# NOTES ON QUANTIFYING MODES OF A SECOND-ORDER TENSOR

## 1. Background

The mechanical behavior of rocks and rock-like materials (concrete, ceramics, etc.) strongly depends on the "loading mode," defined by the values and relative proportions of the eigenvalues of stress, $\{\sigma_1, \sigma_2, \sigma_3\}$. For example, the onset of inelasticity in uniaxial compression,

$$\text{uniaxial compression: } \sigma_1 > 0 \text{ and } \sigma_2 = \sigma_3 = 0 \tag{1.1}$$

is usually significantly lower than it is in uniaxial tension

$$\text{uniaxial tension: } \sigma_1 < 0 \text{ and } \sigma_2 = \sigma_3 = 0 \tag{1.2}$$

For either of these modes, material strength is usually significantly increased by superposition of confining pressure. Moreover, the intensity of stress required to induce failure varies with the value of middle eigenvalue relative to the maximum and minimum eigenvalues.

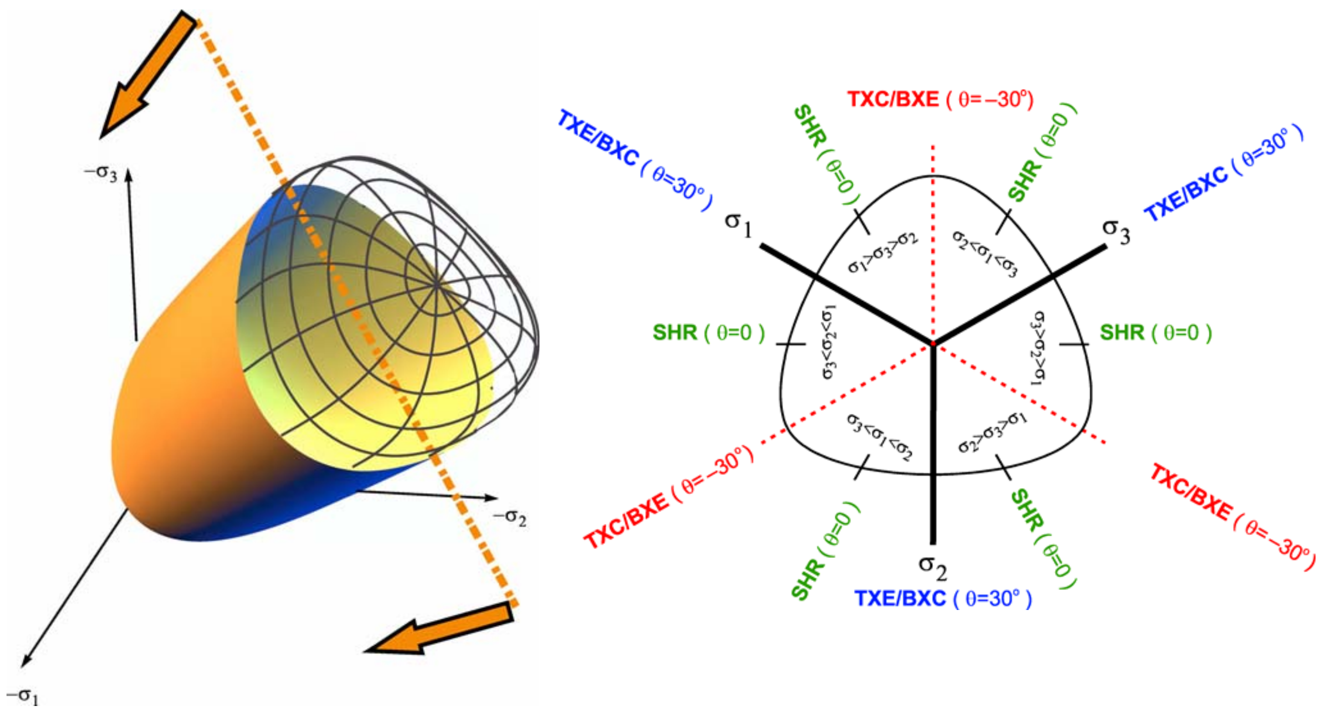


*Figure 1.1. "Yield" surface for a typical isotropic geomechanics model (left), and a constant pressure cross section of this surface (right). The cage is present when the model includes a cap for porosity.*

Figure 1.1 shows a typical yield surface shape for an isotropic geo-material. For isotropic geo-materials (or for *any* isotropic material), the yield function depends only on the principal stresses, and the value of the yield function must be unchanged for any reordering of the eigenvalues. This leads to $120°$ periodic and reflective symmetry of the yield surface about the [111] hydrostat axis in stress space where all three principal stresses are equal, It is therefore natural to introduce cylindrical coordinates $\{r, \theta, z\}$ for which the $z$-axis points along the hydrostat and $\{r, \theta\}$ are polar coordinates on any constant pressure (constant $z$) cross-section of the yield surface. The angular coordinate is called the **Lode angle**, and the radial coordinate equals the magnitude of the stress deviator. The Lode cylindrical coordinates associated with a stress tensor $\sigma_{ij}$ are found by first computing the "mechanics" invariants:

$$J_1 = \sigma_{kk} = \text{trace of stress, tr}\sigma \tag{1.3}$$

$$J_2 = \tfrac{1}{2}S_{im}S_{mi} = \tfrac{1}{2}\text{tr}\boldsymbol{S}^2 \tag{1.4}$$

$$J_3 = \tfrac{1}{3}S_{im}S_{mn}S_{ni} = \tfrac{1}{3}\text{tr}\boldsymbol{S}^3 \tag{1.5}$$

where repeated indices are understood to be summed from 1 to 3, and $\boldsymbol{S}$ is the stress deviator, defined

$$S_{ij} = \sigma_{ij} - \tfrac{1}{3}\sigma_{kk}\delta_{ij} \tag{1.6}$$

in which $\delta_{ij}$ is the Kronecker delta. Because stress is symmetric and because $\boldsymbol{S}$ is deviatoric (i.e., zero trace), alternative formulas for the second and third invariants are

$$J_2 = \tfrac{1}{2}||\boldsymbol{S}||^2 \tag{1.7}$$

and

$$J_3 = \det\boldsymbol{S} \tag{1.8}$$

Once the mechanics invariants $(J_1, J_2, J_3)$ are found, the Lode cylindrical coordinates are given by

$$r = \sqrt{2J_2} = ||\boldsymbol{S}|| \tag{1.9}$$

$$\theta = \tfrac{1}{3}\sin^{-1}\left[\left(\frac{J_3}{2}\right)\left(\frac{3}{J_2}\right)^{3/2}\right] \tag{1.10}$$

$$z = \frac{J_1}{\sqrt{3}} \tag{1.11}$$

The ArcSin is the principal arcsin ranging from $-90°$ to $+90°$, and therefore $\theta$ ranges from $-30°$ to $+30°$. The full octahedral profile can be generated by appropriately rotating or reflecting this sextant.

Of course, the angle from which the Lode angle is measured is a matter of personal preference. Our definition is appealing because its numerical sign varies with the position of the middle eigenvalue relative to the other eigenvalues, as illustrated in Fig. 1.2. As indicated in the caption, triaxial extension (TXE) or biaxial compression (BXC) corresponds to $\theta = -30°$, while triaxial compression (TXC) or biaxial extension (BXE) corresponds to $\theta = +30°$.

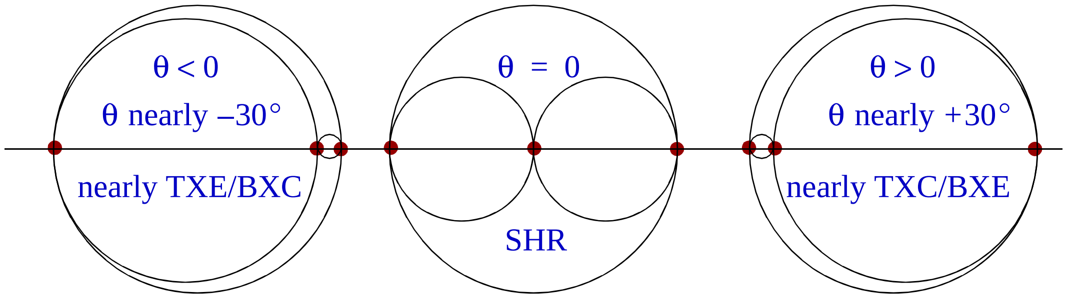


*Figure 1.2. Mohr diagrams showing variation of the Lode angle with the middle eigenvalue. The small dots are principal stresses. Directions to the right on the horizontal axis correspond to increased compression. A TXE/BXC state is axisymmetric with the single "axial" eigenvalue less compressive — more extensional — than the double "lateral" eigenvalue (equivalently, the axial stress is more compressive than the lateral stress). A SHR state has the middle eigenvalue in the center (corresponding to a state of pure shear, perhaps with superimposed confining pressure). A TXC/BXE state is axisymmetric with the single "axial" eigenvalue more compressive than the double "lateral eigenvalue (equivalently, the lateral stresses are more extensional than the axial stress).*

A constant pressure (constant $z$) cross-section of a yield surface is called an **octahedral profile**, as illustrated in the right side of Fig. 1.1. A constant $\theta$ cross-section (i.e., a plot of $r$ vs. $z$) is a **meridional profile**. Since $r$ is the magnitude of the stress deviator and since $z$ is related to pressure $p$ by $z = \sqrt{3}p$, a meridional profile essentially shows how shear strength varies with pressure. For simple von Mises theory, the failure criterion depends only on $J_2$. Equivalently, von Mises failure depends only on $r$, thus making the yield surface a cylinder for which the meridional profile is a horizontal straight line and the octahedral profile is a a circle. For rocks, the meridional profile increases with $z$, indicating increased shear strength with pressure, until the cap is reached, at which point shear strength decreases with pressure, thus giving a "teardrop" shaped meridional profile for porous rock. For rocks, the octahedral profile is strongly triangular at low pressures.

## Lode angle visualization

Recalling Fig. 1.2, a simple color plot of the Lode angle could use "hot" colors where $\theta < 0$ and "cool" colors where $\theta > 0$ to depict variation of the middle eigenvalue and therefore visually indicate which elements are closer to TXE than to TXC. As is true for any cylindrical coordinate system, however, the angular coordinate is undefined when $r=0$. Therefore, one could get the wrong impression in a simple Lode angle color plot because there could be many regions in the plot that have received an arbitrarily assigned value for the Lode angle. Furthermore, for isotropic plasticity theory, the value of the Lode angle is important only for those elements at or near yield. In other words, we want elastically deforming elements to receive less attention in a Lode angle color plot.

For isotropic failure models, the yield criterion can always be cast in the form

$$r = r_Y(\theta, z) \tag{1.12}$$

Therefore, to simultaneously avoid ambiguity in Lode angle plots when $r = 0$ and to give greater attention to those elements at or near yield, we could use Lode angle to define *color*, but allow the ratio

$$\frac{r}{r_Y(\theta, z)} \tag{1.13}$$

to define *brightness*. Elements at yield (where the ratio equals unity) would be fully bright, but those well below the yield threshold would not be visible. An even better "brightness" function would multiply the one in Eq. (1.13) by a Lode-angle-importance factor, giving a brightness function of

$$\frac{r}{r_Y(\theta, z)}\left[2\left(1 - \frac{r_Y(-30°, z)}{r_Y(+30°, z)}\right)\right] \tag{1.14}$$

With this brightness function, the color for Lode angle would be most visible for elements that are not only at yield, but are also *on a part of the yield surface where the value of the Lode angle strongly affects the result*. For strongly triangular octahedral profiles, where the Lode angle matters most,

$$\left[2\left(1 - \frac{r_Y(-30°, z)}{r_Y(+30°, z)}\right)\right] \approx 1 \tag{1.15}$$

For nearly circular octahedral profiles, where the Lode angle is unimportant,

$$\left[2\left(1 - \frac{r_Y(-30°, z)}{r_Y(+30°, z)}\right)\right] \approx 0 \tag{1.16}$$

# 2. Visualization of stiffness eigenmodes

In classical plasticity, the stress rate is related to the strain rate by

$$\dot{\sigma}_{ij} = C_{ijkl}\dot{\varepsilon}_{kl} \text{ during elastic loading} \tag{2.1}$$

$$\dot{\sigma}_{ij} = T_{ijkl}\dot{\varepsilon}_{kl} \text{ during plastic loading} \tag{2.2}$$

A loading increment is elastic if the trial stress state is below the yield surface. Otherwise, the increment is plastic. During plastic legs, the following scalar is *always* positive (even if softening):

$$N_{ij}C_{ijkl}\dot{\varepsilon}_{kl} > 0 \tag{2.3}$$

where $N_{ij}$ is the normal to the yield surface. The normal to the yield surface gives us a directionality that will be used in determining an appropriate “sense” for eigenmodes of the tangent stiffness.

A plasticity model can return a tangent stiffness (either $C$ or $T$), which can be cast as a $6 \times 6$ matrix. The eigenvectors (or eigenvectors of the “stretch” part of the stiffness found from a polar decomposition) are actually second-order tensors indicating the “mode” associated with each eigenvalue. For a softening material, the mode associated with the lowest eigenvalue (lowest stiffness) is the most critical one. During plastic legs, eigenvectors of the tangent stiffness are defined

$$T_{ijkl}Y_{kl} = \lambda Y_{ij} \tag{2.4}$$

Visualizing the eigentensor $Y_{ij}$ associated with the smallest eigenvalue is a means of determining the nature of strain or stress increments associated with the most “dangerous” form of loading. However, if $Y_{ij}$ is an eigentensor, then so is $-Y_{ij}$. Changing the numerical sign of the eigentensor will also change the sign of the Lode angle. This leads to ambiguity in determining whether or not the critical mode is more of a TXE or TXC nature. This problem can be easily resolved by using Eq. (2.3) to define the “direction” or “sense” of the eigentensor. Specifically, $Y_{ij}$ should be oriented such that

$$N_{ij}C_{ijkl}Y_{ij} > 0 \tag{2.5}$$

In other words, the eigenmode for plastic loading should be oriented in a direction that would actually lead to plastic loading. An oppositely oriented eigenmode would correspond to elastic unloading. An analog of this statement is illustrated in Fig. 2.1.

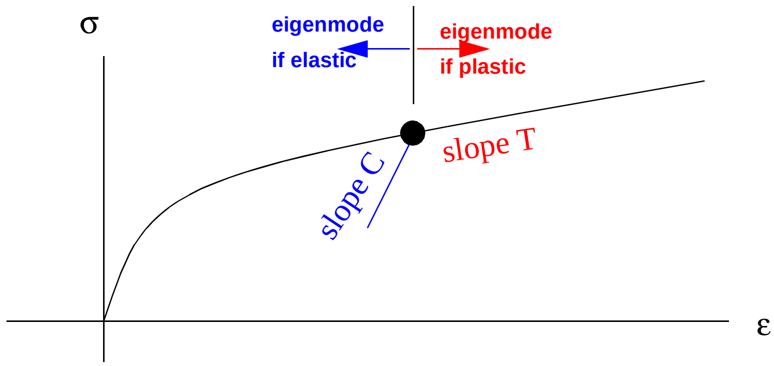


*Figure 2.1. One-dimensional analog of the sense criterion of Eq. (2.5). The eigenmode should point to the right if plastic, but to the left if the loading increment is elastic.*