SPECIAL LINEAR SYSTEMS OF EQUATIONS

- **Symmetric positive definite matrices.**
- **The LDL^T decomposition; The Cholesky factorization**
- **Banded systems**

Positive-Definite Matrices

- A real matrix is said to be positive definite if

$$(Au, u) > 0 \text{ for all } u \neq 0 \; u \in \mathbb{R}^n$$

- Let A be a real positive definite matrix. Then there is a scalar $\alpha > 0$ such that

$$(Au, u) \geq \alpha \|u\|_2^2,$$

- Consider now the case of Symmetric Positive Definite (SPD) matrices.

- Consequence 1: A is nonsingular

- Consequence 2: the eigenvalues of A are (real) positive

A few properties of SPD matrices

➤ Diagonal entries of A are positive

➤ Each principal submatrix ($A(1:k, 1:k)$ in matlab notation) is SPD

➤ For any $n \times k$ matrix X of rank k, the matrix $X^T A X$ is SPD.

➤ The mapping :

$$x, y \quad \rightarrow \quad (x, y)_A \equiv (Ax, y)$$

is a proper inner product on $\mathbb{R}^n$. The associated norm, denoted by $\|.\|_A$, is called the energy norm:

$$\|x\|_A = (Ax, x)^{1/2} = \sqrt{x^T A x}$$

More terminology

➤ A matrix is Positive Semi-Definite if

$$(Au, u) \geq 0 \text{ for all } u \in \mathbb{R}^n$$

➤ Eigenvalues of symmetric positive semi-definite matrices are real nonnegative, i.e., ...

➤ ... A can be singular [If not, A is SPD]

➤ A matrix is said to be Negative Definite if $-A$ is positive definite. Similar definition for Negative Semi-Definite

➤ A matrix that is neither positive semi-definite nor negative semi-definite is indefinite

✍ Show that if $A^T = A$ and $(Ax, x) = 0 \forall x$ then $A = 0$

✍ Show: A is indefinite iff $\exists\ x, y : (Ax, x)(Ay, y) < 0$

The LDL^T and Cholesky factorizations

Consider the LU factorization of an SPD matrix A. Let $D = diag(U)$.

$$A = LU = LD \underbrace{(D^{-1}U)}_{M^T} \equiv LDM^T$$

- Both L and M are unit lower triangular
- Consider $\boxed{L^{-1}AL^{-T} = DM^TL^{-T}}$
- Matrix on the right is upper triangular. But it is also symmetric. Therefore $M^TL^{-T} = I$ and so $M = L$
- The diagonal entries of D are positive [Proof: consider $L^{-1}AL^{-T} = D$]. In the end:

$$A = LDL^T = GG^T \text{ where } G = LD^{1/2}$$

➤ Cholesky factorization is a specialization of the LU factorization for the SPD case. Several variants exist.

First algorithm: row-oriented LDLT

Adapted from Gaussian Elimination [Work only on upper triang. part]

1. For $k = 1 : n-1$ Do:
2. For $i = k+1 : n$ Do:
3. $piv := a(k,i)/a(k,k)$
4. $a(i, i:n) := a(i, i:n) - piv * a(k, i:n)$
5. End
6. End

➤ This will give the U matrix of the LU factorization. Therefore $D = diag(U)$, $L = D^{-1}U$.

Row-Cholesky (outer product form)

Scale the rows as the algorithm proceeds. Line 4 becomes

$$a(i,:) := a(i,:) - [a(k,i)/\sqrt{a(k,k)}] * \left[a(k,:)/\sqrt{a(k,k)}\right]$$

ALGORITHM : 1. Outer product Cholesky

1. For $k = 1 : n$ Do:
2. $\quad A(k, k:n) = A(k, k:n)/\sqrt{A(k,k)}$;
3. $\quad$ For $i := k+1 : n$ Do :
4. $\quad\quad A(i, i:n) = A(i, i:n) - A(k, i) * A(k, i:n);$
5. $\quad$ End
6. End

- Result: Upper triangular matrix U such $A = U^T U$.

Example:

$$A = \begin{pmatrix} 1 & -1 & 2 \\ -1 & 5 & 0 \\ 2 & 0 & 9 \end{pmatrix}$$

- ✍ **Is A symmetric positive definite?**
- ✍ **What is the LDL^T factorization of A ?**
- ✍ **What is the Cholesky factorization of A ?**

Column Cholesky.

Let $A = GG^T$ with G = lower triangular. Then equate j-th columns:

$$a(i,j) = \sum_{k=1}^{j} g(j,k) g^T(k,i) \rightarrow$$

$$\begin{aligned} A(:,j) &= \sum_{k=1}^{j} G(j,k) G(:,k) \\ &= G(j,j) G(:,j) + \sum_{k=1}^{j-1} G(j,k) G(:,k) \rightarrow \end{aligned}$$

$$G(j,j) G(:,j) = A(:,j) - \sum_{k=1}^{j-1} G(j,k) G(:,k)$$

- Assume that first $j-1$ columns of G already known.
- Compute unscaled column-vector:

$$v = A(:,j) - \sum_{k=1}^{j-1} G(j,k)G(:,k)$$

- Notice that $v(j) \equiv G(j,j)^2$.
- Compute $\sqrt{v(j)}$ scale v to get j-th column of G.

ALGORITHM : 2. Column Cholesky

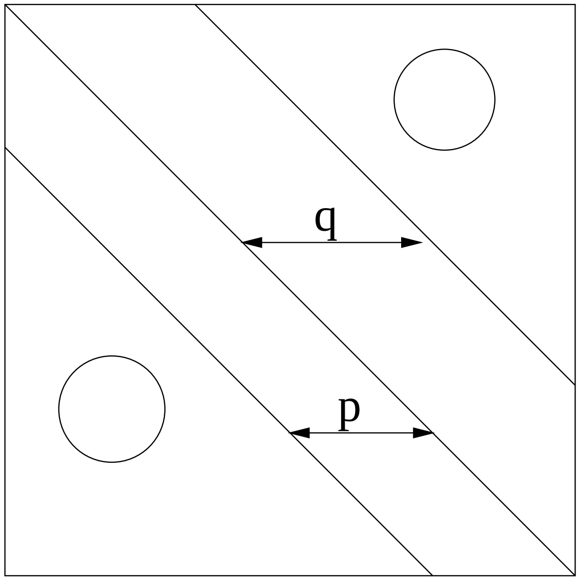
1. For $j = 1 : n$ do
2. For $k = 1 : j - 1$ do
3. $A(j : n, j) = A(j : n, j) - A(j, k) * A(j : n, k)$
4. EndDo
5. If $A(j, j) \le 0$ ExitError("Matrix not SPD")
6. $A(j, j) = \sqrt{A(j, j)}$
7. $A(j + 1 : n, j) = A(j + 1 : n, j)/A(j, j)$
8. EndDo

Example:

$$A = \begin{pmatrix} 1 & -1 & 2 \\ -1 & 5 & 0 \\ 2 & 0 & 9 \end{pmatrix}$$

Banded matrices

- Banded matrices arise in many applications
- A has upper bandwidth q if $a_{ij} = 0$ for $j - i > q$
- A has lower bandwidth p if $a_{ij} = 0$ for $i - j > p$



- Simplest case: tridiagonal ➤ $p = q = 1$.

➤ First observation: Gaussian elimination (no pivoting) preserves the initial banded form. Consider first step of Gaussian elimination:

2. For $i = 2 : n$ Do:
3. $a_{i1} := a_{i1}/a_{11}$ (pivots)
4. For $j := 2 : n$ Do :
5. $a_{ij} := a_{ij} - a_{i1} * a_{1j}$
6. End
7. End

➤ If A has upper bandwidth q and lower bandwidth p then so is the resulting $[L/U]$ matrix. ➤ Band form is preserved (induction)

✍ Operation count?

What happens when partial pivoting is used?

If A has lower bandwidth p, upper bandwidth q, and if Gaussian elimination with partial pivoting is used, then the resulting U has upper bandwidth $p+q$. L has at most $p+1$ nonzero elements per column (bandedness is lost).

➤ **Simplest case: tridiagonal** ➤ $p = q = 1$.

Example:

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 2 & 1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 1 & 0 \\ 0 & 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & 2 & 1 \end{pmatrix}$$