CMSC 207- Lecture 2
CHAPTER 1: SPEAKING MATHEMATICALLY – Contd.

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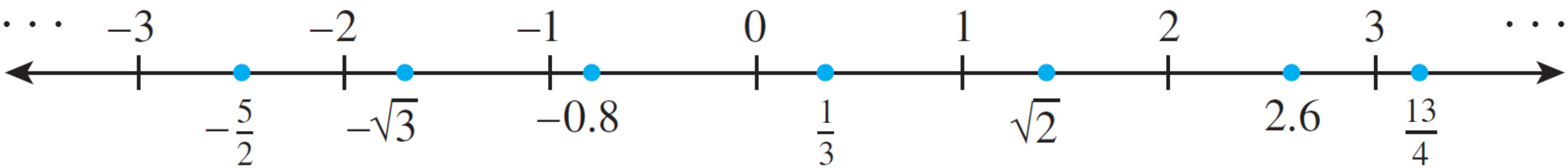
The Language of Sets-1

The set of real numbers are represented as the set of all points on a line, as shown below.



The number 0 corresponds to a middle point, called the ***origin*.**

A unit of distance is marked off, and each point to the right of the origin corresponds to a positive real number found by computing its distance from the origin. Similarly, each point to the left of the origin corresponds to a negative real number found by computing the distance from the origin.

The Language of Sets-2

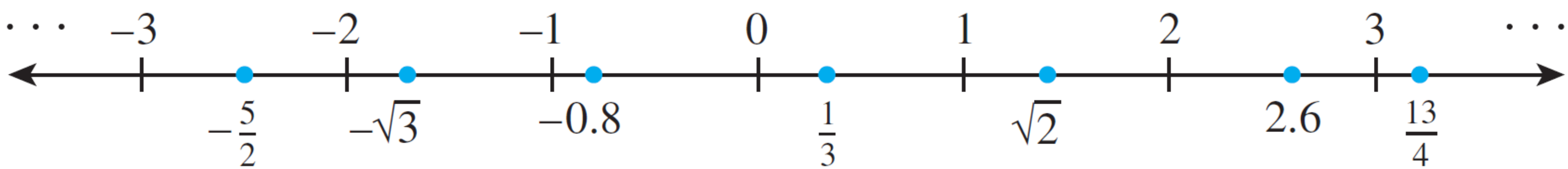
- The set of real numbers is therefore divided into three parts: the set of positive real numbers, the set of negative real numbers, and the number 0. Speaking Mathematically (Speaking Formally),

$$\mathbf{R = R^{+} \cup R^{-} \cup \{0\}}$$

- ***<u>Important</u>: 0 is neither positive nor negative*.**

The Language of Sets-3

Following shows a real number line with labels given for a few real numbers corresponding to points on the line.



The real number line is ***continuous*** because it doesn't contain any holes or gaps.

The set of integers corresponds to a collection of points located at fixed intervals along the real number line. Therefore, the real number line can also represent integers.

The Language of Sets-4

Hence, every integer is a real number, and because the integers are all separated from each other, the set of integers is called ***discrete***. The name ***discrete mathematics*** comes from the distinction between continuous and discrete mathematical objects.

Another way to specify a set is called the:

set-builder notation:

Let S denotes a set, and P(x) be a property that elements of S may or may not satisfy

Define a new set such that it is the set of all elements x in S such that property P(x) is true (satisfied) for each one of them:

$$\{x \in S \mid P(x)\}$$

In-class Assignment-1 (take 8 minutes to complete)

Given that **R** denotes the set of all real numbers, **Z** the set of all integers, and **Z⁺** the set of all positive integers, describe each of the following sets.

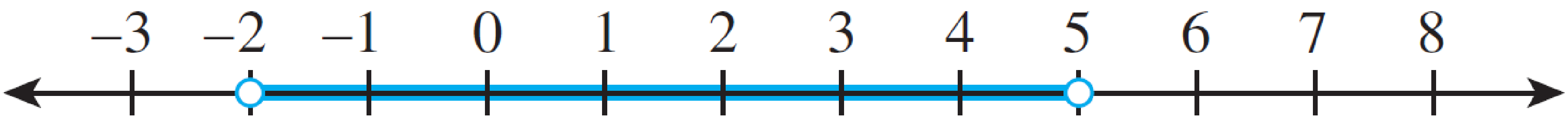
a. $$\{x \in \mathbf{R} \mid -2 < x < 5\}$$

b. $$\{x \in \mathbf{Z} \mid -2 < x < 5\}$$

c. $$\{x \in \mathbf{Z}^{+} \mid -2 < x < 5\}$$

Check Your Solution: In-class Assignment 1

a. $\{x \in \mathbf{R} \mid -2 < x < 5\}$ is the open interval of real numbers (strictly) between –2 and 5 (non-inclusive or exclusive of -2 and 5). It is pictured as: follows:



b. $\{x \in \mathbf{Z} \mid -2 < x < 5\}$ is the set of all integers (strictly) (exclusive). It is equal to the set:

{–1, 0, 1, 2, 3, 4}.

c. Since all the integers in **Z**$^+$ are positive,

$$\{x \in \mathbf{Z}^+ \mid -2 < x < 5\} = \{1, 2, 3, 4\}.$$

Subsets-1

A basic relation between sets is the subset relation

If A and B are two sets, then A is called a subset of B, written as **$A \subseteq B$,** if and only if, every element in set A is also present in set B.

So $A \subseteq B$ implies that For all elements x, if $x \in A$ then $x \in B$ (Formally, $A \subseteq B \equiv \forall x, x \in A \rightarrow x \in B$)

Here, $\equiv$ is Equivalent Symbol.

Also, analogous statements are:

Set A is contained in set B

Set B contains set A

Subsets-2

For a set *A* not to be a subset of a set *B* means that there is at least one element of set *A*, and the element is not present in set *B*.

Speaking Mathematically:

$A \not\subseteq B$ means that There is at least one element x such that $x \in A$ and $x \notin B$.

Formally, $\exists x, x \in A \wedge x \notin B$.

If A and B are sets, then A is a proper subset of B, if, and only if, every element of set A is in set B but there is at least one element of B that is not in A.

Formally, $A \subset B$ (Set A is a proper subset of B)

$$\equiv (\forall x, x \in A \rightarrow x \in B) \wedge (\exists y, y \in B \wedge y \notin A)$$

In-class Assignment-2
(take 8 minutes to complete)

Which of the following are true statements?

a. $y \in \{x, y, z\}$ **b.** $\{y\} \in \{x, y, z\}$ **c.** $y \subseteq \{x, y, z\}$

d. $\{y\} \subseteq \{x, y, z\}$ **e.** $\{y\} \subseteq \{\{x\}, \{y\}\}$ **f.** $\{y\} \in \{\{x\}, \{y\}\}$

Solution:

Only (**a**), (**d**), and (**f**) are true.

For (**b**) to be true, the set $\{x, y, z\}$ would have to contain the element $\{y\}$. But the only elements of $\{x, y, z\}$ are x, y, and z, and y is not equal to $\{y\}$. Hence (**b**) is false.

Example 4 – *Solution*

For (**c**) to be true, the number y would have to be a set and every element in the set y would have to be an element of {x, y, z}. This is not the case, so (**c**) is false.

For (**e**) to be true, every element in the set containing only the number y would have to be an element of the set whose elements are {x} and {y}. But y is not equal to either {x} or {y}, and so (**e**) is false.

Cartesian Products

• Notation

Given elements a and b, the symbol (a, b) denotes the **ordered pair** consisting of a and b together with the specification that a is the first element of the pair and b is the second element. Two ordered pairs (a, b) and (c, d) are equal if, and only if, $a = c$ and $b = d$. Symbolically:

$$(a, b) = (c, d) \quad \text{means that} \quad a = c \text{ and } b = d.$$

Example 5 – *Ordered Pairs*

a. Is (1, 4) = (4, 1)?

b. Is $\left(3, \frac{5}{10}\right) = \left(\sqrt{9}, \frac{1}{2}\right)$?

c. What is the first element of (2, 2)?

Solution:

a. No. By definition of equality of ordered pairs,

$(1, 4) = (4,1)$ if, and only if, $1 = 4$ and $4 = 1$.

But $1 \neq 4$, and so the ordered pairs are not equal.

Example 5 – *Solution*

b. Yes. By definition of equality of ordered pairs,

$\left(3, \frac{5}{10}\right) = \left(\sqrt{9}, \frac{1}{2}\right)$ if, and only if, $3 = \sqrt{9}$ and $\frac{5}{10} = \frac{1}{2}$.

Because these equations are both true, the ordered pairs are equal.

c. In the ordered pair (2, 2), the first and the Second elements are both 2.

Cartesian Products

• Definition

Given sets A and B, the **Cartesian product of $\boldsymbol{A}$ and $\boldsymbol{B}$**, denoted $\boldsymbol{A \times B}$ and read "A cross B," is the set of all ordered pairs (a, b), where a is in A and b is in B. Symbolically:

$$\boldsymbol{A} \times \boldsymbol{B} = \{(a, b) \mid a \in A \text{ and } b \in B\}.$$

Example 6 – *Cartesian Products*

Let $A = \{1, 2, 3\}$ and $B = \{u, v\}$.

a. Find $A \times B$

b. Find $B \times A$

c. Find $B \times B$

d. How many elements are in $A \times B$, $B \times A$, and $B \times B$?

Example 6 – *Solution*

a. $A \times B = \{(1, u), (2, u), (3, u), (1, v), (2, v), (3, v)\}$

b. $B \times A = \{(u, 1), (u, 2), (u, 3), (v, 1), (v, 2), (v, 3)\}$

c. $B \times B = \{(u, u), (u, v), (v, u), (v, v)\}$

d. $A \times B$ has six elements. Note that this is the number of elements in A times the number of elements in B.

$B \times A$ has six elements, the number of elements in B times the number of elements in A. $B \times B$ has four elements, the number of elements in B times the number of elements in B.

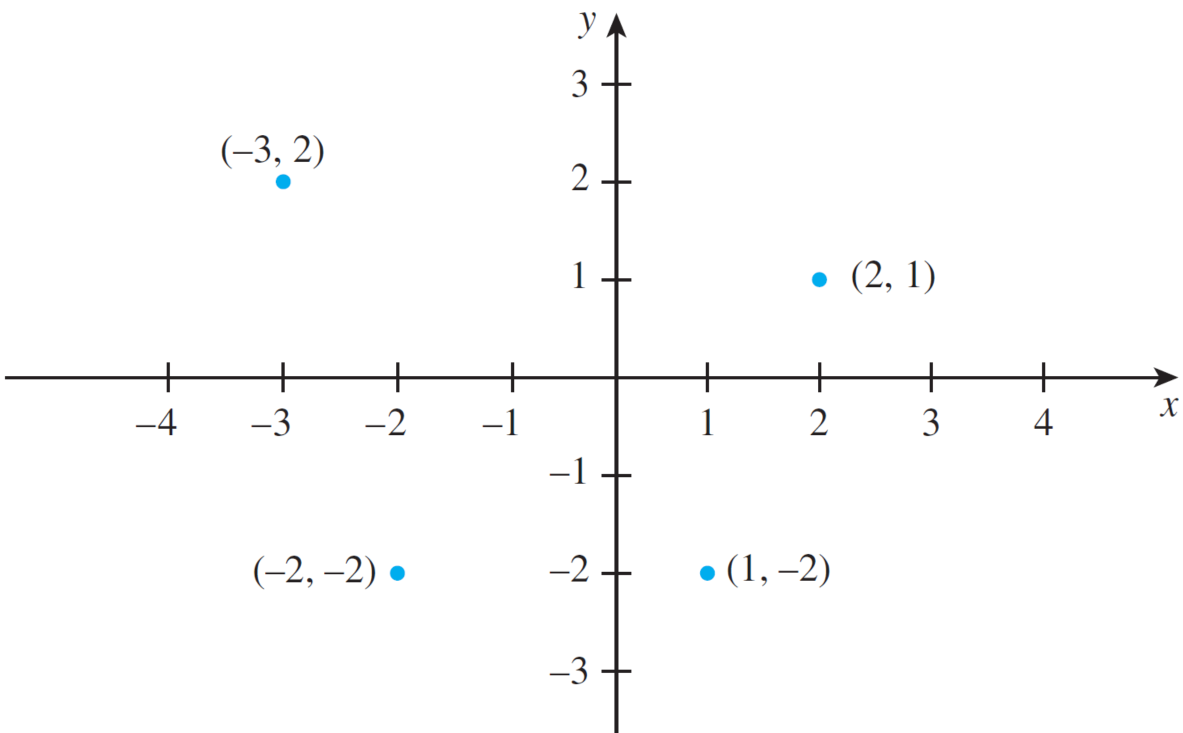
The Meaning of: $\mathbf{R} \times \mathbf{R}$

$\mathbf{R} \times \mathbf{R}$ is the set of all ordered pairs $\boldsymbol{(x, y)}$ where both x and y are real numbers.

- If horizontal and vertical axes are drawn on a plane, and a unit length is marked off, then each ordered pair in $\mathbf{R} \times \mathbf{R}$ corresponds to a unique point in the plane, with the first, and second elements of the pair indicating, respectively, the horizontal and vertical positions of the point.

The Meaning of: R × R Continued.

The term **Cartesian plane** is often used to refer to a plane with this coordinate system, as illustrated in **Figure 1.2.1.**



A Cartesian Plane

Figure 1.2.1

The Language of Relations and Functions

Let us use the notation *x R y* as a shorthand for the sentence "*x* is related to *y*." Then

$$
\begin{array}{lll}
0\ R\ 1 & \text{since} & 0 < 1, \\
0\ R\ 2 & \text{since} & 0 < 2, \\
0\ R\ 3 & \text{since} & 0 < 3, \\
1\ R\ 2 & \text{since} & 1 < 2, \\
1\ R\ 3 & \text{since} & 1 < 3, \quad \text{and} \\
2\ R\ 3 & \text{since} & 2 < 3.
\end{array}
$$

On the other hand, if the notation $x \not R\ y$ represents the sentence "*x* is not related to *y*," then

$$
\begin{array}{lll}
1 \not R\ 1 & \text{since} & 1 \not< 1, \\
2 \not R\ 1 & \text{since} & 2 \not< 1, \quad \text{and} \\
2 \not R\ 2 & \text{since} & 2 \not< 2.
\end{array}
$$