ME 402/512 Continuum Mechanics — Closed Book Exam 1

Points in brackets denote weight.

1. [20] Simplify the following expressions (if possible):

(i) $\delta_{ij}a_j$ (ii) $\delta_{ij}x_i x_j$ (iii) $a_{ij}x_i x_j$ with $a_{ij} = a_{ji}$

(iv) $a_{ij}x_i x_j$ with $a_{ij} = -a_{ji}$ (v) $(x_i x_j),_j$ (vi) $(x_i x_j),_i$

(vii) $(\delta_{ij} + c_{ij})(\delta_{ik} + c_{ik}) - \delta_{jk} - \delta_{mn}c_{mj}c_{nk}$

2. Let F be a scalar function of a vector field $\mathbf{v}$.

(i) [8] Define what is meant by the gradient of F with respect to $\mathbf{v}$.

(ii) [5] What is meant by an orthonormal basis?

(iii) [7] Use your definition in (i) to obtain an expression in indicial form for the gradient if an orthonormal basis is used.

(iv) [10] Use your result from (iii) to determine the gradient with respect to $\mathbf{v}$ of

(a) $F = \mathbf{v} \cdot \mathbf{v}$ (b) $F = \mathbf{v} \cdot \mathbf{T} \cdot \mathbf{v}$ for a constant tensor $\mathbf{T}$.

Give your answers in both direct and indicial notation.

3. Suppose that $\int_R \phi \nabla dV = \int_{\partial R} \phi \mathbf{n}\, dS$

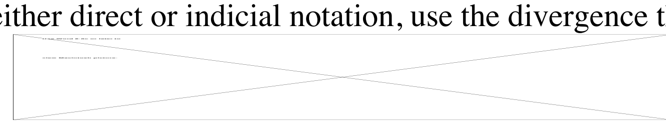
(i) [6] In this context what do each of the following terms denote: ∇, R, ∂R, dV, dS and $\mathbf{n}$

(ii) [4] What conclusion can you draw by letting $\phi = 1$.

(iii) [7] Use the original equation and derive the corresponding equation where the function is a vector rather than a scalar.

(iv) [3] Use the result of (iii) to derive the divergence theorem for a vector field.

(v) 10] With either direct or indicial notation, use the divergence theorem to show that



4. Material elements in the undeformed and deformed configuration are related by .

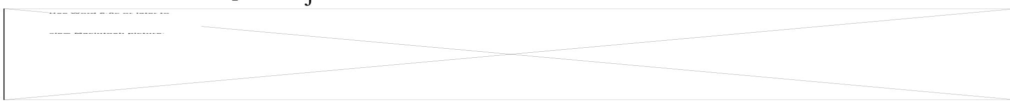
(i) [8] From this definition for $\mathbf{F}$, give $\mathbf{F}$ in terms of a gradient.

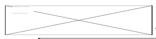
equivalent

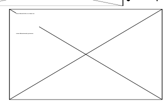
(ii) [12] If L denotes a line segment in the deformed configuration, derive an expression for $\frac{d}{dt}\int_L \mathbf{v} \cdot d\mathbf{r}$ in which the time derivative is inside the integral.

Numbers in brackets denote weights for parts of problems.

1. (a) [5] If a tensor is represented as $\mathbf{T} = a\mathbf{e}_1 \otimes \mathbf{e}_3 + b\mathbf{e}_2 \otimes \mathbf{e}_2$ give the matrix of its components in the basis $\mathbf{e}_i \otimes \mathbf{e}_j$.

Suppose 

(b) [5] Determine . What is the natural basis for **G** based on this expression.

(c) [10] Determine  and associate each free index with either a row or a column of the resulting matrix.

2. Suppose x^i represents a general curvilinear coordinate system.

(a) [5] How are covariant and contravariant bases defined?

(b) [5] Give general representations for vectors and tensors using all possible combinations of bases.

(c) [10] Define the transformation relations between the bases and derive the equation which relates covariant and contravariant components of a vector.

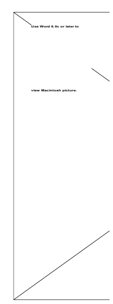
3. [30] State Euler's First and Second Laws. Outline the assumptions and steps necessary to arrive at Cauchy's First and Second Equations of Motion.

4. Consider the planar deformation $x_1 = a X_1$, $x_2 = X_2 + bX_1$ where the base vectors are related as shown in the sketch.

(a) [10] Obtain the displacement vector, **u**. Express this vector two ways: (i) In terms of X_A and $\mathbf{E}_A$, and (ii) in terms of x_i and $\mathbf{e}_i$.

(b) [15] Obtain the components (specify the basis) of **F**, $\mathbf{C} = \mathbf{F}^T \cdot \mathbf{F}$ and $\mathbf{E} = \frac{1}{2}(\mathbf{C} - \mathbf{I})$

(c) [5] If $\mathbf{F} = \mathbf{R} \cdot \mathbf{U}$ and $\mathbf{U}^2 = \mathbf{C}$ how would you obtain **R**.



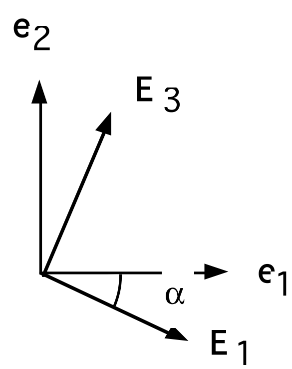
Weights for problems are given in square brackets.

1. [20] For each of the following, give a definition and provide an illustrative example: (i) tensor product, (ii) second order tensor, (iii) indicial notation, (iv) gradient of a vector, and (v) invariant of a second order tensor.

2. [20] Suppose **v** = F(r)**r**, i.e., **v** is a vector formed from the product of the position vector, **r**, and a scalar function, F, which depends on the magnitude of **r** [ ].
(i) Determine (a) , (b) curl **v**, and (c) in terms of F and its derivative.
(ii) If a surface is defined by F(r) = c, a constant, obtain a unit vector normal to the surface.

3. [20] Describe rather fully how you would obtain $\mathbf{T}^{1/3}$. Is it necessary to place any restrictions on the nature of the second order tensor, **T**?

4. [20] Derive the transformation equations that relate components of vectors and tensors in two bases. Suppose two bases are related as shown in the sketch. A vector is given by . Using your derived relation, determine the expression for the vector in the $\mathbf{E}_A$ basis.



E_2 =- e_3

5. [20] Given that f is a scalar function, and that volume and surface integrals are related as follows: . Define all terms in the equation and use the equation to derive the divergence theorem for vectors.

ME 402/512 Continuum Mechanics Closed Book Exam 2

Problem weights are given in square brackets.

1. [10] (i) What is the form that boundary conditions must take? Give an example of each possibility.

[10] (ii) What is the Principle of Material Frame Indifference? Give one example of an equation that satisfies the principle and an example that doesn't.

ME 512 Only (iii) In curvilinear coordinates, how are bases constructed? Give all the resulting representations for vectors and tensors. Describe the rules to be followed for the indicial notation.

ME 402 Only (iii) [10]Describe the procedure for taking a time derivative inside the volume integral expressed in the current configuration.

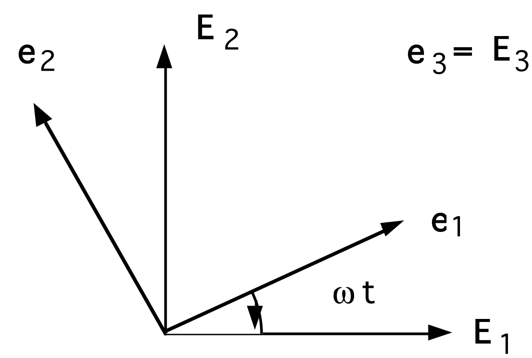
2. [10] (i) Start with the definition of a gradient. Obtain expressions for **F** and where .

[10] (ii) Derive the relation between **e** and **F** where

3. [25] Start with Euler's First and Second Laws. Outline the assumptions and steps required to arrive at Cauchy's First and Second Equations of Motion.

4. For constant θ, the base vectors $\mathbf{e}_i$ and $\mathbf{E}_A$ are related as shown. For constant α and β, a deformation is given by

$$x_1 = X_1 + \beta t X_2$$
$$x_2 = X_2$$
$$x_3 = X_3 e^{-\alpha t}$$



[15] (i) Obtain the components of **F** with respect to the bases:
$\mathbf{E}_A \otimes \mathbf{E}_B$, $\mathbf{e}_i \otimes \mathbf{E}_B$, $\mathbf{e}_i \otimes \mathbf{e}_j$. Is the volume changing? Why?

[10] (ii) Determine **v**, $\mathbf{L} = \mathbf{v}\nabla$, and $\dot{\mathbf{F}}$.

ME 402/512 Continuum Mechanics Closed Book Final Exam

Problem weights are given in square brackets.

1. (A) [8] Many expressions can be written in direct or indicial. Give the equivalent other notation for each of the following:

(B) [8] Indicate whether the result is a scalar, vector or second order tensor and give the corresponding expression in indicial form for each of the following (capital letters are second order tensors; lower case letters are vectors, is the third-order alternating tensor):

2. If the components of **T** in the basis are .

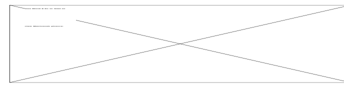
(i) [12] Determine the principal basis and the components of **T** in that basis.

(ii) [12] What are the components of $\mathbf{T}^{1/2}$ in the basis.

3. (**ME 512 only**) [20] Start with a basic definition and obtain a general expression for the gradient of a vector in terms of components and base vectors in a curvilinear coordinate system. Your final expression should have a "Christoffel" type symbol. Derive the corresponding expression for physical components of the gradient in terms of the physical components of the vector.

3. (**ME 402 only**) [20] Start with a basic definition for a gradient of a vector function with respect to a vector and derive the corresponding expression in indicial notation. What

is the gradient of with respect to **v**.

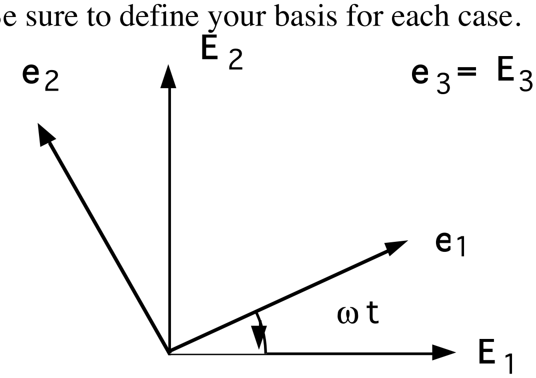
4. [20] The first law of thermodynamics implies that  where U denotes the internal energy, q - the temperature, s - the entropy, r - mass density, $\mathbf{e}$ - total strain, $\mathbf{e}^{p}$ - plastic strain and $\boldsymbol{\sigma}$ - stress. Suppose
$U = \frac{1}{2\rho}(\mathbf{e} - \mathbf{e}^{p}):\mathbf{E}:(\mathbf{e} - \mathbf{e}^{p}) \; + \; c[1 + \frac{s}{s^*} + \frac{1}{2}\left(\frac{s}{s^*}\right)^2]$ where c and s* are constants.

(i) Determine $\boldsymbol{\sigma}$, q and $\dfrac{\partial U}{\partial \mathbf{e}^{p}}$.

(ii) Using your result from (i) express U in terms of strain and q instead of strain and s to obtain a function U*.

(iii) Perform a contact (Legendre) transformation with respect to temperature and entropy to obtain the Helmholtz Free energy A in terms of strain and q. What is an interpretation for $\frac{\partial A}{\partial \theta}$? Is there a similar interpretation for $\frac{\partial U^*}{\partial \theta}$? Do A and U* have the same independent variables? Are they the same functions?

5. [20] Consider the deformation described by $x_1 = X_1 + atX_2$, $x_2 = X_2 + btX_1$, $x_3 = X_3$ superposed on the rotation given by $\theta = \omega t$ where a and b are constants and t denotes time. Determine $\mathbf{F} = \mathbf{r}\nabla_o$, $\mathbf{v} = \dot{\mathbf{r}}$, $\mathbf{L} = \mathbf{v}\nabla$, and indicate how you would obtain the rotation $\mathbf{R}$. Be sure to define your basis for each case.

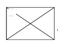
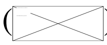


Weights for problems are given in square brackets. Allocate time accordingly.

1. [20] Use indicial notation to prove the following:

$$(\mathbf{u} \times \mathbf{v}) \times \nabla = (\mathbf{u} \cdot \nabla)\mathbf{v} - (\mathbf{v} \cdot \nabla)\mathbf{u} + (\mathbf{v}\nabla) \cdot \mathbf{u} - (\mathbf{u}\nabla) \cdot \mathbf{u}$$

2. [20] Provide the simplest possible expressions for each of the following:

(i) , (ii) $e_{ijk}u_j u_k$, (iii) tr (**I**), (iv) $C_{13}($$)$, (v) tr $(\mathbf{T}^s \cdot \mathbf{T}^a)$

in which **r** is the position vector, **I** is the identity tensor, **u** is a vector, **T** is any second order tensor, $\mathbf{T}^s$ is the symmetric part of **T**, and $\mathbf{T}^a$ is the skew-symmetric part of **T**.

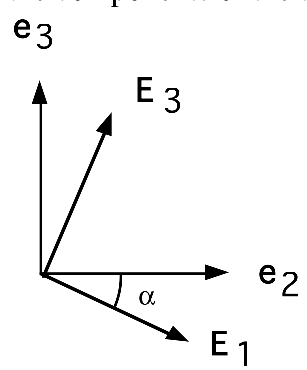
3. [15] What is meant by each of the following (give fairly complete descriptions):

(i) an invariant of a second-order tensor,

(ii) an eigenpair of a second-order tensor,

(iii) strain tensor.

4. [20] Derive the transformation equations that relate components of tensors in two bases. Suppose two bases are related as shown in the sketch. A tensor is given by $\mathbf{T} = 4\mathbf{e}_1 \otimes \mathbf{e}_1 + 3\mathbf{e}_1 \otimes \mathbf{e}_3 - 2\mathbf{e}_2 \otimes \mathbf{e}_1 + 2\mathbf{e}_3 \otimes \mathbf{e}_3$. Using your derived relation, determine the expression for the components of the tensor in the $\mathbf{E}_A$ basis.



$$E_2 = -e_1$$

5. (i) [5]Give the general definition of a gradient of a vector field with respect to a vector.

(ii)[5]Given that the deformation of a continuous medium is described by $d\mathbf{r} = \mathbf{F} \cdot d\mathbf{R}$ use your definition of a gradient to derive the relationship between the gradient based on **R** and the gradient based on **r**.

(iii) [3] Use the definition of a gradient to obtain the explicit expression $\mathbf{F} = \mathbf{r}\nabla_o$.

(iv) [6] Suppose $x_1 = A(X_1)^2 + BX_2$, $x_2 = CX_1X_2$, $x_3 = X_3$. Determine the components of **F**. What tensor basis is associated with these components?

(v) [6] Determine the components of the Lagrangian strain tensor

$\mathbf{E} = (\mathbf{F}^T \cdot \mathbf{F} - \mathbf{I})/2$. What tensor basis is associated with these components?

ME 402/512 Continuum Mechanics Closed Book Exam 2

Weights for problems are given in square brackets. Allocate time accordingly.

1. (i) [5] Use indicial notation to prove that tr($\mathbf{A}\cdot\mathbf{B}\cdot\mathbf{C}$) = tr ($\mathbf{C}\cdot\mathbf{A}\cdot\mathbf{B}$).

(ii) [15] The deformation gradient $\mathbf{F}$ is defined such that $d\mathbf{r} = \mathbf{F}\cdot d\mathbf{R}$. What are $d\mathbf{r}$ and $d\mathbf{R}$? Use the definition to obtain an expression for $\mathbf{F}$ and to obtain the relationship between gradients with respect to $\mathbf{r}$ and $\mathbf{R}$.

2. [20] Give the assumptions and outline the derivation of the relationship which states that the traction is a linear transformation of the unit normal to a surface.

3. (i) [10] Describe the possible combinations of boundary conditions for a continuous body.

(ii) [5] A part of a rigid body, A, contains a frictionless surface defined by the arc of a circle, $r = c$, with c a constant. A portion of a body, B, is pressed against this surface. What is the boundary condition for body, B, for the portion of the boundary in contact with the frictionless surface. (Give a sketch).

(iii) [5] Another part of the surface of body, B, defined by the equation $f(x, y, z) = 0$ is acted on by a fluid which sustains a static pressure P. What is the boundary condition for this part of the surface?

4. [20] Give the principle of conservation of mass in integral form. Then take a time derivative to obtain the "continuity" equation.

ME 512 students do problem 5 and <u>not</u> 6. ME 402 students do problem 5 <u>or</u> 6.

5. Let x^i denote curvilinear coordinates.

(i) [5] How are covariant and contravariant base vectors constructed?

(ii) [5] Relate one set of these base vectors to an orthonormal set of base vectors associated with rectangular Cartesian coordinates.

(iii) [5] How are covariant and contravariant components of a vector defined.

(iv) [5] What is the relationship between the contravariant and covariant components of a vector?

6. (i) [5] What is the definition of the spin tensor?

(ii) [5] What is the definition of the vorticity tensor?

(iii) [10] Derive the relationship between the spin and vorticity tensors.

Weights for problems are given in square brackets. Allocate time accordingly.

1. (i) [10] Derive the transformation relation between components of a second order tensor expressed with the use of two orthonormal bases. What does orthonormal mean?
(ii) [10] What is the definition of an invariant? Given an example and use your transformation relation to prove that your example is an invariant.

2. (i) [10] What is an eigenpair associated with a tensor, **T**? What is meant by the terms "principal values" and "principal directions" of a tensor? If **T** is symmetric and real what can you say about the principal values and principal directions?
(ii) [10] Obtain the eigensystem for the tensor, **T**, whose components are given as follows:

$$T_{ij} \Rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

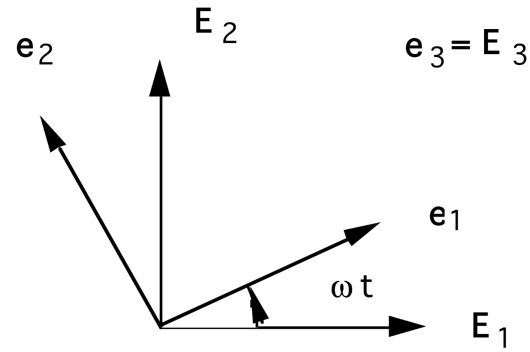
How would you obtain $\mathbf{T}^{1/4}$?

3. [20] Suppose only one component of strain, e, is nonzero so that the internal energy can be given as $u = Ee^2/2 + a + bs + cs^2/2$ in which E,a,b,c are constants and s is the entropy.
(i) Determine s and T, the functions conjugate to e and s, respectively.
(ii) Use the expression for T to determine the internal energy, u*, as a function of e and T. What is $\partial u^* / \partial T$?
(iii) Obtain the Helmholtz Free Energy, A, by performing a "contact transformation" on u with respect to T and s. What should you expect to get for $\partial A / \partial T$?. What do you get for $\partial A / \partial T$ for your function, A?

4. Consider the deformation described by $x_1 = X_1 + at(X_1)^2$, $x_2 = X_2 + btX_1$, $x_3 = X_3$ superposed on the rotation given by $q = wt$ where a, b and w are constants and t denotes time.

(i) [16] Determine $\mathbf{F} = \mathbf{r}\nabla_o$, $\mathbf{v} = \dot{\mathbf{r}}$, $\mathbf{L} = \mathbf{v}\nabla$, . Be sure to define your basis for each case.

(ii) [4] Sketch a segment of material in the undeformed and deformed configuration. Is the volume differential dV changing with time? Why?



Note: ME 512 students must do 5(a) and not 5(b). ME 402 students can do 5(a) or 5(b).

5(a) (i) [12] Start with a basic definition and obtain a general expression for the gradient of a tensor function of position in terms of components and base vectors in a curvilinear coordinate system. Your final expression should have "Christoffel" symbols. Obtain an expression for the divergence of a tensor.

(ii) [8] What is meant by "physical component?. How would you use the physical components of tensors in part (i)? How would you obtain the physical components of the divergence in part (i)?

5(b) (i) [8] Start with a basic definition for a gradient of a tensor function of position and derive the corresponding expression in indicial notation. Obtain an expression for the divergence of a tensor.

(ii) [12] Start with a basic definition for a gradient of a scalar function of a second-order tensor and derive the corresponding expression in indicial notation. What is the gradient of $(\mathbf{s}:\mathbf{s})^{1/2}$ with respect to $\mathbf{s}$, a second order tensor.

ME 412/512 Continuum Mechanics Closed Book Exam 1

Weights for problems are given in square brackets. Allocate time accordingly.

1. [20] Use indicial notation and an orthonormal basis for the following:

(a) For vectors **u** and **v**, show that

$$\text{(i)}\ (\mathbf{u}\cdot\mathbf{v})\nabla = \mathbf{u}\cdot(\mathbf{v}\nabla) + \mathbf{v}\cdot(\mathbf{u}\nabla)$$

$$\text{(ii)}\ (\mathbf{u}\cdot\mathbf{v})\nabla = (\mathbf{v}\nabla)\cdot\mathbf{u} + (\mathbf{u}\nabla)\cdot\mathbf{v} - \mathbf{u}\times(\mathbf{v}\times\nabla) - \mathbf{v}\times(\mathbf{u}\times\nabla)$$

(b) For tensors **A, B** and **C** verify that $\mathrm{tr}(\mathbf{A}\cdot\mathbf{B}\cdot\mathbf{C}) = \mathrm{tr}(\mathbf{B}\cdot\mathbf{C}\cdot\mathbf{A})$

2. [20] (i) Derive the transformation relation for components of a second-order tensor.

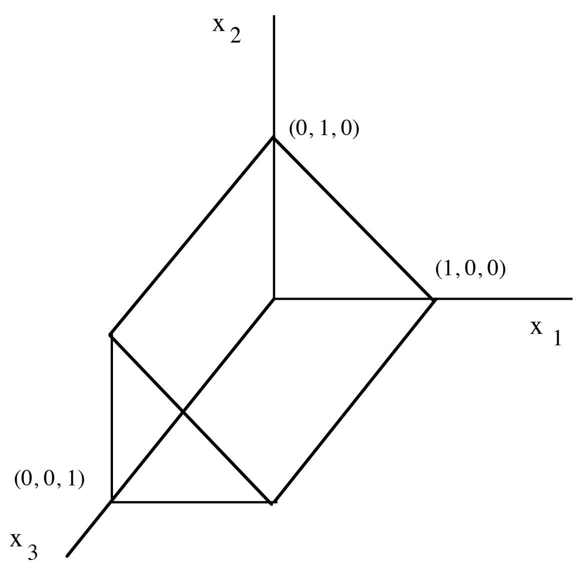
(ii) Give the result in both indical and matrix forms.

(iii) Construct your transformation matrix for two bases related as follows:

$$\mathbf{E}_1 = \frac{1}{\sqrt{14}}(\mathbf{e}_1 - 2\mathbf{e}_2 + 3\mathbf{e}_3) \qquad \mathbf{E}_2 = \frac{1}{\sqrt{5}}(2\mathbf{e}_1 + \mathbf{e}_2) \qquad \mathbf{E}_3 = \mathbf{E}_1\times\mathbf{E}_2$$

3. [15] What is an invariant of a second-order tensor. In general, how many independent invariants are there. Give two examples of sets of independent invariants.

4. [25] A region is defined in the following sketch:



Suppose $\phi = x_1x_2$. Carry out the integrals to show that $\int_R \phi\nabla dV = \int_{\partial R} \phi\mathbf{n}dS$ holds.

5. [20] Suppose $\mathbf{r} = x_i\mathbf{e}_i$, $\mathbf{R} = X_A\mathbf{E}_A$ and $\mathbf{E}_i = \mathbf{e}_i$. ∇_o denotes the gradient with respect to **R**, and ∇ the gradient with respect to **r**. Consider the deformation

$$x_1 = X_1 - X_2 \qquad x_2 = X_2 + X_1 \qquad x_3 = X_3$$

Let $\phi = ax_1^2 + bx_1x_2$ with a and b constants.

Determine $\phi\nabla_o$ and $\phi\nabla$.

ME 412/512 Continuum Mechanics Closed Book Exam 2

Weights for problems are given in square brackets. Allocate time accordingly.

1. (a) [5] If a tensor is represented as $\mathbf{T} = a\mathbf{e}_1 \otimes \mathbf{e}_3 + b\mathbf{e}_2 \otimes \mathbf{e}_2$ give the matrix of its components in the basis $\mathbf{e}_i \otimes \mathbf{e}_j$.

(b) [15] Suppose the eigenvalues of a tensor are 0, -2 and 3, and the eigenvectors corresponding to the first two eigenvalues are $\mathbf{p}_1 = \frac{1}{\sqrt{2}}(\mathbf{e}_2 - \mathbf{e}_3)$ and $\mathbf{p}_2 = \frac{1}{\sqrt{6}}(2\mathbf{e}_1 - \mathbf{e}_2 - \mathbf{e}_3)$. What is the third eigenvector? Give the components of the tensor with respect to the $\mathbf{e}_i \otimes \mathbf{e}_j$ basis.

2. The mapping between differentials of position vectors in the original and current configurations is given by $d\mathbf{r} = \mathbf{F} \cdot d\mathbf{R}$.

(a) [10] How is a differential volume element defined? If J = det(**F**) derive the mapping that relates differential volume elements in the two configurations.

(b) [10] Let ∇_0 denote a gradient with respect to **R** and ∇ the gradient with respect to **r**. Use the basic definition of a gradient and the relation $d\mathbf{r} = \mathbf{F} \cdot d\mathbf{R}$ to derive the relationship that allows one to convert from one gradient to the other.

(c) [5] Derive an equivalent expression for $\frac{d}{dt}\int_R F(t)dV$ but with the time derivative inside the integral.

3. [15] Give the basic formulation that defines either the Eulerian or Lagrangian strain tensor. Then derive the equation that relates the strain tensor to **F**.

4. [25] Outline the arguments that prove the existence of the Cauchy stress tensor $\boldsymbol{\sigma}$.

5. (i) [5] Describe the possible combinations of boundary conditions for a continuous body.

(ii) [5] A part of a rigid body, A, contains a frictionless surface defined by the arc of a circle, r = c, with c a constant. A portion of a second body, B, is pressed against this surface. What is the boundary condition for body, B, for the portion of the boundary in contact with the frictionless surface. (Give a sketch).

(iii) [5] Another part of the surface of body, B, defined by the equation f(x, y, z) = 0 is acted on by a fluid which sustains a static pressure P. What is the boundary condition for this part of the surface?

ME 412/512 Continuum Mechanics Closed Book Final Exam

Weights for problems are given in square brackets. Allocate time accordingly.
Total number of points is 120. Work to maximize point count, not to finish the exam.

1. (i) $\delta_{ij}a_j$ (ii) $\delta_{ij}x_ix_j$ (iii) $(x_ix_j)_{,j}$ (iv) $(\delta_{ij} + a_{ij})(\delta_{ik} + a_{ik}) - \delta_{jk} - \delta_{mn}a_{mj}a_{nk}$

(b) [10] What is the Principle of Material Frame Indifference? Give one example of an equation that satisfies the principle and an example that doesn't.

(c) [5] Material elements in the undeformed and deformed configuration are related by $d\mathbf{r} = \mathbf{F} \cdot d\mathbf{R}$. If L denotes a line segment in the deformed configuration and $\mathbf{u}$ is a function of time t, derive an equivalent expression for $\frac{d}{dt}\int_L \mathbf{u} \cdot d\mathbf{r}$ in which the time derivative is inside

2. Let G be a scalar function of a vector field $\mathbf{v}$.

(i) [5] Define what is meant by the gradient of G with respect to $\mathbf{v}$.

(ii) [5] What is meant by an orthonormal basis?

(iii) [5] Use your definition in (i) to obtain an expression in indicial form for the gradient if an orthonormal basis is used.

(iv) [10] Use your result from (iii) to determine the gradient with respect to $\mathbf{v}$ of

(a) $G = \mathbf{v} \cdot \mathbf{v}$ (b) $G = \dfrac{\mathbf{v} \cdot \mathbf{T} \cdot \mathbf{v}}{(\mathbf{v} \cdot \mathbf{v})^{1/2}}$ for a constant tensor, **T.**

Give your answers in both direct and indicial notation.

3. Suppose that the equation $\int_R \phi \nabla dV = \int_{\partial R} \phi \mathbf{n} dS$ is given.

(i) [6] In this context what do each of the following terms denote: $\nabla, R, \partial R, dV, dS$ and $\mathbf{n}$.

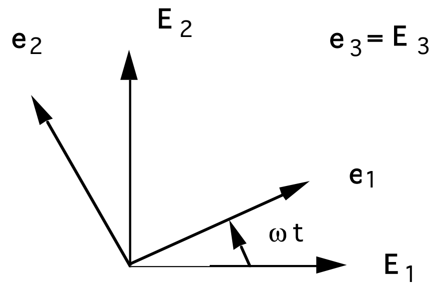
(ii) [10] Use the given equation to derive the corresponding equation where the function, ϕ, is a tensor rather than a scalar.

(iii) [4] Use the result of (ii) to derive the divergence theorem for a tensor field.

4. Consider the deformation described by $x_1 = X_1 + \alpha t X_2$, $x_2 = X_2(1 + \beta t X_1)$, $x_3 = X_3$ superposed on the rotation given by $\theta = \omega t$ where α and β are constants and t denotes time.

(i) [15] Determine $\mathbf{F} = \mathbf{r}\nabla_o$, $\mathbf{v} = \dot{\mathbf{r}}$, $\mathbf{L} = \mathbf{v}\nabla$, . Be sure to define your basis for each set of components.

(ii) [5] Describe rather thoroughly how you would obtain the rotation $\mathbf{R}$ and the right stretch $\mathbf{U}$.



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5. Suppose x^i represents a general curvilinear coordinate system.

(i) [5] How are covariant and contravariant bases defined?

(ii) [5] How are covariant and contravariant components of a vector defined?

(iii) [10] Start with a basic definition and obtain a general expression for the gradient of a vector in terms of components and base vectors in a curvilinear coordinate system. Your final expression should include a "Christoffel" type symbol.

(iv) [4] What are physical components of vectors and tensors.

(v) [6] Indicate how you obtain the physical components of the gradient of a vector in terms of the physical components of the vector.

ME 402/512 Continuum Mechanics Closed Book Exam 1

Weights for problems are given in square brackets. Allocate time accordingly.

1. The components of a tensor, **T**, and a vector, **v**, with respect to the orthonormal basis $\mathbf{e}_i$ are given as follows:

$$[T]_{\mathbf{e}_i \otimes \mathbf{e}_j} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \qquad \{v\}_{\mathbf{e}_i} = \begin{Bmatrix} -3 \\ 2 \\ -1 \end{Bmatrix}$$

(i) [5] Determine the components of the symmetric and skew-symmetric parts of **T.**

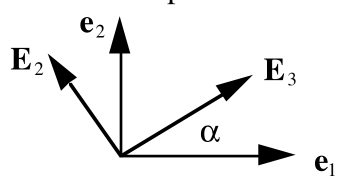
(ii) [5] Obtain $\mathbf{T}\cdot\mathbf{v}$ and $\mathbf{v}\cdot\mathbf{T}$.

(iii) [5] Obtain w_i where $w_i = \varepsilon_{ijk}T_{jk}$.

(iv) [5] Obtain tr(**T**) and **I:T** where **I** is the identity tensor.

(v) [5] Determine det [**T**].

(v) [10] A new orthonormal basis, $\mathbf{E}_A$, is oriented with respect to $\mathbf{e}_i$ as shown in the sketch. Both pairs of base vectors lie in the plane and both bases form a right-handed system. Obtain the 2-3 component of **T** in this new basis.



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2. The gradient theorem for a scalar function is given by $\int_R \phi\nabla dV = \int_{\partial R} \phi\mathbf{n}dS$.

(i) [5] What conclusions can you draw if the function is unity, i.e., $\phi = 1$.

(ii) [10] Derive the corresponding gradient and divergence theorems for vectors.

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3. A definition of a gradient of a vector, $\mathbf{v}\nabla$, is given by $d\mathbf{v} = (\mathbf{v}\nabla)\cdot d\mathbf{r} \quad \forall\, d\mathbf{r}$.

(i) [5] What is meant by each of the terms $\mathbf{v}$, $d\mathbf{v}$, $\mathbf{r}$, $d\mathbf{r}$, and $\forall$.

(ii) [8] Use this definition to derive the expression for the gradient in a rectangular Cartesian coordinate system.

(iii) [7] Apply your result to find the gradient of the following vector:

$$\mathbf{v} = x_2^2\mathbf{e}_1 + x_1x_2\mathbf{e}_2 + \sin(x_1^2x_3)\mathbf{e}_3$$

(iv) [10] Evaluate the line integral, $\int_{P_1}^{P_2} \mathbf{v}\cdot d\mathbf{r}$, for the straight line between the points P_1 and P_2 which have coordinates (1, 2, 0) and (1, 4, 0), respectively.

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4. [20] With respect to a symmetric second-order tensor, what is meant by each of the following terms: (i) eigenvector, (ii) eigenvalue, (iii) eigenpair, (iv) characteristic polynomial, (v) characteristic equation, (vi) principal directions, (vii) principal values, (viii) invariants, (ix) Cayley-Hamilton Theorem, and (x) a non-integer power.

ME 402/512 Continuum Mechanics Closed Book Exam 2

Weights for problems are given in square brackets. Allocate time accordingly. A total of 115 points are available so work on those problems you find easiest to maximize your point total.

1. The mapping between differentials of position vectors in the original and current configurations is given by $d\mathbf{r} = \mathbf{F} \cdot d\mathbf{R}$. Let ∇_0 denote a gradient with respect to $\mathbf{R}$ and ∇ the gradient with respect to $\mathbf{r}$.

(i) [5] Use the basic definition of a gradient to derive an expression for $\mathbf{F}$ involving one of the gradients.

(ii) [10] Derive the relationship that allows one to convert from one gradient to the other.

(iii) [10] Obtain two expressions for the time derivative of $\mathbf{F}$, one involving ∇_0 and one involving ∇.

(iv) [10] Obtain the time derivative of $\mathbf{F}^{-1}$.

3. [25] Give the basic formulation that defines either the Eulerian or Lagrangian strain tensor. Then derive the equation that relates the strain tensor to $\mathbf{F}$. Derive the relationship that gives a physical interpretation for one of the diagonal components of the strain tensor you defined.

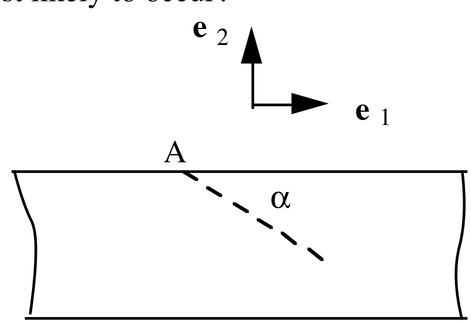
3. The stress tensor, $\boldsymbol{\sigma}$, has been solved for a problem and the components at point A are as follows:

(i) [7] Determine the traction on the top surface of the body. Point A is located at the surface.

(ii) [7] Obtain the traction on a surface which forms an angle α with the top surface as indicated in the figure.

(iii) [6] If failure is governed by the maximum value of the normal component of the traction, at which angle is failure most likely to occur?

$$\boldsymbol{\sigma}\Big|_{\mathbf{e}_i \otimes \mathbf{e}_j} = \begin{bmatrix} 8 & 3 & 0 \\ 3 & 4 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$



4. Suppose a deformation in the X_1 - X_2 plane is given as follows for the case where $\mathbf{e}_i = \mathbf{E}_i$: $x_1 = X_1$ and $x_2 = X_2 - 2a(t)X_1$.

(i) [5] Provide a sketch which illustrates the deformation.

(ii) [5] Obtain the displacement field $\mathbf{u} = \mathbf{r} - \mathbf{R}$.

(iii) [5] Obtain $\mathbf{v}$ in terms of x_i and $\mathbf{e}_i$.

(iv) [5] Obtain $\mathbf{L} = \mathbf{v}\nabla$, and then $\mathbf{D}$ and $\mathbf{W}$.

(v) [5] Obtain $\mathbf{F}$.

(vi) [10] Obtain $\mathbf{e} = \frac{1}{2}[\mathbf{I} - \mathbf{F}^{-T} \cdot \mathbf{F}^{-1}]$.

Weights for problems are given in square brackets. Allocate time accordingly.

Note. Do only the appropriate version of Problem 5.

1. [20] With regard to the deformation of a continuous medium, express with words (and with sketches) what is meant by each of the following expressions. Also define each of the distinct tensors and symbols.

$$\text{(i) } d\mathbf{r} = \mathbf{F}\cdot d\mathbf{R}, \quad \text{(ii) } \mathbf{F} = \mathbf{R}\cdot\mathbf{U}, \quad \text{(iii) } \mathbf{D} = (\mathbf{v}\nabla)_{\text{sym}}, \quad \text{(iv) } \mathbf{L} = \mathbf{D} + \mathbf{W}$$

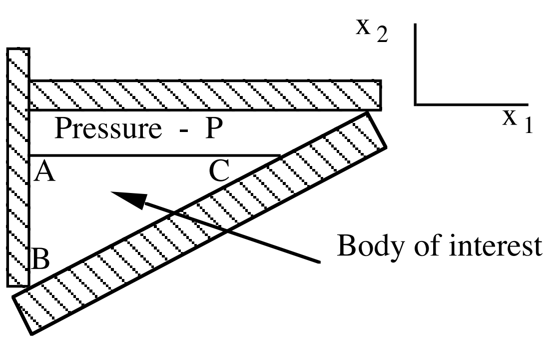
2. [20] Prove that the eigenvectors of a symmetric, real tensor, $\mathbf{T}$, associated with two distinct eigenvalues are orthogonal.

3. (i) [5] What is the Cauchy stress tensor $\boldsymbol{\sigma}$?

(ii) [5] What are Cauchy's equations of motion?

(ii) [10] Outline the arguments which prove that such a tensor must exist and which provides a derivation of Cauchy's equations.

4. (i) [5] Suppose a body is confined by frictionless surfaces along the planes x_3 = -a and x_3 = a. The body is glued along the surface A-B-C to a rigid base. On the top surface A-C, an air pressure of magnitude P is applied. Give the boundary conditions for the body.



(ii) [5] The boundary conditions constitute an essential part of the description of a completely formulated problem. In continuum mechanics what other equations must be specified?

5. (ME 512 only) Suppose x^i represents a general coordinate system.
The covariant components of the metric tensor are known to be

$$g_{ij} \Rightarrow \begin{bmatrix} (x^2)^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{(x^1)^2} \end{bmatrix}$$

(i) [5] Is the coordinate system orthogonal? Why? Is the coordinate system curvilinear? Why? Will the contravariant components of the position vector be x^i? Why?

(ii) [5] What are the contravariant and mixed components of the metric or identity tensor? What are the components of the metric tensor in a rectangular Cartesian system?

(iii) [5] The covariant components of a vector $\mathbf{v}$ are $v_i \Rightarrow (1, 2, -1)$? What are the contravariant components? What is $\mathbf{v} \cdot \mathbf{v}$? Will the gradient of $\mathbf{v}$ be the zero tensor? Why?

(iv) [5] The contravariant components of a tensor $\mathbf{T}$ are $T^{ij} \Rightarrow \begin{bmatrix} 1 & 0 & 3 \\ 0 & 2 & 4 \\ 3 & 4 & 5 \end{bmatrix}$.

What are the covariant components? What is the determinant of $\mathbf{T}$?

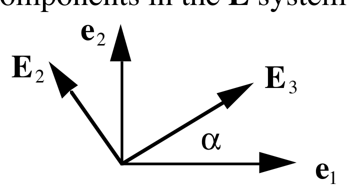
(v) [10] Start with a basic definition and obtain a general expression for the gradient of a vector in terms of components and base vectors in a curvilinear coordinate system. Your final expression should include a "Christoffel" type symbol.

5. (ME 412 only) Consider two orthonormal bases, $\mathbf{e}_i$ and $\mathbf{E}_A$.

(i) [10] Derive the transformation relations that relate components of a second-order tensor in these two systems.

(ii) [10] The components of $\mathbf{T}$ in the $\mathbf{e}_i$ system are $T_{ij} \Rightarrow \begin{bmatrix} 1 & 0 & 3 \\ 0 & 2 & 4 \\ 3 & 4 & 5 \end{bmatrix}$.

What are the components in the $\mathbf{E}$ system if the systems are oriented as follows:



What is the determinant of $\mathbf{T}$?

(iii) [10] From the basic definition, obtain the general expression for the gradient of a tensor.

Weights for problems are given in square brackets. Problems 1, 2 and 3 are each worth 30 points and problem 4 is worth 15 for a total or 105. Allocate time accordingly.

1. The mapping between differentials of position vectors in the original and current configurations is given by $d\mathbf{r} = \mathbf{F} \cdot d\mathbf{R}$. Let ∇_0 denote a gradient with respect to $\mathbf{R}$ and ∇ the gradient with respect to $\mathbf{r}$.

(i) [5] Use the basic definition of a gradient to derive an expression for $\mathbf{F}$ involving one of the gradients.

(ii) [5] Derive the relationship that allows one to convert from one gradient to the other.

(iii) [5] Obtain two expressions for the time derivative of $\mathbf{F}$, one involving ∇_0 and one involving ∇.

(iv) [5] Obtain the time derivative of $\mathbf{F}^{-1}$.

(v) [5] If L denotes a line segment in the deformed configuration and $\mathbf{u}$ is a function of $\mathbf{r}$ and time t, derive an equivalent expression for $\frac{d}{dt}\int_L \mathbf{u} \cdot d\mathbf{r}$ in which the time derivative is inside the integral.

(vi) [5] If $\oint \mathbf{u} \cdot d\mathbf{r} = 0$ for all closed paths, derive the necessary restriction that must be satisfied by **u.**

2. (i) [5] What is the Cauchy stress tensor $\boldsymbol{\sigma}$?

(ii) [5] What are Cauchy's equations of motion?

(iii) [10] Outline the arguments which prove that a stress tensor must exist and which provides a derivation of Cauchy's equations.

(iv) Use indicial notation to prove that

(a) [5] $(\mathbf{r} \times \boldsymbol{\sigma}) \cdot \nabla = \mathbf{r} \times (\boldsymbol{\sigma} \cdot \nabla) + C_{13}(\mathbf{I} \times \boldsymbol{\sigma})$.

(b) [5] If $C_{13}(\mathbf{I} \times \boldsymbol{\sigma}) = \mathbf{0}$ then $\boldsymbol{\sigma} = \boldsymbol{\sigma}^T$.

3. Let **f** be a vector function of a vector field **v**.

(i) [5] Define what is meant by the gradient of **f** with respect to **v**.

(ii) [5] What is meant by an orthonormal basis?

(iii) [10] Use your definition in (i) to obtain an expression in indicial form for the gradient if an orthonormal basis is used.

(iv) [10] Use your result from (iii) to determine the gradient with respect to **v** of

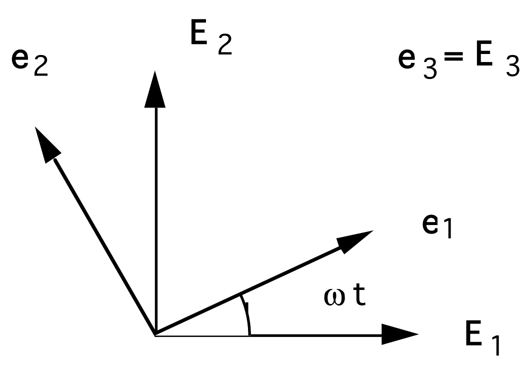
(a) $\mathbf{f} = \mathbf{T} \cdot \mathbf{v}$ (b) $\mathbf{f} = \dfrac{\mathbf{T} \cdot \mathbf{v}}{(\mathbf{v} \cdot \mathbf{T} \cdot \mathbf{v})^{1/2}}$ for a constant tensor **T.**

Give your answers in both direct and indicial notation.

4. [15] Consider the deformation described by

$$x_1 = X_1 + atX_2$$
$$x_2 = X_2$$
$$x_3 = X_3$$

superposed on the rotation given by $\theta = \omega t$ where a and b are constants and t denotes time.
Illustrate the deformation. For a generic material point determine the velocity and acceleration in terms of x_I and $\mathbf{e}_i$.



Weights for problems are given in square brackets. Allocate time accordingly. 110 points are available. Your score will be the sum that you obtain.

1. Express each of the following in both indicial and expanded forms:
(i) [5] $\phi = \mathrm{tr}(\boldsymbol{\sigma} \cdot \boldsymbol{\sigma})$ (Is ϕ a scalar, a vector or a tensor?)
(ii) [5] $\boldsymbol{\sigma} \cdot \nabla + \rho \mathbf{b} = \rho \mathbf{a}$.

2. [5] Simplify the following by using the product rule when taking the derivative. Give the original and simplified form in direct notation. $(x_i x_j)_{,j}$

3. [15] Derive the transformation relations in indicial form for components of a second-order tensor expressed in terms of two orthonormal bases, $\mathbf{e}_i$ and $\mathbf{E}_A$. Then develop the matrix form of the transformation relation. Show clearly the structure of the transformation matrix.

4. Suppose a deformation in the X_1 - X_2 plane is given as follows: $\mathbf{e}_1 = \mathbf{E}_1$, $\mathbf{e}_2 = \mathbf{E}_2$, $x_1 = aX_1 + 2bX_2$, $x_2 = aX_2$ where a and b are constants. The origins coincide.

(i) [5] Provide a sketch which illustrates the deformation.
(ii) [5] What is meant by the displacement field **u**? What is it for this case?
(iii) [5] One of the strain tensors is defined by the equation
$ds^2 - dS^2 = 2d\mathbf{R} \cdot \mathbf{E} \cdot d\mathbf{R}$. Define each of the terms in the definition.
(iv) [5] From the definition, derive an expression for **E** involving the deformation gradient.
(v) [5] Obtain **E** for the given deformation.
(vi) [5] Does the deformation include a rotation? Why?

5. (i) [5] What is a constitutive equation? Give an example.
(ii) [10] What is the Principal of Material Frame Indifference? Give an example of a constitutive equation that satisfies the principal, and one that doesn't.

6. Suppose x^i represents a general coordinate system. The covariant components of the metric tensor are known to be

$$g_{ij} \Rightarrow \begin{bmatrix} (x^2)^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \cos(x^1) \end{bmatrix}$$

(i) [5] Is the coordinate system orthogonal? Why? Is the coordinate system curvilinear? Why? Will the contravariant components of the position vector be x^i? Why?

(ii) [5] What are the contravariant and mixed components of the metric or identity tensor? What are the components of the metric tensor in a rectangular Cartesian system?

(iii) [5] The contravariant components of a tensor **T** are $T^{ij} \Rightarrow \begin{bmatrix} 2 & 3 & 1 \\ 3 & 4 & 0 \\ 1 & 0 & 5 \end{bmatrix}$.

What are the covariant components? What is the determinant of **T**?

(iv) [10] Start with a basic definition and obtain a general expression for the gradient of a vector in terms of components and base vectors in a curvilinear coordinate system. Your final expression should include a "Christoffel" type symbol.

7. Suppose **e** denotes the Eulerian strain. Define the strain deviator, $\mathbf{e}^d$, by the equation $\mathbf{e}^d = \mathbf{e} - \frac{1}{3}\mathrm{tr}(\mathbf{e})\mathbf{I}$ in which **I** is the identity.

(i) [5] What is meant by the terms eigenvalue and eigenvector of **e**?

(ii) [5] Obtain the relationship, if any, that relates the eigenvalues and eigenvectors of **e** to the eigenvalues and eigenvectors of $\mathbf{e}^d$?