ASSIGNMENT 4

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ME 512 - Continuum Mechanics

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For the basis $\boldsymbol{e_i}$, the components of $\boldsymbol{T}$ are: $\begin{bmatrix} 3 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 3 \end{bmatrix}$

1 Find the components of $\boldsymbol{T^2}$ and $\boldsymbol{T^3}$ for the $\boldsymbol{e_i}$ basis:

- $\boldsymbol{T^2} \Rightarrow \overset{e-e}{[T]}\overset{e-e}{[T]} = \begin{bmatrix} 10 & 0 & -6 \\ 0 & 1 & 0 \\ -6 & 0 & 10 \end{bmatrix}$
- $\boldsymbol{T^3} \Rightarrow \overset{e-e}{[T]}\overset{e-e}{[T]}\overset{e-e}{[T]} = \begin{bmatrix} 36 & 0 & -28 \\ 0 & 1 & 0 \\ -28 & 0 & 36 \end{bmatrix}$

2 Find $I_T = tr(\boldsymbol{T}), II_T = tr(\boldsymbol{T^2}), III_T = tr(\boldsymbol{T^3})$:

- $I_T = 3 + 1 + 3 = 7$
- $II_T = 10 + 1 + 10 = 21$
- $III_T = 36 + 1 + 36 = 73$

3 Find the eigenvalues of $\boldsymbol{T}$ and the eigenvectors. Construct a principal basis ($\boldsymbol{p_a}$) expressed in terms of $\boldsymbol{e_i}$

- The eigen problem: $\boldsymbol{T} \cdot \boldsymbol{p} = \boldsymbol{\lambda p} \Rightarrow \overset{e-e}{[T]}\overset{e}{\{p\}} = \lambda\overset{e}{\{p\}}$ *or* $\Big[\overset{e-e}{[T]} - \lambda[I]\Big]\overset{e}{\{p\}} = \{0\}$
  - where:
    $\lambda =$ eigenvalue
    $\{p\} =$ eigenvector
  - a non trivial solution (non zero) for $\overset{e}{\{p\}}$ in $\Big[\overset{e-e}{[T]} - \lambda[I]\Big]\overset{e}{\{p\}} = \{0\}$ only exists if the determinant: $det\Big(\Big[[T] - \lambda[I]\Big]\Big) = 0$. The real values of λ that satisfy this are the eigenvalues of $[T]$.
- the equation: $det\Big(\Big[[T] - \lambda[I]\Big]\Big) = 0$; is titled the characteristic equation of $\Big[[T] - \lambda[I]\Big]$
- a characteristic equation can be shown in terms of a polynomial function (via Leibniz' rule), therefore:
  - characteristic equation: $det\Big(\Big[[T] - \lambda[I]\Big]\Big) = 0$
  - characteristic polynomial: $\lambda^3 - I^*\lambda^2 - II^*\lambda - III^* = -P(\lambda) = 0$

- find the characteristic invariants of $[T]$:
  - characteristic equation: $det\left(\Big[[T]-\lambda[I]\Big]\right)=\lambda^3-I^*\lambda^2-II^*\lambda-III^*=-P(\lambda)=0$ where a * indicates a **characteristic** invariant
  - $I_T^*=I_T=tr\left(\overset{e-e}{[T]}\right)=7$
  - $II_T^*=\frac{1}{2}(II_T-I_T^2)=det\begin{bmatrix}T_{11} & T_{12}\\ T_{21} & T_{22}\end{bmatrix}+det\begin{bmatrix}T_{11} & T_{13}\\ T_{31} & T_{33}\end{bmatrix}+det\begin{bmatrix}T_{22} & T_{23}\\ T_{32} & T_{33}\end{bmatrix}=-14$
  - $III_T^*=\frac{1}{6}(I_T^3-3I_TII_T+2III_T)=det\left(\overset{e-e}{[T]}\right)=8$
- input invariants into the characteristic equation for $\overset{e-e}{[T]}\Rightarrow\lambda^3-7*\lambda^2+14*\lambda-8=0$
- solve the characteristic equation to obtain the three eigenvalues (roots of the cubic polynomial):
  - $(\lambda-4)(\lambda-1)(\lambda-2)=0$
  - $\lambda_1=4 \qquad \lambda_2=2 \qquad \lambda_3=1$
- determine the eigenvectors (one for each eigenvalue); assume one component of $\{p\}=1$ and determine remaining two components. If this assumption results in a vectors that does not satisfy: $\Big[[T]-\lambda[I]\Big]\{p\}=\{0\}$; then assume a value of 0 for the component of $\{p\}$.

$$\Big[[T]-\lambda[I]\Big]\{p\}=\{0\}\Rightarrow\begin{bmatrix}3-\lambda_i & 0 & -1\\ 0 & 1-\lambda_i & 0\\ -1 & 0 & 3-\lambda_i\end{bmatrix}\begin{Bmatrix}p_{i,1}\\ p_{i,2}\\ p_{i,3}\end{Bmatrix}=\begin{Bmatrix}0\\0\\0\end{Bmatrix}$$

  - for $\lambda_1=4:\langle p_1\rangle=\langle 1,0,-1\rangle$; Normalize $\Rightarrow\langle\frac{1}{\sqrt{2}},0,-\frac{1}{\sqrt{2}}\rangle$
  - for $\lambda_2=2:\langle p_2\rangle=\langle 1,0,1\rangle$; Normalize $\Rightarrow\langle\frac{1}{\sqrt{2}},0,\frac{1}{\sqrt{2}}\rangle$
  - for $\lambda_3=1:\langle p_3\rangle=\langle p_1\rangle\times\langle p_2\rangle=\langle 0,-1,0\rangle$; only two of the vectors are independent, the third can be calculated from the two independent vectors to form an orthonormal set of vectors.
- the principal basis: $\boldsymbol{p_a}$ in terms of $\boldsymbol{e_i}$: $\quad \boldsymbol{p_a}\Rightarrow\begin{array}{c}\boldsymbol{p_1}=\frac{1}{\sqrt{2}}\boldsymbol{e_1}-\frac{1}{\sqrt{2}}\boldsymbol{e_3}\\ \boldsymbol{p_2}=\frac{1}{\sqrt{2}}\boldsymbol{e_1}+\frac{1}{\sqrt{2}}\boldsymbol{e_3}\\ \boldsymbol{p_3}=-1\boldsymbol{e_2}\end{array}$

4 Find the components of $\boldsymbol{T},\boldsymbol{T^2}$ and $\boldsymbol{T^3}$ in the $\boldsymbol{p_a}$ basis and calculate respective invariants:

- $\overset{p-e}{[a]}=\begin{bmatrix}\frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}}\\ 0 & -1 & 0\end{bmatrix}$
- $\boldsymbol{T}\Rightarrow\overset{p-p}{[T]}=\overset{p-e}{[a]}\overset{e-e}{[T]}\overset{e-p}{[a]}=\begin{bmatrix}4 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 1\end{bmatrix}$
- $\boldsymbol{T^2}\Rightarrow\overset{p-p}{[T]}\overset{p-p}{[T]}=\begin{bmatrix}16 & 0 & 0\\ 0 & 4 & 0\\ 0 & 0 & 1\end{bmatrix}$
- $\boldsymbol{T^3}\Rightarrow\overset{p-p}{[T]}\overset{p-p}{[T]}\overset{p-p}{[T]}=\begin{bmatrix}64 & 0 & 0\\ 0 & 8 & 0\\ 0 & 0 & 1\end{bmatrix}$

- $I_T = 4 + 2 + 1 = 7$
- $II_T = 16 + 4 + 1 = 21$
- $III_T = 64 + 8 + 1 = 73$ ... same as before, just like they should be.

5 Set up the transformation matrix between $\boldsymbol{p_a}$ and $\boldsymbol{e_i}$

- $\overset{e-p}{[a]} = tr\left(\overset{p-e}{[a]}\right) = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & -1 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix}$
- $\boldsymbol{T} \Rightarrow \overset{e-e}{[T]} = \overset{e-pp-pp-e}{[a]\,[T]\,[a]} = \begin{bmatrix} 3 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 3 \end{bmatrix}$

6

(a) Obtain the values of the invariants in the principal basis $\hat{I_T}, I\hat{I_T}, I\hat{I}I_T$

- $\hat{I_T} = tr\left(\overset{p-p}{[T]}\right) = 7$
- $I\hat{I_T} = \frac{1}{2}(II_T - I_T^2) = -14$
- $I\hat{I}I_T = \frac{1}{6}(I_T^3 - 3I_T II_T + 2III_T) = det\left(\overset{p-p}{[T]}\right) = 8$

(b) Show that the Cayley-Hamilton theorem holds using components in the $\boldsymbol{e_i}$ system.

- $\boldsymbol{T^2} \Rightarrow \overset{e-ee-e}{[T]\,[T]} = \begin{bmatrix} 10 & 0 & -6 \\ 0 & 1 & 0 \\ -6 & 0 & 10 \end{bmatrix}$
- $\boldsymbol{T^3} \Rightarrow \overset{e-ee-ee-e}{[T]\,[T]\,[T]} = \begin{bmatrix} 36 & 0 & -28 \\ 0 & 1 & 0 \\ -28 & 0 & 36 \end{bmatrix}$
- characteristic invariants: $I_T^* = 7, II_T^* = -14, III_T^* = 8$
- Cayley-Hamilton Theorem for the $\boldsymbol{e_i}$system, or in any system (which is why it may be written in direct notation):
  $\boldsymbol{T^3} - I_T^*\boldsymbol{T^2} - II_T^* - III_T^*\boldsymbol{I} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

(c) With the use of components in either system, show that $I\hat{I}I_T = det(\boldsymbol{T})$

- $\frac{1}{6}\left[I_T^3 - 3I_T II_T + 2III_T\right] = \frac{1}{6}\left[7^3 - 3 * 7 * 21 + 2 * 73\right] = 8$
- $det\left([T]\right) = 8$

7 Find the components of the tensor $\boldsymbol{T}^{1/2}$ in the $\boldsymbol{e_i}$ system, i.e., find $T^{1/2}_{ij}T^{1/2}_{jk}$:

- from the spectral theorem, when a tensor of eigenvectors and a diagonal of eigenvalues are formed:
  - $\{\boldsymbol{p_1}, \boldsymbol{p_2}, \boldsymbol{p_3}\} \Rightarrow \overset{e-e}{[P]} \Rightarrow P_{ij} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & -1 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix}$
  - $\{\lambda_1, \lambda_2, \lambda_3\}[I] = [\Lambda] = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
- $T_{ik} = T^{\frac{1}{2}}_{ij}T^{\frac{1}{2}}_{jk} \Rightarrow \overset{e-e}{[P]}^{-1}[\Lambda]\overset{e-e}{[P]} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & -1 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix} \begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & -1 & 0 \end{bmatrix}$

8 Find the components of $\boldsymbol{T}^{-1}$ in the $\boldsymbol{e_i}$ system:

- $\boldsymbol{T^{-1}} = \left(\boldsymbol{T^2} - \hat{I}_T\boldsymbol{T} - I\hat{I}_T\boldsymbol{I}\right) / \left(I\hat{I}I_T\right) \Rightarrow \frac{[T]^{cf}}{det([T])}$
- $\boldsymbol{T^{-1}} \Rightarrow \left(\begin{bmatrix} 10 & 0 & -6 \\ 0 & 1 & 0 \\ -6 & 0 & 10 \end{bmatrix} - 7\begin{bmatrix} 3 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 3 \end{bmatrix} + 14\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)/8 = \begin{bmatrix} 3/8 & 0 & 1/8 \\ 0 & 1 & 0 \\ 1/8 & 0 & 3/8 \end{bmatrix}$
- $\overset{p-p}{[T]}^{-1} = \overset{p-e}{[a]}\,\overset{e-e}{[T]}^{-1}\,\overset{e-p}{[a]} = \begin{bmatrix} 0.25 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$