Discrete Mathematics with Applications, 4th edition
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Solutions for Supplementary Exercises: Chapter 4

1. **Section 4.1**

   Starting Point: Suppose a, b, and c are any *[particular but arbitrarily chosen]* integers such that $a \mid b$ and $a \mid c$.

   Conclusion to be Shown: $a^2 \mid (5b^2 + 7c^3)$

2. **Section 4.1**

   Proof: Suppose a is any even integer. *[We must show that $5(a+3)$ is odd.]* By definition of even,

   $$a = 2r \quad \text{for some integer } r.$$

   Thus

   $$\begin{aligned} 5(a+3) &= 5(2r+3) && \text{by substitution} \\ &= 10r + 15 \\ &= 10r + 14 + 1 \\ &= 2(5r+7) + 1 && \text{by basic algebra.} \end{aligned}$$

   Let $s = 5r + 7$. Then s is an integer because products and sums of integers are integers. Therefore, $5(a+3)$ equals 2 times an integer plus one, and thus $5(a+3)$ is odd by definition of odd *[as was to be shown]*.

3. **Section 4.1**

   Counterexample: Let $a = -1$ and $b = -2$. Then $ab = (-1)(-2) = 2 > 1$ but neither a nor b is greater than 1. (*This is one counterexample among many.*)

4. **Section 4.2**

   Proof: Suppose r and s are any rational numbers. *[We must show that $2r + 3s$ is rational.]* By definition of rational, there exist integers a, b, c,and d such that

   $$r = \frac{a}{b} \quad \text{and} \quad s = \frac{c}{d} \quad \text{and} \quad b \neq 0 \quad \text{and} \quad d \neq 0.$$

   Then

   $$\begin{aligned} 2r + 3s &= 2\left(\frac{a}{b}\right) + 3\left(\frac{c}{d}\right) && \text{by substitution} \\ &= \frac{2a + 3c}{bd} && \text{by basic algebra.} \end{aligned}$$

   But $2a + 3c$ and bd are both integers because products and sums of integers are integers, and $bd \neq 0$ by the zero product property. Thus $2r + 3s$ is a ratio of integers with a nonzero denominator, and hence $2r + 3s$ is rational *[as was to be shown]*.

5. **Section 4.3**

   Proof: Suppose a and b are any integers such that $3 \mid a$ and $a \mid b$. *[We must show that $3 \mid b$.]* By definition of divisibility, there exist integers k,and m such that

   $$a = 3k \quad \text{and} \quad b = am.$$

   Then

   $$\begin{aligned} b &= am \\ &= (3k)m && \text{by substitution} \\ &= 3(km) && \text{by basic algebra.} \end{aligned}$$

   But km is an integer because it is a product of integers. Hence b is equal to 3 times some integer, and thus 3 divides b by definition of divisibility *[as was to be shown]*.

6. **Section 4.4**: Prove that for all real numbers x and y, $||x| - |y|| \leq |x + y|$.

   Proof: Suppose a and b are any real numbers.

   Case 1 ($|a| \geq |b|$ ***):*** By the triangle inequality ((Theorem 4.4.6) with $x = a - b$ and $y = b$, we have

$$|(a - b) + b| \leq |a - b| + |b|.$$

   But $a = (a - b) + a$ by basic algebra, and so

$$|a| \leq |a - b| + |b|.$$

   Subtracting $|b|$ from both sides gives

$$|a| - |b| \leq |a - b|.$$

   Since $|a| \geq |b|$, $|a| - |b| \geq 0$, and thus $||a| - |b|| = |a| - |b|$. Hence

$$||a| - |b|| \leq |a - b|.$$

   Case 2 ($|a| < |b|$ ***):*** By the result of case 1,

$$||b| - |a|| \leq |b - a|.$$

   But, by Lemma 4.4.5,

$$||a| - |b|| = |-(|a| - |b|)| = ||b| - |a|| \quad \text{and} \quad |a - b| = |-(a - b)| = |b - a|.$$

   Thus, by substitution,

$$||a| - |b|| \leq |a - b|.$$

   Conclusion: Cases 1 and 2 have shown that regardless of whether $|a| \geq |b|$ or $|a| < |b|$,

$$||a| - |b|| \leq |a - b|$$

   [as was to be shown].

7. **Section 4.4**

   Proof:

   Suppose m and n are any *[particular but arbitrarily chosen]* integers such that $m \ mod \ 5 = 2$ and $n \ mod \ 6 = 3$.

   Then the remainder obtained when m is divided by 5 is 2 and the remainder obtained when n is divided by 6 is 3, and so there exist integers q and r such that $m = 5q + 2$ and $n = 6r + 3$. It follows that

$$\begin{array}{rcll} mn & = & (5q + 2)(6r + 3) & \text{by substitution} \\ & = & 30qr + 15q + 12r + 6 & \\ & = & 3(10qr + 5q + 4r + 2) & \text{by algebra.} \end{array}$$

   Because products and sums of integers are integers, $10qr + 5q + 4r + 2$ is an integer, and hence $mn = 3 \cdot$(an integer).

   Thus, by definition of divisibility, $mn \ mod \ 3 = 0$.

8. **Section 4.5**: Is the following statement true or false: For all real numbers x, $\lceil x \rceil^2 = \lceil x^2 \rceil$. Prove the statement if it is true and give a counterexample if it is false.

   The statement is false.

   Counterexample: Let $x = 2.1$. Then

$$\lceil x \rceil^2 = \lceil 2.1 \rceil^2 = 3^2 = 9.$$

   On the other hand,

$$\lceil x^2 \rceil = \lceil (2.1)^2 \rceil = \lceil 4.41 \rceil = 5.$$

   Since $9 \neq 5$, $\lceil x \rceil^2 \neq \lceil x^2 \rceil$.

9. **Section 4.6**

   The mistake is that the negation of the statement to be proved was written incorrectly. The negation of "The difference of any irrational number minus any rational number is irrational" is not "The difference of any irrational number minus any rational number is rational." The correct negation is "There exist at least one irrational number and at least one rational number whose difference is rational." Deducing a contradiction from a statement that is not the negation of the given statement does not prove that the given statement is true.

10. **Section 4.7**

    Proof by contradiction:

    Suppose not. That is, suppose $4+5\sqrt{2}$ is rational. *[We must show that this supposition leads logically to a contradiction.]*

    By definition of rational, there exist integers a and b with $4+5\sqrt{2} = \dfrac{a}{b}$ and $b \neq 0$. Solving for $\sqrt{2}$ gives

    $$\sqrt{2} = \frac{\frac{a}{b}+4}{5} = \frac{a+20b}{5b}.$$

    But $a+20b$ and $5b$ are integers (because products and sums of integers are integers) and $5b \neq 0$ (by the zero product property).

    Therefore by definition of rational, $\sqrt{2}$ is rational. This contradicts Theorem 4.7.1 which states that $\sqrt{2}$ is irrational. Hence the supposition is false. In other words $4+5\sqrt{2}$ is irrational.

11. **Section 4.7**

    The mistake is that the negation of the statement to be proved was written incorrectly. The negation of "The square of any irrational number is rational" is not "The square of any irrational number is irrational." The correct negation is "There exist at least one irrational number whose square is rational." Deducing a contradiction from a statement that is not the negation of the given statement does not prove that the given statement is true.

12. **Section 4.8**

    $34391 = 6728 \cdot 5 + 751$ and so $\gcd(34391, 6728) = \gcd(6728, 751)$

    $6728 = 751 \cdot 8 + 720$ and so $\gcd(6728, 751) = \gcd(751, 720)$

    $751 = 720 \cdot 1 + 31$ and so $\gcd(751, 720) = \gcd(720, 31)$

    $720 = 31 \cdot 23 + 7$ and so $\gcd(720, 31) = \gcd(31, 7)$

    $31 = 7 \cdot 4 + 3$ and so $\gcd(31, 7) = \gcd(7, 3)$

    $7 = 3 \cdot 2 + 1$ and so $\gcd(7, 3) = \gcd(3, 1)$

    $3 = 1 \cdot 3 + 0$ and so $\gcd(3, 1) = \gcd(1, 0)$

    But $\gcd(1, 0) = 1$. So $\gcd(34391, 6728) = 1$, and thus 34,391 and 6,728 are relatively prime.