15. Conservation of Mass and Incompressibility

15.1 Initial Comments

In this chapter we provide various forms for the statement that mass is conserved. Then we consider incompressibility and volume preserving deformations.

15.2 Alternative Forms for Conservation of Mass

The mass densities in the reference and spatial configurations are denoted by ρ_0 and ρ, respectively. The reference mass density does not depend on the time, t, but it may depend on position, $\boldsymbol{R}$. If we consider any subregion, R_0^*, of a body in the reference configuration, the corresponding subregion in the spatial configuration is denoted as R^*. The assumption that mass is conserved results in the equation

$$\int_{R_0^*} \rho_0 \, dV_0 = \int_{R} \rho \, dV \qquad \forall \quad R_0^* \tag{15-1}$$

Since the subregion is arbitrary, it follows that an alternative form for conservation of mass is

$$\rho_0 dV_0 = \rho dV \tag{15-2}$$

Recall that $dV = J dV_0$. Then (15-2) implies that

$$\rho_0 = J\rho \tag{15-3}$$

Since ρ_0 does not depend on time, yet another expression for conservation of mass is

$$\frac{d}{dt}(J\rho) = 0 \tag{15-4}$$

Since we know the expression for the derivative of J, and after dividing by J, we obtain

$$\frac{d}{dt}\rho + \rho\left(\boldsymbol{v}\cdot\bar{\bar{\nabla}}\right) = 0 \tag{15-5}$$

With ρ taken to be a function of $\boldsymbol{r}$ and t, and alternative form for (15-5) is

$$\frac{\partial}{\partial t}\rho + \left(\rho\bar{\bar{\nabla}}\right)\cdot\boldsymbol{v} + \rho\left(\boldsymbol{v}\cdot\bar{\bar{\nabla}}\right) = 0 \tag{15-6}$$

or

$$\frac{\partial}{\partial t}\rho + \left(\rho\boldsymbol{v}\right)\cdot\bar{\bar{\nabla}} = 0 \tag{15-7}$$

This last form is known as the equation of continuity.

An important consequence of the equation of mass, as expressed by (15-2), is that the time derivative can be taken inside the integral in the spatial configuration according to the following rule:

$$\frac{d}{dt}\int_R (f)\rho\, dV = \int_R \left(\frac{d}{dt} f\right)\rho\, dV \tag{15-8}$$

Note that this operation requires that the part of the integrand of interest is the coefficient of ρdV . [Exercise: Prove (15-8)].

15.3 Incompressibility

A material is incompressible, if the volume of a material element does not change under any loading. This condition implies that the following equivalent constraints are satisfied:

$$dV = dV_0 \tag{15-9}$$

or

$$J = 1 \tag{15-10}$$

or

$$\frac{d}{dt} J = 0 \tag{15-11}$$

or

$$\boldsymbol{v} \cdot \bar{\bar{\nabla}} = 0 \tag{15-12}$$

If, in addition, conservation of mass is invoked, then

$$\rho = \rho_0 \tag{15-13}$$

or

$$\frac{d}{dt}\rho = \frac{\partial}{\partial t}\rho + \left(\rho\bar{\bar{\nabla}}\right)\cdot \boldsymbol{v} = 0 \tag{15-14}$$

Note that it is possible that the deformation field of a compressible material may be volume preserving.

15.4 Summary

Conservation of mass may be stated in any of the following equivalent forms:

$$\int_{R_0^*} \rho_0 \, dV_0 = \int_R \rho \, dV \quad \forall \quad R_0^* \qquad\qquad \rho_0 dV_0 = \rho dV$$

$$\rho_0 = J\rho \qquad \frac{d}{dt}(J\rho) = 0 \qquad \frac{d}{dt}(J\rho) = 0 \qquad \frac{\partial}{\partial t}\rho + (\rho \boldsymbol{v}) \cdot \bar{\bar{\nabla}} = 0 \tag{15-15}$$

Incompressibility results in the following equivalent forms:

$$dV = dV_0 \qquad J = 1 \qquad \frac{d}{dt}J = 0 \qquad \boldsymbol{v} \cdot \bar{\bar{\nabla}} = 0 \qquad \rho = \rho_0 \qquad \frac{\partial}{\partial t}\rho + \left(\rho \bar{\bar{\nabla}}\right) \cdot \boldsymbol{v} = 0 \tag{15-16}$$

and the time derivative and integral can be interchanged when

$$\frac{d}{dt}\int_R (f)\rho \, dV = \int_R \left(\frac{d}{dt} f\right)\rho \, dV \tag{15-17}$$