Chapter 1

INTRODUCTION TO THE FINITE ELEMENT METHOD

1.1 Historical perspective: the origins of the finite element method

The finite element method constitutes a general tool for the numerical solution of partial differential equations in engineering and applied science. Historically, all major practical advances of the method have taken place since the early 1950s in conjunction with the development of digital computers. However, interest in approximate solutions of field equations dates as far back in time as the development of the classical field theories (e.g. elasticity, electro-magnetism) themselves. The work of Lord Rayleigh (1870) and W. Ritz (1909) on variational methods and the weighted-residual approach taken by B. G. Galerkin (1915) and others form the theoretical framework to the finite element method. With a bit of a stretch, one may even claim that Schellbach's approximate solution to Plateau's problem (find a surface of minimum area enclosed by a given closed curve in three dimensions), which dates back to 1851 is a rudimentary application of the finite element method.

Most researchers agree that the era of the finite element method begins with a lecture presented in 1941 by R. Courant to the American Association for the Advancement of Science. In his work, Courant used the Ritz method and introduced the pivotal concept of spatial discretization for the solution of the classical torsion problem. Courant did not pursue his idea further, since computers were still largely unavailable for his research.

More than a decade later Ray W. Clough, Jr. of the University of California, Berkeley, and his colleagues essentially reinvented the finite element method as a natural extension of matrix structural analysis and published their first work in 1956. Clough, who is credited with coining the term "finite element", has spent the summers of 1952 and 1953 at Boeing under the supervision of M.J. Turner working on modeling of the vibration in a wing structure and it is this work that he led to his formulation of finite elements for plate structures. An apparently simultaneous effort by John Argyris at the University of London independently led to another successful introduction of the method. It should come as no surprise that, to a large extent, the finite element method appears to owe its reinvention to structural engineers. In fact, the consideration of a complicated system as an assemblage of simple components (elements) on which the method relies is very natural in the analysis of structural systems.

In the few years following its introduction to the engineering community, the finite element method has attracted the attention of applied mathematicians, particularly those interested in numerical solution of partial differential equations. In 1973, G. Strang and G.J. Fix authored the first conclusive treatise on mathematical aspects of the method, focusing exclusively on its application to the solution of problems emanating from standard variational theorems.

The finite element has been subject to intense research, both at the mathematical and technical level, and thousands of scientific articles and hundreds of books about it have been authored. By the beginning of the 1990s, the method clearly dominated the numerical solution of problems in the fields of structural analysis, structural mechanics and solid mechanics. Moreover, the finite element method currently competes in popularity with the finite difference method in the areas of heat transfer and fluid mechanics.

1.2 Introductory remarks on the concept of discretization

The basic goal of discretization is to provide an approximation of an infinite dimensional system by a system that can be fully defined with a finite number of "degrees of freedom". To clarify the notion of dimensionality, consider a deformable body in the three-dimensional Euclidean space, for which the position of a typical particle with reference to a fixed coordinate system is defined by means of a vector $\mathbf{x}$, as in Figure 1.1. This is an infinite dimensional system with respect to the position of all of its particle points. If the same body is assumed to be rigid, then it is a finite dimensional system with only six degrees of freedom.

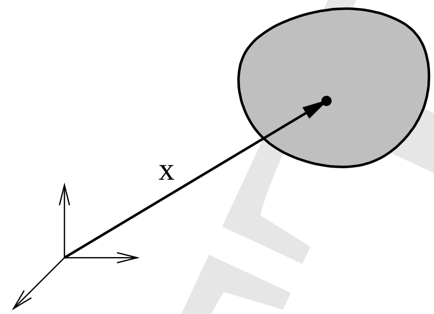

Figure 1.1: *An infinite degree-of-freedom system*

A dimensional reduction of the above system is accomplished by placing a (somewhat severe) restriction on the admissible motions that the body may undergo.

Finite dimensional approximations are very important from the computational standpoint, because they often allow for analytical and/or numerical solutions to problems that would otherwise be intractable. There exist various methods that can reduce infinite dimensional systems to approximate finite dimensional counterparts. Here we consider three of those methods, namely the physically motivated structural analogue substitution method, the finite difference method and the finite element method.

1.2.1 Structural analogue substitution method

Consider the oscillation of a liquid in a manometer. This system can be approximated ("lumped") by means of a single degree-of-freedom mass-spring system, as in Figure 1.2. Clearly, such an approximation is largely intuitive and cannot precisely capture the complexity of the original system (viscosity of the liquid, surface tension effects, geometry of the manometer walls).

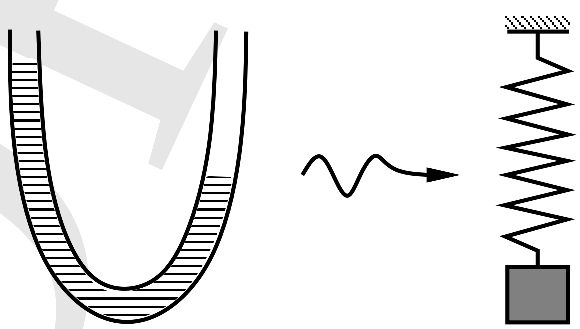

Figure 1.2: *A simple example of the structural analogue method*

The structural analogue substitution method, whenever applicable, generally provides coarse approximations to complex systems. However, its degree of sophistication (hence, also the fidelity of its results) can vary widely. The "network analysis" of Kron in the 1930s and 1940s is generally viewed as a typical example of the structural analogue approach.

1.2.2 Finite difference method

Consider the ordinary differential equation

$$\begin{aligned} k\frac{d^2u}{dx^2} &= f \text{ in } (0,L) \ , \\ u(0) &= u_0 \ , \\ u(L) &= u_L \ , \end{aligned} \tag{1.1}$$

where k is a constant and $f = f(x)$ is a smooth function. Let N points be chosen in the interior of the domain $(0, L)$, each of them equidistant from its immediate neighbors. An algebraic (or "difference") approximation to the second derivative may be computed as

$$\left.\frac{d^2u}{dx^2}\right|_l \doteq \frac{u_{l+1} - 2u_l + u_{l-1}}{\Delta x^2} \ , \tag{1.2}$$

with error $o(\Delta x^2)$. Indeed, assuming that the solution $u(x)$ is at least four times continuously

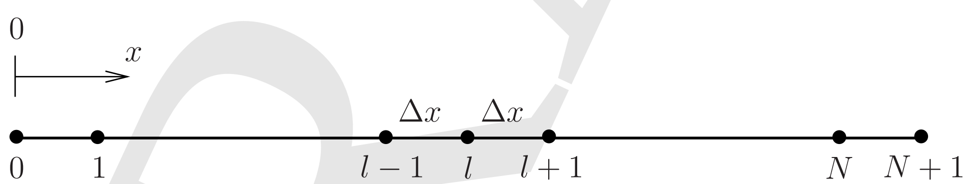

Figure 1.3: *The finite difference method in one dimension*

differentiable and employing twice a Taylor series expansion with remainder around a typical point l in Figure 1.3, write

$$u_{l+1} = u_l + \Delta x\left.\frac{du}{dx}\right|_l + \frac{\Delta x^2}{2!}\left.\frac{d^2u}{dx^2}\right|_l + \frac{\Delta x^3}{3!}\left.\frac{d^3u}{dx^3}\right|_l + \frac{\Delta x^4}{4!}\left.\frac{d^4u}{dx^4}\right|_{l+\theta_1} \ ; \ 0 \leq \theta_1 \leq 1 \ , \tag{1.3}$$

$$u_{l-1} = u_l - \Delta x\left.\frac{du}{dx}\right|_l + \frac{\Delta x^2}{2!}\left.\frac{d^2u}{dx^2}\right|_l - \frac{\Delta x^3}{3!}\left.\frac{d^3u}{dx^3}\right|_l + \frac{\Delta x^4}{4!}\left.\frac{d^4u}{dx^4}\right|_{l-\theta_2} \ ; \ 0 \leq \theta_2 \leq 1 \ . \tag{1.4}$$

Adding the above equations results in

$$\left.\frac{d^2u}{dx^2}\right|_l = \frac{u_{l+1} - 2u_l + u_{l-1}}{\Delta x^2} - \frac{\Delta x^2}{4!}\left(\left.\frac{d^4u}{dx^4}\right|_{l+\theta_1} + \left.\frac{d^4u}{dx^4}\right|_{l-\theta_2}\right) , \tag{1.5}$$

so that ignoring the second term of the right-hand side, the proposed approximation to the second derivative of u is recovered. Applying the difference equation (1.2) to nodal points $1, 2, ..., N$, and accounting for the boundary conditions $(1.1)_{2,3}$ gives rise to a system of N linear algebraic equations

$$\begin{aligned} u_2 - 2u_1 &= \frac{f_1 \Delta x^2}{k} - u_0 , \\ u_{l+1} - 2u_l + u_{l-1} &= \frac{f_l \Delta x^2}{k} \qquad , \qquad l = 2, ..., N-1 , \\ -2u_N + u_{N-1} &= \frac{f_N \Delta x^2}{k} - u_L , \end{aligned} \tag{1.6}$$

with unknowns u_l, $l = 1, 2, \ldots, N$. Again, an infinite-dimensional problem with respect to the value of u in the domain $(0, L)$ is transformed by the above method into an N-dimensional problem.

Clearly, the state equations are (approximately) satisfied only at discrete points $1, 2, ..., N$. Also, the boundary conditions are enforced directly when writing the discrete counterparts of the state equations in the nodes that reside next to the boundaries. It is easy to see that finite difference methods run into difficulties when dealing with complex boundaries due to the need for spatial regularity of the grid.

1.2.3 Finite element method

Revisit the problem in the previous section and consider the same discretization as in Figure 1.3. Consider the line segment between points l and $l+1$. This is now the domain of the finite element e. In this domain, we assume that u varies linearly, as shown in Figure 1.4, and attains values u_l at point l and u_{l+1} at point $l+1$.

The normal flux $q = -k\frac{du}{dn}$, where n denotes the outward unit normal to the element domain is equal to

$$q_l^e = -k\left.\frac{du}{dn}\right|_l^e \doteq -k\frac{u_l - u_{l+1}}{\Delta x} \tag{1.7}$$

and

$$q_{l+1}^e = -k\left.\frac{du}{dn}\right|_{l+1}^e \doteq -k\frac{u_{l+1} - u_l}{\Delta x} \tag{1.8}$$

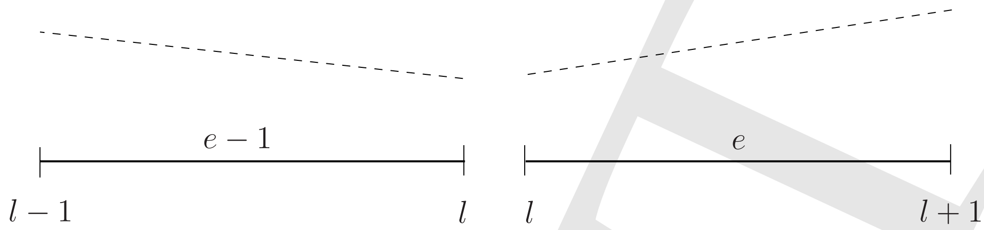

Figure 1.4: *A one-dimensional finite element approximation*

at points l and $l+1$, respectively. These two equations can be written in matrix form as

$$-\frac{k}{\Delta x}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} u_l \\ u_{l+1} \end{bmatrix} = \begin{bmatrix} q_l^e \\ q_{l+1}^e \end{bmatrix} . \tag{1.9}$$

An analogous matrix equation can be written for element $e-1$, whose domain lies between points $l-1$ and l, and takes the form

$$-\frac{k}{\Delta x}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} u_{l-1} \\ u_l \end{bmatrix} = \begin{bmatrix} q_{l-1}^{e-1} \\ q_l^{e-1} \end{bmatrix} . \tag{1.10}$$

Now, adding the first equation of (1.9) to the second equation of (1.10) yields

$$\frac{k}{\Delta x}(u_{l+1} - 2u_l + u_{l-1}) \ = \ q_l^e + q_l^{e-1} . \tag{1.11}$$

To approximate the right-hand side of (1.11), first note that the terms q_l^e and q_l^{e-1} represent fluxes on opposite sides of the same point l, namely, recalling (1.7) and (1.8),

$$q_l^e + q_l^{e-1} \ = \ k\frac{du}{dx}\bigg|_l^e - k\frac{du}{dx}\bigg|_l^{e-1} . \tag{1.12}$$

At the same time, one may rewrite the differential equation as $k\dfrac{du}{dx} = \int f\,dx$. Hence, if the total force $f_{total} = \int_0^L f\,dx$ is distributed to the points $0, 1, \ldots, N+1$ so that to point l corresponds a force $\tilde{f}_l$, then the jump in the normal derivative $k\dfrac{du}{dx}$ at l is exactly $\tilde{f}_l$, therefore (1.11) attains the form

$$\frac{k}{\Delta x}(u_{l+1} - 2u_l + u_{l-1}) \ = \ \tilde{f}_l . \tag{1.13}$$

Then, the complete finite element system becomes

$$\begin{aligned} u_2 - 2u_1 &= \frac{\tilde{f}_1 \Delta x}{k} - u_0 , \\ u_{l+1} - 2u_l + u_{l-1} &= \frac{\tilde{f}_l \Delta x}{k} \qquad , \qquad l = 2, ..., N-1 , \\ -2u_N + u_{N-1} &= \frac{\tilde{f}_N \Delta x}{k} - u_L , \end{aligned} \tag{1.14}$$

This is the so-called *direct approach* to formulating the finite element equations. Upon comparing (1.6) and (1.14), it is concluded that the two sets of equations are identical to within the definition of the force term. Yet, these equations were derived by way of fundamentally different approximations.

It will be established that in finite element methods the state equations are satisfied in an integral sense over the whole domain with respect to a set of (simple) admissible functions. Also, it will be seen that boundary conditions can be handled trivially.

1.3 Classifications of partial differential equations

Consider a scalar partial differential equation (PDE) of the general form

$$F(x, y, ...u, u_{,x}, u_{,y}, ...u_{,xx}, u_{,xy}, u_{,yy}, ...) \; = \; 0 \; , \tag{1.15}$$

where x, y, $\ldots$ are the independent variables, and $u = u(x, y, \ldots)$ is the dependent variable. In addition, write

$$u_{,x} \; = \; \frac{\partial u}{\partial x} \quad , \quad u_{,xx} \; = \; \frac{\partial^2 u}{\partial x^2} \quad , \quad \text{etc} \; . \tag{1.16}$$

The *order* of the PDE is defined as the order of the highest derivative of u in (1.15). Also, a PDE is *linear* if the function F is linear in u and in all of its derivatives, with coefficients depending on the independent variables $x, y, \ldots$.

Examples:

$$\begin{array}{ll} 3u_{,x} + u_{,y} - u = 0 & \text{(linear -- first order)} \; , \\ xu_{,xx} + \frac{1}{y} u_{,yy} - 3u = 0 & \text{(linear -- second order)} \; , \\ u_{,xx}^2 + u_{,yy} = 0 & \text{(non-linear -- second order)} \; , \\ u_{,x} \, u_{,xxx}^2 + u_{,yy} = 0 & \text{(non-linear -- third order)} \; . \end{array}$$

For the purpose of the forthcoming developments, consider linear second-order partial differential equations of the general form

$$au_{,xx} + bu_{,xy} + cu_{,yy} \; = \; d \; , \tag{1.17}$$

where not all a, b, c are equal to zero. In addition, let a, b, c be functions of x, y only, whereas d can be a function of $x, y, u, u_{,x}, u_{,y}$.

Equations of the form (1.17) can be categorized as follows:

(a) Elliptic equations ($b^2 - 4ac < 0$)

A typical example of an elliptic equation is the two-dimensional version of the Laplace (Poisson) equation used in modeling various phenomena (e.g., heat conduction, electrostatics), namely

$$u_{,xx} + u_{,yy} = f \qquad ; \qquad f = f(x, y) \ ,$$

for which $a = c = 1$ and $b = 0$.

(b) Parabolic equations ($b^2 - 4ac = 0$)

The equation of transient linear heat conduction in one dimension,

$$k u_{,xx} = u_{,t} \qquad ; \qquad k = k(x) \ ,$$

where $a = -k$ and $b = c = 0$, is a representative example of a parabolic equation.

(c) Hyperbolic equations ($b^2 - 4ac > 0$)

The one-dimensional linear wave equation,

$$\alpha^2 u_{,xx} - u_{,tt} = 0 \ ,$$

where $a = \alpha^2$, $b = 0$ and $c = -1$, falls in this class of equations.

Extension of the above classification to more general types of partial differential equations than those of the form (1.17) is not always an easy task. The elliptic, hyperbolic or parabolic nature of a partial differential equation is associated with the particular form of its characteristic curves. These are curves along which certain derivatives of a solution to the differential equation exhibit discontinuities.

The type of a partial differential equation determines the overall character of the expected solution. Broadly speaking, elliptic differential equations exhibit solutions which are as smooth as its coefficients allow. On the other hand, the solutions to parabolic differential equations tend to smooth out any initial discontinuities, while the solutions to hyperbolic partial differential equations preserve any initial discontinuities. To a great extent, the type of the partial equation dictates the choice of methodology used in its numerical approximation by the finite element method.

<u>Remarks</u>:

- Partial differential equations of mixed type are possible, such as the classical one-dimensional *convection-diffusion* equation of the form

$$u_{,t} + \alpha u_{,x} = \epsilon u_{,xx} \quad ; \quad \alpha \geq 0 \quad , \quad \epsilon \geq 0 \; .$$

  The above equation is of hyperbolic type if $\epsilon = 0$ and $\alpha > 0$ (i.e., when the diffusive term is suppressed), since

$$\begin{aligned} \alpha^2 u_{,xx} &= \alpha(\alpha u_{,x})_{,x} = \alpha(-u_{,t})_{,x} \\ &= \alpha(-u_{,x})_{,t} = -(\alpha u_{,x})_{,t} \\ &= -(-u_{,t})_{,t} = u_{,tt} \end{aligned}$$

  implies that its solution satisfies the previously mentioned wave equation. On the other hand, for $\epsilon > 0$ and $\alpha = 0$ the convective part vanishes and the equation is purely parabolic and coincides with the previously mentioned one-dimensional transient heat conduction equation. The dominant character in the convection-diffusion equation is controlled by the relative values of parameters α and ϵ.

- The type of a partial differential equation may be spatially dependent, as with the following example:

$$u_{,xx} + x u_{,yy} = 0 \; ,$$

  where $a = 1$, $b = 0$ and $c = x$, so that the equation is elliptic for $x > 0$, parabolic for $x = 0$ and hyperbolic for $x < 0$.

1.4 Suggestions for further reading

Section 1.1

[1] C.A. Felippa. An appreciation of R. Courant's 'Variational methods for the solution of problems of equilibrium and vibrations', 1943. *Int. J. Num. Meth. Engr.*, 37:2159–2187, 1994. [*This reference contains the original article on the finite element method by Courant, preceded by an interesting introduction by C. Felippa.*]

[2] R.W. Clough, Jr. The finite element method after twenty-five years: A personal view. *Comp. Struct*, 12:361–370, 1980. [*This reference offers a unique view of the finite element method by one of its inventors*].

[3] P.G. Ciarlet and J.L. Lions, editors. *Finite Element Methods (Part 1)*, volume II of *Handbook of Numerical Analysis*. North-Holland, Amsterdam, 1991. [*The first article in this handbook presents a comprehensive introduction to the history of the finite element method, authored by J.T. Oden*].

Section 1.2

[1] O.C. Zienkiewicz and R.L. Taylor. *The Finite Element Method; Basic Formulation and Linear Problems*, volume 1. McGraw-Hill, London, 4th edition, 1989. [*Chapter 1 of this book is devoted to an introductory discussion of discretization*].

[2] G. Kron. Numerical solutions of ordinary and partial differential equations by means of equivalent circuits. *J. Appl. Phys.*, 16:172–186, 1945. [*This is an interesting use of an electrical circuits analogue method to obtain approximate solutions of differential equations*].

Section 1.3

[1] F. John. *Partial Differential Equations.* Springer-Verlag, New York, 4th edition, 1985. [*Chapter 2 contains a mathematical discussion of the classification of linear second-order partial differential equations in connection with their characteristics*].