HOMEWORK #2 SOLUTION

COMPUTATIONAL MECHANICS
ME 404-504

① A DERIVE THE WEAK FORM

$$-(xu')' + 2u = -1.8\pi\cos(1.8\pi x) + \sin(1.8\pi x)(2 + (1.8\pi)^2 x)$$

SUBJECT TO

$$u(0)=0, \quad u'(1) = 1.8\pi\cos(1.8\pi)$$

FIRST, DEFINE THE RESIDUAL

$$r = -(xu')' + 2u - f,$$

WITH $f = -1.8\pi\cos(1.8\pi x) + \sin(1.8\pi x)(2 + (1.8\pi)^2 x)$

NEXT,

$$\int_0^1 rv\,dx = 0,$$

$$\int_0^1 \left(-(xu')' + 2u - f\right) v\,dx$$

INTEGRATION BY PARTS YIELDS

$$\Big[-(xu')v\Big]_0^1 + \int_0^1 xu'v' + 2uv - fv\,dx = 0$$

GIVEN $u(0)=0$, THIS REQUIRES THAT $v(0)=0$ IN ORDER FOR v TO BE AN ADMISSIBLE TEST FUNCTION.

$$\int_0^1 xu'v' + 2uv - fv\,dx - v(1)\,T_1 = 0$$

WHERE $T_1 = 1.8\pi\cos(1.8\pi) = 1\cdot u'(1)$.

C THE GALERKIN METHOD NOT SATISFYING THE NATURAL B.C'S

$$\hat{U}_N = \sum_{j=1}^{N} \alpha_j \phi_j, \quad \phi_j(0) = 0.$$

FROM THE WEAK FORM, PART A,

$$\mathbb{K}_{jk} = \int_0^1 (x\phi_j'\phi_k' + 2\phi_j\phi_k)\,dx$$

$$f_k = \int_0^1 \phi_k f \, dx + \phi_k(1) T_1$$

FOR CONSISTENCY, CHOOSE

$$\phi_1 = x(1-x)^2, \quad \phi_2 = (x-1)^2 - 1, \quad \phi_3 = x$$

THESE ARE ADMISSIBLE SINCE THE ESSENTIAL B.C'S ARE SATISFIED. THE NATURAL B.C'S ARE AUTOMATICALLY SATISFIED BY SOLUTIONS TO THE ORIGINAL PROBLEM.

1 TERM:

$$\mathbb{K} = \frac{11}{210}, \quad f = 0.3811, \quad \alpha_1 = \frac{f}{\mathbb{K}} = 7.2765$$

2 TERM:

NOTE, THESE ARE THE SAME AS PART b.iv. THE STRONG & WEAK FORMS YIELD THE SAME $\mathbb{K}$!

$$\mathbb{K} = \begin{bmatrix} \frac{11}{210} & -\frac{1}{15} \\ -\frac{1}{15} & \frac{7}{5} \end{bmatrix}, \quad \underset{\sim}{f} = \begin{Bmatrix} 0.3811 \\ 0.9952 \end{Bmatrix}, \quad \mathbb{K}^{-1}\underset{\sim}{f} = \underset{\sim}{\alpha} = \begin{Bmatrix} 8.7091 \\ 1.1256 \end{Bmatrix}$$

THE FORCE TERMS, HOWEVER, DIFFER!

3 TERM

$$\mathbb{K} = \begin{bmatrix} \frac{11}{210} & \frac{-1}{15} & \frac{-1}{60} \\ \frac{-1}{15} & \frac{7}{5} & \frac{-7}{6} \\ \frac{-1}{60} & \frac{-7}{6} & \frac{7}{6} \end{bmatrix}, \quad \underset{\sim}{f} = \begin{Bmatrix} 0.3811 \\ 0.9952 \\ -0.9445 \end{Bmatrix}, \quad \mathbb{K}^{-1}\underset{\sim}{f} = \underset{\sim}{\alpha} = \begin{Bmatrix} 17.2377 \\ 6.374 \\ 5.8107 \end{Bmatrix}$$

$$\hat{U}_N = \sum_{j=1}^{N} \alpha_j \phi_j.$$

3

LET $\hat{u}_2 = u_0 + T_1 \cdot x + \sum_{j=1}^{2} \alpha_j \phi_j$

ϕ_j IS THUS SUBJECT TO $\phi_i(0)=0$ & $\phi_i'(1)=0$

IN THIS CASE, $u_0 = 0$.

SUBSTITUTION OF $\hat{u}_{L2}$ INTO THE SYSTEM EQUATION YIELDS

$$-\left(x\sum_{j=1}^{2}\alpha_j\phi_j'\right)' + 2\sum_{j=1}^{2}\alpha_j\phi_j = f - \underbrace{\left[-T_1 + 2u_0 + 2T_1x\right]}_{}$$

$$\mathcal{L}[u_0+T_1x] = -\left(x\left(u_0+T_1x\right)'\right)' + 2(u_0+T_1x)$$

DEFINE $f^* = f - [-T_1 + 2u_0 + 2T_1x]$. THE WORK FUNCTION THEN BECOMES, WITH $\underline{v = \sum_{k=1}^{2}\beta_k\psi_k}$,

$$\sum_{k=1}^{2}\beta_k\left[\sum_{j=1}^{2}\alpha_j\int_0^1\psi_k\,\mathcal{L}[\phi_j] - \psi_k f^*\,dx\right] = 0$$

LET $\phi_1 = x(1-x)^2$ & $\phi_2 = (x-1)^2 - 1 \Rightarrow \phi_1' = 3x^2 - 4x + 1,\ \phi_2' = 2x - 2$

$\int\phi_1 dx = \frac{1}{4}x^4 - \frac{2}{3}x^2 + \frac{1}{2}x^2,\ \int\phi_2 dx = \frac{1}{3}x^3 - x^2$

i. THE COLLOCATION METHOD

$$\psi_k = \delta(x-x_k)$$

$$K_{jk} = \int_0^1\delta(x-x_k)\left(-(x\phi_j')' + 2\phi_j\right)dx$$

$$= -\left(x_k\phi_j'(x_k)\right)' + 2\phi_j(x_k) = \begin{cases} 2x_k^3 - 13x_k^2 + 10x_k - 1 & j=1 \\ 2x_k^2 - 8x_k + 2 & j=2 \end{cases}$$

$$f_k = \int_0^1\delta(x-x_k)f^*dx$$

$$= f^*(x_k)$$

LET $x_1 = \frac{1}{3}$ & $x_2 = \frac{2}{3}$, $K = \begin{bmatrix} \frac{26}{27} & -\frac{4}{9} \\ \frac{13}{27} & -\frac{22}{9} \end{bmatrix}$, $\underset{\sim}{f} = \begin{Bmatrix} 15.312 \\ -10.6563 \end{Bmatrix} \Rightarrow K^{-1}\underset{\sim}{f} = \underset{\sim}{\alpha} = \begin{Bmatrix} 19.7042 \\ 8.2405 \end{Bmatrix}$

$$\hat{U}_N = \underbrace{1.8\pi\cos(1.8\pi)x}_{u_h = T_1x} + \underbrace{19.7042}_{\alpha}\underbrace{x(1-x)^2}_{\phi} + \underbrace{8.2405}_{}\underbrace{((x-1)^2-1)}_{}$$

ii THE SUB-DOMAIN METHOD

$$\psi_1 = \begin{cases} 1 & 0 \le x \le 1/2 \\ 0 & 1/2 < x \le 1 \end{cases} \qquad \psi_2 = \begin{cases} 0 & 0 \le x < 1/2 \\ 1 & 1/2 \le x \le 1 \end{cases}$$

WE CAN DEFINE THE SUBDOMAINS

$$\Omega_1: \; x \in [0, 1/2]$$

$$\Omega_2: \; x \in [1/2, 1]$$

$$\mathbb{K}_{jk} = \int_{\Omega_k} (-x\phi_j')' + 2\phi_j \, d\Omega_k = (-x\phi_j')\Big|_{\Omega_k} + \int_{\Omega_k} 2\phi_j \, d\Omega_k$$

$$f_k = \int_{\Omega_k} f^* \, d\Omega_k = -\frac{10}{9\pi}\cos\left(\frac{9}{5}\pi x\right) - \frac{9\pi}{5} x\cos\left(\frac{9}{5}\pi x\right) + \frac{9\pi}{5} x\cos\left(\frac{9}{5}\pi\right) - \frac{9\pi}{5}x^2\cos\left(\frac{5}{5}\pi\right)$$

NOTE, THE SAME BASIS FUNCTIONS FROM PART i ARE USED. ONLY ψ CHANGE

$$\mathbb{K} = \begin{bmatrix} \frac{-23}{96} & \frac{1}{12} \\ \frac{-7}{96} & \frac{-17}{12} \end{bmatrix}, \quad \underset{\sim}{f} = \begin{Bmatrix} 4.5228 \\ -9.0302 \end{Bmatrix} \Rightarrow \mathbb{K}^{-1}\underset{\sim}{f} = \underset{\sim}{\alpha} = \begin{Bmatrix} 16.9644 \\ 5.5011 \end{Bmatrix}$$

$$\hat{U}_N = 1.8\pi\cos(1.8\pi)x + 16.9644(x(1-x)^2) + 5.5011((x-1)^2-1)$$

iii The Least Squares Method

$$\psi_k = \frac{\partial r}{\partial \alpha_k} = \mathcal{L}(\phi_k) = -(x\phi_k')' + 2\phi_k$$

$$K_{jk} = \int_0^1 (-(x\phi_k')' + 2\phi_k)(-(x\phi_j')' + 2\phi_j)\,dx$$

$$f_k = \int_0^1 (-(x\phi_k')' + 2\phi_k)\, f^* \,dx$$

$$\mathbb{K} = \begin{bmatrix} \frac{74}{105} & \frac{4}{15} \\ \frac{4}{15} & \frac{24}{5} \end{bmatrix}, \quad \underset{\sim}{f} = \begin{Bmatrix} 11.247 \\ 28.778 \end{Bmatrix} \Rightarrow \mathbb{K}^{-1} \underset{\sim}{f} = \underset{\sim}{\alpha} = \begin{Bmatrix} 13.984 \\ 5.218 \end{Bmatrix}$$

$$\hat{U}_N = 1.8\pi \cos(1.8\pi)x + 13.984\left(x(1-x)^2\right) + 5.218\left((x-1)^2 - 1\right)$$

iv **THE GALERKIN METHOD**

$$\psi_k = \phi_k$$

$$\mathbb{K}_{jk} = \int_0^1 \phi_k \left(-(x\phi_j')' + 2\phi_j \right) dx$$

$$f_k = \int_0^1 \phi_k f^* \, dx$$

$$\mathbb{K} = \begin{bmatrix} \frac{11}{210} & -\frac{1}{15} \\ -\frac{1}{15} & \frac{7}{5} \end{bmatrix}, \quad \underset{\sim}{f} = \begin{Bmatrix} 6.0457 \\ 6.333 \end{Bmatrix} \Rightarrow \mathbb{K}^{-1}\underset{\sim}{f} = \underset{\sim}{\alpha} = \begin{Bmatrix} 15.424 \\ 5.258 \end{Bmatrix}$$

$$\hat{u}_N = 1.8\pi \cos(1.8\pi)\, x + 15.424 \left(x(1-x)^2 \right) + 5.258 \left((x-1)^2 - 1 \right)$$

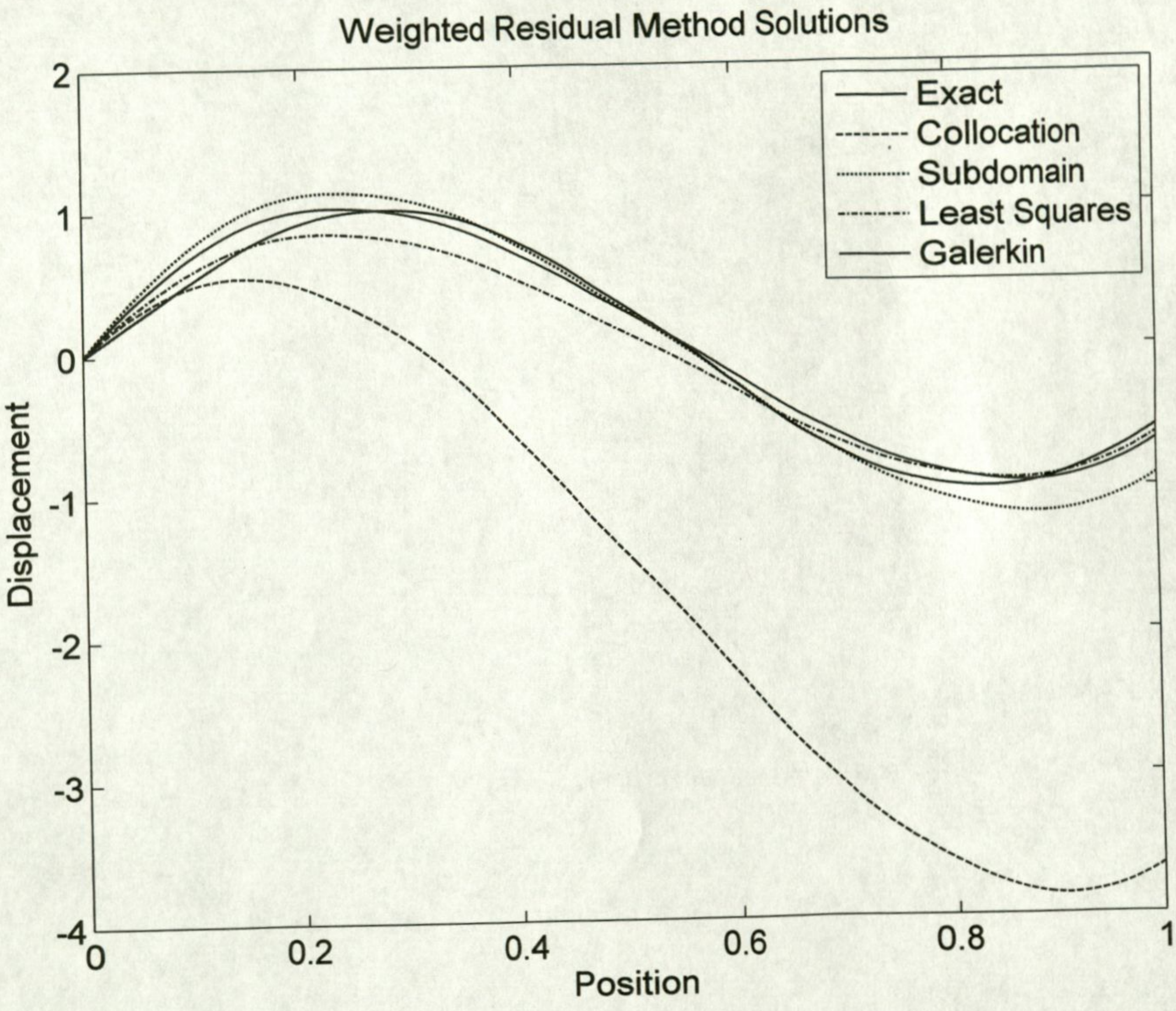
THE ERROR NORMS ARE GIVEN AS

$$\|e(x)\|_{L_2} = \sqrt{\int_0^1 (e(x))^2 \, dx}$$

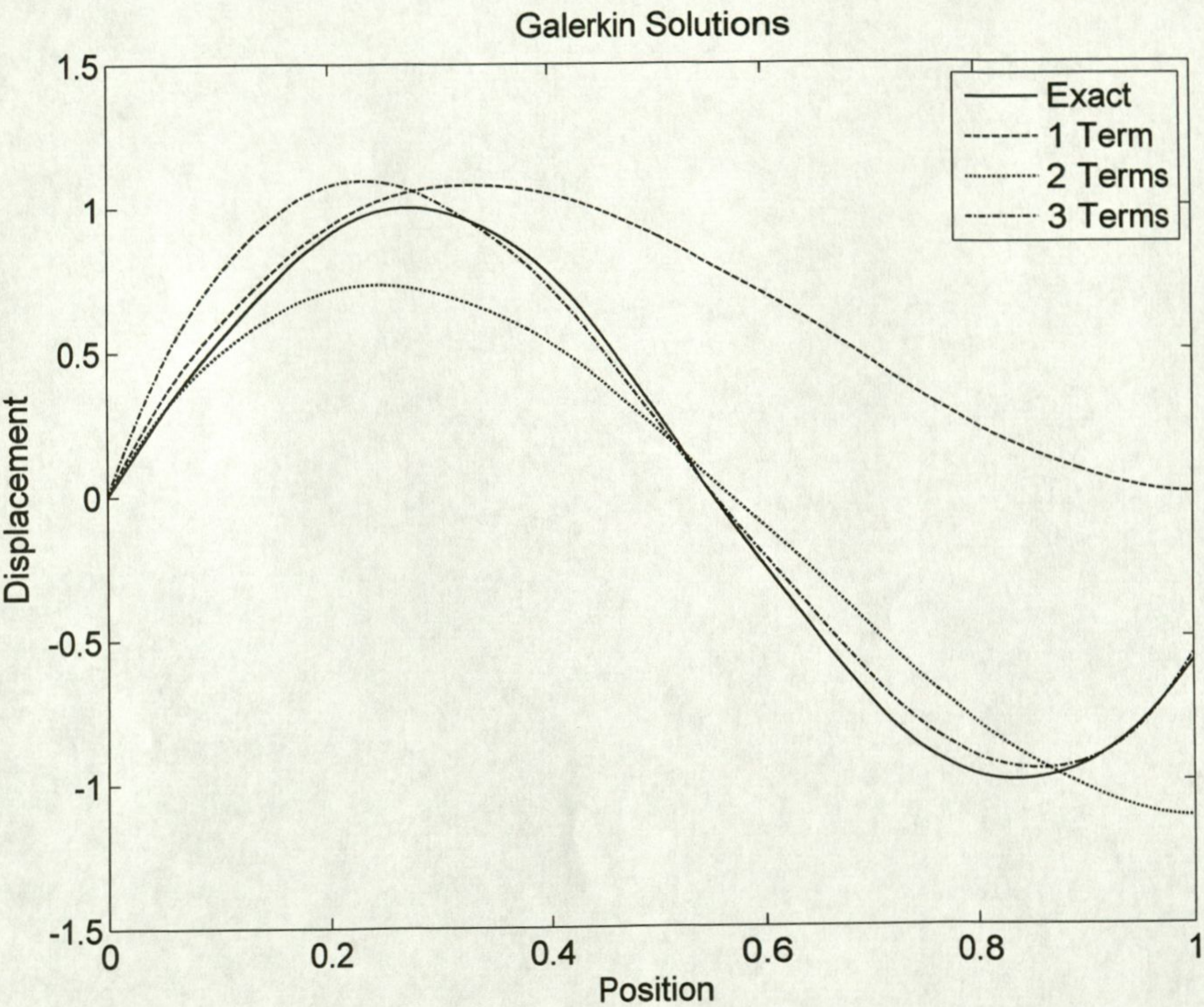
$$\|e(x)\|_E = \sqrt{\frac{1}{2}\int_0^1 x\,(e'(x))^2 \, dx}$$

METHOD	$\hat{u}_N(1)$	$x\hat{u}_N'(1)$	$\|e(x)\|_{L_2}$	$\|e(x)\|_E$
COLLOCATION	1.205	4.575	1.023	1.477
SUBDOMAIN	-0.926	4.575	0.154	0.483
LEAST SQUARES	-0.644	4.575	0.135	0.46
GALERKIN (b)	-0.683	4.575	0.089	0.413
GALERKIN (c)				
1 TERM	0	0	0.738	1.376
2 TERM	-1.126	0	0.203	1.25
3 TERM	-0.563	5.811	0.1	0.318
EXACT	-0.588	4.575	—	—

THE ERROR IN THE FLUX AT $x=1$ FOR PART C IS NOT SURPRISING DUE TO THE CHOICE IN BASIS FUNCTIONS. ALSO, NOTE THAT THE GALERKIN METHOD MINIMIZES THE ERROR AS EXPECTED, AND THAT INCREASING N LEADS TO LOWER ERRORS.



Weighted Residual Method Solutions
Exact
Collocation
Subdomain
Least Squares
Galerkin
Displacement
Position
2
1
0
-1
-2
-3
-4
0
0.2
0.4
0.6
0.8
1



Galerkin Solutions
1.5
1
0.5
0
-0.5
-1
-1.5
0
0.2
0.4
0.6
0.8
1
Displacement
Position
Exact
1 Term
2 Terms
3 Terms

HOMEWORK #3 SOLUTION SET

① a) $-(u')' = 1, \quad 0<x<1, \quad u(0)=u(1)=0$

i) THE WORK FUNCTION

$$\pi[u] = \frac{1}{2}\int_0^1 (u')^2 - 1\cdot u \, dx$$

$$u = \hat{u}_N = \sum_{j=1}^{N} \alpha_j \phi_j$$

$$\boxed{\pi[\hat{u}_N] = \frac{1}{2}\int_0^1 \Big(\sum_{j=1}^{N} \alpha_j \phi_j'\Big)^2 - \sum_{j=1}^{N} \alpha_j \phi_j \, dx}$$

ii) THE RESIDUAL FUNCTION

$$r = \mathcal{L}u - f = -(u')' - 1$$

$$\int_0^1 r v \, dx = \int_0^1 v\left(-(u')' - 1\right) dx = 0$$

$$u = \hat{u}_N = \sum_{j=1}^{N} \alpha_j \phi_j, \quad v = \sum_{k=1}^{N} \beta_k \psi_k$$

$$\boxed{\sum_{k=1}^{N} \beta_k \left[\sum_{j=1}^{N} \alpha_j \int_0^1 \psi_k (-\phi_j'') \, dx - \int_0^1 \psi_k \, dx\right] = 0}$$

iii) THE WEAK FORMULATION

FROM THE RESIDUAL FUNCTION,

$$\int_0^1 v\left(-(u')' - 1\right) dx = -u'v\Big|_0^1 + \int_0^1 v'u' - v \, dx = 0$$

$$\int_0^1 v'u' - v \, dx = 0$$

$$u = \hat{u}_N = \sum_{j=1}^{N} \alpha_j \phi_j, \quad v = \sum_{k=1}^{N} \beta_k \psi_k$$

$$\boxed{\sum_{k=1}^{N} \beta_k \left[\sum_{j=1}^{N} \alpha_j \int_0^1 \phi_j' \psi_k' \, dx - \int_0^1 \psi_k \, dx\right] = 0}$$

iv) $$\|e\|_{L_2} = \sqrt{\int_0^1 (\hat{u}_N - u)^2 \, dx}$$

v) $$\|e\|_E = \sqrt{\frac{1}{2}\int_0^1 (\hat{u}_N' - u')^2 \, dx}$$

b) $-(u')' = f, \quad 0 < x < 1, \quad u(0)=1, \; u'(1)=1$

i) THE WORK FUNCTION

LET $u = u_b + \overbrace{\sum_{j=1}^{N} a_j \phi_j}^{\hat{u}_N}, \quad u_b = 1+x$. NOTE, $u_b'' = 0$.

$$\pi[\hat{u}_N] = \frac{1}{2}\int_0^1 (u')^2 - fu\,dx$$

$$\boxed{\pi[\hat{u}_N] = \frac{1}{2}\int_0^1 \Big(\sum_{j=1}^{N} a_j \phi_j'\Big)^2 - f\sum_{j=1}^{N} a_j\phi_j\,dx}$$

$-\cancel{u}u'' + u' = 1$

$u_b = x-1$

$u_b = 1$

$u_b'' = 0$

ii) THE RESIDUAL FUNCTION

$r = \mathcal{L}u - f. \quad u = u_b + \overbrace{\sum_{j=1}^{N} a_j \phi_j}^{\hat{u}_N}, \quad u_b = 1+x. \quad v = \sum_{k=1}^{N} \beta_k \psi_k$

$$\int_0^1 rv\,dx = \int_0^1 v\mathcal{L}\hat{u}_N - vf + v\cancelto{0}{\mathcal{L}u_b}\,dx = 0$$

$$\boxed{\sum_{k=1}^{N}\beta_k\Big[\sum_{j=1}^{N} a_j \int_0^1 \psi_k\big(-(\phi_j')'\big)\,dx - \int_0^1 f\psi_k\,dx\Big] = 0}$$

iii) THE WEAK FORM

THIS SATISFIES ONLY ESSENTIAL B.C.'s.

FROM $\int_0^1 rv\,dx = \int_0^1 v(-u'') - vf\,dx = 0$, WITH $u = 1 + \sum_{j=1}^{N} a_j\phi_j$

$-vu'\big|_0^1 + \int_0^1 v'u' - vf\,dx = 0 \qquad v = \sum_{k=1}^{N}\beta_k\psi_k$

$$\boxed{\sum_{k=1}^{N}\beta_k\Big[\sum_{j=1}^{N} a_j\int_0^1 \phi_j'\psi_k'\,dx - \int_0^1 \psi_k f\,dx - \psi_k(1)\,\cancel{u'}(1)\Big] = 0}$$

iv) $\|e\|_{L_2} = \sqrt{\int_0^1 (\hat{u}_N - u)^2\,dx}$

v) $\|e\|_E = \sqrt{\frac{1}{2}\int_0^1 (\hat{u}_N' - u')^2\,dx}$

c) $-(Ku')' + Bu' + Cu = f, \quad 0<x<1, \quad u'(0)=1, \quad u'(1)=1$

i) THE WORK FUNCTION

$$\pi[u] = \frac{1}{2}\int_0^1 K(u')^2 + Bu'u + Cu^2 - fu \, dx$$

$$u = u_b + \hat{u}_N, \quad u_b = x, \quad \hat{u}_N = \sum_{j=1}^{N} a_j \phi_j. \quad \text{NOTE, } \mathcal{L}u_b \neq 0.$$

$$-(K\hat{u}_N')' + B\hat{u}_N' + C\hat{u}_N = f \underbrace{- B - Cx}_{-\mathcal{L}u_b}$$

$$\boxed{\pi[\hat{u}_N] = \frac{1}{2}\int_0^1 K\Big(\sum_{j=1}^{N} a_j\phi_j'\Big)^2 + B\Big(\sum_{j=1}^{N} a_j\phi_j'\Big)\Big(\sum_{j=1}^{N} a_j\phi_j\Big) + C\Big(\sum_{j=1}^{N} a_j\phi_j\Big)^2 - (f - B - Cx)\Big(\sum_{j=1}^{N} a_j\phi_j\Big)\, dx}$$

ii) THE RESIDUAL FUNCTION

$$r = \mathcal{L}u - f \qquad u = x + \hat{u}_N, \quad \hat{u}_N = \sum_{j=1}^{N} a_j\phi_j. \Rightarrow r = \mathcal{L}\hat{u}_N + \mathcal{L}x - f.$$

$$\int_0^1 rv\,dx = \int_0^1 v\big(-(K\hat{u}_N')' + B\hat{u}_N' + C\hat{u}_N - f + B + Cx\big)dx = 0$$

$$v = \sum_{k=1}^{N} \beta_k \psi_k$$

$$\boxed{\sum_{k=1}^{N} \beta_k\Big[\sum_{j=1}^{N} a_j \int_0^1 \psi_k(-K\phi_j'' + B\phi_j' + C\phi_j)\,dx - \int_0^1 \psi_k(f - B - Cx)\,dx\Big] = 0}$$

iii) THE WEAK FORM

FROM $\int_0^1 v(-(Ku')' + Bu' + Cu - f)\,dx = 0,$

$$-vKu'\Big|_0^1 + \int_0^1 Kv'u' + Bvu' + Cvu - fv\,dx = 0$$

$$u = \sum_{j=1}^{N} a_j\phi_j, \quad v = \sum_{k=1}^{N} \beta_k\psi_k$$

$$\boxed{\sum_{k=1}^{N} \beta_k\Big[\sum_{j=1}^{N} a_j \int_0^1 K\phi_j'\psi_k' + B\phi_j'\psi_k + C\phi_j\psi_k\,dx - \int_0^1 \psi_k f\,dx - \psi_k(1)\cdot K + \psi_k(0)K\Big] = 0}$$

iv) $\boxed{\|e\|_{L^2} = \sqrt{\int_0^1 (u - u)^2 dx}}$

v) $\boxed{\|e\|_E = \sqrt{\frac{1}{2}\int_0^1 K(u' - u')^2 + B(u' - u')(u - u) + C(u - u)^2\, dx}}$

d) $-(3xu')' = \sin x,\quad 0<x<1,\quad u(0)=1,\quad u(1)=1$

i) THE WORK FUNCTION

$$\pi[u] = \frac{1}{2}\int_0^1 3x(u')^2 - u\sin x \, dx$$

$$u = 1 + \hat{u}_N, \quad \hat{u}_N = \sum_{j=1}^{N} a_j \phi_j .$$

$$\boxed{\pi[\hat{u}_N] = \frac{1}{2}\int_0^1 3x\Big(\sum_{j=1}^{N} a_j\phi_j'\Big)^2 - \Big(\sum_{j=1}^{N} a_j\phi_j\Big)\sin x \, dx}$$

ii) THE RESIDUAL FUNCTION

$r = \mathcal{L}u - f.$ $\quad u = u_b + u_N,\quad u_b = 1,\quad u_N = \sum_{j=1}^{N} a_j\phi_j .$ NOTE, $\mathcal{L}u_b = 0.$

$$\int_0^1 rv = \int_0^1 v\Big(-\Big(3x\sum_{j=1}^{N} a_j\phi_j'\Big)' - \sin x\Big) dx = 0 \qquad \text{LET } v = \sum_{k=1}^{N}\beta_k\psi_k$$

$$\boxed{\sum_{k=1}^{N}\beta_k\Big[\sum_{j=1}^{N} a_j\int_0^1\psi_k\big(-(3x\phi_j')'\big)dx - \int_0^1\psi_k\sin x\,dx\Big] = 0.}$$

iii) THE WEAK FORM

FROM $\int_0^1 v\big(-(3xu')' - \sin x\big)dx = 0$

$$-\cancel{3xu'v\Big|_0^1} + \int_0^1 3xv'u' - v\sin x \, dx = 0.$$

$$u = \sum_{j=1}^{N} a_j\phi_j, \quad v = \sum_{k=1}^{N}\beta_k\psi_k$$

$$\boxed{\sum_{k=1}^{N}\beta_k\Big[\sum_{j=1}^{N} a_j\int_0^1 3x\psi_k'\phi_j'\,dx - \int_0^1\psi_k\sin x\,dx\Big] = 0}$$

iv) $\|e\|_{L2} = \sqrt{\int_0^1 (u-\tilde{u})^2 dx}$

v) $\|e\|_E = \sqrt{\frac{1}{2}\int_0^1 3x(u'-\tilde{u}')^2 dx}$

(2) FOR THE WORK FUNCTION,

U MUST SATISFY BOTH ESSENTIAL & NATURAL B.C.'S

FOR THE RESIDUAL FUNCTION,

U MUST SATISFY BOTH ESSENTIAL & NATURAL B.C.'S

V MUST SATISFY HOMOGENEOUS VERSIONS OF THE GIVEN B.C.'S

FOR THE WEAK FORM,

U MUST SATISFY ESSENTIAL B.C.'S

FOR ALL, U & V MUST BE SQUARE INTEGRABLE

(3) THE DIRECT FORM (SOLVING THE WORK FUNCTION) OFTEN RESULTS IN A NONLINEAR SET OF COUPLED EQUATIONS ($(\sum_{j=1}^{\tilde{n}} \alpha_j \phi_j')^2$ FOR INSTANCE). THE RESIDUAL FUNCTION EXPLICITLY SATISFIES ALL B.C.'S (THOUGH SOMETIMES YIELDS $\mathcal{L}u_0$ TERMS AS EFFECTIVE FORCES. THE WEAK FORM ONLY SATISFIES ESSENTIAL B.C.'S, AND CAN INTRODUCE FLUX TERMS ($Ku'v|_0^1$) AS EFFECTIVE FORCES.

HW #4 SOLUTION SET

FROM CLASS:

$$L = \tfrac{1}{2} m_1 \dot{x}_1^2 + \tfrac{1}{2} m_2 (\dot{x}_1 + \dot{x}_2)^2 + \tfrac{1}{2} m_3 \dot{x}_3^2 - \tfrac{1}{2} k_1 x_1^2 - \tfrac{1}{2} k_2 x_2^2 - \tfrac{1}{2} k_3 x_3^2$$

$$W_{NC} = F(x_1 + x_2) - F_c(x_1 + x_2) - F_c x_3$$

LAGRANGE:

$$\frac{\partial}{\partial t}\frac{\partial L}{\partial \dot{x}_i} - \frac{\partial L}{\partial x_i} = \frac{\partial W_{NC}}{\partial x_i}$$

$F = 10 \sin \pi t$

$F_c = \frac{4}{3} E \sqrt{r}\, \delta^{3/2}$, $\quad \gamma = \gamma_0 - x_1 - x_2 - x_3$, $\quad \delta = \begin{cases} -\gamma & \gamma < 0 \\ 0 & \gamma > 0 \end{cases}$

$$\underbrace{\begin{bmatrix} m_1 + m_2 & m_2 & 0 \\ m_2 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix}}_{M} \underbrace{\begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{Bmatrix}}_{\ddot{x}} + \underbrace{\begin{bmatrix} k_1 & 0 & 0 \\ 0 & k_2 & 0 \\ 0 & 0 & k_3 \end{bmatrix}}_{K} \underbrace{\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}}_{x} = \underbrace{\begin{Bmatrix} F - F_c \\ F - F_c \\ -F_c \end{Bmatrix}}_{f}$$

IN STATE SPACE FORM

$$\underbrace{\begin{bmatrix} M & 0 \\ 0 & I \end{bmatrix}}_{m} \dot{y} = \underbrace{\begin{bmatrix} 0 & -K \\ I & 0 \end{bmatrix}}_{k} y + \underbrace{\begin{Bmatrix} f \\ 0 \end{Bmatrix}}_{F}, \quad y = \begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ x_1 \\ x_2 \\ x_3 \end{Bmatrix}$$

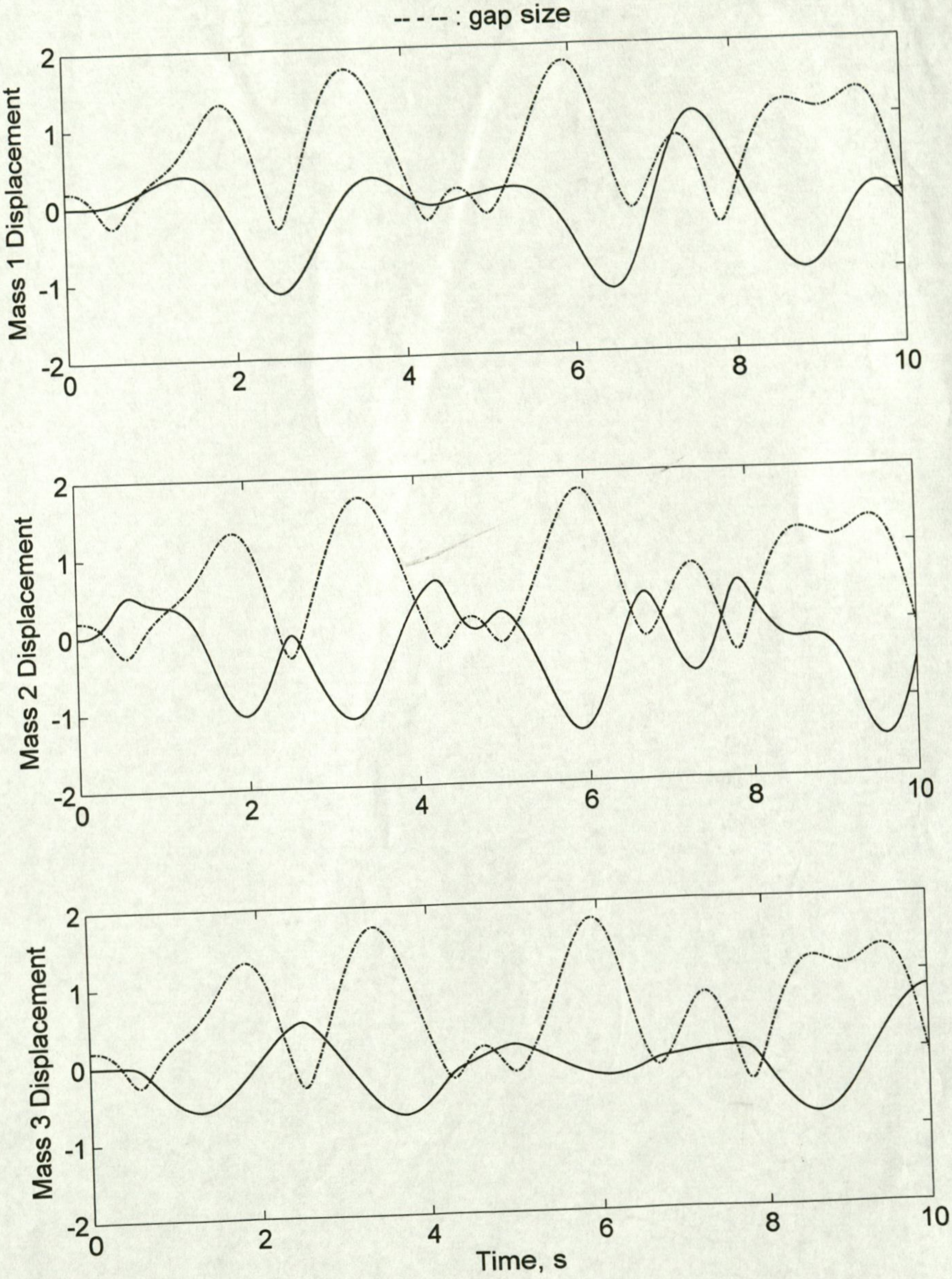
$$\dot{y} = m^{-1} k\, y + m^{-1} F$$
$$= A y + F$$

5/14/14 → 2–30 → 5:30

Presentation

```
close all
clear all

m1 = 4;
m2 = 1;
m3 = 3;
K = 12 ;
gamma0 = 0.2 ;
F = @(t,o) 10*sin(o*t) ;
delta = @(gamma) (sign(gamma)-1)./2.*gamma ;
Fc = @(gamma) 4/3*100*(delta(gamma))^1.5 ;

M = [m1+m2 m2 0; m2 m2 0; 0 0 m3] ;
K = [K 0 0; 0 K 0 ; 0 0 K] ;
AMatrix = [M zeros(3,3); zeros(3,3) eye(3,3)]\ ...
  [zeros(3,3) -K; eye(3,3) zeros(3,3)] ;

Force = @(t,y,o) [M zeros(3,3); zeros(3,3) eye(3,3)]\ ...
  [F(t,o)-Fc(gamma0-y(4)-y(5)-y(6));
   F(t,o)-Fc(gamma0-y(4)-y(5)-y(6));
  -Fc(gamma0-y(4)-y(5)-y(6)); 0; 0; 0] ;

params.IC = [0;0;0;0;0;0] ;
tstart = 0 ;
params.dt = 1e-3 ;
tfinal = 10 ;
params.tol = 1e-3 ;
params.dtmin = 1e-9 ;
params.dtmax = 1e-1 ;
params.output_res = 1e-2 ;
params.output_flag = 1 ;
params.method = 5 ;


[Y, T, ~, ~] = IMEX_a(@(t,y,UD)Force(t,y,pi), ...
                 AMatrix, tstart, tfinal, params) ;

figure(1)
subplot(311)
hold on
plot(T,gamma0-Y(:,4)-Y(:,5)-Y(:,6),'-.k')
plot(T,Y(:,4))
box on
axis([0 10 -2 2])
ylabel('Mass 1 Displacement','FontSize',14)
title('-- - -- : gap size','FontSize',14)
subplot(312)
hold on
plot(T,gamma0-Y(:,4)-Y(:,5)-Y(:,6),'-.k')
plot(T,Y(:,5)+Y(:,4))
box on
```

```
axis([0 10 -2 2])
ylabel('Mass 2 Displacement','FontSize',14)
subplot(313)
hold on
plot(T,gamma0-Y(:,4)-Y(:,5)-Y(:,6),'-.k')
plot(T,Y(:,6))
box on
axis([0 10 -2 2])
ylabel('Mass 3 Displacement','FontSize',14)
xlabel('Time, s','FontSize',14)
set(gcf,'Position',[450 250 550 800])

os = linspace(0.1,20,200)*pi ;
xlabel('Frequency, Hz','FontSize',14)
ylabel('Impact Velocity','FontSize',14)
tfinal = 100 ;
figure(2)
hold on
for cntr = 1:length(os)
  [Y, T, ~, ~] = IMEX_a(@(t,y,UD)Force(t,y,os(cntr)), ...
                  AMatrix, tstart, tfinal, params) ;

  gap = gamma0-Y(:,4)-Y(:,5)-Y(:,6);
  for cnt = floor(length(gap)/2):length(gap)-1
    if gap(cnt) > 0 && gap(cnt+1) < 0
      plot(os(cntr)/pi,Y(cnt,1)+Y(cnt,2)+Y(cnt,3),'.')
    end
  end

end
```



- - - - : gap size
Mass 1 Displacement
Mass 2 Displacement
Mass 3 Displacement
2
1
0
-1
-2
0
2
4
6
8
10
Time, s

Computational Mechanics

Department of Mechanical Engineering
ME404/504
Spring Semester, 2014

FEA Tutorial Project #3

Continuing to help you develop proficiency in finite element methods, the second tutorial is focused on thermal modeling. In particular, for the first part, I would like for you to pay careful attention to the verification and validation section of the tutorial. It's one thing to be able to get an answer using FEA, but if you understand the underlying theory and have confidence that the answer is correct, that's a much more powerful result.

Similar to the first assignment, go to the tutorials website:
https://confluence.cornell.edu/display/SIMULATION/ANSYS+Learning+Modules

1. In the "Introductory Tutorials" section, your first tutorial is either problem 1 of the homework for "Cantilever Beam Modal Analysis" or "Modal Analysis of the Wing." Work through the tutorial and **turn in either plots from problem 1 (for the beam) or 1) a table of the first ten natural frequencies for three different meshes, plus 2) a plot of the first two natural frequencies from the verification and validation page for a single mesh of your choice (for the wing)**.

Your assignments are due at the start of class on Monday, March 31st.

Department of Mechanical Engineering
ME404/504
Spring Semester, 2014

Computational Mechanics

FEA Tutorial Project #4

Continuing to help you develop proficiency in finite element methods, the last tutorial is left to your own research interests. Similar to the previous assignments, go to the tutorials website:

https://confluence.cornell.edu/display/SIMULATION/ANSYS+Learning+Modules

1. In the "Advanced Tutorials" section, work through **one** of these tutorials:
   a. High Resolution FE Model of Bone
   b. Hertz Contact Mechanics
   c. Wind Turbine Blade
   d. Advanced FEA for Large Telescope Truss
   e. Linear Column Buckling

   Work through the tutorial to the verification and validation section. Turn in a plot that is representative of your results, along with a paragraph of what you liked and didn't like about the tutorial.

Your assignments are due at the start of class on Monday, April 21st.

Statement of Academic Integrity

As this exam is given as a take-home exam, I would like to remind everyone of the expectations for academic integrity. From the scholastic regulations listed in the UNM catalog,

> Each student is expected to maintain the highest standards of honesty and integrity in academic and professional matters. The University reserves the right to take disciplinary action, up to and including dismissal, against any student who is found guilty of academic dishonesty or otherwise fails to meet these standards.
>
> Academic dishonesty includes, but is not limited to, dishonesty in quizzes, tests, or assignments; claiming credit for work not done or done by others; and nondisclosure or misrepresentation in filling out applications or other university records.
>
> When a violation of the regulation occurs in connection with a course, seminar, or any other academic activity under the direction of a faculty member, that faculty member is authorized to take whatever action is deemed appropriate, [including the giving of an] "F" in the course and the involuntary withdrawal of the student from the class.

For this exam, you are expected to work individually and to not discuss the exam with any of your colleagues: both fellow students in and out of the class as well as seeking advice from other professors/subject matter experts. You **are** allowed to consult your notes and textbook.

By signing below, you acknowledge that you have read and understood the academic integrity statement, and that you understand the potential penalties for cheating on this exam. Please keep this coversheet attached to your exam.

Mohiuddin Ahmad

(Print Name)

M. uddin

(Signature)

4/16/14

(Date)

1	16
2	14
3	19
4	17
5	33
EC	12
	111

Class grade as of 4/30

A+

Name Mohiuddin Ahmad

Computational Mechanics

Department of Mechanical Engineering
ME404/504
Spring Semester, 2014

Exam

1. Concepts/Definitions (16 points): In two to three sentences, define and describe the following concepts:

A. What was Leibniz's contribution to the field of mechanics?

- Developed Integral & differential calculus & the idea of continuity.
- He invented the differential & integral sign
- Developed method of separation of variables
- Law of conservation of kinetic energy → Developed step wheeled calculator.
- Expressed number in binary

B. Define Virtual Displacement and Virtual Work, and discuss their relevance to the field of mechanics.

Among all the possible displacements a particle may follow under the action of a force, virtual displacement is the one that minimize the action. The work done by virtual displacement is virtual work. They controls the equilibrium of a mechanical system and is fundamental for analytical mechanics.

C. Give an example of a Scleronomic, Holonomic constraint, and enumerate why the example is scleronomic, and why it is holonomic.

A simple pendulum is an example of scleronomic holonomic constraint. The x & y coordinates at any instant are related by the radius, hence holonomic (degree of freedom is reduced). It is also scleronomic as ~~total~~ energy stays constant.

D. Classify each of the following examples as either essential or natural boundary conditions:

Geometric Essential

Neumann Natural

$JG\frac{d\theta}{dr}\Big|_{x=0} = \kappa\theta(0)$ Robin → Natural

2. Variational Calculus (14 points): Derive the solution to each of the following problems via the following steps:

a. Calculate the derivatives for the first Euler-Lagrange equation
b. Apply the Euler-Lagrange equation to determine the differential equation governing the solution
c. Apply the boundary conditions and solve the resulting equations

2.1. $I(y) = \int_0^1 x(y(x))^4 + 2x^2(y(x))^3 y'(x)dx$

Subject to $y(0) = 0$ and $y(1) = 1$

2.2. $I(y) = \int_0^1 (1 + (y'(x))^2)^{1/2} dx$

Subject to $y(0) = 0$ and $y(1) = 2$

For maximum credit, show all work.

2.1 a) $L = xy^4 + 2x^2y^3y'$, $\frac{dL}{dy} = 4xy^3 + 2x^2y'3y^2 = 4xy^3 + 6x^2y^2y'$ ✓

$\frac{dL}{dy'} = 2x^2y^3$, $\frac{d}{dx}\left(\frac{dL}{dy'}\right) = 2x^2 3y^2y' + 2y^3 2x = 4xy^3 + 6x^2y^2y'$ ✓

b) Euler-Lagrange equation $\frac{d}{dx}\left(\frac{dL}{dy'}\right) - \frac{dL}{dy} = 0$

$\Rightarrow 4xy^3 + 6x^2y^2y' - (4xy^3 + 6x^2y^2y') = 0$

$0 = 0$ ✓

It is true for any value of x & y. Therefore, there exists no $y = f(x)$ relation that will minimize the system.

Provided B.C.'s satisfied.

2.2 a) $L = \sqrt{1+y'^2}$, $\frac{dL}{dy'} = \frac{1}{2\sqrt{1+y'^2}} \times 2y' = \frac{y'}{\sqrt{1+y'^2}}$ ✓

$\frac{dL}{dy} = 0$, then no need for $\frac{d}{dx}\left(\frac{dL}{dy}\right)$ ✓ Good.

b) Then, $\frac{d}{dx}\left(\frac{dL}{dy'}\right) = 0 \Rightarrow \frac{dL}{dy'} = C \Rightarrow \frac{y'}{\sqrt{1+y'^2}} = C$ ✓

$\Rightarrow y'^2 = C^2(1+y'^2) = D + Dy'^2$ $[D = C^2]$

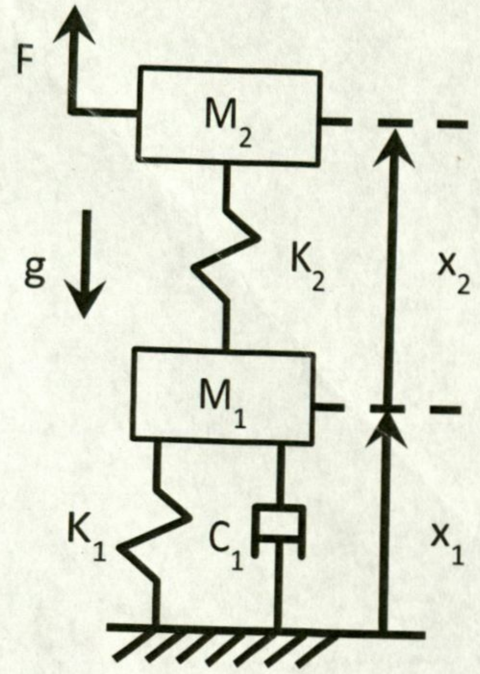
$\Rightarrow (D-1)y'^2 = D \Rightarrow y'^2 = \frac{D}{D-1} \Rightarrow y' = E$, $E = \sqrt{\frac{D}{D-1}}$ = constant

c) $\therefore y = Ex + C_2$ $y(0) = 0 \Rightarrow C_2 = 0$, $y(1) = 2$

$\Rightarrow 2 = E$

Therefore, $y = 2x$. ✓

3. Lagrangian Dynamics (20 points): Derive the equations of motion for the following system using Lagrange's principle, and cast in matrix form, using the variables and coordinates given.



The force $F = \sin(\omega_E t)$. Note that x_1 is an absolute coordinate while x_2 is a relative coordinate. Also, there is no friction, but there is gravity. In your solution, take the following steps:

A. Classify the forces (Active, Constraint, Non-Conservative)
B. Determine the potential and kinetic energy, and the non-conservative forces
C. Apply Lagrange's equation
D. Place in matrix form

For maximum credit, show all work.

(a) Active force: spring forces, No constraint force   Gravity?

Non conservative force: F & damper force

(b) Potential energy: $\frac{1}{2}K_1x_1^2 + \frac{1}{2}K_2x_2^2 + M_1 g x_1 + M_2 g(x_1+x_2) = V$

Kinetic energy: $\frac{1}{2}M_1\dot{x}_1^2 + \frac{1}{2}M_2(\dot{x}_2+\dot{x}_1)^2 = T$

Non conservative force: $-C_1\dot{x}_1 + F$

(c) $\mathcal{L} = T - V = \frac{1}{2}M_1\dot{x}_1^2 + \frac{1}{2}M_2(\dot{x}_1+\dot{x}_2)^2 - \frac{1}{2}K_1x_1^2 - \frac{1}{2}K_2x_2^2 - M_1 g x_1 - M_2 g x_1 - M_2 g x_2$

$\frac{\partial \mathcal{L}}{\partial x_1} = -K_1x_1 - M_1 g - M_2 g$, $\frac{\partial \mathcal{L}}{\partial x_2} = -K_2x_2 - M_2 g$

$\frac{\partial \mathcal{L}}{\partial \dot{x}_1} = M_1\dot{x}_1 + M_2(\dot{x}_1+\dot{x}_2) = (M_1+M_2)\dot{x}_1 + M_2\dot{x}_2$

$$\frac{\partial L}{\partial \dot{x}_2} = M_2(\dot{x}_1 + \dot{x}_2)$$

$$\therefore \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right) = (M_1+M_2)\ddot{x}_1 + M_2\ddot{x}_2$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_2}\right) = M_2(\ddot{x}_1 + \ddot{x}_2)$$

(c) Lagrange's Equation:

$Q_{nc_p} = \frac{\delta W}{\delta x_p}$ Here work done by non-conservative forces $= F(x_1+x_2) + c\dot{x}_1(-x_1)$

$$Q_{nc,x_1} = \frac{\delta W}{\delta x_1} = F - c\dot{x}_1, \quad Q_{nc,x_2} = \frac{\delta W}{\delta x_2} = F, \quad \text{Here } F = \sin\omega_E t.$$

Therefore, Lagrange's equations will be,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right) - \frac{\partial L}{\partial x_1} = Q_{nc,x_1}$$

$$\Rightarrow (M_1+M_2)\ddot{x}_1 + M_2\ddot{x}_2 + Kx_1 + (M_1+M_2)g = \sin\omega_E t - c\dot{x}_1$$

$$\Rightarrow (M_1+M_2)\ddot{x}_1 + M_2\ddot{x}_2 + c\dot{x}_1 + K_1x_1 = \sin\omega_E t - (M_1+M_2)g \quad \text{----- (1)}$$

and $$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_2}\right) - \frac{\partial L}{\partial x_2} = Q_{nc,x_2}$$

$$\Rightarrow M_2\ddot{x}_1 + M_2\ddot{x}_2 + K_2x_2 + M_2 g = \sin\omega_E t$$

$$\Rightarrow M_2\ddot{x}_1 + M_2\ddot{x}_2 + K_2x_2 = \sin\omega_E t - M_2 g \quad \text{----- (2)}$$

(d) In matrix form,

$$\begin{bmatrix} M_1+M_2 & M_2 \\ M_2 & M_2 \end{bmatrix}\begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} c & 0 \\ 0 & 0 \end{bmatrix}\begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} + \begin{bmatrix} K_1 & 0 \\ 0 & K_2 \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} \cancel{\sin\omega_E t} - (M_1+M_2)g \\ \sin\omega_E t - M_2 g \end{Bmatrix}$$

F only applied to the 2nd mass...

4. Computational Methods (20 points): Consider the following differential equation:

$$-(\sinh x\, u')' + xu = e^x$$

$$0 < x < 1, \qquad u'(0) = T_0, \qquad u(1) = 0$$

State u & v (if appropriate), and find the function in terms of ϕ & φ (if appropriate) for

i. The work function
ii. The residual function
iii. The weak formulation
iv. The L_2 error norm (with exact solution u)
v. The Energy error norm (with exact solution u)

For maximum credit, show all work.

(i) $k = \sinh x$

Work function : $\frac{1}{2}\int_0^1 \left(\sinh x (U')^2 - xU^2 - 2e^x U\right) dx$ good

(ii) Residual function: $\int_0^1 r v\, dx = 0$

$\Rightarrow \int_0^1 (\mathcal{L}u - f)\, v\, dx = 0$

$\Rightarrow \int_0^1 [-(\sinh x\, u')' + xu - e^x]\, v\, dx = 0$

What are u & v? Are superposition functions used?

(iii) The weak formulation :

$u'(0) = T_0, \; u(1) = 0$

$\therefore v(1) = 0.$

$\int_0^1 [-(\sinh x\, u')' + xu - e^x]\, v\, dx = 0$

$\Rightarrow -\int_0^1 (\sinh x\, u')' v\, dx + \int_0^1 xuv\, dx - \int_0^1 e^x v\, dx = 0$

$\Rightarrow -v \sinh x\, u' \Big|_0^1 + \int \sinh x\, u'v'\, dx + \int_0^1 xuv\, dx - \int_0^1 e^x v\, dx = 0$

$\Rightarrow \int_0^1 \sinh x\, u'v'\, dx + \int_0^1 xuv\, dx - \int_0^1 e^x v\, dx - \underbrace{v(1)}_{0} \sinh(1)\, u'(1) + v(0) \underbrace{\sinh(0)}_{0} u'(0) = 0.$

$\Rightarrow \int_0^1 \sinh x\, u'v'\, dx + \int_0^1 xuv\, dx - \int_0^1 e^x v\, dx = 0$

This is the weak formulation.

(iv) If error $e(x) = \hat{u} - u$

Then, $\|e\|_{L2} = \sqrt{\int_0^1 e^2\,dx} = \sqrt{\int_0^1 (\hat{u}-u)^2\,dx}$

(v) $\|e\|_e = \sqrt{\frac{1}{2}\int_0^1 \sinh x\,(e')^2 - x e^2 \cdot dx}$

$= \sqrt{\frac{1}{2}\int_0^1 \sinh x\,(\hat{u}'-u')^2 - x(\hat{u}-u)^2\,dx}$

5. Method of Weighted Residuals (30 points): Consider the following boundary value problem:

$$-(J(x)G\theta')' = \sin x, \qquad 0 < x < 1$$

$$J(x) = J_0(1+x)$$

with boundary conditions

$$-GJ_0(1+x)\theta'' - GJ_0\theta' = \sin x.$$

$$J(0)G\theta'(0) = T_0, \qquad \theta(1) = 0.$$

a. Find the approximate **two** term solution via the residual function for this system using the Collocation method. In solving for the approximate solution, use an appropriate function to generate homogeneous boundary conditions at both ends of the domain. Your basis functions should be simple polynomials of your choice provided that they are admissible.
b. Find the approximate **two** term solution via the residual function for this system using the Least Squares method. In solving for the approximate solution, use an appropriate function to generate homogeneous boundary conditions at both ends of the domain. Your basis functions should be simple polynomials of your choice provided that they are admissible.
c. Derive the weak form of the system equation
d. Find the **two** term approximate solutions via the Galerkin method using the weak form of the problem that only explicitly satisfies the essential boundary condition. Your basis functions should be simple polynomials of your choice provided that they are admissible and that they make sense given the boundary conditions.
e. Calculate the L2 and Energy norms, and use a table to present the results for the solutions from parts a, b, and c.
f. Plot your approximate solutions from parts a, b, and c.
g. Discuss your results.

a) Residual function: $\int (\mathcal{L}\theta - f)\, v\, dx = 0$

let, $\hat{\theta} = (x-1)\frac{T_0}{J_0 G} + \sum_{j=1}^{N} \alpha_j \phi_j = \underbrace{(x-1)T}_{\hat{\theta}_b} + \underbrace{\sum_{j=1}^{N} \alpha_j \phi_j}_{\theta_n}$

Here, $T = \frac{T_0}{J_0 G}$ & $\phi'(0) = 0$, $\phi(1) = 0$

Then, $\int (\mathcal{L}(\theta) - f)\, v\, dx = \int [\mathcal{L}(\hat{\theta}_N) - (f - \mathcal{L}(\hat{\theta}_b))]\, v\, dx = 0$

let, $f^* = f - \mathcal{L}(\hat{\theta}_b) = \sin x + T_0$

Now, $\int [\mathcal{L}(\hat{\theta}_N) - f^*]\, v\, dx = 0$, let, $v = \sum_{k=1}^{N} \beta_k \psi_k$

$\Rightarrow \sum \alpha_j \int_0^1 \mathcal{L}[\phi_j(x)]\, \psi_k\, dx - \int_0^1 f^*_v \psi_k\, dx = 0$ $\quad \psi'(0) = 0$, $\psi(1) = 0$.

Then, $K_{jk} = \int_0^1 \mathcal{L}[\varphi_j^\circ(x)]\, \psi_k \, dx$

$\underset{\sim}{f} = \int_0^1 f^* \psi_k \, dx$

Let, Now, $\varphi_j^\circ = \cos\left((2j-1)\frac{\pi}{2}x\right)$

$\varphi_j' = -(2j-1)\sin\left[(2j-1)\frac{\pi}{2}x\right]$

$\varphi_j'' = -(2j-1)^2\cos\left[(2j-1)\frac{\pi}{2}x\right]$

Then.

$$\mathcal{L}(\varphi_j^\circ) = -J_0 G(1+x)\varphi_j^{\circ\,''} - J_0 G \varphi_j'$$

$$= J_0 G(2j-1)^2(1+x)\cos\left[(2j-1)\frac{\pi}{2}x\right] + J_0 G(2j-1)\sin\left[(2j-1)\frac{\pi}{2}x\right]$$

for two term solution, $x_1 = \frac{1}{3}$, $x_2 = \frac{2}{3}$.

$$K_{11}^\circ = \mathcal{L}\left[\varphi_1(x_1)\right] = J_0 G(2-1)^2(1+\tfrac{1}{3})\cos\left[(2-1)\tfrac{\pi}{2}\times\tfrac{1}{3}\right] + J_0 G(2-1)\sin\left[(2-1)\tfrac{\pi}{2}\cdot\tfrac{1}{3}\right]$$

$$= 1.655\, J_0 G$$

$$K_{12} = \mathcal{L}\left[\varphi_1(x_2)\right] = J_0 G(1+\tfrac{2}{3})\cos\left[\tfrac{\pi}{2}\cdot\tfrac{2}{3}\right] + J_0 G\sin\left(\tfrac{\pi}{2}\cdot\tfrac{2}{3}\right)$$

$$= 1.699\, G J_0$$

$$K_{21} = \mathcal{L}(\varphi_2(x_1)] = J_0 G\left[(4-1)^2(1+\tfrac{1}{3})\cos\left[(4-1)\tfrac{\pi}{2}\cdot\tfrac{1}{3}\right] + (4-1)\sin\left[(4-1)\tfrac{\pi}{2}\cdot\tfrac{1}{3}\right]\right]$$

$$= 3 G J_0$$

$$K_{22} = \mathcal{L}\left[\varphi_2(x_2)\right] = J_0 G\left[9(1+\tfrac{2}{3})\cos\left(3\cdot\tfrac{\pi}{2}\cdot\tfrac{2}{3}\right] + 3\sin\left(3\cdot\tfrac{\pi}{2}\cdot\tfrac{2}{3}\right]\right]$$

$$= -15\, G J$$

$$\underset{\sim}{f} = \int f^* \psi_k \, dx = \int f^* \delta(x - x_k)\, dx = f^*(x_k)$$

$\therefore$

$f_1 = \sin(\tfrac{1}{3}) + T_0 = 0.327 + T_0$

$f_2 = \sin(\tfrac{2}{3}) + T_0 = 0.618 + T_0$.

Therefore:

$$J_0 G \begin{bmatrix} 1.655 & 1.699 \\ -13 & -15 \end{bmatrix} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \end{Bmatrix} = \begin{Bmatrix} 0.327 + T_0 \\ 0.618 + T_0 \end{Bmatrix}$$

Solving,
$$\begin{Bmatrix} \alpha_1 \\ \alpha_2 \end{Bmatrix} = \frac{1}{GJ_0} \begin{Bmatrix} 0.199 \\ 0.002 \end{Bmatrix} + \frac{T_0}{GJ_0} \begin{Bmatrix} 0.558 \\ 6.077 \end{Bmatrix}$$

* assuming $T_0 = 0$ & $GJ_0 = 1$ $\begin{Bmatrix} \alpha_1 \\ \alpha_2 \end{Bmatrix} = \begin{Bmatrix} 0.199 \\ 0.002 \end{Bmatrix}$

good

Hence,
$$\hat{\theta} = 0.199 \cos(\tfrac{\pi}{2} x) + 0.002 \cos(\tfrac{3\pi}{2} x)$$

(b) least square methods

$$K_{ij} = \int \mathcal{L}(\varphi_j) \mathcal{L}(\varphi_i) \, d\Omega$$, assuming $GJ_0 = 1$ & $T_0 = 0$

$\therefore$
$$K_{11} = \int_0^1 \left[(1+x)\cos(\tfrac{\pi}{2}x) + \sin(\tfrac{\pi}{2}x)\right]^2 dx = 2.3176$$

$$K_{12} = K_{21} = \int_0^1 \left[(1+x)\cos(\tfrac{\pi}{2}x) + \sin(\tfrac{\pi}{2}x)\right]\left[9(1+x)\cos(\tfrac{3\pi}{2}x) + 3\sin(\tfrac{3\pi}{2}x)\right] dx$$
$$= -6.3274$$

$$K_{22} = \int_0^1 \left[9(1+x)\cos(\tfrac{3\pi}{2}x) + 3\sin(\tfrac{3\pi}{2}x)\right]^2 dx = 104.859$$

$$\underset{\sim}{f_j} = \int_0^1 \mathcal{L}(\varphi_j) f \, dx$$

$$f_1 = \int_0^1 \left[(1+x)\cos(\tfrac{\pi}{2}x) + \sin(\tfrac{\pi}{2}x)\right] \sin x \, dx = 1.5$$

$$f_2 = \int_0^1 \left[9(1+x)\cos(\tfrac{3\pi}{2}x) + 3\sin(\tfrac{3\pi}{2}x)\right] dx = -3.5884$$

Then,
$$\begin{bmatrix} 2.317 & -6.3274 \\ -6.3274 & 104.859 \end{bmatrix} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \end{Bmatrix} = \begin{Bmatrix} 1.5 \\ -3.5884 \end{Bmatrix}$$

$$\Rightarrow \begin{Bmatrix} \alpha_1 \\ \alpha_2 \end{Bmatrix} = \begin{Bmatrix} 0.663 \\ 0.006 \end{Bmatrix}$$

Therefore, $\hat{\theta} = 0.663\cos(\frac{\pi}{2}x) + 0.066\cos(\frac{3\pi}{2}x)$

(c) Weak form:

$$\int\left(-(J_0G(1+x)\theta')' - \sin x\right) v\,dx = 0$$

$$\Rightarrow -\int (J_0G(1+x)\theta')' v\,dx - \int \sin x\, v\,dx = 0$$

$$\Rightarrow -J_0G(1+x)\theta' v\Big|_0^1 + \int J_0G(1+x)\theta' v'\,dx - \int \sin x\, v\,dx = 0$$

$$\Rightarrow \int_0^1 J_0G(1+x)\theta' v'\,dx - \int_0^1 \sin x\, v\,dx - J_0G(2)\theta'(1)\overset{0}{\cancel{v(1)}} + \underbrace{J_0G\,\theta'(0)}_{T_0}v(0) = 0$$

$$\Rightarrow \int_0^1 GJ_0(1+x)\theta' v'\,dx - \int_0^1 \sin x\, v\,dx + T_0 v(0) = 0$$

This is the weak form.

(d) Let, $\theta = \sum_{j=1}^{n} \alpha_j \phi_j$, $\phi(1) = 0$ | $v = \sum_{k=1}^{N} \beta_k \phi_k$

Then, from weak form,

$$\sum \alpha_j \int_0^1 GJ_0(1+x)\phi_j'\phi_k' - \int_0^1 \sin x\,\phi_k\,dx + T_0\phi_k(0) = 0$$

$$K_{ij} = \int_0^1 GJ_0(1+x)\phi_i'\phi_j'\,dx, \quad \underset{\sim}{f} = \int_0^1 \sin x\,\phi_k\,dx - T_0\phi_k(0).$$

Let $\phi_1 = (1-x)$, $\phi_2 = x^2(1-x)$, assuming $GJ_0 = 1$ & $T_0 = 0$

$\phi_1' = -1$ $\quad \phi_2' = 2x - 3x^2$

$$K_{11} = \int_0^1 (1+x)(-1)(-1)\,dx = 1.5, \quad K_{12} = K_{21} = \int_0^1 (1+x)(-1)(2x-3x^2)\,dx$$

$$= 0.083.$$

$$K_{22} = \int_0^1 (1+x)(2x-3x^2)^2\,dx$$

$$= 0.233.$$

$$f_1 = \int_0^1 \sin x \times (1-x)\,dx = 0.1585$$

$$f_2 = \int_0^1 \sin x \times (x^2 - x^3)\,dx = 0.046$$

Then,
$$\begin{bmatrix} 1.5 & 0.083 \\ 0.083 & 0.233 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{Bmatrix} 0.1585 \\ 0.046 \end{Bmatrix}$$

$$\therefore \begin{Bmatrix} \alpha_1 \\ \alpha_2 \end{Bmatrix} = \begin{Bmatrix} 0.097 \\ 0.163 \end{Bmatrix}$$

Therefore: $\hat{\theta} = 0.097(1-x) + 0.163(x^2 - x^3)$

(e) The exact solution of the field equation is determined using $\sin x = x - \frac{x^3}{3!}$. The exact solution is,

$$\theta = \frac{1}{24}\left[11x - \frac{11x^2}{2} - \frac{x^3}{3} + \frac{x^4}{4}\right] - \frac{11}{24}\ln(1+x) + 0.092$$

Nice work ✓

Then for collocation method:

$$\text{error } e(x) = \hat{\theta} - \theta = 0.199\cos\left(\tfrac{\pi}{2}x\right) + 0.002\cos\left(\tfrac{3\pi}{2}x\right) - \theta$$

$\therefore$ L_2 Norm $\|e(x)\|_{L_2} = \sqrt{\int_0^1 (e(x))^2\,dx} = \sqrt{4.95} = 2.22$

E Norm $\|e(x)\|_e = \sqrt{\frac{1}{2}\int_0^1 (1+x)[e'(x)]^2\,dx}$

here,
$$e'(x) = -0.199 \times \tfrac{\pi}{2}\sin\left(\tfrac{\pi}{2}x\right) - 0.002 \times \frac{3\pi}{2}\sin\left(\tfrac{3\pi}{2}x\right) - \frac{1}{24}\left[11 - 11x - x^2 + x^3\right] + \frac{11}{24(1+x)}.$$

$$= -0.312\sin\left(\tfrac{\pi}{2}x\right) - 0.009\sin\left(\tfrac{3\pi}{2}x\right) - \frac{1}{24}\left[11 - 11x - x^2 + x^3\right] + \frac{11}{24(1+x)}$$

$\therefore \|e(x)\|_e = 0.093$

For least square method:

$$e(x) = 0.663\cos(\pi/2\, x) + 0.006\cos\left(\frac{3\pi}{2}x\right) - \frac{1}{24}\left[11x - \frac{11x^2}{2} - \frac{x^3}{3} + \frac{x^4}{4}\right] + \frac{11}{24}\ln(1+x) - 0.092$$

$$e'(x) = -1.04\sin(\pi/2\, x) - 0.028\sin(3\pi/2\, x) - \frac{1}{24}\left[11 - 11x - x^2 + x^3\right] + \frac{11}{24(1+x)}$$

Then

$$\|e\|_{L_2} = \sqrt{\int_0^1 (e(x))^2\,dx} = 0.59$$

$$\|e\|_e = \sqrt{\frac{1}{2}\int_0^1 (1+x)(e'(x))^2\,dx} = 0.571$$

For Galerkin method using weak form:

$$e(x) = 0.097(1-x) + 0.163(x^2 - x^3) - \frac{11}{24}x + \frac{11}{48}x^2 + \frac{x^3}{72} - \frac{x^4}{96} + \frac{11}{24}\ln(1+x) - 0.092$$

$$= -0.555x + 0.392x^2 - 0.149x^3 - \frac{x^4}{96} + \frac{11}{24}\ln(1+x) + 0.005$$

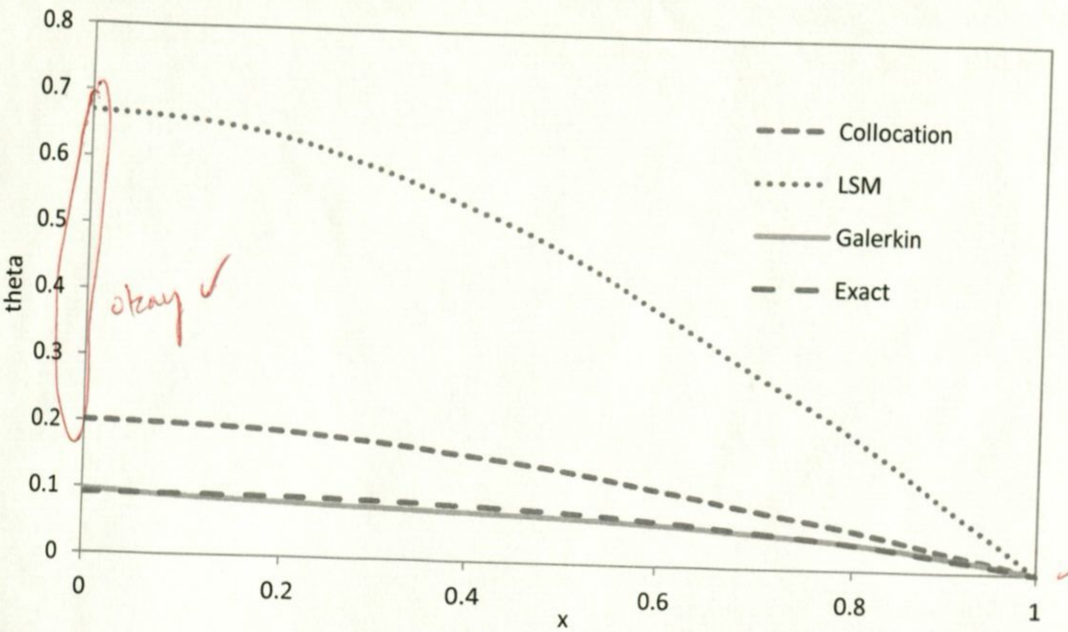
$$e'(x) = -0.555 + 0.784x - 0.447x^2 - \frac{x^3}{24} + \frac{11}{24(1+x)}$$

$$\therefore \|e\|_{L_2} = \sqrt{\int_0^1 (e(x))^2\,dx} = 0.006$$

$$\|e\|_e = \sqrt{\frac{1}{2}\int_0^1 (1+x)(e'(x))^2\,dx} = 0.028$$

Method	L_2 norm	E norm
Collocation	2.22	0.093
Least square	0.59	0.571
Galerkin	0.006	0.028

f) Plot for solution in a, b, & c.



g) Galerkin method from weak form gives the most accurate solution. It can be seen in figure above or in error norm table.

For all the methods, it can be seen from the plot that, they maintain closely the boundary condition. [$\theta'(0)=0$, $\theta(1)=0$].

Note: After finishing this problem, I noticed that you prefered simple polynomials for basis function. I didn't have much time to redo the problem.

Exact solution is not given. So, to solve it, I simplified $\sin x$ by taylor series expansion.

Extra Credit. Finite Element Method (12 points): Derive the elemental stiffness matrix and force vector for problem 5 using piecewise linear basis functions, and calculate its value as a function of global position x.

field equation : $-\left(J_0 G(1+x)\,\theta'\right)' = \sin x$

weak form : $\int GJ_0(1+x)\,\theta' v'\,dx - \int \sin x\, v\,dx + T_0\, v(0) = 0$

N=2

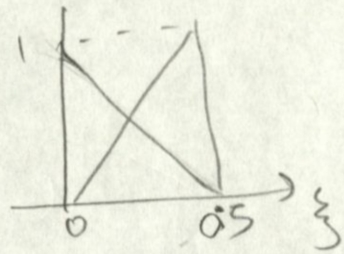
$$\mathbb{K}^{e}_{JK} = \int_0^h GJ_0(1+x)\,\psi_J^{e\prime}\,\psi_K^{e\prime}\,d\xi \;, \quad f^e_K = \int_0^h \sin x\,\psi^e_K\,d\xi - T_0\psi_K(0)$$

Now, $x = x_J + \xi$, $\psi_A = 1 - \xi/h$, $\psi_B = \xi/h$, $\psi_B' = \frac{1}{h}$

For element 1: good. $\psi_A' = -1/h$

$x_j = 0$

$\therefore\; x = \xi,$



Then $K^1_{11} = GJ_0\int_0^h (1+\xi)\times\left(-\frac{1}{h}\right)\left(\frac{-1}{h}\right)d\xi$

$$= \frac{GJ_0}{h^2}\int_0^h (1+\xi)\,d\xi = \frac{GJ_0}{h^2}\left(\xi + \frac{\xi^2}{2}\right)_0^h$$

$$= \frac{GJ_0}{0.5^2}\left(0.5 + \frac{0.5^2}{2}\right) = 2.5\,GJ_0$$

$$K^1_{12} = K^1_{21} = -\frac{GJ_0}{h^2}\int_0^h (1+\xi)\,d\xi = -2.5\,GJ_0$$

$$K^1_{22} = GJ_0\int_0^1 (1+\xi)(1/h)(1/h)\,d\xi = \frac{GJ_0}{h^2}\int_0^1 (1+\xi)\,d\xi$$

$$= 2.5\,GJ_0$$

$$\therefore\; \mathbb{K}^1 = 2.5\,GJ_0\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}$$

$$f^1_1 = \int_0^{0.5}\sin\xi\times\left(1 - \xi/0.5\right)d\xi - T_0\,(1) = 0.041 - T_0$$

$$f^1_2 = \int_0^{0.5}\sin\xi\cdot\frac{\xi}{0.5}\cdot d\xi - T_0\cdot 0 = 0.082$$

$$f = \begin{Bmatrix} 0.041 - T_0 \\ 0.0812 \end{Bmatrix}$$

element 2:

$$x = x_j + \xi = 0.5 + \xi$$

$$\therefore K_{11}^2 = GJ_0 \int_0^{0.5} (1 + 0.5 + \xi)\left(-\frac{1}{h}\right)\left(-\frac{1}{h}\right) d\xi = \frac{GJ_0}{h^2} \int_0^{0.5} (1.5 + \xi)\, d\xi$$

$$= GJ_0 \times 3.5$$

$$K_{12}^2 = K_{21}^2 = \frac{GJ_0}{-h^2} \int_0^{0.5} (1.5 + \xi)\, d\xi = -3.5\, GJ_0$$

$$K_{22} = 3.5\, GJ_0$$

$$\therefore \mathbb{K}^2 = 3.5\, GJ_0 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$f_1^2 = \int_0^{0.5} \sin(0.5 + \xi)\left(1 - \xi/0.5\right) d\xi - T_0 \psi(0)$$

$$= -0.0878 - T_0$$

$$f_2^2 = \int_0^{0.5} \sin(0.5 + \xi)\left(\xi/0.5\right) d\xi - T_0 \times 0$$

$$= 0.22367$$

$$\underset{\sim}{f}^2 = \begin{Bmatrix} -0.0878 - T_0 \\ 0.224 \end{Bmatrix}$$

$$\therefore \text{global stiffness matrix } \mathbb{K} = \begin{bmatrix} 2.5\, GJ_0 & -2.5\, GJ_0 & \\ -2.5\, GJ_0 & 2.5\, GJ_0 + 3.5\, GJ_0 & -3.5\, GJ_0 \\ & -3.5\, GJ_0 & 3.5\, GJ_0 \end{bmatrix}$$

$$= GJ_0 \begin{bmatrix} 2.5 & -2.5 & 0 \\ -2.5 & 6 & -3.5 \\ 0 & -3.5 & 3.5 \end{bmatrix}$$

$$\underline{f} = \begin{Bmatrix} 0{\cdot}041 - T_0 \\ 0{\cdot}0812 - 0{\cdot}0878 - T_0 \\ 0{\cdot}224 \end{Bmatrix} = \begin{Bmatrix} 0{\cdot}041 - T_0 \\ -0{\cdot}006 - T_0 \\ 0{\cdot}224 \end{Bmatrix}$$

Good. Next time solve for element j instead → Much easier to consider an arbitrary number of elements that way.

$$K_{ij}^{e} = \int_0^h (\xi + x_e)\, \psi_i^{e\prime}\, \psi_j^{e\prime}\, d\xi$$

$$= \int (\xi)\,(1/h)^2 (-1)^{i+j}\, d\xi + x_e \int (1/h)^2 (-1)^{i+j}\, d\xi$$

$$= \left(\frac{1}{h^2}\right)(-1)^{i+j} \cdot \left[\frac{1}{2}\xi^2 + \underbrace{h(e-1)}_{x_j}\cdot \xi\right]_0^h$$

$$= \left(1/h^2\right)(-1)^{i+j} \left[\frac{h^2}{2} + h^2(e-1)\right]$$

$$= (1/h^2)(-1)^{i+j} \cdot h^2 (e - 1/2) = (-1)^{i+j}(e - 1/2)$$

$$\mathbb{k}_e = \begin{bmatrix} e - 1/2 & 1/2 - e \\ 1/2 - e & e - 1/2 \end{bmatrix}$$

Global:

$$K = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & & & \\ -\frac{1}{2} & \frac{1}{2}+\frac{3}{2} & -\frac{3}{2} & & \\ & -\frac{3}{2} & \frac{3}{2}+\frac{5}{2} & & \\ & & & \left[(n-1)-\frac{1}{2} + n-\frac{1}{2}\right] & \frac{1}{2}-N \\ & & & \frac{1}{2}-N & N-\frac{1}{2} \end{bmatrix}$$

for $E = 1:N$

$K($