# 4. POSITION VECTORS, RECTANGULAR CARTESIAN COORDINATES AND OTHER SPECIAL CURVILINEAR COORDINATE SYSTEMS

## 4.1 Initial Comments

In this chapter we introduce the concept of a Euclidean point space, again with no reference to a coordinate system. Then we introduce the conventional rectangular coordinate system and show that this is alternative way to define points in space. Finally, two other common coordinate systems are presented partly to illustrate the complexities that they exhibit.

## 4.2 Euclidean Point Space

A Euclidean point space is identified with two important assertions: (i) a reference point "O" can be identified, and (ii) any orthonormal basis can be considered the same for all points. The particular vector that relates a generic point "P" to the reference point is called the position vector

$$\boldsymbol{r}^P = r_i^P \boldsymbol{e}_i \tag{4-1}$$

as illustrated in Fig. 4-1(a). This vector is treated as any other vector. Frequently the superscript "P" is dropped and if an orthonormal basis is used, the components are labeled $x_i$ so that alternative forms for (4-1) are

$$\boldsymbol{r} = r_i \boldsymbol{e}_i = x_i \boldsymbol{e}_i \tag{4-2}$$

If two points, "A" and "B" are related through a vector $\boldsymbol{\rho}^{A-B}$, the conventional form for vector addition yields

$$\boldsymbol{r}^B = \boldsymbol{r}^A + \boldsymbol{\rho}^{A-B} \tag{4-3}$$

as indicated in Fig. 4-1(b).

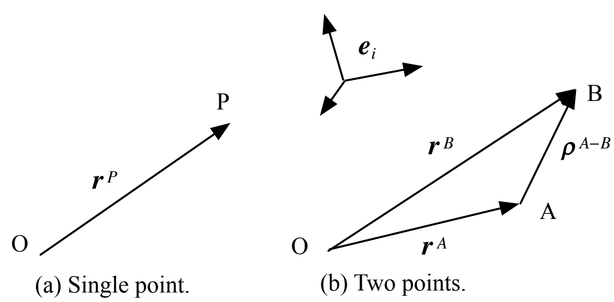



Fig. 4-1. Position vectors in a Eucllidean point space.

There are spaces that are not Euclidean such as the surface of a sphere so the assumptions that a single reference point exists and that the basis can be considered the same everywhere are worth emphasizing.

## 4.3 Rectangular Cartesian Coordinate System

We label the familiar rectangular Cartesian coordinates $x_i$ as sketched in Fig. 4-2. We note immediately that with such coordinates, a reference point "O", the origin, is automatically assumed and the coordinates identify directions. The adjective "rectangular" implies that any one coordinate line is orthogonal to the other two, and the adjective "Cartesian" implies that coordinate values are a measure of distance from the origin. The coordinates of a generic point "P" are labeled $x_i^P$. Typically the coordinates of a generic point are just given as "$x_i$". We will follow this convention even though the notation is self-contradictory in that we are using $x_i$ to denote two different items, the coordinate axes in one sense, and the coordinates of a point in the other.

Frequently unit vectors are identified with each coordinate axis (the old ***i, j, k*** approach) such as $\boldsymbol{e}_i$ shown in Fig. 4-2. But the coordinate axes are already identifying directions so in this sense the introduction of a related basis is not introducing new information. The real reason for introducing a physical basis is that it makes the representation of vectors and transformation relations so much easier than if only coordinate systems are used. This point was emphasized by the approach used in the previous chapter where base vectors were introduced without coordinates

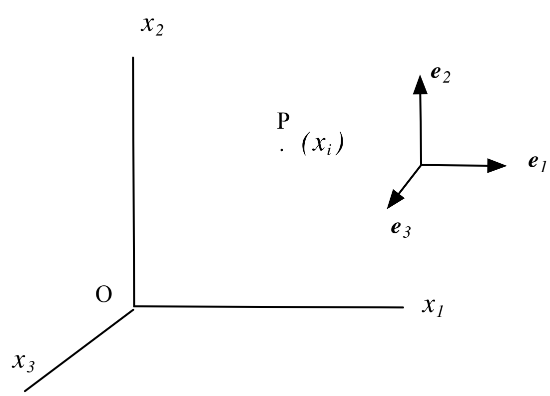



Fig. 4-2. Rectangular Cartesian coordinates with associated base vectors.

## 4.4 General Coordinate Systems

One of the most valuable advantages of using a particular coordinate system is to be able to specify the boundary of a body my merely saying the value of one coordinate is constant. Therefore we now give a brief introduction for how one might construct appropriate base vectors for a particular coordinate system under the assumption that the space is Euclidean, i.e., we can assume the same orthonormal basis exists at all points and a position vector in this system is available, or a rectangular Cartesian system can be used as a starting point.

Let the coordinates in our alternative system be labeled $y_i$ and assume that these coordinates can be expressed in terms of $x_i$, and *vice versa*, or

$$y_i = f_i(x_1, x_2, x_3) \quad \text{and} \quad x_i = f_i^*(y_1, y_2, y_3) \tag{4-4}$$

in which $f_i$ and $f_i^*$ are functions of the specified variables. Recall that the position vector is

$$\boldsymbol{r} = x_i \boldsymbol{e}_i = g_i(y_1, y_2, y_3)\boldsymbol{e}_i \tag{4-5}$$

The base vectors associated with the coordinate system $y_i$ are defined to be

$$\boldsymbol{g}_i = \frac{\partial \boldsymbol{r}}{\partial y_i} = \frac{\partial f_j^*}{\partial y_i}\boldsymbol{e}_j = \frac{\partial f_1^*}{\partial y_i}\boldsymbol{e}_1 + \frac{\partial f_2^*}{\partial y_i}\boldsymbol{e}_2 + \frac{\partial f_3^*}{\partial y_i}\boldsymbol{e}_3 \tag{4-6}$$

Therefore we have the rows necessary to construct the transformation matrix

$$\left[\overset{g-e}{a}\right] = \begin{bmatrix} \dfrac{\partial f_1^*}{\partial y_1} & \dfrac{\partial f_2^*}{\partial y_1} & \dfrac{\partial f_3^*}{\partial y_1} \\ \dfrac{\partial f_1^*}{\partial y_2} & \dfrac{\partial f_2^*}{\partial y_2} & \dfrac{\partial f_3^*}{\partial y_2} \\ \dfrac{\partial f_1^*}{\partial y_3} & \dfrac{\partial f_2^*}{\partial y_3} & \dfrac{\partial f_3^*}{\partial y_3} \end{bmatrix} \tag{4-7}$$

Since the base vectors $\boldsymbol{g}_i$ are not orthonormal in general, the matrix of (4-7) must be inverted so that $\boldsymbol{e}_i$ can be expressed in terms of $\boldsymbol{g}_i$.

Often a general coordinate system will be orthogonal in which case the basis $\boldsymbol{g}_i$ will also be orthogonal but not normal. In this case unit vectors are obtained by dividing by the magnitudes, $h_i$, to obtain what are called "physical" base vectors

$$\begin{aligned} &\underline{\boldsymbol{g}}_1 = \frac{\boldsymbol{g}_1}{h_1} && \underline{\boldsymbol{g}}_2 = \frac{\boldsymbol{g}_2}{h_2} && \underline{\boldsymbol{g}}_3 = \frac{\boldsymbol{g}_3}{h_3} \\ &h_1 = \sqrt{\boldsymbol{g}_1 \cdot \boldsymbol{g}_1} && h_2 = \sqrt{\boldsymbol{g}_2 \cdot \boldsymbol{g}_2} && h_3 = \sqrt{\boldsymbol{g}_3 \cdot \boldsymbol{g}_3} \end{aligned} \tag{4-8}$$

Now the two bases, $\boldsymbol{e}_i$ and $\underline{\boldsymbol{g}}_i$ are orthonormal and the transformation matrix will be orthogonal. Note that conventional indicial notation could not be used for the equations in (4-8).

### 4.5 Cylindrical Coordinate System

Here we illustrate the equations of the previous subsection with a coordinate system that most of us are familiar with, namely cylindrical coordinates. For the sake of convenience, let us use the notation

$$x_i \Rightarrow (x, y, z) \qquad y_i \Rightarrow (r, \theta, z) \tag{4-9}$$

The functions $f_i$ and $f_i^*$ are

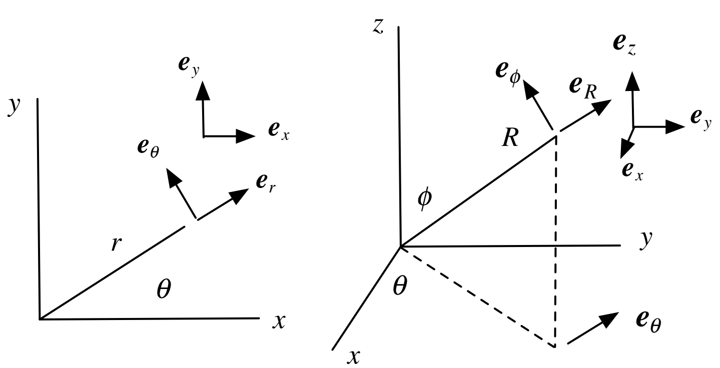
$$r = f_1 = \sqrt{x^2 + y^2} \qquad x = f_1^* = r\cos\theta$$
$$\theta = f_2 = tan^{-1}\left(\frac{y}{x}\right) \qquad y = f_2^* = r\sin\theta \tag{4-10}$$
$$z = f_3 = z \qquad z = f_3^* = z$$

Now we use (4-6) to obtain

$$\begin{aligned}
\boldsymbol{g}_r &= \frac{\partial f_1^*}{\partial r}\boldsymbol{e}_1 + \frac{\partial f_2^*}{\partial r}\boldsymbol{e}_2 + \frac{\partial f_3^*}{\partial r}\boldsymbol{e}_3 = cos\theta\boldsymbol{e}_x + sin\theta\boldsymbol{e}_y \\
\boldsymbol{g}_\theta &= \frac{\partial f_1^*}{\partial \theta}\boldsymbol{e}_1 + \frac{\partial f_2^*}{\partial \theta}\boldsymbol{e}_2 + \frac{\partial f_3^*}{\partial \theta}\boldsymbol{e}_3 = -r\,sin\theta\boldsymbol{e}_x + r\,cos\theta\boldsymbol{e}_y \\
\boldsymbol{g}_z &= \frac{\partial f_1^*}{\partial z}\boldsymbol{e}_1 + \frac{\partial f_2^*}{\partial z}\boldsymbol{e}_2 + \frac{\partial f_3^*}{\partial z}\boldsymbol{e}_3 = \boldsymbol{e}_z
\end{aligned} \tag{4-11}$$

Note that $\boldsymbol{g}_\theta$ has magnitude $r$. If we convert to unit base vectors and use the conventional notation, we obtain the relations easily obtained geometrically, and shown in Fig. 4-3(a), as

$$\boldsymbol{e}_r = cos\theta\boldsymbol{e}_x + sin\theta\boldsymbol{e}_y \quad \boldsymbol{e}_\theta = -sin\theta\boldsymbol{e}_x + cos\theta\boldsymbol{e}_y \quad \boldsymbol{e}_z = \boldsymbol{e}_z \tag{4-12}$$



(a) Cylindrical coordinate system. (b) Spherical coordinate system

Fig. 4-3. Cylindrical and spherical coordinates.

We also observe that these base vectors are orthogonal so the transformation matrix is orthogonal and the inverse relations become

$$\boldsymbol{e}_x = cos\theta\boldsymbol{e}_r - sin\theta\boldsymbol{e}_\theta \quad \boldsymbol{e}_y = sin\theta\boldsymbol{e}_r + cos\theta\boldsymbol{e}_\theta \quad \boldsymbol{e}_z = \boldsymbol{e}_z \tag{4-13}$$

Now we use these relations to show that the position vector expressed in the two bases is

$$\boldsymbol{r} = x\boldsymbol{e}_x + y\boldsymbol{e}_y + z\boldsymbol{e}_z = r\boldsymbol{e}_r + z\boldsymbol{e}_z \tag{4-14}$$

Therefore the components of the position vector in the cylindrical system are

$$r_r = r \quad r_\theta = 0 \quad r_z = z \tag{4-15}$$

The components of the position vector are not equal to the coordinates. The notation $\boldsymbol{r}$, and not $\boldsymbol{x}$, is used for the position vector to help avert the trap of automatically inserting coordinates of a point for the components of the position vector. In fact, the coordinates of a point equal the components of the position vector only for a rectangular Cartesian system, which is a very special case.

## 4.6 Spherical Coordinate System

Here present the corresponding brief summary of the relevant equations for the other coordinate system that should be in everyone's repertoire, namely spherical coordinates. Let us use the notation

$$x_i \Rightarrow (x,y,z) \qquad y_i \Rightarrow (R,\theta,\phi) \tag{4-16}$$

The functions $f_i$ and $f_i^*$ are

$$\begin{aligned} R &= f_1 = \sqrt{x^2+y^2+z^2} & x &= f_1^* = R\cos\theta\sin\phi \\ \theta &= f_2 = \tan^{-1}\left(\frac{y}{x}\right) & \mathrm{y} &= f_2^* = R\sin\theta\sin\phi \\ \phi &= f_3 = \tan^{-1}\left(\frac{\sqrt{x^2+y^2}}{z}\right) & z &= f_3^* = R\cos\phi \end{aligned} \tag{4-17}$$

Now we use (4-6) to obtain

$$\begin{aligned} \boldsymbol{g}_R &= \frac{\partial f_1^*}{\partial R}\boldsymbol{e}_1 + \frac{\partial f_2^*}{\partial R}\boldsymbol{e}_2 + \frac{\partial f_3^*}{\partial R}\boldsymbol{e}_3 = \cos\theta\sin\phi\boldsymbol{e}_x + \sin\theta\sin\phi\boldsymbol{e}_y + \cos\phi\boldsymbol{e}_z \\ \boldsymbol{g}_\theta &= \frac{\partial f_1^*}{\partial \theta}\boldsymbol{e}_1 + \frac{\partial f_2^*}{\partial \theta}\boldsymbol{e}_2 + \frac{\partial f_3^*}{\partial \theta}\boldsymbol{e}_3 = -R\sin\theta\sin\phi\boldsymbol{e}_x + R\cos\theta\sin\phi\boldsymbol{e}_y \\ \boldsymbol{g}_\phi &= \frac{\partial f_1^*}{\partial \phi}\boldsymbol{e}_1 + \frac{\partial f_2^*}{\partial \phi}\boldsymbol{e}_2 + \frac{\partial f_3^*}{\partial \phi}\boldsymbol{e}_3 = R\cos\theta\cos\phi\boldsymbol{e}_x + R\sin\theta\cos\phi\boldsymbol{e}_y - R\sin\phi\boldsymbol{e}_z \end{aligned} \tag{4-18}$$

We note that the base vectors are orthogonal but not of unit length. Their magnitudes are

$$h_R = 1 \qquad h_\theta = R\sin\phi \qquad h_\phi = R \tag{4-19}$$

so that the unit vectors become

$$\begin{aligned} \boldsymbol{e}_R &= \cos\theta\sin\phi\boldsymbol{e}_x + \sin\theta\sin\phi\boldsymbol{e}_y + \cos\phi\boldsymbol{e}_z \\ \boldsymbol{e}_\theta &= -\sin\theta\boldsymbol{e}_x + \cos\theta\boldsymbol{e}_y \\ \boldsymbol{e}_\phi &= \cos\theta\cos\phi\boldsymbol{e}_x + \sin\theta\cos\phi\boldsymbol{e}_y - \sin\phi\boldsymbol{e}_z \end{aligned} \tag{4-20}$$

and these vectors are shown in Fig. 4-3(b). The corresponding transformation matrices will be orthogonal. When the matrix is used, the position vector becomes

$$\boldsymbol{r} = x\boldsymbol{e}_x + y\boldsymbol{e}_y + z\boldsymbol{e}_z = R\boldsymbol{e}_R \tag{4-21}$$

and the components of the position vector in the spherical system are

$$r_R = R \qquad r_\theta = 0 \qquad r_\phi = 0 \tag{4-22}$$

Again the components are not equal to the coordinates.

**4.7 Closing Comments**

Here we have shown the close connection between the use of orthonormal bases and rectangular Cartesian coordinates. As shown in the previous chapter for non-orthonormal bases, and in this chapter for non rectangular Cartesian coordinates, either one of these choices introduces significantly more complicated relations that we have just touched upon. In particular, with curvilinear coordinates such as cylindrical and spherical, the base vectors vary with position. This is a very important consideration in obtaining gradients.

Unless specified otherwise, we will work with orthonormal bases that do not vary with position. The implication is that the approach is greatly simplified over what it might be. Nevertheless, if our derivations are in direct notation, they hold for all bases and coordinate systems. The complication comes when specific expressions are to be developed in these coordinate systems as suggested by the introductory material in Subsections 4.5 and 4.6.