CMSC 207- Lecture 9
CHAPTER 3: The Logic of Quantified Statements (Contd.)

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Statements with Multiple Quantifiers

When a statement contains more than one quantifier, then the actions suggested by the quantifiers are being performed in the order in which the quantifiers occur.

For instance, consider a statement of the form

$\forall x$ in set D, $\exists y$ in set E such that x and y satisfy property $P(x, y)$.

Statements with Multiple Quantifiers

Consider a statement containing both ∀ and ∃, where the ∃ comes before the ∀:

∃ an x in D such that $\forall y$ in E, x and y satisfy property $P(x, y)$.

To show that a statement of this form is true: Find one single element (call it x) in ***domain, D*** with the following property:

- After you have found your x, select any element whatsoever from ***domain, E***. Call the element y.
- Show that x together with the y satisfy property ***P*(*x*, *y*).**

Statements with Multiple Quantifiers

To establish the truth of a statement of the form

$$\exists x \text{ in } D \text{ such that } \forall y \text{ in } E, P(x, y)$$

it is required to find just one particular *x* in *D* that makes P(x, y) true regardless of *y* in *E* that in being selected.

Example 3 – *Interpreting Multiply-Quantified Statements*

A college cafeteria line has four stations: **salads, main courses, desserts, and beverages**.

The salad station offers a choice of green salad or fruit salad; the main course station offers spaghetti or fish; the dessert station offers pie or cake; and the beverage station offers milk, soda, or coffee. Three students, Uta, Tim, and Yuen, go through the line and make the following choices:

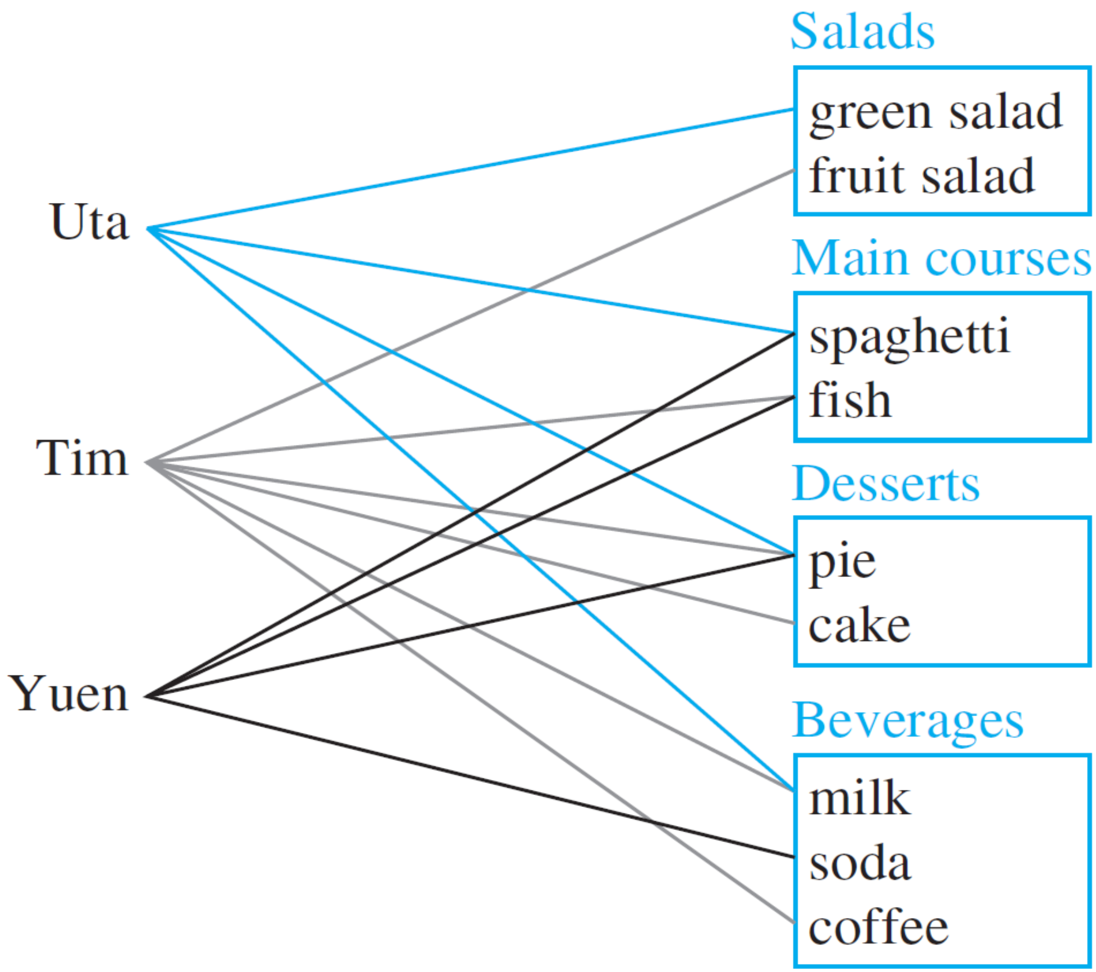
Uta: green salad, spaghetti, pie, milk

Tim: fruit salad, fish, pie, cake, milk, coffee

Yuen: spaghetti, fish, pie, soda

Example 3 – *Interpreting Multiply-Quantified Statements*

These choices are illustrated in Figure



Example 3 – *Interpreting Multiply-Quantified Statements*

Write each of following statements informally and find its truth value.

a. ∃ an item *I* such that ∀ students *S*, *S* chose *I*.

b. ∃ a student *S* such that ∀ items *I*, *S* chose *I*.

c. ∃ a student *S* such that ∀ stations *Z*, ∃ an item *I* in *Z* such that *S* chose *I*.

d. ∀ students *S* and ∀ stations *Z*, ∃ an item *I* in *Z* such that *S* chose *I*.

Example 3 – *Solution*

a. There is an item that was chosen by every student. This is true; every student chose pie.

b. There is a student who chose every available item. This is false; no student chose all nine items.

c. There is a student who chose at least one item from every station. This is true; both Uta and Tim chose at least one item from every station.

d. Every student chose at least one item from every station. This is false; Yuen did not choose a salad.

Example 4 – *Translating Multiply-Quantified Statements from Informal to Formal Language*

The **reciprocal** of a real number *a* is a real number *b* such that *ab* = 1. The following two statements are true. Rewrite them formally using quantifiers and variables:

a. Every nonzero real number has a reciprocal.

b. There is a real number with no reciprocal.

Solution:

The number 0 has no reciprocal.

a. ∀ nonzero real numbers *u*, ∃ a real number *v* such that *uv* = 1.

b. ∃ a real number *c* such that ∀ real numbers *d*, *cd* ≠ 1.

Negations of Multiply-Quantified Statements

It is possible to use the same rules to negate multiply-quantified statements that are being used to negate simpler quantified statements.

Using negation, the universal quantifier changes to existential quantifier, and vice versa. Therefore,

$\sim(\forall x$ in D, $P(x)) \equiv \exists x$ in D such that $\sim P(x)$.

And **$\sim(\exists x$ in D such that $P(x)) \equiv \forall x$ in D, $\sim P(x)$.**

Negations of Multiply-Quantified Statements

Apply these laws to find:

$$\sim(\forall x \text{ in } D, \exists y \text{ in } E \text{ such that } P(x, y))$$

by moving in stages from left to right along the sentence.

***First version of negation*:** $\exists x$ in D such that $\sim(\exists y$ in E such that $P(x, y))$.

***Final version of negation*:** $\exists x$ in D such that $\forall y$ in E, $\sim P(x, y)$.

Negations of Multiply-Quantified Statements

Similarly, to find:

$\sim(\exists x$ in D such that $\forall y$ in E, $P(x, y))$,

we have

***First version of negation*:**

$\forall x$ in D, $\sim(\forall y$ in E, $P(x, y))$.

***Final version of negation*:**

$\forall x$ in D, $\exists y$ in E such that $\sim P(x, y)$.

Negations of Multiply-Quantified Statements

These facts can be summarized as follows:

Negations of Multiply-Quantified Statements

$\sim(\forall x \text{ in } D, \exists y \text{ in } E \text{ such that } P(x, y)) \equiv \exists x \text{ in } D \text{ such that } \forall y \text{ in } E, \sim P(x, y).$

$\sim(\exists x \text{ in } D \text{ such that } \forall y \text{ in } E, P(x, y)) \equiv \forall x \text{ in } D, \exists y \text{ in } E \text{ such that } \sim P(x, y).$

Order of Quantifiers

Consider the following two statements:

∀ people *x*, ∃ a person *y* such that *x* loves *y*.

∃ a person *y* such that ∀ people *x*, *x* loves *y*.

Except for the order of the quantifiers, these statements are identical.

However, ***the first*** means that given any person, it is possible to find someone whom that person loves, whereas ***the second*** means that there is one amazing individual who is loved by all people.

Order of Quantifiers

The two sentences illustrate an extremely important property about multiply-quantified statements:

> In a statement containing both $\forall$ and $\exists$, changing the order of the quantifiers usually changes the meaning of the statement.

If one quantifier immediately follows another quantifier *of the same type*, then the order of the quantifiers does not affect the meaning.

Formal Logical Notation

In some areas of computer science, logical statements are expressed in purely symbolic notation.

The notation involves using predicates to describe all properties of variables and omitting the words ***such that*** in existential statements.

The formalism also depends on the following facts:

"$\forall x$ in D, $P(x)$" can be written as **"$\forall x(x \text{ in } D \rightarrow P(x))$,"** and **"$\exists x$ in D such that $P(x)$"** can be written as: **"$\exists x(x \text{ in } D \wedge P(x))$."**

Arguments with Quantified Statements

If some property is true of *everything* in a set, then it is true of *any particular* thing in the set.

- **Universal instantiation** is *the* fundamental tool of **deductive reasoning**. Deductive reasoning is also known as the reasoning by necessity. In deductive reasoning, there is a set of logical arguments that forms a conclusion. Deductive reasoning takes Premises down to Conclusion.

•The rule of universal instantiation can be combined with modus ponens to obtain the valid form of argument called ***universal modus ponens***.

Universal Modus Ponens

Formal Version	*Informal Version*
$\forall x$, if $P(x)$ then $Q(x)$.	If x makes $P(x)$ true, then x makes $Q(x)$ true.
$P(a)$ for a particular a.	a makes $P(x)$ true.
• $Q(a)$.	• a makes $Q(x)$ true.

- Rewrite the following argument using quantifiers, variables, and predicate symbols. Is this argument valid? Why?

If an integer is even, then its square is even.

k is a particular integer that is even.

- k^2 is even.

- Solution:
- The major premise of this argument can be rewritten as: $\forall x$, if x is an even integer then x^2 is even.

- Let $E(x)$ be the predicate: "x is an even integer," Then $E(x^2)$ is the predicate: "x^2 is an even integer," and let k stand for a particular integer that is even.
- Then the argument has the following form:

$\forall x$, if $E(x)$ then $E(x^2)$.

$E(k)$, for a particular k.

- $E(k^2)$.

- This argument has the form of universal modus ponens and is therefore valid.

- Here is a proof that the sum of any two even integers is even.
- It makes use of the definition of even integer, namely, that an integer is *even* if, and only if, it equals twice some integer. (Or, more formally: $\forall$ integers x, x is even if, and only if, $\exists$ an integer k such that $\boldsymbol{x = 2k}$.)
- Suppose m and n are particular but arbitrarily chosen even integers. Then (1) $\boldsymbol{m = 2r}$ for some integer r, and (2) $\boldsymbol{n = 2s}$ for some integer s.

- Hence

$$m + n = 2r + 2s \quad \text{by substitution}$$
$$= 2(r + s)^{(3)} \quad \text{by factoring out the 2.}$$

- Now (4) $r + s$ is an integer, and so (5) $2(r + s)$ is even.
- Thus $m + n$ is even.

- The following expansion of the proof shows how each of the numbered steps is justified by arguments that are valid by universal modus ponens.

(1) If an integer is even, then it equals twice some integer. *m* is a particular even integer.

- ***m equals twice some integer r.***

(2) If an integer is even, then it equals twice some integer. *n* is a particular even integer.

- ***n* equals twice some integer *s*.**

(3) If a quantity is an integer, then it is a real number. Now, *r* and *s* are particular integers.

- ***r and s are real numbers.***

For all *a, b*, and *c*, if *a, b*, and *c* are real numbers,
then $ab + ac = a(b + c)$.

2, *r*, and *s* are particular real numbers.

- $\boldsymbol{2r + 2s = 2(r + s)}$.

(4) For all *u* and *v*, if *u* and *v* are integers, then $u + v$ is an integer. *r* and *s* are two particular integers. ***Hence, r + s is an integer.***

(5) If a number equals twice some integer, then that number is even.

$2(r + s)$ equals twice the integer $r + s$.

- ***2(r + s) is even.***

- Another crucially important rule of inference is ***universal modus tollens.*** Its validity results from ***combining universal instantiation with modus tollens***.
- Universal modus tollens is at the heart of ***proof of contradiction,*** which is one of the most important methods of mathematical argument.

Universal Modus Tollens

Formal Version	*Informal Version*
$\forall x$, if $P(x)$ then $Q(x)$.	If x makes $P(x)$ true, then x makes $Q(x)$ true.
$\sim Q(a)$, for a particular a.	a does not make $Q(x)$ true.
• $\sim P(a)$.	• a does not make $P(x)$ true.

- Rewrite the following argument using quantifiers, variables, and predicate symbols. Write the major premise in conditional form. Is this argument valid? Why?

All human beings are mortal.

Zeus is not mortal.

- ***Zeus is not human.***

- Solution:
- The major premise can be rewritten as:

$$\forall x, \text{ if } x \text{ is human then } x \text{ is mortal.}$$

- Let ***H*** **(*x*)** be "***x* is human**," let ***M*** **(*x*)** be "***x* is mortal**," and let ***Z*** stand for ***Zeus***.
- The argument becomes:

$$\forall x, \text{ if } H(x) \text{ then } M(x)$$

$$\sim M(Z)$$

$$\bullet \quad \sim H(Z).$$

- This argument has the form of universal modus tollens, and is therefore valid.

- An argument is valid if, and only if, the truth of its conclusion follows ***necessarily*** from the truth of its premises (that means, if p is true, then q is true or $\mathbf{p \rightarrow q}$).
- In an argument, the premises represent the evidences or factual claims for proving the conclusion as a claim or inferential claim.
- The formal definition is as follows:

• Definition

To say that an *argument form* is **valid** means the following: No matter what particular predicates are substituted for the predicate symbols in its premises, if the resulting premise statements are all true, then the conclusion is also true. An *argument* is called **valid** if, and only if, its form is valid.

- Consider the statement

All integers are rational numbers.

- Or, formally,

$\forall$ ***integers n, n is a rational number.***

- Picture the set of all integers and the set of all rational numbers as disks – which are called the Disk Diagrams.

- The truth of the given statement is represented by placing the integers disk entirely inside the rational numbers disk, as shown in Figure 3.4.1

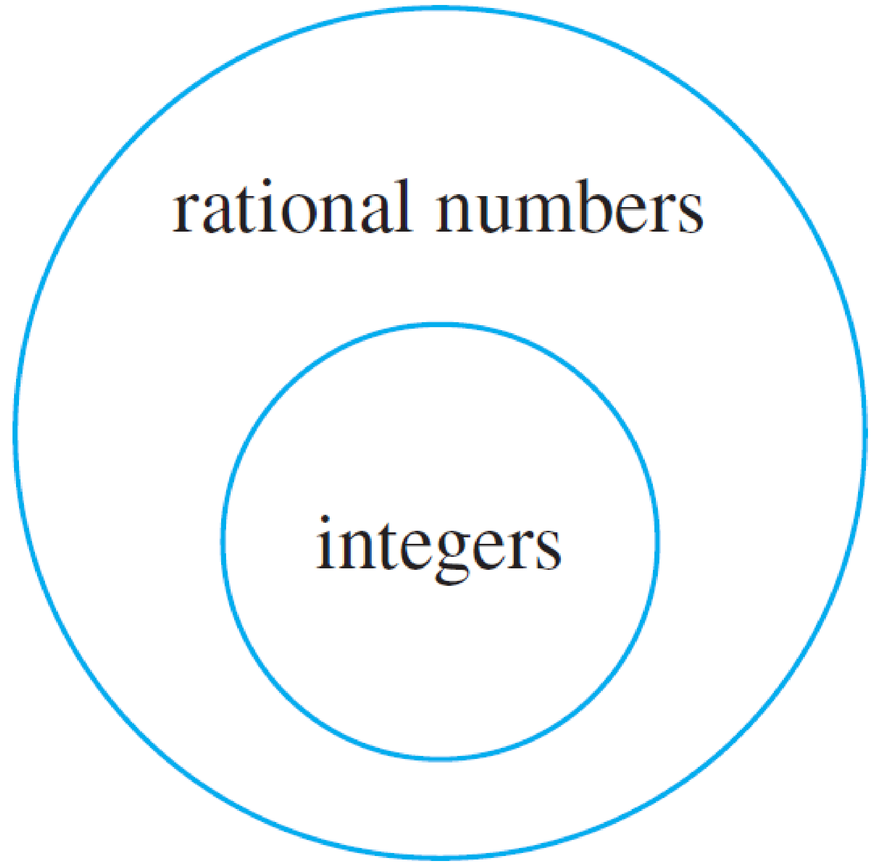


Figure 3.4.1

- To test the validity of an argument diagrammatically, represent the truth of both premises with diagrams.
- Then analyze the diagrams to see whether they necessarily represent the truth of the conclusion as well.

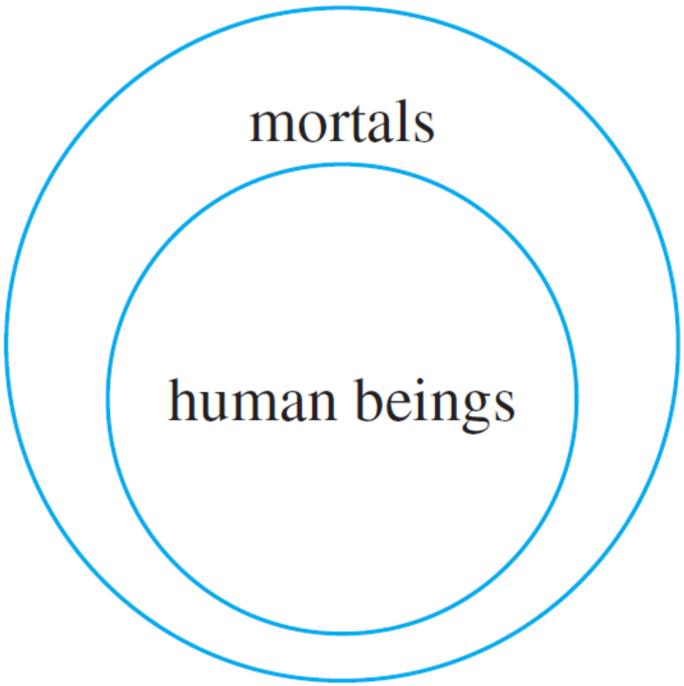
•Use a diagram to show the ***invalidity*** of the following argument:

All human beings are mortal.

Felix is mortal.

- ***Felix is a human being.***

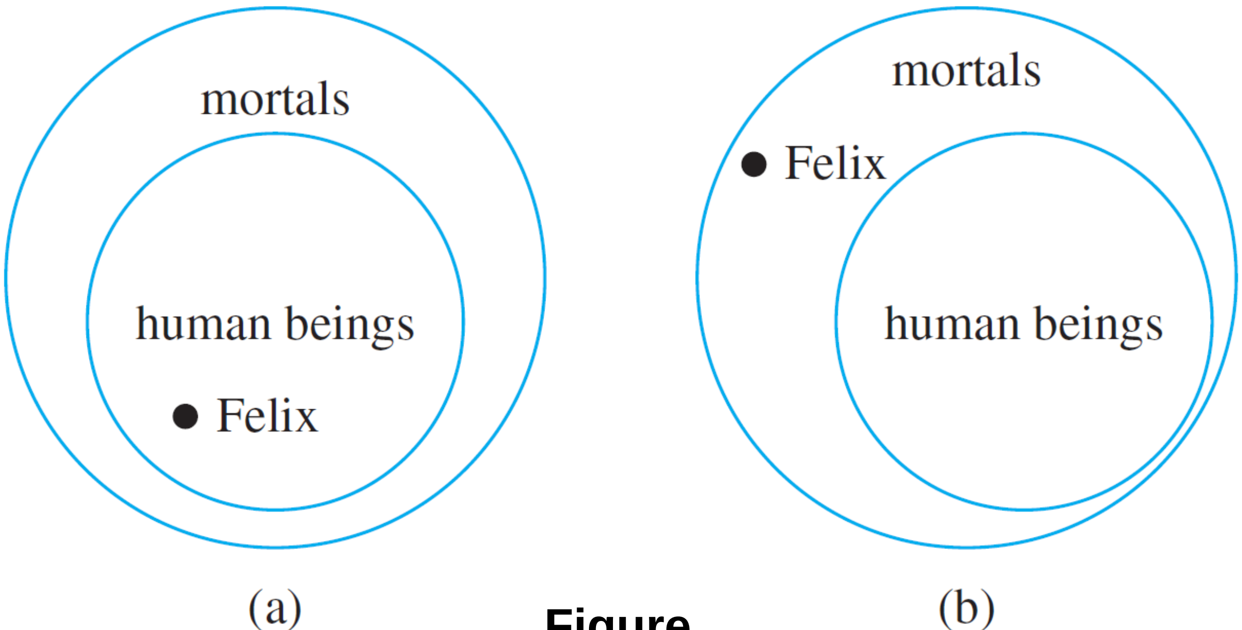
•The major and minor premises are represented diagrammatically below.



Major premise

Minor premise

• All that is known is that the Felix dot is located *somewhere* inside the mortals disk. Where it is located with respect to the human beings disk cannot be determined. Either one of the situations shown in Figure might be the case.



Figure

- The conclusion “**Felix is a human being**” is true in the first case but not in the second (Felix might, for example, be a horse).
- Because the conclusion does not necessarily follow from the premises, the argument is invalid.

- The argument of Example 6 would be valid if the major premise were replaced by its converse. But since a universal conditional statement is not logically equivalent to its converse, such a replacement cannot be made.
- We say that this argument exhibits the **converse error**.

Converse Error (Quantified Form)

Formal Version	*Informal Version*
$\forall x$, if $P(x)$ then $Q(x)$.	If x makes $P(x)$ true, then x makes $Q(x)$ true.
$Q(a)$ for a particular a.	a makes $Q(x)$ true.
$\bullet$ $P(a)$. $\leftarrow$ invalid conclusion	$\bullet$ a makes $P(x)$ true. $\leftarrow$ invalid conclusion

•The following form of argument would be valid if a conditional statement were logically equivalent to its inverse. But it is not, and the argument form is invalid. Here, the argument exhibits the **inverse error**.

Inverse Error (Quantified Form)

Formal Version	*Informal Version*
$\forall x$, if $P(x)$ then $Q(x)$.	If x makes $P(x)$ true, then x makes $Q(x)$ true.
$\sim P(a)$, for a particular a.	a does not make $P(x)$ true.
• $\sim Q(a)$. ← invalid conclusion	• a does not make $Q(x)$ true. ← invalid conclusion

•Use disk diagrams to test the following argument for validity:

No polynomial functions have horizontal asymptotes.

This function has a horizontal asymptote.

• This function is not a polynomial function.

- A good way to represent the major premise diagrammatically is shown in Figure 3.4.6, two disks—a disk for polynomial functions and a disk for functions with horizontal asymptotes—that do not overlap at all.

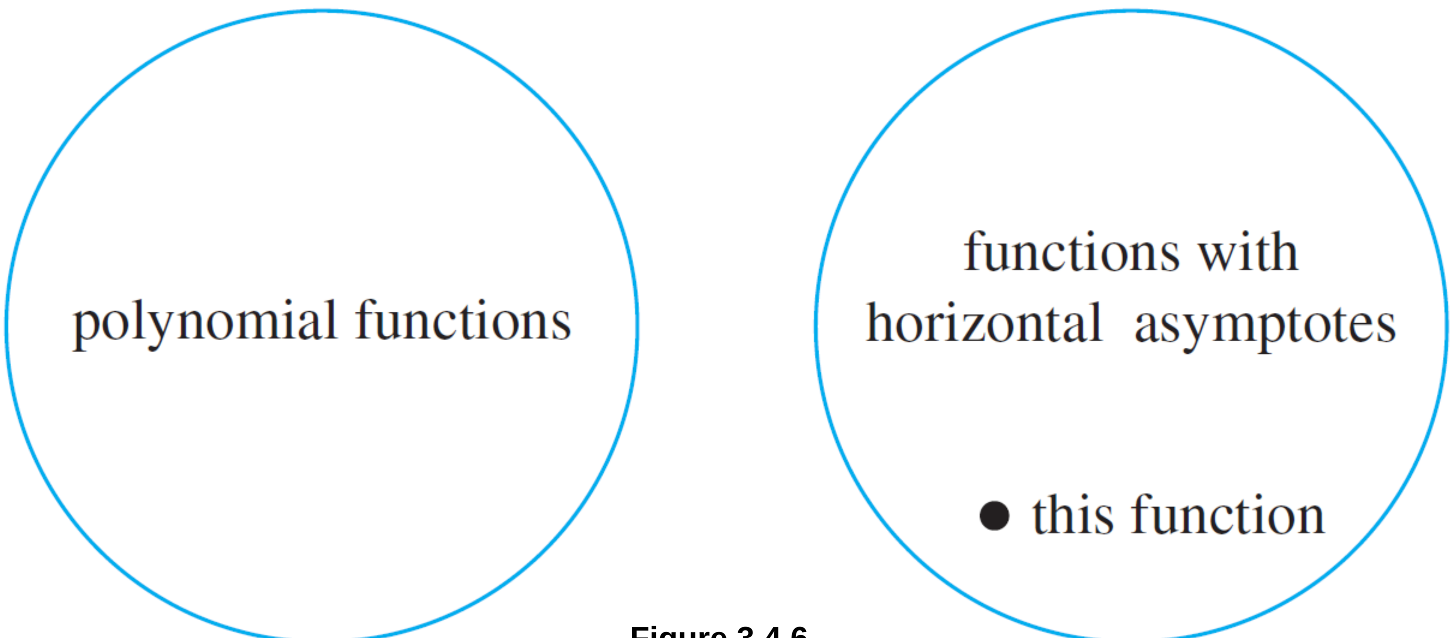


Figure 3.4.6

- The minor premise is represented by placing a dot labeled “this function” inside the disk for functions with horizontal asymptotes.
- The diagram shows that “this function” must lie outside the polynomial functions disk, and so the truth of the conclusion necessarily follows from the truth of the premises.
- Hence the argument is valid.

- An alternative approach to this example is to transform the statement "No polynomial functions have horizontal asymptotes" into the equivalent form " $\forall x$, if x is a polynomial function, then x does not have a horizontal asymptote."

- If this is done, the argument can be seen to have the form

$$\forall x, \text{ if } P(x) \text{ then } Q(x).$$

$$\sim Q(a), \text{ for a particular } a.$$

$$\bullet \sim P(a).$$

where ***P*** **(*x*)** is "***x*** **is a polynomial function**" and ***Q*** **(*x*)** is "***x*** **does not have a horizontal asymptote**."

- This is valid by ***universal modus tollens***.

- Universal modus ponens, and modus tollens were obtained by combining universal instantiation with modus ponens and modus tollens.
- In the same way, additional forms of arguments involving universally quantified statements can be obtained by combining universal instantiation with other of the valid argument forms.

•Consider the following argument:

$$p \rightarrow q$$
$$q \rightarrow r$$
$$\bullet \quad p \rightarrow r$$

•This argument form can be combined with universal instantiation to obtain the following valid argument form.

Universal Transitivity

Formal Version	*Informal Version*
$\forall x\, P(x) \rightarrow Q(x).$	Any x that makes $P(x)$ true makes $Q(x)$ true.
$\forall x\, Q(x) \rightarrow R(x).$	Any x that makes $Q(x)$ true makes $R(x)$ true.
$\bullet\ \forall x\, P(x) \rightarrow R(x).$	• Any x that makes $P(x)$ true makes $R(x)$ true.