2. a) 1-D steady-state heat conduction in cylindrical coord.

problem Data

i) domain: $r_1 \le r \le r_2$

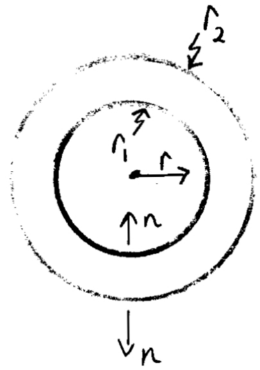
ii) forcing func.: $Q = 0$

iii) coef. func.: $\hat{k} = \hat{k}_0 \Rightarrow$ constant

iv) BCTs : a) $T\big|_{r=r_1} = T_1$

b) $T\big|_{r=r_2} = T_2$

$\Big\}$ Dirichlet type BCs



Governing Eqn.

- $\dfrac{1}{r}\dfrac{d}{dr}\left(k r \dfrac{dT}{dr}\right) = 0$

- because $k = k_0$, pull this out

$$\frac{\hat{k}_0}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right) = 0$$

- multiply by r/k_0

$$\frac{d}{dr}\left(r\frac{dT}{dr}\right) = 0$$

- solve the first order-linear ODE

$$r\frac{dT}{dr} = C_1$$

$$dT = C_1\frac{dr}{r}$$

$$T = C_1 \ln(r) + C_2$$

- solve for integration constants

at $r = r_1$, $T_1 = C_1 \ln(r_1) + C_2$

at $r = r_2$, $T_2 = C_1 \ln(r_2) + C_2$

$$C_2 = T_1 - C_1 \ln(r_1) \qquad (1)$$

$$T_2 = C_1 \ln(r_2) + \left(T_1 - C_1 \ln(r_1)\right) \quad \rightarrow \text{sub (1) into } T_2$$

$$T_2 - T_1 = C_1 \ln\left(r_2/r_1\right) \quad \rightarrow \text{rearrange}$$

$$C_1 = \frac{T_2 - T_1}{\ln\left(r_2/r_1\right)}$$

$$\therefore\ T(r) = \left(\frac{T_2 - T_1}{\ln(r_2/r_1)}\right) \ln(r_2) + \left(T_2 - \left(\frac{T_2 - T_1}{\ln(r_2/r_1)}\right) \ln(r_1)\right)$$

- $T(r)$ is of class C^∞

  $T(r)$ is a strong solution

2.b) <u>data</u>

i) $r_1 \leq r \leq r_2$

ii) $Q = 0$

iii) $\hat{k} = \hat{k}_0$

iv)

a) $T\Big|_{r=r_2} = T_2 \quad \rightarrow$ Dirichlet

b) $Q^*\Big|_{r=r_1} = Q_1^* \quad \rightarrow$ Neumann

$*$where $\dfrac{dT}{dr} = \dfrac{Q^*}{n\hat{k}}$

- from previous prob.

$$\frac{dT}{dr} = \frac{C_1}{r}$$

$$T = C_1 \ln(r) + C_2$$

- solve for C_1

  - at inner boundary, $n = -1$

$$\left.\frac{dT}{dr}\right|_{r=r_1} = -\frac{Q_1^*}{\hat{k}_0} = \frac{C_1}{r_1}$$

$$C_1 = -\frac{r_1 Q_1^*}{\hat{k}_0}$$

- solve for C_2
  - at outer boundary

$$T_2 = \frac{-r_1 Q_1^*}{k_0^A} \ln(r_2) + C_2$$

$$C_2 = T_2 + \frac{r_1 Q_1^*}{k_0^A} \ln(r_2)$$

$$\therefore \quad T(r) = -\frac{r_1 Q_1^*}{k_0^A} \ln(r) + T_2 + \frac{r_1 Q_1^*}{k_0^A} \ln(r_2)$$

$$= \frac{r_1 Q_1^*}{k_0^A} \ln\left(\frac{r_2}{r}\right) + T_2$$

- $T(r)$ is of class C^∞

  $T(r)$ is a strong solution

2.c) <u>data</u>

i) $r_1 \le r \le r_2$

ii) $Q = 0$

iii) $k^A = k_0^A$

iv) a) $Q^*\big|_{r=r_1} = Q_1^*$

b) $Q^*\big|_{r=r_2} = Q_2^*$

- again, from 2.a)

$$\frac{dT}{dr} = \frac{C_1}{r}$$

$$T = C_1 \ln(r) + C_2$$

↘

- solve for BC at $r = r_1$

$$\left.\frac{dT}{dr}\right|_{r=r_1} = -\frac{Q_1^*}{k_0^A} = \frac{C_1}{r_1}$$

$$C_1 = -\frac{r_1 Q_1^*}{k_0^A} \qquad (1)$$

- solve for BC at $r = r_2$

$$\left.\frac{dT}{dr}\right|_{r=r_2} = \frac{Q_2^*}{k_0^A} = \frac{C_1}{r_2}$$

$$C_1 = \frac{r_2 Q_2^*}{k_0^A} \qquad (2)$$

- from the problem data, cl calculated

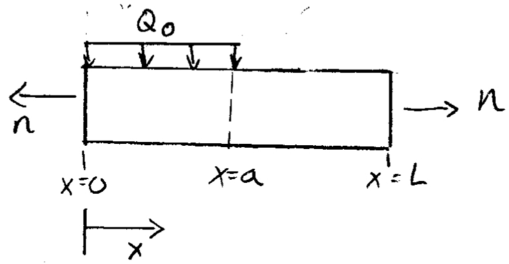
$$C_1 = -\frac{r_1 Q_1^*}{k_0^A} = \frac{r_2 Q_2^*}{k_0^A} \Longrightarrow Q_1^* = -\frac{r_2 Q_2^*}{r_1}$$

  - this cannot be true $\forall\ Q_1^*$ & Q_2^* & is $\therefore$ not well posed

- $T = C_1 \ln(r) + C_2$

  -> C_2 is undefined based on the problem data, $\therefore$ the solution is not unique & not well posed

1-D steady-state heat conduction in Cartesian Coord.



(i) $0 \le x \le L$
(ii) $Q = Q_0 \cdot H[a-x]$
(iii) $k = k_0$
(iv)

a) $T\Big|_{x=0} = T_0$

b) $Q^*\Big|_{x=L} = k_0 \dfrac{dT}{dx}\Big|_{x=L} = Q_L^* n$

Governing Eqn.

$$\frac{d}{dx}\left(k_0 \frac{dT}{dx}\right) = -Q_0 \cdot H[a-x]$$

$$k_0 \frac{dT}{dx} = -Q_0 \langle a-x \rangle + C_1$$

$$T = -\frac{Q_0 \langle a-x \rangle^2}{2k_0} + \frac{C_1 x}{k_0} + C_2$$

b) $k_0 \dfrac{dT}{dx}\Big|_{x=L} = Q_L^* = -Q_0 \langle a-x \rangle + C_1$

$$C_1 = Q_L^* + Q_0 \langle a-x \rangle$$

a) $T\Big|_{x=0} = T_0 = -\dfrac{Q_0 \langle a-0 \rangle^2}{2k_0} + \dfrac{Q_L^* + Q_0 \langle a-0 \rangle}{k_0} x + C_2$

$$\therefore T(x) = -\frac{Q_0 \langle a-x \rangle^2}{2k_0} + \frac{(Q_L^* + Q_0 \langle a-x \rangle)\, x}{k_0} + \frac{Q_0 \langle a-0 \rangle^2}{2k_0} - \frac{Q_L^* + Q_0 \langle a-0 \rangle x}{k_0}$$

☆ T is a strong solution of class C^1 – because $\frac{dT}{dx}$ is continuous, but $\frac{d^2T}{dx}$ will contain $H[a-x]$ and be discontinuous (But defined everywhere)

→ Solution is weak

1-D steady state heat conduction in Cartesian Coord.

i) $0 \le x \le L$

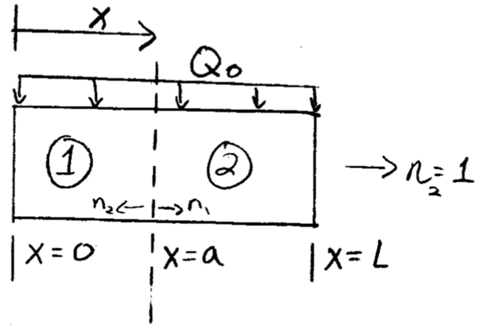
ii) $Q = Q_0$ → const. forcing func.

iii) $k^A = k_1^A + (k_2^A - k_1^A) \cdot H[x - a]$ → discontinuous coef. func

iv)

a) $T_1 \Big|_{x=0} = T_1^0$

b) $Q^* \Big|_{x=L} = k_2^A \dfrac{dT_2}{dx}\Big|_{x=L} = Q_L^* \, n$



Governing Eqn:

$$\frac{d}{dx}\left(k^A \frac{dT}{dx}\right) + Q_0 = 0$$

• solve in each subdomain:

① $\dfrac{d}{dx}\left(k_1^A \dfrac{dT_1}{dx}\right) = -Q_0$

$k_1^A \dfrac{dT_1}{dx} = -Q_0 x + C_1$

$T_1 = -\dfrac{Q_0 x^2}{2k_1^A} + \dfrac{C_1 x}{k_1^A} + C_2$

② $\dfrac{d}{dx}\left(k_2^A \dfrac{dT_2}{dx}\right) = -Q_0$

$k_2^A \dfrac{dT_2}{dx} = -Q_0 x + C_3$

$T_2 = -\dfrac{Q_0 x^2}{2k_2^A} + \dfrac{C_3 x}{k_2^A} + C_4$

• assuming continuity of T, derive relationship across the boundary at $x = a$

$$\int_{a-\varepsilon}^{a+\varepsilon} \left(\frac{d}{dx}\left(k^A \frac{dT}{dx}\right) + Q_0\right) dx = k^A \frac{dT}{dx}\Big|_{a-\varepsilon}^{a+\varepsilon} + Q_0 x \Big|_{a-\varepsilon}^{a+\varepsilon}$$

→ solve on both sides of boundary then let $\varepsilon \to 0$

• $a+\varepsilon$ is in ②

• $a-\varepsilon$ is in ①

$$k_2^A \frac{dT_2}{dx}\Big|_{x=a} - k_1^A \frac{dT_1}{dx}\Big|_{x=a} + \underbrace{\cancel{(Q_0 a - Q_0 a)}}_{\to 0} = 0$$

c) $\therefore \; k_2^A \dfrac{dT_2}{dx}\Big|_{x=a} = k_1^A \dfrac{dT_1}{dx}\Big|_{x=a}$ ⇒ $\dfrac{dT}{dx}$ is discontinuous

· from b)

$$k_2^A \left.\frac{dT_2}{dX}\right|_{x=L} = Q_L^* = -Q_0 L + C_3$$

$$\boxed{C_3 = Q_L^* + Q_0 L}$$

· from c)

$$k_1^A \left.\frac{dT_1}{dx}\right|_{x=a} = k_2^A \left.\frac{dT_2}{dx}\right|_{x=a}$$

$$-Q_0 a + C_1 = -Q_0 a + Q_L^* + Q_0 L$$

$$\boxed{C_1 = Q_L^* + Q_0 L}$$

· from a)

$$T_1\Big|_{x=0} = \boxed{T_1^0 = C_2}$$

· assuming continuity of T at $x=a$

$$T_2\Big|_{x=a} = T_1\Big|_{x=a}$$

$$\frac{-Q_0 a^2}{2k_2^A} + \frac{(Q_L^* + Q_0 L)a}{k_2^A} + C_4 = -\frac{Q_0 a^2}{2k_1^A} + \frac{(Q_L^* + Q_0 L)a}{k_1^A} + T_1^0$$

$$\boxed{C_4 = \frac{Q_0 a^2}{2}\left(\frac{1}{k_2^A - k_1^A}\right) + (Q_L^* + Q_0 L)a\left(\frac{1}{k_1^A - k_2^A}\right) + T_1^0}$$

$$\therefore \quad T_1(x) = -\frac{Q_0 X^2}{2k_1^A} + \frac{(Q_L^* + Q_0 L)x}{k_1^A} + T_1^0$$

$$T_2(x) = -\frac{Q_0 X^2}{2k_2^A} + \frac{(Q_L^* + Q_0 L)x}{k_1^A} + C_4$$

$$T(x) = T_1 + (T_2 - T_1)\cdot H[x-a]$$

★ $T(x)$ is a weak solution to the 2^{nd} order ODE because d^2T/dx^2 is not defined everywhere. $T(x)$ is of Class C^0, because $T(x)$ is Continuous but dT/dx is not

Manufacture a solution

i) domain: $0 \le x \le L$

ii) coef func. → given: $k^A(x) = k_0^A + (k_L^A - k_0^A)\cdot H[x-a] + k_2^A \sin\left(\frac{\pi x}{2L}\right)$

iii) choose $T(x)$:

Let $T(x) = Lx + T_0 \ln(x+L)$

- evaluate primary variable at domain bdry.

$$T\Big|_{x=0} = T_0 \ln(L)$$

$$T\Big|_{x=L} = L^2 + T_0 \ln(2L)$$

- evaluate flux at bdrys.

$$\frac{dT}{dx} = L + \frac{T_0}{x+L}$$

$$k^A \frac{dT}{dx}\Big|_{x=0} = -Q_0^* = k_0\left(L + \frac{T_0}{L}\right)$$

$$k^A \frac{dT}{dx}\Big|_{x=L} = Q_L^* = k^A\left(x + \frac{T_0}{x+1}\right)$$

$$= \left(k_0^A + (k_L^A - k_0^A) + k_2^A \sin\left(\frac{\pi}{2}\right)\right)\left(L + \frac{T_0}{2L}\right)$$

- evaluate forcing func. that satisfies the gov. diff eqn.

$$\frac{d}{dx}\left(k \frac{dT}{dx}\right) + Q(x) = 0$$

$$\frac{dk}{dx}\frac{dT}{dx} + k\frac{d^2T}{dx^2} + Q(x) = 0$$

- $\frac{d^2T}{dx^2} = -\frac{T_0}{(x+1)^2}$

- $\frac{dk^A}{dx} = (k_\ell^A - k_0^A)\langle x-a\rangle + \frac{k_2^A \pi}{2L}\cos\left(\frac{\pi x}{2L}\right)$

$$\therefore \quad Q(x) = -\frac{dk}{dx}\frac{dT}{dx} - k\frac{d^2T}{dx^2}$$

$$= -\left[\left(k_l^A - k_0^A\right)\langle x-a\rangle + \frac{k_2^A \pi}{2L}\cos\left(\frac{\pi x}{2L}\right)\right]\left(L + \frac{T_0}{x+L}\right) -$$

$$\left[k_0^A + \left(k_L^A - k_0^A\right) H[x-a] + k_2^A \sin\left(\frac{\pi x}{2L}\right)\right]\left(-\frac{T_0}{(x+1)^2}\right)$$