The focus of this assignment is to consider, in some detail, a direct method for solving the linear algebraic problem with measures of error.

1. Provide a written summary of the relevant theory related to matrix multiplication, the Gram-Schmidt procedure for obtaining an orthonormal basis for a vector space, and the QR algorithm for solving a linear algebraic equation. Be sure to include definitions of terms. Insert the key lines of your code within the text and describe in words how the code relates to the theory.

2. Construct two simple matrices $[A]_{4x3}$ and $[B]_{3x2}$. Show that you get the same result using the following two approaches to obtain the product:

$$[C]=[A][B]=\begin{bmatrix} \langle A\rangle^{1}\{B\}^{1} & \langle A\rangle^{1}\{B\}^{2} & \cdots \\ \langle A\rangle^{2}\{B\}^{1} & \langle A\rangle^{2}\{B\}^{2} & \cdots \\ \cdots & \cdots & \cdots \end{bmatrix}=\{A\}^{1}\langle B\rangle^{1}+\{A\}^{2}\langle B\rangle^{2}+\cdots$$

3. Construct a vector space by specifying four vectors of length n = 4 where the fourth vector is a linear combination of the first three. Now re-order the vectors so that your fourth vector is one of the first three. Write a program that implements the Gram-Schmidt procedure for obtaining an orthonormal basis. What is the dimension of the vector space based on your results of the Gram-Schmidt procedure? Express one of your original vectors as a linear combination of the orthonormal base vectors, i.e., find the components of the vector with respect to the new basis.

4. Now choose four independent vectors (the first three can be the same as those chosen in Problem 3, if independent). Let these vectors be the columns of a matrix [A].

(i) Apply your Gram-Schmidt algorithm to obtain the matrix [Q]. As a check find $[Q][Q]^{T}$. What should the result be?

(ii) Find the matrix $[R]=[Q]^{T}[A]$. Is the matrix upper triangular?

(iii) Write a program that performs the back substitution and provides the solution to the linear algebraic equation $[R]\{x\}=\{\hat{b}\}$ where $\{\hat{b}\}=[Q]^{T}\{b\}$.

(iv) Assume a solution $\{x\}^{ex}$ and compute a vector $\{b\}=[A]\{x\}^{ex}$. Apply your QR algorithm to find the solution to $[A]\{x\}=\{b\}$. Compute scalar measures of error with one based on the exact solution and one based on the residual.

(vi) Perform one iterative improvement for your solution. Does each measure of error decrease?

(vii) Use your program to find the inverse of [A]. How good is your result as indicated by an error based on the "magnitude" of a matrix as defined by the Frobenius norm?