Chapter 2

The Euler-Lagrange equation

In this chapter, we will give necessary conditions for an extremum of a function of the type

$$I(x)=\int_a^b F\left(x(t),x'(t),t\right)dt,$$

with various types of boundary conditions. The necessary condition is in the form of a differential equation that the extremal curve should satisfy, and this differential equation is called the *Euler-Lagrange* equation.

We begin with the simplest type of boundary conditions, where the curves are allowed to vary between two fixed points.

2.1 The simplest optimisation problem

The *simplest optimisation problem* can be formulated as follows:

Let $F(\alpha,\beta,\gamma)$ be a function with continuous first and second partial derivatives with respect to (α,β,γ). Then find $x\in C^1[a,b]$ such that $x(a)=y_a$ and $x(b)=y_b$, and which is an extremum for the function

$$I(x)=\int_a^b F\left(x(t),x'(t),t\right)dt. \tag{2.1}$$

In other words, the simplest optimisation problem consists of finding an extremum of a function of the form (2.5), where the class of admissible curves comprises all smooth curves joining two *fixed* points; see Figure 2.1. We will apply the necessary condition for an extremum (established in Theorem 1.4.2) to the solve the simplest optimisation problem described above.

Theorem 2.1.1 *Let $S=\{x\in C^1[a,b]\mid x(a)=y_a \text{ and } x(b)=y_b\}$, and let $I:S\to\mathbb{R}$ be a function of the form*

$$I(x)=\int_a^b F\left(x(t),x'(t),t\right)dt.$$

If I has an extremum at $x_0\in S$, then x_0 satisfies the Euler-Lagrange equation*:*

$$\frac{\partial F}{\partial\alpha}\left(x_0(t),x_0'(t),t\right)-\frac{d}{dt}\left(\frac{\partial F}{\partial\beta}\left(x_0(t),x_0'(t),t\right)\right)=0,\quad t\in[a,b]. \tag{2.2}$$

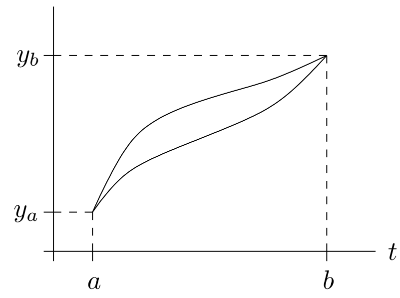

Figure 2.1: Possible paths joining the two fixed points (a, y_a) and (b, y_b).

Proof The proof is long and so we divide it into several steps.

STEP 1. First of all we note that the set S is not a vector space (unless $y_a = 0 = y_b$)! So Theorem 1.4.2 is not applicable directly. Hence we introduce a new linear space X, and consider a new function $\tilde{I} : X \to \mathbb{R}$ which is defined in terms of the old function I.

Introduce the linear space

$$X = \{x \in C^1[a, b] \mid x(a) = x(b) = 0\},$$

with the induced norm from $C^1[a, b]$. Then for all $h \in X$, $x_0 + h$ satisfies $(x_0 + h)(a) = y_a$ and $(x_0 + h)(b) = y_b$. Defining $\tilde{I}(h) = I(x_0 + h)$, for $h \in X$, we note that $\tilde{I} : X \to \mathbb{R}$ has a local extremum at 0. It follows from Theorem 1.4.2 that[1] $D\tilde{I}(0) = 0$.

STEP 2. We now calculate $D\tilde{I}(0)$. We have

$$\begin{aligned}\tilde{I}(h) - \tilde{I}(0) &= \int_a^b F\left((x_0 + h)(t), (x_0 + h)'(t), t\right) dt - \int_a^b F\left(x_0(t), x_0'(t), t\right) dt \\ &= \int_a^b \left[F\left(x_0(t) + h(t), x_0'(t) + h'(t), t\right) dt - F\left(x_0(t), x_0'(t), t\right)\right] dt.\end{aligned}$$

Recall that from Taylor's theorem, if F possesses partial derivatives of order 2 in a ball B of radius r around the point $(\alpha_0, \beta_0, \gamma_0)$ in $\mathbb{R}^3$, then for all $(\alpha, \beta, \gamma) \in B$, there exists a $\Theta \in [0, 1]$ such that

$$\begin{aligned}F(\alpha, \beta, \gamma) &= F(\alpha_0, \beta_0, \gamma_0) + \left((\alpha - \alpha_0)\frac{\partial}{\partial\alpha} + (\beta - \beta_0)\frac{\partial}{\partial\beta} + (\gamma - \gamma_0)\frac{\partial}{\partial\gamma}\right) F\bigg|_{(\alpha_0,\beta_0,\gamma_0)} + \\ &\frac{1}{2!}\left((\alpha - \alpha_0)\frac{\partial}{\partial\alpha} + (\beta - \beta_0)\frac{\partial}{\partial\beta} + (\gamma - \gamma_0)\frac{\partial}{\partial\gamma}\right)^2 F\bigg|_{(\alpha_0,\beta_0,\gamma_0)+\Theta((\alpha,\beta,\gamma)-(\alpha_0,\beta_0,\gamma_0))}.\end{aligned}$$

Hence for $h \in X$ such that $\|h\|$ is small enough,

$$\begin{aligned}\tilde{I}(h) - \tilde{I}(0) &= \int_a^b \left[\frac{\partial F}{\partial\alpha}\left(x_0(t), x_0'(t), t\right) h(t) + \frac{\partial F}{\partial\beta}\left(x_0(t), x_0'(t), t\right) h'(t)\right] dt + \\ &\frac{1}{2!}\int_a^b \left(h(t)\frac{\partial}{\partial\alpha} + h'(t)\frac{\partial}{\partial\beta}\right)^2 F\bigg|_{(x_0(t)+\Theta(t)h(t), x_0'(t)+\Theta(t)h'(t), t)} dt.\end{aligned}$$

It can be checked that there exists a $M > 0$ such that

$$\left|\frac{1}{2!}\int_a^b \left(h(t)\frac{\partial}{\partial\alpha} + h'(t)\frac{\partial}{\partial\beta}\right)^2 F\bigg|_{(x_0(t)+\Theta(t)h(t), x_0'(t)+\Theta(t)h'(t), t)} dt\right| \le M\|h\|^2,$$

[1] Note that by the 0 in the right hand side of the equality, we mean the zero map, namely the continuous linear map from X to $\mathbb{R}$, which is defined by $h \mapsto 0$ for all $h \in X$.

and so $D\tilde{I}(0)$ is the map

$$h \mapsto \int_a^b \left[\frac{\partial F}{\partial \alpha}(x_0(t), x_0'(t), t)\, h(t) + \frac{\partial F}{\partial \beta}(x_0(t), x_0'(t), t)\, h'(t) \right] dt. \tag{2.3}$$

STEP 3. Next we show that if the map in (2.3) is the zero map, then this implies that (2.2) holds. Define

$$A(t) = \int_a^t \frac{\partial F}{\partial \alpha}(x_0(\tau), x_0'(\tau), \tau)\, d\tau.$$

Integrating by parts, we find that

$$\int_a^b \frac{\partial F}{\partial \alpha}(x_0(t), x_0'(t), t)\, h(t)dt = -\int_a^b A(t)h'(t)dt,$$

and so from (2.3), it follows that $D\tilde{I}(0) = 0$ implies that

$$\int_a^b \left[-A(t) + \frac{\partial F}{\partial \beta}(x_0(t), x_0'(t), t) \right] h'(t)dt = 0 \text{ for all } h \in X.$$

STEP 4. Finally, using Lemma 1.4.5, we obtain

$$-A(t) + \frac{\partial F}{\partial \beta}(x_0(t), x_0'(t), t) = k \text{ for all } t \in [a, b].$$

Differentiating with respect to t, we obtain (2.3). This completes the proof of Theorem 2.1.1. ■

Note that the Euler-Lagrange equation is only a *necessary* condition for the existence of an extremum (see the remark following Theorem 1.4.2). However, in many cases, the Euler-Lagrange equation by itself is enough to give a complete solution of the problem. In fact, the existence of an extremum is sometimes clear from the context of the problem. If in such scenarios, there exists only one solution to the Euler-Lagrange equation, then this solution must a fortiori be the point for which the extremum is achieved.

Example. Let $S = \{x \in C^1[0,1] \mid x(0) = 0 \text{ and } x(1) = 1\}$. Consider the function $I : S \to \mathbb{R}$ given by

$$I(x) = \int_0^1 \left(\frac{d}{dt}x(t) - 1 \right)^2 dt.$$

We wish to find $x_0 \in S$ that minimizes I. We proceed as follows:

STEP 1. We have $F(\alpha, \beta, \gamma) = (\beta - 1)^2$, and so $\dfrac{\partial F}{\partial \alpha} = 0$ and $\dfrac{\partial F}{\partial \beta} = 2(\beta - 1)$.

STEP 2. The Euler-Lagrange equation (2.2) is now given by

$$0 - \frac{d}{dt}(2(x_0'(t) - 1)) = 0 \quad \text{for all } t \in [0, 1].$$

STEP 3. Integrating , we obtain $2(x_0'(t) - 1) = C$, for some constant C, and so $x_0' = \frac{C}{2} + 1 =: A$. Integrating again, we have $x_0(t) = At + B$, where A and B are suitable constants.

STEP 4. The constants A and B can be determined by using that fact that $x_0 \in S$, and so $x_0(0) = 0$ and $x_0(a) = 1$. Thus we have

$$\begin{aligned} A0 + B &= 0, \\ A1 + B &= 1, \end{aligned}$$

which yield $A = 1$ and $B = 0$.

So the unique solution x_0 of the Euler-Lagrange equation in S is $x_0(t) = t$, $t \in [0, 1]$; see Figure 2.2.

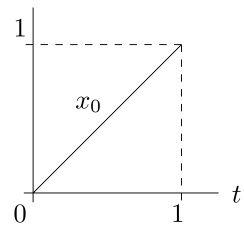

Figure 2.2: Minimizer for I.

Now we argue that the solution x_0 indeed minimizes I. Since $(x'(t) - 1)^2 \geq 0$ for all $t \in [0, 1]$, it follows that $I(x) \geq 0$ for all $x \in C^1[0, 1]$. But

$$I(x_0) = \int_0^1 (x_0'(t) - 1)^2 dt = \int_0^1 (1-1)^2 dt = \int_0^1 0 dt = 0.$$

As $I(x) \geq 0 = I(x_0)$ for all $x \in S$, it follows that x_0 minimizes I. $\diamond$

Definition. The solutions of the Euler-Lagrange equation (2.3) are called *critical curves*.

The Euler-Lagrange equation is in general a second order differential equation, but in some special cases, it can be reduced to a first order differential equation or where its solution can be obtained entirely by evaluating integrals. We indicate some special cases in Exercise 3 on page 31, where in each instance, F is independent of one of its arguments.

Exercises.

1. Let $S = \{x \in C^1[0, 1] \mid x(0) = 0 = x(1)\}$. Consider the map $I : S \to \mathbb{R}$ given by

$$I(x) = \int_0^1 (x(t))^3 dt, \quad x \in S.$$

Using Theorem 2.1.1, find the critical curve $x_0 \in S$ for I. Does I have a local extremum at x_0?

2. Write the Euler-Lagrange equation when F is given by

   (a) $F(\alpha, \beta, \gamma) = \sin \beta$,
   (b) $F(\alpha, \beta, \gamma) = \alpha^3 \beta^3$,
   (c) $F(\alpha, \beta, \gamma) = \alpha^2 - \beta^2$,
   (d) $F(\alpha, \beta, \gamma) = 2\gamma\beta - \beta^2 + 3\beta\alpha^2$.

3. Prove that:

   (a) If $F(\alpha, \beta, \gamma)$ does not depend on α, then the Euler-Lagrange equation becomes

   $$\frac{\partial F}{\partial \beta}(x(t), x'(t), t) = c,$$

   where C is a constant.

   (b) If F does not depend on β, then the Euler-Lagrange equation becomes

   $$\frac{\partial F}{\partial \alpha}(x(t), x'(t), t) = 0.$$

   (c) If F does not depend on γ and if x_0 is twice-differentiable in $[a, b]$, then the Euler-Lagrange equation becomes

   $$F(x(t), x'(t), t) - x'(t)\frac{\partial F}{\partial \beta}(x(t), x'(t), t) = C,$$

   where C is a constant.
   Hint: What is $\frac{d}{dt}\left(F(x(t), x'(t), t) - x'(t)\frac{\partial F}{\partial \beta}(x(t), x'(t), t)\right)$?

4. Find the curve which has minimum length between $(0, 0)$ and $(1, 1)$.

5. Let $S = \{x \in C^1[0, 1] \mid x(0) = 0 \text{ and } x(1) = 1\}$. Find critical curves in S for the functions $I : S \to \mathbb{R}$, where I is given by:

   (a) $I(x) = \displaystyle\int_0^1 x'(t)dt$

   (b) $I(x) = \displaystyle\int_0^1 x(t)x'(t)dt$

   (c) $I(x) = \displaystyle\int_0^1 (x(t) + tx'(t))dt$

   for $x \in S$.

6. Find critical curves for the function

   $$I(x) = \int_1^2 t^3(x'(t))^2 dt$$

   where $x \in C^1[1, 2]$ with $x(1) = 5$ and $x(2) = 2$.

7. Find critical curves for the function

   $$I(x) = \int_1^2 \frac{(x'(t))^3}{t^2} dt$$

   where $x \in C^1[1, 2]$ with $x(1) = 1$ and $x(2) = 7$.

8. Find critical curves for the function

   $$I(x) = \int_0^1 \left[2tx(t) - (x'(t))^2 + 3x'(t)(x(t))^2\right] dt$$

   where $x \in C^1[0, 1]$ with $x(0) = 0$ and $x(1) = -1$.

9. Find critical curves for the function

$$I(x) = \int_0^1 \left[2(x(t))^3 + 3t^2 x'(t)\right] dt$$

where $x \in C^1[0,1]$ with $x(0) = 0$ and $x(1) = 1$. What if $x(0) = 0$ and $x(1) = 2$?

10. Consider the copper mining company mentioned in the introduction. If future money is discounted continuously at a constant rate r, then we can assess the present value of profits from this mining operation by introducing a factor of e^{-rt} in the integrand of (1.37). Suppose that $\alpha = 4$, $\beta = 1$, $r = 1$ and $P = 2$. Find a critical mining operation x_0 such that $x_0(0) = 0$ and $x_0(T) = Q$.

2.2 Calculus of variations: some classical problems

Optimisation problems of the type considered in the previous section were studied in various special cases by many leading mathematicians in the past. These were often solved by various techniques, and these gave rise to the branch of mathematics known as the 'calculus of variations'. The name comes from the fact that often the procedure involved the calculation of the 'variation' in the function I when its argument (which was typically a curve) was changed, and then passing limits. In this section, we mention two classical problems, and indicate how these can be solved using the Euler-Lagrange equation.

2.2.1 The brachistochrone problem

The calculus of variations originated from a problem posed by the Swiss mathematician Johann Bernoulli (1667-1748). He required the form of the curve joining two fixed points A and B in a vertical plane such that a body sliding down the curve (under gravity and no friction) travels from A to B in minimum time. This problem does not have a trivial solution; the straight line from A to B is not the solution (this is also intuitively clear, since if the slope is high at the beginning, the body picks up a high velocity and so its plausible that the travel time could be reduced) and it can be verified experimentally by sliding beads down wires in various shapes.

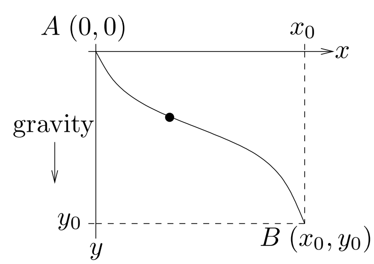

Figure 2.3: The brachistochrone problem.

To pose the problem in mathematical terms, we introduce coordinates as shown in Figure 2.3, so that A is the point $(0,0)$, and B corresponds to (x_0, y_0). Assuming that the particle is released from rest at A, conservation of energy gives $\frac{1}{2}mv^2 - mgy = 0$, where we have taken the zero potential energy level at $y = 0$, and where v denotes the speed of the particle. Thus the speed is given by $v = \frac{ds}{dt} = \sqrt{2gy}$, where s denotes arc length along the curve. From Figure 2.4, we see that an element of arc length, δs is given approximately by $((\delta x)^2 + (\delta y)^2)^{\frac{1}{2}}$.

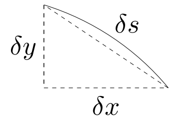

Figure 2.4: Element of arc length.

Hence the time of descent is given by

$$T = \int_{\text{curve}} \frac{ds}{\sqrt{2gy}} = \frac{1}{\sqrt{2g}} \int_0^{y_0} \sqrt{\frac{1+\left(\frac{dx}{dy}\right)^2}{y}}\,dy.$$

Our problem is to find the path $\{x(y),\ y \in [0, y_0]\}$, satisfying $x(0) = 0$ and $x(y_0) = x_0$, which minimizes T, that is, to determine the minimizer for the function $I : S \to \mathbb{R}$, where

$$I(x) = \frac{1}{\sqrt{2g}} \int_0^{y_0} \left(\frac{1 + (x'(y))^2}{y}\right)^{\frac{1}{2}} dy, \quad x \in S,$$

and $S = \{x \in C^1[0, y_0] \mid x(0) = 0 \text{ and } x(y_0) = x_0\}$. Here[2] $F(\alpha, \beta, \gamma) = \sqrt{\frac{1+\beta^2}{\gamma}}$ is independent of α, and so the Euler-Lagrange equation becomes

$$\frac{d}{dy}\left(\frac{x'(y)}{\sqrt{1 + (x'(y))^2}} \frac{1}{\sqrt{y}}\right) = 0.$$

Integrating with respect to y, we obtain

$$\frac{x'(y)}{\sqrt{1 + (x'(y))^2}} \frac{1}{\sqrt{y}} = C,$$

where C is a constant. It can be shown that the general solution of this differential equation is given by

$$x(\Theta) = \frac{1}{2C^2}(\Theta - \sin\Theta) + \tilde{C}, \quad y(\Theta) = \frac{1}{2C^2}(1 - \cos\Theta),$$

where $\tilde{C}$ is another constant. The constants are chosen so that the curve passes through the points $(0, 0)$ and (x_0, y_0).

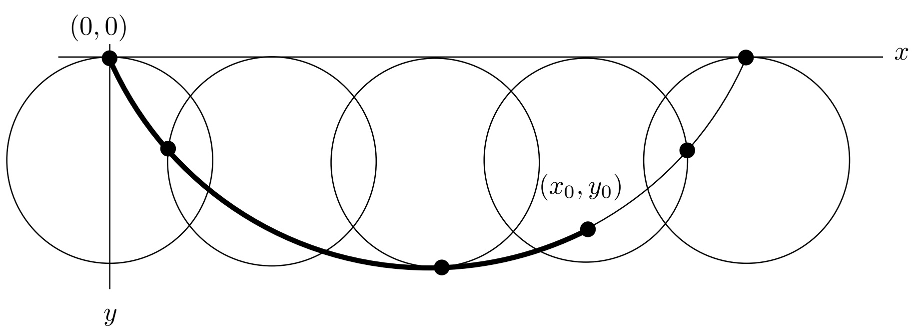

Figure 2.5: The cycloid through $(0, 0)$ and (x_0, y_0).

This curve is known as a *cycloid*, and in fact it is the curve described by a point P in a circle that rolls without slipping on the x axis, in such a way that P passes through (x_0, y_0); see Figure 2.5.

[2]Strictly speaking, the F here does *not* satisfy the demands made in Theorem 2.1.1. Notwithstanding this fact, with some additional argument, the solution given here can be fully justified.

2.2.2 Minimum surface area of revolution

The problem of minimum surface area of revolution is to find among all the curves joining two given points (x_0, y_0) and (x_1, y_1), the one which generates the surface of minimum area when rotated about the x axis.

The area of the surface of revolution generated by rotating the curve y about the x axis is

$$S(y) = 2\pi \int_{x_0}^{x_1} y(x)\sqrt{1+(y'(x))^2}dx.$$

Since the integrand does not depend explicitly on x, the Euler-Lagrange equation is

$$F(y(x), y'(x), x) - y'(x)\frac{\partial F}{\partial \beta}(y(x), y'(x), x) = C,$$

where C is a constant, that is,

$$y\sqrt{1+(y')^2} - y\frac{(y')^2}{\sqrt{1+(y')^2}} = C.$$

Thus $y = C\sqrt{1+(y')^2}$, and it can be shown that this differential equation has the general solution

$$y(x) = C\cosh\left(\frac{x+C_1}{C}\right). \tag{2.4}$$

This curve is called a *catenary*. The values of the arbitrary constants C and C_1 are determined by the conditions $y(x_0) = y_0$ and $y(x_1) = y_1$. It can be shown that the following three cases are possible, depending on the positions of the points (x_0, y_0) and (x_1, y_1):

1. If a single curve of the form (2.4) passes through the points (x_0, y_0) and (x_1, y_1), then this curve is the solution of the problem; see Figure 2.6.

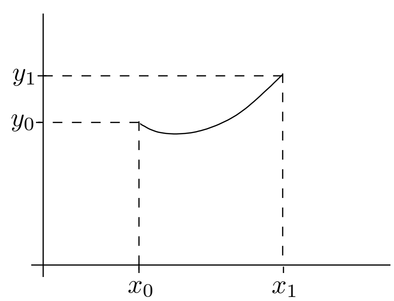

Figure 2.6: The catenary through (x_0, y_0) and (x_1, y_1).

2. If two critical curves can be drawn through the points (x_0, y_0) and (x_1, y_1), then one of the curves actually corresponds to the surface of revolution if minimum area, and the other does not.

3. If there does not exist a curve of the form (2.4) passing through the points (x_0, y_0) and (x_1, y_1), then there is no surface in the class of smooth surfaces of revolution which achieves the minimum area. In fact, if the location of the two points is such that the distance between them is sufficiently large compared to their distances from the x axis, then the area of the surface consisting of two circles of radius y_0 and y_1 will be less than the area of any surface of revolution generated by a smooth curve passing through the points; see Figure 2.7.

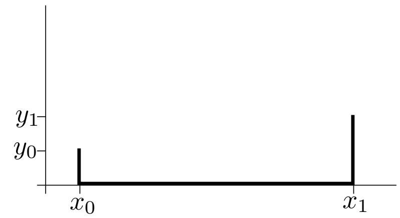

Figure 2.7: The polygonal curve $(x_0, y_0) - (x_0, 0) - (x_1, 0) - (x_1, y_1)$.

This is intuitively expected: imagine a soap bubble between concentric rings which are being pulled apart. Initially we get a soap bubble between these rings, but if the distance separating the rings becomes too large, then the soap bubble breaks, leaving soap films on each of the two rings. This example shows that a critical curve need not always exist in the class of curves under consideration.

2.3 Free boundary conditions

Besides the simplest optimisation problem considered in the previous section, we now consider the optimisation problem with *free boundary conditions* (see Figure 2.8).

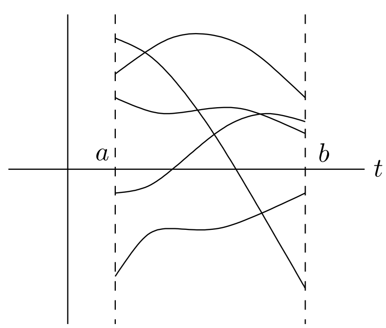

Figure 2.8: Free boundary conditions.

Let $F(\alpha, \beta, \gamma)$ be a function with continuous first and second partial derivatives with respect to (α, β, γ). Then find $x \in C^1[a, b]$ which is an extremum for the function

$$I(x) = \int_a^b F\left(x(t), \frac{dx}{dt}(t), t\right) dt. \tag{2.5}$$

Theorem 2.3.1 *Let $I : C^1[a, b] \to \mathbb{R}$ be a function of the form*

$$I(x) = \int_a^b F\left(x(t), \frac{dx}{dt}(t), t\right) dt, \quad x \in C^1[a, b],$$

where $F(\alpha, \beta, \gamma)$ is a function with continuous first and second partial derivatives with respect to (α, β, γ). If I has a local extremum at x_0, then x_0 satisfies the Euler-Lagrange equation:

$$\frac{\partial F}{\partial \alpha}\left(x_0(t), \frac{dx_0}{dt}(t), t\right) - \frac{d}{dt}\left(\frac{\partial F}{\partial \beta}\left(x_0(t), \frac{dx_0}{dt}(t), t\right)\right) = 0, \quad t \in [a, b], \tag{2.6}$$

together with the transversality conditions

$$\frac{\partial F}{\partial \beta}\left(x_0(t), \frac{dx_0}{dt}(t), t\right)\bigg|_{t=a} = 0 \quad \textit{and} \quad \frac{\partial F}{\partial \beta}\left(x_0(t), \frac{dx_0}{dt}(t), t\right)\bigg|_{t=b} = 0. \tag{2.7}$$

Proof

STEP 1. We take $X = C^1[a,b]$ and compute $DI(x_0)$. Proceeding as in the proof of Theorem 2.1.1, it is easy to see that

$$DI(x_0)(h) = \int_a^b \left[\frac{\partial F}{\partial \alpha}(x_0(t), x_0'(t), t)\, h(t) + \frac{\partial F}{\partial \beta}(x_0(t), x_0'(t), t)\, h'(t)\right] dt,$$

$h \in C^1[a,b]$. Theorem 1.4.2 implies that this linear functional must be the zero map, that is, $(DI(x_0))(h) = 0$ for all $h \in C^1[a,b]$. In particular, it is also zero for all h in $C^1[a,b]$ such that $h(a) = h(b) = 0$. But recall that in STEP 3 and STEP 4 of the proof of Theorem 2.1.1, we proved that if

$$\int_a^b \left[\frac{\partial F}{\partial \alpha}(x_0(t), x_0'(t), t)\, h(t) + \frac{\partial F}{\partial \beta}(x_0(t), x_0'(t), t)\, h'(t)\right] dt = 0 \tag{2.8}$$

for all h in $C^1[a,b]$ such that $h(a) = h(b) = 0$, then this implies that he Euler-Lagrange equation (2.6) holds.

STEP 2. Integration by parts in (2.8) now gives

$$\begin{aligned} DI(x_0)(h) &= \int_a^b \left[\frac{\partial F}{\partial \alpha}(x_0(t), x_0'(t), t) - \frac{d}{dt}\left(\frac{\partial F}{\partial \beta}(x_0(t), x_0'(t), t)\right)\right] h(t)dt + && (2.9)\\ &\quad \frac{\partial F}{\partial \beta}(x_0(t), x_0'(t), t)\, h(t)\bigg|_{t=a}^{t=b} && (2.10)\\ &= 0 + \frac{\partial F}{\partial \beta}(x_0(t), x_0'(t), t)\bigg|_{t=b} h(b) - \frac{\partial F}{\partial \beta}(x_0(t), x_0'(t), t)\bigg|_{t=a} h(a). \end{aligned}$$

The integral in (2.9) vanishes since we have shown in STEP 1 above that (2.6) holds. Thus the condition $DI(x_0) = 0$ now takes the form

$$\frac{\partial F}{\partial \beta}(x_0(t), x_0'(t), t)\bigg|_{t=b} h(b) - \frac{\partial F}{\partial \beta}(x_0(t), x_0'(t), t)\bigg|_{t=a} h(a) = 0,$$

from which (2.7) follows, since h is arbitrary. This completes the proof. ∎

Exercises.

1. Find all curves $y = y(x)$ which have minimum length between the lines $x = 0$ and the line $x = 1$.

2. Find critical curves for the following function, when the values of x are free at the endpoints:

$$I(x) = \int_0^1 \frac{1}{2}\left[(x'(t))^2 + x(t)x'(t) + x'(t) + x(t)\right] dt.$$

Similarly, we can also consider the *mixed* case (see Figure 2.9), when one end of the curve is fixed, say $x(a) = y_a$, and the other end is free. Then it can be shown that the curve x satisfies the Euler-Lagrange equation, the transversality condition

$$\frac{\partial F}{\partial \beta}(x_0(t), x_0'(t), t)\bigg|_{t=a} h(a) = 0$$

at the free end point, and $x(a) = y_a$ serves as the other boundary condition.

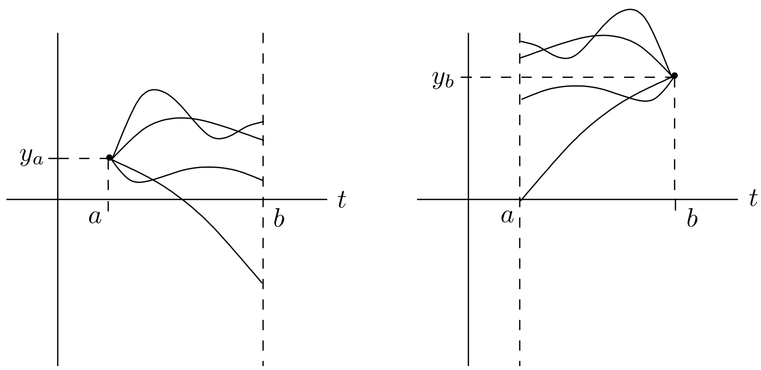

Figure 2.9: Mixed cases.

We can summarize the results by the following: critical curves for (2.5) satisfy the Euler-Lagrange equation (2.6) and moreover there holds

$$\frac{\partial F}{\partial \beta}(x_0(t), x_0'(t), t) = 0 \text{ at the free end point.}$$

Exercises.

1. Find the curve $y = y(x)$ which has minimum length between $(0, 0)$ and the line $x = 1$.
2. The cost of a manufacturing process in an industry is described by the function I given by
$$I(x) = \int_0^1 \left[\frac{1}{2}(x'(t))^2 + x(t)\right] dt,$$
for $x \in C^1[0, 1]$ with $x(0) = 1$.
   (a) If also $x(1) = 0$, find a critical curve x_* for I, and find $I(x_*)$.
   (b) If $x(1)$ is not specified, find a critical curve x_{**} for I, and find $I(x_{**})$.
   (c) Which of the values $I(x_*)$ and $I(x_{**})$ found in parts above is larger? Explain why you would expect this, assuming that x_* and x_{**} in fact minimize I on the respective domains specified above.
3. Find critical curves for the following functions:
   (a) $I(x) = \displaystyle\int_0^{\frac{\pi}{2}} \left[(x(t))^2 - (x'(t))^2\right] dt$, $x(0) = 0$ and $x\left(\frac{\pi}{2}\right)$ is free.
   (b) $I(x) = \displaystyle\int_0^{\frac{\pi}{2}} \left[(x(t))^2 - (x'(t))^2\right] dt$, $x(0) = 1$ and $x\left(\frac{\pi}{2}\right)$ is free.
4. Determine the curves that maximize the function $I : S \to \mathbb{R}$, where $I(x) = \displaystyle\int_0^1 \cos(x'(t))dt$, $x \in S$ and $S = \{x \in C^1[0, 1] \mid x(0) = 0\}$. What are the curves that minimize I?

2.4 Generalization

The results in this chapter can be generalized to the case when the integrand F is a function of more than one independent variable: if we wish to find extremum values of the function

$$I(x_1, \dots, x_n) = \int_a^b F\left(x_1(t), \dots, x_n(t), \frac{dx_1}{dt}(t), \dots, \frac{dx_n}{dt}(t), t\right) dt,$$

where $F(\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n, \gamma)$ is a function with continuous partial derivatives of order ≤ 2, and $x_1, \dots, x_n$ are continuously differentiable functions of the variable t, then following a similar analysis as before, we obtain n Euler-Lagrange equations to be satisfied by the optimal curve, that is,

$$\frac{\partial F}{\partial \alpha_k}(x_{1*}(t), \dots, x_{n*}(t), x'_{1*}(t), \dots, x'_{n*}(t), t) - \frac{d}{dt}\left(\frac{\partial F}{\partial \beta_k}(x_{1*}(t), \dots, x_{n*}(t), x'_{1*}(t), \dots, x'_{n*}(t), t)\right) = 0,$$

for $t \in [a, b]$, $k \in \{1, \dots, n\}$. Also at any end point where x_k is free,

$$\frac{\partial F}{\partial \beta_k}\left(x_{1*}(t), \dots, x_{n*}(t), \frac{dx_{1*}}{dt}(t), \dots, \frac{dx_{n*}}{dt}(t), t\right) = 0.$$

Exercise. Find critical curves of the function

$$I(x_1, x_2) = \int_0^{\frac{\pi}{2}} \left[(x'_1(t))^2 + (x'_2(t))^2 + 2x_1(t)x_2(t)\right] dt$$

such that $x_1(0) = 0$, $x_1\left(\frac{\pi}{2}\right) = 1$, $x_2(0) = 0$, $x_2\left(\frac{\pi}{2}\right) = 1$.

Remark. Note that with the above result, we can also solve the problem of finding extremal curves for a function of the type

$$I(x) : \int_a^b F\left(x(t), \frac{dx}{dt}(t), \dots, \frac{d^n x}{dt^n}(t), t\right) dt,$$

for over all (sufficiently differentiable) curves x defined on an interval $[a, b]$, taking values in $\mathbb{R}$. Indeed, we may introduce the auxiliary functions

$$x_1(t) = x(t), \quad x_2(t) = \frac{dx}{dt}(t), \quad \dots, \quad x_n(t) = \frac{d^n x}{dt^n}(t), \quad t \in [a, b],$$

and consider the problem of finding extremal curves for the new function $\tilde{I}$ defined by

$$\tilde{I}(x_1, \dots, x_n) = \int_a^b F(x_1(t), x_2(t), \dots, x_n(t), t) dt.$$

Using the result mentioned in this section, we can then solve this problem. Note that we eliminated *high* order derivatives at the price of converting the *scalar* function into a *vector*-valued function. Since we can always do this, this is one of the reasons in fact for considering functions of the type (1.28) where no high order derivatives occur.

2.5 Optimisation in function spaces versus that in $\mathbb{R}^n$

To understand the basic 'infinite-dimensionality' in the problems of optimisation in function spaces, it is interesting to see how they are related to the problems of the study of functions of n real variables. Thus, consider a function of the form

$$I(x)=\int_a^b F\left(x(t),\frac{dx}{dt}(t),t\right)dt,\quad x(a)=y_a,\quad x(b)=y_b.$$

Here each curve x is assigned a certain number. To find a related function of the sort considered in classical analysis, we may proceed as follows. Using the points

$$a=t_0,t_1,\dots,t_n,t_{n+1}=b,$$

we divide the interval $[a,b]$ into $n+1$ equal parts. Then we replace the curve $\{x(t),\ t\in[a,b]\}$ by the polygonal line joining the points

$$(t_0,y_a),(t_1,x(t_1)),\dots,(t_n,x(t_n)),(t_{n+1},y_b),$$

and we approximate the function I at x by the sum

$$I_n(x_1,\dots,x_n)=\sum_{k=1}^{n}F\left(x_k,\frac{x_k-x_{k-1}}{h_k},t_k\right)h_k, \tag{2.11}$$

where $x_k=x(t_k)$ and $h_k=t_k-t_{k-1}$. Each polygonal line is uniquely determined by the ordinates $x_1,\dots,x_n$ of its vertices (recall that $x_0=y_a$ and $x_{n+1}=y_b$ are fixed), and the sum (2.11) is therefore a function of the n variables $x_1,\dots,x_n$. Thus as an approximation, we can regard the optimisation problem as the problem of finding the extrema of the function $I_n(x_1,\dots,x_n)$.

In solving optimisation problems in function spaces, Euler made extensive use of this 'method of finite differences'. By replacing smooth curves by polygonal lines, he reduced the problem of finding extrema of a function to the problem of finding extrema of a function of n variables, and then he obtained exact solutions by passing to the limit as $n\to\infty$. In this sense, functions can be regarded as 'functions of infinitely many variables' (that is, the infinitely many values of $x(t)$ at different points), and the calculus of variations can be regarded as the corresponding analog of differential calculus of functions of n real variables.