**Chapter 5 – Additional Problems with Solution – Helpful for the Homework, and Chapter Quiz on Chapter 5**

**Problem 1:**

Use mathematical induction to prove that for all integers $n \geq 0$,

$$1 + 3 + 3^2 + \cdots + 3^n = \frac{3^{n+1} - 1}{2}.$$

**Solution:**

Note that $1 + 3 + 3^2 + \cdots + 3^n = \sum_{i=0}^{n} 3^i$.

***Show that $P(0)$ is true***: $P(0)$ is true because the left-hand side is $\sum_{i=0}^{0} 3^i = 3^0 = 1$, and the right-hand side is $\frac{3^{0+1} - 1}{2} = \frac{3 - 1}{2} = 1$ also.

***Show that for all integers $k \geq 0$, if $P(k)$ is true then $P(k+1)$ is true***: Let $k$ be any integer with $k \geq 0$, and suppose that

$$1 + 3 + 3^2 + \cdots + 3^k = \frac{3^{k+1} - 1}{2}. \qquad \leftarrow \quad \begin{array}{l} P(k) \\ \text{inductive hypothesis} \end{array}$$

We must show that

$$1 + 3 + 3^2 + \cdots + 3^{k+1} = \frac{3^{(k+1)+1} - 1}{2}.$$

Or, equivalently, that

$$1 + 3 + 3^2 + \cdots + 3^{k+1} = \frac{3^{k+2} - 1}{2} \qquad \leftarrow \quad P(k+1)$$

.

(a) Now the left-hand side of $P(k+1)$ is

$$\begin{aligned}
1+3+3^2+\cdots+3^{k+1} &= 1+3+3^2+\cdots+3^k+3^{k+1} \\
&\quad \text{by making the next-to-last term explicit} \\
&= \frac{3^{k+1}-1}{2}+3^{k+1} \\
&\quad \text{by inductive hypothesis} \\
&= \frac{3^{k+1}-1}{2}+\frac{2\cdot 3^{k+1}}{2} \\
&\quad \text{by creating a common denominator} \\
&= \frac{3\cdot 3^{k+1}-1}{2} \\
&\quad \text{by adding the fractions and combining like terms} \\
&= \frac{3^{k+2}-1}{2} \\
&\quad \text{by a law of exponents,}
\end{aligned}$$

which equals the right-hand side of $P(k+1)$.

Thus the left-hand and right-hand sides of $P(k+1)$ are equal *[as was to be shown]*.

**Problem 2:**
Prove by Mathematical Induction:

The sum of the first $n$ positive odd integers is $n^2$, that is, $1 + 3 + 5 + \cdots + (2n - 1) = n^2$
**Solution:**
Let $P(n)$ denote the proposition that the sum of the first $n$ odd positive integers is $n^2$

***BASIS STEP: P(1)*** states that the sum of the first one odd positive integer is $1^2$. This is true because the sum of the first odd positive integer is 1. The basis step is complete.

***INDUCTIVE STEP:*** To complete the inductive step we must show that the proposition $P(k) \to P(k + 1)$ is true for every positive integer $k$. To do this, we first assume the inductive hypothesis. The inductive hypothesis is the statement that $P(k)$ is true for an arbitrary positive integer $k$, that is,
$1 + 3 + 5 + \cdots + (2k - 1) = k^2$.
(Note that the $k$th odd positive integer is $(2k - 1)$.
To show that $\forall k(P(k) \to P(k + 1))$ is true, we must show that if $P(k)$ is true (the inductive hypothesis), then $P(k + 1)$ is true. Note that $P(k + 1)$ is the statement that
$1 + 3 + 5 + \cdots + (2k - 1) + (2k + 1) = (k + 1)^2$.
So, assuming that $P(k)$ is true, it follows that
$1 + 3 + 5 + \cdots + (2k - 1) + (2k + 1) = [1 + 3 + \cdots + (2k - 1)] + (2k + 1)$
$= k^2 + (2k + 1)$
$= k^2 + 2k + 1$
$= (k + 1)^2$.

This shows that *P(k +* 1*)* follows from *P(k)*. Note that we used the inductive hypothesis *P(k)* in the second equality to replace the sum of the first *k* odd positive integers by $k^2$.

**Problem 3:**

Use mathematical induction to show that

$$1 + 2 + 2^2 + \cdots + 2^n = 2^{n+1} - 1$$

for all nonnegative integers $n$.

*Solution:* Let $P(n)$ be the proposition that $1 + 2 + 2^2 + \cdots + 2^n = 2^{n+1} - 1$ for the integer $n$.

*BASIS STEP:* $P(0)$ is true because $2^0 = 1 = 2^1 - 1$. This completes the basis step.

*INDUCTIVE STEP:* For the inductive hypothesis, we assume that $P(k)$ is true for an arbitrary nonnegative integer $k$. That is, we assume that

$$1 + 2 + 2^2 + \cdots + 2^k = 2^{k+1} - 1.$$

To carry out the inductive step using this assumption, we must show that when we assume that $P(k)$ is true, then $P(k+1)$ is also true. That is, we must show that

$$1 + 2 + 2^2 + \cdots + 2^k + 2^{k+1} = 2^{(k+1)+1} - 1 = 2^{k+2} - 1$$

assuming the inductive hypothesis $P(k)$. Under the assumption of $P(k)$, we see that

$$\begin{aligned} 1 + 2 + 2^2 + \cdots + 2^k + 2^{k+1} &= (1 + 2 + 2^2 + \cdots + 2^k) + 2^{k+1} \\ &\stackrel{\text{IH}}{=} (2^{k+1} - 1) + 2^{k+1} \\ &= 2 \cdot 2^{k+1} - 1 \\ &= 2^{k+2} - 1. \end{aligned}$$

Note that we used the inductive hypothesis in the second equation in this string of equalities to replace $1 + 2 + 2^2 + \cdots + 2^k$ by $2^{k+1} - 1$. We have completed the inductive step.

Because we have completed the basis step and the inductive step, by mathematical induction we know that $P(n)$ is true for all nonnegative integers $n$. That is, $1 + 2 + \cdots + 2^n = 2^{n+1} - 1$ for all nonnegative integers $n$. ◀

**Problem 4:**

Use mathematical induction to prove that $7^{n+2} + 8^{2n+1}$ is divisible by 57 for every nonnegative integer $n$.

*Solution:* To construct the proof, let $P(n)$ denote the proposition: "$7^{n+2} + 8^{2n+1}$ is divisible by 57."

*BASIS STEP:* To complete the basis step, we must show that $P(0)$ is true, because we want to prove that $P(n)$ is true for every nonnegative integer. We see that $P(0)$ is true because $7^{0+2} + 8^{2\cdot 0+1} = 7^2 + 8^1 = 57$ is divisible by 57. This completes the basis step.

*INDUCTIVE STEP:* For the inductive hypothesis we assume that $P(k)$ is true for an arbitrary nonnegative integer $k$; that is, we assume that $7^{k+2} + 8^{2k+1}$ is divisible by 57. To complete the inductive step, we must show that when we assume that the inductive hypothesis $P(k)$ is true, then $P(k+1)$, the statement that $7^{(k+1)+2} + 8^{2(k+1)+1}$ is divisible by 57, is also true.

The difficult part of the proof is to see how to use the inductive hypothesis. To take advantage of the inductive hypothesis, we use these steps:

$$\begin{aligned}
7^{(k+1)+2} + 8^{2(k+1)+1} &= 7^{k+3} + 8^{2k+3} \\
&= 7 \cdot 7^{k+2} + 8^2 \cdot 8^{2k+1} \\
&= 7 \cdot 7^{k+2} + 64 \cdot 8^{2k+1} \\
&= 7(7^{k+2} + 8^{2k+1}) + 57 \cdot 8^{2k+1}.
\end{aligned}$$

We can now use the inductive hypothesis, which states that $7^{k+2} + 8^{2k+1}$ is divisible by

**57. So,** $7(7^{k+2} + 8^{2k+1})$, is **divisible by 57. Also,** $57 \cdot 8^{2k+1}$ **is divisible by 57.**