CMSC 207- Lecture 19
CHAPTER 7: Functions (7.3 & 7.4)

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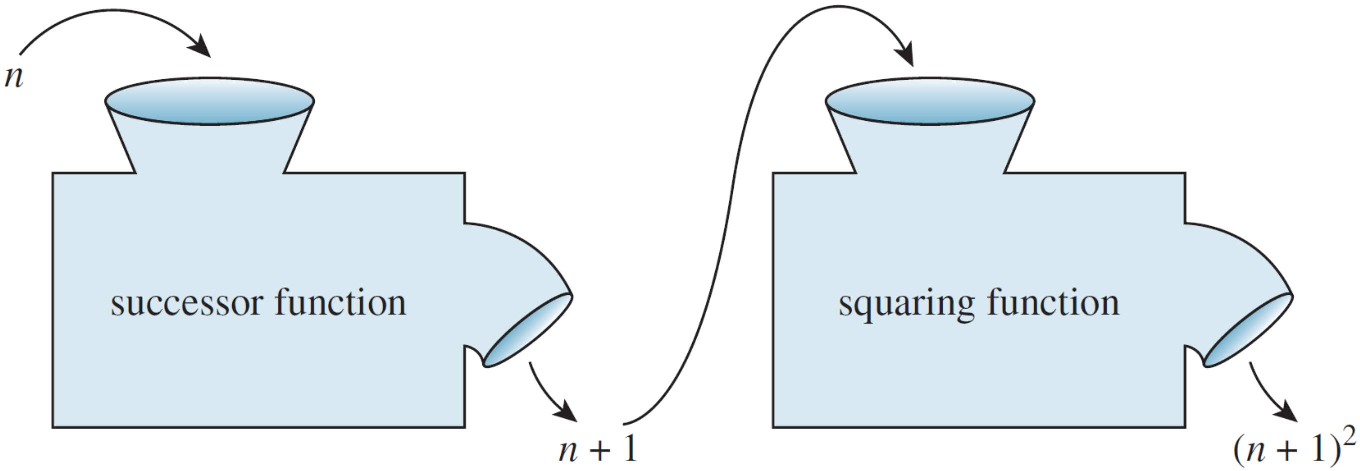
Composition of Functions

- Consider two functions, the successor function and the squaring function, defined from **Z** (the set of integers) to **Z**, and imagine that each is represented by a machine.
- If the two machines are connected such that the output from the successor function is used as input to the squaring function, then they work together to operate as one larger function machine.
- In this larger machine, an integer n is first increased by 1 to obtain $n + 1$; then the quantity $n + 1$ is squared to obtain $(n + 1)^2$.

Composition of Functions

- Illustrated in the following drawing.



- Combining functions in this way is called ***composing*** them; the resulting function is called the ***composition*** of the two functions.

Composition of Functions

- Composition can be formed only if the output of the first function is acceptable input to the second function.

That is, the range of the first function must be contained in the domain of the second function.

• Definition

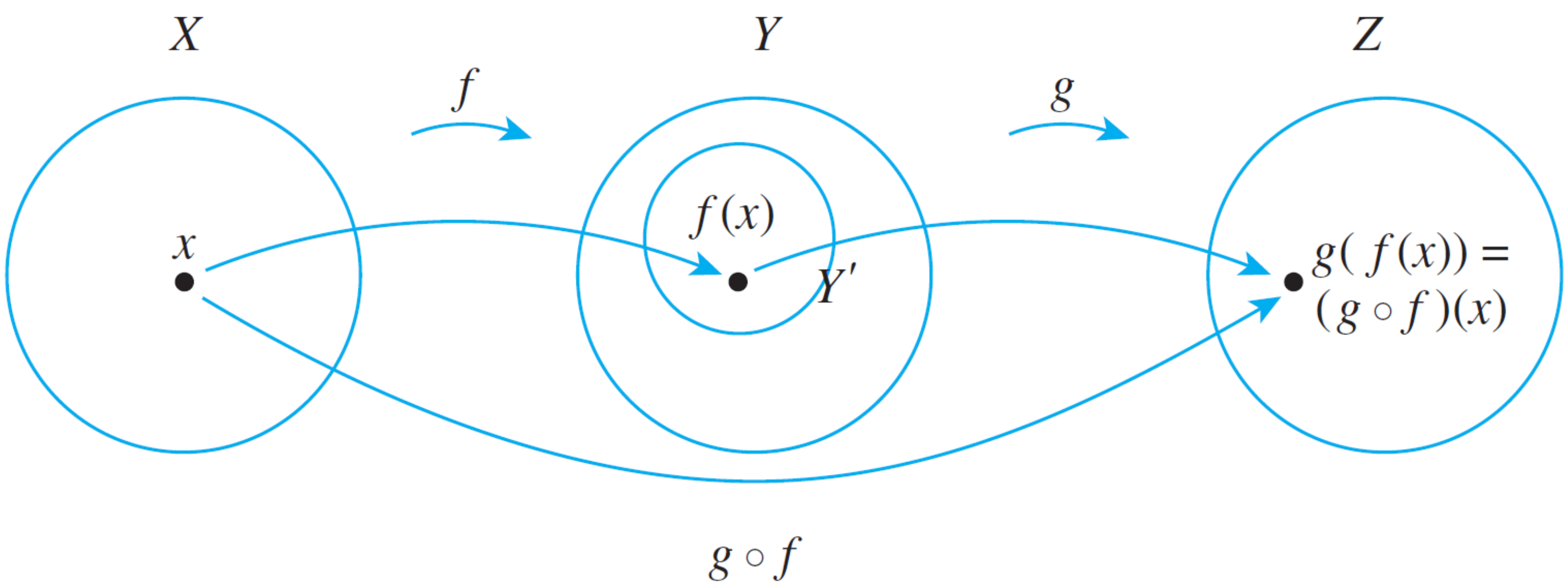
Let $f: X \to Y'$ and $g: Y \to Z$ be functions with the property that the range of f is a subset of the domain of g. Define a new function $g \circ f: X \to Z$ as follows:

$$(g \circ f)(x) = g(f(x)) \quad \text{for all } x \in X,$$

where $g \circ f$ is read "g circle f" and $g(f(x))$ is read "g of f of x." The function $g \circ f$ is called the **composition of f and g.**

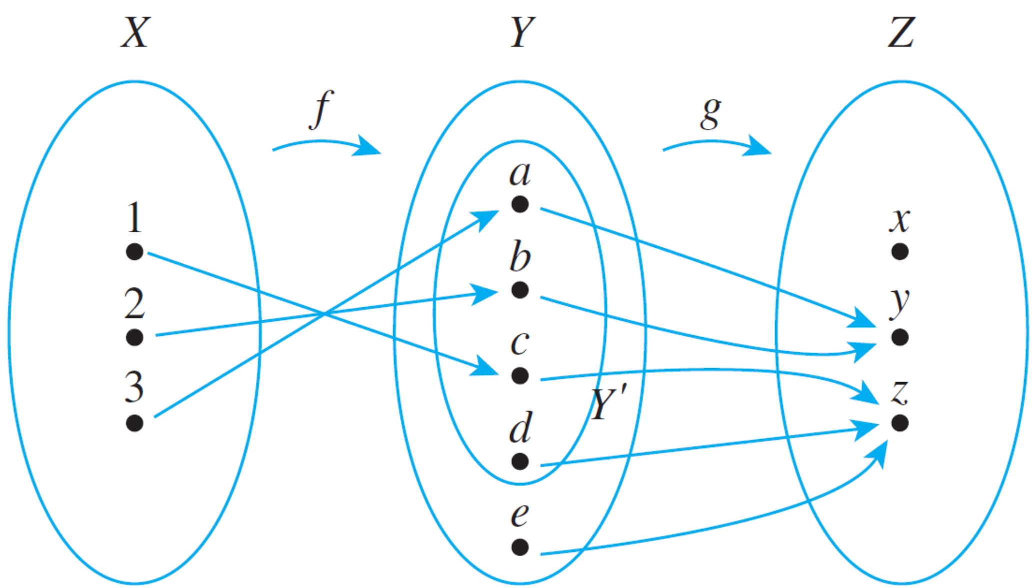
Composition of Functions

- Definition is shown schematically as follows.



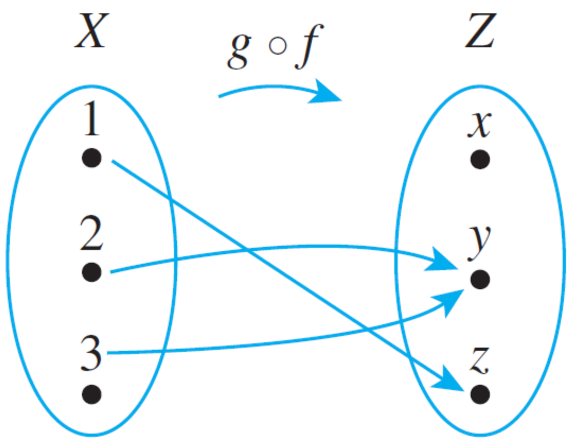
Example 2 – *Composition of Functions Defined on Finite Sets*

- Let $X = \{1, 2, 3\}$, $Y' = \{a, b, c, d\}$, $Y = \{a, b, c, d, e\}$, and
$Z = \{x, y, z\}$. Define functions $f: X \to Y'$ and $g: Y \to Z$ by the arrow diagrams below.



Example 2 – *Solution*

- To find the arrow diagram for $g \circ f$, just trace the arrows all the way across from X to Z through Y. The result is shown below.



$$(g \circ f)(1) = g(f(1)) = g(c) = z$$

$$(g \circ f)(2) = g(f(2)) = g(b) = y$$

$$(g \circ f)(3) = g(f(3)) = g(a) = y$$

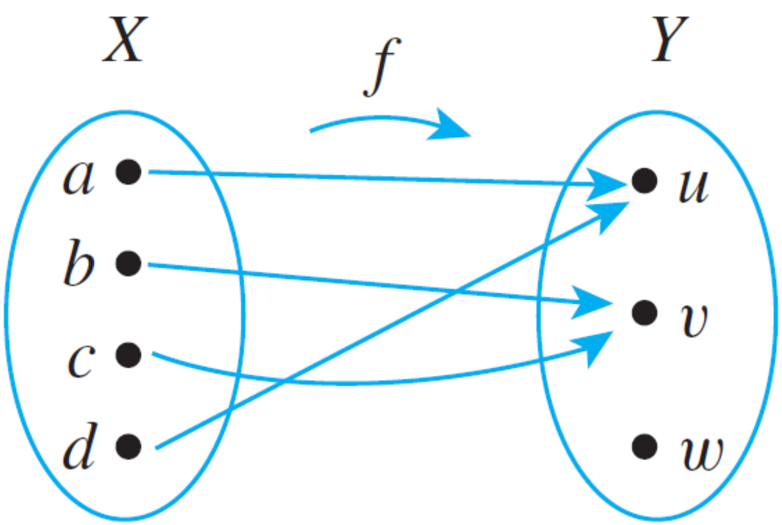
- **The range of $g \circ f$ is $\{y, z\}$.**

Identity Function

- We have known that the identity function on a set X, I_X, is the function from X to X defined by the formula $I_X(x) = x \quad \text{for all } x \in X.$
- That is, the identity function on X sends each element of X to itself. What happens when an identity function is composed with another function?

Example 3 – *Composition with the Identity Function*

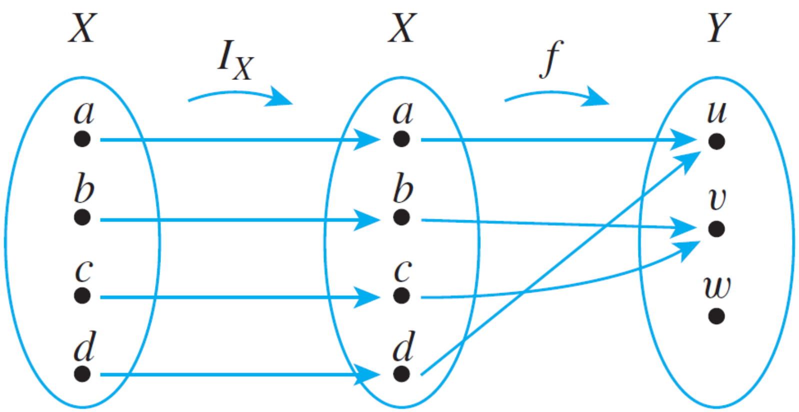
- Let $X = \{a, b, c, d\}$ and $Y = \{u, v, w\}$, and suppose f: $X \rightarrow Y$ is given by the arrow diagram.



- Find $\boldsymbol{f} \circ \boldsymbol{I_X}$ and $\boldsymbol{I_Y} \circ \boldsymbol{f}$.

Example 3 – *Solution*

- The values of $f \circ I_X$ are obtained by tracing through the arrow diagram shown below.



$$(f \circ I_X)(a) = f(I_X(a)) = f(a) = u$$
$$(f \circ I_X)(b) = f(I_X(b)) = f(b) = v$$
$$(f \circ I_X)(c) = f(I_X(c)) = f(c) = v$$
$$(f \circ I_X)(d) = f(I_X(d)) = f(d) = u$$

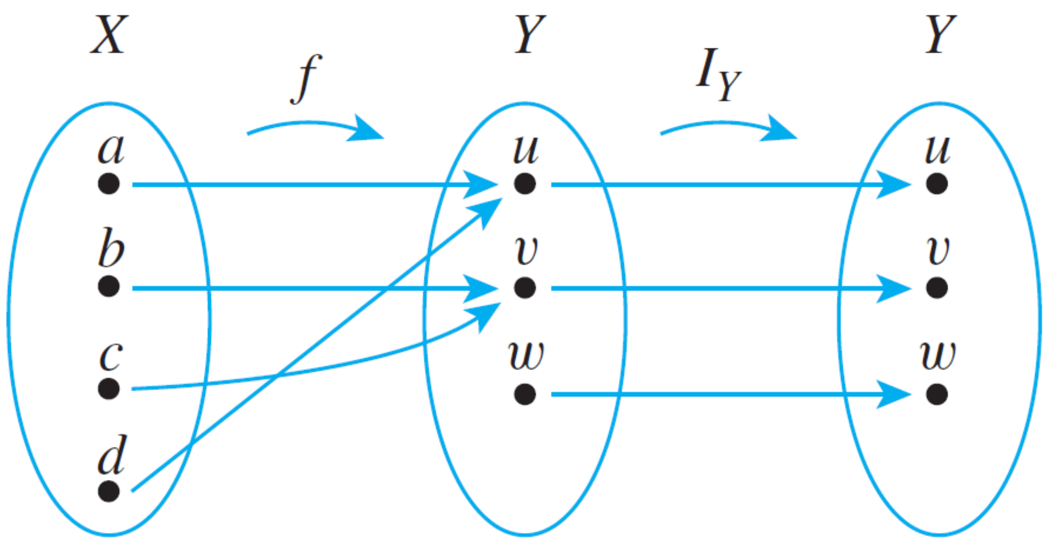
$$(f \circ I_X)(x) = f(x).$$

- Note that for all elements *x* in *X*,

Example 3 – *Solution*

- By definition of equality of functions, this means that $f \circ I_X = f$.

Similarly, the equality $I_Y \circ f = f$ can be verified by tracing through the arrow diagram below for each x in X and noting that in each case, $(I_Y \circ f)(x) = f(x)$.



Composition of Functions

- More generally, the composition of any function with an identity function equals the function.

Theorem 7.3.1 Composition with an Identity Function

If f is a function from a set X to a set Y, and I_X is the identity function on X, and I_Y is the identity function on Y, then

$$\text{(a)}\quad f \circ I_X = f \quad \text{and} \quad \text{(b)}\quad I_Y \circ f = f.$$

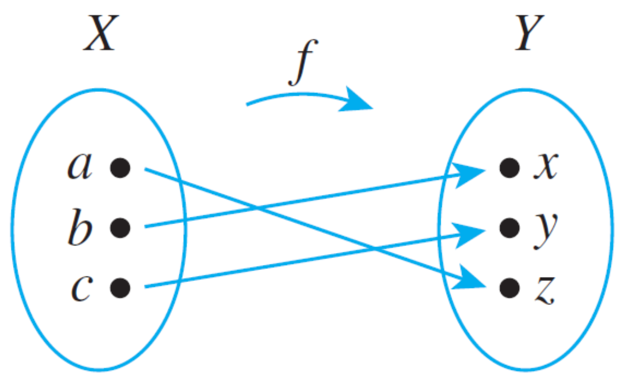
- Now let f be a function from a set X to a set Y, and suppose f has an inverse function f^{-1}. We have known that f^{-1} is the function from Y to X with the property that

$$f^{-1}(y) = x \quad \Leftrightarrow \quad f(x) = y.$$

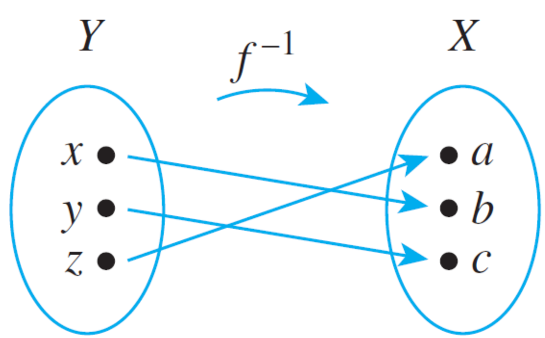
- What happens when f is composed with f^{-1}? Or when f^{-1} is composed with f?

Example 4 – *Composing a Function with Its Inverse*

- Let $X = \{a, b, c\}$ and $Y = \{x, y, z\}$. Define $f: X \rightarrow Y$ by the following arrow diagram.



- Then f is one-to-one and onto. Thus f^{-1} exists and is found by tracing the arrows backwards, as shown below.



Example 4 – *Composing a Function with Its Inverse*

- Now $f^{-1} \circ f$ is found by following the arrows from X to Y by f and back to X by f^{-1}.

$$(f^{-1} \circ f)(a) = f^{-1}(f(a)) = f^{-1}(z) = a$$
$$(f^{-1} \circ f)(b) = f^{-1}(f(b)) = f^{-1}(x) = b$$

and

$$(f^{-1} \circ f)(c) = f^{-1}(f(c)) = f^{-1}(y) = c.$$

- Thus the composition of f and f^{-1} sends each element to itself. Therefore, $f^{-1} \circ f = I_X$. & $f \circ f^{-1} = I_Y$.
- Here, Ix and Iy are Identity functions on Domain & Co-domain, respectively.

Composition of Functions

- More generally, the composition of any function with its inverse (if it has one) is an identity function. Intuitively, the function sends an element in its domain to an element in its co-domain and the inverse function sends it back again, so the composition of the two sends each element to itself.
- This reasoning is formalized in Theorem 7.3.2.

Theorem 7.3.2 Composition of a Function with Its Inverse

If $f: X \to Y$ is a one-to-one and onto function with inverse function $f^{-1}: Y \to X$, then

$$\text{(a)}\quad f^{-1} \circ f = I_X \quad \text{and} \quad \text{(b)}\quad f \circ f^{-1} = I_Y.$$

Composition of One-to-One Functions

- What happens, when two one-to-one functions are composed?
- For example, let $X = \{a, b, c\}$, $Y = \{w, x, y, z\}$, and $Z = \{1, 2, 3, 4, 5\}$, and define one-to-one functions f: $X \rightarrow Y$ and g: $Y \rightarrow Z$ as shown in the arrow diagrams of Figure 7.3.1.

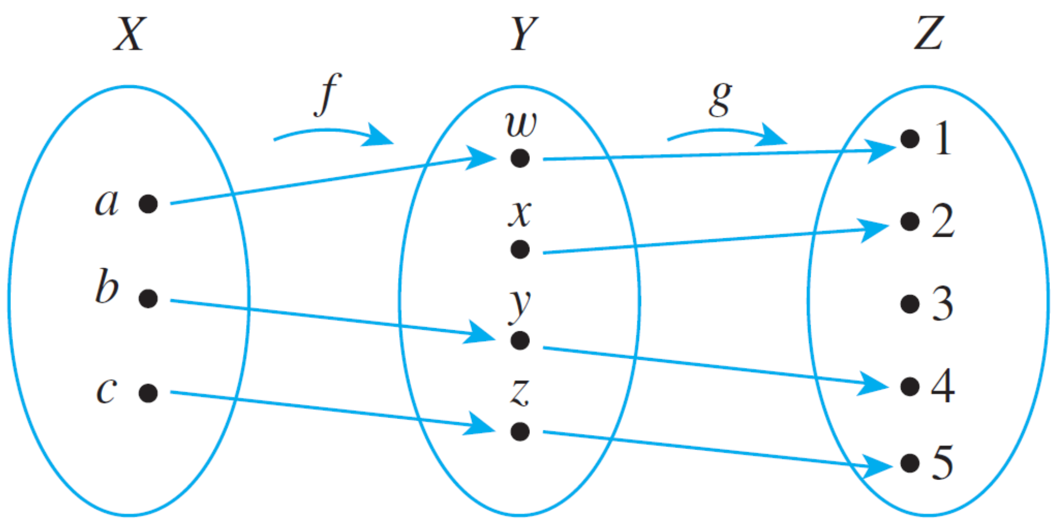


Figure 7.3.1

Composition of One-to-One Functions

• Then $g \circ f$ is the function with the arrow diagram shown in Figure 7.3.2.

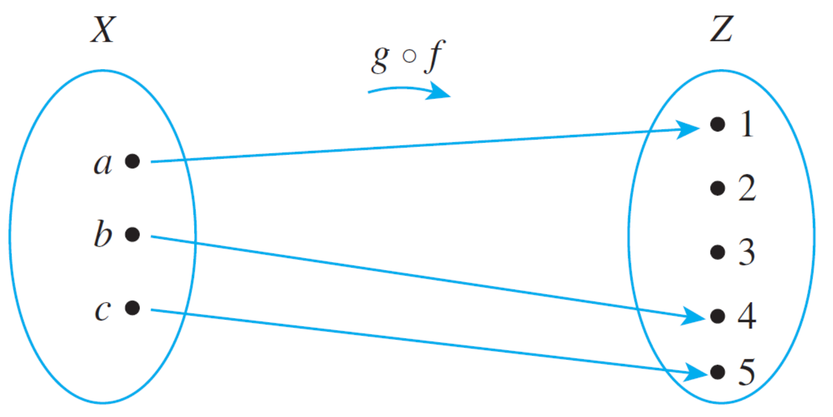


Figure 7.3.2

• From the diagram it is clear that the composition of two one-to-one functions is one-to-one.

Composition of One-to-One Functions

- Compositions of two one-to-one functions is always one-to-one.

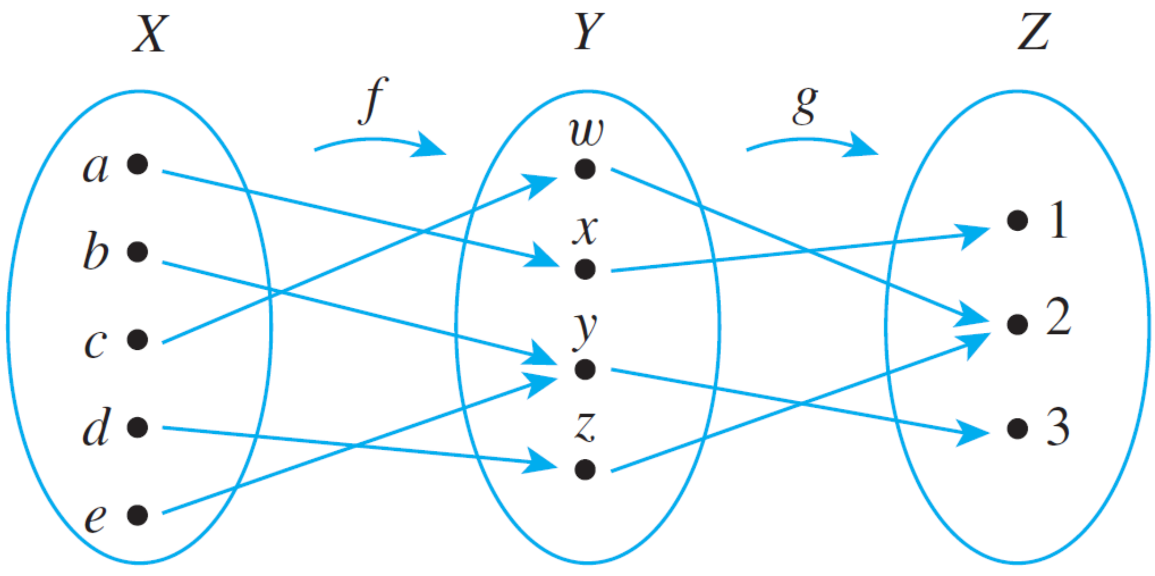
Theorem 7.3.3

If $f\colon X \to Y$ and $g\colon Y \to Z$ are both one-to-one functions, then $g \circ f$ is one-to-one.

- **In-class Practice Exercise #1**

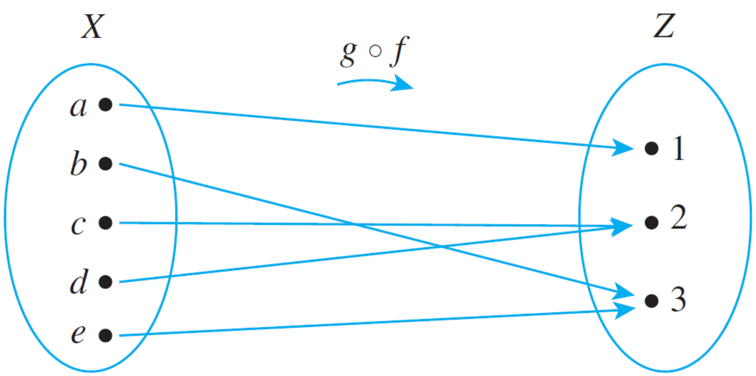
Composition of Onto Functions

- Now consider what happens when two onto functions are composed. For example, let $X = \{a, b, c, d, e\}$, $Y = \{w, x, y, z\}$, and $Z = \{1, 2, 3\}$. Define onto functions f and g by the following arrow diagrams.



Composition of Onto Functions

- Then $g \circ f$ is the function with the arrow diagram shown below.
- It is clear from the diagram that $g \circ f$ is onto.



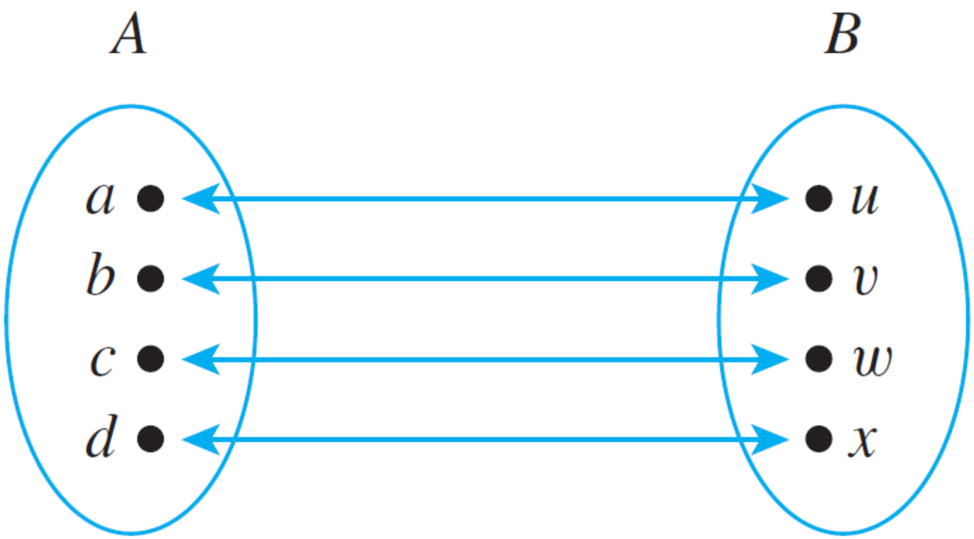
- So the composition of any two onto functions (that can be composed) is onto.

Theorem 7.3.4

If $f: X \to Y$ and $g: Y \to Z$ are both onto functions, then $g \circ f$ is onto.

Cardinality with Applications to Computability

- Two finite sets whose elements can be paired by a one-to-one correspondence have the *same size*. This is illustrated by the following diagram.



The elements of set A can be put into one-to-one correspondence with the elements of B.

Cardinality with Applications to Computability

- Now a **finite set** is one that has no elements at all or that can be put into one-to-one correspondence with a set of the form $\{1, 2, \ldots, n\}$ for some positive integer n.
- An **infinite set** is a nonempty set that **cannot be put** into one-to-one correspondence with $\{1, 2, \ldots, n\}$ for any positive integer n.

• Definition

Let A and B be any sets. ***A* has the same cardinality as *B*** if, and only if, there is a one-to-one correspondence from A to B. In other words, A has the same cardinality as B if, and only if, there is a function f from A to B that is one-to-one and onto.

Cardinality with Applications to Computability

- The following theorem gives some basic properties of cardinality.

Theorem 7.4.1 Properties of Cardinality

For all sets A, B, and C:

a. **Reflexive property of cardinality:** A has the same cardinality as A.

b. **Symmetric property of cardinality:** If A has the same cardinality as B, then B has the same cardinality as A.

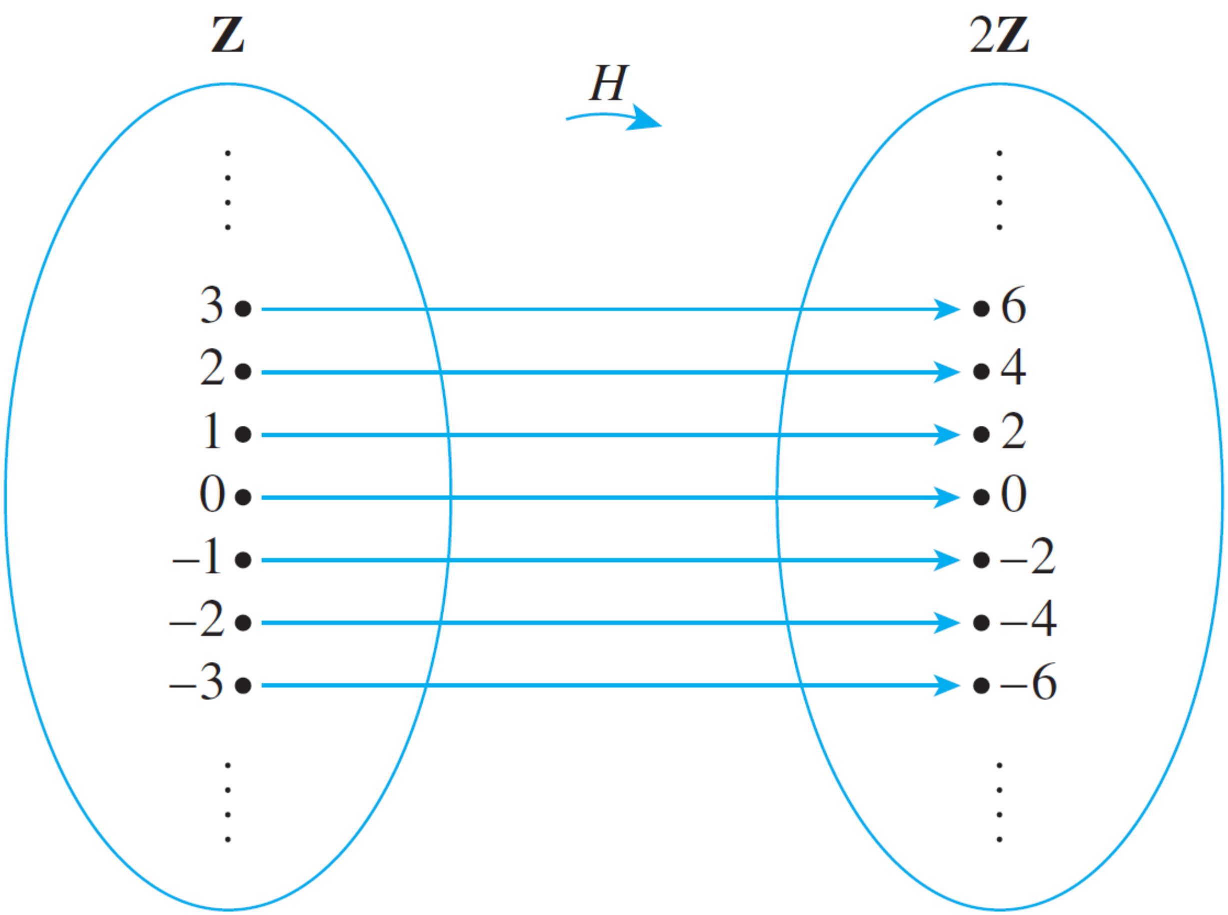
c. **Transitive property of cardinality:** If A has the same cardinality as B and B has the same cardinality as C, then A has the same cardinality as C.

Example – *An Infinite Set and a Proper Subset Can Have the Same Cardinality*

- Let 2**Z** be the set of all even integers. Prove that 2**Z** and **Z** have the same cardinality.
- Solution:
- Consider the function *H* from **Z** to 2**Z** defined as follows:
- $H(n) = 2n \text{ for all } n \in \mathbf{Z}.$

Example – *Solution*

- A (partial) arrow diagram for *H* is shown below.



Example – *Solution*

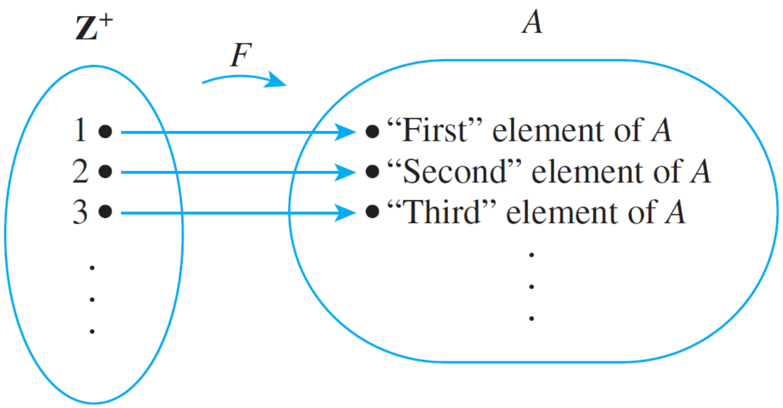
- To show that H is one-to-one, suppose $H(n_1) = H(n_2)$ for some integers n_1 and n_2.
- Then $\mathbf{2n_1 = 2n_2}$ by definition of H, and dividing both sides by 2 gives $n_1 = n_2$. Hence h is one-to-one.
- To show that H is onto, suppose m is any element of 2**Z**. Then m is an even integer, and so $m = 2k$ for some integer k.
- It follows that $H(k) = 2k = m$. Thus there exists k in **Z** with $H(k) = m$, and hence H is onto.
- Therefore, by definition of cardinality, **Z** and 2**Z** have the same cardinality.

Countable Sets

- The set $\mathbf{Z}^+$ of counting numbers {1, 2, 3, 4, . . .} is the most basic of all infinite sets.
- A set A having the same cardinality as this set is called *countably infinite*.
- The reason is that the one-to-one correspondence between the two sets can be used to "count" the elements of A: If F is a one-to-one and onto function from $\mathbf{Z}^+$ to A, then $F(1)$ can be designated as the first element of A, $F(2)$ as the second element of A, $F(3)$ as the third element of A, and so forth.

Countable Sets

This is illustrated graphically in Figure 7.4.1.



"Counting" a Countably Infinite Set **Figure 7.4.1**

Because F is one-to-one, no element is ever counted twice, and because it is onto, every element of A is counted eventually.

- **Definition**

A set is called **countably infinite** if, and only if, it has the same cardinality as the set of positive integers $\mathbf{Z}^+$. A set is called **countable** if, and only if, it is finite or countably infinite. A set that is not countable is called **uncountable.**

Example – *Countability of Z, the Set of All Integers*

- Show that the set **Z** of all integers is countable.
- **Solution:**

The set **Z** of all integers is certainly not finite, so if it is countable, it must be because it is **countably infinite**.

To show that **Z** is countably infinite, find a function from the positive integers $\mathbf{Z}^+$ to **Z** that is one-to-one and onto.

Example – *Solution*

- The trick is to start in the middle and work outward systematically. Let the first integer be 0, the second 1, the third –1, the fourth 2, the fifth –2, and so forth as shown in Figure 7.4.2, starting at 0 and swinging outward in back and forth arcs from positive to negative integers and back again, picking up one additional integer at each swing.

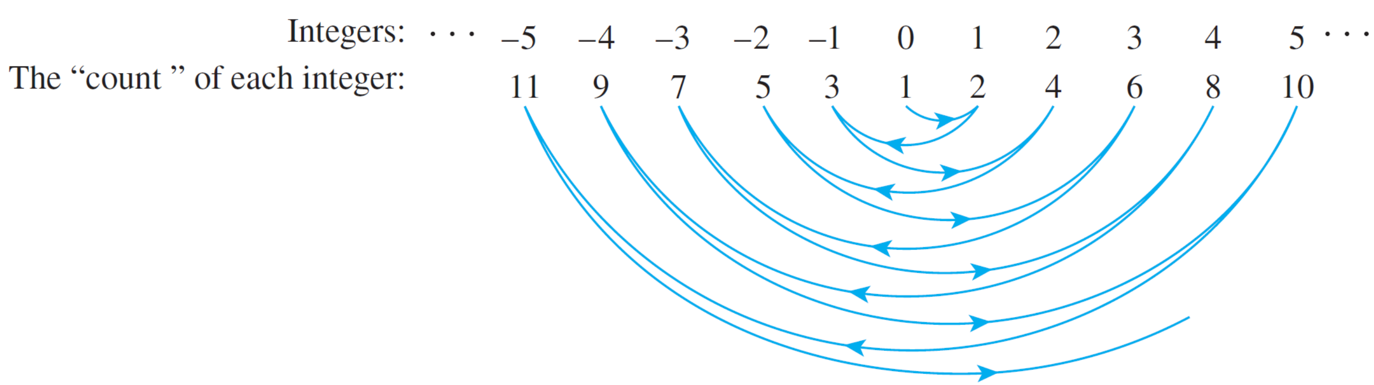


Figure 7.4.2 **“Counting” the Set of All Integers**

Example – *Solution*

•It is clear from the diagram that no integer is counted twice (so the function is one-to-one) and every integer is counted eventually (so the function is onto).

•Consequently, this diagram defines a function from $\mathbf{Z}^+$ to $\mathbf{Z}$ that is one-to-one and onto. It follows by definition of cardinality that $\mathbf{Z}^+$ has the same cardinality as $\mathbf{Z}$. Thus $\mathbf{Z}$ is **countably infinite** and hence countable**.** The function can also be described by the explicit formula:

$$F(n) = \begin{cases} \dfrac{n}{2} & \text{if } n \text{ is an even positive integer} \\ -\dfrac{n-1}{2} & \text{if } n \text{ is an odd positive integer.} \end{cases}$$

In-class Practice Exercise #2