Department of Mechanical Engineering
ME404/504
Spring Semester, 2014
Due 3/17/14
3/24.

Computational Mechanics

Homework #2

Method of Weighted Residuals. Consider the following boundary value problem:

$$-(xu')' + 2u = -1.8\pi\cos(1.8\pi x) + \sin(1.8\pi x)\,(2 + (1.8\pi)^2 x), \qquad 0 < x < 1$$

with boundary conditions

$$u(0) = 0, \qquad u'(1) = 1.8\pi\cos(1.8\pi).$$

a. Find the approximate two term solution via the work function for this system using:
   i. The Collocation method
   ii. The Subdomain method
   iii. The Least Squares method
   iv. The Galerkin method

   work function.

   In solving for the approximate solution, use an appropriate function to generate homogeneous boundary conditions at both ends of the domain. You may choose your own basis functions.
b. Derive the weak form of the system equation
c. Find the 1, 2, and 3 term approximate solutions via the Galerkin method using the weak form of the problem that only explicitly satisfies the essential boundary condition.
d. Compare all approximate solutions of both parts a and c in terms of the exact solution $u(x) = \sin(1.8\pi x)$ both pointwise by plotting and with respect to the L_2 norm and Energy norm.
e. Evaluate the value of u and the flux xu' at $x = 1$ for all of the solutions.
f. Discuss your results. Use a table to present your results for the norms in part d and the quantities in part e.

Homework #2 Mohiuddin Ahmad

Problem:

$$-(xu')' + 2u = -1.8\pi\cos(1.8\pi x) + \sin(1.8\pi x)\left(2 + (1.8\pi)^2 x\right)$$

$0 < x < 1$

BC: $u(0)=0$, $u'(1) = 1.8\pi\cos(1.8\pi)$.

Solution:

(a) let the approximate solution is,

$$\hat{U}_N = \sin(1.8\pi x) + \sum_{j=1}^{N} \alpha_j \varphi_j$$, which satisfies the BC, if $\varphi(0)=0$, $\varphi'(1)=0$.

Therefore, $f^* = f - \mathcal{L}(\sin 1.8\pi x)$

$= f + (x\, 1.8\pi\cos 1.8\pi x)' - 2\sin 1.8\pi x$

$= f + (-x(1.8\pi)^2\sin 1.8\pi x) + 1.8\pi\cos 1.8\pi x - 2\sin 1.8\pi x$

$= -1.8\pi\cos(1.8\pi x) + \sin(1.8\pi x)(2 + (1.8\pi)^2 x) - x(1.8\pi)^2\sin 1.8\pi x + 1.8\pi\cos 1.8\pi x - 2\sin 1.8\pi x$

$= 0$

a	19/20
b	5/5
c	6/6
d f	10/9
	40/40

Very nice work.

Therefore, $[\alpha]$ will be zero or infinite solution depending on the value of $\mathbb{K}$.

$$\mathbb{K}_{jk} = \int \mathcal{L}(\varphi_j(x))\, \psi_k\, dx$$

(i) The collocation method:

$$\psi_k = \delta(x - x_k).$$

Then, $$\mathbb{K}_{jk} = \int \mathcal{L}(\varphi_j(x))\, \delta(x - x_k)\, dx.$$

$$= \mathcal{L}(\varphi_j(x_k)).$$

for two term solution, let, $x_1 = 1/3$, $x_2 = 2/3$. Then,

Now,

let, $\phi_j = \sin[(2j-1)\frac{\pi}{2} x]$, which satisfies,

$\phi(0) = 0$

$\phi'(1) = 0$.

Then, $\phi_j' = (2j-1)\frac{\pi}{2} \cos((2j-1)\frac{\pi}{2} x]$

$\phi_j'' = -((2j-1)\frac{\pi}{2})^2 \sin[(2j-1)\frac{\pi}{2} x]$

Then,

$$\mathcal{L}(\phi_j(x)) = -(x\phi_j')' + 2\phi_j$$

$$= -x\phi_j'' - \phi_j' + 2\phi_j$$

$$= +x((2j-1)\frac{\pi}{2})^2 \sin[(2j-1)\frac{\pi}{2}x] - ((2j-1)\frac{\pi}{2}) \cos((2j-1)\frac{\pi}{2}x) + 2\sin((2j-1)\frac{\pi}{2}x]$$

$$= -(2j-1)\frac{\pi}{2}\cos[(2j-1)\frac{\pi}{2}x] + \sin([2j-1)\frac{\pi}{2}x]\left(2 + ((2j-1)\frac{\pi}{2})^2 x\right) \quad \text{------ (1)}$$

$\therefore$ $K_{11} = \mathcal{L}(\phi_1(1/3))$

$$= -\frac{\pi}{2}\cos(\frac{\pi}{2} \times \frac{1}{3}) + \sin(\frac{\pi}{2} \cdot \frac{1}{3})\left[2 + (\frac{\pi}{2})^2 \cdot \frac{1}{3}\right]$$

$= 0.05$

$K_{12} = \mathcal{L}(\phi_1(2/3))$

$$= -\frac{\pi}{2}\cos(\frac{\pi}{2} \times \frac{2}{3}) + \sin(\frac{\pi}{2} \times \frac{2}{3})\left[2 + (\frac{\pi}{2})^2 \times \frac{2}{3}\right]$$

$= 2.37$

$$K_{21} = -\frac{3\pi}{2}\cos(\frac{3\pi}{2} \times \frac{1}{3}) + \sin(\frac{3\pi}{2} \times \frac{1}{3})\left[2 + (\frac{3\pi}{2})^2 \times \frac{1}{3}\right]$$

$= 9.4$

$$K_{22} = -\frac{3\pi}{2}\cos\left(\frac{3\pi}{2} * \frac{2}{3}\right) + \sin\left(\frac{3\pi}{2} * \frac{2}{3}\right)\left[2 + \left(\frac{3\pi}{2}\right)^2 * \frac{2}{3}\right]$$

$$= 4.71.$$

here, $\frac{K_{11}}{K_{21}} \neq \frac{K_{12}}{K_{22}}$. Therefore, only possible solution for, $\begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix}\begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is, $\begin{Bmatrix} \alpha_1 \\ \alpha_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$.

Therefore,

approximate solution from collocation method,

$$\hat{U}_N = \sin(1.8\pi x)$$

(ii) The subdomain method:

Yes, but slow why. i.e. $\psi_k = ?$

From subdomain method, $K_{12} = K_{21} = 0$, always.

Therefore,

$$\left.\begin{aligned} K_{11}\alpha_1 &= 0 \Rightarrow \alpha_1 = 0 \\ K_{22}\alpha_2 &= 0 \Rightarrow \alpha_2 = 0 \end{aligned}\right\} \text{whatever the value of } \mathbb{K} \text{ matrix.}$$

Then, $\hat{U}_N = \sin(1.8\pi x)$

(iii) The least square method:

$$\mathbb{K}_{ij} = \int \mathcal{L}(\phi_i)\mathcal{L}(\phi_j)\, dx$$

using equation (1),

$$K_{11} = \int_0^1 \left[-\frac{\pi}{2}\cos\left(\frac{\pi}{2}x\right) + \sin\left(\frac{\pi}{2}x\right)\left[2 + \left(\frac{\pi}{2}\right)^2 x\right]\right]^2 dx$$

$$= 5.10$$

$$K_{12} = \int_0^1 \left[-\frac{\pi}{2}\cos\left(\frac{\pi}{2}x\right) + \sin\left(\frac{\pi}{2}x\right)\left[2 + \left(\frac{\pi}{2}\right)^2 x\right]\right]\left[-\frac{3\pi}{2}\cos\left(\frac{3\pi}{2}x\right) + \sin\left(\frac{3\pi}{2}x\right)\left[2 + \left(\frac{3\pi}{2}\right)^2 x\right]\right] dx$$

$$= -4.16 = K_{21}.$$

$$K_{22} = \int_0^1 \left(-\frac{3\pi}{2}\cos\left(\frac{3\pi}{2}x\right) + \sin\left(\frac{3\pi}{2}x\right)\left[2+\left(\frac{3\pi}{2}\right)^2 x\right]^2 dx.$$

$$= -55.99$$

as, $\frac{K_{11}}{K_{21}} \neq \frac{K_{12}}{K_{22}}$. The only possible solution is

$\alpha_1 = \alpha_2 = 0$., Then $\hat{U}_N = \sin(1.8\pi x)$.

(IV) The Galerkin method:

From Galerkin method:

$$K_{ij} = \int_0^1 \mathcal{L}(\varphi_j)\, \varphi_i \, dx$$

$$\therefore \quad K_{11} = \int_0^1 \left[-\frac{\pi}{2}\cos\left(\frac{\pi}{2}x\right) + \sin\left(\frac{\pi}{2}x\right)\left[2+\left(\frac{\pi}{2}\right)^2 x\right]\right] \sin\left((2i-1)\frac{\pi}{2}x\right) dx.$$

$$= 1.37$$

$$K_{12} = \int_0^1 \left[-\frac{\pi}{2}\cos\left(\frac{\pi}{2}x\right) + \sin\left(\frac{\pi}{2}x\right)\left[2+\left(\frac{\pi}{2}\right)^2 x\right]\right] \sin\frac{3\pi}{2}x \, dx$$

$$= -0.75$$

$$K_{21} = \int_0^1 \left[-\frac{3\pi}{2}\cos\frac{3\pi}{2}x + \sin\left(\frac{3\pi}{2}x\right)\left[2+\left(\frac{3\pi}{2}\right)^2 x\right]\right] \sin\left(\frac{\pi}{2}x\right) dx$$

$$= -0.75$$

$$K_{22} = \int_0^1 \left[-\frac{3\pi}{2}\cos\left(\frac{3\pi x}{2}\right) + \sin\left(\frac{3\pi x}{2}\right)\left[2+\left(\frac{3\pi}{2}\right)^2 x\right]\right] \sin\left(\frac{3\pi x}{2}\right) dx$$

$$= 6.3$$

as, $\frac{K_{11}}{K_{21}} \neq \frac{K_{12}}{K_{22}}$, the only possible solution is,

$\alpha_1 = \alpha_2 = 0$.

Therefore, $\hat{U}_N = \sin(1.8\pi x)$.

All the four methods give the same solution,

$$\hat{u}_N = \sin(1.8\pi x)$$

(b). Weak form of the system equation

$$\pi(v) = \int (L(u) - f)\, v\, dx$$

$$= \int \left[(xu')' + 2u - f\right] v\, dx$$

$$= -\int_0^1 (xu')'\, v\, dx + \int_0^1 2uv\, dx - \int_0^1 fv\, dx$$

$$= -\left(v\int_0^1 (xu')\, dx + \int_0^1 2uv\, dx\right) - \int_0^1 fv\, dx$$

$$= -(xu')\, v\Big|_0^1 + \int_0^1 xu'v'\, dx + \int_0^1 2uv\, dx - \int fv\, dx$$

$$= -u'(1)\, v(1) + \int_0^1 xu'v'\, dx + \int_0^1 2uv\, dx - \int fv\, dx$$

$$= \int_0^1 xu'v'\, dx + \int_0^1 2uv\, dx - \int fv\, dx - u'(1)\, v(1)$$

This is the weak form of the system equation.

(c) [illegible]

(C) Let, $\hat{u}_N = \sum \alpha_j \varphi_j$

let, $\varphi_j = x^j$ which satisfy the essential boundary condition $u(0)=0$.

Then, and, $v = \sum \beta_K \varphi_K$.

$$\pi(u) = \int_0^1 x \sum \beta_K \varphi_K' \sum \alpha_j \varphi_j' \, dx + \int_0^1 2 \sum \beta_K \varphi_K \sum \alpha_j \cdot \varphi_j \, dx$$

$$- \int f \sum \beta_K \varphi_K \, dx - u'(1) \cdot \sum \beta_K \varphi_K(1) = 0.$$

$$\Rightarrow \quad \sum \alpha_j \left[\int_0^1 x \varphi_j' \varphi_K' \, dx + 2 \int_0^1 \varphi_j \varphi_K \, dx \right]$$

$$- \left[\int_0^1 f \varphi_K \, dx + u'(1) \varphi_K(1) \right] = 0.$$

Here, $K_{ij} = \int_0^1 x \varphi_i' \varphi_j' \, dx + 2 \int \varphi_j \varphi_i \, dx$

& $\underset{\sim}{f} = \int_0^1 f \varphi_K \, dx + u'(1) \varphi_K(1)$.

(i) One term approximation: $\varphi_1 = x$, $\varphi_1' = 1$

$$K_{11} = \int_0^1 x \cdot (1)(1) \, dx + 2 \int_0^1 x \cdot x \cdot dx$$

$$= \left. \frac{x^2}{2} \right|_0^1 + 2 \left. \frac{x^3}{3} \right|_0^1$$

$$= \frac{1}{2} + \frac{2}{3} = 1.17.$$

Then, $\alpha_{11} = \dfrac{f_{11}}{K_{11}}$.

$$f_1 = \int_0^1 \left[-1.8\pi \cos(1.8\pi x) + \sin(1.8\pi x)\left[2 + (1.8\pi)^2 x\right]\right] x \, dx$$
$$+ 1.8\pi \cos(1.8\pi)$$
$$= -5.52 + 1.8\pi \cos(1.8\pi)$$
$$= -0.944$$

Therefore $\alpha = \dfrac{-0.944}{1.17} = -0.807$

Therefore, one term solution is, $\hat{U}_N = -0.807x$

(ii) Two term approximation:

$K_{11} = 1.17$, $f_1 = -0.944$

$$K_{12} = K_{21} = \int_0^1 x.(1)(2x)\,dx + 2\int_0^1 x \times x^2\,dx$$
$$= 2\frac{x^3}{3}\Big|_0^1 + 2\frac{x^4}{4}\Big|_0^1$$
$$= 2/3 + 1/2$$
$$= 1.17$$

$$K_{22} = \int_0^1 x(2x)^2\,dx + 2\int_0^1 x^4\,dx$$
$$= 4\frac{x^4}{4}\Big|_0^1 + 2\frac{x^5}{5}\Big|_0^1$$
$$= 1 + \frac{2}{5}$$
$$= 1.4$$

$$f_2 = \int_0^1 \left[-1.8\pi \cos(1.8\pi x) + \sin(1.8\pi x)\left[2 + (1.8\pi)^2 x\right]\right] x^2\,dx$$
$$+ 1.8\pi \cos(1.8\pi)$$
$$= -5.47 + 1.8\pi \cos(1.8\pi)$$
$$= -0.894$$

Now,

$$[\alpha] = [K] \setminus \underset{\sim}{f}$$

$$\begin{Bmatrix} \alpha_1 \\ \alpha_2 \end{Bmatrix} = \begin{bmatrix} 1.17 & 1.17 \\ 1.17 & 1.4 \end{bmatrix}^{-1} \begin{bmatrix} -0.944 \\ -0.894 \end{bmatrix}$$

$$= \begin{Bmatrix} -1.024 \\ 0.217 \end{Bmatrix}$$

Therefore, $U_N = -1.024x + 0.217x^2$

(iii) Three term approximation:

$$K_{13} = K_{31} = \int_0^1 x \cdot (1)(3x^2)\,dx + 2\int_0^1 (x)(x^3)\,dx$$

$$= 3 \cdot \frac{x^4}{4}\Big|_0^1 + 2 \cdot \frac{x^5}{5}\Big|_0^1$$

$$= 3/4 + 2/5$$

$$= 1.15$$

$$K_{23} = K_{32} = \int_0^1 x(2x)(3x^2)\,dx + 2\int x^2 x^3\,dx$$

$$= 6\,\frac{x^5}{5}\Big|_0^1 + 2\,\frac{x^6}{6}\Big|_0^1$$

$$= 6/5 + 2/6$$

$$= 1.53$$

$$K_{33} = \int_0^1 x \cdot (3x^2)^2\,dx + 2\int_0^1 (x^3)^2\,dx$$

$$= 9 \cdot \frac{x^6}{6}\Big|_0^1 + 2\,\frac{x^7}{7}\Big|_0^1$$

$$= 9/6 + 2/7$$

$$= 1.785$$

$$f_3 = \int_0^1 \left(-1.8\pi\cos(1.8\pi x) + \sin(1.8\pi x)\left[2+(1.8\pi)^2 x\right]\right] x^3\,dx + 1.8\pi\cos(1.8\pi)$$

$$= -5.051 + 1.8\pi\cos(1.8\pi)$$

$$= -0.476$$

Therefore,

$$\begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{Bmatrix} = \begin{bmatrix} 1.17 & 1.17 & 1.15 \\ 1.17 & 1.4 & 1.53 \\ 1.15 & 1.53 & 1.785 \end{bmatrix}^{-1} \begin{Bmatrix} -0.944 \\ -0.894 \\ -0.476 \end{Bmatrix}$$

$$= \begin{Bmatrix} 8.18 \\ -22.52 \\ 13.76 \end{Bmatrix}$$

Therefore, the solution is,

$$\hat{U}_N = 8.18x - 22.52x^2 + 13.76x^3.$$

(d) The solutions obtained by

(i) Collocation, (ii) Subdomain, (iii) least square and (iv) Galerkin method is $= \sin(1.8\pi x)$ which is exactly equal to the exact solution.

From Galerkin method on weak form:

1 term approximation, $\hat{U}_N = -.807x$

2 term " $\hat{U}_N = -1.024x + 0.217x^2$

3 " " $\hat{U}_N = 8.18x - 22.52x^2 + 13.76x^3$.

(d) <u>L_2 norm :</u>

$$e(x) = U_* - \hat{U}_N$$

$$\|e(x)\|_{L_2} = \sqrt{\int_0^L (e(x))^2 dx}$$

$$\|e(x)\|_e = \sqrt{\frac{1}{2}\int_0^L K(x)(e'(x))^2 dx}$$

For collocation, subdomain, least square and Galerkin method, exact solution = approximate solution resulting all norms equal to zero.

For weak form galerkin method :

<u>One term solution :</u>

$$\hat{U}_N = -0.807x$$

$$U_* = \sin(1.8\pi x)$$

$$e(x) = \sin(1.8\pi x) + 0.807x \quad , \quad e'(x) = 1.8\pi\cos(1.8\pi x) + 0.807$$

$$\therefore \|e(x)\|^2_{L_2} = \int_0^1 [\sin(1.8\pi x) + 0.807x]^2 dx$$

$$= 0.498 \quad \Rightarrow \|e\|_{L_2} = 0.706$$

$$\|e(x)\|_e^2 = \frac{1}{2}[\int_0^1 K(x)(e'(x))^2 dx + 2\int_0^1 (e(x))^2 dx$$

good

$$= \frac{1}{2}[\int_0^1 x[\quad 1.8\pi\cos(1.8\pi x) + 0.807]^2 dx$$

$$+ 2\int (\sin 1.8\pi x + 0.807x)^2 dx$$

$$\Rightarrow \|e(x)\|_e = 1.855$$

$$\|U_*\|_{L_2} = \sqrt{\int_0^1 (U_*)^2 dx}$$

$$U_* = \sin 1.8\pi x$$

$$U_*' = 1.8\pi \cos(1.8\pi x)$$

$$\Rightarrow \|U_*\|_{L_2}^2 = \int_0^1 (\sin(1.8\pi x))^2 dx$$

$$= 0.542 \quad \Rightarrow \|U_*\|_{L_2} = 0.736$$

$$\therefore \frac{\|e\|_{L_2}}{\|U_*\|_{L_2}} = \frac{0.706}{0.736} = 0.96 = 96\%$$

$$\|U_*\|_e^2 = \frac{1}{2}\left[\int_0^1 k(x)(U_*')^2 dx + 2\int_0^1 (\sin(1.8\pi x))^2 dx\right.$$

$$= \frac{1}{2}(1.8\pi)^2 \times \int_0^1 x(\cos(1.8\pi x))^2 dx + 2\int_0^1 \{\sin 1.8\pi x\}^2 dx$$

$$=$$

$$= 3.82$$

$$\therefore \|U_*\|_e = 1.955$$

Then, $\dfrac{\|e\|_e}{\|U_*\|_e} = \dfrac{1.855}{1.955} = 94.88$

Two term solution:

$$\hat{U}_N = -1.024x + 0.217x^2$$

$$e(x) = \sin(1.8\pi x) + 1.024x - 0.217x^2.$$

$$\|e(x)\|_{L_2}^2 = \int_0^1 \left[\sin(1.8\pi x) + 1.024x - 0.217x^2\right]^2 dx$$

$$= 0.5382$$

$$\Rightarrow \|e(x)\|_{L_2} = 0.734$$

$$e'(x) = 1.8\pi\cos(1.8\pi x) + 1.024 - 0.434x$$

$$\|e(x)\|_e^2 = \frac{1}{2}\left(\int_0^1 k(x)\,(e'(x))^2\,dx + 2\int_0^1 (e(x))^2\,dx\right.$$

$$= \frac{1}{2}\int_0^1 x\,[1.8\pi\cos(1.8\pi x) + 1.024 - 0.434x]^2\,dx + 2\int_0^1 (\quad in\,1.8\pi x + 1.024x - 0.217x^2)^2\,dx$$

$$\|e(x)\|_e = 1.85$$

$$\therefore \quad \frac{\|e(x)\|_{L_2}}{\|u_*\|_{L_2}} = \frac{0.734}{0.736} = 99.7\,\%$$

$$\& \quad \frac{\|e(x)\|_e}{\|u_*\|_e} = \frac{1.85}{1.955} = 94.62\,\%$$

<u>Three term solution:</u>

$$e(x) = \sin(1.8\pi x) - 8.18x + 22.52x^2 - 13.76x^3$$

$$e'(x) = 1.8\pi\cos(1.8\pi x) - 8.18 + 45.04x - 41.28x^2$$

$$\|e(x)\|_{L_2}^2 = \int_0^1 [\sin(1.8\pi x) - 8.18x + 22.52x^2 - 13.76x^3]^2\,dx$$

$$\|e(x)\|_{L_2} = 0.128$$

$$\|e(x)\|_e^2 = \frac{1}{2}\int_0^1 x\,[1.8\pi\cos(1.8\pi x) - 8.18 + 45.04x - 41.28x^2]^2\,dx + 2\int_0^1 (\sin 1.8\pi x - 8.18x + 22.52x^2 - 13.76x^3)^2\,dx$$

$$\therefore \|e(x)\|_e = 0.498$$

Then,

$$\frac{\|e\|_{L_2}}{\|u_*\|_{L_2}} = \frac{0.128}{0.736} = 17.39\,\%$$

$$\frac{\|e\|_e}{\|u_*\|_e} = \frac{0.498}{1.955} = 25.47\,\%$$

(e) For collocation, Subdomain, least square and Galerkin method:

$$\hat{u}_N = \sin(1.8\pi x)$$

$$\therefore \hat{u}_N(1) = \sin(1.8\pi) = -0.588$$

$$x\hat{u}_N'(1) = 1.8\pi\cos(1.8\pi) = 4.575$$

From weak form:

For 1 term approximation:

$$\hat{U}_N = -0.807x.$$

$$\therefore \hat{U}_N(1) = -0.807$$

$$\& \quad x\hat{u}_N'(1) = -0.807$$

For 2 term approximation:

$$\hat{U}_N = -1.024x + 0.217x^2$$

$$\hat{U}_N' = -1.024 + 0.434x$$

$$\therefore \hat{U}_N(1) = -1.024 + 0.217 = -0.807$$

$$\hat{U}_N'(1) = -1.024 + 0.434 = -0.59$$

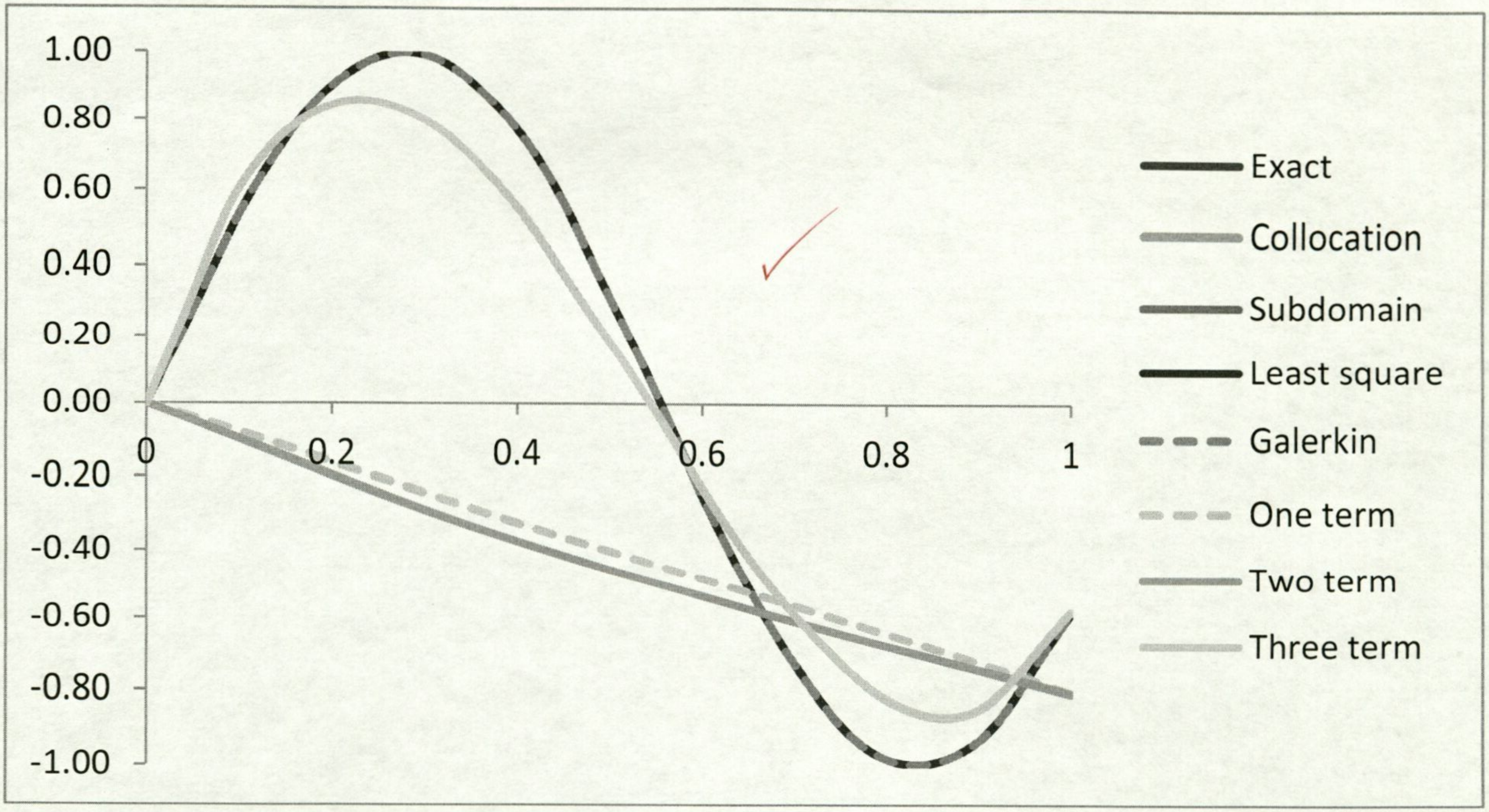
For 3 term approximation:

$$\hat{U}_N = 8.18x - 22.52x^2 + 13.76x^3$$

$$\hat{U}_N' = 8.18 - 45.04x + 41.28x^2$$

$$\therefore \hat{U}_N(1) = -0.58$$

$$\hat{U}_N'(1) = 4.42$$



1.00
0.80
0.60
0.40
0.20
0.00
-0.20
-0.40
-0.60
-0.80
-1.00
0
0.2
0.4
0.6
0.8
1
Exact
Collocation
Subdomain
Least square
Galerkin
One term
Two term
Three term

Method	terms	‖e(x)‖L2	‖U‖L2	‖e(x)‖L2/‖U‖L2	‖e(x)‖e	‖U‖e	‖e(x)‖e/‖U‖e	u(1)	xu'(1)
Collocation		0	0.736	0	0	0.365	0	-0.588	4.575
Subdomain		0	0.736	0	0	0.365	0	-0.588	4.575
Least square		0	0.736	0	0	0.365	0	-0.588	4.575
Galerkin		0	0.736	0	0	0.365	0	-0.588	4.575
Galerkin (weak form)	one	0.706	0.736	96%	1.71	0.365	94.88%	-0.807	-0.807
	two	0.734	0.736	99.70%	1.7	0.365	94.6	-0.807	-0.59
	three	0.128	0.736	17.39%	0.482	0.365	25.4	-0.58	4.42

If the trial solution is choosen considering the BC's, sometimes it is possible to get an approximate solution equal to the exact solution as happen for all ~~four~~ method of residuals here.

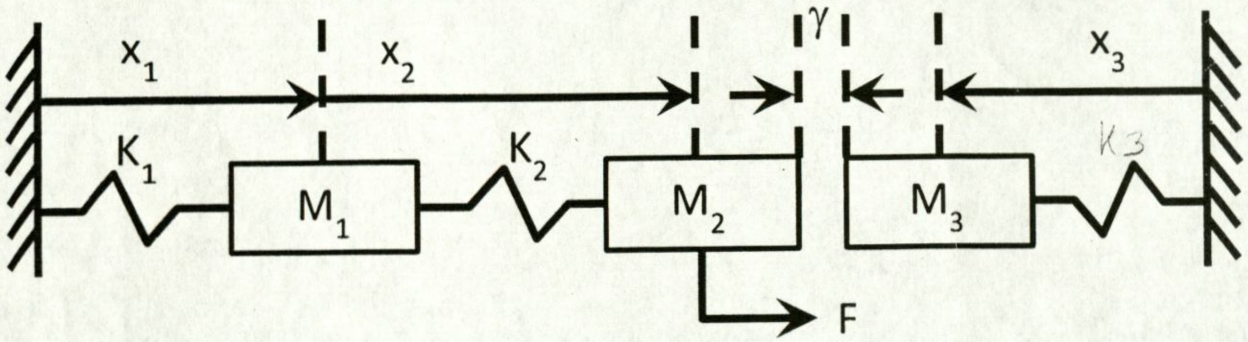
u & xu' at $x=1$ is very close to exact solution when three terms are considered. During this time norms are also small compared to one or two term solution. If a sin or cos function is choosen, it might be possible to get an approximate solution closely equal to exact solution during first term.

Computational Mechanics

Department of Mechanical Engineering
ME404/504
Spring Semester, 2014
Due May 5th

Homework #4

Time Dependent Problems. The purpose of this assignment is to develop your own numerical time integration code, and to apply it to a problem with a strong nonlinearity. Consider the following system:



with rest initial conditions (i.e. $x_1(0) = x_2(0) = x_3(0) = 0$, along with their time derivatives). The gap between masses 2 and 3 is defined as γ. Contact between masses two and three is defined to be Hertzian:

$$F_C = {}^4/_3 E\sqrt{r^*}\,\delta^{3/2},$$

$$\delta = \begin{cases} 0 & \gamma > 0 \\ -\gamma & \gamma \le 0 \end{cases}$$

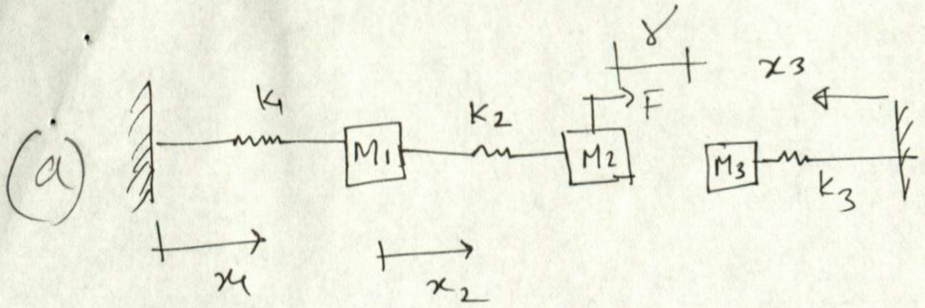
a) Derive the equations of motion for the system using a Lagrangian approach. (Consider verifying your equations with a Newtonian approach).
b) For $F = 10\sin \pi t$, $E\sqrt{r^*} = 100$, $M_1 = 4$, $M_2 = 1$, $M_3 = 3$, $K_1 = K_2 = K_3 = 12$, and $\gamma = 0.2$ when the system is at rest, simulate the response of the system using an IMEX method for 10 seconds.
   i. Plot the displacements of all three masses as functions of time.
   ii. Demonstrate convergence by refining your time step. 1e-6
c) Discuss your results.

Extra Credit. Generate a frequency response plot for this system by varying the forcing frequency from 0.1π to 20π. Plot the impact velocities between the two masses (observed over the first 100 seconds of simulation) as a function of frequency.

HW#4
Mohiuddin Ahmad

41/40

(a)



$$V = PE = \frac{1}{2}(k_1x_1^2 + k_2x_2^2 + k_3x_3^2)$$

$$T = KE = \frac{1}{2}\left(M_1\dot{x}_1^2 + M_2\dot{x}_1^2 + M_2\dot{x}_2^2 + 2M_2\dot{x}_1\dot{x}_2\right) + \frac{1}{2}M_3\dot{x}_3^2\Big)$$

work done by external forces, $W = F(x_1+x_2) - F_c(x_1+x_2) - F_c\,x_3$

$$\mathcal{L} = T - V = \frac{1}{2}\left(m_1\dot{x}_1^2 + m_2\dot{x}_1^2 + m_2\dot{x}_2^2 + 2m_2\dot{x}_1\dot{x}_2 + m_3\dot{x}_3^2 - k_1x_1^2 - k_2x_2^2 - k_3x_3^2\right)$$ ~~where $(x_1+x_2) > \gamma$~~

~~$F(x_1+x_2) < 1$, where $x_1+x_2 < \gamma$.~~

Then,

$$Q_{NC,x_1} = \frac{\partial W}{\partial x_1} = F - F_c$$

$$Q_{NC,x_2} = \frac{\partial W}{\partial x_2} = F - F_c$$

$$Q_{NC,x_3} = \frac{\partial W}{\partial x_3} = -F_c$$

$\therefore \frac{\partial L}{\partial x_1} = -k_1x_1, \quad \frac{\partial L}{\partial \dot{x}_1} = m_1\dot{x}_1 + m_2\dot{x}_1 + m_2\dot{x}_2$

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{x}_1}\right) = m_1\ddot{x}_1 + m_2\ddot{x}_2 + m_2\ddot{x}_1$$

\# $\frac{\partial L}{\partial x_2} = -k_2x_2, \quad \frac{\partial L}{\partial \dot{x}_2} = m_2\dot{x}_2 + m_2\dot{x}_1, \quad \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{x}_2}\right) = m_2\ddot{x}_2 + m_2\ddot{x}_1$

\# $\frac{\partial L}{\partial x_3} = -k_3x_3, \quad \frac{\partial L}{\partial \dot{x}_3} = m_3\dot{x}_3 \quad \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{x}_3}\right) = m_3\ddot{x}_3$

Lagrangian Eqn: $\frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}_i}\right) - \frac{\partial \mathcal{L}}{\partial x_i} = Q_{NC,x_i}$

Then,

$$m_1\ddot{x}_1 + m_2\ddot{x}_2 + m_2\ddot{x}_1 + k_1x_1 = F - F_c$$

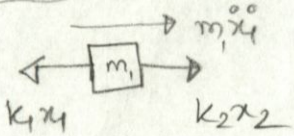
$$m_2\ddot{x}_2 + m_2\ddot{x}_1 + k_2x_2 = F - F_c$$

$$m_3\ddot{x}_3 + k_3x_3 = -F_c.$$

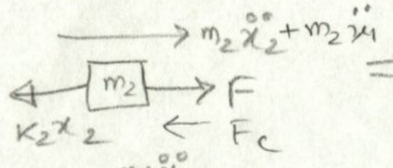
In matrix form:

$$\begin{bmatrix} m_1+m_2 & m_2 & 0 \\ m_2 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{Bmatrix} + \begin{bmatrix} k_1 & 0 & 0 \\ 0 & k_2 & 0 \\ 0 & 0 & k_3 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} F-F_c \\ F-F_c \\ -F_c \end{Bmatrix}$$

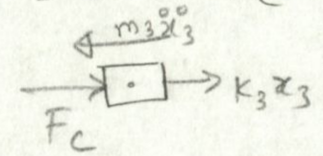
Verifying using Newtons Approach:



$\Rightarrow \quad k_1x_1 + m_1\ddot{x}_1 - k_2x_2 = 0 \quad ------(1)$



$\Rightarrow \quad m_2\ddot{x}_1 + m_2\ddot{x}_2 + F_c + k_2x_2 - F = 0 \quad ---(11)$



$\Rightarrow \quad m_3\ddot{x}_3 + k_3x_3 + F_c = 0 \quad ------(111)$

From Eqn (11), $\quad k_2x_2 = F - F_c - m_2(\ddot{x}_1 + \ddot{x}_2)$

Then From (1)

$$m_1\ddot{x}_1 + k_1x_1 - F + F_c + m_2\ddot{x}_1 + m_2\ddot{x}_2 = 0$$

$$\Rightarrow \quad \ddot{x}_1(m_1+m_2) + \ddot{x}_2 m_2 + k_1x_1 = F - F_c$$

In matrix form

$$\begin{bmatrix} m_1+m_2 & m_2 & 0 \\ m_2 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{Bmatrix} + \begin{bmatrix} k_1 & 0 & 0 \\ 0 & k_2 & 0 \\ 0 & 0 & k_3 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} F-F_c \\ F-F_c \\ -F_c \end{Bmatrix}$$

Therefore, Lagrangian & Newton method yields exactly same set of equations.

+1 for clarifying this in class.

(b) $\mathbb{M} = \begin{bmatrix} m_1+m_2 & m_2 & 0 \\ m_2 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix}$ $\mathbb{K} = \begin{bmatrix} k_1 & 0 & 0 \\ 0 & k_2 & 0 \\ 0 & 0 & k_3 \end{bmatrix}$, $\underset{\sim}{f} = \begin{Bmatrix} F-F_c \\ F-F_c \\ -F_c \end{Bmatrix}$

let

$$\underset{\sim}{q} = \begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ x_1 \\ x_2 \\ x_3 \end{Bmatrix}, \quad \underset{\sim}{\mathring{q}} = \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \\ \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{Bmatrix}$$

Now,

$$\mathbb{B}\underset{\sim}{\mathring{q}} + \mathbb{D}\underset{\sim}{q} = \underset{\sim}{e}$$

here,

$$\mathbb{B} = \begin{bmatrix} \mathbb{M} & 0 \\ 0 & \mathbb{I} \end{bmatrix}, \quad \mathbb{D} = \begin{bmatrix} 0 & \mathbb{K} \\ -\mathbb{I} & 0 \end{bmatrix}, \quad \underset{\sim}{e} = \begin{Bmatrix} \underset{\sim}{f} \\ 0 \end{Bmatrix}$$

$$\therefore \quad \underset{\sim}{\mathring{q}} = \underbrace{-\mathbb{B}^{-1}\mathbb{D}}_{\mathbb{A}}\, \underset{\sim}{q} + \underbrace{\mathbb{B}^{-1}e}_{\underset{\sim}{F}}$$

$$\underset{\sim}{\mathring{q}} = \mathbb{A}\underset{\sim}{q} + \underset{\sim}{F}$$

Here

$$\mathbb{B} = \begin{bmatrix} m_1+m_2 & m_2 & 0 & 0 & 0 & 0 \\ m_2 & m_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & m_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbb{D} = \begin{bmatrix} 0 & 0 & 0 & k_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & k_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & k_3 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \end{bmatrix}, \quad \underset{\sim}{e} = \begin{Bmatrix} F-F_c \\ F-F_c \\ -F_c \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

Determine $\mathbb{A} = -\mathbb{B}^{-1}\mathbb{D}$, $\underset{\sim}{\mathbb{F}} = \mathbb{B}^{-1}\underset{\sim}{e}$

b (i) Displacement of three masses

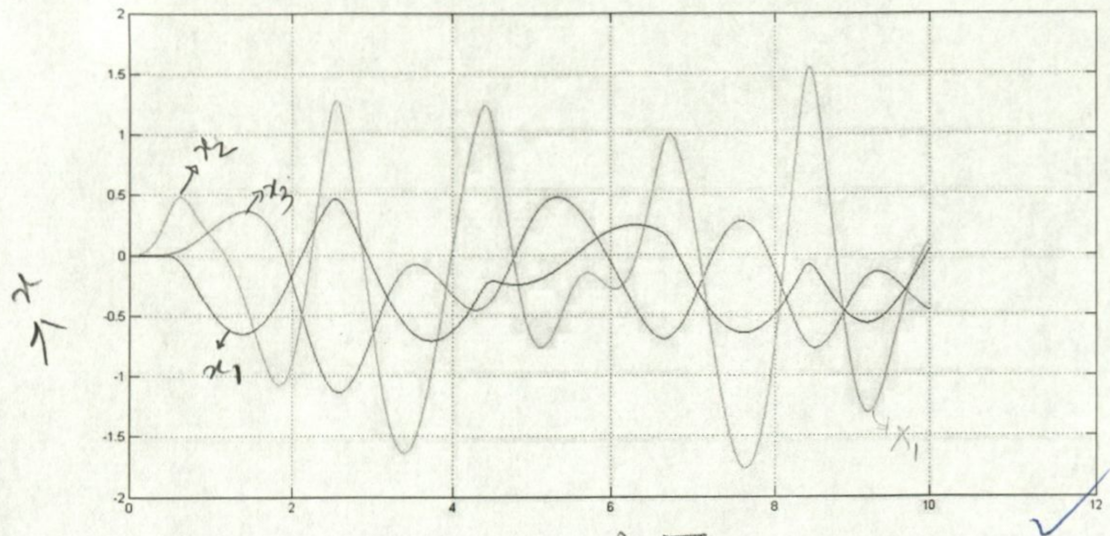
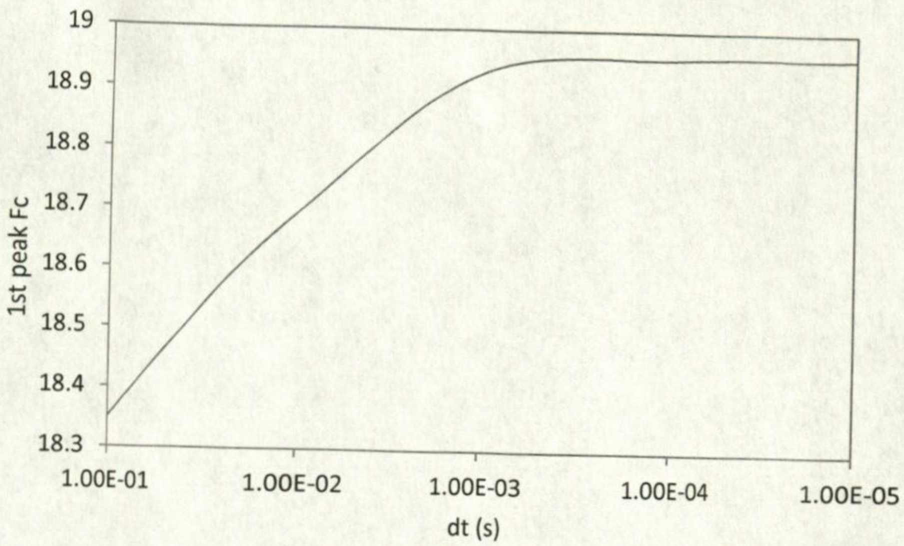


Figure-1 → Time

(11) Convergence



(c) Discussion:

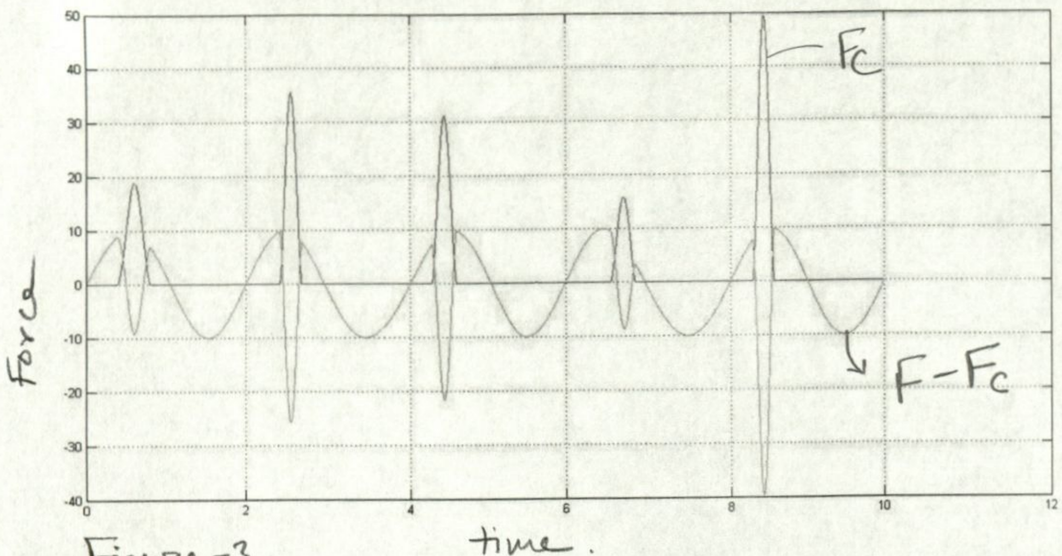


Figure-3

As $\omega = \pi$, $T = \frac{2\pi}{\pi} = 2s$.

Therefore, in 10s we expect to have 5 impact. Figure above shown that mass 2 & 3 have five impacts. The impact force is suppoed to last very short duration. This is exactly happening here.

Figure 1 shows that, x_1 is not affected by the impact that much. Its maximum amplitude stay almost same.

Appendix

MATLAB code:

```
clear all,clc
B=[5 1 0 0 0 0;1 1 0 0 0 0;0 0 3 0 0 0;0 0 0 1 0 0; 0 0 0 0 1 0;0 0 0 0 0 1];
 D=[0 0 0 12 0 0;0 0 0 0 12 0;0 0 0 0 0 12;-1 0 0 0 0 0;0 -1 0 0 0 0;0 0 -1 0
0 0];
 H=inv(B);
 A=-H*D;
 T=10;
 dt=1e-4;
 n=T/dt+1;
 H2=dt*A;
 H3=eye(6)-H2;
 C=inv(H3);
 y=[];
 y(:,1)=zeros(6,1);
 e=zeros(6,1);
 f=[];
 F_c=[];
 y_star=[];
  for i=2:n
      t(i)=i*dt;
     gama(i)=.2-y(4,i-1)-y(5,i-1)-y(6,i-1);
     if gama(i)>=0
        del(i)=0;
     else
        del(i)=-gama(i);
     end
%      if gama(i)<0
%         del(i)=-gama(i);
%      else
%         del(i)=0;
%      end
     F_c(i)=133.33*del(i)^1.5;
     F(i)=10*sin(pi*t(i));
     F_1(i)=F(i)-F_c(i);
     F_2(i)=F_1(i);
     F_3(i)=-F_c(i);
     e=[F_1(i) F_2(i) F_3(i) 0 0 0]';
     f=H*e;
     y_star=y(:,i-1)+dt*f;
     y(:,i)=C*y_star;

  end
figure();
plot(t,y(4,:),'r',t,y(5,:),'g',t,y(6,:),'b')
grid on
% figure()
% plot(t,F)
% plot(t,F_c,'r',t,F_1,'g')
% grid on
%[p,q]=findpeaks(F_c)
%plot(t,gama);
% grid on
%y23=y(3,:)+y(2,:);
%plot(t,y23);
%plot(t,y(1,:),'r',t,y(2,:),'g',t,y(3,:),'b');
%grid on;
```