CBE 521
ADVANCED TRANSPORT PHENOMENA
HOMEWORK 2

BRANDON LAMPE

1. Problem 3: Steady Parallel Rectilinear Flow

1.1. Part b: Show the solution can be written as the sum of particular and complementary (homogeneous) solutions, where $v = v_p + v_c$.

Statement 1. *The complete solution is:*

$$\frac{\partial^2 v}{\partial \xi^2} + \frac{\partial^2 v}{\partial \eta^2} = -1 \tag{1}$$

the problem domain:

$$v(\pm 1, \eta) = 0 \tag{2}$$

$$v(\xi, \pm r) = 0 \tag{3}$$

Statement 2. *Must make sense of problem domain definitions:*

$$x \in [0, 2B] \tag{4}$$

$$y \in [0, 2W] \tag{5}$$

$$r \equiv \frac{W}{B} \tag{6}$$

$$\xi \equiv \frac{x}{B} = \frac{2B}{B} = 2 \implies -1 \le \xi \le 1 \tag{7}$$

$$\eta \equiv \frac{y}{B} = \frac{2W}{B} = \frac{2rB}{B} = 2r \implies -r \le \eta \le r \tag{8}$$

Statement 3. *Separate the differential equation into particular and homogeneous parts, and let the particular solution (v_p) satisfy the boundary conditions given in equation 2:*

$$\frac{\partial^2 v_p}{\partial \xi^2} = -1 \tag{9}$$

$$v_p(\pm 1, \eta) = 0 \tag{10}$$

Statement 4. *The complementary part of the problem is then given by:*

$$\frac{\partial^2 v_c}{\partial \xi^2} + \frac{\partial^2 v_c}{\partial \eta^2} = 0 \tag{11}$$

$$\tag{12}$$

The complementary solution must satisfy the following:

$$v - v_p = v_c \implies \frac{\partial^2 v}{\partial \xi^2} + \frac{\partial^2 v}{\partial \eta^2} - \frac{\partial^2 v_p}{\partial \xi^2} = -1 - (-1) = 0 \tag{13}$$

The boundary conditions must then be:

$$v(\pm 1, \eta) - v_p(\pm 1, \eta) = v_c(\pm 1, \eta) = 0 - 0 = 0 \tag{14}$$

$$v(\xi, \pm r) - v_p(\xi, \pm r) = v_c(\xi, \pm r) \implies 0 - c_1 = c_1 \tag{15}$$

Date: Sept. 29, 2015.

Therefore:

$$c_1 = v_c(\xi, \pm r) = -v_p(\xi, \pm r) \tag{16}$$

1.2. **Part c: Show the particular solution:**

Statement 5. *The solution to equation 9 is obtained by integrating twice and solving for the boundary conditions:*

$$v_p(\xi = \pm 1, \eta) = -\frac{\xi^2}{2} + \frac{1}{2} = \frac{1}{2}(1 - \xi^2) \tag{17}$$

1.3. **Part d: Obtain the complementary solution:**

Statement 6. *Considering only the homogeneous form of the equation will allow for a system of basis functions that satisfy the given boundary conditions, and a solution method similar to that for the Laplace equation will be utilized. Additionally, the boundary conditions of the homogeneous form will be modified such that the boundary conditions for v are satisfied (i.e., $v_c(\xi, \pm r) = -v_p(\xi, \pm r)$). The Homogeneous form of the equation is:*

$$\frac{\partial^2 v_c}{\partial \xi^2} + \frac{\partial^2 v_c}{\partial \eta^2} = 0 \tag{18}$$

$$v_c(\pm 1, \eta) = 0$$

$$v_c(\xi, \pm r) = -v_p(\xi, \pm r) = \frac{\xi^2}{2} - \frac{1}{2}$$

Statement 7. *Seperate the variables and assume $v_c(\xi, \eta)$ consists of two independent functions $\Xi(\xi)$ and $H(\eta)$:*

$$v_c(\xi, \eta) = \Xi(\xi)\, H(\eta)$$

therefore, using the product rule, the partial derivatives are:

$$v_{c,\xi\xi} = \Xi^{''} H$$

$$v_{c,\eta\eta} = H^{''} \Xi$$

Statement 8. *Substitute the above definitions into equation 18 and the following (equation 19) is obtained. If the left and right sides of the above equation are equal for all ξ and η, then λ must be a constant.*

$$\Xi^{''}(\xi) H(\eta) = -H^{''}(\eta) \Xi(\xi) = \lambda = \frac{\Xi^{''}(\xi)}{\Xi(\xi)} = -\frac{H^{''}(\eta)}{H(\eta)} \tag{19}$$

Statement 9. *Because λ is constant, equation 19 may be rewritten in terms of two ordinary differential equations.*

$$\Xi^{''} - \Xi \lambda^2 = 0 \tag{20}$$

$$H^{''} + H \lambda^2 = 0 \tag{21}$$

Statement 10. *By translating the boundary conditions to be in terms of the separation variables and dividing them to be in terms of a single variable results in the following:*

$$\begin{aligned}
v_c(+1, \eta) &= \Xi(+1) H(\eta) = 0 \qquad \forall\, \eta \in [-r, +r] \implies \Xi(+1) = 0 \\
v_c(-1, \eta) &= \Xi(-1) H(\eta) = 0 \qquad \forall\, \eta \in [-r, +r] \implies \Xi(-1) = 0 \\
v_c(\xi, -r) &= \Xi(\xi) H(-r) = 0 \qquad \forall\, \xi \in [-1, +1] \implies H(-r) = 0 \\
v_c(\xi, +r) &= \Xi(\xi) H(+r) = 0 \qquad \forall\, \xi \in [-1, +1] \implies H(+r) = 0
\end{aligned}$$

Statement 11. *The equations (Eq. 21) is now an ordinary differential equations with homogeneous boundary conditions having the characteristic equation $m^2 + \lambda^2 = 0$. The solution of this system is in form of an infinite sequence of eigenfunctions (H_n) and eigenvalues (λ_n).*

$$H^{''} + H \lambda^2 = 0 \tag{22}$$

$$H(+r) = 0$$

$$H(-r) = 0$$

Statement 12. *The solution to 21 can be written in the form:* $H = e^{\alpha\eta}\left(Asin(\beta\eta) + Bcos(\beta\eta)\right)$. *Here* $\alpha = 0$ *and* $\beta = \lambda$. *Substitution of* $(\eta + r)$ *for* η *in the equation result only in a shift in the eigenspace and allows for a solution when the boundary conditions are applied:*

$$H = Asin(\beta(\eta + r)) + Bcos(\beta((\eta + r))$$
$$H(\eta = -r) = 0$$
$$H(\eta = +r) = 0$$

Statement 13. *Solving for the boundary conditions at* $\eta = -r$ *results in* $B = 0$. *This leaves* $H = Asin(\lambda(\eta + r))$, *and when solved at the boundary* $\eta = +r$:

$$H(+r) = 0 = Asin(\lambda 2r) \tag{23}$$

If $A = 0$, *the solution reduces to the trivial solution* $H = 0$. *For a non-trivial solution* $sin(\lambda(2r)) = 0$ *and by letting* $\lambda(2r) = (n+1)\pi$, $n = 0, 1, 2, ...$ *a solution for* λ *may be obtained:*

$$\lambda_n = \frac{(n+1)\pi}{2r}, \; n = 0, 1, 2, 3, ... \tag{24}$$

$$H_n(\eta) = sin\left(\frac{(n+1)\pi(\eta + r)}{2r}\right), \; n = 0, 1, 2, 3... \tag{25}$$

Statement 14. *Solve the remaining ODE (Equation 20) with boundary conditions:*

$$\Xi^{''} - \Xi\lambda^2 = 0 \tag{26}$$
$$\Xi(\xi = +1) = 0$$
$$\Xi(\xi = -1) = 0$$

Statement 15. *The remaining equation has the characteristic equation of* $m^2 - \lambda^2 = 0$. *By performing similar steps to those shown in statements 11 and 12 above, it may be shown that* $\Xi = Asinh(\lambda(\xi + 1))$ *and for a nontrivial solution:*

$$\lambda_m = \frac{(m+1)\pi\xi + 1}{2}, \; m = 0, 1, 2, 3... \tag{27}$$

$$\Xi_m(\xi) = sinh\left(\frac{(m+1)\pi(\xi + 1)}{2}\right), \; m = 0, 1, 2, 3... \tag{28}$$

Statement 16. *the complementary solution is then obtained when the partial second derivatives of* H_n *and* Ξ_m *are calculated and the solution is summed over* m *and* n. *This results in:*

$$v_c\left(\xi, \eta\right) = \Xi\left(\xi\right) H\left(\eta\right) = \frac{\partial^2 v_c}{\partial \xi^2} + \frac{\partial^2 v_c}{\partial \eta^2} =$$
$$-\left(\left(\frac{(m+1)\pi(1+1)}{2}\right)^2 \left(\frac{(n+1)\pi(1+r)}{2r}\right)^2\right) \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} sinh\left(\frac{(m+1)\pi(\xi+1)}{2}\right) sin\left(\frac{(n+1)\pi(\eta+r)}{2r}\right)$$

1.4. **Part e: Show that as the aspect ratio r grows large, teh velocity distribution approaches that of the particular soluation v_p.**

Statement 17. *Because the aspect ration* r *is in the denominator of* v_c, *as* $r \to \infty$, $v_c \to 0$. *This results in:*

$$r \to \infty : v = v_p + v_c = v_p + 0 = v_p \tag{29}$$