CMSC 207- Lecture 16
CHAPTER 6: Set Theory (6.1 & 6.2)

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Subsets: Proof and Disproof

- We begin by rewriting for a set *A* to be a subset of a set *B* as a formal universal conditional statement:

$$A \subseteq B \quad \Leftrightarrow \quad \forall x, \text{ if } x \in A \text{ then } x \in B.$$

- The negation is, therefore, existential:

$$\mathrm{A} \not\subseteq \mathrm{B} \quad \Leftrightarrow \quad \exists x \text{ such that } x \in A \text{ and } x \notin B.$$

Subsets: Proof and Disproof

- A *proper subset* of a set is a subset that is not equal to its containing set. Thus

A is a **proper subset** of B $\Leftrightarrow$

(1) $A \subseteq B$, and
(2) there is at least one element in B that is not in A.

In-class Exercise 1

- Let $A = \{1\}$ and $B = \{1, \{1\}\}$.
  a. Is $A \subseteq B$?
  b. If so, is A a proper subset of B?

Subsets: Proof and Disproof

- Because we define what it means for one set to be a subset of another by means of a universal conditional statement, we can use the method of direct proof to establish a subset relationship. Such a proof is called an *element argument* and is the fundamental proof technique of set theory.

Element Argument: The Basic Method for Proving That One Set Is a Subset of Another

Let sets X and Y be given. To prove that $X \subseteq Y$,

1. **suppose** that x is a particular but arbitrarily chosen element of X,
2. **show** that x is an element of Y.

Example 2 – *Proving and Disproving Subset Relations*

- Define sets *A* and *B* as follows:

$$A = \{m \in \mathbf{Z} \mid m = 6r + 12 \text{ for some } r \in \mathbf{Z}\}$$

$$B = \{n \in \mathbf{Z} \mid n = 3s \text{ for some } s \in \mathbf{Z}\}.$$

- **a.** Outline a proof that $A \subseteq B$.

- **b.** Prove that $A \subseteq B$.

- **c.** Disprove that $B \subseteq A$.

Example 2 – *Solution*

- **a. Proof Outline:**
  Suppose x is a particular but arbitrarily chosen element of A.

  $\vdots$

- Therefore, x is an element of B.
- **b. Proof:**
  Suppose x is a particular but arbitrarily chosen element of A.

[We must show that $x \in B$. By definition of B, this means we must show that $x = 3 \cdot$ (some integer).]

Example 2 – *Solution*

- By definition of A, there is an integer r such that $x = 6r + 12$.
- *[Given that $x = 6r + 12$, can we express x as $3 \cdot$(some integer)? I.e., does $6r + 12 = 3 \cdot$(some integer)? Yes, $6r + 12 = 3 \cdot (2r + 4)$.]* *[We must check that s is an integer.]*
- Let $s = 2r + 4$.
- Then s is an integer because products and sums of integers are integers.

Example 2 – *Solution*

- Also

$$3s = 3(2r + 4) = 6r + 12 = x,$$

[which is what was to be shown].

- Thus, by definition of *B*, *x* is an element of *B*,
- **c.** To disprove a statement means to show that it is false, and to show it is false that $B \subseteq A$, you must find an element of *B* that is not an element of *A*.

Example 2 – *Solution*

- By the definitions of A and B, this means that you must find an integer x of the form $3 \cdot$ (some integer) that cannot be written in the form $6 \cdot$ (some integer) $+ 12$.
- A little experimentation reveals that various numbers do the job. For instance, you could let $x = 3$. Then $x \in B$ because $3 = 3 \cdot 1$, but $x \notin A$ because there is no integer r such that $3 = 6r + 12$. For if there were such an integer, then $6r + 12 = 3$ by assumption

Example 2 – *Solution*

- $$\begin{aligned} \Rightarrow \quad 2r+4 &= 1 && \text{by dividing both sides by 3} \\ \Rightarrow \quad 2r &= 3 && \text{by subtracting 4 from both sides} \\ \Rightarrow \quad r &= 3/2 && \text{by dividing both sides by 2,} \end{aligned}$$
- but 3/2 is not an integer. Thus $3 \in B$ but $3 \notin A$, and so $B \not\subseteq A$.
- **Classroom Practice Exercise 2**

Set Equality

- We have known that by the axiom of extension, sets *A* and *B* are equal if, and only if, they have exactly the same elements.

• Definition

Given sets A and B, A **equals** B, written $\boldsymbol{A = B}$, if, and only if, every element of A is in B and every element of B is in A.

Symbolically:

$$A = B \quad \Leftrightarrow \quad A \subseteq B \text{ and } B \subseteq A.$$

Example 3 – *Set Equality*

- Define sets *A* and *B* as follows:

$$A = \{m \in \mathbf{Z} \mid m = 2a \text{ for some integer } a\}$$
$$B = \{n \in \mathbf{Z} \mid n = 2b - 2 \text{ for some integer } b\}$$

- Is *A* = *B*?
- Solution:
- Yes. To prove this, both subset relations $A \subseteq B$ and $B \subseteq A$ must be proved.

Example 3 – *Solution*

- **Part 1, Proof That $A \subseteq B$:**
- Suppose x is a particular but arbitrarily chosen element of A. *[We must show that $x \in B$. By definition of B, this means we must show that $x = 2 \cdot$ (some integer) $- 2$.]*
- By definition of A, there is an integer a such that $x = 2a$. *[Given that $x = 2a$, can x also be expressed as $2 \cdot$ (some integer) $- 2$? i.e., is there an integer, say b, such that*
$2a = 2b - 2$? Solve for b to obtain $b = (2a + 2)/2 = a + 1$.

Example 3 – *Solution*

- Let $b = a + 1$. *[First check that b is an integer.]*
- Then b is an integer because it is a sum of integers. *[Then check that $x = 2b – 2$.]*
- Also $2b – 2 = 2(a + 1) – 2 = 2a + 2 – 2 = 2a = x$,
- Thus, by definition of B, x is an element of B
- *[which is what was to be shown]*.
- **Part 2, Proof That $B \subseteq A$:**
- Similarly, we can prove that $B \subseteq A$. Hence $A = B$.

Venn Diagrams

- If sets A and B are represented as regions in the plane, relationships between A and B can be represented by pictures, called **Venn diagrams,** that were introduced by the British mathematician John Venn in 1881.
- For instance, the relationship $A \subseteq B$ can be pictured in one of two ways, as shown in Figure 6.1.1.

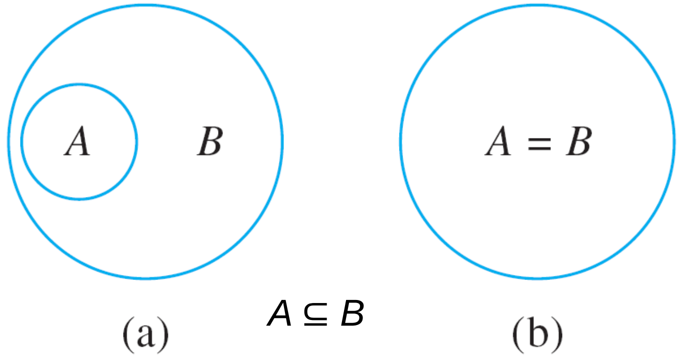
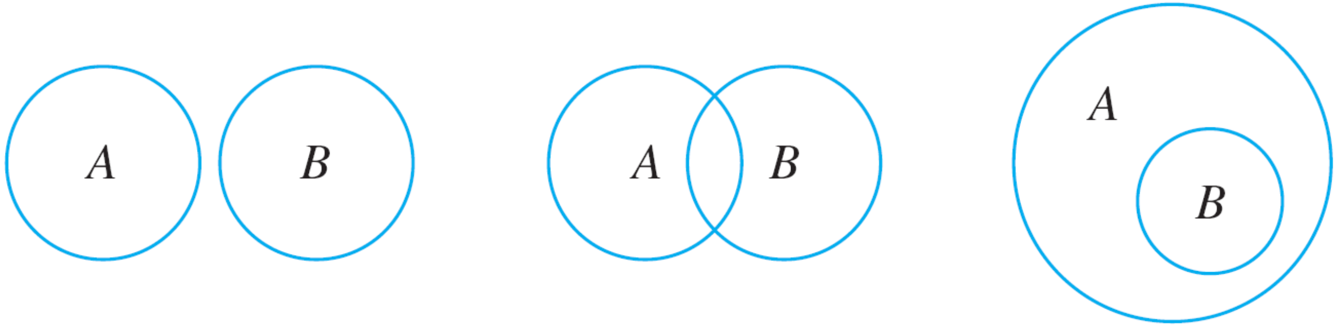


Figure 6.1.1

Venn Diagrams

- The relationship $A \not\subseteq B$ can be represented in three different ways with Venn diagrams, as shown in Figure 6.1.2.



(a) (b) (c)

$A \not\subseteq B$

Figure 6.1.2

Example 4 – *Relations among Sets of Numbers*

- Since **Z**, **Q**, and **R** denote the sets of integers, rational numbers, and real numbers, respectively, **Z** is a subset of **Q** because every integer is rational (any integer *n* can be written in the form $\frac{n}{1}$).
- **Q** is a subset of **R** because every rational number is real (any rational number can be represented as a length on the number line).
- **Z** is a proper subset of **Q** because there are rational numbers that are not integers (for example, $\frac{1}{2}$).

- **Q** is a proper subset of **R** because there are real numbers that are not rational (for example, $\sqrt{2}$).
- This is shown diagrammatically in Figure 6.1.3.

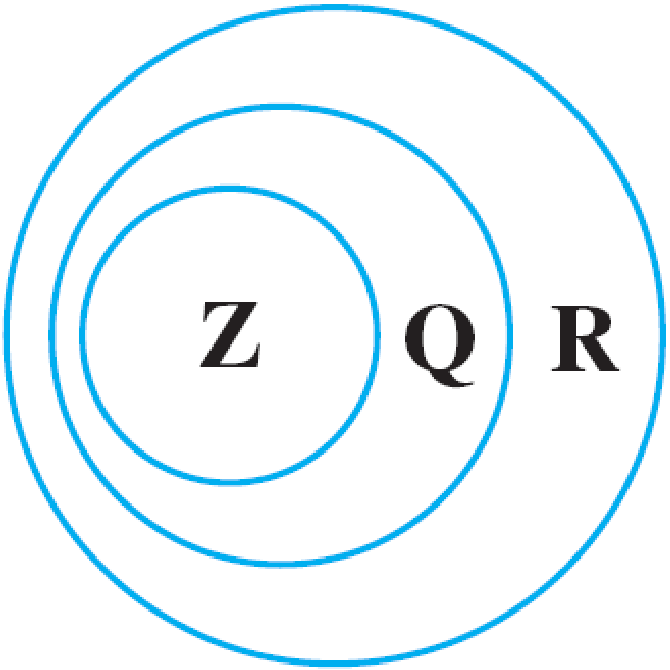


Figure 6.1.3

Operations on Sets

• Definition

Let A and B be subsets of a universal set U.

1. The **union** of A and B, denoted $\boldsymbol{A \cup B}$**,** is the set of all elements that are in at least one of A or B.

2. The **intersection** of A and B, denoted $\boldsymbol{A \cap B}$**,** is the set of all elements that are common to both A and B.

3. The **difference** of B minus A (or **relative complement** of A in B), denoted $\boldsymbol{B - A}$**,** is the set of all elements that are in B and not A.

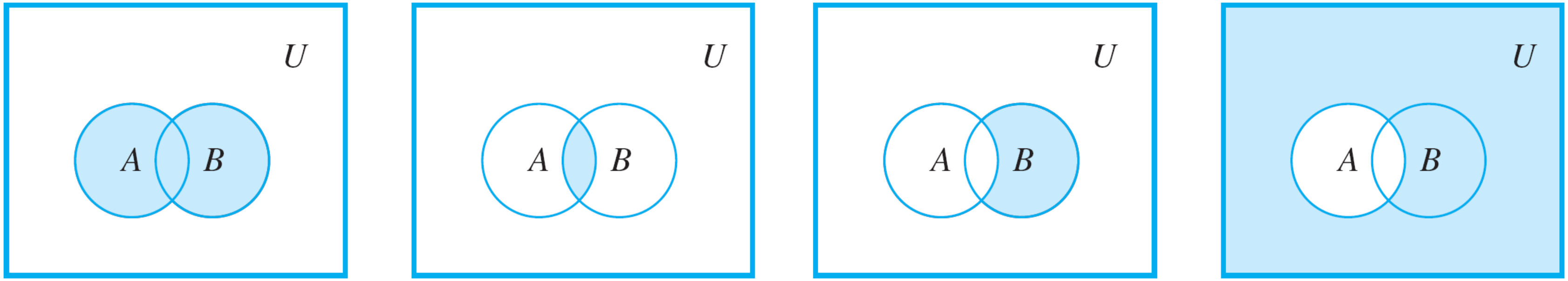
4. The **complement** of A, denoted $\boldsymbol{A^c}$**,** is the set of all elements in U that are not in A.

Symbolically:

$$
\begin{aligned}
A \cup B &= \{x \in U \mid x \in A \text{ or } x \in B\},\\
A \cap B &= \{x \in U \mid x \in A \text{ and } x \in B\},\\
B - A &= \{x \in U \mid x \in B \text{ and } x \notin A\},\\
A^c &= \{x \in U \mid x \notin A\}.
\end{aligned}
$$

Operations on Sets

- Venn diagram representations for union, intersection, difference, and complement are shown in Figure 6.1.4.



Shaded region represents $A \cup B$.

Shaded region represents $A \cap B$.

Shaded region represents $B - A$.

Shaded region represents A^c.

Figure 6.1.4

Example 5 – *Unions, Intersections, Differences, and Complements*

- Let the universal set be the set $U = \{a, b, c, d, e, f, g\}$ and let $A = \{a, c, e, g\}$ and $B = \{d, e, f, g\}$. Find $A \cup B$, $A \cap B$, $B - A$, and A^c.

- Solution:

$$A \cup B = \{a, c, d, e, f, g\}$$

$$A \cap B = \{e, g\}$$

$$B - A = \{d, f\}$$

$$A^c = \{b, d, f\}$$

Operations on Sets

- The definitions of unions and intersections for more than two sets are very similar to the definitions for two sets.

• Definition

Unions and Intersections of an Indexed Collection of Sets

Given sets $A_0, A_1, A_2, \ldots$ that are subsets of a universal set U and given a nonnegative integer n,

$$\bigcup_{i=0}^{n} A_i = \{x \in U \mid x \in A_i \text{ for at least one } i = 0, 1, 2, \ldots, n\}$$

$$\bigcup_{i=0}^{\infty} A_i = \{x \in U \mid x \in A_i \text{ for at least one nonnegative integer } i\}$$

$$\bigcap_{i=0}^{n} A_i = \{x \in U \mid x \in A_i \text{ for all } i = 0, 1, 2, \ldots, n\}$$

$$\bigcap_{i=0}^{\infty} A_i = \{x \in U \mid x \in A_i \text{ for all nonnegative integers } i\}.$$

Operations on Sets

- An alternative notation for $\bigcup_{i=0}^{n} A_i$ is $A_0 \cup A_1 \cup \ldots \cup A_n$, and an alternative notation for $\bigcap_{i=0}^{n} A_i$ is $A_0 \cap A_1 \cap \ldots \cap A_n$.

The Empty Set

- A set that has no elements is unique, and given a special name. We call it the **empty set** (or **null set**) and denote it by the symbol **Ø**.
- Thus $\mathbf{\{1, 3\} \cap \{2, 4\} = \emptyset}$ and $\mathbf{\{\boldsymbol{x} \in R \mid \boldsymbol{x}^2 = -1\} = \emptyset}$.

Example 8 – *A Set with No Elements*

- **Describe the set:** $D = \{x \in \mathbf{R} \mid 3 < x < 2\}.$
- **Solution:**

We have known that $a < x < b$ means that $a < x$ and $x < b$. So D consists of all real numbers that are both greater than 3 and less than 2.

- Since there are no such numbers, D has no elements, and so $D = \emptyset$.

Partitions of Sets

- In many applications of set theory, sets are divided up into non-overlapping (or *disjoint*) pieces. Such a division is called a *partition*.

• Definition

Two sets are called **disjoint** if, and only if, they have no elements in common.
Symbolically:

$$A \text{ and } B \text{ are disjoint} \quad \Leftrightarrow \quad A \cap B = \emptyset.$$

Example 9 – *Disjoint Sets*

- Let $A = \{1, 3, 5\}$ and $B = \{2, 4, 6\}$. Are A and B disjoint?
- Solution:

Yes. By inspection A and B have no elements in common, or, in other words, $\{1, 3, 5\} \cap \{2, 4, 6\} = \emptyset$.

Partitions of Sets

- **Definition**

Sets A_1, A_2, $A_3 \ldots$ are **mutually disjoint** (or **pairwise disjoint** or **nonoverlapping**) if, and only if, no two sets A_i and A_j with distinct subscripts have any elements in common. More precisely, for all $i, j = 1, 2, 3, \ldots$

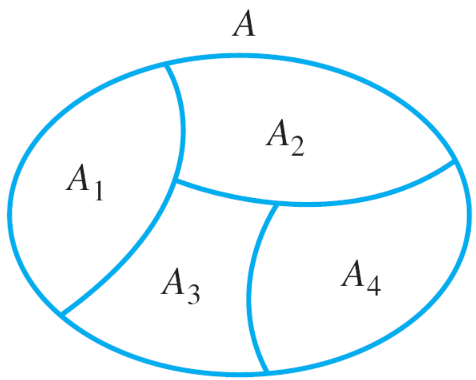
$$A_i \cap A_j = \emptyset \quad \text{whenever } i \neq j.$$

Example 10 – *Mutually Disjoint Sets*

- **a.** Let $A_1 = \{3, 5\}$, $A_2 = \{1, 4, 6\}$, and $A_3 = \{2\}$. Are A_1, A_2, and A_3 mutually disjoint?
- **b.** Let $B_1 = \{2, 4, 6\}$, $B_2 = \{3, 7\}$, and $B_3 = \{4, 5\}$. Are B_1, B_2, and B_3 mutually disjoint?
- Solution:
- **a.** Yes. A_1 and A_2 have no elements in common, A_1 and A_3 have no elements in common, and A_2 and A_3 have no elements in common.
- **b.** No. B_1 and B_3 both contain 4.

Partitions of Sets

- **Suppose *A*, *A_1*, *A_2*, *A_3*, and *A_4* are the sets of points represented by the regions shown in Figure 6.1.5.**



A Partition of a Set

Figure 6.1.5

- Then A_1, A_2, A_3, and A_4 are subsets of A, and
- $A = A_1 \cup A_2 \cup A_3 \cup A_4$.

Partitions of Sets

- Suppose further that boundaries are assigned to the regions representing A_1, A_2, A_3, and A_4 in such a way that these sets are mutually disjoint.
- Then A is called a *union of mutually disjoint subsets*, and the collection of sets $\{A_1, A_2, A_3, A_4\}$ is said to be a *partition* of A.

• Definition

A finite or infinite collection of nonempty sets $\{A_1, A_2, A_3 \ldots\}$ is a **partition** of a set A if, and only if,

1. A is the union of all the A_i
2. The sets $A_1, A_2, A_3, \ldots$ are mutually disjoint.

- **In-class Assignment #3**

In-class Assignment #3

- **a.** Let $A = \{1, 2, 3, 4, 5, 6\}$, $A_1 = \{1, 2\}$, $A_2 = \{3, 4\}$, and $A_3 = \{5, 6\}$. Is $\{A_1, A_2, A_3\}$ a partition of A?
- **b.** Let **Z** be the set of all integers and let

$$T_0 = \{n \in \mathbf{Z} \mid n = 3k, \text{ for some integer } k\},$$

$$T_1 = \{n \in \mathbf{Z} \mid n = 3k + 1, \text{ for some integer } k\}, \text{ and}$$

$$T_2 = \{n \in \mathbf{Z} \mid n = 3k + 2, \text{ for some integer } k\}.$$

- Is $\{T_0, T_1, T_2\}$ a partition of **Z**?
- **Hints:** Apply the Quotient-Remainder theorem

Power Sets

• **Definition**

Given a set A, the **power set** of A, denoted $\mathscr{P}(A)$, is the set of all subsets of A.

Example 12 – *Power Set of a Set*

- Find the power set of the set $\{x, y\}$. That is, find $\mathscr{P}(\{x, y\})$.
- Solution:
- $\mathscr{P}(\{x, y\})$ is the set of all subsets of $\{x, y\}$. We know that $\emptyset$ is a subset of every set, and so $\emptyset \in \mathscr{P}(\{x, y\})$.
- Also any set is a subset of itself, so $\{x, y\} \in \mathscr{P}(\{x, y\})$. The only other subsets of $\{x, y\}$ are $\{x\}$ and $\{y\}$, so $\mathscr{P}(\{x, y\}) = \{\emptyset, \{x\}, \{y\}, \{x, y\}\}$.

Cartesian Products

- **Definition**

Let n be a positive integer and let $x_1, x_2, \ldots, x_n$ be (not necessarily distinct) elements. The **ordered n-tuple, $(x_1, x_2, \ldots, x_n)$,** consists of $x_1, x_2, \ldots, x_n$ together with the ordering: first x_1, then x_2, and so forth up to x_n. An ordered 2-tuple is called an **ordered pair,** and an ordered 3-tuple is called an **ordered triple.**

Two ordered n-tuples $(x_1, x_2, \ldots, x_n)$ and $(y_1, y_2, \ldots, y_n)$ are **equal** if, and only if, $x_1 = y_1, x_2 = y_2, \ldots, x_n = y_n$.

Symbolically:

$$(x_1, x_2, \ldots, x_n) = (y_1, y_2, \ldots, y_n) \quad \Leftrightarrow \quad x_1 = y_1, x_2 = y_2, \ldots, x_n = y_n.$$

In particular,

$$(a, b) = (c, d) \quad \Leftrightarrow \quad a = c \text{ and } b = d.$$

Example 13 – *Ordered n-tuples*

- **a.** Is $(1, 2, 3, 4) = (1, 2, 4, 3)$?
- **b.**

  Is $\left(3, (-2)^2, \frac{1}{2}\right) = \left(\sqrt{9}, 4, \frac{3}{6}\right)$?
- Solution:
- **a.** No. By definition of equality of ordered 4-tuples,

  $$(1,\ 2,\ 3,\ 4) = (1,\ 2,\ 4,\ 3) \Leftrightarrow 1 = 1,\ 2 = 2,\ 3 = 4,\ \text{and } 4 = 3$$

- But $3 \neq 4$, and so the ordered 4-tuples are not equal.

Example 13 – *Solution*

- **b.** Yes. By definition of equality of ordered
$$\left(3, (-2)^2, \frac{1}{2}\right) = \left(\sqrt{9}, 4, \frac{3}{6}\right) \quad \Leftrightarrow \quad 3 = \sqrt{9} \text{ and } (-2)^2 = 4 \text{ and } \frac{1}{2} = \frac{3}{6}.$$
- Because these equations are all true, the two ordered triples are equal.

Cartesian Products

- **Definition**

Given sets $A_1, A_2, \ldots, A_n$, the **Cartesian product** of $A_1, A_2, \ldots, A_n$ denoted $\boldsymbol{A_1 \times A_2 \times \ldots \times A_n}$, is the set of all ordered n-tuples $(a_1, a_2, \ldots, a_n)$ where $a_1 \in A_1, a_2 \in A_2, \ldots, a_n \in A_n$.

Symbolically:

$$A_1 \times A_2 \times \cdots \times A_n = \{(a_1, a_2, \ldots, a_n) \mid a_1 \in A_1, a_2 \in A_2, \ldots, a_n \in A_n\}.$$

In particular,

$$A_1 \times A_2 = \{(a_1, a_2) \mid a_1 \in A_1 \text{ and } a_2 \in A_2\}$$

is the Cartesian product of A_1 and A_2.

Example 14 – *Cartesian Products*

- Let $A_1 = \{x, y\}$, $A_2 = \{1, 2, 3\}$, and $A_3 = \{a, b\}$.

a. Find $A_1 \times A_2$.

b. Find $(A_1 \times A_2) \times A_3$.

c. Find $A_1 \times A_2 \times A_3$.

- **Solution:**

- **a.** $A_1 \times A_2 = \{(x, 1), (x, 2), (x, 3), (y, 1), (y, 2), (y, 3)\}$

- **b.** The Cartesian product of A_1 and A_2 is a set, so it may be used as one of the sets making up another Cartesian product. This is the case for $(A_1 \times A_2) \times A_3$.

Example 14 – *Solution*

$$(A_1 \times A_2) \times A_3 = \{(u, v) \mid u \in A_1 \times A_2 \text{ and } v \in A_3\}$$ by definition of Cartesian product

$$= \{((x, 1), a), ((x, 2), a), ((x, 3), a), ((y, 1), a),$$

$$((y, 2), a), ((y, 3), a), ((x, 1), b), ((x, 2), b), ((x, 3), b),$$

$$((y, 1), b), ((y, 2), b), ((y, 3), b)\}$$

- **c.** The Cartesian product $A_1 \times A_2 \times A_3$ is superficially similar to, but is not quite the same mathematical object as, $(A_1 \times A_2) \times A_3$. $(A_1 \times A_2) \times A_3$ is a set of ordered pairs of which one element is itself an ordered pair, whereas $A_1 \times A_2 \times A_3$ is a set of ordered triples.

Example 14 – *Solution*

- By definition of Cartesian product,

$$A_1 \times A_2 \times A_3 = \{(u, v, w) \mid u \in A_1, v \in A_2, \text{and } w \in A_3\}$$

$$= \{(x, 1, a), (x, 2, a), (x, 3, a), (y, 1, a), (y, 2, a),$$

$$(y, 3, a), (x, 1, b), (x, 2, b), (x, 3, b), (y, 1, b),$$

$$(y, 2, b), (y, 3, b)\}.$$

Properties of Sets

- We begin by listing some set properties that involve subset relations.

Theorem 6.2.1 Some Subset Relations

1. *Inclusion of Intersection:* For all sets A and B,

$$\text{(a) } A \cap B \subseteq A \quad \text{and} \quad \text{(b) } A \cap B \subseteq B.$$

2. *Inclusion in Union:* For all sets A and B,

$$\text{(a) } A \subseteq A \cup B \quad \text{and} \quad \text{(b) } B \subseteq A \cup B.$$

3. *Transitive Property of Subsets:* For all sets A, B, and C,

$$\text{if } A \subseteq B \text{ and } B \subseteq C, \text{ then } A \subseteq C.$$

Properties of Sets

- Procedural versions of the definitions of the other set operations are derived similarly and are summarized below.

Procedural Versions of Set Definitions

Let X and Y be subsets of a universal set U and suppose x and y are elements of U.

1. $x \in X \cup Y \quad \Leftrightarrow \quad x \in X$ or $x \in Y$
2. $x \in X \cap Y \quad \Leftrightarrow \quad x \in X$ and $x \in Y$
3. $x \in X - Y \quad \Leftrightarrow \quad x \in X$ and $x \notin Y$
4. $x \in X^c \quad \Leftrightarrow \quad x \notin X$
5. $(x, y) \in X \times Y \quad \Leftrightarrow \quad x \in X$ and $y \in Y$

Set Identities

- An **identity** is an equation that is universally true for all elements in some set. For example, the equation $a + b = b + a$ is an identity for real numbers because it is true for all real numbers a and b. The collection of set properties in the next theorem consists entirely of set identities. That is, they are equations that are true for all sets in some universal set.

Theorem 6.2.2 Set Identities

Let all sets referred to below be subsets of a universal set U.

1. *Commutative Laws:* For all sets A and B,

$$\text{(a) } A \cup B = B \cup A \quad \text{and} \quad \text{(b) } A \cap B = B \cap A.$$

Set Identities

cont'd

2. *Associative Laws:* For all sets A, B, and C,

$$\text{(a) } (A \cup B) \cup C = A \cup (B \cup C) \quad \text{and}$$
$$\text{(b) } (A \cap B) \cap C = A \cap (B \cap C).$$

3. *Distributive Laws:* For all sets, A, B, and C,

$$\text{(a) } A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \quad \text{and}$$
$$\text{(b) } A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$$

4. *Identity Laws:* For all sets A,

$$\text{(a) } A \cup \emptyset = A \quad \text{and} \quad \text{(b) } A \cap U = A.$$

5. *Complement Laws:*

$$\text{(a) } A \cup A^c = U \quad \text{and} \quad \text{(b) } A \cap A^c = \emptyset.$$

6. *Double Complement Law:* For all sets A,

$$(A^c)^c = A.$$

7. *Idempotent Laws:* For all sets A,

$$\text{(a) } A \cup A = A \quad \text{and} \quad \text{(b) } A \cap A = A.$$

Set Identities

cont'd

8. *Universal Bound Laws:* For all sets A,

$$\text{(a) } A \cup U = U \quad \text{and} \quad \text{(b) } A \cap \emptyset = \emptyset.$$

9. *De Morgan's Laws:* For all sets A and B,

$$\text{(a) } (A \cup B)^c = A^c \cap B^c \quad \text{and} \quad \text{(b) } (A \cap B)^c = A^c \cup B^c.$$

10. *Absorption Laws:* For all sets A and B,

$$\text{(a) } A \cup (A \cap B) = A \quad \text{and} \quad \text{(b) } A \cap (A \cup B) = A.$$

11. *Complements of U and $\emptyset$:*

$$\text{(a) } U^c = \emptyset \quad \text{and} \quad \text{(b) } \emptyset^c = U.$$

12. *Set Difference Law:* For all sets A and B,

$$A - B = A \cap B^c.$$

Proving Set Identities

- As we have known, Two sets are equal $\Leftrightarrow$ each is a subset of the other.

Basic Method for Proving That Sets Are Equal

Let sets X and Y be given. To prove that $X = Y$:

1. Prove that $X \subseteq Y$.
2. Prove that $Y \subseteq X$.

Proving Set Identities

- Suppose *A* and *B* are arbitrarily chosen sets.

Theorem 6.2.2(3)(a) A Distributive Law for Sets

For all sets A, B, and C,

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C).$$

Theorem 6.2.2(9)(a) A De Morgan's Law for Sets

For all sets A and B, $(A \cup B)^c = A^c \cap B^c$.

Proving Set Identities

- The set property given in the next theorem says that if one set is a subset of another, then their intersection is the smaller of the two sets and their union is the larger of the two sets.

Theorem 6.2.3 Intersection and Union with a Subset

For any sets A and B, if $A \subseteq B$, then

$$\text{(a) } A \cap B = A \quad \text{and} \quad \text{(b) } A \cup B = B.$$

The Empty Set

- The crucial fact is that **the negation of a universal statement is existential:** If a set *B* is not a subset of a set *A*, then there exists an element in *B* that is not in *A*. But if *B* has no elements, then no such element can exist.

Theorem 6.2.4 A Set with No Elements Is a Subset of Every Set

If E is a set with no elements and A is any set, then $E \subseteq A$.

- How many sets with no elements are there? <u>**Only one**</u>**.**

Example 5 – *A Proof for a Conditional Statement*

- **Prove that for all sets *A*, *B*, and *C*, if $A \subseteq B$ and $B \subseteq C^c$, then $A \cap C = \emptyset$.**
- **Solution:**
- **Since the statement to be proved is both universal and conditional, you start with the method of direct proof: Suppose** *A*, *B*, and *C* are arbitrarily chosen sets that satisfy the condition: $A \subseteq B$ and $B \subseteq C^c$.
- **Show** that $A \cap C = \emptyset$.

Example 5 – *Solution*

- Suppose there is an element x in $A \cap C$.
- By definition of intersection, $x \in A$ and $x \in C$. Then, since $A \subseteq B$, $x \in B$ by definition of subset. Also, since $B \subseteq C^c$, then $x \in C^c$ by definition of subset again. It follows by definition of complement that $x \notin C$. Thus, $x \in C$ and $x \notin C$, which is a contradiction.
- **So the supposition that there is an element x in $A \cap C$ is false, and thus $A \cap C = \emptyset$ *[as was to be shown]*.**