CMSC 207- Lecture 6
CHAPTER 2: The Logic of Compound Statements
Section 2.3

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Valid and Invalid Arguments

An argument is a sequence of statements ending in a conclusion. In this section we show how to determine whether an argument is valid—that is, whether the conclusion follows *necessarily* from the preceding statements. We will show that this determination depends only on the form of an argument, not on its content.

For example, the argument

If Socrates is a man, then Socrates is mortal.

Socrates is a man.

- ***Socrates is mortal.***

Valid and Invalid Arguments

The abstract form is:

If *p* then *q*

p

• *q*

When considering the abstract form of an argument, think of *p* and *q* as variables for which statements may be substituted.

An argument form is called *valid* if, and only if, whenever statements are substituted that make ***all the premises true***, the ***conclusion is also true***.

Valid and Invalid Arguments

- **Definition**

An **argument** is a sequence of statements, and an **argument form** is a sequence of statement forms. All statements in an argument and all statement forms in an argument form, except for the final one, are called **premises** (or **assumptions** or **hypotheses**). The final statement or statement form is called the **conclusion.** The symbol • , which is read "therefore," is normally placed just before the conclusion.

To say that an *argument form* is **valid** means that no matter what particular statements are substituted for the statement variables in its premises, if the resulting premises are all true, then the conclusion is also true. To say that an *argument* is **valid** means that its form is valid.

When an argument is valid and its premises are true, the truth of the conclusion is said to be *inferred* or *deduced* from the truth of the premises. If a conclusion "isn't necessarily so," then it isn't a valid deduction.

Valid and Invalid Arguments

Testing an Argument Form for Validity

1. Identify the premises and conclusion of the argument form.

2. Construct a truth table showing the truth values of all the premises and the conclusion.

3. A row of the truth table in which all the premises are true is called a **critical row.** If there is a critical row in which the conclusion is false, then it is possible for an argument of the given form to have true premises and a false conclusion, and so the argument form is invalid. If the conclusion in *every* critical row is true, then the argument form is valid.

Example 1 – *Determining Validity or Invalidity*

Determine whether the following argument form is valid or invalid by drawing a truth table, indicating which columns represent the premises and which represent the conclusion, and annotating the table with a sentence of explanation.

When you fill in the table, you only need to indicate the truth values for the conclusion in the rows where all the premises are true (the critical rows) because the truth values of the conclusion in the other rows are irrelevant to the validity or invalidity of the argument.

Example 1 – *Determining Validity or Invalidity* cont'd

$$p \rightarrow q \vee \sim r$$

$$q \rightarrow p \wedge r$$

$$\bullet \; p \rightarrow r$$

Solution:

The truth table shows that even though there are several situations in which the premises and the conclusion are all true (rows 1, 7, and 8), there is one situation (row 4) where the premises are true and the conclusion is false.

Example 1 – *Solution*

cont'd

p	q	r	$\sim r$	$q \vee \sim r$	$p \wedge r$	$p \to q \vee \sim r$ (premises)	$q \to p \wedge r$ (premises)	$p \to r$ (conclusion)
T	T	T	F	T	T	T	T	T
T	T	F	T	T	F	T	F	
T	F	T	F	F	T	F	T	
T	F	F	T	T	F	T	T	F
F	T	T	F	T	F	T	F	
F	T	F	T	T	F	T	F	
F	F	T	F	F	F	T	T	T
F	F	F	T	T	F	T	T	T

This row shows that an argument of this form can have true premises and a false conclusion. Hence this form of argument is invalid.

Modus Ponens and Modus Tollens

An argument form consisting of two premises and a conclusion is called a **syllogism.** The first and second premises are called the **major premise** and **minor premise**, respectively.

The most famous form of syllogism in logic is called **modus ponens.** It has the following form:

If *p* then *q*.

p

• *q*

Modus Ponens and Modus Tollens

Modus Ponens is a valid form of argument.

To prove, we construct a truth table for the premises and conclusion.

		premises		conclusion	
p	q	$p \rightarrow q$	p	q	
T	T	T	T	T	⟵ critical row
T	F	F	T		
F	T	T	F		
F	F	T	F		

Modus Ponens and Modus Tollens

The first row is the only one in which both premises are true, and the conclusion in that row is also true. Hence the argument form is valid.

Next consider another valid argument form called **modus tollens**. It has the following form:

If *p* then *q*.

∼*q*

• ∼*p*

Example 2 – *Recognizing Modus Ponens and Modus Tollens*

Use modus ponens or modus tollens to fill in the blanks of the following arguments so that they become valid inferences.

a. If there are more pigeons than there are pigeonholes, then at least two pigeons roost in the same hole.
There are more pigeons than there are pigeonholes.

- ____.

b. If 870,232 is divisible by 6, then it is divisible by 3.
870,232 is not divisible by 3.

- ____.

Example 2 – *Solution*

a. At least two pigeons roost in the same hole. by modus ponens

b. 870,232 is not divisible by 6. by modus tollens

Additional Valid Argument Forms: Rules of Inference

A **rule of inference** is a form of argument that is valid. Thus modus ponens and modus tollens are both rules of inference.

The following are additional examples of rules of inference that are frequently used in deductive reasoning.

Example 3 – *Generalization*

The following argument forms are valid:

a. p

$\bullet\ p \vee q$

b. q

$\bullet\ p \vee q$

These argument forms are used for making generalizations. For instance, according to the first, if p is true, then, more generally, "p or q" is true for *any* other statement q.

As an example, suppose you are given the job of counting the upperclassmen at your school. You ask what class Anton is in and are told he is a junior.

Example 3 – *Generalization*

cont'd

You reason as follows:

Anton is a junior.

- (more generally) Anton is a junior or Anton is a senior.

Knowing that upperclassman means junior or senior, you add Anton to your list.

Example 4 – *Specialization*

The following argument forms are valid:

a. $p \wedge q$
$\bullet\ p$

b. $p \wedge q$
$\bullet\ q$

These argument forms are used for specializing. When classifying objects according to some property, you often know much more about them than whether they do or do not have that property.

When this happens, you discard extraneous information as you concentrate on the particular property of interest.

Example 4 – *Specialization*

cont'd

For instance, suppose you are looking for a person who knows graph algorithms to work with you on a project. You discover that Ana knows both numerical analysis and graph algorithms. You reason as follows:

Ana knows numerical analysis and Ana knows graph algorithms.

- (in particular) Ana knows graph algorithms.

Accordingly, you invite her to work with you on your project.

Example 5 – *Elimination*

The following argument forms are valid:

a. $p \vee q$

$\sim q$

$\bullet\ p$

b. $p \vee q$

$\sim p$

$\bullet\ q$

These argument forms say that when you have only two possibilities and you can rule one out, the other must be the case. For instance, suppose you know that for a particular number *x*,

$$x - 3 = 0 \quad \text{or} \quad x + 2 = 0.$$

Example 5 – *Elimination*

cont'd

If you also know that x is not negative, then $x \neq -2$, so

$$x + 2 \neq 0.$$

By elimination, you can then conclude that

- $x - 3 = 0.$

Example 6 – *Transitivity*

The following argument form is valid:

$p \rightarrow q$

$q \rightarrow r$

$\therefore\ p \rightarrow r$

Many arguments in mathematics contain chains of if-then statements.

From the fact that one statement implies a second and the second implies a third, you can conclude that the first statement implies the third.

Example 6 – *Transitivity*

cont'd

Here is an example:

If 18,486 is divisible by 18, then 18,486 is divisible by 9.

If 18,486 is divisible by 9, then the sum of the digits of 18,486 is divisible by 9.

- If 18,486 is divisible by 18, then the sum of the digits of 18,486 is divisible by 9.

Example 7 – *Proof by Division into Cases*

The following argument form is valid:

$$p \vee q$$

$$p \rightarrow r$$

$$q \rightarrow r$$

• r

It often happens that you know one thing or another is true. If you can show that in either case a certain conclusion follows, then this conclusion must also be true.

For instance, suppose you know that x is a particular nonzero real number.

Example 7 – *Proof by Division into Cases* cont'd

Any real number is positive, negative, or zero. Thus (by elimination) you know that x is positive or x is negative.

You can deduce that $x^2 > 0$ by arguing as follows:

x is positive or x is negative.

If x is positive, then $x^2 > 0$.

If x is negative, then $x^2 > 0$.

$\therefore$ $x^2 > 0$.

Example 8 – *Application: A Complex Deduction*

You are about to leave for school in the morning and discover that you don't have your glasses. You know the following statements are true:

a. If I was reading the newspaper in the kitchen, then my glasses are on the kitchen table.

b. If my glasses are on the kitchen table, then I saw them at breakfast.

c. I did not see my glasses at breakfast.

d. I was reading the newspaper in the living room or I was reading the newspaper in the kitchen.

Example 8 – *Application: A More Complex Deduction*

cont'd

e. If I was reading the newspaper in the living room then my glasses are on the coffee table.

Where are the glasses?

Solution:

Let *RK* = I was reading the newspaper in the kitchen.

GK = My glasses are on the kitchen table.

SB = I saw my glasses at breakfast.

RL = I was reading the newspaper in the living room.

GC = My glasses are on the coffee table.

Example 8 – *Solution*

cont'd

Here is a sequence of steps you might use to reach the answer, together with the rules of inference that allow you to draw the conclusion of each step:

1.

$RK \rightarrow GK$ by (a)

$GK \rightarrow SB$ by (d)

- $RK \rightarrow SB$ by transitivity

2.

$RK \rightarrow SB$ by the conclusion of (1)

$\sim SB$ by (c)

- $\sim RK$ by modus tollens

Example 8 – *Solution*

cont'd

3. $RL \vee RK$ by (d)

   $\sim RK$ by the conclusion of (2)

   - RL by elimination

4. $RL \to GC$ by (e)

   RL by the conclusion of (3)

   - GC by modus ponens

Thus the glasses are on the coffee table.

Fallacies

A **fallacy** is an error in reasoning that results in an invalid argument. Three common fallacies are **using ambiguous premises,** and treating them as if they were unambiguous, **circular reasoning** (assuming what is to be proved without having derived it from the premises), and **jumping to a conclusion** (without adequate grounds).

In this section we discuss two other fallacies, called ***converse error*** and ***inverse error***, which give rise to arguments that superficially resemble those that are valid by modus ponens and modus tollens but are not, in fact, valid.

Fallacies

For an argument to be valid, every argument of the same form whose premises are all true must have a true conclusion. It follows that for an argument to be invalid means that there is an argument of that form whose premises are all true and whose conclusion is false.

Example 9 – *Converse Error*

Show that the following argument is invalid:

If Zeke is a cheater, then Zeke sits in the back row.

Zeke sits in the back row.

• Zeke is a cheater.

Solution:

The general form of the previous argument is as follows:

$$\boldsymbol{p \rightarrow q}$$

$$\boldsymbol{q}$$

$$\bullet\ \boldsymbol{p}$$

Fallacies

The fallacy underlying this invalid argument form is called the **converse error** because the conclusion of the argument would follow from the premises if the premise: $p \to q$ were replaced by its converse.

Such a replacement is not allowed, however, because a conditional statement is not logically equivalent to its converse. Converse error is also known as the ***fallacy of affirming the consequent***.

Another common error in reasoning is called the *inverse error*.

Example 10 – *Inverse Error*

Consider the following argument:

If interest rates are going up, stock market prices will go down.

Interest rates are not going up.

- ∴ Stock market prices will not go down.

Note that this argument has the following form:

$$\boldsymbol{p \rightarrow q}$$

$$\boldsymbol{\sim p}$$

$$\therefore \boldsymbol{\sim q}$$

Example 10 – *Inverse Error*

cont'd

The fallacy underlying this invalid argument form is called the **inverse error** because the conclusion of the argument would follow from the premises if the premise $\boldsymbol{p \to q}$ **were replaced by its inverse**.

Such a replacement is not allowed, however, because a conditional statement is not logically equivalent to its inverse. Inverse error is also known as the ***fallacy of denying the antecedent***.

Example 11 – *A Valid Argument with a False Premise and a False Conclusion*

The argument below is valid by modus ponens. But its major premise is false, and so is its conclusion.

If Bonnie Tyler is a rock star, then Bonnie Tyler has black hair.

Bonnie Tyler is a rock star.

• Bonnie Tyler has black hair.

Example 12 – *An Invalid Argument with True Premises and a True Conclusion*

The argument below is invalid by the converse error, but it has a true conclusion.

If Rockville is a big city, then Rockville has tall buildings.

Rockville has tall buildings.

- Rockville is a big city.

Fallacies

• Definition

An argument is called **sound** if, and only if, it is valid *and* all its premises are true. An argument that is not sound is called **unsound**.

Contradictions and Valid Arguments

The concept of logical contradiction can be used to make inferences through a technique of reasoning called the *contradiction rule*. Suppose p is some statement whose truth you wish to deduce.

Contradiction Rule

If you can show that the supposition that statement p is false leads logically to a contradiction, then you can conclude that p is true.

Example 13 – *Contradiction Rule*

Show that the following argument form is valid:

$\sim p \to c$, where c is a contradiction

• p

Solution:

Construct a truth table for the premise and the conclusion of this argument.

			premises	conclusion
p	$\sim p$	c	$\sim p \to$ c	p
T	F	F	T	T
F	T	F	F	

There is only one critical row in which the premise is true, and in this row the conclusion is also true. Hence this form of argument is valid.

Contradictions and Valid Arguments

The ***contradiction rule*** is the logical heart of the method of ***proof by contradiction***.

A slight variation also provides the basis for solving many logical puzzles by eliminating contradictory answers: ***If an assumption leads to a contradiction, then that assumption must be false.***

Summary of Rules of Inference

Table 2.3.1 summarizes some of the most important rules of inference.

Modus Ponens	$p \to q$ p • q	**Elimination**	**a.** $p \lor q$ $\sim q$ • p **b.** $p \lor q$ $\sim p$ • q
Modus Tollens	$p \to q$ $\sim q$ • $\sim p$	**Transitivity**	$p \to q$ $q \to r$ • $p \to r$
Generalization	**a.** p • $p \lor q$ **b.** q • $p \lor q$	**Proof by Division into Cases**	$p \lor q$ $p \to r$ $q \to r$ • r
Specialization	**a.** $p \land q$ • p **b.** $p \land q$ • q		
Conjunction	p q • $p \land q$	**Contradiction Rule**	$\sim p \to c$ • p

Valid Argument Forms

Table 2.3.1