Fundamental Formulas in Plasticity Theory

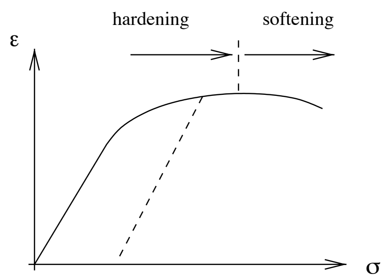


Figure 1: Typical stress-strain curve

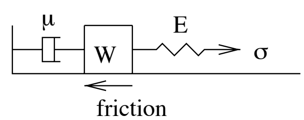


Figure 2: Mechanical model

Yield Condition

$f = f(\underline{\sigma}, T, \underline{\xi})$	yield function
$f < 0$	elastic response
$f > 0$	elastic/inelastic response
$\underline{\sigma}$	stress tensor
T	temperature
$\underline{\xi}$	array of internal variables

The yield function defines the state of stress which separates purely elastic behvior of a material from elastic/plastic behavior. The surface represented by $f = 0$ in stress-space is the yield surface. Points inside this surface (on the same side of the surface as the origin) are points within the elastic range of the material. Points outside the surface represent states of stress which cause plastic (inelastic) deformation of the material.

Additive Decompostion

$$\epsilon_{ij} = \epsilon^e_{ij} + \epsilon^p_{ij}$$
$$\dot{\epsilon}_{ij} = \dot{\epsilon}^e_{ij} + \dot{\epsilon}^p_{ij}$$

Although this may seem obvious, it is, in fact, an assumption. When the elastic strains are large, it can be shown that it is not a good assumption. However, if the elastic strains are small it is acceptable.

Internal variables

$$\dot{\xi}_\alpha = g_\alpha(\underline{\sigma}, T, \underline{\xi}) \qquad \text{Evolution equation}$$

Internal variables are related to the structure of the material and are history dependent. They describe hardening and softening characteristics (see below).

Flow Rule

$$\begin{aligned} \dot{\epsilon}^p_{ij} &= \phi \frac{\partial g}{\partial \sigma_{ij}} && \text{g = flow potential} \\ \dot{\epsilon}^p_{ij} &= \phi \frac{\partial f}{\partial \sigma_{ij}} && \text{Associative flow law} \end{aligned}$$

The gradient defines the relative relationship between the components of $\dot{\epsilon}_{ij}$, and ϕ defines the magnitude of flow. Because plastic flow does not exist for $f < 0$

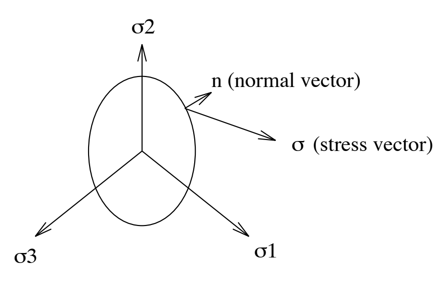
$$\phi = \frac{1}{\mu} \langle f \rangle$$

is an appropriate choice for ϕ. The quantity μ has the role of viscosity and incorporates the dimension of time into the formulation. It too can be a function of the current state of stress, temperature, and internal variables.

Two important conditions:

$$\frac{\partial f}{\partial \sigma_{ij}} \dot{\sigma}_{ij} \quad \begin{array}{ll} > 0 & \text{Loading} \\ < 0 & \text{Unloading} \end{array}$$

$$\sum_\alpha \frac{\partial f}{\partial \xi_\alpha} \dot{\xi}_\alpha = \quad \begin{array}{ll} > 0 & \text{Softening} \\ < 0 & \text{Hardening} \end{array}$$



Yield function in stress space

Note, because $f > 0$ indicates inelastic behavior and $f < 0$ elastic behavior, any increase in f due to changes in the internal variables creates a more inelastic situation.

It is often assumed that changes in the internal variables are due only to plastic deformation; hence, when $f < 0$, $\dot{\xi}_\alpha = 0$. Furthermore, it is often assumed that the time rate of change of all internal variables is linearly propotional to the time rate of change of the plastic strain. Therefore, it is appropriate to use the function ϕ (as defined above for the flow rule) in the evolution equations as follows:

$$\boxed{\begin{aligned} \dot{\xi}_\alpha &= g_\alpha(\underline{\sigma}, T, \underline{\xi}) \\ &= \phi h_\alpha(\underline{\sigma}, T, \underline{\xi}) \end{aligned}}$$

A convenient hardening function can now be defined as:

$$\boxed{\begin{array}{rcll} \sum \dfrac{\partial f}{\partial \xi_\alpha} \dot{\xi}_\alpha &=& \phi \sum \dfrac{\partial f}{\partial \xi_\alpha} h_\alpha & \\ &=& -\phi H & \\ \text{where } H &=& -\sum \dfrac{\partial f}{\partial \xi_\alpha} h_\alpha & \\ \text{hence: } H &>& 0 & \text{hardening} \\ H &<& 0 & \text{softening} \end{array}}$$

REFERENCE:

Almost without exception, the notation in these notes follow those used in:

Lubliner, Jacob; *PLASTICITY THEORY*, Macmillan Publishing Company; New York; 1990.

EGT: April, 1998

EXAMPLE 1:

One-dimensional Case

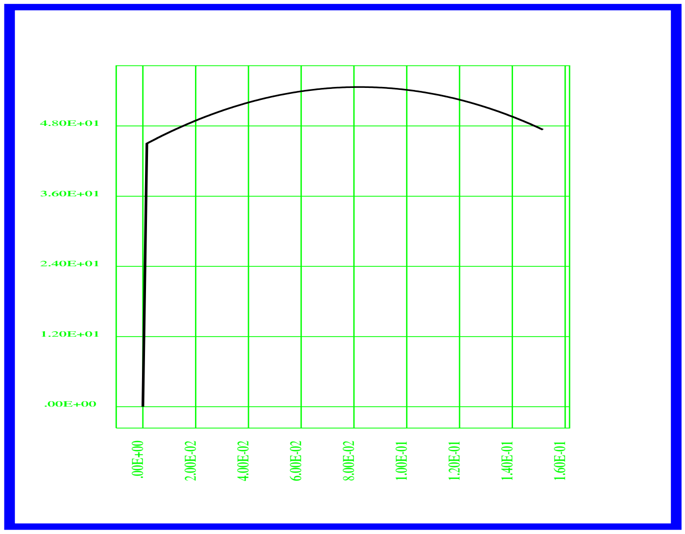


Figure 3: Stress-strain curve

The curve shown corresponds to the following data:

$$
\begin{aligned}
E &= 30(10^3) && \text{Modulus of Elasticity (ksi)} \\
\sigma_y &= 45.0 + 240.0\xi_1 - 15000\xi_2 && \text{Yield stress (ksi)} \\
\sigma_e &= \sqrt{\sigma\sigma} && \text{Effective stress} \\
f &= (\sigma_e - \sigma_y) && \text{Yield function} \\
\dot{\epsilon}^p &= \phi\frac{\partial f}{\partial \sigma} && \\
&= \tfrac{1}{\mu}\left\langle \sigma_e - \sigma_y \right\rangle \frac{\sigma}{\sigma_e} && \text{Flow rule} \\
\dot{\xi}_1 &= \phi h_1 && \\
&= \tfrac{1}{\mu}\left\langle \sigma_e - \sigma_y \right\rangle && \text{Evolution eq. for } \xi_1 \\
\dot{\xi}_2 &= \phi h_2 && \\
&= = \tfrac{1}{\mu}\left\langle \sigma_e - \sigma_y \right\rangle 2.0\xi_1 && \text{Evolution eq. for } \xi_2 \\
H &= 240.0 - 1500.0(2.0\xi_1) && \text{Hardening function}
\end{aligned}
$$

EXAMPLE 2:

Use the one-dimensional data presented in the last example to develop a full three-dimensional theory.

To convert from a one-dimensional theory to a three dimensional theory, equations for stress and strain-rate for the axial components of these tensors, must be converted to equations which represent all components. When stress and strain-rate appear in scalar equations, they must be replaced by appropriate invariants of these tensors. The final equations must reduce to the one-dimensional equations when applied to a uniaxial state of stress.

Elastic constants:

E must remain the same and a second constant must be added. Poisson's ratio is a logical choice to be used with E, but it is also customary to use either Lame's constants, or the shear and bulk moduli as the two independent elastic constants.

Yield stress:

The yield stress is a scalar quantity which defines the boundary between elastic and elastic/plastic behavior. Because it describes a physically measured phenomenon, it must appear in a three-dimensional theory for the purpose of describing the same phenomena. It may, however, be converted to another scalar value that predicts exactly the same yield stress for uniaxial conditions.

Yield function:

The scalar σ_e used in f must be converted to an invariant of the stress tensor for three-dimensional analyses. The exact choice of the invariant will define the three-dimensional yield theory being proposed. Whatever invariant is used (and the corresponding representation of σ_y) the yield function must reduce to the one dimensional expression for uniaxial loadings. The most common invariant used for metals is J_2 or one its equivalents such as the effective stress defined as:

$$\sigma_e = \sqrt{\tfrac{3}{2}\sigma'_{ij}\sigma'_{ij}}$$

where

$$\sigma'_{ij} = \sigma_{ij} - \sigma_{kk}\delta_{ij}$$

The effective stress defined in this manner reduces to the axial stress in one-dimension; hence, the yield function remains exactly the same, namely:

$$f = (\sigma_e - \sigma_y)$$

The flow rule:

The one dimensional flow rule can be extended to a three-dimensional flow rule by simply replacing stress and plastic strain-rates with their tensor equivalents. Thus:

$$\dot{\epsilon}^p_{ij} = \phi \frac{\partial f}{\partial \sigma_{ij}}$$

where

$$\begin{aligned}
\frac{\partial f}{\partial \sigma_{ij}} &= \frac{\partial \sigma_e}{\partial \sigma_{ij}} \\
&= \frac{\partial}{\partial \sigma_{ij}} \left(\frac{3}{2}\sigma'_{mn}\sigma'_{mn}\right)^{\frac{1}{2}} \\
&= \frac{1}{2}\left(\frac{3}{2}\sigma'_{mn}\sigma'_{mn}\right)^{-\frac{1}{2}} \frac{\partial}{\partial \sigma_{ij}} \left(\frac{3}{2}\sigma'_{pq}\sigma'_{pq}\right) \\
&= \left(\frac{1}{2\sigma_e}\right) \frac{3}{2} \left(2\sigma'_{pq}\right) \frac{\partial \sigma'_{pq}}{\partial \sigma_{ij}} \\
&= \frac{3}{2}\frac{\sigma'_{pq}}{\sigma_e}\frac{\partial \sigma_{ij}}{\partial \sigma_e} \\
&= \frac{3}{2}\frac{\sigma'_{ij}}{\sigma_e}
\end{aligned}$$

hence

$$\dot{\epsilon}^p_{ij} = \frac{1}{\mu}\langle \sigma_e - \sigma_y \rangle \frac{3}{2}\frac{\sigma'_{ij}}{\sigma_e}$$

Note, in the above the following was used:

$$\sigma'_{pq} = \sigma_{pq} - \frac{1}{3}\sigma_{kk}\delta_{pq}$$

thus

$$\begin{aligned}
\frac{\partial \sigma'_{pq}}{\partial \sigma_{ij}} &= \frac{\partial \sigma_{pq}}{\partial \sigma_{ij}} - \frac{1}{3}\left(\frac{\partial \sigma_{kk}}{\partial \sigma_{ij}}\right)\delta_{pq} \\
&= \delta_{ip}\delta_{jq} - \frac{1}{3}\left(\delta_{ik}\delta_{jk}\right)\delta_{pq} \\
&= \delta_{ip}\delta_{jq} - \frac{1}{3}\delta_{ij}\delta_{pq}
\end{aligned}$$

giving

$$\begin{aligned}
\sigma'_{pq}\frac{\partial \sigma'_{pq}}{\partial \sigma_{ij}} &= \sigma'_{ij} - \frac{1}{3}\left(\delta_{ij}\delta_{pq}\right)\sigma'_{pq} \\
&= \sigma'_{ij}
\end{aligned}$$

Evolution equations and hardening function

The scalar evolution equations and hardening functions remain exactly the same with the understanding that σ_e is now defined as it was above.

SUMMARY

$$
\begin{array}{rcll}
E & = & 30(10^3) & \text{Modulus of Elasticity (ksi)} \\
\nu & = & 0.3 & \text{Poisson's ratio} \\
\sigma_y & = & 45.0 + 240.0\xi_1 - 15000\xi_2 & \text{Yield stress (ksi)} \\
\sigma_e & = & \sqrt{\frac{3}{2}\sigma'_{ij}\sigma'_{ij}} & \text{Effective stress} \\
f & = & (\sigma_e - \sigma_y) & \text{Yield function} \\
\dot{\epsilon}^p & = & \phi\frac{\partial f}{\partial \sigma} & \\
 & = & \frac{1}{\mu}\left\langle \sigma_e - \sigma_y \right\rangle \frac{3}{2}\frac{\sigma'_{ij}}{\sigma_e} & \text{Flow rule} \\
\dot{\xi}_1 & = & \phi h_1 & \\
 & = & \frac{1}{\mu}\left\langle \sigma_e - \sigma_y \right\rangle & \text{Evolution eq. for } \xi_1 \\
\dot{\xi}_2 & = & \phi h_2 & \\
 & = & = \frac{1}{\mu}\left\langle \sigma_e - \sigma_y \right\rangle 2.0\xi_1 & \text{Evolution eq. for } \xi_2 \\
H & = & 240.0 - 1500.0(2.0\xi_1) & \text{Hardening function}
\end{array}
$$

Incremental Equations of Plasticity in Finite Element Form

(1) Equilibrium

$$\sum_e \int_{V_e} [N'] \{\Delta\sigma\} \, dV = \{\Delta F\}$$

(2) Elastic constitutive equations

$$\{\Delta\sigma\} = [C] \{\Delta\epsilon\} - [C] \{\Delta\epsilon^p\}$$

(3) Flow rule

$$\{\Delta\epsilon^p\} = \Delta\lambda \left\{ \frac{\partial f}{\partial\sigma} \right\}$$

Note:
$\Delta\lambda = \Phi \Delta t =$ e.g. $= \frac{1}{2\mu} \Delta t \langle f \rangle$

(4) Consistency condition

$$\Delta f = \lfloor \frac{\partial f}{\partial \sigma} \rfloor \{\Delta\sigma\} + \frac{\partial f}{\partial e} \Delta e = 0$$

Stress must remain on yield surface. Only one internal variable, e, is assumed.

(5) Evolution equation for e

$$\begin{aligned} \Delta e &= \Delta\epsilon^p_{\text{eff}} \\ &= \sqrt{\tfrac{2}{3} \lfloor \Delta\epsilon^p \rfloor \{\Delta\epsilon^p\}} \\ &= \Delta\lambda \sqrt{\tfrac{2}{3} \lfloor a \rfloor \{a\}} \\ &= \Delta\lambda \, h \end{aligned}$$

where

$$\{a\} = \left\{ \frac{\partial f}{\partial\sigma} \right\}$$

Note: $\{a\}$ is a direction vector, indicating the direction of plastic flow based on an associated flow law. We will use this notation in what follows.

(6) Hardening parameter

$$H = -\frac{\partial f}{\partial e} h$$

Eq. (3) → Eq. (2)

$$\{\Delta\sigma\} = [C]\,\{\Delta\epsilon\} - \Delta\lambda\,[C]\,\{a\} \tag{7}$$

Eq. (6) → Eq. (4)

$$\lfloor a \rfloor\,\{\Delta\sigma\} - \Delta\lambda\ H = 0 \tag{8}$$

From Eq. (8) and (7)

$$\Delta\lambda = \frac{\lfloor a \rfloor\,[C]\,\{\Delta\epsilon\}}{\lfloor a \rfloor\,[C]\,\{a\} + H} \tag{9}$$

To obtain this equation, multiply Eq. (7) by $\lfloor a \rfloor$ and then substitute Eq. (8) into it.

From Eq. (9) and (7)

$$\{\Delta\sigma\} = [C]\left[[I] + \frac{\{a\}\,\lfloor a \rfloor\,[C]}{\lfloor a \rfloor\,[C]\,\{a\} + H}\right]\{\Delta\epsilon\} \tag{10}$$

Define $[C_{ep}]$ such that:

$$\{\Delta\sigma\} = [C_{ep}]\,\{\Delta\epsilon\} \tag{11}$$

Incremental Solution Algorithm

The method outlined below is for the special case of a Von Mises type yield function with a constant hardening parameter. Other types of yield functions or non-constant hardening parameters require more sophisticated solution algorithms. The method shown uses the "radial return" method. All external loads are given as functions of time; hence, time will be used to designate the incremental character of the solution process. However, the method is for time-independent plasticity.

At time equal t:

$\{u\}_t$, $\{\sigma\}_t$, e_t, and $\{F\}_t$ are known.

$\sum \int [N']^T \{\sigma\}_t dV = \{F\}$ i.e. system is in equilibrium.

Increment to time equal $t + \Delta t$:

Determine correct stiffness matrix:

1. Create $[K]_t$ using either $[C]$ or $[C_{ep}]$ depending on the current state of stress and the last calculated loading or unloading state.

2. Solve (Global):
$$[K]_t \{\Delta u\} = \{\Delta F\}$$

3. At local (quadrature) level, for all points, determine:
$$\begin{aligned}
\{\Delta \epsilon\} &= [N'] \{\Delta u\} && \text{Total strain increment} \\
\{\Delta \sigma\} &= [C] \{\Delta \epsilon\} && \text{Assume increment is elastic} \\
R &= \lfloor a \rfloor \{\Delta \sigma\} && \text{loading indicator}
\end{aligned}$$

4. If any loading indicator did not agree with assumed value when calculating $[K]$, return to Step 1, else continue.

Begin iterations:

1. Create $[K]_t$ using either $[C]$ or $[C_{ep}]$ depending on the current state of stress and the last calculated loading or unloading state.

2. Determine:
$$\{\Delta R\} = \{F\}_t + \{\Delta F\} - \sum_e \int_{V_e} [N'] \{\sigma\}_t dV$$
Note: $\{\Delta R\} = \{\Delta F\}$ if current state of stress is in equilibrium.

3. Solve (Global):
$$[K]_t \{\Delta u\} = \{\Delta R\}$$

4. At local (quadrature) level, determine:

$$\begin{aligned}
\{\Delta\epsilon\} &= [N']\{\Delta u\} && \text{Total strain increment}\\
\{\Delta\sigma\} &= [C]\{\Delta\epsilon\} && \text{Assume increment is elastic}\\
\{\sigma\}_B &= \{\sigma\}_t + \{\Delta\sigma\} && \text{predictor step}\\
f_B &= \left(1 - \frac{\sigma_y}{\sigma_e}\right) && \text{Predicted yield function}
\end{aligned}$$

if $f_B > 0$, then correct:

$$\begin{aligned}
\Delta f &= -f_B && \text{Needed change in } f\\
\Delta f &= \lfloor a \rfloor \{\delta\sigma\} - \Delta\lambda H && \text{See Eq. 4}\\
\delta\sigma &= -[C]\{\epsilon^p\} = -\Delta\lambda [C]\{a\} &&\\
\delta f &= -\Delta\lfloor a \rfloor[C]\{a\} - \Delta H &&\\
\Delta\lambda &= \frac{f_B}{\lfloor a \rfloor[C]\{a\} + H} &&\\
\{\sigma\}_c &= \{\sigma\}_t - \Delta[C]\{a\} && \text{Corrected stress}
\end{aligned}$$

endif

5. Calculate:

$$\{\Delta R\} = \{F\}_t + \{\Delta F\} - \sum_e \int_{V_e} [N']\{\sigma\}_c \, dV$$

6. if

$$\lfloor \Delta R \rfloor \{\Delta R\} < \text{convergence tolerance}$$

then:

$\{\sigma\}_c$ is in equilibrium everywhere.

$f \leq 0$ everywhere.

$\{\sigma\}_{t+\Delta t} = \{\sigma\}_c$

$\{u\}_{t+\Delta t} = \{u\} + \{\Delta u\}$

$e_{t+\Delta t} = e_t + \Delta h$ see Eq. 5

Ready for next increment of load.

else:

$\{R\} = \{R\} + \{\Delta F\}$

Return to Step 3.

Calculation of H for a Bilinear-type Material

Assume E and E_t in the following figure have been obtained experimentally.

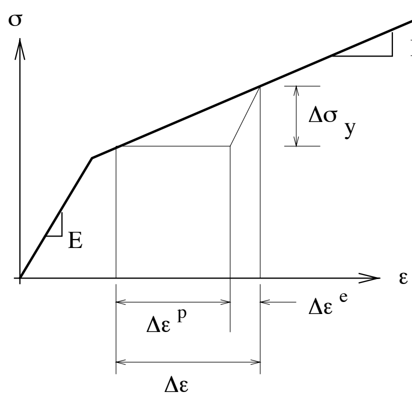


Figure 4: Bilinear-type $\sigma - \epsilon$ curve

YIELD FUNCTION:

$$f = \sigma_e - \sigma_y$$

where the effective stress in 1D is simply:

$$\sigma_e = \sqrt{\sigma\sigma}$$

FLOW RULE:

$$\begin{aligned} \dot{\epsilon}^p &= \phi\frac{\partial f}{\partial \sigma} \\ &= \phi\frac{\partial \sigma_e}{\partial \sigma} \\ &= \phi\frac{\sigma}{\sigma_e} \end{aligned}$$

HARDENING:

$$\sigma_y = \sigma_{y_o} + \beta e$$

where β is a constant and e is the internal derived from

EVOLUTION EQUATION:

$$\begin{aligned} \dot{e} &= \dot{\epsilon}^p_{\text{eff}} \\ &= \sqrt{\dot{\epsilon}^p\dot{\epsilon}^p} \\ &= \phi\sqrt{\frac{\sigma}{\sigma_e}\frac{\sigma}{\sigma_e}} \\ &= \phi \end{aligned}$$

H is defined by:

$$\begin{aligned}
\frac{\partial f}{\partial e}\dot{e} &= -\frac{\partial \sigma_y}{\partial e}\dot{e} \\
&= -\frac{\partial \sigma_y}{\partial e}\phi \\
&= H\phi
\end{aligned}$$

hence:

$$H = -\frac{\partial \sigma_y}{\partial e}$$

To obtain H in terms of E and E_t shown in Fig. 4,

$$\begin{aligned}
\Delta\sigma_y &= H\Delta\epsilon^p \\
&= H\left[\Delta\epsilon - \Delta\epsilon^e\right] \\
&= H\left[\Delta\epsilon - \Delta\epsilon^e\right] \\
&= H\left[\frac{\Delta\sigma_y}{E_t} - \frac{\Delta\sigma_y}{E}\right] \\
\Delta\sigma_y &= H\Delta\sigma_y\left[\frac{1}{E_t} - \frac{1}{E}\right] \\
1 &= H\left[\frac{1}{E_t} - \frac{1}{E}\right] \\
1 &= H\left[\frac{E - E_t}{E_t E}\right]
\end{aligned}$$

$$\boxed{H = \left(\frac{EE_t}{E - E_t}\right)}$$