ASSIGNMENT 1

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ME 512 - Continuum Mechanics

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1 Given:

$$T_{pq} \Rightarrow \begin{bmatrix} -4 & 2 & 3 \\ 2 & 3 & 1 \\ 4 & 3 & 1 \end{bmatrix}; \qquad u_i \Rightarrow (0,-1,2); \qquad v_i \Rightarrow (1,-2,-3).$$

1.a What index (indices) should be associated with w?

1.b For each case obtain the set of numbers associated with w.

1.c For each case construct the corresponding matrix expression.

(i) $w_? = T_{ik}v_k$

a: w_i

b: $w_i = T_{i1}v_1 + T_{i2}v_2 + T_{i3}v_3$
$w_1 = T_{11}v_1 + T_{12}v_2 + T_{13}v_3$
$w_2 = T_{21}v_1 + T_{22}v_2 + T_{23}v_3$
$w_3 = T_{31}v_1 + T_{32}v_2 + T_{33}v_3$
$w_i \Rightarrow (-17,-7,-5)$

c: $\{w\} = [T]\{v\}$ or $< w > = < v > [T]$

$$\{w\} = \begin{Bmatrix} (-4*1)+(2*-2)+(3*-3) \\ (2*1)+(3*-2)+(1*-3) \\ (4*1)+(3*-2)+(1*-3) \end{Bmatrix} = \begin{Bmatrix} -17 \\ -7 \\ -5 \end{Bmatrix}$$

(ii) $w_? = u_k v_p$

a: w_{kp}

b: $w_{kp} = u_k v_p$
$w_{11} = u_1 v_1$

$$w_{kp} \Rightarrow \begin{bmatrix} 0 & 0 & 0 \\ -1 & 2 & 3 \\ 2 & -4 & -6 \end{bmatrix} \text{*note: no summation}$$

c: $$[w] = \{u\} < v > = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 2 & 3 \\ 2 & -4 & -6 \end{bmatrix}$$

(iii) $w_? = T_{km}v_k u_m$

a: w; no free index, therefore a scalar

b: $w = v_1T_{11}u_1 + v_2T_{21}u_1 + v_3T_{31}u_1 + v_1T_{12}u_2 + v_2T_{22}u_2 + v_1T_{32}u_2 + v_1T_{13}u_3 + v_2T_{23}u_3 + v_3T_{33}u_3 \Rightarrow 9$

c: $w =< v > [T]\{u\}$

$$< v > [T] =< 1, -2, -3 > \begin{bmatrix} -4 & 2 & 3 \\ 2 & 3 & 1 \\ 4 & 3 & 1 \end{bmatrix} =< -20, -13, -2 >$$

$$w =< -20, -13, -2 > \begin{Bmatrix} 0 \\ -1 \\ 2 \end{Bmatrix} = 9$$

(iv) $w_? = T_{ps}u_p$

a: w_s

b: $w_s = u_1T_{1s} + u_2T_{2s} + u_3T_{3s}$
$w_1 = u_1T_{11} + u_2T_{21} + u_3T_{31}$
$w_2 = u_1T_{12} + u_2T_{22} + u_3T_{32}$
$w_s \Rightarrow (6, 3, 1)$

c: $$< w >=< u > [T] =< 0, -1, 2 > \begin{bmatrix} -4 & 2 & 3 \\ 2 & 3 & 1 \\ 4 & 3 & 1 \end{bmatrix} =< 6, 3, 1 >$$

(v) $w_? = T_{rs}u_s v_r$

a: w; no free index, therefore a scalar

b: $w = u_rT_{rs}u_s$
$w = u_1T_{11}v_1 + u_1T_{12}v_2 + u_1T_{13}v_3 + u_2T_{21}v_1 + u_2T_{22}v_2 + u_2T_{23}v_3 + u_3T_{31}v_1 + u_3T_{32}v_2 + u_3T_{33}v_3 \Rightarrow -3$

c: $w =< u > [T]\{v\}$
from (iv): $< u > [T] =< 6, 3, 1 >$

$$w =< 6, 3, 1 > \begin{Bmatrix} 1 \\ -2 \\ -3 \end{Bmatrix} = -3$$

(vi) $w_? = T_{nn}$

a: w; no free index, therefore a scalar

b: $w = T_{nn}$
$w = T_{11} + T_{22} + T_{33} \Rightarrow 0$

c: $w = tr[T] = 0$

(vii) $w_? = T_{pq}T_{qr}$

a: w_{pr}

b: $$w_{pr} = \begin{bmatrix} T_{11}T_{11} + T_{12}T_{21} + T_{13}T_{31} & T_{11}T_{12} + T_{12}T_{22} + T_{13}T_{32} & T_{11}T_{13} + T_{12}T_{23} + T_{13}T_{33} \\ T_{21}T_{11} + T_{22}T_{21} + T_{23}T_{31} & T_{21}T_{12} + T_{22}T_{22} + T_{23}T_{32} & T_{21}T_{13} + T_{22}T_{23} + T_{23}T_{33} \\ T_{31}T_{11} + T_{32}T_{21} + T_{33}T_{31} & T_{31}T_{12} + T_{32}T_{22} + T_{33}T_{32} & T_{31}T_{13} + T_{32}T_{23} + T_{33}T_{33} \end{bmatrix} \Rightarrow \begin{bmatrix} 32 & 7 & -7 \\ 2 & 16 & 10 \\ -6 & 20 & 16 \end{bmatrix}$$

c: $$[w] = [T][T] = \begin{bmatrix} -4 & 2 & 3 \\ 2 & 3 & 1 \\ 4 & 3 & 1 \end{bmatrix} \begin{bmatrix} -4 & 2 & 3 \\ 2 & 3 & 1 \\ 4 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 32 & 7 & -7 \\ 2 & 16 & 10 \\ -6 & 20 & 16 \end{bmatrix}$$

(viii) $w_? = T_{pq}T_{pr} = T^T_{rp}T_{pq}$

a: w_{rq}

b: $w_{pr} = \begin{bmatrix} T^T_{11}T_{11}+T^T_{12}T_{21}+T^T_{13}T_{31} & T^T_{11}T_{12}+T^T_{12}T_{22}+T^T_{13}T_{32} & T^T_{11}T_{13}+T^T_{12}T_{23}+T^T_{13}T_{33} \\ T^T_{21}T_{11}+T^T_{22}T_{21}+T^T_{23}T_{31} & T^T_{21}T_{12}+T^T_{22}T_{22}+T^T_{23}T_{32} & T^T_{21}T_{13}+T^T_{22}T_{23}+T^T_{23}T_{33} \\ T^T_{31}T_{11}+T^T_{32}T_{21}+T^T_{33}T_{31} & T^T_{31}T_{12}+T^T_{32}T_{22}+T^T_{33}T_{32} & T^T_{31}T_{13}+T^T_{32}T_{23}+T^T_{33}T_{33} \end{bmatrix} \Rightarrow \begin{bmatrix} 36 & 10 & -6 \\ 10 & 22 & 12 \\ -6 & 12 & 11 \end{bmatrix}$

c: $[w] = [T]^T[T] = \begin{bmatrix} -4 & 2 & 4 \\ 2 & 3 & 3 \\ 3 & 1 & 1 \end{bmatrix} \begin{bmatrix} -4 & 2 & 3 \\ 2 & 3 & 1 \\ 4 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 36 & 10 & -6 \\ 10 & 22 & 12 \\ -6 & 12 & 11 \end{bmatrix}$

(ix) $w_? = T_{AB}T_{AB} = T^T_{BA}T_{AB}$

a: w; no free index, therefore a scalar

b: $w \Rightarrow 36 + 22 + 11 = 69$; trace of final matrix in (viii)

c: $w = tr[[T]^T[T]] = 69$

(x) $w_? = T_{pq}T_{qp}$

a: w; no free index, therefore a scalar

b: $w \Rightarrow 32 + 16 + 16 = 64$; trace of final matrix in (vii)

c: $[w] = tr[[T][T]] = 64$

2 What is wrong with each of the following indicial equations:

2.a $w_i = b_{ik}u_iv_k$

- No free indices exist; therefore, the result (w) should be a scalar instead of a vector.
- Correct form: $w = u_ib_{ik}v_k$ (prefered set order).

2.b $\phi = b_{ik}u_i$

- ϕ should be a set of three numbers, not a single scalar.
- Correct form: $\phi_k = u_ib_{ik}$.

2.c $\phi_{jp} = R_{ijkl}T_{kl}u_p$

- ϕ should be a set of 27 numbers (3^3), not a set of 9 numbers.
- Correct form: $\phi_{ijp} = R_{ijkl}T_{kl}u_p$.

3 Show that the ϵ - δ identity holds when the free indices assume the following values:

ϵ - δ identity: $\epsilon_{ijk}\epsilon_{irs} = \delta_{jr}\delta_{ks} - \delta_{js}\delta_{kr}$

(j, k, r, s) =

- (1, 1, 1, 1):
  $\epsilon_{i11}\epsilon_{i11} = \epsilon_{111}\epsilon_{111} + \epsilon_{211}\epsilon_{211} + \epsilon_{311}\epsilon_{311} = 0 + 0 + 0 = 0$
  $\delta_{11}\delta_{11} - \delta_{11}\delta_{11} = 1 - 1 = 0$

- (1, 1, 1, 2):
  $\epsilon_{i11}\epsilon_{i12} = \epsilon_{111}\epsilon_{112} + \epsilon_{211}\epsilon_{212} + \epsilon_{311}\epsilon_{312} = 0 + 0 + 0 = 0$
  $\delta_{11}\delta_{12} - \delta_{12}\delta_{11} = 0 - 0 = 0$
- (1, 1, 1, 3):
  $\epsilon_{i11}\epsilon_{i13} = \epsilon_{111}\epsilon_{113} + \epsilon_{211}\epsilon_{213} + \epsilon_{311}\epsilon_{313} = 0 + 0 + 0 = 0$
  $\delta_{11}\delta_{13} - \delta_{13}\delta_{11} = 0 - 0 = 0$
- (1, 1, 2, 1):
  $\epsilon_{i11}\epsilon_{i21} = \epsilon_{111}\epsilon_{121} + \epsilon_{211}\epsilon_{221} + \epsilon_{311}\epsilon_{321} = 0 + 0 + 0 = 0$
  $\delta_{12}\delta_{11} - \delta_{11}\delta_{12} = 0 - 0 = 0$
- (1, 1, 2, 2):
  $\epsilon_{i11}\epsilon_{i22} = \epsilon_{111}\epsilon_{122} + \epsilon_{211}\epsilon_{222} + \epsilon_{311}\epsilon_{322} = 0 + 0 + 0 = 0$
  $\delta_{12}\delta_{12} - \delta_{12}\delta_{12} = 0 - 0 = 0$
- (1, 1, 2, 3):
  $\epsilon_{i11}\epsilon_{i23} = \epsilon_{111}\epsilon_{123} + \epsilon_{211}\epsilon_{223} + \epsilon_{311}\epsilon_{323} = 0 + 0 + 0 = 0$
  $\delta_{12}\delta_{13} - \delta_{13}\delta_{12} = 0 - 0 = 0$
- (1, 2, 2, 3):
  $\epsilon_{i12}\epsilon_{i23} = \epsilon_{112}\epsilon_{123} + \epsilon_{212}\epsilon_{223} + \epsilon_{312}\epsilon_{323} = 0 + 0 + 0 = 0$
  $\delta_{12}\delta_{13} - \delta_{13}\delta_{22} = 0 - 0 = 0$

4 Show that the alternating symbol-determinant identity holds when (l, m, n) = (1, 2, 3)

Alternating symbol-determinant identity: $\epsilon_{ijk}a_{il}a_{jm}a_{kn} = \epsilon_{lmn}|[a]|$

- $\epsilon_{ijk}a_{il}a_{jm}a_{kn} =$
  $\epsilon_{111}a_{1l}a_{1m}a_{1n} + \epsilon_{211}a_{2l}a_{1m}a_{1n} + \epsilon_{311}a_{3l}a_{1m}a_{1n} +$
  $\epsilon_{121}a_{1l}a_{2m}a_{1n} + \epsilon_{221}a_{2l}a_{2m}a_{1n} + \epsilon_{321}a_{3l}a_{2m}a_{1n} +$
  $\epsilon_{131}a_{1l}a_{3m}a_{1n} + \epsilon_{231}a_{2l}a_{3m}a_{1n} + \epsilon_{331}a_{3l}a_{3m}a_{1n} +$

  $\epsilon_{112}a_{1l}a_{1m}a_{2n} + \epsilon_{212}a_{2l}a_{1m}a_{2n} + \epsilon_{312}a_{3l}a_{1m}a_{2n} +$
  $\epsilon_{122}a_{1l}a_{2m}a_{2n} + \epsilon_{222}a_{2l}a_{2m}a_{2n} + \epsilon_{322}a_{3l}a_{2m}a_{2n} +$
  $\epsilon_{132}a_{1l}a_{3m}a_{2n} + \epsilon_{232}a_{2l}a_{3m}a_{2n} + \epsilon_{332}a_{3l}a_{3m}a_{2n} +$

  $\epsilon_{113}a_{1l}a_{1m}a_{3n} + \epsilon_{213}a_{2l}a_{1m}a_{3n} + \epsilon_{313}a_{3l}a_{1m}a_{3n} +$
  $\epsilon_{123}a_{1l}a_{2m}a_{3n} + \epsilon_{223}a_{2l}a_{2m}a_{3n} + \epsilon_{323}a_{3l}a_{2m}a_{3n} +$
  $\epsilon_{133}a_{1l}a_{3m}a_{3n} + \epsilon_{233}a_{2l}a_{3m}a_{3n} + \epsilon_{333}a_{3l}a_{3m}a_{3n}$

- assign value to Alternating Symbol:

$$\epsilon_{ijk}a_{il}a_{jm}a_{kn} = \begin{pmatrix} 0+0+0+ \\ 0+0-a_{3l}a_{2m}a_{1n}+ \\ 0+a_{2l}a_{3m}a_{1n}+0+ \\ \\ 0+0+a_{3l}a_{1m}a_{2n}+ \\ 0+0+0+ \\ -a_{1l}a_{3m}a_{2n}+0+0+ \\ \\ 0-a_{2l}a_{1m}a_{3n}+0+ \\ a_{1l}a_{2m}a_{3n}+0+0+ \\ 0+0+0 \end{pmatrix}$$

- replace (l, m, n) with (1, 2, 3):
  $-a_{31}a_{22}a_{13} + a_{21}a_{32}a_{13} + a_{31}a_{12}a_{23} - a_{11}a_{32}a_{23} - a_{21}a_{12}a_{33} + a_{11}a_{22}a_{33}$

- rearrange (note: factors have been arranged such that the first indices are 123):
  $\epsilon_{ijk}a_{i1}a_{j2}a_{k3} = \big(a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}\big)$
- $\epsilon_{lmn}|[a]|$ for (l, m, n) = (1, 2, 3):
  $\epsilon_{123}|[a]| = \big(a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}\big)$

$\therefore \epsilon_{ijk}a_{i1}a_{j2}a_{k3} = \epsilon_{123}|[a]|$

5 Using the alternating symbol-determinant identity, prove the determinant of the product of two matrices equals the product of the determinants of the matrices.

E.g., $|[a][b]| = |[a]||[b]|$

- determine $[a][b]$:
  $$[a][b] = [c] \Rightarrow \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} & a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} & a_{11}b_{13} + a_{12}b_{23} + a_{13}b_{33} \\ a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} & a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32} & a_{21}b_{13} + a_{22}b_{23} + a_{23}b_{33} \\ a_{31}b_{11} + a_{32}b_{21} + a_{33}b_{31} & a_{31}b_{12} + a_{32}b_{22} + a_{33}b_{32} & a_{31}b_{13} + a_{32}b_{23} + a_{33}b_{33} \end{bmatrix}$$

- calculate determinant:
  if $[a][b] = [c]$, then $|[a][b]| = |[c]| \Rightarrow \big(c_{11}c_{22}c_{33} + c_{12}c_{23}c_{31} + c_{13}c_{21}c_{32} - c_{13}c_{22}c_{31} - c_{11}c_{23}c_{32} - c_{12}c_{21}c_{33}\big)$
- calculate determinant with ϵ_{ijk}:
  $$\epsilon_{ijk}c_{1i}c_{2j}c_{3k} = \begin{pmatrix} \epsilon_{111}c_{11}c_{21}c_{31} + \epsilon_{121}c_{11}c_{22}c_{31} + \epsilon_{131}c_{11}c_{23}c_{31} + \\ \epsilon_{211}c_{12}c_{21}c_{31} + \epsilon_{221}c_{12}c_{22}c_{31} + \epsilon_{231}c_{12}c_{23}c_{31} + \\ \epsilon_{311}c_{13}c_{21}c_{31} + \epsilon_{321}c_{13}c_{22}c_{31} + \epsilon_{331}c_{13}c_{23}c_{31} + \\ \\ \epsilon_{112}c_{11}c_{21}c_{32} + \epsilon_{122}c_{11}c_{22}c_{32} + \epsilon_{132}c_{11}c_{23}c_{32} + \\ \epsilon_{212}c_{12}c_{21}c_{32} + \epsilon_{222}c_{12}c_{22}c_{32} + \epsilon_{232}c_{12}c_{23}c_{32} + \\ \epsilon_{312}c_{13}c_{21}c_{32} + \epsilon_{322}c_{13}c_{22}c_{32} + \epsilon_{332}c_{13}c_{23}c_{32} + \\ \\ \epsilon_{113}c_{11}c_{21}c_{33} + \epsilon_{123}c_{11}c_{22}c_{33} + \epsilon_{133}c_{11}c_{23}c_{33} + \\ \epsilon_{213}c_{12}c_{21}c_{33} + \epsilon_{223}c_{12}c_{22}c_{33} + \epsilon_{233}c_{12}c_{23}c_{33} + \\ \epsilon_{313}c_{13}c_{21}c_{33} + \epsilon_{323}c_{13}c_{22}c_{33} + \epsilon_{333}c_{13}c_{23}c_{33} \end{pmatrix}$$
  $$|[c]| \Rightarrow \epsilon_{ijk}c_{1i}c_{2j}c_{3k} = \begin{pmatrix} 0 + 0 + 0 + \\ 0 + 0 + c_{12}c_{23}c_{31} + \\ 0 - c_{13}c_{22}c_{31} + 0 + \\ \\ 0 + 0 - c_{11}c_{23}c_{32} + \\ 0 + 0 + 0 + \\ c_{13}c_{21}c_{32} + 0 + 0 + \\ \\ 0 + c_{11}c_{22}c_{33} + 0 + \\ -c_{12}c_{21}c_{33} + 0 + 0 + \\ 0 + 0 + 0 \end{pmatrix} = \begin{pmatrix} c_{11}c_{22}c_{33} + c_{12}c_{23}c_{31} + c_{13}c_{21}c_{32} \\ -c_{13}c_{22}c_{31} - c_{11}c_{23}c_{32} - c_{12}c_{21}c_{33} \end{pmatrix}$$

$$
\epsilon_{ijk}c_{1i}c_{2j}c_{3k} = \begin{pmatrix}
(a_{11}b_{11}+a_{12}b_{21}+a_{13}b_{31})\,(a_{21}b_{12}+a_{22}b_{22}+a_{23}b_{32})\,(a_{31}b_{13}+a_{32}b_{23}+a_{33}b_{33})+ \\
(a_{11}b_{12}+a_{12}b_{22}+a_{13}b_{32})\,(a_{21}b_{13}+a_{22}b_{23}+a_{23}b_{33})\,(a_{31}b_{11}+a_{32}b_{21}+a_{33}b_{31})+ \\
(a_{11}b_{13}+a_{12}b_{23}+a_{13}b_{33})\,(a_{21}b_{11}+a_{22}b_{21}+a_{23}b_{31})\,(a_{31}b_{12}+a_{32}b_{22}+a_{33}b_{32})- \\
\\
(a_{11}b_{13}+a_{12}b_{23}+a_{13}b_{33})\,(a_{21}b_{12}+a_{22}b_{22}+a_{23}b_{32})\,(a_{31}b_{11}+a_{32}b_{21}+a_{33}b_{31})- \\
(a_{11}b_{11}+a_{12}b_{21}+a_{13}b_{31})\,(a_{21}b_{13}+a_{22}b_{23}+a_{23}b_{33})\,(a_{31}b_{12}+a_{32}b_{22}+a_{33}b_{32})- \\
(a_{11}b_{12}+a_{12}b_{22}+a_{13}b_{32})\,(a_{21}b_{11}+a_{22}b_{21}+a_{23}b_{31})\,(a_{31}b_{13}+a_{32}b_{23}+a_{33}b_{33})
\end{pmatrix}
$$

- expand polynomials:

$$
\epsilon_{ijk}c_{1i}c_{2j}c_{3k} = \begin{pmatrix}
+a_{11}a_{22}a_{33}b_{11}b_{22}b_{33}+a_{11}a_{22}a_{33}b_{12}b_{23}b_{31}+a_{11}a_{22}a_{33}b_{13}b_{21}b_{32} \\
-a_{11}a_{22}a_{33}b_{13}b_{22}b_{31}-a_{11}a_{22}a_{33}b_{11}b_{23}b_{32}-a_{11}a_{22}a_{33}b_{12}b_{21}b_{33} \\
\\
+a_{12}a_{23}a_{31}b_{11}b_{22}b_{33}+a_{12}a_{23}a_{31}b_{12}b_{23}b_{31}+a_{12}a_{23}a_{31}b_{13}b_{21}b_{32} \\
-a_{12}a_{23}a_{31}b_{13}b_{22}b_{31}-a_{12}a_{23}a_{31}b_{11}b_{23}b_{32}-a_{12}a_{23}a_{31}b_{12}b_{21}b_{33} \\
\\
+a_{13}a_{21}a_{32}b_{11}b_{22}b_{33}+a_{13}a_{21}a_{32}b_{11}b_{22}b_{33}+a_{13}a_{21}a_{32}b_{13}b_{21}b_{32} \\
-a_{13}a_{21}a_{32}b_{13}b_{22}b_{31}-a_{13}a_{21}a_{32}b_{11}b_{23}b_{32}-a_{13}a_{21}a_{32}b_{12}b_{21}b_{33} \\
\\
-a_{13}a_{22}a_{31}b_{11}b_{22}b_{33}-a_{13}a_{22}a_{31}b_{11}b_{22}b_{33}-a_{13}a_{22}a_{31}b_{13}b_{21}b_{32} \\
+a_{13}a_{22}a_{31}b_{13}b_{22}b_{31}+a_{13}a_{22}a_{31}b_{11}b_{23}b_{32}+a_{13}a_{22}a_{31}b_{12}b_{21}b_{33} \\
\\
-a_{11}a_{23}a_{32}b_{11}b_{22}b_{33}-a_{11}a_{23}a_{32}b_{11}b_{22}b_{33}-a_{11}a_{23}a_{32}b_{13}b_{21}b_{32} \\
+a_{11}a_{23}a_{32}b_{13}b_{22}b_{31}+a_{11}a_{23}a_{32}b_{11}b_{23}b_{32}+a_{11}a_{23}a_{32}b_{12}b_{21}b_{33} \\
\\
-a_{12}a_{21}a_{33}b_{11}b_{22}b_{33}-a_{12}a_{21}a_{33}b_{11}b_{22}b_{33}-a_{12}a_{21}a_{33}b_{13}b_{21}b_{32} \\
+a_{12}a_{21}a_{33}b_{13}b_{22}b_{31}+a_{12}a_{21}a_{33}b_{11}b_{23}b_{32}+a_{12}a_{21}a_{33}b_{12}b_{21}b_{33}
\end{pmatrix}
$$

- calculate $|[a]|$ with ϵ_{ijk}:

$$
\epsilon_{ijk}a_{i1}a_{j2}a_{k3} = \begin{pmatrix}
\epsilon_{111}a_{11}a_{21}a_{31}+\epsilon_{121}a_{11}a_{22}a_{31}+\epsilon_{131}a_{11}a_{23}a_{31}+ \\
\epsilon_{211}a_{12}a_{21}a_{31}+\epsilon_{221}a_{12}a_{22}a_{31}+\epsilon_{231}a_{12}a_{23}a_{31}+ \\
\epsilon_{311}a_{13}a_{21}a_{31}+\epsilon_{321}a_{13}a_{22}a_{31}+\epsilon_{331}a_{13}a_{23}a_{31}+ \\
\\
\epsilon_{112}a_{11}a_{21}a_{32}+\epsilon_{122}a_{11}a_{22}a_{32}+\epsilon_{132}a_{11}a_{23}a_{32}+ \\
\epsilon_{212}a_{12}a_{21}a_{32}+\epsilon_{222}a_{12}a_{22}a_{32}+\epsilon_{232}a_{12}a_{23}a_{32}+ \\
\epsilon_{312}a_{13}a_{21}a_{32}+\epsilon_{322}a_{13}a_{22}a_{32}+\epsilon_{332}a_{13}a_{23}a_{32}+ \\
\\
\epsilon_{113}a_{11}a_{21}a_{33}+\epsilon_{123}a_{11}a_{22}a_{33}+\epsilon_{133}a_{11}a_{23}a_{33}+ \\
\epsilon_{213}a_{12}a_{21}a_{33}+\epsilon_{223}a_{12}a_{22}a_{33}+\epsilon_{233}a_{12}a_{23}a_{33}+ \\
\epsilon_{313}a_{13}a_{21}a_{33}+\epsilon_{323}a_{13}a_{22}a_{33}+\epsilon_{333}a_{13}a_{23}a_{33}
\end{pmatrix}
$$

$$
|[a]| \Rightarrow \epsilon_{ijk}a_{i1}a_{j2}a_{k3} = \begin{pmatrix}
0+0+0+ \\
0+0+a_{12}a_{23}a_{31}+ \\
0-a_{13}a_{22}a_{31}+0+ \\
\\
0+0-a_{11}a_{23}a_{32}+ \\
0+0+0+ \\
a_{13}a_{21}a_{32}+0+0+ \\
\\
0+a_{11}a_{22}a_{33}+0+ \\
-a_{12}a_{21}a_{33}+0+0+ \\
0+0+0
\end{pmatrix} = \begin{pmatrix}
a_{11}a_{22}a_{33}+a_{12}a_{23}a_{31}+a_{13}a_{21}a_{32} \\
-a_{13}a_{22}a_{31}-a_{11}a_{23}a_{32}-a_{12}a_{21}a_{33}
\end{pmatrix}
$$

- calculate $|[b]|$ with ϵ_{pqr}:

$$\epsilon_{pqr}b_{p1}b_{q2}b_{r3} = \begin{pmatrix} \epsilon_{111}b_{11}b_{21}b_{31} + \epsilon_{121}b_{11}b_{22}b_{31} + \epsilon_{131}b_{11}b_{23}b_{31}+ \\ \epsilon_{211}b_{12}b_{21}b_{31} + \epsilon_{221}b_{12}b_{22}b_{31} + \epsilon_{231}b_{12}b_{23}b_{31}+ \\ \epsilon_{311}b_{13}b_{21}b_{31} + \epsilon_{321}b_{13}b_{22}b_{31} + \epsilon_{331}b_{13}b_{23}b_{31}+ \\ \\ \epsilon_{112}b_{11}b_{21}b_{32} + \epsilon_{122}b_{11}b_{22}b_{32} + \epsilon_{132}b_{11}b_{23}b_{32}+ \\ \epsilon_{212}b_{12}b_{21}b_{32} + \epsilon_{222}b_{12}b_{22}b_{32} + \epsilon_{232}b_{12}b_{23}b_{32}+ \\ \epsilon_{312}b_{13}b_{21}b_{32} + \epsilon_{322}b_{13}b_{22}b_{32} + \epsilon_{332}b_{13}b_{23}b_{32}+ \\ \\ \epsilon_{113}b_{11}b_{21}b_{33} + \epsilon_{123}b_{11}b_{22}b_{33} + \epsilon_{133}b_{11}b_{23}b_{33}+ \\ \epsilon_{213}b_{12}b_{21}b_{33} + \epsilon_{223}b_{12}b_{22}b_{33} + \epsilon_{233}b_{12}b_{23}b_{33}+ \\ \epsilon_{313}b_{13}b_{21}b_{33} + \epsilon_{323}b_{13}b_{22}b_{33} + \epsilon_{333}b_{13}b_{23}b_{33} \end{pmatrix}$$

$$|[b]| \Rightarrow \epsilon_{pqr}b_{p1}b_{q2}b_{r3} = \begin{pmatrix} 0+0+0+ \\ 0+0+b_{12}b_{23}b_{31}+ \\ 0-b_{13}b_{22}b_{31}+0+ \\ \\ 0+0-b_{11}b_{23}b_{32}+ \\ 0+0+0+ \\ b_{13}b_{21}b_{32}+0+0+ \\ \\ 0+b_{11}b_{22}b_{33}+0+ \\ -b_{12}b_{21}b_{33}+0+0+ \\ 0+0+0 \end{pmatrix} = \begin{pmatrix} b_{11}b_{22}b_{33} + b_{12}b_{23}b_{31} + b_{13}b_{21}b_{32} \\ -b_{13}b_{22}b_{31} - b_{11}b_{23}b_{32} - b_{12}b_{21}b_{33} \end{pmatrix}$$

- then multiply $(\epsilon_{ijk}a_{i1}a_{j2}a_{k3})$*$(\epsilon_{pqr}b_{p1}b_{q2}b_{r3})$:
  $\epsilon_{ijk}a_{i1}a_{j2}a_{k3}\epsilon_{pqr}b_{p1}b_{q2}b_{r3} = \begin{pmatrix} a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \\ -a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} \end{pmatrix} * \begin{pmatrix} b_{11}b_{22}b_{33} + b_{11}b_{22}b_{33} + b_{13}b_{21}b_{32} \\ -b_{13}b_{22}b_{31} - b_{11}b_{23}b_{32} - b_{12}b_{21}b_{33} \end{pmatrix}$

- expand polynomials:

$$\epsilon_{ijk}a_{i1}a_{j2}a_{k3}\epsilon_{pqr}b_{p1}b_{q2}b_{r3} = \begin{pmatrix} +a_{11}a_{22}a_{33}b_{11}b_{22}b_{33} + a_{11}a_{22}a_{33}b_{12}b_{23}b_{31} + a_{11}a_{22}a_{33}b_{13}b_{21}b_{32} \\ -a_{11}a_{22}a_{33}b_{13}b_{22}b_{31} - a_{11}a_{22}a_{33}b_{11}b_{23}b_{32} - a_{11}a_{22}a_{33}b_{12}b_{21}b_{33} \\ \\ +a_{12}a_{23}a_{31}b_{11}b_{22}b_{33} + a_{12}a_{23}a_{31}b_{12}b_{23}b_{31} + a_{12}a_{23}a_{31}b_{13}b_{21}b_{32} \\ -a_{12}a_{23}a_{31}b_{13}b_{22}b_{31} - a_{12}a_{23}a_{31}b_{11}b_{23}b_{32} - a_{12}a_{23}a_{31}b_{12}b_{21}b_{33} \\ \\ +a_{13}a_{21}a_{32}b_{11}b_{22}b_{33} + a_{13}a_{21}a_{32}b_{11}b_{22}b_{33} + a_{13}a_{21}a_{32}b_{13}b_{21}b_{32} \\ -a_{13}a_{21}a_{32}b_{13}b_{22}b_{31} - a_{13}a_{21}a_{32}b_{11}b_{23}b_{32} - a_{13}a_{21}a_{32}b_{12}b_{21}b_{33} \\ \\ -a_{13}a_{22}a_{31}b_{11}b_{22}b_{33} - a_{13}a_{22}a_{31}b_{11}b_{22}b_{33} - a_{13}a_{22}a_{31}b_{13}b_{21}b_{32} \\ +a_{13}a_{22}a_{31}b_{13}b_{22}b_{31} + a_{13}a_{22}a_{31}b_{11}b_{23}b_{32} + a_{13}a_{22}a_{31}b_{12}b_{21}b_{33} \\ \\ -a_{11}a_{23}a_{32}b_{11}b_{22}b_{33} - a_{11}a_{23}a_{32}b_{11}b_{22}b_{33} - a_{11}a_{23}a_{32}b_{13}b_{21}b_{32} \\ +a_{11}a_{23}a_{32}b_{13}b_{22}b_{31} + a_{11}a_{23}a_{32}b_{11}b_{23}b_{32} + a_{11}a_{23}a_{32}b_{12}b_{21}b_{33} \\ \\ -a_{12}a_{21}a_{33}b_{11}b_{22}b_{33} - a_{12}a_{21}a_{33}b_{11}b_{22}b_{33} - a_{12}a_{21}a_{33}b_{13}b_{21}b_{32} \\ +a_{12}a_{21}a_{33}b_{13}b_{22}b_{31} + a_{12}a_{21}a_{33}b_{11}b_{23}b_{32} + a_{12}a_{21}a_{33}b_{12}b_{21}b_{33} \end{pmatrix}$$

- therefore:
  $\epsilon_{ijk}c_{1i}c_{2j}c_{3k} = \epsilon_{ijk}a_{i1}a_{j2}a_{k3}\epsilon_{pqr}b_{p1}b_{q2}b_{r3} \Rightarrow |[a]||[b]| = |[c]| = |[a][b]|$

6 Use the cofactor matrix approach to find the inverse of [T].

- Given:
  $[T] = \begin{bmatrix} -4 & 2 & 3 \\ 2 & 3 & 1 \\ 4 & 3 & 1 \end{bmatrix}$

- calculate determinant and cofactor matrix:

$$|[T]| = -14; \qquad [T^{cf}] = \begin{bmatrix} T^{cf}_{11} & T^{cf}_{12} & T^{cf}_{13} \\ T^{cf}_{21} & T^{cf}_{22} & T^{cf}_{23} \\ T^{cf}_{31} & T^{cf}_{32} & T^{cf}_{33} \end{bmatrix}$$

- where the minor matrices are:

$$T^{cf}_{11} = \begin{vmatrix} 3 & 1 \\ 3 & 1 \end{vmatrix} = 0; \qquad T^{cf}_{12} = -\begin{vmatrix} 2 & 1 \\ 4 & 1 \end{vmatrix} = 2; \qquad T^{cf}_{13} = \begin{vmatrix} 2 & 3 \\ 4 & 3 \end{vmatrix} = -6;$$

$$T^{cf}_{21} = -\begin{vmatrix} 2 & 3 \\ 3 & 1 \end{vmatrix} = 7; \qquad T^{cf}_{22} = \begin{vmatrix} -4 & 3 \\ 4 & 1 \end{vmatrix} = -16; \qquad T^{cf}_{23} = -\begin{vmatrix} -4 & 2 \\ 4 & 3 \end{vmatrix} = 20;$$

$$T^{cf}_{31} = \begin{vmatrix} 2 & 3 \\ 3 & 1 \end{vmatrix} = -7; \qquad T^{cf}_{32} = -\begin{vmatrix} -4 & 3 \\ 2 & 1 \end{vmatrix} = 10; \qquad T^{cf}_{33} = \begin{vmatrix} -4 & 2 \\ 2 & 3 \end{vmatrix} = -16;$$

- calculate the transpose of the cofactor:
  $[T^{cf}]^T = \begin{bmatrix} 0 & 7 & -7 \\ 2 & -16 & 10 \\ -6 & 20 & -16 \end{bmatrix}$; known as the "adjoint"

- calculate the inverse:
  $[T]^{-1} = \frac{1}{|[T]|}[T^{cf}]^T = \begin{bmatrix} 0 & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{7} & -\frac{8}{7} & -\frac{5}{7} \\ \frac{3}{7} & -\frac{10}{7} & \frac{8}{7} \end{bmatrix}$