Homework #3 Mohiuddin Ahmad ①

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Problem 1:

(a) $-(u')' = 1, \quad 0<x<1, \quad u(0) = u(1) = 0$

(i) Work function,

$$\pi(U) = \frac{1}{2}\int_{\Omega} (U')^2 d\Omega - \int_{\Omega} U \, d\Omega$$ ✓

using $U = \hat{U}_N = \sum_{j=1}^{N} \alpha_j \phi_j$ ✓ $\quad \phi(0) = \phi(1) = 0$

Then,

$$\pi(\hat{U}_N) = \frac{1}{2}\int_0^1 \left(\sum \alpha_j \phi_j'\right)^2 dx - \int_0^1 \left(\sum \alpha_j \phi_j\right) dx$$ ✓

(ii) The residual function:

$$\int_0^1 r v \, dx = 0$$

$$\Rightarrow \int_0^1 (-u''-1)\, v \, dx = 0$$ ✓

using $\hat{u}_N = \sum_{j=1}^{N} \alpha_j \phi_j$, $u_0 = 0$ ✓

$v = \sum_{k=1}^{N} \beta_k \psi_k$,

$$\Rightarrow \int_0^1 \left(-\sum_{j=1}^{N} \alpha_j \phi_j'' - 1\right)\left(\sum_{k=1}^{N} \beta_k \psi_k\right) dx = 0$$ ✓

$$\Rightarrow -\sum \alpha_j \int_0^1 \phi_j'' \psi_k \, dx - \int_0^1 \psi_k \, dx = 0.$$

(iii) Weak formulation:

$$\int_0^1 (-u''-1)\, v \, dx = 0$$

$$\Rightarrow -v u' \Big|_0^1 + \int_0^1 u'v' \, dx - \int_0^1 v \, dx = 0$$

$$\therefore \int_0^1 u'v' \, dx - \int_0^1 v \, dx - v u' \Big|_0^1 = 0.$$ ✓

here, $u(0) = u(1) = 0$

$v(0)$ and $v(1)$ unknown and will be equal to zero if Galerkin method is used.

Then, weak form is, $\int_0^1 u'v'dx - \int_0^1 v\,dx - v\frac{du}{dx}\Big|_0^1 = 0.$

(iv) L₂ norm:

$$\|e\|_{L_2} = \sqrt{\int_0^1 (U-u)^2 dx}$$ ✓

$$= \sqrt{\int_0^1 \Big(\sum_{j=1}^{n} \alpha_j \phi_j - u\Big)^2 dx}.$$

(v) Energy norm:

$$\pi(U) = \frac{1}{2}\int_0^1 (u')^2 dx - \int_0^1 u\,dx$$

Then, $$\|e\|_e = \sqrt{\frac{1}{2}\int_0^1 (e')^2 dx}$$ ✓

(b) $-(u')' = f$, $0 < x < 1$, $u(0) = 1$, $u'(1) = 1$

$\Rightarrow -u'' = f.$

$\mathcal{L}[u_b] \neq 0$ now...

(i) The work function:

$$\pi(\hat{U}_N) = \frac{1}{2}\int_0^1 (\hat{U}_N')^2 dx - \int_0^1 f\,\hat{U}_N\,dx$$

$$= \frac{1}{2}\int_0^1 \Big(\sum \alpha_j \phi_j'\Big)^2 dx - \int_0^1 f \sum_j \alpha_j \phi_j\,dx$$

Let, $1+x$ would work too...

$$U = 1 + \frac{x^2}{2} + \sum_{j=1}^{N} \alpha_j \phi_j$$

$$= U_b + \hat{U}_N.$$

$\phi(0) = \phi(1) = 0.$

(ii) Residual function:

$$\int_0^1 (-u''-f)v\,dx=0$$

$$\Rightarrow -\int_0^1 \sum \alpha_j \varphi_j''\ \psi_k\,dx - \int_0^1 (1+f)\,\psi_k\,dx = 0$$

(iii) Weak formulation

$$\int_0^1 (-u''-f)v\,dx=0$$

$$\Rightarrow -\int_0^1 u''v\,dx - \int_0^1 fv\,dx = 0$$

$$\Rightarrow -vu'\Big|_0^1 + \int_0^1 v'u'\,dx - \int_0^1 fv\,dx = 0$$

As $u(0)=1$, v will have a homogenious BC.

$v(0)=0$. Then,

$$\int_0^1 u'v'\,dx - \int fv\,dx - v(1)\cancelto{1}{u'(1)} + \cancelto{0}{v(0)u'(0)} = 0$$

$$\Rightarrow \int_0^1 u'v'\,dx - \int fv\,dx - v(1) = 0$$

This is the weak form.

(iv) L_2 norm: $\|e\|_{L_2} = \sqrt{\int_0^1 (U-u)^2\,dx}$

(v) Energy norm: $e = U-u$

$$\|e\|_e = \sqrt{\frac{1}{2}\int (e')^2\,dx}$$

(c) $-(ku')' + Bu' + cu = f, \quad 0 < x < 1, \quad u'(0) = 1, \; u'(1) = 1$] Then $\varphi'(0) = \varphi'(1) = 0$

(i) The work function:

$$\pi(U) = \frac{1}{2}\int_0^1 [(KU')^2 + 2BU'U + CU^2 - 2fU] dx$$

(ii) Residual function,

$$\int_0^1 (Lu - f) v \, dx = 0$$

$$U = \overbrace{\frac{\sin 2\pi x}{2\pi}}^{U_B} + \overbrace{\sum \alpha_j \phi_j}^{\hat{U}_N}$$

$$U' = \cos 2\pi x + \sum \alpha_j \phi_j'$$

$$U'' = \underbrace{-(2\pi) \sin 2\pi x}_{} + \sum \alpha_j \phi_j''$$

or $U_b = x$

$U_b' = 1$

$U_b'' = 0$

$$\Rightarrow \int_0^1 (-Ku'' + BU' + cu - f) v \, dx = 0$$

$$\Rightarrow \int_0^1 (-KU''v + Bu'v + cuv - fv) \, dx = 0$$

$$\Rightarrow \int_0^1 \Big(\underbrace{-K \sum_{j=1}^N \alpha_j \phi_j'' \varphi_k + B \sum_{j=1}^N \alpha_j \phi_j' \varphi_k + C \sum_{j=1}^N \alpha_j \phi_j \varphi_k}_{L\hat{U}_N} \underbrace{+ (2\pi) \sin 2\pi x \, K\varphi_k + B \cos 2\pi x \, \varphi_k + \frac{C \sin 2\pi x}{2\pi} \varphi_k - f\varphi_k}_{f^*} \Big) dx = 0$$

$$\Rightarrow \int_0^1 (L\hat{U}_N - f^*) = 0$$

(iii) The weak formulation:

$$\int_0^1 (-ku'' + BU' + cu - f) v \, dx = 0$$

$$\Rightarrow -Ku'v\Big|_0^1 + \int_0^1 (Ku'v' + Bu'v + cuv - fv) \, dx = 0$$

$$\int_0^1 (Ku'v' + Bu'v + cuv - fv) dx - Kv(1) + Kv(0) = 0$$

this is the weak formulation.

(iv) L_2 error norm:

$$\|e\|_{L_2} = \sqrt{\int_0^1 \left(\frac{\sin 2\pi x}{2\pi} + \sum_{j=1}^{N} \alpha_j \phi_j - u\right)^2 dx}$$ ✓

$$e = \left(\frac{\sin 2\pi x}{2\pi} + \sum_{j=1}^{N} \alpha_j \phi_j - u\right)$$

(v) e norm:

$$\|e\|_e = \sqrt{\frac{1}{2}\int_0^1 \left((e')^2 + 2B\underbrace{\cancel{u'u}}_{e'e} + \underbrace{Cu^2}_{e^2}\right) dx}$$

(d) $-(3xu')' = \sin x \quad , \quad 0<x<1 , \quad u(0)=1, u(1)=1$

$\Rightarrow -3xu'' - 3u' = \sin x$

(i) $$\pi(u) = \frac{1}{2}\int_0^1 (3xu')^2 dx - \int_0^1 (\sin x) u \, dx$$

let, $\hat{v} = \cos(2\pi x) + \sum_{j=1}^{N} \alpha_j \phi_j \quad , \quad \phi(0) = \phi(1) = 0$

or $v_0 = 1 \ldots$

Then, $$\pi(\hat{v}_N) = \frac{1}{2}\int_0^1 3x\left(\sum_{j=1}^{N} \alpha_j \phi_j'\right)^2 dx - \int_0^1 (\sin x) \sum_{j=1}^{N} \alpha_j \phi_j \, dx$$

(ii) The residual function

$$\int_0^1 (\underbrace{-3xu'' - 3u'}_{L\hat{u}_N} - \sin x)\, v \, dx = 0$$

$$\Rightarrow \int_0^1 \Big(-3x \sum_{j=1}^{n} \alpha_j \phi_j'' \psi_k - 3 \sum \alpha_j \phi_j' \psi_k + \underbrace{(2\pi)^2 \cos(2\pi x) \psi_k + (2\pi) \sin(2\pi x) \psi_k - \sin x \, \psi_k}_{f^*}\Big) dx = 0$$

$$\Rightarrow \int_0^1 (L\hat{u}_N - f^*) = 0$$ ✓

(iii) The weak formulation:

$$\int_0^1 \left(-(3xu')' - \sin x\right) v\,dx = 0$$

$$\Rightarrow \quad -3xu'v\Big|_0^1 + \int_0^1 3xu'v'dx - \int_0^1 \sin x \; v\,dx = 0$$

$$\Rightarrow \quad \int_0^1 3xu'v'dx - \int_0^1 \sin x \; v\,dx - 3u'(0)\,v(0) = 0$$

Here, $v(0) = v(1) = 0$. Therefore, the weak form is,

$$\int_0^1 3xu'v'dx - \int_0^1 \sin x\, v\,dx$$

(iv) L_2 norm:

$$\|e\|_{L_2} = \sqrt{\int_0^1 (e')^2 dx} \qquad e' = \left(\cos 2\pi x + \sum \alpha_j \phi_j - u\right)$$

(v) $$\|e\|_e = \sqrt{\frac{1}{2}\int_0^1 3x(e')^2}$$

Problem 2:

For natural BC: $\int_\Omega (u')^2 d\Omega < \alpha$, boundary limits for u are given.

For v: if u is not homogenious, v needs to be homogenious, or $v(0)$ or $v(1) = 0$.

\# For work function and weak formulation u should be homogenious whereas in residual function, it will also have the boundary terms.

Differentiate b/t natural & essential BC's

3. For <u>problem 1:</u>

Direct method gives the actual solution of the field equation, although it is not possible all the time. Method of weighted residuals uses a weighted function. depending on the weight function, solution may be different. Finite element method gives solution close to the exact solution, if number of elements are choosen properly.

→ Based on weak form

<u>For field equation</u> $-(u')'=1$

$$u' = -x + c_1$$
$$u = -\frac{x^2}{2} + c_1 x + c_2$$

$$u(0) = 0 \Rightarrow c_2 = 0$$
$$u(1) = 0 \Rightarrow -\frac{1}{2} + c_1 = 0$$
$$\Rightarrow c_1 = \frac{1}{2}$$

$$\therefore \quad u = -\frac{x^2}{2} + \frac{x}{2}$$

From weighted residuals,

$$\int_0^1 (-u''-1)\, v\, dx = 0$$

$$\Rightarrow -\int_0^1 (\Sigma \alpha_j \varphi_j'' + 1)\, \psi_k \, dx = 0.$$

if collocation method is used, $\psi_k = \delta(x - x_k)$

For one term solution,

$$-\int_0^1 (\alpha \varphi(x) + 1)\, \delta(x - x_k)\, dx = 0$$

$$\Rightarrow \alpha \varphi(x_k) + 1 = 0$$

$$\Rightarrow \alpha = \frac{-1}{\varphi(x_k)} = -2.$$

if $\varphi(x) = x$

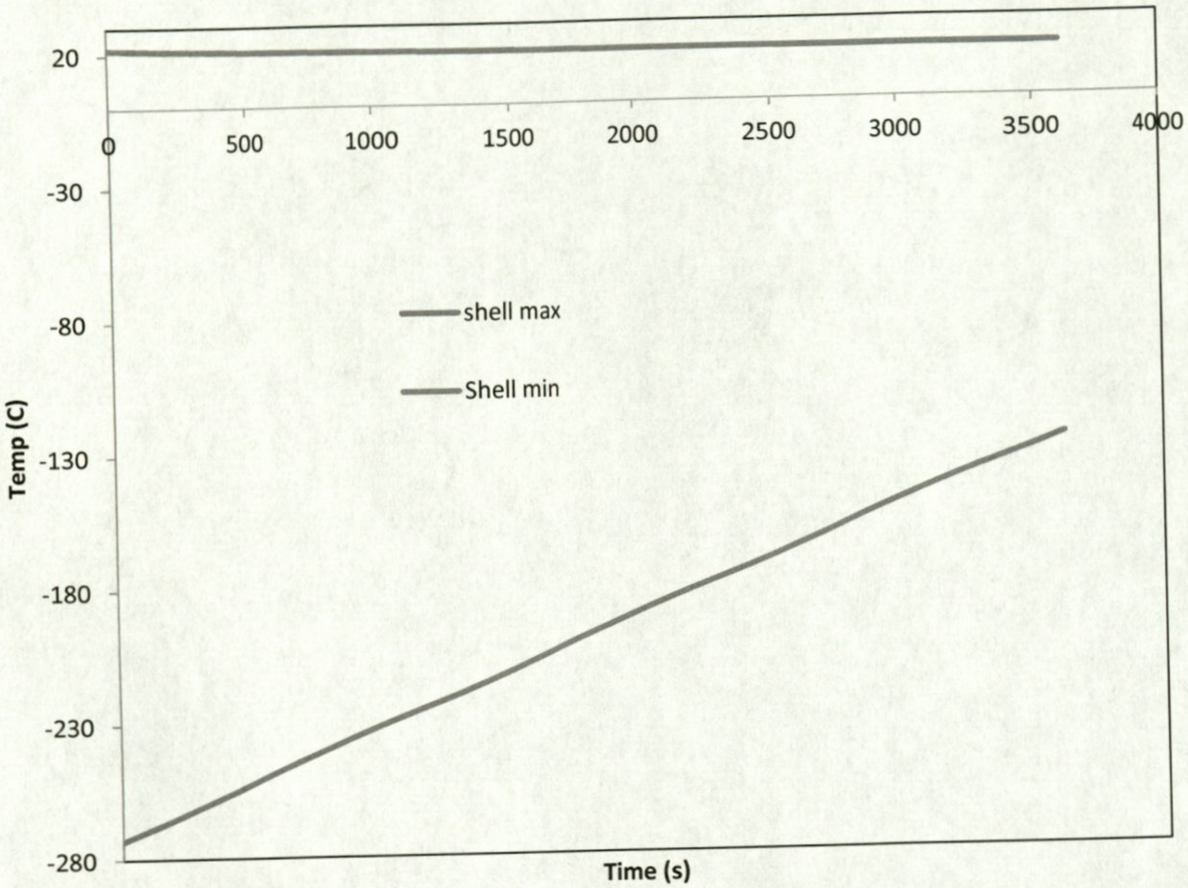
$x_k = \frac{1}{2}$

$\therefore \quad u = -2x$

depending on number of terms, the solution differs.

* Depending on number of element, the solution may differ from exact one.

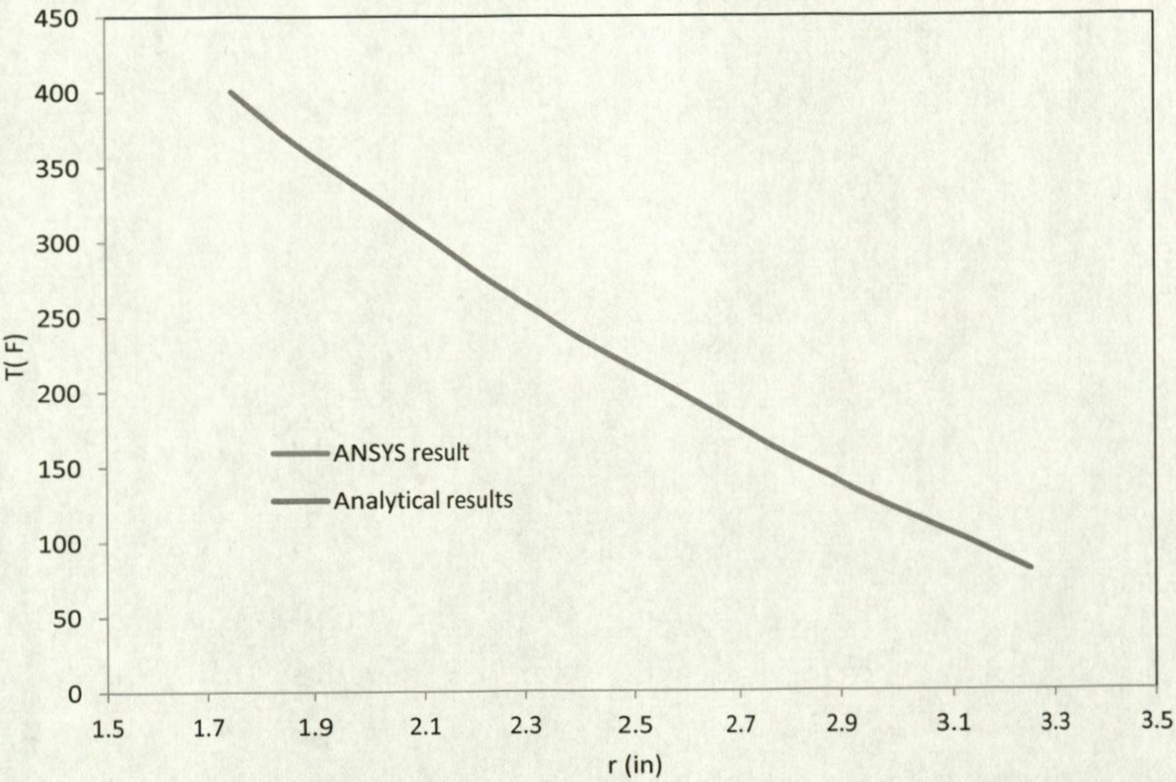
Problem 2: Radiation between surfaces



FEA Tutorial Project #2 Mohiuddin Ahmad 02/26/2014

Problem 1: Heat conduction of a cylinder

Verification of ANSYS result



HOMEWORK #3

COMPUTATIONAL MECHANICS

ME 404/504

A. DERIVE THE WEAK FORMULATION

$$((1-x^2)u')' + 12u = 0, \quad 0<x<1$$

$$u(0)=0, \quad u(1)=1$$

THE RESIDUAL, $r = \mathcal{L}u - f$ IS

$$r = ((1-x^2)u')' + 12u.$$

FROM THE WORK FUNCTION,

$$\pi[u] = \frac{1}{2}\int_0^1 vr\,dx$$

$$= \frac{1}{2}\int_0^1 v(((1-x^2)u')' + 12u)\,dx$$

AS WE'RE SEEKING SOLUTIONS FOR $r=0$, WE CAN JUST CONSIDER THE INTEGRAL & NEGLECT THE $\frac{1}{2}$ TERM. INTEGRATION BY PARTS YIELDS

$$v(1-x^2)u'\Big|_0^1 + \int_0^1 v'(x^2-1)u' + 12vu\,dx = 0$$

ADMISSIBLE v REQUIRE ONLY THAT $v(0)=0$, AND $v(1)=0$ SINCE THE BOUNDARY IS PRESCRIBED AND WE CANT HAVE A VIRTUAL DISPLACEMENT AT THIS POINT, GIVING

$$\int_0^1 v'(x^2-1)u' + 12vu\,dx = 0$$

ALSO, NOTE $v(1-x^2)u'\Big|_{x=1}$ YIELDS $v(1)(1-1^2)u'(1) = 0$.

FOR SOLUTION, START WITH THE GALERKIN DISCRETIZATION

$$\hat{U}_N = \sum_{j=1}^{N} \alpha_j \phi_j(x)$$

$$V = \sum_{i=1}^{N} \beta_i \phi_i(x)$$

SUBSTITUTION INTO THE WEAK FORMULATION YIELDS

$$\int_0^1 \left(\sum_{i=1}^{N} \beta_i \phi_i'\right)(x^2-1)\left(\sum_{j=1}^{N} \alpha_j \phi_j'\right) + 12\left(\sum_{i=1}^{N} \beta_i \phi_i\right)\left(\sum_{j=1}^{N} \alpha_j \phi_j\right) dx$$

$$\sum_{i=1}^{N} \beta_i \left[\sum_{j=1}^{N} \alpha_j \int_0^1 (x^2-1)\phi_i'\phi_j' + 12\phi_i\phi_j \, dx \right]$$

THIS GIVES

$$\mathbb{K}_{ij} = \int_0^1 (x^2-1)\phi_i'\phi_j' + 12\phi_i\phi_j \, dx$$

$$f_i = 0$$

NEXT, LET US LOOK AT THIS FROM A LOCAL PERSPECTIVE INSTEAD OF GLOBALLY.

$$\mathbb{K}_{ij}^e = \int_0^h \left((\xi + x_i)^2 - 1\right) \psi_i^{e\prime} \psi_j^{e\prime} + 12\, \psi_i^e \psi_j^e \, d\xi$$

$$f_i^e = 0$$

B. PIECEWISE LINEAR ELEMENTS

THE BASIS FUNCTIONS ARE GIVEN LOCALLY AS

$$\psi_1^e = 1 - \frac{\xi}{h}, \quad \psi_1^{e\prime} = \frac{-1}{h}$$

$$\psi_2^e = \frac{\xi}{h}, \quad \psi_2^{e\prime} = \frac{1}{h}$$

THE LOCAL STIFFNESS ELEMENT TERMS ARE THEN:

$$K_{11}^e = \int_0^h ((\xi + x_e)^2 - 1)\frac{1}{h^2} + 12(1 - \frac{\xi}{h})^2 d\xi$$

$$= \int_0^h \frac{1}{h^2}(\xi^2 + 2x_e\xi + x_e^2 - 1) + 12(\frac{1}{h^2}\xi^2 - \frac{2}{h}\xi + 1)\, d\xi$$

$$= \left[\frac{1}{h^2}\frac{13}{3}\xi^3 - \frac{1}{h^2}(12h - x_e)\xi^2 + \frac{1}{h^2}(x_e^2 - 1 + 12h^2)\xi\right]_0^h$$

NOTE, $x_e = (e-1)h$, $e = 1, 2, \ldots, N-1$, AND $h = \frac{1}{N-1}$, WHERE N IS THE NUMBER OF GLOBAL BASIS FUNCTIONS, AND THERE ARE $N-1$ ELEMENTS IN THE DISCRETIZATION.

$$K_{11}^e = \left[\frac{13}{3}h - 12h + (e-1)h + (e-1)^2h - \frac{1}{h} + 12h\right] = \frac{13}{3}h - \frac{1}{h} + (e^2 - e)h$$

$$K_{12}^e = K_{21}^e = \int_0^h ((\xi + x_e)^2 - 1)(\frac{-1}{h^2}) + 12(\frac{\xi}{h})(1 - \frac{\xi}{h})\, d\xi$$

$$= \int_0^h \frac{-1}{h^2}(\xi^2 + 2x_e\xi + x_e^2 - 1) + 12\frac{\xi}{h} - 12\frac{\xi^2}{h^2}\, d\xi$$

$$= \left[\frac{1}{h^2}\frac{-13}{3}\xi^3 + \frac{1}{h^2}(6h - x_e)\xi^2 + \frac{1}{h^2}(1 - x_e^2)\xi\right]_0^h = \frac{5}{3}h + \frac{1}{h} - (e^2 - e)h$$

$$K_{22}^e = \int_0^h ((\xi + x_e)^2 - 1)\frac{1}{h^2} + 12\frac{\xi^2}{h^2}\, d\xi$$

$$= \int_0^h \frac{1}{h^2}(\xi^2 + 2x_e\xi + x_e^2 - 1) + 12\frac{\xi^2}{h^2}\, d\xi$$

$$= \left[\frac{1}{h^2}\frac{13}{3}\xi^3 + \frac{1}{h^2}x_e\xi^2 + \frac{1}{h^2}(x_e^2 - 1)\xi\right]_0^h = \frac{13}{3}h - \frac{1}{h} + (e^2 - e)h$$

AND, OF COURSE, $f_i^e = 0$.

THUS,

$$\mathbb{K}^e = \begin{bmatrix} \frac{13}{3}h - \frac{1}{h} + (e^2-e)h & \frac{5}{3}h + \frac{1}{h} - (e^2-e)h \\ \frac{5}{3}h + \frac{1}{h} - (e^2-e)h & \frac{13}{3}h - \frac{1}{h} + (e^2-e)h \end{bmatrix}, \qquad \underset{\sim}{f}^e = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

ASSEMBLY OF THE GLOBAL MATRIX GIVES

$$\mathbb{K} = \begin{bmatrix} \left(\frac{13}{3}h - \frac{1}{h}\right) & \left(\frac{5}{3}h + \frac{1}{h}\right) & 0 & 0 & \cdots & 0 \\ \left(\frac{5}{3}h + \frac{1}{h}\right) & \left(\frac{13}{3}h - \frac{1}{h}\right) + \left(\frac{13}{3}h - \frac{1}{h} + 2h\right) & \left(\frac{5}{3}h + \frac{1}{h} - 2h\right) & 0 & \cdots & 0 \\ 0 & \left(\frac{5}{3}h + \frac{1}{h} - 2h\right) & \ddots & & & \vdots \\ 0 & 0 & & \ddots & & \vdots \\ \vdots & \vdots & & & \left(\frac{13}{3}h - \frac{1}{h} + (N^2-5N+6)h\right) + \left(\frac{13}{3}h - \frac{1}{h} + (N^2-3N+2)h\right) & \left(\frac{5}{3}h + \frac{1}{h} - (N^2-3N+2)h\right) \\ 0 & 0 & & & \left(\frac{5}{3}h + \frac{1}{h} - (N^2-3N+2)h\right) & \left(\frac{13}{3}h - \frac{1}{h} + (N^2-3N+2)h\right) \end{bmatrix}$$

SINCE $u(0)=0$ AND $u(1)=1$, THIS PRESCRIBES α_1 AND α_N: $\alpha_1 = 0$, $\alpha_N = 1$

THIS MAKES THE FIRST AND LAST ROWS OF $\mathbb{K}$ REDUNDANT. SINCE $\alpha_1 = 0$, THE FIRST COLUMN BECOMES $\underset{\sim}{0}$. WITH $\alpha_N = 1$, HOWEVER, THE LAST COLUMN BECOMES AN EFFECTIVE FORCE, $\underset{\sim}{f}^* = \left\{0\ 0 \cdots 0\ -\left(\frac{5}{3}h + \frac{1}{h} - (N^2-3N+2)h\right)\right\}^T$

$$\begin{bmatrix} \left(\frac{32}{3}h - \frac{2}{h}\right) & \left(\frac{-1}{3}h + \frac{1}{h}\right) & 0 & & 0 \\ \left(\frac{-1}{3}h + \frac{1}{h}\right) & \left(\frac{50}{3}h - \frac{2}{h}\right) & \left(\frac{-13}{3}h + \frac{1}{h}\right) & & \vdots \\ 0 & \left(\frac{-13}{3}h + \frac{1}{h}\right) & \ddots & & \\ & & & \ddots & 0 \\ & & & & \left(\frac{5}{3}h + \frac{1}{h} - (N^2-5N+6)h\right) \\ 0 & \cdots & 0 & \left(\frac{5}{3}h + \frac{1}{h} - (N^2-5N+6)h\right) & \left(\frac{26}{3}h - \frac{2}{h} + (2N^2-8N+8)h\right) \end{bmatrix} \begin{Bmatrix} \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \vdots \\ \alpha_{N-2} \\ \alpha_{N-1} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ -5/3h - 1/h + (N^2-3N+2)h \end{Bmatrix}$$

THE SOLUTIONS & ERROR CALCULATIONS ARE DISCUSSED IN PART D.

C. Quadratic Elements

The basis functions are given locally as

$$\psi_1^e = \frac{2}{h^2}\left(\xi - \frac{h}{2}\right)\left(\xi - h\right), \qquad \psi_1^{e\prime} = \frac{4}{h^2}\xi - \frac{3}{h}$$

$$\psi_2^e = \frac{4}{h^2}(\xi)(\xi - h), \qquad \psi_2^{e\prime} = \frac{8}{h^2}\xi - \frac{4}{h}$$

$$\psi_3^e = \frac{2}{h^2}(\xi)\left(\xi - \frac{h}{2}\right), \qquad \psi_3^{e\prime} = \frac{4}{h^2}\xi - \frac{1}{h}$$

The local stiffness element terms are then:

$$K_{11}^e = \int_0^h \left((\xi + x_i)^2 - 1\right)\left(\frac{4}{h^2}\xi - \frac{3}{h}\right)^2 + 12\left(\frac{2}{h^2}\right)^2\left(\xi^2 - \frac{3h}{2}\xi + \frac{h^2}{2}\right)^2 d\xi$$

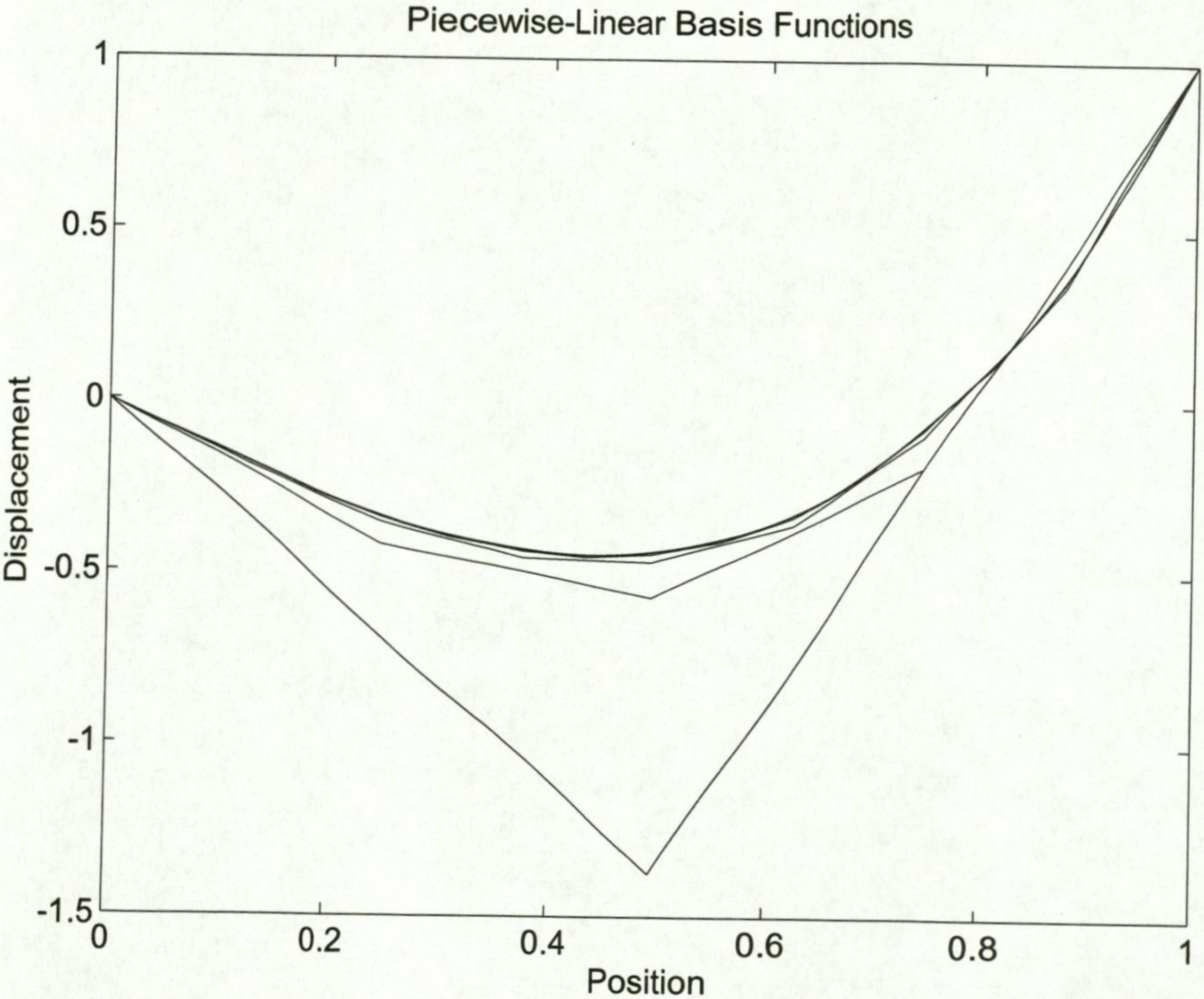
$$= \int_0^h \left(\xi^2 + 2x_i\xi + x_i^2 - 1\right)\left(\frac{16}{h^4}\xi^2 - \frac{12}{h^3}\xi - \frac{9}{h^2}\right) + \frac{48}{h^4}\left(\xi^4 - 3h\xi^3 + \frac{7h^2}{4}\xi^2 - \frac{3h^3}{2}\xi + \frac{h^4}{4}\right) d\xi$$

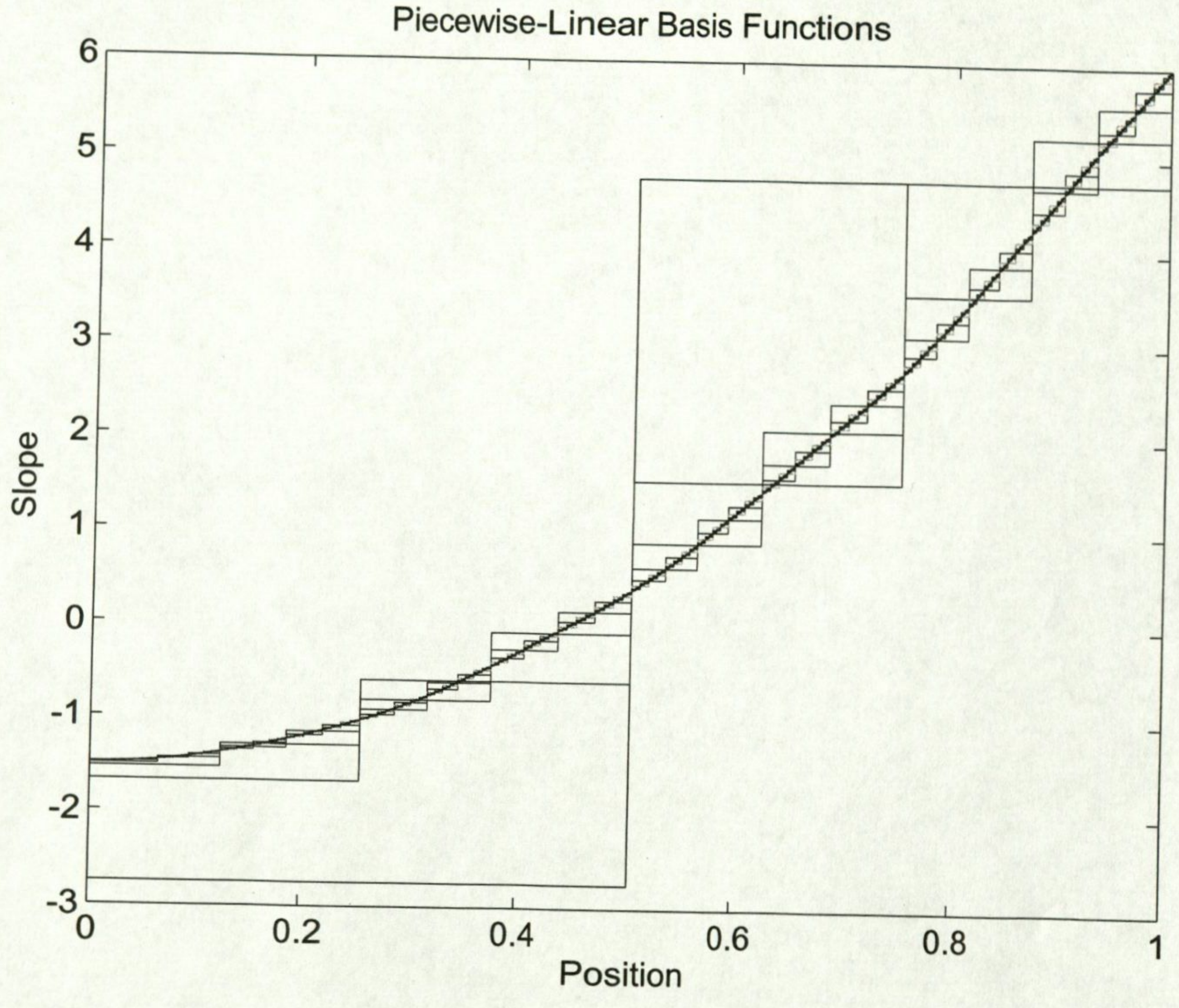
As these expressions become complicated quickly, with many opportunities to introduce errors, the $\mathbb{K}_{ij}^e$ terms are best found symbolically:

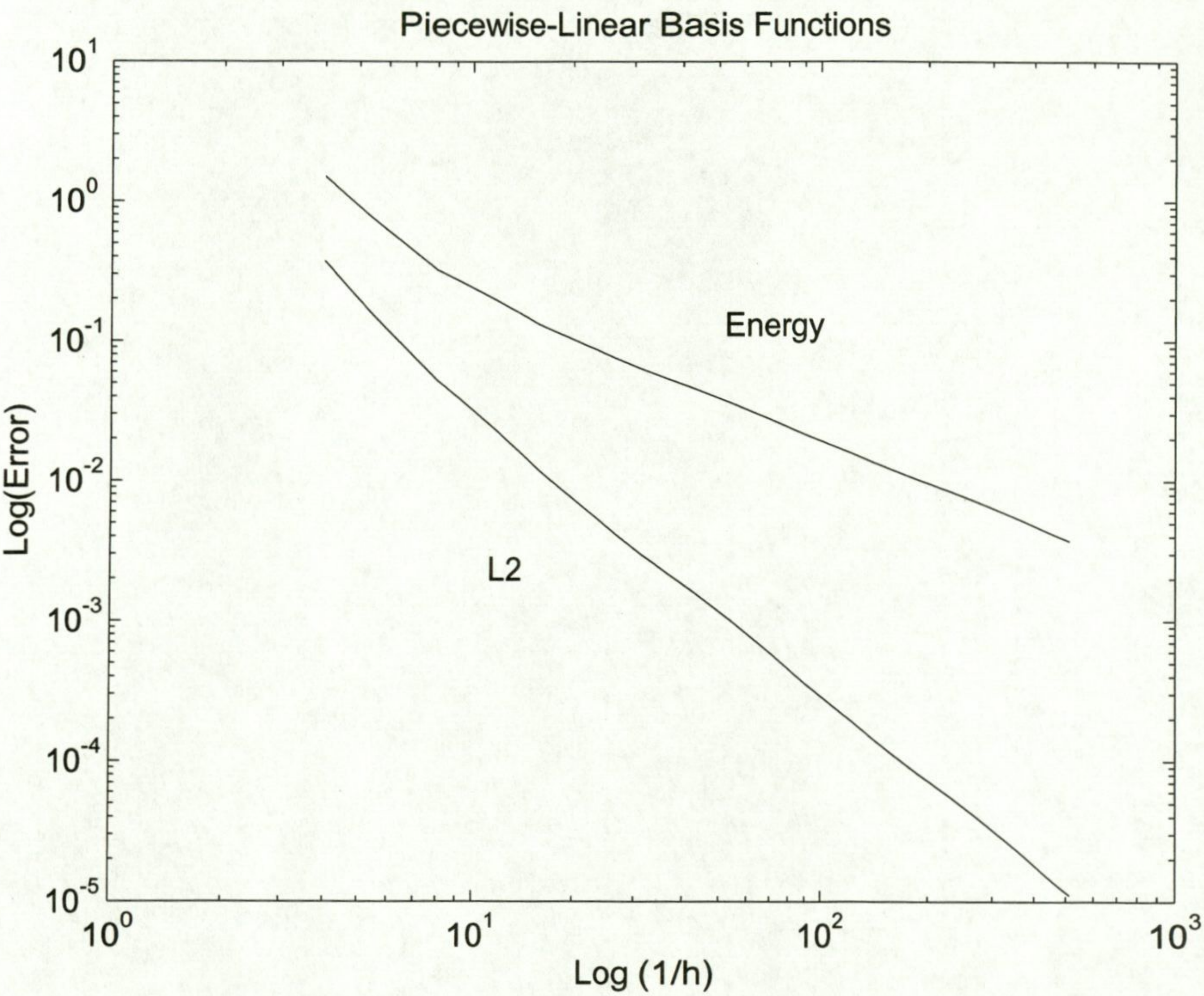
$$\mathbb{K}^e = \begin{bmatrix} \frac{9h}{5} + x_i + \frac{1}{h}\left(\frac{7x_i^2}{3} - \frac{7}{3}\right) & \frac{4x_i}{3} - \frac{2h}{5} + \frac{1}{h}\left(\frac{8x_i^2}{3} - \frac{8}{3}\right) & \frac{x_i}{3} - \frac{h}{5} + \frac{1}{h}\left(\frac{x_i^2}{3} - \frac{1}{3}\right) \\ \frac{4x_i}{3} - \frac{2h}{5} + \frac{1}{h}\left(\frac{8x^2}{3} - \frac{8}{3}\right) & \frac{128h}{15} + \frac{16x_i}{3} + \frac{1}{h}\left(\frac{16x_i^2}{3} - \frac{16}{3}\right) & \frac{14h}{15} + 4x_i + \frac{1}{h}\left(\frac{8x_i^2}{3} - \frac{8}{3}\right) \\ \frac{x_i}{3} - \frac{h}{5} + \frac{1}{h}\left(\frac{x_i^2}{3} - \frac{1}{3}\right) & \frac{14h}{15} + 4x_i + \frac{1}{h}\left(\frac{8x_i^2}{3} - \frac{8}{3}\right) & \frac{47h}{15} + \frac{11x_i}{3} + \frac{1}{h}\left(\frac{7x_i^2}{3} - \frac{7}{3}\right) \end{bmatrix}$$

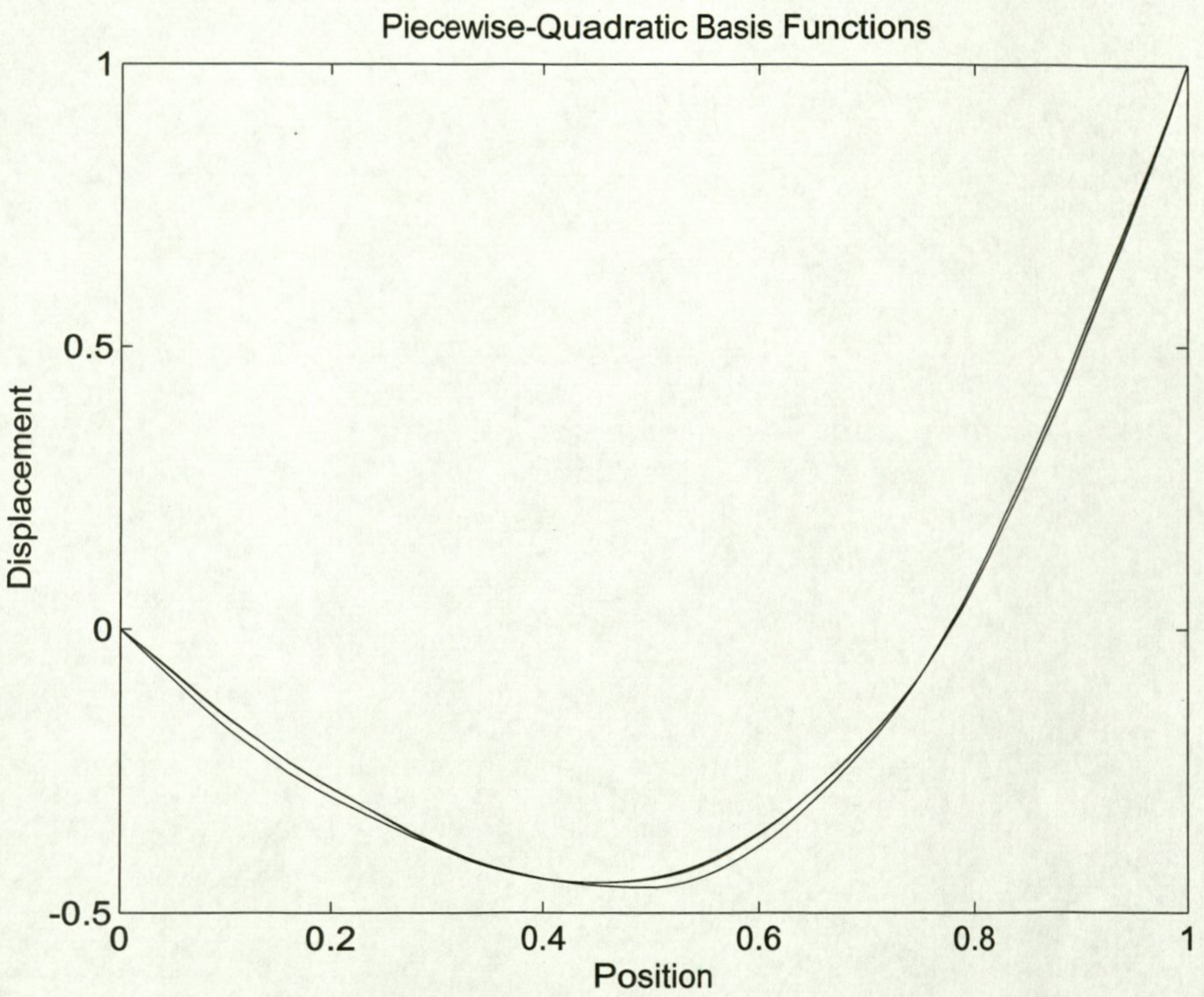
```
For piecewise-linear basis functions,
1/h       L2 norm     Energy norm
_______________________________________
  4     0.3708314      1.4953691
  8     0.0518883      0.3212293
 16     0.0116885      0.1303000
 32     0.0028518      0.0616107
 64     0.0007087      0.0303395
128     0.0001769      0.0150829
256     0.0000442      0.0075020
512     0.0000111      0.0037174

For piecewise-quadratic basis functions,
1/h       L2 norm     Energy norm
_______________________________________
  4     0.0129709      0.0841213
  8     0.0017719      0.0198927
 16     0.0002356      0.0049045
 32     0.0000304      0.0012219
 64     0.0000039      0.0003052
128     0.0000005      0.0000763
256     0.0000001      0.0000191
512     0.0000000      0.0000048
>>
```

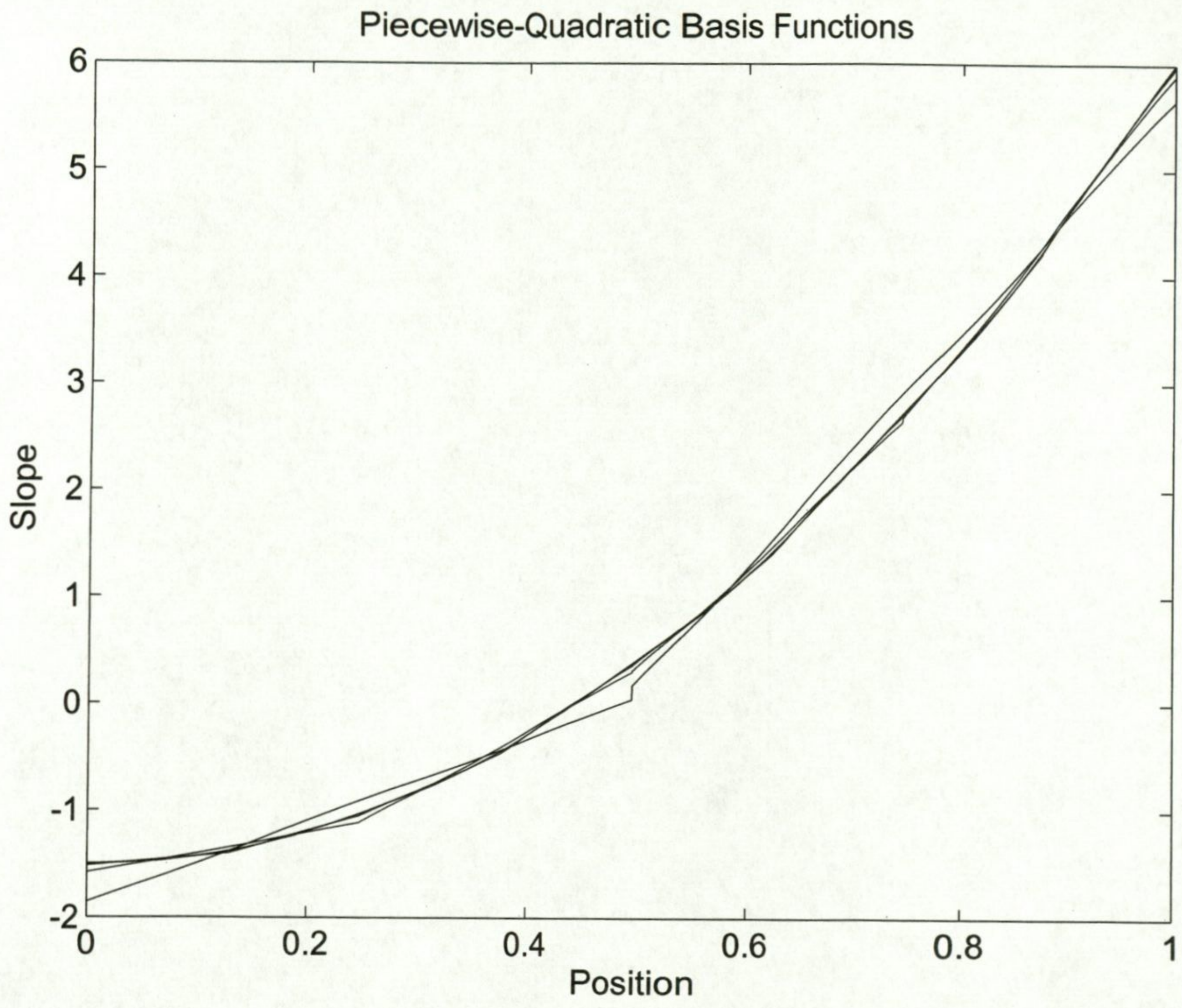

Piecewise-Linear Basis Functions
Displacement
Position
1
0.5
0
-0.5
-1
-1.5
0
0.2
0.4
0.6
0.8
1


Piecewise-Linear Basis Functions
Slope
Position
6
5
4
3
2
1
0
-1
-2
-3
0
0.2
0.4
0.6
0.8
1

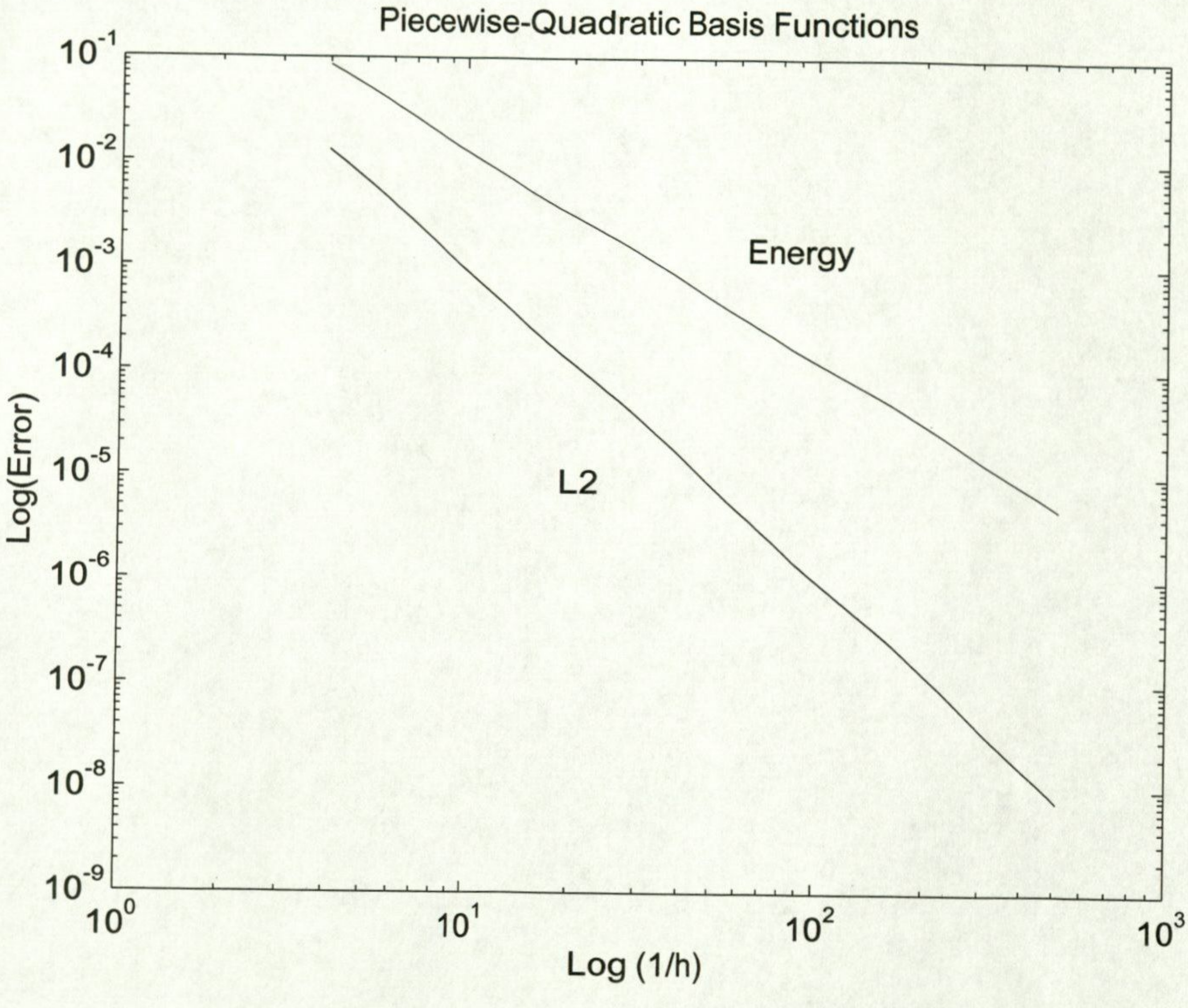

Piecewise-Linear Basis Functions
Log(Error)
Log (1/h)
Energy
L2
10^{1}
10^{0}
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{0}
10^{1}
10^{2}
10^{3}



Piecewise-Quadratic Basis Functions
Displacement
Position
1
0.5
0
-0.5
0
0.2
0.4
0.6
0.8
1



Piecewise-Quadratic Basis Functions
Slope
Position
6
5
4
3
2
1
0
-1
-2
0
0.2
0.4
0.6
0.8
1



Piecewise-Quadratic Basis Functions
Energy
L2
Log(Error)
Log (1/h)
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
10^{-7}
10^{-8}
10^{-9}
10^{0}
10^{1}
10^{2}
10^{3}

D. DISCUSSION

FIRST, THE ERROR EQUATIONS FOR BOTH METHODS:

$$\|e\|_{L2} = \|\hat{U}_{2N} - \hat{U}_N\|_{L2} = \sqrt{\int_0^L (\hat{U}_{2N} - \hat{U}_N)\, dx}$$

$$\|e\|_E = \|\hat{U}_{2N} - \hat{U}_N\|_E = \sqrt{\tfrac{1}{2}\int_0^L (1-x^2)(\hat{U}_{2N}' - \hat{U}_N') + 12(\hat{U}_{2N} - \hat{U}_N)\, dx}$$

IN YOUR DISCUSSION OF THE ERROR, YOU SHOULD NOTE THAT $\|e\|_{L2}$ SCALES WITH h^{k+1} AND $\|e\|_E$ WITH h^k, WHERE k IS THE ORDER OF THE BASIS FUNCTION USED. ON A LOG-LOG PLOT, THIS MEANS THAT $\|e\|_{L2}$ IS A STRAIGHT LINE OF SLOPE 2 FOR LINEAR FUNCTIONS & 3 FOR QUADRATIC FUNCTIONS WHILE $\|e\|_E$ HAS SLOPE 1 & 2 FOR THE SAME BASIS FUNCTIONS RESPECTIVELY.

ALSO NOTE THE CONVERGENCE OF THE GRAPHED SOLUTION. QUADRATIC FUNCTIONS VISIBLY CONVERGE WITH JUST 4 ELEMENTS, WHILE LINEAR FUNCTIONS NEED AT LEAST 16!

```
close all
clear all
clc

% Define the number of elements to use:
Ns = [2 4 8 16 32 64 128 256 512] ;

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%
%%   Piecewise-linear elements
%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

% From my analytical derivation, the elemental stiffness matrix is:
Ke = @(e,h)  [13*h/3-1/h+(e^2-e)*h 5/3*h+1/h-(e^2-e)*h;
              5/3*h+1/h-(e^2-e)*h  13/3*h-1/h+(e^2-e)*h] ;

% Setup figures for plotting
figure(1)
clf
set(gca,'FontSize',14)
set(gcf,'color','white')
box on
xlabel('Position','FontSize',14)
ylabel('Displacement','FontSize',14)
title('Piecewise-Linear Basis Functions','FontSize',14)

figure(2)
clf
set(gca,'FontSize',14)
set(gcf,'color','white')
box on
xlabel('Position','FontSize',14)
ylabel('Slope','FontSize',14)
title('Piecewise-Linear Basis Functions','FontSize',14)

errE = zeros(length(Ns)-1,1) ;
errL = zeros(length(Ns)-1,1) ;
for cntN = 1:length(Ns)
  N = Ns(cntN) ;        % Number of elements
  h = 1/N ;             % Element size
  K = zeros(N+1,N+1) ; % Setup the global stiffness matrix
  for cnt = 1:N         % The assembly procedure:
    K(cnt:cnt+1,cnt:cnt+1) = Ke(cnt,h)+K(cnt:cnt+1,cnt:cnt+1) ;
  end
  % Apply BC
  % u(0) = 0
  % u(1) = 1
  K = K(2:end,2:end) ;
  f = -1*K(1:end-1,end) ;
  K = K(1:end-1,1:end-1) ;
```

```
% Plot the errors
figure(3)
clf
set(gca,'FontSize',14)
set(gcf,'color','white')
box on
plot(NS,errL,'b')
hold on
plot(NS,errE,'r')
set(gca,'XScale','log','YScale','log')
xlabel('Log (1/h)','FontSize',14)
ylabel('Log(Error)','FontSize',14)
title('Piecewise-Linear Basis Functions','FontSize',14)


%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%
%%   Piecewise-quadratic elements
%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

% Use Matlab to determine the elemental stiffness matrix for the
% piecewise-quadratic basis functions:

% syms xi e h
% % x_i = (e-1)*h
% psi1 = 2/h^2*(xi-h/2)*(xi-h);
% psi2 = 4/h^2*xi*(xi-h);
% psi3 = 2/h^2*xi*(xi-h/2);
% dpsi1 = 4/h^2*xi-3/h;
% dpsi2 = 8/h^2*xi-4/h;
% dpsi3 = 4/h^2*xi-1/h;
% K11 = int(((xi+(e-1)*h)^2-1)*dpsi1*dpsi1+12*psi1*psi1,xi,0,h);
% K12 = int(((xi+(e-1)*h)^2-1)*dpsi1*dpsi2+12*psi1*psi2,xi,0,h);
% K13 = int(((xi+(e-1)*h)^2-1)*dpsi1*dpsi3+12*psi1*psi3,xi,0,h);
% K22 = int(((xi+(e-1)*h)^2-1)*dpsi2*dpsi2+12*psi2*psi2,xi,0,h);
% K23 = int(((xi+(e-1)*h)^2-1)*dpsi2*dpsi3+12*psi2*psi3,xi,0,h);
% K33 = int(((xi+(e-1)*h)^2-1)*dpsi3*dpsi3+12*psi3*psi3,xi,0,h);
% K = [K11 K12 K13; K12 K22 K23; K13 K23 K33] ;

% The above K yields the local stiffness matrix:
Ke = @(e,h) [h*((7*e^2)/3-(11*e)/3+47/15)-7/(3*h), ...
                            h*((8*e^2)/3-4*e+14/15)-8/(3*h), ...
                                          -h*(-e^2/3+e/3+1/5)-1/(3*h);
             h*((8*e^2)/3-4*e+14/15)-8/(3*h), ...
                            h*((16*e^2)/3-(16*e)/3+128/15)-16/(3*h), ...
                                     -h*(-(8*e^2)/3+(4*e)/3+2/5)-8/(3*h);
             -h*(-e^2/3+e/3+1/5)-1/(3*h), ...
                            -h*(-(8*e^2)/3+(4*e)/3+2/5)-8/(3*h), ...
                                          h*((7*e^2)/3-e+9/5)-7/(3*h)] ;
```

```
    idx = ceil(x(cntr)/h) ; % Determine the element that x(cntr) is in
    % As a reminder to myself:
    % psi1 = 2/h^2*(xi-h/2)*(xi-h);
    % psi2 = 4/h^2*xi*(xi-h);
    % psi3 = 2/h^2*xi*(xi-h/2);
   y(cntr) = alpha(2*(idx-1)+1)*2/h^2*(x(cntr)-(idx-1)*h-h/2) ...
                                               *(x(cntr)-(idx-1)*h-h) ...
    +alpha(2*(idx-1)+2)*4/h^2*(x(cntr)-(idx-1)*h)*(x(cntr)-(idx-1)*h-h) ...
    +alpha(2*(idx-1)+3)*2/h^2*(x(cntr)-(idx-1)*h)*(x(cntr)-(idx-1)*h-h/2) ;
  end
  y(end) = alpha(end) ;

  % Plot the displacements
  figure(4)
  hold all
  plot(x,y)
  % Plot the slopes
  figure(5)
  hold all
  % The lazy but effective way of calculating slopes:
  v = diff(y)/x(2) ;
  plot(x(1:end-1)+x(2)/2,v)
  % Calculate error norms
  if cntN > 1
    errL(cntN-1) = sqrt(sum((y-yold).^2)*x(2)) ;
    errE(cntN-1) = sqrt(1/2*(sum((1-(x(1:end-1)+x(2)/2).^2).* ...
               ((diff(y)-diff(yold))./x(2)).^2')+12*sum((y-yold).^2))*x(2)) ;
  end
  yold = y ;
end

% Print error results
fprintf('\nFor piecewise-quadratic basis functions,\n')
NS = zeros(length(Ns)-1,1) ;
fprintf('1/h        L2 norm       Energy norm\n')
fprintf('________________________________\n')
for cntr = 1:length(Ns)-1
  NS(cntr) = Ns(cntr+1) ;
  fprintf('%3d %12.7f %14.7f',NS(cntr),errL(cntr),errE(cntr)) ;
  fprintf('\n')
end

% Plot errors
figure(6)
clf
set(gca,'FontSize',14)
set(gcf,'color','white')
box on
plot(NS,errL,'b')
hold on
plot(NS,errE,'r')
```

HOMEWORK #1 SOLUTION SET

COMPUTATIONAL MECHANIC
ME 404/504

1.A. $$I(y) = \int_1^{32} x^2 (y'(x))^6 \, dx$$

$$y(1) = 1, \quad y(32) = 2$$

EULER-LAGRANGE:

$$\frac{d}{dx}\frac{\partial f}{\partial y'} - \frac{\partial f}{\partial y} = 0$$

$$f(y, y'; x) = x^2 (y'(x))^6$$

PART a) DERIVATIVES

$$\frac{\partial f}{\partial y} = 0$$

$$\frac{\partial f}{\partial y'} = 6x^2 (y')^5$$

$\frac{d}{dx}\frac{\partial f}{\partial y'}$ NOT NECESSARY SINCE $\frac{\partial f}{\partial y} = 0$.

PART b) EULER-LAGRANGE

$$\frac{d}{dx}\left(6x^2 (y')^5\right) = 0$$

INTEGRATION GIVES

$x^2 (y')^5 = \alpha$, WHERE α IS AN UNKNOWN CONSTANT

$$y'(x) = \frac{\beta}{x^{2/5}} \qquad (\beta = \alpha^{1/5})$$

$$y(x) = c_1 x^{3/5} + c_2$$

PART c) BOUNDARY CONDITIONS

THE UNKNOWN CONSTANTS c_1 & c_2 ARE NOW DETERMINED VIA

$$y(1) = 1 = c_1 + c_2 \Rightarrow c_2 = 1 - c_1$$

$$y(32) = 2 = 8c_1 + c_2 \Rightarrow c_1 = \frac{1}{7}, \quad c_2 = \frac{6}{7}$$

$$y(x) = \frac{1}{7} x^{3/5} + \frac{6}{7}$$

1.B. $I(y) = \int_0^{\pi} \left((y(x))^2 - (y'(x))^2\right) dx$

$$y(0) = 0, \quad y(\pi) = 0$$

$$f(y, y', x) = y^2 - y'^2$$

PART a) DERIVATIVES

$$\frac{\partial f}{\partial y} = 2y$$

$$\frac{\partial f}{\partial y'} = -2y'$$

$$\frac{d}{dx}\frac{\partial f}{\partial y'} = -2y''$$

PART b) EULER-LAGRANGE

$$\frac{d}{dx}\frac{\partial f}{\partial y'} - \frac{\partial f}{\partial y} = 0$$

$$-2y'' - 2y = 0$$

$$y(x) = A\sin(x) + B\cos(x)$$

PART c) BOUNDARY CONDITIONS

$y(0) = 0 \Rightarrow B = 0$

$y(\pi) = 0 \Rightarrow A$ IS ARBITRARY

$y(x) = A\sin(x)$

NOTE, y IS A MAXIMIZER OF $I(y)$. THERE IS NO TRUE MINIMIZER OF $I(y)$. TO SHOW THIS, CONSIDER

$y_*(x, \gamma) = \sin(\gamma x)$

$y_*(x, 1)$, THE MAXIMIZER, GIVES $I(y_*(x,1)) = 0$

$y_*(x, 100)$, BY CONTRAST, GIVES $I(y_*(x,100)) = -1.571 \cdot 10^4$

THUS, THERE IS NO TRUE MINIMIZER AS γ CAN ALWAYS BE CHOSEN TO FIND A LOWER VALUE OF $I(y_*(x,\gamma))$.

PART b) ORTHOGONALITY TEST

THE NATURAL FREQUENCIES ARE FOUND VIA

$$\det(\mathbb{K}-\omega^2\mathbb{M})=0$$

$$\det\begin{bmatrix} 2g/L-\omega^2 & -g/L \\ -\omega^2 & g/L-\omega^2 \end{bmatrix}=0$$

$$(2g/L-\omega^2)(g/L-\omega^2)-g/L\,\omega^2=0$$

$$2(g/L)^2-4g/L\,\omega^2+\omega^4=0$$

$$\omega^2=2g/L\pm\sqrt{4(g/L)^2-2(g/L)^2}$$

$$\omega_1=0.7654\sqrt{g/L}\,,\quad \omega_2=1.8478\sqrt{g/L}$$

WITH CORRESPONDING MODE SHAPES

$$\left[\mathbb{K}-\omega^2\mathbb{M}\right]\begin{Bmatrix}U_1\\U_2\end{Bmatrix}=\begin{Bmatrix}0\\0\end{Bmatrix}$$

$$(2g/L-\omega^2)U_1=g/L\,U_2$$

Mode 1: $\underset{\sim}{U}_1=\begin{Bmatrix}0.7071\\1\end{Bmatrix}$ Mode 2: $U_2=\begin{Bmatrix}-0.7071\\1\end{Bmatrix}$

ORTHOGONALITY CAN BE CHECKED VIA

$$\underset{\sim}{U}_i^T\,\mathbb{M}\,\underset{\sim}{U}_j=c\delta_{ij}$$

$$\{0.7071\quad 1\}\,Mg\begin{bmatrix}2 & -1\\0 & 1\end{bmatrix}\begin{Bmatrix}-0.7071\\1\end{Bmatrix}\overset{?}{=}0$$

$$\begin{Bmatrix} 0.7071 & 1 \end{Bmatrix} \begin{Bmatrix} -2.4142 \\ 1 \end{Bmatrix} = -0.7071 \neq 0$$

THUS THE MODE SHAPES ARE NOT ORTHOGONAL

PART c) POTENTIAL & KINETIC ENERGY, & NON-CONSERVATIVE FORCES

POTENTIAL ENERGY

$$V = Mgh_1 + Mgh_2$$

$$V = MgL(1-\cos\theta_1) + MgL(2-\cos\theta_1 - \cos\theta_2)$$

KINETIC ENERGY

$$T = \tfrac{1}{2}(ML\dot{\theta})^2 + \tfrac{1}{2}M(L\dot{\theta}_1 + L\dot{\theta}_2)^2$$

$$T = \tfrac{1}{2}\cdot 2ML^2\dot{\theta}_1^2 + ML^2\dot{\theta}_1\dot{\theta}_2 + \tfrac{1}{2}ML^2\dot{\theta}_2^2$$

NON-CONSERVATIVE FORCES

$$Q_{NC} = 0$$

NOTE THAT V CONTAINS NONLINEAR TERMS, LET US USE A SECOND ORDER TAYLOR SERIES, $\cos(q) = 1 - \frac{q^2}{2}$.

$$V = MgL\theta_1^2 + \tfrac{1}{2}MgL\theta_2^2$$

PART d) LAGRANGE'S EQUATION

$$L = T - V$$

$$L = \tfrac{1}{2}\cdot 2ML^2\dot{\theta}_1^2 + \tfrac{1}{2}ML^2\dot{\theta}_1\dot{\theta}_2 + \tfrac{1}{2}ML^2\dot{\theta}_2^2 - MgL\theta_1^2 - \tfrac{1}{2}MgL\theta_2^2$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = Q_{NC_i}$$

$i=1$: $2ML^2\ddot{\theta}_1 + ML^2\ddot{\theta}_2 + 2MgL\theta_1 = 0$

$i=2$: $ML^2\ddot{\theta}_1 + ML^2\ddot{\theta}_1 + MgL\theta_2 = 0$

IN MATRIX FORM

$$ML^2\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{Bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{Bmatrix} + MgL\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

PART e) ORTHOGONALITY TEST

AS BEFORE, THE NATURAL FREQUENCIES ARE FOUND VIA

$$\det\left[\mathbb{K} - \omega^2 \mathbb{M}\right] = 0$$

THIS YIELDS

$$\omega_1 = 0.7654\sqrt{g/L}, \quad \omega_2 = 1.8478\sqrt{g/L},$$

THE SAME AS IN THE NEWTONIAN FRAMEWORK.

THE MODE SHAPES:

$$\left[\mathbb{K} - \omega^2 \mathbb{M}\right]\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$(2gL - \omega^2 L^2)\, u_1 = \omega^2 L\, u_2$$

$$\underset{\sim}{u}_1 = \begin{Bmatrix} 0.7071 \\ 1 \end{Bmatrix} \qquad \underset{\sim}{u}_2 = \begin{Bmatrix} -0.7071 \\ 1 \end{Bmatrix}$$

$$\underset{\sim}{u}_i^T \mathbb{M}\, \underset{\sim}{u}_j = c\,\delta_{ij}$$

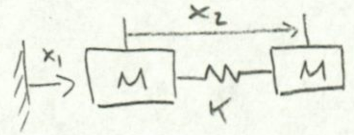
$$\begin{bmatrix} \underset{\sim}{u}_1^T \mathbb{M}\, \underset{\sim}{u}_1 & \underset{\sim}{u}_2^T \mathbb{M}\, \underset{\sim}{u}_1 \\ \underset{\sim}{u}_1^T \mathbb{M}\, \underset{\sim}{u}_2 & \underset{\sim}{u}_2^T \mathbb{M}\, \underset{\sim}{u}_2 \end{bmatrix} = \begin{bmatrix} 3.4142 & 0 \\ 0 & 0.5858 \end{bmatrix} \checkmark$$

PART f) DISCUSS ORTHOGONALITY

IN PROBLEMS SUCH AS VIBRATION THAT LEND THEMSELVES TO BEING SOLVED VIA SUPERPOSITION, ORTHOGONALITY ALLOWS US TO ISOLATE EACH MODE SHAPE IN ORDER TO DEVELOP A MODAL EQUATION OF MOTION:

$$\ddot{\eta}_j + 2\zeta\omega\dot{\eta}_j + \omega^2\eta_j = N_j$$

THIS IS ALSO OBSERVED IN FIELDS SUCH AS FLUID MECHANICS (PROPER ORTHOGONAL DECOMPOSITION & GALERKIN METHODS). THUS, IT HAS FAR REACHING APPLICATIONS/USES.

2. 

2A. CLASSIFY THE FORCES

THERE IS ONLY ONE FORCE, THE SPRING FORCE BETWEEN THE TWO MASSES. THIS IS A CONSERVATIVE/ACTIVE FORCE, AND WILL CONTRIBUTE TO THE POTENTIAL ENERGY, V.

2B. POTENTIAL & KINETIC ENERGY, & NON-CONSERVATIVE FORCES.

POTENTIAL: $V = \frac{1}{2} K x_2^2$

KINETIC: $T = \frac{1}{2} M_1 \dot{x}_1^2 + \frac{1}{2} M_2 (\dot{x}_1 + \dot{x}_2)^2$

$$T = \frac{1}{2}(M_1 + M_2)\dot{x}_1^2 + \frac{1}{2}(2M_2)\dot{x}_1\dot{x}_2 + \frac{1}{2} M_2 \dot{x}_2^2$$

NON-CONSERVATIVE: $Q_{NC} = 0$

2C. LAGRANGE'S EQUATION

$$L = T - V$$

$$L = \frac{1}{2}(M_1 + M_2)\dot{x}_1^2 + \frac{1}{2}(2M_2)\dot{x}_1\dot{x}_2 + \frac{1}{2} M_2 \dot{x}_2^2 - \frac{1}{2} K x_2^2$$

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{x}_i}\right) - \frac{\partial L}{\partial x_i} = Q_{NC}$$

$i=1$: $(M_1 + M_2)\ddot{x}_1 + M_2 \ddot{x}_2 = 0$

$i=2$: $M_2 \ddot{x}_1 + M_2 \ddot{x}_2 + K x_2 = 0$

2D. MATRIX FORM

$$\begin{bmatrix} M_1 + M_2 & M_2 \\ M_2 & M_2 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & K \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

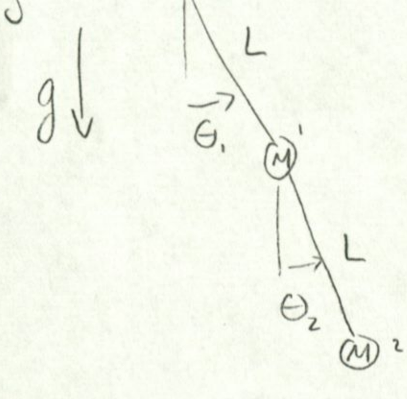
OBSERVE, THIS CAN BE REDUCED FURTHER USING THE FIRST EQUATION

$$\ddot{x}_1 = -\frac{M_2}{M_1 + M_2}\ddot{x}_2$$

THIS YIELDS

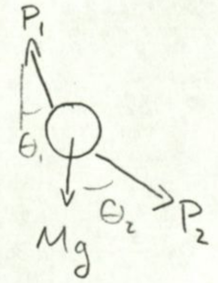
$$\frac{M_1 M_2}{M_1 + M_2}\ddot{x}_2 + K x_2 = 0 \qquad \text{OR} \qquad \ddot{x}_2 + \omega^2 x_2 = 0$$

WITH $\mu = \frac{M_1 M_2}{M_1 + M_2}$, AND $\omega^2 = \frac{K}{\mu}$.



Part a) Newtonian Approach

Free Body Diagrams:

1: 

2: 

$$\sum M_1 = J\ddot{\theta}_2 + M\underset{\sim}{a}d \qquad \sum F_{x_1} = ML\ddot{\theta}_1\cos\theta_1 \quad X$$

$$\underbrace{ML^2\ddot{\theta}_2}_{J\ddot{\theta}_2} + \underbrace{ML\ddot{\theta}_1\cdot L}_{M\underset{\sim}{a}d} = \underbrace{-MgL\sin\theta_2}_{\sum M_1} \qquad \cancel{ML\ddot{\theta}_1\cos\theta_1 = -P_1\sin\theta_1 + P_2\sin\theta_2}$$

$$ML^2(\ddot{\theta}_1 + \ddot{\theta}_2) + MgL\sin\theta_2 = 0 \qquad ML\ddot{\theta}_1\cos\theta_1 + P_1\sin\theta_1 - P_2\sin\theta_2 = 0$$

These coupled nonlinear equations are linearized via

$$\sin(q) = q, \qquad \cos(q) = 1$$

$$ML^2\ddot{\theta}_1 + ML^2\ddot{\theta}_2 + MgL\theta_2 = 0$$
$$ML\ddot{\theta}_1 + P_1\theta_1 - P_2\theta_2 = 0$$

We can deduce that $P_1 = 2Mg$ & $P_2 = Mg$, yielding

$$ML\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{Bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{Bmatrix} + Mg\begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix}\begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$