12. STRAIN TENSORS

12.1 Initial Comments

Here we define two strain tensors that are commonly used. First the tensors are expressed in terms of the deformation gradient and then the strains are expressed in terms of the displacement gradient.

12.2 Eulerian and Lagrangian Strain Tensors

Definitions of Strain

The Lagrangian, $\boldsymbol{E}$, and Eulerian, $\boldsymbol{e}$, strain tensors are defined such that

$$ds^2 - dS^2 = d\boldsymbol{r}\cdot d\boldsymbol{r} - d\boldsymbol{R}\cdot d\boldsymbol{R} = 2d\boldsymbol{r}\cdot\boldsymbol{e}\cdot d\boldsymbol{r} = 2d\boldsymbol{R}\cdot\boldsymbol{E}\cdot d\boldsymbol{R} \quad \forall \; d\boldsymbol{R} \text{ or } d\boldsymbol{r} \tag{12-1}$$

If we use the material, or referential, or Lagrangian form of $d\boldsymbol{r} = \boldsymbol{F}\cdot d\boldsymbol{R} = d\boldsymbol{R}\cdot\boldsymbol{F}^T$, then

$$d\boldsymbol{R}\cdot\boldsymbol{E}\cdot d\boldsymbol{R} = \frac{1}{2}\left[d\boldsymbol{r}\cdot d\boldsymbol{r} - d\boldsymbol{R}\cdot d\boldsymbol{R}\right] = \frac{1}{2}d\boldsymbol{R}\cdot\left[\boldsymbol{F}^T\cdot\boldsymbol{F} - I\right]\cdot d\boldsymbol{R} \;\; \forall \;\; d\boldsymbol{R} \tag{12-2}$$

with the result that

$$\boldsymbol{E} = \frac{1}{2}\left[\boldsymbol{F}^T\cdot\boldsymbol{F} - I\right] = \frac{1}{2}\left[\boldsymbol{U}^2 - I\right] = \frac{1}{2}\left[\boldsymbol{C} - I\right] \tag{12-3}$$

Conversely, if the Eulerian, or spatial, form of $d\boldsymbol{R} = \boldsymbol{F}^{-1}\cdot d\boldsymbol{r} = d\boldsymbol{r}\cdot\boldsymbol{F}^{-T}$ is used, then

$$d\boldsymbol{r}\cdot\boldsymbol{e}\cdot d\boldsymbol{r} = \frac{1}{2}\left[d\boldsymbol{r}\cdot d\boldsymbol{r} - d\boldsymbol{R}\cdot d\boldsymbol{R}\right] = \frac{1}{2}d\boldsymbol{r}\cdot\left[\boldsymbol{I} - \boldsymbol{F}^{-T}\cdot\boldsymbol{F}^{-1}\right]\cdot d\boldsymbol{r} \;\; \forall \;\; d\boldsymbol{r} \tag{12-4}$$

which yields

$$\boldsymbol{e} = \frac{1}{2}\left[\boldsymbol{I} - \boldsymbol{F}^{-T}\cdot\boldsymbol{F}^{-1}\right] = \frac{1}{2}\left[\boldsymbol{I} - \boldsymbol{V}^{-2}\right] = \frac{1}{2}\left[\boldsymbol{I} - \boldsymbol{B}^{-1}\right] \tag{12-5}$$

We note that the two strain tensors satisfy the relation

$$\boldsymbol{E} = \boldsymbol{F}^T\cdot\boldsymbol{e}\cdot\boldsymbol{F} \tag{12-6}$$

Recall from the previous chapter that the spectral decompositions of the right and left stretch tensors are

$$\boldsymbol{U} = \sum_{i=1}^{3}\Lambda_i\boldsymbol{N}_i\otimes\boldsymbol{N}_i \qquad \boldsymbol{V} = \sum_{i=1}^{3}\Lambda_i\boldsymbol{n}_i\otimes\boldsymbol{n}_i \tag{12-7}$$

It follows that the spectral decompositions of the strain tensors are:

$$\boldsymbol{E} = \frac{1}{2}\sum_{i=1}^{3}\left(\Lambda_i^2 - 1\right)\boldsymbol{N}_i\otimes\boldsymbol{N}_i \qquad \boldsymbol{e} = \frac{1}{2}\sum_{i=1}^{3}\left(1 - \frac{1}{\Lambda_i^2}\right)\boldsymbol{n}_i\otimes\boldsymbol{n}_i \tag{12-8}$$

Note in each case that when the principal stretches are unity, the strain tensors are null tensors.

Superimposed Rotation, $\boldsymbol{Q}$

We consider the Lagrangian and Eulerian strain tensors whose new forms under a rigid-body rotation are

$$\begin{aligned}
\boldsymbol{E}^* &= \tfrac{1}{2}\left[\boldsymbol{F}^{*T}\cdot\boldsymbol{F}^* - \boldsymbol{I}\right] = \tfrac{1}{2}\left[\boldsymbol{F}^T\cdot\boldsymbol{Q}^T\cdot\boldsymbol{Q}\cdot\boldsymbol{F}-\boldsymbol{I}\right] \\
&= \tfrac{1}{2}\left[\boldsymbol{F}^T\cdot\boldsymbol{F}-\boldsymbol{I}\right] = \boldsymbol{E} \\
\boldsymbol{e}^* &= \tfrac{1}{2}\left[\boldsymbol{I}-\boldsymbol{F}^{*-T}\cdot\boldsymbol{F}^{*-1}\right] = \tfrac{1}{2}\left[\boldsymbol{I}-\boldsymbol{Q}\cdot\boldsymbol{F}^{-T}\cdot\boldsymbol{F}^{-1}\cdot\boldsymbol{Q}^T\right] \\
&= \tfrac{1}{2}\boldsymbol{Q}\cdot\left[\boldsymbol{I}-\boldsymbol{Q}\cdot\boldsymbol{F}^{-T}\cdot\boldsymbol{F}^{-1}\right]\cdot\boldsymbol{Q}^T = \boldsymbol{Q}\cdot\boldsymbol{e}\cdot\boldsymbol{Q}^T
\end{aligned} \tag{12-9}$$

We see that the Lagrangian strain is a material tensor, while the Eulerian strain is spatial.

A Note on Components of Tensors with a Mixed Basis

Consider the deformation gradient as an example. If we use a single basis, then the components of the transpose of the tensor are defined as follows:

$$\boldsymbol{F} = \langle\boldsymbol{E}\rangle\overset{E-E}{[F]}\otimes\{\boldsymbol{E}\} \qquad \boldsymbol{F}^T = \langle\boldsymbol{E}\rangle\overset{E-E}{[F^T]}\otimes\{\boldsymbol{E}\} \qquad \overset{E-E}{[F^T]} = \overset{E-E}{[F]}^T \tag{12-10}$$

in other words, the matrix form of the components of transpose is simply the transpose of the matrix of components of the original tensor.

Now suppose a mixed basis is used. The trick is to transform to a pure basis, take the transpose and then transform back. The original tensor is expressed as follows:

$$\boldsymbol{F} = \langle\boldsymbol{e}\rangle\overset{e-E}{[F]}\otimes\{\boldsymbol{E}\} = \langle\boldsymbol{E}\rangle\overset{E-e}{[a]}\overset{e-E}{[F]}\otimes\{\boldsymbol{E}\} = \langle\boldsymbol{E}\rangle\overset{E-E}{[F]}\otimes\{\boldsymbol{E}\} \tag{12-11}$$

Then the transpose is

$$\boldsymbol{F}^T = \langle\boldsymbol{E}\rangle\overset{E-E}{[F]}^T\otimes\{\boldsymbol{E}\} = \langle\boldsymbol{E}\rangle\overset{e-E}{[F]}^T\overset{E-e}{[a]}^T\otimes\{\boldsymbol{E}\} = \langle\boldsymbol{E}\rangle[F^T]\otimes\overset{e-E}{[a]}\{\boldsymbol{E}\} \tag{12-12}$$

with the final result that

$$\boldsymbol{F}^T = \langle\boldsymbol{E}\rangle\overset{E-e}{[F^T]}\otimes\{\boldsymbol{e}\} \qquad \overset{E-e}{[F^T]} = \overset{e-E}{[F]}^T \tag{12-13}$$

If a mixed basis is used the component matrix of the transposes tensor is simply the transpose of the component matrix with an interchange of the mixed basis.

Components of Strain Tensors

Recall that the deformation tensor is given by

$$\boldsymbol{F} = \boldsymbol{r}\bar{\nabla}_0 = \frac{\partial x_i}{\partial X_j}\boldsymbol{e}_i \otimes \boldsymbol{E}_j = \overset{e-E}{F}_{ij}\,\boldsymbol{e}_i \otimes \boldsymbol{E}_j$$

$$\boldsymbol{F}^T = \overset{e-E}{F}_{ji}\,\boldsymbol{E}_i \otimes \boldsymbol{e}_j \qquad \overset{e-E}{F}_{ij} = \frac{\partial x_i}{\partial X_j} \tag{12-14}$$

and its inverse by

$$\boldsymbol{F}^{-1} = \boldsymbol{R}\bar{\nabla} = \frac{\partial X_i}{\partial x_j}\boldsymbol{E}_i \otimes \boldsymbol{e}_j = \overset{E-e}{F^{-1}}_{ij}\,\boldsymbol{E}_i \otimes \boldsymbol{e}_j$$

$$\boldsymbol{F}^{-T} = \overset{E-e}{F^{-1}}_{ji}\,\boldsymbol{e}_i \otimes \boldsymbol{E}_j \qquad \overset{E-e}{F^{-1}}_{ij} = \frac{\partial X_i}{\partial x_j} \tag{12-15}$$

It follows that

$$\boldsymbol{E} = E_{ij}\boldsymbol{E}_i \otimes \boldsymbol{E}_j \qquad E_{ij} = \frac{1}{2}\left[\overset{e-E}{F}_{ki}\,\overset{e-E}{F}_{kj} - \delta_{ij}\right] = \frac{1}{2}\left[\frac{\partial x_k}{\partial X_i}\frac{\partial x_k}{\partial X_j} - \delta_{ij}\right]$$

$$\boldsymbol{e} = e_{ij}\boldsymbol{e}_i \otimes \boldsymbol{e}_j \qquad \mathrm{e}_{ij} = \frac{1}{2}\left[\delta_{ij} - \overset{e-E}{F^{-1}}_{ki}\,\overset{e-E}{F^{-1}}_{kj}\right] = \frac{1}{2}\left[\delta_{ij} - \frac{\partial X_k}{\partial x_i}\frac{\partial X_k}{\partial x_j}\right] \tag{12-16}$$

Note that the "natural" tensor bases for $\boldsymbol{E}$ and $\boldsymbol{e}$ are $\boldsymbol{E}_i \otimes \boldsymbol{E}_j$ and $\boldsymbol{e}_i \otimes \boldsymbol{e}_j$, respectively.

12.3 Physical Interpretation of Eulerian and Lagrangian Strain Components

Consider a physical element $d\boldsymbol{R} = dX_1\boldsymbol{E}_1$ in the original configuration. Then (12-1) becomes

$$ds^2 - dS^2 = ds^2 - dX_1^2 = 2dX_1\boldsymbol{E}_1 \cdot \boldsymbol{E} \cdot dX_1\boldsymbol{E}_1 = 2dX_1^2 E_{11} \tag{12-17}$$

and the interpretation of the component of strain, E_{11}, from (12-17) is

$$E_{11} = \frac{1}{2}\left[\Lambda_{st,1}^2 - 1\right] \qquad \Lambda_{st,1} = \frac{ds}{|dX_1|} \tag{12-18}$$

in which $\Lambda_{st,1}$ is defined to be the stretch of a fiber originally in the direction of $\boldsymbol{E}_1$. Similar interpretations hold for the other diagonal components of the Lagrangian strain tensor.

Now consider an element that ends up as $d\boldsymbol{r} = dx_1\boldsymbol{e}_1$ in the final configuration. Again, using the appropriate form of (12-1) we get

$$ds^2 - dS^2 = dx_1^2 - dS^2 = 2dx_1\boldsymbol{e}_1 \cdot \boldsymbol{e} \cdot dx_1\boldsymbol{e}_1 = 2e_{11}dx_1^2 \tag{12-19}$$

from which the physical interpretation of e_{11} becomes

$$e_{11} = \frac{1}{2}\left[1 - \frac{1}{\lambda_{st,1}^2}\right] \qquad \lambda_{st,1} = \frac{|dx_1|}{dS} \tag{12-20}$$

where $\lambda_{st,1}$ is defined to be the stretch of a fiber that ends in the direction of $\boldsymbol{e}_1$.

Now we develop the physical interpretation of a shear, or off diagonal, component of the Lagrangian strain tensor. We choose the material element in the original configuration to be $d\boldsymbol{R} = dX_1\boldsymbol{E}_1 + dX_2\boldsymbol{E}_2$. Since $\boldsymbol{E}$ is symmetric so that $E_{12} = E_{21}$, (12-1) yields

$$ds^2 - dS^2 = 2d\boldsymbol{R} \cdot \boldsymbol{E} \cdot d\boldsymbol{R} = 2\left[E_{11}dX_1^2 + 2E_{12}dX_1dX_2 + E_{22}dX_2^2\right] \tag{12-21}$$

With the use of $dS^2 = dX_1^2 + dX_2^2$ and (12-18) we obtain

$$\begin{aligned} ds^2 &= (2E_{11}+1)dX_1^2 + 4E_{12}dX_1dX_2 + (2E_{22}+1)dX_2^2 \\ &= \Lambda_{st,1}^2 dX_1^2 + 4E_{12}dX_1dX_2 + \Lambda_{st,2}^2 dX_2^2 \end{aligned} \tag{12-22}$$

The material element in the deformed configuration is

$$d\boldsymbol{r} = ds_1\boldsymbol{u}_1 + ds_2\boldsymbol{u}_2 = \Lambda_{st,1}dX_1\boldsymbol{u}_1 + \Lambda_{st,2}dX_2\boldsymbol{u}_2 \tag{12-23}$$

in which $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ are unit vectors as indicated in Fig. 12-1. The application of the law of cosines yields an alternative equation for ds^2, namely

$$\begin{aligned} ds^2 &= ds_1^2 + ds_2^2 - 2ds_1ds_2\cos\beta \\ &= \Lambda_{st,1}^2 dX_1^2 + \Lambda_{st,2}^2 dX_2^2 - 2\Lambda_{st,1}\Lambda_{st,2}dX_1dX_2\cos\beta \end{aligned} \tag{12-24}$$

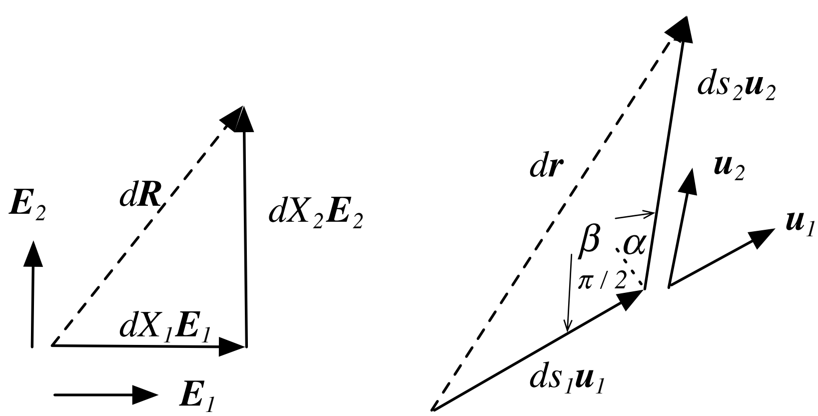

Fig. 12-1. Sketch for interpreting shear strain.

Since the right sides of (12-22) and (12-24) must be the same for all choices of dX_1 and dX_2 we obtain the following representation for the shear component

$$4E_{12} = -2\Lambda_{st,1}\Lambda_{st,2}\cos\beta \tag{12-25}$$

If we let $\beta = \alpha + \pi/2$ where α denotes the change from the right angle that the initial elements formed, then

$$E_{12} = \frac{1}{2}\Lambda_{st,1}\Lambda_{st,2}\sin\alpha \tag{12-26}$$

For infinitesimal deformations, (12-26) reduces to

$$E_{12} \approx \alpha / 2 \tag{12-27}$$

In a similar manner a physical interpretation can be developed for a shear component of the Eulerian strain.

12.4 Strains as Functions of the Displacement Vector

Frequently the deformation is described with the use of the displacement vector for a material point that is defined by

$$\boldsymbol{u} = \boldsymbol{\rho}^{O-o} + \boldsymbol{r} - \boldsymbol{R} \tag{12-28}$$

as shown in Fig. 12-2.

The gradients of $\boldsymbol{u}$ with respect to $\boldsymbol{R}$ and $\boldsymbol{r}$ are used so frequently that we will designate these gradients as follows:

$$\boldsymbol{H} = \boldsymbol{u}\bar{\bar{\nabla}}_0 \qquad \boldsymbol{h} = \boldsymbol{u}\bar{\bar{\nabla}} \tag{12-29}$$

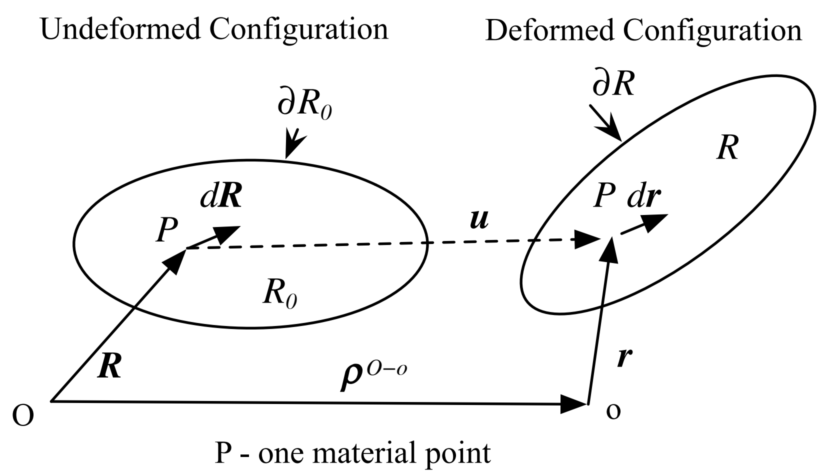

Fig. 12-2. Sketch showing the definition of the displacement vector $\boldsymbol{u}$.

Since $\boldsymbol{\rho}^{O-o}$ is a constant vector and with the use of (12-28) it follows that

$$\boldsymbol{H} = \boldsymbol{F} - \boldsymbol{I} \qquad \boldsymbol{h} = \boldsymbol{I} - \boldsymbol{F}^{-1} \tag{12-30}$$

or

$$\boldsymbol{F} = \boldsymbol{H} + \boldsymbol{I} \qquad \boldsymbol{F}^{-1} = \boldsymbol{I} - \boldsymbol{h} \tag{12-31}$$

After taking the transposes we obtain

$$\begin{aligned} \boldsymbol{C} = \boldsymbol{U}^2 = \boldsymbol{F}^T \cdot \boldsymbol{F} = \boldsymbol{H} + \boldsymbol{H}^T + \boldsymbol{H}^T \cdot \boldsymbol{H} + \boldsymbol{I} \\ \boldsymbol{B}^{-1} = \boldsymbol{V}^{-2} = \boldsymbol{F}^{-T} \cdot \boldsymbol{F}^{-1} = \boldsymbol{I} - \boldsymbol{h} - \boldsymbol{h}^T + \boldsymbol{h}^T \cdot \boldsymbol{h} \end{aligned} \tag{12-32}$$

and expressions for the Lagrangian and Eulerian strain tensors from (12-3) and (12-5) are

$$\boldsymbol{E} = \frac{1}{2}\left[\boldsymbol{H} + \boldsymbol{H}^T + \boldsymbol{H}^T \cdot \boldsymbol{H}\right] \qquad \boldsymbol{e} = \frac{1}{2}\left[\boldsymbol{h} + \boldsymbol{h}^T - \boldsymbol{h}^T \cdot \boldsymbol{h}\right] \tag{12-33}$$

In terms of components and base tensors, (12-33) becomes

$$\begin{aligned} \boldsymbol{E} &= \frac{1}{2}\left[\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i} + \frac{\partial u_k}{\partial X_i}\frac{\partial u_k}{\partial X_j}\right]\boldsymbol{E}_i \otimes \boldsymbol{E}_j \\ \boldsymbol{e} &= \frac{1}{2}\left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{\partial u_k}{\partial x_i}\frac{\partial u_k}{\partial x_j}\right]\boldsymbol{e}_i \otimes \boldsymbol{e}_j \end{aligned} \tag{12-34}$$

Note that for the Lagrangian strain the displacement vector is assumed to be a function of $\boldsymbol{R}$ and the components are given with respect to the basis $\boldsymbol{E}_i$, whereas for the Eulerian strain $\boldsymbol{u}$ must be specified as a function of $\boldsymbol{r}$ with components in the $\boldsymbol{e}_i$ system.

12.5 Infinitesimal Deformations

An infinitesimal deformation is defined to exist when the principal stretches and the angle in the Euler-Rodriguez formula for the rotation tensor satisfies the conditions:

$$\Lambda_i = 1 + \varepsilon_i \qquad |\varepsilon_i| << 1 \qquad |\theta| << 1 \tag{12-35}$$

First we look at the implications with regard to the stretches. The use of (12-35) yields

$$\begin{aligned} \boldsymbol{U}^{sm} &= \sum_{i=1}^{3}(1+\varepsilon_i)\boldsymbol{N}_i \otimes \boldsymbol{N}_i = \boldsymbol{I} + \sum_{i=1}^{3}\varepsilon_i \boldsymbol{N}_i \otimes \boldsymbol{N}_i \\ \boldsymbol{V}^{sm} &= \sum_{i=1}^{3}(1+\varepsilon_i)\boldsymbol{n}_i \otimes \boldsymbol{n}_i = \boldsymbol{I} + \sum_{i=1}^{3}\varepsilon_i \boldsymbol{n}_i \otimes \boldsymbol{n}_i \end{aligned} \tag{12-36}$$

where the subscript “sm” denotes “small”. We note that

$$\Lambda_i^2 - 1 \approx 2\varepsilon_i \qquad \frac{1}{\Lambda_i^2} \approx 1 - 2\varepsilon_i \qquad 1 - \frac{1}{\Lambda_i^2} \approx 2\varepsilon_i \qquad \ln(\Lambda_i) \approx \varepsilon_i \tag{12-37}$$

and the spectral representations for the strain tensors become

$$\boldsymbol{E}^{sm} = \boldsymbol{L}_U^{sm} = \sum_{i=1}^{3}\varepsilon_i \boldsymbol{N}_i \otimes \boldsymbol{N}_i \qquad \boldsymbol{e}^{sm} = \boldsymbol{L}_V^{sm} = \sum_{i=1}^{3}\varepsilon_i \boldsymbol{n}_i \otimes \boldsymbol{n}_i \tag{12-38}$$

Recall that the Euler-Rodriguez formula for representing the rotation tensor is

$$\boldsymbol{R} = \cos\theta(\boldsymbol{I} - \boldsymbol{a} \otimes \boldsymbol{a}) + \boldsymbol{a} \otimes \boldsymbol{a} + \sin\theta(\boldsymbol{\varepsilon} \cdot \boldsymbol{a}) \tag{12-39}$$

If the rotation is small, i.e., $|\theta| << 1$, then $cos\theta \approx 1-\theta^2/2+\cdots \approx 1$ and $sin\theta \approx \theta$. The rotation tensor for small deformation becomes

$$\boldsymbol{R}^{sm} = \boldsymbol{I} + \boldsymbol{\omega}^{sm} \qquad \boldsymbol{\omega}^{sm} = \theta(\boldsymbol{\varepsilon}\cdot\boldsymbol{a}) \tag{12-40}$$

The formula relating the principal vectors of $\boldsymbol{U}$ and $\boldsymbol{V}$, or of $\boldsymbol{E}$ and $\boldsymbol{e}$, is

$$\boldsymbol{n}_i = \boldsymbol{R}^{sm}\cdot\boldsymbol{N}_i \approx \boldsymbol{N}_i \tag{12-41}$$

Then

$$\boldsymbol{U}^{sm} = \boldsymbol{V}^{sm} \qquad \boldsymbol{E}^{sm} = \boldsymbol{L}_U^{sm} = \boldsymbol{e}^{sm} = \boldsymbol{L}_V^{sm} \tag{12-42}$$

i.e., all strain tensors become identical. The reason for the factor $1/2$ in the definitions given in (12-3) and (12-5) was to provide this coalescence under the assumption of small deformations.

The deformation gradients in the two configurations are related through $(\ )\bar{\nabla}_0 \approx (\ )\bar{\nabla}\cdot\boldsymbol{F}$. But for small deformations and for this equation, we can make the simplification that

$$\begin{aligned}(\ )\bar{\nabla}_0 &\approx (\ )\bar{\nabla}\cdot\boldsymbol{R}^{sm}\cdot\boldsymbol{U}^{sm} \approx (\ )\bar{\nabla}\cdot\boldsymbol{I}\cdot I\\ (\ )\bar{\nabla}_0 &= (\ )\bar{\nabla} \equiv (\ )\bar{\nabla}^{sm}\end{aligned} \tag{12-43}$$

i.e., there is only a single gradient.

As before, introduce the displacement vector

$$\boldsymbol{u} = \boldsymbol{r} - \boldsymbol{R} \tag{12-44}$$

One form for the small deformation gradient is

$$\boldsymbol{F}^{sm} = \boldsymbol{I} + \boldsymbol{h} \qquad \boldsymbol{h} = \boldsymbol{u}\bar{\nabla}^{sm} \tag{12-45}$$

But if we use the small deformation assumptions for the deformation gradient, then

$$\begin{aligned}\boldsymbol{F}^{sm} &= \boldsymbol{R}^{sm}\cdot\boldsymbol{U}^{sm} = (\boldsymbol{I}+\boldsymbol{\omega}^{sm})\cdot\left[\boldsymbol{I}+\sum_{i=1}^{3}\varepsilon_i\boldsymbol{N}_i\otimes\boldsymbol{N}_i\right]\\ &\approx \boldsymbol{I}+\boldsymbol{\omega}^{sm}+\sum_{i=1}^{3}\varepsilon_i\boldsymbol{N}_i\otimes\boldsymbol{N}_i + H.O.T.\end{aligned} \tag{12-46}$$

where H.O.T. means higher order terms and, for this case, explicitly means quadratic terms involving $\varepsilon_i\theta$ that can be neglected with respect to the first-order terms. It follows from the expression for small strains in (12-42), and the use of (12-45) and (12-46) that

$$\boldsymbol{h} = \boldsymbol{\omega}^{sm} + \boldsymbol{e}^{sm} \tag{12-47}$$

We observe that the first term on the right side of (12-47) is skew symmetric and the second term is symmetric. Therefore, by equating symmetric and skew-symmetric parts, we obtain

$$\boldsymbol{e}^{sm} = \tfrac{1}{2}\left(\boldsymbol{h} + \boldsymbol{h}^T\right) \qquad \boldsymbol{\omega}^{sm} = \tfrac{1}{2}\left(\boldsymbol{h} - \boldsymbol{h}^T\right) \tag{12-48}$$

or in indicial form

$$e_{ij}^{sm} = \tfrac{1}{2}\left(u_{i,j} + u_{j,i}\right) \qquad \omega_{ij}^{sm} = \tfrac{1}{2}\left(u_{i,j} - u_{j,i}\right) \tag{12-49}$$

Now we provide a physical interpretation of the infinitesimal strain and rotation tensors in the x-y plane. Define two differentials of position in the x-y plane and the original configuration to be $d\boldsymbol{R}^{(1)} = dx\boldsymbol{e}_x$ and $d\boldsymbol{R}^{(2)} = dy\boldsymbol{e}_y$. In the deformed configuration, these elements become $d\boldsymbol{r}^{(1)}$ and $d\boldsymbol{r}^{(2)}$, respectively. From (12-45), the deformation gradient is

$$\boldsymbol{F}^{sm} = \left(1 + \frac{\partial u_x}{\partial x}\right)\boldsymbol{e}_x \otimes \boldsymbol{e}_x + \left(\frac{\partial u_x}{\partial y}\right)\boldsymbol{e}_x \otimes \boldsymbol{e}_y + \left(\frac{\partial u_y}{\partial x}\right)\boldsymbol{e}_y \otimes \boldsymbol{e}_x + \left(1 + \frac{\partial u_y}{\partial y}\right)\boldsymbol{e}_y \otimes \boldsymbol{e}_y \tag{12-50}$$

It follows that

$$\begin{aligned} d\boldsymbol{r}^{(1)} &= \boldsymbol{F}^{sm} \cdot d\boldsymbol{R}^{(1)} = \left(1 + \frac{\partial u_x}{\partial x}\right)\boldsymbol{e}_x dx + \left(\frac{\partial u_y}{\partial x}\right)\boldsymbol{e}_y dx \\ d\boldsymbol{r}^{(2)} &= \boldsymbol{F}^{sm} \cdot d\boldsymbol{R}^{(2)} = \left(\frac{\partial u_x}{\partial y}\right)\boldsymbol{e}_x dy + \left(1 + \frac{\partial u_y}{\partial y}\right)\boldsymbol{e}_y dy \end{aligned} \tag{12-51}$$

These differentials of position are displayed in the sketch of Fig. 12-3.

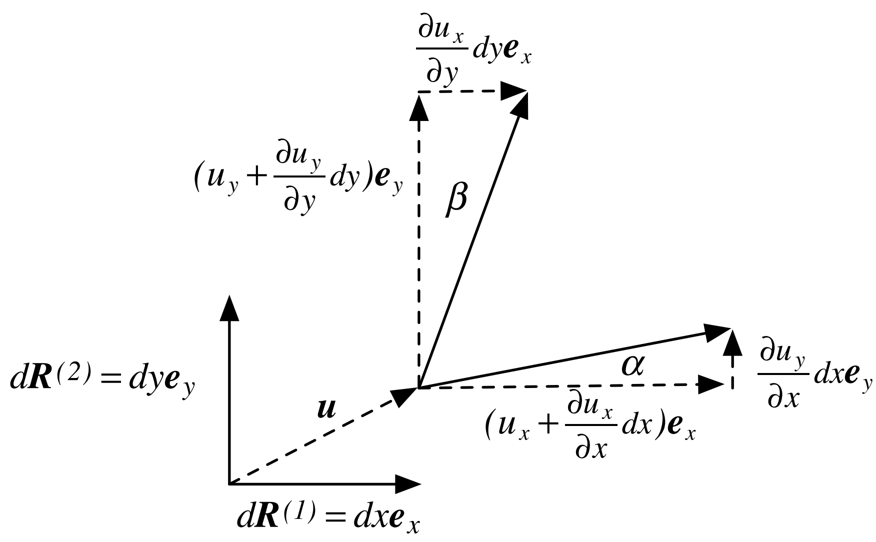

Fig. 12-3. Sketch of infinitesimal deformation in the x-y plane.

The square of the stretch in the $\boldsymbol{e}_x$-direction is

$$\Lambda_{st,x}^2 = \frac{d\boldsymbol{r}^{(1)} \cdot d\boldsymbol{r}^{(1)}}{dx^2} = \frac{\left(1 + \frac{\partial u_x}{\partial x}\right)^2 dx^2 + \left(\frac{\partial u_y}{\partial x}\right)^2 dx^2}{dx^2} = 1 + 2\frac{\partial u_x}{\partial x} + H.O.T. \quad (12\text{-}52)$$

where H.O.T. indicates higher-order terms that are considered negligible. The corresponding component of strain is $e_{xx} = (\Lambda_{st,x}^2 - 1)/2$ which leads us to the expressions for the two normal components of strain:

$$e_{xx} = \frac{\partial u_x}{\partial x} \qquad e_{yy} = \frac{\partial u_y}{\partial y} \quad (12\text{-}53)$$

where the expression for e_{yy} is found by analogy. The shear component of strain is

$$e_{xy} = \frac{1}{2}(\alpha + \beta) \qquad \alpha = \frac{\partial u_y}{\partial x} \qquad \beta = \frac{\partial u_x}{\partial y} \quad (12\text{-}54)$$

The equation for the shear strain requires a note of caution. Historically, the infinitesimal shear strain, now often called the engineering shear strain, was defined to be

$$\gamma_{xy} = \alpha + \beta \quad (12\text{-}55)$$

The term e_{xy} is the component of the infinitesimal strain tensor, but γ_{xy} is not. If γ_{xy} is used inadvertently in the transformation relations for components of the strain tensor, the error may be difficult to detect.

The infinitesimal rotation becomes

$$\omega_{xy}^{sm} = \frac{1}{2}(\alpha - \beta) \quad (12\text{-}56)$$

and the axis of rotation is $\boldsymbol{a} = \boldsymbol{e}_z$.

12.6 Summary

The Lagrangian strain is

$$\boldsymbol{E} = \frac{1}{2}\left[\boldsymbol{F}^T \cdot \boldsymbol{F} - \boldsymbol{I}\right] = \frac{1}{2}\left[\boldsymbol{U}^2 - \boldsymbol{I}\right] = \frac{1}{2}\left[\boldsymbol{C} - \boldsymbol{I}\right] \quad (12\text{-}57)$$

and the Eulerian strain is

$$\boldsymbol{e} = \frac{1}{2}\left[\boldsymbol{I} - \boldsymbol{F}^{-T} \cdot \boldsymbol{F}^{-1}\right] = \frac{1}{2}\left[\boldsymbol{I} - \boldsymbol{V}^{-2}\right] = \frac{1}{2}\left[\boldsymbol{I} - \boldsymbol{B}^{-1}\right] \quad (12\text{-}58)$$

The spectral decompositions of $\boldsymbol{U}$ and $\boldsymbol{V}$ are

$$\boldsymbol{U} = \sum_{i=1}^{3} \Lambda_i \boldsymbol{N}_i \otimes \boldsymbol{N}_i \qquad \boldsymbol{V} = \sum_{i=1}^{3} \Lambda_i \boldsymbol{n}_i \otimes \boldsymbol{n}_i \quad (12\text{-}59)$$

and the spectral decompositions of $\boldsymbol{E}$ and $\boldsymbol{e}$ are

$$\boldsymbol{E}=\frac{1}{2}\sum_{i=1}^{3}\left(\Lambda_i^2-1\right)\boldsymbol{N}_i\otimes\boldsymbol{N}_i \qquad \boldsymbol{e}=\frac{1}{2}\sum_{i=1}^{3}\left(1-\frac{1}{\Lambda_i^2}\right)\boldsymbol{n}_i\otimes\boldsymbol{n}_i \tag{12-60}$$

For arbitrary bases, and when the deformation is given by $\boldsymbol{r}=\boldsymbol{r}(\boldsymbol{R})$, the Lagrangian and Eulerian strain tensors are

$$\begin{aligned}&\boldsymbol{E}=E_{ij}\boldsymbol{E}_i\otimes\boldsymbol{E}_j \qquad E_{ij}=\frac{1}{2}\left[\overset{e-E}{F}_{ki}\overset{e-E}{F}_{kj}-\delta_{ij}\right]=\frac{1}{2}\left[\frac{\partial x_k}{\partial X_i}\frac{\partial x_k}{\partial X_j}-\delta_{ij}\right]\\&\boldsymbol{e}=e_{ij}\boldsymbol{e}_i\otimes\boldsymbol{e}_j \qquad \mathrm{e}_{ij}=\frac{1}{2}\left[\delta_{ij}-\overset{e-E}{F^{-1}}_{ki}\overset{e-E}{F^{-1}}_{kj}\right]=\frac{1}{2}\left[\delta_{ij}-\frac{\partial X_k}{\partial x_i}\frac{\partial X_k}{\partial x_j}\right]\end{aligned} \tag{12-61}$$

If the deformation is described through the use of a displacement vector $\boldsymbol{u}$, the strain tensors are given as functions of gradients of displacement. If $\boldsymbol{u}=\boldsymbol{u}(\boldsymbol{R})$, the Lagrangian strain is

$$\begin{aligned}&\boldsymbol{E}=\frac{1}{2}\left[\boldsymbol{H}+\boldsymbol{H}^T+\boldsymbol{H}^T\cdot\boldsymbol{H}\right] \qquad \boldsymbol{H}=\boldsymbol{u}\bar{\nabla}_0\\&\boldsymbol{E}=\frac{1}{2}\left[\frac{\partial u_i}{\partial X_j}+\frac{\partial u_j}{\partial X_i}+\frac{\partial u_k}{\partial X_i}\frac{\partial u_k}{\partial X_j}\right]\boldsymbol{E}_i\otimes\boldsymbol{E}_j\end{aligned} \tag{12-62}$$

Similarly, if the displacement is expressed as a function of the spatial position, $\boldsymbol{u}=\boldsymbol{u}(\boldsymbol{r})$, the Eulerian strain is

$$\begin{aligned}&\boldsymbol{e}=\frac{1}{2}\left[\boldsymbol{h}+\boldsymbol{h}^T-\boldsymbol{h}^T\cdot\boldsymbol{h}\right] \qquad \boldsymbol{h}=\boldsymbol{u}\bar{\nabla}\\&\boldsymbol{e}=\frac{1}{2}\left[\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}-\frac{\partial u_k}{\partial x_i}\frac{\partial u_k}{\partial x_j}\right]\boldsymbol{e}_i\otimes\boldsymbol{e}_j\end{aligned} \tag{12-63}$$

For infinitesimal deformations, we have the simplifications that

$$\begin{aligned}&\boldsymbol{H}=\boldsymbol{h} \qquad \boldsymbol{u}\bar{\nabla}_0=\boldsymbol{u}\bar{\nabla}\\&\boldsymbol{E}=\boldsymbol{e}=\frac{1}{2}\left[\boldsymbol{H}+\boldsymbol{H}^T\right]=\frac{1}{2}\left[\frac{\partial u_i}{\partial X_j}+\frac{\partial u_j}{\partial X_i}\right]\boldsymbol{E}_i\otimes\boldsymbol{E}_j=\frac{1}{2}\left[\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right]\boldsymbol{e}_i\otimes\boldsymbol{e}_j\end{aligned} \tag{12-64}$$

and the rotation becomes

$$\boldsymbol{R}^{sm}=\boldsymbol{I}+\boldsymbol{\omega}^{sm} \qquad \boldsymbol{\omega}^{sm}=\tfrac{1}{2}\left(u_{i,j}-u_{j,i}\right)\boldsymbol{e}_i\otimes\boldsymbol{e}_j \tag{12-65}$$

Next we move on to the situation where deformations are time dependent and another set of variables that must be defined.