10. DEFORMATION

10.1 Initial Comments

Now we initiate the formal study of continuum mechanics, which we define to be the study of the equations that describe the deformation of a continuous body subjected to static or dynamic physical and thermal loads. By continuous we mean that no new free surfaces (cracks or voids) are created from the loading process. The implication is that gradients are assumed to exist and are unique.

This Chapter focuses on how the deformation of a body is described, on the existence of two position vectors and the corresponding two gradients and the definition of the fundamental tensor called the deformation gradient. This gradient is used to relate line elements, area elements and volume elements for two configurations. Finally important relationships involving the gradient operators and the deformation gradient are derived.

10.2 Deformation

We start by defining the domain of body in an undeformed state. The body is said to occupy a region, R_0, with boundary, ∂R_0, and the material points are defined by the position vector

$$\boldsymbol{R} = X_i \boldsymbol{E}_i \tag{10-1}$$

with limits on the components X_i that are consistent with the boundaries of the body. The basis, $\boldsymbol{E}_i$, and origin, O, are chosen for convenience for describing the body in the undeformed configuration.

In a deformed configuration the body occupies a region, R, with boundary, ∂R, and material points are defined by the position vector

$$\boldsymbol{r} = x_i \boldsymbol{e}_i \tag{10-2}$$

with limits on the components x_i that are consistent with the boundaries of the body. The basis, $\boldsymbol{e}_i$, and origin, o, is chosen for convenience for describing the body in the deformed configuration. The two bases may or may not coincide. Unless stated otherwise, both bases are assumed not to vary with position.

If we consider any material point, $\boldsymbol{R}$, the corresponding position of the same material point is given by the function, $\boldsymbol{r}(\boldsymbol{R})$, i.e., by a vector function of a vector. Material points in the undeformed configuration are a mapped to a new location in the deformed configuration. The function is said to be one-to-one and onto which means that one material point maps to a unique point in the deformed state. The implication is that the inverse mapping must exist, $\boldsymbol{R}(\boldsymbol{r})$. However, it must be understood that even if the mapping one way is expressed analytically, it may not be possible to find the inverse map. For most large-scale problems the mapping functions are determined numerically.

A sketch indicating the meaning of these variables is given in Fig. 10-1. Also shown are the differentials of the position vectors, $d\boldsymbol{R}$ and $d\boldsymbol{r}$.

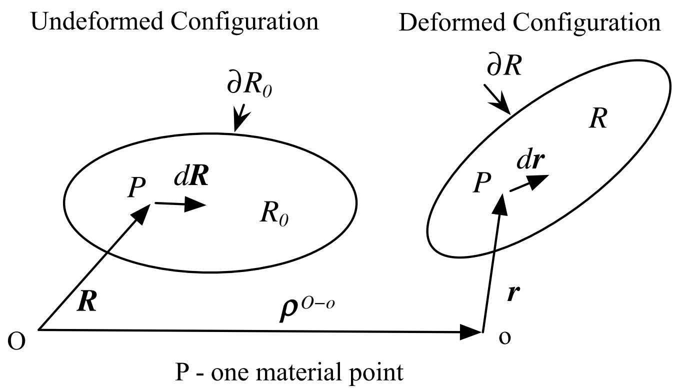

Fig. 10-3. Notation for deformation of a material body.

10.3 Deformation Gradient

The deformation and its inverse are defined as vector functions of vectors defined as follows:

$$\boldsymbol{r} = \boldsymbol{r}(\boldsymbol{R}) \qquad \boldsymbol{R} = \boldsymbol{R}(r) \tag{10-3}$$

Using the definition of a gradient it follows that

$$d\boldsymbol{r} = (\boldsymbol{r}\bar{\bar{\nabla}}_0)\cdot d\boldsymbol{R} \qquad d\boldsymbol{R} = (\boldsymbol{R}\bar{\bar{\nabla}})\cdot d\boldsymbol{r} \tag{10-4}$$

in which

$$()\bar{\bar{\nabla}}_0 \equiv ()\bar{\bar{\nabla}}_{\boldsymbol{R}} \qquad ()\bar{\bar{\nabla}} \equiv ()\bar{\bar{\nabla}}_{\boldsymbol{r}} \tag{10-5}$$

A simpler notation is to define the deformation gradients, $\boldsymbol{F}$ and $\boldsymbol{G}$, to be

$$\boldsymbol{F} = \boldsymbol{r}\bar{\bar{\nabla}}_0 \qquad \boldsymbol{G} = \boldsymbol{R}\bar{\bar{\nabla}} \tag{10-6}$$

and then

$$d\boldsymbol{r} = \boldsymbol{F}\cdot d\boldsymbol{R} \qquad d\boldsymbol{R} = \boldsymbol{G}\cdot d\boldsymbol{r} \tag{10-7}$$

It follows from (10-7) that

$$\boldsymbol{G} = \boldsymbol{F}^{-1} \tag{10-8}$$

so the use of $\boldsymbol{G}$ is dropped.

If representation by components and base vectors of (10-1) and (10-2) are used, then alternative forms of (10-3) are given by

$$x_i = x_i(X_{(j)}) \qquad X_i = X_i(x_{(j)}) \tag{10-9}$$

where the notation (j) is used to indicate the free index notation is suspended. It follows that the deformation gradient and its inverse are then given by

$$\boldsymbol{F}=\frac{\partial x_i}{\partial X_j}\boldsymbol{e}_i\otimes\boldsymbol{E}_j \qquad \boldsymbol{F}^{-1}=\frac{\partial X_i}{\partial x_j}\boldsymbol{E}_i\otimes\boldsymbol{e}_j \tag{10-10}$$

Note that these two tensors are described naturally with the use of mixed bases and components. As always, any one basis can be converted to the other to obtain components with respect to the $\boldsymbol{e}_i\otimes\boldsymbol{e}_j$ basis or $\boldsymbol{E}_i\otimes\boldsymbol{E}_j$.

To verify that one is the inverse of the other consider the product

$$\begin{aligned}\boldsymbol{F}\cdot\boldsymbol{F}^{-1}&=\frac{\partial x_i}{\partial X_j}\boldsymbol{e}_i\otimes\boldsymbol{E}_j\cdot\frac{\partial X_k}{\partial x_l}\boldsymbol{E}_k\otimes\boldsymbol{e}_l=\frac{\partial x_i}{\partial X_j}\frac{\partial X_k}{\partial x_l}\delta_{jk}\boldsymbol{e}_i\otimes\boldsymbol{e}_l\\&=\frac{\partial x_i}{\partial X_k}\frac{\partial X_k}{\partial x_l}\boldsymbol{e}_i\otimes\boldsymbol{e}_l=\frac{\partial x_i}{\partial x_l}\boldsymbol{e}_i\otimes\boldsymbol{e}_l=\delta_{il}\boldsymbol{e}_i\otimes\boldsymbol{e}_l=\boldsymbol{e}_i\otimes\boldsymbol{e}_i=\boldsymbol{I}\end{aligned} \tag{10-11}$$

where the chain rule has been used to go from the first line to the second.

The component forms can also be used to show that

$$\boldsymbol{R}\bar{\bar{\nabla}}_0=\frac{\partial X_i}{\partial X_j}\boldsymbol{E}_i\otimes\boldsymbol{E}_j=\delta_{ij}\boldsymbol{E}_i\otimes\boldsymbol{E}_j \tag{10-12}$$

or

$$\boldsymbol{R}\bar{\bar{\nabla}}_0=\boldsymbol{I} \qquad \boldsymbol{r}\bar{\bar{\nabla}}=\boldsymbol{I} \tag{10-13}$$

where the second relation is obtained in a similar manner.

Example Deformations

Suppose a deformation is defined as follows:

$$x_1=aX_1 \qquad x_2=bX_2 \qquad x_3=cX_3 \tag{10-14}$$

with (a,b,c) a set of constants. According to the signs of the constants, the deformation is a uniform extension or contraction in each direction. The components of the deformation gradient are

$$\frac{\partial x_i}{\partial X_j}\Rightarrow\begin{bmatrix}a&0&0\\0&b&0\\0&0&c\end{bmatrix} \tag{10-15}$$

The deformation does not change from one material point to another. This is an example of homogeneous deformation.

An alternative approach is to use the point "O" as the origin for both position vectors. For this case, we label the position vector for a deformed material point as

$$\tilde{\boldsymbol{r}}=\boldsymbol{r}-\boldsymbol{\rho}^{O-0} \tag{10-16}$$

where $\boldsymbol{\rho}^{O-o}$ denotes the vector from one origin to the other. Since this gradient is a constant, its gradient is zero, so that either one can be used to obtain the deformation gradient. The choice of the point "o" for the origin is simpler for this problem. In addition there is a rotation that, for this special case, is captured by using the basis $\boldsymbol{e}_i$ as indicated by the sketch of the deformation in Fig. 10-2.

If $(\hat{a},\hat{b},\hat{c})$ denotes an additional set of constants, then a generalization of the homogeneous deformation of (10-14) to an inhomogeneous deformation is the following:

$$x_1 = aX_1 + \hat{a}X_1^2 \qquad x_2 = bX_2 + \hat{b}X_2^2 \qquad x_3 = cX_3 + \hat{c}X_3^2 \tag{10-17}$$

for which the components of the deformation gradient are

$$\frac{\partial x_i}{\partial X_j} \Rightarrow \begin{bmatrix} a+2\hat{a}X_1 & 0 & 0 \\ 0 & b+2\hat{b}X_2 & 0 \\ 0 & 0 & c+2\hat{c}X_3 \end{bmatrix} \tag{10-18}$$

Now the deformation depends on the location of the material point.

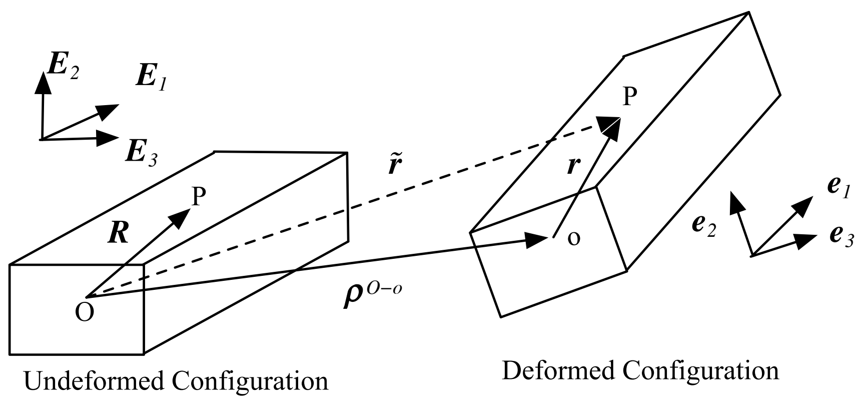

Fig. 10-2. Notation associated with the deformation of a block.

We note that the inverse relations in which the components X_i are expressed in terms of x_i is easy to obtain for (10-14), but for the relatively simple case of (10-17) the inverse transformation is subject to the ambiguity of a choice of sign from the quadratic equation.

Even though the deformation of (10-17) is inhomogeneous, it is still relatively simple in that the nonzero components of the deformation gradient in (10-18) lie on the diagonal. A more general deformation that is more realistic is the following:

$$x_1 = aX_1 + \hat{a}X_1X_2 \qquad x_2 = bX_2 + \hat{b}X_1X_3 \qquad x_3 = cX_3 + \hat{c}X_3^2 \tag{10-19}$$

with the components of the deformation gradient given by

$$\frac{\partial x_i}{\partial X_j} \Rightarrow \begin{bmatrix} a+\hat{a}X_2 & \hat{a}X_1 & 0 \\ \hat{b}X_3 & b & \hat{b}X_1 \\ 0 & 0 & c+2\hat{c}X_3 \end{bmatrix} \tag{10-20}$$

As shown in the next Chapter, components such as those in (10-20) include a rotation that is in addition to that exhibited by the different orientations of the two sets of vase vectors.

10.4 Deformations of Line, Area and Volume Elements

Line Elements

The differential of a coordinate measuring distance along a line element, dS_0, in the original configuration is defined from the equation

$$d\boldsymbol{R} = \boldsymbol{t}_0 dS_0 \tag{10-21}$$

in which $\boldsymbol{t}_0$ is a unit vector tangent to the line as shown in Fig. 10-3. But the unit vector is simply

$$\boldsymbol{t}_0 = \frac{d\boldsymbol{R}}{dS_0} = \frac{d\boldsymbol{R}}{|d\boldsymbol{R}|} = \frac{d\boldsymbol{R}}{(d\boldsymbol{R}\cdot d\boldsymbol{R})^{1/2}} \tag{10-22}$$

It follows that

$$dS_0 = \boldsymbol{t}_0 \cdot d\boldsymbol{R} = (d\boldsymbol{R}\cdot d\boldsymbol{R})^{1/2} \tag{10-23}$$

Similarly in the deformed configuration, corresponding equations are

$$d\boldsymbol{r} = \boldsymbol{t}ds \qquad\qquad ds = \boldsymbol{t}\cdot d\boldsymbol{r} = (d\boldsymbol{r}\cdot d\boldsymbol{r})^{1/2} \tag{10-24}$$

in which ds and $\boldsymbol{t}$ are the differential of a line element and the unit tangent vector, respectively. But $d\boldsymbol{r} = \boldsymbol{F}\cdot d\boldsymbol{R}$ and the use of (10-24) indicates that the line elements are related by

$$ds = \{(\boldsymbol{F}\cdot d\boldsymbol{R})\cdot(\boldsymbol{F}\cdot d\boldsymbol{R})\}^{1/2} = \{d\boldsymbol{R}\cdot \boldsymbol{F}^T\cdot \boldsymbol{F}\cdot d\boldsymbol{R}\}^{1/2} = \{\boldsymbol{t}_0\cdot \boldsymbol{F}^T\cdot \boldsymbol{F}\cdot \boldsymbol{t}_0\}^{1/2} dS_0 \tag{10-25}$$

In the development, if we use $d\boldsymbol{R} = \boldsymbol{F}^{-1}\cdot d\boldsymbol{r}$ it can be shown that equivalent expressions are

$$dS_0 = \{(\boldsymbol{F}^{-1}\cdot d\boldsymbol{r})\cdot(\boldsymbol{F}^{-1}\cdot d\boldsymbol{r})\}^{1/2} = \{d\boldsymbol{r}\cdot \boldsymbol{F}^{-T}\cdot \boldsymbol{F}-\boldsymbol{1}\cdot d\boldsymbol{r}\}^{1/2} = \{\boldsymbol{t}\cdot \boldsymbol{F}^{-T}\cdot \boldsymbol{F}^{-1}\cdot \boldsymbol{t}\}^{1/2} ds \tag{10-26}$$

in which $\boldsymbol{F}^{-T}$ denotes the transpose of the inverse.

We define the stretch of an element of original length dS_0 to be

$$\Lambda_{st} = \frac{ds}{dS_0} \tag{10-27}$$

If the element does not change length, then $\Lambda_{st} = 1$. If the element decreases in length,

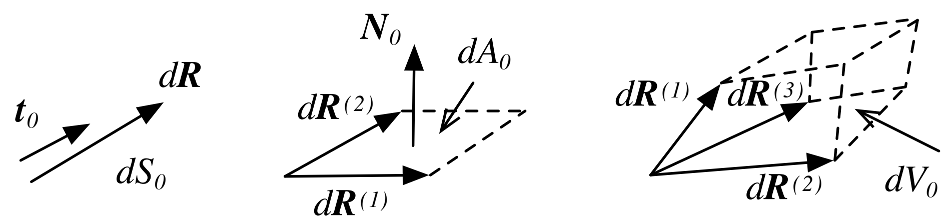

Fig. 10-3. Sketches of line, area and volume elements in the original configuration.

then $\Lambda_{st} < 1$, and $\Lambda_{st} > 1$ if the element length increases. Since a material element cannot be negative, the restriction $\Lambda_{st} > 0$ holds. With the use of (10-25) it follows that

$$\Lambda_{st} = \{ \boldsymbol{t}_0 \cdot \boldsymbol{F}^T \cdot \boldsymbol{F} \cdot \boldsymbol{t}_0 \}^{1/2} \tag{10-28}$$

a relation we will use later.

<u>Area Elements</u>

Consider two differential line elements $d\boldsymbol{R}^{(1)}$ and $d\boldsymbol{R}^{(2)}$ in the original configuration as shown in Fig. 10-3. With a unit vector $\boldsymbol{N}_0$ defined by the right-hand rule associated with a cross product, the vector area element, $d\boldsymbol{A}_0$, and the scalar area element, dA_0 , are defined by

$$d\boldsymbol{A}_0 = d\boldsymbol{R}^{(1)} \times d\boldsymbol{R}^{(2)} \qquad d\boldsymbol{A}_0 = \boldsymbol{N}_0 dA_0 \qquad dA_0 = \boldsymbol{N}_0 \cdot (d\boldsymbol{R}^{(1)} \times d\boldsymbol{R}^{(2)}) \tag{10-29}$$

Corresponding area elements are defined in the deformed configuration:

$$d\boldsymbol{a} = d\boldsymbol{r}^{(1)} \times d\boldsymbol{r}^{(2)} \qquad d\boldsymbol{a} = \boldsymbol{n} da \qquad da = \boldsymbol{n} \cdot (d\boldsymbol{r}^{(1)} \times d\boldsymbol{r}^{(2)}) \tag{10-30}$$

Recall that

$$d\boldsymbol{r}^{(1)} = \boldsymbol{F} \cdot d\boldsymbol{R}^{(1)} \qquad d\boldsymbol{r}^{(2)} = \boldsymbol{F} \cdot d\boldsymbol{R}^{(2)} \tag{10-31}$$

Substitute (10-31) into (10-30) to obtain

$$\boldsymbol{n} da = (\boldsymbol{F} \cdot d\boldsymbol{R}^{(1)}) \times (\boldsymbol{F} \cdot d\boldsymbol{R}^{(2)}) \tag{10-32}$$

Premultiply by $\boldsymbol{F} \cdot \boldsymbol{N}_0$ and use $(\boldsymbol{F} \cdot \boldsymbol{N}_0) \cdot \boldsymbol{n} = \boldsymbol{N}_0 \cdot \boldsymbol{F}^T \cdot \boldsymbol{n}$ in (10-32) with the result that

$$\boldsymbol{N}_0 \cdot \boldsymbol{F}^T \cdot \boldsymbol{n} da = (\boldsymbol{F} \cdot \boldsymbol{N}_0) \cdot \left[(\boldsymbol{F} \cdot d\boldsymbol{R}^{(1)}) \times (\boldsymbol{F} \cdot d\boldsymbol{R}^{(2)}) \right] \tag{10-33}$$

Now we use the definition of a determinant expressed in direct notation and given in (6-18) to show that

$$(\boldsymbol{F} \cdot \boldsymbol{N}_0) \cdot \left[(\boldsymbol{F} \cdot d\boldsymbol{R}^{(1)}) \times (\boldsymbol{F} \cdot d\boldsymbol{R}^{(2)}) \right] = det(\boldsymbol{F}) \boldsymbol{N}_0 \cdot \left[d\boldsymbol{R}^{(1)} \times d\boldsymbol{R}^{(2)} \right] \tag{10-34}$$

Since the determinant of $\boldsymbol{F}$ is used extensively, it is conventional to let

$$J = det(\boldsymbol{F}) \tag{10-35}$$

Then with the use of (10-29) and (10-33), we obtain

$$\boldsymbol{N}_0 \cdot \boldsymbol{F}^T \cdot \boldsymbol{n}da = JdA_0 = (\boldsymbol{N}_0 \cdot \boldsymbol{N}_0)JdA_0 \tag{10-36}$$

an equation that must hold for any choice of material elements and, hence, for arbitrary $\boldsymbol{N}_0$. The result is that (10-36) yields the following equivalent forms for relating area elements:

$$\begin{aligned} &\boldsymbol{F}^T \cdot \boldsymbol{n}da = \boldsymbol{N}_0 JdA_0 && \boldsymbol{F}^T \cdot d\boldsymbol{a} = Jd\boldsymbol{A}_0 \\ &d\boldsymbol{A}_0 = \frac{1}{J}\boldsymbol{F}^T \cdot d\boldsymbol{a} && d\boldsymbol{a} = J\boldsymbol{F}^{-T} \cdot d\boldsymbol{A}_0 \\ &dA_0 = \frac{1}{J}(\boldsymbol{N}_0 \cdot \boldsymbol{F}^T \cdot \boldsymbol{n})da && da = J(\boldsymbol{n} \cdot \boldsymbol{F}^{-T} \cdot \boldsymbol{N}_0)dA_0 \end{aligned} \tag{10-37}$$

Collectively, these equivalent equations are known as Nanson's Relation

Volume Elements

Now we choose three line elements in the original configuration $d\boldsymbol{R}^{(1)}, d\boldsymbol{R}^{(2)}$ and $d\boldsymbol{R}^{(3)}$ that form a volume element as shown in Fig. 10-3 and defined as follows:

$$dV_0 = d\boldsymbol{R}^{(1)} \cdot (d\boldsymbol{R}^{(2)} \times d\boldsymbol{R}^{(3)}) \tag{10-38}$$

with the direction of any one of the elements changed, if necessary, to ensure that $dV_0 > 0$. The corresponding volume element in the deformed configuration is

$$dV = d\boldsymbol{r}^{(1)} \cdot (d\boldsymbol{r}^{(2)} \times d\boldsymbol{r}^{(3)}) \tag{10-39}$$

With the use of the deformation gradient the deformed volume element becomes

$$dV = (\boldsymbol{F} \cdot d\boldsymbol{R}^{(1)}) \cdot \left[(\boldsymbol{F} \cdot d\boldsymbol{R}^{(2)}) \times (\boldsymbol{F} \cdot d\boldsymbol{R}^{(3)})\right] \tag{10-40}$$

Again, we use the definition of a determinant expressed in direct notation as given in (6-18) to show that

$$dV = det(\boldsymbol{F})\left[\boldsymbol{R}^{(1)} \cdot (d\boldsymbol{R}^{(2)} \times d\boldsymbol{R}^{(3)})\right] \tag{10-41}$$

Then (10-35) is used to give the following simple relation between volume elements:

$$dV = JdV_0 \tag{10-42}$$

Since a volume of mass cannot change sign, it follows that the deformation gradient must satisfy the condition that $J > 0$ and therefore, $\boldsymbol{F}^{-1}$ exists.

10.5 Transformation between Gradient Operators

Suppose we have a general function, f, that can be considered a function of either $\boldsymbol{R}$ or $\boldsymbol{r}$. Then the total differential is given by either of the following two expressions:

$$df = f\bar{\nabla} \cdot d\boldsymbol{r} = f\bar{\nabla}_0 \cdot d\boldsymbol{R} \tag{10-43}$$

But with the use of (10-7) we have

$$f\bar{\nabla} \cdot d\boldsymbol{r} = f\bar{\nabla} \cdot \boldsymbol{F} \cdot d\boldsymbol{R} \tag{10-44}$$

When (10-44) is substituted in (10-43), and the result must hold for arbitrary $d\boldsymbol{R}$, we obtain the following relations between the two gradient operators:

$$f\bar{\nabla}_0 = f\bar{\nabla} \cdot \boldsymbol{F} \qquad\qquad f\bar{\nabla} = f\bar{\nabla}_0 \cdot \boldsymbol{F}^{-1} \tag{10-45}$$

The function, f, may be a scalar, vector or a tensor.

Suppose f is a function of $\boldsymbol{R}$ so we can easily obtain $f\bar{\nabla}_0$. One way to obtain $f\bar{\nabla}$ is to use the inverse transformation $\boldsymbol{R} = \boldsymbol{R}(\boldsymbol{r})$ so that f is a function of $\boldsymbol{r}$. However, the inverse transformation may be too difficult to obtain, or simply not available. The alternative approach is to simply use the second of (10-45).

10.6 Identities Involving the Deformation Gradients and Divergence Operators

Here we develop two important identities that follow from the use of Nanson's Relation and the divergence theorem. For any subregion, R^*, in the deformed body the surface integral with argument, $\boldsymbol{n}$, satisfies

$$\int_{\partial R^*} \boldsymbol{n}\, da = \boldsymbol{0} \tag{10-46}$$

Use (10-37), or Nanson's Relation, to convert to the corresponding surface integral of the same material points in the original configuration:

$$\int_{\partial R_0^*} J\boldsymbol{F}^{-T} \cdot \boldsymbol{N}_0\, dA_0 = \boldsymbol{0} \tag{10-47}$$

Now apply the divergence theorem to obtain an equation involving the volume integral:

$$\int_{R_0^*} (J\boldsymbol{F}^{-T}) \cdot \bar{\nabla}_0\, dV_0 = \boldsymbol{0} \tag{10-48}$$

Recall that this equation for arbitrary region. If the argument is not zero, there must be some region where the argument is positive and some region where it is negative. Change the region R_0^* to be the subregion where the argument is positive. Then the integral will be greater than zero that violates the constraint of (10-48). When a similar argument is used for the subregion where the argument is negative, the implication of (10-48) for arbitrary subregion is the identity

$$(\boldsymbol{JF}^{-T})\cdot\bar{\bar{\nabla}}_0=\boldsymbol{0} \tag{10-49}$$

Next we start with the surface integral identity in the original configuration

$$\int_{\partial R_0^*}\boldsymbol{N}_0\,dA_0=\boldsymbol{0} \tag{10-50}$$

and use Nanson's Relation to convert to a surface integral in the current configuration. Then the application of the divergence theorem yields the second identity

$$\left(\tfrac{1}{J}\boldsymbol{F}^T\right)\cdot\bar{\bar{\nabla}}=\boldsymbol{0} \tag{10-51}$$

These identities are used when the various forms of the equations of motion are derived

10.7 Summary

The deformation gradient is the foundation for continuum mechanics as it appears in almost every measure of strain and in the various forms of the equations of motion. In this Chapter we have introduced the following important relations, all of which involve the deformation or the deformation gradient. The most basic relations are

$$\begin{aligned}
&\boldsymbol{r}=\boldsymbol{r}(\boldsymbol{R}) && \boldsymbol{R}=\boldsymbol{R}(r)\\
&\boldsymbol{F}=\boldsymbol{r}\bar{\bar{\nabla}}_0 && \boldsymbol{F}^{-1}=\boldsymbol{R}\bar{\bar{\nabla}}\\
&d\boldsymbol{r}=\boldsymbol{F}\cdot d\boldsymbol{R} && d\boldsymbol{R}=\boldsymbol{F}^{-1}\cdot d\boldsymbol{r}\\
&\boldsymbol{F}=\frac{\partial x_i}{\partial X_j}\boldsymbol{e}_i\otimes\boldsymbol{E}_j && \boldsymbol{F}^{-1}=\frac{\partial X_i}{\partial x_j}\boldsymbol{E}_i\otimes\boldsymbol{e}_j\\
&\boldsymbol{R}\bar{\bar{\nabla}}_0=\boldsymbol{I} && \boldsymbol{r}\bar{\bar{\nabla}}=\boldsymbol{I}
\end{aligned} \tag{10-52}$$

The relations involving line, area and volume elements are

$$\begin{aligned}
&dS_0=\boldsymbol{t}_0\cdot d\boldsymbol{R} && d\boldsymbol{r}=\boldsymbol{t}ds\\
&dS_0=(d\boldsymbol{R}\cdot d\boldsymbol{R})^{1/2} && ds=(d\boldsymbol{r}\cdot d\boldsymbol{r})^{1/2}\\
&\Lambda_{st}=\frac{ds}{dS_0}=\{\boldsymbol{t}_0\cdot\boldsymbol{F}^T\cdot\boldsymbol{F}\cdot\boldsymbol{t}_0\}^{1/2}\\
&dV=JdV_0 && J=det(\boldsymbol{F})\\
&\boldsymbol{N}_0dA_0=\frac{1}{\boldsymbol{J}}\boldsymbol{F}^T\cdot\boldsymbol{n}da && \boldsymbol{n}da=\boldsymbol{JF}^{-T}\cdot\boldsymbol{N}_0dA_0
\end{aligned} \tag{10-53}$$

And finally, the relation between the two gradients, and the two identities involving the gradients are

$$\begin{aligned}
&()\bar{\bar{\nabla}}_0=()\bar{\bar{\nabla}}\cdot\boldsymbol{F} && ()\bar{\bar{\nabla}}=()\bar{\bar{\nabla}}_0\cdot\boldsymbol{F}^{-1}\\
&(\boldsymbol{JF}^{-T})\cdot\bar{\bar{\nabla}}_0=\boldsymbol{0} && \left(\tfrac{1}{J}\boldsymbol{F}^T\right)\cdot\bar{\bar{\nabla}}=\boldsymbol{0}
\end{aligned} \tag{10-54}$$