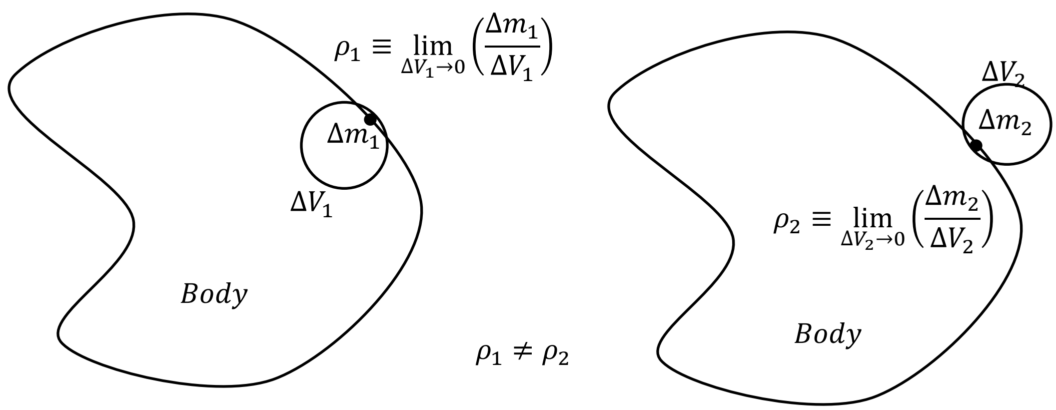
Assignment #1 - Solution

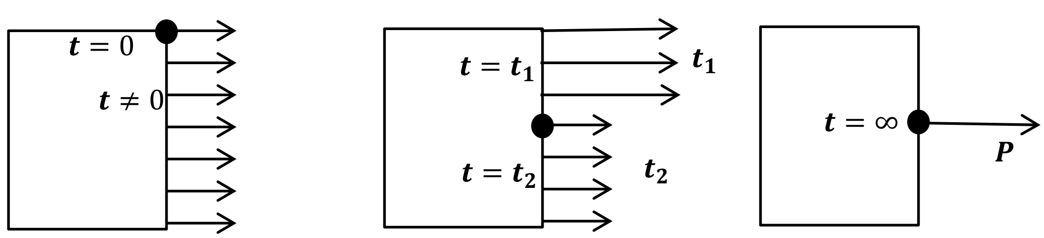
1.1 Show why mass density, $\rho \equiv \lim_{\Delta V \to 0}\left(\frac{\Delta m}{\Delta V}\right)$, is undefined on the boundary of a homogeneous material body.

Solution: If ΔV_1, taken as the volume surrounding a point on the surface of the body, approaches the point on the boundary from the inside of the body, the contained mass Δm_1 will be larger than if ΔV_2 is taken as the volume approaching the point from outside the body, in which case the contained mass Δm_2 will be smaller (even zero, perhaps). Therefore, because the way in which the volume approaches zero affects the limit, the density ρ is not uniquely defined. If instead we evaluate the point at which density is desired within the body, the limit has a chance of being uniquely defined if the mass distribution were sufficiently uniform within the body. At the atomic scale, the concept of density always breaks down.



1.2 Describe and sketch three physical situations in which the traction vector $\boldsymbol{t}$, defined in Eq. 1.1, is undefined at a point.

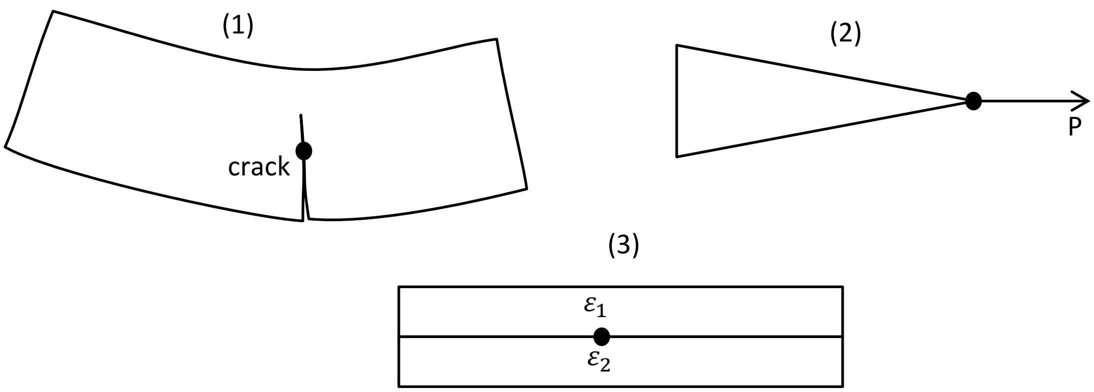
Solution: The traction vector is defined as $\boldsymbol{t} \equiv \lim_{\Delta A \to 0}\left(\frac{\Delta \boldsymbol{F}}{\Delta A}\right)$. The sketches below show that the limit is not uniquely defined (1) at corners; (2) at jumps in traction on a smooth surface; and (3) at applied point loads. In the first two cases, the limit depends upon the way in which the area goes to zero. In the third case, the limit goes to infinity. Also, the concept of traction breaks down at the crystallite, molecular, and atomic scales – essentially at the mesoscale and below.



1.3 Describe and sketch three physical situations in which the strain component at a point ε_{xx} defined in Eq. 1.2, is undefined.

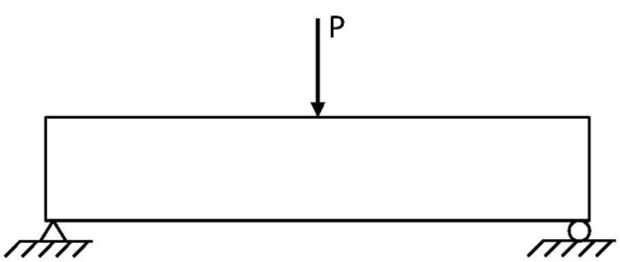
Solution: Axial strain is defined as $\varepsilon_{XX} \equiv \lim_{\Delta x \to 0}\left(\frac{\Delta u}{\Delta X}\right) = \frac{\partial u}{\partial X}$.

(1) If the displacement u is discontinuous (say, across a discrete crack), the limit does not exist.
(2) If the limit goes to infinity, the limit does not exist.
(3) If the axial strain jumps in the transverse direction, say due to differential thermal expansion of two materials, the limit does not exist on the interface.
(4) At the mesoscale and below, "strain" is insufficiently smooth to be well defined.



1.4 The centrally-loaded beam shown in Fig. 1.2 has a 24" span from support to support, is 6" deep, and is 3" thick. If the maximum allowable shear stress is 15,000 PSI, what load, P, is permitted using

a. Timoshenko beam theory, ignoring stress concentrations.
b. Plane stress finite element analysis. Assume all loads and reactions are uniformly distributed tractions over a length of one inch along the beam length, and a width of 3" transverse to the beam axis.
c. Concentrated loads and reactions.



Solution:

(a) $M = \frac{PL}{4}$; $V = \frac{P}{2}$; $I = \frac{bh^3}{12} = \frac{3"\times 6"^3}{12} = 54in^4$

At center of beam, $\sigma_{allow} = \frac{Mc}{I} = \frac{\frac{PL}{4}\times c}{I} = P\frac{L\times\frac{h}{2}}{4I} = P\frac{24"\times 6"}{8\times 54in^4} = P \times 0.3333in^{-2}$.

Considering Mohr's circle, $\tau_{allow} = 15{,}000PSI = \frac{\sigma_{max}}{2} = \frac{P \times 0.3333in^{-2}}{2}$;

$$P_{n(bending)} = \frac{2 \times 15{,}000PSI}{0.3333in^{-2}} = 90.0\ kips$$

The maximum shear stress away from the ends of the beam occur at the centroid, and

$\tau_{allow} = 15{,}000PSI = \frac{VQ}{Ib} = \frac{3V}{2A} = \frac{3 \times \frac{P}{2}}{2 \times b \times h} = \frac{P}{24in^2}$;

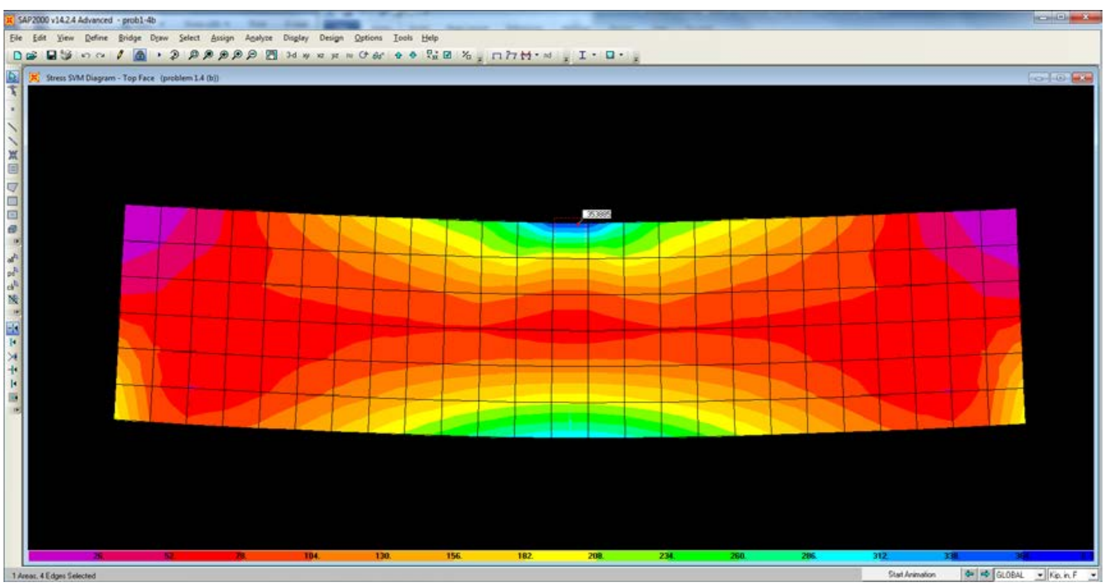
$$P_{n(shear)} = 24in^2 \times 15{,}000PSI = 360kips$$

So bending controls, and the nominal strength is

$\boldsymbol{P_n = 90.0\ kips}$. (This solution ignores stress concentrations near the applied loads.)

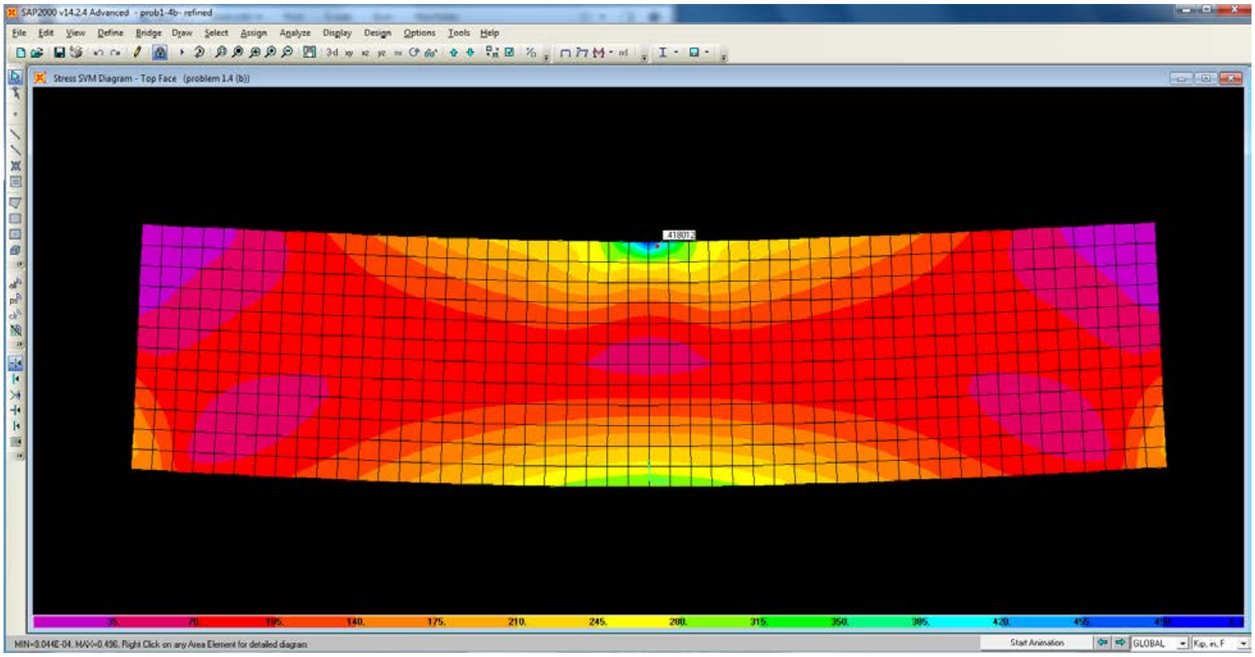
(b) The finite element analysis below, with 1"x1" finite elements, is loaded with 1 kip vertical load, applied as two 0.5 kip equivalent nodal loads applied to the top of the central top element. The maximum equivalent Von Mises stress (a measure of maximum equivalent uniaxial stress that is to be compared to the uniaxial tensile strength of the material $\sigma_{Yield} = 2\tau_{Yield}$) is 0.354 ksi for a 1-kip applied load. (The reactions at bottom left and bottom right are applied as a pair of 0.25kip upward equivalent nodal loads applied to the bottom of the leftmost and rightmost finite elements at the bottom of the beam.)

Thus, with linear scaling, $\tau_{allow} = 15{,}000PSI = P_{n(FEM)} \times \frac{(0.354\ ksi/kip)}{2}$; and $\boldsymbol{P_{n(FEM)} = 84.75kips}$, 6% less than what is predicted by Timoshenko beam theory.

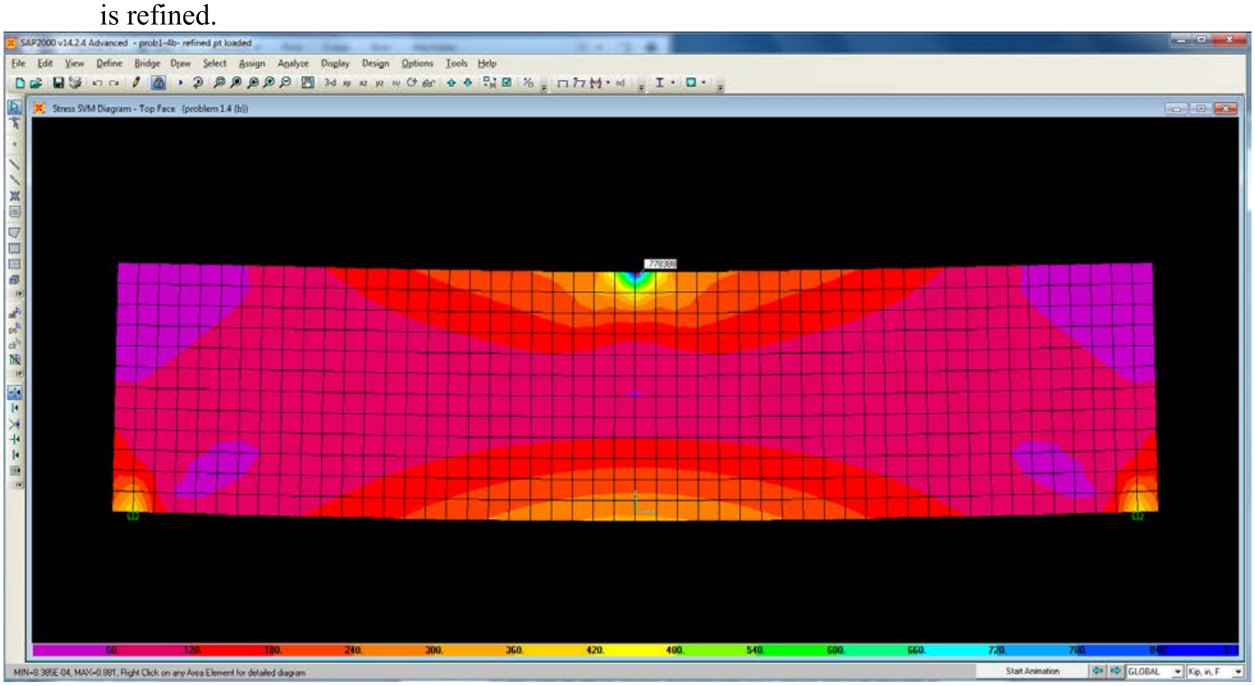


With mesh refinement, using 0.5"x0.5" finite elements, with appropriate equivalent nodal loads (.25 kip, 0.5 kip, 0.25 kip applied to three consecutive nodes at the top middle of the beam) the maximum Von Mises stress under the applied load at the top of the beam increases to 0.482 ksi, as shown below, so $\tau_{allow} = 15{,}000PSI = P_{n(FEM)} \times \frac{(0.482\ ksi/kip)}{2}$, giving a nominal strength of $\boldsymbol{P_{n(FEM)} = 62.2\ kips}$, 31% less than what is predicted by Timoshenko beam theory.

One sees that linear elastic finite element analysis is fraught with modeling issues and problems with interpretation of the results.



(c) With the load been applied to just one node, as shown in the figure below, the issue of interpretation of results is more problematic. The maximum stress in the beam ($\sigma_{max} = 0.87$ ksi in the figure below), at the location of the applied point loads, increases without limit as the mesh is refined.



1.5 Assume that the 6” by 12” square by ¼” thick glass plate shown in Fig. 1.7 is in a condition of plane stress. The plate has a 2” diameter hole at its center. The glass has a tensile strength of 10,000 PSI. If we apply equal and opposite uniformly distributed loads of P to opposite ends of the glass plate as shown, what is the load at which the glass plate will break? (Use a finite element program or a solution from the literature to solve this problem.)

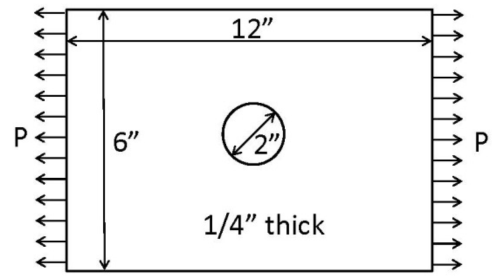


Figure 0.1. Glass plate.

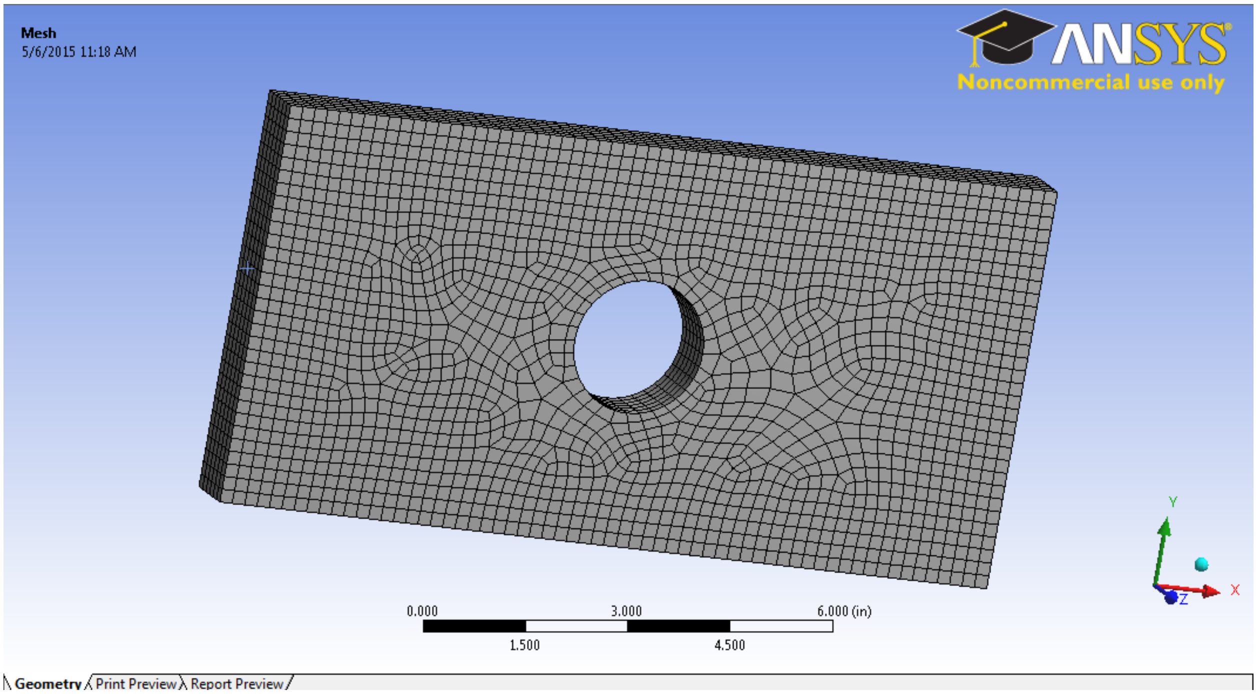
Solution:
From the literature (for example, Boresi, Schmidt and Sidebottom, Advanced Mechanics of Materials, 5th Edition) for a plate of finite width D with a centrally-located hole of radius ρ, with $\frac{\rho}{D} = \frac{1"}{6"}$ the stress concentration factor is $K = 2.35$; and if we assume the hole is small compared to the width of the plate, then the stress concentration factor is $K = 3.0$. Taking $K = 3.0$, and using stress based upon net area,

$$\sigma_{max} = 10.0\ KSI = 3.0 \times \frac{P}{(6"\text{-}2") \times 0.25"};\ P_n = 10.0\ KSI \times \frac{4" \times 0.25"}{3.0} = 3.33\ kips\ .$$

Using the finite element method, with an applied stress of 1.0 KSI, we find that the maximum principal stress is 3.6216 KSI, as shown below. So

$$P_n = 1.0\ KSI \times \frac{10.0\ KSI}{3.6216\ \text{KSI}} \times 6" \times 0.25" = 4.14\ kips\ .$$

The finite element mesh and stress results are shown on the following page.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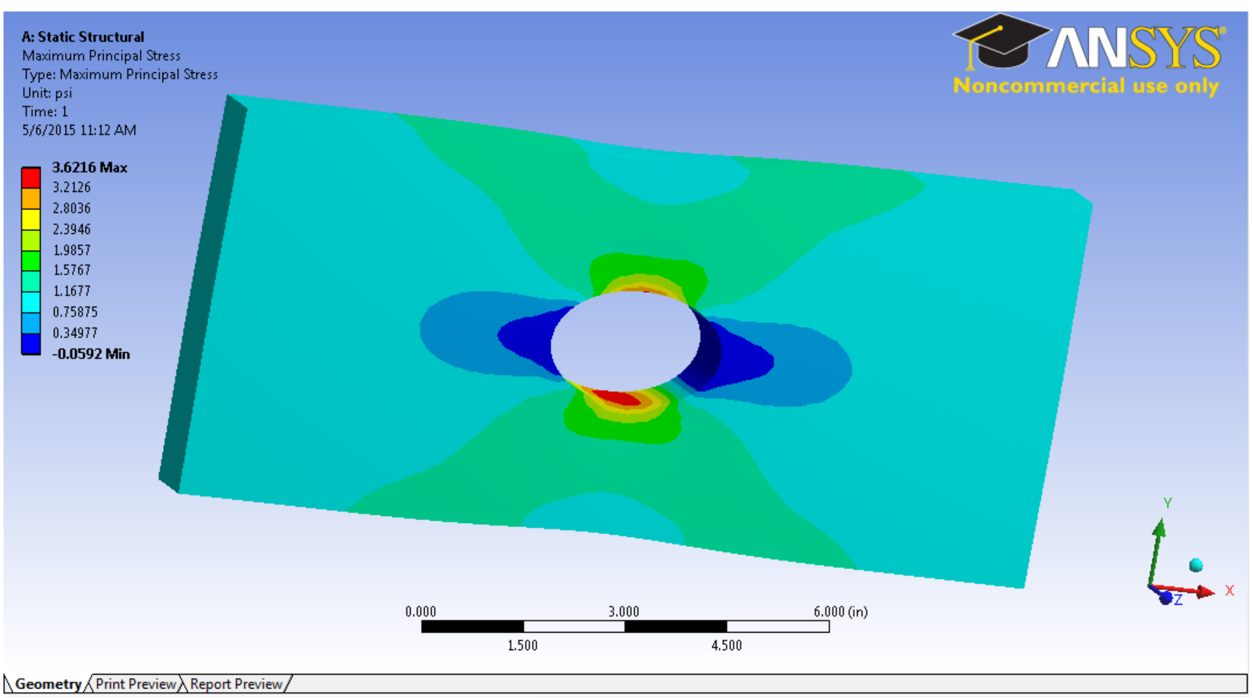
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1.6 Immediately after the tensile strength of the glass is reached in Problem 1.5, two cracks propagate from the top and bottom of the hole in a direction transverse to the direction of applied load P. When the two cracks are each 0.5" long, what is the maximum stress in the plate?

Solution: The stresses at the crack tips are theoretically infinite. (Although "stress" has no meaning at the crack tips, as the conditions for taking the limit as area goes to zero do not exist at the crack tip.)