Preconditioning techniques

- Basic concepts
- Gauss-Seidal and SSOR as preconditioners
- Incomplete LU factorizations
- Level-of-fill and threshold-based methods.
- Multi-elimination, ARMS
- See Chapter 10 of text for details.

Preconditioning – Basic principles

Basic idea is to use the Krylov subspace method on a modified system such as

$$\boxed{M^{-1}Ax = M^{-1}b.}$$

- The matrix $M^{-1}A$ need not be formed explicitly; only need to solve $Mw = v$ whenever needed.
- Consequence: fundamental requirement is that it should be easy to compute $M^{-1}v$ for an arbitrary vector v.

Left, Right, and Split preconditioning

Left preconditioning

$$M^{-1}Ax = M^{-1}b$$

Right preconditioning

$$AM^{-1}u = b, \text{ with } x = M^{-1}u$$

Split preconditioning: M is factored as $M = M_L M_R$.

$$M_L^{-1}AM_R^{-1}u = M_L^{-1}b, \text{ with } x = M_R^{-1}u$$

An observation. Introduction to Preconditioning

➤ Take a look back at basic relaxation methods: Jacobi, Gauss-Seidel, SOR, SSOR, ...

➤ Iterations of the form $x^{(k+1)} = Mx^{(k)} + f$ where M is of the form $M = I - P^{-1}A$. For example for SSOR,

$$P_{SSOR} = (D - \omega E)D^{-1}(D - \omega F)$$

➤ Referred to as the SSOR preconditioner

➤ The iteration $x^{(k+1)} = Mx^{(k)} + f$ is attempting to solve $(I - M)x = f$. Since $M \equiv I - P^{-1}A$ this system can be rewritten as $P^{-1}Ax = P^{-1}b$

In other words:

Relaxation iter. $\Longleftrightarrow$ *Preconditioned Fixed Point Iter.*

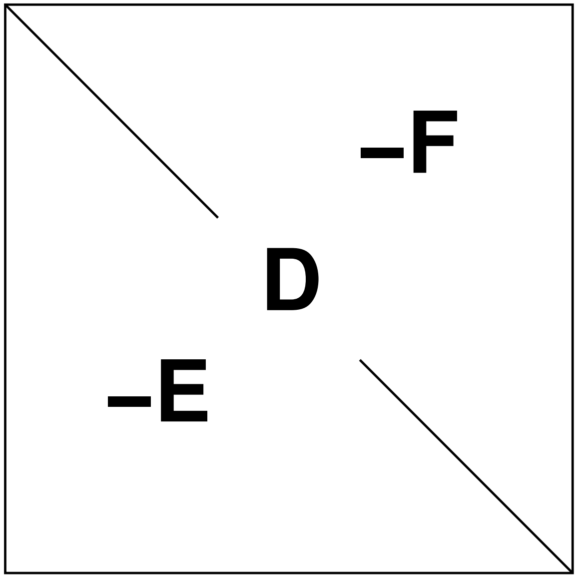
Standard preconditioners

- Simplest preconditioner: M = Diag(A) ➤ poor convergence.
- Next to simplest: SSOR.

$$\boxed{M = (D - \omega E)D^{-1}(D - \omega F)}$$

- Still simple but often more efficient: ILU(0).
- ILU(p) – ILU with level of fill p – more complex.
- Class of ILU preconditioners with threshold
- Class of approximate inverse preconditioners
- Class of Multilevel ILU preconditioners
- Algebraic Multigrid Preconditioners

The SOR/SSOR preconditioner



- SOR preconditioning

$$M_{SOR} = (D - \omega E)$$

- SSOR preconditioning

$$M_{SSOR} = (D-\omega E)D^{-1}(D-\omega F)$$

- $M_{SSOR} = LU$, $L =$ lower unit matrix, $U =$ upper triangular. One solve with $M_{SSOR} \approx$ same cost as a MAT-VEC.

- k-step SOR (resp. SSOR) preconditioning:

$$\boxed{k \text{ steps of SOR (resp. SSOR)}}$$

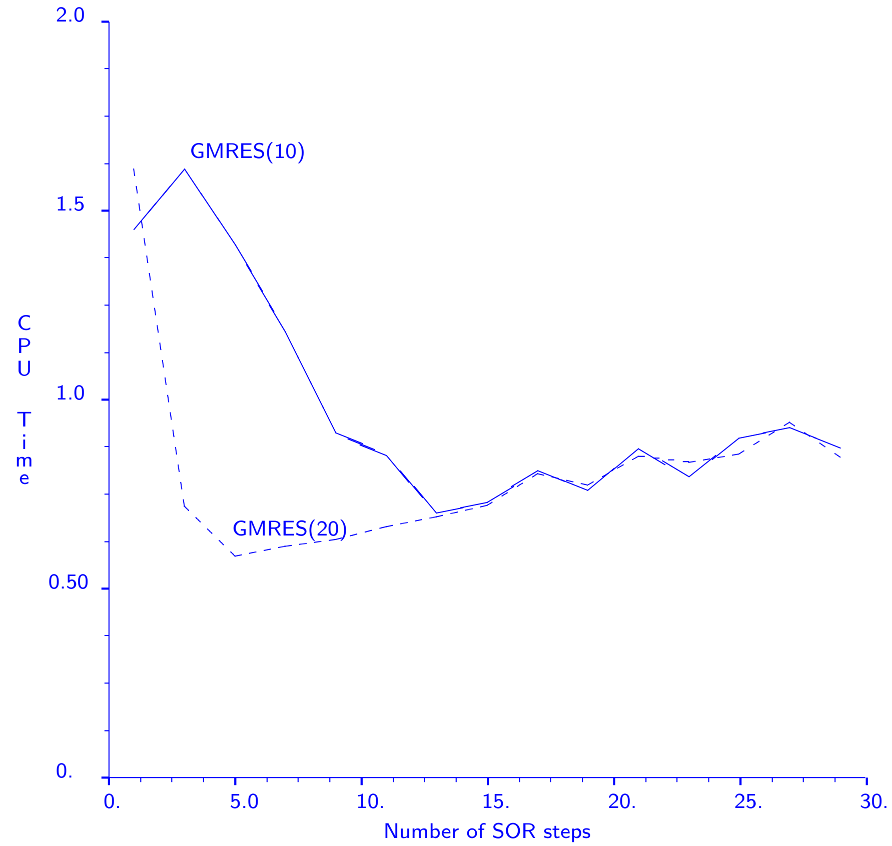
- Questions: Best ω? For preconditioning can take $\omega = 1$

$$\boxed{M = (D - E)D^{-1}(D - F)}$$

Observe: $M = LU + R$ with $R = ED^{-1}F$.

- Best k? $k = 1$ is rarely the best. Substantial difference in performance.

Iteration times versus k for SOR(k) preconditioned GMRES



ILU(0) and IC(0) preconditioners

- **Notation:** $NZ(X) = \{(i,j) \mid X_{i,j} \neq 0\}$
- Formal definition of ILU(0):

$$\begin{aligned} & A = LU + R \\ & NZ(L) \bigcup NZ(U) = NZ(A) \\ & r_{ij} = 0 \text{ for } (i,j) \ \in \ NZ(A) \end{aligned}$$

- This does not define $ILU(0)$ in a unique way.

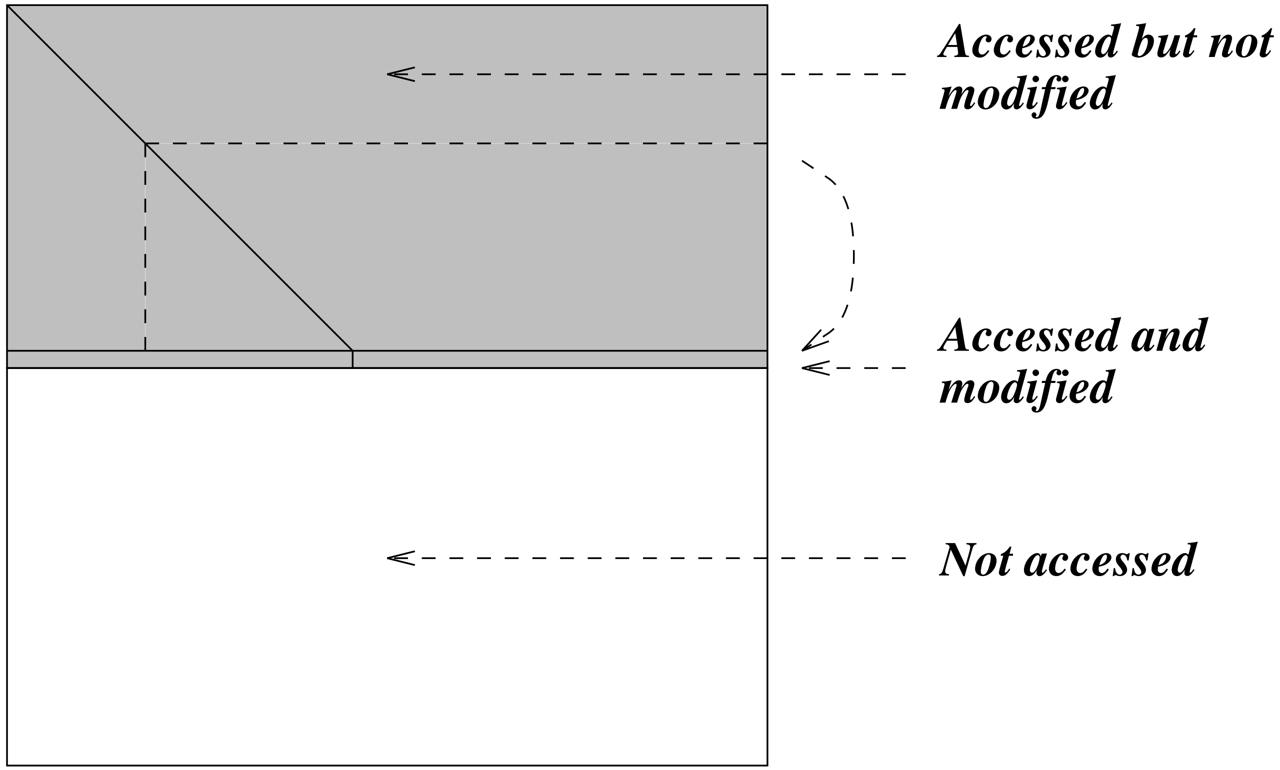
<u>Constructive definition:</u> Compute the LU factorization of A but drop any fill-in in L and U outside of Struct(A).

- ILU factorizations are often based on i, k, j version of GE.

What is the IKJ version of GE?

ALGORITHM : 1. *Gaussian Elimination – IKJ Variant*

1. For $i = 2, \ldots, n$ Do:
2. For $k = 1, \ldots, i-1$ Do:
3. $a_{ik} := a_{ik}/a_{kk}$
4. For $j = k+1, \ldots, n$ Do:
5. $a_{ij} := a_{ij} - a_{ik} * a_{kj}$
6. EndDo
7. EndDo
8. EndDo


Accessed but not modified
Accessed and modified
Not accessed

ILU(0) – zero-fill ILU

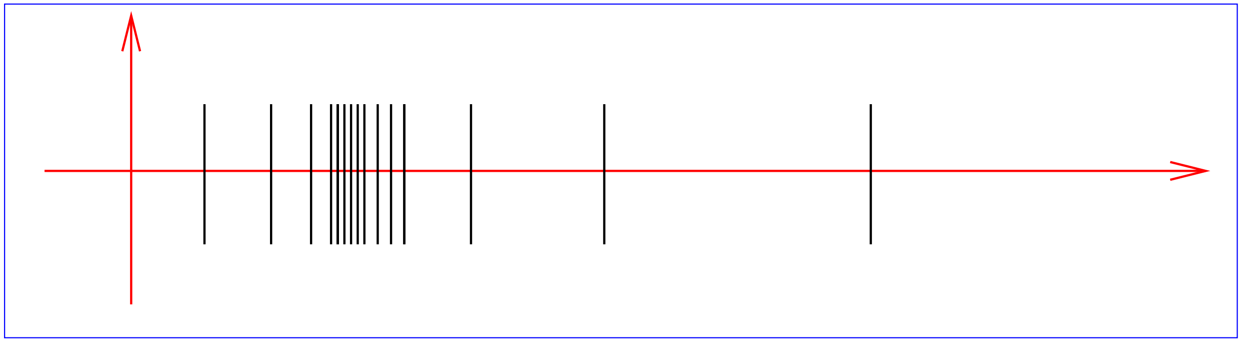
ALGORITHM : 2. *ILU(0)*

For $i = 1, \ldots, N$ *Do:*
  For $k = 1, \ldots, i-1$ *and if* $(i,k) \in NZ(A)$ *Do:*
    Compute $a_{ik} := a_{ik}/a_{kj}$
    For $j = k+1, \ldots$ *and if* $(i,j) \in NZ(A)$, *Do:*
    compute $a_{ij} := a_{ij} - a_{ik}a_{k,j}$.
  EndFor
EndFor

➤ When A is SPD then the ILU factorization $=$ Incomplete Choleski factorization – IC(0). Meijerink and Van der Vorst [1977].

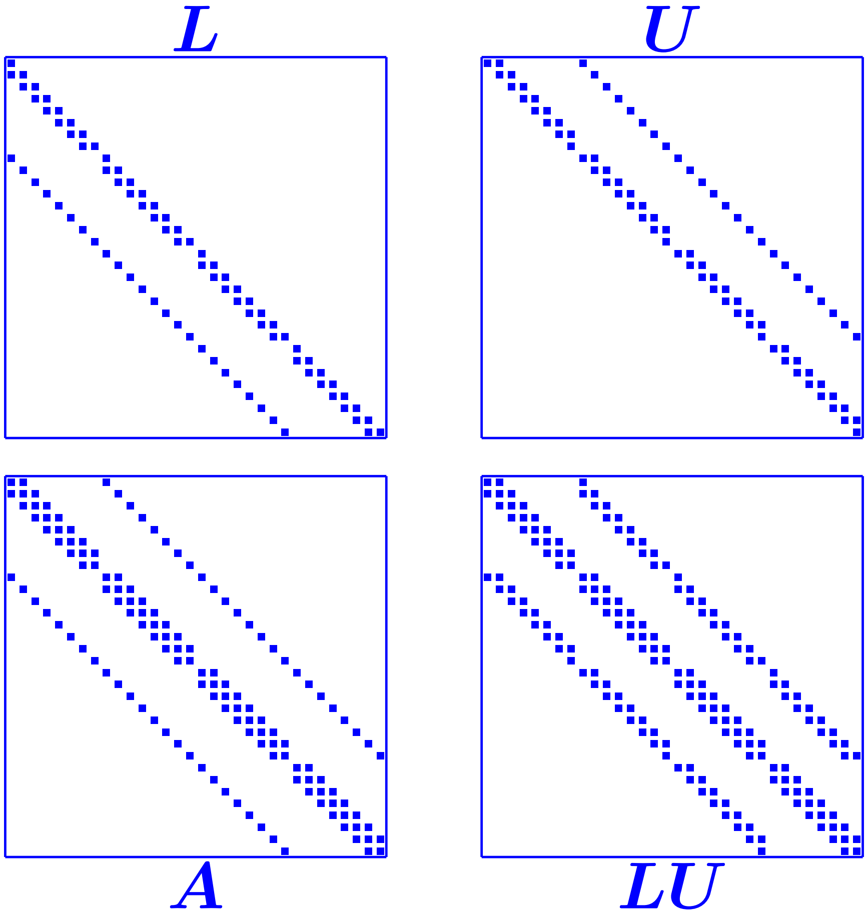
Typical eigenvalue distribution

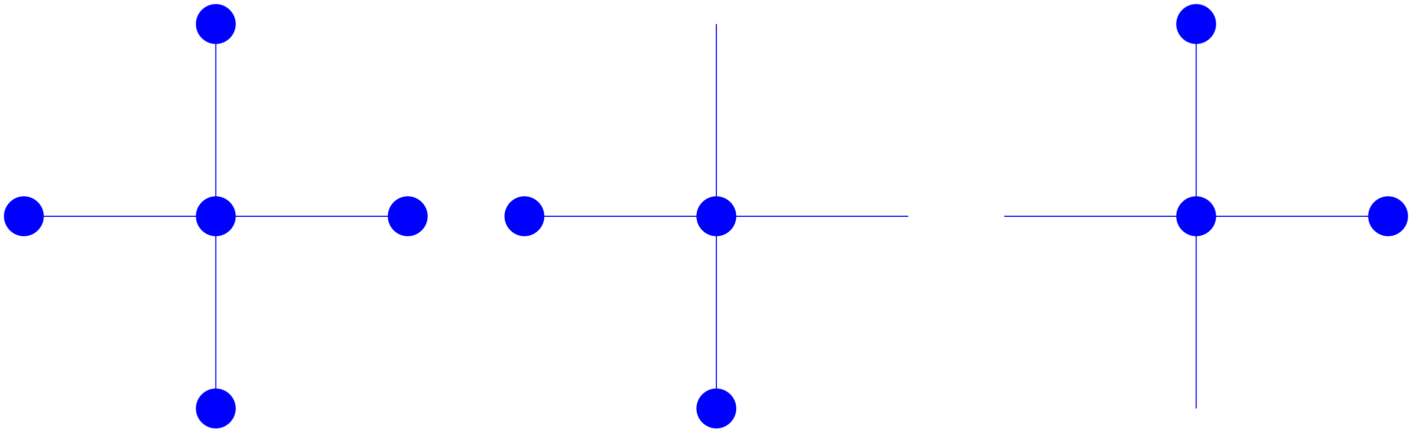
- More than anything else, what determines the convergence of an iterative method is the distribution of the eigenvalues of the matrix.
- Need to consider eigenvalues of preconditioned matrix $M^{-1}A$



- Clustering around 1 results in fast convergence

Pattern of ILU(0) for 5-point matrix





Stencil of $\boldsymbol{A}$ Stencil of $\boldsymbol{L}$ Stencil of $\boldsymbol{U}$

Higher order ILU factorization

➤ Higher accuracy incomplete Choleski: for regularly structured problems, IC(p) allows p additional diagonals in L.

➤ Can be generalized to irregular sparse matrices using the notion of level of fill-in [Watts III, 1979]

- Initially $Lev_{ij} = \begin{cases} 0 & \text{for } a_{ij} \neq 0 \\ \infty & \text{for } a_{ij} == 0 \end{cases}$
- At a given step i of Gaussian elimination:

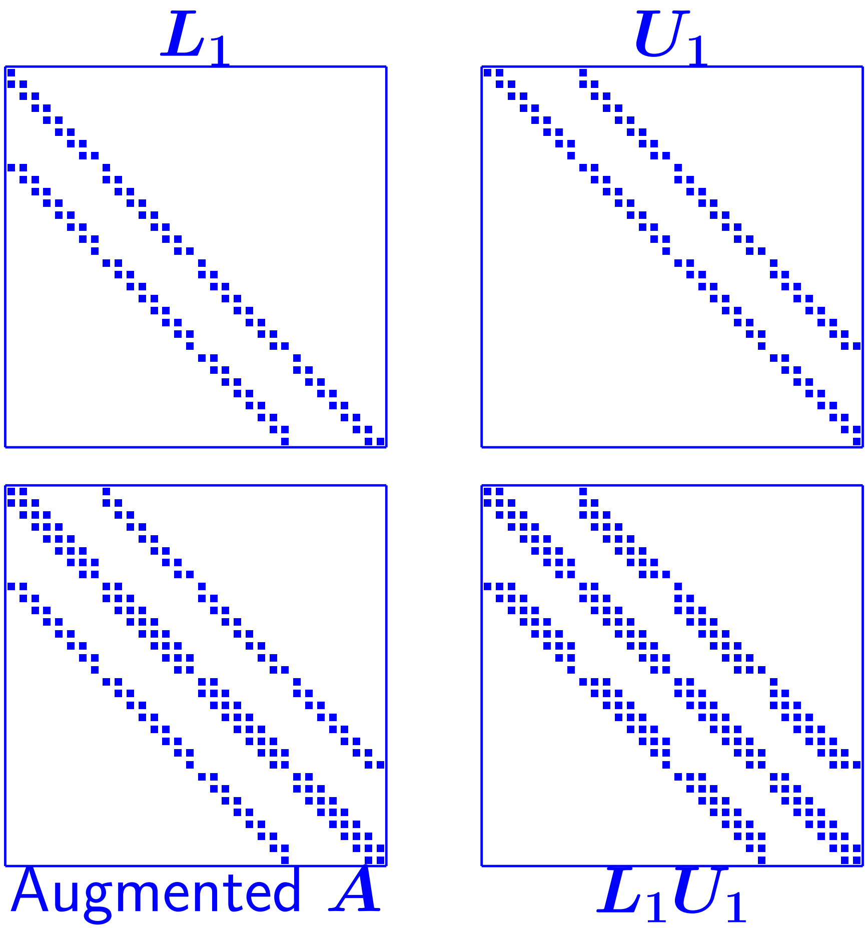
$$\boxed{Lev_{kj} = \min\{Lev_{kj}; Lev_{ki} + Lev_{ij} + 1\}}$$

➤ ILU(p) Strategy = drop anything with level of fill-in exceeding p.

* Increasing level of fill-in <u>usually</u> results in more accurate ILU and...

* ...typically in fewer steps and fewer arithmetic operations.

ILU(1)



ALGORITHM : 3. *ILU(p)*

For $i = 2, N$ *Do*

For each $k = 1, \ldots, i-1$ *and if* $a_{ij} \neq 0$ *do*

Compute $a_{ik} := a_{ik}/a_{jj}$

Compute $a_{i,*} := a_{i,*} - a_{ik}a_{k,*}$.

Update the levels of $a_{i,*}$

In row i*: if* $lev(a_{ij}) > p$ *set* $a_{ij} = 0$

EndFor

EndFor

➤ The algorithm can be split into a symbolic and a numerical phase.
Level-of-fill ➤ in Symbolic phase

ILU with threshold – generic algorithms

ILU(p) factorizations are based on structure only and not numerical values ➤ potential problems for non M-matrices.

➤ One remedy: ILU with threshold – (generic name ILUT.)

Two broad approaches:

First approach [derived from direct solvers]: use any (direct) sparse solver and incorporate a dropping strategy. [Munksgaard ('78), Osterby & Zlatev, Sameh & Zlatev'90, D. Young, & al. (Boeing) etc...]

Second approach : [derived from 'iterative solvers' viewpoint]

1. use a (row or colum) version of the (i,k,j) version of GE;
2. apply a drop strategy for the elment l_{ik} as it is computed;
3. perform the linear combinations to get a_{i*}. Use full row expansion of a_{i*};
4. apply a drop strategy to fill-ins.

ILU with threshold: $ILUT(k, \epsilon)$

- Do the i, k, j version of Gaussian Elimination (GE).
- During each i-th step in GE, discard any pivot or fill-in whose value is below $\epsilon \|row_i(A)\|$.
- Once the i-th row of $L + U$, (L-part + U-part) is computed retain only the k largest elements in both parts.

➤ Advantages: controlled fill-in. Smaller memory overhead.

➤ Easy to implement – much more so than preconditioners derived from direct solvers.

➤ can be made quite inexpensive.

Other preconditioners

Many other techniques have been developed:

- Approximate inverse methods
- Polynomial preconditioners
- Multigrid - type methods
- Incomplete LU based on Crout factorization
- Multi-elimination and multilevel ILU (ARMS)