1. ONE-DIMENSIONAL MODEL PROBLEMS

1.1 INTRODUCTION

We begin by providing a summary of differential equations that govern one-dimensional problems of engineering interest. We summarize the derivations as a mechnism for introducing terminology that must be used carefully in subsequent sections. Only those subsections covering the field of interest needs to be covered prior to continuing with he ubsequent chapters.

Any numerical procedure should be verified by comparing numerical and analytical solutions. This chapter provides a set of one-dimensional equations with solutions for heat conduction, and motion in bars and beams. Also included are problems with axial and spherical symmetry. This set of governing equations includes types of differential equations that are basic and cover a much broader range of applications than those presented here, provided an appropriate interpretation of variables is made in the context of the particular field.

In each case, a brief derivation is given for the governing differential equations that include time dependence. However, in this chapter only steady-state or time-independent solutions are given; transient solutions are provided in the next chapter. Special attention is given to boundary conditions because an inappropriate specification of boundary conditions is one of the most common source of errors in input to numerical programs. For numerical analysis, it is good practice to convert to dimensionless parameters but ultimately, solutions must be interpreted in terms of physical parameters used to describe a given problem.

We will see that even though one-dimensional problems are elementary in a certain sense, the problems do provide the means for introducing new concepts that can seem extremely complicated when presented in higher dimensions. The chapter closes with a section on more general topics that are of special significance for the choice of a numerical procedure. In general, many of the ideas carry over directly to problems in higher spatial dimensions.

1.2 HEAT CONDUCTION

1.2.1 Governing Equations

The simplest form of one-dimensional heat conduction arises naturally in the case of an insulated bar. The same situation holds in three dimensions if there are no variations in cross planes normal to the direction of conduction, a condition that might occur with heat conduction through a wall. These two configurations are illustrated in Fig. 1.2-1 where the segment in the wall is shown as a uniform bar with dotted lines. The usual rectangular Cartesian coordinates are denoted by x, y and z, with the associated set of orthonormal base vectors $\mathbf{e}_x$, $\mathbf{e}_y$ and $\mathbf{e}_z$, respectively.

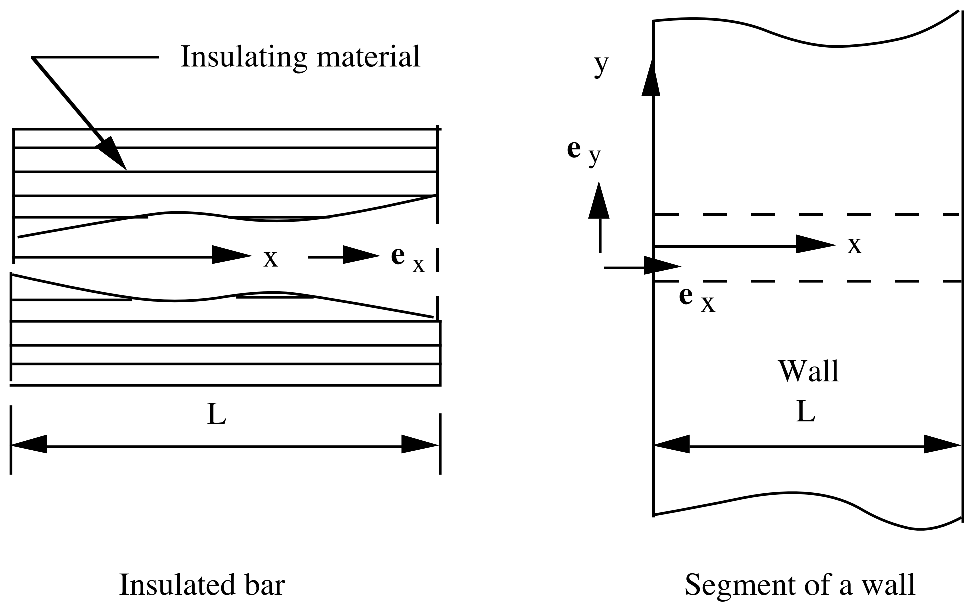


Fig. 1.2-1. One-dimensional problem of an insulated bar or segment of a wall.

To derive the governing equations, suppose x is a spatial coordinate used to define points along the axis of an insulated bar of variable cross-sectional area A(x) and length L. Assume no dependence on y and z, i.e., assume the temperatures of all points on any cross-sectional plane are the same and denoted by T(x,t), with t representing time. Then, the internal thermal energy, U^T, per unit length is

$$U^T = A\rho C(T - T_0) = \bar{\rho} C(T - T_0) \tag{1.2-1}$$

where T_0 is the temperature at which the internal energy is taken to be zero, $T \geq T_0$, C is the specific heat, ρ is the mass density and $\bar{\rho} = \rho A$ is the mass per unit length.

The first law of thermodynamics (Conservation of Energy) states that the rate at which internal energy increases must equal the rate at which heat is added to a body or any segment of a body. Suppose a heat source per unit length, $Q(x,t)$, acts along the body. It is customary to adopt the convention that heat added to a body is positive. Consider an arbitrary segment of bar defined by $x_1 \le x \le x_2$ as indicated in Fig. 1.2-2 with an expanded view shown next in Fig. 1.2-3. The rate at which heat is transfered through the ends of the segment from the remaining part of the bar is defined to be Q^*_1 and Q^*_2, for the left and right ends of the segment, respectively. Since there is no dependence on y and z, heat flow per unit area is defined to be q_1^* and q_2^* so that

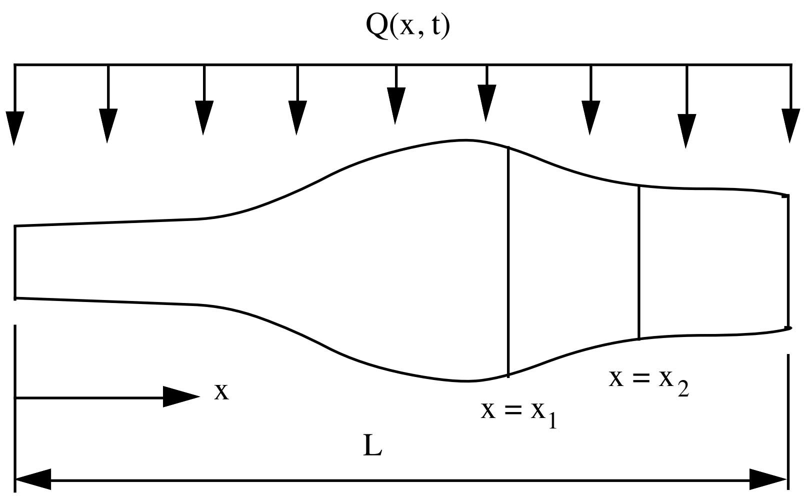


Fig. 1.2-2. Notation for heat conduction in a bar.

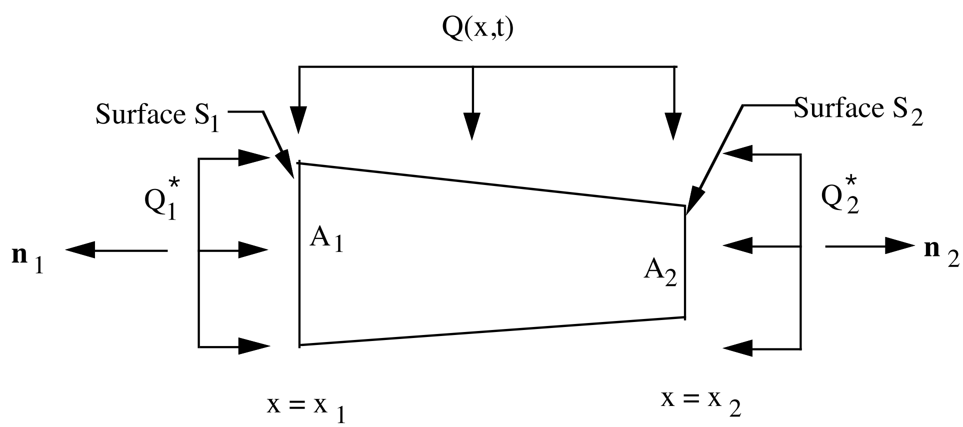


Fig. 1.2-3. Sketch showing notation for a small segment of a bar.

$$Q_1^* = A_1 q_1^* , \qquad Q_2^* = A_2 q_2^* \tag{1.2-2}$$

The mathematical expression of the first law of thermodynamics in one dimension is

$$\int_{x_1}^{x_2} \frac{dU^T}{dt} dx = \int_{x_1}^{x_2} Q\, dx + A_1 q_1^* + A_2 q_2^* = \int_{x_1}^{x_2} Q\, dx + (Aq^*)\big|_{x=x_1} + (Aq^*)\big|_{x=x_2} \tag{1.2-3}$$

This equation should certainly be satisfied in the limit as x_1 approaches x_2. The integrals then contribute nothing to the equation so the result is

$$\lim_{x_1 \to x_2} [(Aq^*)\big|_{x=x_1} + (Aq^*)\big|_{x=x_2}] = 0 \tag{1.2-4}$$

In the limit, the two surfaces have the same cross-sectional area; the only difference is the outer unit normal vector, $\mathbf{n}$. In one dimension, $\mathbf{n} = n_X \mathbf{e}_X$. Since there is only one component, replace n_X with n. On the surface $x = x_1$, the normal is $\mathbf{n}_1 = n_{1x}\mathbf{e}_x$ with $n_{1x} = -1$ and on the surface $x = x_2$, the normal is $\mathbf{n}_2 = n_{2x}\mathbf{e}_x$ with $n_{2x} = 1$. If it is assumed that q* depends on $\mathbf{n}$ and if (1.2-4) is to be satisfied for all choices of x_1 and x_2, then there must be a vector, called the **flux vector**, $\mathbf{q}$, that provides a linear transformation between q* and $\mathbf{n}$ through the following dot product:

$$q^* = -\mathbf{q} \cdot \mathbf{n} \tag{1.2-5}$$

Therefore, one of the implications of the first law of thermodynamics is the existence of the flux vector. The minus sign in (1.2-5) is selected by convention to denote the choice that $\mathbf{q} \cdot \mathbf{n}$ is the heat flow per unit area out of the face. In general, the heat flux vector is written as $\mathbf{q} = q_x\mathbf{e}_x + q_y\mathbf{e}_y + q_z\mathbf{e}_z$. In one dimension, the only nonzero component is q_x, which will simply be labeled q, so that $\mathbf{q} = q\mathbf{e}_x$. An alternative way to interpret (1.2-5) is that, if the bar is cut, opposite faces exhibit equal and opposite fluxes, q*. Now, with the use of (1.2-5), the limiting case of (1.2-4) is automatically satisfied. Next, substitute (1.2-5) in (1.2-3) to obtain

$$\int_{x_1}^{x_2} \frac{dU^T}{dt} dx = \int_{x_1}^{x_2} Q\, dx - (Anq)\big|_{x=x_1} - (Anq)\big|_{x=x_2} \tag{1.2-6}$$

Use the divergence theorem (Appendix A.6) to replace the last two terms with an integral and obtain the following expression:

$$\int_{x_1}^{x_2} [\frac{dU^T}{dt} + \frac{\partial(qA)}{\partial x} - Q] dx = 0 \tag{1.2-7}$$

which must hold $\forall$ x_1 and x_2. (The symbol $\forall$ is frequently used to denote "for all" or "for every choice.") From the fundamental theorem of continuum mechanics given in Appendix A.1, the integrand must be zero and the governing differential equation is therefore

$$\frac{dU^T}{dt} = -\frac{\partial(qA)}{\partial x} + Q \tag{1.2-8}$$

If $\bar{\rho}C$ is not a function of time, and if x and t are independent variables, then the use of (1.2-1) yields

$$c\frac{\partial T}{\partial t} = -\frac{\partial(qA)}{\partial x} + Q\,, \qquad c = \bar{\rho}C \tag{1.2-9}$$

So far there are two dependent variables, the secondary variable or flux, q, and the primary variable or temperature, T, but only one governing equation. To obtain a complete set of equations, a constitutive equation must be introduced that relates these two variables. For heat conduction, the basic constitutive equation is called **Fourier's law**, which states that

$$q = -K\frac{\partial T}{\partial x} \tag{1.2-10}$$

where K is called the thermal conductivity. The substitution of (1.2.1-10) in (1.2.1-9) yields

$$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + Q = c\frac{\partial T}{\partial t}\,, \qquad k = KA \quad \text{and} \quad c = \bar{\rho}C \tag{1.2-11}$$

This equation is known as the transient (time dependent) **heat conduction equation**. The usual approach is to solve (1.2-11) for temperature, and then obtain the flux from (1.2-10).

Alternative notations are often used to denote partial derivatives. One of the most common is to use a comma followed by an x to show a partial derivative with respect to x, and a similar notation for a derivative with respect to t. With this notation, (1.2-11) becomes

$$\left(kT_{,x}\right)_{,x} + Q = cT_{,t}\,, \qquad k = KA \quad \text{and} \quad c = \bar{\rho}C \tag{1.2-12}$$

Yet another notation is used for the derivative with respect to time and that is a superposed dot: $T_{,t} = \dot{T}$. We will use both notations.

Steady-state heat conduction is the case where the problem is independent of time in which case the equations reduce to the equivalent forms of

$$(qA)_{,x} - Q = 0\,, \qquad q = -KT_{,x} \tag{1.2-13$_1$}$$

or

$$\left(kT_{,x}\right)_{,x} + Q = 0\,, \qquad k = KA \tag{1.2-13$_2$}$$

To obtain the solution to a differential equation a body of information called the **data** must be specified. For the transient problem of (1.2-12) the **data** consist of the following:

(i) the domain, $0 \le x \le L$ and $t \ge 0$;
(ii) the cross-sectional area, A(x), a geometrical parameter;
(iii) the conductivity, K(x), a material parameter;
(iv) the mass per unit length, $\bar{\rho}$, and specific heat, C, material parameters;
(v) the heat generating function, Q(x,t);
(vi) the boundary conditions; and,
(vii) the initial conditions.

For the steady-state problem of (1.2-13), Q is only a function of x and Items (iv) and (vii) are not required.

The **data** come from a physical description of the problem, and it is the engineer's task to provide this information. From a mathematical viewpoint, there are restrictions on the data if the problem is to be **well posed** which means that a solution **exists**, that the solution is **unique**, and that only small changes in the solution occur with small changes in the data (**stability with respect to data**). For the vast majority of cases, a careful setup by an engineer will result in a well posed problem; a lack of well posedness usually implies a mistake has been made.

One example that results in ill posedness is the choice of functions A(x) or K(x) that exhibit zeros or negative values for a segment of the domain. Such a situation is not physically realistic, but sometimes such choices are made inadvertently and not realized until a numerical solution is sought. A more common source of ill posedness comes from a poor selection of boundary conditions. The next subsection concentrates on this aspect of problem formulation, and examples are given to illustrate some of the points.

Exact solutions to (1.2-12) or (1.2-13) satisfy the boundary conditions, interior jump conditions as required, and the intial conditions if the problem is a transient one. Frequently, exact solutions are difficult to obtain so approximate solutions are sought. In this Chapter we focus on exact solutions and defer consideration of approximate solutions to later.

1.2.2 Boundary Conditions

Consider the heat conduction equation of the form given in (1.2-12). Boundary conditions always involve the primary function (T) and derivatives (fluxes) of the primary function up to one order less than the differential equation. Since the heat conduction equation is a second-order differential equation in x, the boundary conditions must provide T, the first derivative ($T_{,x}$), or a combination of T and its derivative. Typical combinations are given next.

<u>(1) Dirichlet Boundary Conditions</u>

If the primary variable is specified on the boundaries (x = 0 and x = L) the boundary conditions are said to be of the Dirichlet type.

<u>Example:</u>

Consider a problem for which the data are the following:

(i) the domain, $0 \le x \le L$;
(ii) the area, A(x) =1;
(iii) the conductivity, K(x) specified so that with A=1, $k(x) = k_o$, a constant;
(iv) the heat source, Q(x) = bx with b a constant; and
(v) the temperature, $T = T_0$ at x = 0 and $T = T_L$ at x = L.

The governing differential equation reduces to $k_o T_{,xx} + bx = 0$. The complete solution consists of the homogeneous, T_h, and particular, T_p, parts:

$$T = T_h + T_p\,, \qquad T_h = c_1 + c_2 x \qquad \text{and} \qquad T_p = -\frac{b}{k_o}\frac{x^3}{6} \tag{1.2-14}$$

where c_1 and c_2 are arbitrary constants to be determined from the boundary conditions. Application of the boundary conditions yields the complete solution to the boundary value problem:

$$T = T_o + \left(T_L - T_o + \frac{b}{k_o}\frac{L^3}{6}\right)\frac{x}{L} - \frac{b}{k_o}\frac{x^3}{6} \qquad (1.2\text{-}15)$$

<u>(2) Mixed Boundary Conditions</u>

Mixed boundary conditions consist of prescribing the primary variable over part of the boundary and the flux over the other part. Specifically q* can be given at x = 0 and T at x = L or T at x = 0 and q* at x = L. To relate q* to the derivative of T, recall that q* = -qn and $q = -KT_{,x}$. Also n = -1 for x = 0 and n = 1 for x = L.

<u>Example:</u>

Suppose the data are the following:

(i) the domain, $0 \le x \le L$;
(ii) the area, A(x) =1;
(iii) the conductivity, $K(x) = k_o$, a constant, $(k = k_o)$;
(iv) the heat source, Q(x) = bsin(πx/L) with b a constant; and
(v) the boundary flux, $q^* = q^*_0$ at x=0 and the temperature, $T = T_L$ at x = L.

The governing differential equation reduces to $k_o T_{,xx} + b\sin(\pi x/L) = 0$. The general solution is

$$T = T_h + T_p\,, \qquad T_h = c_1 + c_2 x \qquad \text{and} \qquad T_p = \frac{b}{k_o}\frac{L^2}{\pi^2}\sin\frac{\pi x}{L} \qquad (1.2\text{-}16)$$

where c_1 and c_2 are constants to be determined from the boundary conditions. To apply the flux boundary condition, note that $q^* = -qn = q = -kT_{,x}$ at x = 0 or

$$q_0^* = -k_o\left(c_2 + \frac{b}{k_o}\frac{L}{\pi}\cos\frac{\pi x}{L}\right)_{x=0} = -k_o c_2 - \frac{bL}{\pi} \qquad (1.2\text{-}17)$$

After solving for c_2 and using the other boundary condition, the solution is

$$T = T_L - \left(\frac{q_0^*}{k_o} + \frac{b}{k_o}\frac{L}{\pi}\right)(x - L) + \frac{b}{k_o}\frac{L^2}{\pi^2}\sin\frac{\pi x}{L} \qquad (1.2\text{-}18)$$

An alternative approach is to use $(1.2\text{-}13_1)$:

$$(qA)_{,x} - Q = 0\,, \qquad q = -KT_{,x} \tag{1.2-19}$$

If the first equation is integrated, the result is $q = \int Q dx + c$ with c the constant of integration. Since one of the boundary conditions is for the flux, q, the constant c can be determined explicitly. If flux is the variable of interest, then there is no need to integrate the second equation to obtain temperature. Time independent problems for which the boundary conditions are of the form where the secondary variable can be obtained without obtaining the primary variable are said to be **statically determinate**; otherwise the problems are **statically indeterminate**.

(3) Neumann Boundary Conditions

If the flux is prescribed at all points on the boundary, the problem is said to be of the Neumann type.

Example:

Consider a problem with the following data:

(i) the domain, $0 \le x \le L$;
(ii) the area, $A(x) = 1$;
(iii) the conductivity, $K(x) = k_o$, a constant, ($k = k_o$);
(iv) the heat source, $Q(x) = be^{-cx/L}$ with b and c constants; and
(v) the boundary fluxes, $q^* = q_0^*$ at $x = 0$ and $q^* = q_L^*$ at $x = L$.

Integrate the equation $(qA)_{,x} - Q = 0$ to obtain

$$q = -b\frac{L}{c}e^{-cx/L} + c_1 \tag{1.2-20}$$

Now, there exists the dilemma that there are two boundary conditions on flux but only one constant of integration, c_1. Suppose c_1 is chosen so that the boundary condition at x = 0 is satisfied: $qn|_{x=0} = -q_0^*$. Since n = -1 at x = 0, the result is $c_1 = q^*_0 + b\frac{L}{c}$. Now consider the boundary condition at x = L: $qn|_{x=L} = -q_L^*$. With n = 1 at x = L, the result is

$$q_0^* + q_L^* + b\frac{L}{c}(1 - e^{-c}) = 0 \tag{1.2-21}$$

In general, it is possible to specify arbitrary values for q_0^* and q_L^*. The conclusion from (1.2-21) is that a solution does not exist. The problem is an example of one that is **not well posed**.

To understand why there must be a constraint imposed for this type of boundary condition, consider the total heat, Q_T, added to the system by the source Q(x):

$$Q_T = \int_0^L Q\,dx = b\frac{L}{c}(1 - e^{-c}) \qquad (1.2\text{-}22)$$

When (1.2-22) is substituted in (1.2-21), the result is

$$q_0^* + q_L^* + Q_T = 0 \qquad (1.2\text{-}23)$$

which is merely the statement that for a well-posed problem, the total heat added to the system must be zero, a result consistent with the first law of thermodynamics for time independent problems.

Now suppose q_L^* meets the constraint of (1.2-23) so a unique solution for the flux exists. Integrate $q = -KT_{,x}$ to obtain the temperature:

$$T = -\frac{1}{k_o}\left[\left(q_0^* + b\frac{L}{c}\right)x + b\frac{L^2}{c2}e^{-\frac{c}{L}x}\right] + c_2 \qquad (1.2\text{-}24)$$

The constant c_2 is arbitrary because there is no boundary condition on T. Now an infinite number of solutions for temperature exist so the problem remains **ill posed**. To make the problem well posed, T can be arbitrarily assigned a value for some x, not necessarily at the boundary. The physical implication is that all temperatures can be uniformly shifted up or down without causing a violation of either the governing differential equation or the boundary conditions.

(4) Robin Boundary Conditions

Suppose a fluid is flowing over the ends of the bar. Then a more appropriate form for the boundary conditions is a prescribed value for a linear combination of the temperature and flux. In general, when a linear combination of primary and secondary variables is given, the boundary condition is said to be of the **Robin type**.

Suppose Robin conditions are given at each end. Then the boundary conditions are of the form

$$\alpha_0 T + \beta_0 q = \gamma_0 \qquad\qquad \alpha_L T + \beta_L q = \gamma_L \qquad (1.2\text{-}25)$$

for x = 0 and x = L, respectively. The parameters $(\alpha_0, \beta_0, \gamma_0)$ and $(\alpha_L, \beta_L, \gamma_L)$ are obtained from a physical interpretation of a given problem. Clearly, all of the previous types of boundary conditions are special cases of Robin boundary conditions. Therefore, if one is writing a general purpose computer program, the Robin type is the one to use. Of course, a check should be made to ensure that the boundary conditions do not lead to ill posedness.

1.2.3 Handling Discontinuities and Point Sources

Suppose we have the situation that at $x = x_d$, There is a discontinuity in the cross-sectional conductivity

$$K^A = KA \tag{1.2-26}$$

due to a jump in the conductivity, K, or cross-sectional area, A, or both. For the sake of generality, suppose a point source of heat also exists at this point as represented by

$$Q = Q_d \delta\left[x - x_d\right] \tag{1.2-27}$$

A straight forward way to handle the problem is to separate the domain into left part $0 \le x < x_d$ and a right part $x_d < x \le L$. We obtain a solution to the heat conduction equation for the two subdomains which we label T^L and T^R. The left solution contains the two integrations constants c_1 and c_2, and the right solution contains the two integration constants c_3 and c_4. We have the two boundary conditions at $x = 0$ and $x = L$, so two additional equations are required at the point $x = x_d$. The first is continuity of temperature, or

$$T^L\Big|_{x=x_d} = T^R\Big|_{x=x_d} \tag{1.2-28}$$

The second equation is obtained by performing a spatial integration of the terms in the governing equation as follows:

$$\int_{x_d-\varepsilon}^{x_d+\varepsilon} \left(KAT_{,x}\right)_{,x} dx + \int_{x_d-\varepsilon}^{x_d+\varepsilon} Q^s dx + \int_{x_d-\varepsilon}^{x_d+\varepsilon} Q_d \delta\left[x - x_d\right] dx = 0 \tag{1.2-29}$$

in which Q^s is the strong part of the forcing function; i.e., the part that does not contain the Dirac-delta function. As $\varepsilon \to 0$ the second and third integrals become

$$\lim_{\varepsilon \to 0} \int_{x_d-\varepsilon}^{x_d+\varepsilon} Q^s dx = 0 \qquad \lim_{\varepsilon \to 0} \int_{x_d-\varepsilon}^{x_d+\varepsilon} Q_d \delta[x - x_d]\, dx = Q_d \tag{1.2-30}$$

Now we carry out the first integral

$$\begin{aligned} \int_{x_d-\varepsilon}^{x_d+\varepsilon} (KAT,_x),_x dx &= (KAT,_x)\Big|_{x_d-\varepsilon}^{x_d+\varepsilon} \\ &= (K^R A^R T^R,_x)\Big|_{x=x_d} - (K^L A^L T^L,_x)\Big|_{x=x_d} \end{aligned} \tag{1.2-31}$$

When the possibility of the point source is included, we obtain the fourth equation that must be used to evaluate the integration constants:

$$(K^L A^L T^L,_x)\Big|_{x=x_d} = (K^R A^R T^R,_x)\Big|_{x=x_d} + Q_d \tag{1.2-32}$$

If $Q_d = 0$ and $A^L K^L = A^R K^R$, then $T,_x$ is continuous and T is at least of class C^1 and is a "strong" solution, i.e., T and its first and second derivatives exist. Otherwise $T,_x$ is discontinuous and T is a weak solution, i.e., its second derivative contains a Dirac-delta function.

1.2.4 The Addition of Radiation

Suppose a bar is not insulated but, instead, exists in an environment surrounded by a body with a fixed temperature, T_f, as indicated in Fig. 1.2-4. The process of convection will remove heat from the bar at a rate, Q_c, a term that be considered analogous to Q in the governing differential equations of (1.2-11) for transient conduction and (1.2-13) for steady state problems. Q_c is assumed to be linearly proportional to the difference of the temperature of the bar, T, and of the fixed temperature with a factor, b:

$$Q_c = b(T - T_f) \tag{1.2-33}$$

In particular, this is not a constitutive equation and b is not a material parameter. The coefficient b is problem dependent and involves both volumetric and surface material. The transient equation becomes

$$(kT,_x),_x - Q_c + Q = cT,_t\,, \qquad k = KA \quad \text{and} \quad c = \overline{\rho}C \tag{1.2-34}$$

in which the minus sign is used with Q_c to denote the assumption that heat is being removed rather than added. Since T_f is a constant, the term bT_f is usually incorporated with Q. Then the governing equations become

$$(kT_{,x})_{,x} - bT + Q = cT_{,t} \ , \qquad k = KA \quad \text{and} \quad c = \overline{\rho}C \qquad (1.2\text{-}35)$$

for transient problems. For the steady-state case, the equation simplifies to

$$(qA)_{,x} + bT - Q = 0 \ , \qquad q = -KT_{,x} \qquad (1.2\text{-}36_1)$$

or

$$\left(kT_{,x}\right)_{,x} - bT + Q = 0 \qquad (1.2\text{-}36_2)$$

Suppose $k = k_o$, a constant. Then the solution to the homogeneous equation $k_o T_{,xx} - bT = 0$ is

$$T_h = c_1 \cosh rx + c_2 \sinh rx \ , \qquad r = \sqrt{\frac{b}{k_o}} \qquad (1.2\text{-}37)$$

where c_1 and c_2 are integration constants. This solution is quite different from the case when $b = 0$.

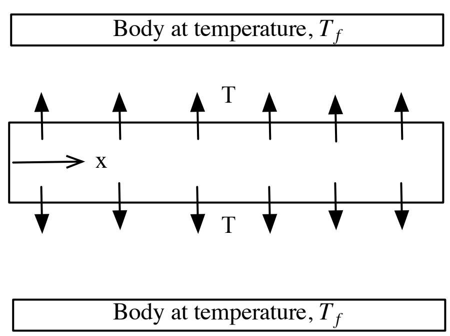


Fig. 1.2-4. Heat loss due to convection.

<u>Example:</u>

Suppose the data are the following:

(i) the domain, $0 \le x \le L$;
(ii) the area, $A(x) = 1$;
(iii) the conductivity, $K(x) = k_o$, a constant, $(k = k_o)$;
(iv) the heat source, $Q = Q_o$, a constant; and

(v) the boundary fluxes, $q^* = q_0^*$ at $x = 0$ and $q^* = q_L^*$ at $x = L$.

Note that the boundary conditions are of the Neumann type which results in ill-posedness for conventional heat conduction. The particular solution is $T_p = Q_o/b$. The complete solution is $T = T_h + T_p$ so that the first derivative is $T_{,x} = c_1 r \sinh rx + c_2 r \cosh rx$. Application of the flux boundary conditions at x = 0 and x = L result in

$$(qn)\big|_{x=0} = \left[-k_o T_{,x}(-1)\right]\big|_{x=0} = -q_0^* \qquad \text{or} \qquad c_2 = -\frac{q_0^*}{rk_o} \tag{1.2-38}$$

and

$$\begin{aligned} &(qn)\big|_{x=L} = \left[-k_o T_{,x}(1)\right]\big|_{x=L} = q_L^* \\ &\text{or} \quad k_o(c_1 r \sinh rL + c_2 \cosh rL) = q_L^* \end{aligned} \tag{1.2-39}$$

respectively. The coefficient, c_1, is obtained from the last equation. Note that with this differential equation, the use of flux boundary conditions results in a well-posed problem.

1.2.5 The Addition of Advection

Consider the situation where heat transport is assisted by the motion of a fluid through a medium. The problem can arise if the heat-conducting material contains pores through which a fluid or a gas is moving as represented symbolically in Fig. 1.2-5.

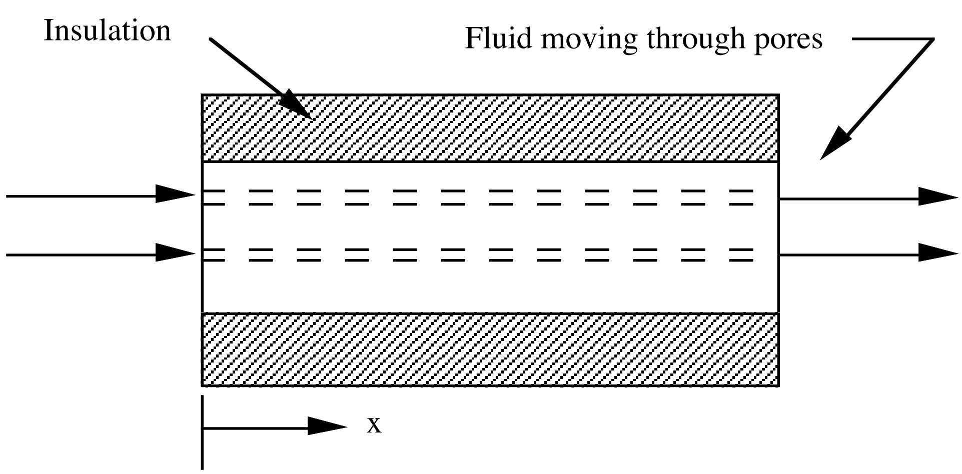


Fig. 1.2-5. Heat flux assisted by fluid transport.

Suppose the fluid is moving with velocity v^f. The result is tha heat is being moved through the bar proportional to the fluid velocity. We denote this heat transfer as an additional flux q^f where

$$q^f = v^f T \tag{1.2-40}$$

Equation (1.2-12) governing transient heat conduction now has an additional term:

$$(kT,_x),_x - v^f T,_x + Q = cT,_t \tag{1.2-41}$$

For steady-state heat conduction, $T,_t = 0$, and the governing equation reduces to

$$(kT,_x),_x - v^f \mathrm{T},_\mathrm{x} + Q = 0 \tag{1.2-42}$$

The first term, $(kT,_x),_x$, is called **diffusion** and the second term, $v^f T,_x$, is **advection** so that (1.2-35) is often called the **diffusion-advection equation**. Normally, the velocity parameter v^f is modified to incorporate geometrical properties associted with details of the fluid channels.

1.3 STRESS ANALYSIS FOR A BAR

For any problem in stress analysis there are always three equations; the equation of motion, a constitutive equation relating stress and strain, and a kinematic equation relating strain to a primary displacement variable. In the next subsection each of these equations is summarized for the case of uniaxial motion in a bar, and the implications of various combinations of boundary conditions are discussed.

1.3.1 Governing Equations for an Elastic Bar

Consider a bar of cross-sectional area A(x) and length L as shown in Fig. 1.3-1. Assume no dependence on y or z. Let u denote the displacement of any cross-sectional plane in the bar when the bar is deformed as shown in Fig. 1.3-2. Then $v = \dot{u}$ is the velocity and $a = \dot{v}$ denotes the acceleration.

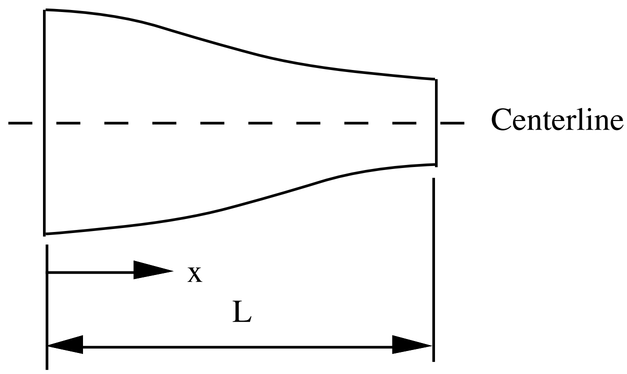


Fig. 1.3-1. Bar with variable cross-sectional area.

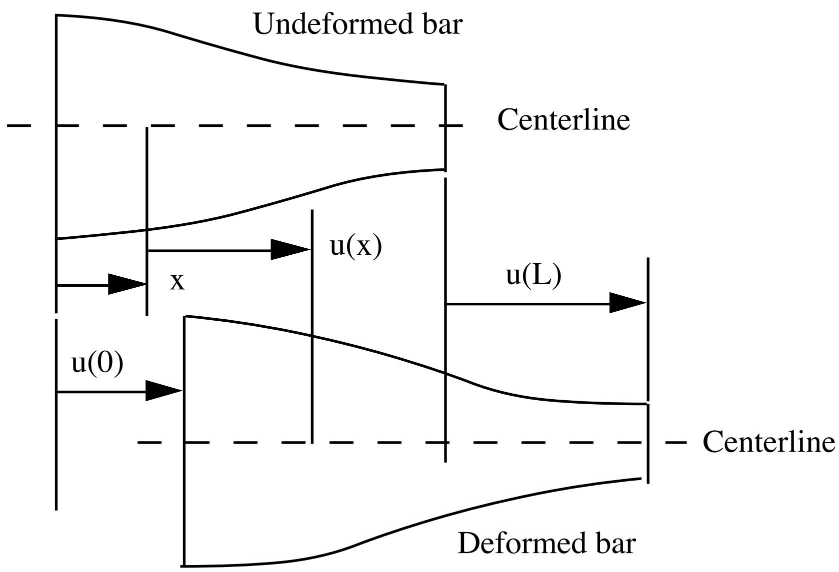


Fig. 1.3-2. Displacement of various points along a bar.

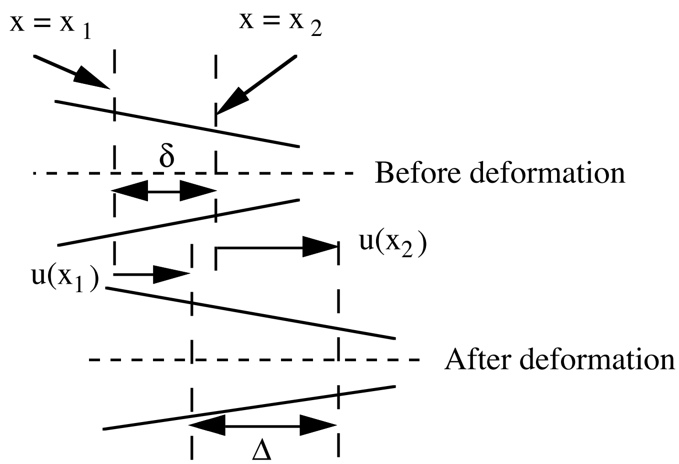


Fig. 1.3-3. Displacement of cross-sectional planes initially a distance δ apart.

Consider two cross-sectional planes at $x = x_1$ and $x = x_2$, a distance initially $\delta = x_2 - x_1$ apart as shown in Fig. 1.3-3. After deformation, these same two planes are a distance $\Delta = \delta + u(x_2) - u(x_1)$ apart. The strain, e_1, at the location $x = x_1$ is defined to be

$$\begin{aligned}
e_1 &= \lim_{\delta\to 0}\frac{\Delta-\delta}{\delta} \\
&= \lim_{\delta\to 0}\frac{u(x_2)-u(x_1)}{\delta} \\
&= \lim_{\delta\to 0}\frac{u(x_1)+u,_x(x_1)[x_2-x_1]+u,_{xx}(x_1)[x_2-x_1]^2/2!+\cdots-u(x_1)}{\delta} \\
&= u,_x(x_1)
\end{aligned} \qquad (1.3\text{-}1)$$

in which Taylor series has been used. A conversion to x as the generic variable, instead of x_1, yields the kinematic relationship giving strain in terms of the displacement gradient:

$$e(x) = u,_x \qquad (1.3\text{-}2)$$

Next, we derive the governing equation of motion. In the process, we argue that a field variable called stress must exist and the traction on a surface consists of a linear product of stress and the unit normal.

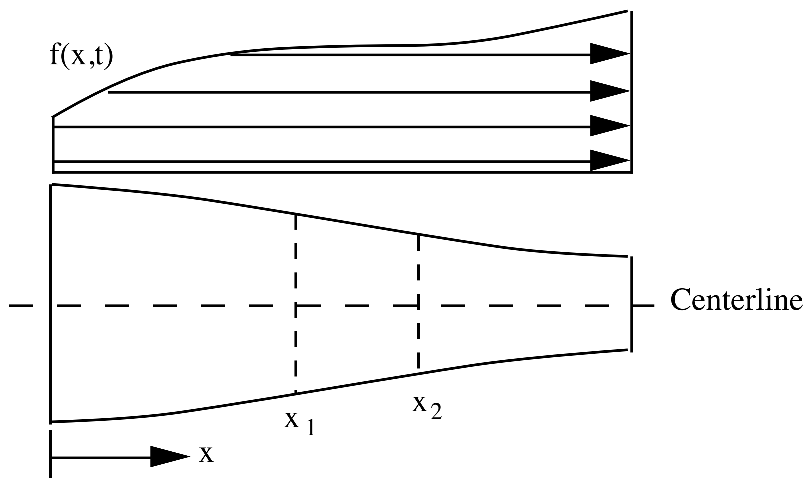


Fig. 1.3-4. Force distribution acting on bar.

Suppose a force per unit length f(x,t) acts on a bar as shown above in Fig. 1.3-4. The sketch of the forcing function is intended to emphasize the point that the forcing function acts in the axial direction, not transversely as is the case for a beam, and that the force can vary along the bar. Pass imaginary cuts through the body at $x = x_1$ and $x = x_2$ and consider the portion $x_1 \le x \le x_2$ as a separate body. Assume that the effect of one part of the body on the other can be represented as a traction τ per unit cross-sectional area and that the resultant force, F, acting on the cross-section is

$$F = \tau A \qquad (1.3\text{-}3)$$

The resultant of the applied force acting on the lateral surface of this segment is

$$R = \int_{x_1}^{x_2} f(x,t)dx \qquad (1.3\text{-}4)$$

Any body force, such as the one due to gravity, can be incorporated in R. The resultant of the applied force and the forces acting over the transverse cuts are shown in Fig. 1.3-5.

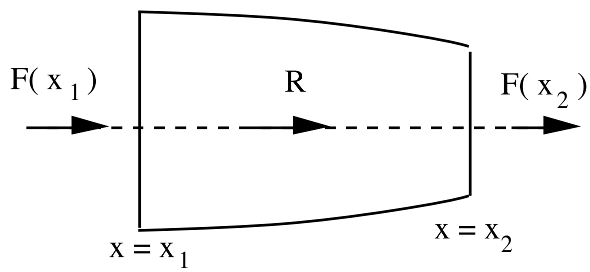


Fig. 1.3-5. Resultant forces acting on a segment of the bar.

The analogue of Newton's law for a segment of a continuous body is called Euler's law:

$$R + F(x_1) + F(x_2) = \int_{x_1}^{x_2} \rho A \ddot{u}\,dx \tag{1.3-5}$$

where $\ddot{u}$ denotes the acceleration of the points in the cross-sectional plane at x and ρ is the mass density. The left side is the total force applied to the segment while the right side is the inertia. This relation is postulated to hold for all segments, i.e.,$\forall$ x_1 and x_2. Suppose we take the limit as $x_1 \to x_2$ of all terms in (1.3-5). Since the integration range vanishes and we expect R, ρ, A and $\ddot{u}$ to vary continuously, it follows that

$$\lim_{x_1 \to x_2} R = 0\ , \qquad \lim_{x_1 \to x_2} \int_{x_1}^{x_2} \rho A \ddot{u}\,dx = 0 \tag{1.3-6}$$

there remains

$$\lim_{x_1 \to x_2} F(x_1) + F(x_2) = 0 \tag{1.3-7}$$

or

$$\lim_{x_1 \to x_2} [\tau A]_{x_1} + [\tau A]_{x_2} = 0 \tag{1.3-8}$$

But, the area is generally a continuous function so that

$$\lim_{x_1 \to x_2} A\big|_{x_1} = A\big|_{x_2} \tag{1.3-9}$$

which leaves

$$\lim_{x_1 \to x_2} \tau\big|_{x_1} = -\tau\big|_{x_2} \tag{1.3-10}$$

If the assumption of Cauchy is made that τ is a function of the unit normal, then the implication of (1.3-10) is that τ is a linear function of n. The factor of proportionality is called the stress, σ, (stress tensor, or flux), i.e., we have proven the existence of a stress function just as the existence of the flux vector was proven in the context of heat conduction. Therefore,

$$\tau = \sigma n \tag{1.3-11}$$

so that if the stress is known, (1.3-11) can be utilized to obtain the traction on any surface with normal defined by a positive or negative value of n.

For a particular cross section with area, A, define the stress resultant, Σ, to be

$$\Sigma = \sigma A \tag{1.3-12}$$

Then, from (1.3-3) (1.3-11) and (1.3-12), alternative equivalent expressions for the force acting on a transverse cut are the following:

$$F = \tau A\ , \qquad F = \sigma n A \qquad \text{and} \qquad F = \Sigma n \tag{1.3-13}$$

With the objective of deriving the equation of motion, we substitute (1.3-13) in (1.3-5) to obtain

$$R + \Sigma n\big|_{x_1} + \Sigma n\big|_{x_2} = \int_{x_1}^{x_2} \rho A \ddot{u} dx \tag{1.3-14}$$

If the definition of R from (1.3-4) is used, and if the divergence theorem is invoked for those terms involving the stress resultant, then (1.3-14) becomes

$$\int_{x_1}^{x_2} [\,\Sigma,_x + f(x,t) - \rho A \ddot{u}\,] dx = 0 \tag{1.3-15}$$

which must hold for all choices of x_1 and x_2. From Appendix A.1, we conclude that the integrand must be zero so the governing differential equation of motion for a bar is the following

$$\Sigma,_x + f(x,t) = \rho A \ddot{u} \tag{1.3-16}$$

Up to now, no mention has been made concerning the nature of the material. Consider the class of materials that can be considered linearly elastic. If E is **Young's modulus**, then the **constitutive (stress-strain) relation** is

$$\sigma = Ee \tag{1.3-17}$$

Let k be the **stiffness** and $\bar{\rho}$ the mass per unit length:

$$k = EA \qquad \text{and} \qquad \bar{\rho} = \rho A \tag{1.3-18}$$

If the governing equations are written separately, the result is

$$\Sigma,_x + f = \bar{\rho}\ddot{u}\ , \qquad \sigma = Ee \qquad \text{and} \qquad e = u,_x \tag{1.3-19}$$

or, if the equations are combined in one equation in terms of u only, then

$$(ku,_x),_x + f = \bar{\rho}\ddot{u} \tag{1.3-20}$$

Initial values for u and $\dot{u}$ must be given and boundary conditions are usually of the form that either u is prescribed or the force $F = n\Sigma$ is prescribed at each end. An alternative form to the force boundary condition is to divide by A and work in terms of the traction ($\tau = n\sigma = nEu,_x$).

Under certain conditions, this one-dimensional formulation can actually be the governing equation for a three-dimensional problem, e.g., a slab in which case A can be chosen as a unit area. This one-dimensional treatment is appropriate for the three-dimensional problem if no function depends on the transverse coordinates y and z.

1.3.2 Solutions to the Static Problem

Consider the static form of (1.3-19) or (1.3-20), i.e., assume all variables are independent of t. Then the governing equations are

$$(\sigma A),_x + f = 0\ , \qquad \sigma = Eu,_x \tag{1.3-21$_1$}$$

in which the constitutive equation and the strain-displacement equation have been combined, or

$$(ku,_x),_x + f = 0 \tag{1.3-21$_2$}$$

with k = AE. The data for the problem consist of:

(i) the coefficient function k(x) [or A(x) and E(x)];
(ii) the forcing function f(x);
(iii) the domain of the solution $0 \le x \le L$; and
(iv) the boundary conditions at $x = 0, L$.

Solutions to the boundary value problem satisfy (1.3-21) and the boundary conditions.

Boundary Conditions

A large number of problems fall under the following possibilities:

(i) $u(0) = u_0$ and $u(L) = u_L$ (Dirichlet)

(ii) $\left.\begin{array}{l} u(0) = u_0 \text{ and } \tau(L) = \tau_L \\ \tau(0) = \tau_0 \text{ and } u(L) = u_L \end{array}\right\}$ (mixed) (1.3-22)

(iii) $\tau(0) = \tau_0$ and $\tau(L) = \tau_L$ (Neumann)

where u_0, u_L, τ_0 and τ_L are prescribed values. Recall that $\tau = \sigma n$ and in anticipation of obtaining a solution to (1.3-21) let

$$F_1(x) = \int_0^x f(\tilde{x})d\tilde{x}\,, \qquad F_2(x) = \int_0^x \frac{1}{k} F_1(\hat{x})d\hat{x} \tag{1.3-23}$$

where $\tilde{x}$ and $\hat{x}$ are variables of integration. Consider the formulation given by ($1.3\text{-}21_1$). After one integration, the result can be expressed by any of the following forms:

$$\sigma A + F_1 = c_1 \qquad \text{or} \qquad EAu,_x + F_1 = c_1 \qquad \text{or} \qquad u,_x + \frac{F_1}{k} = \frac{c_1}{k} \tag{1.3-24}$$

where $\sigma = Eu,_x$ and $k = EA$ have been used to obtain the second and third equations, respectively. Another integration yields

$$u(x) + F_2(x) = c_1\overline{k}^{-1}(x) + c_2 \tag{1.3-25}$$

where we introduce an integrated effect of the inverse of k as follows:

$$\overline{k}^{-1} = \int_0^x \frac{1}{k(\hat{x})} d\hat{x} \tag{1.3-26}$$

The boundary conditions are used to evaluate the coefficients c_1 and c_2. Next we investigate the implications of the form of the boundary conditions on the evaluation of these coefficients.

Case (i): Dirichlet

With $u(0) = u_0$, the application of (1.3-25) at x = 0 yields

$$u_0 + F_2(0) = c_1 \overline{k}^{-1}(0) + c_2$$

Since $F_2(0) = 0$ and $\overline{k}^{-1}(0) = 0$, necessarily the coefficient c_2 satisfies $c_2 = u_0$. Now the use of (1.3-25) at x = L provides the equation

$$u(L) + F_2(L) = c_1 \overline{k}^{-1}(L) + u_0 \qquad \text{or} \qquad c_1 = [u_L - u_0 + F_2(L)] / \overline{k}^{-1}(L)$$

Thus, we conclude for Dirichlet conditions that

$$u(x) = c_1 \overline{k}^{-1}(x) + c_2 - F_2(x) \quad \text{and} \quad \sigma = Eu_{,x} = \frac{E}{k}[c_1 - F_1(x)] \tag{1.3-27}$$

For this case the displacement, u, had to be determined (to obtain c_1) before the stress could be obtained. A problem such as this, where the stress (flux) can be determined only by first obtaining the displacement (primary variable), is said to be **statically indeterminate**.

Case (ii): Mixed

Suppose the boundary condition on the traction, τ, is specified at x = 0. Then

$$\tau(0) = \tau_0 = n\sigma(0) = -\sigma(0)$$

From (1.3-24) with x = 0, the coefficient c_1 is given by

$$c_1 = F_1(0) + A(0)\sigma(0) = -A_0\tau_0$$

where $A_0 = A(0)$. We now use (1.3-24) to solve for the stress, σ, for any x:

$$\sigma = \frac{c_1}{A} - \frac{F_1}{A} = -\frac{[\tau_0 A_0 + F_1(x)]}{A(x)} \tag{1.3-28}$$

We conclude that the stress can be determined everywhere without the need for obtaining the displacement. Such a problem is said to be **statically determinate**. The remaining boundary condition on u is used to obtain the second integration constant, c_2, and is necessary to obtain the complete analytical expression for u. Similar arguments hold if the mixed boundary conditions include the traction prescribed at x=L.

Case (iii): Neumann

As for Case (ii), the boundary condition $\tau(0) = \tau_0$ can be used to evaluate c_1, with the result given in (1.3-28). Now, suppose the traction boundary condition at the end x = L is also imposed. The result is

$$\tau_L = n\sigma(L) = \sigma(L) = -\frac{[\tau_0 A_0 + F_1(L)]}{A(L)}$$

or

$$A_0\tau_0 + A_L\tau_L + F_1(L) = 0 \tag{1.3-29}$$

where $A_L = A(L)$. This implies that the boundary values τ_0 and τ_L cannot be arbitray. For this case τ_0 and τ_L must meet the criterion of overall equilibrium; otherwise the problem is not well posed as discussed in Section 1.2. Furthermore, there is now no condition on u so that c_2 cannot be determined. For this case u is not unique; it can only be determined to within an arbitrary constant which corresponds to an arbitrary rigid body translation.

A boundary condition of this type will cause, for example, the stiffness matrix in a finite element formulation to be singular. A solution cannot be obtained unless the matrix is modified. This is an example of why it is important to understand the role of boundary conditions in obtaining analytical solutions to simple problems.

Solutions for Smooth Data

If the coefficient functions and the forcing function are of class C^∞, i.e., the functions and all their derivatives are continuous (see Appendix A.8), then the data are said to be smooth. Then, it follows that u and σ, as solutions of (1.3-21_1) or (1.3-21_2) are also of class C^∞. When we seek a solution to the problem, we must look to just those functions belonging to C^∞. This is a very restrictive condition and such problems might be considered classical.

Typical examples of problems in this class are cases where k and f are polynomials in x. Except for the fact that the expression for $\overline{k}^{-1}$ can become complicated, solutions are easily obtained for this elementary differential equation. These are the types of problems typically considered in college textbooks on ordinary and partial differential equations and, therefore, will not be pursued futher. Unfortunately, most engineering problems do not fall in this category. Examples of situations where functions are not smooth are point forces, jumps in material properties from laminated materials, and abrupt changes in cross-sectional areas. In the next subsection, we demonstrate that obtaining solutions is no longer a trivial matter even for one-dimensional problems. As shown later, one of the great attributes of the finite element method is the ease with which problems with nonsmooth functions can be handled.

1.3.3 Solutions for Nonsmooth Data

Discontinuous Forcing Function and Smooth Stiffness:

A natural and convenient method for describing functions that are not smooth is to use generalized functions. Mathematically, the corresponding material is often called distribution theory. It is assumed that the reader is familiar with the approach at the level offered in most undergraduate texts on mechanics of materials. This is the level of the summary provided in Appendix A.8.

Consider the case of a step force applied at x = a. Then our basic equilibrium equation becomes

$$(ku_{,x})_{,x} = -F_o H[x - a] \qquad 0 < x < L \tag{1.3-30}$$

where H[x - a] is the Heaviside (step) function, and F_o is a constant describing the magnitude of the force. As before, we integrate twice to obtain

$$ku_{,x} = -F_o < x - a >^1 + c_1 \,, \qquad u = \int \frac{c_1}{k} dx - g(x) + c_2 \tag{1.3-31}$$

where c_1 and c_2 are constants of integration, and $< x - a >^1$ denotes the linear ramp function. The function g(x) is defined by

$$g(x) = F_o \int_a^x \frac{(\hat{x} - a)}{k(\hat{x})} d\hat{x} \tag{1.3-32}$$

If the coefficient, k, is constant then the integral can be performed immediately to arrive at

$$g(x) = \frac{F_o}{k} \frac{< x - a >^2}{2} = \frac{F_o}{2k} H[x - a](x - a)^2 \tag{1.3-33}$$

where $<x - a>^2$ is the quadratic ramp function. For this situation where k is a special case of a smooth function, it is seen that u is continuous, $u_{,x}$ is continuous, but $u_{,xx}$ is not. The function u therefore has the continuity classification of $u \in C^1$. If the point x = a is excluded from the domain, then the solution $u \in C^\infty$.

Point Forcing Function and Smooth Stiffness

Suppose P is a point force applied at x = a. The governing differential equation is

$$(ku_{,x})_{,x} = -P\delta[x - a]\ , \qquad 0 < x < L\ , \qquad 0 < a < L \tag{1.3-34}$$

Integrate once to obtain

$$ku_{,x} = -PH[x - a] + c_1 \tag{1.3-35}$$

with c_1 an integration constant. Immediately we see that $u_{,x}$ is discontinuous. Integrating again with c_2 as a second integration constant shows that the displacement, u, is given by

$$u = \int \frac{c_1}{k} d\hat{x} - \int \frac{PH[\hat{x}-a]}{k(\hat{x})} d\hat{x} + c_2 \tag{1.3-36}$$

With the use of (1.3-35), the values of $u_{,x}$ immediately to the left and right of the point x = a are determined to be

$$u_{,x}\big|_{x=a^-} = \frac{c_1}{k(a)} \quad \text{and} \quad u_{,x}\big|_{x=a^+} = \frac{1}{k(a)}\left[-P + c_1\right] \tag{1.3-37}$$

The jump in $u_{,x}$ at x=a, denoted as $[u_{,x}]_a$, is

$$[u_{,x}]_a = u_{,x}\big|_{x=a^+} - u_{,x}\big|_{x=a^-} = -\frac{P}{k(a)} \tag{1.3-38}$$

so that we have a jump in the strain and, hence, the stress. Here, we see that $u \in C^0$. If point a is excluded from the domain, then $u \in C^\infty$.

The alternative approach for obtaining a solution to this problem without using generalized functions is as follows:

(i) solve the differential equation for 0 < x < a and a < x < L, which yields four integration constants;
(ii) apply two boundary conditions at x=0 and x=L;
(iii) apply the continuity condition $u\big|_{a^+} = u\big|_{a^-}$; and
(iv) apply the jump condition of (1.3-38).

Discontinuous Stiffness and Smooth Forcing Function

Let the coefficient, k, in the governing differential equation be the piecewise constant function $k = k_0 + k_1 H[x - a]$ which, for an arbitrary forcing function, is

$$(ku_{,x})_{,x} = -f\ , \qquad 0 < x < L\ , \qquad 0 < a < L \tag{1.3-39}$$

After one integration, it follows that

$$ku_{,x} = -\int f dx + c_1$$

or, solving for $u_{,x}$, we have

$$u_{,x} = \frac{1}{k_0 + k_1 H[x-a]}\left\{-\int f dx + c_1\right\} \tag{1.3-40}$$

Define the function, g_a, according to

$$g_a \equiv -\int_o^a f dx + c_1 \tag{1.3-41}$$

Then, we can express $u_{,x}$ at points on either side of x = a as

$$u_{,x}\Big|_{x=a^-} = \frac{g_a}{k_0} \qquad \text{and} \qquad u_{,x}\Big|_{x=a^+} = \frac{g_a}{k_0 + k_1} \tag{1.3-42}$$

Hence the jump in $u_{,x}$ at x = a becomes

$$[u_{,x}]_a = \frac{g_a}{k_0(k_0 + k_1)}\left[k_0 - (k_0 + k_1)\right] = \frac{-k_1\ g_a}{k_0\ (k_0 + k_1)} \tag{1.3-43}$$

Since there is only a jump in the value of $u_{,x}$, an integration yields a function, u, that is continuous, i.e., $u \in C^0$. For example, if one notes that

$$\frac{1}{k_0 + k_1 H(x-a)} = \frac{1}{k_0(k_0 + k_1)}\left(k_0 + k_1 - k_1 H[x-a]\right) \tag{1.3-44}$$

then (1.3-40) can be integrated directly.

<u>Strong Formulation and Strong Solutions</u>

The governing differential equation and the specific jump conditions in data at discrete points is called the strong formulation. (Note: The use of generalized functions to express the coefficient and forcing functions is merely an indirect way of specifying the jump or discontinuity conditions.) The solution to the differential equation that also satisfies the jump and boundary conditions is called the strong solution. For each subdomain defined between the points at which discontinuity conditions hold, the solution is of class C^∞. This degree of smoothness is not required for weak solutions, which will be discussed in detail in Chapter 6.

1.3.4 The Embedded Bar

Another class of problem that can be treated with one-dimensional methods is exemplified by a bar embedded in a medium. Examples are piles in earth media, reinforcing rods in concrete, and fiber-reinforced composites. If axial forces are applied to the bar, the effect of the surrounding medium can often be approximated with distributed springs that provide a retarding spring force, k_s, per unit length of bar and for a unit displacement u. The situation is shown schematically in Fig. 1.3-6.

The altered equation of motion can be obtained by assuming that there is an addition to the forcing function, f^a, which acts in the direction opposite to the displacement, u, so that (1.3-16) becomes

$$(ku,_x),_x + f + f^a = \overline{\rho}\ddot{u} \, , \qquad f^a = -k_s u \tag{1.3-45}$$

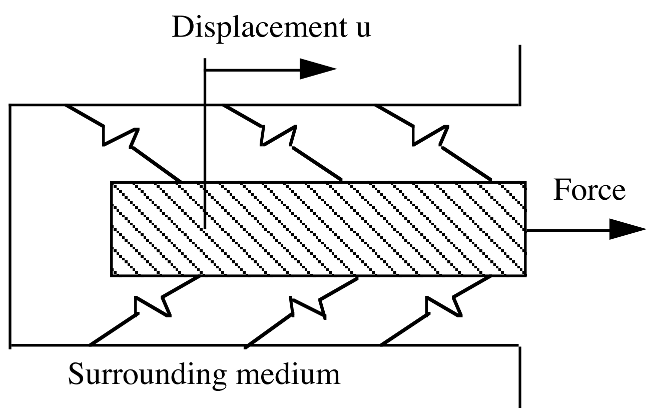


Fig. 1.3-6. A bar embedded in a surrounding medium.

The resulting equation of motion and equilibrium equation in terms of u are

$$\begin{aligned} &(ku,_x),_x - k_s u + f = \bar{\rho}\ddot{u} && \text{(transient)} \\ &(ku,_x),_x - k_s u + f = 0 && \text{(steady state)} \end{aligned} \tag{1.3-46}$$

Note that the static equation is of a form identical to that of steady-state heat conduction with lateral convection (1.2-29_2). Therefore, statements made concerning the nature of the solution in that context hold here as well. In particular, if stress boundary conditions are specified on each end, then the problem remains well posed. The physical reason is that the restraining force simulated through springs on the lateral surface now provide a mechanism for maintaining equilibrium of the bar.

The parameter k_s used to characterize the distributed springs is not a material parameter. Instead it is a parameter that must be determined experimentally or deduced from an exact three-dimensional solution to the complete problem. Generally, k_s will depend on the material properties of both the bar and the surrounding medium, and on the geometry of the problem. Also, it is quite possible for k_s to depend on x although it is generally treated as a constant.

If the embedding medium also contains a viscous term (with a viscous coefficient of c_V), the governing transient equation becomes

$$(ku,_x),_x - c_v\dot{u} - k_s u + f = \bar{\rho}\ddot{u} \tag{1.3-47}$$

The term involving the first time derivative of u is of the same order as the transient part of the heat conduction equation. However, here both first and second time derivatives of the primary variable are present.

If the constitutive equation of the bar, itself, is **viscoelastic** instead of just elastic, then the stress depends on the strain rate as well as the strain:

$$\sigma = Eu,_x + E_v\dot{u},_x \tag{1.3-48}$$

in which E_v is the viscoelastic material parameter. If we define a viscoelastic stiffness to be $k_v = AE_v$, then (1.3-47) is replaced with

$$(ku,_x),_x + (k_v\dot{u},_x),_x - c_v\dot{u} - k_s u + f = \bar{\rho}\ddot{u} \tag{1.3-49}$$

which is the equation of motion for a viscoelastic bar embedded in an elastic-viscous medium. The equation of motion for a free, viscoelastic bar is obtained by setting $c_v = 0$ and $k_s = 0$.

1.4 STRESS ANALYSIS FOR A BEAM

1.4.1 Euler-Bernoulli Theory

Consider a beam with the orientation shown in Fig. 1.4-1 The applied force and moment per unit length are given by F(x,t) and M(x,t), respectively. Consider a typical segment of length x_2 - x_1. Let the displacement of the center of mass be w(x,t) in the direction of y, and let the rotation of the element be θ(x,t) with the positive sense chosen to be in the z-direction using the right-hand rule (See Fig. 1.4-2).

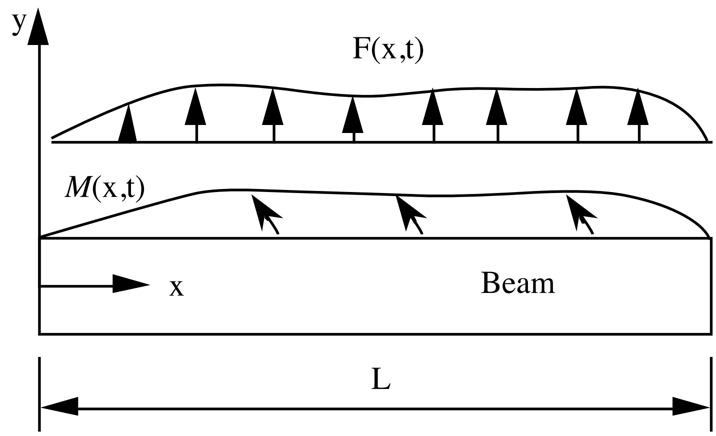


Fig. 1.4-1. Sketch of beam without boundary conditions.

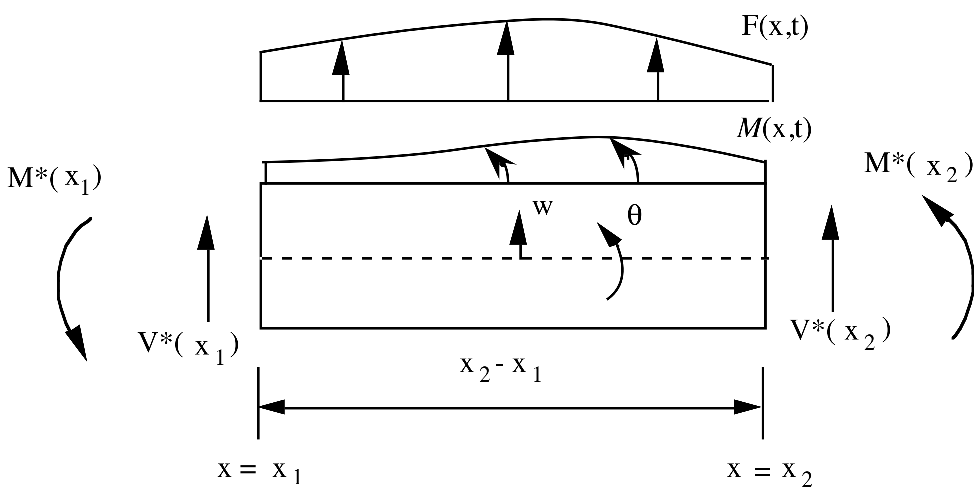


Fig. 1.4-2. Beam segment, notation and positive convention.

Assume that the effects of the remainder of the beam on the element can be represented though internal generalized forces that we denote as the moment M* and transverse shear V*. Euler's first law for transverse motion of the element is:

$$V^*(x_1) + V^*(x_2) + \int_{x_1}^{x_2} F(x,t)\, dx = \int_{x_1}^{x_2} \bar{\rho}\ddot{w}(x,t)\, dx \qquad (1.4\text{-}1)$$

where $\bar{\rho}$ is the mass per unit length and $\ddot{w}$ is the acceleration of a mass point in the y direction. Assume that the transverse force, V*, on each face depends on the unit normal to the respective face. Take the limit as x_1 approaches x_2. In a procedure identical to that used to argue that there must exist a flux vector, **q**, in heat conduction and a stress, σ, for the bar problem, we determine that for the beam there must exist a generalized stress, called the transverse shear, V, which represents the linear transformation between the transverse force, V*, and the unit normal:

$$V^* = Vn \qquad (1.4\text{-}2)$$

with n = -1 on a left face and n = 1 on a right face. Now, the first two terms of (1.4-1) become

$$V^*(x_1) + V^*(x_2) = [Vn]_{x=x_1} + [Vn]_{x=x_2} = \int_{x_1}^{x_2} V_{,x}\, dx \qquad (1.4\text{-}3)$$

in which the divergence theorem has been used. Then (1.4-1) becomes

$$\int_{x_1}^{x_2} [V_{,x} + F(x,t) - \bar{\rho}\ddot{w}]dx = 0 \qquad (1.4\text{-}4)$$

which is to hold for all x_1 and x_2. Since the limits of integration are arbitrary (Appendix A.1), the integrand must be zero so that

$$V_{,x} + F = \bar{\rho}\ddot{w} \qquad (1.4\text{-}5)$$

which is the partial differential equation governing transverse motion of a beam.

Euler's second law for rotary motion about the center of mass of the element is

$$M^*(x_1) + M^*(x_2) + \left[V^*(x_2) - V^*(x_1)\right]\frac{(x_2-x_1)}{2} + \int_{x_1}^{x_2} \mathcal{M}(x,t)dx = \int_{x_1}^{x_2} I_r\ddot{\theta}dx \qquad (1.4\text{-}6)$$

where $\ddot{\theta}$ is the angular acceleration, I_r is the rotational inertia per unit length, and the element is assumed short enough so that the force distribution does not contribute to the moment. Again, the procedure is followed of first considering the limit as x_1 approaches x_2 with the assumption that M* is a function of the unit normal. As before, we are forced to conclude that the relationship is linear. It is conventional to let the factor of proportionately be the bending moment, M, so that

$$M^* = Mn \qquad (1.4\text{-}7)$$

The resulting positive convention for both bending moment and shear is shown in Fig. 1.4-3.

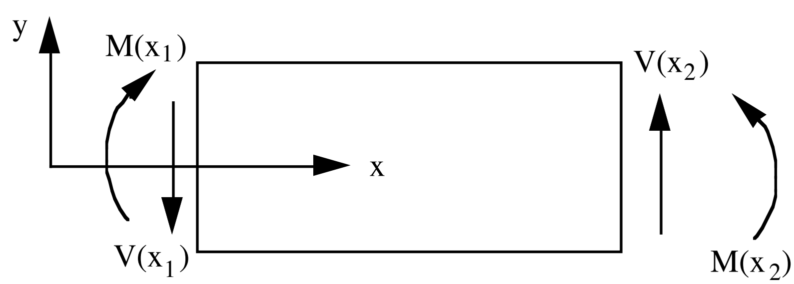


Fig. 1.4-3. Positive convention for bending moment and shear.

The first two terms of (1.4-6) become

$$M^*(x_1) + M^*(x_2) = [Mn]_{x_1} + [Mn]_{x_2} = \int_{x_1}^{x_2} M_{,x}\, dx \qquad (1.4\text{-}8)$$

in which the divergence theorem has been applied. The next term in (1.4-6) is approximated as follows:

$$\left[V^*(x_2) - V^*(x_1)\right]\frac{(x_2-x_1)}{2} = [nV|_{x_2} - nV|_{x_1}]\frac{(x_2-x_1)}{2}$$
$$= \left[\frac{V(x_2)+V(x_1)}{2}\right](x_2 - x_1) \cong \int_{x_1}^{x_2} V\,dx \qquad (1.4\text{-}9)$$

in which the approximation is used to obtain a representation involving an integral. Now consider the second term on the right side of (1.4-6) and let

$$I = \int y^2 dA = r_g^2 A \qquad (1.4\text{-}10)$$

be the second moment of area for a typical cross-section of area A. Then r_g is called the **radius of gyration** for the cross-section, and the rotational inertia per unit length is

$$I_r = \int \rho y^2 dA = \overline{\rho} r_g^2 \qquad (1.4\text{-}11)$$

For a beam of rectangular cross-section of width b and height H, a straightforward calculation shows that

$$I = \frac{1}{12} bH^3\,, \qquad A = bH \qquad \text{and} \qquad I_r = \frac{\rho}{12} bH^3 = \overline{\rho}\,\frac{H^2}{12} \qquad (1.4\text{-}12)$$

where we make use of the fact that, by definition, $\overline{\rho} = \rho bH$. With the substitution of (1.4-8), (1.4-9) and (1.4-11) in (1.4-6), the result is

$$\int_{x_1}^{x_2} [V + M,_x + \mathcal{M} - I_r\ddot{\theta}]\,dx = 0 \qquad (1.4\text{-}13)$$

which must hold for arbitrary x_1 and x_2. Hence, the integrand must vanish, and the result is a partial differential equation governing the rotational motion of the beam. This equation and the equation governing transverse motion are summarized together as follows:

$$M,_x + V + \mathcal{M} = I_r\ddot{\theta} \qquad \text{and} \qquad V,_x + F = \overline{\rho}\,\ddot{w} \qquad (1.4\text{-}14)$$

The equations must be solved subject to the boundary conditions that either M or θ is prescribed at each end and either V or w is prescribed at each end.

The assumption of an elastic body implies the following generalized stress-strain relations:

$$M = EI\kappa \qquad \text{and} \qquad V = GAe_s \tag{1.4-15}$$

in which E is **Young's modulus**, G is the **shear modulus**, κ is the **bending strain** and e_s is the **transverse shear strain**. A kinematical argument involving the assumption that transverse fibers originally straight and perpendicular to the centerline remain straight but not necessarily perpendicular to the deformed centerline result in the strain deformation relations (see Fig. 1.4-4)

$$\kappa = \theta,_x \qquad \text{and} \qquad e_s = w,_x - \theta \tag{1.4-16}$$

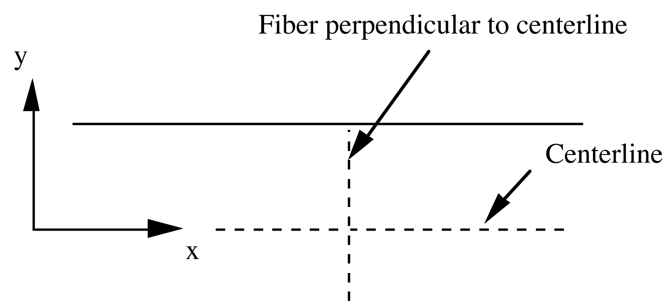
The conventional Euler-Bernouilli beam theory is obtained under the **Kirchhoff Hypothesis** which states that fibers originally straight and perpendicular to the center line remain straight and perpendicular to the center line. This implies no shear strain or

$$\theta = w,_x \qquad \text{(Kirchhoff Hypothesis)} \tag{1.4-17}$$

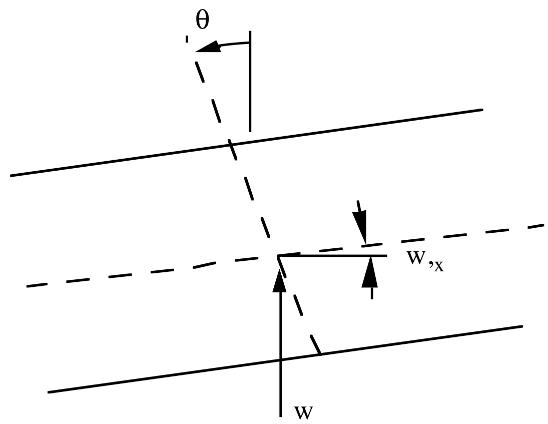
The hypothesis has proven to be extremely good for thin beams, i.e., L/H > 5. The substitution of (1.4-17) in (1.4-16) and (1.4-15) yields

$$M = EIw,_{xx} \qquad \text{and} \qquad V = 0 \tag{1.4-18}$$

However, unless $M,_x + M = I_r\ddot{\theta}$ which is highly unlikely, V cannot be zero and there is an inherent contradiction in conventional beam theory. The elastic stress-strain relation implies V=0 while the equation of motion implies $V \neq 0$.



Underformed beam element



Deformed beam element

Fig. 1.4-4. Sketch showing deformation parameters.

Suppose the first of (1.4-14) is used to determine V:

$$V = -M,_x - \mathcal{M} + I_r\ddot{\theta} = -M,_x - \mathcal{M} + I_r\ddot{w},_x \tag{1.4-19}$$

where (1.4-17) has been used. Substitute (1.4-18) in the second of (1.4-14). Then one governing equation of motion is obtained, namely

$$M,_{xx} - (I_r\ddot{w},_x),_x + \bar{\rho}\ddot{w} = F - \mathcal{M},_x \tag{1.4-20}$$

If, in addition, the rotary inertia term, $(I_r\ddot{w},_x),_x$, is negligible, the result is the Euler-Bernoulli beam equation

$$M,_{xx} + \bar{\rho}\ddot{w} = F - \mathcal{M},_x \tag{1.4-21}$$

which can be combined with the constitutive equation

$$M = EIw,_{xx} \tag{1.4-22}$$

to yield

$$(EIw,_{xx}),_{xx} + \bar{\rho}\ddot{w} = F - \mathcal{M},_x \tag{1.4-23}$$

The static case is the conventional beam equation

$$(EIw,_{xx}),_{xx} = F - \mathcal{M},_x \tag{1.4-24}$$

which is sometimes written in the following alternative forms (with *M* ignored):

$$M,_{xx} = F\,, \qquad M = EIw,_{xx} \tag{1.4-25}$$

or

$$V,_{x} = -F\,, \quad M,_{x} = -V \quad \text{and} \quad M = EIw,_{xx} \tag{1.4-26}$$

These equations must be solved subject to the boundary conditions

$$\left.\begin{array}{lll} nM = M^*_b & \text{or} & w,_x = \theta^*_b \\ nV = V^*_b & \text{or} & w = w^*_b \end{array}\right\} \text{ at } x = 0 \text{ and } x = L \tag{1.4-27}$$

with V^*_b, M^*_b, θ^*_b and w^*_b denoting prescribed boundary values for transverse force, moment, rotation and transverse displacement, respectively.

1.4.2 Exact Solutions to the Static Problem

Statically Determinate Problems

Recall from (1.4-25) that if *M* = 0, the governing equations for the Euler-Bernouilli beam are

$$M,_{xx} = F \quad \text{and} \quad M = EIw,_{xx} \tag{1.4-28}$$

with the shear given by

$$V = -M,_x \tag{1.4-29}$$

Integrate the first of (1.4-28) twice to obtain

$$M,_x = \int F d\hat{x} + c_1\,, \qquad M = \int\!\!\int F d\hat{x} d\tilde{x} + c_1 x + c_2 \tag{1.4-30}$$

where c_1 and c_2 are the integration constants. It follows that if: (i) one boundary condition is on shear and one on moment, or (ii) there are two boundary conditions on the moment, then c_1 and c_2 can be determined without solving for the displacement w. The shear and moment functions can then be obtained without integrating to obtain w. Such classes of problems are said to be **statically determinate** because the **generalized stresses**, M and V, are determined directly from equilibrium and do not involve the **bending stiffness** EI.

Example Problem:

The data for the problem shown in Fig. 1.4-5 consist of the following:
(i) the domain, $0 \le x \le L$;
(ii) the forcing function, $F = F_o H[x - L/2]$ with F_o a constant;
(iii) the beam stiffness, EI, constant; and
(iv) the boundary conditions, $w(0) = 0$, $M(0) = 0$, $w(L) = 0$, and $M(L) = 0$.

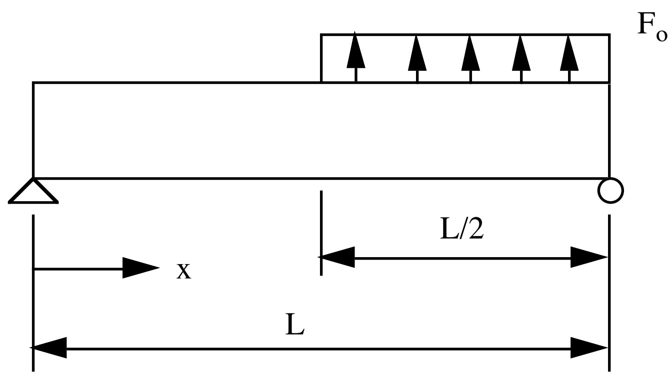


Fig. 1.4-5. Example problem that is statically determinate.

Two of the boundary conditions are on M, wherefore the problem is statically determinate. The solution is obtained by integrating the moment equilibrium equation, which is the first of (1.4-28), twice to obtain

$$M,_x = F_o < x - \tfrac{L}{2} >^1 + c_1 \qquad \text{and} \qquad M = \tfrac{F_o}{2} < x - \tfrac{L}{2} >^2 + c_1 x + c_2 \qquad (1.4\text{-}31)$$

Now, applying the boundary conditions, there follows

$$M(L) = \tfrac{F_o}{2}(L - \tfrac{L}{2})^2 + c_1 L = \frac{F_o L^2}{8} + c_1 L = 0 \;, \qquad c_1 = -\frac{F_o L}{8} \qquad (1.4\text{-}32)$$

Therefore, the solutions for M and V are

$$M = \tfrac{F_o}{2} < x - \tfrac{L}{2} >^2 - \frac{F_o L}{8} x \;, \qquad V = -M,_x = \frac{F_o L}{8} - F < x - \tfrac{L}{2} >^1 \qquad (1.4\text{-}33)$$

Figure 1.4-6 shows a sketch of the solution.

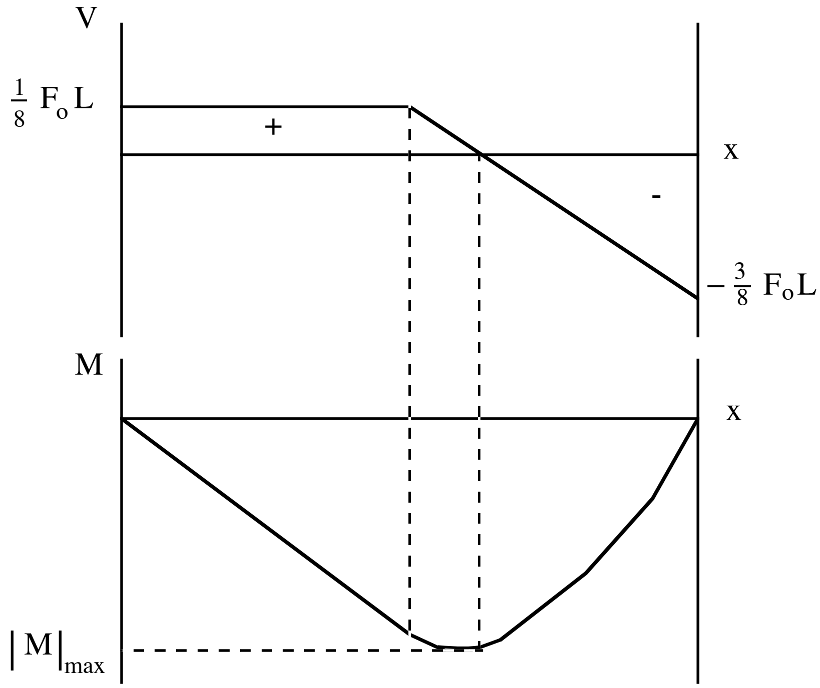


Fig. 1.4-6. Graphs of the solution for M and V.

Statically Indeterminate Problems

If either one or no boundary condition is prescribed on M or V, then the complete displacement problem must be solved to evaluate the integration constants and to obtain M and V as functions of x.

Example Problem:

The data for the problem shown in Fig. 1.4-7 consist of the following:

(i) the domain, $0 \le x \le L$;

(ii) the forcing function, $F(x) = -P\,\delta[x - \frac{3}{4}L]$ with P a constant;

(iii) the beam stiffness, EI, constant; and

(iv) the boundary conditions, $w(0) = 0$, $w_{,x}(0) = 0$, $w(L) = 0$, and $M(L) = 0$.

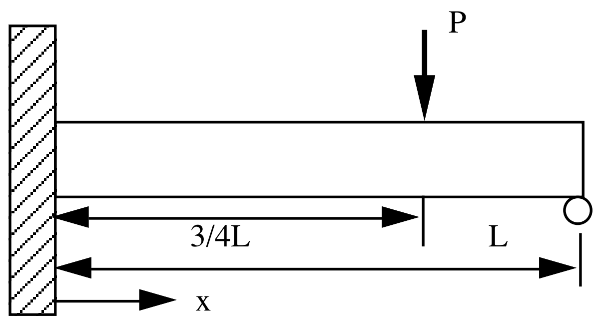


Fig. 1.4-7. A statically indeterminate problem.

The governing equation for this problem is

$$(EIw_{,xx})_{,xx} = -P\,\delta[x - \tfrac{3}{4}L] \tag{1.4-34}$$

By integrating twice, the solution for $w_{,xx}$ is obtained in which c_1 and c_2 are integration constants. At this point we can solve for c_2 in terms of c_1 by using the boundary condition M(L) = 0. The result is

$$EIw_{,xx} = -P < x - \tfrac{3}{4}L >^1 + c_1 x + c_2 \,, \qquad c_2 = \tfrac{PL}{4} - c_1 L \tag{1.4-35}$$

One more integration provides another integration constant c_3. Use of the boundary condition $w_{,x}(0) = 0$ yields

$$EIw_{,x} = -\tfrac{P}{2} < x - \tfrac{3}{4}L >^2 + c_1(\tfrac{x^2}{2} - Lx) + \tfrac{PL}{4}x \tag{1.4-36}$$

A final integration introduces a fourth integration constant, c_4, in the expression

$$EIw = -\tfrac{P}{6} < x - \tfrac{3}{4}L >^3 + c_1(\tfrac{x^3}{6} - L\tfrac{x^2}{2}) + \tfrac{PL}{8}x^2 + c_4 \tag{1.4-35}$$

To complete the solution, solve for c_1 and c_4 from the boundary conditions w (0) = 0 and w(L) = 0. The details are left as an exercise for the reader.

Ill-posed Problems

Suppose the shear is prescribed at each end of the beam. After one integration of the static beam equation, the result is

$$M,_x = \int F dx + c_1 \qquad \text{or} \qquad V = -\int F dx - c_1 \tag{1.4-36}$$

There is only one constant of integration

$$V(0) = -c_1 \qquad \Rightarrow \qquad V(L) = -\int_0^L F dx - c_1 \tag{1.4-37}$$

or

$$V(L) - V(0) + \int_0^L F dx = 0 \tag{1.4-38}$$

This last equation is merely a statement of overall equilibrium. If it is satisfied then either w(0) or w(L) can be specified arbitrarily and a meaningful solution can be found. For numerical solutions, if one of these displacements is not prescribed, the stiffness matrix will be singular. If overall equilibrium is not satisfied, then the problem is really one of dynamics and no attempt should be made to analyze the problem using the equilibrium equation.

1.4.3 Timoshenko Theory

Timoshenko theory does not invoke the Kirchhoff hypothesis given in (1.4-17). The result is the removal of the contradiction involving the transverse shear but at the price of maintaining two kinematic variables, θ and w. The governing equations are:

$$\left.\begin{array}{ll} M,_x + V + \mathcal{M} = I_r \ddot{\theta} \ , & \text{Transverse equation of motion} \\ V,_x + F = \overline{\rho} \ddot{w} \ , & \text{Rotational equation of motion} \\ M = EI\theta,_x \ , & \text{Moment constitutive equation} \\ V = GA(w,_x - \theta) \ , & \text{Shear constitutive equation} \end{array}\right\} \tag{1.4-39}$$

The physical interpretation of the deformation inherent in the theory is that straight fibers originally perpendicular to the centerline remain straight but not necessarily perpendicular to the centerline in the deformed configuration (see Fig. 1.4-4). The theory implicitly implies that the transverse shear strain is constant over the cross-section whereas a parabolic distribution with zero shear on the top and bottom surfaces is a more realistic distribution for most problems. To account for the unrealistic shear distribution, a correction factor, κ, is normally introduced in the shear constitutive equation

$$V = \kappa GA(w,_x - \theta) \tag{1.4-40}$$

with a value $\kappa = 5/6$ typically used.

Suppose the constitutive equations are substituted in the equations of motion. The result is

$$\left.\begin{aligned}(\mathrm{EI}\theta,_x),_x + \kappa\mathrm{GA}(w,_x - \theta) + M &= \mathrm{I}_r\ddot{\theta}\\ [\kappa\mathrm{GA}(w,_x - \theta)],_x + \mathrm{F} &= \bar{\rho}\ddot{w}\end{aligned}\right\} \tag{1.4-41}$$

The boundary conditions are normally (i) either V or w and (ii) either M or θ prescribed at each end. Initial conditions consist of values for θ and w and their time derivatives

Consider the static case with only a transverse forcing function F(x) applied. The governing set of equations becomes

$$\begin{aligned}&\mathrm{M},_x + \mathrm{V} = 0\ , &&\mathrm{V},_x + \mathrm{F} = 0\\ &\mathrm{M} = \mathrm{EI}\theta,_x\ , &&\mathrm{V} = \kappa\mathrm{GA}(w,_x - \theta)\end{aligned} \tag{1.4-42}$$

<u>Example:</u>

Consider a cantilever beam with a point force acting on the right end as shown in Fig. 1.4-8. The data for the problem are:

(i) the domain, $0 \le x \le L$;

(ii) the forcing function F(x) = 0;

(iii) the constant beam stiffness, EI; and

(iv) the boundary conditions $w(0) = 0$, $w,_x(0) = 0$, $M(L) = 0$, and $V(L) = P$.

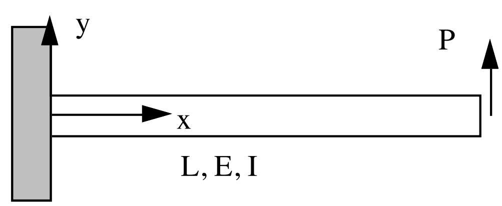


Fig. 1.4-8. Model problem for Timoshenko Theory.

The solution to (1.4-42) is

$$w = \frac{P}{EI}\left(\frac{1}{2}Lx^2 - \frac{1}{6}x^3\right) + \frac{Px}{\kappa GA} \tag{1.4-43}$$

where the first term is the solution of Euler-Bernouilli beam theory and the second term is the modification provided by Timoshenko theory. Details of obtaining the solution are left as an exercise.

1.5 PROBLEMS WITH AXIAL AND SPHERICAL SYMMETRY

1.5.1 Cylindrical and Spherical Coordinates

Let x, y and z denote rectangular Cartesian coordinates. Then **cylindrical coordinates,** r, θ and z, are defined by the relations:

$$\left.\begin{array}{lll} r = \sqrt{x^2 + y^2}\ , & \theta = \arctan\dfrac{y}{x}\ , & z = z \\ x = r\cos\theta\ , & y = r\sin\theta\ , & z = z \end{array}\right\} \qquad (1.5\text{-}1)$$

The relationships between Cartesian and cylindrical coordinates are illustrated in Fig. 1.5-1. Similarly, as shown in Fig. 1.5-2, **spherical coordinates** R, ϕ and θ are defined by

$$\left.\begin{array}{lll} R = \sqrt{x^2 + y^2 + z^2}\ , & \phi = \arctan\dfrac{\sqrt{x^2+y^2}}{z}\ , & \theta = \arctan\dfrac{y}{x} \\ x = R\sin\phi\cos\theta\ , & y = R\sin\phi\sin\theta\ , & z = R\cos\phi \end{array}\right\} \qquad (1.5\text{-}2)$$

If the boundaries of domains coincide with constant values of any of these coordinates, then it is usually most convenient to formulate the governing equations and to obtain solutions in terms of the corresponding coordinate variables. If the solution variables depend only on the cylindrical coordinate, r, for example, the problem is one dimensional and is said to be cylindrically or **axially symmetric**. Another common class of one dimensional problems exists if the solution variables depend only on R in which case the problem is **spherically symmetric**.

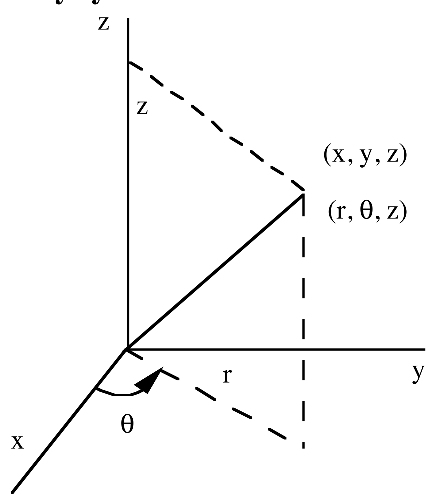


Fig. 1.5-1 Cylindrical coordinates.

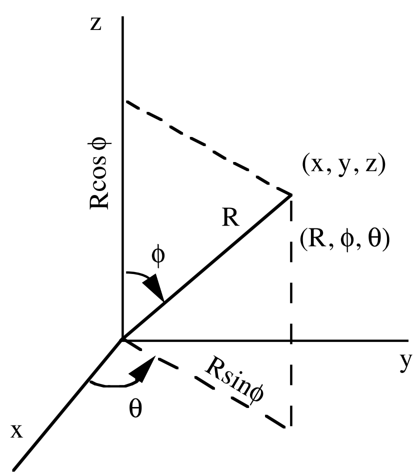


Fig. 1.5-2 Spherical coordinates.

1.5.2 Heat Conduction

The details of a general derivation of the heat conduction equations in cylindrical or spherical coordinates are not given. Instead, the pertinent governing equations for heat flow in the radial direction are given for each case.

Cylindrical Coordinates

As in the one dimensional formulation in a Cartesian system given by (1.2-9) and (1.2-10), the governing equations consist of the energy equation and the constitutive equation. For axial symmetry, there is only heat flow in the radial direction, r. The energy equation in cylindrical coordinates becomes

$$cT,_t = -(q,_r + \tfrac{q}{r}) + Q \ , \qquad c = \overline{\rho} C \tag{1.5-3}$$

which exhibits a term (q/r) not present in rectangular Cartesian coordinates. Fourier's law is

$$q = -KT,_r \tag{1.5-4}$$

When combined, the single heat conduction equation for temperature is

$$cT,_t = (KT,_r),_r + \tfrac{K}{r} T,_r + Q \tag{1.5-5}$$

Not only is one of the coefficients a variable, there is the possibility of a singularity at r = 0. If the domain includes r = 0, i.e., if the body is a solid cylinder, then special care must be taken at the origin. The appropriate boundary condition necessary for a well-posed problem is that of no flux, or $T,_r(0) = 0$, at r = 0.

Spherical Coordinates

For heat conduction in the R direction for spherically symmetric problems the energy and constitutive equations are:

$$cT,_t = -(q,_R + \frac{2q}{R}) + Q \ , \qquad q = -KT,_R \tag{1.5-6}$$

Note that the only difference from the cylindrical case is a factor of 2 for the term proportional to q/R. The combined heat conduction equation is

$$cT,_t = (KT,_R),_R + \frac{2K}{R} T,_R + Q \tag{1.5-7}$$

with the same comment holding at R = 0 as that made for the cylindrical case.

1.5.3 Stress Analysis

With regard to stress analysis, two or more components of the strain and stress tensors must be introduced even under the assumption of symmetry. Since the equations of continuum mechanics are beyond the expected background for this text, the equations will merely be given with the objective of showing that the final governing equation is indeed one-dimensional and of the class appropriate for this chapter.

Cylindrical Coordinates

For bodies in which isotropic elasticity and axial symmetry hold, the stress analysis problem involves one component of displacement, u, in the r direction, the radial, e_{rr}, and tangential, $e_{\theta\theta}$, components of strain, and the radial, σ_{rr}, and tangential, $\sigma_{\theta\theta}$, components of stress. The equation of motion in the r direction is

$$\rho u,_{tt} = \tfrac{1}{r}(r\sigma_{rr}),_r - \tfrac{1}{r}\sigma_{\theta\theta} \tag{1.5-8}$$

The strain-displacement relations are

$$e_{rr} = u,_r \ , \qquad e_{\theta\theta} = \frac{u}{r} \tag{1.5-9}$$

and the constitutive equations assume the following form:

$$\sigma_{rr} = E_1 e_{rr} + E_2 e_{\theta\theta} \ , \qquad \sigma_{\theta\theta} = E_2 e_{rr} + E_1 e_{\theta\theta} \tag{1.5-10}$$

The material parameters E_1 and E_2 are related to Young's modulus, E, and Poisson's ratio, ν, as follows:

$$\left.\begin{aligned} E_1 &= \frac{E}{1-\nu^2} \ , && E_2 = \nu E_1 \ ; && \text{Plane stress} \\ E_1 &= \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} \ , && E_2 = \frac{\nu}{(1-\nu)} E_1 \ ; && \text{Plane strain} \end{aligned}\right\} \tag{1.5-11}$$

When the above equations are combined, the resulting equation of motion is

$$\rho u,_{tt} = E_1(u,_{rr} + \tfrac{1}{r}u,_r - \tfrac{1}{r^2}u) = \tfrac{E_1}{r^2}[r(ru,_r),_r - u] \tag{1.5-12}$$

which again represents a partial differential equation of the category to be considered.

Spherical Coordinates

For bodies in which isotropic elasticity and spherical symmetry hold, the stress analysis problem involves one component of displacement, u, in the R direction, the normal components of strain, e_{RR}, $e_{\phi\phi}$ and $e_{\theta\theta}$, and stress, σ_{rr}, $\sigma_{\phi\phi}$ and $\sigma_{\theta\theta}$. The equation of motion in the R direction is

$$\rho u,_{tt} = \frac{1}{R^2}(R^2\sigma_{RR}),_R - \frac{1}{R}(\sigma_{\phi\phi} + \sigma_{\theta\theta}) \tag{1.5-13}$$

the strain-displacement relations are

$$e_{RR} = u,_R \ , \qquad e_{\phi\phi} = \frac{1}{R}u \ , \qquad e_{\theta\theta} = \frac{1}{R}u \tag{1.5-14}$$

and the constitutive equation assumes the form

$$\left.\begin{aligned} \sigma_{RR} &= E_1 e_{RR} + E_2(e_{\phi\phi} + e_{\theta\theta}) \\ \sigma_{\phi\phi} &= E_1 e_{\phi\phi} + E_2(e_{\theta\theta} + e_{RR}) \\ \sigma_{\theta\theta} &= E_1 e_{\theta\theta} + E_2(e_{RR} + e_{\phi\phi}) \end{aligned}\right\} \tag{1.5-15}$$

in which

$$E_1 = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} \ , \qquad E_2 = \frac{\nu}{(1-\nu)}E_1 \tag{1.5-16}$$

When combined, the above equations yield the following equation of motion in terms of the single function u:

$$\rho u,_{tt} = E_1\left(u,_{RR} + \frac{2}{R}u,_R - \frac{2}{R^2}u\right) = \frac{E_1}{R^2}[(R^2 u,_R),_R - u] \tag{1.5-17}$$

which again is of the form being considered in this book.

1.6 RELATED TOPICS

1.6.1 General Form of Differential Equations

The heat conduction equation and the equation of motion for bars and axially or spherically symmetric problems are contained within the general form of the differential equation given by

$$\rho u,_{tt} + cu,_t = (ku,_x),_x - au,_x - bu + f(x,t) \qquad (1.6\text{-}1)$$

The assumption of linearity implies that the coefficients ρ, c, k, a and b cannot be functions of u or of any of its derivatives. These coefficients can be functions of time but such an occurrence is relatively rare and will not be considered. However, a dependence on the spatial coordinate appears frequently and must be taken into consideration. For axially or spherically symmetric problems, the coefficients are functions of the spatial coordinate as a consequence of the respective geometries. For all classes of problems, the coefficients may be functions of x because of changes in material properties. For bar problems, either heat conduction or motion equations, the coefficients may change as a result of changes in the cross-sectional area.

For time-dependent problems, initial conditions must be specified. If the density $\rho = 0$, the function u(x, 0) must be given and, if $\rho \neq 0$, the derivative of the function, $\dot{u}(x, 0)$, must also be specified. In general, initial conditions are needed up to one less than the highest time derivative in the differential equation. The usual form of the boundary conditions is that at each boundary point, either the function or its derivative with respect to x is specified. Occasionally, a linear combination of the function and its derivative is the appropriate form. For the static problem, sometimes the boundary conditions will be such that either no solution exists, or an infinite number of solutions exist. These are examples of ill posedness.

The form of the equation of motion for the Euler-Bernouilli conventional beam theory, with no applied moment function, is

$$(\bar{k}w,_{xx}),_{xx} + \bar{\rho}\ddot{w} = F(x,t) \qquad (1.6\text{-}2)$$

Again the coefficients, $\bar{k}$ and $\bar{\rho}$, may be functions of space and time but usually the coefficients are constants or, at worst, piecewise constants. Initial conditions for w and its rate must be given. At each boundary point, two conditions must be specified. Appropriate combinations are (i) either the function or its third derivative and (ii) either the first or second derivative. Occasionally a linear combination of the function and its third derivative (a transverse spring is an example), or of the first and second derivatives (a torsional spring) are specified. Suitable combinations are required to ensure a unique solution to the time-independent problem.

1.6.2 General Method for Treating Discontinuities

Consider a limited form of the static problem described by

$$(ku,_x),_x - bu + f = 0 \ , \qquad 0 < x < L \tag{1.6-3}$$

Suppose the forcing function is a point force of magnitude P applied at x = a. If the Dirac delta function is used, then f(x) = Pδ[x-a]. A method for obtaining an analytical solution is to break the problem into two parts; the first for 0 < x < a and the second for a < x < L. On each part the forcing function is smooth, so a solution using standard methods can be obtained with two constants of integration associated with each part. Label these two solutions $u^{(1)}$ and $u^{(2)}$. The boundary condition at x = 0 is applied to $u^{(1)}$ and the boundary condition at x = L to $u^{(2)}$. Continuity on the solution is always assumed so $u^{(1)}(a) = u^{(2)}(a)$. One additional equation is required to obtain the required number of equations to determine the integration constants. To obtain the appropriate equation, integrate each term in (1.6-3) across an interval a - ε to a + ε with ε an arbitrarily small positive number:

$$\int_{a-\varepsilon}^{a+\varepsilon}(ku,_x),_x\,dx - \int_{a-\varepsilon}^{a+\varepsilon} bu dx + \int_{a-\varepsilon}^{a+\varepsilon} P\,\delta[x-a]\,dx = 0 \tag{1.6-4}$$

With the assumption that b is continuous at x = a, and in the limit as $\varepsilon \to 0$, the second integral goes to zero because u is continuous. By definition of the Dirac delta function, the third term is simply P. Consequently, we conclude that

$$\lim_{\varepsilon\to 0}(ku,_x)\Big|_{a-\varepsilon}^{a+\varepsilon} + P = 0 \tag{1.6-5}$$

or

$$k(a)u^{(2)},_x(a) - k(a)u^{(1)},_x(a) + P = 0 \tag{1.6-6}$$

which is the fourth equation to be used to evaluate the integration constants.

If the forcing function contains discontinuities of the Heaviside type, or if f(x) is continuous, then the contribution from the forcing function will be zero for any interval with $\varepsilon \to 0$. The conclusion is that the derivative of u is continuous for all forcing terms except the Dirac delta function. However suppose k is discontinuous at x = a. Then only the first term from (1.6-4) provides a contribution and the equation corresponding to (1.6-6) relating the jump in the derivative of u is

$$k^{(2)}(a)u^{(2)}{}_{,x}(a) - k^{(1)}(a)u^{(1)}{}_{,x}(a) = 0 \qquad (1.6\text{-}7)$$

in which $k^{(2)}(a)$ is interpreted as the value of k just to the right of the discontinuity and $k^{(1)}(a)$ is the value just to the left. A similar analysis to determine the jump in the derivative of u can be performed if b is discontinuous.

1.6.3 Self Adjointness

For the one-dimensional case, instead of treating heat conduction and motion in a bar as separate problems we observe that both forms are contained in the following differential equation:

$$(ku_{,x})_{,x} - a u_{,x} - bu + f = \bar{\rho}\ddot{u} + \bar{c}\dot{u} \qquad (1.6\text{-}8)$$

For time-independent problems, the governing differential equation becomes

$$L(u) + f = 0\ , \qquad L(u) = (ku_{,x})_{,x} - a u_{,x} - bu \qquad (1.6\text{-}9)$$

in which $L()$ denotes the linear differential operator associated with the differential equation. Suppose the domain is $0 < x < L$, and the boundary conditions are homogeneous. Consider a function w(x) which satisfies the same homogenous boundary conditions. Multiply each term in (1.6-9) by w and integrate over the domain. Then, perform an integration by parts on the first term. The result is

$$\int_0^L w[(ku_{,x})_{,x} - a u_{,x} - bu]\,dx = w(ku_{,x})\Big|_0^L - \int_0^L [w_{,x}\,ku_{,x} + w a u_{,x} + bwu]\,dx \quad (1.6\text{-}10)$$

Suppose another integration by parts is performed on both the first and second terms so that u factors out of the expression in the integrand. The result is

$$\begin{aligned} & w(ku_{,x})\Big|_0^L - \int_0^L [w_{,x}\,ku_{,x} + w a u_{,x} + bwu]\,dx \\ & \quad = -w_{,x}\,ku\Big|_0^L + w[ku_{,x} - a u]\Big|_0^L + \int_0^L [uL^*(w)] \end{aligned} \qquad (1.6\text{-}11)$$

in which the **adjoint operator** $L^*()$ acting on w is

$$L^*(\mathrm{w}) = (\mathrm{kw}_{,\mathrm{x}})_{,\mathrm{x}} + (a\mathrm{w})_{,\mathrm{x}} - \mathrm{bw} \tag{1.6-12}$$

If $L^*() = L()$, the differential operator is said to be **self adjoint**. If $a \neq 0$, the given example is not self adjoint; the differential operator is self adjoint if $a = 0$.

Suppose the differential operator is self adjoint. In addition, suppose the boundary conditions on w are chosen adjoint to those of u in the sense that if the flux is prescribed with respect to u, then w = 0, and if u is prescribed then the flux term $\mathrm{kw}_{,\mathrm{x}}$ is taken to be zero. Then the differential equation is said to be self adjoint. Sometimes it is a boundary condition and not the differential operator that make a problem not self adjoint.

In general, suppose $L()$ denotes a differential operator of order m, an even integer. With homogeneous boundary conditions and after m/2 integrations by parts, if

$$\int_0^{\mathrm{L}} \mathrm{w}L(\mathrm{u})\mathrm{dx} = \int_0^{\mathrm{L}} \mathrm{u}L(\mathrm{w})\mathrm{dx} \tag{1.6-13}$$

holds, the differential operator is said to be **self adjoint**.

As a practical example, with four integrations by parts, it can be shown that the differential operator associated with conventional beam theory

$$L(\mathrm{w}) = (\mathrm{EIw}_{,\mathrm{xx}})_{,\mathrm{xx}} \tag{1.6-14}$$

is a self-adjoint operator. The proof is left as an exercise.

1.6.4 Energy Formulation

Frequently the governing equations are given in terms of a **principle of minimum potential energy**, which states that the equilibrium equation is satisfied at that point when the potential energy attains a minimum value. An expression for the potential energy can only be obtained if the differential operator is self adjoint. The potential energy is the difference between the **internal energy** and the **work**. The potential energy is derived by multiplying each term in the differential equation by the dependent variable and integrating over the domain. Integration by parts is performed as necessary with each term until a quadratic form of the dependent variable and each of its derivatives is achieved. The internal energy (the strain energy for stress problems) is the sum of the quadratic terms multiplied by $[(-1)^{m/2}]/2$ where m is the order of the differential equation and is an even number for self-adjoint problems. The work is the sum of terms involving a product of an applied force and the dependent variable including an applied flux (force for displacement problems) at a boundary.

As an example, consider the governing differential equation for the time independent case where the differential operator is given by (1.6-9) with $a = 0$:

$$(ku,_x),_x - bu + f = 0 \tag{1.6-15}$$

First, multiply by u, integrate over the domain, and perform one integration by parts to obtain

$$\int_0^L u[(ku,_x),_x - bu + f]dx = \int_0^L [-u,_x ku,_x - ubu + uf]dx + u(ku,_x)\Big|_0^L \tag{1.6-16}$$

The integration by parts is necessary to derive the quadratic term involving the derivative of the dependent variable. The first two terms are multiplied by $(1/2)(-1)^{m/2}$ for m = 2 to obtain the **internal energy**, U:

$$U = \tfrac{1}{2}\int_0^L [k(u,_x)^2 + bu^2]dx \tag{1.6-17}$$

which is said to be **positive definite** because U > 0 for any nonzero u or $u,_x$ with the assumptions that k and b are positive everywhere. The work, W, done by the applied forces is

$$W = \int_0^L uf\,dx + f_0^* u_0 + f_L^* u_L \tag{1.6-18}$$

in which $f_0^* = -(ku,_x)_{x=0}$ and $f_L^* = (ku,_x)_{x=L}$ are the applied fluxes at x = 0 and x = L, respectively. If a flux is not prescribed on the boundary, then the corresponding boundary term is not included. The **potential energy** is the difference

$$P(u) = U(u) - W(u) \tag{1.6-19}$$

Often, the expression for potential energy is assumed to be the primary description of a problem, and the governing equation is obtained by minimizing the potential energy. To illustrate such an approach, we take the **variation** of P with respect to u. The variation consists of taking the differential of each integrand, and replacing the differential of u with what is called the variation δu. The variation, δu, is considered to be arbitrary except for those points (boundaries) where u is specified; at those points $\delta u = 0$. The steps of taking the variation of P with respect to u are given as follows:

$$\begin{aligned}
\delta_u P &= \frac{1}{2}\int_0^L \delta[k(u,_x)^2 + bu^2]dx - \int_0^L \delta[uf]\,dx - \delta(f_0^* u_0) - \delta(f_L^* u_L) \\
&= \int_0^L [ku,_x \delta u,_x + bu\delta u]\,dx - \int_0^L \delta uf\,dx - f_0^*\delta u_0 - f_L^*\delta u_L \qquad (1.6\text{-}20) \\
&= (ku,_x - nA\tau)\delta u\Big|_0^L - \int_0^L \delta u[ku,_{xx} - bu + f]dx
\end{aligned}$$

At a boundary point either $\delta u = 0$ or the coefficient $(ku,_x - nA\tau) = 0$, i.e., either the displacement, u, is prescribed in which case the variation of u is zero, or the flux, τ, is prescribed and the boundary condition is $ku,_x = nA\tau$. Since δu is arbitrary over all interior points, the fundamental lemma of variational calculus (Appendix A.2) is invoked with the result that the coefficient of δu in the integrand must be zero, or $(ku,_x),_x - bu + f = 0$ which is the desired governing equation. A basic difference from the fundamental principle of continuum mechanics, which was used previously in the derivation of the governing differential equation, is that with the calculus of variations the limits of integration are fixed rather than being arbitrary.

If the non-self-adjoint equation had been considered, then the expression for the corresponding internal energy will contain a term, $a uu,_x$, in the integrand with the consequence that the integrand is not positive definite and a minimum principle cannot be established.

Similarly, for conventional beam theory, the potential energy is readily shown to be

$$P(w) = \frac{1}{2}\int_0^L EI(w,_{xx})^2\,dx - W$$
$$W = \int_0^L wF\,dx + M_0^*(w,_x)_0 + M_L^*(w,_x)_L + V_0^* w_0 + V_L^* w_L \tag{1.6-18}$$

A work term from the boundary is deleted if the flux is not prescribed.

For those problems for which a potential energy exists, it is often more convenient to begin with the potential energy and to apply a minimizing procedure to obtain approximate solutions. However, the procedure is not completely general, so we have opted not to follow the energy approach.

1.7 CONCLUDING REMARKS

In this Chapter, an attempt has been made to show that the differential equations given by

$$\bar{\rho}u,_{tt} + \bar{c}u,_t = (ku,_x),_x - bu,_x - au + f(x,t) \qquad \text{and} \qquad (\bar{k}w,_{xx}),_{xx} + \bar{\rho}\ddot{w} = F(x,t)$$

cover a large number of engineering problems, many of which have not been discussed for the sake of brevity. As examples of other cases that could have been considered, we mention torsion of a bar and diffusion of moisture through ground media. The derivation of the equations in the context of interest is important because of the need to physically interpret the variables. Examples given in this chapter include heat flux, stress, internal forces and bending moments. Special care must be given to boundary conditions because of the possibility of ill posedness.

In the use of computer codes, an error diagnostic can often be traced to an improperly specified boundary condition. Even for one-dimensional problems, the need for numerical solutions becomes apparent for cases when the forcing function and coefficients in the equations are discontinuous functions of the spatial variable.

A brief discussion of self adjointness and the consequences for the existence of a potential function has been provided. Many books on the finite element method begin with a potential function in which case self adjointness is implicitly assumed to hold. However, as will be shown later, the existence of a potential function is not necessary for the application of the finite element method.

The next chapter provides transient solutions to model problems, which can be used as part of the verification for a numerical algorithm. However, if transient problems are not of interest, Chapter 2 can be skipped with little loss in continuity with the remainder of the text.

1.8 EXERCISES

1. For each of the following problems (i) give the data, (ii) solve for u and σ, and (iii) provide sketches of u, $u_{,x}$ and σ as functions of x. Classify each solution as classical, strong or weak. If a problem is not well posed, state the reason.

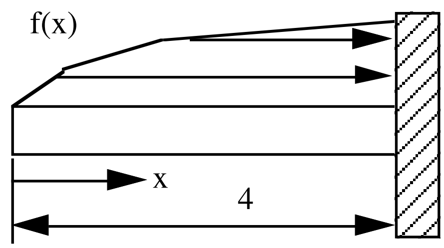
(1.1)

$k = 1$

$A = 1$

(i) $f = 4\frac{x}{L}$

(ii) $f = 4(\frac{x}{L})^2$



(1.2) Displacement of end is δ.

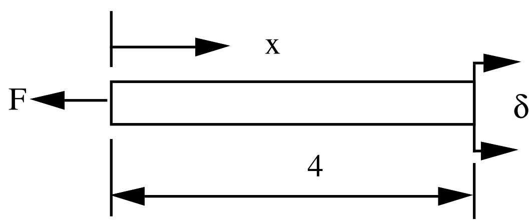
$F = 4 \qquad \delta = \frac{1}{4}$

(i) $A = 1$

$k = 1 + H[x - 1]$

(ii) $k = 1$

$A = 1 + 3H[x - 1]$

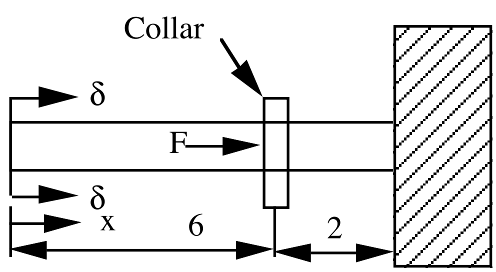


(1.3)

Displacement – δ

$F = 4 \qquad \delta = \frac{1}{4}$

$k = 1 \qquad A = 1$

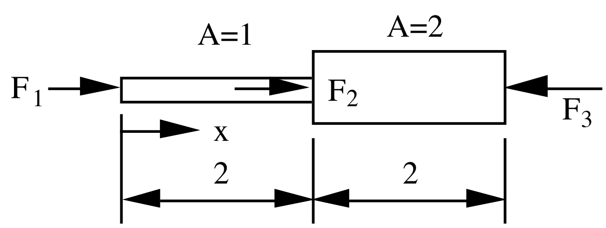


(1.4)

$k = 1$

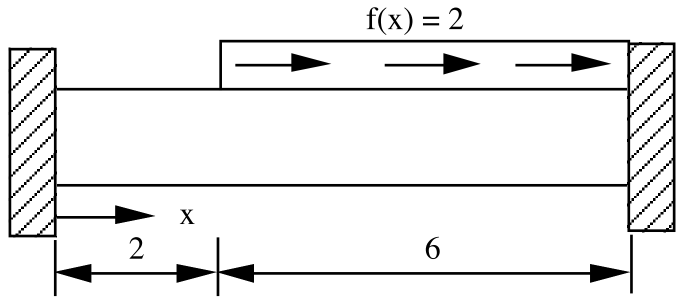
(i) $F_1 = 2,\ F_2 = 3,\ F_3 = 5$

(ii) $F_1 = 2,\ F_2 = 1,\ F_3 = 5$



(1.5)

(i)
$E = 1 \quad \forall\, x$
$A = 1 \quad \forall\, x$
(ii)
$E = 1 \quad \forall\, x$
$A = 1 + H[x - 6]$



2. Repeat the cases of Problem 1 for the governing differential equation

$$(ku_{,x})_{,x} - bu + f(x) = 0$$

with $b = 1$.

3. Consider the diffusion-advection equation of (1.2-35):

$$\left(kT_{,x}\right)_{,x} - aT_{,x} + Q = 0 \ , \qquad 0 < x < L$$

with a and k constants. Show that an appropriate dimensionless variable for relating the relative amounts of diffusion and advection is $a^* = aL/k$. Obtain a solution for Q = 0 and boundary conditions of T(0) = 1 and T(L) = 0. Plot solutions for various values of a^*. Show that an apparent boundary layer develops for large values of a^*.

4. Verify that the solution to the cantilever beam problem for the Timoshenko Theory is given by (1.4-43).

5. Consider the linear differential operator $L(u) = c_2(x)u_{,xx} + c_1(x)u_{,x} - c_0(x)u$ where $c_2(x) > 0$ and $c_0(x) > 0$. Determine the relationship between c_0, c_1 and c_2 that must hold if the operator is to be self adjoint. What is the corresponding quadratic form and internal energy? Which of the following operators are self adjoint? For those that are self adjoint, determine for which cases an internal energy can be defined (positive definite).

(i) $u_{,xx} - u$	(ii) $k(x)u_{,xx}$	(iii) $(ku_{,x})_{,x} - u$
(iv) $(ku_{,x})_{,x} + u$	(v) $u_{,xx} - u_{,x} - u$	(vi) $(2-x)u_{,xx} - u_{,x} - u$

6. Show that the operator of (1.6-14) is self adjoint.