18. Initial and Boundary Conditions

18.1 Initial Comments

To be specific, suppose we consider an isotropic elastic body so that the governing equations in terms of the displacement vector, $\boldsymbol{u}(t,\boldsymbol{r})$, are:

<u>The Strain-Displacement Relations</u>

$$\boldsymbol{e} = \tfrac{1}{2}\left[\boldsymbol{h} + \boldsymbol{h}^T + \boldsymbol{h}^T \cdot \boldsymbol{h}\right] \qquad \boldsymbol{h} = \boldsymbol{u}\bar{\tilde{\nabla}} \tag{18-1}$$

which represents 6 scalar equations.

<u>The Equations of Motion</u>

$$\boldsymbol{\sigma} \cdot \bar{\tilde{\nabla}} + \rho \boldsymbol{b} = \rho \boldsymbol{a} \qquad \boldsymbol{\sigma} = \boldsymbol{\sigma}^T \tag{18-2}$$

which represents 3 scalar equations.

<u>Constitutive Equations</u>

A simple example is isotropic elasticity as given by

$$P_T = Be_v \qquad \boldsymbol{\sigma}^{dev} = 2G\boldsymbol{e}^{dev} \qquad \boldsymbol{\sigma} = P_T \boldsymbol{I} + \boldsymbol{\sigma}^{dev} \tag{18-3}$$

which represents 6 scalar equations. In addition we have the 6 scalar equations for

<u>Velocity and Acceleration</u>

$$\boldsymbol{v} = \dot{\boldsymbol{u}} \qquad \boldsymbol{a} = \dot{\boldsymbol{v}} \tag{18-4}$$

We have a total of 21 equations to solve for the 9 components of $\boldsymbol{u}, \boldsymbol{v}$ and $\boldsymbol{a}$, 6 components of strain and 6 components of stress, a total of 21 components.

The boundary value problem is completed by specifying the initial and boundary conditions.

18.2 Initial Conditions

In terms of the displacement our set of partial differential equations is second order in time. Therefore, initial values of both displacement and its first time derivative, velocity, must be specified at every point in the domain. With the assumption that $\boldsymbol{R} = \boldsymbol{r}\big|_{t=0}$, the initial values are

$$\boldsymbol{u}(0,\boldsymbol{R}) = \boldsymbol{u}^*(\boldsymbol{R}) \qquad \boldsymbol{v}(0,\boldsymbol{R}) = \boldsymbol{v}^*(\boldsymbol{R}) \tag{18-5}$$

The functions $\boldsymbol{u}^*(\boldsymbol{R})$ and $\boldsymbol{v}^*(\boldsymbol{R})$ must be obtained from the description of the problem. Note that if nothing is said about initial conditions, it is often assumed that these functions are zero, a bad habit to fall into.

18.3 Boundary Conditions

Boundary conditions represent a topic that is woefully absent as a topic in many books on continuum mechanics. Here we present the topic by listing the types of boundary conditions from the most common to least common. We assume the problem is time-dependent and that the primary variable is the displacement vector, $\boldsymbol{u}$. The key point to remember is that the displacement vector, or traction vector, or some combination of the two vectors, must be specified as a function of time for every point on the surface, or boundary, of the body under consideration.

<u>Neumann Boundary Condition</u>

The displacement is prescribed for every point on the boundary:

$$\boldsymbol{u} = \boldsymbol{u}^b(\boldsymbol{r},t) \qquad \boldsymbol{r} \in \partial R^b \tag{18-6}$$

in which ∂R^b is the boundary of the body and $\boldsymbol{u}^b(\boldsymbol{r},t)$ is a prescribed function of time for all points on the boundary. Sometimes the velocity is prescribed. Then the displacement is obtained by performing an integration in time, either analytically or numerically.

<u>Neumann Boundary Condition</u>

The traction is prescribed for every point on the boundary:

$$\tau = \boldsymbol{\tau}^b(\boldsymbol{r},t) \qquad \boldsymbol{r} \in \partial R^b \tag{18-7}$$

in which ∂R^b is the boundary of the body and $\boldsymbol{\tau}^b(\boldsymbol{r},t)$ is a prescribed function of time for all points on the boundary. Neumann boundary conditions in the mathematical literature are associated with derivatives of the primary variable, $\boldsymbol{u}$. The specification of traction falls in this category since traction can be expressed in terms of components of stress, and stress is expressed in terms of strain through the constitutive equation, and strain is given as a function of the gradient of displacement.

<u>Mixed Surface Boundary Condition</u>

The displacement is prescribed for every point on the portion ∂R^u of the boundary, and the traction is prescribed for every point on the portion ∂R^τ of the boundary. The union of the two portions of the surface form the complete boundary. Mathematically, this boundary condition is stated as follows:

$$\boldsymbol{u} = \boldsymbol{u}^b(\boldsymbol{r},t) \text{ for } \boldsymbol{r} \in \partial R^u \quad \text{and} \quad \tau = \boldsymbol{\tau}^b(\boldsymbol{r},t) \text{ for } \boldsymbol{r} \in \partial R^\tau; \qquad \partial R^u \cup \partial R^\tau = \partial R^b \tag{18-8}$$

<u>Mixed Component (MC) Boundary Condition</u>

The key idea for this type of boundary condition is that for a point on the surface, one component of traction, and the other orthogonal components of displacement, or vice versa, is prescribed. One way to describe this type of boundary condition is to use two orthogonal projections, as described by the second-order tensors $\boldsymbol{P}^{(1)}$ and $\boldsymbol{P}^{(2)}$ with the properties that

$$\boldsymbol{P}^{(1)} \cdot \boldsymbol{P}^{(1)} = \boldsymbol{P}^{(1)} \qquad \boldsymbol{P}^{(2)} \cdot \boldsymbol{P}^{(2)} = \boldsymbol{P}^{(2)} \qquad \boldsymbol{P}^{(1)} \cdot \boldsymbol{P}^{(2)} = \boldsymbol{0}$$
$$\boldsymbol{P}^{(2)} \cdot \boldsymbol{P}^{(1)} = \boldsymbol{0} \qquad \boldsymbol{P}^{(1)} + \boldsymbol{P}^{(2)} = \boldsymbol{I} \tag{18-9}$$

Then this type of boundary condition is described as follows:

$$\boldsymbol{P}^{(1)} \cdot \boldsymbol{u} + \boldsymbol{P}^{(2)} \cdot \tau = \boldsymbol{\gamma}(r,t) \qquad \boldsymbol{r} \in \partial R^{MC} \tag{18-10}$$

The function $\boldsymbol{\gamma}(\boldsymbol{r},t)$ and the projection operators must be specified on that portion of the boundary, ∂R^{MC}, on which the mixed component boundary condition holds.

An example of this type of boundary condition is the situation where a portion of the surface of the body is pressed against a rigid frictionless planar surface with unit normal $\boldsymbol{n}$. The normal component of displacement of the body is zero, and because the surface is frictionless, the two orthogonal components of traction are zero.

Let $\boldsymbol{t}$ be any unit vector perpendicular to $\boldsymbol{n}$ and choose $\boldsymbol{p} = \boldsymbol{n} \times \boldsymbol{t}$. Then if we choose the projection operators to be

$$\boldsymbol{P}^{(1)} = \boldsymbol{n} \otimes \boldsymbol{n} \qquad \boldsymbol{P}^{(2)} = \boldsymbol{t} \otimes \boldsymbol{t} + \boldsymbol{p} \otimes \boldsymbol{p} \tag{18-11}$$

(Problem: Verify that (18-9) is satisfied.) Then (18-10) becomes

$$\boldsymbol{P}^{(1)} \cdot \boldsymbol{u} + \boldsymbol{P}^{(2)} \cdot \boldsymbol{\tau} = \boldsymbol{0} \quad \text{or} \quad u_n = 0,\ \tau_t = 0,\ \tau_p = 0 \tag{18-12}$$

Robin Boundary (RB) Condition

The essential idea of this type of boundary condition is that a linear combination of displacement and traction is prescribed:

$$C_1 \boldsymbol{u} + C_2 \boldsymbol{\tau} = \boldsymbol{\Gamma}(\boldsymbol{r},t) \qquad \boldsymbol{r} \in \partial R^{RB} \tag{18-13}$$

This should be considered a symbolic equation because the coefficients may vary with location and time, and may be different for the various components. A simple example of when this boundary condition holds is the case of when springs are attached to points on the surface. Then the normal component of traction is proportional to the normal component of traction. The springs may be introduced as a model to simulate the effect of a surrounding medium. An example would be a structure embedded in the earth.

Boundary Condition at Infinity

Some problems involve domains that extend indefinitely in one or more directions. An example is a force applied locally on the surface of an infinite half space, or on the surface of the earth. One of the boundary conditions must be a condition that either the traction or the displacement must approach zero as a radial coordinate becomes large.

Boundary Condition at the Origin

Frequently, the use of particular coordinate systems imply that if certain functions are to remain finite, a restriction must be placed on the primary variable. An example of such a condition occurs with cylindrical coordinates for which the components of strain involve the radial and circumferential components of displacement divided by the radial coordinate. For the strains to remain finite at the origin, a physically realistic requirement, the implication is that the radial and circumferential components of displacement must be zero at the origin. A similar situation arises with the use of spherical coordinates.

18.4 Summary Comments

A part of the data that must be provided in the description of a boundary-value problem consists of the initial and boundary conditions. Often, some of these are not specified under the assumption that the prescribed functions are zero. This lack of specificity is a lazy habit that can lead to a fundamental error. This is especially true in the case of boundary conditions since it is often assumed that boundary conditions fall in only the first three categories outlined above. (Look at how boundary conditions are described in your favorite reference book). Here we have attempted to provide a complete description.

The data for a problem consists of the following:

1. The domain in space and time.
2. The coefficient functions in the governing partial differential equation.
3. The forcing function.
4. The initial conditions.
5. The boundary conditions.

This chapter has focused on the last two items but it is useful to think of these two as part of the general problem of providing the data.