14. Interchanging Time Derivatives with Gradients and Integrals

14.1 Initial Comments

Once time is included as a variable, interchanging time derivatives with gradients is often necessary, as is taking the time derivative inside an integral when the limits of integration depend on time. By transforming to the material configuration and then back to the spatial configuration, we show that the concept is quite simple. However, some algebraic details must be taken care of.

14.2 Interchanges in the Material or Reference Configuration

Let f denote a generic function that may be a scalar, a vector or a tensor, and let $\odot$ denote a generic operation such as nothing, a dot product or the curl operator. Because t and $\boldsymbol{R}$ are independent operators, the time derivative can be interchanged with the reference gradient operator to yield

$$\frac{d}{dt}\left[(f\odot)\bar{\bar{\nabla}}_0\right]=\left(\dot{f}\odot\right)\bar{\bar{\nabla}}_0 \tag{14-1}$$

The limits of integration for line, area and volume integrals in the reference configuration are independent of time because we are following material points that do not change in the reference configuration. Therefore for line, area and volume integrals we know that

$$\begin{aligned}
&\frac{d}{dt}\left[\int_{\partial R_L} f\odot d\boldsymbol{R}\right]=\int_{\partial R_L}\dot{f}\odot d\boldsymbol{R}\\
&\frac{d}{dt}\left[\int_{\partial R} f\odot d\boldsymbol{A}\right]=\int_{\partial R}\dot{f}\odot d\boldsymbol{A}\\
&\frac{d}{dt}\left[\int_{R} f\, dV\right]=\int_{R}\dot{f}\, dV
\end{aligned} \tag{14-2}$$

The operations that are automatic in the material configuration are not so simple for gradient and integral operators in the spatial configuration. We will make extensive use of the transformation between gradient operators and the rates of deformation summarized as follows:

$$\begin{aligned}
&(f\odot)\bar{\bar{\nabla}}=(f\odot)\bar{\bar{\nabla}}_0\cdot\boldsymbol{F}^{-1} && (f\odot)\bar{\bar{\nabla}}_0=(f\odot)\bar{\bar{\nabla}}\cdot\boldsymbol{F}\\
&\dot{\boldsymbol{F}}=\boldsymbol{L}\cdot\boldsymbol{F} && \dot{\boldsymbol{F}}^T=\boldsymbol{F}^T\cdot\boldsymbol{L}^T\\
&\dot{\boldsymbol{F}}^{-1}=-\boldsymbol{F}^{-1}\cdot\boldsymbol{L} && \dot{\boldsymbol{F}}^{-T}=-\boldsymbol{L}^T\cdot\boldsymbol{F}^{-T}\\
&d\boldsymbol{A}_0=\frac{1}{J}\boldsymbol{F}^T\cdot d\boldsymbol{a} && d\boldsymbol{a}=J\boldsymbol{F}^{-T}\cdot d\boldsymbol{A}_0\\
&dV=JdV_0 && \dot{J}=J\left(tr(\boldsymbol{L})\right)
\end{aligned} \tag{14-3}$$

14.3 Interchange of Time Derivative and the Spatial Gradient

Suppose we want to take the time derivative of $(f \odot)\bar{\nabla}$ and express it in terms of $(\dot{f} \odot)\bar{\nabla}$. We require the relationships for the transformation of gradients and use the derivative of $\boldsymbol{F}^{-1}$ to obtain

$$\begin{aligned} \frac{d}{dt}\left[(f \odot)\bar{\nabla}\right] &= \frac{d}{dt}\left[(f \odot)\bar{\nabla}_0 \cdot \boldsymbol{F}^{-1}\right] \\ &= \left[(\dot{f} \odot)\bar{\nabla}_0 \cdot \boldsymbol{F}^{-1} + (f \odot)\bar{\nabla}_0 \cdot \dot{\boldsymbol{F}}^{-1}\right] \\ &= \left[(\dot{f} \odot)\bar{\nabla}_0 \cdot \boldsymbol{F}^{-1} - (f \odot)\bar{\nabla}_0 \cdot \boldsymbol{F}^{-1} \cdot \boldsymbol{L}\right] \end{aligned} \tag{14-4}$$

Now we transform the gradient back to the spatial form to get our final form

$$\frac{d}{dt}\left[(f \odot)\bar{\nabla}\right] == \left[(\dot{f} \odot)\bar{\nabla} - (f \odot)\bar{\nabla} \cdot \boldsymbol{L}\right] \tag{14-5}$$

However, note that implicitly $f = f[\boldsymbol{r}(t), t]$ so the appropriate form of the time derivative must be used.

14.4 Interchange of Time Derivative and Integrals

<u>Line Integral</u>

We follow a similar procedure with each category of spatial integral. First consider the time derivative of a line integral

$$\begin{aligned} \frac{d}{dt}\left[\int_{\partial R_L} f \odot d\boldsymbol{r}\right] &= \frac{d}{dt}\left[\int_{\partial R_{0L}} \left(f \odot \boldsymbol{F}^{-1}\right) \cdot d\boldsymbol{R}\right] \\ &= \int_{\partial R_{0L}} \left(\dot{f} \odot \boldsymbol{F}^{-1} + f \odot \dot{\boldsymbol{F}}^{-1}\right) \cdot d\boldsymbol{R} \\ &= \int_{\partial R_{0L}} \left(\dot{f} \odot \boldsymbol{F}^{-1} - f \odot \boldsymbol{L} \cdot \boldsymbol{F}^{-1}\right) \cdot d\boldsymbol{R} \end{aligned} \tag{14-6}$$

We transform back to get

$$\frac{d}{dt}\left[\int_{\partial R_L} f \odot d\boldsymbol{r}\right] = \int_{\partial R_L} \left(\dot{f} \odot \boldsymbol{I} - f \odot \boldsymbol{L}\right) \cdot d\boldsymbol{r} \tag{14-7}$$

Area Integral

For the area integral we use Nanson's relation:

$$\frac{d}{dt}\left[\int_{\partial R} f \odot \boldsymbol{n}\, da\right] = \frac{d}{dt}\left[\int_{\partial R_o} f \odot \boldsymbol{J}\boldsymbol{F}^{-T} \cdot \boldsymbol{N}\, dA_0\right]$$
$$= \int_{\partial R_0} \left(\dot{f} \odot \boldsymbol{J}\boldsymbol{F}^{-T} + f \odot \dot{\boldsymbol{J}}\boldsymbol{F}^{-T} + f \odot \boldsymbol{J}\dot{\boldsymbol{F}}^{-T}\right) \cdot \boldsymbol{N}\, dA_0$$

Use expressions for the derivatives of $\boldsymbol{J}$ and $\boldsymbol{F}^{-T}$ in the integrand:

$$\frac{d}{dt}\left[\int_{\partial R} f \odot \boldsymbol{n}\, da\right] = \int_{\partial R_0} \left(\dot{f} \odot \boldsymbol{J}\boldsymbol{F}^{-T} + f \odot J(\boldsymbol{v} \cdot \bar{\nabla})\boldsymbol{F}^{-T} - f \odot J\boldsymbol{L}^T \cdot \boldsymbol{F}^{-T}\right) \cdot \boldsymbol{N}\, dA_0 \qquad (14\text{-}8)$$

Now transform back to the current configuration to obtain the final result

$$\frac{d}{dt}\left[\int_{\partial R} f \odot \boldsymbol{n}\, da\right] = \int_{\partial R_0} \left(\dot{f} \odot \boldsymbol{I} + f \odot (\boldsymbol{v} \cdot \bar{\nabla})\boldsymbol{I} - f \odot \boldsymbol{L}^T\right) \cdot \boldsymbol{n}\, da \qquad (14\text{-}9)$$

Volume Integral

Similarly, for the volume integral

$$\frac{d}{dt}\left[\int_R f\, dV\right] = \frac{d}{dt}\left[\int_{R_0} fJ\, dV_0\right] = \int_{R_0} \left(\dot{f}J + f\dot{J}\right) dV_0 = \int_{R_0} \left(\dot{f}J + fJtr(\boldsymbol{L})\right) dV_0 \qquad (14\text{-}10)$$

Transform back to obtain

$$\frac{d}{dt}\left[\int_R f\, dV\right] = \int_R \left(\dot{f} + f\, tr(\boldsymbol{L})\right) dV \qquad (14\text{-}11)$$

14.5 Reynolds Transport Theorem

Three-Dimensional Form

As an application of these equations we prove a special case of Reynolds Transport Theorem where the velocity is restricted to being the velocity of material points, the form used throughout these notes. The theorem states that

$$\frac{d}{dt}\int_R f\, dV = \int_R \frac{\partial f}{\partial t} dV + \int_{\partial R} f\left(\boldsymbol{v} \cdot \boldsymbol{n}\right) da \qquad (14\text{-}12)$$

To prove this theorem we start with (14-11) and rewrite the right side as

$$\int_R \left(\dot{f} + f\, tr(\boldsymbol{L})\right) dV = \int_R \left(\frac{\partial f}{\partial t} + f\bar{\bar{\nabla}} \cdot \boldsymbol{v} + f\left(\boldsymbol{v} \cdot \bar{\bar{\nabla}}\right)\right) dV$$
$$= \int_R \left(\frac{\partial f}{\partial t} + (f\boldsymbol{v}) \cdot \bar{\bar{\nabla}}\right) dV \qquad (14\text{-}13)$$

Next we apply the divergence theorem to the second term on the right side to obtain

$$\int_R (f\boldsymbol{v}) \cdot \bar{\bar{\nabla}}\, dV = \int_{\partial R} (f\boldsymbol{v}) \cdot \boldsymbol{n}\, da \qquad (14\text{-}14)$$

The combination of (14-11), (14-13) and (14-14) yields (14-12).

<u>One-Dimensional Leibniz Integral Rule</u>

As an application of (14-12) consider the special case of $x_1 = x, x_2 = y, x_3 = z$, and $f = f(x,t)$ with the domain defined by $0 \le y \le 1, 0 \le z \le 1$, and $\mathrm{a(t)} \le \mathrm{x} \le \mathrm{b(t)}$. The left side of (14-12) becomes

$$\frac{d}{dt}\int_R f\, dV = \frac{d}{dt}\int_{a(t)}^{b(t)} f(x,t)\, dx \qquad (14\text{-}15)$$

while the right side reduces to

$$\int_R \frac{\partial f}{\partial t} dV + \int_{\partial R} f\left(\boldsymbol{v} \cdot \boldsymbol{n}\right) da = \int_{a(t)}^{b(t)} \frac{\partial f(x,t)}{\partial t} dx + \left[f(x,t)\left(\boldsymbol{v} \cdot \boldsymbol{e}_x\right)\right]\Big|_{x=b} + \left[f(x,t)\left(\boldsymbol{v} \cdot -\boldsymbol{e}_x\right)\right]\Big|_{x=a} \qquad (14\text{-}16)$$

But

$$\left(\boldsymbol{v} \cdot \boldsymbol{e}_x\right)_{x=b} = \dot{b} \qquad \left(\boldsymbol{v} \cdot -\boldsymbol{e}_x\right)_{x=a} = \dot{a} \qquad (14\text{-}17)$$

The final result is

$$\frac{d}{dt}\int_{a(t)}^{b(t)} f(x,t)\, dx = \int_{a(t)}^{b(t)} \frac{\partial f(x,t)}{\partial t} dx + f\left\{b(t),t\right\}\frac{\partial b}{\partial t} - f\left\{a(t),t\right\}\frac{\partial a}{\partial t} \qquad (14\text{-}18)$$

which is the one-dimensional Leibniz Integral Rule for differentiation under the integral sign.

14.5 Summary

The key relations derived in this chapter are:

$$\begin{aligned}
&\frac{d}{dt}\big[(f\odot)\bar{\nabla}\big] = \big[(\dot{f}\odot)\bar{\nabla} - (f\odot)\bar{\nabla}\cdot \boldsymbol{L}\big] \\
&\frac{d}{dt}\left[\int_{\partial R_L} f\odot d\boldsymbol{r}\right] = \int_{\partial R_L}\big(\dot{f}\odot \boldsymbol{I} - f\odot \boldsymbol{L}\big)\cdot d\boldsymbol{r} \\
&\frac{d}{dt}\left[\int_{\partial R} f\odot \boldsymbol{n}\, da\right] = \int_{\partial R_0}\big(\dot{f}\odot \boldsymbol{I} + f\odot(\boldsymbol{v}\cdot\bar{\nabla})\boldsymbol{I} - f\odot \boldsymbol{L}^T\big)\cdot \boldsymbol{n}\, da \\
&\frac{d}{dt}\left[\int_R f\, dV\right] = \int_R\big(\dot{f} + f\, tr(\boldsymbol{L})\big)dV
\end{aligned} \tag{14-19}$$

An application, when combined with divergence theorem is Reynolds' Transport Theorem of which one example is Leibniz' one-dimensional integral rule:

$$\begin{aligned}
&\frac{d}{dt}\int_R f\, dV = \int_R \frac{\partial f}{\partial t} dV + \int_{\partial R} f\big(\boldsymbol{v}\cdot\boldsymbol{n}\big)\, da \\
&\frac{d}{dt}\int_{a(t)}^{b(t)} f(x,t)\,dx = \int_{a(t)}^{b(t)} \frac{\partial f(x,t)}{\partial t} dx + f\{b(t),t\}\frac{\partial b}{\partial t} - f\{a(t),t\}\frac{\partial a}{\partial t}
\end{aligned} \tag{14-20}$$