CE598 Assignment #5
Do Problems 5.11 – 5.15
Due October 1, 2015

5.11 A linear-elastic uniaxial bar is stretched axially. The Young's modulus E and cross-sectional area A are constant with respect to the axial coordinate X. Assuming a material space $X \in \mathcal{R}^0 \in \mathbb{R}^1$, determine the pairwise force function f in terms of E, A, and the peridynamic horizon δ. Assume $f = k\frac{|\xi+\eta|-|\xi|}{|\xi|}$, where k is a constant, to be expressed in terms of E, A, and L.

Solution: We require the classical strain energy per unit volume to be the same as the peridynamic strain energy per unit volume.

$$\mathfrak{U}_{classical} = \frac{1}{2}\sigma\epsilon = \frac{E}{2}\epsilon^2$$

and assuming constant strain ϵ, $\eta = \epsilon\xi$ and $\epsilon = \frac{|\xi+\eta|-|\xi|}{|\xi|}$so

$$\mathfrak{U}_{peridynamic} = \int_{\xi'=-\delta}^{\xi'=\delta}\left(\frac{1}{2}\int_{\epsilon^*=0}^{\epsilon} f \times (\xi' d\epsilon^*)\right)Ad\xi' = \frac{A}{2}\int_{\xi'=-\delta}^{\xi'=\delta}\left(\int_{\epsilon^*=0}^{\epsilon} k\xi'\epsilon^* d\epsilon^*\right)d\xi'$$

(Note that the strain energy in each bond has been divided by two because only half of each bond's energy is associated with each of the interacting particles.)

$$\mathfrak{U}_{peridynamic} = \frac{A}{2}2\int_{\xi'=0}^{\xi'=\delta}\left(\int_{\epsilon^*=0}^{\epsilon} k\xi'\epsilon^* d\epsilon^*\right)d\xi' = A\int_{\xi'=0}^{\xi'=\delta}\left(k\xi'\frac{\epsilon^2}{2}\right)d\xi' = \frac{Ak\epsilon^2}{2}\int_{\xi'=0}^{\xi'=\delta}\xi' d\xi' = \frac{Ak\delta^2\epsilon^2}{4}.$$

Equating the classical and peridynamic strain energy densities,

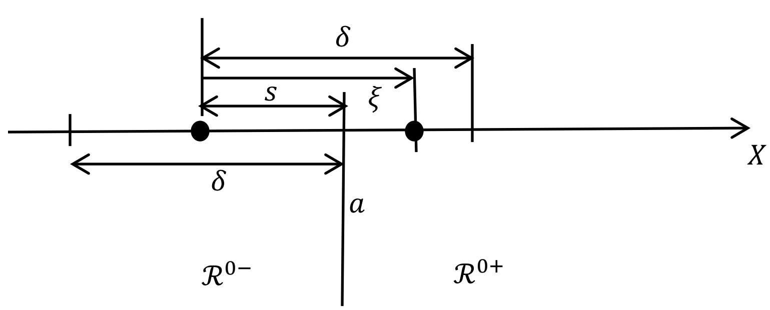
$$\frac{Ak\delta^2\epsilon^2}{4} = \frac{E}{2}\epsilon^2,$$

and if the strain is nonzero,

$$Ak\delta^2 = 2E \text{ or } k = \frac{2E}{A\delta^2}.$$

$$\text{So} = \frac{2E}{A\delta^2}\left(\frac{|\xi+\eta|-|\xi|}{|\xi|}\right).$$

Alternately, assuming constant strain ϵ, we can partition $\mathcal{R}^0$ at cross-section a into two subdomains $\mathcal{R}^{0-}$ and $\mathcal{R}^{0+}$ and then integrate the peridynamic forces acting between all pairs of particles within $\mathcal{R}^{0-}$ and $\mathcal{R}^{0+}$, and equate this force to $A\sigma$:



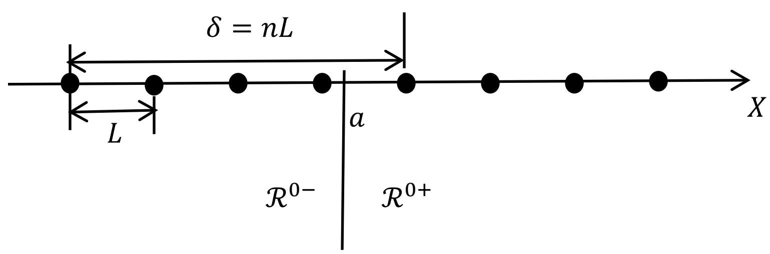
$$F = \int_{s=-\delta}^{0}\int_{\xi=s}^{\delta} f(Ad\xi)(Ads) = A^2\int_{s=-\delta}^{0}\int_{\xi=s}^{\delta}(k\epsilon)d\xi ds\,;$$

$$F = A^2 k\epsilon \int_{s=-\delta}^{0} (\delta - s)\, ds = A^2 k\epsilon \left[\delta s - \frac{s^2}{2}\right]_{s=-\delta}^{0} = A^2 k\epsilon \frac{\delta^2}{2};$$

$$\text{So } \sigma = \frac{F}{A} = Ak\epsilon\frac{\delta^2}{2}; E = \frac{\sigma}{\epsilon} = Ak\frac{\delta^2}{2}; k = \frac{2E}{A\delta^2}.$$

5.12 A linear-elastic uniaxial bar is stretched axially. The Young's modulus E and cross-sectional area A are constant with respect to the axial coordinate X. Assuming a material lattice $X \subset \mathcal{R}^0 \subset \mathbb{Z}^1$, with initial reference lattice spacing L, determine the pairwise force function, f, in terms of E, A, and L, if (a) the peridynamic horizon is $\delta = L$; (b) $\delta = 2L$; (c) $\delta = 3L$; (d) $\delta = nL$. In all cases, assume that $f = k\frac{|\xi+\eta|-|\xi|}{|\xi|}$. k is a constant, to be expressed in terms of E, A, and L.

Solution:
Assume a constant strain ϵ, and partition $\mathcal{R}^0$ at cross-section a into two subdomains $\mathcal{R}^{0-}$ and $\mathcal{R}^{0+}$ and then sum the peridynamic forces acting between all pairs of particles within $\mathcal{R}^{0-}$ and $\mathcal{R}^{0+}$, and equate this force to $A\sigma$:



Volume per particle is $\Delta V = AL = A\left(\frac{\delta}{n}\right)$.
The number of bonds crossing any particular cross section a is:
Leftmost particle in $\mathcal{R}^{0-}$ with bonds in $\mathcal{R}^{0-}$: 1 bond
Next to leftmost particle in $\mathcal{R}^{0-}$ with bonds in $\mathcal{R}^{0-}$: 2 bonds

$$\vdots$$

Rightmost particle in $\mathcal{R}^{0-}$ with bonds in $\mathcal{R}^{0-}$: n bonds
So the number of bonds crossing section a is $N_b = 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$

$$F = fA\left(\frac{\delta}{n}\right)A\left(\frac{\delta}{n}\right)\frac{n(n+1)}{2} = \sigma A;$$
$$k\epsilon\left(\frac{\delta}{n}\right)^2 A\frac{n(n+1)}{2} = \sigma;$$
$$k\left(\frac{\delta}{n}\right)^2 A\frac{n(n+1)}{2} = \frac{\sigma}{\epsilon} = E;$$
$$k = \frac{2En^2}{A\delta^2 n(n+1)} = \frac{2En}{A\delta^2(n+1)};$$

So the answers to the questions are:

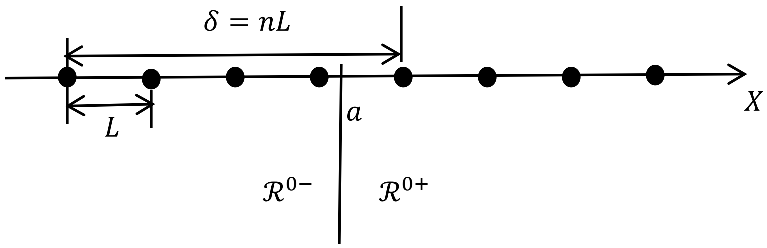
(a) $\delta = L$: $k = \frac{2E1}{A\delta^2(1+1)} = \frac{E}{A\delta^2}$;
(b) $\delta = 2L$: $k = \frac{2E2}{A\delta^2(1+2)} = \frac{4E}{3A\delta^2}$;
(c) $\delta = 3L$: $k = \frac{2E3}{A\delta^2(1+3)} = \frac{3E}{2A\delta^2}$;
(d) $\delta = nL$: $k = \frac{2E\infty}{A\delta^2(1+\infty)} = \frac{2E}{A\delta^2}$ (Which matches the continuum peridynamics solution.)

5.13 Repeat Problem 5.12, this time assuming that pairwise force function f varies as $f = k\frac{|\xi+\eta|-|\xi|}{|\xi|}\left(1 - \left|\frac{\xi}{\delta}\right|\right)\ \forall\ \xi \le \delta$, and $f = 0\ \forall\ \xi > \delta$.

Solution:

So the pairwise force function is $f = k\epsilon\left(1 - \left|\frac{\xi}{\delta}\right|\right)$, where ϵ is the strain, assumed constant with respect to ξ.

Partition $\mathcal{R}^0$ at cross-section a into two subdomains $\mathcal{R}^{0-}$ and $\mathcal{R}^{0+}$ and then sum the peridynamic forces acting between all pairs of particles within $\mathcal{R}^{0-}$ and $\mathcal{R}^{0+}$, and equate this force to $A\sigma$:



Volume per particle is $\Delta V = AL = A\left(\frac{\delta}{n}\right)$.

The number of bonds crossing any particular cross section a is:

1st (leftmost) particle in $\mathcal{R}^{0-}$ with bonds in $\mathcal{R}^{0-}$: 1 bond $\xi = \delta = nL$; $f_1 = k\epsilon\left(1 - \left|\frac{\delta}{\delta}\right|\right) = k\epsilon 0$

2nd particle in $\mathcal{R}^{0-}$ with bonds in $\mathcal{R}^{0-}$: 2 bonds $\xi = \delta$ and $\xi = \delta - L$; $f_2 = k\epsilon\left(1 - \left|\frac{\delta-L}{\delta}\right|\right) = k\epsilon\frac{L}{\delta}$

$\vdots$

ith particle in $\mathcal{R}^{0-}$ with bonds in $\mathcal{R}^{0-}$: i bonds with $f_i = k\epsilon\frac{(0+1+\cdots+(i-1))L}{\delta}$

Adding up the forces on each particle in $\mathcal{R}^{0-}$ (where i represents the i th term in the sum):

$$F = \left(A\frac{\delta}{n}\right)^2\left[k\epsilon\frac{(0)L}{\delta} + k\epsilon\frac{(0+1)L}{\delta} + k\epsilon\frac{(0+1+2)L}{\delta} + \cdots + k\epsilon\frac{(0+1+\cdots+(i-1))L}{\delta} + \cdots + k\epsilon\frac{(0+1+\cdots+(n-1))L}{\delta}\right];$$

$$F = \left(A\frac{\delta}{n}\right)^2\sum_{i=1}^{n}\left[k\epsilon\frac{\sum_{j=0}^{i-1}(jL)}{\delta}\right] = \left(A\frac{\delta}{n}\right)^2\frac{k\epsilon L}{\delta}\sum_{i=1}^{n}\left[\sum_{j=0}^{i-1}(j)\right] = \left(A\frac{\delta}{n}\right)^2\frac{k\epsilon L}{\delta}\sum_{i=1}^{n}\left[\frac{(i-1)i}{2}\right] = \sigma A;$$

$$\left(A\frac{\delta}{n}\right)^2\frac{kL}{\delta}\sum_{i=1}^{n}\left[\frac{(i-1)i}{2}\right] = \frac{\sigma}{\epsilon}A = EA;\ k = \frac{En^2}{AL\delta\sum_{i=1}^{n}\left[\frac{(i-1)i}{2}\right]} = \frac{En^2}{\frac{AL\delta}{2}\left(\sum_{i=1}^{n} i^2 - \sum_{i=1}^{n} i\right)}.$$

Note that $\sum_{i=1}^{n} i^2 = \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}$ and $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$, so

$$k = \frac{2En^2}{AL\delta\left(\frac{n^3}{3}+\frac{n^2}{2}+\frac{n}{6}-\frac{n(n+1)}{2}\right)} = \frac{2En^2}{AL\delta\left(\frac{n^3}{3}-\frac{n}{3}\right)} = \frac{6En^2}{AL\delta(n^3-n)} = \frac{6En^3}{A\delta^2(n^3-n)}.$$

So the answers to the parts of the problem are:

(a) $\delta = 1L$: $k = \frac{6En^3}{A\delta^2(n^3-n)} = \infty$;

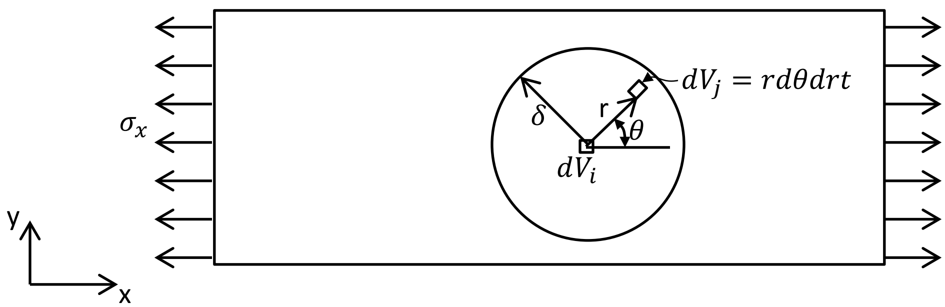
(b) $\delta = 2L$: $k = \frac{6En^3}{A\delta^2(n^3-n)} = \frac{6E2^3}{A\delta^2(2^3-2)} = \frac{8En^2}{A\delta^2}$;

(c) $\delta = 3L$: $k = \frac{6En^3}{A\delta^2(n^3-n)} = \frac{6E3^3}{A\delta^2(3^3-3)} = \frac{162E}{24\delta^2\delta} = \frac{27E}{4A\delta^2}$;

(d) $\delta = nL$: $k = \frac{6En^3}{A\delta^2(n^3-n)}$. (As $n \to \infty$, $k \to \frac{6E}{A\delta^2}$.)

5.14 In a planar ordinary continuum bond-based peridynamic material body with thickness t, the magnitude of the pairwise force function is $f = kt\frac{\|\boldsymbol{\xi}+\boldsymbol{\eta}\|-\|\boldsymbol{\xi}\|}{\|\boldsymbol{\xi}\|} = ktS$, where S is the bond stretch, for all bonds within the peridynamic horizon δ. Assuming homogeneous deformation, determine the Young's modulus E and Poisson's ratio ν of the equivalent linear elastic classical material, assuming (a) plane stress conditions; (b) plane strain conditions.

Solution: Let's consider a plane stress bar in uniaxial tension:



The strategy is to make the classical strain energy density $U_{classical}$ the same as the peridynamic strain energy density $U_{peridynamic}$. As the shear strain $\gamma_{xy} = 0$, we have $\begin{Bmatrix}\sigma_{xx}\\ \sigma_{yy}\end{Bmatrix} = [D]\begin{Bmatrix}\epsilon_{xx}\\ \epsilon_{yy}\end{Bmatrix}$ where $[D] = \frac{E}{1-\nu^2}\begin{bmatrix}1 & \nu\\ \nu & 1\end{bmatrix}$ for plane stress, and $[D] = \frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix}(1-\nu) & \nu\\ \nu & (1-\nu)\end{bmatrix}$ for plane strain.

The classical strain energy density is $U_{classical} = \frac{1}{2}\lfloor\sigma_{xx} \quad \sigma_{yy}\rfloor\begin{Bmatrix}\epsilon_{xx}\\ \epsilon_{yy}\end{Bmatrix} = \frac{1}{2}\lfloor\epsilon_{xx} \quad \epsilon_{yy}\rfloor[D]\begin{Bmatrix}\epsilon_{xx}\\ \epsilon_{yy}\end{Bmatrix}$.

The strain energy stored in a single peridynamic bond of length r is $du_{bond} = \frac{(Sr)f}{2}\,dV_i dV_j =$, with one-half of this strain energy associated with each of the interacting particles, so if $f = ktS$, $du_{bond} = \frac{1}{2}\left[\frac{(Sr)f}{2}\,dV_i dV_j\right] = \frac{1}{4}\left[(Sr)ktS dV_i dV_j\right]$. But

$S = \lfloor cos^2\theta \quad sin^2\theta\rfloor\begin{Bmatrix}\epsilon_{xx}\\ \epsilon_{yy}\end{Bmatrix}$, so $du_{bond} = \frac{rkt}{4}\left[\lfloor\epsilon_{xx} \quad \epsilon_{yy}\rfloor\begin{Bmatrix}cos^2\theta\\ sin^2\theta\end{Bmatrix}\lfloor cos^2\theta \quad sin^2\theta\rfloor\begin{Bmatrix}\epsilon_{xx}\\ \epsilon_{yy}\end{Bmatrix}\right]dV_i dV_j$, or

$du_{bond} = \frac{rkt}{4}\lfloor\epsilon_{xx} \quad \epsilon_{yy}\rfloor\begin{bmatrix}cos^4\theta & cos^2\theta sin^2\theta\\ cos^2\theta sin^2\theta & sin^4\theta\end{bmatrix}\begin{Bmatrix}\epsilon_{xx}\\ \epsilon_{yy}\end{Bmatrix}dV_i dV_j$. We integrate bond energies:

$$U_{peridynamic} =$$

$$= \frac{1}{dV_i}\int_{\mathcal{H}} du_{bond} = \frac{1}{dV_i}\int_{r=0}^{\delta}\int_{\theta=-\pi}^{\pi}\frac{rkt}{4}\lfloor\epsilon_{xx} \quad \epsilon_{yy}\rfloor\begin{bmatrix}cos^4\theta & cos^2\theta sin^2\theta\\ cos^2\theta sin^2\theta & sin^4\theta\end{bmatrix}\begin{Bmatrix}\epsilon_{xx}\\ \epsilon_{yy}\end{Bmatrix}rd\theta dr dV_i$$

$U_{peridynamic} = \frac{kt}{4}\lfloor\epsilon_{xx} \quad \epsilon_{yy}\rfloor\int_{r=0}^{\delta}\int_{\theta=-\pi}^{\pi}\begin{bmatrix}cos^4\theta & cos^2\theta sin^2\theta\\ cos^2\theta sin^2\theta & sin^4\theta\end{bmatrix}r^2 d\theta dr\begin{Bmatrix}\epsilon_{xx}\\ \epsilon_{yy}\end{Bmatrix}$, so

$$U_{peridynamic} = \lfloor\epsilon_{xx} \quad \epsilon_{yy}\rfloor\frac{\pi kt\delta^3}{48}\begin{bmatrix}3 & 1\\ 1 & 3\end{bmatrix}\begin{Bmatrix}\epsilon_{xx}\\ \epsilon_{yy}\end{Bmatrix}.$$

For plane stress conditions,

$$U_{classical} = \frac{1}{2}\lfloor\epsilon_{xx} \quad \epsilon_{yy}\rfloor \frac{E}{1-\nu^2}\begin{bmatrix}1 & \nu \\ \nu & 1\end{bmatrix}\begin{Bmatrix}\epsilon_{xx} \\ \epsilon_{yy}\end{Bmatrix} = U_{peridynamic} = \lfloor\epsilon_{xx} \quad \epsilon_{yy}\rfloor \frac{\pi k t \delta^3}{48}\begin{bmatrix}3 & 1 \\ 1 & 3\end{bmatrix}\begin{Bmatrix}\epsilon_{xx} \\ \epsilon_{yy}\end{Bmatrix}$$

and for arbitrary strains, we require that $\frac{1}{2}\frac{E}{1-\nu^2}\begin{bmatrix}1 & \nu \\ \nu & 1\end{bmatrix} = \frac{\pi k t \delta^3}{48}\begin{bmatrix}3 & 1 \\ 1 & 3\end{bmatrix}$.

This is four equations (two independent equtions): $\frac{48E}{2\pi k t \delta^3(1-\nu^2)}\begin{bmatrix}1 & \nu \\ \nu & 1\end{bmatrix} = \begin{bmatrix}3 & 1 \\ 1 & 3\end{bmatrix}$. So $\frac{48E}{2\pi k t \delta^3(1-\nu^2)} = 3$ and $\frac{48E\nu}{2\pi k t \delta^3(1-\nu^2)} = 1$; therefore $\nu = \frac{1}{3}$, and $k = \frac{9E}{\pi t \delta^3}$.

For plane strain conditions, and for arbitrary strains, we require that $\frac{1}{2}\frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix}(1-\nu) & \nu \\ \nu & (1-\nu)\end{bmatrix} = \frac{\pi k t \delta^3}{48}\begin{bmatrix}3 & 1 \\ 1 & 3\end{bmatrix}$ or

$\frac{24E}{\pi k t \delta^3(1+\nu)(1-2\nu)}\begin{bmatrix}(1-\nu) & \nu \\ \nu & (1-\nu)\end{bmatrix} = \begin{bmatrix}3 & 1 \\ 1 & 3\end{bmatrix}$, so $\frac{24E(1-\nu)}{\pi k t \delta^3(1+\nu)(1-2\nu)} = 3$; $\frac{24E\nu}{\pi k t \delta^3(1+\nu)(1-2\nu)} = 1$

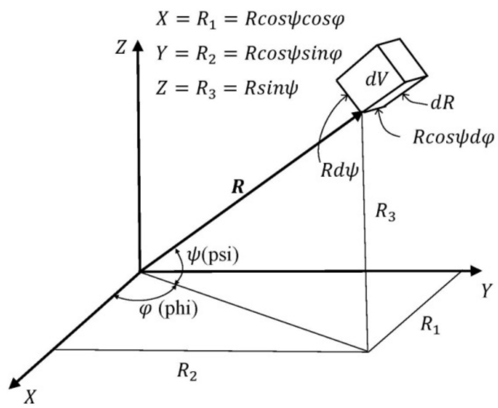
Thus $1-\nu = 3\,\nu$ and $\nu = \frac{1}{4}$. Finally, $= \frac{24E\nu}{\pi t \delta^3(1+\nu)(1-2\nu)} = \frac{24E\frac{1}{4}}{\pi t \delta^3\left(1+\frac{1}{4}\right)\left(1-2\cdot\frac{1}{4}\right)} = \frac{6E}{\pi t \delta^3\left(\frac{5}{4}\right)\left(\frac{1}{2}\right)} = \frac{48E}{5\pi t \delta^3}$.

In conclusion, for plane stress, $= \frac{1}{3}$, and $k = \frac{9E}{\pi t \delta^3}$,

and for plane strain, $\nu = \frac{1}{4}$ and $k = \frac{48E}{5\pi t \delta^3}$.

5.15 In a three-dimensional ordinary continuum bond-based peridynamic material body, the magnitude of the pairwise force function is $f = k\frac{\|\xi+\boldsymbol{\eta}\|-\|\xi\|}{\|\xi\|} = kS$, where S is the bond stretch for all bonds within the peridynamic horizon δ. Assuming homogeneous deformation, determine the Young's modulus E and Poisson's ratio ν of the equivalent linear elastic classical material.

Solution:



Cartesian and spherical coordinate systems.

$$[D_{6\times6}] = \frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix} (1-\nu) & \nu & 0 & \nu & 0 & 0 \\ \nu & (1-\nu) & 0 & \nu & 0 & 0 \\ 0 & 0 & \frac{(1-2\nu)}{2} & 0 & 0 & 0 \\ \nu & \nu & 0 & (1-\nu) & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{(1-2\nu)}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{(1-2\nu)}{2} \end{bmatrix}. \qquad (9.1)$$

and omitting shear stresses and strains $\gamma_{XY}, \gamma_{YZ}, \gamma_{XZ}$ (assuming the XYZ axes are principal axes):

$$\begin{Bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{zz}\end{Bmatrix} = \frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix}(1-\nu) & \nu & \nu \\ \nu & (1-\nu) & \nu \\ \nu & \nu & (1-\nu)\end{bmatrix}\begin{Bmatrix}\epsilon_{xx}\\ \epsilon_{yy}\\ \epsilon_{zz}\end{Bmatrix}, \text{ or } \{\sigma\} = [D_{3\times3}]\{\epsilon\}$$

The classical strain energy density is $U_{classical} = \frac{1}{2}\lfloor\epsilon\rfloor[D]\{\epsilon\}$.

For a given bond, $= {n_X}^2\varepsilon_{XX} + {n_Y}^2\varepsilon_{YY} + {n_Z}^2\varepsilon_{ZZ} + n_X n_Y\gamma_{XY} + n_Y n_Z\gamma_{YZ} + n_Z n_X\gamma_{XZ}$,

where $n_X = cos(\varphi)cos(\psi); n_Y = sin(\varphi)cos(\psi)\ ; n_Z = sin(\psi)$.

In spherical coordinates, $S = cos^2(\varphi)cos^2(\psi)(\varphi)\varepsilon_{XX} + sin^2(\varphi)cos^2(\psi)\varepsilon_{YY} + sin^2(\psi)\varepsilon_{ZZ}$, or

$$S = \lfloor cos^2(\varphi) \quad sin^2(\varphi) \quad sin^2(\psi)\rfloor\begin{Bmatrix}\varepsilon_{XX}\\ \varepsilon_{YY}\\ \varepsilon_{ZZ}\end{Bmatrix}.$$

$$du_{bond} = \frac{1}{2}\left(\frac{fSr}{2}\right) = \frac{kS^2r}{4}.$$

$$du_{bond} = \frac{rk}{4}\lfloor\epsilon_{xx} \quad \epsilon_{yy} \quad \epsilon_{zz}\rfloor\left[\begin{Bmatrix}cos^2(\varphi)cos^2(\psi)\\ sin^2(\varphi)cos^2(\psi)\\ sin^2(\psi)\end{Bmatrix}\lfloor cos^2(\varphi)cos^2(\psi) \quad sin^2(\varphi)cos^2(\psi) \quad sin^2(\psi)\rfloor\right]\begin{Bmatrix}\epsilon_{xx}\\ \epsilon_{yy}\\ \epsilon_{zz}\end{Bmatrix}dV_i dV_j$$

$$du_{bond} = \frac{rk}{4}\lfloor\epsilon_{xx} \quad \epsilon_{yy} \quad \epsilon_{zz}\rfloor\begin{bmatrix}cos^2(\varphi)cos^2(\psi)cos^2(\varphi)cos^2(\psi) & cos^2(\varphi)cos^2(\psi)sin^2(\varphi)cos^2(\psi) & cos^2(\varphi)cos^2(\psi)sin^2(\psi)\\ sin^2(\varphi)cos^2(\psi)cos^2(\varphi)cos^2(\psi) & sin^2(\varphi)cos^2(\psi)sin^2(\varphi)cos^2(\psi) & sin^2(\varphi)cos^2(\psi)sin^2(\psi)\\ sin^2(\psi)cos^2(\varphi)cos^2(\psi) & sin^2(\psi)sin^2(\varphi)cos^2(\psi) & sin^2(\psi)sin^2(\psi)\end{bmatrix}\begin{Bmatrix}\epsilon_{xx}\\ \epsilon_{yy}\\ \epsilon_{zz}\end{Bmatrix}dV_i dV_j$$

The classical strain energy density is $U_{classical} = \lfloor\epsilon\rfloor \frac{1}{2}[D_{3\times3}]\{\epsilon\}$.

The peridynamic strain energy density is $U_{peridynamic} = \int_{\mathcal{H}} du_{bond}$, or

$$U_{peridynamic} = \frac{1}{dV_i}\int\int\int \frac{rk}{4}\lfloor\epsilon_{xx} \quad \epsilon_{yy} \quad \epsilon_{zz}\rfloor \begin{bmatrix} cos^2(\varphi)cos^2(\psi)cos^2(\varphi)cos^2(\psi) & cos^2(\varphi)cos^2(\psi)sin^2(\varphi)cos^2(\psi) & cos^2(\varphi)cos^2(\psi)sin^2(\psi) \\ sin^2(\varphi)cos^2(\psi)cos^2(\varphi)cos^2(\psi) & sin^2(\varphi)cos^2(\psi)sin^2(\varphi)cos^2(\psi) & sin^2(\varphi)cos^2(\psi)sin^2(\psi) \\ sin^2(\psi)cos^2(\varphi)cos^2(\psi) & sin^2(\psi)sin^2(\varphi)cos^2(\psi) & sin^2(\psi)sin^2(\psi) \end{bmatrix} \begin{Bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \epsilon_{zz} \end{Bmatrix} dV_i dV_j$$

and $V_j = dr \times rd\psi \times rcos\psi d\varphi = r^2 cos\psi dr d\psi d\varphi$, so

$$U_{peridynamic} = \lfloor\epsilon_{xx} \quad \epsilon_{yy} \quad \epsilon_{zz}\rfloor \frac{k}{4}$$

$$\int_{r=0}^{\delta}\int_{\varphi=0}^{2\pi}\int_{\psi=\frac{-\pi}{2}}^{\frac{\pi}{2}} \begin{bmatrix} cos^2(\varphi)cos^2(\psi)cos^2(\varphi)cos^2(\psi) & cos^2(\varphi)cos^2(\psi)sin^2(\varphi)cos^2(\psi) & cos^2(\varphi)cos^2(\psi)sin^2(\psi) \\ sin^2(\varphi)cos^2(\psi)cos^2(\varphi)cos^2(\psi) & sin^2(\varphi)cos^2(\psi)sin^2(\varphi)cos^2(\psi) & sin^2(\varphi)cos^2(\psi)sin^2(\psi) \\ sin^2(\psi)cos^2(\varphi)cos^2(\psi) & sin^2(\psi)sin^2(\varphi)cos^2(\psi) & sin^2(\psi)sin^2(\psi) \end{bmatrix} r^3 cos\psi dr d\psi d\varphi \begin{Bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \epsilon_{zz} \end{Bmatrix}$$

Equating the classical and peridynamic strain energy densities, for arbitrary strain,

$$\frac{k}{4}\int_{r=0}^{\delta}\int_{\varphi=0}^{2\pi}\int_{\psi=\frac{-\pi}{2}}^{\frac{\pi}{2}} \begin{bmatrix} cos^2(\varphi)cos^2(\psi)cos^2(\varphi)cos^2(\psi) & cos^2(\varphi)cos^2(\psi)sin^2(\varphi)cos^2(\psi) & cos^2(\varphi)cos^2(\psi)sin^2(\psi) \\ sin^2(\varphi)cos^2(\psi)cos^2(\varphi)cos^2(\psi) & sin^2(\varphi)cos^2(\psi)sin^2(\varphi)cos^2(\psi) & sin^2(\varphi)cos^2(\psi)sin^2(\psi) \\ sin^2(\psi)cos^2(\varphi)cos^2(\psi) & sin^2(\psi)sin^2(\varphi)cos^2(\psi) & sin^2(\psi)sin^2(\psi) \end{bmatrix} r^3 cos\psi dr d\psi d\varphi$$

$$= \frac{E}{2(1+\nu)(1-2\nu)} \begin{bmatrix} (1-\nu) & \nu & \nu \\ \nu & (1-\nu) & \nu \\ \nu & \nu & (1-\nu) \end{bmatrix}$$

Integrating with respect to r:

$$\frac{\frac{\delta^4}{4}k}{4}\int_{\varphi=0}^{2\pi}\int_{\psi=\frac{-\pi}{2}}^{\frac{\pi}{2}} \begin{bmatrix} cos^2(\varphi)cos^2(\psi)cos^2(\varphi)cos^2(\psi) & cos^2(\varphi)cos^2(\psi)sin^2(\varphi)cos^2(\psi) & cos^2(\varphi)cos^2(\psi)sin^2(\psi) \\ sin^2(\varphi)cos^2(\psi)cos^2(\varphi)cos^2(\psi) & sin^2(\varphi)cos^2(\psi)sin^2(\varphi)cos^2(\psi) & sin^2(\varphi)cos^2(\psi)sin^2(\psi) \\ sin^2(\psi)cos^2(\varphi)cos^2(\psi) & sin^2(\psi)sin^2(\varphi)cos^2(\psi) & sin^2(\psi)sin^2(\psi) \end{bmatrix} cos\psi d\psi d\varphi$$

$$= \frac{E}{2(1+\nu)(1-2\nu)} \begin{bmatrix} (1-\nu) & \nu & \nu \\ \nu & (1-\nu) & \nu \\ \nu & \nu & (1-\nu) \end{bmatrix}$$

Next, integrating with respect to ψ:

$$\frac{\delta^4 k}{16}\int_{\varphi=0}^{2\pi} \begin{bmatrix} \frac{16}{15}cos^2(\varphi)cos^2(\varphi) & \frac{16}{15}cos^2(\varphi)sin^2(\varphi) & \frac{4}{15}cos^2(\varphi) \\ \frac{16}{15}sin^2(\varphi)cos^2(\varphi) & \frac{16}{15}sin^2(\varphi)sin^2(\varphi) & \frac{4}{15}sin^2(\varphi) \\ \frac{4}{15}cos^2(\varphi) & \frac{4}{15}sin^2(\varphi) & \frac{2}{5} \end{bmatrix} d\varphi = \frac{E}{2(1+\nu)(1-2\nu)} \begin{bmatrix} (1-\nu) & \nu & \nu \\ \nu & (1-\nu) & \nu \\ \nu & \nu & (1-\nu) \end{bmatrix};$$

Finally, integrate with respect to φ:

$$\frac{\delta^4 k}{16} \begin{bmatrix} \frac{4\pi}{5} & \frac{4\pi}{15} & \frac{4\pi}{15} \\ \frac{4\pi}{15} & \frac{4\pi}{5} & \frac{4\pi}{15} \\ \frac{4\pi}{15} & \frac{4\pi}{15} & \frac{4\pi}{5} \end{bmatrix} = \frac{E}{2(1+\nu)(1-2\nu)} \begin{bmatrix} (1-\nu) & \nu & \nu \\ \nu & (1-\nu) & \nu \\ \nu & \nu & (1-\nu) \end{bmatrix}$$

Equating the 11 and 12 terms: $\frac{\delta^4 k}{16}\left(\frac{4\pi}{5}\right) = \frac{E}{2(1+\nu)(1-2\nu)}(1-\nu)$ and $\frac{\delta^4 k}{16}\left(\frac{4\pi}{15}\right) = \frac{E}{2(1+\nu)(1-2\nu)}(\nu)$

Therefore, $\left(\frac{1}{3}\right)\frac{E}{2(1+\nu)(1-2\nu)}(1-\nu) = \frac{E}{2(1+\nu)(1-2\nu)}(\nu)$ or $\frac{(1-\nu)}{3} = \nu$ or $1 = 4\nu$ or $= \frac{1}{4}$.

Finally, $\frac{\delta^4 k}{16}\left(\frac{4\pi}{15}\right) = \frac{E}{2\left(1+\frac{1}{4}\right)\left(1-2\times\frac{1}{4}\right)}\left(\frac{1}{4}\right) = \frac{E}{2\left(\frac{5}{4}\right)\left(\frac{1}{2}\right)}\left(\frac{1}{4}\right) = \frac{E}{5}$ so $k = \frac{E15\times16}{4\pi5\delta^4} = \frac{12E}{\pi\delta^4}$;

$$\boxed{\text{In summary, } \nu = \frac{1}{4} \text{ and } k = \frac{12E}{\pi\delta^4}}$$