Ass. 10 Brandon Lampe 1

- problems 1 to 3 were turned in during class

4. $\Gamma = \oint_{C_L} \underline{V} \cdot d\underline{r}$ => circulation around a closed curve 'C'

4. i show: $\frac{d\Gamma}{dt} = \oint_{C_L} \underline{a} \cdot d\underline{r} + \oint_{C_L} \underline{V} \cdot \underline{\underline{L}} \cdot d\underline{r}$

$\rightarrow \frac{d\Gamma}{dt} = \frac{d}{dt}\left[\oint_{C_L} \underline{V} \cdot d\underline{r}\right]$; note: $d\underline{r} = \underline{\underline{F}} \cdot d\underline{R}$.

$= \frac{d}{dt}\left[\oint_{C_{L_0}} \underline{V} \cdot \underline{\underline{F}} \cdot d\underline{R}\right]$ → $\underline{R}$ is independent of t

$= \oint_{C_{L_0}} \left\{\frac{d}{dt}\left(\underline{V} \cdot \underline{\underline{F}}\right)\right\} \cdot d\underline{R}$

$= \oint_{C_{L_0}} \left\{\frac{d\underline{V}}{dt} \cdot \underline{\underline{F}} + \underline{V} \cdot \dot{\underline{\underline{F}}}\right\} \cdot d\underline{R}$ → $\frac{d\underline{V}}{dt} = \underline{a}$

→ $\dot{\underline{\underline{F}}} = \underline{\underline{F}} \cdot \underline{\underline{L}}$

$= \oint_{C_{L_0}} \left\{\underline{a} \cdot \underline{\underline{F}} + \underline{V} \cdot \underline{\underline{F}} \cdot \underline{\underline{L}}\right\} \cdot d\underline{R}$

$= \oint_{C_L} \left\{\underline{a} + \underline{V} \cdot \underline{\underline{L}}\right\} \cdot d\underline{r}$ → sub $d\underline{r}$ for $\underline{\underline{F}} \cdot d\underline{R}$ + trans. back

$= \oint_{C_L} \underline{a} \cdot d\underline{r} + \oint_{C_L} \underline{V} \cdot \underline{\underline{L}} \cdot d\underline{r}$

4. ii show: $\oint_C \underline{v} \cdot \underline{\underline{L}} \cdot d\underline{r} = \oint_C \underline{v} \cdot d\underline{v} = 0$

recall: $\underline{\underline{L}} = \underline{v}\,\overleftarrow{\nabla}$

$d\underline{v} = \underline{v}\,\overleftarrow{\nabla} \cdot d\underline{r}$

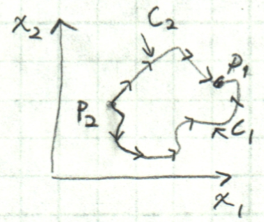
then: $\oint_c \underline{v} \cdot \underline{\underline{L}} \cdot d\underline{r} = \oint_c \underline{v} \cdot \underline{v}\,\overleftarrow{\nabla} \cdot d\underline{r}$

$\oint_c \underline{v} \cdot d\underline{v} = \oint_c \underline{v} \cdot \underline{v}\,\overleftarrow{\nabla} \cdot d\underline{r}$

- the line integral is path independent between any two points.

- therefore, for a closed path, the line integral must be zero

e.g.,



- for the line integral between points 'P_1' + 'P_2', the paths 'C_1' + 'C_2' are equal.

$$\int_{C_1} f \odot d\underline{r} = \int_{C_2} f \odot d\underline{r}$$

$$\int_{C_1} f \odot d\underline{r} - \int_{C_2} f \odot dr = 0 ,$$

- this expressed as one closed curve:

$$\oint_{C_1 + (-C_2)} f \odot d\underline{r} = 0$$

4. iii $\Gamma = \oint_C \underline{v} \cdot d\underline{r}$

Let $\underline{v} = \phi \overleftarrow{\nabla}$

show that $\Gamma = 0$

Recall: $d\phi = \phi \overleftarrow{\nabla} \cdot d\underline{r} \rightarrow \frac{d\phi}{d\underline{r}} = \phi \overleftarrow{\nabla}$

then: $\oint_c \phi \overleftarrow{\nabla} \cdot dr = \oint_c \frac{d\phi}{dr} \cdot dr = \oint_c d\phi$

- for a closed path, the upper and lower limits are equal; say p

$$\oint d\phi = \int_p^p d\phi = \phi \Big|_p^p = \phi(p) - \phi(p) = 0$$

4. iv show: $\dot{\underline{v}} = \dot{\phi} \overleftarrow{\nabla} - \underline{v} \cdot \underline{\underline{L}}$ for $\underline{v} = \phi \overleftarrow{\nabla}$

in current config, need to transform to original config because it is not dep. on t

$$\dot{\underline{v}} = \frac{dv}{dt} = \frac{d}{dt}(\phi \overleftarrow{\nabla}) = \frac{d}{dt}(\phi \overleftarrow{\nabla}_0 \cdot \underline{\underline{F}}^{-1})$$

$$= \left(\frac{d\phi}{dt}\right) \overleftarrow{\nabla}_0 \cdot \underline{\underline{F}}^{-1} + \phi \overleftarrow{\nabla}_0 \cdot \dot{\underline{\underline{F}}}^{-1}$$

recall: $(\,)\overleftarrow{\nabla} = (\,)\overleftarrow{\nabla}_0 \cdot \underline{\underline{F}}^{-1}$

$\dot{\underline{\underline{F}}}^{-1} = -\underline{\underline{F}}^{-1} \cdot \underline{\underline{L}}$

$$= \dot{\phi} \overleftarrow{\nabla} + \phi \overleftarrow{\nabla}_0 (-\underline{\underline{F}}^{-1} \cdot \underline{\underline{L}})$$

$$= \dot{\phi} \overleftarrow{\nabla} - \phi \overleftarrow{\nabla}_0 \cdot \underline{\underline{F}}^{-1} \cdot \underline{\underline{L}}$$

$$= \dot{\phi} \overleftarrow{\nabla} - \phi \overleftarrow{\nabla} \cdot \underline{\underline{L}}$$

$$= \dot{\phi} \overleftarrow{\nabla} - \underline{v} \cdot \underline{\underline{L}}$$

5.i) Show: $\frac{d}{dt}(\underline{n}\,da) = (\underline{v}\cdot\overleftarrow{\nabla})\,\underline{n}\,da - \underline{\underline{L}}^T\cdot\underline{n}\,da$

from Nanson's relation: $\underline{n}\,da = J\,\underline{\underline{F}}^{-T}\cdot\underline{N}_0\,dA_0$

$$\frac{d}{dt}(\underline{n}\,da) = \frac{d}{dt}\left(J\,\underline{\underline{F}}^{-T}\cdot\underline{N}_0\,dA_0\right)$$

$$\dot{\underline{n}}\,da + \underline{n}\,\dot{da} = \frac{d}{dt}\left(J\,\underline{\underline{F}}^{-T}\right)\cdot\underline{N}_0\,dA_0$$

$\rightarrow \underline{N}_0 + dA_0$ are ind. wrt time

$$= \left(\dot{J}\,\underline{\underline{F}}^{-T} + J\,\dot{\underline{\underline{F}}}^{-T}\right)\cdot\underline{N}_0\,dA_0$$

Recall: $\dot{\underline{\underline{F}}}^{-T} = -\underline{\underline{F}}^{-T}\cdot\underline{\underline{L}}^T$

$\dot{J} = J\,tr(\underline{\underline{L}}) = J\,\underline{v}\cdot\overleftarrow{\nabla}$

$$= \left(J\,\underline{v}\cdot\overleftarrow{\nabla}\,\underline{\underline{F}}^{-T} + J(-\underline{\underline{F}}^{-T}\cdot\underline{\underline{L}}^T)\right)\cdot\underline{N}_0\,dA_0$$

Recall: $J\,dA_0\,\underline{N}_0 = \underline{\underline{F}}^T\cdot\underline{n}\,da$

$\therefore J\,dA_0\,\underline{\underline{F}}^{-T}\cdot\underline{N}_0 = \underline{n}\,da$

$$= \left(J\,\underline{v}\cdot\overleftarrow{\nabla}\,\underline{\underline{F}}^{-T} - J\,\underline{\underline{F}}^{-T}\cdot\underline{\underline{L}}^T\right)\cdot\underline{N}_0\,dA_0$$

$$= \left(\underline{v}\cdot\overleftarrow{\nabla} - \underline{\underline{L}}^T\right)\cdot\underline{n}\,da$$

$$= (\underline{v}\cdot\overleftarrow{\nabla})\,\underline{n}\,da - \underline{\underline{L}}\cdot\underline{n}\,da$$

5.ii) Show: $\frac{d}{dt}\left(\int f\,\underline{n}\,da\right) = \int\left(\frac{df}{dt}\,\underline{n} + f(\underline{v}\cdot\overleftarrow{\nabla})\underline{n} - f\,\underline{\underline{L}}^T\cdot\underline{n}\right)da$

→ 'da' is dependent on time; therefore, 'da' needs to be transformed to the original basis. Then the time derivative may be made part of the integrand.

↓

5. ii) cont. ; Recall: $J\,\underline{\underline{F}}^{-T}\cdot\underline{N}_0\,dA_0 = \underline{n}\,da$ -> Nanson's

$$\dot{J} = J\,\underline{\underline{L}} = J(\underline{V}\cdot\overleftarrow{\nabla})$$

$$\dot{\underline{\underline{F}}}^{-T} = -\underline{\underline{F}}^{-T}\cdot\underline{\underline{L}}^{T}$$

$$\frac{d}{dt}\left(\int f\,\underline{n}\,da\right) = \frac{d}{dt}\left[\int f J\,\underline{\underline{F}}^{-T}\cdot\underline{N}_0\,dA_0\right]$$

→ because A_0 is ind. of t; $\underline{N}_0$ is also ind. of t

$$= \int \frac{d}{dt}\left(f J F^{-T}\right)\cdot\underline{N}\,dA_0$$

$$= \int\left(\frac{df}{dt}J\underline{\underline{F}}^{-T} + f\dot{J}\underline{\underline{F}}^{-T} + fJ\dot{\underline{\underline{F}}}^{-T}\right)\cdot\underline{N}_0\,dA_0$$

$$= \int\left(\frac{df}{dt}J\underline{\underline{F}}^{-T} + fJ(\underline{V}\cdot\overleftarrow{\nabla})\underline{\underline{F}}^{-T} - fJ\underline{\underline{F}}^{-T}\cdot\underline{\underline{L}}^{T}\right)\cdot\underline{N}_0\,dA_0$$

$$= \int\left(\frac{df}{dt}\underline{n} + f(\underline{V}\cdot\overleftarrow{\nabla})\underline{n} - \underbrace{f\underline{n}\cdot\underline{\underline{L}}^{T}}_{?}\right)da$$

- would $f\underline{n}\cdot\underline{\underline{L}}^{T} = f\underline{\underline{L}}\cdot\underline{n}$, should $\underline{\underline{L}}^{T}$ be swapped with $\underline{\underline{L}}$ to retain the same final vector, unless $\underline{\underline{L}}$ is symm., but I don't think it is.

? -> $\displaystyle\int\left(\frac{df}{dt}\underline{n} + f(\underline{V}\cdot\overleftarrow{\nabla})\underline{n} - f\underline{\underline{L}}\cdot\underline{n}\right)da$

is this correct?