ASSIGNMENT 3

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ME 512 - Continuum Mechanics

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1 Suppose the following:

$$T_{pq} \Rightarrow \begin{bmatrix} -1 & 2 & 3 \\ 2 & -2 & 2 \\ 4 & 3 & 4 \end{bmatrix} \qquad u_i \Rightarrow (1, -2, 2) \qquad v_i \Rightarrow (-2, 1, -3)$$

Notes on the nomenclature:

- $\boldsymbol{I}$ denotes the identity tensor.
- Upper case letters indicate a second order tensor and lower case letters indicate a vector.
- An implied basis of $\boldsymbol{e_i}$ and $\boldsymbol{e_i} \otimes \boldsymbol{e_j}$ as used for vectors and tensors, respectively.

$$\boldsymbol{T} = \begin{matrix} -1(\boldsymbol{e_1} \otimes \boldsymbol{e_1}) + 2(\boldsymbol{e_1} \otimes \boldsymbol{e_2}) + 3(\boldsymbol{e_1} \otimes \boldsymbol{e_3}) \\ +2(\boldsymbol{e_2} \otimes \boldsymbol{e_1}) - 2(\boldsymbol{e_2} \otimes \boldsymbol{e_2}) + 2(\boldsymbol{e_2} \otimes \boldsymbol{e_3}) \\ +4(\boldsymbol{e_3} \otimes \boldsymbol{e_1}) + 3(\boldsymbol{e_3} \otimes \boldsymbol{e_2}) + 4(\boldsymbol{e_3} \otimes \boldsymbol{e_3}) \end{matrix} \qquad \boldsymbol{u} = 1\boldsymbol{e_1} - 2\boldsymbol{e_2} + 2\boldsymbol{e_3} \qquad \boldsymbol{v} = -2\boldsymbol{e_1} + 1\boldsymbol{e_2} - 3\boldsymbol{e_3}$$

- or more succinctly as:
  $\boldsymbol{T} = T_{pq}(\boldsymbol{e_p} \otimes \boldsymbol{e_q}) \qquad \boldsymbol{u} = u_i\boldsymbol{e_i} \qquad \boldsymbol{v} = v_i\boldsymbol{e_i}$

(a) $? = \boldsymbol{u} \bullet \boldsymbol{v}$

(i) $w = \boldsymbol{u} \bullet \boldsymbol{v}$ (a scalar)

(ii) $w = \boldsymbol{u} \bullet \boldsymbol{v}$
$w = u_i v_i = u_1 v_1 + u_2 v_2 + u_3 v_3$
$w = < u > \{v\}$

(iii) $w = -10$

(b) $? = \boldsymbol{T} \bullet \boldsymbol{u}$

(i) $w_i\boldsymbol{e_i} \Rightarrow \boldsymbol{T} \bullet \boldsymbol{u} \Rightarrow T_{ij}(\boldsymbol{e_i} \otimes \boldsymbol{e_j}) \bullet v_k\boldsymbol{e_k} = T_{ij}v_k\delta_{jk}\boldsymbol{e_i} = T_{ij}v_j\boldsymbol{e_i}$ (components and base vector)

(ii) $\boldsymbol{w} = \boldsymbol{T} \bullet \boldsymbol{u}$
$w_i = T_{ij}u_j$
$\{w\} = [T]\{u\}$

(iii) $\boldsymbol{w} = 1\boldsymbol{e_1} + 10\boldsymbol{e_2} + 6\boldsymbol{e_3}$

(c) $? = \boldsymbol{u} \bullet \boldsymbol{T^T}$

(i) $w_i \boldsymbol{e_i} \Rightarrow \boldsymbol{u} \bullet \boldsymbol{T^T} = \boldsymbol{T} \bullet \boldsymbol{u}$ (components and base vector)

(ii) $\boldsymbol{w} = \boldsymbol{T} \bullet \boldsymbol{u}$
$w_i = T_{ij} u_j$
$\{w\} = [T]\{u\}$

(iii) $\boldsymbol{w} = 1\boldsymbol{e_1} + 10\boldsymbol{e_2} + 6\boldsymbol{e_3}$

(d) $? = \boldsymbol{v} \bullet \boldsymbol{T} \bullet \boldsymbol{u}$

(i) $w = \boldsymbol{v} \bullet \boldsymbol{T} \bullet \boldsymbol{u} = \boldsymbol{T} \cdot\cdot (\boldsymbol{u} \otimes \boldsymbol{v})$ (dot product with two vectors is a scalar)

(ii) $w = \boldsymbol{v} \bullet \boldsymbol{T} \bullet \boldsymbol{u}$
$w = T_{pq} v_p u_q$
$w = \langle v \rangle [T] \{u\}$

(iii) $w = -10$

(e) $? = \boldsymbol{u} \otimes \boldsymbol{v}$

(i) ? $\Rightarrow$ the "tensor product" or "dyadic multiplication" between two base vectors operates on an arbitrary vector e.g., $(\boldsymbol{u} \otimes \boldsymbol{v}) \bullet \boldsymbol{z} = \boldsymbol{u}(\boldsymbol{v} \bullet \boldsymbol{z})$.

(ii) $\boldsymbol{u} \otimes \boldsymbol{v}$ $\boldsymbol{u_i} \otimes \boldsymbol{v_i}$
$\{\boldsymbol{u}\} \otimes \langle \boldsymbol{v} \rangle$

(iii) $\begin{bmatrix} (\boldsymbol{u_1} \otimes \boldsymbol{v_1}) & (\boldsymbol{u_1} \otimes \boldsymbol{v_2}) & (\boldsymbol{u_1} \otimes \boldsymbol{v_3}) \\ (\boldsymbol{u_2} \otimes \boldsymbol{v_1}) & (\boldsymbol{u_2} \otimes \boldsymbol{v_2}) & (\boldsymbol{u_2} \otimes \boldsymbol{v_3}) \\ (\boldsymbol{u_3} \otimes \boldsymbol{v_1}) & (\boldsymbol{u_3} \otimes \boldsymbol{v_2}) & (\boldsymbol{u_3} \otimes \boldsymbol{v_3}) \end{bmatrix}$

(f) $? = \boldsymbol{I} \cdot\cdot \boldsymbol{T}$

(i) $w = \boldsymbol{I} \cdot\cdot \boldsymbol{T}$ (The inner product between two tensors results in a scalar)

(ii) $w = \boldsymbol{I} \cdot\cdot \boldsymbol{T} = \boldsymbol{T^{tr}}$
$w = T_{pp}$
$w = tr([T])$

(iii) $w = 1$

(g) Determine the components of $\boldsymbol{T^{sym}}$ and $\boldsymbol{T^{skw}}$ of $\boldsymbol{T}$. Part (iv) of problem 1

- The symmetric part of $\boldsymbol{T}$:
  $$\boldsymbol{T^{sym}} = \frac{1}{2}\left[[\boldsymbol{T}] + [\boldsymbol{T}]^{\boldsymbol{T}}\right] \Rightarrow \begin{bmatrix} -1 & 2 & 7/2 \\ 2 & -2 & 5/2 \\ 7/2 & 5/2 & 4 \end{bmatrix}$$
- The skew-symmetric part of $\boldsymbol{T}$:
  $$\boldsymbol{T^{skw}} = \frac{1}{2}\left[[\boldsymbol{T}] - [\boldsymbol{T}]^{\boldsymbol{T}}\right] \Rightarrow \begin{bmatrix} 0 & 0 & -1/2 \\ 0 & 0 & -1/2 \\ 1/2 & 1/2 & 0 \end{bmatrix}$$

(h) Determine the components b_i, c_i, and d_i.
Part (v) of problem 1

- $b_i = \frac{1}{2}\mathcal{E}_{ijk}T_{jk}$ or $\boldsymbol{b} = \frac{1}{2}\overset{3}{\boldsymbol{\mathcal{E}}} \cdot \cdot \boldsymbol{T}$

$$b_i \Rightarrow \begin{array}{rcr} b_1 = \frac{1}{2}(T_{23} - T_{32}) & = & -1/2 \\ b_2 = \frac{1}{2}(T_{13} - T_{31}) & = & -1/2 \\ b_3 = \frac{1}{2}(T_{12} - T_{21}) & = & 0 \end{array}$$

- $c_i = \frac{1}{2}\mathcal{E}_{ijk}T^{sym}_{jk}$ or $\boldsymbol{c} = \frac{1}{2}\overset{3}{\boldsymbol{\mathcal{E}}} \cdot \cdot \boldsymbol{T^{sym}}$

$$c_i \Rightarrow \begin{array}{rcr} c_1 = \frac{1}{2}(T^{sym}_{23} - T^{sym}_{32}) & = & 0 \\ c_2 = \frac{1}{2}(T^{sym}_{13} - T^{sym}_{31}) & = & 0 \\ c_3 = \frac{1}{2}(T^{sym}_{12} - T^{sym}_{21}) & = & 0 \end{array}$$

- $d_i = \frac{1}{2}\mathcal{E}_{ijk}T^{skw}_{jk}$ or $\boldsymbol{d} = \frac{1}{2}\overset{3}{\boldsymbol{\mathcal{E}}} \cdot \cdot \boldsymbol{T^{skw}}$

$$d_i \Rightarrow \begin{array}{rcr} d_1 = \frac{1}{2}(T^{skw}_{23} - T^{skw}_{32}) & = & 1/2 \\ d_2 = \frac{1}{2}(T^{skw}_{13} - T^{skw}_{31}) & = & 1/2 \\ d_3 = \frac{1}{2}(T^{skw}_{12} - T^{skw}_{21}) & = & 0 \end{array}$$

2 Show that $\boldsymbol{v} = (\boldsymbol{v} \cdot \boldsymbol{n})\boldsymbol{n} + \boldsymbol{n} \times (\boldsymbol{v} \times \boldsymbol{n})$ holds for all $\boldsymbol{n}$ and that this represents a resolution (or projection) of $\boldsymbol{v}$ into vectors parallel and perpendicular to $\boldsymbol{n}$, where $\boldsymbol{n}$ is a unit vector:

(i) set the right term $\boldsymbol{n} \times (\boldsymbol{v} \times \boldsymbol{n}) = \boldsymbol{u} \times (\boldsymbol{v} \times \boldsymbol{w})$ and solve;

(ii) $\boldsymbol{v} \times \boldsymbol{w} \Rightarrow v_i\boldsymbol{e_i} \times w_i\boldsymbol{e_i} = v_i w_i \mathcal{E}_{ijk}\boldsymbol{e_k}$;

(iii) $\boldsymbol{u} \times (\boldsymbol{v} \times \boldsymbol{w}) \Rightarrow u_l\boldsymbol{e_l} \times v_i w_i \mathcal{E}_{ijk}\boldsymbol{e_k} = u_l v_i w_j \mathcal{E}_{ijk}\mathcal{E}_{klm}\boldsymbol{e_m}$;

(iv) replace $\mathcal{E}_{ijk}\mathcal{E}_{klm}$ using the $\mathcal{E} - \delta$ identity:
free indices: i, j, l, m; therefore: $\mathcal{E}_{ijk}\mathcal{E}_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$;

(v) $(\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})u_l v_i w_j \boldsymbol{e_m}$;

(vi) remove Kronecker Deltas that contain a repeated (dummy) index: $u_i v_i w_m \boldsymbol{e_m} - u_j w_j v_m \boldsymbol{e_m}$;

(vii) replace with original values and write in direct notation: $(\boldsymbol{n} \cdot \boldsymbol{v})\boldsymbol{n} - (\boldsymbol{n} \cdot \boldsymbol{n})\boldsymbol{v} = 0$

(viii) where in (vii) $\boldsymbol{n} \cdot \boldsymbol{v} = v^n$, then $v^n\boldsymbol{n} = \boldsymbol{v}$, and $\boldsymbol{n} \cdot \boldsymbol{n} = 1$

(ix) the left term $(\boldsymbol{v} \cdot \boldsymbol{n})\boldsymbol{n} = \|\boldsymbol{v}\|\boldsymbol{n} = \boldsymbol{v}$

(x) therefore $\boldsymbol{v} = (\boldsymbol{v} \cdot \boldsymbol{n})\boldsymbol{n} + \boldsymbol{n} \times (\boldsymbol{v} \times \boldsymbol{n})$

(xi) Additionally, the resolution of $\boldsymbol{v}$ parallel to $\boldsymbol{n}$ is: $(\boldsymbol{v} \cdot \boldsymbol{n})\boldsymbol{n}$ as shown in (ix)
the resolution of $\boldsymbol{v}$ perpendicular to $\boldsymbol{n}$ is shown in (vii): $\boldsymbol{n} \times (\boldsymbol{v} \times \boldsymbol{n}) = (\boldsymbol{n} \cdot \boldsymbol{v})\boldsymbol{n} - (\boldsymbol{n} \cdot \boldsymbol{n})\boldsymbol{v} = \|v\|\boldsymbol{n} - \boldsymbol{v}$

3 Suppose T and U are second-order tensors.

(a) Show $tr(\boldsymbol{T} \cdot \boldsymbol{U}) = tr(\boldsymbol{T^T} \cdot \boldsymbol{U})$ if either $\boldsymbol{T}$ or $\boldsymbol{U}$ are symmetric.

- if $\boldsymbol{T}$ is symmetric then: $\boldsymbol{T} \Rightarrow T_{ij} = T^T_{ij} = T_{ij}$
- define $\boldsymbol{A}$ such that $A_{ik} = T_{ij}U_{jk} = T^T_{ij}U_{jk}$
- therefore $tr(\boldsymbol{A}) = tr(\boldsymbol{T} \cdot \boldsymbol{U}) = tr(\boldsymbol{T^T} \cdot \boldsymbol{U})$

(b) Show $tr(\boldsymbol{T} \cdot \boldsymbol{U}) = 0$ if one of the tensors is skew-symmetric and the other is symmetric.

- symmetric tensor $\Rightarrow T_{ij} = T_{ji}$
- skew-symmetric tensor $\Rightarrow U_{ij} = -U_{ji}$
- $tr(\boldsymbol{T} \cdot \boldsymbol{U}) \Rightarrow T_{ij}U_{ij} = -T_{ij}U_{ji} = -T_{ji}U_{ji}$; this can only be true if $tr(\boldsymbol{T} \cdot \boldsymbol{U}) = 0$

4

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5 Given:

$$\begin{array}{cccc} & \boldsymbol{E_1} & \boldsymbol{E_1} & \boldsymbol{E_1} \\ \boldsymbol{e_1} & \pi/2 & \pi/4 & 3\pi/4 \\ \boldsymbol{e_2} & \pi/4 & \pi/3 & \pi/3 \\ \boldsymbol{e_3} & \pi/4 & 2\pi/3 & 2\pi/3 \end{array} \qquad \overset{e}{\{v\}} = \begin{Bmatrix} 1 \\ -2 \\ 3 \end{Bmatrix} \qquad \overset{e-e}{[T]} = \begin{bmatrix} 2 & 0 & -3 \\ 0 & 6 & 0 \\ -3 & 0 & 4 \end{bmatrix}$$

(a)

- $\boldsymbol{e_i}$ in terms of $\boldsymbol{E_A}$: $\quad \boldsymbol{e_i} \Rightarrow \begin{aligned} \boldsymbol{e_1} &= \cos(\pi/2)\boldsymbol{E_1} + \cos(\pi/4)\boldsymbol{E_2} + \cos(3\pi/4)\boldsymbol{E_3} \\ \boldsymbol{e_2} &= \cos(\pi/4)\boldsymbol{E_1} + \cos(\pi/3)\boldsymbol{E_2} + \cos(\pi/3)\boldsymbol{E_3} \\ \boldsymbol{e_3} &= \cos(\pi/4)\boldsymbol{E_1} + \cos(2\pi/3)\boldsymbol{E_2} + \cos(2\pi/3)\boldsymbol{E_3} \end{aligned}$

- $\boldsymbol{E_A}$ in terms of $\boldsymbol{e_i}$: $\quad \boldsymbol{E_A} \Rightarrow \begin{aligned} \boldsymbol{E_1} &= \cos(\pi/2)\boldsymbol{e_1} + \cos(\pi/4)\boldsymbol{e_2} + \cos(\pi/4)\boldsymbol{e_3} \\ \boldsymbol{E_2} &= \cos(\pi/4)\boldsymbol{e_1} + \cos(\pi/3)\boldsymbol{e_2} + \cos(2\pi/3)\boldsymbol{e_3} \\ \boldsymbol{E_3} &= \cos(3\pi/4)\boldsymbol{e_1} + \cos(\pi/3)\boldsymbol{e_2} + \cos(2\pi/3)\boldsymbol{e_3} \end{aligned}$

- Verify that $\boldsymbol{E_A}$ is a right-handed orthonormal system:
  - Right-handed and orthogonal because: $\begin{aligned} \boldsymbol{E_1} \times \boldsymbol{E_2} &= \boldsymbol{E_3} \\ \boldsymbol{E_2} \times \boldsymbol{E_3} &= \boldsymbol{E_1} \\ \boldsymbol{E_3} \times \boldsymbol{E_1} &= \boldsymbol{E_2} \end{aligned}$
  - Normal (i.e., unit length of one) because: $\begin{aligned} \boldsymbol{E_1} \cdot \boldsymbol{E_1} &= 1 \\ \boldsymbol{E_2} \cdot \boldsymbol{E_2} &= 1 \\ \boldsymbol{E_3} \cdot \boldsymbol{E_3} &= 1 \end{aligned}$

(b) Obtain the transformation matrix:

- $\overset{e-E}{[a]} = \begin{bmatrix} 0 & \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & 1/2 & 1/2 \\ \sqrt{2}/2 & -1/2 & -1/2 \end{bmatrix}$

- $\overset{e-E}{[a]^T} = \overset{E-e}{[a]} = \begin{bmatrix} 0 & \sqrt{2}/2 & \sqrt{2}/2 \\ \sqrt{2}/2 & 1/2 & -1/2 \\ -\sqrt{2}/2 & 1/2 & -1/2 \end{bmatrix}$

- because the transformation matrix is orthonormal: $\overset{e-E}{[a]}\overset{E-e}{[a]} = [I]$

(c) Find the components of $\boldsymbol{v}$ in the $\boldsymbol{E_A}$ system:

- $\overset{E}{\{v\}} = \overset{E-e}{[a]}\overset{e}{\{v\}} = \begin{Bmatrix} \sqrt{2}/2 \\ (\sqrt{2}-5)/2 \\ (-\sqrt{2}-5)/2 \end{Bmatrix}$

- $\overset{e}{\{v\}} = \overset{e-E}{[a]}\overset{E}{\{v\}} = \begin{Bmatrix} 1 \\ -2 \\ 3 \end{Bmatrix}$

(d) Find the components of $\boldsymbol{T}$ in the $\boldsymbol{E_A}$ system:

- $\overset{E-E}{[T]} = \overset{E-e}{[a]}\overset{e-e}{[T]}\overset{e-E}{[a]} = \begin{bmatrix} 5.00 & -0.79 & 2.21 \\ -0.79 & 5.62 & 1.50 \\ 2.21 & 1.50 & 1.38 \end{bmatrix}$

(e) Find the mixed components of $\overset{E-e}{[T]}$ and $\overset{e-E}{[T]}$:

- $\overset{E-e}{[T]} = \overset{E-e}{[a]}\overset{e-e}{[T]} = \overset{E-E}{[T]}\overset{E-e}{[a]} = \overset{e-E}{[T]}^T = \begin{bmatrix} -2.12 & 4.24 & 2.82 \\ 2.91 & 3.00 & -4.12 \\ 0.09 & 3.00 & 0.12 \end{bmatrix}$

- $\overset{e-E}{[T]} = \overset{e-E}{[a]}\overset{E-E}{[T]} = \overset{e-e}{[T]}\overset{e-E}{[a]} = \overset{E-e}{[T]}^T = \begin{bmatrix} -2.12 & 2.91 & 0.09 \\ 4.24 & 3.00 & 3.00 \\ 2.82 & -4.12 & 0.12 \end{bmatrix}$

6 Obtain the transformation matrices for the transforming components from:

- assuming each basis is orthonormal

(a) the $\boldsymbol{e_i}$ basis to the $\boldsymbol{E_A}$ basis:

- $\boldsymbol{E_A} \Rightarrow \begin{aligned} \boldsymbol{E_1} &= \cos(\alpha)\boldsymbol{e_1} + \cos(90^\circ-\alpha)\boldsymbol{e_2} + \cos(90^\circ)\boldsymbol{e_3} \\ \boldsymbol{E_2} &= \cos(\alpha+90^\circ)\boldsymbol{e_1} + \cos(90^\circ)\boldsymbol{e_2} + \cos(90^\circ)\boldsymbol{e_3} \\ \boldsymbol{E_3} &= \cos(90^\circ)\boldsymbol{e_1} + \cos(90^\circ)\boldsymbol{e_2} + \cos(0)\boldsymbol{e_3} \end{aligned}$

- $\overset{E-e}{[a]} = \begin{bmatrix} \cos(\alpha) & \sin(\alpha) & 0 \\ -\sin(\alpha) & \cos(\alpha) & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(b) the $\boldsymbol{e_i}$ basis to the $\boldsymbol{E_A}$ basis:

- $\boldsymbol{E_A} \Rightarrow \begin{aligned} \boldsymbol{E_1} &= \cos(0)\boldsymbol{g_1} + \cos(90^\circ)\boldsymbol{g_2} + \cos(90^\circ)\boldsymbol{g_3} \\ \boldsymbol{E_2} &= \cos(90^\circ)\boldsymbol{g_1} + \cos(-\beta)\boldsymbol{g_2} + \cos(-\beta-90^\circ)\boldsymbol{g_3} \\ \boldsymbol{E_3} &= \cos(90^\circ)\boldsymbol{g_1} + \cos(\beta-90^\circ)\boldsymbol{g_2} + \cos(90^\circ+\beta)\boldsymbol{g_3} \end{aligned}$

- $\overset{E-g}{[a]} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\beta) & -\sin(\beta) \\ 0 & \sin(\beta) & \cos(\beta) \end{bmatrix}$

(c) the $\boldsymbol{e_i}$ basis to the $\boldsymbol{E_A}$ basis:

- $\overset{e-g}{[a]} = \overset{e-E}{[a]}\overset{E-g}{[a]} = \overset{E-e}{[a]^T}\overset{E-g}{[a]} = \begin{bmatrix} \cos\alpha & -\sin\alpha\cos\beta & \sin\alpha\cos\beta \\ \sin\alpha & \cos\alpha\cos\beta & -\cos\alpha\sin\beta \\ 0 & \sin\beta & \cos\beta \end{bmatrix}$

- to check that this transformation is still orthonormal: $\overset{e-g}{[a]}\overset{g-e}{[a]} = [I]$