CMSC 207- Lecture 23
CHAPTER 8: Relations (8.5)

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Antisymmetry

- We have defined three properties of relations: **reflexivity**, **symmetry**, and **transitivity**. A fourth property of relations is called ***antisymmetry***.
- In terms of the arrow diagram of a relation, saying that a relation is **antisymmetric** is the same as saying that whenever there is an arrow going from one element to another *distinct* element, there is *not* an arrow going back from the second to the first.

Antisymmetry

- **Definition**

Let R be a relation on a set A. R is **antisymmetric** if, and only if,

$$\text{for all } a \text{ and } b \text{ in } A, \quad \text{if } a\ R\ b \text{ and } b\ R\ a \text{ then } a = b.$$

- By taking the negation of the definition, a relation R is ***not* antisymmetric** if, and only if, there are elements a and b in A such that $a\ R\ b$ and $b\ R\ a$ but $a \neq b$.

Example 2 – *Testing for Antisymmetry of "Divides" Relations*

- Let R_1 be the "***divides***" relation on the set of all positive integers, and let R_2 be the "***divides***" relation on the set of all integers.

$$\text{For all } a, b \in Z^+, \qquad a \; R_1 \; b \quad \Leftrightarrow \quad a \mid b.$$
$$\text{For all } a, b \in Z, \qquad a \; R_2 \; b \quad \Leftrightarrow \quad a \mid b.$$

- **a.** Is R_1 antisymmetric? Prove or give a counterexample.
- **b.** Is R_2 antisymmetric? Prove or give a counterexample.

Example 2 – *Solution*

- **a.** R_1 is antisymmetric.

•Proof:

Suppose a and b are positive integers such that $a\ R_1\ b$ and $b\ R_1\ a$. *[We must show that $a = b$.]* By definition of R_1, $a \mid b$ and $b \mid a$.

- Thus, by definition of divides, there are integers k_1 and k_2 with $b = k_1 a$ and $a = k_2 b$. It follows that

$$b = k_1 a = k_1(k_2 b) = (k_1 k_2) b.$$

- Dividing both sides by b gives

$$k_1 k_2 = 1.$$

Example 2 – *Solution*

- Now since *a* and *b* are both integers k_1 and k_2 are both positive integers also.
- But the only product of two positive integers that equals 1 is $1 \cdot 1$.
- Thus $$k_1 = k_2 = 1$$
- and so $$a = k_2 b = 1 \cdot b = b.$$
- *[This is what was to be shown.]*

Example 2 – *Solution*

- **b.** R_2 is **not antisymmetric**.
  - **Counterexample:**
  - Let $a = 2$ and $b = -2$. Then $a \mid b$ *[since* $-2 = (-1) \cdot 2$*]* and $b \mid a$ *[*since $2 = (-1)(-2)$*]*.
  - Hence $a \ R_2 \ b$ and $b \ R_2 \ a$ but $a \neq b$.

Partial Order Relations

- A relation that is reflexive, antisymmetric, and transitive is called a *partial order relation*.

• Definition

Let R be a relation defined on a set A. R is a **partial order relation** if, and only if, R is reflexive, antisymmetric, and transitive.

- Two fundamental partial order relations are the "**less than or equal to**" relation on a set of real numbers and the "**subset**" relation on a set of sets.

Example 4 – *A "Divides" Relation on a Set of Positive Integers*

- Let **|** be the "**divides**" relation on a set *A* of positive integers. That is, for all ***a*, *b*** $\in$ ***A***,

$$\boxed{a \mid b \quad \Leftrightarrow \quad b = ka \text{ for some integer } k.}$$

- Prove that | is a partial order relation on *A*.

- Solution:

| is reflexive: *[We must show that for all* $a \in A$*, a | a.]* Suppose $a \in A$. Then $a = 1 \cdot a$, so *a* | *a* by definition of divisibility.

Example 4 – *Solution*

- **| is antisymmetric:** *[We must show that for all a, b ∈ A, if a | b and b | a then a = b.]* The proof of this is virtually identical to that of Example 2(a).
- **/ is transitive:** To show transitivity means to show that for all $a, b, c \in A$, if $a \mid b$ and $b \mid c$ then $a \mid c$. But this was proved as Theorem 4.3.3.
- Since | is reflexive, antisymmetric, and transitive, | is a partial order relation on A.

Theorem 4.3.3 Transitivity of Divisibility

For all integers a, b, and c, if a divides b and b divides c, then a divides c.

Partial Order Relations

• Notation

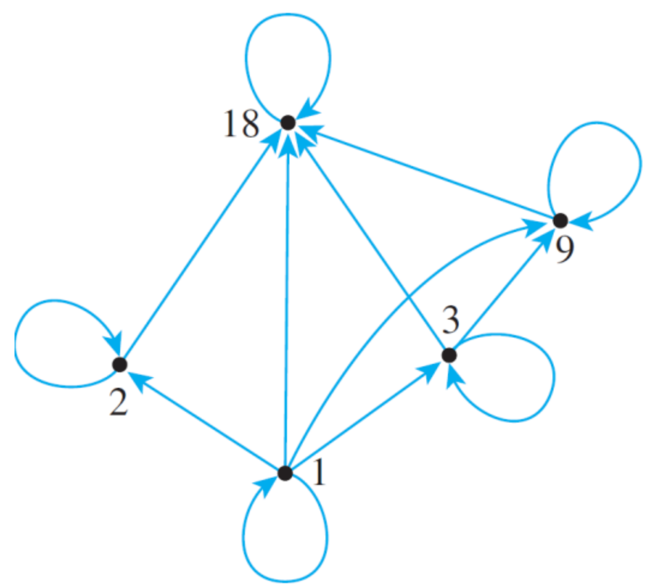
Because of the special paradigmatic role played by the $\leq$ relation in the study of partial order relations, the symbol $\preceq$ is often used to refer to a general partial order relation, and the notation $x \preceq y$ is read "x is less than or equal to y" or "y is greater than or equal to x."

Hasse Diagrams

- Let $A = \{1, 2, 3, 9, 18\}$ and consider the "divides" relation on A: For all $a, b \in A$,

$$a \mid b \quad \Leftrightarrow \quad b = ka \text{ for some integer } k.$$

- The directed graph of this relation has the following appearance:



Hasse Diagrams

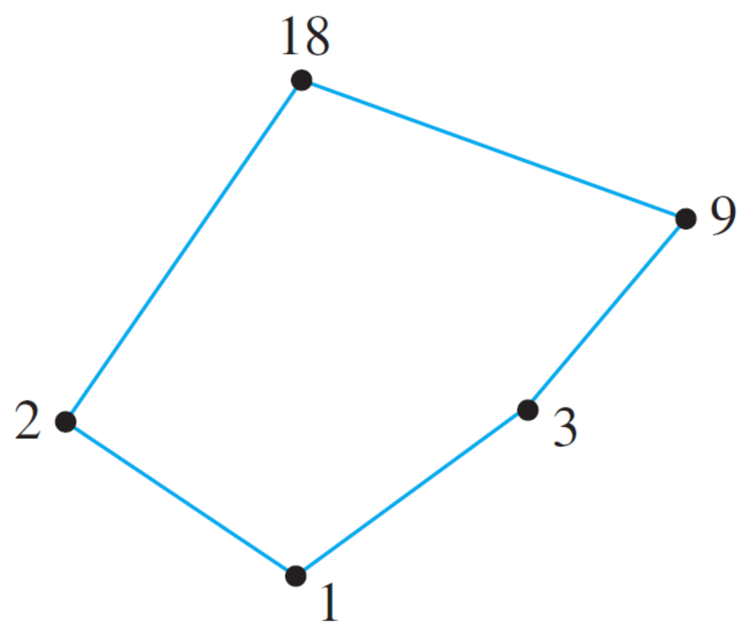
- There is a loop at every vertex, all other arrows point in the same direction (upward), and any time there is an arrow from one point to a second and from the second point to a third, there is an arrow from the first point to the third.
- Given any **partial order relation** defined on a finite set, it is possible to draw the directed graph in such a way that all of these properties are satisfied.

Hasse Diagrams

- This makes it possible to associate a somewhat simpler graph, called a **Hasse diagram** with a partial order relation defined on a finite set.
- To obtain a Hasse diagram, proceed as follows:
- Start with a directed graph of the relation, placing vertices on the page so that all arrows point upward. **Then eliminate**
- 1. the loops at all the vertices,

Hasse Diagrams

- 2. all arrows whose existence is implied by the transitive property,
- 3. the direction indicators on the arrows.
- For the relation given previously, the **Hasse diagram** is as follows:



Example 7 – *Constructing a Hasse Diagram*

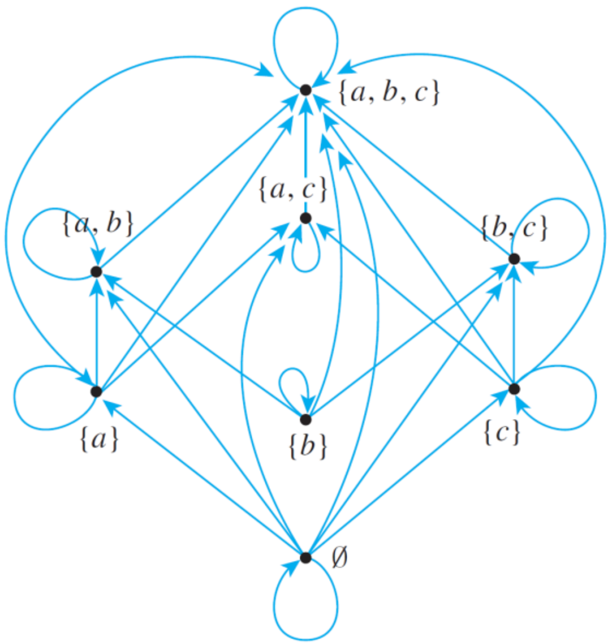
- Consider the "subset" relation, $\subseteq$, on the set $\mathscr{P}$ ({a, b, c}). That is, for all sets *U* and *V* in $\mathscr{P}$ ({a, b, c}),

$$U \subseteq V \quad \Leftrightarrow \quad \forall x, \text{ if } x \in U \text{ then } x \in V.$$

- Construct the Hasse diagram for this relation.

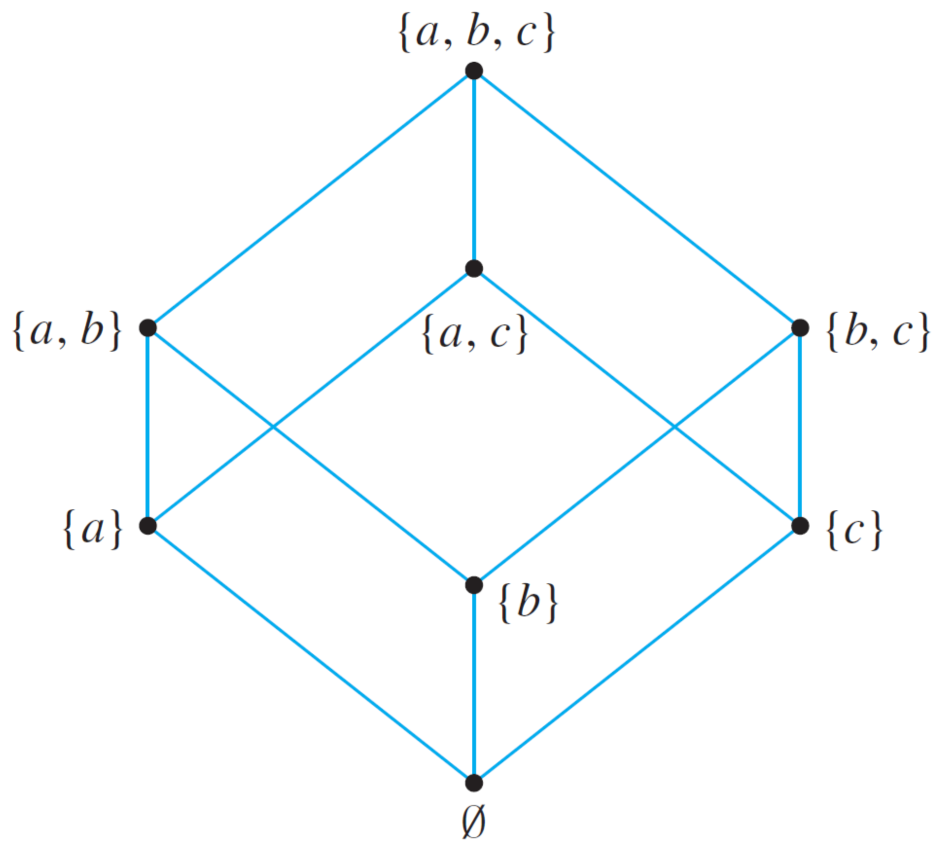
Solution:

- Draw the directed graph of the relation in such a way that all arrows except loops point upward.



Example 7 – *Solution*

- Then strip away all loops, unnecessary arrows, and direction indicators to obtain the **Hasse diagram**.



Hasse Diagrams

- To recover the directed graph of a relation from the **Hasse diagram**, just reverse the instructions given previously, using the knowledge that the original directed graph was sketched so that all arrows pointed upward:

1. Reinsert the direction markers on the arrows making all arrows point upward.
2. Add loops at each vertex.
3. For each sequence of arrows from one point to a second and from that second point to a third, add an arrow from the first point to the third.

In-class Assignment #1

- Consider the “subset” relation on $\mathscr{P}$ (S) for the following set S. Draw the **Hesse diagram** for each relation.
- **S = {0, 1}**

In-class Assignment #2

- Let A = {3, 4, 5, 6, 7, 8, 9} and define a binary relation R_3 on A as follows: For all x, y $\in$ A, xRy if and only if $\mathbf{x \mid y}$. List the pairs in the relation. Draw a Hasse diagram for the partially ordered set.