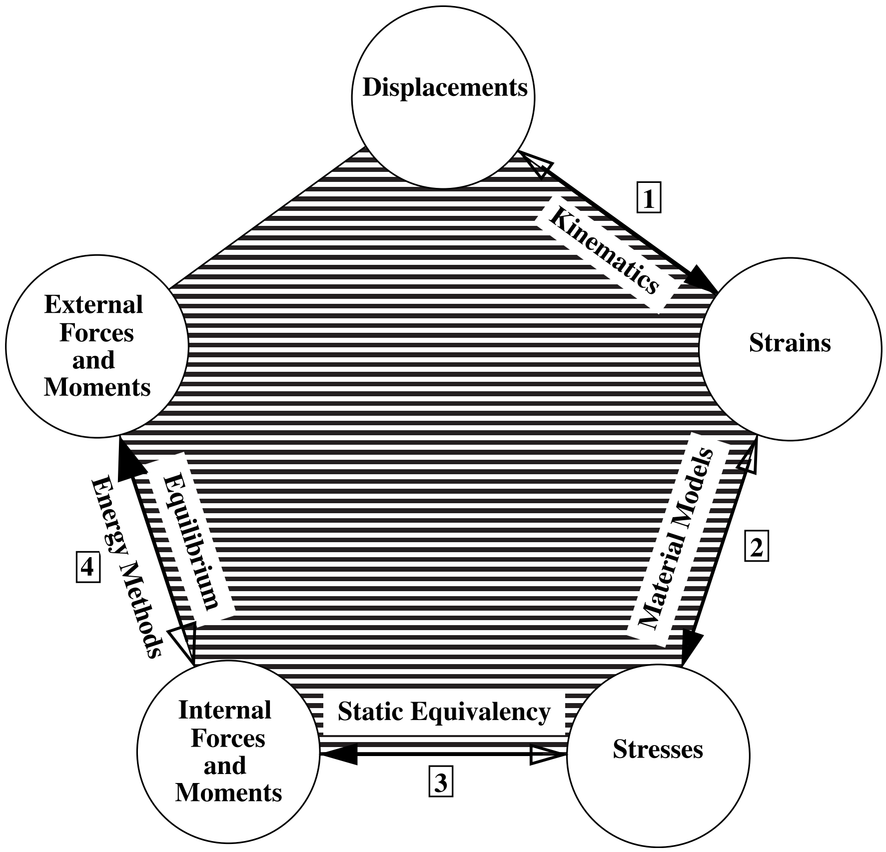
Energy Methods

- Minimum-energy principles are an alternative to statement of equilibrium equations.

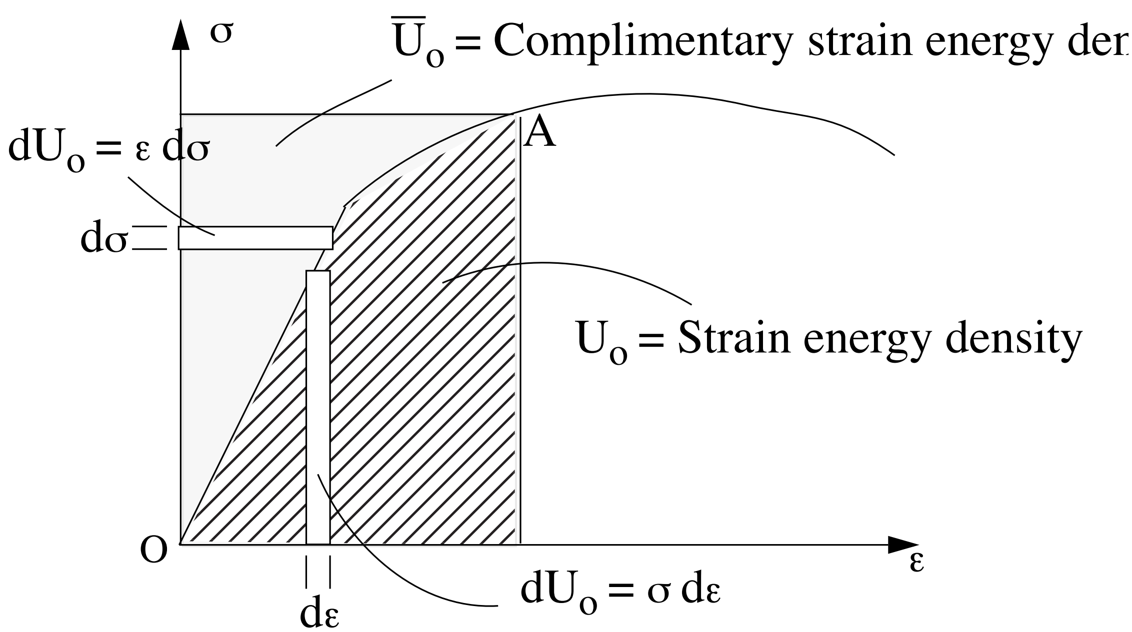


The learning objectives in this chapter is:

- Understand the perspective and concepts in energy methods.

Strain Energy

- The energy stored in a body due to deformation is called the *strain energy*.
- The strain energy per unit volume is called the *strain energy density* and is the area underneath the stress-strain curve up to the point of deformation.



Strain Energy: $U = \int_V U_o \; dV$ [

Strain Energy Density: $U_o = \int_0^{\varepsilon} \sigma d\varepsilon$

Units: N-m / m^3, Joules / m^3, in-lbs / in^3, or ft-lb/ft.^{3}

Complimentary Strain Energy Density: $\overline{U}_o = \int_0^{\sigma} \varepsilon d\sigma$

Linear Strain Energy Density

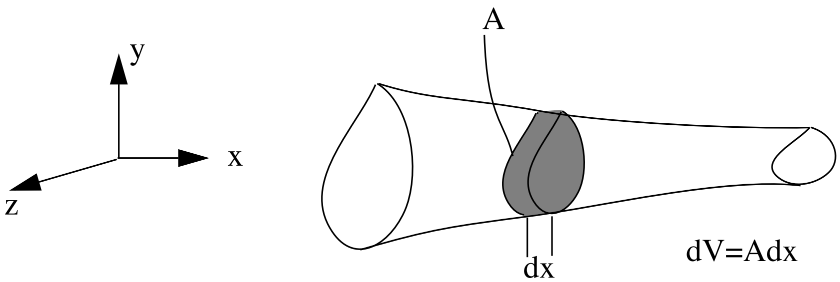
Uniaxial tension test: $U_o = \int_0^{\varepsilon} \sigma d\varepsilon = \int_0^{\varepsilon} (E\varepsilon) d\varepsilon = \frac{E\varepsilon^2}{2} = \frac{1}{2}\sigma\varepsilon$

$$U_o = \frac{1}{2}\tau\gamma$$

- Strain energy and strain energy density is a scaler quantity.

$$\boxed{U_o = \frac{1}{2}[\sigma_{xx}\varepsilon_{xx} + \sigma_{yy}\varepsilon_{yy} + \sigma_{zz}\varepsilon_{zz} + \tau_{xy}\gamma_{xy} + \tau_{yz}\gamma_{yz} + \tau_{zx}\gamma_{zx}]}$$

1-D Structural Elements



Axial strain energy

- All stress components except σ_{xx} are zero.

$$\sigma_{xx} = E\varepsilon_{xx} \qquad \varepsilon_{xx} = \frac{du}{dx}(x)$$

$$U_A = \int_V \frac{1}{2}E\varepsilon_{xx}^2 dV = \int_L \left[\int_A \frac{1}{2}E\left(\frac{du}{dx}\right)^2 dA\right] dx = \int_L \left[\frac{1}{2}\left(\frac{du}{dx}\right)^2 \int_A E dA\right] dx$$

$$U_A = \int_L U_a \, dx \qquad U_a = \frac{1}{2}EA\left(\frac{du}{dx}\right)^2$$

- U_a is the strain energy per unit length.

$$\bar{U}_A = \int_L \bar{U}_a \, dx \qquad \bar{U}_a = \frac{1}{2}\frac{N^2}{EA}$$

Torsional strain energy

- All stress components except $\tau_{x\theta}$ in polar coordinate are zero

$$\tau_{x\theta} = G\gamma_{x\theta} \qquad \gamma_{x\theta} = \rho\frac{d\phi}{dx}(x)$$

$$U_T = \int_V \frac{1}{2}G\gamma_{x\theta}^2 dV = \int_L\left[\int_A \frac{1}{2}G\left(\rho\frac{d\phi}{dx}\right)^2 dA\right]dx = \int_L\left[\frac{1}{2}\left(\frac{d\phi}{dx}\right)^2\int_A G\rho^2 dA\right]dx$$

$$U_T = \int_L U_t \; dx \qquad U_t = \frac{1}{2}GJ\left(\frac{d\phi}{dx}\right)^2$$

- U_t is the strain energy per unit length.

$$\bar{U}_T = \int_L \bar{U}_t \; dx \qquad \bar{U}_t = \frac{1}{2}\frac{\boldsymbol{T}^2}{GJ}$$

Strain energy in symmetric bending about z-axis

There are two non-zero stress components, σ_{xx} and τ_{xy}.

$$\sigma_{xx} = E\varepsilon_{xx} \qquad \varepsilon_{xx} = -y\frac{d^2v}{dx^2}$$

$$U_B = \int_V \frac{1}{2}E\varepsilon_{xx}^2 dV = \int_L\left[\int_A \frac{1}{2}E\left(y\frac{d^2v}{dx^2}\right)^2 dA\right]dx = \int_L\left[\frac{1}{2}\left(\frac{d^2v}{dx^2}\right)^2\int_A Ey^2 dA\right]dx$$

$$U_B = \int_L U_b \; dx \qquad U_b = \frac{1}{2}EI_{zz}\left(\frac{d^2v}{dx^2}\right)^2$$

- where U_b is the bending strain energy per unit length.

$$\bar{U}_B = \int_L \bar{U}_b \; dx \qquad \bar{U}_b = \frac{1}{2}\frac{\boldsymbol{M_z}^2}{EI_{zz}}$$

The strain energy due to shear in bending is: $U_S = \int_V \frac{1}{2}\tau_{xy}\gamma_{xy} \; dV = \int_V \frac{1}{2}\frac{\tau_{xy}^2}{E} \; dV$

As $\tau_{max} \ll \sigma_{max}$ $\qquad U_s \ll U_B$

Table 7.1 Energy densities

	Strain energy density per unit length	Complimentary strain energy density per unit length
Axial	$U_a = \frac{1}{2}EA\left(\frac{du}{dx}\right)^2$	$\overline{U}_a = \frac{1}{2}\frac{N^2}{EA}$
Torsion of circular shafts	$U_t = \frac{1}{2}GJ\left(\frac{d\phi}{dx}\right)^2$	$\overline{U}_t = \frac{1}{2}\frac{T^2}{GJ}$
Symmetric bending of beams	$U_b = \frac{1}{2}EI_{zz}\left(\frac{d^2v}{dx^2}\right)^2$	$\overline{U}_b = \frac{1}{2}\frac{M_z^2}{EI_{zz}}$

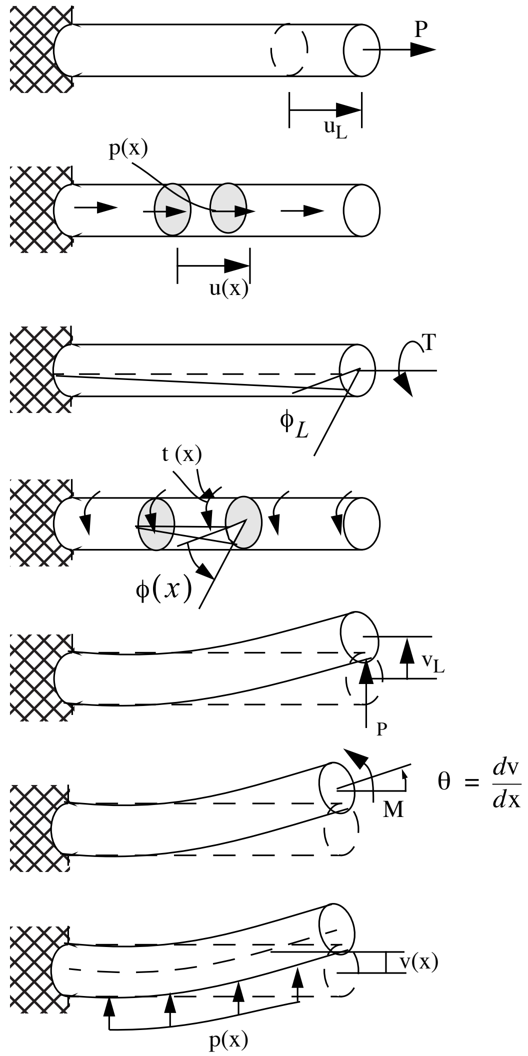
Work

- If a force moves through a distance, then work has been done by the force.

$$dW = F du$$

- Work done by a force is conservative if it is path independent.
- Non-linear systems and non-conservative systems are two independent description of a system.

Loading Mode | Work



$$\delta W = P\delta u_L$$

$$\delta W = \int_0^L p(x)\delta u(x)dx$$

$$\delta W = T\delta\phi_L$$

$$\delta W = \int_0^L t(x)\delta\phi(x)dx$$

$$\delta W = P\delta v_L$$

$$\delta W = M\delta\theta_L$$

$$\delta W = \int_0^L p(x)\delta v(x)dx$$

- Any variable that can be used for describing deformation is called the generalized displacement.
- Any variable that can be used for describing the cause that produces deformation is called the generalized force.

Virtual Work

- Virtual work methods are applicable to linear and non-linear systems, to conservative as well as non-conservative systems.

The principle of virtual work:

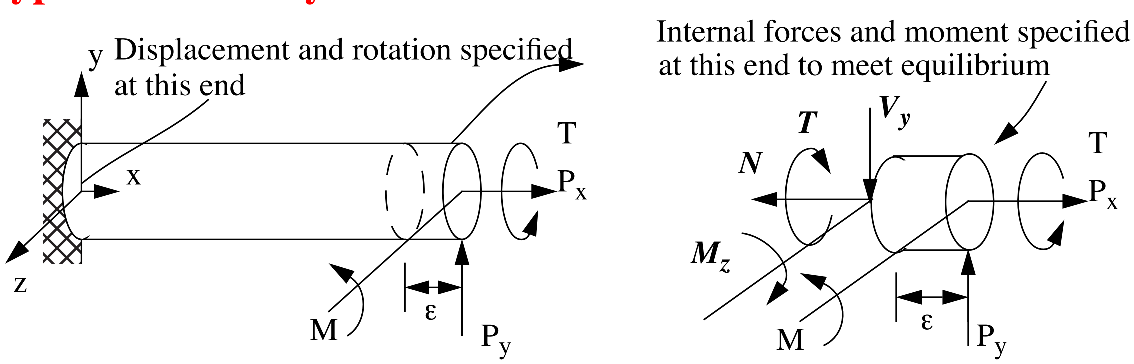
The total virtual work done on a body at equilibrium is zero.

$$\delta W = 0$$

- Symbol δ will be used to designate a virtual quantity

$$\delta W_{ext} = \delta W_{int}$$

Types of boundary conditions



Kinematic variable (Primary variable) | Statical variable (Secondary variable)

$$u \overset{\text{or}}{\longleftrightarrow} \boldsymbol{N}$$

$$\phi \overset{\text{or}}{\longleftrightarrow} \boldsymbol{T}$$

$$v \overset{\text{or}}{\longleftrightarrow} \boldsymbol{V_y}$$

$$\theta = \frac{dv}{dx} \overset{\text{or}}{\longleftrightarrow} \boldsymbol{M_z}$$

Geometric boundary conditions (Kinematic boundary conditions) (Essential boundary conditions):

Condition specified on kinematic (primary) variable at the boundary.

Statical boundary conditions (Natural boundary conditions)

Condition specified on statical (secondary)variable at the boundary.

Kinematically admissible functions

- Functions that are continuous and satisfies all the kinematic boundary conditions are called *kinematically admissible functions*.
- actual displacement solution is always a kinematically admissible function
- Kinematically admissible functions are not required to correspond to solutions that satisfy equilibrium equations.

Statically admissible functions

- Functions that satisfy satisfies all the static boundary conditions, satisfy equilibrium equations at all points, and are continuous at all points except where a concentrated force or moment is applied are called *statically admissible functions*.
- Actual internal forces and moments are always statically admissible.
- Statically admissible functions are not required to correspond to solutions that satisfy compatibility equations.

7.3 Determine a class of kinematically admissible displacement functions for the beam shown in Fig. P7.3.

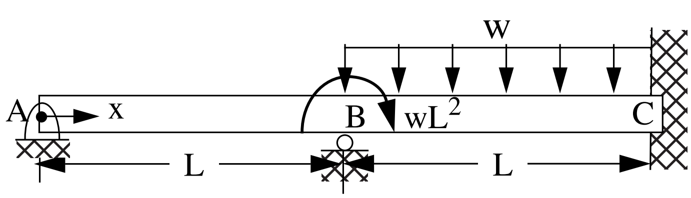
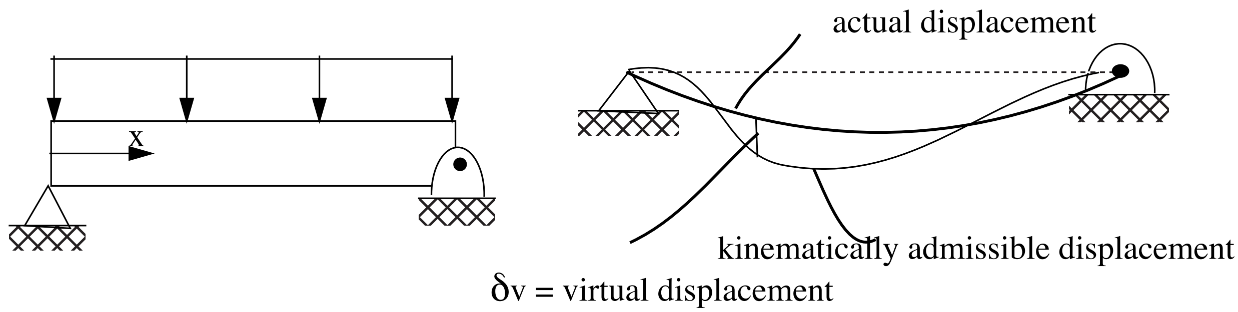


Fig. P7.3

7.4 For the beam and loading shown in Fig. P7.3 determine a statically admissible bending moment.

Virtual displacement method

- The virtual displacement is an infinitesimal imaginary kinematically admissible displacement field imposed on a body.



- Of all the virtual displacements the one that satisfies the virtual work principle is the actual displacement field.

Virtual Force Method

- The virtual force is an infinitesimal imaginary statically admissible force field imposed on a body.
- Of all the virtual force fields the one that satisfies the virtual work principle is the actual force field.

7.7 The roller at P shown in Fig. P7.7 slides in the slot due to the force F = 20kN. Both bars have a cross-sectional area of A = 100 mm^2 and a modulus of elasticity E = 200 GPa. Bar AP and BP have lengths of L_{AP}= 200 mm and L_{BP}= 250 mm respectively. Determine the axial stress in the member AP by virtual displacement method.

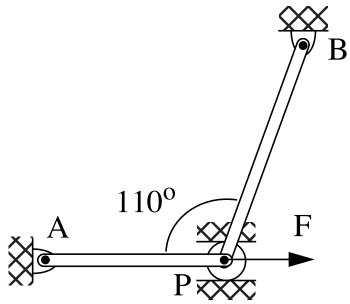


Fig. P7.7

7.8 A force F = 20kN is appled to pin shown in Fig. P7.8. Both bars have a cross-sectional area of A = 100 mm^2 and a modulus of elasticity E = 200 GPa. Bar AP and BP have lengths of L_{AP}= 200 mm and L_{BP}= 250 mm respectively. Using virtual force method determine the movement of pin in the direction of force F.

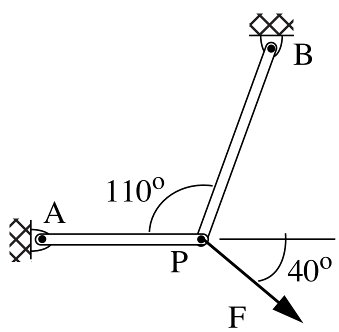


Fig. P7.8