

Exercise 1.1 One dimensional wave equation (transverse vibrations of an infinite string)

*One dimensional wave equation (transverse vibrations of an **infinite** string)*

Initial-value problem (IVP)

Find $u = u(x, t) = ?$ satisfying for $x \in (-\infty, +\infty)$ and $t \in [0, \infty)$:

$$\text{PDE: } u_{tt} = c^2 u_{xx}, \quad \text{ICs: } \begin{cases} u(x, 0) = f(x), \\ u_t(x, 0) = g(x). \end{cases}$$

(Here, c is the speed of wave propagation.)

Find the so-called *d'Alembert solution*. To this end:

- Replace x and t by new canonical coordinates $\xi = x + c$, $\eta = x - c$ to obtain a canonical form $u_{\xi\eta} = 0$.
- Solve the transformed equation (by two straightforward integrations).
- Transform back to the original coordinates to get a general solution.
- Substitute the general solution into the ICs.

Consider two cases of ICs:

1. $f(x) = 1$ for $x \in (-1, 1)$ and $f(x) = 0$ otherwise, and $g(x) \equiv 0$;
2. $f(x) \equiv 0$ (initial position at equilibrium) and $g(x) = \sin(x)$ (initial velocity as in a piano string).

Exercise 1.2 One dimensional wave equation (transverse vibrations of a finite string)

*One dimensional wave equation (transverse vibrations of a **finite** string)*

Initial-boundary-value problem (IBVP)

Find $u = u(x, t) = ?$ satisfying for $x \in [0, L]$ and $t \in [0, \infty)$:

$$\text{PDE: } u_{tt} = c^2 u_{xx}, \quad \text{BCs: } \begin{cases} u(0, t) = 0, \\ u(L, t) = 0, \end{cases} \quad \text{ICs: } \begin{cases} u(x, 0) = f(x), \\ u_t(x, 0) = g(x). \end{cases}$$

Hints and comments:

- The PDE as well as BCs are linear and homogeneous.
- The solution can be found by the standard technique of *separation of variables* as the infinite sum: $u(x, t) = \sum_{i=1}^{\infty} a_i X_i(x) T_i(t)$.
- $X_i(x)$ will be fundamental shapes of the so-called *standing waves*.

Consider two cases of ICs:

1. $f(x) = \sin(\pi x/L) + \frac{1}{2} \sin(3\pi x/L)$ and $g(x) \equiv 0$;
2. $f(x) \equiv 0$ and $g(x) = \sin(3\pi x/L)$.