Chapter 9

HYPERBOLIC DIFFERENTIAL EQUATIONS

The classical Bubnov-Galerkin finite element method is optimal in the sense of the best approximation property for elliptic partial differential equations. In many problems of mechanics and convective heat transfer where convection dominates diffusion, this method ceases to be optimal. Rather, its solutions exhibit spurious oscillations in the dependent variable which tend to increase depending on the relative strength of the convective component. It is clear that another method has to be used in order to circumvent this problem. A concise discussion of this issue is the subject of the present chapter.

9.1 The one-dimensional convection-diffusion equation

The limitations of the classical Bubnov-Galerkin method and an alternative approach designed to address these limitations are discussed here in the context of the one-dimensional convection-diffusion equation

$$u_{,t} + \alpha u_{,x} = \epsilon u_{,xx} \qquad ; \qquad \alpha \geq 0 \quad , \quad \epsilon \geq 0 \; , \tag{9.1}$$

which was already encountered in Chapter 1. The steady solution of this equation in the domain $(0, L)$ with boundary conditions $u(0) = 0$ and $u(L) = \bar{u} > 0$ is

$$u(x) = \frac{1 - e^{\frac{\alpha}{\epsilon}x}}{1 - e^{\frac{\alpha}{\epsilon}L}} \, \bar{u} \; . \tag{9.2}$$

The non-dimensional number $Pe = \dfrac{\alpha}{\epsilon} L$, is known as the *Péclet number* and provides a measure of relative significance of convection and diffusion, such that convection dominates

if $Pe \gg 1$ and diffusion dominates if $Pe \ll 1$. Recalling the definition of the Péclet number, one may rewrite (9.2) as

$$u(x) \;=\; \frac{1 - e^{Pe\frac{x}{L}}}{1 - e^{Pe}}\,\bar{u}\;. \tag{9.3}$$

It is clear from (9.3) that when diffusion dominates, then $u(x) \doteq \frac{x}{L}\bar{u}$. This is because, in this case the exponentials in (9.3) have exponents that are much smaller than one, thus can be accurately approximated by a first order Taylor expansion. In contrast, when convection dominates, then the solution is nearly zero throughout the domain except for a thin boundary layer close to $x = L$ where it increases sharply to $\bar{u}$. The latter is due to the fact that in this case the exponential terms in (9.3) are much larger than one, so that $u(x) \doteq e^{Pe(\frac{x}{L}-1)}\bar{u}$. It is precisely this steep boundary layer that the classical Bubnov-Galerkin method fails to accurately resolve, as will be argued shortly.

Before proceeding further, it is important to note that the one-dimensional convection-diffusion equation is not substantially different in nature from the three-dimensional Navier-Stokes equations which govern the motion of a compressible Newtonian fluid, and which can be expressed as

$$-\nabla p + (\lambda + \mu)\nabla(\nabla \cdot \mathbf{v}) + \mu\nabla \cdot (\nabla \mathbf{v}) + \rho\mathbf{b} \;=\; \rho(\frac{\partial \mathbf{v}}{\partial t} + \frac{\partial \mathbf{v}}{\partial \mathbf{x}}\mathbf{v})\;. \tag{9.4}$$

Here, $p = p(\rho)$ is the pressure, $\mathbf{v}$ is the fluid velocity, ρ is the mass density, $\mathbf{b}$ is the applied body force, and λ, μ are material constants. The second and third terms of the left-hand side are diffusive and the last term of the right-hand side is convective. A similar conclusion can be reached for the incompressible case. In the Navier-Stokes equations, the non-dimensional parameter that quantifies the relative significance of convection and diffusion is the *Reynolds number* Re, defined as $Re = \frac{\rho\|\mathbf{v}\|}{\mu}L$. Again, the classical Bubnov-Galerkin method performs poorly for $Re \gg 1$, while it yields good results for $Re \ll 1$.

Returning to the one-dimensional steady convection-diffusion equation, one may start by applying the Bubnov-Galerkin method and subsequently discretize the resulting equations by $N + 1$ equally-sized finite elements with linear interpolation functions for the dependent variable u, see Figure 9.1. This readily leads to the system of linear algebraic equations

$$\alpha\frac{1}{2h}(u_{I+1} - u_{I-1}) \;=\; \epsilon\frac{1}{h^2}(u_{I+1} - 2u_I + u_{I-1}) \quad , \quad I = 1, 2, \ldots, N\;, \tag{9.5}$$

where $h = \frac{L}{N+1}$. In fact, these equations coincide with those obtained by applying the centered-difference method directly on the differential equation. One may rewrite the above

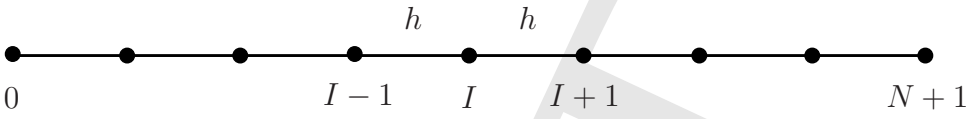

Figure 9.1: *Finite element discretization for the one-dimensional convection-diffusion equation*

equations in the form

$$au_{I-1} + bu_I + cu_{I+1} \;=\; 0 \quad , \quad I = 1, 2, \ldots, N \;, \tag{9.6}$$

where $a = -(\frac{\alpha}{2} + \frac{\epsilon}{h})$, $b = \frac{2\epsilon}{h}$, and $c = \frac{\alpha}{2} - \frac{\epsilon}{h}$. The system of equations in (9.6) can be expressed in matrix form as

$$\begin{bmatrix} b & c & & & & \\ a & b & c & & & \\ & a & b & c & & \\ & & \ddots & & & \\ & & & a & b & c \\ & & & & a & b \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_{N-1} \\ u_N \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ -c\bar{u} \end{bmatrix} . \tag{9.7}$$

Clearly, the system (9.7) is non-symmetric as long as $\alpha \neq 0$. If $c = 0$ (i.e., if $\frac{\alpha h}{2\epsilon} = 1$), then one gets a "sharp" solution of the form $u_I = 0$ for $I = 0, 1, \ldots, N$ and $u_{N+1} = \bar{u}$, which is an accurate approximation of the exact solution, see Figure 9.2. Likewise, noting that

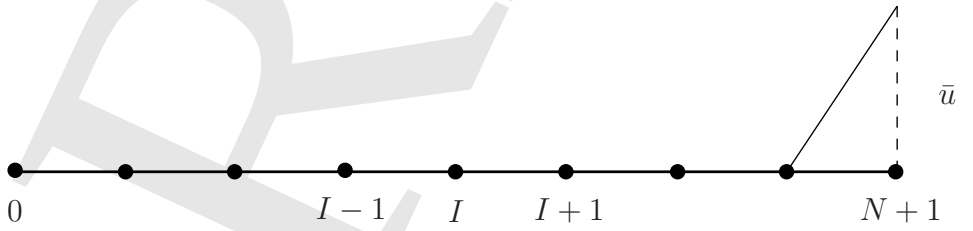

Figure 9.2: *Finite element solution for the one-dimensional convection-diffusion equation for* $c = 0$

$b > 0$ always, it is clear from (9.7) that the numerical solution exhibits no oscillations when $c < 0$, while it will exhibit spurious oscillations around zero when $c > 0$, see Figure 9.3. In conclusion, one may be able to accurately resolve the analytical solution so long as $c \leq 0$, or if, equivalently, the so-called *grid Péclet number* $Pe_h = \frac{\alpha h}{2\epsilon}$ is less or equal to one. For

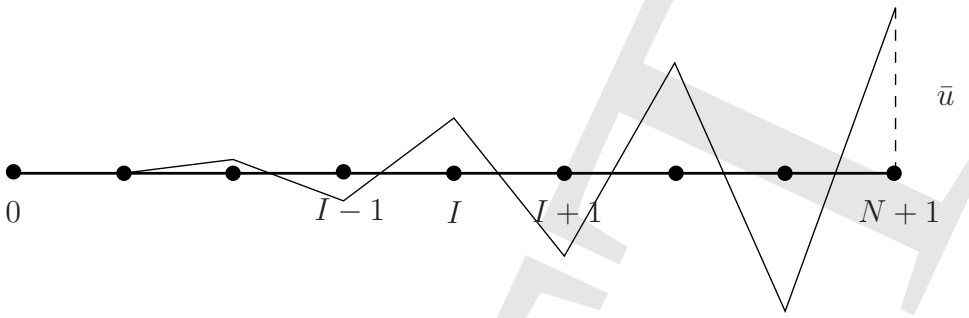

Figure 9.3: *Finite element solution for the one-dimensional convection-diffusion equation for* $c > 0$

$Pe \gg 1$, satisfying the condition $Pe_h \leq 1$ may require a prohibitively small h. This is precisely why the Bubnov-Galerkin method is not a practical formulation for convection-dominated problems.

To remedy the oscillatory behavior of the Bubnov-Galerkin method, one may choose instead to employ an *upwinding method*. This can be understood by considering again equation (9.6) and writing it for $I = N$ in the form

$$au_{N-1} + bu_N = -c\bar{u} < 0 \ , \tag{9.8}$$

where it is assumed that $c > 0$ (which implies oscillations of the Bubnov-Galerkin solution). With reference to (9.8), one may argue that the term au_{N-1} is small compared to the others, and b is positive, hence $u_N < 0$. Similarly, one may write (9.6) for $I = N - 1$ and use the same argument to conclude that $u_{N-1} > 0$, etc., which essentially explains the presence of oscillations when $c > 0$. Since this problem is obviously caused by the convective (as opposed to the diffusive) part of the equation, one idea is to modify the spatial interpolation of the convective term by forcing it to use information which is taken to be preferentially upstream (i.e., skew the interpolation toward the part of the domain where the solution is relatively constant). To this end, equation (9.5) may be replaced by

$$\alpha\frac{1}{h}(u_I - u_{I-1}) \ = \ \epsilon\frac{1}{h^2}(u_{I+1} - 2u_I + u_{I-1}) \quad , \quad I = 1, 2, \ldots, N \ , \tag{9.9}$$

which it tantamount to using an *upwind difference* approximation $\left.\frac{du}{dx}\right|_I \doteq \frac{1}{h}(u_I - u_{I-1})$ as opposed to a centered difference $\left.\frac{du}{dx}\right|_I \doteq \frac{1}{2h}(u_{I+1} - u_{I-1})$ for the convective term. One may interpret this upwind difference as the sum of the corresponding centered difference and an

artificial viscosity term. Indeed, note that

$$\frac{1}{h}(u_I - u_{I-1}) - \frac{1}{2h}(u_{I+1} - u_{I-1}) \;=\; -\frac{1}{2h}(u_{I-1} - 2u_I + u_{I+1}) \;. \tag{9.10}$$

Clearly, the right-hand side of (9.10) is a term that would contribute additional diffusion, as it corresponds to the discrete Laplace operator with a *grid diffusion* constant $k_h = \dfrac{h}{2}$. This is not necessarily an undesirable feature as it is well-known that the centered-difference method under-diffuses (i.e., its convergence is from below in the appropriate energy norm, see Chapter 7).

It may be shown in the context of the one-dimensional convection-diffusion equation that there exists an optimal amount of diffusion that can be added to the problem by way of upwinding to render the numerical solution exact at the nodes of a uniformly discretized domain. This corresponds to grid diffusion $k_{h,opt} = \dfrac{h}{2}\left[\coth Pe_h - \dfrac{1}{Pe_h}\right]$.

The upwinding method can be also interpreted as a Petrov-Galerkin method in which the weighting functions for a given node are preferentially weighing its upwind domain, see Figure 9.4 for a schematic depiction. To further appreciate this point, write the weak form

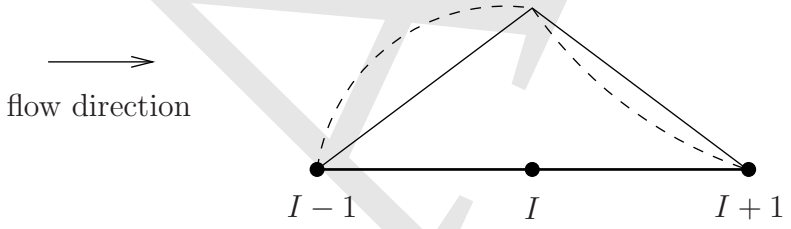

Figure 9.4: *A schematic depiction of the upwind Petrov-Galerkin method for the convection-diffusion equation (continuous line: Bubnov-Galerkin, broken line: Petrov-Galerkin)*

of the steady convection-diffusion equation as

$$\int_0^L w(\alpha u_{,x} - \epsilon u_{,xx})\, dx \;=\; 0 \;. \tag{9.11}$$

where the weighting function w satisfies $w_1(0) = w_1(L) = 0$. Using integration by parts, the weak form (9.11) can be rewritten as

$$\int_0^L w\alpha u_{,x}\; dx \;+ \int_0^L w_{,x}\, \epsilon u_{,x}\; dx \;=\; 0 \;. \tag{9.12}$$

Recalling now that upwinding can be interpreted as artificial diffusion with constant $k_{h,opt}$, one may modify (9.12) so that it takes the form

$$\int_0^L w\alpha u_{,x}\; dx \;+ \int_0^L w_{,x}\, (\epsilon + k_{h,opt}) u_{,x}\; dx \;=\; 0 \;. \tag{9.13}$$

If the total diffusive contribution in (9.13) is expressed as

$$w_{,x}\,(\epsilon + k_{h,opt})u_{,x} \;=\; \tilde{w}_{,x}\,\epsilon u_{,x} \;\;, \tag{9.14}$$

then $\tilde{w}$ is the new weighting function for the diffusive part of the convection-diffusion equation in the spirit of the Petrov-Galerkin method. In this case, $\tilde{w} = \dfrac{\epsilon + k_{h,opt}}{\epsilon} w$.

Multi-dimensional generalizations of upwind finite element methods need special attention. This is because upwinding should only be effected in the direction of the flow (*streamline upwinding*). This is because the introduction of diffusion in directions other than the flow direction (*crosswind diffusion*) generates excessive errors.

9.2 Linear elastodynamics

The problem of linear elastostatics described in detail in Section 7.3 is extended here to include the effects of inertia. The resulting equations of motion take the form

$$\begin{aligned}
\nabla\cdot\boldsymbol{\sigma} \;+\mathbf{f} \;&=\; \rho\ddot{\mathbf{u}} && \text{in } \Omega\times I \;, \\
\boldsymbol{\sigma}\mathbf{n} \;&=\; \bar{\mathbf{t}} && \text{on } \Gamma_q\times I \;, \\
\mathbf{u} \;&=\; \bar{\mathbf{u}} && \text{on } \Gamma_u\times I \;, \\
\mathbf{u}(x_1,x_2,x_3,0) \;&=\; \mathbf{u}_0(x_1,x_2,x_3) && \text{in } \Omega \;, \\
\mathbf{v}(x_1,x_2,x_3,0) \;&=\; \mathbf{v}_0(x_1,x_2,x_3) && \text{in } \Omega \;,
\end{aligned} \tag{9.15}$$

where $\mathbf{u} = \mathbf{u}(x_1,x_2,x_3,t)$ is the unknown displacement field, ρ is the mass density, and $I = (0,T)$, with $T > 0$ being a given time. Also, $\mathbf{u}_0$ and $\mathbf{v}_0$ are the prescribed initial displacement and velocity fields. Clearly, equations $(9.15)_{2,3}$ are two sets of boundary conditions set on Γ_q and Γ_u, respectively, which are assumed to hold throughout the time interval I. Likewise, two sets of initial conditions are set in $(9.15)_{4,5}$ for the whole domain Ω at time $t = 0$. The strong form of the resulting initial/boundary-value problem is stated as follows: given functions $\mathbf{f}$, ρ, $\bar{\mathbf{t}}$, $\bar{\mathbf{u}}$, $\mathbf{u}_0$ and $\mathbf{v}_0$, as well as a constitutive equation (7.3) for $\boldsymbol{\sigma}$, find $\mathbf{u}$ in $\Omega\times I$, such that the equations (9.15) are satisfied.

A Galerkin-based weak form of the linear elastostatics problem has been derived in Section 7.3. In the elastodynamics case, the only substantial difference involves the inclusion of the term $\int_\Omega \mathbf{w}\cdot\rho\ddot{\mathbf{u}}\,d\Omega$, as long as one adopts the semi-discrete approach. As a result, the weak form at a given time can be expressed as

$$\int_\Omega \mathbf{w}\cdot\rho\ddot{\mathbf{u}}\,d\Omega + \int_\Omega \nabla_s\mathbf{w} : \boldsymbol{\sigma}\,d\Omega \;=\; \int_\Omega \mathbf{w}\cdot\mathbf{f}\,d\Omega + \int_{\Gamma_q} \mathbf{w}\cdot\bar{\mathbf{t}}\,d\Gamma \;. \tag{9.16}$$

Following the development in Section 7.3, the discrete counterpart of (9.16) can be written as

$$\int_\Omega [\mathbf{w}_h]^T \rho [\ddot{\mathbf{u}}_h]\, d\Omega + \int_\Omega \langle \boldsymbol{\epsilon}(\mathbf{w}_h)\rangle^T [\mathbf{D}] \langle \boldsymbol{\epsilon}(\mathbf{u}_h)\rangle \, d\Omega \;=\; \int_\Omega [\mathbf{w}_h]^T [\mathbf{f}]\, d\Omega + \int_{\Gamma_q} [\mathbf{w}_h]^T [\bar{\mathbf{t}}]\, d\Gamma \;. \quad (9.17)$$

A Galerkin approximation of (9.17) at the element level using the nomenclature of (7.29-7.42) leads to a system of ordinary differential equations of the form

$$[\mathbf{M}^e][\ddot{\mathbf{u}}^e] + [\mathbf{K}^e][\mathbf{u}^e] \;=\; [\mathbf{F}^e] + [\mathbf{F}^{\text{int},e}] \;, \quad (9.18)$$

where all quantities have already been defined in Section 7.3 except for the element mass matrix $[\mathbf{M}^e]$ which is given by

$$[\mathbf{M}^e] \;=\; \int_{\Omega^e} [\mathbf{N}^e]^T \rho [\mathbf{N}^e]\, d\Omega \;. \quad (9.19)$$

Clearly, the mass matrix is symmetric and also positive-definite, provided $\rho > 0$. Following a standard procedure, the contribution of the forcing vector $[\mathbf{F}^{\text{int},e}]$ due to interelement tractions is neglected upon assembly of the global equations. As a result, the equations (9.18) give rise to their assembled counterparts in the form

$$[\mathbf{M}][\hat{\ddot{\mathbf{u}}}] + [\mathbf{K}][\hat{\mathbf{u}}] \;=\; [\mathbf{F}] \;, \quad (9.20)$$

where $\hat{\mathbf{u}}$ is the global unknown displacement vector[1]. The preceding equations are, of course, subject to initial conditions that can be written in vectorial form as $\hat{\mathbf{u}}(0) = \hat{\mathbf{u}}_0$ and $\hat{\mathbf{v}}(0) = \hat{\mathbf{v}}_0$.

The most commonly employed method for the numerical solution of the system of coupled linear second-order ordinary differential equations (9.20) is the *Newmark method*. This is based on a time series expansion of $\hat{\mathbf{u}}$ and $\hat{\mathbf{v}} = \hat{\dot{\mathbf{u}}}$. Concentrating on the time interval $(t_n, t_{n+1}]$, the Newmark method is defined by the equations

$$\begin{aligned} [\hat{\mathbf{u}}_{n+1}] \;&=\; [\hat{\mathbf{u}}_n] + [\hat{\mathbf{v}}_n]\Delta t_n + \frac{1}{2}\Big\{(1-2\beta)[\hat{\mathbf{a}}_n] + 2\beta[\hat{\mathbf{a}}_{n+1}]\Big\}\Delta t_n^2 \;, \\ [\hat{\mathbf{v}}_{n+1}] \;&=\; [\hat{\mathbf{v}}_n] + \Big\{(1-\gamma)[\hat{\mathbf{a}}_n] + \gamma[\hat{\mathbf{a}}_{n+1}]\Big\}\Delta t_n \;, \end{aligned} \quad (9.21)$$

where $\Delta t_n = t_{n+1} - t_n$, $[\hat{\mathbf{a}}] = [\hat{\ddot{\mathbf{u}}}]$, and β, γ are parameters chosen such that

$$0 \le \beta \le \frac{1}{2} \quad , \quad 0 < \gamma \le 1 \;. \quad (9.22)$$

[1] The overhead "hat" symbol is used to distinguish between the vector field $\mathbf{u}$ and the solution vector $\hat{\mathbf{u}}$ emanating from the finite element approximation of the vector field $\mathbf{u}$.

The special case $\beta = \dfrac{1}{4}$, $\gamma = \dfrac{1}{2}$ corresponds to the trapezoidal rule. Likewise, the special case $\beta = 0$, $\gamma = \dfrac{1}{2}$ corresponds to the centered-difference rule.

It is clear that the Newmark equations (9.21) define a whole family of time integrators. It is important to distinguish this family of integrators into two categories, namely implicit and explicit integrators, corresponding to $\beta > 0$ and $\beta = 0$, respectively.

The general *implicit Newmark method* may be implemented as follows: first, solve $(9.21)_1$ for $[\hat{\mathbf{a}}_{n+1}]$, namely write

$$[\hat{\mathbf{a}}_{n+1}] \;=\; \frac{1}{\beta\Delta t_n^2}\Big\{[\hat{\mathbf{u}}_{n+1}] - [\hat{\mathbf{u}}_n] - [\hat{\mathbf{v}}_n]\Delta t_n\Big\}) - \frac{1-2\beta}{2\beta}[\hat{\mathbf{a}}_n] \;. \tag{9.23}$$

Then, substitute (9.23) into the semi-discrete form (9.20) evaluated at t_{n+1} to find that

$$\left\{\frac{1}{\beta\Delta t_n^2}[\mathbf{M}] + [\mathbf{K}]\right\}[\hat{\mathbf{u}}_{n+1}] \;=\; [\mathbf{F}_{n+1}] + [\mathbf{M}]\left\{([\hat{\mathbf{u}}_n] + [\hat{\mathbf{v}}_n]\Delta t_n)\frac{1}{\beta\Delta t_n^2} + \frac{1-2\beta}{2\beta}[\hat{\mathbf{a}}_n]\right\} \;. \tag{9.24}$$

After solving (9.24) for $[\hat{\mathbf{u}}_{n+1}]$, one may compute the acceleration $[\hat{\mathbf{a}}_{n+1}]$ from (9.23) and the velocity $[\hat{\mathbf{v}}_{n+1}]$ from $(9.21)_2$. It can be shown that the implicit Newmark method is unconditionally stable.

The general *explicit Newmark method* may be implemented as follows: starting from the semi-discrete equations (9.20) evaluated at t_{n+1}, one may substitute $[\hat{\mathbf{u}}_{n+1}]$ from $(9.21)_1$ to find that

$$[\mathbf{M}][\hat{\mathbf{a}}_{n+1}] \;=\; -[\mathbf{K}]\Big\{[\hat{\mathbf{u}}_n] + [\hat{\mathbf{v}}_n]\Delta t_n + \frac{1}{2}[\hat{\mathbf{a}}_n]\Delta t_n^2\Big\} + [\mathbf{F}_{n+1}] \;. \tag{9.25}$$

If $[\mathbf{M}]$ is rendered diagonal (see discussion in Chapter 8), then $[\hat{\mathbf{a}}_{n+1}]$ can be determined without solving any coupled linear algebraic equations. Then, the velocities $[\hat{\mathbf{v}}_{n+1}]$ are immediately computed from $(9.21)_2$. Also, the displacements $[\hat{\mathbf{u}}_{n+1}]$ are computed from $(9.21)_1$ independently of the accelerations $[\hat{\mathbf{a}}_{n+1}]$.

The general solution of the homogeneous counterpart of (9.20) is of the form

$$\hat{\mathbf{u}} \;=\; \sum_{i=1}^{N} c_i e^{i\omega_i t}\boldsymbol{\Phi}_i \;, \tag{9.26}$$

where N is the dimension of the vector $[\hat{\mathbf{u}}]$ and c_i are constants, while $\boldsymbol{\Phi}_i$ are N-dimensional vectors and ω_i are scalar parameters to be determined. Substituting a typical vector of (9.26) into the homogeneous counterpart of (9.20) leads to

$$\sum_{i=1}^{N} e^{i\omega_i t}\left\{[\mathbf{K}] - \omega_i^2[\mathbf{M}]\right\}\boldsymbol{\Phi}_i \;=\; \mathbf{0} \;. \tag{9.27}$$

It follows from (9.27) that $(\omega_i^2, \boldsymbol{\Phi}_i)$ can be extracted from the eigenvalue problem

$$\big([\mathbf{K}] - \omega_i^2[\mathbf{M}]\big)\,\boldsymbol{\Phi}_i \;=\; \mathbf{0}\;. \tag{9.28}$$

Letting

$$[\boldsymbol{\Phi}] \;=\; [\boldsymbol{\Phi}_1\,\boldsymbol{\Phi}_2\,\ldots\,\boldsymbol{\Phi}_N]\;, \tag{9.29}$$

the preceding eigenvalue problem can be expressed as

$$[\mathbf{M}][\boldsymbol{\Phi}][\boldsymbol{\Omega}] \;=\; [\mathbf{K}][\boldsymbol{\Phi}]\;, \tag{9.30}$$

where $\boldsymbol{\Omega}$ is a diagonal $N \times N$ matrix that contains all eigenvalues ω_i^2, $i = 1, 2, \ldots, N$.

The solution of the non-homogeneous problem (9.20) may be obtained by variation of parameters, according to which the solution vector is written as

$$[\hat{\mathbf{u}}] \;=\; [\boldsymbol{\Phi}][\hat{\mathbf{y}}]\;, \tag{9.31}$$

therefore

$$[\mathbf{M}][\boldsymbol{\Phi}][\ddot{\hat{\mathbf{y}}}] + [\mathbf{K}][\boldsymbol{\Phi}][\hat{\mathbf{y}}] \;=\; [\mathbf{F}]\;. \tag{9.32}$$

Upon premultiplying the preceding equation by $[\boldsymbol{\Phi}]^T$, one finds that

$$[\boldsymbol{\Phi}]^T[\mathbf{M}][\boldsymbol{\Phi}][\ddot{\hat{\mathbf{y}}}] + [\boldsymbol{\Phi}]^T[\mathbf{K}][\boldsymbol{\Phi}][\hat{\mathbf{y}}] \;=\; [\boldsymbol{\Phi}]^T[\mathbf{F}]\;. \tag{9.33}$$

Appealing to the standard diagonalization property already derived for for the parabolic case in Section 8.1, the equations (9.33) are decoupled and written as

$$m_i\ddot{y}_i + k_i y_i \;=\; g_i \quad , \quad i = 1, 2, \ldots, N\;, \tag{9.34}$$

where $g_i = [\boldsymbol{\Phi}_i]^T[\mathbf{F}]$.

The stability of the explicit Newmark method ($\beta = 0$) can be investigated for the homogeneous scalar equation

$$m\ddot{u} + ku \;=\; 0 \tag{9.35}$$

derived from (9.34) by a simple change in notation. For this equation, the Newmark equations (9.21) can be written as

$$\begin{aligned} u_{n+1} \;&=\; u_n + v_n\Delta t + \frac{1}{2}a_n\Delta t^2 \\ v_{n+1} \;&=\; v_n + [(1-\gamma)a_n + \gamma a_{n+1}]\Delta t \end{aligned} \tag{9.36}$$

or, taking into account (9.35) at t_n and t_{n+1},

$$\begin{aligned} u_{n+1} &= u_n + v_n\Delta t + \frac{1}{2}\{-\frac{k}{m}u_n\}\Delta t^2 \\ v_{n+1} &= v_n - \frac{k}{m}\big[(1-\gamma)u_n + \gamma u_{n+1}\big]\Delta t \\ &= v_n - \frac{k}{m}\Big[(1-\gamma)u_n + \gamma\big\{u_n + v_n\Delta t + \frac{1}{2}\{-\frac{k}{m}u_n\}\Delta t^2\big\}\Big]\Delta t \ . \end{aligned} \tag{9.37}$$

Equations (9.37) can be put in matrix form as

$$\left[\begin{array}{c} u_{n+1} \\ v_{n+1} \end{array}\right] = \left[\begin{array}{cc} 1-\frac{1}{2}\alpha & \Delta t \\ -\frac{\alpha}{\Delta t} + \frac{1}{2}\gamma\frac{\alpha^2}{\Delta t} & 1-\gamma\alpha \end{array}\right] \left[\begin{array}{c} u_n \\ v_n \end{array}\right] \ , \tag{9.38}$$

where $\alpha = \frac{k}{m}\Delta t^2$.

It is easy to show that the stability of the explicit Newmark method depends on the spectral properties of the *amplification matrix* $[\mathbf{r}]$, defined with reference to (9.38) as

$$[\mathbf{r}] = \left[\begin{array}{cc} 1-\frac{1}{2}\alpha & \Delta t \\ -\frac{\alpha}{\Delta t} + \frac{1}{2}\gamma\frac{\alpha^2}{\Delta t} & 1-\gamma\alpha \end{array}\right] . \tag{9.39}$$

Specifically, for the method to be stable both eigenvalues of $[\mathbf{r}]$ need to less than or equal to one in absolute value. Here, these eigenvalues are given by

$$\lambda_{1,2} = 1 - \frac{1}{2}\left[(\frac{1}{2}+\gamma)\alpha \pm \sqrt{\Big\{(\frac{1}{2}+\gamma)\alpha\Big\}^2 - 4\alpha}\right] \ . \tag{9.40}$$

For the most common case of $\gamma = 0.5$ (which corresponds to the centered-difference method), the eigenvalues of the amplification matrix reduce to

$$\lambda_{1,2} = 1 - \frac{1}{2}\left[\alpha \pm \sqrt{\alpha^2 - 4\alpha}\right] \ , \tag{9.41}$$

which can be easily shown to be less than or equal to one in absolute value if $\alpha \le 4$. This, in turn, implies that the critical step-size Δt_c for explicit Newmark with $\gamma = 0.5$ is

$$\Delta t_c = \frac{2}{\sqrt{\frac{k}{m}}} \ . \tag{9.42}$$

In a multi-dimensional setting, the preceding condition becomes

$$\Delta t_c = \frac{2}{\max_i \sqrt{\frac{k_i}{m_i}}} \ , \tag{9.43}$$

where k_i and m_i are the diagonalized stiffness and mass components, respectively. Clearly, condition (9.43) places restrictions to the step-size and, therefore, dictates the cost of the explicit computations. As in the parabolic problem of Section 8.2, a simple scaling argument shows that the critical step is of order $o(h)$, where h is a linear measure of mesh size.