Assignment 2

ME-500: Numerical Methods in Mechanical Engineering

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1 Summary

This assignment focused on forming problems and obtaining solutions to the second order differential equation known as the heat equation. Simple (one dimension) problems were formed in both cylindrical and cartesian coordinates as:

$$\frac{1}{r}\frac{d}{dr}(kr\frac{dT}{dr}) = 0\,(1) \qquad\qquad \frac{d}{dx}(k\frac{dT}{dx}) = 0\,(2)$$

The following are definitions of words used in this assignment (in no particular order).

Delta: ($\delta[x-a]$, where a is the reference value) a function often termed the *Dirac Delta* or *impulse symbol*. The function is defined:

$$\int_{a-\epsilon}^{a+\epsilon} \delta[x-a]dx = 1\,\forall\,\epsilon > 0$$

This definition results in the following:

$$\delta[x-a] = \begin{Bmatrix} 0 & if & x \neq a \\ \infty & if & x = a \end{Bmatrix}$$

This function is not differentiable and the integral of the *Delta* function is the

Strong: a function is said to be *strong* if it is defined everywhere in the problem domain and in the context of the problem. That is, if the function is the solution to a second order differential equation, then the function would be a strong solution only if the function itself, its first derivative, and its second derivative are defined with respect to the independent variable(s). The term *strong* implies that a Dirac Delta (δ) is not needed in the function definition. Additionally, *strong* does not imply the function must be continuous, e.g., Heaviside functions are admissible.

Weak: a function is said to be *weak* if it requires the use of the Delta for it to be defined. In the context of a solution to a second-order differential equation, the function is said to be *weak* if the function itself, its first derivative, or its second derivative require the Delta function for a description.

Class: or function class (C) regarding the solution to a differential equation, i.e., a function. This provides a description of the function and its derivatives.

Class	Cont. Derivatives	Examples
C^0	none	Primary variable is smooth, e.g., Heaviside function
C^1	first	Primary variable and first derivative are smooth, e.g., Ramp function
C^2	first and second	Primary variable and its first two derivatives are smooth
C^∞	infinite	Primary variable and all derivatives are smooth, e.g., T= sin(x)

Heaviside: ($H[x-a]$, where a is the reference value) a discontinuous function, but it is defined everywhere in the domain (i.e., does not necessarily result in a *weak* function).

$$H[x-a] = \begin{Bmatrix} 1 & if & x \geq a \\ 0 & if & x < a \end{Bmatrix}$$

Ramp: ($< x - a >$, where a is the reference value) a continuous function with the slope of x. The derivative of the *Ramp* function is the Heaviside function.

$$< x - a >= \begin{Bmatrix} x - a & if & x \geq a \\ 0 & if & x < a \end{Bmatrix}$$

Dirichlet: a boundary condition imposed on the primary independent variable.

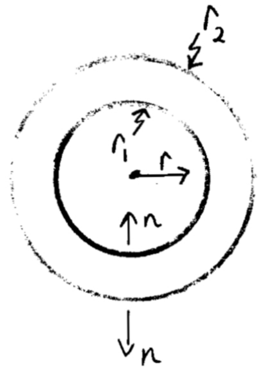
Neumann: a boundary condition imposed on a derivative of the primary independent variable.

Robin: a weighted combination of Dirichlet and Neumann type boundary conditions.

Forcing: a nonzero function that results in a nonhomogeneous differential equation. The solution to a differential equation with a nonzero forcing function may be determined using superposition of solutions to the homogenous and forcing (particular) terms.

Coeficient: or differential coefficient of a function. Is the defined multplier to the derivative of the primary variable, e.g., for $q = -k\frac{dT}{dx}$, k is the coeficient function, which multiplies the differential.

2. a) 1-D steady-state heat conduction in cylindrical coord.



problem Data

i) domain: $r_1 \le r \le r_2$

ii) forcing func.: $Q = 0$

iii) coef. func.: $\hat{k} = \hat{k}_0 \Rightarrow$ constant

iv) BCTs: a) $T\big|_{r=r_1} = T_1$

b) $T\big|_{r=r_2} = T_2$ } Dirichlet type BCs

Governing Eqn.

- $$\frac{1}{r}\frac{d}{dr}\left(k r \frac{dT}{dr}\right) = 0$$
- because $k = k_0$, pull this out

$$\frac{\hat{k}_0}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right) = 0$$

- multiply by r/k_0

$$\frac{d}{dr}\left(r\frac{dT}{dr}\right) = 0$$

- solve the first order-linear ODE

$$r\frac{dT}{dr} = C_1$$

$$dT = C_1 \frac{dr}{r}$$

$$T = C_1 \ln(r) + C_2$$

- solve for integration constants

at $r = r_1$, $T_1 = C_1 \ln(r_1) + C_2$

at $r = r_2$, $T_2 = C_1 \ln(r_2) + C_2$

$$C_2 = T_1 - C_1 \ln(r_1) \qquad (1)$$

$$T_2 = C_1 \ln(r_2) + \left(T_1 - C_1 \ln(r_1)\right) \quad \rightarrow \text{sub (1) into } T_2$$

$$T_2 - T_1 = C_1 \ln\left(\frac{r_2}{r_1}\right) \quad \rightarrow \text{rearrange}$$

$$C_1 = \frac{T_2 - T_1}{\ln\left(\frac{r_2}{r_1}\right)}$$

$$\therefore \; T(r) = \left(\frac{T_2 - T_1}{\ln\left(\frac{r_2}{r_1}\right)}\right) \ln(r_2) + \left(T_2 - \left(\frac{T_2 - T_1}{\ln\left(\frac{r_2}{r_1}\right)}\right) \ln(r_1)\right)$$

- $T(r)$ is of class C^∞

  $T(r)$ is a strong solution

2.b) <u>data</u>

i) $r_1 \le r \le r_2$

ii) $Q = 0$

iii) $\hat{k} = \hat{k}_0$

iv)

a) $T\big|_{r=r_2} = T_2 \quad \rightarrow$ Dirichlet

b) $Q^*\big|_{r=r_1} = Q_1^* \quad \rightarrow$ Neumann

$*$ where $\frac{dT}{dr} = \frac{Q^*}{n\hat{k}}$

- from previous prob.

$$\frac{dT}{dr} = \frac{C_1}{r}$$

$$T = C_1 \ln(r) + C_2$$

- solve for C_1

  - at inner boundary, $n = -1$

$$\left.\frac{dT}{dr}\right|_{r=r_1} = -\frac{Q_1^*}{\hat{k}_0} = \frac{C_1}{r_1}$$

$$C_1 = -\frac{r_1 Q_1^*}{\hat{k}_0}$$

- solve for C_2
  - at outer boundary

$$T_2 = \frac{-r_1 Q_1^*}{k_0^A} \ln(r_2) + C_2$$

$$C_2 = T_2 + \frac{r_1 Q_1^*}{k_0^A} \ln(r_2)$$

$$\therefore \quad T(r) = -\frac{r_1 Q_1^*}{k_0^A} \ln(r) + T_2 + \frac{r_1 Q_1^*}{k_0^A} \ln(r_2)$$

$$= \frac{r_1 Q_1^*}{k_0^A} \ln\left(\frac{r_2}{r}\right) + T_2$$

- $T(r)$ is of class C^∞

  $T(r)$ is a strong solution

2.c) <u>data</u>

i) $r_1 \leq r \leq r_2$

ii) $Q = 0$

iii) $k^A = k_0^A$

iv) a) $Q^*\big|_{r=r_1} = Q_1^*$

b) $Q^*\big|_{r=r_2} = Q_2^*$

- again, from 2.a)

$$\frac{dT}{dr} = \frac{C_1}{r}$$

$$T = C_1 \ln(r) + C_2$$

- solve for BC at $r = r_1$

$$\left.\frac{dT}{dr}\right|_{r=r_1} = -\frac{Q_1^*}{k_0^A} = \frac{C_1}{r_1}$$

$$C_1 = -\frac{r_1 Q_1^*}{k_0^A} \qquad (1)$$

- solve for BC at $r = r_2$

$$\left.\frac{dT}{dr}\right|_{r=r_2} = \frac{Q_2^*}{k_0^A} = \frac{C_1}{r_2}$$

$$C_1 = \frac{r_2 Q_2^*}{k_0^A} \qquad (2)$$

- from the problem data, el calculated

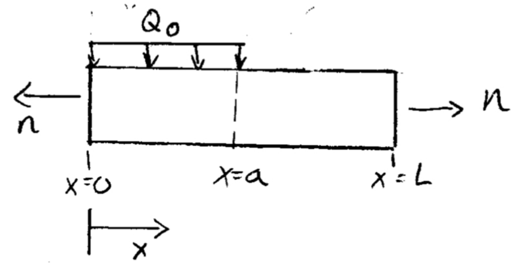
$$C_1 = -\frac{r_1 Q_1^*}{k_0^A} = \frac{r_2 Q_2^*}{k_0^A} \Longrightarrow Q_1^* = -\frac{r_2 Q_2^*}{r_1}$$

  - this cannot be true $\forall$ Q_1^* + Q_2^* + is $\therefore$ not well posed

- $T = C_1 \ln(r) + C_2$

  → C_2 is undefined based on the problem data, $\therefore$ the solution is not unique + not well posed

1-D steady-state heat conduction in Cartesian Coord.



(i) $0 \le x \le L$
(ii) $Q = Q_0 \cdot H[a-x]$
(iii) $k = k_0$
(iv)
a) $T\Big|_{x=0} = T_0$

b) $Q^*\Big|_{x=L} = k_0 \dfrac{dT}{dx}\Big|_{x=L} = Q_L^* n$

Governing Eqn.

$$\frac{d}{dx}\left(k_0 \frac{dT}{dx}\right) = -Q_0 \cdot H[a-x]$$

$$k_0 \frac{dT}{dx} = -Q_0 \langle a-x \rangle + C_1$$

$$T = -\frac{Q_0 \langle a-x \rangle^2}{2k_0} + \frac{C_1 x}{k_0} + C_2$$

b) $k_0 \dfrac{dT}{dx}\Big|_{x=L} = Q_L^* = -Q_0 \langle a-x \rangle + C_1$

$$C_1 = Q_L^* + Q_0 \langle a-x \rangle$$

a) $T\Big|_{x=0} = T_0 = -\dfrac{Q_0 \langle a-0 \rangle^2}{2k_0} + \dfrac{Q_L^* + Q_0 \langle a-0 \rangle}{k_0} x + C_2$

$$\therefore T(x) = -\frac{Q_0 \langle a-x \rangle^2}{2k_0} + \frac{(Q_L^* + Q_0 \langle a-x \rangle)x}{k_0} + \frac{Q_0 \langle a-0 \rangle^2}{2k_0} - \frac{Q_L^* + Q_0 \langle a-0 \rangle x}{k_0}$$

☆ T is a strong solution of class C^1 - because $\frac{dT}{dx}$ is continuous, but $\frac{d^2T}{dx}$ will contain $H[a-x]$ and be discontinuous (But defined everywhere)

→ <u>Solution is weak</u>

1-D steady state heat conduction in Cartesian Coord.

i) $0 \le X \le L$

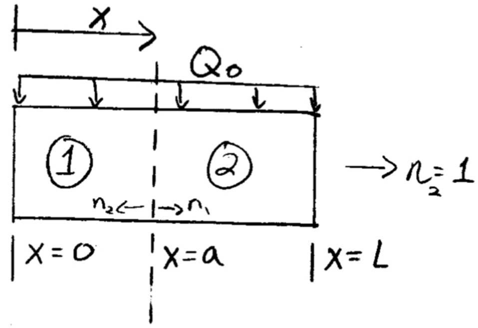
ii) $Q = Q_0$ → const. forcing func.

iii) $k^A = k_1^A + (k_2^A - k_1^A) \cdot H[X - a]$ → discontinuous coef. func

iv)

a) $$T_1 \Big|_{X=0} = T_1^0$$

b) $$Q^* \Big|_{X=L} = k_2^A \frac{dT_2}{dX}\Big|_{X=L} = Q_L^* \, n$$



Governing Eqn:

$$\frac{d}{dX}\left(k^A \frac{dT}{dX}\right) + Q_0 = 0$$

• solve in each subdomain:

① $$\frac{d}{dX}\left(k_1^A \frac{dT_1}{dx}\right) = -Q_0$$

$$k_1^A \frac{dT_1}{dx} = -Q_0 X + C_1$$

$$T_1 = -\frac{Q_0 X^2}{2k_1^A} + \frac{C_1 X}{k_1^A} + C_2$$

② $$\frac{d}{dx}\left(k_2^A \frac{dT_2}{dX}\right) = -Q_0$$

$$k_2^A \frac{dT_2}{dx} = -Q_0 X + C_3$$

$$T_2 = -\frac{Q_0 X^2}{2k_2^A} + \frac{C_3 X}{k_2^A} + C_4$$

• assuming continuity of T, derive relationship across the boundary at $x = a$

$$\int_{a-\varepsilon}^{a+\varepsilon} \left(\frac{d}{dx}\left(k^A \frac{dT}{dx}\right) + Q_0\right) dx = k^A \frac{dT}{dx}\Big|_{a-\varepsilon}^{a+\varepsilon} + Q_0 X \Big|_{a-\varepsilon}^{a+\varepsilon}$$

→ solve on both sides of boundary then let $\varepsilon \to 0$

• $a+\varepsilon$ is in ②

• $a-\varepsilon$ is in ①

$$k_2^A \frac{dT_2}{dx}\Big|_{x=a} - k_1^A \frac{dT_1}{dx}\Big|_{x=a} + \cancel{(Q_0 a - Q_0 a)}^{\,0} = 0$$

c) $$\therefore\ k_2^A \frac{dT_2}{dx}\Big|_{x=a} = k_1^A \frac{dT_1}{dx}\Big|_{x=a} \quad \Rightarrow \quad \frac{dT}{dx} \text{ is discontinuous}$$

3-0235 — 50 SHEETS — 5 SQUARES
3-0236 — 100 SHEETS — 5 SQUARES
3-0237 — 200 SHEETS — 5 SQUARES
3-0137 — 200 SHEETS — FILLER

COMET

· from b)

$$k_2^A \left.\frac{dT_2}{dX}\right|_{x=L} = Q_L^* = -Q_0 L + C_3$$

$$\boxed{C_3 = Q_L^* + Q_0 L}$$

· from c)

$$k_1^A \left.\frac{dT_1}{dx}\right|_{x=a} = k_2^A \left.\frac{dT_2}{dx}\right|_{x=a}$$

$$-Q_0 a + C_1 = -Q_0 a + Q_L^* + Q_0 L$$

$$\boxed{C_1 = Q_L^* + Q_0 L}$$

· from a)

$$T_1\Big|_{x=0} = \boxed{T_1^0 = C_2}$$

· assuming continuity of T at x=a

$$T_2\Big|_{x=a} = T_1\Big|_{x=a}$$

$$\frac{-Q_0 \cancel{a^2}}{\cancel{2} k_2^A} + \frac{(Q_L^* + \cancel{Q_0 L})a}{\cancel{k_2^A}} + \cancel{C_4} = -\frac{Q_0 \cancel{a^2}}{\cancel{2} k_1^A} + \frac{(Q_L^* + \cancel{Q_0 L})a}{\cancel{k_1^A}} + T_1^0$$

$$\boxed{C_4 = \frac{Q_0 a^2}{2}\left(\frac{1}{k_2^A - k_1^A}\right) + (Q_L^* + Q_0 L)a\left(\frac{1}{k_1^A - k_2^A}\right) + T_1^0}$$

$$\therefore \quad T_1(x) = -\frac{Q_0 X^2}{2 k_1^A} + \frac{(Q_L^* + Q_0 L)x}{k_1^A} + T_1^0$$

$$T_2(x) = -\frac{Q_0 X^2}{2 k_2^A} + \frac{(Q_L^* + Q_0 L)x}{k_1^A} + C_4$$

$$T(x) = T_1 + (T_2 - T_1)\cdot H[x-a]$$

★ $T(x)$ is a weak solution to the 2^{nd} order ODE because d^2T/dx^2 is not defined everywhere. $T(x)$ is of Class C^0, because $T(x)$ is continuous but dT/dx is not

<u>Manufacture a solution</u>

i) domain : $0 \leq x \leq L$

ii) coef func. → given: $k^A(x) = k_0^A + (k_L^A - k_0^A) \cdot H[x-a] + k_2^A \sin\left(\frac{\pi x}{2L}\right)$

iii) choose $T(x)$:

$$\text{Let } T(x) = Lx + T_0 \ln(x+L)$$

- evaluate primary variable at domain bdry.

$$T\Big|_{x=0} = T_0 \ln(L)$$

$$T\Big|_{x=L} = L^2 + T_0 \ln(2L)$$

- evaluate flux at bdrys.

$$\frac{dT}{dx} = L + \frac{T_0}{x+L}$$

$$k^A \frac{dT}{dx}\Big|_{x=0} = -Q_0^* = k_0\left(L + \frac{T_0}{L}\right)$$

$$k^A \frac{dT}{dx}\Big|_{x=L} = Q_L^* = k^A\left(x + \frac{T_0}{x+1}\right)$$

$$= \left(k_0^A + (k_L^A - k_0^A) + k_2^A \sin\left(\frac{\pi}{2}\right)\right)\left(L + \frac{T_0}{2L}\right)$$

- evaluate forcing func. that satisfies the gov. diff eqn.

$$\frac{d}{dx}\left(k \frac{dT}{dx}\right) + Q(x) = 0$$

$$\frac{dk}{dx}\frac{dT}{dx} + k\frac{d^2T}{dx^2} + Q(x) = 0$$

- $\frac{d^2T}{dx^2} = -\frac{T_0}{(x+1)^2}$

- $\frac{dk^A}{dx} = (k_l^A - k_0^A)\langle x-a \rangle + \frac{k_2^A \pi}{2L}\cos\left(\frac{\pi x}{2L}\right)$

$$\therefore \quad Q(x) = -\frac{dk}{dx}\frac{dT}{dx} - k\frac{d^2T}{dx^2}$$

$$= -\left[\left(k_l^A - k_0^A\right)\langle x-a\rangle + \frac{k_2^A \pi}{2L}\cos\left(\frac{\pi x}{2L}\right)\right]\left(L + \frac{T_0}{x+L}\right) -$$

$$\left[k_0^A + \left(k_L^A - k_0^A\right)H[x-a] + k_2^A \sin\left(\frac{\pi x}{2L}\right)\right]\left(-\frac{T_0}{(x+l)^2}\right)$$