Matrix-Vector products and Triangular systems

- Matrix-vector products
- Background on linear systems
- Triangular systems
- Sparse Right-hand side.

Sparse matrices – data structure in C

- Recall:

```
typedef struct SpaFmt {
/*---------------------------------------------
| C-style CSR format - used internally
| for all matrices in CSR format
|---------------------------------------------*/
  int n;
  int *nzcount;  /* length of each row */
  int **ja;      /*  to store column indices  */
  double **ma;   /*  to store nonzero entries */
} CsMat, *csptr;
```

- Can store rows of a matrix (CSR) or its columns (CSC)
- Let us first recall how to perform the operation $y = A * x$ (matvecs) – seen earlier

Matvec – row version

```
void matvec( csptr mata, double *x, double *y )
{
   int i, k, *ki;
   double *kr;
   for (i=0; i<mata->n; i++) {
      y[i] = 0.0;
      kr = mata->ma[i];
      ki = mata->ja[i];
      for (k=0; k<mata->nzcount[i]; k++)
         y[i] += kr[k] * x[ki[k]];
   }
}
```

Matvec – Column version

```
void matvecC( csptr mata, double *x, double *y )
{
  int n = mata->n, i, k, *ki;
  double *kr;
  for (i=0; i<n; i++)
    y[i] = 0.0;
  for (i=0; i<n; i++) {
    kr = mata->ma[i];
    ki = mata->ja[i];
    for (k=0; k<mata->nzcount[i]; k++)
      y[ki[k]] += kr[k] * x[i];
  }
}
```

Background: Linear systems

The Problem: $\boldsymbol{A}$ is an $\boldsymbol{n \times n}$ matrix, and $\boldsymbol{b}$ a vector of $\mathbb{R}^n$. Find $\boldsymbol{x}$ such that:

$$\boxed{\boldsymbol{Ax = b}}$$

➤ $\boldsymbol{x}$ is the unknown vector, $\boldsymbol{b}$ the right-hand side, and $\boldsymbol{A}$ is the coefficient matrix

Example:

$$\begin{cases} 2x_1 + 4x_2 + 4x_3 = 6 \\ x_1 + 5x_2 + 6x_3 = 4 \\ x_1 + 3x_2 + x_3 = 8 \end{cases} \quad \text{or} \quad \begin{bmatrix} 2 & 4 & 4 \\ 1 & 5 & 6 \\ 1 & 3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 8 \end{bmatrix}$$

- Standard mathematical solution by Cramer's rule:

$$\boxed{x_i = \det(A_i)/\det(A)}$$

$A_i =$ matrix obtained by replacing i-th column by b.

- Note: This formula is useless in practice beyond $n = 3$ or $n = 4$.

- Three situations:

1. The matrix A is nonsingular. There is a unique solution given by $x = A^{-1}b$.
2. The matrix A is singular and $b \in \mathrm{Ran}(A)$. There are infinitely many solutions.
3. The matrix A is singular and $b \notin \mathrm{Ran}(A)$. There are no solutions.

Triangular linear systems

Example:

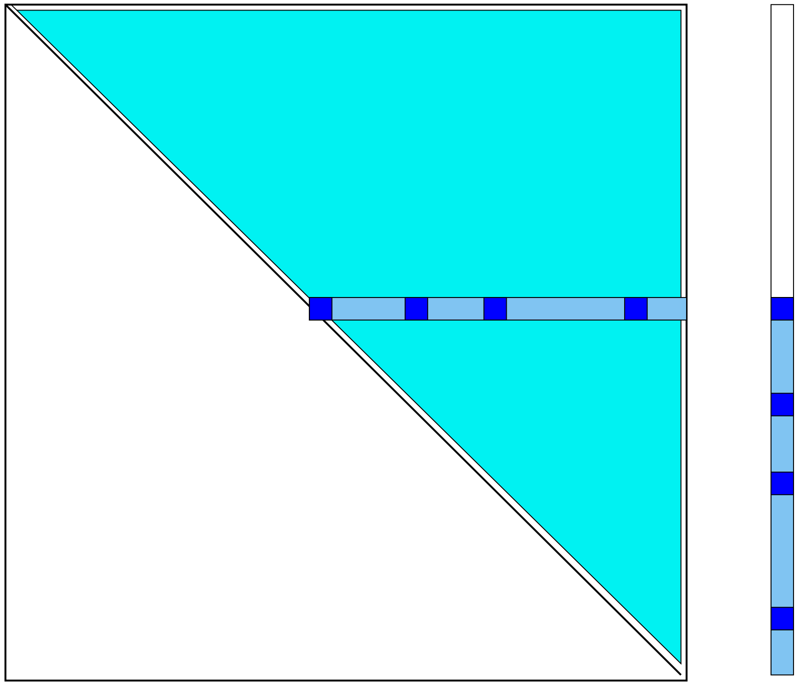
$$\begin{bmatrix} 2 & 4 & 4 \\ 0 & 5 & -2 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}$$

Back-Substitution
Row version

For $i = n : -1 : 1$ do:
  $t := b_i$
  For $j = i + 1 : n$ do
    $t := t - a_{ij}x_j$
  End
  $x_i = t/a_{ii}$
End

✍ Operation count?

Illustration for sparse case (Sparse A, dense b)



- Assumes diagonal entry stored first in inverted form

```
void Usol(csptr mata, double *b, double *x)
{
  int i, k, *ki;
  double *ma;
  for (i=mata->n-1; i>=0; i--) {
    ma = mata->ma[i];
    ki = mata->ja[i];
    x[i] = b[i] ;
// Note: diag. entry avoided
    for (k=1; k<mata->nzcount[i]; k++)
      x[i] -= ma[k] * x[ki[k]];
    x[i] *= ma[0];
  }
}
```

- Operation count?

Column version

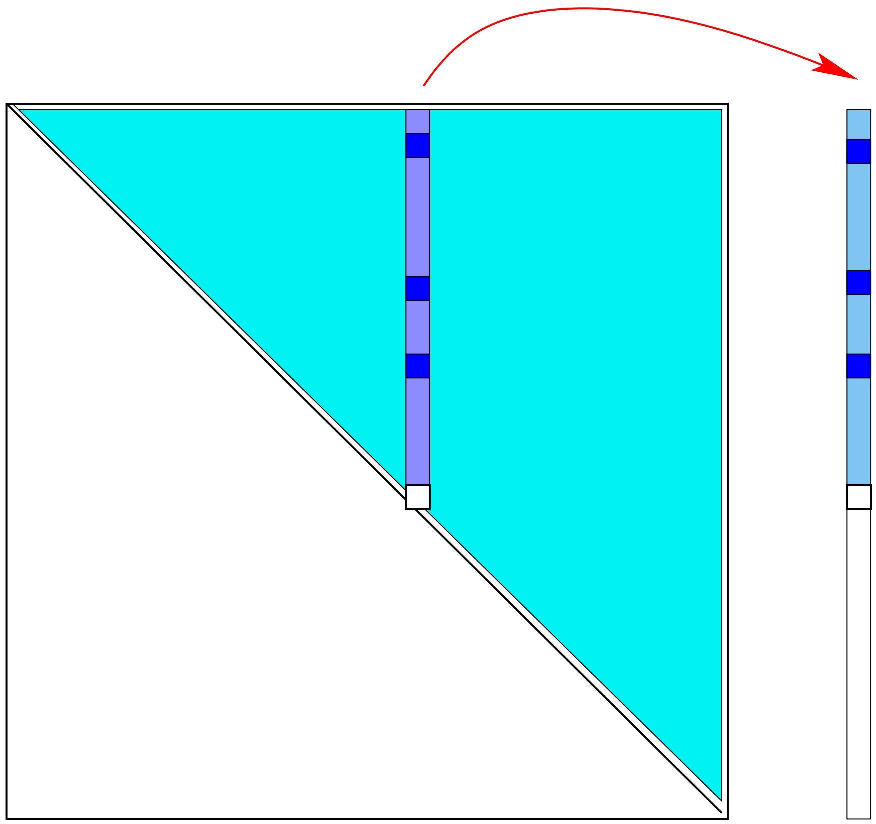
➤ Column version of back-subsitution:

Back-Substitution
Column version

For $j = n : -1 : 1$ do:
  $x_j = b_j / a_{jj}$
  For $i = 1 : j - 1$ do
    $b_i := b_i - x_j * a_{ij}$
  End
End

✍ Justify the above algorithm [Show that it does indeed give the solution]

Illustration for sparse case (Sparse $\boldsymbol{A}$, dense $\boldsymbol{b}$)



➤ Assumes diagonal entry stored first in inverted form

```
void UsolC(csptr mata, double *b, double *x)
{
  int i, k, *ki;
  double *ma;
  for (i=mata->n-1; i>=0; i--) {
        ja = U->ja[i];
        ma = U->ma[i];
        x[i] *= ma[0];
// Note: diag. entry avoided
        for( j = 1; j < U->nzcount[i]; j++ )
            x[ja[j]] -= ma[j] * x[i];
  }
}
```

✍ Operation count ?

Sparse A and sparse b

Illustration: Consider solving $Lx = b$ in the situation:

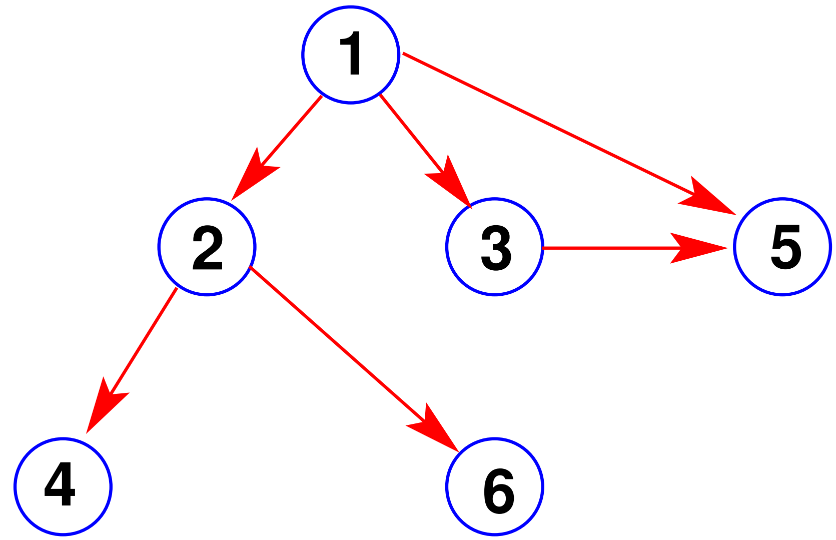
$$L = \begin{array}{|c|c|c|c|c|c|}\hline * & & & & & \\ \hline * & * & & & & \\ \hline * & & * & & & \\ \hline & * & & * & & \\ \hline * & & * & & * & \\ \hline & * & & & & * \\ \hline \end{array} \qquad b = \begin{array}{|c|}\hline * \\ \hline \\ \hline \\ \hline \\ \hline \\ \hline \\ \hline \end{array}$$

✍ Show progress of the pattern of $x = L^{-1}b$ by performing symbolically a column solve for system $Lx = b$.

✍ Show how this pattern can be determined with Topological sorting. Generalize to any sparse b.

Sparse A and sparse b: Example

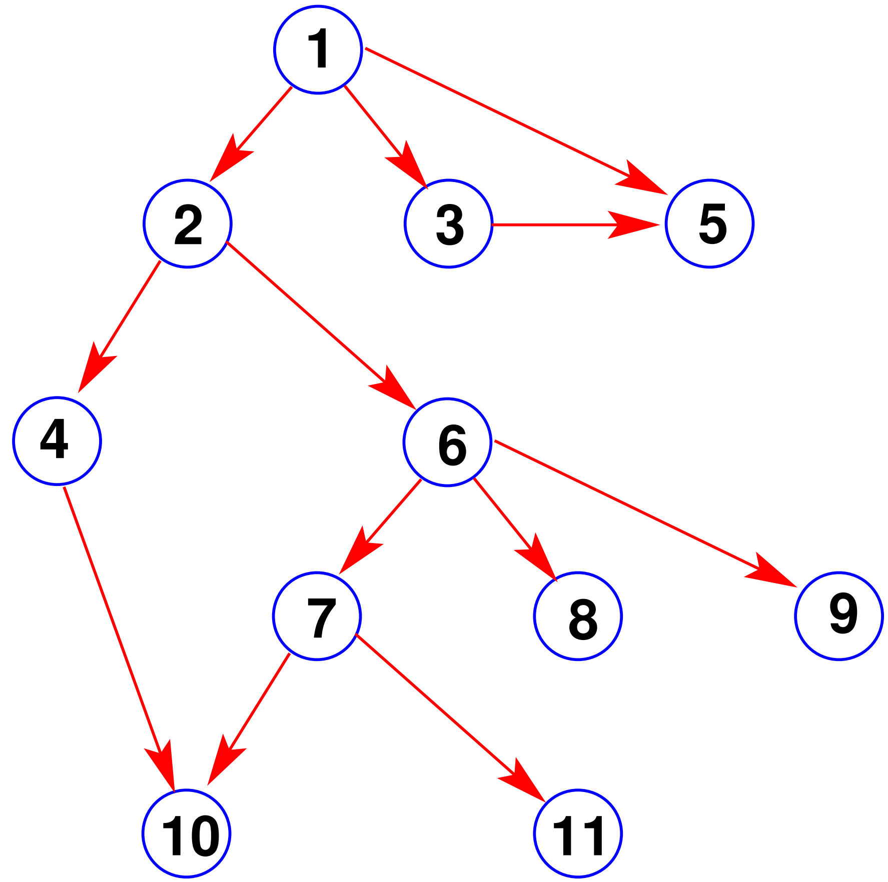
- Consider triangular system in previous example.
- Graph of matrix shown in next figure
- Sets dependencies between tasks



- Root: node 1 (see right-hand side $\boldsymbol{b}$)
- Post-order traversal: 6, 4, 2, 5, 3, 1
- Reverse: 1, 3, 5, 2, 4, 6
- In many cases, this leads to a short traversal

✍ Example: remove link $\mathbf{1} \longrightarrow \mathbf{2}$ and redo

✍ Consider a triangular system with the following graph where b has nonzero entries in positions 3 and 7



✍ Same question if b has a nonzero entry in position 1.

✍ Explore sparsity of solution in each case.

LU factorization from sparse triangular solves

➤ LU factorization built one column at a time. At step k:

We want: $\underbrace{L_k}_{n\times n}\underbrace{U_k}_{n\times k} = \underbrace{A_k}_{n\times k} \quad (\equiv A(1:n,1:k))$

$$\left[\begin{array}{ccc|c|ccc} 1 & & & & & & \\ * & 1 & & & & & \\ * & * & 1 & & & & \\ * & * & * & 1 & & & \\ * & * & * & ? & 1 & & \\ * & * & * & ? & & 1 & \\ * & * & * & ? & & & 1 \end{array}\right] \left[\begin{array}{ccc|c} x & x & x & ? \\ & x & x & ? \\ & & x & . \\ & & & ? \\ & & & 0 \\ & & & 0 \\ & & & 0 \end{array}\right] = A_k$$

➤ In blue: has been determined. In red: to be determined

- Step 0: Set the terms ? in L_k to zero. Result $\equiv \tilde{L}_k$
- Step 1 : Solve $\tilde{L}_k w = a_k$ [Sparse $\tilde{L}_k$, sparse RHS]
- Step 2: set

$$u = \left|\begin{array}{c} w_1 \\ w_2 \\ \vdots \\ w_k \\ \hline 0 \\ \vdots \\ 0 \\ 0 \end{array}\right| \qquad z = \frac{1}{w_k} \left|\begin{array}{c} 0 \\ \vdots \\ 0 \\ 0 \\ \hline w_{k+1} \\ w_{k+2} \\ \vdots \\ w_n \end{array}\right|$$

- Then $L_k U_k = A_k$ with

$$\underbrace{\begin{bmatrix} 1 & & & \vline & & \vline & & & \\ * & 1 & & \vline & & \vline & & & \\ * & * & 1 & \vline & & \vline & & & \\ * & * & * & \vline & 1 & \vline & & & \\ * & * & * & \vline & z_{k+1} & \vline & 1 & & \\ * & * & * & \vline & \vdots & \vline & & 1 & \\ * & * & * & \vline & z_n & \vline & & & 1 \end{bmatrix}}_{L_k} ; \quad \underbrace{\begin{bmatrix} x & x & x & \vline & u_1 \\ & x & x & \vline & u_2 \\ & & x & \vline & \vdots \\ & & & \vline & u_k \\ & & & \vline & 0 \\ & & & \vline & 0 \\ & & & \vline & 0 \end{bmatrix}}_{U_k}$$

- Verification: Note $L_k = \tilde{L}_k + z e_k^T$; Also $\tilde{L}_k z = z$
- Must verify only $L_k U_k(:, k) = a_k$, i.e., $L_k u = a_k$

$$\begin{aligned} L_k u &= (\tilde{L}_k + z e_k^T) u = \tilde{L}_k (I + z e_k^T) u \\ &= \tilde{L}_k (u + w_k z) = \tilde{L}_k w = a_k \end{aligned}$$

- Key step: solve triangular system
- In sparse case: sparse triangular system with sparse right-hand side
- Use topological sorting at each step
- Scheme derived from this known as 'left-looking' sparse LU –
- Also known as 'Gilbert and Peierls' approach
- Reference: J. R. Gilbert and T. Peierls, Sparse partial pivoting in time proportional to arithmetic operations, SIAM J. Sci. Statist. Comput., 9 (1988), pp. 862-874

✍ Benefit of this approach: Partial pivoting is easy. Show how you would do it.