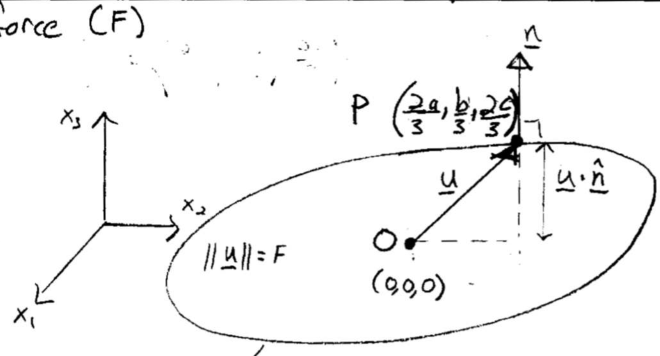
3) unable to verify Stoke's theorem

1) Find the component of the force (F) normal to ϕ at point P:



$$\underline{n}^P = \phi^P \overleftarrow{\nabla} = \text{Vector} \perp \text{ to } \phi \text{ at } P$$

$$= \phi^P_{,x_i} \, \underset{\sim}{e}_i$$

$$= \left\langle \frac{\partial \phi^P}{\partial x_1}, \frac{\partial \phi^P}{\partial x_2}, \frac{\partial \phi^P}{\partial x_3} \right\rangle$$

$$\phi \overleftarrow{\nabla} = \left\langle \frac{2x}{a^2}, \frac{2y}{b^2}, \frac{2z}{c^2} \right\rangle$$

$$\phi^P \overleftarrow{\nabla} = \left\langle \frac{2\left(\frac{2a}{3}\right)}{a^2}, \frac{2\left(\frac{b}{3}\right)}{b^2}, \frac{2\left(\frac{2c}{3}\right)}{c^2} \right\rangle$$

$$\phi^P \overleftarrow{\nabla} = \left\langle \frac{4}{3a}, \frac{2}{3b}, \frac{4}{3c} \right\rangle = \underline{n}^P$$

$$\|\underline{n}^P\| = \sqrt{\left(\frac{4}{3a}\right)^2 + \left(\frac{2}{3b}\right)^2 + \left(\frac{4}{3c}\right)^2}$$

$$\hat{\underline{n}}^P = \left\langle \frac{\underline{n}^P}{\|\underline{n}^P\|} \right\rangle$$

$$\phi = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0$$

$$= a^{-2}x^2 + b^{-2}y^2 + c^{-2}z^2 - 1$$

$$P = \left(\frac{2a}{3}, \frac{b}{3}, \frac{2c}{3} \right)$$

$$O = (0, 0, 0)$$

$$\underline{u} = \overrightarrow{OP}$$

$$= \left\langle \frac{2a}{3} - 0, \frac{b}{3} - 0, \frac{2c}{3} - 0 \right\rangle$$

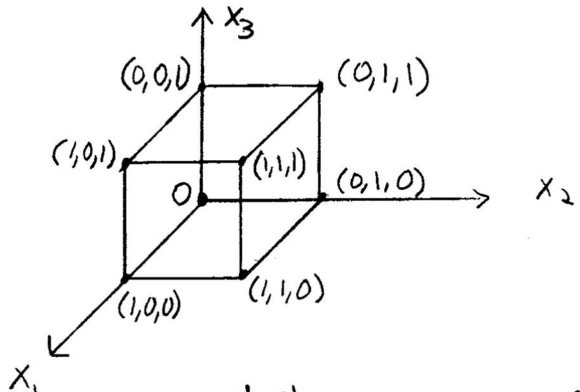
$$= \left\langle \frac{2a}{3}, \frac{b}{3}, \frac{2c}{3} \right\rangle$$

$$\|\underline{u}\| = F$$

- Component of force vector ($\underline{u}$) in the direction $\perp$ to the surface (ϕ) $= \underline{u} \cdot \hat{\underline{n}}^P = \left\langle \frac{2a}{3}, \frac{b}{3}, \frac{2c}{3} \right\rangle \cdot \left\{ \frac{\underline{n}^P}{\|\underline{n}^P\|} \right\}$

2 (i) $$\int_R \underline{r} \cdot \overleftarrow{\nabla}\, dQ = \int_{\partial R} \underline{r} \cdot \underline{n}\, dS$$

(ii)



Limits of Integration

$0 \le x_1 \le 1$
$0 \le x_2 \le 1$
$0 \le x_3 \le 1$

$x_1 = 0$; $\int_0^1 \int_0^1 1\, dx_2\, dx_3 = \int_0^1 x_2 \Big|_0^1 dx_3 = \int_0^1 (1-0)\, dx_3$
$= (1-0) \cdot x_3 \Big|_0^1 = (1-0)[1-0] = 1$

$x_1 = 1$; $\int_0^1 \int_0^1 dx_2\, dx_3 = \int_0^1 (1-0)\, dx_3 = (1-0)(1-0) = 1$

$x_2 = 0$; $\int_0^1 \int_0^1 dx_1\, dx_3 = \int_0^1 (1-0)\, dx_3 = (1-0)(1-0) = 1$

$x_2 = 1$; $\int_0^1 \int_0^1 dx_1\, dx_3 = \int_0^1 (1-0)\, dx_3 = (1-0)(1-0) = 1$

$x_3 = 0$; $\int_0^1 \int_0^1 dx_1\, dx_2 = \int_0^1 (1-0)\, dx_2 = (1-0)(1-0) = 1$

$x_3 = 1$; $\int_0^1 \int_0^1 dx_1\, dx_2 = \int_0^1 (1-0)\, dx_2 = (1-0)(1-0) = 1$

Sum of surface area $= 6$

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4 (i)^{a} Verify divergence theorem:

$$\int_S \underline{V} \cdot \underline{n}\, dA = \int_R \underline{V} \cdot \overleftarrow{\nabla}\, dV$$

$$\underline{V} = \langle A X_2,\ B x_2,\ C X_1 X_3 \rangle$$

$$\underline{V} \cdot \overleftarrow{\nabla} = \frac{\partial V_i}{\partial x_j} \underline{e}_j \cdot \underline{e}_i = V_{i,j}\, \delta_{ji} = V_{i,i}$$

$$= \langle 0,\ B,\ C X_1 \rangle$$

$$\int_R \underline{V} \cdot \overleftarrow{\nabla} dV = \int_0^c \int_0^b \int_0^a (B + C X_1)\, dx_1\, dx_2\, dx_3 = \int_0^c\int_0^b \left[B x_1 \Big|_0^a + \tfrac{1}{2} C X_1^2 \Big|_0^a \right] dx_2\, dx_3$$

$$= \int_0^c \int_0^b \left(Ba + \tfrac{1}{2} C a^2\right) dx_2\, dx_3$$

$$= \int_0^c \left[B a x_2 \Big|_0^b + \tfrac{1}{2} C a^2 x_2 \Big|_0^b \right] dx_3$$

$$= \int_0^c \left(Bab + \tfrac{1}{2} C a^2 b\right) dx_3$$

$$= Bab\, x_3 \Big|_0^c + \tfrac{1}{2} C a^2 b \Big|_0^c$$

$$= \underline{Babc + \tfrac{1}{2} C a^2 bc} = \int_R \underline{V} \cdot \overleftarrow{\nabla} dV$$

$\downarrow$

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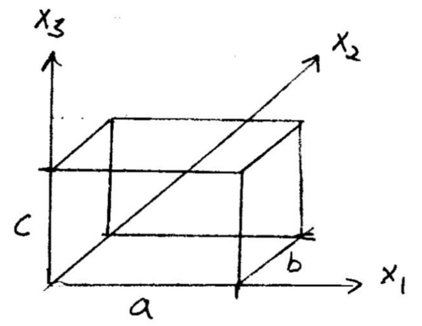
4. $\binom{b}{i}$



$\underline{V} = \langle Ax_2, Bx_2, Cx_1x_3 \rangle$

→ 6 sides of hexahedron

Calc. $\int_S \underline{v} \cdot \underline{n} \, dA$

$x_3 = 0$; $\quad \hat{\underline{n}} = \langle 0, 0, -1 \rangle$; $\underline{V} \cdot \underline{n} = -Cx_1x_3 = 0$

$$\int_S \int 0 \, dx_1 dx_2 = 0$$

↑ integral over the surface

$x_3 = c$, $\quad \hat{\underline{n}} = \langle 0, 0, 1 \rangle$; $\underline{V} \cdot \underline{n} = Cx_1x_3 = Ccx_1$

$$\int_S \int Ccx_1 \, dx_1 dx_2 = \int_0^b \int_0^a Ccx_1 \, dx_1 \, dx_2$$

$$= \int_0^b \left[Cc \tfrac{1}{2} x_1^2 \Big|_0^a \right] dx_2$$

$$= \int_0^b \tfrac{1}{2} Cc a^2 \, dx_2 = \tfrac{1}{2} Cc a^2 x_2 \Big|_0^b$$

$$= \tfrac{1}{2} Cc a^2 b$$

$x_2 = 0$, $\quad \hat{\underline{n}} = \langle 0, -1, 0 \rangle$; $\underline{V} \cdot \underline{n} = -Bx_2 = 0$

$$\int_S \int 0 \, dx_1 \, dx_3 = 0$$

↓

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4) (c)

$X_2 = b$; $\hat{\underline{n}} = \langle 0, 1, 0 \rangle$; $\underline{V} \cdot \underline{n} = B X_2 = Bb$

$$\iint_S Bb \, dx_1 \, dx_3 = \int_0^c \int_0^a Bb \, dx_1 \, dx_3$$

$$= \int_0^c \left[Bb\, x_1 \Big|_0^a \right] dx_3$$

$$= \int_0^c Bba \, dx_3 = Bba X_3 \Big|_0^c$$

$$= Bbac$$

$X_1 = 0$; $\hat{\underline{n}} = \langle -1, 0, 0 \rangle$; $\underline{V} \cdot \underline{n} = -A X_2$

$$\int_0^c \int_0^b -A X_2 \, dx_2 \, dx_3 = \int_0^c \left[-\frac{1}{2} A X_2^2 \Big|_0^b \right] dx_3$$

$$= \int_0^c -\frac{1}{2} A b^2 \, dx_3 = -\frac{1}{2} A b^2 X_3 \Big|_0^c$$

$$= -\frac{1}{2} A b^2 c$$

$X_1 = a$; $\hat{\underline{n}} = \langle 1, 0, 0 \rangle$; $\underline{V} \cdot \underline{n} = A X_2$

$$\int_0^c \int_0^b A x_2 \, dx_2 \, dx_3 = \int_0^c \left[\frac{1}{2} A x_2^2 \Big|_0^b \right] dx_3$$

$$= \int_0^c \frac{1}{2} A b^2 \, dx_3 = \frac{1}{2} A b^2 x_3 \Big|_0^c$$

$$= \frac{1}{2} A b^2 c$$

$$\therefore \int_S \underline{v} \cdot \underline{n} \, dA = 0 + \frac{1}{2} C a^2 bc + 0 + Babc - \frac{1}{2} Ab^2 c + \frac{1}{2} Ab^2 c$$

$$= \underline{Babc + \frac{1}{2} C a^2 bc}$$; div theorem verified

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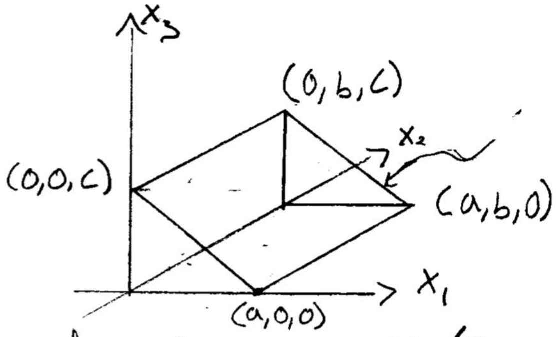
4 (ii)^a



- verify divergence theorem.

$$\int_S \underline{v} \cdot \underline{n}\, dA = \int_R \underline{v} \cdot \overleftarrow{\nabla}\, dV$$

from Previous: $\underline{v} \cdot \overleftarrow{\nabla} = \langle 0, B, Cx_1 \rangle$

$$\int_R \underline{v} \cdot \overleftarrow{\nabla}\, dV = \int \int \int (B + Cx_1)\, dx_1\, dx_2\, dx_3$$

Calc. limits of integration:

$\underline{P} = \langle 0, b, 0 \rangle \qquad \underline{Q} = \langle -a, 0, c \rangle$

$$\underline{P} \times \underline{Q} = \varepsilon_{ijk} P_i Q_j \underset{\sim}{e}_k = \begin{Bmatrix} P_2 Q_3 - P_3 Q_2 \\ P_3 Q_1 - P_1 Q_3 \\ P_1 Q_2 - P_2 Q_1 \end{Bmatrix}$$

normal to sloping plane $= \langle bc, 0, -ab \rangle$

Eqn. of sloping plane: $bc(x_1 - 0) + 0(x_2 - b) - ab(x_3 - 0) = 0$

$$bc\, x_1 - ab\, x_3 = 0$$

upper limit x_3: $\dfrac{\cancel{b} c x_1}{a \cancel{b}} = x_3 \rightarrow x_3 = \frac{c}{a} x_1$

$$\int_0^a \int_0^b \int_0^{\frac{c}{a} x_1} (B + C x_1)\, dx_3\, dx_2\, dx_1$$

$$= \int_0^a \int_0^b \left[B x_3 \Big|_0^{c/a\, x_1} + C x_1 x_3 \Big|_0^{c/a\, x_1} \right] dx_2\, dx_1$$

$$= \int_0^a \int_0^b \left(B \frac{c}{a} x_1 + C \frac{c}{a} x_1^2 \right) dx_2\, dx_1$$

$$= \int_0^a \left[B \frac{c}{a} x_1 x_2 \Big|_0^b + C \frac{c}{a} x_1^2 x_2 \Big|_0^b \right] dx_1$$

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4 (ii) b

$$= \int_0^a \left(B \tfrac{c}{a} b x_1 + C \tfrac{c}{a} b x_1^2 \right) dx_1$$

$$= B \tfrac{c}{a} b \tfrac{1}{2} x_1^2 \Big|_0^a + C \tfrac{c}{a} b \tfrac{1}{3} x_1^3 \Big|_0^a$$

$$= B \tfrac{c}{a} b \tfrac{1}{2} a^2 + C \tfrac{c}{a} b \tfrac{1}{3} a^3$$

$$= \tfrac{1}{2} B b a^2 \tfrac{c}{a} + \tfrac{1}{3} C b a^3 \tfrac{c}{a} = \int_R \underline{v} \cdot \nabla \, dV$$

Calc. $\int_S \underline{v} \cdot \underline{\hat{n}} \, dS$; $\underline{V} = \langle A x_2, B x_2, C x_1 x_3 \rangle$

$x_3 = 0$; $\underline{\hat{n}} = \langle 0, 0, -1 \rangle$; $\underline{v} \cdot \underline{\hat{n}} = C x_1 x_3 = 0$

$$\int_S \int 0 \, dA = 0$$

$x_2 = 0$, $\underline{\hat{n}} = \langle 0, -1, 0 \rangle$; $\underline{v} \cdot \underline{\hat{n}} = -B x_2 = 0$

$x_1 = 0$, $\underline{\hat{n}} = \langle -1, 0, 0 \rangle$, $\underline{v} \cdot \underline{\hat{n}} = -A x_2$

$$\int_0^c \int_0^b -A x_2 \, dx_2 \, dx_3 = \int_0^c \left[-A \tfrac{1}{2} x^2 \Big|_0^b \right] dx_3$$

$$= \int_0^c -A \tfrac{1}{2} b^2 \, dx_3 = -\tfrac{1}{2} A b^2 x_3 \Big|_0^c$$

$$= -\tfrac{1}{2} A b^2 c$$

$x_2 = b$, $\hat{n} = \langle 0, 1, 0 \rangle$, $\underline{v} \cdot \hat{n} = B x_2 = B b$

$$\int_0^b \int_0^{\frac{c}{a} x_1} B b \, dx_3 \, dx_1 = \int_0^b \left[B b x_3 \Big|_0^{\frac{c}{a} x_1} \right] dx_1$$

$$= \int_0^b B b \tfrac{c}{a} x_1 \, dx_1$$

4 (ii) c

$$= \frac{1}{2} B b \frac{c}{a} x_1^2 \Big|_0^b = \frac{1}{2} B b^3 \frac{c}{a}$$

$$X_3 = \frac{c}{a} X_1 \quad \hat{n} = \frac{\langle bc, 0, -ab \rangle}{\sqrt{(bc)^2 + (ab)^2}},$$

didn't use

$$\underline{V} \cdot \hat{n} = \frac{A X_2\, bc}{\sqrt{(bc)^2 + (ab)^2}} - \frac{C x_1 x_3\, ab}{\sqrt{(bc)^2 + (ab)^2}}$$

$$= \left((bc)^2 + (ab)^2\right)^{-1/2} \left[A\, bc\, X_2 - C ab\, X_1 \frac{c}{a} X_1\right]$$

- because this plane is not aligned w/ coord. axes!

$$dS \neq dA \rightarrow G(x_1, x_2, x_3) = z - g(X_1, X_2)$$

then! $\hat{n}\, dS = \nabla G\, dA$

$$x_3 = g(x_1, x_2) = \frac{c}{a} X_1$$

$$\int_S \int \underline{V} \cdot n\, dS = \int_R \int \underline{V} \cdot \left\langle -\frac{\partial g}{\partial x_1}, -\frac{\partial g}{\partial x_2}, +1 \right\rangle dA$$

$$= \int_R \int \langle A x_2, B x_2, C x_1 x_2 \rangle \cdot \left\langle -\frac{c}{a}, 0, 1 \right\rangle dA$$

$$= \int_R \int \left(-A \frac{c}{a} X_2 + C X_1 X_2\right) dA$$

$$= \int_0^b \int_0^a \left(-A \frac{c}{a} X_2 + C X_1 X_2\right) dX_1\, dX_2$$

$$= \int_0^b \left[-A \frac{c}{a} X_2 X_1 \Big|_0^a + C \frac{1}{2} X_1^2 X_2 \Big|_0^a\right] dx_2$$

$$= \int_0^b \left(-A \frac{c}{a} a X_2 + \frac{1}{2} C a^2 X_2\right) dx_2$$

4 (ii) d

$$= -\frac{1}{2} A c x_2^2 \Big|_0^b + \frac{1}{4} C a^2 x_2^2 \Big|_0^b$$

$$= -\tfrac{1}{2} A c b^2 + \frac{1}{4} C a^2 b^2$$

$$\int_S \underline{v} \cdot \underline{\hat{n}} \, dS = -\frac{1}{2} A b^2 c - \frac{1}{2} A b^2 c + \frac{1}{4} C a^2 b^2$$

... which does not equal $\int_R \underline{v} \cdot \vec{\nabla} \, dV$