 illustrates a partition of a set *A* by subsets A_1, A_2, . . . , A_6



$$A_i \cap A_j = \emptyset, \text{ whenever } i \neq j$$
$$A_i \cup A_2 \cup \cdots \cup A_6 = A$$

A Partition of a Set

Figure 8.3.1

The Relation Induced by a Partition

• Definition

Given a partition of a set A, the **relation induced by the partition,** R, is defined on A as follows: For all $x, y \in A$,

$$x \; R \; y \quad \Leftrightarrow \quad \text{there is a subset } A_i \text{ of the partition such that both } x \text{ and } y \text{ are in } A_i.$$

Example 1 – ***Relation Induced by a Partition***

Let A = {0, 1, 2, 3, 4} and consider the following partition of A: {0, 3, 4}, {1}, {2}.

Find the **relation *R*** induced by this partition.

Solution:

Since {0, 3, 4} is a subset of the partition,

0 R 3 because both 0 and 3 are in {0, 3, 4},

3 R 0 because both 3 and 0 are in {0, 3, 4},

Example 1 – *Solution*

$0 \; R \; 4$ because both 0 and 4 are in $\{0, 3, 4\}$,

$4 \; R \; 0$ because both 4 and 0 are in $\{0, 3, 4\}$,

$3 \; R \; 4$ because both 3 and 4 are in $\{0, 3, 4\}$, and

$4 \; R \; 3$ because both 4 and 3 are in $\{0, 3, 4\}$.

Also, $0 \; R \; 0$ because both 0 and 0 are in $\{0, 3, 4\}$

$3 \; R \; 3$ because both 3 and 3 are in $\{0, 3, 4\}$, and

$4 \; R \; 4$ because both 4 and 4 are in $\{0, 3, 4\}$.

Example 1 – *Solution*

Since {1} is a subset of the partition,
1 *R* 1 because both 1 and 1 are in {1},

and since {2} is a subset of the partition,
2 *R* 2 because both 2 and 2 are in {2}.

Hence,

$R = \{(0, 0), (0, 3), (0, 4), (1, 1), (2, 2), (3, 0), (3, 3), (3, 4), (4, 0), (4, 3), (4, 4)\}$.

- **In-class Assignment #1**

The Relation Induced by a Partition

A relation induced by a partition of a set satisfies all three properties: **reflexivity**, **symmetry**, and **transitivity**.

Theorem 8.3.1

Let A be a set with a partition and let R be the relation induced by the partition. Then R is reflexive, symmetric, and transitive.

Definition of an Equivalence Relation

A relation on a set that satisfies the three properties of reflexivity, symmetry, and transitivity is called an ***equivalence relation***.

• Definition

Let A be a set and R a relation on A. R is an **equivalence relation** if, and only if, R is reflexive, symmetric, and transitive.

Thus, according to **Theorem 8.3.1**, the relation induced by a partition is an equivalence relation.

Example 2 – *An Equivalence Relation on a Set of Subsets*

Let X be the set of all nonempty subsets of **{1, 2, 3}**. Then $\boldsymbol{X}$ = **{{1}, {2}, {3}, {1, 2}, {1, 3}, {2, 3}, {1, 2, 3}}**

Define a relation **R** on X as follows: For all A and B in X, A **R** $B \Leftrightarrow$ the least element of A equals the least element of B.

Prove that **R** is an equivalence relation on X.

Example 2 – *Solution*

R is reflexive: Suppose *A* is a nonempty subset of {1, 2, 3}. *[We must show that A* **R** *A.]*

It is true to say that the least element of *A* equals the least element of *A*. Thus, by definition of *R*, *A* **R** *A*.

R is symmetric: Suppose *A* and *B* are nonempty subsets of {1, 2, 3} and *A* **R** *B*. *[We must show that B* **R** *A.]*

Since *A* **R** *B*, the least element of *A* equals the least element of *B*. But this implies that the least element of *B* equals the least element of *A*, and so, by definition of **R**, *B* **R** *A*.

Example 2 – *Solution*

R is transitive: Suppose *A*, *B*, and *C* are nonempty subsets of {1, 2, 3}, *A* **R** *B*, and *B* **R** *C*. *[We must show that A* **R** *C.]*

Since *A* **R** *B*, the least element of *A* equals the least element of *B* and since *B* **R** *C*, the least element of *B* equals the least element of *C*. Thus, the least element of *A* equals the least element of *C*, and so, by definition of **R**, *A* **R** *C*.

Equivalence Classes of an Equivalence Relation

Suppose there is an equivalence relation on a certain set.

If *a* is any particular element of the set, then the subset of all elements that are related to *a* is called the *equivalence class*

- **Definition**

Suppose A is a set and R is an equivalence relation on A. For each element a in A, the **equivalence class of a,** denoted $[a]$ and called the **class of a** for short, is the set of all elements x in A such that x is related to a by R.

In symbols:

$$[a] = \{x \in A \mid x \; R \; a\}$$

Equivalence Classes of an Equivalence Relation

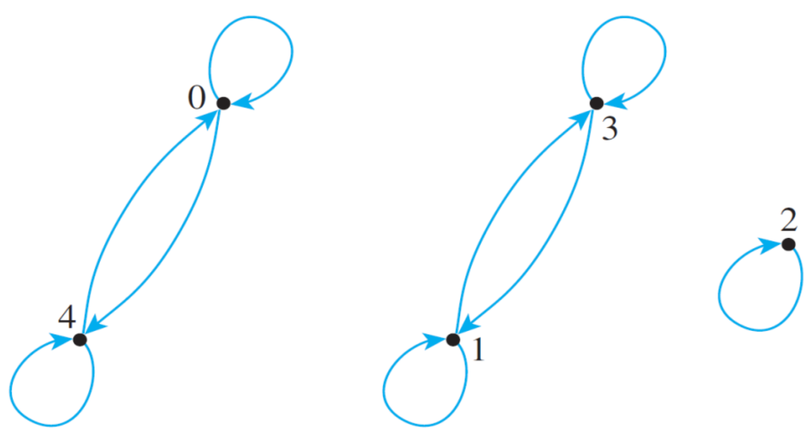
When several equivalence relations on a set are under discussion, the notation $[a]_R$ is often used to denote the equivalence class of a under R. The procedural version of this definition is

$$\text{for all } x \in A, \quad x \in [a] \quad \Leftrightarrow \quad x \; R \; a.$$

Example 5 – *Equivalence Classes of a Relation Given as a set of Ordered Pairs*

Let $\boldsymbol{A} = \{0, 1, 2, 3, 4\}$ and define a relation R on A as follows: $\boldsymbol{R} = \{(0, 0), (0, 4), (1, 1), (1, 3), (2, 2), (3, 1), (3, 3), (4, 0), (4, 4)\}$.

The directed graph for R is as shown below. As can be seen by inspection, R is an equivalence relation on A. Find the distinct equivalence classes of R.



Example 5 – *Solution*

First find the equivalence class of every element of *A*.

$$[0] = \{x \in A \mid x \; R \; 0\} = \{0, 4\}$$

$$[1] = \{x \in A \mid x \; R \; 1\} = \{1, 3\}$$

$$[2] = \{x \in A \mid x \; R \; 2\} = \{2\}$$

$$[3] = \{x \in A \mid x \; R \; 3\} = \{1, 3\}$$

$$[4] = \{x \in A \mid x \; R \; 4\} = \{0, 4\}$$

Note that [0] = [4] and [1] = [3]. Thus the *distinct* equivalence classes of the relation are

{0, 4}, {1, 3}, and {2}.

Equivalence Classes of an Equivalence Relation

The first lemma says that if two elements of *A* are related by an equivalence relation *R*, then their equivalence classes are the same.

Lemma 8.3.2

Suppose A is a set, R is an equivalence relation on A, and a and b are elements of A. If $a\ R\ b$, then $[a] = [b]$.

This lemma says that if a certain condition is satisfied, then [*a*] = [*b*]. Now [*a*] and [*b*] are *sets*, and two sets are equal if, and only if, each is a subset of the other.

Equivalence Classes of an Equivalence Relation

Hence the proof of the lemma consists of two parts: first, a proof that $[a] \subseteq [b]$ and second, a proof that $[b] \subseteq [a]$.

To show each subset relation, it is necessary to show that every element in the left-hand set is an element of the right-hand set.

The second lemma says that any two equivalence classes of an equivalence relation are either mutually disjoint or identical.

Lemma 8.3.3

If A is a set, R is an equivalence relation on A, and a and b are elements of A, then

$$\text{either} \quad [a] \cap [b] = \emptyset \quad \text{or} \quad [a] = [b].$$

Equivalence Classes of an Equivalence Relation

The statement of Lemma 8.3.3 has the form

if *p* then (*q* or *r*),

where *p* is the statement "*A* is a set, *R* is an equivalence relation on *A*, and *a* and *b* are elements of *A*," *q* is the statement "$[a] \cap [b] = \emptyset$," and *r* is the statement "$[a] = [b]$."

Theorem 8.3.4 The Partition Induced by an Equivalence Relation

If A is a set and R is an equivalence relation on A, then the distinct equivalence classes of R form a partition of A; that is, the union of the equivalence classes is all of A, and the intersection of any two distinct classes is empty.

In-class Assignment #2

In the following, the relation R is an equivalence relation on the set A. Find the distinct equivalence classes of R.

A = {a, b, c, d}

R = {(a, a), (b, b), (b, d), (c, c), (d, b), (d, d)}

In-class Assignment #3

Let R be the relation of congruence modulo 7. Which of the following equivalence classes are equal?

- [35], [3], [-7], [12], [0], [-2], [17]
- **Hints:** [0] = [-7] = [35]