CMSC 207- Lecture 5
CHAPTER 2: The Logic of Compound Statements (2.2)

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Conditional Statements

- Let *p* and *q* be statements. A sentence of the form "If *p*, then *q*" is denoted symbolically (formally) by "**p → q**"; ***p*** is called the ***hypothesis***, and ***q*** is called the ***conclusion***. For instance, consider the following statement: If $\underbrace{\text{4,686 is divisible by 6}}_{\text{hypothesis}}$, then $\underbrace{\text{4,686 is divisible by 3}}_{\text{conclusion}}$

- Such a sentence is called ***conditional*** because the truth of statement ***q*** is conditioned on the truth of statement ***p***.

Conditional Statements

- The notation $\boldsymbol{p \rightarrow q}$ indicates that $\rightarrow$ is a connective, like $\wedge$ or $\vee$, that can be used to join statements to create new statements. To define $\boldsymbol{p \rightarrow q}$ as a statement, therefore, we must specify the ***truth values*** for $\boldsymbol{p \rightarrow q}$ ***using a truth table*** as we specified truth values for: $\boldsymbol{p \wedge q}$ and for $\boldsymbol{p \vee q}$ using truth tables.

Conditional Statements

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

Truth Table for $p \leftrightarrow q$

- **Definition**

If p and q are statement variables, the **conditional** of q by p is "If p then q" or "p implies q" and is denoted $p \to q$. It is false when p is true and q is false; otherwise it is true. We call p the **hypothesis** (or **antecedent**) of the conditional and q the **conclusion** (or **consequent**).

Conditional Statements

- A conditional statement that is **true by virtue of the fact that its hypothesis is false** is often called **vacuously true** or **true by default**. Thus the statement "If you show up for work Monday morning, then you will get the job" is vacuously true if you do not show up for work Monday morning. In general, when the "if" part of an if-then statement is false, the statement as a whole is said to be true, regardless of whether the conclusion is true or false.

Example 1 – *A Conditional Statement with a False Hypothesis*

- Consider the statement:

If $0 = 7$ then $1 = 5$.

Since the hypothesis of this statement is false, the statement as a whole is true.

Conditional Statements

- In expressions that include $\rightarrow$ as well as other logical operators such as $\wedge$, $\vee$, and $\sim$, the **order of operations** is that $\rightarrow$ is performed last.

- Thus, according to the specification of order of operations, **$\sim$ is performed first**, then $\wedge$ and $\vee$, and finally $\rightarrow$.

Example 2 – *Truth Table for* $p \vee \sim q \rightarrow \sim p$

- Construct a truth table for the statement form:

  $p \vee \sim q \rightarrow \sim p.$

Solution:

Due to the order of operations, the following two expressions are equivalent: $p \vee \sim q \rightarrow \sim p$ and $(p \vee (\sim q)) \rightarrow (\sim p)$, and this order will determine the construction of the truth table.

First fill in the four possible combinations of truth values for p and q.

Next enter the truth values for $\sim p$ and $\sim q$ using the definition of negation.

Example 2 – *Solution*

- **Next** fill in the $\boldsymbol{p} \vee \sim\boldsymbol{q}$ column using the definition of $\vee$. **Finally,** fill in the $\boldsymbol{p} \vee \sim\boldsymbol{q} \rightarrow \sim\boldsymbol{p}$ column using the definition of $\rightarrow$.
- The only rows in which the **hypothesis** $\boldsymbol{p} \vee \sim\boldsymbol{q}$ is true, and the **conclusion** $\sim\boldsymbol{p}$ is false are the first and second rows.
- So you put F's in those two rows and T's in the other two rows.

		conclusion		hypothesis	
p	q	$\sim p$	$\sim q$	$p \vee \sim q$	$p \vee \sim q \rightarrow \sim p$
T	T	F	F	T	F
T	F	F	T	T	F
F	T	T	F	F	T
F	F	T	T	T	T

Example 3 – *Division into Cases: Showing that $p \vee q \to r \equiv (p \to r) \wedge (q \to r)$*

- Use truth tables to show the logical equivalence of the statement forms, $p \vee q \to r$ and $(p \to r) \wedge (q \to r)$. Annotate the table with a sentence of explanation.

- **Solution:**

First fill in the eight possible combinations of truth values for p, q, and r.

- Then fill in the columns for $\boldsymbol{p \vee q}$, $\boldsymbol{p \to r}$, and $\boldsymbol{q \to r}$ using the definitions of ***or*** and ***if-then***.

Example 3 – *Solution*

- For instance, the $\boldsymbol{p \to r}$ column has **F's** in the second and fourth rows because these are the rows in which p is true, and q is false.
- Next fill in the $\boldsymbol{p \vee q \to r}$ column using the definition of ***if-then***. The rows in which the hypothesis $\boldsymbol{p \vee q}$ is true, and the **conclusion** $\boldsymbol{r}$ is false are the second, fourth, and sixth. So F's go in these rows and T's in all the others.

Example 3 – *Solution*

- The complete table shows that $\boldsymbol{p \vee q \rightarrow r}$, and $\boldsymbol{(p \rightarrow r) \wedge (q \rightarrow r)}$ have the same truth values for each combination of truth values of $\boldsymbol{p}$, $\boldsymbol{q}$, and $\boldsymbol{r}$. Hence, the two statement forms are logically equivalent.

p	q	r	$p \vee q$	$p \rightarrow r$	$q \rightarrow r$	$p \vee q \rightarrow r$	$(p \rightarrow r) \wedge (q \rightarrow r)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	F
T	F	T	T	T	T	T	T
T	F	F	T	F	T	F	F
F	T	T	T	T	T	T	T
F	T	F	T	T	F	F	F
F	F	T	F	T	T	T	T
F	F	F	F	T	T	T	T

$p \vee q \rightarrow r$ and $(p \rightarrow r) \wedge (q \rightarrow r)$ always have the same truth values, so they are logically equivalent

The Negation of a Conditional Statement

- By definition, $p \rightarrow q$ is false if, and only if, its hypothesis, p, is true and its conclusion, q, is false. It follows that

> The negation of "if p then q" is logically equivalent to "p and not q."

- This can be restated symbolically as follows:

$$\sim(p \rightarrow q) \equiv p \wedge \sim q$$

Example 5 – *Negations of If-Then Statements*

- Write negations for each of the following statements:
- **a**. If my car is in the repair shop, then I cannot get to class.
- **b**. If Sara lives in Athens, then she lives in Greece.

•Solution:

- **a**. My car is in the repair shop, and I can get to class.
- **b**. Sara lives in Athens, and she does not live in Greece. (Sara might live in Athens, Georgia; Athens, Ohio; or Athens, Wisconsin.)

The Contrapositive of a Conditional Statement

- One of the most fundamental laws of logic is the equivalence between a conditional statement and its contrapositive.

• Definition

The **contrapositive** of a conditional statement of the form "If p then q" is

$$\text{If } \sim q \text{ then } \sim p.$$

Symbolically,

$$\text{The contrapositive of } p \to q \text{ is } \sim q \to \sim p.$$

- **This is True, since:**

A conditional statement is logically equivalent to its contrapositive.

Example 6 – *Writing the Contrapositive*

- Write each of the following statements in its equivalent contrapositive form:
- **a**. If Johnny can swim across the lake, then Johnny can swim to the island.
- **b**. If today is Easter, then tomorrow is Monday.

- Solution:

a. If Johnny cannot swim to the island, then Johnny cannot swim across the lake.

b. If tomorrow is not Monday, then today is not Easter.

The Converse and Inverse of a Conditional Statement

- The fact that a conditional statement, and its contrapositive are logically equivalent is very important and has wide application. Two other variants of a conditional statement are *not* logically equivalent to the statement.

• Definition

Suppose a conditional statement of the form "If p then q" is given.

1. The **converse** is "If q then p."
2. The **inverse** is "If $\sim p$ then $\sim q$."

Symbolically,

$$\text{The converse of } p \to q \text{ is } q \to p,$$

and

$$\text{The inverse of } p \to q \text{ is } \sim p \to \sim q.$$

Example 7 – *Writing the Converse and the Inverse*

- Write the converse and inverse of each of the following statements:
- **a**. If Johnny can swim across the lake, then Johnny can swim to the island.
- **b**. If today is Easter, then tomorrow is Monday.
- Solution:

a. ***Converse*****:** If Johnny can swim to the island, then Johnny can swim across the lake.

- ***Inverse*****:** If Johnny cannot swim across the lake, then Johnny cannot swim to the island.

Example 7 – *Solution*

- **b**. ***Converse*****:** If tomorrow is Monday, then today is Easter.
- ***Inverse*****:** If today is not Easter, then tomorrow is not Monday.

The Converse and Inverse of a Conditional Statement

1. A conditional statement and its converse are *not* logically equivalent.
2. A conditional statement and its inverse are *not* logically equivalent.
3. The converse and the inverse of a conditional statement are logically equivalent to each other.

Only If and the Biconditional

- "p only if q" means that p can take place *only* if q takes place also. That is, if q does not take place, then p cannot take place.
- Another way to state this is that if p occurs, then q must also occur (by the logical equivalence between a statement and its contrapositive).

• Definition

It p and q are statements,

p **only if** q means "if not q then not p,"

or, equivalently,

"*if* p then q."

Example 8 – *Converting Only If to If-Then*

- Rewrite the following statement in if-then form in two ways, one of which is the contrapositive of the other.
- John will break the world’s record for the mile run only if he runs the mile in under four minutes.
- **Solution:**

***Version 1*:** If John does not run the mile in under four minutes, then he will not break the world’s record.

- ***Version 2*:** If John breaks the world’s record, then he will have run the mile in under four minutes.

Only If and the Biconditional

• Definition

Given statement variables p and q, the **biconditional of p and q** is "p if, and only if, q" and is denoted $p \leftrightarrow q$. It is true if both p and q have the same truth values and is false if p and q have opposite truth values. The words *if and only if* are sometimes abbreviated **iff.**

- **The biconditional has the following truth table:**

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Truth Table for $p \leftrightarrow q$

Only If and the Biconditional

- In order of operations $\leftrightarrow$ is coequal with $\rightarrow$. As with $\wedge$ and $\vee$, the only way to indicate precedence between them is to use parentheses. The full hierarchy of operations for the five logical operators is:

Order of Operations for Logical Operators

1.	$\sim$	Evaluate negations first.
2.	$\wedge, \vee$	Evaluate $\wedge$ and $\vee$ second. When both are present, parentheses may be needed.
3.	$\rightarrow, \leftrightarrow$	Evaluate $\rightarrow$ and $\leftrightarrow$ third. When both are present, parentheses may be needed.

Only If and the Biconditional

- According to the separate definitions of *if* and *only if*, saying "*p* if, and only if, *q*" should mean the same as saying both "*p* if *q*" and "*p* only if *q*."
- The following annotated truth table shows that this is the case:

p	q	$p \to q$	$q \to p$	$p \leftrightarrow q$	$(p \to q) \wedge (q \to p)$
T	T	T	T	T	T
T	F	F	T	F	F
F	T	T	F	F	F
F	F	T	T	T	T

Truth Table Showing that $p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$

$p \leftrightarrow q$ and $(p \to q) \wedge (q \to p)$ always have the same truth values, so they are logically equivalent

Example 9 – *If and Only If*

- Rewrite the following statement as a conjunction of two if-then statements:
  - This computer program is correct if, and only if, it produces correct answers for all possible sets of input data.
- **Solution:**

If this program is correct, then it produces the correct answers for all possible sets of input data; and if this program produces the correct answers for all possible sets of input data, then it is correct.

Necessary and Sufficient Conditions

- The phrases ***necessary condition*** and ***sufficient condition***, as used in formal English, correspond exactly to their definitions in logic.

• Definition

If r and s are statements:

r is a **sufficient condition** for s	means	"if r then s."
r is a **necessary condition** for s	means	"if not r then not s."

- In other words, to say "*r* is a sufficient condition for *s*" means that the occurrence of *r* is *sufficient* to guarantee the occurrence of *s*.

Necessary and Sufficient Conditions

- On the other hand, to say "***r is a necessary condition for s***" means that ***if r does not occur, then s cannot occur either***:

The occurrence of *r* is necessary to obtain the occurrence of *s*. Because of the equivalence between a statement and its contrapositive,

> r is a necessary condition for s also means "if s then r."

- Consequently,

> r is a necessary and sufficient condition for s means "r if, and only if, s."

Example 10 – *Interpreting Necessary and Sufficient Conditions*

- Consider the statement "If John is eligible to vote, then he is at least 18 years of age." **($p \rightarrow q$)**

The condition: "John is at least 18 years of age" is *necessary* for the condition "John is eligible to vote" to be true.

- If John were younger than 18, then he would not be eligible to vote. **($\sim q \rightarrow \sim p$)**