CMSC 207- Lecture 20
CHAPTER 8: Relations (8.1)

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Relations on Sets

A **binary relation** is a subset of a Cartesian product of two sets.

An *n-ary relation* is a subset of a Cartesian product of *n* sets, where *n* is any integer greater than or equal to two.

Such a relation is the fundamental structure used in relational databases. As we focus on binary relations in this text, we use the term *relation* to mean binary relation.

Example – *The Congruence Modulo 2 Relation*

Define a relation E from $\mathbf{Z}$ to $\mathbf{Z}$ as follows: For all $(m, n) \in \mathbf{Z} \times \mathbf{Z}$, $m\ E\ n \quad \Leftrightarrow \quad m - n$ is even.

a. Is $4\ E\ 0$? Is $2\ E\ 6$? Is $3\ E\ (-3)$? Is $5\ E\ 2$?

b. List five integers that are related by E to 1.

c. Prove that if n is any odd integer, then $n\ E\ 1$.

Solution:

a. Yes, $4\ E\ 0$ because $4 - 0 = 4$ and 4 is even.
Yes, $2\ E\ 6$ because $2 - 6 = -4$ and -4 is even.

Example – *Solution*

Yes, $3\ E\ (-3)$ because $3 - (-3) = 6$ and 6 is even.

No, $5 \not{E}\ 2$ because $5 - 2 = 3$ and 3 is not even.

b. There are many such lists. One is

1 because $1 - 1 = 0$ is even,

3 because $3 - 1 = 2$ is even,

5 because $5 - 1 = 4$ is even,

-1 because $-1 - 1 = -2$ is even,

-3 because $-3 - 1 = -4$ is even.

Example – *Solution*

c. Proof:

Suppose n is any odd integer.

Then $n = 2k + 1$ for some integer k. Now by definition of E, $n\ E\ 1$ if, and only if, $n - 1$ is even.

But by substitution,

$$n - 1 = (2k + 1) - 1 = 2k,$$

and since k is an integer, $2k$ is even.

Hence $n\ E\ 1$.

- **In-class Assignment #1**

The Inverse of a Relation

If R is a relation from A to B, then a relation R^{-1} from B to A can be defined by interchanging the elements of all the ordered pairs of R.

• Definition

Let R be a relation from A to B. Define the inverse relation R^{-1} from B to A as follows:

$$R^{-1} = \{(y, x) \in B \times A \mid (x, y) \in R\}.$$

This definition can be written as follows:

For all $x \in A$ and $y \in B$, $\quad (y, x) \in R^{-1} \quad \Leftrightarrow \quad (x, y) \in R.$

Example – *The Inverse of a Finite Relation*

Let $A = \{2, 3, 4\}$ and $B = \{2, 6, 8\}$ and let R be the "divides" relation from A to B: For all $(x, y) \in A \times B$,

$$x \; R \; y \quad \Leftrightarrow \quad x \mid y \qquad\qquad x \text{ divides } y.$$

a. State explicitly which ordered pairs are in R and R^{-1}, and draw arrow diagrams for R and R^{-1}.

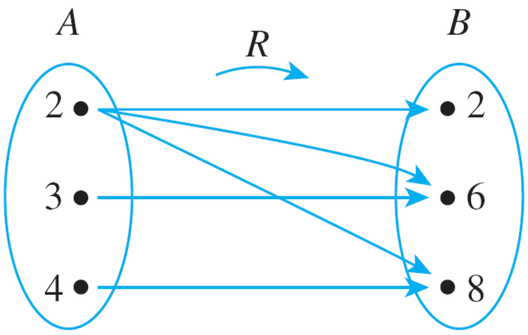
b. Describe R^{-1} in words.

Solution:

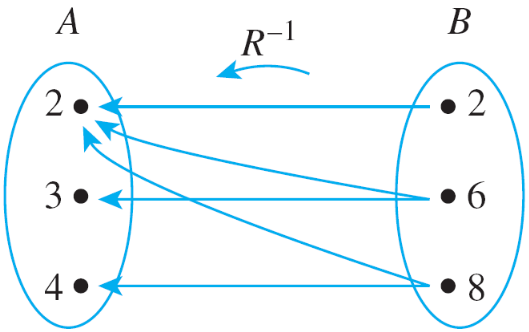
a. $R = \{(2, 2), (2, 6), (2, 8), (3, 6), (4, 8)\}$

$R^{-1} = \{(2, 2), (6, 2), (8, 2), (6, 3), (8, 4)\}$

Example – *Solution*

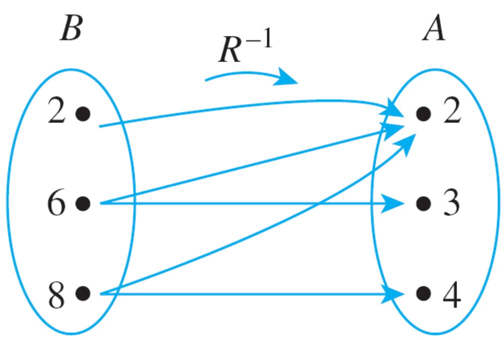


To draw the arrow diagram for R^{-1}, you can copy the arrow diagram for R but reverse the directions of the arrows.



Example – *Solution*

Or you can redraw the diagram so that *B* is on the left.



b. ***R*****$^{-1}$** can be described in words as follows:

For all (y, x) $\in$ $B \times A$, $y\ R^{-1}\ x \Leftrightarrow y$ is a multiple of x.

Directed Graph of a Relation

• Definition

A **relation on a set** A is a relation from A to A.

When a relation *R* is defined *on* a set *A*, the arrow diagram of the relation can be modified so that it becomes a **directed graph**.

Instead of representing *A* as two separate sets of points, represent *A* only once, and draw an arrow from each point of *A* to each related point.

Directed Graph of a Relation

As with an ordinary arrow diagram,

For all points x and y in A,

$$\text{there is an arrow from } x \text{ to } y \quad \Leftrightarrow \quad x \; R \; y \quad \Leftrightarrow \quad (x, y) \in R.$$

If a point is related to itself, a loop is drawn that extends out from the point and goes back to it.

Example – *Directed Graph of a Relation*

Let $A = \{3, 4, 5, 6, 7, 8\}$ and define a relation R on A as follows: For all $x, y \in A$, $x\ R\ y \Leftrightarrow 2 \mid (x - y)$.

Draw the directed graph of R.

Solution:

Here, $3\ R\ 3$ because $3 - 3 = 0$ and $2 \mid 0$, since $0 = 2 \cdot 0$. Thus, there is a loop from 3 to itself.

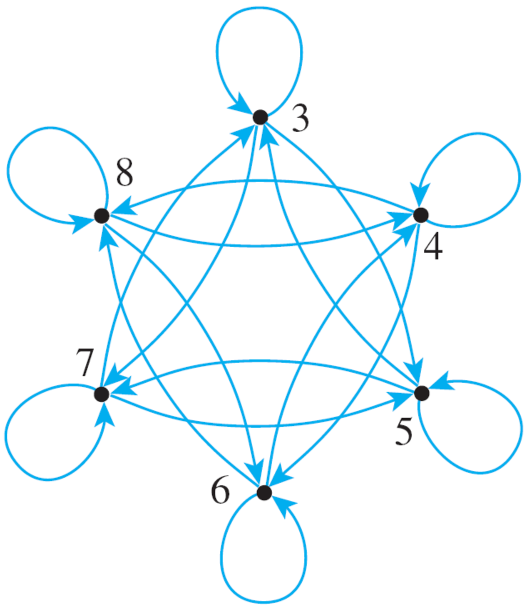
Similarly, there is a loop from 4 to itself, from 5 to itself, and so forth, since the difference of each integer with itself is 0, and $2 \mid 0$.

Example – *Solution*

Also, $3\ R\ 5$ because $3 – 5 = –2 = 2 \cdot (–1)$. And
$5\ R\ 3$ because $5 – 3 = 2 = 2 \cdot 1$.

Hence, there is an arrow from 3 to 5 and also an arrow from 5 to 3.

The other arrows in the directed graph, as shown below, are obtained by similar reasoning.



N-ary Relations and Relational Databases

***N*-ary relations** form the mathematical foundation for relational database theory.

A binary relation is a subset of a Cartesian product between two sets, similarly, an *n-ary* relation is a subset of a Cartesian product of *n* sets.

• Definition

Given sets $A_1, A_2, \ldots, A_n$, an ***n*-ary relation** R on $A_1 \times A_2 \times \cdots \times A_n$ is a subset of $A_1 \times A_2 \times \cdots \times A_n$. The special cases of 2-ary, 3-ary, and 4-ary relations are called **binary, ternary,** and **quaternary relations,** respectively.

- **In-class Assignment #2**