

Chapter 3

Heat Conduction in Cylindrical and Spherical Coordinates

1 Introduction

The method of separation of variables is also useful in the determination of solutions to heat conduction problems in cylindrical and spherical coordinates. A few selected examples will be used for illustration.

Recall that the simplest form of the heat equation in cylindrical coordinates (r, ϕ, z) is

$$\rho C_p \frac{\partial T}{\partial t} = \frac{1}{r} \frac{\partial}{\partial r} (kr \frac{\partial T}{\partial r}) + \frac{1}{r^2} \frac{\partial}{\partial \phi} (k \frac{\partial T}{\partial \phi}) + \frac{\partial}{\partial z} (k \frac{\partial T}{\partial z}) + g$$

And in spherical coordinates (r, ϕ, θ)

$$\rho C_p \frac{\partial T}{\partial t} = \frac{1}{r^2} \frac{\partial}{\partial r} (kr^2 \frac{\partial T}{\partial r}) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} (k \sin \theta \frac{\partial T}{\partial \theta}) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial}{\partial \phi} (k \frac{\partial T}{\partial \phi}) + g$$

2 Fundamental Solutions to Steady State, Unidimensional Problems

As in the case of Cartesian coordinates, analytical solutions are readily obtained for unidirectional problems in cylindrical and spherical coordinates. Solutions to steady unidimensional problems can be readily obtained by elementary methods as shown below.

2.1 Cylindrical Coordinates

Consider the infinite hollow cylinder with inner and outer radii r_1 and r_2 , respectively. At steady state the heat equation in cylindrical coordinates with azimuthal symmetry becomes

$$\frac{d}{dr} (r \frac{dT}{dr}) = 0$$

the general solution of which is

$$T(r) = A \ln r + B$$

where the constants A and B must be determined from the specific boundary conditions involved. This represents the steady state loss of heat through a cylindrical wall. Two particular sets of boundary conditions are now investigated in detail.

Case 1.- Assume the heat flux at $r = r_1$, q_1 is given (Neumann boundary condition) , while the temperature at $r = r_2$ is specified as T_2 (Dirichlet boundary condition). Therefore, at $r = r_1$ one has

$$q(r_1) = q_1 = -k \frac{dT}{dr} \Big|_{r=r_1}$$

Substitution into the general solution given above yields

$$A = -\frac{q_1 r_1}{k}$$

Now, from the boundary condition at $r = r_2$ one has

$$B = T_2 + \frac{q_1 r_1}{k} \ln r_2$$

The required solution is then

$$T(r) = \frac{q_1 r_1}{k} \ln\left(\frac{r_2}{r}\right) + T_2$$

Finally, the unknown temperature at $r = r_1$ is given by

$$T_1 = \frac{q_1 r_1}{k} \ln\left(\frac{r_2}{r_1}\right) + T_2$$

Note that since $\ln(r_2/r_1) > 0$, if $q_1 > 0$ (i.e. heat flows into the cylindrical shell through the *inner* radius) then $T_1 > T_2$.

Case 2.- This case is identical to the first except that boundary conditions are reversed relative to r_1 and r_2 , i.e. one has a Dirichlet boundary condition at $r = r_1$ (Known temperature T_1) and a Neumann boundary condition at $r = r_2$ (Known heat flux q_2). Proceeding as before one gets the solution

$$T(r) = \frac{q_2 r_2}{k} \ln\left(\frac{r_1}{r}\right) + T_1$$

and the unknown temperature at $r = r_2$ is

$$T_2 = \frac{q_2 r_2}{k} \ln\left(\frac{r_1}{r_2}\right) + T_1$$

Since $\ln(r_1/r_2) < 0$, if $q_2 < 0$ is (i.e. heat flows into the cylindrical shell through the *outer* radius) then $T_2 > T_1$.

2.2 Spherical Coordinates

Consider a hollow sphere with inner and outer radii r_1 and r_2 , respectively. At steady state the heat equation in spherical coordinates with azimuthal and poloidal symmetry becomes

$$\frac{d}{dr}(r^2 \frac{dT}{dr}) = 0$$

the general solution of which is

$$T(r) = \frac{A}{r} + B$$

where the constants A and B must be determined from the specific boundary conditions involved. This represents the steady state loss of heat through a spherical shell. As in the previous section, two particular sets of boundary conditions are now investigated in detail.

Case 1.- Assume the heat flux is given at $r = r_1$ as q_1 (Neumann boundary condition) while the temperature is specified as T_2 at $r = r_2$ (Dirichlet boundary condition). Therefore, at $r = r_1$ one has

$$q(r_1) = q_1 = -k \frac{dT}{dr} \Big|_{r=r_1}$$

yielding

$$A = \frac{q_1 r_1^2}{k}$$

The second boundary condition then leads to

$$B = T_2 - \frac{q_1 r_1^2}{k} \frac{1}{r_2}$$

so that the desired solution is

$$T(r) = \frac{q_1 r_1^2}{k} \left(\frac{1}{r} - \frac{1}{r_2} \right) + T_2$$

and the unknown temperature at $r = r_1$ is

$$T_1 = \frac{q_1 r_1^2}{k} \left(\frac{1}{r_1} - \frac{1}{r_2} \right) + T_2$$

Note that the since $(1/r_1 - 1/r_2) > 0$, if $q_1 > 0$ (i.e. heat flows into the spherical shell through the *inner* radius) then $T_1 > T_2$, .

Case 2.- Now assume instead that $T(r_1) = T_1$ is given (Dirichlet) and $q(r_2) = q_2$ is also known (Neumann). With these

$$A = \frac{q_2 r_2^2}{k}$$

and

$$B = T_1 - \frac{q_2 r_2^2}{k} \frac{1}{r_1}$$

so that the desired solution is

$$T(r) = \frac{q_2 r_2^2}{k} \left(\frac{1}{r} - \frac{1}{r_1} \right) + T_1$$

and the unknown temperature at $r = r_2$ is

$$T_2 = \frac{q_2 r_2^2}{k} \left(\frac{1}{r_2} - \frac{1}{r_1} \right) + T_1$$

Note also that since $(1/r_2 - 1/r_1) < 0$, if $q_2 > 0$ (i.e. heat flows into the spherical shell through the *outer* radius) then $T_2 > T_1$.

3 Steady State Multidimensional Problems

Many multidimensional problems can be solved by simple extension of the separation of variables method. As an example consider the problem of steady state heat conduction in a short cylinder (radius r_0 , height L). The goal is to determine the steady state temperature field $T(r, z)$ inside the cylinder subject to specific conditions on its boundaries. The governing equation in this case is

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2} = 0$$

Let the boundary conditions at $r = r_0$ and $z = L$ be

$$T(R, z) = T(r, L) = 0$$

and at $z = 0$,

$$T(r, 0) = T_0$$

An additional but implicit requirement is that the solution remain bounded at $r = 0$.

Now assuming the required solution is of the form

$$T(r, z) = R(r)Z(z)$$

and proceeding as in the Cartesian case, yields the following set of ODEs

$$\frac{d}{dr} \left(r \frac{dR}{dr} \right) + \lambda^2 r R = 0$$

and

$$\frac{\partial^2 Z}{\partial z^2} - \lambda^2 Z = 0$$

where λ^2 is the separation constant representing the eigenvalues of the Sturm-Liouville problem for R , i.e. λ_n are the roots of

$$J_0(\lambda_n r_0) = 0$$

for $n = 1, 2, \dots$

Solving the above, incorporating the boundedness condition as well as the first two boundary conditions and performing a linear combination of the eigensolutions gives

$$T(r, z) = \sum_{n=1}^{\infty} R_n(r) Z_n(z) = \sum_{n=1}^{\infty} a_n J_0(\lambda_n r) \sinh(\lambda_n [L - z])$$

Finally, substituting the boundary condition at $z = 0$ requires that

$$T_0 = \sum_{n=1}^{\infty} [a_n \sinh(\lambda_n L)] J_0(\lambda_n r)$$

which is the Fourier-Bessel series representation of T_0 and leads directly to the Fourier coefficients

$$a_n = \frac{1}{\sinh(\lambda_n L)} \frac{2}{r_0^2 J_1^2(\lambda_n r_0)} \int_0^{r_0} T_0 J_0(\lambda_n r) r dr$$

4 Analytical Solutions to Transient Problems

Analytical solutions to various transient problems in cylindrical and spherical coordinates can be obtained using the method of separation of variables. Several examples are presented below.

4.1 The Quenching Problem for a Cylinder with Fixed Temperature at its Boundary

Consider the quenching problem where a long cylinder (radius $r = b$) initially at $T = f(r)$ whose surface temperature is made equal to zero for $t > 0$.

The heat equation in this case is

$$\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial T}{\partial r} \right) = \frac{1}{\alpha} \frac{\partial T}{\partial t}$$

subject to

$$T = 0$$

at $r = b$ and

$$\frac{\partial T}{\partial r} = 0$$

at $r = 0$.

According to the separation of variables method we assume a solution of the form $T(r, t) = R(r)\Gamma(t)$. Substituting this into the heat equation yields

$$\frac{1}{rR} \frac{d}{dr} \left(r \frac{dR}{dr} \right) = \frac{1}{\alpha\Gamma} \frac{d\Gamma}{dt}$$

This equation only makes sense when both terms are equal to a negative constant $-\lambda^2$. The original problem has now been transformed into two boundary value problems, namely

$$\frac{d\Gamma}{dt} + \alpha\lambda^2\Gamma = 0$$

with general solution

$$\Gamma(t) = C \exp(-\alpha\lambda^2 t)$$

where C is a constant and

$$r^2 \frac{d^2 R}{dr^2} + r \frac{dR}{dr} + r^2 \lambda^2 R = 0$$

subject to $R(b) = 0$ and necessarily bounded at $r = 0$.

The last equation is a special case of **Bessel's equation**, the only physically meaningful solution of which has the form

$$R(r) = A' J_0(\lambda r)$$

where A' is another constant and $J_0(z)$ is the Bessel function of first kind of order zero of the argument given by

$$J_0(z) = 1 - \frac{z^2}{(1!)^2 2^2} + \frac{z^4}{(2!)^2 2^4} - \frac{z^6}{(3!)^2 2^6} + \dots$$

Since $R(b) = 0$ this requires that $J_0(\lambda b) = 0$ which defines the eigenvalues and eigenfunctions for this problem. The eigenvalues are thus given as the roots of

$$J_0(\lambda_n b) = 0$$

Specifically, the first four roots are: $\lambda_1 b = 2.405$, $\lambda_2 b = 5.520$, $\lambda_3 b = 8.654$, and $\lambda_4 b = 11.79$.

A particular solution is then

$$T_n(r, t) = R_n(r)\Gamma(t) = A_n J_0(\lambda_n r) \exp(-\alpha \lambda_n^2 t)$$

where $A_n = A'_n C$ and $n = 1, 2, 3, \dots$

Superposition of particular solutions yields the more general solution

$$T(r, t) = \sum_{n=1}^{\infty} T_n(r, t) = \sum_{n=1}^{\infty} R_n(r)\Gamma(t) = \sum_{n=1}^{\infty} A_n J_0(\lambda_n r) \exp(-\alpha \lambda_n^2 t)$$

All is left is to determine the A_n 's. For this we use the given initial condition, i.e.

$$T(r, 0) = f(r) = \sum_{n=1}^{\infty} A_n J_0(\lambda_n r)$$

This is the **Fourier-Bessel series representation** of $f(r)$ and one can use the orthogonality property of the eigenfunctions to write

$$\begin{aligned} \int_0^b r J_0(\lambda_m r) f(r) dr &= \sum_{n=1}^{\infty} A_n \int_0^b r J_0(\lambda_m r) J_0(\lambda_n r) dr = A_n \int_0^b r J_0^2(\lambda_m r) dr = \\ &= \frac{b^2 A_m}{2} [J_0^2(\lambda_m b) + J_1^2(\lambda_m b)] = \frac{b^2 A_m}{2} J_1^2(\lambda_m b) \end{aligned}$$

where $J_1(z) = -dJ_0(z)/dz$ is the Bessel function of first kind of order one of the argument.

Therefore,

$$A_n = \frac{2}{b^2 J_1^2(\lambda_n b)} \int_0^b r J_0(\lambda_n r) f(r) dr$$

so that the required solution is

$$T(r, t) = \frac{2}{b^2} \sum_{n=1}^{\infty} \frac{J_0(\lambda_n r)}{J_1^2(\lambda_n b)} \exp(-\alpha \lambda_n^2 t) \int_0^b r' J_0(\lambda_n r') f(r') dr'$$

An important special case is obtained when $f(r) = T_i = \text{constant}$. The required solution then becomes

$$T(r, t) = \frac{2T_i}{b} \sum_{n=1}^{\infty} \frac{J_0(\lambda_n r)}{\lambda_n J_1(\lambda_n b)} \exp(-\alpha \lambda_n^2 t)$$

Exercise: Derive the above result.

4.2 The Quenching Problem for a Cylinder with Convective Heat Losses at its Boundary

Many problems involving more complex boundary conditions can also be solved using the separation of variables method. As an example consider the quenching problem where a long cylinder (radius $r = b$) initially at $T = f(r)$ is exposed to a cooling medium at zero temperature which extracts heat uniformly from its surface.

The heat equation in this case is

$$\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial T}{\partial r} \right) = \frac{1}{\alpha} \frac{\partial T}{\partial t}$$

subject to

$$-k \frac{\partial T}{\partial r} = hT$$

at $r = b$, and subject to

$$\frac{\partial T}{\partial r} = 0$$

at $r = 0$.

Separation of variables $T(r, t) = R(r)\Gamma(t)$ gives in this case

$$T(r, t) = \frac{2}{b^2} \sum_{m=1}^{\infty} \exp(-\alpha \beta_m^2 t) \frac{\beta_m^2 J_0(\beta_m r)}{(\beta_m^2 + (\frac{h}{k})^2) J_0^2(\beta_m b)} \int_0^b r' J_0(\beta_m r') f(r') dr'$$

A special case of interest is when the initial temperature is constant $= T_i$ and the surrounding environment is at a non-zero temperature $= T_{\infty}$. In this case the above equation reduces to

$$T(r, t) = T_{\infty} + \frac{2}{b} (T_i - T_{\infty}) \sum_{m=1}^{\infty} \frac{1}{\beta_m} \frac{J_1(\beta_m b) J_0(\beta_m r)}{J_0^2(\beta_m b) + J_1^2(\beta_m b)} e^{-\beta_m^2 \alpha t}$$

where the eigenvalues β_m are obtained from the roots of the following transcendental equation*

$$\beta b J_1(\beta b) - Bi J_0(\beta b) = 0$$

with $Bi = hb/k$.

Exercise: Derive the above result.

4.3 The Quenching Problem for a Sphere with Fixed Temperature at its Boundary

Consider the quenching problem where a sphere (radius $r = b$) initially at $T = f(r)$ whose surface temperature is made equal to zero for $t > 0$.

The heat equation in this case is

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial T}{\partial r} \right) = \frac{1}{\alpha} \frac{\partial T}{\partial t}$$

subject to

$$T = 0$$

at $r = b$ and

$$\frac{\partial T}{\partial r} = 0$$

at $r = 0$.

According to the separation of variables method we assume a solution of the form $T(r, t) = R(r)\Gamma(t)$. Substituting this into the heat equation yields

$$\frac{1}{r^2 R} \frac{d}{dr} \left(r^2 \frac{dR}{dr} \right) = \frac{1}{\alpha \Gamma} \frac{d\Gamma}{dt}$$

Again, this equation only makes sense when both terms are equal to a negative constant $-\lambda^2$. The original problem has now been transformed into two boundary value problems, namely

$$\frac{d\Gamma}{dt} + \lambda^2 \alpha \Gamma = 0$$

with general solution

$$\Gamma(t) = C \exp(-\alpha \lambda^2 t)$$

where C is a constant and

$$\frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{dR}{dr} \right) + \lambda^2 R = 0$$

subject to $R(b) = 0$ and necessarily bounded at $r = 0$. The general solution of the last problem is

$$R(r) = A' \frac{\sin(\lambda r)}{r} + B' \frac{\cos(\lambda r)}{r} = A' \frac{\sin(\lambda r)}{r}$$

where A' and B' are constants and $B' = 0$ since the temperature must be bounded at $r = 0$. Moreover, the boundary condition at $r = b$ yields the eigenvalues

$$\lambda_n = \frac{n\pi}{b}$$

and the eigenfunctions

$$R_n(r) = \frac{A'_n}{r} \sin(\lambda_n r) = \frac{1}{r} \sin\left(\frac{n\pi r}{b}\right)$$

for $n = 1, 2, 3, \dots$

A particular solution is then

$$T_n(r, t) = R_n(r) \Gamma(t) = \frac{A_n}{r} \sin(\lambda_n r) \exp(-\alpha \lambda_n^2 t)$$

where $A_n = A'_n C$ and $n = 1, 2, 3, \dots$

Superposition of particular solutions yields the more general solution

$$T(r, t) = \sum_{n=1}^{\infty} T_n(r, t) = \sum_{n=1}^{\infty} R_n(r) \Gamma(t) = \sum_{n=1}^{\infty} \frac{A_n}{r} \sin(\lambda_n r) \exp(-\alpha \lambda_n^2 t)$$

To determine the A_n 's we use the given initial condition, i.e.

$$T(r, 0) = f(r) = \sum_{n=1}^{\infty} \frac{A_n}{r} \sin(\lambda_n r)$$

this is again a Fourier series representation with the Fourier coefficients given by

$$A_n = \frac{2}{b} \int_0^b f(r') \sin\left(\frac{n\pi r'}{b}\right) r' dr'$$

An important special case results when the initial temperature $f(r) = T_i = \text{constant}$. In this case the Fourier coefficients become

$$A_n = -\frac{T_i b}{n\pi} \cos(n\pi) = -\frac{T_i b}{n\pi} (-1)^n$$

Exercise: Derive the above result.

4.4 The Quenching Problem for a Sphere with Convective Heat Losses at its Boundary

Consider a sphere with initial temperature $T(r, 0) = F(r)$ and dissipating heat by convection into a medium at zero temperature at its surface $r = b$. The heat conduction equation in 1D spherical coordinates is

$$\frac{\partial^2 T}{\partial r^2} + \frac{2}{r} \frac{\partial T}{\partial r} = \frac{1}{\alpha} \frac{\partial T}{\partial t}$$

In terms of the new variable $U(r, t) = rT(r, t)$ the mathematical formulation of the problem is

$$\frac{\partial^2 U}{\partial r^2} = \frac{1}{\alpha} \frac{\partial U}{\partial t}$$

subject to

$$U(0, t) = 0, \quad r = 0$$

$$\frac{\partial U}{\partial r} + \left(\frac{h}{k} - \frac{1}{b}\right)U = 0, \quad r = b$$

and

$$U(r, 0) = rF(r), \quad t = 0$$

This is just like heat 1D conduction from a slab so the solution is

$$T(r, t) = \frac{2}{r} \sum_{m=1}^{\infty} e^{-\alpha \beta_m^2 t} \frac{\beta_m^2 + (h/k - 1/b)^2}{b(\beta_m^2 + (h/k - 1/b)^2) + (h/k - 1/b)} \sin(\beta_m r) \int_{r'=0}^b r' F(r') \sin(\beta_m r') dr'$$

and the eigenvalues β_m are the positive roots of

$$\beta_m b \cot(\beta_m b) + b(h/k - 1/b) = 0$$

Consider now as another example a hollow sphere $a \leq r \leq b$ initially at $F(r)$ and dissipating heat by convection at both its surfaces via heat transfer coefficients h_1 and h_2 . Introducing the transformation $x = r - a$, the problem becomes identical to 1D heat conduction from a slab and the solution is

$$T(r, t) = \frac{1}{r} \sum_{m=1}^{\infty} e^{-\alpha \beta_m^2 t} \frac{1}{N(\beta_m)} R(\beta_m, r) \int_{r'=a}^b r' F(r') R(\beta_m, r') dr'$$

where

$$R(\beta_m, r) = \beta_m \cos(\beta_m[r - a]) + (h_1/k + 1/a) \sin(\beta_m[r - a])$$

and the eigenvalues β_m are the positive roots of

$$\tan(\beta_m[b - a]) = \frac{\beta_m([h_1/k + 1/a] + [h_2/k - 1/b])}{\beta_m^2 - [h_1/k + 1/a][h_2/k - 1/b]}$$

5 Non-homogeneous Problems

Nonhomogeneous problems for the cylinder or sphere can be solved using the same approach used to solve similar problems in Cartesian coordinates. Consider a long cylinder initially at $T = F(r)$ inside which heat is generated at a constant rate g_0 and whose boundary $r = b$ is subjected to $T = 0$. The mathematical statement of the problem consists of the heat equation*

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} + \frac{1}{k} g_0 = \frac{1}{\alpha} \frac{\partial T}{\partial t}$$

with $T = 0$ at $r = b$ and $t > 0$ and $T = F(r)$ at $t = 0$. The problem can be split into a steady state problem giving $T_s(r) = (g_0/4k)(b^2 - r^2)$ and a homogeneous transient problem giving $T_h(r, t)$. The solution to the original problem becomes

$$T(r, t) = T_s(r) + T_h(r, t).$$

Exercise: Solve the above problem.

6 Transient Temperature Nomographs: Heisler Charts

The solutions obtained for 1D nonhomogeneous problems with Neumann boundary conditions in cylindrical and spherical coordinate systems using the method of separation of variables have been collected and assembled in the form of transient temperature nomographs by Heisler. As in the Cartesian case, the charts are a useful baseline against which to validate one's own analytical or numerical computations. The Heisler charts summarize the solutions to the following three important problems.

The first problem is the 1D transient homogeneous heat conduction in a solid cylinder of radius b from an initial temperature T_i and with one boundary insulated and the other subjected to a convective heat flux condition into a surrounding environment at T_∞ .

Introduction of the following nondimensional parameters simplifies the mathematical formulation of the problem. First is the dimensionless distance

$$R = \frac{r}{b}$$

next, the dimensionless time

$$\tau = \frac{\alpha t}{b^2}$$

then the dimensionless temperature

$$\theta(X, \tau) = \frac{T(r, t) - T_\infty}{T_i - T_\infty}$$

and finally, the **Biot number**

$$Bi = \frac{hb}{k}$$

With the new variables, the mathematical formulation of the heat conduction problem becomes

$$\frac{1}{R} \frac{\partial}{\partial R} \left(R \frac{\partial \theta}{\partial R} \right) = \frac{\partial \theta}{\partial \tau}$$

subject to

$$\frac{\partial \theta}{\partial R} = 0$$

at $R = 0$,

$$\frac{\partial \theta}{\partial R} + Bi\theta = 0$$

at $R = 1$, and

$$\theta = 1$$

in $0 \leq R \leq 1$ for $\tau = 0$.

As the second problem consider the cooling of a sphere ($0 \leq r \leq b$) initially at a uniform temperature T_i and subjected to a uniform convective heat flux at its surface into a medium at T_∞ with heat transfer coefficient h . In terms of the dimensionless quantities $Bi = hb/k$, $\tau = \alpha t/b^2$, $\theta = (T(r, t) - T_\infty)/(T_i - T_\infty)$ and $R = r/b$, the mathematical statement of the problem is

$$\frac{1}{R^2} \frac{\partial}{\partial R} \left(R^2 \frac{\partial \theta}{\partial R} \right) = \frac{\partial \theta}{\partial \tau}$$

in $0 < R < 1$, subject to

$$\frac{\partial \theta}{\partial R} = 0$$

at $R = 0$, and

$$\frac{\partial \theta}{\partial R} + Bi\theta = 0$$

at $R = 1$, and

$$\theta = 1$$

in $0 \leq R \leq 1$, for $\tau = 0$.

As before, the solutions to the above problems are well known and are readily available in graphical form (Heisler charts).

7 Solution of Transient Multidimensional Problems by Product Superposition

As in the Cartesian coordinates case, the solutions obtained for unidimensional systems can often be combined by *product superposition* in order to obtain solutions to multidimensional problems. Specifically, the solution to the problem of unsteady state conduction in a finite cylinder (radius r_0 , height $2L$) which is initially at temperature T_i and is subsequently exposed to a quenching environment at temperature T_∞ , can be readily written down in dimensionless form as follows

$$\left[\frac{T(r, z, t) - T_\infty}{T_i - T_\infty} \right]_{finite_cylinder} = \left[\frac{T(r, z, t) - T_\infty}{T_i - T_\infty} \right]_{2L_slab} \times \left[\frac{T(r, z, t) - T_\infty}{T_i - T_\infty} \right]_{infinite_cylinder}$$

8 Numerical Methods

Heat conduction problems in cylindrical and spherical coordinates are readily solved using numerical methods. Finite difference, finite volume and finite element methods can all be applied.

9 Finite Difference Methods

Consider the problem of transient heat conduction in a solid cylinder of radius R with azimuthal symmetry and independent of z

$$\frac{\partial T}{\partial t} = \frac{\alpha}{r} \left[\frac{\partial}{\partial r} \left(r \frac{\partial T}{\partial r} \right) \right]$$

Introducing a mesh of N nodes along the r -direction, r_i with $i = 1, 2, \dots, N$ and $\Delta r = R/(N - 1)$ and a mesh of nodes in time t_j , with $j = 1, 2, \dots$, spacing Δt , and forward differencing in time, a finite difference analog is

$$\frac{T_{i,j+1} - T_{i,j}}{\Delta t} = \alpha \frac{r_{i+1/2} \left(\frac{T_{i+1,j} - T_{i,j}}{\Delta r} \right) - r_{i-1/2} \left(\frac{T_{i,j} - T_{i-1,j}}{\Delta r} \right)}{r_i \Delta r}$$

where $r_{i+1/2}$ is a radial position located halfway between r_{i+1} and r_i , $r_{i-1/2}$ is a radial position located halfway between r_i and r_{i-1} and $T_{i,j} \approx T(r_i, t_j)$. This constitutes an *explicit* method for the direct determination of the unknown temperatures at all nodes at the new time level $n + 1$.

If backward differencing in time is used instead, the result is

$$\frac{T_{i,j+1} - T_{i,j}}{\Delta t} = \alpha \frac{r_{i+1/2} \left(\frac{T_{i+1,j+1} - T_{i,j+1}}{\Delta r} \right) - r_{i-1/2} \left(\frac{T_{i,j+1} - T_{i-1,j+1}}{\Delta r} \right)}{r_i \Delta r}$$

Since one equation is obtained for each node and each equation relates the approximate value of T at the node with those of its two neighboring nodes one has then an *implicit scheme*. The result is a system of interlinked simultaneous algebraic equations with simple tridiagonal structure which is readily solved using standard numerical linear algebra methods.

As a second example consider the problem of transient heat conduction in a solid sphere of radius R with azimuthal and poloidal symmetry

$$\frac{\partial T}{\partial t} = \frac{\alpha}{r^2} \left[\frac{\partial}{\partial r} \left(r^2 \frac{\partial T}{\partial r} \right) \right]$$

Introducing a mesh of N nodes along the r -direction, r_i with $i = 1, 2, \dots, N$ and $\Delta r = R/(N - 1)$ and a mesh of nodes in time t_j , with $j = 1, 2, \dots$, spacing Δt , and forward differencing in time, a finite difference analog is

$$\frac{T_{i,j+1} - T_{i,j}}{\Delta t} = \alpha \frac{r_{i+1/2}^2 \left(\frac{T_{i+1,j} - T_{i,j}}{\Delta r} \right) - r_{i-1/2}^2 \left(\frac{T_{i,j} - T_{i-1,j}}{\Delta r} \right)}{r_i^2 \Delta r}$$

where $T_{i,j} \approx T(r_i, t_j)$. Again, this is an explicit scheme.

If backward differencing in time is used instead, the result is

$$\frac{T_{i,j+1} - T_{i,j}}{\Delta t} = \alpha \frac{r_{i+1/2}^2 \left(\frac{T_{i+1,j+1} - T_{i,j+1}}{\Delta r} \right) - r_{i-1/2}^2 \left(\frac{T_{i,j+1} - T_{i-1,j+1}}{\Delta r} \right)}{r_i^2 \Delta r}$$

Once again, an implicit system of interlinked simultaneous algebraic equations with simple tridiagonal structure is obtained.

As in the Cartesian coordinates case, the explicit method is conditionally stable and the time step must be selected so as to satisfy the CFL condition, namely

$$\Delta t < \frac{\Delta r^2}{2\alpha}$$

10 Boundary Conditions

When a multidimensional problem exhibits cylindrical or spherical symmetry it can be represented as a one dimensional problem in polar coordinates. For example for radially symmetric heat conduction in rods or spheres

$$\frac{\partial T}{\partial t} = \frac{\alpha}{r^\gamma} \frac{\partial}{\partial r} \left(r^\gamma \frac{\partial T}{\partial r} \right) = T_t = \alpha [T_{rr} + \frac{\gamma}{r} T_r]$$

where $\gamma = 1$ for systems with cylindrical symmetry and $\gamma = 2$ for systems with spherical symmetry. Subscript notation for derivatives is used for simplicity of representation.

All ideas presented before can be directly applied here with little modification. However, special care is required to handle the symmetry condition at the origin $r = 0$. For symmetry, it is required that

$$\frac{\partial T}{\partial r} = 0$$

By means of a Maclaurin expansion one can show that at the origin, the following form of the heat equation is valid (when one has symmetry at the origin),

$$T_t = (\gamma + 1)\alpha T_{rr}$$

Introducing a phantom node next to the origin, as in the Cartesian case and expressing the above two equations in terms of their finite difference analogues it is possible to obtain a finite difference formula for the node point located at the origin.

If no symmetry can be assumed the following expressions can be used instead to approximate the Laplacian,

$$\nabla^2 T \approx \frac{4(T_{M,j} - T_{0,j})}{\Delta r^2}$$

for cylindrical systems and

$$\nabla^2 T \approx \frac{6(T_{M,j} - T_{0,j})}{\Delta r^2}$$

for spherical systems. Here $T_{M,j}$ is the nearest-neighbor mean value of T obtained by averaging over all nearest neighbor nodes to the origin. The above approximations can then be used together with the original heat equation

$$\frac{\partial T}{\partial t} = \alpha \nabla^2 T$$

at the origin in order to obtain finite difference formulae for $T_{0,j}$.

11 Finite Element Methods

Consider the problem of one-dimensional steady-state conduction in a hollow cylindrical shell subject to a boundary condition of the second kind on the inner surface ($r = r_1$) and of the first kind on the outside surface ($r = r_2$), i.e.

$$\frac{1}{r} \left[\frac{\partial}{\partial r} \left(r k \frac{\partial T}{\partial r} \right) \right] = 0$$

subject to

$$-k \left(\frac{\partial T}{\partial r} \right)_{r_1} = q_1$$

and

$$T(r_2) = T_2$$

As described before, this problem has an analytical solution given by

$$T(r) = \frac{q_1 r_1}{k} \ln\left(\frac{r_2}{r}\right) + T_2$$

11.1 Variational Statement of the Problem

In the *calculus of variations* it is shown that the problem of finding the function F which minimizes a certain *functional* I subject to constraints is entirely equivalent to that of solving the associated *Euler equation* for F . Let $F = F(x, y, y')$ where $y = y(x)$ and $y' = dy/dx$. The fundamental problem in the calculus of variations consists of finding a specific $y(x)$ which minimizes I such that,

$$I = \int_a^b F(x, y, y') dx$$

subject to

$$y(a) = A$$

and

$$y(b) = B$$

It can then be shown that the function $y(x)$ which minimizes I is the same which is obtained by solving the associated Euler equation of the problem, which is

$$\frac{d}{dx} \left(\frac{\partial F}{\partial y'} \right) - \frac{\partial F}{\partial y} = 0$$

Therefore, in finding $y(x)$ it does not matter whether the variational problem or the differential problem are solved, since the same $y(x)$ solves both.

In the case of the heat conduction problem of the section above, the variational statement of the problem requires finding the function $T(r)$ which minimizes the following *functional*

$$I = \frac{1}{2} \int_0^{2\pi} \int_{r_1}^{r_2} k \left(\frac{dT}{dr} \right)^2 r dr d\theta - \int_0^{2\pi} q_1 T_1 r_1 d\theta$$

subject to the boundary condition

$$T(r_2) = T_2$$

and where T_1 is the temperature on the inner surface of the cylindrical shell.

11.2 The Rayleigh-Ritz Procedure: Single Element Representation

An approach to solving this variational problem is based on the Rayleigh-Ritz procedure. In this procedure one assumes a simple, easily integrable, form for the solution, substitute in the expression for the functional and implement the analytical condition for a minimum. A constructive computer-based implementation of the Rayleigh-Ritz procedure is the essence of the *finite element method*.

It is convenient to start with a single element representation. The *finite element* spans the entire wall thickness of the hollow cylinder. The edges of the finite element are the locations of the inner and outer surfaces (also called *finite element nodes*).

Assume that the solution of the problem can be approximately expressed as

$$T(r) = N_1(r)a_1 + N_2(r)a_2 = \mathbf{N}\vec{a}$$

where a_1 and a_2 are, respectively, the temperatures at the radial positions $r = r_1$ and $r = r_2$. N_1 and N_2 are simple functions of the radial distance given by,

$$N_1 = \frac{r_2 - r}{r_2 - r_1}$$

$$N_2 = \frac{r - r_1}{r_2 - r_1}$$

N_1 and N_2 are known as the *global shape functions* associated with the (nodal) positions r_1 and r_2 . The global shape function matrix \mathbf{N} in this case is simply a row vector

$$\mathbf{N} = \vec{N} = [N_1, N_2]$$

and the *vector of nodal temperatures* is

$$\vec{a} = [a_1, a_2]^T$$

From the above, the temperature gradient is readily obtained as

$$\frac{dT}{dr} = \mathbf{B}\vec{a}$$

where the row matrix $\mathbf{B} = \vec{B}$ is

$$\mathbf{B} = \vec{B} = \left[\frac{dN_1}{dr}, \frac{dN_2}{dr} \right] = [B_1, B_2]$$

Using the vector notation and substituting in the expression for the functional one obtains

$$I = \frac{1}{2} \int_0^{2\pi} \int_{r_1}^{r_2} \vec{a}^T \mathbf{B}^T k \mathbf{B} \vec{a} r dr d\theta - \int_0^{2\pi} a_1 q_1 r_1 d\theta$$

One can now proceed to carry out the integrals. The integral over θ is direct (since none of the quantities in the integrands depends on the angular coordinate). Further, the vector operations can be performed to split the integrand in the remaining integral into four terms, i.e.

$$I = \pi k \int_{r_1}^{r_2} \left[\left(\frac{dN_1}{dr} \right)^2 a_1^2 + \left(\frac{dN_1}{dr} \right) \left(\frac{dN_2}{dr} \right) a_1 a_2 + \left(\frac{dN_2}{dr} \right) \left(\frac{dN_1}{dr} \right) a_2 a_1 + \left(\frac{dN_2}{dr} \right)^2 a_2^2 \right] r dr - 2\pi a_1 q_1 r_1 \quad (1)$$

The individual integrals are easily obtained, i.e.

$$\int_{r_1}^{r_2} \left(\frac{dN_1}{dr} \right)^2 a_1^2 = \frac{r_2 + r_1}{2(r_2 - r_1)} a_1^2$$

$$\int_{r_1}^{r_2} \left(\frac{dN_1}{dr} \right) \left(\frac{dN_2}{dr} \right) a_1 a_2 = -\frac{r_2 + r_1}{2(r_2 - r_1)} a_1 a_2$$

$$\int_{r_1}^{r_2} \left(\frac{dN_2}{dr} \right) \left(\frac{dN_1}{dr} \right) a_2 a_1 = -\frac{r_2 + r_1}{2(r_2 - r_1)} a_2 a_1$$

$$\int_{r_1}^{r_2} \left(\frac{dN_2}{dr} \right)^2 a_2^2 = \frac{r_2 + r_1}{2(r_2 - r_1)} a_2^2$$

Therefore, the functional becomes

$$I = \frac{2\pi(r_2 + r_1)k}{4(r_2 - r_1)} (a_1^2 - 2a_1 a_2 + a_2^2) - 2\pi a_1 q_1 r_1$$

In matrix notation, this is simply

$$I = \frac{1}{2} \vec{a}^T \mathbf{K} \vec{a} - \vec{a}^T \vec{f}$$

where \mathbf{K} is the *global stiffness matrix*

$$\mathbf{K} = \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix} = \frac{\pi k(r_2 + r_1)}{(r_2 - r_1)} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

and \vec{f} is the *global force vector*

$$\vec{f} = \begin{pmatrix} 2\pi q_1 r_1 \\ 0 \end{pmatrix}$$

In the single element approximation there is no distinction between global quantities and element quantities. For example, the global stiffness matrix given above is also the *element stiffness matrix* K^e .

Finally, since the temperature has been specified at the right boundary (i.e. $a_2 = T_2$), the only independent variable affecting the functional is a_1 .

The condition for the functional to have a minimum is

$$\frac{\partial I}{\partial a_1} = 0$$

Therefore, taking the derivative and solving for a_1 one obtains

$$a_1 = T_1 = T_2 + \frac{2q_1 r_1 (r_2 - r_1)}{k(r_2 + r_1)}$$

In matrix notation, this is equivalent to

$$\mathbf{K}\vec{a} = \vec{f}$$

i.e. a linear algebraic equation which can be solved for the unknown a_1 .

11.3 The Rayleigh-Ritz Procedure: Multi-Element Representation

For this case one could proceed as above for the single element case. However, for a multi-element representation, the functional can be simply expressed as the sum of the individual contributions of the m elements considered in the analysis, i.e.

$$I = \frac{1}{2} \vec{a}^T \mathbf{K} \vec{a} - \vec{a}^T \vec{f} = \sum_1^m \left[\frac{1}{2} (\vec{a}^e)^T \mathbf{K}^e \vec{a}^e - (\vec{a}^e)^T \vec{f}^e \right]$$

where the superscript e is used to indicate that the quantities belong to individual elements in the collection. Therefore, we talk about the *element stiffness matrix* \mathbf{K}^e (which was derived earlier for the single element representation), the *element force vector* \vec{f}^e , and the *element vector of nodal temperatures* \vec{a}^e .

This last equation is very important and provides the foundation for the constructive computer implementation of finite element procedures.

The variational statement of the heat conduction problem requires the minimization of the functional, i.e.

$$\frac{\partial I}{\partial a_i} = 0, i = 1, 2, \dots, n$$

where n is the number of nodes associated with the m elements. Therefore, the problem to be solved becomes

$$\mathbf{K}\vec{a} = \vec{f}$$

i.e. a system of linear algebraic equations which can be solved for the unknown a_i 's by standard methods.

In summary, the practical computer implementation of finite element procedures proceeds according to the following steps:

Step 1: Determine the individual element stiffness and forces

Step 2: Assemble the global stiffness and force from the elemental quantities

Step 3: Solve the resulting system of algebraic equations for the nodal temperatures