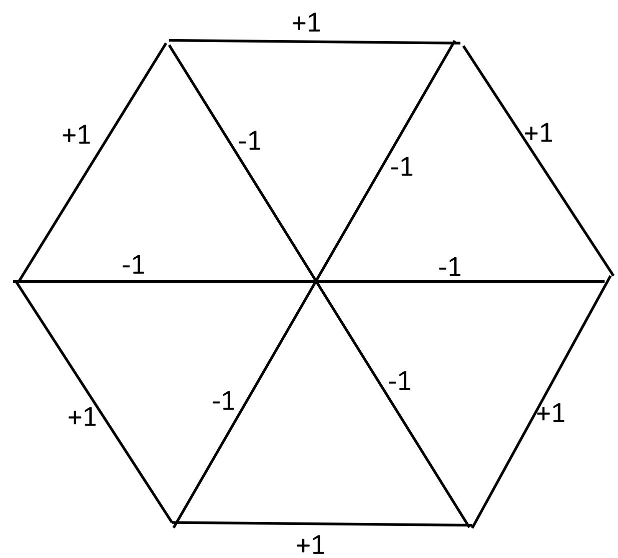
Assignment #2 - Solution

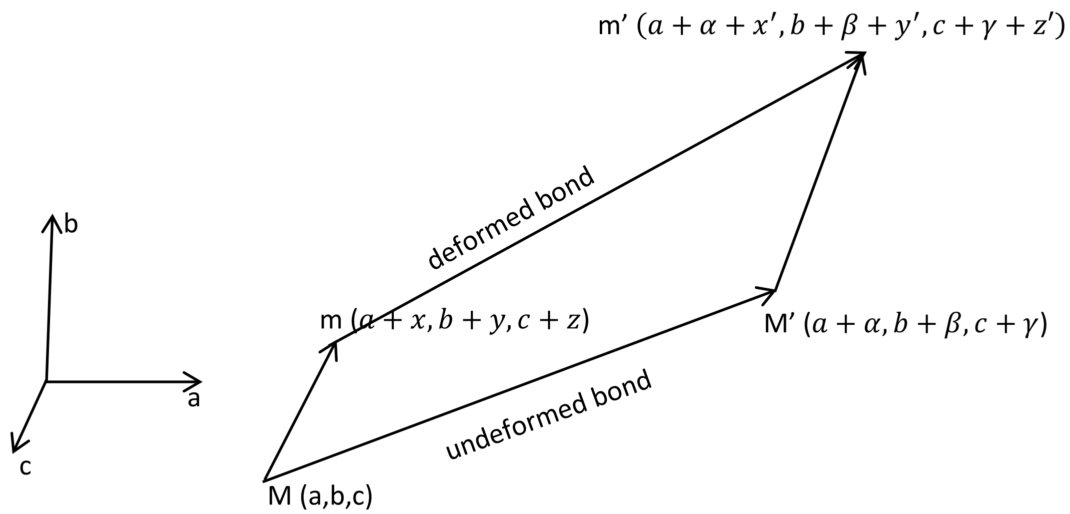
2.1 Navier states that "Between a molecule M, and any M' of the neighboring molecules, there exists an action P, which is the difference of these two forces. In the natural state of the body, all actions P are non-existent, or destroy each other reciprocally, because the molecule M is at rest." Describe and sketch how the pair-wise bond forces, P, between particles need not necessarily be zero in a body at rest, even without external loading.

Solution: The sketch below shows a hexagonal lattice body, with no external loading, in which each particle is in static equilibrium, and the entire body is in static equilibrium, but the bond forces between particles are not null. (Considered as a truss, the structure shown is not statically determinate.)



2.2 Navier’s paper has no sketches to help with understanding. Draw a sketch, showing the material point M and its neighbor M’, both before and after deformation. Annotate the sketch, using the geometric and kinematic symbols used in Navier’s paper.

Solution:



With Taylor series expansions of the displacement components (x, y, z):

$$
\begin{aligned}
x' = x &+ \frac{dx}{da}\alpha + \frac{1}{2}\frac{d^2x}{da^2}\alpha^2 + etc., \\
&+ \frac{dx}{db}\beta + \frac{d^2x}{dadb}\alpha\beta \\
&+ \frac{dx}{dc}\gamma + \frac{1}{2}\frac{d^2d}{db^2}\beta^2 \\
&+ \frac{d^2x}{dadc}\alpha\gamma \\
&+ \frac{d^2x}{dbdc}\beta\gamma \\
&+ \frac{1}{2}\frac{d^2x}{dc^2}\gamma^2
\end{aligned}
\qquad
\begin{aligned}
y' = y &+ \frac{dy}{da}\alpha + \frac{1}{2}\frac{d^2y}{da^2}\alpha^2 + etc., \\
&+ \frac{dy}{db}\beta + \frac{d^2y}{dadb}\alpha\beta \\
&+ \frac{dy}{dc}\gamma + \frac{1}{2}\frac{d^2y}{db^2}\beta^2 \\
&+ \frac{d^2y}{dadc}\alpha\gamma \\
&+ \frac{d^2y}{dbdc}\beta\gamma \\
&+ \frac{1}{2}\frac{d^2y}{dc^2}\gamma^2
\end{aligned}
\qquad
\begin{aligned}
z' = z &+ \frac{dz}{da}\alpha + \frac{1}{2}\frac{d^2z}{da^2}\alpha^2 + etc. \\
&+ \frac{dz}{db}\beta + \frac{d^2z}{dadb}\alpha\beta \\
&+ \frac{dz}{dc}\gamma + \frac{1}{2}\frac{d^2z}{db^2}\beta^2 \\
&+ \frac{d^2z}{dadc}\alpha\gamma \\
&+ \frac{d^2z}{dbdc}\beta\gamma \\
&+ \frac{1}{2}\frac{d^2z}{dc^2}\gamma^2
\end{aligned}
$$

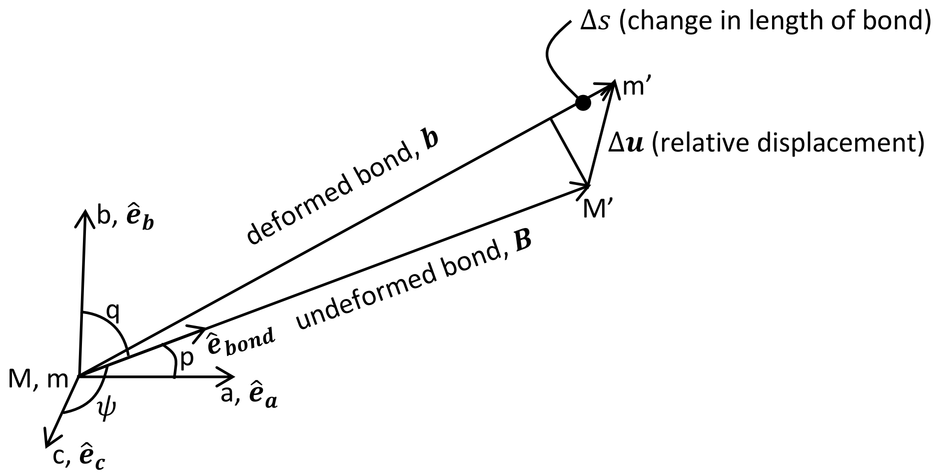
(Taylor developed his famous series approximation in 1715, about one hundred years before Navier’s work.)

2.3 Navier develops an expression for the change in distance between the two points M and M' due to deformation as:

$$\Delta s = (x' - x)cos.p + (y' - y)cos.q + (z' - z)sin.\psi$$

Using modern notation, draw a sketch and derive an equivalent expression for the change in length of the bond Δs.

Solution: Assuming small deformations, and translating the undeformed bond $\boldsymbol{B}$ and the deformed bond $\boldsymbol{b}$ to the origin of the coordinate system, we have the picture below.



The relative displacement vector is $\Delta\boldsymbol{u} = (x' - x)\hat{\boldsymbol{e}}_{\boldsymbol{a}} + (y' - y)\hat{\boldsymbol{e}}_{\boldsymbol{b}} + (z' - z)\hat{\boldsymbol{e}}_{\boldsymbol{c}}$, and the unit vector in the direction of the bond $\boldsymbol{B}$ (and of bond $\boldsymbol{b}$, assuming small displacements), is

$$\hat{\boldsymbol{e}}_{\boldsymbol{bond}} = cos.p\hat{\boldsymbol{e}}_{\boldsymbol{a}} + cos.q\hat{\boldsymbol{e}}_{\boldsymbol{b}} + sin.\psi\hat{\boldsymbol{e}}_{\boldsymbol{c}} ,$$

so the component of $\Delta\boldsymbol{u}$ in the direction of the bond is

$$\Delta s = \Delta\boldsymbol{u} \cdot \hat{\boldsymbol{e}}_{\boldsymbol{bond}} = (x' - x)cos.p + (y' - y)cos.q + (z' - z)sin.\psi$$

(Note: if the direction of the deformed bond $\boldsymbol{b}$ were changed significantly from the direction of the undeformed bond $\boldsymbol{B}$, then the formula for the change in length Δs would be more involved.)

2.4 Navier's theory has one elastic constant, ε. Show that this constant is equal to $\frac{2E}{5}$, where E is Young's modulus.

Solution: In part 7 of Navier's paper, he solved the problem of a cuboidal body subject to uniaxial strain, with no strain in the transverse directions. Let us consider the same problem.
Navier's equations for the tractions on the surface of any body are:

$$X' = \varepsilon\left[\cos.l\left(3\frac{dx'}{da'}+\frac{dy'}{db'}+\frac{dz'}{dc'}\right)+cos.m\left(\frac{dx'}{db'}+\frac{dy'}{da'}\right)+cos.n\left(\frac{dx'}{dc'}+\frac{dz'}{da'}\right)\right],$$
$$Y' = \varepsilon\left[\cos.l\left(\frac{dx'}{db'}+\frac{dy'}{da'}\right)+cos.m\left(\frac{dx'}{da'}+3\frac{dy'}{db'}+\frac{dz'}{dc'}\right)+cos.n\left(\frac{dy'}{dc'}+\frac{dz'}{db'}\right)\right],$$
$$Z' = \varepsilon\left[\cos.l\left(\frac{dx'}{dc}+\frac{dz'}{da'}\right)+cos.m\left(\frac{dy'}{dc'}+\frac{dz'}{db'}\right)+cos.n\left(\frac{dx'}{da'}+\frac{dy'}{db'}+3\frac{dz'}{dc'}\right)\right],$$

But in our problem, the only non-zero strain component is $\epsilon_{axial} = \frac{dx'}{da'}$, so these equations simplify to

$$\begin{aligned} \sigma_{axialX} &= \varepsilon\left[\cos.l\left(3\frac{dx'}{da'}\right)\right], & \sigma_{axialX} &= \varepsilon[3\epsilon_{axialX}], \\ \sigma_{transY} &= \varepsilon\left[cos.m\left(\frac{dx'}{da'}\right)\right], \text{ or } & \sigma_{transY} &= \varepsilon[\epsilon_{axialX}], \quad (*) \\ \sigma_{transZ} &= \varepsilon\left[cos.n\left(\frac{dx'}{da'}\right)\right], & \sigma_{transZ} &= \varepsilon[\epsilon_{axialX}], \end{aligned}$$

So the transverse normal stresses are one-third of the axial stress.
The Navier Cauchy relations between stress and strain (given by Eq. 9.38) are:

$$\begin{Bmatrix}\sigma_{XX}\\ \sigma_{YY}\\ \tau_{XY}\\ \sigma_{ZZ}\\ \tau_{YZ}\\ \tau_{XZ}\end{Bmatrix} = \frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix}(1-\nu) & \nu & 0 & \nu & 0 & 0\\ \nu & (1-\nu) & 0 & \nu & 0 & 0\\ 0 & 0 & \frac{(1-2\nu)}{2} & 0 & 0 & 0\\ \nu & \nu & 0 & (1-\nu) & 0 & 0\\ 0 & 0 & 0 & 0 & \frac{(1-2\nu)}{2} & 0\\ 0 & 0 & 0 & 0 & 0 & \frac{(1-2\nu)}{2}\end{bmatrix}\begin{Bmatrix}\epsilon_{XX}\\ \epsilon_{YY}=0\\ \gamma_{XY}=0\\ \epsilon_{ZZ}=0\\ \gamma_{YZ}=0\\ \gamma_{XZ}=0\end{Bmatrix}$$

So in our uniaxial strain problem,

$$\begin{Bmatrix}\sigma_{XX}\\ \sigma_{YY}\\ \tau_{XY}\\ \sigma_{ZZ}\\ \tau_{YZ}\\ \tau_{XZ}\end{Bmatrix} = \frac{E}{(1+\nu)(1-2\nu)}\begin{Bmatrix}(1-\nu)\epsilon_{XX}\\ (\nu)\epsilon_{XX}\\ 0\\ (\nu)\epsilon_{XX}\\ 0\\ 0\end{Bmatrix} \quad (**)$$

Thus equating the first and second equations of (*) and (**),

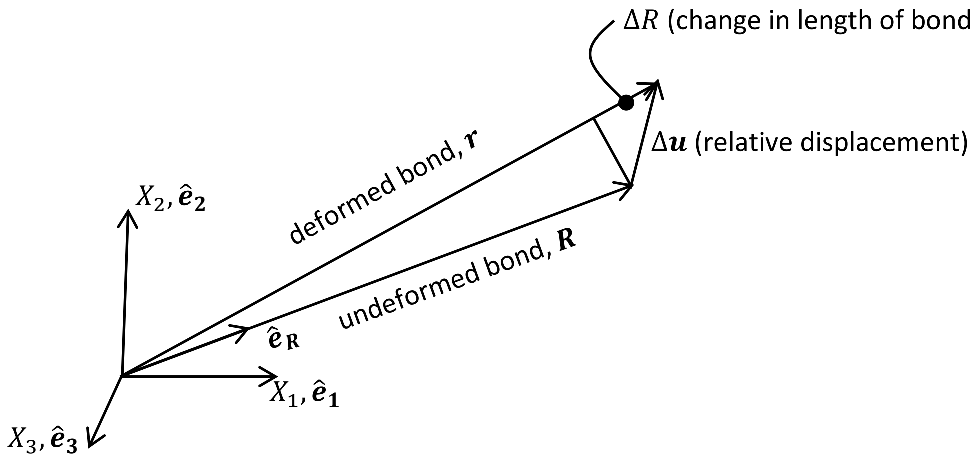
$$\varepsilon[3\epsilon_{axialX}] = \frac{E}{(1+\nu)(1-2\nu)}(1-\nu)\epsilon_{XX} \text{ and } [\epsilon_{axialX}] = \frac{E}{(1+\nu)(1-2\nu)}(\nu)\epsilon_{XX},$$

$$\text{or } 3\varepsilon = \frac{E}{(1+\nu)(1-2\nu)}(1-\nu) \text{ and } \varepsilon = \frac{E}{(1+\nu)(1-2\nu)}(\nu)$$

Solving these two equations simultaneously for ν and ε, we find that $\nu = \frac{1}{4}$ and $\varepsilon = \frac{2E}{5}$.

2.5 Derive Eq. 2.5: $\Delta R = {(u_1R_1 + u_2R_2 + u_3R_3)}/{R} = {(u_iR_i)}/{\sqrt{R_jR_j}}$.

Solution: Assuming small deformations, and translating the undeformed bond $\boldsymbol{R}$ and the deformed bond $\boldsymbol{r}$ to the origin of the coordinate system, we have the picture below.



The relative displacement vector is $\Delta\boldsymbol{u} = u_1\hat{\boldsymbol{e}}_1 + u_2\hat{\boldsymbol{e}}_2 + u_3\hat{\boldsymbol{e}}_3$, and the unit vector in the direction of the bond $\boldsymbol{R}$ (and of bond $\boldsymbol{r}$, assuming small displacements), is

$$\hat{\boldsymbol{e}}_R = \frac{R_1\hat{\boldsymbol{e}}_1 + R_2\hat{\boldsymbol{e}}_2 + R_3\hat{\boldsymbol{e}}_3}{\sqrt{R_1^2 + R_2^2 + R_3^2}},$$

so the component of $\Delta\boldsymbol{u}$ in the direction of the bond is

$$\Delta R = \Delta\boldsymbol{u} \cdot \hat{\boldsymbol{e}}_R = (u_1\hat{\boldsymbol{e}}_1 + u_2\hat{\boldsymbol{e}}_2 + u_3\hat{\boldsymbol{e}}_3) \cdot \left(\frac{R_1\hat{\boldsymbol{e}}_1 + R_2\hat{\boldsymbol{e}}_2 + R_3\hat{\boldsymbol{e}}_3}{\sqrt{R_1^2 + R_2^2 + R_3^2}}\right) =$$

$$= \frac{u_1R_1 + u_2R_2 + u_3R_3}{\sqrt{R_1^2 + R_2^2 + R_3^2}} = {(u_1R_1 + u_2R_2 + u_3R_3)}/{R} = {(u_iR_i)}/{\sqrt{R_jR_j}} \quad .$$

(Note: if the direction of the deformed bond $\boldsymbol{r}$ were changed significantly from the direction of the undeformed bond $\boldsymbol{R}$, then the formula for the change in length ΔR would be more involved.)

2.6 Derive Eq. 2.7 from Eq. 2.6

Solution: Eq. 2.6: $\Delta R = {(U_{i,k}R_k)R_i}\Big/{\sqrt{R_jR_j}}$.

Using indicial notation, and considering the denominator and using the Pythagorean theorem,

$$\sqrt{R_jR_j} = \sqrt{R_1R_1 + R_2R_2 + R_3R_3} = \sqrt{{R_1}^2 + {R_2}^2 + {R_3}^2} = R\ .$$

Using indicial notation, and considering the numerator,

$$(U_{i,k}R_k)R_i = (U_{1,k}R_k)R_1 + (U_{2,k}R_k)R_2 + (U_{3,k}R_k)R_3 =$$
$$(U_{1,1}R_1 + U_{1,2}R_2 + U_{1,3}R_3)R_1 + (U_{2,1}R_1 + U_{2,2}R_2 + U_{2,3}R_3)R_2 + (U_{3,1}R_1 + U_{3,2}R_2 + U_{3,3}R_3)R_3 =$$
$$= (U_{1,1}R_1^2 + (U_{1,2} + U_{2,1})R_1R_2 + (U_{1,3} + U_{3,1})R_1R_3 + U_{2,2}R_2^2 + (U_{2,3} + U_{3,2})R_2R_3 + U_{3,3}R_3^2)$$

So finally,

$$\Delta R = \begin{pmatrix} U_{1,1}R_1^2 + (U_{1,2} + U_{2,1})R_1R_2 + (U_{1,3} + U_{3,1})R_1R_3 \\ +U_{2,2}R_2^2 + (U_{2,3} + U_{3,2})R_2R_3 + U_{3,3}R_3^2 \end{pmatrix} \Big/ R,$$ which is Eq. 2.7 .