CMSC 207- Lecture 13
CHAPTER 5: Sequences, Mathematical Induction, and Recursion (5.2)

Dr. Ahmed Tarek

Mathematical Induction I

- Mathematical induction as a powerful proof technique in computer science has three variations.
- One is Weak Induction, also known as the Mathematical Induction I. The other one is the Strong Induction or Mathematical Induction II. The third one is Structural Induction, which will not be discussed in this course.
- Mathematical Induction is a method for proving that a **property, P(n)** defined for ***integers n*** is true for all values of *n* that are greater than or equal to some initial integer.

Mathematical Induction I

Principle of Mathematical Induction

Let $P(n)$ be a property that is defined for integers n, and let a be a fixed integer. Suppose the following two statements are true:

1. $P(a)$ is true.
2. For all integers $k \geq a$, if $P(k)$ is true then $P(k+1)$ is true.

Then the statement

$$\text{for all integers } n \geq a,\ P(n)$$

is true.

3 Parts in the Proof Process for Mathematical Induction I also known as the Principle of Mathematical Induction:

(a) **Basis:** At first, we show that P(n) is true for base case, **n = a**

(b) **Induction:** Assuming P(K) true for n = k, which is called Induction Hypothesis, we show that P(k+1) is also true based on P(k) true

(c) **Conclusion:** P(a) is true. If P(k) is true, P(k+1) is also true. As P(a) is true, so P(a+1) is true. As P(a+1) is true, so P(a+2) is true. As P(a+2) true, so P(a+3) is true. Proceeding this way, P(n) is true for all n ≥ a. (Q.E.D.)

Mathematical Induction I

In your textbook, Conclusion is omitted. So, proving a statement by Mathematical Induction I became a two-step process. The first step is called the *basis step*, and the second step is called the *inductive step*.

Method of Proof by Mathematical Induction

Consider a statement of the form, "For all integers $n \geq a$, a property $P(n)$ is true." To prove such a statement, perform the following two steps:

Step 1 (basis step): Show that $P(a)$ is true.

Step 2 (inductive step): Show that for all integers $k \geq a$, if $P(k)$ is true then $P(k+1)$ is true. To perform this step,

suppose that $P(k)$ is true, where k is any particular but arbitrarily chosen integer with $k \geq a$.

[This suppostion is called the **inductive hypothesis.***]*

Then

show that $P(k+1)$ is true.

Example 1 – *Sum of the First n Integers*

Use Mathematical Induction I to prove that

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2} \quad \text{for all integers } n \geq 1.$$

Solution:

To construct a proof by induction, you must first identify the property *P*(*n*). In this case, *P*(*n*) is the equation

$$\boxed{1 + 2 + \cdots + n = \frac{n(n+1)}{2}.} \quad \leftarrow \text{the property } (P(n))$$

Example 1 – *Solution*

- In the basis step of the proof, you must show that the property is true for n = 1, or, in other words that P(1) is true.
- Now P(1) is obtained by substituting 1 in place of n in P(n). The left-hand side of P(1) is the sum of all the successive integers starting at 1 and ending at 1. This is just 1. Thus P(1) is

$$\boxed{1 = \frac{1(1+1)}{2}.} \quad \leftarrow \text{basis } (P(1))$$

Example 1 – *Solution*

- This equation is true because the right-hand side is

$$\frac{1(1+1)}{2} = \frac{1 \cdot 2}{2} = 1,$$

which equals the left-hand side.

- In Induction Hypothesis, you assume that $P(k)$ is true, for a particular but arbitrarily chosen integer k with $k \geq 1$. Based on that assumption, you prove that P(k+1) is also true for $k \geq 1$.

Example 1 – *Solution*

- Now, it is required to prove that $P(k + 1)$ is true. **What are $P(k)$ and $P(k + 1)$?** $P(k)$ is obtained by substituting k for every n in $P(n)$.
- Thus $P(k)$ is

$$\boxed{1 + 2 + \cdots + k = \frac{k(k+1)}{2}.} \quad \leftarrow \text{inductive hypothesis} \quad (P(k))$$

- Similarly, $P(k + 1)$ is obtained by substituting the quantity $(k + 1)$ for every n that appears in $P(n)$.

Example 1 – *Solution*

- Thus $P(k + 1)$ is

$$1 + 2 + \cdots + (k + 1) = \frac{(k + 1)((k + 1) + 1)}{2},$$

or, equivalently,

$$\boxed{1 + 2 + \cdots + (k + 1) = \frac{(k + 1)(k + 2)}{2}.} \quad \leftarrow \text{to show } (P(k + 1))$$

Example 1 – *Solution*

- The inductive hypothesis is the supposition that $P(k)$ is true. This supposition has been used to prove that $P(k + 1)$ is also true.
- $P(k + 1)$ is an equation, and the truth of an equation can be shown in many different ways.
- Use the inductive hypothesis along with algebra, and other known facts to transform separately the left-hand and right-hand sides until you see that they are the same.

Example 1 – *Solution*

In this case, the left-hand side of $P(k + 1)$ is

$$1 + 2 + \cdots + (k + 1),$$

which equals

$$(1 + 2 + \cdots + k) + (k + 1)$$

The next-to-last term is k because the terms are successive integers and the last term is $k + 1$.

But by substitution from the inductive hypothesis,

$$(1 + 2 + \cdots + k) + (k + 1)$$

$$= \frac{k(k+1)}{2} + (k + 1)$$

since the inductive hypothesis says that $1 + 2 + \cdots + k = \frac{k(k+1)}{2}$

Example 1 – *Solution*

$$= \frac{k(k+1)}{2} + \frac{2(k+1)}{2}$$

by multiplying the numerator and denominator of the second term by 2 to obtain a common denominator

$$= \frac{k^2 + k}{2} + \frac{2k+2}{2}$$

by multiplying out the two numerators

$$= \frac{k^2 + 3k + 2}{2}$$

by adding fractions with the same denominator and combining like terms.

Example 1 – *Solution*

So the left-hand side of $P(k + 1)$ is $\dfrac{k^2 + 3k + 2}{2}$.

Now, the right-hand side of $P(k + 1)$ is by expanding the numerator.

$$\frac{(k+1)(k+2)}{2} = \frac{k^2 + 3k + 2}{2}$$

Thus, the two sides of $P(k + 1)$ are equal to each other, and so the equation $P(k + 1)$ is true. **Q.E.D**

Theorem 5.2.2 Sum of the First *n* Integers

For all integers $n \geq 1$,

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}.$$

Mathematical Induction I

- **Definition Closed Form**

If a sum with a variable number of terms is shown to be equal to a formula that does not contain either an ellipsis or a summation symbol, we say that it is written **in closed form**.

For example, writing $1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$ expresses the sum $1 + 2 + 3 + \cdots + n$ in closed form.

Example 2 – *Applying the Formula for the Sum of the First n Integers*

a. Evaluate $2 + 4 + 6 + \cdots + 500$.

b. Evaluate $5 + 6 + 7 + 8 + \cdots + 50$.

c. For an integer $h \geq 2$, write $1 + 2 + 3 + \cdots + (h - 1)$ in closed form.

Example 2 – *Solution*

a.

$$2 + 4 + 6 + \cdots + 500 = 2 \cdot (1 + 2 + 3 + \cdots + 250)$$

$$= 2 \cdot \left(\frac{250 \cdot 251}{2}\right)$$ by applying the formula for the sum of the first n integers with $n = 250$

$$= 62,750.$$

b.

$$5 + 6 + 7 + 8 + \cdots + 50 = (1 + 2 + 3 + \cdots + 50) - (1 + 2 + 3 + 4)$$

$$= \frac{50 \cdot 51}{2} - 10$$ by applying the formula for the sum of the first n integers with $n = 50$

$$= 1,265$$

Example 2 – *Solution*

c.

$$1 + 2 + 3 + \cdots + (h-1) = \frac{(h-1)\cdot[(h-1)+1]}{2}$$ by applying the formula for the sum of the first n integers with $n = h - 1$

$$= \frac{(h-1)\cdot h}{2}$$ since $(h-1)+1 = h$.

Virtual Classroom Practice Exercise

Mathematical Induction I

- In a **geometric sequence,** each term is obtained from the preceding one by multiplying by a constant factor.
- If the first term is 1 and the constant factor is r, then the sequence is $1, r, r^2, r^3, \ldots, r^n, \ldots$.
- The sum of the first n terms of this sequence is given by the formula $$\sum_{i=0}^{n} r^i = \frac{r^{n+1}-1}{r-1}$$

for all integers $n \geq 0$ and real numbers r not equal to 1.

Mathematical Induction I

The expanded form of the formula is:

$$r^0 + r^1 + r^2 + \cdots + r^n = \frac{r^{n+1} - 1}{r - 1},$$

and because $r^0 = 1$ and $r^1 = r$, the formula for $n \geq 1$ can be rewritten as

$$1 + r + r^2 + \cdots + r^n = \frac{r^{n+1} - 1}{r - 1}.$$

Example 3 – *Sum of a Geometric Sequence*

Theorem 5.2.3 Sum of a Geometric Sequence

For any real number r except 1, and any integer $n \geq 0$,

$$\sum_{i=0}^{n} r^i = \frac{r^{n+1} - 1}{r - 1}.$$

Solution:

In this example the property *P*(*n*) is again an equation, although in this case it contains a real variable *r*:

$$\boxed{\sum_{i=0}^{n} r^i = \frac{r^{n+1} - 1}{r - 1}.} \leftarrow \text{the property } (P(n))$$

Example 3 – *Solution*

- The proof begins by supposing that *r* is a particular but arbitrarily chosen real number not equal to 1.
- Then the proof continues by mathematical induction on *n*, starting with *n* = 0.
- In the basis step, you show that *P*(0) is true;

$$\boxed{\sum_{i=0}^{0} r^i = \frac{r^{0+1}-1}{r-1}.} \leftarrow \text{basis } (P(0))$$

- **L.H.S. = r^0 = 1 , R.H.S. =** $\dfrac{r^{0+1}-1}{r-1} = \dfrac{r-1}{r-1} = 1$
- In the inductive step, with Induction Hypothesis, suppose *k* is any integer with $k \geq 0$ for which *P*(*k*) is true with *n* = *k*.

Example 3 – *Solution*

So you substitute *k* for each *n* in *P*(*n*):

$$\boxed{\sum_{i=0}^{k} r^i = \frac{r^{k+1} - 1}{r - 1}.} \leftarrow \text{inductive hypothesis } (P(k))$$

Next, you show that *P*(*k* + 1) is also true with

n = *k* + 1.

So you substitute *k* + 1 for each *n* in *P*(*n*):

$$\sum_{i=0}^{k+1} r^i = \frac{r^{(k+1)+1} - 1}{r - 1},$$

Example 3 – *Solution*

- Or, equivalently,

$$\boxed{\sum_{i=0}^{k+1} r^i = \frac{r^{k+2}-1}{r-1}.} \leftarrow \text{to show } (P(k+1))$$

- Start with the left-hand side and transform it step-by-step into the right-hand side using the inductive hypothesis together with algebra and other known facts.

$$\sum_{i=0}^{k+1} r^i = \sum_{i=0}^{k} r^i + r^{k+1}$$ by writing the $(k+1)$st term separately from the first k terms

$$= \frac{r^{k+1}-1}{r-1} + r^{k+1}$$ by substitution from the inductive hypothesis

Example 3 – *Solution*

$$= \frac{r^{k+1} - 1}{r - 1} + \frac{r^{k+1}(r-1)}{r-1}$$

by multiplying the numerator and denominator of the second term by $(r - 1)$ to obtain a common denominator

$$= \frac{(r^{k+1} - 1) + r^{k+1}(r-1)}{r-1}$$

by adding fractions

$$= \frac{r^{k+1} - 1 + r^{k+2} - r^{k+1}}{r-1}$$

by multiplying out and using the fact that $r^{k+1} \cdot r = r^{k+1} \cdot r^1 = r^{k+2}$

$$= \frac{r^{k+2} - 1}{r-1}$$

by canceling the r^{k+1}'s.

which is the right-hand side of $P(k + 1)$

- *Since we have proved the basis step and the inductive step, we conclude that the theorem is true.* ***Virtual Classroom Practice Exercise Follows!***

Proving an Equality

The proofs of the basis and inductive steps in Examples 1 and 3 illustrate two different ways to show that an equation is true for Weak Induction or the Mathematical Induction I:

(1) Transforming the left-hand side, and the right-hand side **independently** until they become equal, and

(2) Transforming one side of the equation until it becomes the same as the other side of the equation. Sometimes people use a method that they believe proves equality but **that is actually is invalid**.

Proving an Equality

For example, to prove the basis step for Theorem 5.2.3, they perform the following steps:

$$\sum_{i=0}^{0} r^i = \frac{r^{0+1} - 1}{r - 1}$$

$$r^0 = \frac{r^1 - 1}{r - 1}$$

$$1 = \frac{r - 1}{r - 1}$$

$$1 = 1$$

Proving an Equality

- The problem with this method is that starting from a statement, and deducing a true conclusion does not prove that the statement is true.
- A true conclusion can also be deduced from a false statement. For instance, the steps below show how to deduce the true conclusion that 1 = 1 from the false statement that 1 = 0:

$$1 = 0 \quad \leftarrow \text{false}$$

$$0 = 1 \quad \leftarrow \text{false}$$

Proving an Equality

Adding:

$$1 + 0 = 0 + 1$$

$$1 = 1 \leftarrow \text{true}$$

- When using Mathematical Induction I (Weak Induction) or, Mathematical Induction II (Strong Induction) to prove formulas, **be sure to use a method that avoids invalid reasoning, both for the basis step and for the inductive step.**

Example 4 – *Applying the Formula for the Sum of a Geometric Sequence*

In each of (**a**) and (**b**) below, assume that ***m*** is an integer that is greater than or equal to 3. Write each of the sums in closed form.

a. $1 + 3 + 3^2 + \cdots + 3^{m-2}$

b. $3^2 + 3^3 + 3^4 + \cdots + 3^m$

Solution:

a.

$$1 + 3 + 3^2 + \cdots + 3^{m-2} = \frac{3^{(m-2)+1} - 1}{3 - 1}$$

by applying the formula for the sum of a geometric sequence with $r = 3$ and $n = m - 2$

$$= \frac{3^{m-1} - 1}{2}.$$

Example 4 – *Solution*

b.

$$3^2 + 3^3 + 3^4 + \cdots + 3^m = 3^2 \cdot (1 + 3 + 3^2 + \cdots + 3^{m-2}) \qquad \text{by factoring out } 3^2$$

$$= 9 \cdot \left(\frac{3^{m-1} - 1}{2}\right) \qquad \text{by part (a).}$$

- **Virtual In-class Practice Exercise**

Algebraic Approach to Prove the Formula for Geometric Sequence

- The formula for the geometric sequence may be shown to be true using a pure algebraic approach instead of Mathematical Induction I.
- Let $S_n = 1 + r + r^2 + \cdots + r^n.$

  So, $r S_n = r + r^2 + r^3 + \cdots + r^{n+1},$

  $$\begin{aligned} r S_n - S_n &= (r + r^2 + r^3 + \cdots + r^{n+1}) - (1 + r + r^2 + \cdots + r^n) \\ &= r^{n+1} - 1. \end{aligned}$$

  But $r S_n - S_n = (r - 1) S_n.$
- Equating, and dividing by $r - 1$ gives,

$$S_n = \frac{r^{n+1} - 1}{r - 1}.$$

Additional Example – Mathematical Induction I (Weak Induction)

- Prove **P(n)**: $\forall n \in Z^+$, $\mathbf{(3^n\text{-}1)}$ is an even integer.
- It is required to prove that, $\forall n \in Z^+$, $\mathbf{(3^n\text{-}1) = 2r}$**,**

where, $r \in Z$

- **Basis: n = 1.** $(3^n-1) = 3-1 = 2 = 2x1$. So, the Basis is true.
- **Induction:** With the Induction Hypothesis, assume P(k) is true for any arbitrary positive integer k. That is $(3^k-1) = 2r$ is true. With that assumption, prove P(k+1), which is $(3^{k+1}-1)$ is an even integer. Now, $(3^{k+1}-1) = (3x\ 3^k-1) = 2x3^k + (3^k-1) = 2x3^k + 2r = 2(3^k + r)$. For any $k \geq 1$, $3^k \geq 3$, and is an integer. Adding, 3^k to r provides with another integer, t.
- Therefore, $(3^{k+1}-1) = 2(3^k + r) = 2t$, and is even. **Q.E.D.**