INTRODUCTION TO CONTINUUM MECHANICS

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CHAPTER 1. TENSOR CALCULUS

1.1 Review of Matrices

Definition: A matrix $[A]_{m \times n}$ represents a set of numbers where m = no. of rows and n = no. of columns:

$$[A]_{mxn} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \tag{1.1-1}$$

Matrix Multiplication: If [A] is an mxn matrix and [B] is an pxq matrix then the multiplication [A] [B] is defined only if n = p.

$$[A]_{mxn}\,[B]_{pxq} = [C]_{mxq}$$

where

$$\left.\begin{aligned} c_{11} &= a_{11}b_{11} + a_{12}b_{21} + ... + a_{1n}b_{n1} \\ c_{12} &= a_{11}b_{12} + a_{12}b_{22} + ... + a_{1n}b_{n2} \\ &\vdots \\ c_{1q} &= a_{11}b_{1q} + a_{12}b_{2q} + ... + a_{1n}b_{nq} \end{aligned}\right\} \tag{1.1-2}$$

$$c_{21} = a_{21}b_{11} + a_{22}b_{21} + ... + a_{2n}b_{n1} \qquad \text{etc.}$$

Matrix Addition: Two matrices can be added only if they have the same numbers of rows and columns.

$$[A]_{mxn} + [B]_{mxn} = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \cdots & a_{1n}+b_{1n} \\ a_{21}+b_{21} & a_{22}+b_{22} & \cdots & a_{2n}+b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}+b_{m1} & a_{m2}+b_{m2} & \cdots & a_{mn}+b_{mn} \end{bmatrix} \tag{1.1-3}$$

Transpose of a Matrix: The transpose of a matrix is obtained by interchanging the rows and columns of a matrix. The transpose of a matrix [A] is denoted by $[A]^T$.

Example: $$[A]=\begin{bmatrix} 3 & 1 & -1 & 2 \\ 2 & 4 & 0 & -3 \\ 1 & 5 & 7 & 1 \end{bmatrix} \qquad [A]^T=\begin{bmatrix} 3 & 2 & 1 \\ 1 & 4 & 5 \\ -1 & 0 & 7 \\ 2 & -3 & 1 \end{bmatrix}$$

Identity Matrix: The identity matrix is a square matrix with unity on the main diagonal and zeros elsewhere

$$[I]_{nxn}=\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} \tag{1.1-4}$$

The identity matrix is the multiplicative identity for matrices.

$$\begin{aligned} [A]_{mxn}[I]_{nxn} &= [A]_{mxn} \\ [I]_{mxm}[A]_{mxn} &= [A]_{mxn} \end{aligned} \tag{1.1-5}$$

Except for special cases, such as row or column matrices, only square matrices will be considered in the following development and hence the subscripts differentiating the number of rows and columns in a matrix can be eliminated.

Inverse of a Matrix: If [A] [B] = [I] then [B] is called the inverse of [A] and is denoted by $[B] = [A]^{-1}$. Likewise, [A] is the inverse of [B] and $[A] = [B]^{-1}$, i.e.

$$[A][A]^{-1}=[A]^{-1}[A]=[I] \tag{1.1-6}$$

Symmetric Matrix: If a_{ij} denotes a typical element of the matrix [A], then [A] is symmetric if $a_{ij} = a_{ji}$. Alternately, [A] is symmetric if

$$[A]=[A]^T \tag{1.1-7}$$

Skew-Symmetric Matrix: If the matrix [A] is skew-symmetric then $a_{ij} = -a_{ji}$ or $[A] = -[A]^T$; specifically

$$[A]_{\text{skew-sym}} = \begin{bmatrix} 0 & a_{12} & a_{13} & \cdots \\ -a_{12} & 0 & a_{23} & \cdots \\ -a_{13} & -a_{23} & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \tag{1.1-8}$$

Note that the main diagonal terms of a skew-symmetric matrix are zero.

Any matrix can be expressed as the sum of a symmetric and a skew-symmetric matrix

$$[A] = [A]_{\text{symm}} + [A]_{\text{skew-symm.}} \tag{1.1-9}$$

where

$$\begin{aligned} [A]_{\text{symm.}} &= 1/2\left[[A] + [A]^T\right] \\ [A]_{\text{skew-symm.}} &= 1/2\left[[A] - [A]^T\right] \end{aligned} \tag{1.1-10}$$

Example:

$$[A] = \begin{bmatrix} 4 & 4 & -1 \\ 2 & 2 & 5 \\ 1 & 3 & 0 \end{bmatrix}$$

$$[A]_{\text{symm}} = \begin{bmatrix} 4 & 3 & 0 \\ 3 & 2 & 4 \\ 0 & 4 & 0 \end{bmatrix} \qquad [A]_{\text{skew-symm}} = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}$$

$$\therefore \qquad [A] = [A]_{\text{symm}} + [A]_{\text{Skew-symm}}$$

Orthogonal Matices: If $[A]^T = [A]^{-1}$ then [A] is said to be orthogonal. If [A] is orthogonal then $[A]\,[A]^T = [I]$

1.2 Determinants and Matrix Inversion

Suppose that we have a set of linear equation of the form

$$[A]_{nxn}\{x\}_{nx1} = \{b\}_{nx1} \tag{1.2-1}$$

If [A] is a 2 x 2 matrix this set of equations becomes

$$\begin{aligned} a_{11}X_1 + a_{12}X_2 &= b_1 \\ a_{21}X_1 + a_{22}X_2 &= b_2 \end{aligned} \tag{1.2-2}$$

We define the determinant of the matrix [A] as

$$\det[A] = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12} \tag{1.2-3}$$

We can solve the set of equations in (1.2-2) by Cramer's rule and obtain the following:

$$X_1 = \frac{\begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}} \qquad X_2 = \frac{\begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}} \tag{1.2-4}$$

Note that the determinant of a matrix is defined only if the matrix is square.

Let $[A_{kl}]$ be the matrix obtained by deleting the k^{th} row and l^{th} column of [A]. The cofactors of the matrix [A] are defined as:

$$C^A_{k\ell} = (-1)^{k+\ell}|A_{k\ell}| \tag{1.2-5}$$

The determinant of $[A]_{nxn}$ can be defined in terms of its cofactors as:

$$|A| = \sum_{j=1}^{n} a_{ij}C^A_{ij} \quad \text{for} \quad i = 1 \text{ or } 2 \text{ or } 3 \text{ or} \ldots n \tag{1.2-6}$$

The above formula is said to be the cofactor expansion of det [A] with respect to one of its rows. It is a recursive relation. The determinant of a matrix with a single component is defined to be the component itself.

To expand the determinant of A with respect to one of its columns we use the formula:

$$|A| = \sum_{i=1}^{n} a_{ij} C_{ij}^{A} \quad \text{for } j=1 \text{ or } 2 \text{ or } \ldots n \tag{1.2-7}$$

Example: If $[A] = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

then $$|A| = a_{11}(-1)^{1+1}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + a_{12}(-1)^{1+2}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$$

$$+ a_{13}(-1)^{1+3}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

(Cofactor expansion with respect to the first row)

or $$|A| = a_{12}(-1)^{1+2}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{22}(-1)^{2+2}\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} + a_{32}(-1)^{3+2}\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}$$

(Cofactor expansion with respect to the second column)

Rules for Determinants

(1) If the matrix $[A]'$ is derived from $[A]$ by interchanging two rows (or columns) then $\det [A]' = -\det [A]$

(2) If two rows (or columns) of a matrix $[A]$ are the same then $\det [A] = 0$

(3) Determinant of a matrix = determinant of its transpose: $\det [A] = \det [A]^T$

(4) If the matrix $[A]'$ is derived from $[A]$ by multiplying a row (or column) by the scalar λ, then $\det [A]' = \lambda \det [A]$

(5) If the Matrix $[A]'$ is derived from $[A]$ by adding a scalar multiple of a row (or column) to another, then $\det [A]' = \det [A]$

(6) If $[A]$ and $[B]$ are square matrices then $\det ([A][B]) = (\det [A])(\det [B])$

(7) $\sum_{j=1}^{n} a_{ij} C^{A}_{kj}$ for $i = 1$ or 2 or ... n and $k \neq i$

$= 0$ (1.2-8)

Similarly $\sum_{i=1}^{n} a_{ij} C^{A}_{ik}$ for $j=1$ or 2 or ...n and $k \neq j$

$= 0$

The <u>adjoint</u> of the matrix $[A]$ is defined as the transpose of the matrix whose elements are the cofactors of $[A]$.

Example: If $[A] = \begin{bmatrix} -3 & 1 & 2 \\ 4 & 5 & -6 \\ 0 & 3 & -1 \end{bmatrix}$

then adjoint $[A] = \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix} = \begin{bmatrix} 13 & 7 & -16 \\ 4 & 3 & -10 \\ 12 & 9 & -19 \end{bmatrix}$

Cofactor matrix $[C^{A}] = [C^{A}_{ij}]$ (1.2-9)

Adjoint matrix $[C^{Ad}] = [C^{A}]^{T}$ (1.2-10)

It can be shown with the use of Eqs. 1.2-6, 1.2-7 and 1.2-8 that

$$[A][C^{Ad}] = \begin{bmatrix} |A| & 0 & 0 & \cdots \\ 0 & |A| & 0 & \cdots \\ 0 & 0 & |A| & \cdots \\ \vdots & \vdots & & \ddots \; |A| \end{bmatrix} = |A| \; [I] \qquad (1.2\text{-}11)$$

Hence $[A]^{-1} = \dfrac{[C^{Ad}]}{|A|}$ (1.2-12)

Returning to the set of linear equations $[A]\ \{X\} = \{b\}$ we have

$[A]^{-1} [A] \{X\} = [A]^{-1} \{b\}$

$$\{X\} = [A]^{-1}\{b\} \qquad (1.2\text{-}13)$$

The above solution exists if and only if $|A| \neq 0$. Alternately, $[A]^{-1}$ exists if and only if $|A| \neq 0$.

$$\text{If } [A]\{X\} = \{0\} \qquad (1.2\text{-}14)$$

then a non-trivial solution for $\{X\}$ exists if and only if $|A| = 0$ in which case $[A]$ is said to be singular.

Derivative of a Determinant

With an expansion using row p the determinant of a matrix $[A]$ can be expressed as:

$$|A| = C^A_{p1}a_{p1} + C_{p2}a_{p2} + \dots + C^A_{pq}a_{pq} + \dots + C^A_{pn}a_{pn}$$

Note that C^A_{p1} does not contain any of the terms from row p or column 1 from the original matrix by definition. Similarly C^A_{pq} does not contain a_{pi} or a_{iq}, $i=1,\cdots,n$

Hence: $\dfrac{\partial C^A_{pi}}{\partial a_{pq}} = 0$ for $i=1,\cdots,n$ and p a fixed index

Note also that $\dfrac{\partial a_{pi}}{\partial a_{pq}} = 0$ if $i \neq q$ and that $\dfrac{\partial a_{pq}}{\partial a_{pq}} = 1$.

Using the chain rule for derivatives we obtain:

$$\frac{\partial|A|}{\partial a_{pq}} = C^A_{p1}\frac{\partial a_{p1}}{\partial a_{pq}} + \frac{\partial C^A_{p2}}{\partial a_{pq}} a_{p1} + C^A_{p2}\frac{\partial a_{p2}}{\partial a_{pq}} + \frac{\partial C^A_{p2}}{\partial a_{pq}} a_{p2}$$

$$+ \dots + C^A_{pq}\frac{\partial a_{pq}}{\partial a_{pq}} + \frac{\partial C^A_{pq}}{\partial a_{pq}} a_{pq} + \dots$$

$$\frac{\partial|A|}{\partial a_{pq}} = 0 + 0 + \cdots C^A_{pq}\cdot 1 + 0 + \cdots + 0 \cdots$$

$$\frac{\partial|A|}{\partial a_{pq}} = C^A_{pq} \text{ for p and q} = 1, \cdots, n \qquad (1.2\text{-}15)$$

Additional Definitions and Properties of Matrices (Insert)

1. $\lambda[A]$ is the matrix obtained when all components of $[A]$ are multiplied by λ. Then

$$\det[\lambda[A]] = \lambda^n \det[A] \qquad (1.2\text{-}16)$$

2. A diagonal matrix is the matrix with the form

$$[D] = \begin{bmatrix} D_{11} & 0 & 0 & \cdots & \\ 0 & D_{22} & 0 & \cdots & \\ 0 & 0 & D_{33} & & \\ \vdots & & & \ddots & \\ & & & & D_{nn} \end{bmatrix} \qquad \text{i.e. } D_{ij} = 0 \text{ for } i \neq j \qquad (1.2\text{-}17)$$

Then

$$\operatorname{Det}[D] = D_{11} D_{22} \cdots D_{nn} \qquad [D]^{-1} = \begin{bmatrix} \frac{1}{D_{11}} & 0 & \cdots & \\ 0 & \frac{1}{D_{22}} & & \\ \vdots & & \ddots & \\ & & & \frac{1}{D_{nn}} \end{bmatrix} \qquad (1.2\text{-}18)$$

3. A scalar matrix is the particular diagonal matrix

$$[S] = \begin{bmatrix} \alpha & 0 & 0 & \\ 0 & \alpha & & \\ 0 & & \ddots & \\ & & & \alpha \end{bmatrix} = \alpha [I] \qquad (1.2\text{-}19)$$

Then

$$\operatorname{Det}[S] = \alpha^n \qquad [S]^{-1} = \frac{1}{\alpha}[I]^{-1} = \frac{1}{\alpha}[I] \qquad (1.2\text{-}20)$$

4. Theorem: $$[[A][B]]^{-1} = [B]^{-1}[A]^{-1} \qquad (1.2\text{-}21)$$

Proof: Multiply on the left by $[A][B]$

$$[[A][B]]\,[[A][B]]^{-1} = [A][B][B]^{-1}[A]^{-1}$$

$$[I] = [A][I][A]^{-1}$$

$$= [A][A]^{-1}$$

$$= [I]$$

EOP

5. Theorem:

$$[[A][B]]^T = [B]^T[A]^T \qquad (1.2\text{-}22)$$

Proof: A typical element on the left side is

$$\left(\sum_j a_{ij}\, b_{jk}\right)^T = \sum_j a_{kj}\, b_{ji}$$

A typical element on the right side is

$$\sum_j b_{ji}\, a_{kj} = \sum_j a_{kj}\, b_{ji}$$

EOP

6. Theorem:

$$[A^T]^{-1} = [A^{-1}]^T \qquad (1.2\text{-}23)$$

Proof: By definition $[A][A^{-1}] = [I]$.

Take the transpose.

$$[A^{-1}]^T[A]^T = [I]^T = [I]$$

Therefore $$[A^{-1}]^T = [A^T]^{-1}$$

EOP.

From now on $[A]^{-T}$ will be used for the inverse of the transpose (or transpose of the inverse)

Section 3 Indicial Notation

Let u_i represent the set (u_1, u_2, u_3). The symbol i is called an index and u represents the set of numbers u_i for i=1,2,3. The index i is called a dummy index because it is only used to identify a particular element of the set.

For example:

$$u_i = (u_1,u_2,u_3) \tag{1.3-1}$$

$$U_k = (U_1,U_2,U_3)$$

Unless otherwise noted a single index shall indicate that there are 3 elements in the set

$$v_m = (v_1,v_2,v_3)$$

Matrices can be expressed in terms of 2 indices.

$$a_{ij} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \text{set of 9 numbers} \tag{1.3-2}$$

$$a_{mn} = \text{same set of numbers as above}$$

$$\left.\begin{array}{l} C_{ijk} \text{ represents a set of 27 numbers} \\ E_{ijk\ell} \text{ represents a set of 81 numbers} \end{array}\right\} \tag{1.3-3}$$

In general the total number of terms in a set = $(3)^{\text{no. of indices}}$.

Summation Convention

A product of 2 sets of 3 numbers in the indicial notation is defined by

$$u_i v_i = u_1 v_1 + u_2 v_2 + u_3 v_3 \tag{1.3-4}$$

The index i is called a dummy or summation index and any repeated index in a product of terms implies summation. The dummy indices can be replaced by any other pair of indices. For example,

$$u_i v_i = u_k v_k = u_1 v_1 + u_2 v_2 + u_3 v_3$$

The analog to this in matrix notation is

$$u_i v_i = \langle u\rangle\{v\} \qquad \text{where } \langle u\rangle = \{u\}^T$$

Another common product of terms is given by

$$w_i = b_{ij}u_j = b_{i1}u_1 + b_{i2}u_2 + b_{i3}u_3 \qquad (1.3\text{-}5)$$
$$= b_{ik}u_k = b_{i\ell}u_\ell$$

"i" is called the free ind and j is the summation ind Here the summation ind can be replaced by any other ind except "i" since more than two identical indices in any one product of terms is not defined. The free ind "i" must appear once in each product. The analog in matrix notation to (1.3-5) is

$$b_{ij}u_j = \left[B\right]\{u\}$$

Another similar but different product is

$$Z_j = b_{ij}u_i = b_{1j}u_1 + b_{2j}u_2 + b_{3j}u_3 \qquad (1.3\text{-}6)$$
$$= u_i b_{ij} = u_k b_{kj} \Rightarrow \langle u\rangle\left[B\right]$$

An example of a product of terms with no free indices is

$$b_{ij}\ u_i\ v_j = b_{kj}\ u_k\ v_j = b_{k\ell}u_k v_\ell \qquad (1.3\text{-}7)$$

Since no free indices exist the product is one number. In matrix notation this number is $\langle u\rangle[B]\{v\}$

Each product of terms must have the same free indices. For example, a set of nine equations can be represented by the set

$$C_{ijk}b_{k\ell}u_j + E_{ijk\ell}T_{jk} = Q_{i\ell} \qquad (1.3\text{-}8)$$

In (1.3-8) the free indices in each of the terms are i and ℓ.

Other examples: $b_{ii} = b_{11} + b_{22} + b_{33}$

$$a_{ijk}d_{ik} = a_{1jk}d_{1k} + a_{2jk}d_{2k} + a_{3jk}d_{3k}$$
$$= a_{1j1}d_{11} + a_{1j2}d_{12} + a_{1j3}d_{13}$$
$$+ a_{2j1}d_{21} + a_{2j2}d_{22} + a_{2j3}d_{23}$$
$$+ a_{3j1}d_{31} + a_{3j2}d_{32} + a_{3j3}d_{33}$$

Section 4 Vectors

A vector space consists of:

1. A set V of objects (vectors)
2. A field F of scalars
3. A rule, called vector addition, which associates with each pair of vectors, $\underset{\sim}{u},\underset{\sim}{v}$, in V a vector $\underset{\sim}{u} + \underset{\sim}{v}$ such that
   (a) Commutivitity exists $\underset{\sim}{u} + \underset{\sim}{v} = \underset{\sim}{v} + \underset{\sim}{u}$
   (b) Associativity exists $\underset{\sim}{u} + (\underset{\sim}{v} + \underset{\sim}{w}) = (\underset{\sim}{u} + \underset{\sim}{v}) + \underset{\sim}{w}$
   (c) There is a unique vector, $\underset{\sim}{0}$, such that $\underset{\sim}{0} + \underset{\sim}{v} = \underset{\sim}{v}$
   (d) There is a unique vector, $-\underset{\sim}{v}$, such that $\underset{\sim}{v} + (-\underset{\sim}{v}) = \underset{\sim}{0}$
4. A rule, scalar multiplication, which associates with each scalar "a" in F and a vector $\underset{\sim}{v}$ in V a vector $a\underset{\sim}{v}$ which is in V and satisfies
   (a) $1\underset{\sim}{v} = \underset{\sim}{v}$
   (b) $(ab)\underset{\sim}{v} = a(b\underset{\sim}{v})$
   (c) $a(\underset{\sim}{u} + \underset{\sim}{v}) = a\underset{\sim}{u} + a\underset{\sim}{v}$
   (d) $(a + b)\underset{\sim}{u} = a\underset{\sim}{u} + b\underset{\sim}{u}$

Euclidean Vector Space

In addition to the rules for a simple vector space a Euclidean Vector Space contains an inner product rule (scalar product) of two vectors which is a scalar. The rules it follows are:

1. $\underset{\sim}{u} \cdot \underset{\sim}{v} = \underset{\sim}{v} \cdot \underset{\sim}{u}$ (commutative)
2. $(\alpha\underset{\sim}{u}) \cdot \underset{\sim}{v} = \alpha(\underset{\sim}{u} \cdot \underset{\sim}{v})$
3. $\underset{\sim}{u} \cdot (\underset{\sim}{v} + \underset{\sim}{w}) = \underset{\sim}{u} \cdot \underset{\sim}{v} + \underset{\sim}{u} \cdot \underset{\sim}{w}$ (distributive)
4. If $\underset{\sim}{u} \cdot \underset{\sim}{v} = 0$ for arbitrary $\underset{\sim}{v}$ then $\underset{\sim}{u} = \underset{\sim}{0}$
5. $\underset{\sim}{u} \cdot \underset{\sim}{u} > 0$ if $\underset{\sim}{u} \neq \underset{\sim}{0}$

Let us restrict ourselves to E3 (Euclidean 3- Space)

A set of 3 vectors, $\underset{\sim}{e}_i$, are linearly independent if

$$\alpha_i \underset{\sim}{e}_i = \underset{\sim}{0} \Rightarrow \alpha_i = (0, 0, 0) \qquad (1.4\text{-}1)$$

Suppose we choose a set ₃ (such that)

$$
\begin{aligned}
\underset{\sim}{e}_1 \cdot \underset{\sim}{e}_1 &= 1 \\
\underset{\sim}{e}_1 \cdot \underset{\sim}{e}_2 &= 0 \\
\underset{\sim}{e}_1 \cdot \underset{\sim}{e}_3 &= 0 \\
\underset{\sim}{e}_2 \cdot \underset{\sim}{e}_1 &= 0 \\
\underset{\sim}{e}_2 \cdot \underset{\sim}{e}_2 &= 1 \\
&\vdots
\end{aligned}
\tag{1.4-2}
$$

Then $\underset{\sim}{e}_i$ represents an orthonormal basis and any vector $\underset{\sim}{v}$ can be expressed as

$$\underset{\sim}{v} = v_i \underset{\sim}{e}_i \tag{1.4-3}$$

$\underset{\sim}{e}_i$ are called the base vectors, v_i are called the components of the vector and $\underset{\sim}{v}$ is the direct notation for the vector.

Section 5 Vector Properties

Let $\delta_{ij} \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (1.5-1)

δ_{ij} is called the Kronecker delta and has the property

$$\delta_{ij} = 1 \text{ if } i = j$$

$$\delta_{ij} = 0 \text{ if } i \neq j$$

$$\delta_{ij} b_j = \delta_{i1} b_1 + \delta_{i2} b_2 + \delta_{i3} b_3 = C_i$$

$$C_1 = \delta_{11} b_1 + \delta_{12} b_2 + \delta_{13} b_3 = b_1 + 0 + 0 = b_1$$

Hence

$$\left.\begin{aligned} \delta_{ij} b_j &= b_i \\ \delta_{ij} b_i &= b_j \\ \delta_{ij} a_{jk} &= a_{ik} \\ \delta_{ik} a_{jk} &= a_{ji} \end{aligned}\right\} \quad (1.5\text{-}2)$$

Note that $\underset{\sim}{e}_i \cdot \underset{\sim}{e}_j = \delta_{ij}$ (1.5-3)

$$\begin{aligned} \underset{\sim}{u} \cdot \underset{\sim}{v} &= (u_i \underset{\sim}{e}_i) \cdot (v_j \underset{\sim}{e}_j) \\ &= u_i v_j \, (\underset{\sim}{e}_i \cdot \underset{\sim}{e}_j) \\ &= u_i v_j (\delta_{ij}) \\ &= u_i v_i \\ &= u_1 v_1 + u_2 v_2 + u_3 v_3 \end{aligned} \quad (1.5\text{-}4)$$

We define the "length" of a vector to be

$$| \underset{\sim}{u} | = (\underset{\sim}{u} \cdot \underset{\sim}{u})^{\frac{1}{2}} = (u_i \cdot u_i)^{\frac{1}{2}} \quad (1.5\text{-}5)$$

and the "angle", ϕ, between two vectors is defined by

$$\underset{\sim}{u} \cdot \underset{\sim}{v} = | \underset{\sim}{u} | \, | \underset{\sim}{v} | \cos \phi$$

$$\text{or } \cos\phi = \frac{u_i v_i}{|\underset{\sim}{u}||\underset{\sim}{v}|} \qquad (1.5\text{-}6)$$

A unit vector in the direction of $\underset{\sim}{u}$ is

$$\underset{\sim}{n}^{u} = \frac{\underset{\sim}{u}}{|\underset{\sim}{u}|} \qquad (1.5\text{-}7)$$

Note that $\underset{\sim}{n}^{u} \cdot \underset{\sim}{n}^{u} = 1$

The direction cosines of $\underset{\sim}{u}$ are the cosines of the angles between $\underset{\sim}{u}$ and $\underset{\sim}{e}_i$

$$\underset{\sim}{u} \cdot \underset{\sim}{e}_1 = |\underset{\sim}{u}||\underset{\sim}{e}_1| \cos\phi_1$$

$$\text{Cos } \phi_1 = \frac{\underset{\sim}{u} \cdot \underset{\sim}{e}_1}{|\underset{\sim}{u}| \cdot 1} = \frac{u_1}{|\underset{\sim}{u}|} = n_1^u \qquad (1.5\text{-}8)$$

Examples: $u_i \Rightarrow (3,1,-1)$

$v_i \Rightarrow (2,2,0)$

$$|\underset{\sim}{u}| = \{(3)(3) + (1)(1) + (-1)(-1)\}^{\frac{1}{2}} = (9 + 1 + 1)^{\frac{1}{2}} = \sqrt{11}$$

$$|\underset{\sim}{v}| = \{(2)(2) + (2)(2) + (0)(0)\}^{\frac{1}{2}} = (4 + 4 + 0) = 2\sqrt{2}$$

$$\text{Cos } \phi = \frac{\{(3)(2) + (1)(2) + (-1)(0)\}}{(\sqrt{11})(2\sqrt{2})} = \frac{(6 + 2 + 0)}{2\sqrt{22}} = \frac{4}{\sqrt{22}}$$

$$n_i^u \Rightarrow \frac{(3,1,-1)}{\sqrt{11}} = \left(\frac{3}{\sqrt{11}}, \frac{1}{\sqrt{11}}, \frac{-1}{\sqrt{11}}\right) \left(\text{Direction Cosines of } \underset{\sim}{u}\right)$$

$$n_i^v \Rightarrow \frac{(2,2,0)}{2\sqrt{2}} = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right) \left(\text{Direction Cosines of } \underset{\sim}{v}\right)$$

<u>Alternating Symbol</u>

$$\varepsilon_{ijk} \begin{cases} = 1 \text{ if } ijk \text{ form a positive permutation} \\ = -1 \text{ if } ijk \text{ form a negative permutation} \\ = 0 \text{ if any 2 indices are the same} \end{cases} \qquad (1.5\text{-}9)$$

1 2 3 => positive permutation

$$\varepsilon_{123} = \varepsilon_{231} = \varepsilon_{312} = 1$$

1 3 2 => negative permutation

$$\varepsilon_{132} = \varepsilon_{213} = \varepsilon_{321} = -1$$

In general

$$\varepsilon_{ijk} = \varepsilon_{jki} = \varepsilon_{kij}$$
$$\varepsilon_{ijk} = -\varepsilon_{ikj} = -\varepsilon_{jik} = -\varepsilon_{kji} \tag{1.5-10}$$

Cross Product

The cross product of 2 vectors is defined by

$$\underset{\sim}{u} \times \underset{\sim}{v} = |\underset{\sim}{u}||\underset{\sim}{v}| \sin\phi \, \underset{\sim}{c} \tag{1.5-11}$$

where ϕ is the angle obtained starting from $\underset{\sim}{u}$ and going to $\underset{\sim}{v}$ and $\underset{\sim}{c}$ is a unit vector obtained by the right-hand rule.

We set up the base vectors in a right-handed orientation such that

$$\begin{aligned} \underset{\sim}{e}_1 \times \underset{\sim}{e}_2 &= \underset{\sim}{e}_3 \\ \underset{\sim}{e}_2 \times \underset{\sim}{e}_3 &= \underset{\sim}{e}_1 \\ \underset{\sim}{e}_3 \times \underset{\sim}{e}_1 &= \underset{\sim}{e}_2 \\ \underset{\sim}{e}_2 \times \underset{\sim}{e}_2 &= 0 \\ &\vdots \end{aligned} \tag{1.5-12}$$

Hence $\underset{\sim}{e}_i \times \underset{\sim}{e}_j = \varepsilon_{ijk} \underset{\sim}{e}_k$ (1.5-13)

$$\begin{aligned} \underset{\sim}{u} \times \underset{\sim}{v} &= (u_i \underset{\sim}{e}_i) \times (v_j \underset{\sim}{e}_j) \\ &= (u_i v_j) \underset{\sim}{e}_i \times \underset{\sim}{e}_j \\ &= u_i v_j \varepsilon_{ijk} \underset{\sim}{e}_k \\ &= \underset{\sim}{b} \\ &= b_k \underset{\sim}{e}_k \end{aligned}$$

or

$$\underset{\sim}{b} = \underset{\sim}{u} \times \underset{\sim}{v} \text{ (direct notation)} \tag{1.5-14}$$

$$b_k = \varepsilon_{ijk}\, u_i v_j \text{ (indicial notation)}$$

$$b_1 = \varepsilon_{321}\, u_3 v_2 + \varepsilon_{231}\, u_2 v_3 + 0 + \ldots 0$$

$$= -u_3 v_2 + u_2 v_3$$

$$b_2 = \varepsilon_{312}\, u_3 v_1 + \varepsilon_{132}\, u_1 v_3 + 0 + \ldots 0$$

$$= u_3 v_1 - u_1 v_3$$

$$b_3 = u_1 v_2 - u_2 v_1$$

Kronecker Delta - Alternating Symbol Identity

It can be shown that

$$\varepsilon_{ijk}\, \varepsilon_{i\ell m} = \delta_{j\ell}\delta_{km} - \delta_{jm}\delta_{k\ell} \tag{1.5-15}$$

Example of the use of the identity:

Expanding the cross product $\underset{\sim}{u} \times (\underset{\sim}{v} \times \underset{\sim}{w})$ we obtain

$$\underset{\sim}{u} \times (\underset{\sim}{v} \times \underset{\sim}{w}) = u\, \underset{\sim}{e}_i \times (v_j \underset{\sim}{e}_j \times w_k \underset{\sim}{e}_k)$$

$$= u_i v_j w_k \underset{\sim}{e}_i \times (\underset{\sim}{e}_j \times \underset{\sim}{e}_k)$$

Recall that $\underset{\sim}{e}_j \times \underset{\sim}{e}_k = \varepsilon_{jk\ell} \underset{\sim}{e}_\ell$ and that $\underset{\sim}{e}_i \times \underset{\sim}{e}_\ell =$

$\varepsilon_{i\ell m} \underset{\sim}{e}_m$

Hence $\underset{\sim}{u} \times (\underset{\sim}{v} \times \underset{\sim}{w}) = u_i v_j w_k \varepsilon_{jk\ell} \varepsilon_{i\ell m} \underset{\sim}{e}_m$

Noting that $\varepsilon_{\ell jk} = \varepsilon_{jk\ell}$ and that $\varepsilon_{i\ell m} = \varepsilon_{\ell mi}$

we substitute into the above equation and obtain

$$\underset{\sim}{u} \times (\underset{\sim}{v} \times \underset{\sim}{w}) = u_i v_j w_k\, \varepsilon_{\ell jk} \varepsilon_{\ell mi} \underset{\sim}{e}_m$$

Substituting from (1.5-15) we obtain

$$\underset{\sim}{u} \times (\underset{\sim}{v} \times \underset{\sim}{w}) = u_i v_j w_k (\delta_{jm}\delta_{ki} - \delta_{ji}\delta_{km}) \underset{\sim}{e}_m$$

$$= u_i v_m w_i \underset{\sim}{e}_m - u_i v_i w_m \underset{\sim}{e}_m$$

$$= (u_i w_i)(v_m \underset{\sim}{e}_m) - (u_i v_i)(w_m \underset{\sim}{e}_m)$$

$$\underset{\sim}{u} \times (\underset{\sim}{v} \times \underset{\sim}{w}) = (\underset{\sim}{u} \cdot \underset{\sim}{w})\underset{\sim}{v} - (\underset{\sim}{u} \cdot \underset{\sim}{v})\underset{\sim}{w} \tag{1.5-16}$$

Utilizing the properties of the Kronecker Delta, it follows that

$$\begin{aligned}\varepsilon_{ijk}\varepsilon_{ijm} &= \delta_{jj}\delta_{km} - \delta_{jm}\delta_{kj} \\ &= 3\,\delta_{km} - \delta_{km} \\ &= 2\,\delta_{km}\end{aligned} \tag{1.5-17}$$

Similarly

$$\varepsilon_{ijk}\varepsilon_{ijk} = 2\delta_{kk} = 6 \tag{1.5-18}$$

Determinant of a Matrix

It can be shown that if a_{ij} represents the components of a matrix then

$$\left.\begin{aligned}\varepsilon_{ijk}a_{i\ell}a_{jm}a_{kn} &= \varepsilon_{\ell mn}\det[A] \\ \varepsilon_{\ell mn}\varepsilon_{ijk}a_{i\ell}a_{jm}a_{kn} &= 6\det[A]\end{aligned}\right\} \tag{1.5-19}$$

Particular Case: From the basic definition of a determinant

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{aligned} &a_{11}a_{22}a_{33} + a_{21}a_{32}a_{13} + a_{12}a_{23}a_{31} \\ &-a_{31}a_{22}a_{13} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}\end{aligned}$$

If we let $\ell = 1$, $m = 2$, and $n = 3$, then the first equation of (1.5-19) becomes

$$\begin{aligned}\varepsilon_{ijk}a_{i1}a_{j2}a_{k3} &= \varepsilon_{ij1}a_{i1}a_{j2}a_{13} \\ &+ \varepsilon_{ij2}a_{i1}a_{j2}a_{23} \\ &+ \varepsilon_{ij3}a_{i1}a_{j2}a_{33}\end{aligned}$$

$$\begin{aligned} &= \varepsilon_{231}a_{21}a_{32}a_{13} + \varepsilon_{321}a_{31}a_{22}a_{13} \\ &+ \varepsilon_{132}a_{11}a_{32}a_{23} + \varepsilon_{312}a_{31}a_{12}a_{23} \\ &+ \varepsilon_{123}a_{11}a_{22}a_{33} + \varepsilon_{213}a_{21}a_{12}a_{33}\end{aligned}$$

$$\begin{aligned} &= a_{21}a_{32}a_{13} - a_{31}a_{22}a_{13} - a_{11}a_{32}a_{23} \\ &+ a_{31}a_{12}a_{23} + a_{11}a_{22}a_{33} - a_{21}a_{12}a_{33}\end{aligned}$$

$$\begin{aligned} &= \varepsilon_{123}\det[A] \\ &= \det[A]\end{aligned}$$

Section 6 Linear Transformations and Tensors

Let L be a linear transformation operating on vectors, i.e.

$$L\,(\alpha \underset{\sim}{v}) = \alpha L(\underset{\sim}{v})$$
$$L\,(\underset{\sim}{v} + \underset{\sim}{w}) = L(\underset{\sim}{v}) + L(\underset{\sim}{w}) \qquad (1.6\text{-}1a)$$

The symbol of a tensor $\underset{\approx}{T}$ is used to denote a general linear transformation of vectors into vectors, i.e.,

$$\underset{\approx}{T}(\underset{\sim}{u}) \equiv \underset{\approx}{T}\cdot\underset{\sim}{u} = \underset{\sim}{v} \qquad \underset{\approx}{T}(\alpha\underset{\sim}{u}) \equiv \underset{\approx}{T}\cdot\alpha\underset{\sim}{u} = \alpha\underset{\sim}{v}$$
$$\underset{\approx}{T}(\underset{\sim}{u}+\underset{\sim}{w}) \equiv \underset{\approx}{T}\cdot(\underset{\sim}{u}+\underset{\sim}{w}) = \underset{\approx}{T}\cdot\underset{\sim}{u} + \underset{\approx}{T}\cdot\underset{\sim}{w} \qquad (1.6\text{-}1b)$$

Now let $\underset{\sim}{u}$ and $\underset{\sim}{v}$ be elements of the vector space V and define a tensor product $\underset{\sim}{u} \otimes \underset{\sim}{v}$ which is an element of the tensor product space $V \otimes V$ such that the following properties are satisfied:

$$\text{(i)}\quad (\underset{\sim}{u} + \underset{\sim}{v}) \otimes \underset{\sim}{w} = (\underset{\sim}{u} \otimes \underset{\sim}{w}) + (\underset{\sim}{v} \otimes \underset{\sim}{w})$$
$$\text{(ii)}\quad \underset{\sim}{w} \otimes (\underset{\sim}{u} + \underset{\sim}{v}) = \underset{\sim}{w} \otimes \underset{\sim}{u} + \underset{\sim}{w} \otimes \underset{\sim}{v} \qquad (1.6\text{-}2)$$
$$\text{(iii)}\quad \alpha(\underset{\sim}{u} \otimes \underset{\sim}{v}) = \alpha\underset{\sim}{u} \otimes \underset{\sim}{v} = \underset{\sim}{u} \otimes \alpha\underset{\sim}{v}$$

The tensor product is related to the scalar product in the following way

$$(\underset{\sim}{u} \otimes \underset{\sim}{v}) \cdot \underset{\sim}{w} = (\underset{\sim}{v} \cdot \underset{\sim}{w})\underset{\sim}{u} \qquad (1.6\text{-}3)$$
$$\underset{\sim}{w} \cdot (\underset{\sim}{u} \otimes \underset{\sim}{v}) = (\underset{\sim}{w} \cdot \underset{\sim}{u})\underset{\sim}{v}$$

The operator $L(\;) = \{(\underset{\sim}{u} \otimes \underset{\sim}{v}).\}(\;)$ transforms vectors into vectors that satisfy (1.6-1) and hence represents a linear operator that transforms vectors into vectors. We call such an operator $(\underset{\sim}{u} \otimes \underset{\sim}{v}) \cdot (\;)$ a tensor of second order.

However, $(\underset{\sim}{u} \otimes \underset{\sim}{v}) \cdot \underset{\sim}{w} = (\underset{\sim}{v} \cdot \underset{\sim}{w})\underset{\sim}{u}$ is a transformation that always yields a vector parallel to $\underset{\sim}{u}$, i.e. it is not a general transformation. A general transformation would be obtained by letting $\underset{\sim}{u}$ and $\underset{\sim}{v}$ be arbitrary.

Rather than considering all possible components for $\underset{\sim}{u}$ and $\underset{\sim}{v}$ in the expression

$$\underset{\sim}{u} \otimes \underset{\sim}{v} = u_i\underset{\sim}{e}_i \otimes v_j\underset{\sim}{e}_j = u_i v_j \underset{\sim}{e}_i \otimes \underset{\sim}{e}_j$$

the symbol $\underset{\approx}{T}$ is introduced to represent a general second order tensor:

$$\underset{\approx}{T} = T_{ij}\,\underset{\sim}{e}_i \otimes \underset{\sim}{e}_j \qquad (1.6\text{-}4)$$

where T_{ij} are the components of the tensor and $\underset{\sim}{e}_i \otimes \underset{\sim}{e}_j$ is the basis for the tensor product space

Consider

$$\underset{\approx}{T} \cdot \underset{\sim}{u} = (T_{ij}\underset{\sim}{e}_i \otimes \underset{\sim}{e}_j) \cdot u_k\underset{\sim}{e}_k$$

Note that $\underset{\sim}{e}_j \cdot u_k\underset{\sim}{e}_k = u_k\delta_{jk} = u_j$

$$\left.\begin{aligned}
\text{Hence } \underset{\approx}{T} \cdot \underset{\sim}{u} &= T_{ij}\, u_j\, \underset{\sim}{e}_i \\
\underset{\sim}{u} \cdot \underset{\approx}{T} &= u_i\underset{\sim}{e}_i \cdot (T_{jk}\underset{\sim}{e}_j \otimes \underset{\sim}{e}_k) \\
&= u_i T_{jk}\delta_{ij}\underset{\sim}{e}_k \\
&= u_j T_{jk}\underset{\sim}{e}_k
\end{aligned}\right\} \qquad (1.6\text{-}5)$$

$$\begin{aligned}
u \cdot (\underset{\approx}{T} \cdot \underset{\sim}{v}) &= (\underset{\sim}{u} \cdot \underset{\approx}{T}) \cdot \underset{\sim}{v} \\
&= u_k\underset{\sim}{e}_k \cdot \left[(T_{ij}\underset{\sim}{e}_i \otimes \underset{\sim}{e}_j) \cdot v_\ell\underset{\sim}{e}_\ell\right] \\
&= u_k\underset{\sim}{e}_k \cdot (T_{ij}v_\ell\delta_{j\ell}\underset{\sim}{e}_i) \\
&= u_k\underset{\sim}{e}_k \cdot (T_{ij}v_j\underset{\sim}{e}_i) \\
&= T_{ij}u_k v_j\delta_{ki} \\
&= T_{ij}u_i v_j
\end{aligned}$$

Higher Order Tensors

An example of a third order tensor is the alternating tensor defined by

$$\underset{\approx}{\varepsilon} = \varepsilon_{ijk}\underset{\sim}{e}_i \otimes \underset{\sim}{e}_j \otimes \underset{\sim}{e}_k \qquad (1.6\text{-}6)$$

$$\underset{\approx}{\varepsilon} \cdot \underset{\sim}{u} = (\varepsilon_{ijk}\underset{\sim}{e}_i \otimes \underset{\sim}{e}_j \otimes \underset{\sim}{e}_k) \cdot u_\ell\underset{\sim}{e}_\ell$$

Since $\underset{\sim}{e}_k \cdot u_\ell\underset{\sim}{e}_\ell = u_\ell\delta_{k\ell} = u_k$, then

$$\underset{\approx}{\varepsilon} \cdot \underset{\sim}{u} = \varepsilon_{ijk}u_k\underset{\sim}{e}_i \otimes \underset{\sim}{e}_j \quad \text{(second order tensor)}$$

The elasticity tensor $\underset{\approx}{c}$ is an example of a fourth order tensor

$$\underset{\approx}{c} = c_{ijk\ell}\underset{\sim}{e}_i \otimes \underset{\sim}{e}_j \otimes \underset{\sim}{e}_k \otimes \underset{\sim}{e}_\ell \qquad (1.6\text{-}7)$$

Composition of Tensors

The composition of tensors $\underset{\approx}{S}$ and $\underset{\approx}{T}$ is defined such that the following holds:

$$(\underset{\approx}{S} \cdot \underset{\approx}{T}) \cdot \underset{\sim}{u} = \underset{\approx}{S} \cdot (\underset{\approx}{T} \cdot \underset{\sim}{u}) \tag{1.6-8}$$

$$= (S_{ij}\underset{\sim}{e}_i \otimes \underset{\sim}{e}_j) \cdot (T_{k\ell} u_\ell \underset{\sim}{e}_k)$$

$$= S_{ij} T_{k\ell} \delta_{jk} u_\ell \underset{\sim}{e}_i$$

$$= S_{ij} T_{j\ell} u_\ell \underset{\sim}{e}_i$$

Hence $\underset{\approx}{S} \cdot \underset{\approx}{T} = S_{ij} T_{j\ell} \underset{\sim}{e}_i \otimes \underset{\sim}{e}_\ell$

Identity Tensor

$$\underset{\approx}{I} = \delta_{ij} \underset{\sim}{e}_i \otimes \underset{\sim}{e}_j \tag{1.6-9}$$

$$= \underset{\sim}{e}_i \otimes \underset{\sim}{e}_i$$

Using the composition rule for tensors it can be shown that

$$\underset{\approx}{T} \cdot \underset{\approx}{I} = \underset{\approx}{I} \cdot \underset{\approx}{T} = \underset{\approx}{T} \tag{1.6-10}$$

The transpose of a tensor $\underset{\approx}{T}$ is denoted by $\underset{\approx}{T}^T$ and defined to be

$$\underset{\approx}{T}^T = T_{ij} \underset{\sim}{e}_j \otimes \underset{\sim}{e}_i$$

$$= T_{ji} \underset{\sim}{e}_i \otimes \underset{\sim}{e}_j \tag{1.6-11}$$

Thm:

The transpose of the composition of two tensors is

$$(\underset{\approx}{S} \cdot \underset{\approx}{T})^T = (\underset{\approx}{T}^T \cdot \underset{\approx}{S}^T) \tag{1.6-12}$$

Proof:

$$\underset{\approx}{S} \cdot \underset{\approx}{T} = S_{ij} T_{jk} \underset{\sim}{e}_i \otimes \underset{\sim}{e}_k$$

$$(\underset{\approx}{S} \cdot \underset{\approx}{T})^T = S_{ij} T_{jk} \underset{\sim}{e}_k \otimes \underset{\sim}{e}_i$$

$$= S^T_{ji} T^T_{kj} \underset{\sim}{e}_k \otimes \underset{\sim}{e}_i$$

$$= T^T_{kj} S_{ji}^T \underset{\sim}{e}_k \otimes \underset{\sim}{e}_i$$

$$= \underset{\approx}{T}^T \cdot \underset{\approx}{S}^T$$

EOP

The inverse of $\underset{\approx}{T}$ is defined such that

$$\underset{\approx}{T} \cdot \underset{\approx}{T}^{-1} = \underset{\approx}{T}^{-1} \cdot \underset{\approx}{T} = \underset{\approx}{I} \tag{1.6-13}$$

Thm: The inverse and transpose operations can be taken in any order

$$(\underset{\approx}{T}^{-1})^T = (\underset{\approx}{T}^T)^{-1} \tag{1.6-14}$$

Proof:

$$(\underset{\approx}{T}^{-1} \cdot \underset{\approx}{T})^T = (\underset{\approx}{T} \cdot \underset{\approx}{T}^{-1})^T = \underset{\approx}{I}^T = \underset{\approx}{I}$$

The use of eq. (1.6-12) on the first term yields

$$\underset{\approx}{T}^T \cdot (\underset{\approx}{T}^{-1})^T = \underset{\approx}{I}$$

Using the fact that $\underset{\approx}{T}^T \cdot (\underset{\approx}{T}^T)^{-1} = \underset{\approx}{I}$ and noting that these relationships hold for arbitrary $\underset{\approx}{T}^T$ we obtain

$$(\underset{\approx}{T}^{-1})^T = (\underset{\approx}{T}^T)^{-1}$$

EOP

Note:

Three common notations have been introduced to illustrate relations between functions that represent physical phenomena. The choice of a particular notation depends on the particular application, e.g. proof of theorems, numerical computations, and its familiarity to the user. A simple example representing these notations follows:

$\underset{\approx}{T} \cdot \underset{\sim}{u} = \underset{\sim}{v}$ Direct Notation

$T_{ij} u_j \underset{\sim}{e}_i = v_i \underset{\sim}{e}_i$ Indicial Notation

or $T_{ij} u_j = v_i$

$[T]\{u\} = \{v\}$ Matrix Notation

Section 7 Tensor Properties

$\underset{\approx}{T}$ is symmetric if $\underset{\approx}{T} = \underset{\approx}{T}^T$ or if (1.7-1)

$$\underset{\approx}{T} \cdot \underset{\sim}{v} = \underset{\sim}{v} \cdot \underset{\approx}{T} \quad \text{for all } \underset{\sim}{v}$$

$\underset{\approx}{T}$ is skew-symmetric (anti-symmetric) if $\underset{\approx}{T} = -\underset{\approx}{T}^T$ or if

$$\underset{\sim}{u} \cdot \underset{\approx}{T} \cdot \underset{\sim}{u} = 0 \quad \text{for all } \underset{\sim}{u} \qquad (1.7\text{-}2)$$

For any tensor $\underset{\approx}{T}$ we can define

$$\underset{\approx}{T}_{symm} = 1/2\ (\underset{\approx}{T} + \underset{\approx}{T}^T)$$

$$\underset{\approx}{T}_{anti\text{-}symm} = 1/2\ (\underset{\approx}{T} - \underset{\approx}{T}^T) \qquad (1.7\text{-}3)$$

$$\underset{\sim}{T} = \underset{\approx}{T}_{symm.} + \underset{\approx}{T}_{anti\text{-}symm}$$

Determinant of a Tensor

The determinant of a tensor is written as $||\underset{\approx}{T}||$, det $(\underset{\approx}{T})$ and is defined such that the following holds $\forall$ (for all) $\underset{\sim}{u}, \underset{\sim}{v}, \underset{\sim}{w}$

$$(\underset{\approx}{T} \cdot \underset{\sim}{u}) \cdot \left[(\underset{\approx}{T} \cdot \underset{\sim}{v}) \times (\underset{\approx}{T} \cdot \underset{\sim}{w})\right] = \det(\underset{\approx}{T})\left[\underset{\sim}{u} \cdot (\underset{\sim}{v} \times \underset{\sim}{w})\right] \qquad (1.7\text{-}4)$$

$$\underset{\sim}{u} \cdot (\underset{\sim}{v} \times \underset{\sim}{w}) = \varepsilon_{ijk} u_i v_j w_k$$

$$\underset{\approx}{T} \cdot \underset{\sim}{w} = T_{ij} w_j \underset{\sim}{e}_i$$

$$\underset{\approx}{T} \cdot \underset{\sim}{v} = T_{\ell m} v_m \underset{\sim}{e}_\ell$$

$$\underset{\approx}{T} \cdot \underset{\sim}{u} = T_{pq} u_q \underset{\sim}{e}_p$$

Hence $(\underset{\approx}{T} \cdot \underset{\sim}{v}) \times (\underset{\approx}{T} \cdot \underset{\sim}{w}) = T_{\ell m} v_m T_{ij} w_j \varepsilon_{\ell ik} \underset{\sim}{e}_k$

$$(\underset{\approx}{T} \cdot \underset{\sim}{u}) \cdot \left[(\underset{\approx}{T} \cdot \underset{\sim}{v}) \times (\underset{\approx}{T} \cdot \underset{\sim}{w})\right] = T_{pq} u_q T_{\ell m} v_m T_{ij} w_j \varepsilon_{\ell ik} \delta_{pk}$$

Since $\varepsilon_{\ell ik} \delta_{pk} = \varepsilon_{\ell ip}$

$$(\underset{\approx}{T} \cdot \underset{\sim}{u}) \cdot \left[(\underset{\approx}{T} \cdot \underset{\sim}{v}) \times (\underset{\approx}{T} \cdot \underset{\sim}{w})\right] = \varepsilon_{\ell ip} T_{\ell m} T_{ij} T_{pq} u_q v_m w_j$$

After some manipulation of the indices to get the form $u_i v_j w_k$, the right hand side of the equation becomes

$$\varepsilon_{\ell qp} T_{\ell j} T_{qk} T_{pi} u_i v_j w_k$$

Noting that $\underset{\sim}{u}$, $\underset{\sim}{v}$ and $\underset{\sim}{w}$ are arbitrary and using (1.7-4) we arrive at

$$\varepsilon_{\ell qp} T_{\ell j} T_{qk} T_{pi} = \det(\underset{\approx}{T})\ \varepsilon_{ijk}$$

But from eq. (1.5-19)

$$\varepsilon_{\ell qp} T_{\ell j} T_{qk} T_{pi} = \varepsilon_{ijk} \det [T]$$

Hence $\det [\underset{\approx}{T}] = \det [T]$ (1.7-5)

where T represents the components of $\underset{\approx}{T}$ in an orthonormal basis.

Contraction Operator

Given a tensor

$$\underset{\approx}{\underset{\vdots}{R}} = R_{ijk\cdots}\, \underset{\sim}{e}_i \otimes \underset{\sim}{e}_j \otimes \underset{\sim}{e}_k \otimes \ldots\ldots$$

The operation

$$C_{\Phi\psi}\left[\underset{\approx}{\underset{\vdots}{R}}\right] = R_{ijk}\cdots \underset{\sim}{e}_i \cdots \overbrace{\underset{\Phi}{\underset{\sim}{e}} \cdots\cdots \underset{\psi}{\underset{\sim}{e}}}^{\text{Dot}} \cdots \qquad (1.7\text{-}6)$$

means the dot product of the Φth and ψth base vectors is taken and the tensor product symbols retained for the remaining base vectors, if any.

Example : $C_{12}\left[T_{ij}\underset{\sim}{e}_i \otimes \underset{\sim}{e}_j\right]$

$$= T_{ij}\underset{\sim}{e}_i \cdot \underset{\sim}{e}_j$$

$$= T_{ij}\delta_{ij} \qquad (1.7\text{-}7_1)$$

$$= T_{ii}$$

$= \text{tr}\,(\underset{\approx}{T})$ called the trace of $\underset{\approx}{T}$

Example : $C_{23}\,(\underset{\approx}{T} \otimes \underset{\sim}{u})$

$$= C_{23}\,(T_{ij}\underset{\sim}{e}_i \otimes \underset{2}{\underset{\sim}{e}_j} \otimes u_k \underset{3}{\underset{\sim}{e}_k})$$

$$= T_{ij}\underset{\sim}{e}_i u_k (\underset{\sim}{e}_j \cdot \underset{\sim}{e}_k) \qquad (1.7\text{-}7_2)$$

$$= T_{ij}\underset{\sim}{e}_i u_k \delta_{jk}$$

$$= T_{ij} u_j \underset{\sim}{e}_i$$

$$= \underset{\approx}{T} \cdot \underset{\sim}{u}$$

Example: $C_{12}\,(\underset{\approx}{S}\cdot\underset{\approx}{T}) = \mathrm{tr}\,(\underset{\approx}{S}\cdot\underset{\approx}{T})$

$$= C_{12}(S_{ij}T_{jk}\underset{\sim}{e}_i \otimes \underset{\sim}{e}_k)$$

$$= S_{ij}T_{jk}\underset{\sim}{e}_i \cdot \underset{\sim}{e}_k \qquad (1.7\text{-}7_3)$$

$$= S_{ij}T_{jk}\delta_{ik}$$

$$= S_{ij}T_{ji}$$

Example: $C_{35}C_{46}(\underset{\approx}{C} \otimes \underset{\approx}{E})$

$$= C_{35}C_{46}\left[C_{ijk\ell}\underset{\sim}{e}_i \otimes \underset{\sim}{e}_j \otimes \overset{3}{\underset{\sim}{e}_k} \otimes \overset{4}{\underset{\sim}{e}_\ell} E_{mn} \otimes \overset{5}{\underset{\sim}{e}_m} \otimes \overset{6}{\underset{\sim}{e}_n}\right]$$

$$= C_{ijk\ell}E_{mn}\underset{\sim}{e}_i \otimes \underset{\sim}{e}_j\delta_{km}\delta_{\ell n} \qquad (1.7\text{-}7_4)$$

$$= C_{ijk\ell}E_{k\ell}\underset{\sim}{e}_i \otimes \underset{\sim}{e}_j$$

Note that each contraction reduces the order of the tensor by two.

Thm: Suppose $\underset{\approx}{T}^s$ is symmetric and $\underset{\approx}{S}^a$ is anti- symmetric. Then

$$\mathrm{tr}\,(\underset{\approx}{T}^s \cdot \underset{\approx}{S}^a) = 0 \qquad (1.7\text{-}8)$$

Proof:

$$\mathrm{tr}\,(\underset{\approx}{T}^s \cdot \underset{\approx}{S}^a) = T^s_{ij}S^a_{ji}$$

But $T^s_{ij} = T^s_{ji}$ and $S^a_{ji} = -\,S^a_{ij}$

So $T^s_{ij}S_{ji}{}^a = T^s_{ji}(-S^a_{ij}) = -\,T^s_{ji}S^a_{ij}$

But the only number that equals its negative is the number zero. EOP

Axial Vector

Let $\underset{\approx}{S}^a$ be an anti-symmetric tensor

Then $\underset{\sim}{a}^s$, called the axial vector of $\underset{\approx}{S}^a$, is defined by

$$\underset{\sim}{a}^s = 1/2\; C_{24}C_{35}(\underset{\approx}{\varepsilon} \otimes \underset{\approx}{S}^a) \quad \text{tensor notation}$$

$$a^s_i = 1/2\; \varepsilon_{ijk}S^a_{jk} \quad \text{indicial notation} \qquad (1.7\text{-}9)$$

$$a^s_1 = 1/2\left[\varepsilon_{123}S_{23}{}^a + \varepsilon_{132}S^a_{32}\right]$$

$$= 1/2 \left[S^a_{23} - S^a_{32} \right]$$
$$= S^a_{23}$$

Likewise $a^s_2 = S^a_{31}$

$$a^s_3 = S^a_{12}$$

The inverse relation is obtained by multiplying the indicial form of (1.7-9) by $\varepsilon_{i\ell m}$:

$$\varepsilon_{i\ell m} a^s_i = 1/2\ \varepsilon_{i\ell m}\varepsilon_{ijk} S^a_{jk}$$

Note that $\varepsilon_{i\ell m}\varepsilon_{ijk} = \delta_{\ell j}\delta_{mk} - \delta_{\ell k}\delta_{mj}$

Hence
$$\begin{aligned} \varepsilon_{i\ell m} a^s_i &= 1/2\ (\delta_{\ell j}\delta_{mk} - \delta_{\ell k}\delta_{mj}) S^a_{jk} \\ &= 1/2\ (S^a_{\ell m} - S^a_{m\ell}) \qquad (1.7\text{-}10) \\ &= S^a_{\ell m} \end{aligned}$$

Determinant of a Product of Tensors

Consider the composition $\underset{=}{T} = \underset{=}{R} \cdot \underset{=}{S}$ when substituted in (1.7-4)

$$(\underset{=}{R} \cdot \underset{=}{S} \cdot \underset{\sim}{u}) \cdot \left[(\underset{=}{R} \cdot \underset{=}{S} \cdot \underset{\sim}{v}) \times (\underset{=}{R} \cdot \underset{=}{S} \cdot \underset{\sim}{w}) \right] = \det(\underset{=}{R} \cdot \underset{=}{S}) \left[\underset{\sim}{u} \cdot (\underset{\sim}{v} \times \underset{\sim}{w}) \right].$$

Let

$$\underset{\sim}{u}^* = \underset{=}{S} \cdot \underset{\sim}{u} \qquad \underset{\sim}{v}^* = \underset{=}{S} \cdot \underset{\sim}{v} \qquad \underset{\sim}{w}^* = \underset{=}{S} \cdot \underset{\sim}{w}$$

Then the left side becomes

$$(\underset{=}{R} \cdot \underset{\sim}{u}^*) \cdot \left[(\underset{=}{R} \cdot \underset{\sim}{v}^*) \times (\underset{=}{R} \cdot \underset{\sim}{w}^*) \right] = \det(\underset{=}{R}) \quad \underset{\sim}{u}^* \cdot (\underset{\sim}{v}^* \times \underset{\sim}{w}^*)$$

$$= \det(\underset{=}{R}) \quad \underset{=}{S} \cdot \underset{\sim}{u} \cdot \left[(\underset{=}{S} \cdot \underset{\sim}{v}) \times (\underset{=}{S} \cdot \underset{\sim}{w}) \right]$$

$$= \det(\underset{=}{R}) \det(\underset{=}{S}) \left[\underset{\sim}{u} \cdot (\underset{\sim}{v} \times \underset{\sim}{w}) \right]$$

Thus

$$\det(\underset{=}{R} \cdot \underset{=}{S}) = \det(\underset{=}{R}) \det(\underset{=}{S}) \qquad (1.7\text{-}11)$$

A similar result holds for matrices.

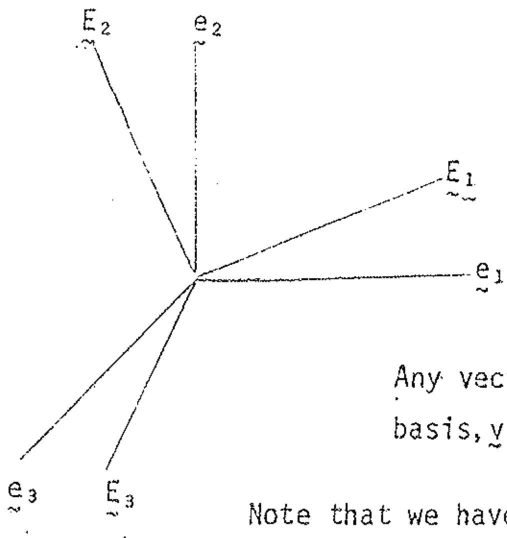
Theorem Involving the Trace

For any three vectors $\underset{\sim}{u}$, $\underset{\sim}{v}$ and $\underset{\sim}{w}$

$$(\underset{=}{A} \cdot \underset{\sim}{u}) \cdot (\underset{\sim}{v} \times \underset{\sim}{w}) + \underset{\sim}{u} \cdot \left[(\underset{=}{A} \cdot \underset{\sim}{v}) \times \underset{\sim}{w} \right] + \underset{\sim}{u} \cdot \left[\underset{\sim}{v} \times (\underset{=}{A} \cdot \underset{\sim}{w}) \right] = \operatorname{tr} \underset{=}{A} \quad \underset{\sim}{u} \cdot (\underset{\sim}{v} \times \underset{\sim}{w})$$

Proof: Expand and show using indicial notation.

Section 8 Transformation of Bases



The figure illustrates two orthonormal right handed bases.

Any vector $\underset{\sim}{v}$ can be represented using either basis, $\underset{\sim}{v} = v_i \underset{\sim}{e}_i = v_{\hat{A}} \underset{\sim}{E}_A$ (1.8-1)

Note that we have represented the vector $\underset{\sim}{v}$ wrt (with respect to) two different bases, namely $\underset{\sim}{e}_i$ and $\underset{\sim}{E}_A$. The components of the vector $\underset{\sim}{v}$, namely v_i and $v_{\hat{A}}$, will be different if the bases $\underset{\sim}{e}_i$ and $\underset{\sim}{E}_A$ are different.

The transformation relation expresses the components of a vector, tensor, etc. wrt a desired basis in terms of the known components in a different basis.

Suppose that the components v_i are given and we want to find $V_{\hat{A}}$. We dot the terms of eq. (1.8-1) on the right wrt $\underset{\sim}{E}_B$.

Then $v_i \underset{\sim}{e}_i \cdot \underset{\sim}{E}_B = v_{\hat{A}} \underset{\sim}{E}_A \cdot \underset{\sim}{E}_B$

$$= v_{\hat{A}} \delta_{AB}$$

$$= v_{\hat{B}}$$

Let $a_{\hat{B}i} = \underset{\sim}{E}_{\grave{B}} \cdot \underset{\sim}{e}_i$ (1.8-2)

Then $v_{\hat{B}} = a_{\hat{B}i} v_i$ (1.8-3)

Now suppose the components $v_{\hat{A}}$ are known.

$$(\underset{\sim}{v} = v_i \underset{\sim}{e}_i) \cdot \underset{\sim}{e}_j = (v_{\hat{A}} E_A) \cdot \underset{\sim}{e}_j$$

$$v_i \delta_{ij} = v_{\hat{A}} a_{\hat{A}j}$$

$$v_j = a_{\hat{A}j} v_{\hat{A}} \qquad (1.8\text{-}4)$$

A transformation relation between the two sets of base vectors can be obtained as follows:

$$v_i \underset{\sim}{e}_i = v_{\hat{A}} \underset{\sim}{E}_A$$

$$= a_{\hat{A}i} v_i \underset{\sim}{E}_A \qquad \forall\ v_i$$

Hence $\underset{\sim}{e}_i = a_{\hat{A}i} \underset{\sim}{E}_A$

Similarly $\underset{\sim}{E}_A = a_{\hat{A}i}\, e_i$ (1.8-5)

To show the orthogonality property of the matrix $a_{\hat{A}i}$, consider $v_i = a_{\hat{A}i} v_A$

$$= a_{\hat{A}i} a_{\hat{A}j} v_j$$

$$= \delta_{ij} v_j \qquad \forall\ v_j$$

Hence

$$a_{\hat{A}i} a_{\hat{A}j} = \delta_{ij} \qquad (1.8\text{-}6)$$

or

$$a^T_{i\hat{A}} a_{\hat{A}j} = \delta_{ij}$$

and it follows that

$$a^T_{i\hat{A}} = (a_{\hat{A}j})^{-1} \qquad (1.8\text{-}7)$$

which satisfies the definition of an orthogonal matrix.

Similarly $a_{\hat{A}i} a_{\hat{B}i} = \delta_{\hat{A}\hat{B}}$ (1.8-8)

Take the determinant of both sides of (1.8-6) to obtain

$$|a_{\hat{A}i}|^2 = |\delta_{ij}| = 1 \quad \text{or} \quad |a_{\hat{A}i}| = \pm 1 \qquad (1.8\text{-}9)$$

For right-handed bases

$$|a_{\hat{A}i}| = +1 \qquad (1.8\text{-}10)$$

and this condition should always be checked when working on a problem.

Perform a similar analysis for a tensor which can be represented in either basis:

$$\underset{=}{T} = T_{ij}\, \underset{\sim}{e}_i \otimes \underset{\sim}{e}_j = T_{\hat{A}\hat{B}}\, \underset{\sim}{E}_A \otimes \underset{\sim}{E}_B \qquad (1.8\text{-}11)$$

Consider

$$\underset{=}{T} \cdot \underset{\sim}{e}_k = T_{ik}\underset{\sim}{e}_i = T_{\hat{A}\hat{B}}\, \underset{\sim}{E}_A\, a_{\hat{B}k}$$

and

$$\underset{\sim}{e}_m \cdot \underset{=}{T} \cdot \underset{\sim}{e}_k = T_{mk} = T_{\hat{A}\hat{B}}\, a_{\hat{B}k}\, a_{\hat{A}m}$$

or

$$T_{ij} = T_{\hat{A}\hat{B}}\, a_{\hat{A}i}\, a_{\hat{B}j} = a^T_{i\hat{A}}\, T_{\hat{A}\hat{B}}\, a_{\hat{B}j} \qquad (1.8\text{-}12)$$

In matrix notation

$$[T] = [a]^T [\hat{T}] [a] \qquad (1.8\text{-}13)$$

The procedure can be performed in reverse to obtain $T_{\hat{A}\hat{B}}$ if the components T_{ij} are known. The result is

$$T_{\hat{A}\hat{B}} = T_{ij}\, a_{\hat{A}i}\, a_{\hat{B}j} \qquad (1.8\text{-}14)$$

or

$$[\hat{T}] = [a][T][a]^T \qquad (1.8\text{-}15)$$

A mixed basis can also be used, i.e.,

$$\underset{=}{T} = T_{i\hat{A}}\, \underset{\sim}{e}_i \otimes \underset{\sim}{E}_A = T_{\hat{A}i}\, \underset{\sim}{E}_A \otimes \underset{\sim}{e}_i \qquad (1.8\text{-}16)$$

so that four sets of components are possible: T_{ij}, $T_{\hat{A}\hat{B}}$, $T_{i\hat{A}}$ and $T_{\hat{A}i}$. It is easily shown that

$$\begin{aligned} T_{i\hat{A}} &= T_{ij}\, a_{\hat{A}j} \\ &= T_{\hat{B}j}\, a_{\hat{B}i}\, a_{\hat{A}j} \\ &= T_{\hat{B}\hat{A}}\, a_{\hat{B}i} \quad \text{etc.} \end{aligned} \qquad (1.8\text{-}17)$$

Many aspects of continuum mechanics involve orthogonal tensors which can also be considered as tensors that describe rotations. Since the components of such tensors are intimately related to matrices such as $a_{\hat{A}i}$, an effort will be made here to show the relationship.

Define a tensor

$$\underset{=}{Q} = \underset{\sim}{E}_i \otimes \underset{\sim}{e}_i = \delta_{ij}\, \underset{\sim}{E}_i \otimes \underset{\sim}{e}_j \qquad (1.8\text{-}18)$$

i.e., the mixed components consist of the Kronecker delta. Utilize (1.8-5) to obtain the following alternative representations:

$$\underset{=}{Q} = a_{\hat{i}j}\, \underset{\sim}{e}_j \otimes \underset{\sim}{e}_i = a^T_{i\hat{j}}\, \underset{\sim}{e}_i \otimes \underset{\sim}{e}_j \qquad (1.8\text{-}19)$$

$$= a_{\hat{j}i}\, \underset{\sim}{E}_i \otimes \underset{\sim}{E}_j = a^T_{i\hat{j}}\, \underset{\sim}{E}_i \otimes \underset{\sim}{E}_j$$

Note that $\underset{=}{Q}$ has the same components with respect to the tensor basis $\underset{\sim}{e}_i \otimes \underset{\sim}{e}_j$ as well as $\underset{\sim}{E}_i \otimes \underset{\sim}{E}_j$.

Consider the operation

$$\underset{=}{Q} \cdot \underset{\sim}{e}_\ell = \underset{\sim}{E}_i \otimes \underset{\sim}{e}_i \cdot \underset{\sim}{e}_\ell = \underset{\sim}{E}_\ell \qquad (1.8\text{-}20)$$

i.e., $\underset{=}{Q}$ "rotates" the basis $\underset{\sim}{e}_\ell$ into the basis $\underset{\sim}{E}_\ell$.

Conversely,

$$\underset{=}{Q}^T \cdot \underset{\sim}{E}_\ell = \underset{\sim}{e}_i \otimes \underset{\sim}{E}_i \cdot \underset{\sim}{E}_\ell = \underset{\sim}{e}_\ell \qquad (1.8\text{-}21)$$

Note that the components of $\underset{=}{Q}^T$ in either of the "pure" bases are the terms $a_{\hat{A}i}$.

This shows that $\underset{\approx}{Q}$ transforms each of the base vectors $\underset{\sim}{e}_\ell$ into the base vectors $\underset{\sim}{E}_\ell$, and $\underset{\approx}{Q}^T$ performs the inverse transformation. Now consider

$$\underset{\approx}{Q} \cdot \underset{\approx}{Q}^T = \underset{\sim}{E}_i \otimes \underset{\sim}{e}_i \cdot (\underset{\sim}{e}_j \otimes \underset{\sim}{E}_j)$$

$$= \delta_{ij} \underset{\sim}{E}_i \otimes \underset{\sim}{E}_j$$

$$= \underset{\approx}{I}$$

Hence $\underset{\approx}{Q}^T = \underset{\approx}{Q}^{-1}$ and $\underset{\approx}{Q}$ is an orthogonal tensor

Suppose we are given $\underset{\sim}{u}$ and $\underset{\sim}{v}$. Define two different vectors by

$$\bar{\underset{\sim}{u}} = \underset{\approx}{Q} \cdot \underset{\sim}{u}$$

$$\bar{\underset{\sim}{v}} = \underset{\approx}{Q} \cdot \underset{\sim}{v} = \underset{\sim}{v} \cdot \underset{\approx}{Q}^T \qquad (1.8\text{-}22)$$

and consider the dot product

$$\bar{\underset{\sim}{v}} \cdot \bar{\underset{\sim}{u}} = (\underset{\sim}{v} \cdot \underset{\approx}{Q}^T) \cdot (\underset{\approx}{Q} \cdot \underset{\sim}{u})$$

$$= \underset{\sim}{v} \cdot (\underset{\approx}{I} \cdot \underset{\sim}{u})$$

$$= \underset{\sim}{v} \cdot \underset{\sim}{u} \qquad (1.8\text{-}23)$$

Equation (1.8-23) shows that the lengths of the vectors are preserved under an orthogonal transformation and that the angle between the vectors is preserved. Since $\underset{\sim}{u}$ and $\underset{\sim}{v}$ are arbitrary, they can represent vectors between any two points in a rigid body. The tensor $\underset{\approx}{Q}$ can then be interpreted as a means of describing rigid body rotation.

From the definition of $\bar{\underset{\sim}{v}}$, it follows that

$$\bar{\underset{\sim}{v}} = \underset{\approx}{Q} \cdot \underset{\sim}{v}$$

$$\bar{v}_i \underset{\sim}{E}_i = \underset{\sim}{E}_i \otimes \underset{\sim}{e}_i \cdot v_k \underset{\sim}{e}_k$$

$$= \underset{\sim}{E}_i v_i \qquad (1.8\text{-}24)$$

or

$$\bar{v}_i = v_i$$

This shows that the components of $\bar{\underset{\sim}{v}}$ wrt the $\underset{\sim}{E}_i$ basis equal the components of the different vector $\underset{\sim}{v}$ wrt the $\underset{\sim}{e}_i$ basis.

Consider a tensor $\underset{\approx}{P}$ whose inverse exists and define a tensor $\overline{\underset{\approx}{T}}$ such that

$$\overline{\underset{\approx}{T}} = \underset{\approx}{P} \cdot \underset{\approx}{T} \cdot \underset{\approx}{P}^{-1} \qquad (1.8\text{-}25)$$

Then $\overline{\underset{\approx}{T}}$ and $\underset{\approx}{T}$ are said to be similar. If $\overline{\underset{\approx}{T}} = \underset{\approx}{T}$ for all choices of $\underset{\approx}{P} = \underset{\approx}{Q}$ where $\underset{\approx}{Q}$ is orthogonal, i.e., for $\underset{\approx}{Q}^{-1} = \underset{\approx}{Q}^T$, then $\underset{\approx}{T}$ is said to be isotropic, i.e., the components of $\underset{\approx}{T}$ are the same in any orthonormal bases. The only isotropic second order tensor is

$$\underset{\approx}{T}^{iso} = \alpha\underset{\approx}{I} \qquad (1.8\text{-}26)$$

Section 9 Eigenvalues and Eigenvectors

Consider real, symmetric tensors and suppose there exists a vector $\underset{\sim}{v}$ such that

$$\underset{\approx}{T} \cdot \underset{\sim}{v} = \lambda \underset{\sim}{v} \tag{1.9-1}$$

Then, if $\underset{\sim}{v}$ does exist it is said to be an eigenvector and λ is an eigenvalue. Note that

$$\underset{\sim}{u} = \frac{\pm \underset{\sim}{v}}{|\underset{\sim}{v}|} \tag{1.9-2}$$

is also an eigenvector of the tensor $\underset{\approx}{T}$.

Thm: All eigenvalues of a real, symmetric tensor are real.

Proof: Let x and y be complex numbers and let $\overline{x}$ indicate the complex conjugate of x. It can be shown that

$$\overline{xy} = \overline{x}\ \overline{y}$$

Hence $\overline{\underset{\approx}{T} \cdot \underset{\sim}{v}} = \overline{\lambda \underset{\sim}{v}}$

$$\overline{\underset{\approx}{T}} \cdot \overline{\underset{\sim}{v}} = \overline{\lambda}\,\overline{\underset{\sim}{v}}$$

Note that since $\underset{\approx}{T}$ is real, $\overline{\underset{\approx}{T}} = \underset{\approx}{T}$.

Then $\quad \underset{\sim}{v} \cdot \underset{\approx}{T} \cdot \overline{\underset{\sim}{v}} = \overline{\lambda}\,\overline{\underset{\sim}{v}} \cdot \underset{\sim}{v}$

$$\overline{\underset{\sim}{v}} \cdot \underset{\approx}{T} \cdot \underset{\sim}{v} = \lambda \overline{\underset{\sim}{v}} \cdot \underset{\sim}{v}$$

Using the fact that $\underset{\approx}{T}$ is symmetric we can subtract corresponding terms in the above two equations and obtain

$$0 = (\overline{\lambda} - \lambda)\overline{\underset{\sim}{v}} \cdot \underset{\sim}{v}$$

Since $\overline{\underset{\sim}{v}} \cdot \underset{\sim}{v}$ is a positive definite quantity the above equation implies that $\overline{\lambda} = \lambda$ and hence λ must be real. EOP

Thm: If λ_1 and λ_2 are two different eigenvalues then the two different eigenvectors $\underset{\sim}{v}_1$ and $\underset{\sim}{v}_2$ are orthogonal.

Proof: $\underset{\sim}{v}_2 \cdot (\underset{\approx}{T} \cdot \underset{\sim}{v}_1 = \lambda_1 \underset{\sim}{v}_1) \Rightarrow \underset{\sim}{v}_2 \cdot \underset{\approx}{T} \cdot \underset{\sim}{v}_1 = \lambda_1 \underset{\sim}{v}_1 \cdot \underset{\sim}{v}_2$

$$\underset{\sim}{v}_1 \cdot (\underset{\approx}{T} \cdot \underset{\sim}{v}_2 = \lambda_2 \underset{\sim}{v}_2) \Rightarrow \underset{\sim}{v}_1 \cdot \underset{\approx}{T} \cdot \underset{\sim}{v}_2 = \lambda_2 \underset{\sim}{v}_1 \cdot \underset{\sim}{v}_2$$

Using the symmetry of $\underset{\approx}{T}$ we can subtract corresponding terms in the above two equations and get

$$0 = (\lambda_1 - \lambda_2)\underset{\sim}{v}_1 \cdot \underset{\sim}{v}_2$$

Since $\lambda_1 \neq \lambda_2$ then $\underset{\sim}{v}_1 \cdot \underset{\sim}{v}_2 = 0$ and $\underset{\sim}{v}_1$ and $\underset{\sim}{v}_2$ must be orthogonal EOP

In general, a second order tensor is associated with three eigenvalues. If we suppose that they are distinct then $\underset{\sim}{v}_1$, $\underset{\sim}{v}_2$, $\underset{\sim}{v}_3$ are mutually orthogonal.

We can construct a new set of eigenvectors

$$\underset{\sim}{\rho}_1 = \frac{\pm \underset{\sim}{v}_1}{|\underset{\sim}{v}_1|} \quad \underset{\sim}{\rho}_2 = \frac{\pm \underset{\sim}{v}_2}{|\underset{\sim}{v}_2|} \quad \underset{\sim}{\rho}_3 = \frac{\pm \underset{\sim}{v}_3}{|\underset{\sim}{v}_3|} \tag{1.9-3}$$

such that $\underset{\sim}{\rho}_1 \times \underset{\sim}{\rho}_2 = \underset{\sim}{\rho}_3$ (right-handed system). Then the basis $\underset{\sim}{\rho}_1, \underset{\sim}{\rho}_2, \underset{\sim}{\rho}_3$ is called the principal basis with the directions of the vectors called the principal directions.

Let $\underset{\sim}{\rho}_1$ be an eigenvector of $\underset{\approx}{T}$.

$$\underset{\approx}{T} \cdot \underset{\sim}{\rho}_1 = \lambda \underset{\sim}{\rho}_1$$

Expanding the left hand side of the above equation in the principal basis

$$\begin{aligned}(T^{\rho}_{ij} \underset{\sim}{\rho}_i \otimes \underset{\sim}{\rho}_j) \cdot \underset{\sim}{\rho}_1 &= T^{\rho}_{ij} \underset{\sim}{\rho}_i \delta_{j1} \\ &= T^{\rho}_{i1} \underset{\sim}{\rho}_i \\ &= T^{\rho}_{11} \underset{\sim}{\rho}_1 + T^{\rho}_{21} \underset{\sim}{\rho}_2 + T^{\rho}_{31} \underset{\sim}{\rho}_3 \\ &= \lambda_1 \underset{\sim}{\rho}_1\end{aligned}$$

Hence $T^{\rho}_{11} = \lambda_1$ and $T^{\rho}_{21} = T^{\rho}_{31} = 0$ and since $\underset{\approx}{T}$ is symmetric $T^{\rho}_{12} = T^{\rho}_{13} = 0$. Using the same procedure for $\underset{\sim}{\ell}_2$ and $\underset{\sim}{\ell}_3$ results in

$$\underset{\approx}{T} = T^{\rho}_{11}\underset{\sim}{\ell}_1 \otimes \underset{\sim}{\ell}_1 + T^{\rho}_{22}\underset{\sim}{\ell}_2 \otimes \underset{\sim}{\ell}_2 + T^{\rho}_{33}\underset{\sim}{\ell}_3 \otimes \underset{\sim}{\ell}_3$$

or

$$T^{\rho}_{ij} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \tag{1.9-4}$$

Note that the components of $\underset{\approx}{T}$ form a diagonal matrix when the principal basis is used but this is not necessarily true for other bases.

Theorem: For a real symmetric tensor, there is no loss in generality if the eigenvectors are assumed to be real.

Proof: Suppose $\underset{\sim}{v}$ is complex and λ is an eigenvalue. Then

$$\underset{\sim}{v} = \underset{\sim}{v}_r + i\,\underset{\sim}{v}_i$$

where $i = \sqrt{-1}$ and both $\underset{\sim}{v}_r$ and $\underset{\sim}{v}_i$ are real. By definition

$$\underset{\approx}{T} \cdot \underset{\sim}{v} = \lambda \underset{\sim}{v}$$

$$\underset{\approx}{T} \cdot (\underset{\sim}{v}_r + i\,\underset{\sim}{v}_i) = \lambda(\underset{\sim}{v}_r + i\,\underset{\sim}{v}_i)$$

Since $\underset{\approx}{T}$ and λ are real ∴

$$\underset{\approx}{T} \cdot \underset{\sim}{v}_r = \lambda \underset{\sim}{v}_r$$

$$\underset{\approx}{T} \cdot \underset{\sim}{v}_i = \lambda \underset{\sim}{v}_i$$

which implies that $\underset{\sim}{v}_i = \alpha\, \underset{\sim}{v}_r$ for any scalar α. Thus

$$\underset{\sim}{v} = (1 + \alpha i)\underset{\sim}{v}_r$$

EOP

Let $\underset{\sim}{P}_A$ be the principal basis of $\underset{=}{T}$, i.e.

$$\underset{=}{T} = T^p_{AB}\, \underset{\sim}{P}_A \otimes \underset{\sim}{P}_B = T_{ij}\, \underset{\sim}{e}_i \otimes \underset{\sim}{e}_j \tag{1.9-5}$$

where

$$T^p_{AB} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \tag{1.9-6}$$

Let

$$a^p_{Ai} = \underset{\sim}{P}_A \cdot \underset{\sim}{e}_i \tag{1.9-7}$$

Then

$$T_{ij} = T^p_{AB}\, a^p_{Ai}\, a^p_{Bj} \qquad T^p_{AB} = T_{ij}\, a^p_{Ai}\, a^p_{Bj} \tag{1.9-8}$$

Consider

$$\underset{=}{T}^2 = \underset{=}{T} \cdot \underset{=}{T} \tag{1.9-9}$$

$$\underset{=}{T}^2 \cdot \underset{\sim}{P}_A = \underset{=}{T} \cdot \underset{=}{T} \cdot \underset{\sim}{P}_A = \underset{=}{T} \cdot \lambda_{(A)} \underset{\sim}{P}_A = \lambda^2_{(A)} \underset{\sim}{P}_A$$

where an index with a bracket means the summation convention is suspended. This result shows that $\underset{\sim}{P}_A$ represents the eigenvectors for $\underset{=}{T}^2$ and the eigenvalues are simply the square of the eigenvalues of $\underset{=}{T}$. A corresponding result holds for other integer powers.

The concept is extended to fractional powers as follows: As an example define $\underset{=}{T}^f$ to be

$$\underset{=}{T}^f = T^{p^f}_{AB}\, E_A \otimes \underset{\sim}{P}_B \tag{1.9-10}$$

where

$$T^{p^f}_{AB} = \begin{bmatrix} \lambda_1^f & 0 & 0 \\ 0 & \lambda_2^f & 0 \\ 0 & 0 & \lambda_3^f \end{bmatrix} \tag{1.9-11}$$

where f is the fractional power. To obtain components in an arbitrary basis, the transformation matrix must be used. Unless all eigenvalues are positive, it is quite possible to have imaginary terms resulting from this operation.

A tensor $\underline{\underline{T}}$ is said to be positive definite if $\underset{\sim}{v} \cdot \underline{\underline{T}} \cdot \underset{\sim}{v}$ is positive for all $\underset{\sim}{v} \neq \underset{\sim}{0}$. Represent $\underset{\sim}{v}$ in terms of the principal basis of $\underline{\underline{T}}$, i.e., let

$$\underset{\sim}{v} = v_A \underset{\sim}{p}_A \tag{1.9-12}$$

Then

$$\underline{\underline{T}} \cdot \underset{\sim}{v} = v_A \underline{\underline{T}} \cdot \underset{\sim}{p}_A = v_A \lambda_{(A)} \underset{\sim}{p}_A \tag{1.9-13}$$

and

$$\underset{\sim}{v} \cdot \underline{\underline{T}} \cdot \underset{\sim}{v} = v_B \underset{\sim}{p}_B \cdot v_A \lambda_{(A)} \underset{\sim}{p}_A \tag{1.9-14}$$

$$= \lambda_{(A)} v_A v_A$$

Since v_A are arbitrary, the only way this product can always be positive is if all eigenvalues of $\underline{\underline{T}}$ are positive.

Section 10 Invariant Properties of Tensors

Consider two orthonormal bases, $\underset{\sim}{E}_A$ and $\underset{\sim}{e}_i$. Then the transformation relation is

$$T_{\hat{A}\hat{B}} = a_{\hat{A}i} a_{\hat{B}j} T_{ij}$$

Now consider tr $(\underset{\approx}{T})$

$$\begin{aligned} \operatorname{tr}(\underset{\approx}{T}) &= T_{\hat{A}\hat{A}} \\ &= a_{\hat{A}i} a_{\hat{A}j} T_{ij} \qquad (1.10\text{-}1) \\ &= \delta_{ij} T_{ij} \\ &= T_{ii} \end{aligned}$$

Hence, the sum of the diagonal components are the same for any basis and tr $(\underset{\approx}{T})$ is called an invariant

If we define

$$\underset{\approx}{T}^2 = \underset{\approx}{T} \cdot \underset{\approx}{T}$$

and $\underset{\approx}{T}^3 = \underset{\approx}{T} \cdot \underset{\approx}{T}^2$

then tr $(\underset{\approx}{T}^2)$, tr $(\underset{\approx}{T}^3)$, etc. are also invariants.

We define the three basic invariants of the tensor $\underset{\approx}{T}$ as

$$\begin{aligned} I_T &= \operatorname{tr}(\underset{\approx}{T}) = T_{ii} \\ II_T &= \operatorname{tr}(\underset{\approx}{T}^2) = T_{ij} T_{ji} \qquad (1.10\text{-}2) \\ III_T &= \operatorname{tr}(\underset{\approx}{T}^3) = T_{ij} T_{jk} T_{ki} \end{aligned}$$

Recall the standard eigenvalue equation

$$\underset{\approx}{T} \cdot \underset{\sim}{v} = \lambda \underset{\sim}{v}$$

or $(\underset{\approx}{T} - \lambda \underset{\approx}{I}) \cdot \underset{\sim}{v} = 0$

For a non-trivial solution to exist for $\underset{\sim}{v}$, a necessary and sufficient condition is that

$$\det(\underset{\approx}{T} - \lambda \underset{\approx}{I}) = 0 \qquad (1.10\text{-}3)$$

Equation (1.10-3) is called the characteristic equation.

From equation (1.5-19) we observed that

$$\varepsilon_{ijk} \varepsilon_{\ell mn} a_{i\ell} a_{jm} a_{kn} = 6 \det [A]$$

Hence $6 \det(\underset{\sim}{T} - \lambda\underset{\sim}{I}) = \varepsilon_{ijk}\varepsilon_{\ell mn}(T_{i\ell} - \lambda\delta_{i\ell})(T_{jm} - \lambda\delta_{jm})$
$(T_{kn}-\lambda\delta_{kn})$

$$= \varepsilon_{ijk}\varepsilon_{\ell mn}\left[T_{i\ell}\, T_{jm}T_{kn}\right] - \lambda\varepsilon_{ijk}\varepsilon_{\ell mn}(\delta_{i\ell}T_{jm}T_{kn} + T_{i\ell}\delta_{jm}T_{kn} + T_{i\ell}T_{jm}\delta_{kn})$$

$$+ \lambda^2\varepsilon_{ijk}\varepsilon_{\ell mn}(\delta_{i\ell}\delta_{jm}T_{kn} + \delta_{i\ell}T_{jm}\delta_{kn} + T_{i\ell}\delta_{jm}\delta_{kn})$$

$$- \lambda^3\, \varepsilon_{ijk}\varepsilon_{\ell mn}\delta_{i\ell}\delta_{jm}\delta_{kn}$$

Examining the above equation term by term we observe that with the use of the Kronecker delta, alternating symbol identity

$$6 \det(\underset{\approx}{T} - \lambda\underset{\approx}{I}) = 6 \det\left[\underset{\approx}{T}\right] - 3\lambda(T_{jj}T_{kk} - T_{ij}T_{ji}) + 3\lambda^2(2T_{kk}) - 6\lambda^3$$

Now define

$$\hat{I}_T = I_T$$

$$\hat{II}_T = \tfrac{1}{2}(II_T - I_T^{\,2}) \qquad (1.10\text{-}4)$$

$$\hat{III}_T = \det\left[\underset{\approx}{T}\right] = 1/6\left[I_T^{\,3} - 3I_T II_T + 2III_T\right]$$

(The last expression for $\hat{III}_T$ will be derived later from the Cayley-Hamilton Thm)

Then the characteristic equation becomes the following

$$\lambda^3 - \hat{I}_T\lambda^2 - \hat{II}_T\lambda - \hat{III}_T = 0 \qquad (1.10\text{-}5)$$

Note that this is a third order polynomial in λ with solutions λ_1, λ_2, λ_3.

When we use the principal basis we can express the invariants as

$$I_T = \lambda_1 + \lambda_2 + \lambda_3$$

$$II_T = \lambda_1^2 + \lambda_2^2 + \lambda_3^2$$

$$III_T = \lambda_1^3 + \lambda_2^3 + \lambda_3^3 \qquad (1.10\text{-}6)$$

$$\text{Hence} \quad \hat{I}_T = \lambda_1 + \lambda_2 + \lambda_3$$

$$\hat{II}_T = -(\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1) \qquad (1.10\text{-}7)$$

$$\hat{III}_T = \lambda_1\lambda_2\lambda_3$$

Cayley - Hamilton Theorem

A tensor (matrix) satisfies its own characteristic equation, i.e.

$$\underset{\approx}{T}^3 - \hat{I}_T\underset{\approx}{T}^2 - \hat{II}_T\underset{\approx}{T} - \hat{III}_T\underset{\approx}{I} = \underset{\approx}{0} \qquad (1.10\text{-}8) \; *$$

Taking the trace of each term gives

$$III_T - \hat{I}_T II_T - \hat{II}_T I_T - 3\hat{III}_T = 0$$

If we solve for $\hat{III}_T$ in the above equation and substitute for the values $\hat{I}_T$ and $\hat{II}_T$ we see that it agrees with the expression for $\hat{III}_T$ in (1.10-4)

Multiply each term in (1.10-8) by $\underset{\approx}{T}$

$$\underset{\approx}{T}^4 - \hat{I}_T\underset{\approx}{T}^3 - \hat{II}_T\underset{\approx}{T}^2 - \hat{III}_T\underset{\approx}{T} = \underset{\approx}{0}$$

If we take the trace of each term we observe that tr $(\underset{\approx}{T}^4)$ is a function of I_T, II_T and III_T

$$\text{tr}\,(\underset{\approx}{T}^4) = f(I_T, II_T, III_T) \qquad (1.10\text{-}9)$$

The above equation implies that there are only three independent invariants of a second order tensor.

If we multiply each term in (1.10-8) by $\underset{\approx}{T}^{-1}$ we can solve for $\underset{\approx}{T}^{-1}$ as follows:

$$\underset{\approx}{T}^{-1} = \frac{\underset{\approx}{T}^2 - \hat{I}_T\underset{\approx}{T} - \hat{II}_T\underset{\approx}{I}}{\hat{III}_T} \qquad (1.10\text{-}10)$$

where $\hat{III}_T$ can be replaced by det $[\underset{\approx}{T}]$. Thus, $\underset{\approx}{T}^{-1}$ is defined only if det $[\underset{\approx}{T}]$ is not zero.

* This theorem is easily proven if $\underset{\approx}{T}$ is expressed in terms of its principal basis.

We wish to solve for the eigenvectors $\underset{\sim}{v}$ corresponding to the equation

$$(\underset{\sim}{T} - \lambda \underset{\sim}{I}) \cdot \underset{\sim}{v} = \underset{\sim}{0}$$

or

$$(T_{ij} - \lambda\delta_{ij}) \cdot v_j = 0 \quad \text{(indicial notation)}$$

Expanding we obtain

$$\begin{aligned} (T_{11} - \lambda)v_1 + T_{12}v_2 + T_{13}v_3 &= 0 \\ T_{21}v_1 + (T_{22} - \lambda)v_2 + T_{23}v_3 &= 0 \\ T_{31}v_1 + T_{32}v_2 + (T_{33} - \lambda)v_3 &= 0 \end{aligned} \qquad (1.10\text{-}11)$$

For a solution to exist for these three equations

$$\begin{vmatrix} T_{11}-\lambda & T_{12} & T_{13} \\ T_{21} & T_{22}-\lambda & T_{23} \\ T_{31} & T_{32} & T_{33}-\lambda \end{vmatrix} = 0 \qquad (1.10\text{-}12)$$

If λ_1, λ_2 and λ_3 are distinct then we can solve for $\underset{\sim}{v}_1, \underset{\sim}{v}_2$ and $\underset{\sim}{v}_3$ for each eigenvalue, that is, with $\lambda=\lambda_1$, obtain $\underset{\sim}{v}=\underset{\sim}{v}^{(1)}$, with $\lambda=\lambda_2$ solve for $\underset{\sim}{v}=\underset{\sim}{v}^{(2)}$ and with $\lambda=\lambda_3$ solve for $\underset{\sim}{v}=\underset{\sim}{v}^{(3)}$. Normalize and change signs if necessary to obtain an orthonormal basis.

If $\lambda_1 = \lambda_2$ and λ_3 is distinct, then we can obtain $\underset{\sim}{v}^{(3)}$ (the eigenvector associated with λ_3). For $\underset{\sim}{v}^{(1)}$ and $\underset{\sim}{v}^{(2)}$ the equations are the same since $\lambda_1 = \lambda_2$ so a degree of arbitrariness can be used.

The components of the vector $\underset{\sim}{v}^{(1)}$ can be chosen so that $\underset{\sim}{v}^{(1)}$ is any vector perpendicular to $\underset{\sim}{v}^{(3)}$. Then $\underset{\sim}{v}^{(2)}$ can be chosen so that the three vectors are orthonormal.

If all the λ's are the same then any direction represents an eigenvector and any 3 orthogonal vectors will be solutions to the equation.

Section 11 Mohr's Circle

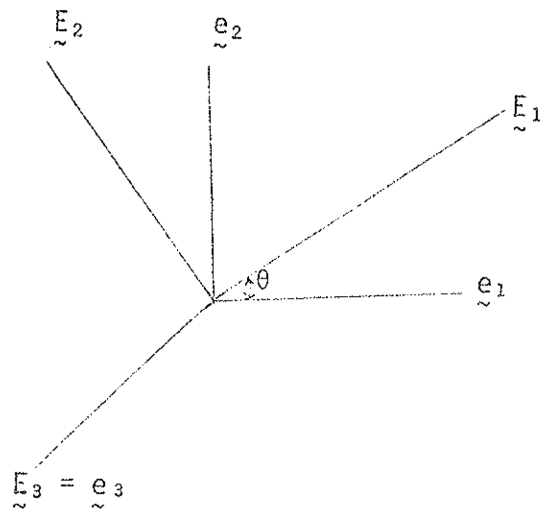
Given a set of components in an orthonormal system and given that the $\underset{\sim}{e}_3$ direction is a principal direction

$$T_{ij} = \begin{bmatrix} T_{11} & T_{12} & 0 \\ T_{21} & T_{22} & 0 \\ 0 & 0 & T_{33} \end{bmatrix}$$

We can then construct the transformation matrix

$$a_{\hat{A}i} = \underset{\sim}{E}_A \cdot \underset{\sim}{e}_i$$

that corresponds to the figure below.



$$a_{\hat{A}i} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Observe that

$$\det |a_{\hat{A}i}| = \cos^2\theta - (-\sin^2\theta) = 1$$

and that

$$a_{\hat{A}i} a_{\hat{A}j} = \delta_{ij}$$

We want to perform a coordinate transformation such that the following holds

$$T_{\hat{A}\hat{B}} = a_{\hat{A}i}a_{\hat{B}j}T_{ij}$$
$$= a_{\hat{A}1}a_{\hat{B}1}T_{11} + a_{\hat{A}2}a_{\hat{B}2}T_{12} + \cdots$$

Note that $T_{13} = T_{23} = T_{31} = T_{32} = 0$ and that $T_{12} = T_{21}$

Then

$$T_{\hat{1}\hat{1}} = (a_{\hat{1}1})^2T_{11} + 2a_{\hat{1}1}a_{\hat{1}2}T_{12} + (a_{\hat{1}2})^2T_{22}$$
$$T_{\hat{1}\hat{2}} = a_{\hat{1}1}a_{\hat{2}1}T_{11} + a_{\hat{1}2}a_{\hat{2}1}T_{21} + a_{\hat{1}1}a_{\hat{2}2}T_{12} + a_{\hat{1}2}a_{\hat{2}2}T_{22}$$
$$T_{\hat{2}\hat{2}} = (a_{\hat{2}1})^2T_{11} + 2a_{\hat{2}1}a_{\hat{2}2}T_{12} + (a_{\hat{2}2})^2T_{22}$$

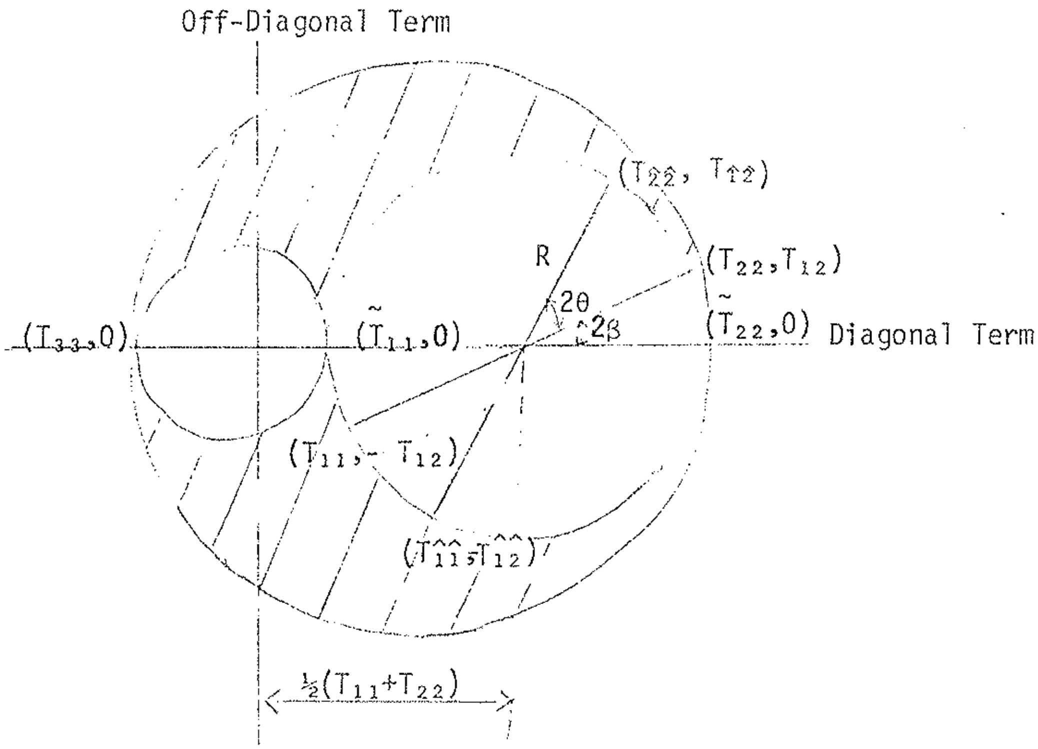
or

$$T_{\hat{1}\hat{1}} = \cos^2\theta T_{11} + 2\cos\theta\sin\theta T_{12} + \sin^2\theta T_{22}$$
$$= \frac{(1+\cos 2\theta)}{2}T_{11} + \sin 2\theta T_{12} + \frac{(1-\cos 2\theta)}{2}T_{22}$$
$$= \tfrac{1}{2}(T_{11} + T_{22}) - \frac{(T_{22} - T_{11})}{2}\cos 2\theta + \sin 2\theta T_{12}$$

$$T_{\hat{1}\hat{2}} = -\cos\theta\sin\theta T_{11} + (\cos^2\theta - \sin^2\theta)T_{12} + \sin\theta\cos\theta T_{22}$$
$$= \frac{(T_{22}-T_{11})}{2}\sin 2\theta + T_{12}\cos 2\theta$$

$$T_{\hat{2}\hat{2}} = \sin^2\theta T_{11} - 2\cos\theta\sin\theta T_{12} + \cos^2\theta T_{22}$$
$$= \frac{(1-\cos 2\theta)}{2}T_{11} - \sin 2\theta T_{12} + \frac{(1+\cos 2\theta)}{2}T_{22}$$
$$= \tfrac{1}{2}(T_{11} + T_{22}) + \left(\frac{T_{22}-T_{11}}{2}\right)\cos 2\theta - \sin 2\theta T_{12}$$

Let's now construct the Mohr's Circle as shown below



From this figure it follows that

$$T_{11} = \tfrac{1}{2}(T_{11} + T_{22}) - R\cos 2\beta$$
$$T_{12} = R\sin 2\beta$$
$$T_{22} = \tfrac{1}{2}(T_{11} + T_{22}) + R\cos 2\beta$$

Hence

$$T_{\hat{1}\hat{1}} = \tfrac{1}{2}(T_{11} + T_{22}) - R\cos(2\theta + 2\beta)$$

Recall that $\cos(2\theta + 2\beta) = \cos 2\theta \cos 2\beta - \sin 2\theta \sin 2\beta$ and that

$R\cos 2\beta = \dfrac{T_{22} - T_{11}}{2}$ and $R\sin 2\beta = T_{12}$

$$T_{\hat{1}\hat{1}} = \tfrac{1}{2}(T_{11} + T_{22}) - R\cos 2\theta \cos 2\beta + R\sin\theta \sin 2\beta$$
$$= \tfrac{1}{2}(T_{11} + T_{22}) - \frac{(T_{22} - T_{11})}{2}\cos 2\theta + T_{12}\sin 2\theta$$

$$T_{\hat{1}\hat{2}} = R\sin(2\theta + 2\beta)$$

$$= R\sin 2\theta\cos 2\beta + R\cos 2\theta\sin 2\beta$$

$$= \frac{T_{22}-T_{11}}{2}\sin 2\theta + T_{12}\cos 2\theta$$

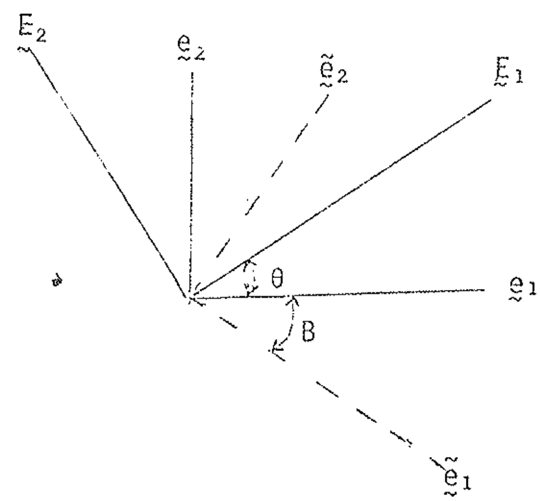
$$T_{\hat{2}\hat{2}} = \tfrac{1}{2}(T_{11} + T_{22}) + R\cos(2\theta + 2\beta)$$

$$= \tfrac{1}{2}(T_{11} + T_{22}) + R\cos 2\theta\cos 2\beta - R\sin 2\theta\sin 2\beta$$

$$= \tfrac{1}{2}(T_{11} + T_{22}) + \frac{(T_{22}-T_{11})}{2}\cos 2\theta - T_{12}\sin 2\theta$$

Observe that the above equations are identical to the transformation relations for $T_{\hat{1}\hat{1}}$, $T_{\hat{1}\hat{2}}$, and $T_{\hat{2}\hat{2}}$ derived previously.

We have shown how Mohr's circle relates the tensor components in one orthogonal basis, say $\underset{\sim}{E}_i$, to those components in another orthogonal basis, say $\underset{\sim}{e}_i$. If we denote the principal basis by $\tilde{\underset{\sim}{e}}_i$ the relationship between the bases can be seen below. The corresponding principal values $\tilde{T}_{11}$ and $\tilde{T}_{22}$ are also shown on Mohr's circle.



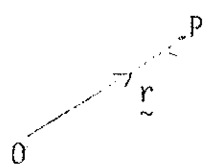
Since Mohr's circle corresponds to a rotation about a principal axis, additional Mohr's circles can be introduced. The smallest circle in the Mohr's figure corresponds to a rotation about $\tilde{\underset{\sim}{e}}_2$, the largest circle corresponds to a rotation about $\tilde{\underset{\sim}{e}}_1$, and the other circle corresponds to a rotation about $\underset{\sim}{e}_3$. Also, for any two orthogonal directions, t and s, the components T_{tt} and T_{ts} must lie in the shaded region of the Mohr's circle (without proof).

Finally, note that Mohr's circle holds for any symmetric tensor.

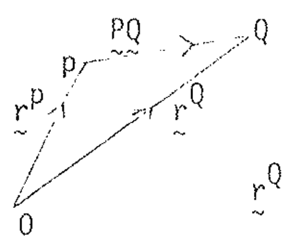
Section 12 Coordinate Systems

Consider the space of elementary geometry. The space consists of elements called points and a directed line segment between any two points is called a geometrical vector. The rules of geometrical vectors satisfy the rules of a Euclidian vector space.

Choose a reference point 0 and consider the vector $\underset{\sim}{r}$ (position vector) to some point P



The set of all geometrical vectors can be constructed from the set of all position vectors as shown below



$$\underset{\sim}{r}^Q = \underset{\sim}{r}^P + \underset{\sim\sim}{PQ}$$

The set of all position vectors (or the set of all geometrical vectors plus the reference point 0) is known as a Euclidean point space.

Points in a geometrical space can also be defined with the use of coordinates x_i, i.e. with each point, a unique set of numbers (x_1, x_2, x_3) is assigned. This unique assignment is called a one-to-one correspondence.

Now if a function F (tensor of any order) varies from one point to the next, then

$$F = F(\underset{\sim}{r}) \tag{1.12-1}$$

or

$$F = F(x_i)$$

Either of these notations is acceptable, i.e., use

(a) A set of base vectors and a reference point

or

(b) A coordinate system

Note that it is sometimes convenient to have the basis change from point to point. If the base vectors are chosen ∋

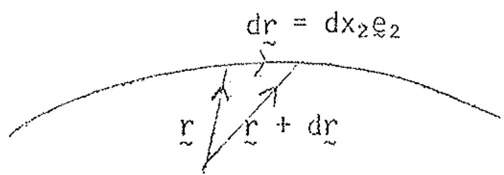
$$\underset{\sim}{e}_i = \frac{\partial \underset{\sim}{r}}{\partial x_i} \tag{1.12-2}$$

or

$$d\underset{\sim}{r} = dx_i \, \underset{\sim}{e}_i$$

then the basis is a natural basis of the coordinate system. This implies that for curvilinear coordinates, $\underset{\sim}{e}_i$ may not be dimensionless nor orthonormal.

The second part of (1.12-2) is illustrated below where the curve is obtained by making x_1 and x_3 constant and letting x_2 vary



For rectangular cartesian coordinates (RCC) the base vectors do not change from point to point and (1.12-2) is satisfied with the origin at the point O and

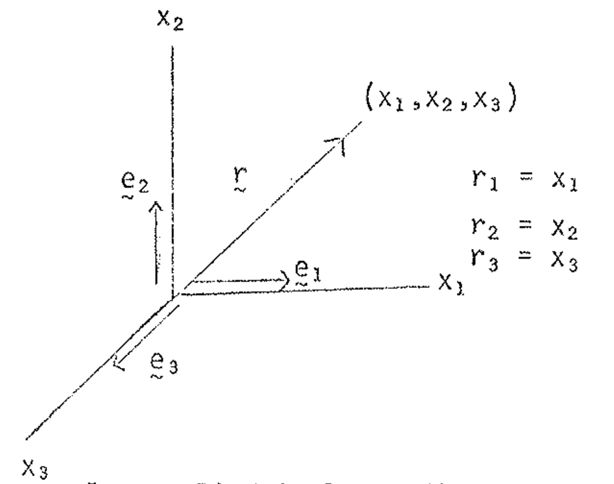
$$\begin{aligned} \underset{\sim}{r} &= r_i \underset{\sim}{e}_i \\ & x_i \underset{\sim}{e}_i \end{aligned} \tag{1.12-3}$$

Observe that

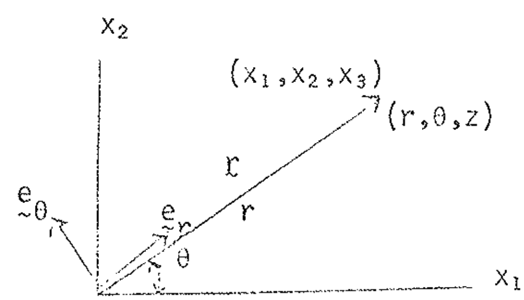
$$\frac{\partial \underset{\sim}{r}}{\partial x_j} = \frac{\partial x_i}{\partial x_j} \underset{\sim}{e}_i + x_i \frac{\partial \underset{\sim}{e}_i}{\partial x_j}$$

but $\dfrac{\partial \underset{\sim}{e}_i}{\partial x_j} = 0$ so $$\frac{\partial \underset{\sim}{r}}{\partial x_j} = \delta_{ij}\underset{\sim}{e}_i$$
$$= \underset{\sim}{e}_j$$

The rectangular cartesian coordinate system is shown below



In a cylindrical coordinate system we have



The components of $\underset{\sim}{r}$ in the cylindrical coordinate system are

$$r_{\hat{A}} = (r,o,z)$$

or

$$\underset{\sim}{r} = r\,\underset{\sim}{e}_r + z\underset{\sim}{e}_z$$

This represents the more typical case of where the components of $\underset{\sim}{r}$ are not identical to the coordinates. For this specific case, the θ dependence of $\underset{\sim}{r}$ is contained in the base vector $\underset{\sim}{e}_r$, i.e., $\underset{\sim}{e}_r = \underset{\sim}{e}_r(\theta)$. Also $\underset{\sim}{e}_\theta = \underset{\sim}{e}_\theta(\theta)$ so that derivatives of these base vectors with respect to θ are not zero. Thus an RCC system represents a very special case with unique and simple relations that must be replaced when any other coordinate system is used.

Section 13 Gradient, Divergence, Curl and Related Operators

Consider a scalar function of position $\phi(\underset{\sim}{r})$ (or $\phi(x_i)$). Then the gradient of ϕ (grad ϕ, $\phi\nabla$) is defined to be a vector ∋

$$d\phi = (\phi\nabla)\cdot d\underset{\sim}{r} \tag{1.13-1}$$

$$\forall\ d\underset{\sim}{r} \text{ at a point } \underset{\sim}{r}$$

Let $\phi\nabla = \underset{\sim}{w} = w_i \underset{\sim}{e}_i$

Then $\phi\nabla\cdot d\underset{\sim}{r} = (w_i \underset{\sim}{e}_i)\cdot dx_j \underset{\sim}{e}_j = w_i dx_i$

Expanding $d\phi$ as a total diffential

$$d\phi = \frac{\partial\phi}{\partial x_i}dx_i = \phi,_i\, dx_i \tag{1.13-2}$$

$$= \frac{\partial\phi}{\partial x_1}dx_1 + \frac{\partial\phi}{\partial x_2}dx_2 + \frac{\partial\phi}{\partial x_3}dx_3$$

Since $d\underset{\sim}{r}$ is arbitrary in (1.13-1) we can combine (1.13-1) and (1.13-2) to get

$$\phi\nabla = \phi,_i\, \underset{\sim}{e}_i \tag{1.13-3}$$

Similarly, grad $\underset{\sim}{v}$ ($\underset{\sim}{v}\nabla$, $\underset{\sim}{v}\otimes\nabla$) is defined to be a second order tensor*

∋

$$d\underset{\sim}{v} = (\underset{\sim}{v}\nabla)\cdot d\underset{\sim}{r} \tag{1.13-4}$$

$$\forall\ d\underset{\sim}{r} \text{ at } \underset{\sim}{r}$$

Let $\underset{\sim}{v}\nabla = T_{ij}\underset{\sim}{e}_i \otimes \underset{\sim}{e}_j$

Then

$$(\underset{\sim}{v}\nabla)\cdot d\underset{\sim}{r} = (T_{ij}\underset{\sim}{e}_i \otimes \underset{\sim}{e}_j)\cdot dx_j \underset{\sim}{e}_j$$

$$= T_{ij}dx_j \underset{\sim}{e}_i$$

* As an alternative approach, let $\phi = \underset{\sim}{u}\cdot\underset{\sim}{v}$ where $\underset{\sim}{u}$ is a constant but arbitrary vector. Then

$d\phi = (\phi\nabla)\cdot d\underset{\sim}{r} = \underset{\sim}{u}\cdot d\underset{\sim}{v} = \underset{\sim}{u}\cdot(\underset{\sim}{v}\nabla)\cdot d\underset{\sim}{r}$

Since $d\phi$ is a scalar, $\underset{\sim}{v}\nabla$ must be a second order tensor.

Additional note: Suppose $\phi = C$, a constant, defines a surface to which $\underset{\sim}{r}$ is restricted, i.e., $d\underset{\sim}{r}$ is a vector tangent to the surface. Then $d\phi = \phi\nabla\cdot d\underset{\sim}{r} = 0$ which implies $\phi\nabla$ is a vector normal to the surface.

Using the definition of $d\underset{\sim}{v}$ for a right-handed Cartesian coordinate system gives

$$d\underset{\sim}{v} = d(v_i \underset{\sim}{e}_i)$$
$$= dv_i (\underset{\sim}{e}_i)$$
$$= \frac{\partial v_i}{\partial x_j} dx_j \underset{\sim}{e}_i$$
$$= v_{i,j} dx_j \underset{\sim}{e}_i$$

Equating terms yields

$$T_{ij} = V_{i,j}$$

Substituting back into the definition of $\underset{\sim}{v}\nabla$ gives

$$\underset{\sim}{v}\nabla = v_{i,j} \underset{\sim}{e}_i \otimes \underset{\sim}{e}_j \qquad (1.13\text{-}5)$$

An alternative definition of grad $\underset{\sim}{v}$ is $\nabla \underset{\sim}{v}$ where

$$d\underset{\sim}{v} = d\underset{\sim}{r} \cdot (\nabla \underset{\sim}{v})$$

Performing the same type of operations as before we obtain

$$\nabla \underset{\sim}{v} = v_{j,i} \underset{\sim}{e}_i \otimes \underset{\sim}{e}_j$$
$$= (\underset{\sim}{v}\nabla)^T \qquad (1.13\text{-}6)$$

We can also define the gradient of a second order tensor (grad $\underset{\approx}{T}$, $\underset{\approx}{T}\nabla$, $\underset{\approx}{T} \otimes \nabla$),

$$d\underset{\approx}{T} = (\underset{\approx}{T}\nabla) \cdot d\underset{\sim}{r} \qquad (1.13\text{-}7)$$

Using similar procedures as before we can show that

$$\underset{\approx}{T}\nabla = T_{ij,k} \underset{\sim}{e}_i \otimes \underset{\sim}{e}_j \otimes \underset{\sim}{e}_k \qquad (1.13\text{-}8)$$

Divergences

The divergence of a vector can be expressed in terms of the contraction operation and the gradient as

$$
\begin{aligned}
\text{Div } \underset{\sim}{v} &= C_{12}(\underset{\sim}{v}\nabla) \\
&= C_{12}(v_{i,j}\underset{\sim}{e}_i \otimes \underset{\sim}{e}_j) \\
&= v_{i,j}\delta_{ij} \\
&= v_{i,i} && (1.13\text{-}9) \\
&= \text{tr } (\underset{\sim}{v}\nabla) \\
&= \underset{\sim}{v}\cdot\nabla \\
&= \frac{\partial v_i}{\partial x_i}
\end{aligned}
$$

The divergence of a second order tensor is defined $\ni$

$$
\begin{aligned}
\text{Div } \underset{\approx}{T} &= C_{23}(\underset{\approx}{T}\nabla) \\
&= C_{23}(T_{ij,k}\underset{\approx}{e}_i \otimes \underset{\approx}{e}_j \otimes e_k) \\
&= T_{ij,k}\underset{\sim}{e}\delta_{jk} \\
&= T_{ij,j}\underset{\sim}{e}_i && (1.13\text{-}10) \\
&= \underset{\approx}{T}\cdot\nabla
\end{aligned}
$$

Curl of a Vector

The curl is defined $\ni$

$$
\begin{aligned}
\text{Curl } \underset{\sim}{v} &= \underset{\sim}{v} \times \nabla \\
&= \underset{\sim}{v}_{,j} \times \underset{\sim}{e}_j && (1.13\text{-}11) \\
&= (v_i\underset{\sim}{e}_i)_{,j} \times \underset{\sim}{e}_j
\end{aligned}
$$

$$= v_{i,j}\underset{\sim}{e}_i \times \underset{\sim}{e}_j$$

$$= \varepsilon_{ijk} v_{i,j}\underset{\sim}{e}_k$$

Note that $(\nabla \times \underset{\sim}{v}) = -(\underset{\sim}{v} \times \nabla)$ (1.13-12)

Divergence of the Curl

The divergence of the curl of a vector can be expressed as

$$\begin{aligned} \text{Div (curl } \underset{\sim}{v}) &= (\text{curl } \underset{\sim}{v}) \cdot \nabla \\ &= (\varepsilon_{ijk} v_{i,j}\underset{\sim}{e}_k)\cdot\nabla \\ &= (\varepsilon_{ijk} v_{i,j}\underset{\sim}{e}_k)_{,\ell}\cdot\underset{\sim}{e}_\ell \\ &= \varepsilon_{ijk} v_{i,j\ell}\delta_{k\ell} \\ &= \varepsilon_{ijk} v_{i,jk} \end{aligned} \tag{1.13-13}$$

Since $v_{i,jk} = v_{i,kj}$ and $\varepsilon_{ijk} = -\varepsilon_{ikj}$ the sum of terms in the last equation is zero so that

$$\text{Div (curl } \underset{\sim}{v}) = 0$$

The curl of a second order tensor can be defined as follows:

$$\begin{aligned} \underset{\approx}{T} \times \nabla &= (\underset{\approx}{T})_{,k} \times \underset{\sim}{e}_k \\ &= (T_{ij}\underset{\sim}{e}_i \otimes \underset{\sim}{e}_j)_{,k} \times \underset{\sim}{e}_k \\ &= (T_{ij}\underset{\sim}{e}_i \otimes \varepsilon_{jk\ell}\underset{\sim}{e}_\ell)_{,k} \\ &= \varepsilon_{jk\ell} T_{ij,k}\underset{\sim}{e}_i \otimes \underset{\sim}{e}_\ell \end{aligned} \tag{1.13-14}$$

The divergence of the gradient of a scalar function $\phi(x_i)$ is defined as

$$\text{Div (grad } \phi) = (\phi\nabla)\cdot\nabla$$

$$= \phi\nabla^2 \qquad \text{(called the Laplacian)}$$

$$= (\phi_{,i}\underset{\sim}{e}_i)_{,j}\cdot\underset{\sim}{e}_j \qquad (1.13\text{-}15)$$

$$= \phi_{,ij}\delta_{ij}$$

$$= \phi_{,ii}$$

$$= \frac{\partial^2\phi}{\partial x_1{}^2} + \frac{\partial^2\phi}{\partial x_2{}^2} + \frac{\partial^2\phi}{\partial x_3{}^2}$$

The divergence of the gradient of a vector can be defined in similar terms as

$$\text{Div (grad } \underset{\sim}{v}) = (\underset{\sim}{v}\nabla)\cdot\nabla$$

$$= \underset{\sim}{v}\nabla^2 \qquad (1.13\text{-}16)$$

$$= v_{i,jj}\underset{\sim}{e}_i$$

Two Definitions of Gradient

A. W. MARRIS[1]

A STUDY of the literature of vectorial mechanics discloses that two different definitions of gradient are used and implied. This dichotomy can cause confusion and perhaps errors, so that it seems worthwhile to present the situation in a Note. We will designate the definitions by A and B.

Definition A will be taken from Brandt [1],[2] pp. 182 and 189. The gradient of the vector $\mathbf{v}$ is the dyadic point function

$$\nabla\mathbf{v} \equiv \operatorname{grad}\mathbf{v} = \mathbf{i}\frac{\partial\mathbf{v}}{\partial x} + \mathbf{j}\frac{\partial\mathbf{v}}{\partial y} + \mathbf{k}\frac{\partial\mathbf{v}}{\partial z} \tag{1}$$

where $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ are Cartesian unit vectors.

The definition (1) may be written in index notation as

$$\operatorname{grad}\mathbf{v} = \operatorname{grad}(v_\alpha\mathbf{e}^\alpha) = \mathbf{e}^\beta\frac{\partial}{\partial x^\beta}(v_\alpha\mathbf{e}^\alpha) = v_{\alpha;\beta}\mathbf{e}^\beta\mathbf{e}^\alpha = v_{\beta;\alpha}\mathbf{e}^\alpha\mathbf{e}^\beta \tag{2}$$

where $\mathbf{e}^\alpha\mathbf{e}^\beta$ are base vectors and the semicolon denotes covariant differentiation. Just as the gradient of a scalar (zero-order tensor) is a vector (first-order tensor) so the gradient of a vector (first-order tensor) is a second-order tensor.

The gradient of a second-order tensor is given by

$$\operatorname{grad}\mathbf{t} = \operatorname{grad}(t_{\alpha\beta}\mathbf{e}^\alpha\mathbf{e}^\beta) = \mathbf{e}^\gamma\frac{\partial}{\partial x^\gamma}(t_{\alpha\beta}\mathbf{e}^\alpha\mathbf{e}^\beta) = t_{\alpha\beta;\gamma}\mathbf{e}^\gamma\mathbf{e}^\alpha\mathbf{e}^\beta = t_{\beta\gamma;\alpha}\mathbf{e}^\alpha\mathbf{e}^\beta\mathbf{e}^\gamma \tag{3}$$

and so on, for higher-order tensors.

In terms of definition A, one has for the spatial increments of $\mathbf{v}$ and $\mathbf{t}$

$$d\mathbf{v} = d\mathbf{r}\cdot\operatorname{grad}\mathbf{v} = v_{\beta;\alpha}dx^\alpha\mathbf{e}^\beta \tag{4}$$

$$d\mathbf{t} = d\mathbf{r}\cdot\operatorname{grad}\mathbf{t} = t_{\beta\gamma;\alpha}dx^\alpha\mathbf{e}^\beta\mathbf{e}^\gamma \tag{5}$$

In particular, material acceleration written in spatial coordinates is given by

$$\dot{\mathbf{v}} = \frac{\partial\mathbf{v}}{\partial t} + \mathbf{v}\cdot\operatorname{grad}\mathbf{v} \tag{6}$$

while the divergence of a second-order tensor is given as

$$\operatorname{div}\mathbf{t} = \operatorname{grad}\cdot\mathbf{t} = \mathbf{e}^\gamma\cdot\frac{\partial}{\partial x^\gamma}(t_{\alpha\beta}\mathbf{e}^\alpha\mathbf{e}^\beta) = g^{\alpha\gamma}t_{\alpha\beta;\gamma}\mathbf{e}^\beta \tag{7}$$

$g^{\alpha\gamma}$ being the contravariant metric of the coordinate system. In Cartesian coordinates,

$$\operatorname{div}\mathbf{t} = t_{\alpha\beta,\alpha}\mathbf{e}^\beta \tag{8}$$

the comma denoting partial differentiation.

This definition A stems presumably from the elementary use of the operator ∇ operating on a scalar point function ϕ; thus

$$\nabla\phi = \operatorname{grad}\phi = \mathbf{e}^\alpha\frac{\partial\phi}{\partial x^\alpha} \tag{9}$$

is unnatural on at least two counts. Traditionally, the student of elementary calculus is indoctrinated into placing the increment *after* the differential rather than before it as in (4) and (5). Thus he tends to learn

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[2] Numbers in brackets designate References at end of Note.

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References

1 Brandt, L., *Vector and Tensor Analysis*, Wiley, New York, 1947.

2 Eringen, A. C., *Nonlinear Theory of Continuous Media*, McGraw-Hill, New York, 1962.

3 Truesdell, C., and Noll, W., "Nonlinear Field Theories of Mechanics," *Handbuch der Physik*, Flügge, S., ed., Vol. 111/3, 1965.

4 Truesdell, C., "Principles of Continuum Mechanics," *Colloquium Lectures in Pure and Applied Science*, No. 5, Socony Mobil Oil Company, Field Research Laboratory, Dallas, Texas, Feb. 1960, p. 30, equation (41).

5 Truesdell, C., and Toupin, R., "The Classical Field Theories," *Handbuch der Physik*, Flügge, S., ed., Vol. 111/1, 1960, p. 543.

$$y = f(x)$$
$$dy = f'(x)dx$$

rather than

$$dy = dxf'(x)$$

which is analogous to the form (4) and (5). The second difficulty is perhaps more significant. Quantities are *called* gradients which are *not* gradients according to definition A. In classical finite elasticity, we consider the deformation from a reference state specified by material coordinates X^1, X^2, X^3 to a configuration state specified by spatial coordinates x^1, x^2, x^3. The matrix

$$\{F^\alpha{}_A\} = \left\{\frac{\partial x^\alpha}{\partial X^A}\right\} = \{x,_A{}^\alpha\} \qquad (10)$$

is termed the *deformation gradient*. In point of fact, it is the *transpose* of the gradient of a vector having Cartesian components x^α with respect to Cartesian coordinates X^A. It is altogether appropriate, in teaching fluid mechanics, to take the material derivative of the deformation gradient to obtain the "velocity gradient" (Eringen [2], p. 69):

$$\dot{F}_A{}^\alpha = v_{;\beta}{}^\alpha x,_A{}^\beta \qquad (11)$$

so that

$$\dot{F}_A{}^\alpha\big|_{t=0} = v_{;\beta}{}^\alpha \qquad (12)$$

But then one has to apologize to the students and tell them that the velocity gradient $v_{;\beta}{}^\alpha$ is not the velocity gradient according to definition A, but rather, its transpose.

This precedent from finite elasticity is compatible with definition B of gradient, which we take from Truesdell and Noll [3] p. 15. In the notation of this treatise, "the gradient of a vector field $\mathbf{v} = \mathbf{v}(\mathbf{x})$ is the second-order tensor field, which will be denoted by $\nabla \mathbf{v} = \nabla \mathbf{v}(\mathbf{x})$, defined as the linear transformation that assigns to a vector $\mathbf{a}$ the vector given by the following rule:

$$(\nabla \mathbf{v})\mathbf{a} = \lim_{s\to 0} \frac{\mathbf{v}(\mathbf{x} + s\mathbf{a}) - \mathbf{v}(\mathbf{x})}{s} \qquad (13)''$$

In the notation of equation (2), this definition yields

$$\text{grad } \mathbf{v} = \text{grad } (v_\alpha \mathbf{e}^\alpha) = \left\{\frac{\partial}{\partial x^\beta}[v_\alpha \mathbf{e}^\alpha]\right\}\mathbf{e}^\beta = v_{\alpha;\beta}\mathbf{e}^\alpha\mathbf{e}^\beta \qquad (14)$$

The gradient of a second-order tensor according to definition B is given by

$$\text{grad } \mathbf{t} = \text{grad } (t_{\alpha\beta}\mathbf{e}^\alpha\mathbf{e}^\beta) = \left\{\frac{\partial}{\partial x^\gamma}(t_{\alpha\beta}\mathbf{e}^\alpha\mathbf{e}^\beta)\right\}\mathbf{e}^\gamma = t_{\alpha\beta;\gamma}\mathbf{e}^\alpha\mathbf{e}^\beta\mathbf{e}^\gamma \qquad (15)$$

and so on, for higher-order tensors, reference [3], p. 39.

On the basis of definition B, the results corresponding to (4)–(7) are

$$d\mathbf{v} = \text{grad } \mathbf{v}\cdot d\mathbf{r} = v_{\alpha;\beta}dx^\beta \mathbf{e}^\alpha \qquad (16)$$

$$d\mathbf{t} = \text{grad } \mathbf{t}\cdot d\mathbf{r} = t_{\alpha\beta;\gamma}dx^\gamma \mathbf{e}^\alpha\mathbf{e}^\beta \qquad (17)$$

$$\dot{\mathbf{v}} = \frac{\partial \mathbf{v}}{\partial t} + \text{grad } \mathbf{v}\cdot\mathbf{v} \qquad (18)$$

while

$$\text{div } \mathbf{t} = \text{grad}\cdot\mathbf{t} = \left\{\frac{\partial}{\partial x^\gamma}(t_{\alpha\beta}\mathbf{e}^\alpha\mathbf{e}^\beta)\right\}\cdot\mathbf{e}^\gamma = g^{\beta\gamma}t_{\alpha\beta;\gamma}\mathbf{e}^\alpha \qquad (19)$$

or in Cartesian coordinates,

$$\text{div } \mathbf{t} = t_{\alpha\beta;\beta}\mathbf{e}^\alpha \qquad (20)$$

The definition B of gradient, in requiring that the vector increment follow the derivative rather than precede it, is by way of being analogous to the elementary notation. Definition B is the gradient that is implied when we call $F_A{}^\alpha$ the deformation gradient in finite elasticity.

While the writer prefers definition B, i.e., to define gradient on the premise that the base vectors corresponding to the coordinates with respect to which the gradient is taken must be set on the extreme right [as opposed to the extreme left, as in definition A], one's purpose here is not so much to cite one definition in preference to the other but rather to indicate pitfalls than can result from indiscriminate use of both definitions, and plead that one definition may be adopted universally. To the claim that definition A is the Gibbsian definition and universal, one must reply that definition B is implied when we speak of displacement gradients in elasticity.

Definitions A and B give the same value, of course, for the gradient of a scalar quantity. Since the gradient of a vector according to definition A is the transpose of the gradient according to definition B, the rotation associated with definition A is in the opposite sense to that associated with definition B. In particular, if, as is usually done, we define

$$\text{curl } \mathbf{v} = (\text{grad})_A \times \mathbf{v} \qquad (21)$$

then

$$\text{curl } \mathbf{v} = \mathbf{v} \times (\text{grad})_B \qquad (22)$$

The difference must undoubtedly be felt by one studying elastic rotations based on a deformation gradient normally defined according to definition B.

The divergence of a vector is the same on both definitions. The divergence of a second-order tensor will only be the same on the two definitions provided the tensor is symmetric in a neighborhood including the point. Writing $\{\text{div } \mathbf{t}\}_A$ and $\{\text{div } \mathbf{t}\}_B$ for the divergence of an unsymmetric second-order tensor based on definitions A and B, and given by equations (7) and (19), respectively, we have

$$\{\text{div } \mathbf{t}\}_A - \{\text{div } \mathbf{t}\}_B = \text{curl } \mathbf{t}_x \qquad (23)$$

where curl is defined as in (21), and $\mathbf{t}_x$ is the *vector* of the unsymmetric tensor $\mathbf{t}$; i.e., if $\mathbf{t} = t_{\alpha\beta}\mathbf{e}^\alpha\mathbf{e}^\beta$, then $\mathbf{t}_x = t_{\alpha\beta}\mathbf{e}^\alpha \times \mathbf{e}^\beta$.

In continuum mechanics and fluid mechanics, we require the divergence of the stress tensor for substitution in the Euler-Cauchy equations of motion of a continuous medium

$$\rho\dot{\mathbf{v}} = \rho\mathbf{b} + \text{div } \mathbf{t} \qquad (24)$$

where ρ is the density, and $\mathbf{b}$ is the body force per unit mass. In the usual problems met with, the stress tensor is symmetric and no ambiguity occurs in the interpretation of div $\mathbf{t}$. However, instances are possible in which the stress tensor will not be symmetric; for example, the case of certain electrically polarized materials in which an applied electromagnetic field can produce body couples and couple stresses.

If we define the stress so that in a stress component $t_{\alpha\beta}$ the first index α represents the component of the normal to the surface element upon which the stress acts, and the second index represents the direction of the component force; i.e., so that the stress vector $\mathbf{t}_\nu$ on the surface element whose normal is $\mathbf{n}$ is given by

$$\mathbf{t}_\nu = \mathbf{n}\cdot\mathbf{t}$$
$$t_{\nu\beta} = n^\alpha t_{\alpha\beta} \qquad (25)$$

then equation (24) presumes the definition (7) or (8) for div $\mathbf{t}$ based on definition A of gradient. (Eringen [2] p. 104, equation 32.10.)

But, here again, there is considerable ambiguity in the literature. Truesdell [4, 5] defines the stress tensor to be such that in the component $t_{\alpha\beta}$ the first index α represents the direction of the component force and the second index represents the direction of the component of the normal $\mathbf{n}$ to the surface upon which the stress acts. (This definition illuminates the stress tensor $\mathbf{t}$ as the linear transformation relating the *stress vector* $\mathbf{t}_\nu$ whose components are the stresses on the surface element whose normal is $\mathbf{n}$; i.e., $\mathbf{t}_\nu = \mathbf{t}\cdot\mathbf{n}$.) In terms of the latter definition of the stress, equation (24) requires the form (19) or (20) for div $\mathbf{t}$ based on definition B of the gradient.

Section 14. Gauss' Divergence Theorem

Several theorems, of which the Divergence Theorem is one, are related through elementary mathematical operations. Of this set of theorems, we pick one of the easiest to prove in detail, and then derive the others sequentially.

Green's Theorem:

Let R be a bounded region with piecewise smooth, orientable surface ∂R. Let $\phi(\underset{\sim}{r})$ be a scalar function defined over $R + \partial R$ of class C^1. Then

$$\int_R \phi \nabla \, dV = \int_{\partial R} \phi \, \underset{\sim}{n} \, ds \qquad (1.14\text{-}1)$$

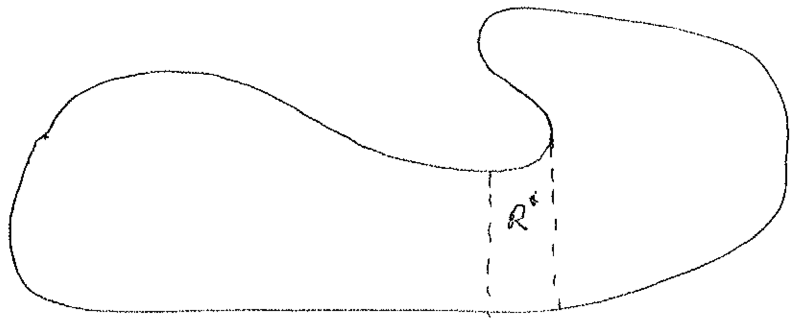
where
dV = volume element
ds = area element on surface
$\underset{\sim}{n}$ = outward directed unit normal vector.

Proof: Separate the region R into a sum of regions each of which is simple with a smooth surface. The word simple implies that the boundary surface can be described with single-valued functions. Introduce a rectangular cartesian coordinate system. If it can be shown that

$$\int_R \frac{\partial \phi}{\partial x_3} dV = \int_{\partial R} \phi \, n_3 \, ds \qquad (1.14\text{-}2)$$

it follows that a similar relation holds for the other two components and hence (1.14-1) holds.

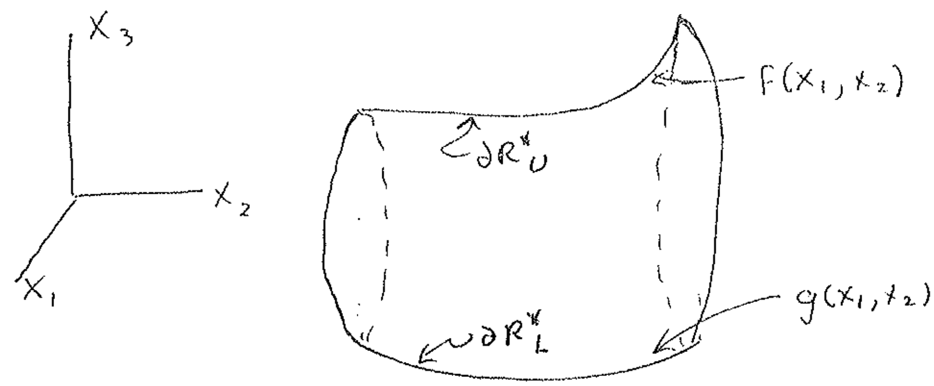
Consider the region shown below:



The subregion R^* is simple. This means that it is possible to define the upper ∂R_U^* and lower ∂R_L^* surfaces by means of single valued functions as follows:

$$\begin{aligned} \partial R_u^* &: \quad x_3 = f(x_1, x_2) \\ \partial R_L^* &: \quad x_3 = g(x_1, x_2) \end{aligned} \qquad (1.14\text{-}3)$$

as shown in the sketch below:



On the lateral surface, $n_3 = 0$ so there is no contribution to the surface integral of (1.14-2). Consider the first term in (1.14-2) for the subregion:

$$\int_{R^*} \frac{\partial \varphi}{\partial x_3}\, dv = \iint \int_g^f \frac{\partial \varphi}{\partial x_3}\, dx_3\, dx_1\, dx_2 \qquad (1.14\text{-}4)$$

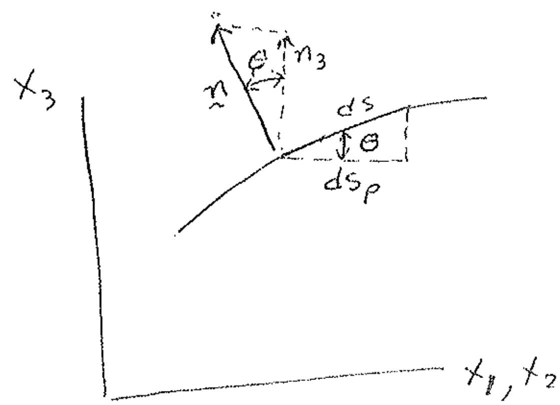
$$= \iint \langle \varphi[x_1, x_2, f(x_1,x_2)] - \varphi[x_1, x_2, g(x_1,x_2)] \rangle\, dx_1 dx_2$$

Now consider the second term in (1.14-2)

$$\int_{\partial R^*} \phi\, n_3\, ds = \int_{\partial R_U^*} \phi\, n_3\, ds + \int_{\partial R_L^*} \phi\, n_3\, ds \tag{1.14-5}$$

An area element in the $x_1 - x_2$ plane is

$$\begin{aligned} dS_p &= dx_1\, dx_2 \\ &= ds \cos\theta \end{aligned} \tag{1.14-6}$$



as shown in the sketch. Also, on the upper and lower surfaces

$$\begin{aligned} \cos\theta &= n_3 \quad \text{on } \partial R_U^* \\ \cos\theta &= -n_3 \quad \text{on } \partial R_L^* \end{aligned} \tag{1.14-7}$$

It follows from (1.14-4), (1.14-5), (1.14-6) and (1.14-7) that

$$\begin{aligned} \int_{\partial R^*} \phi\, n_3\, ds &= \int_{\partial R_U^*} \phi \cos\theta\, ds - \int_{\partial R_L^*} \phi \cos\theta\, ds \\ &= \iint \phi[x_1, x_2, f(x_1, x_2)]\, dx_1\, dx_2 - \iint \phi[x_1, x_2, g(x_1, x_2)]\, dx_1\, dx_2 \\ &= \int_{R^*} \frac{\partial \phi}{\partial x_3}\, dV \end{aligned} \tag{1.14-8}$$

Thus (1.14-2) holds for any subregion, and consequently for the complete region R. A similar construction can be utilized for the other two components which demonstrates that (1.14-1) holds.

EOP

Gradient Theorem

Let R be a bounded region whose boundary, ∂R, is a piecewise, smooth, orientable surface. Let $\underset{\sim}{v}(\underset{\sim}{r})$ be defined over $R + \partial R$ and of class C^1. Then

$$\int_R \underset{\sim}{v} \nabla \, dV = \int_{\partial R} \underset{\sim}{v} \otimes \underset{\sim}{n} \, ds \qquad (1.14\text{-}9)$$

Proof:

In (1.14-1) let

$$\phi = \underset{\sim}{u} \cdot \underset{\sim}{v} \qquad (1.14\text{-}10)$$

where $\underset{\sim}{u}$ is a constant but arbitrary vector. Then (1.14-1) yields

$$\underset{\sim}{u} \cdot \int_R \underset{\sim}{v} \nabla \, dV = \underset{\sim}{u} \cdot \int_{\partial R} \underset{\sim}{v} \otimes \underset{\sim}{n} \, ds$$

and, since $\underset{\sim}{u}$ is arbitrary, (1.14-9) follows. EOP.

Gauss' Divergence Theorem

This theorem is merely a corolary of the previous theorem. If the trace of each term in (1.14-9) is taken, then

$$\int_R \underset{\sim}{v} \cdot \nabla \, dV = \int_{\partial R} \underset{\sim}{v} \cdot \underset{\sim}{n} \, ds \qquad (1.14\text{-}11)$$

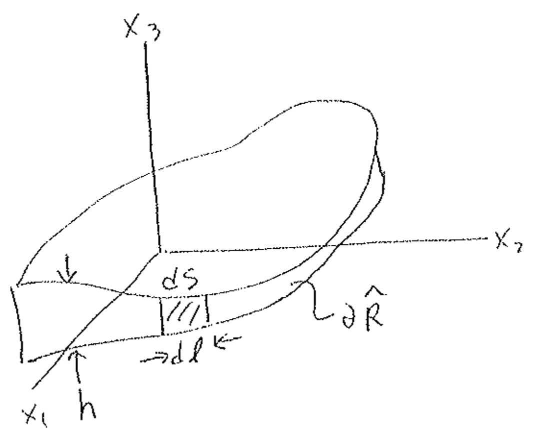
which is the Divergence Theorem for vectors.

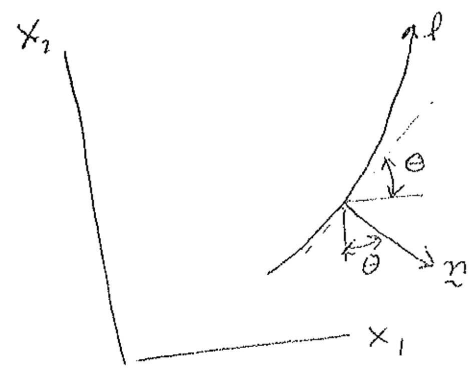
Section 15 Other Integral Relations

Green's Theorem in Two Dimensions

Suppose $\phi \equiv \phi(x_1, x_2)$. Consider a body with a lateral, $\partial\hat{R}$, surface ($n_3 = 0$) whose dimension in the n_3-direction is $h(x_1, x_2)$. The contribution to the surface integral of (1.14-1) from the upper and lower surfaces will cancel out so that (1.14-1) reduces to

$$\int_{R_2} \phi \nabla \, dV = \int_{\partial\hat{R}} \phi \, \underset{\sim}{n} \, ds \qquad (1.15\text{-}1)$$





$$dV = h \, dx_1 dx_2 \qquad ds = h \, dl \qquad (1.15\text{-}1)$$

Eq. (1.15-1) becomes

$$\iint_{R_2} (\phi_{,1} \underset{\sim}{e}_1 + \phi_{,2} \underset{\sim}{e}_2) \, h \, dx_1 dx_2 = \int_{\partial\hat{R}} \phi (n_1 \underset{\sim}{e}_1 + n_2 \underset{\sim}{e}_2) \, h \, dl \qquad (1.15\text{-}2)$$

But

$$dl \cos\theta = dx_1 \qquad n \cos\theta = -n_2 \qquad n = 1 \qquad (1.15\text{-}3)$$
$$dl \sin\theta = dx_2 \qquad n \sin\theta = n_1$$

and

$$\underset{\sim}{n} \, dl = \underset{\sim}{e}_1 dx_2 - \underset{\sim}{e}_2 dx_1 \qquad (1.15\text{-}4)$$

Green's theorem in two dimensions becomes

$$\iint_{R_2} (\phi_{,1}\, \underset{\sim}{e}_1 + \phi_{,2}\, \underset{\sim}{e}_2)\, h\, dx_1 dx_2 = \oint_{\partial \hat{R}} \phi h (\underset{\sim}{e}_1 dx_2 - \underset{\sim}{e}_2 dx_1) \qquad (1.15\text{-}5)$$

or if h is constant

$$\iint_{R_2} (\phi_{,1}\, \underset{\sim}{e}_1 + \phi_{,2}\, \underset{\sim}{e}_2)\, dx_1 dx_2 = \oint_{\partial \hat{R}} \phi (\underset{\sim}{e}_1 dx_2 - \underset{\sim}{e}_2 dx_1) \qquad (1.15\text{-}6)$$

Divergence Theorem in Two Dimensions

As before, let $\phi = \underset{\sim}{u} \cdot \underset{\sim}{v}$ for constant but arbitrary $\underset{\sim}{u}$. Then (1.15-6) yields the Divergence Theorem in two dimensions (after some vector manipulations):

$$\iint_{R_2} \left(\frac{\partial V_1}{\partial x_1} + \frac{\partial V_2}{\partial x_2}\right) dx_1 dx_2 = \oint_{\partial R_2} (V_1 dx_2 - V_2 dx_1) \qquad (1.15\text{-}7)$$

In one dimension, the divergence theorem reduces to

$$\int_a^b \frac{\partial V}{\partial x} dx = V \Big|_a^b \qquad (1.15\text{-}8)$$

Three-Dimensional Form of Integration by Parts

Let $\phi = fg$. Then

$$\phi \nabla = g(f\nabla) + f(g\nabla) \qquad (1.15\text{-}9)$$

The use of Green's Theorem (1.14-1) yields

$$\int_R g(f\nabla)\, dV = \int_{\partial R} fg \underset{\sim}{n}\, ds - \int_R f(g\nabla)\, dV \qquad (1.15\text{-}10)$$

Gradient Theorem for Tensors

In (1.14-9), let $\underset{\sim}{v} = \underset{\sim}{u} \cdot \underset{\approx}{T}$ for an arbitrary but constant tensor. The result is

$$\int_R \underset{\approx}{T} \, \nabla \, dV = \int_{\partial R} \underset{\approx}{T} \otimes \underset{\sim}{n} \, ds \qquad (1.15\text{-}11)$$

Divergence Theorem for Tensors

Apply the contraction operator C_{23} to (1.15-11) to obtain

$$\int_R \underset{\approx}{T} \cdot \nabla \, dV = \int_{\partial R} \underset{\approx}{T} \cdot \underset{\sim}{n} \, ds \qquad (1.15\text{-}12)$$

Let $\underset{\approx a}{R}$ denote a tensor of arbitrary order. A procedure completely analogous to that used to derive (1.15-9) and (1.15-10) can be followed to derive the gradient and divergence theorems for arbitrary tensors:

$$\int_R \underset{\approx a}{R} \, \nabla \, dV = \int_{\partial R} \underset{\approx}{R} \otimes \underset{\sim}{n} \, ds \qquad (1.15\text{-}13)$$

$$\int_R \underset{\approx a}{R} \cdot \nabla \, dV = \int_{\partial R} \underset{\approx a}{R} \cdot \underset{\sim}{n} \, ds \qquad (1.15\text{-}14)$$

Curl Theorem

Let

$$\underset{\sim}{v} = \underset{\sim}{u} \times \underset{\sim}{w} \qquad (1.15\text{-}15)$$

where $\underset{\sim}{w}$ is a vector function and $\underset{\sim}{u}$ is constant but arbitrary. Then

$$(\underset{\sim}{u} \times \underset{\sim}{w}) \cdot \underset{\sim}{n} = \underset{\sim}{u} \cdot (\underset{\sim}{w} \times \underset{\sim}{n})$$

$$(\underset{\sim}{u} \times \underset{\sim}{w}) \otimes \nabla = \underset{\sim}{u} \times (\underset{\sim}{w} \otimes \nabla) - \underset{\sim}{w} \times (\underset{\sim}{u} \otimes \nabla)$$

$$= \underset{\sim}{u} \times (\underset{\sim}{w} \otimes \nabla)$$

since $\underset{\sim}{u} \otimes \nabla = \underset{\approx}{0}$. It can be shown that

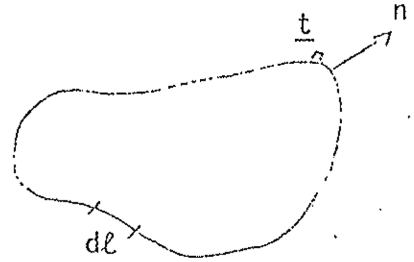
$$(\underset{\sim}{u} \times \underset{\sim}{w}) \cdot \nabla = \underset{\sim}{u} \cdot (\underset{\sim}{w} \times \nabla) \qquad (1.15\text{-}16)$$

Substitute the above results in the divergence theorem, and use the fact that $\underset{\sim}{u}$ is arbitrary to obtain

$$\int_R \underset{\sim}{w} \times \nabla \, dV = \int_{\partial R} \underset{\sim}{w} \times \underset{\sim}{n} \, ds \qquad (1.15\text{-}17)$$

Stoke's Theorem

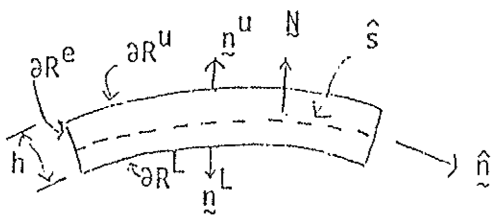
(Top view)



Consider the illustration below where the dotted line represents the surface of interest

In the top view, $\underset{\sim}{t}$ represents the tangent vector at a point, $\underset{\sim}{n}$ represents the normal vector at the point and $d\ell$ is a differential element of length.

(Section View)



In the lower figure the upper, lower, and edge surface boundaries are represented by ∂R^u, ∂R^L, and ∂R^e, respectively. The unit normals to the upper and lower surfaces are, respectively, $\underset{\sim}{n}^u$ and $\underset{\sim}{n}^L$. The surface shown by the dotted line is symbolized by $\hat{S}$ which has a unit normal $\underset{\sim}{N}$.

Note that as the thickness of the body, h, approaches zero

$$\left.\begin{aligned} \partial R^u &= \partial R^L = \hat{s} \\ \underset{\sim}{n}^u &= \underset{\sim}{N} \\ \underset{\sim}{n}^L &= -\underset{\sim}{N} \end{aligned}\right\} \quad \text{as } h \to 0$$

Start with the curl and divergence theorems:

$$\int_R \underset{\sim}{w} \times \nabla \, dV = \int_{\partial R} \underset{\sim}{w} \times \underset{\sim}{n} \, ds \qquad (a)$$

$$\int_R \underset{\sim}{w} \cdot \nabla \, dV = \int_{\partial R} \underset{\sim}{w} \cdot \underset{\sim}{n} \, ds \qquad (b)$$

Let the region degenerate to a surface. The corresponding theorems are:

$$\int_{\hat{S}} \underset{\sim}{w} \times \nabla \, d\hat{S} = \oint_{\partial \hat{S}} \underset{\sim}{w} \times \hat{\underset{\sim}{n}} \, d\ell \qquad (a^*)$$

$$\int_{\hat{S}} \underset{\sim}{w} \cdot \nabla \, d\hat{S} = \oint_{\partial \hat{S}} \underset{\sim}{w} \cdot \hat{\underset{\sim}{n}} \, d\ell \qquad (b^*)$$

In (a^*) let $\underset{\sim}{w} = \underset{\sim}{N}$. $\underset{\sim}{N} \times \hat{\underset{\sim}{n}} = \underset{\sim}{t}$ $\quad \oint_{\partial \hat{S}} \underset{\sim}{t} \, d\ell = 0$

The integral relation (a^*) holds $\forall \hat{S}$. $\Rightarrow$ $\underset{\sim}{N} \times \nabla = \underset{\sim}{0}$ everywhere.

In (b^*) let $\underset{\sim}{w} = \underset{\sim}{v} \times \underset{\sim}{N}$

$$\underset{\sim}{w} \cdot \nabla = (\underset{\sim}{v} \times \underset{\sim}{N}) \cdot \nabla = \underset{\sim}{v} \cdot \underbrace{(\underset{\sim}{N} \times \nabla)}_{\underset{\sim}{0}} - (\underset{\sim}{v} \times \nabla) \cdot \underset{\sim}{N}$$

$$\underset{\sim}{w} \cdot \hat{\underset{\sim}{n}} = (\underset{\sim}{v} \times \underset{\sim}{N}) \cdot \hat{\underset{\sim}{n}} = \underset{\sim}{v} \cdot (\underset{\sim}{N} \times \hat{\underset{\sim}{n}}) = \underset{\sim}{v} \cdot \underset{\sim}{t}$$

$\underset{\sim}{t} \, d\ell = d\underset{\sim}{r}$ on the boundary.

Therefore

$$-\int_{\hat{S}} (\underset{\sim}{v} \times \nabla) \cdot \underset{\sim}{N} \, d\hat{S} = \oint_{\partial \hat{S}} \underset{\sim}{v} \cdot d\underset{\sim}{r}$$

which is Stokes' Theorem.

Applications of Stoke's Theorem:

(a) Compatibility — restrictions on strain and rotation that ensure a displacement field is continuous.

Derivation. If $\underset{\sim}{u}$ is continuous, then

$$\oint d\underset{\sim}{u} = \underset{\sim}{0} \quad \text{for all closed loops}$$

$$= \oint \underset{\sim}{u}\nabla \cdot d\underset{\sim}{r}$$

$$= -\int_{\hat{S}} [(\underset{\sim}{u}\nabla \times \nabla)] \cdot \underset{\sim}{N} \, d\hat{S} \qquad \forall \hat{S}$$

$\Rightarrow \quad \underset{\sim}{u}\nabla \times \nabla = \underset{\approx}{0} \quad \Rightarrow$ second derivatives of $\underset{\sim}{u}$ are interchangeable $\Rightarrow \underset{\sim}{u} \in C^2$

But

$$\underset{\sim}{u}\nabla = \underset{\approx}{e} + \underset{\approx}{\omega} \qquad \underset{\approx}{e} \text{ - strain}, \quad \underset{\approx}{\omega} \text{ - rotation}.$$

$\therefore$

$$(\underset{\approx}{e} + \underset{\approx}{\omega}) \times \nabla = \underset{\approx}{0}$$

Direct Notation

$$\underset{\approx}{\varepsilon} : [(\underset{\approx}{e}+\underset{\approx}{\omega})\nabla : \underset{\approx}{\varepsilon}] = \underset{\sim}{0}$$

Convert to indicial notation:

$$[(e_{ij} + \omega_{ij})\, \underset{\sim}{e}_i \otimes \underset{\sim}{e}_j]_{,k} \times \underset{\sim}{e}_k = \underset{\approx}{0}$$

$$\varepsilon_{jkl}(e_{ij,k} + \omega_{ij,k})\, \underset{\sim}{e}_i \otimes \underset{\sim}{e}_l = \underset{\approx}{0}$$

or

$$\varepsilon_{jkl}(e_{ij,k} + \omega_{ij,k}) = 0$$

Let

$i=1,\ l=1 \qquad \varepsilon_{231}(e_{12,3} + \omega_{12,3} - e_{13,2} - \omega_{13,2}) = 0$

or

$$e_{12,3} + \omega_{12,3} - e_{13,2} - \omega_{13,2} = 0 \qquad (i)$$

$i=1\ l=2$ and $i=1, l=3$ yield the same eqn.

Just permutate indices to obtain a total of <u>3</u> independent equations of compatibility:

$$e_{23,1} + \omega_{23,1} - e_{21,3} - \omega_{21,3} = 0 \qquad (ii)$$

$$e_{31,2} + \omega_{31,2} - e_{32,1} - \omega_{32,1} = 0 \qquad (iii)$$

Note: The conventional compatibility equations are obtained by eliminating ω_{ij}. Result is 3 independent eqs. involving second derivatives of strain.

(b) Major Symmetry of Elasticity Tensor

Suppose the internal energy U is a function of the strain components $e_{11}, e_{22}, e_{33}, e_{12}, e_{21}, \ldots$

Define a generalized vector $\underset{\sim}{r} = e_{11} \underset{\sim}{e}_1 + \cdots + e_{23} \underset{\sim}{e}_9$ (9-space) Then $d\underset{\sim}{r} = de_{11} \underset{\sim}{e}_1 + \cdots + de_{23} \underset{\sim}{e}_9$

The internal energy is assumed to be continuous for all paths in strain space:

$$\oint dU = 0$$

$$= \oint U \nabla_r \cdot d\underset{\sim}{r}$$

$$= -\int (U \nabla_r \times \nabla_r) \cdot \underset{\sim}{N}\, dS \qquad \text{Stoke's Th'm.}$$

must hold $\forall$ closed loops. Therefore

$$U \nabla_r \times \nabla_r = \underset{\approx}{0}$$

which implies the mixed partial derivatives are equal:

$$\frac{\partial^2 U}{\partial e_{11} \partial e_{22}} = \frac{\partial^2 U}{\partial e_{22} \partial e_{11}} \quad \text{etc.}$$

or

$$\frac{\partial^2 U}{\partial e_{ij} \partial e_{kl}} = \frac{\partial^2 U}{\partial e_{kl} \partial e_{ij}} \qquad *$$

Suppose $U = \frac{1}{2} C_{ijkl}\, e_{ij}\, e_{kl}$

Then $*$ implies

$$C_{ijkl} = C_{klij}$$

i.e., $\underset{\equiv}{C}$ must satisfy major symmetry.

Chapter 2 Kinematics of a Continuous Medium

Section 1 Concepts of a Continuous Medium

"Medium" is analogous to "matter" and can be either a solid, liquid, or a gas. A "continuous medium" (continuum) refers to a substance in which a simplifying assumption is made that the molecular structure of the matter can be disregarded.

We desire to use mathematics in the sense of continuous functions, derivatives, etc. Presumably, these functions describe the behavior of the medium in some "averaged" sense. Thus, it is assumed that matter can be considered distributed continuously (or mapped) onto a region in a Euclidean point space. Then the governing functions and their derivatives are considered continuous in the region except for a finite number of surfaces of discontinuity (welded surfaces, etc.), and the material is considered to be a continuous medium.

There are two common ways of describing the motion of a material:

a. Referential Description (Material, Lagrangian)

In this description the independent variables are the time, t, and the position, $\underset{\sim}{R}$, of a material point (particle), X, at some reference time (usually $t = 0$). This is the method usually used in solid mechanics.

b. Spatial Description (Eulerian)

For this case, the independent variables are the time, t, and the position, $\underset{\sim}{r}$, of a material point at time t. This approach is used in fluid mechanics.

The coordinates of the point $\underset{\sim}{R}$ are denoted by X_A and the coordinates of the point $\underset{\sim}{r}$ are denoted by x_i.

If an RCC system is used in each case then

$$R_A = X_A \quad \text{and} \quad r_i = x_i \tag{2.1-1}$$

Note that the RCC systems assumed in (2.1-1) may be different for the two cases.

As a historical note, (from Truesdell), Euler is credited with introducing material coordinates and d'Alembert pioneered the use of spatial coordinates.

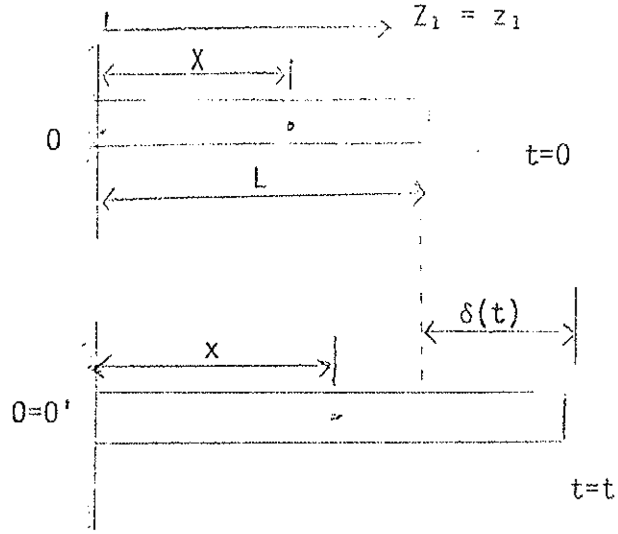
A sketch illustrating the interpretation of $\underset{\sim}{R}$ and $\underset{\sim}{r}$ is shown below. Note that the two configurations do not have to be described with the same origin.

The motion of a material point, X, can be described as

$$\underset{\sim}{r} = \underset{\sim}{r}(\underset{\sim}{R}, t) \tag{2.1-2}$$

where $\underset{\sim}{r}$ and $\underset{\sim}{R}$ are the positions of the same material point.

Example 1:



Z_1, z_1 - coordinate systems

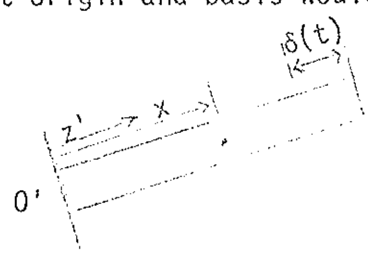
X, x - coordinates of a point

Hence, the position in terms of spatial coordinates is

$$x = X\left(1 + \frac{\delta(t)}{L}\right)$$

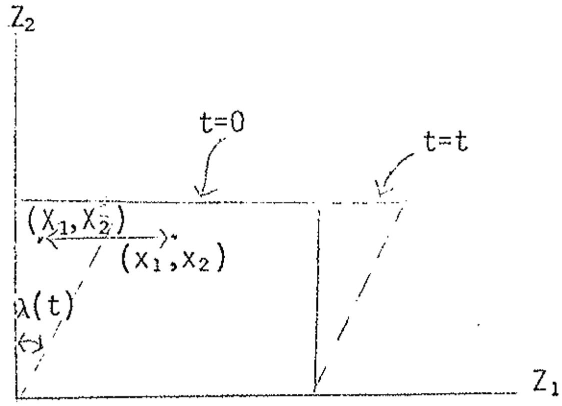
or $\underset{\sim}{r} = \underset{\sim}{r}(\underset{\sim}{R}, t)$

Note that since only elongation of the bar occured we used the same origin and basis in the spatial description as in the referential description. However, if rotation and translation of the bar occurs then a different origin and basis would be desirable.



In the figure shown above the same relationship between spatial and referential descriptions holds.

Example 2:



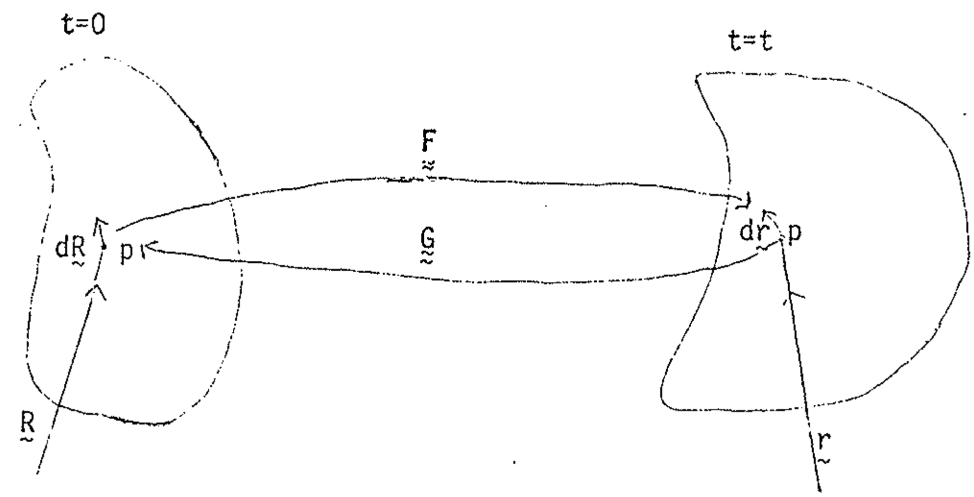
We can relate the points (X_1,X_2) and (x_1,x_2) in the following way

$$x_2 = X_2$$
$$x_1 = X_1 + \lambda(t)X_2$$

For this example both coordinate descriptions have the same origin. As before, if the body translates and rotates a different origin and basis may be chosen. Whether the origin and basis stay the same depends upon the particular situation.

Section 2 Referential and Spatial Gradients

Consider the following illustration



where the medium has undergone some deformation from time t=0 to t=t. This deformation includes rigid body motion. The coordinates of the body on the left are X_A with base vectors $\underset{\sim}{E}_A$ and the respective coordinates and base vectors for the body on the right are x_i and $\underset{\sim}{e}_i$.

A function ϕ that depends on $\underset{\sim}{R}$ can also be expressed as a function of $\underset{\sim}{r}$ through the use of (2.1-2), that is

$$\phi(\underset{\sim}{R},t) = \phi(\underset{\sim}{r}(\underset{\sim}{R},t),t) = \phi(\underset{\sim}{r},t) \qquad (2.2\text{-}1)$$

Let $d\phi$ denote the differential of ϕ associated with the same two neighboring points in either configuration. Gradient operators associated with the undeformed $(\,)\nabla_0$ and deformed $(\,)\nabla$ configurations are defined such that

$$d\phi = \phi(\underset{\sim}{R},t)\nabla_0 \cdot d\underset{\sim}{R} = \phi(\underset{\sim}{r},t)\nabla \cdot d\underset{\sim}{r} \qquad (2.2\text{-}2)$$

Implicit in these definitions is the assumption that t is held fixed so $d\phi$ does not represent the total differential of ϕ. It follows from (2.2-2) that

The gradient operator wrt the undeformed (reference) configuration is defined as

$$(\ )\nabla_0 = \frac{\partial(\ )}{\partial X_A} \otimes \underset{\sim}{E}_A = (\ )_{,A} \otimes \underset{\sim}{E}_A \tag{2.2-3a}$$

where the subscript "o" implies the reference configuration.

The gradient operator wrt the deformed configuration is

$$(\ )\nabla = \frac{\partial(\ )}{\partial x_i} \otimes \underset{\sim}{e}_i = (\ )_{,i} \otimes \underset{\sim}{e}_i \tag{2.2-3b}$$

The following identities are easily shown:

$$\left.\begin{aligned} \underset{\sim}{R}\nabla_0 &= (X_A\underset{\sim}{E}_A)_{,B} \otimes \underset{\sim}{E}_B = \delta_{AB}\underset{\sim}{E}_A \otimes \underset{\sim}{E}_B \\ &= \underset{\sim}{E}_A \otimes \underset{\sim}{E}_A = \underset{\approx}{I} \\ \text{and} \qquad \underset{\sim}{R}\cdot\nabla_0 &= \mathrm{tr}(\underset{\sim}{R}\nabla_0) = \mathrm{tr}(\underset{\approx}{I}) = 3 \end{aligned}\right\} \tag{2.2-4}$$

Similarly

$$\underset{\sim}{r}\nabla = \underset{\approx}{I}$$

and

$$\underset{\sim}{r}\cdot\nabla = 3$$

Now let

$$\underset{\sim}{r}\nabla_0 = \underset{\approx}{F} \tag{2.2-5}$$

and

$$\underset{\sim}{R}\nabla = \underset{\approx}{G}$$

From (2.2-3) we can obtain

$$\left.\begin{aligned} d\underset{\sim}{r} &= (\underset{\sim}{r}\nabla_0)\cdot d\underset{\sim}{R} = \underset{\approx}{F}\cdot d\underset{\sim}{R} \\ \text{and similarly} \quad d\underset{\sim}{R} &= \underset{\approx}{G}\cdot d\underset{\sim}{r} \end{aligned}\right\} \tag{2.2-6}$$

$\underset{\approx}{F}$ and $\underset{\approx}{G}$ are called mapping functions.

Using (2.2-6) gives

$$\begin{aligned} d\underset{\sim}{r} &= (\underset{\approx}{F}\cdot\underset{\approx}{G})\cdot d\underset{\sim}{r} \\ &= \underset{\approx}{I}\cdot d\underset{\sim}{r} \qquad \forall\ d\underset{\sim}{r} \end{aligned}$$

$$\begin{aligned} \text{Hence} \quad \underset{\approx}{I} &= \underset{\approx}{F}\cdot\underset{\approx}{G} \\ \text{or} \quad \underset{\approx}{G} &= \underset{\approx}{F}^{-1} \end{aligned} \tag{2.2-7}$$

$\underset{\approx}{F}$ is called the deformation gradient.

As a recap we can write

$$\left.\begin{aligned} \underset{\sim}{r}\nabla_0 &= \underset{\approx}{F} \\ \underset{\sim}{R}\nabla &= \underset{\approx}{F}^{-1} \\ d\underset{\sim}{r} &= \underset{\approx}{F}\cdot d\underset{\sim}{R} \\ d\underset{\sim}{R} &= \underset{\approx}{F}^{-1} d\underset{\sim}{r} \end{aligned}\right\} \tag{2.2-8}$$

Expanding $\underset{\approx}{F}$ we obtain

$$\begin{aligned} \underset{\approx}{F} = \underset{\sim}{r}\nabla_0 &= \frac{\partial}{\partial X_A}(x_i \underset{\sim}{e}_i) \otimes \underset{\sim}{E}_A \\ &= \frac{\partial x_i}{\partial X_A}\underset{\sim}{e}_i \otimes \underset{\sim}{E}_A = F_{iA}\underset{\sim}{e}_i \otimes \underset{\sim}{E}_A \end{aligned} \tag{2.2-9}$$

Note that we are using two different sets of base vectors in describing $\underset{\approx}{F}$.

We can perform a coordinate transformation and obtain

$$\underset{\approx}{F} = \frac{\partial x_i}{\partial x_A} \; a_{Bi} \underset{\sim}{E}_B \otimes \underset{\sim}{E}_A \tag{2.2-10}$$

The inverse of $\underset{\approx}{F}$ can be expressed in the form

$$\underset{\approx}{F}^{-1} = \frac{\partial X_A}{\partial x_i} \; \underset{\sim}{E}_A \otimes \underset{\sim}{e}_i \tag{2.2-11}$$

We observe that

$$\begin{aligned}
(\;)\nabla_0 &= \frac{\partial}{\partial X_A} (\;) \otimes \underset{\sim}{E}_A = \frac{\partial}{\partial x_i} (\;) \frac{\partial x_i}{\partial X_A} \otimes \underset{\sim}{E}_A \\
&= \left(\frac{\partial}{\partial x_i}(\;) \otimes \underset{\sim}{e}_i\right) \cdot \left(\frac{\partial x_j}{\partial X_A} \underset{\sim}{e}_j \otimes \underset{\sim}{E}_A\right) \\
&= \{(\;)\nabla\} \cdot \{\underset{\approx}{F}\}
\end{aligned}$$

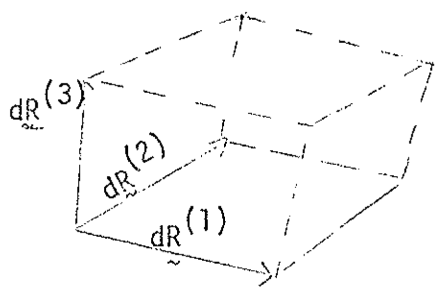
$$\text{Hence } (\;)\nabla_0 = \{(\;)\nabla\} \cdot \underset{\approx}{F} \tag{2.2-12}$$

$$\text{or } (\;)\nabla = \{(\;)\nabla_0\} \cdot \underset{\approx}{F}^{-1}$$

Section 3 Relations between Volume Elements

We wish to relate volume elements at t=0 to volume elements at t=t in which the volume elements contain the same material points.

Pick three differential vectors $d\underset{\sim}{R}^{(1)}$, $d\underset{\sim}{R}^{(2)}$ and $d\underset{\sim}{R}^{(3)}$ that are not co-planar but not necessarily orthogonal.



The volume element at time t=0 can be expressed as

$$dV_0 = d\underset{\sim}{R}^{(1)}\cdot(d\underset{\sim}{R}^{(2)} \times d\underset{\sim}{R}^{(3)}) \tag{2.3-1}$$

The relationship of the differential vectors at time t=t is

$$d\underset{\sim}{r}^{(1)} = \underset{\approx}{F}\cdot d\underset{\sim}{R}^{(1)}$$
$$\vdots \qquad\qquad \vdots$$
$$d\underset{\sim}{r}^{(3)} = \underset{\approx}{F}\cdot d\underset{\sim}{R}^{(3)}$$

Hence the volume element at time t=t becomes

$$\begin{aligned} dV &= d\underset{\sim}{r}^{(1)}\cdot(d\underset{\sim}{r}^{(2)} \times d\underset{\sim}{r}^{(3)}) \\ &= (\underset{\approx}{F}\cdot d\underset{\sim}{R}^{(1)})\cdot\{(\underset{\approx}{F}\cdot d\underset{\sim}{R}^{(2)}) \times (\underset{\approx}{F}\cdot d\underset{\sim}{R}^{(3)})\} \end{aligned}$$

Recalling from (1.7-4) the definition of the determinant of a tensor we obtain

$$dV = (\det \underset{\approx}{F})d\underset{\sim}{R}^{(1)}\cdot(d\underset{\sim}{R}^{(2)} \times d\underset{\sim}{R}^{(3)}) \tag{2.3-2}$$

Let

$$J = \det \underset{\approx}{F} \tag{2.3-3}$$

Then

$$dV = J dV_0 \tag{2.3-4}$$

Recall from eq. 2.2-10

$$\underset{\approx}{F} = \frac{\partial x_i}{\partial X_A} a_{Bi} \underset{\approx}{E}_B \otimes \underset{\approx}{E}_A$$

Since the determinant of a tensor equals the determinant of its components for an RCC system, we have

$$\det \underset{\approx}{F} = \det \left(\frac{\partial x_i}{\partial X_A} a_{Bi} \right)$$

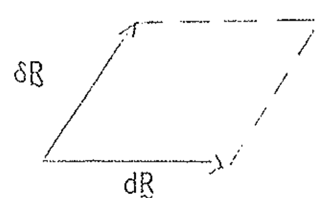
Also the determinant of a product equals the product of the determinants and the determinant of the coordinate transformation matrix equals one for the right hand systems. Hence

$$\det \underset{\approx}{F} = \det \left(\frac{\partial x_i}{\partial X_A}\right) \det \left(a_{Bi}\right)$$

$$\det \underset{\approx}{F} = \det \left(\frac{\partial x_i}{\partial X_A}\right) \tag{2.3-5}$$

<u>Area Elements</u>

Consider two differential vectors that are not co-linear



Then an area element in its undeformed state is defined to be

$$\left.\begin{aligned} d\underset{\sim}{S}_0 &= \underset{\sim}{N} dS_0 = d\underset{\sim}{R} \times \delta \underset{\sim}{R} \\ \text{or} \qquad dS_0 &= \underset{\sim}{N} \cdot (d\underset{\sim}{R} \times \delta \underset{\sim}{R}) \end{aligned}\right\} \tag{2.3-6}$$

where $\underset{\sim}{N}$ is the unit normal vector to the plane of the element.

In the deformed state the area element is defined as

$$d\underset{\sim}{S} = \underset{\sim}{n}dS = d\underset{\sim}{r} \times \delta\underset{\sim}{r}$$

or $$dS = \underset{\sim}{n}\cdot(d\underset{\sim}{r} \times \delta\underset{\sim}{r})$$

where $\underset{\sim}{n}$ is the appropriate normal vector.

Substitution for $d\underset{\sim}{r}$ and $\delta\underset{\sim}{r}$ yields

$$\underset{\sim}{n}dS = (\underset{\approx}{F}\cdot d\underset{\sim}{R}) \times (\underset{\approx}{F}\cdot\delta\underset{\sim}{R})$$

Pre-multiplying by $\underset{\approx}{F}\cdot\underset{\sim}{N}$ gives

$$(\underset{\approx}{F}\cdot\underset{\sim}{N})\cdot\underset{\sim}{n}ds = (\underset{\approx}{F}\cdot\underset{\sim}{N})\cdot[(\underset{\approx}{F}\cdot d\underset{\sim}{R}) \times (\underset{\approx}{F}\cdot\delta\underset{\sim}{R})]$$
$$= \det(\underset{\approx}{F})\underset{\sim}{N}\cdot(d\underset{\sim}{R} \times \delta R)$$

or $\underset{\sim}{N}\cdot\underset{\approx}{F}^T\cdot\underset{\sim}{n}dS = J\underset{\sim}{N}\cdot(\underset{\sim}{N}dS_0)$

The above equation must hold for all $\underset{\sim}{N}$ (or equivalently, for all $d\underset{\sim}{R}$ and $\delta\underset{\sim}{R}$) so we can eliminate $\underset{\sim}{N}$ and substitute for $\underset{\sim}{n}dS$ and $\underset{\sim}{N}dS_0$ to obtain

$$\left.\begin{aligned} &\underset{\approx}{F}^T\cdot d\underset{\sim}{S} = Jd\underset{\sim}{S}_0 \\ \text{Hence}\quad &d\underset{\sim}{S} = J\underset{\approx}{F}^{-T}\cdot d\underset{\sim}{S}_0 \\ \text{or}\quad &d\underset{\sim}{S}_0 = \frac{1}{J}\underset{\approx}{F}^T\cdot d\underset{\sim}{S} \end{aligned}\right\} \qquad (2.3\text{-}7)$$

(Note: $\underset{\approx}{F}^{-T} = (\underset{\approx}{F}^{-1})^T$).

Equation 2.3-7 is known as Nanson's Relation.

Recall (1.15-7)

$$Q = \int_{\partial R^*} \underset{\sim}{n} ds \qquad \forall \text{ subregions } R^*$$

Substituting from (2.3-7) gives

$$\underset{\sim}{Q} = \int_{\partial R^*_0} J\underset{\approx}{F}^{-T} \cdot d\underset{\sim}{S}_0 = \int_{\partial R^*_0} J\underset{\approx}{F}^{-T} \cdot \underset{\sim}{N} dS_0$$

Using the divergence theorem we can transform the surface integral into a volume integral and get

$$Q = \int_{R^*_0} (J\underset{\approx}{F}^{-T}) \cdot \nabla_0 dV_0$$

Since the subregion R^* is arbitrary, the subregion R^*_0 is also arbitrary and hence this equation implies

$$(J\underset{\approx}{F}^{-T}) \cdot \nabla_0 = \underset{\sim}{Q}$$

or similarly $\quad (\frac{1}{J} \underset{\approx}{F}^{T}) \cdot \nabla = Q \qquad (2.3\text{-}8)$

Section 4 Time Derivatives of Functions and Integrals

Recall that the position vector in the referential description is independent of time whereas in the spatial description the position vector is a function of time. Hence,

$$\frac{dR}{dt} = 0$$

and $$\frac{d\underset{\sim}{r}}{dt} = \underset{\sim}{v} \tag{2.4-1}$$

provided the origins of $\underset{\sim}{R}$ and $\underset{\sim}{r}$ coincide.

The time derivative of a scalar function Φ given as a function of $\underset{\sim}{R}$ and t is defined to be

$$\frac{d\Phi}{dt} = \lim_{\Delta t \to 0} \frac{\Phi(\underset{\sim}{R}, t+\Delta t) - \Phi(\underset{\sim}{R}, t)}{\Delta t}$$

$$= \frac{\partial \Phi}{\partial t} = \dot{\Phi} \tag{2.4-2}$$

Now suppose $\Phi = \Phi\ (\underset{\sim}{r}, t)$. Then the total differential is

$$\hat{d}\Phi = (d\Phi)_t + (d\Phi)_{\underset{\sim}{r}}$$

In the above expression the first term on the right side of the equation is computed holding $\underset{\sim}{r}$ fixed while the second term is computed holding t fixed.

We can express the differentials in the above equation as

$$(d\Phi)_t = \frac{\partial \Phi}{\partial t}\, dt$$

and

$$(d\Phi)_{\underset{\sim}{r}} = (\Phi\nabla)\cdot d\underset{\sim}{r}$$

Hence the total differential becomes

$$d\Phi = \frac{\partial \Phi}{\partial t}\, dt + (\Phi\nabla)\cdot d\underset{\sim}{r}$$

The total derivative with respect to t is obtained by dividing this expression by Δt and letting $\Delta t \rightarrow 0$. The result is

$$\frac{\hat{d}\Phi}{dt} = \frac{\partial \Phi}{\partial t} + (\Phi \nabla) \cdot \underset{\sim}{v} \tag{2.4-3}$$

$$= \mathring{\Phi}$$

For vector functions we can obtain the following:

$$\text{If } \underset{\sim}{u} = \underset{\sim}{u}(\underset{\sim}{R},t) \text{ then } \mathring{\underset{\sim}{u}} = \frac{\partial \underset{\sim}{u}}{\partial t} \tag{2.4-4}$$

$$\text{If } \underset{\sim}{u} = \underset{\sim}{u}(\underset{\sim}{r},t) \text{ then } \mathring{\underset{\sim}{u}} = \frac{\partial \underset{\sim}{u}}{\partial t} + (\underset{\sim}{u}\nabla) \cdot \underset{\sim}{v}$$

$\mathring{\underset{\sim}{u}}(\underset{\sim}{r},t)$ is called the material derivative of $\underset{\sim}{u}$ and $\frac{\partial \underset{\sim}{u}}{\partial t}(\underset{\sim}{r},t)$ is called the spatial derivative of $\underset{\sim}{u}$.

Consider the time derivative of an integral.

$$\frac{d}{dt}\int_R F(t)\, dV$$

Since the limits of integration for R are a function of time, it is convenient to transform the integral to the reference system in which the limits are not functions of time. Then the time derivative can be taken inside the integral to get

$$\frac{d}{dt}\int_R F(t)dV = \int_{R_0} \frac{d}{dt}\{JF(t)\}dV_0$$

Now a transformation back to the spatial system can be made with the result

$$\frac{d}{dt}\int_R F(t)dV = \int_R \frac{d}{dt}\{JF(t)\}\cdot\frac{1}{J}\,dV \tag{2.4-5}$$

Interchange of Time Derivatives with Gradients, Volume Integrals & Surface Integrals

$$\frac{d}{dt}\left[(\;)\nabla_0\right] = \left[\frac{d}{dt}(\;)\right]\nabla_0$$

$$\begin{aligned}
\frac{d}{dt}\left[(\;)\nabla\right] &= \frac{d}{dt}\left[(\;)\nabla_0\cdot \underset{=}{F}^{-1}\right] \\
&= \frac{d}{dt}\left[(\;)\nabla_0\right]\cdot \underset{=}{F}^{-1} + (\;)\nabla_0\cdot \dot{\underset{=}{F}}^{-1} \\
&= \left[\frac{d}{dt}(\;)\right]\nabla_0\cdot \underset{=}{F}^{-1} + (\;)\nabla\cdot \dot{\underset{=}{F}}\cdot \underset{=}{F}^{-1} \\
&= \left[\frac{d}{dt}(\;)\right]\nabla - (\;)\nabla\cdot \underset{=}{L}
\end{aligned}$$

$$\begin{aligned}
\frac{d}{dt}\int_{\partial R}(\;)\,d &= \frac{d}{dt}\int_{\partial R}(\;)\,\underset{\sim}{n}\cdot\underset{\sim}{n}\,ds \\
&= \frac{d}{dt}\int_{\partial R_0}(\;)\,\underset{\sim}{n}\cdot J\underset{=}{F}^{-T}\cdot\underset{\sim}{N}\,dS_0 \\
&= \int_{\partial R_0}\frac{d}{dt}\left[(\;)\,\underset{\sim}{n}\cdot J\underset{=}{F}^{-T}\right]\cdot\underset{\sim}{N}\,dS_0 \\
&= \int_{\partial R}\frac{d}{dt}\left[(\;)\,\underset{\sim}{n}\cdot J\underset{=}{F}^{-T}\right]\cdot\frac{1}{J}\underset{=}{F}^{T}\cdot\underset{\sim}{n}\,ds
\end{aligned}$$

$$\frac{d}{dt}\int_R(\;)\,dV = \quad\cdots$$ From (2.4-5)

Section 5 Rate of Deformation

Let $\underset{\approx}{L} = \underset{\sim}{v}\nabla$ (2.5-1)

where ∇ is the spatial velocity gradient.

We can define $\underset{\approx}{L}$ to be the sum of a symmetric and a skew-symmetric tensor

$$\underset{\approx}{L} = \underset{\approx}{D} + \underset{\approx}{W}$$

where $$\underset{\approx}{D} = \tfrac{1}{2}\,(\underset{\approx}{L} + \underset{\approx}{L}^T) \tag{2.5-2}$$

and $$\underset{\approx}{W} = \tfrac{1}{2}\,(\underset{\approx}{L} - \underset{\approx}{L}^T)$$

$\underset{\approx}{D}$ is known as the rate of deformation tensor and $\underset{\approx}{W}$ is called the vorticity tensor. The axial vector associated with $\underset{\approx}{W}$ is called the vorticity vector.

Taking the time derivative of $\underset{\approx}{F}$ gives

$$\dot{\underset{\approx}{F}} = \frac{d}{dt}\,(\underset{\sim}{r}\nabla_0) = \left(\frac{d\underset{\sim}{r}}{dt}\right)\nabla_0 = \underset{\sim}{v}\nabla_0$$

Substituting from (2.2-9) and (2.5-1) gives

$$\dot{\underset{\approx}{F}} = \underset{\sim}{v}\nabla_0 = \underset{\sim}{v}\nabla\cdot\underset{\approx}{F} = \underset{\approx}{L}\cdot\underset{\approx}{F}$$

Hence

$$\underset{\approx}{L} = \dot{\underset{\approx}{F}}\cdot\underset{\approx}{F}^{-1} \tag{2.5-3}$$

To obtain $\dot{\underset{\approx}{F}}^{-1}$ we observe that

$$\frac{d}{dt}\,[\underset{\approx}{F}\cdot\underset{\approx}{F}^{-1} = \underset{\approx}{I}] \Rightarrow \dot{\underset{\approx}{F}}\cdot\underset{\approx}{F}^{-1} + \underset{\approx}{F}\cdot\dot{\underset{\approx}{F}}^{-1} = \underset{\approx}{0}$$

or $\quad \dot{\underset{\approx}{F}}^{-1} = -\underset{\approx}{F}^{-1}\cdot\dot{\underset{\approx}{F}}\cdot\underset{\approx}{F}^{-1}$

$$= -\underset{\approx}{F}^{-1}\cdot\underset{\approx}{L}\cdot\underset{\approx}{F}\cdot\underset{\approx}{F}^{-1}$$

$$\dot{\underset{\approx}{F}}^{-1} = -\underset{\approx}{F}^{-1}\cdot\underset{\approx}{L} \qquad (2.5\text{-}4)$$

Alternately $\dot{\underset{\approx}{F}}^{-T} = -\underset{\approx}{L}^{T}\cdot\underset{\approx}{F}^{-T}$

Thm: $\frac{dJ}{dt} = J\,tr(\underset{\approx}{L}) = J(v\cdot\nabla) = J\,tr(\underset{\approx}{D}) \qquad (2.5\text{-}5)$

Proof: We can easily show that $tr(\underset{\approx}{L})$ equals $tr(\underset{\approx}{D})$ since

$$\underset{\approx}{L} = \underset{\approx}{D} + \underset{\approx}{W}$$

or $\quad tr(\underset{\approx}{L}) = tr(\underset{\approx}{D}) + tr(\underset{\approx}{W})$

and $tr(\underset{\approx}{W}) = 0$ since $\underset{\approx}{W}$ is skew-symmetric

By definition we know that

$$tr\,\underset{\approx}{L} = tr(\underset{\sim}{v} \otimes \nabla) = \underset{\sim}{v}\cdot\nabla$$

and $\quad J = \det \underset{\approx}{F}$

where $\quad \underset{\approx}{F} = F_{iA}\underset{\sim}{e}_i \otimes \underset{\sim}{E}_A$

We can express the time derivative of J as

$$\dot{J} = \frac{\partial J}{\partial F_{iA}}\frac{\partial F_{iA}}{\partial t} = \frac{\partial J}{\partial F_{ia}}\dot{F}_{iA} = C^F_{iA}\,\dot{F}_{iA} \qquad (2.5\text{-}6)$$

where C^F_{iA} is the cofactor matrix of F where use has been made of (1.2-15)

We recall from (1.2-15) that

$$C^F_{iA} = (C^{Fd}_{iA})^T$$ transpose of adjoint matrix from (1.2-10)

(2.5-7)

$$= (F^{-1}_{iA})^T \det \underset{=}{F}$$ from (1.2-17)

Then

$$\dot{J} = \det \underset{=}{F} \, (F^{-1}_{iA})^T \dot{F}_{iA}$$

$$= J \, F^{-1}_{Ai} \dot{F}_{iA}$$

$$= J \operatorname{tr}(\underset{=}{F}^{-1} \cdot \dot{\underset{=}{F}})$$

$$= J \operatorname{tr}(\underset{=}{F}^{-1} \cdot \underset{=}{L} \cdot \underset{=}{F} = J \operatorname{tr} \underset{=}{L}$$ EOP

<u>Alternative Proof</u> Pg. 81, Bowen

Let $\underset{\sim}{u}$, $\underset{\sim}{v}$ and $\underset{\sim}{w}$ be constant but arbitrary vectors. By the definition of a determinant

$$J \ \underset{\sim}{u} \cdot (\underset{\sim}{v} \times \underset{\sim}{w}) = (\underset{=}{F} \cdot \underset{\sim}{u}) \cdot [(\underset{=}{F} \cdot \underset{\sim}{v}) \times (\underset{=}{F} \cdot \underset{\sim}{w})$$

$$\dot{J} \ \underset{\sim}{u} \cdot (\underset{\sim}{v} \times \underset{\sim}{w}) = (\dot{\underset{=}{F}} \cdot \underset{\sim}{u}) \cdot [(\underset{=}{F} \cdot \underset{\sim}{v}) \times (\underset{=}{F} \cdot \underset{\sim}{w})]$$
$$+ (\underset{=}{F} \cdot \underset{\sim}{u}) \cdot [(\dot{\underset{=}{F}} \cdot \underset{\sim}{v}) \times (\underset{=}{F} \cdot \underset{\sim}{w})]$$
$$+ (\underset{=}{F} \cdot \underset{\sim}{u}) \cdot [(\underset{=}{F} \cdot \underset{\sim}{v}) \times (\dot{\underset{=}{F}} \cdot \underset{\sim}{w})]$$

$$= \underset{=}{L} \cdot (\underset{=}{F} \cdot \underset{\sim}{u}) \cdot [(\underset{=}{F} \cdot \underset{\sim}{v}) \times (\underset{=}{F} \cdot \underset{\sim}{w})]$$
$$+ (\underset{=}{F} \cdot \underset{\sim}{u}) \cdot [(\underset{=}{L} \cdot \underset{=}{F} \cdot \underset{\sim}{v}) \times (\underset{=}{F} \cdot \underset{\sim}{w})]$$
$$+ (\underset{=}{F} \cdot \underset{\sim}{u}) \cdot [(\underset{=}{F} \cdot \underset{\sim}{v}) \times (\underset{=}{L} \cdot \underset{=}{F} \cdot \underset{\sim}{w})$$

$$= (\operatorname{tr} \underset{=}{L})(\underset{=}{F} \cdot \underset{\sim}{u}) \cdot [(\underset{=}{F} \cdot \underset{\sim}{v}) \times (\underset{=}{F} \cdot \underset{\sim}{w})]$$ From trace theorem on page 24a

$$= (\operatorname{tr} \underset{=}{L}) \ J \ \underset{\sim}{u} \cdot (\underset{\sim}{v} \times \underset{\sim}{w})$$

$$\dot{J} = J \operatorname{tr} \underset{=}{L}$$ EOP

Bowen, R.M., Lectures on Continuum Mechanics, SC-M-67-326, May, 1967, Sandia Laboratories, Albuquerque, NM

Section 6 Strains and Strain Rates

Consider the illustration below

Let $\underset{\approx}{E}$ = Lagrangian or material strain tensor

and $\underset{\approx}{e}$ = Eulerian or spatial strain tensor

The strain tensors are defined $\ni$

$$
\begin{aligned}
d\underset{\sim}{r}\cdot d\underset{\sim}{r} - d\underset{\sim}{R}\cdot d\underset{\sim}{R} &= ds^2 - dS^2 \\
&= 2d\underset{\sim}{r}\cdot\underset{\approx}{e}\cdot d\underset{\sim}{r} \qquad (2.6\text{-}1) \\
&= 2d\underset{\sim}{R}\cdot\underset{\approx}{E}\cdot d\underset{\sim}{R}
\end{aligned}
$$

$\forall$ choices of $d\underset{\sim}{R}$ or $d\underset{\sim}{r}$

Normally, the strain tensors are functions of different position vectors, i.e.

$$\underset{\approx}{e} = \underset{\approx}{e}(\underset{\sim}{r},t) \text{ and } \underset{\approx}{E} = \underset{\approx}{E}\,(\underset{\sim}{R},t)$$

From (2.2-6) we recall

$$d\underset{\sim}{r} = \underset{\approx}{F}\cdot d\underset{\sim}{R}$$

or $$d\underset{\sim}{r} = d\underset{\sim}{R}\cdot\underset{\approx}{F}^T$$

Hence

$$d\underset{\sim}{r}\cdot d\underset{\sim}{r} - d\underset{\sim}{R}\cdot d\underset{\sim}{R} = d\underset{\sim}{R}\cdot \underset{\approx}{F}^T\cdot \underset{\approx}{F}\cdot d\underset{\sim}{R} - d\underset{\sim}{R}\cdot \underset{\approx}{I}\cdot d\underset{\sim}{R}$$

$$= d\underset{\sim}{R}\cdot(\underset{\sim}{F}^T\cdot\underset{\sim}{F} - \underset{\sim}{I})\cdot d\underset{\sim}{R} \qquad \forall\ d\underset{\sim}{R}$$

Equating the previous expression and (2.6-1) gives

$$\underset{\approx}{E} = \tfrac{1}{2}\,(\underset{\approx}{F}^T\cdot\underset{\approx}{F} - \underset{\approx}{I}) \tag{2.6-2$_1$}$$

Similar procedures can be carried out to obtain $\underset{\approx}{e}$ in the following form

$$\underset{\approx}{e} = \tfrac{1}{2}\,(\underset{\approx}{I} - \underset{\approx}{F}^{-T}\cdot\underset{\approx}{F}^{-1}) \tag{2.6-2$_2$}$$

Recall that $\underset{\approx}{F}$ can be expressed as

$$\underset{\approx}{F} = \frac{\partial x_i}{\partial X_A}\ \underset{\sim}{e}_i \otimes \underset{\sim}{E}_A$$

Hence

$$\underset{\approx}{F}^T = \frac{\partial x_j}{\partial X_B}\ \underset{\sim}{E}_B \otimes \underset{\sim}{e}_j$$

and

$$\underset{\approx}{F}^T\cdot\underset{\approx}{F} = \frac{\partial x_i}{\partial X_A}\frac{\partial x_i}{\partial X_B}\ \underset{\sim}{E}_B \otimes \underset{\sim}{E}_A$$

Note that the identity tensor can be written

$$\underset{\approx}{I} = \delta_{AB}\ \underset{\sim}{E}_B \otimes \underset{\sim}{E}_A$$

Substituting into (2.6-2) gives

$$E_{AB} = \tfrac{1}{2}\,\left(\frac{\partial x_i}{\partial X_A}\frac{\partial x_i}{\partial X_B} - \delta_{AB}\right) \tag{2.6-3}$$

and

$$e_{ij} = \tfrac{1}{2}\left(\delta_{ij} - \frac{\partial X_A}{\partial x_i}\frac{\partial X_A}{\partial x_j}\right)$$

Expanding E_{AB} we can obtain

$$E_{11} = \tfrac{1}{2}\left[\frac{\partial x_1}{\partial X_1}\frac{\partial x_1}{\partial X_1} + \frac{\partial x_2}{\partial X_1}\frac{\partial x_2}{\partial X_1} + \frac{\partial x_3}{\partial X_1}\frac{\partial x_3}{\partial X_1} - 1\right]$$

Expressions for other components of the strain tensors E_{AB} and e_{ij} can be similarly obtained.

Strain Rates

Using (2.6-2) we can express the time derivative of the strain tensor as

$$\begin{aligned}
\dot{\underset{\approx}{E}} &= \tfrac{1}{2}\,(\dot{\underset{\approx}{F}}^T\cdot\underset{\approx}{F} + \underset{\approx}{F}^T\cdot\dot{\underset{\approx}{F}}) \\
&= \tfrac{1}{2}\,(\underset{\approx}{F}^T\cdot\underset{\approx}{L}^T\cdot\underset{\approx}{F} + \underset{\approx}{F}^T\cdot\underset{\approx}{L}\cdot\underset{\approx}{F}) \\
&= \underset{\approx}{F}^T\cdot\tfrac{1}{2}(\underset{\approx}{L}^T + \underset{\approx}{L})\cdot\underset{\approx}{F}
\end{aligned}$$

$$\dot{\underset{\approx}{E}} = \underset{\approx}{F}^T\cdot\underset{\approx}{D}\cdot\underset{\approx}{F}$$

Similarly

$$\dot{\underset{\approx}{e}} = -\tfrac{1}{2}[\dot{\underset{\approx}{F}}^{-T}\cdot\underset{\approx}{F}^{-1} + \underset{\approx}{F}^{-T}\cdot\dot{\underset{\approx}{F}}^{-1}]$$

With the use of (2.5-4) we obtain

$$\dot{\underset{\approx}{e}} = \tfrac{1}{2}[\underset{\approx}{L}^T\cdot\underset{\approx}{F}^{-T}\cdot\underset{\approx}{F}^{-1} + \underset{\approx}{F}^{-T}\cdot\underset{\approx}{F}^{-1}\cdot\underset{\approx}{L}]$$

From (2.6-2) we can write

$$\underset{\approx}{F}^{-T}\cdot\underset{\approx}{F}^{-1} = \underset{\approx}{I} - 2\underset{\approx}{e}$$

Hence $$\dot{\underset{\approx}{e}} = \tfrac{1}{2}[\underset{\approx}{L}^T\cdot(\underset{\approx}{I} - 2\underset{\approx}{e}) + (\underset{\approx}{I} - 2\underset{\approx}{e})\cdot\underset{\approx}{L}]$$

Expanding into individual terms and after substituting for $\underset{\approx}{D}$ we arrive at

$$\dot{\underset{\approx}{e}} = \underset{\approx}{D} - \underset{\approx}{L}^T\cdot\underset{\approx}{e} - \underset{\approx}{e}\cdot\underset{\approx}{L}$$

Summarizing, we obtain the strain rates for the material and spatial descriptions as

$$\dot{\underset{\approx}{E}} = \underset{\approx}{F}^T\cdot\underset{\approx}{D}\cdot\underset{\approx}{F} \tag{2.6-4}$$

and $$\dot{\underset{\approx}{e}} = \underset{\approx}{D} - \underset{\approx}{L}^T\cdot\underset{\approx}{e} - \underset{\approx}{e}\cdot\underset{\approx}{L}$$

Three items should be noted from these results:

(i) Neither of the strain rates defined above equal the rate of deformation.

(ii) Both strain rate tensors are symmetric

(iii)

$$ds^2 - dS^2 = 2\, d\underset{\sim}{R}\cdot\underset{\approx}{E}\cdot d\underset{\sim}{R}$$

Hence $$\frac{d}{dt}(ds^2 - dS^2) = 2d\underset{\sim}{R}\cdot\dot{\underset{\approx}{E}}\cdot d\underset{\sim}{R}$$

$$\frac{d}{dt}(ds^2) - 0 = 2d\underset{\sim}{R}\cdot\underset{\approx}{F}^T\cdot\underset{\approx}{D}\cdot\underset{\approx}{F}\cdot d\underset{\sim}{R}$$

Recalling that $\underset{\approx}{F}\cdot d\underset{\sim}{R} = d\underset{\sim}{r} = d\underset{\sim}{R}\cdot\underset{\approx}{F}^T$ we obtain

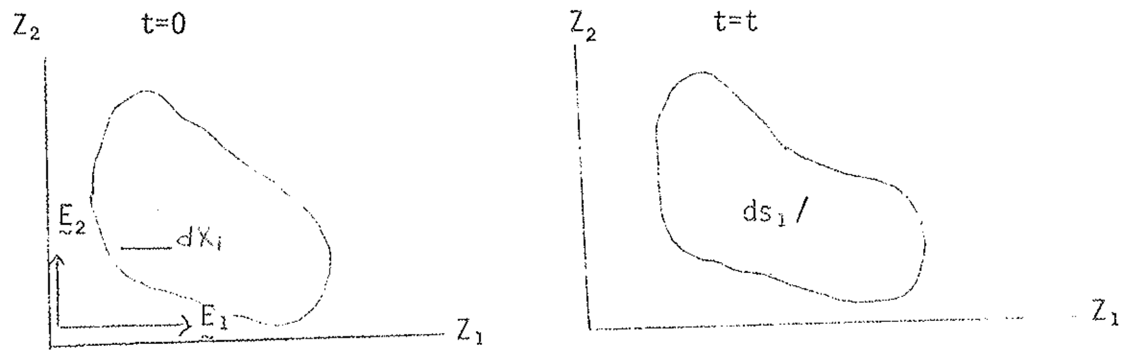
$$\begin{aligned}\frac{d}{dt}(ds^2) &= 2d\underset{\sim}{r}\cdot\underset{\approx}{D}\cdot d\underset{\sim}{r} \\ &= 2d\underset{\sim}{R}\cdot\dot{\underset{\approx}{E}}\cdot d\underset{\sim}{R}\end{aligned} \tag{2.6-5}$$

It can be shown that the two strain tensors are related by the following expression

$$\underset{\approx}{F}^T\cdot\underset{\approx}{e}\cdot\underset{\approx}{F} = \underset{\approx}{E} \tag{2.6-6}$$

Section 7 Physical Interpretation of Strains

Consider a body in which an element dX_1, originally parallel to $\underset{\sim}{E}_1$, undergoes a deformation. The same element in the deformed state has length ds_1 and is not necessarily parallel to $\underset{\sim}{e}_1$.



From (2.6-1) we recall

$$ds^2 - dS^2 = 2d\underset{\sim}{R}\cdot\underset{\approx}{E}\cdot d\underset{\sim}{R}$$

After substitution for $d\underset{\sim}{S}$ and $d\underset{\sim}{R}$ we obtain

$$ds_1^2 - (dx_1)^2 = 2\ dX_1\underset{\sim}{E}_1\cdot\underset{\approx}{E}\cdot dX_1\underset{\sim}{E}_1$$

$$= 2\ E_{11}(dX_1)^2$$

Let $\Lambda_1 = \frac{ds_1}{dX_1}$ be known as the "stretch" of the element originally parallel to $\underset{\sim}{E}_1$. Hence, we can substitute into the above expression and get

$$\tfrac{1}{2}[(\Lambda_1)^2 - 1] = E_{11}$$

Now let $\Lambda_1 = 1 + \varepsilon$

Then $\quad E_{11} = \tfrac{1}{2}\,(2\varepsilon + \varepsilon^2) = \varepsilon + \frac{\varepsilon^2}{2}$

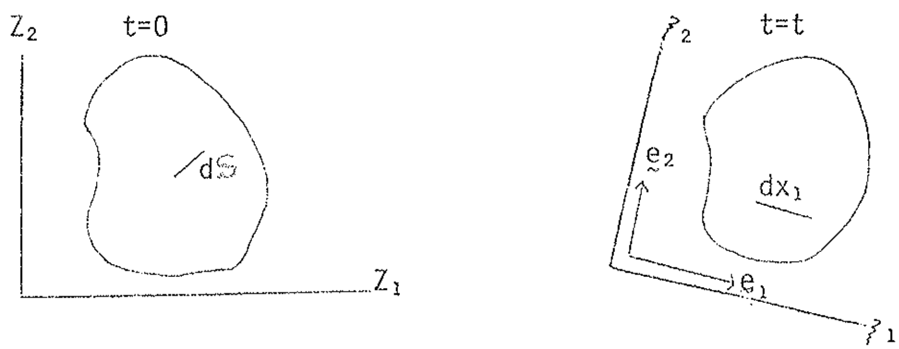
If $\varepsilon << 1$ then for infinitesimal theory

$$E_{11} = \varepsilon$$

The definition of Λ_1 is physically meaningful in the traditional one dimensional definition since

$$\varepsilon = \frac{ds_1 - dX_1}{dX_1} = \frac{ds_1}{dX_1} - 1 = \Lambda_1 - 1$$

Consider the same body shown previously, but isolate an element of length dx_1 parallel to $\underset{\sim}{e}_1$ in the deformed state. The same element in the undeformed state has length dS, and is not necessarily parallel to $\underset{\sim}{E}_1$.



Then $ds^2 - dS^2 = 2d\underset{\sim}{r} \cdot \underset{\approx}{e} \cdot d\underset{\sim}{r}$

or $(dx_1)^2 - dS^2 = 2dx_1\underset{\sim}{e}_1 \cdot \underset{\approx}{e} \cdot dx_1\underset{\sim}{e}_1$

$$= 2\, e_{11}(dx_1)^2$$

Let $\lambda_1 = \frac{dx_1}{dS}$ be the "stretch" of an element which is parallel to $\underset{\sim}{e}$ in the deformed state.

A substitution into the above expression yields

$$e_{11} = \tfrac{1}{2}\,(1 - \frac{1}{\lambda_1^2})$$

Now let $\lambda_1 = 1 + \tilde{\varepsilon}$

Then

$$e_{11} = \tfrac{1}{2}\left(\frac{\lambda_1^2-1}{\lambda_1^2}\right)$$

$$= \tfrac{1}{2}\left[\frac{2\tilde{\varepsilon}+\tilde{\varepsilon}^2}{1+2\tilde{\varepsilon}+\tilde{\varepsilon}^2}\right]$$

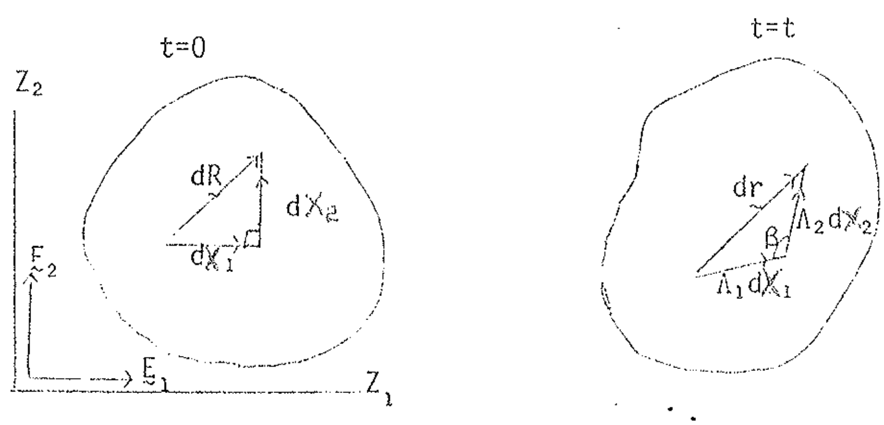
If $\tilde{\varepsilon}<<1$ then the higher order terms can be neglected and the $2\tilde{\varepsilon}$ in the denominator can be neglected in comparison with 1. Hence, we obtain

$$e_{11} = \tilde{\varepsilon}$$

Note that both definitions of strain that we have used yield the same results for infinitesimal deformations.

Shearing Strain

Consider a body with two elements originally perpendicular that undergoes a deformation



Then $\qquad d\underset{\sim}{R} = dX_1\underset{\sim}{E}_1 + dX_2\underset{\sim}{E}_2$

and $\qquad dS^2 = d\underset{\sim}{R}\cdot d\underset{\sim}{R}$

$$= (dX_1)^2 + (dX_2)^2$$

Recall from (2.6-1)

$$ds^2 - dS^2 = 2d\underset{\sim}{R}\cdot\underset{\approx}{E}\cdot d\underset{\sim}{R}$$

After a substitution we obtain

$$ds^2 - dS^2 = 2[dX_1E_{11}dX_1 + 2dX_1E_{12}dX_2 + dX_2E_{22}dX_2]$$

or $$ds^2 = (2E_{11} + 1)dX_1^2 + 4E_{12}dX_1dX_2 + (2E_{22} + 1)dX_2^2$$

Applying the law of cosines to the above figure associated with the deformed state yields

$$ds^2 = \Lambda_1^2dX_1^2 + \Lambda_2^2dX_2^2 - 2\Lambda_1\Lambda_2dX_1dX_2 \cos\beta$$

Since dX_1 and dX_2 are arbitrary we can equate co-efficients in the above equations to obtain

$$2E_{11} = \Lambda_1^2 - 1$$
$$2E_{22} = \Lambda_2^2 - 1$$
$$4E_{12} = -2\Lambda_1\Lambda_2 \cos \beta$$

If we let $\beta = \frac{\pi}{2} + \alpha$ (i.e., α is the angle change from a right angle), then

$$\cos \beta = \cos(\frac{\pi}{2} + \alpha) = -\sin \alpha$$

Then

$$E_{12} = \tfrac{1}{2} \Lambda_1\Lambda_2 \sin \alpha$$

In the infinitesimal case $\Lambda_1 = 1 + \varepsilon_1$ and $\Lambda_2 = 1 + \varepsilon_2$ where $\varepsilon_1 << 1$, $\varepsilon_2 << 1$ and $\alpha << 1$. Then Λ_1 and Λ_2 are both essentially unity and $\sin\alpha \cong \alpha$.

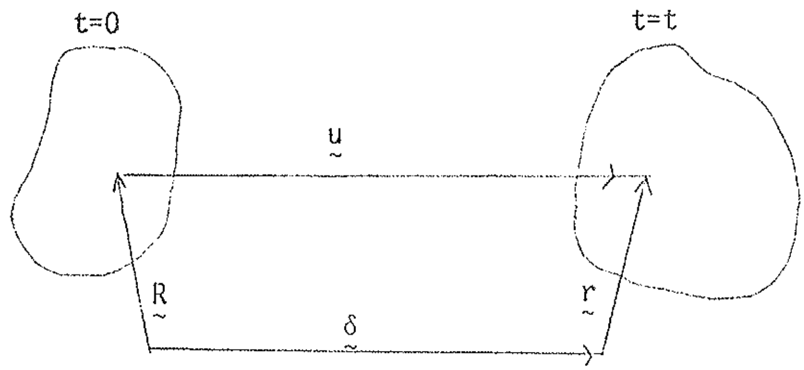
Hence $$E_{12} = \tfrac{1}{2}(1)(1)(\alpha) = \tfrac{1}{2}\alpha$$

There is another definition for shear strain called "engineering shear strain" which is defined to be

$$\gamma_{12} = \alpha$$

Section 8 Strain-Displacement Relations, Infinitesimal Strains and Rotations

Once again, consider a point in a body that undergoes a deformation through some period of time.



In the above illustration $\underset{\sim}{\delta}$ is the vector relating the origins of the two reference systems and $\underset{\sim}{u}$ is the displacement vector of the point of interest.

Recall that $\underset{\sim}{u}$ has two different representations

$$\underset{\sim}{u} = \underset{\sim}{u}(\underset{\sim}{R},t) \quad \text{or } \underset{\sim}{u} = \underset{\sim}{u}(\underset{\sim}{r},t)$$

Since $\underset{\sim}{\delta}$ only relates the origins of the reference system we can assume that $\underset{\sim}{\delta}$ is not a function of time or position.

Refering to the above figure we obtain

$$\underset{\sim}{u} = \underset{\sim}{\delta} + \underset{\sim}{r} - \underset{\sim}{R} \tag{2.8-1}$$

Now recall $\underset{\approx}{F} = \underset{\sim}{r}\nabla_0$

Hence $\underset{\approx}{F} = (-\underset{\sim}{\delta} + \underset{\sim}{R} + \underset{\sim}{u})\nabla_0$

Since $\underset{\sim}{\delta}$ is not a function of position we note that $\underset{\sim}{\delta}\nabla_0 = \underset{\approx}{0}$.

If we observe that $\underset{\sim}{R}\nabla_0 = \underset{\approx}{I}$ and define a tensor $\underset{\approx}{H}$ (the reference configuration displacement gradient) such that

$$\underset{\sim}{u}\nabla_0 = \underset{\approx}{H} \tag{2.8-2}$$

then we arrive at

$$\underset{\approx}{F} = \underset{\approx}{I} + \underset{\approx}{H} \tag{2.8-3}$$

Similarly,

$$\begin{aligned} \underset{\approx}{F}^{-1} &= \underset{\sim}{R}\nabla \\ &= (\underset{\sim}{\delta} + \underset{\sim}{r} - \underset{\sim}{u})\nabla \end{aligned}$$

Noting that $\underset{\sim}{\delta}\nabla = \underset{\approx}{0}$ and $\underset{\sim}{r}\nabla = \underset{\approx}{I}$ we obtain

$$\underset{\approx}{F}^{-1} = \underset{\approx}{I} - \underset{\sim}{u}\nabla$$

If we denote the displacement gradient with respect to the spatial configuration by

$$\underset{\approx}{h} = \underset{\sim}{u}\nabla \tag{2.8-4}$$

then $$\underset{\approx}{F}^{-1} = \underset{\approx}{I} - \underset{\approx}{h} \tag{2.8-5}$$

Returning to the tensor $\underset{\approx}{H}$ we have

$$\underset{\approx}{H} = u\nabla_0$$

Using (2.2-12) we can express $\underset{\approx}{H}$ as $\underset{\approx}{H} = (\underset{\sim}{u}\nabla)\cdot\underset{\approx}{F}$

or

$$\begin{aligned} \underset{\approx}{H} &= \underset{\approx}{h}\cdot\underset{\approx}{F} \\ &= \underset{\approx}{h}\cdot(\underset{\approx}{I} + \underset{\approx}{H}) \\ &= \underset{\approx}{h}\cdot(\underset{\approx}{I} + (\underset{\approx}{h}\cdot(\underset{\approx}{I} + \underset{\approx}{H}))) \end{aligned} \tag{2.8-6}$$

If the expansion is formally continued, we obtain

$$\underset{\approx}{H} = \underset{\approx}{h} + \underset{\approx}{h} \cdot \underset{\approx}{h} + \underset{\approx}{h} \cdot \underset{\approx}{h} \cdot \underset{\approx}{h} + \cdots$$

By defining $\underset{\approx}{h}^2$ as $\underset{\approx}{h} \cdot \underset{\approx}{h}$ and similarly for higher order terms this expression becomes

$$\underset{\approx}{H} = \underset{\approx}{h} + \underset{\approx}{h}^2 + \underset{\approx}{h}^3 + \underset{\approx}{h}^4 + \cdots \qquad (2\text{-}8\text{-}7)$$

From (2.8-4) and (2.2-12) we obtain

$$\begin{aligned} \underset{\approx}{h} &= \underset{\sim}{u}\nabla \\ &= (\underset{\sim}{u}\nabla_0) \cdot \underset{\approx}{F}^{-1} \end{aligned}$$

Substituting from (2.8-2) and (2.8-5) gives

$$\begin{aligned} \underset{\approx}{h} &= \underset{\approx}{H} \cdot (\underset{\approx}{I} - \underset{\approx}{h}) \\ &= \underset{\approx}{H} \cdot (\underset{\approx}{I} - (\underset{\approx}{H} \cdot (\underset{\approx}{I} - \underset{\approx}{h}))) \end{aligned}$$

As before, we can expand $\underset{\approx}{h}$ to obtain

$$\underset{\approx}{h} = \underset{\approx}{H} - \underset{\approx}{H}^2 + \underset{\approx}{H}^3 - \underset{\approx}{H}^4 + \cdots \qquad (2.8\text{-}8)$$

Hence, from (2.8-7) and (2.8-8) we arrive at

$$\begin{aligned} \underset{\approx}{F} &= \underset{\approx}{I} + \underset{\approx}{H} = \underset{\approx}{I} + \underset{\approx}{h} + \underset{\approx}{h}^2 + \underset{\approx}{h}^3 + \underset{\approx}{h}^4 + \cdots \\ \underset{\approx}{F}^{-1} &= \underset{\approx}{I} - \underset{\approx}{h} = \underset{\approx}{I} - \underset{\approx}{H} + \underset{\approx}{H}^2 - \underset{\approx}{H}^3 + \underset{\approx}{H}^4 - \cdots \end{aligned} \qquad (2.8\text{-}9)$$

Note that if we had treated the tensors $\underset{\approx}{h}$ and $\underset{\approx}{H}$ simply as variables and had expanded $\underset{\approx}{F}$ and $\underset{\approx}{F}^{-1}$ into infinite series we would have obtained the same results as in (2.8-9). Also note that (2.8-9) represent convergent power series only if the maximum absolute values of the eigenvalues of $\underset{\approx}{h}$ and $\underset{\approx}{H}$ are less than one.

The use of (2.6-2) for the strain tensor leads to

$$\underset{\approx}{E} = \tfrac{1}{2}(\underset{\approx}{F}^T \cdot \underset{\approx}{F} - \underset{\approx}{I}) = \tfrac{1}{2}[(\underset{\approx}{I} + \underset{\approx}{H}^T)\cdot(\underset{\approx}{I} + \underset{\approx}{H}) - \underset{\approx}{I}]$$

$$\underset{\approx}{E} = \tfrac{1}{2}[\underset{\approx}{H} + \underset{\approx}{H}^T + \underset{\approx}{H}^T \underset{\approx}{H}] \qquad (2.8\text{-}10)$$

Similarly $\quad \underset{\approx}{e} = \tfrac{1}{2}[\underset{\approx}{h} + \underset{\approx}{h}^T - \underset{\approx}{h}^T \cdot \underset{\approx}{h}]$

In indicial notation the above relations are

$$E_{AB} = \tfrac{1}{2}[u_{A,B} + u_{B,A} + u_{C,A}u_{C,B}]$$

$$e_{ij} = \tfrac{1}{2}[u_{i,j} + u_{j,i} - u_{k,i}u_{k,j}] \qquad (2.8\text{-}11)$$

Hence $\quad E_{11} = \tfrac{1}{2}[u_{1,1} + u_{1,1} + (u_{1,1})^2 + (u_{2,1})^2 + (u_{3,1})^2]$

Infinitesimal Displacement Gradients

The components of $\underset{\approx}{H}$ or $\underset{\approx}{h}$ are all considered to be infinitesimal. This implies that the dot product $\underset{\approx}{H}^T \cdot \underset{\approx}{H}$ can be neglected in comparison with $\underset{\approx}{H} + \underset{\approx}{H}^T$. Similarly, $\underset{\approx}{h}^T \cdot \underset{\approx}{h}$ can be neglected in comparison with $\underset{\approx}{h} + \underset{\approx}{h}^T$.

Note that the above statements are not exactly correct when a component of $\underset{\approx}{H}$ (or $\underset{\approx}{h}$) equals zero. For this case, any corresponding value of $\underset{\approx}{H}^T \cdot \underset{\approx}{H}$ (or $\underset{\approx}{h}^T \cdot \underset{\approx}{h}$) may be significant (in comparison with zero).

We also assume that the components of $\underset{\approx}{H}$ are much less than unity so that

$$\underset{\approx}{F} = \underset{\approx}{I} + \underset{\approx}{H}$$

$$\simeq \underset{\approx}{I}$$

Hence, in infinitesimal theory

$$\underset{\approx}{E} \simeq \underset{\approx}{E}^{i} = \tfrac{1}{2}(\underset{\approx}{H} + \underset{\approx}{H}^T)$$

$$\underset{\approx}{e} \simeq \underset{\approx}{e}^{\ell} = \tfrac{1}{2}\,(\underset{\approx}{h} + \underset{\approx}{h}^{T}) \tag{2.8-12}$$

and
$$\begin{aligned} \underset{\approx}{H} &= \underset{\approx}{u}\nabla_0 \\ &= \underset{\approx}{u}\nabla\cdot\underset{\approx}{F} \\ &= \underset{\approx}{h}\cdot(\underset{\approx}{I} + \underset{\approx}{H}) \end{aligned}$$

or
$$\underset{\approx}{H} \simeq \underset{\approx}{h} \tag{2.8-13$_1$}$$

Therefore the gradient operator is the same with respect to either configuration, i.e.

$$\underset{\approx}{E}^{\ell} = \underset{\approx}{e} \tag{2.8-13$_2$}$$

Infinitesimal displacement gradients implies that there are infinitesimal strains and rotations but infinitesimal strains does not imply that there are infinitesimal displacement gradients, i.e., for example, the case of rigid body translation.

Consider the following

<u>Example:</u> $h_{ij} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$$e_{ij} = \tfrac{1}{2}\,[h_{ij} + h_{ji} - h_{ki}h_{kj}]$$

Hence

$$[e] = \tfrac{1}{2} < \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 4 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} >$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Therefore we have a case where a large displacement gradient results in zero strain.

Recall that

$$\underset{\approx}{h} = \underset{\sim}{u}\nabla = \tfrac{1}{2}(\underset{\sim}{u}\nabla + \nabla\underset{\sim}{u}) + \tfrac{1}{2}(\underset{\sim}{u}\nabla - \nabla\underset{\sim}{u})$$

$$= \underset{\approx}{e}^{i} + \underset{\approx}{\omega}$$

where $\underset{\approx}{\omega} = \tfrac{1}{2}(\underset{\approx}{h} - \underset{\approx}{h}^{T})$ (2.8-14)

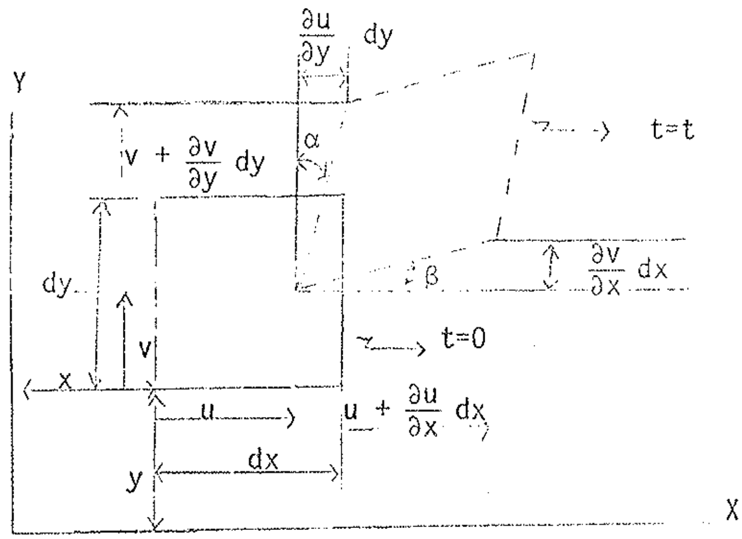
$\underset{\approx}{\omega}$ is known as the infinitesimal rotation tensor

We can express the gradients of the displacement vector in the following form

$$\begin{bmatrix} u_{1,1} & u_{1,2} & u_{1,3} \\ u_{2,1} & u_{2,2} & u_{2,3} \\ u_{3,1} & u_{3,2} & u_{3,3} \end{bmatrix} = \begin{bmatrix} u_{,x} & u_{,y} & u_{,z} \\ v_{,x} & v_{,y} & v_{,z} \\ w_{,x} & w_{,y} & w_{,z} \end{bmatrix}$$

where the displacements u,v and w correspond to the directions x,y and z respectively.

Consider a body that undergoes an infinitesimal deformation over a period of time.



From the above figure and the definition of infinitesimal strains it follows that the strain components are

$$\varepsilon_x = \frac{\partial u}{\partial x}$$

$$\varepsilon_y = \frac{\partial v}{\partial y}$$

$$\varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = \frac{1}{2}(\alpha + \beta)$$

and the infinitesimal nonzero rotation component is

$$\omega_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right)$$

$$= \frac{1}{2}(\alpha - \beta)$$

Section 9 Rigid Body Motion

Rigid body motion implies that $\underset{\approx}{E} = \underset{\approx}{0}$ for all time and for all material points. Hence

$$2\underset{\approx}{E} = \underset{\approx}{F}^T \cdot \underset{\approx}{F} - \underset{\approx}{I} = \underset{\approx}{0} \qquad (2.9\text{-}1)$$

and

$$\underset{\approx}{F}^T \cdot \underset{\approx}{F} = \underset{\approx}{I}$$

or

$$\underset{\approx}{F}^T = \underset{\approx}{F}^{-1}$$

Therefore $\underset{\approx}{F}$ is an orthogonal tensor under rigid body motion.

For this case let

$$\underset{\approx}{F} = \underset{\approx}{R} \qquad (2.9\text{-}2)$$

where $\underset{\approx}{R}$ is known as the rotation tensor.

We want to show that $\underset{\approx}{R}$ is not a function of position. Since

$$(\underset{\approx}{R}^T \cdot \underset{\approx}{R}) = \underset{\approx}{I}$$

it follows that

$$J = \det(\underset{\approx}{F}) = \det(\underset{\approx}{R}) = 1$$

and from Eq. (2.3-8) with $\underset{\approx}{F}^{-T} = \underset{\approx}{R}$
we get

$$\underset{\approx}{R} \cdot \nabla_0 = \underset{\sim}{0} \qquad (2.9\text{-}3)$$

Note: There is some question at this point as to whether or not this is sufficient. The condition $\underset{\approx}{R}\nabla_0 = \underset{\approx}{0}$ would definitely be sufficient. Hence $\underset{\approx}{R}$ is the same for every point in a rigid body although it may be a function of time.

Now recall

$$d\underset{\sim}{r} = \underset{\approx}{F}\cdot d\underset{\sim}{R}$$

$$= \underset{\approx}{R}\cdot d\underset{\sim}{R}$$

Integrating along a path between two points within a rigid body we obtain

$$\int_A^B d\underset{\sim}{r} = \int_A^B \underset{\approx}{R}\cdot d\underset{\sim}{R} = \underset{\approx}{R}\cdot\int_A^B d\underset{\sim}{R}$$

Note that we can bring $\underset{\approx}{R}$ outside the integral sign since it is not a function of position. An illustration of this rigid body is shown below

t=0 t=t

Hence

$$\int_A^B d\underset{\sim}{R} = \underset{\sim}{\rho}_0{}^{AB} \qquad \text{and} \qquad \int_A^B d\underset{\sim}{r} = \underset{\sim}{\rho}^{AB}$$

and

$$\left.\begin{aligned} \underset{\sim}{\rho}^{AB} &= \underset{\approx}{R}\cdot\underset{\sim}{\rho}_0{}^{AB} \\ \underset{\sim}{\rho}_0{}^{AB} &= \underset{\approx}{R}^T\cdot\underset{\sim}{\rho}^{AB} \end{aligned}\right\} \qquad (2.9\text{-}4)$$

The time derivative of $\underset{\sim}{\rho}^{AB}$ yields

$$\dot{\underset{\sim}{\rho}}^{AB} = \dot{\underset{\approx}{R}}\cdot\underset{\sim}{\rho}_0{}^{AB} + \underset{\approx}{R}\cdot\dot{\underset{\sim}{\rho}}_0{}^{AB}$$

By observing that $\dot{\underset{\sim}{\rho}}_0{}^{AB} = 0$ and substituting for $\underset{\sim}{\rho}_0{}^{AB}$ we obtain

$$\dot{\underset{\sim}{\rho}}^{AB} = \dot{\underset{\approx}{R}}\cdot\underset{\approx}{R}^T\cdot\underset{\sim}{\rho}^{AB} \qquad (2.9\text{-}5)$$

Recall that

$$\underset{\approx}{R}\cdot\underset{\approx}{R}^T = \underset{\approx}{I}$$

After taking the time derivative of both sides of the above equation we get

$$\dot{\underset{\approx}{R}}\cdot\underset{\approx}{R}^T + \underset{\approx}{R}\cdot\dot{\underset{\approx}{R}}^T = \underset{\approx}{0}$$

Now let

$$\underset{\approx}{\Omega} = \dot{\underset{\approx}{R}}\cdot\underset{\approx}{R}^T \qquad (2.9\text{-}6)$$

Then

$$\underset{\approx}{\Omega}^T = \underset{\approx}{R}\cdot\dot{\underset{\approx}{R}}^T$$

Hence, after a substitution into the previous equation

$$\underset{\approx}{\Omega} + \underset{\approx}{\Omega}^T = \underset{\approx}{0}$$

or

$$\underset{\approx}{\Omega} = -\underset{\approx}{\Omega}^T \qquad (2.9\text{-}7)$$

and $\underset{\approx}{\Omega}$ is skew-symmetric.

Hence

$$\dot{\underset{\sim}{\rho}}^{AB} = \underset{\approx}{\Omega}\cdot\underset{\sim}{\rho}^{AB} \qquad (2.9\text{-}8)$$

Now let

$$\Omega_{ij} = -\varepsilon_{ijk}\,\omega_k \qquad (2.9\text{-}9)$$

$$\underset{\approx}{\Omega} = -C_{34}(\underset{\approx}{\varepsilon} \otimes \underset{\sim}{\omega})$$

where $\underset{\sim}{\omega}$ is called the axial vector of $\underset{\approx}{\Omega}$ or the angular velocity vector.

In indicial form (2.9-8) becomes

$$\dot{\rho}_i^{AB} = \Omega_{ij}\,\rho_j^{AB}$$

$$= -\varepsilon_{ijk}\rho_j^{AB}\omega_k$$

We observe that the above equation is the indicial form of the cross product:

$$\dot{\underset{\sim}{\rho}}^{AB} = - \underset{\sim}{\rho}^{AB} \times \underset{\sim}{\omega} \qquad (2.9\text{-}10)$$
$$= \underset{\sim}{\omega} \times \underset{\sim}{\rho}^{AB}$$

If we express $\underset{\sim}{\rho}^{AB}$ in terms of the position vectors we observe

$$\underset{\sim}{\rho}^{AB} = \underset{\sim}{r}^{A} - \underset{\sim}{r}^{B}$$

and

$$\dot{\underset{\sim}{\rho}}^{AB} = \dot{\underset{\sim}{r}}^{A} - \dot{\underset{\sim}{r}}^{B} \qquad (2.9\text{-}11)$$
$$= \underset{\sim}{v}^{A} - \underset{\sim}{v}^{B}$$

where $\underset{\sim}{v}$ is the velocity vector.

The substitution of (2.9-11) into (2.9-10) yields the standard equation relating the velocity of two points in a rigid body:

$$\underset{\sim}{v}^{B} = \underset{\sim}{v}^{A} + \underset{\sim}{\omega} \times \underset{\sim}{\rho}^{AB} \qquad (2.9\text{-}12)$$

The time derivative of (2.9-12) yields the corresponding acceleration equation.

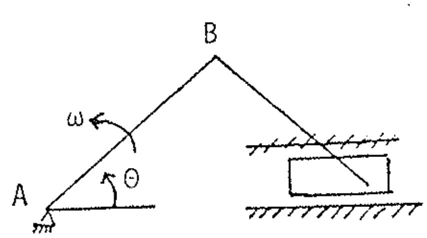
In general, we note that

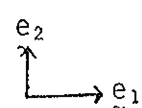
$$\dot{\underset{\sim}{\rho}} = \underset{\sim}{\omega} \times \underset{\sim}{\rho}$$

where ρ is any vector joining two points in the rigid body.

Example:

Crank-Slider Mechanism





$$\underset{\sim}{\omega} = \omega\, \underset{\sim}{e}_3$$
$$\underset{\sim}{v}^{A} = \underset{\sim}{0}$$
$$\underset{\sim}{\rho}^{AB} = \rho^{AB}(\cos\Theta\, \underset{\sim}{e}_1 + \sin\Theta\, \underset{\sim}{e}_2)$$

$$\dot{\underset{\sim}{\rho}}^{AB} = \underset{\sim}{\omega} \times \underset{\sim}{\rho}^{AB} = \underset{\sim}{v}^{B} - \underset{\sim}{v}^{A}$$
$$\underset{\sim}{v}^{B} = \omega \underset{\sim}{e}_3 \times \underset{\sim}{\rho}^{AB}$$
$$= \omega\, \rho^{AB}(-\sin\Theta\, \underset{\sim}{e}_1 + \cos\Theta\, \underset{\sim}{e}_2)$$
$$|\underset{\sim}{v}^{B}| = \omega\, \rho^{AB}$$

Sect. 9 Relations Involving the Polar Decomposition of $\underline{\underline{F}}$

Polar Decomposition Theorem

$\underline{\underline{F}}$ is assumed to be nonsingular. Then $\underline{\underline{F}}$ admits the unique representations

$$\underline{\underline{F}} = \underline{\underline{R}} \cdot \underline{\underline{U}} = \underline{\underline{V}} \cdot \underline{\underline{R}} \qquad (2.9-1)$$

referred to as the right and left decompositions, respectively, where $\underline{\underline{U}}$ and $\underline{\underline{V}}$ are positive definite symmetric tensors and $\underline{\underline{R}}$ is an orthogonal tensor. It follows that

$$\underline{\underline{U}}^2 = \underline{\underline{F}}^T \cdot \underline{\underline{F}} \qquad \underline{\underline{V}}^2 = \underline{\underline{F}} \cdot \underline{\underline{F}}^T$$

Let

$$\underline{\underline{C}} = \underline{\underline{F}}^T \cdot \underline{\underline{F}} \qquad \underline{\underline{B}} = \underline{\underline{F}} \cdot \underline{\underline{F}}^T \qquad (2.9-2)$$

Then

$$\underline{\underline{U}} = \underline{\underline{C}}^{1/2} \qquad \underline{\underline{V}} = \underline{\underline{B}}^{1/2} \qquad (2.9-3)$$

Now let

$$\underline{\underline{R}} = \underline{\underline{F}} \cdot \underline{\underline{U}}^{-1} \qquad \underline{\underline{R}}^* = \underline{\underline{V}}^{-1} \cdot \underline{\underline{F}} \qquad (2.9-4)$$

To verify that $\underline{\underline{R}}$ is orthogonal, consider

$$\begin{aligned}
\underline{\underline{R}}^T \cdot \underline{\underline{R}} &= \underline{\underline{U}}^{-T} \cdot \underline{\underline{F}}^T \cdot \underline{\underline{F}} \cdot \underline{\underline{U}}^{-1} \\
&= \underline{\underline{U}}^{-1} \cdot \underline{\underline{F}}^T \cdot \underline{\underline{F}} \cdot \underline{\underline{U}}^{-1} \\
&= \underline{\underline{U}}^{-1} \cdot \underline{\underline{C}} \cdot \underline{\underline{U}}^{-1} \\
&= \underline{\underline{U}}^{-1} \cdot \underline{\underline{U}}^2 \cdot \underline{\underline{U}}^{-1} \\
&= \underline{\underline{I}}
\end{aligned}$$

$$\begin{aligned}
\underline{\underline{R}}^* \cdot \underline{\underline{R}}^{*T} &= \\
&\vdots \\
&= \underline{\underline{I}}
\end{aligned}$$

To prove that $\underline{\underline{R}} \cdot \underline{\underline{U}}$ is a unique decomposition, suppose that

$$\underline{\underline{F}} = \underline{\underline{R}} \cdot \underline{\underline{U}} = \underline{\underline{R}}_o \cdot \underline{\underline{U}}_o \quad \text{where } \underline{\underline{R}}_o \text{ and } \underline{\underline{U}}_o$$

represents another decomposition. Now

$$\underline{\underline{F}}^T \cdot \underline{\underline{F}} = \underline{\underline{U}}^2 = \underline{\underline{U}}_o^2 \quad , \text{ hence } \quad \underline{\underline{U}} = \underline{\underline{U}}_o$$

and from $\underline{\underline{R}} \cdot \underline{\underline{U}} = \underline{\underline{R}}_o \cdot \underline{\underline{U}}_o$ it follows that $\underline{\underline{R}} = \underline{\underline{R}}_o$.

To prove that $\underline{\underline{R}}^* = \underline{\underline{R}}$, write

$$\begin{aligned} \underline{\underline{F}} &= \underline{\underline{V}} \cdot \underline{\underline{R}}^* \\ &= \underline{\underline{R}}^* \cdot \underline{\underline{R}}^{*T} \cdot \underline{\underline{V}} \cdot \underline{\underline{R}}^* \\ &= \underline{\underline{R}} \cdot \underline{\underline{U}} \end{aligned}$$

Use the fact that $\underline{\underline{R}}$ and $\underline{\underline{U}}$ are unique to conclude that

$$\underline{\underline{R}}^* = \underline{\underline{R}} \qquad (2.9-5)$$

and

$$\underline{\underline{R}}^{*T} \cdot \underline{\underline{V}} \cdot \underline{\underline{R}}^* = \underline{\underline{U}} \qquad (2.9-6)$$

$$\underline{\underline{V}} = \underline{\underline{R}} \cdot \underline{\underline{U}} \cdot \underline{\underline{R}}^T$$

Relationships That Involve the Polar Decomposition of $\underline{\underline{F}}$

$$\underline{\underline{F}} = \underline{\underline{R}} \cdot \underline{\underline{U}} = \underline{\underline{V}} \cdot \underline{\underline{R}}$$

$$\underline{\underline{C}} = \underline{\underline{U}}^2 \qquad \text{Right Cauchy-Green Tensor}$$

$$\underline{\underline{B}} = \underline{\underline{V}}^2 \qquad \text{Left Cauchy-Green Tensor}$$

Then

$$\underline{\underline{E}} = \tfrac{1}{2}(\underline{\underline{C}} - \underline{\underline{I}}) \qquad \underline{\underline{\epsilon}} = \tfrac{1}{2}(\underline{\underline{I}} - \underline{\underline{B}}^{-1})$$

$\underline{\underline{U}}, \underline{\underline{V}}$ - right & left stretch tensors.

Recall

$$d\underset{\sim}{r} = \underline{\underline{F}} \cdot d\underset{\sim}{R} = \underline{\underline{R}} \cdot \underline{\underline{U}} \cdot d\underset{\sim}{R}$$

Let

$$d\underset{\sim}{R}^* = \underline{\underline{U}} \cdot d\underset{\sim}{R} \qquad \text{stretch}$$

$$d\underset{\sim}{r} = \underline{\underline{R}} \cdot d\underset{\sim}{R}^* \qquad \text{rotation}$$

Let

$$d\underset{\sim}{r}^* = \underline{\underline{R}} \cdot d\underset{\sim}{R} \qquad \text{rotation.}$$

$$d\underset{\sim}{r} = \underline{\underline{V}} \cdot d\underset{\sim}{r}^* \qquad \text{stretch}$$

Thus the operation of $\underline{\underline{F}} = \underline{\underline{R}} \cdot \underline{\underline{U}}$ $(\underline{\underline{V}} \cdot \underline{\underline{R}})$ can be thought of as a stretch followed by a rotation (rotation followed by a stretch).

Let $\Lambda_{(\alpha)}, \underset{\sim}{N}_\alpha$ be the eigenpair for $\underline{\underline{U}}$

$\lambda_{(\alpha)}, \underset{\sim}{n}_\alpha$ be the eigenpair for $\underline{\underline{V}}$

Then

$$\underline{\underline{U}} = \Lambda_{(\alpha)} \underset{\sim}{N}_\alpha \otimes \underset{\sim}{N}_\alpha \qquad \underline{\underline{V}} = \lambda_{(\alpha)} \underset{\sim}{n}_\alpha \otimes \underset{\sim}{n}_\alpha$$

But

$$\underline{\underline{V}} = \underline{\underline{R}} \cdot \underline{\underline{U}} \cdot \underline{\underline{R}}^T = \Lambda_{(\alpha)} (\underline{\underline{R}} \cdot \underset{\sim}{N}_\alpha) \otimes (\underset{\sim}{N}_\alpha \cdot \underline{\underline{R}}^T)$$

The operation of an orthogonal tensor on a set of vectors produces a new set with lengths unchanged and angles between the vectors maintained. Thus the eigenvectors of $\underset{\approx}{V}$ are

$$\underset{\sim}{n}_\alpha = \underset{\approx}{R} \cdot \underset{\sim}{N}_\alpha$$

and

$$\lambda_{(\alpha)} = \Lambda_{(\alpha)}$$

It follows that the rotation tensor is given by

$$\underset{\approx}{R} = \underset{\sim}{n}_\alpha \otimes \underset{\sim}{N}_\alpha$$

Relationships Involving Rates

Recall that

$$\underline{\underline{F}} = \underline{\underline{R}} \cdot \underline{\underline{U}}$$

$$\dot{\underline{\underline{F}}} = \underline{\underline{L}} \cdot \underline{\underline{F}} = \dot{\underline{\underline{R}}} \cdot \underline{\underline{U}} + \underline{\underline{R}} \cdot \dot{\underline{\underline{U}}}$$

$$\underline{\underline{F}}^{-1} = \underline{\underline{U}}^{-1} \cdot \underline{\underline{R}}^T$$

$$\underline{\underline{L}} = \underline{\underline{D}} + \underline{\underline{W}} = \dot{\underline{\underline{F}}} \cdot \underline{\underline{F}}^{-1}$$ $\underline{\underline{W}}$ - vorticity

$$= \dot{\underline{\underline{R}}} \cdot \underline{\underline{R}}^T + \underline{\underline{R}} \cdot \dot{\underline{\underline{U}}} \cdot \underline{\underline{U}}^{-1} \cdot \underline{\underline{R}}^T$$

$$\underline{\underline{L}}^T = \underline{\underline{R}} \cdot \dot{\underline{\underline{R}}}^T + \underline{\underline{R}} \cdot \underline{\underline{U}}^{-1} \cdot \dot{\underline{\underline{U}}} \cdot \underline{\underline{R}}^T$$

Let

$$\underline{\underline{\Omega}} = \dot{\underline{\underline{R}}} \cdot \underline{\underline{R}}^T$$ Spin

$\underline{\underline{\Omega}}$ - skew-symmetric

$$\underline{\underline{D}}^* = \frac{1}{2} (\dot{\underline{\underline{U}}} \cdot \underline{\underline{U}}^{-1} + \underline{\underline{U}}^{-1} \cdot \dot{\underline{\underline{U}}})$$

Then

$$\underline{\underline{D}} = \frac{1}{2} (\underline{\underline{L}} + \underline{\underline{L}}^T) = \underline{\underline{R}} \cdot \underline{\underline{D}}^* \cdot \underline{\underline{R}}^T \quad \text{or} \quad \underline{\underline{D}}^* = \underline{\underline{R}}^T \cdot \underline{\underline{D}} \cdot \underline{\underline{R}}$$

$$\underline{\underline{W}} = \frac{1}{2} (\underline{\underline{L}} - \underline{\underline{L}}^T) = \underline{\underline{\Omega}} + \frac{1}{2} \underline{\underline{R}} \cdot (\dot{\underline{\underline{U}}} \cdot \underline{\underline{U}}^{-1} - \underline{\underline{U}}^{-1} \cdot \dot{\underline{\underline{U}}}) \cdot \underline{\underline{R}}^T$$

$$\mathbf{F} = \mathbf{V}\cdot\mathbf{R} \qquad \mathbf{F}^{-1} = \mathbf{R}^T\cdot\mathbf{V}^{-1}$$

$$\dot{\mathbf{F}} = \mathbf{L}\cdot\mathbf{F} = \dot{\mathbf{V}}\cdot\mathbf{R} + \mathbf{V}\cdot\dot{\mathbf{R}} \qquad \dot{\mathbf{R}} = \mathbf{\Omega}\cdot\mathbf{R}$$

$$\mathbf{L} = \mathbf{D} + \mathbf{W} = \dot{\mathbf{F}}\cdot\mathbf{F}^{-1}$$

$$= \dot{\mathbf{V}}\cdot\mathbf{V}^{-1} + \mathbf{V}\cdot\mathbf{\Omega}\cdot\mathbf{R}\cdot\mathbf{R}^T\cdot\mathbf{V}^{-1}$$

$$= (\dot{\mathbf{V}} + \mathbf{V}\cdot\mathbf{\Omega})\cdot\mathbf{V}^{-1}$$

$$(\mathbf{D} + \mathbf{W})\cdot\mathbf{V} = \dot{\mathbf{V}} + \mathbf{V}\cdot\mathbf{\Omega}$$ Take transpose

$$\mathbf{V}\cdot(\mathbf{D} - \mathbf{W}) = \dot{\mathbf{V}} - \mathbf{\Omega}\cdot\mathbf{V}$$ subtract

$$\mathbf{D}\cdot\mathbf{V} - \mathbf{V}\cdot\mathbf{D} + \mathbf{W}\cdot\mathbf{V} + \mathbf{V}\cdot\mathbf{W} = \mathbf{V}\cdot\mathbf{\Omega} + \mathbf{\Omega}\cdot\mathbf{V}$$

$$\mathbf{D}\cdot\mathbf{V} - \mathbf{V}\cdot\mathbf{D} = \mathbf{V}\cdot(\mathbf{\Omega} - \mathbf{W}) + (\mathbf{\Omega} - \mathbf{W})\cdot\mathbf{V}$$

Let ω, w and z be the axial vectors of $\mathbf{\Omega}$, $\mathbf{W}$ + $\mathbf{D}\cdot\mathbf{V}-\mathbf{V}\cdot\mathbf{D}$

i.e., $\mathbf{\Omega} = \boldsymbol{\varepsilon}\cdot\omega \qquad \mathbf{W} = \boldsymbol{\varepsilon}\cdot w \qquad \mathbf{D}\cdot\mathbf{V} - \mathbf{V}\cdot\mathbf{D} = \boldsymbol{\varepsilon}\cdot z$

$$\boldsymbol{\varepsilon}\cdot z = \mathbf{V}\cdot\boldsymbol{\varepsilon}\cdot(\omega - w) + \boldsymbol{\varepsilon} : (\omega - w)\otimes\mathbf{V}$$

$$\varepsilon_{ijk} z_k = V_{il}\,\varepsilon_{ljk}(\omega_k - w_k) + \varepsilon_{iml}(\omega_l - w_l)V_{mj}$$

Multiply by ε_{ijp}

$$\varepsilon_{ijk}\,\varepsilon_{ijp} = 2\delta_{pk}$$

$$\varepsilon_{ijp}\,\varepsilon_{ljk} = \varepsilon_{jpi}\,\varepsilon_{jkl} = \delta_{kp}\delta_{il} - \delta_{pl}\delta_{ik}$$

$$\varepsilon_{ijp}\,\varepsilon_{iml} = \delta_{jm}\delta_{pl} - \delta_{jl}\delta_{pm}$$

$$2z_p = V_{ii}(\omega_p - w_p) - V_{pi}(\omega_i - w_i) - (\omega_j - w_j)V_{pj} + (\omega_p - w_p)V_{jj}$$

$$z_p = \left[(V_{ii})\,\delta_{pj} - V_{pj}\right](\omega_j - w_j)$$

or $$z = \left[(\operatorname{tr}\mathbf{V})\,\mathbf{I} - \mathbf{V}\right](\omega - w)$$

or

$$\omega = w + \left[(\operatorname{tr}\mathbf{V})\,\mathbf{I} - \mathbf{V}\right]^{-1}\cdot z$$

Hence: $\mathbf{\Omega}$ is expressed in terms of $\mathbf{W}$; $\mathbf{V}$ + $\mathbf{D}$

Consider a differential of material $d\underset{\sim}{x}$.

The associated unit vector is $\underset{\sim}{n} = \dfrac{d\underset{\sim}{x}}{ds}$ $\quad ds = (d\underset{\sim}{x} \cdot d\underset{\sim}{x})^{1/2}$

Recall $\quad d\underset{\sim}{x} = \underset{=}{F} \cdot d\underset{\sim}{X}$

Take time derivative $\quad \dot{\overline{d\underset{\sim}{x}}} = \dot{\underset{=}{F}} \cdot d\underset{\sim}{X}$

$$= \underset{=}{L} \cdot \underset{=}{F}^{-1} \cdot d\underset{\sim}{X}$$

$$= \underset{=}{L} \cdot d\underset{\sim}{x}$$

$$d\dot{\underset{\sim}{s}} = \frac{1}{2ds}\left(\dot{\overline{d\underset{\sim}{x}}} \cdot d\underset{\sim}{x} + d\underset{\sim}{x} \cdot \dot{\overline{d\underset{\sim}{x}}} \right)$$

$$= \frac{1}{2ds}\, d\underset{\sim}{x} \cdot (\underset{=}{L}^T + \underset{=}{L}) \cdot d\underset{\sim}{x}$$

$$= \frac{1}{2ds}\, d\underset{\sim}{x} \cdot \underset{=}{D} \cdot d\underset{\sim}{x}$$

$$\dot{\underset{\sim}{n}} = \frac{\dot{\overline{d\underset{\sim}{x}}}}{ds} - \frac{d\underset{\sim}{x}}{ds}\, \frac{\dot{\overline{ds}}}{ds}$$

$$= \underset{=}{L} \cdot \underset{\sim}{n} - \underset{\sim}{n} (\underset{\sim}{n} \cdot \underset{=}{D} \cdot \underset{\sim}{n})$$

$$= \left[\underset{=}{D} + \underset{=}{W} - (\underset{\sim}{n} \cdot \underset{=}{D} \cdot \underset{\sim}{n}) \underset{=}{i} \right] \cdot \underset{\sim}{n}$$

Now suppose $\underset{\sim}{n}$ is an eigenvector of $\underset{=}{D}$: $\underset{=}{D} \cdot \underset{\sim}{n} = \lambda \underset{\sim}{n}$

Then

$$\dot{\underset{\sim}{n}} = \underset{=}{W} \cdot \underset{\sim}{n} \qquad \text{(Theorem of Gosiewski)}$$

Define left and right spin tensors ($\underline{\underline{\Omega}}_L$ and $\underline{\underline{\Omega}}_R$) by the following:

$$\dot{\underset{\sim}{n}}_\alpha = \underline{\underline{\Omega}}_L \cdot \underset{\sim}{n}_\alpha \qquad \dot{\underset{\sim}{N}}_\alpha = \underline{\underline{\Omega}}_R \cdot \underset{\sim}{N}_\alpha$$

Then

$$\dot{\underline{\underline{R}}} = \dot{\underset{\sim}{n}}_\alpha \otimes \underset{\sim}{N}_\alpha + \underset{\sim}{n}_\alpha \otimes \dot{\underset{\sim}{N}}_\alpha$$

$$= (\underline{\underline{\Omega}}_L \cdot \underset{\sim}{n}_\alpha) \otimes \underset{\sim}{N}_\alpha + \underset{\sim}{n}_\alpha \otimes \underline{\underline{\Omega}}_R \cdot \underset{\sim}{N}_\alpha$$

$$\underline{\underline{\Omega}} = \dot{\underline{\underline{R}}} \cdot \underline{\underline{R}}^T = \dot{\underline{\underline{R}}} \cdot (\underset{\sim}{N}_\beta \otimes \underset{\sim}{n}_\beta)$$

$$= (\underline{\underline{\Omega}}_L \cdot \underset{\sim}{n}_\alpha) \otimes \underset{\sim}{n}_\alpha + \underset{\sim}{n}_\alpha (\underset{\sim}{N}_\beta \cdot \underline{\underline{\Omega}}_R \cdot \underset{\sim}{N}_\alpha) \otimes \underset{\sim}{n}_\beta$$

$$= \underline{\underline{\Omega}}_L + \underline{\underline{R}} \cdot \underset{\sim}{N}_\alpha (\underset{\sim}{N}_\alpha \cdot \underline{\underline{\Omega}}_R^T \cdot \underset{\sim}{N}_\beta) \otimes \underset{\sim}{N}_\beta \cdot \underline{\underline{R}}^T$$

$$= \underline{\underline{\Omega}}_L + \underline{\underline{R}} \cdot \underline{\underline{\Omega}}_R^T \cdot \underline{\underline{R}}^T$$

$$= \underline{\underline{\Omega}}_L - \underline{\underline{R}} \cdot \underline{\underline{\Omega}}_R \cdot \underline{\underline{R}}^T.$$

Theorem of Gosiewski: (Pg. 173, Jaunzemis)

If $\underset{\sim}{d}_\alpha$ is the principal basis of $\underline{\underline{D}}$, then

$$\dot{\underset{\sim}{d}}_\alpha = \underline{\underline{W}} \cdot \underset{\sim}{d}_\alpha$$

Define left and right orientation tensors ($\mathbf{R}_L$ and $\mathbf{R}_R$) by the following:

$$\mathbf{R}_L = \mathbf{n}_\alpha \otimes \mathbf{E}_\alpha \qquad\qquad \mathbf{R}_R = \mathbf{N}_\alpha \otimes \mathbf{E}_\alpha$$

where $\mathbf{E}_\alpha$ represents an orthonormal basis $\Rightarrow \dot{\mathbf{E}}_\alpha = \mathbf{0}$

Then

$$\dot{\mathbf{R}}_L = \dot{\mathbf{n}}_\alpha \otimes \mathbf{E}_\alpha \qquad\qquad \dot{\mathbf{R}}_R = \dot{\mathbf{N}}_\alpha \otimes \mathbf{E}_\alpha$$

$$= (\boldsymbol{\Omega}_L \cdot \mathbf{n}_\alpha) \otimes \mathbf{E}_\alpha \qquad\qquad = (\boldsymbol{\Omega}_R \cdot \mathbf{N}_\alpha) \otimes \mathbf{E}_\alpha$$

and

$$\dot{\mathbf{R}}_L \cdot \mathbf{R}_L^T = (\boldsymbol{\Omega}_L \cdot \mathbf{n}_\alpha) \otimes \mathbf{E}_\alpha \cdot (\mathbf{E}_\beta \otimes \mathbf{n}_\beta) \qquad \dot{\mathbf{R}}_R \cdot \mathbf{R}_R^T = (\boldsymbol{\Omega}_R \cdot \mathbf{N}_\alpha) \otimes \mathbf{E}_\alpha \cdot (\mathbf{E}_\beta \otimes \mathbf{N}_\beta)$$

$$= (\boldsymbol{\Omega}_L \cdot \mathbf{n}_\alpha) \otimes \mathbf{n}_\alpha \qquad\qquad = (\boldsymbol{\Omega}_R \cdot \mathbf{N}_\alpha) \otimes \mathbf{N}_\alpha$$

$$= \boldsymbol{\Omega}_L \qquad\qquad = \boldsymbol{\Omega}_R$$

Note that

$$\mathbf{R} = \mathbf{R}_L \cdot \mathbf{R}_R^T$$

$$\dot{\mathbf{R}} = \dot{\mathbf{R}}_L \cdot \mathbf{R}_R^T + \mathbf{R}_L \cdot \dot{\mathbf{R}}_R^T$$

$$= \boldsymbol{\Omega}_L \cdot \mathbf{R}_L \cdot \mathbf{R}_R^T + \mathbf{R}_L \cdot \mathbf{R}_R^T \cdot \boldsymbol{\Omega}_R^T$$

$$= \boldsymbol{\Omega}_L \cdot \mathbf{R} - \mathbf{R} \cdot \boldsymbol{\Omega}_R$$

$$\dot{\mathbf{R}} \cdot \mathbf{R}^T = \boldsymbol{\Omega}_L - \mathbf{R} \cdot \boldsymbol{\Omega}_R \cdot \mathbf{R}^T$$

$$\boldsymbol{\Omega} = \boldsymbol{\Omega}_L - \mathbf{R} \cdot \boldsymbol{\Omega}_R \cdot \mathbf{R}^T$$

An Alternative Expression for $\underline{\underline{R}}$

Recall that

$$\underline{\underline{R}} = \underset{\sim}{n}_\alpha \otimes \underset{\sim}{N}_\alpha \qquad \underline{\underline{R}}_R = \underset{\sim}{N}_\alpha \otimes \underset{\sim}{E}_\alpha \quad \text{or} \quad \underset{\sim}{N}_\alpha = \underset{\sim}{E}_\alpha \cdot \underline{\underline{R}}_R^T$$

$$= \underset{\sim}{n}_\alpha \otimes \underset{\sim}{E}_\alpha \cdot \underline{\underline{R}}_R^T$$

$$= \underset{\sim}{n}_\alpha \otimes (\underset{\sim}{E}_\alpha \cdot \underline{\underline{R}}_R^T \cdot \underset{\sim}{E}_\beta)\, \underset{\sim}{E}_\beta$$

Let

$$\underline{\underline{R}} = \underset{\sim}{r}_\beta \otimes \underset{\sim}{E}_\beta$$

where

$$\underset{\sim}{r}_\beta = (\underset{\sim}{E}_\alpha \cdot \underline{\underline{R}}_R^T \cdot \underset{\sim}{E}_\beta)\, \underset{\sim}{n}_\alpha$$

$$= (\underset{\sim}{E}_\beta \cdot \underline{\underline{R}}_R \cdot \underset{\sim}{E}_\alpha)\, \underset{\sim}{n}_\alpha$$

and

$$\dot{\underset{\sim}{r}}_\beta = \underline{\underline{\Omega}} \cdot \underset{\sim}{r}_\beta$$

Check

$$\dot{\underset{\sim}{r}}_\beta = (\underset{\sim}{E}_\beta \cdot \dot{\underline{\underline{R}}}_R \cdot \underset{\sim}{E}_\alpha)\, \underset{\sim}{n}_\alpha + (\underset{\sim}{E}_\beta \cdot \underline{\underline{R}}_R \cdot \underset{\sim}{E}_\alpha)\, \dot{\underset{\sim}{n}}_\alpha$$

$$= \left[\underset{\sim}{E}_\beta \cdot (\underline{\underline{\Omega}}_R \cdot \underset{\sim}{N}_\gamma) \otimes \underset{\sim}{E}_\gamma \cdot \underset{\sim}{E}_\alpha\right] \underset{\sim}{n}_\alpha + (\underset{\sim}{E}_\beta \cdot \underline{\underline{R}}_R \cdot \underset{\sim}{E}_\alpha)\, \underline{\underline{\Omega}}_L \cdot \underset{\sim}{n}_\alpha$$

$$= (\underset{\sim}{r}_\beta \cdot \underline{\underline{R}}) \cdot \underline{\underline{\Omega}}_R \cdot (\underset{\sim}{n} \cdot \underline{\underline{R}})\, \underset{\sim}{n}_\alpha + \underline{\underline{\Omega}}_L \cdot \underset{\sim}{r}_\beta$$

$$= \underset{\sim}{r}_\beta \cdot \underline{\underline{R}} \cdot \underline{\underline{\Omega}}_R \cdot \underline{\underline{R}}^T + \underline{\underline{\Omega}}_L \cdot \underset{\sim}{r}_\beta$$

$$= -\underline{\underline{R}} \cdot \underline{\underline{\Omega}}_R \cdot \underline{\underline{R}}^T \cdot \underset{\sim}{r}_\beta + \underline{\underline{\Omega}}_L \cdot \underset{\sim}{r}_\beta$$

$$= (\underline{\underline{\Omega}}_L - \underline{\underline{R}} \cdot \underline{\underline{\Omega}}_R \cdot \underline{\underline{R}}^T) \cdot \underset{\sim}{r}_\beta$$

$$= \underline{\underline{\Omega}} \cdot \underset{\sim}{r}_\beta$$

Logarithmic Strains and Their Derivatives

In addition to the right and left stretch tensors, define a third tensor

$$\hat{\underline{\underline{U}}} = \underline{\underline{R}}_R^T \cdot \underline{\underline{U}} \cdot \underline{\underline{R}}_R = \underline{\underline{R}}_L^T \cdot \underline{\underline{V}} \cdot \underline{\underline{R}}_L$$

Then

$$\underline{\underline{U}} = \Lambda_{(\alpha)} \underline{N}_\alpha \otimes \underline{N}_\alpha \qquad \underline{\underline{V}} = \Lambda_{(\alpha)} \underline{n}_\alpha \otimes \underline{n}_\alpha \qquad \hat{\underline{\underline{U}}} = \Lambda_{(\alpha)} \underline{E}_\alpha \otimes \underline{E}_\alpha$$

Corresponding logarithmic strains would be

$$\underline{\underline{\mathcal{L}}}_R = \ln \Lambda_{(\alpha)} \underline{N}_\alpha \otimes \underline{N}_\alpha = \ln \underline{\underline{U}} \qquad \underline{\underline{\mathcal{L}}}_L \underline{\underline{V}} = \ln \Lambda_{(\alpha)} \underline{n}_\alpha \otimes \underline{n}_\alpha = \ln \underline{\underline{V}} \qquad \hat{\underline{\underline{\mathcal{L}}}} = \ln \Lambda_{(\alpha)} \underline{E}_\alpha \otimes \underline{E}_\alpha = \ln \hat{\underline{\underline{U}}}$$

Then

$$\dot{\hat{\underline{\underline{\mathcal{L}}}}} = \frac{\dot{\Lambda}_{(\alpha)}}{\Lambda_{(\alpha)}} \underline{E}_\alpha \otimes \underline{E}_\alpha$$

Define

$$\hat{\underline{\underline{D}}} = \frac{1}{2} \left(\dot{\hat{\underline{\underline{U}}}} \cdot \hat{\underline{\underline{U}}}^{-1} + \hat{\underline{\underline{U}}}^{-1} \cdot \dot{\hat{\underline{\underline{U}}}} \right)$$

$$= \frac{\dot{\Lambda}_{(\alpha)}}{\Lambda_{(\alpha)}} \underline{E}_\alpha \otimes \underline{E}_\alpha$$

$$= \dot{\hat{\underline{\underline{\mathcal{L}}}}}$$

But

$$\dot{\hat{\underline{\underline{U}}}} = \dot{\underline{\underline{R}}}_R^T \cdot \dot{\underline{\underline{U}}} \cdot \underline{\underline{R}}_R + \dot{\underline{\underline{R}}}_R^T \cdot \underline{\underline{U}} \cdot \underline{\underline{R}}_R + \underline{\underline{R}}_R^T \cdot \underline{\underline{U}} \cdot \dot{\underline{\underline{R}}}_R$$

$$= \dot{R}_R^T \cdot \dot{\underline{\underline{U}}} \cdot \underline{\underline{R}}_R - \underline{\underline{R}}_R^T \cdot \underline{\underline{\Omega}}_R \cdot \underline{\underline{U}} \cdot \underline{\underline{R}}_R + \underline{\underline{R}}_R^T \cdot \underline{\underline{U}} \cdot \underline{\underline{\Omega}}_R \cdot \underline{\underline{R}}_R$$

$$\dot{\hat{\underline{\underline{U}}}} \cdot \hat{\underline{\underline{U}}}^{-1} = \underline{\underline{R}}_R^T \cdot \dot{\underline{\underline{U}}} \cdot \underline{\underline{U}}^{-1} \cdot \underline{\underline{R}}_R - \underline{\underline{R}}_R^T \cdot \underline{\underline{\Omega}}_R \cdot \underline{\underline{R}}_R + \underline{\underline{R}}_R^T \cdot \underline{\underline{U}} \cdot \underline{\underline{\Omega}}_R \cdot \underline{\underline{U}}^{-1} \cdot \underline{\underline{R}}_R$$

$$\hat{\underline{\underline{D}}} = \underline{\underline{R}}_R^T \cdot \underline{\underline{D}}^{\gamma} \cdot \underline{\underline{R}}_R + \frac{1}{2} \underline{\underline{R}}_R^T \cdot \left(\underline{\underline{U}} \cdot \underline{\underline{\Omega}}_R \cdot \underline{\underline{U}}^{-1} - \underline{\underline{U}}^{-1} \cdot \underline{\underline{\Omega}}_R \cdot \underline{\underline{U}} \right) \cdot \underline{\underline{R}}_R$$

$$= \underline{\underline{R}}_L^T \cdot \underline{\underline{D}} \cdot \underline{\underline{R}}_L + \frac{1}{2} \underline{\underline{R}}_L^T \cdot \left(\underline{\underline{F}} \cdot \underline{\underline{\Omega}}_R \cdot \underline{\underline{U}}^{-1} - \underline{\underline{U}}^{-1} \cdot \underline{\underline{\Omega}}_R \cdot \underline{\underline{F}}^T \right) \cdot \underline{\underline{R}}_L$$

Expressions involving derivatives of the other logarithmic strains are much more complicated. For example consider the following:

$$\dot{\underset{=}{\mathcal{L}}}_R = \underset{=}{A} + \underset{=}{B} + \underset{=}{C}$$

where

$$\underset{=}{A} = \frac{\dot{\Lambda}_{(\alpha)}}{\Lambda_{(\alpha)}} \underset{\sim}{N}_\alpha \otimes \underset{\sim}{N}_\alpha$$

$$\underset{=}{B} = \ln\Lambda_{(\alpha)} \, \dot{\underset{\sim}{N}}_\alpha \otimes \underset{\sim}{N}_\alpha = \ln\Lambda_{(\alpha)} \, (\underset{=}{\Omega}_R \cdot \underset{\sim}{N}_\alpha) \otimes \underset{\sim}{N}_\alpha$$

$$= \underset{=}{\Omega}_R \cdot (\ln\Lambda_{(\alpha)} \, \underset{\sim}{N}_\alpha \otimes \underset{\sim}{N}_\alpha)$$

$$= \underset{=}{\Omega}_R \cdot \underset{=}{\mathcal{L}}_R$$

$$\underset{=}{C} = \ln\Lambda_{(\alpha)} \, \underset{\sim}{N}_\alpha \otimes \dot{\underset{\sim}{N}}_\alpha = \ln\Lambda_{(\alpha)} \, \underset{\sim}{N}_\alpha \otimes \underset{=}{\Omega}_R \cdot \underset{\sim}{N}_\alpha$$

$$= \ln\Lambda_{(\alpha)} \, \underset{\sim}{N}_\alpha \otimes \underset{\sim}{N}_\alpha \cdot \underset{=}{\Omega}_R^T$$

$$= -\underset{=}{\mathcal{L}}_R \cdot \underset{=}{\Omega}_R$$

$$\therefore \quad \dot{\underset{=}{\mathcal{L}}}_R = \frac{\dot{\Lambda}_{(\alpha)}}{\Lambda_{(\alpha)}} \underset{\sim}{N}_\alpha \otimes \underset{\sim}{N}_\alpha + \underset{=}{\Omega}_R \cdot \underset{=}{\mathcal{L}}_R - \underset{=}{\mathcal{L}}_R \cdot \underset{=}{\Omega}_R$$

Similarly

$$\dot{\underset{=}{\mathcal{L}}}_L = \frac{\dot{\Lambda}_{(\alpha)}}{\Lambda_{(\alpha)}} \underset{\sim}{n}_\alpha \otimes \underset{\sim}{n}_\alpha + \underset{=}{\Omega}_L \cdot \underset{=}{\mathcal{L}}_L - \underset{=}{\mathcal{L}}_L \cdot \underset{=}{\Omega}_L$$

In a similar manner

$$\dot{\underset{=}{U}} = \dot{\lambda}_{(\alpha)} \underset{-}{N}_\alpha \otimes \underset{-}{N}_\alpha + \underset{=}{\Omega}_R \cdot \underset{=}{U} - \underset{=}{U} \cdot \underset{=}{\Omega}_R$$

$$\dot{\underset{=}{U}} \cdot \underset{=}{U}^{-1} = \frac{\dot{\lambda}_{(\alpha)}}{\lambda_{(\alpha)}} \underset{-}{N}_\alpha \otimes \underset{-}{N}_\alpha + \underset{=}{\Omega}_R - \underset{=}{U} \cdot \underset{=}{\Omega}_R \cdot \underset{=}{U}^{-1}$$

$$\underset{=}{U}^{-1} \cdot \dot{\underset{=}{U}} = \frac{\dot{\lambda}_{(\alpha)}}{\lambda_{(\alpha)}} \underset{-}{N}_\alpha \otimes \underset{-}{N}_\alpha - \underset{=}{\Omega}_R + \underset{=}{U}^{-1} \cdot \underset{=}{\Omega}_R \cdot \underset{=}{U}$$

$$\underset{=}{D}^* = \frac{1}{2}(\dot{\underset{=}{U}} \cdot \underset{=}{U}^{-1} + \underset{=}{U}^{-1} \cdot \dot{\underset{=}{U}}) = \frac{\dot{\lambda}_{(\alpha)}}{\lambda_{(\alpha)}} \underset{-}{N}_\alpha \otimes \underset{-}{N}_\alpha + \frac{1}{2}(\underset{=}{U}^{-1} \cdot \underset{=}{\Omega}_R \cdot \underset{=}{U} - \underset{=}{U} \cdot \underset{=}{\Omega}_R \cdot \underset{=}{U}^{-1})$$

If $\underset{=}{\Omega}_R = \underset{=}{0}$ then $\dot{\underset{=}{\epsilon}}_R = \underset{=}{D}^*$

which is the result shown by Gurtin and Spear (1983).

Note: The diagonal components of $\underset{=}{D}^*$ with respect to the basis $\underset{-}{N}_\alpha \otimes \underset{-}{N}_\beta$ are the terms $\dot{\lambda}_{(\alpha)}/\lambda_{(\alpha)}$. The off-diagonal components with respect to the same basis are the off-diagonal components (there are no diagonal components) of $\frac{1}{2}(\underset{=}{U}^{-1} \cdot \underset{=}{\Omega}_R \cdot \underset{=}{U} - \underset{=}{U} \cdot \underset{=}{\Omega}_R \cdot \underset{=}{U}^{-1})$ and hence, if $\underset{=}{U}$ is known, $\underset{=}{\Omega}_R$ can be obtained.

Gurtin, M.E., and Spear, K., 1983, "On the Relationship Between the Logarithmic Strain Rate and the Stretching Tensor", *Int'l J. of Solids & Structures*, Vol. 19, No. 5, pp. 437-444.

Recall that

$$\underline{\underline{W}} = \underline{\underline{\Omega}} + \frac{1}{2}\underline{\underline{R}} \cdot (\dot{\underline{\underline{U}}} \cdot \underline{\underline{U}}^{-1} - \underline{\underline{U}}^{-1} \cdot \dot{\underline{\underline{U}}}) \cdot \underline{\underline{R}}^T$$

Use the expression for $\dot{\underline{\underline{U}}}$ in terms of $\underline{\underline{\Omega}}_R$ to write

$$\underline{\underline{W}} = \underline{\underline{\Omega}} + \underline{\underline{R}} \cdot \underline{\underline{\Omega}}_R \cdot \underline{\underline{R}}^T - \frac{1}{2}\underline{\underline{R}} \cdot (\underline{\underline{U}} \cdot \underline{\underline{\Omega}}_R \cdot \underline{\underline{U}}^{-1} - \underline{\underline{U}}^{-1} \cdot \underline{\underline{\Omega}}_R \cdot \underline{\underline{U}}) \cdot \underline{\underline{R}}^T$$

$$= \underline{\underline{\Omega}} + \underline{\underline{R}} \cdot (\underline{\underline{\Omega}}_R - \frac{1}{2}\underline{\underline{U}} \cdot \underline{\underline{\Omega}}_R \cdot \underline{\underline{U}}^{-1} + \frac{1}{2}\underline{\underline{U}}^{-1} \cdot \underline{\underline{\Omega}}_R \cdot \underline{\underline{U}}) \cdot \underline{\underline{R}}^T$$

Also recall that

$$\underline{\underline{D}} = \underline{\underline{R}} \cdot \underline{\underline{D}}^{\kappa} \cdot \underline{\underline{R}}^T$$

Then

$$\underline{\underline{D}} = \frac{\dot{\lambda}_{(\alpha)}}{\lambda_{(\alpha)}} \underline{\underline{R}} \cdot \underline{N}_\alpha \otimes \underline{N}_\alpha \cdot \underline{\underline{R}}^T + \frac{1}{2}\underline{\underline{R}} \cdot (\underline{\underline{U}}^{-1} \cdot \underline{\underline{\Omega}}_R \cdot \underline{\underline{U}} - \underline{\underline{U}} \cdot \underline{\underline{\Omega}}_R \cdot \underline{\underline{U}}^{-1}) \cdot \underline{\underline{R}}^T$$

$$= \frac{\dot{\lambda}_{(\alpha)}}{\lambda_{(\alpha)}} \underline{n}_\alpha \otimes \underline{n}_\alpha + \frac{1}{2}(\underline{\underline{F}}^{-T} \cdot \underline{\underline{\Omega}}_R \cdot \underline{\underline{F}}^T - \underline{\underline{F}} \cdot \underline{\underline{\Omega}}_R \cdot \underline{\underline{F}}^{-1})$$

Procedure for Time Integration in a Finite Element Code

At each step in an integration procedure, $\underline{\underline{F}}$ or variants of $\underline{\underline{F}}$ such as $\underline{\underline{R}}$ and $\underline{\underline{U}}$ must be obtained. Essentially 9 pieces of information are required. These can be (1) the components of $\underline{\underline{F}}$, (2) the components of $\underline{\underline{R}}$ and $\underline{\underline{U}}$ (or $\underline{\underline{V}}$), or (3) the right and left bases, $\underset{\sim}{N}_\alpha$ and $\underset{\sim}{n}_\alpha$, and the principal stretches $\Lambda_{(\alpha)}$.

Suppose that the information available is $\underline{\underline{D}}$ and $\underline{\underline{W}}$. Taylor and Flanagan (1987) propose the following:

1. Use $\quad \underset{\sim}{\omega} = \underset{\sim}{w} - 2\left[\underline{\underline{V}} - 2(\mathrm{tr}\,\underline{\underline{V}})\,\underline{\underline{I}}\right]^{-1} \cdot \underline{\underline{Z}}$

   to update $\underset{\sim}{\omega}$ and hence $\underline{\underline{\Omega}}$

2. Use $\quad \dot{\underline{\underline{R}}} = \underline{\underline{\Omega}} \cdot \underline{\underline{R}}$ to update $\underline{\underline{R}}$

3. Use $\quad \dot{\underline{\underline{V}}} = (\underline{\underline{D}} + \underline{\underline{W}})\,\underline{\underline{V}} - \underline{\underline{V}} \cdot \underline{\underline{\Omega}}$ to update $\underline{\underline{V}}$

   With $\underline{\underline{V}}$, $\Lambda_{(\alpha)}$ and $\underset{\sim}{n}_\alpha$ are available.
   With $\underline{\underline{R}}$, $\underset{\sim}{N}_{(\alpha)}$ is available so the information associated with $\underline{\underline{F}}$ is known.

Taylor, L.M., and Flanagan, D.P., 1987, PRONTO 2D - A Two-Dimensional Transient Solid Dynamics Program, Sandia Report SAND86-0594•UC-32, Sandia National Laboratories, Albuquerque, NM 87185

Logarithmic Strain with $\underline{\underline{\Omega}}_R = 0$

Let
$$\underline{\underline{\mathcal{L}}}_R^* = \ln \underline{\underline{U}}$$

Then
$$\dot{\underline{\underline{\mathcal{L}}}}_R^* = \underline{\underline{D}}^*$$

where
$$\underline{\underline{D}}^* = \text{sym} \frac{d}{dt} \ln \underline{\underline{U}}$$
$$= \frac{1}{2} (\dot{\underline{\underline{U}}} \cdot \underline{\underline{U}}^{-1} + \underline{\underline{U}}^{-1} \cdot \dot{\underline{\underline{U}}})$$
$$= \underline{\underline{R}}^T \cdot \underline{\underline{D}} \cdot \underline{\underline{R}}$$

As a comparison, consider
$$\dot{\underline{\underline{E}}} = \underline{\underline{F}}^T \cdot \underline{\underline{D}} \cdot \underline{\underline{F}}$$
$$= \underline{\underline{U}} \cdot \underline{\underline{R}}^T \cdot \underline{\underline{D}} \cdot \underline{\underline{R}} \cdot \underline{\underline{U}}$$
$$= \underline{\underline{U}} \cdot \underline{\underline{D}}^* \cdot \underline{\underline{U}}$$

Recall the stretch parameters $\Lambda_1, \Lambda_2, \Lambda_3$

Let $\underset{\sim}{e}_A$ be a principal basis of $\underline{\underline{U}}$, and hence of $\underline{\underline{U}}^2$, and hence of $\underline{\underline{F}}^T \cdot \underline{\underline{F}}$, and hence of $\underline{\underline{E}}$, and hence of $\underline{\underline{\mathcal{L}}}^*$.

In this basis
$$E_{AB} \Rightarrow \frac{1}{2} \begin{bmatrix} \Lambda_1^2 - 1 & & 0 \\ & \Lambda_2^2 - 1 & \\ 0 & & \Lambda_3^2 - 1 \end{bmatrix}$$

$$U_{AB} \Rightarrow \begin{bmatrix} \Lambda_1 & & \\ & \Lambda_2 & \\ & & \Lambda_3 \end{bmatrix}$$

$$\mathcal{L}_{AB}^* \Rightarrow \begin{bmatrix} \ln \Lambda_1 & & \\ & \ln \Lambda_2 & \\ & & \ln \Lambda_3 \end{bmatrix}$$

Consider the case where $\Lambda_1 = 2$, i.e

$\frac{ds}{dX_1} = 2$ for a fiber oriented originally along the X_1-axis. Then

$\Lambda_1 = 2 \qquad E_{11} = \frac{3}{2} \qquad U_{11} = 2 \qquad \mathcal{L}^*_{11} = \ln 2$

Now consider $\frac{ds}{dX_1} = \frac{1}{2}$. Then

$\Lambda_1 = \frac{1}{2} \qquad E_{11} = -\frac{3}{8} \qquad U_{11} = \frac{1}{2} \qquad \mathcal{L}^*_{11} = -\ln 2$

Thus the logarithmic strain has a natural symmetry with regard to stretching and compression that other strain measures do not have.

Eigenvalue Problem for Unsymmetric Tensors

Frequently, tensors that are not symmetric must be investigated and an eigensystem analysis can prove as useful as it is for symmetric tensors. Examples of such tensors are $\underset{=}{F}$, $\underset{=}{h} = \underset{\sim}{u}\nabla$, $\underset{=}{H} = \underset{\sim}{u}\nabla_0$.

Similar to the case for symmetric tensors, the eigenvalue problem consists of finding solutions to the equation

$$\underset{=}{A} \cdot \underset{\sim}{v} = \lambda \underset{\sim}{v}$$

A solution exists provided

$$\det(\underset{=}{A} - \lambda \underset{=}{I}) = 0$$

and the solutions are the eigenpairs $(\lambda_i, \underset{\sim}{r}_i)$ where $\underset{\sim}{r}_i$ are known as the right eigenvectors of $\underset{=}{A}$ and λ_i are the eigenvalues. Consider the problem

$$\underset{\sim}{u} \cdot \underset{=}{A} = \lambda \underset{\sim}{u} \quad \text{or} \quad \underset{=}{A}^T \cdot \underset{\sim}{u} = \lambda \underset{\sim}{u}$$

A solution to this equation exists provided

$$\det(\underset{=}{A}^T - \lambda \underset{=}{I}) = 0$$

The eigenvalues λ_i that result are identical to the previous set, but the eigenvectors $\underset{\sim}{l}_i$, called the left eigenvectors, are different from $\underset{\sim}{r}_i$.

By definition

$$\underset{=}{A} \cdot \underset{\sim}{r}_1 = \lambda_1 \underset{\sim}{r}_1$$

Take the inner product

$$\underset{\sim}{l}_1 \cdot \underset{=}{A} \cdot \underset{\sim}{r}_1 = \lambda_1 \underset{\sim}{l}_1 \cdot \underset{\sim}{r}_1$$

Choose

$$\underset{\sim}{l}_1 \cdot \underset{\sim}{r}_1 = 1 \qquad \underset{\sim}{l}_2 \cdot \underset{\sim}{r}_2 = 1 \qquad \underset{\sim}{l}_3 \cdot \underset{\sim}{r}_3 = 1$$

where the last two equations are chosen similarly.

To show orthogonality, consider

$$\underset{\sim}{l}_2 \cdot \underset{=}{A} \cdot \underset{\sim}{r}_1 = \underset{\sim}{l}_2 \cdot \lambda_1 \underset{\sim}{r}_1$$

$$= \lambda_2 \underset{\sim}{l}_2 \cdot \underset{\sim}{r}_1$$

If $\lambda_1 \neq \lambda_2$, this equation can only be satisfied if $\underset{\sim}{l}_2 \cdot \underset{\sim}{r}_1 = 0$. Thus, for unsymmetric tensors the left and right eigenvectors satisfy the general orthogonality relation

$$\underset{\sim}{l}_2 \cdot \underset{\sim}{r}_1 = 0 \qquad \underset{\sim}{l}_2 \cdot \underset{\sim}{r}_3 = 0$$

$$\underset{\sim}{l}_3 \cdot \underset{\sim}{r}_2 = 0 \qquad \underset{\sim}{l}_3 \cdot \underset{\sim}{r}_1 = 0$$

$$\underset{\sim}{l}_1 \cdot \underset{\sim}{r}_3 = 0 \qquad \underset{\sim}{l}_1 \cdot \underset{\sim}{r}_2 = 0$$

which, together with the normalizing condition, can be summarized in the equations

$$\underset{\sim}{l}_i \cdot \underset{\sim}{r}_j = \delta_{ij} \qquad \underset{\sim}{l}_i \cdot \underset{=}{A} \cdot \underset{\sim}{r}_j = \lambda_{(i)} \delta_{ij}$$

Now $\underset{\sim}{l}_i$ and $\underset{\sim}{r}_i$ can each be considered a basis although each is not necessarily orthonormal. Then a representation for $\underset{=}{A}$ is

$$\underset{=}{A} = \lambda_{(i)} \underset{\sim}{r}_i \otimes \underset{\sim}{l}_i \,.$$

By direct substitution it follows that

$$\underset{=}{A} \cdot \underset{\sim}{r}_j = \lambda_{(i)} \underset{\sim}{r}_i \delta_{ij} = \lambda_{(j)} \underset{\sim}{r}_j$$

$$\underset{\sim}{l}_j \cdot \underset{=}{A} = \lambda_{(i)} \delta_{ij} \underset{\sim}{l}_i = \lambda_{(j)} \underset{\sim}{l}_j$$

As an example, consider the deformation tensor

$$\underset{=}{F} = \underset{=}{R} \cdot \underset{=}{U} = \underset{=}{R} \cdot \Lambda_{(\alpha)} \underset{\sim}{N}_\alpha \otimes \underset{\sim}{N}_\alpha = \Lambda_{(\alpha)} \underset{\sim}{n}_\alpha \otimes \underset{\sim}{N}_\alpha$$

$$\underset{=}{F} \cdot \underset{\sim}{N}_\beta = \Lambda_{(\beta)} \underset{\sim}{n}_\beta$$

$$\underset{=}{F} \cdot \underset{\sim}{n}_\beta = \Lambda_{(\alpha)} \underset{\sim}{n}_\alpha (\underset{\sim}{N}_\alpha \cdot \underset{\sim}{n}_\beta)$$

so that neither $\underset{\sim}{N}_\alpha$ nor $\underset{\sim}{n}_\alpha$ are the right eigenvectors of $\underset{=}{F}$. Recall that $\underset{\sim}{N}_\alpha$ and $\underset{\sim}{n}_\alpha$ are the eigenvectors of $\underset{=}{F}^T \cdot \underset{=}{F}$ and $\underset{=}{F} \cdot \underset{=}{F}^T$, respectively.

A completely analogous situation holds for $\underset{=}{H} = \underset{\sim}{u} \nabla_0$, i.e., the right and left eigenvectors of $\underset{=}{H}$ will differ from those for $\underset{=}{H}^T \cdot \underset{=}{H}$. The need for an appropriate representation is necessary to show that $\underset{=}{H}^T \cdot \underset{=}{H}$ may be neglected under certain infinitesimal assumptions in the expression for the Lagrangian strain tensor

$$\underset{=}{E} = \frac{1}{2} (\underset{=}{H} + \underset{=}{H}^T + \underset{=}{H}^T \cdot \underset{=}{H}) = \frac{1}{2} (\underset{=}{F}^T \cdot \underset{=}{F} - \underset{=}{I})$$

Thus the eigenvectors of $\underset{=}{E}$ are $\underset{\sim}{N}_\alpha$, but these are not necessarily the eigenvectors of $(\underset{=}{H} + \underset{=}{H}^T)$ or of $\underset{=}{H}^T \cdot \underset{=}{H}$.

Chapter 3 - General Principles of Continuum Mechanics

Section 1 - Conservation of Mass

One of the basic postulates of continuum mechanics is that mass is conserved, i.e.,

$$\int_{R_0} \rho_0 dV_0 = \int_R \rho dV \quad \text{for any region } R_0 \tag{3.1-1}$$

where ρ_0 is defined as the mass density per unit undeformed volume and ρ as the mass density per unit deformed volume. It is assumed that $\rho > o$ everywhere in the medium. We also note the R_0 and R contain the same material points. Equation 3.1-1 can be expressed in several alternate methods which are derived in this section.

From Eq. 2.3-4 we substitute for dV and obtain

$$\int_R \rho dV = \int_{R_0} \rho J dV_0 \quad .$$

hence

$$\rho_0 = \rho J$$

Since Eq. 3.1-1 holds for all R_0 we can write

$$\rho_0 dV_0 = \rho dV$$

or

$$\frac{\rho_0}{\rho} = J = \frac{dV}{dV_0}$$

Noting that $\rho_0 dV_0$ is not a function of time we obtain

$$\frac{d}{dt}(\rho_0 dV_0) = 0$$

$$= \frac{d}{dt}(\rho dV)$$

$$= \frac{d}{dt}(\rho J) dV_0$$

or $\quad \frac{d}{dt}(\rho J) = 0$ since $dV_0 \neq 0$

Hence $\quad \frac{d\rho}{dt} J + \rho \frac{dJ}{dt} = 0$

Substituting from Eq. 2.5-5 we obtain

$$J[\frac{d\rho}{dt} + \rho(\underset{\sim}{v} \cdot \nabla)] = 0$$

Since it was assumed that $\rho > o$ for all t then $J > o$ (and hence $\underset{\approx}{F}^{-1}$ exists) and

$$\frac{d\rho}{dt} + \rho(\underset{\sim}{v} \cdot \nabla) = 0$$

If we assume $\rho = \rho(\underset{\sim}{r}, t)$ and use the definition of the material time derivative we obtain

$$\frac{\partial \rho}{\partial t} + (\rho \nabla) \cdot \underset{\sim}{v} + \rho(\underset{\sim}{v} \cdot \nabla) = 0$$

or

$$\frac{\partial \rho}{\partial t} + (\rho \underset{\sim}{v}) \cdot \nabla = 0$$

This last expression is known as the continuity equation.

<u>Summary of Alternate Expressions for the Conservation of Mass Postulate</u>

$$\left.\begin{aligned}
&\int_{R_0} \rho_0 dV_0 = \int_R \rho dV \\
&\rho dV = \rho_0 dV_0 \\
&J = \frac{\rho_0}{\rho} \\
&\frac{d}{dt}(\rho dV) = 0 = \frac{d}{dt}(\rho J) \\
&\frac{d\rho}{dt} + \rho(\underset{\sim}{v} \cdot \nabla) = 0
\end{aligned}\right\} \qquad (3.1\text{-}2)$$

$$\frac{\partial \rho}{\partial t} + (\rho \underset{\sim}{v}) \cdot \nabla = 0$$

When considering incompressible materials. i.e., materials in which a volume element does not change, the following observations are noted.

$$\frac{d}{dt}(dV) = 0$$

Hence
$$\frac{d}{dt}(J) = 0$$

since $dV = J dV_0$ and dV_0 is not a function of time.

Also
$$\underset{\sim}{v} \cdot \nabla = 0$$

since $\frac{dJ}{dt} = 0 = J(\underset{\sim}{v} \cdot \nabla)$ and $J > o$.

Using the expressions in Eq. 3.1-2 we also note

$$\frac{d\rho}{dt} = 0$$

Thus, conservation of mass and any one of the following expressions describe an incompressible material:

$$\left.\begin{aligned} &\frac{d}{dt}(dV) = 0 \\ &\frac{d}{dt}(J) = 0 \\ &(\underset{\sim}{v} \cdot \nabla) = tr(\underset{\approx}{D}) = 0 \\ &\frac{d\rho}{dt} = 0 \end{aligned} \quad \text{Incompressible Material} \right\} \quad (3.1\text{-}3)$$

The choice of a particular form of Eq. 3.1-2 or Eq. 3.1-3 is essentially one of convenience for a given application.

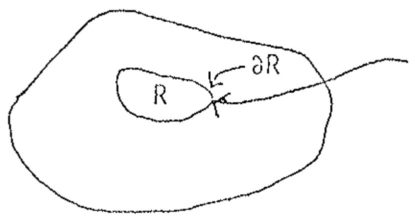
An important consequence is that

$$\frac{d}{dt}\int_R (\ )\rho\, dV = \int_R \frac{d}{dt}(\ )\rho\, dV$$

Section 2 Reynold's Transport Theorem

Thm: $$\frac{d}{dt}\int_R \Psi\rho\,dV = \int_R \frac{\partial}{\partial t}(\rho\Psi)dV + \int_{\partial R} \rho\Psi\underset{\sim}{v}\cdot\underset{\sim}{n}ds \qquad (3.2\text{-}1)$$

where R is any subregion of a body (or control volume) as illustrated below



and Ψ is a scalar function defined on a per unit mass basis.

One physical interpretation of the theorem is that if Ψ is the heat per unit mass, the rate of total heat increase in a control volume equals the rate of heat generated within the body plus the rate at which heat flows across the boundary.

Proof:

We can change the limits of the integral on the left side of Eq. (3.2-1) from R to R_0 so that the time derivative can be brought inside the integral sign since the limits of R_0 do not depend on time.

$$\frac{d}{dt}\int_R \Psi\rho\,dV = \frac{d}{dt}\int_{R_0} \Psi\rho J\,dV_0$$

$$= \int_{R_0} \frac{d}{dt}(\Psi\rho J)dV_0$$

Changing back to limits over R we obtain

$$\frac{d}{dt}\int_R \Psi\rho dV = \int_R \frac{d}{dt}(\Psi\rho J)\ \frac{1}{J}dV \qquad (3.2\text{-}2)$$

Note that

$$\frac{1}{J}\frac{d}{dt}(\Psi\rho J) = \frac{d}{dt}(\Psi\rho) + \frac{\Psi\rho}{J}\dot{J}$$

and

$$\dot{J} = J(\underset{\sim}{v}\cdot\nabla)$$

Using the material derivative definition and substituting for $\dot{J}$, we get

$$\frac{d}{dt}(\Psi\rho) + \frac{\Psi\rho}{J}\dot{J} = \frac{\partial}{\partial t}(\Psi\rho) + (\Psi\rho)\nabla\cdot\underset{\sim}{v} + \Psi\rho(\underset{\sim}{v}\cdot\nabla) \qquad (3.2\text{-}3)$$

The last two terms in the above expression combine to give

$$(\Psi\rho)\nabla\cdot\underset{\sim}{v} + \Psi\rho(\underset{\sim}{v}\cdot\nabla) = (\Psi\rho\underset{\sim}{v})\cdot\nabla$$

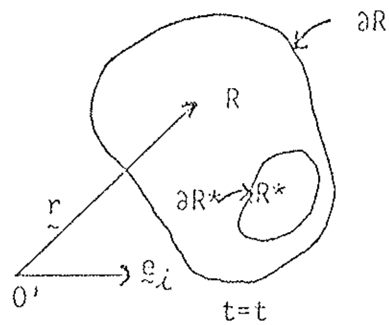
Using the divergence theorem we note that

$$\int_R (\Psi\rho\underset{\sim}{v})\cdot\nabla dV = \int_{\partial R}\Psi\rho\underset{\sim}{v}\cdot\underset{\sim}{n}ds \qquad (3.2\text{-}4)$$

Combining Eq. 3.3-2, 3.3-3, and 3.3-4 the desired result is obtained.

EOP

Section 3 Euler's First and Second Laws



Consider the above figure at time t=t. The region of the whole body is R with a boundary ∂R while R* denotes a subregion of the body with boundary ∂R*. The origin of the system is O' and the base vectors are $\underset{\sim}{e}_i$. Recall that the base vectors plus O' constitute a reference frame. Now assume that the reference frame is Galilean, i.e., fixed relative to the stars.

If we designate $\underset{\sim}{u}$ as the displacement at any point in the body then we recall from Sec. 2.8.

$$
\begin{aligned}
\underset{\sim}{u} &= \underset{\sim}{r} - \underset{\sim}{R} \\
\underset{\sim}{v} &= \dot{\underset{\sim}{u}} = \dot{\underset{\sim}{r}} \\
\underset{\sim}{a} &= \ddot{\underset{\sim}{u}} = \ddot{\underset{\sim}{r}}
\end{aligned}
\tag{3.3-1}
$$

Now let $\underset{\sim}{F}$ be the total force acting on R and $\underset{\sim}{L}$ be the total torque wrt the point O' acting on the body. Note that the two sources of torque are applied couples acting on the body and moments due to the applied forces.

We define the total momentum of the body R to be

$$
\underset{\sim}{P} = \int_R \rho \underset{\sim}{v} dV \tag{3.3-2}
$$

The moment of momentum or angular momentum of the body R is

$$\underset{\sim}{H} = \int_R \underset{\sim}{r} \times \underset{\sim}{v} \, \rho dV \tag{3.3-3}$$

Euler's first and second laws are postulated in the following form.

$$\underset{\sim}{F} = \overset{\circ}{\underset{\sim}{P}}$$

and

$$\underset{\sim}{L} = \overset{\circ}{\underset{\sim}{H}} \tag{3.3-4}$$

Consider the time derivative of the momentum

$$\overset{\circ}{\underset{\sim}{P}} = \frac{d}{dt} \int_R \rho \underset{\sim}{v} dV$$

If we transform the limits in the above integral from R to R_0, we can bring the time derivative inside the integral sign. Taking the appropriate derivatives and then changing back to the R limits we obtain

$$\overset{\circ}{\underset{\sim}{P}} = \int_R \rho \, \underset{\sim}{a} \, dV$$

Similarly, the time derivative of the moment of momentum is

$$\overset{\circ}{\underset{\sim}{H}} = \frac{d}{dt} \int_R (\underset{\sim}{r} \times \underset{\sim}{v}) \rho \, dV$$

Performing the same type of transformations as for the momentum case we can write

$$\overset{\circ}{\underset{\sim}{H}} = \int_R \frac{d}{dt} (\underset{\sim}{r} \times \underset{\sim}{v}) \rho \, dV$$

or $$\dot{\underset{\sim}{H}} = \int_R [(\dot{\underset{\sim}{r}} \times \underset{\sim}{v}) + (\underset{\sim}{r} \times \underset{\sim}{a})]\rho dV$$

However, $(\dot{\underset{\sim}{r}} \times \underset{\sim}{v}) = (\underset{\sim}{v} \times \underset{\sim}{v}) = \underset{\sim}{0}$

Hence, alternate expressions to (3.3-4) are

$$\underset{\sim}{F} = \int_R \rho \, \underset{\sim}{a} \, dV$$
$$\underset{\sim}{L} = \int_R (\underset{\sim}{r} \times \underset{\sim}{a}) \, \rho dV \tag{3.3-5}$$

Since any region R* can also be considered a body, we assume that Euler's laws also apply to R*.

Hence

$$\underset{\sim}{f} = \int_{R^*} \rho \underset{\sim}{a} dV$$
$$\underset{\sim}{\ell} = \int_{R^*} (\underset{\sim}{r} \times \underset{\sim}{a}) \rho dV \tag{3.3-6}$$

where $\underset{\sim}{f}$ is the resultant force on R* and $\underset{\sim}{\ell}$ is the resultant torque on R*.

Section 4 Stress Principle of Euler and Cauchy

Assume that the total force $\underset{\sim}{f}$ acting on a body is composed of a contact force $\underset{\sim}{f}_c$ and an external body force $\underset{\sim}{f}_b$. This body force is due to external sources, such as gravity.

Hence $$\underset{\sim}{f} = \underset{\sim}{f}_c + \underset{\sim}{f}_b \tag{3.4-1}$$

Now assume that the body force is determined from a vector field $\underset{\sim}{b}\,(\underset{\sim}{r},\underset{\sim}{t})$ which is defined to be the external body force per unit mass. Then

$$\underset{\sim}{f}_b = \int_{R^*} \underset{\sim}{b}\,\rho dV \tag{3.4-2}$$

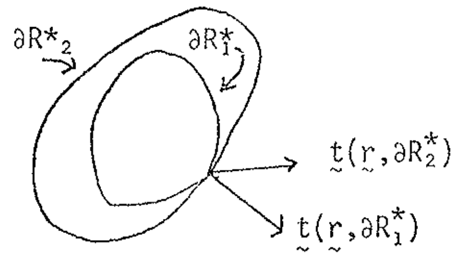
Also assume that the effect of the material external to R* on the body R* can be characterized by a vector field representing a force per unit area acting on ∂R^* and called the stress vector, $\underset{\sim}{t}$. In general, it is assumed that $\underset{\sim}{t}$ depends on the point and on the region characterized by R* (or ∂R^*).

Hence $$\underset{\sim}{t} = \underset{\sim}{t}\,(\underset{\sim}{r}, \partial R^*) \tag{3.4-3}$$

When $\partial R^* = \partial R$, $\underset{\sim}{t}$ is called the surface traction

Then $$\underset{\sim}{f}_c = \int_{\partial R^*} \underset{\sim}{t}(\underset{\sim}{r},\partial R^*)ds \tag{3.4-4}$$

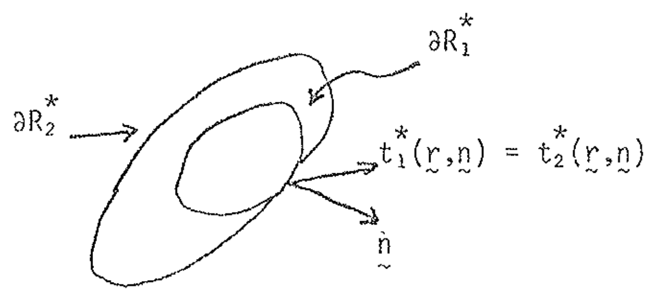
The assumption that $\underset{\sim}{t}$ is a function of ∂R^* implies that different stress vectors can exist at a point with the same unit normal if the regions on which the stress vectors act are different. This is illustrated in the following figure.



Such an assumption would yield a very complicated theory and is also physically not plausible. Thus, it is postulated that the stress vector $\underset{\sim}{t}$ depends only on $\underset{\sim}{r}$ and $\underset{\sim}{n}$ where $\underset{\sim}{n}$ is the unit normal vector at the specified point:

$$\underset{\sim}{t} = \underset{\sim}{t}\,(\underset{\sim}{r}, \underset{\sim}{n}) \tag{3.4-5}$$

Such a restriction yields the situation illustrated below:



Section 5 Cauchy's Fundamental Theorem on Stress

We want to show that $\underset{\sim}{t}$ is a linear transformation of $\underset{\sim}{n}$.

From Eq. (3.3-6) we can write

$$\underset{\sim}{f} = \int_{R^*} \rho \underset{\sim}{a}\, dV$$

Substituting for $\underset{\sim}{f}$ from Section 3.4 we obtain

$$\int_{\partial R^*} \underset{\sim}{t}\, dS + \int_{R^*} \rho \underset{\sim}{b} dV = \int_{R^*} \rho \underset{\sim}{a} dV \tag{3.5-1}$$

Recall from Sec. 1.6 the requirements of a linear transformation. In particular, $\underset{\sim}{L}$ is linear transformation of $\underset{\sim}{u}$ if

$$\underset{\sim}{L}(\underset{\sim}{u} + \underset{\sim}{w}) = \underset{\sim}{L}(\underset{\sim}{u}) + \underset{\sim}{L}(\underset{\sim}{w})$$

and

$$\underset{\sim}{L}(c\underset{\sim}{u}) = c\underset{\sim}{L}(\underset{\sim}{u})$$

These conditions imply that

$$\underset{\sim}{L}(\alpha\underset{\sim}{x} + \beta\underset{\sim}{y} + \gamma\underset{\sim}{z}) = \alpha\underset{\sim}{L}(\underset{\sim}{x}) + \beta\underset{\sim}{L}(\underset{\sim}{y}) + \gamma\underset{\sim}{L}(\underset{\sim}{z})$$

or

$$\underset{\sim}{v} = \underset{\approx}{A}\cdot\underset{\sim}{u} \quad \text{where } \underset{\sim}{x}, \underset{\sim}{y}, \underset{\sim}{z} \text{ are particular choices of } \underset{\sim}{u} \tag{3.5-2}$$

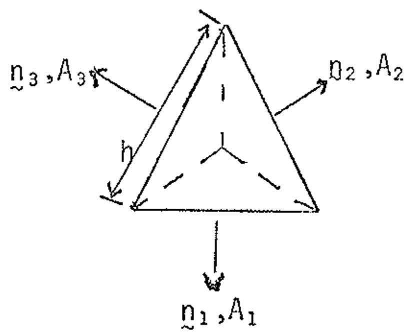
Cauchy's fundamental theorem on stress is that $\underset{\sim}{t}$ is a linear transformation of $\underset{\sim}{n}$, i.e.,

$$\underset{\sim}{t} = \underset{\approx}{\sigma}\cdot\underset{\sim}{n} \tag{3.5-3}$$

where $\underset{\approx}{\sigma}$ is known as the Cauchy stress tensor, or

$$\underset{\sim}{t}(\alpha\underset{\sim}{u}_1 + \beta\underset{\sim}{u}_2) = \alpha\underset{\sim}{t}(\underset{\sim}{u}_1) + \beta\underset{\sim}{t}(\underset{\sim}{u}_2)$$

Consider the tetrahedron shown below



The unit normal vectors and areas of three sides of the tetrahedron are shown in the sketch while the plane that coincides with the plane of this page has unit normal vector $\underset{\sim}{n}$ and area A. Consider this figure to be the region R*.

Now let the tetrahedron shrink in size to as small as we like so that the position vectors to points on any of the surfaces are equal. That is

$$\underset{\sim}{r}_1 = \underset{\sim}{r}_2 = \underset{\sim}{r}_3 = \underset{\sim}{r}$$

to within arbitrarily small vectors $\underset{\sim}{\varepsilon}_1$, $\underset{\sim}{\varepsilon}_2$, and $\underset{\sim}{\varepsilon}_3$.

However, the unit normal vectors will not be equal

$$\underset{\sim}{n}_1 \neq \underset{\sim}{n}_2 \neq \underset{\sim}{n}_3 \neq \underset{\sim}{n}$$

In Eq. (3.5-1), the surface integral of the tetrahedron can be expressed as the sum of four integrals with integrands

$$\begin{aligned} \underset{\sim}{t} &= \underset{\sim}{t}(\underset{\sim}{n}) \\ \underset{\sim}{t}_1 &= \underset{\sim}{t}(\underset{\sim}{n}_1) \\ \underset{\sim}{t}_2 &= \underset{\sim}{t}(\underset{\sim}{n}_2) \\ \underset{\sim}{t}_3 &= \underset{\sim}{t}(\underset{\sim}{n}_3) \end{aligned} \qquad (3.5\text{-}4_1)$$

with the dependence on $\underset{\sim}{r}$ suppressed for convenience. If we let the length of the sides of the tetrahedron approach zero then the volume integrals in Eq. (3.5-1) are of order h^3 while the surface integral is of order h^2. Hence, the volume integrals can be ignored with respect to the surface integrals as $h \rightarrow o$.

Therefore

$$\lim_{h \to o} \int_{\partial R^*} \underset{\sim}{t} ds = \underset{\sim}{0} = \underset{\sim}{t}A + \underset{\sim}{t}_1 A_1 + \underset{\sim}{t}_2 A_2 + \underset{\sim}{t}_3 A_3$$

or

$$\underset{\sim}{t} = -\left(\frac{A_1}{A} \underset{\sim}{t}_1 + \frac{A_2}{A} \underset{\sim}{t}_2 + \frac{A_3}{A} \underset{\sim}{t}_3\right) \qquad (3.5\text{-}4_2)$$

From Eq. (1.15-7) recall that

$$\int_{\partial R^*} \underset{\sim}{n} ds = \underset{\sim}{0} = A\underset{\sim}{n} + A_1 \underset{\sim}{n}_1 + A_2 \underset{\sim}{n}_2 + A_3 \underset{\sim}{n}_3$$

Hence

$$\underset{\sim}{n} = -\left(\frac{A_1}{A} \underset{\sim}{n}_1 + \frac{A_2}{A} \underset{\sim}{n}_2 + \frac{A_3}{A} \underset{\sim}{n}_3\right)$$

Since $\underset{\sim}{t} = \underset{\sim}{t}(\underset{\sim}{n})$ we obtain

$$\underset{\sim}{t} = \underset{\sim}{t}\left(-\frac{A_1}{A} \underset{\sim}{n}_1 - \frac{A_2}{A} \underset{\sim}{n}_2 - \frac{A_3}{A} \underset{\sim}{n}_3\right) \qquad (3.5\text{-}4_3)$$

and since $\underset{\sim}{t}(\underset{\sim}{n}_1) = \underset{\sim}{t}_1$, $\underset{\sim}{t}(\underset{\sim}{n}_2) = \underset{\sim}{t}_2$ and $\underset{\sim}{t}(\underset{\sim}{n}_3) = \underset{\sim}{t}_3$,

The combination of Eqs. $(3.5\text{-}4_1)$, $(3.5\text{-}4_2)$ and $(3.5\text{-}4_3)$ yields

$$\underset{\sim}{t}\left(-\frac{A_1}{A} \underset{\sim}{n}_1 - \frac{A_2}{A} \underset{\sim}{n}_2 - \frac{A_3}{A} \underset{\sim}{n}_3\right) = -\frac{A_1}{A} \underset{\sim}{t}(\underset{\sim}{n}_1) - \frac{A_2}{A} \underset{\sim}{t}(\underset{\sim}{n}_2) - \frac{A_3}{A} \underset{\sim}{t}(\underset{\sim}{n}_3)$$

which satisfies the requirements of Eq. (3.5-2). Hence $\underset{\sim}{t}$ is a linear transformation of $\underset{\sim}{n}$ or

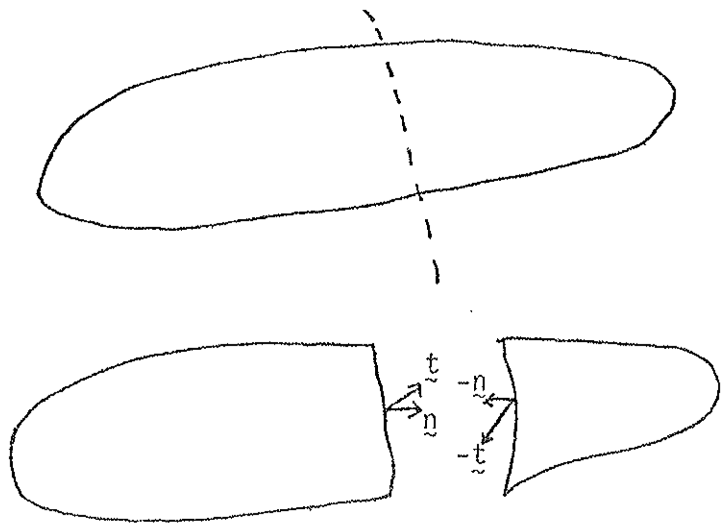
$$\underset{\sim}{t} = \underset{\approx}{\sigma} \cdot \underset{\sim}{n}$$

Cauchy's Lemma

The stress vector $\underset{\sim}{t}$ has the following property

$$\underset{\sim}{t}(\underset{\sim}{r}, \underset{\sim}{n}) = -\underset{\sim}{t}(\underset{\sim}{r}, -\underset{\sim}{n}) \qquad (3.5\text{-}5)$$

This relationship is evident from Eq. (3.5-3). It is also apparent from a section cut through an arbitrary body with the stress vector sketched on opposite sides of the section as shown:



Section 6 Cauchy's First and Second Equations of Motion

Euler expressed the equations of motion in the following form

$$\int_{\partial R^*} \underset{\sim}{t} ds + \int_{R^*} \underset{\sim}{b} \rho dV = \int_{R^*} \rho \ddot{\underset{\sim}{u}} dV$$

$$\int_{\partial R^*} \underset{\sim}{r} \times \underset{\sim}{t} ds + \int_{R^*} \underset{\sim}{r} \times \underset{\sim}{b} \, \rho dV = \int_{R^*} \underset{\sim}{r} \times \ddot{\underset{\sim}{u}} \, \rho dV \qquad (3.6\text{-}1)$$

where the vectors $\underset{\sim}{t}$ and $\underset{\sim}{b}$ have been defined in Section 3.4. These equations result from the assumptions made on the nature of the forces acting on ∂R^*. In particular, it is assumed that no body couples exist.

Using the equation $\underset{\sim}{t} = \underset{\approx}{\sigma} \cdot \underset{\sim}{n}$ in the surface integrals of (3.6-1) and applying the divergence theorem we obtain

$$\int_{\partial R^*} \underset{\approx}{\sigma} \cdot \underset{\sim}{n} ds = \int_{R^*} \underset{\approx}{\sigma} \cdot \nabla dV$$

Since the region R* is arbitrary, Euler's first equation yields Cauchy's first equation of motion

$$\underset{\approx}{\sigma} \cdot \nabla + \rho \underset{\sim}{b} = \rho \ddot{\underset{\sim}{u}} \qquad (3.6\text{-}2)$$

Substituting for $\ddot{\underset{\sim}{u}}$ gives

$$\underset{\approx}{\sigma} \cdot \nabla + \rho \underset{\sim}{b} = \rho [\frac{\partial v}{\partial t} + (\underset{\sim}{v} \nabla) \cdot \underset{\sim}{v}]$$

Note the non-linear term involving the acceleration in the above equation.

The surface integral in Euler's second equation becomes

$$\int_{\partial R^*} \underset{\sim}{r} \times \underset{\sim}{t} ds = \int_{\partial R^*} \underset{\sim}{r} \times \underset{\approx}{\sigma} \cdot \underset{\sim}{n} ds = \int_{R^*} (\underset{\sim}{r} \times \underset{\approx}{\sigma}) \cdot \nabla dV$$

An expansion of the last expression gives

$$\int_{R^*} (\underset{\sim}{r} \times \underset{\approx}{\sigma})\cdot\nabla\, dV = \int_{R^*} \{\underset{\sim}{r} \times (\underset{\approx}{\sigma}\cdot\nabla) + C_{13}[(\nabla\underset{\sim}{r}) \times \underset{\approx}{\sigma}]\}dV$$

Noting that $\nabla\underset{\sim}{r} = \underset{\approx}{I}$, Euler's second law becomes

$$\int_{R^*} \{C_{13}(\underset{\approx}{I} \times \underset{\sim}{\sigma}) + \underset{\sim}{r} \times [(\underset{\sim}{\sigma}\cdot\nabla) + \rho\underset{\sim}{b} - \rho\ddot{\underset{\sim}{u}}]\}dV = \underset{\sim}{0}$$

The term within the square brackets equals $\underset{\sim}{0}$ from (3.6-2). Hence

$$C_{13}(\underset{\approx}{I} \times \underset{\approx}{\sigma}) = 0$$

By expanding the terms in the parentheses we get

$$C_{13}(\delta_{ij}\underset{\sim}{e}_i \otimes \underset{\sim}{e}_j \times \sigma_{k\ell}\underset{\sim}{e}_k \otimes \underset{\sim}{e}_\ell) = 0$$

The cross product operation yields

$$C_{13}(\delta_{ij}\underset{\sim}{e}_i \otimes \sigma_{k\ell}\varepsilon_{jkm}\underset{\sim}{e}_m \otimes \underset{\sim}{e}_\ell) = 0$$

Hence

$$\delta_{ij}\sigma_{k\ell}\varepsilon_{jkm}\delta_{i\ell}\underset{\sim}{e}_m = 0$$

and

$$\delta_{ij}\sigma_{ki}\varepsilon_{jkm}\underset{\sim}{e}_m = 0$$

$$\sigma_{kj}\varepsilon_{jkm}\underset{\sim}{e}_m = 0$$

If we expand the above equation we obtain

$$\sigma_{12} = \sigma_{21}$$

$$\sigma_{13} = \sigma_{31}$$

$$\sigma_{23} = \sigma_{32}$$

Hence $\underset{\approx}{\sigma}$ is symmetric and so

$$\underset{\approx}{\sigma} = \underset{\approx}{\sigma}^T \qquad (3.6\text{-}3)$$

Section 7 Use of Material Description*

We define a stress vector $\hat{\underset{\sim}{t}}$ such that

$$\hat{\underset{\sim}{t}}dS_0 = \underset{\sim}{t}dS \tag{3.7-1}$$

where dS_0 is an area element in the undeformed state.

The Piola-Kirchhoff stress tensor of the first kind, $\hat{\underset{\approx}{\tau}}$, is defined such that

$$\hat{\underset{\approx}{\tau}} \cdot \underset{\sim}{N} = \hat{\underset{\sim}{t}} \tag{3.7-2}$$

Using Nanson's relations we can write

$$\underset{\sim}{n}dS = J\, \underset{\approx}{F}^{-T} \cdot \underset{\sim}{N}\, dS_0$$

Combining Eq. (3.7-1), Nanson's relations, and the definition of the Cauchy stress tensor gives

$$\hat{\underset{\approx}{\tau}} \cdot \underset{\sim}{N}\, dS_0 = \underset{\approx}{\sigma} \cdot \underset{\sim}{n} dS = \underset{\approx}{\sigma} \cdot J \underset{\approx}{F}^{-T} \cdot \underset{\sim}{N}\, dS_0$$

The above expression must hold for all $\underset{\sim}{N}\, dS_0$, hence

$$\left.\begin{aligned} \hat{\underset{\approx}{\tau}} &= J \underset{\approx}{\sigma} . \underset{\approx}{F}^{-T} \\ \text{or} \qquad \underset{\approx}{\sigma} &= \frac{1}{J} \hat{\underset{\approx}{\tau}} \cdot \underset{\approx}{F}^{T} \end{aligned}\right\} \tag{3.7-3}$$

Cauchy's second equation then becomes

$$\hat{\underset{\approx}{\tau}} \cdot \underset{\approx}{F}^{T} = \underset{\approx}{F} \cdot \hat{\underset{\approx}{\tau}}^{T} \tag{3.7-4}$$

Substituting for $\underset{\approx}{\sigma}$, Cauchy's first equation can be written

$$\left(\frac{1}{J} \hat{\underset{\approx}{\tau}} . \underset{\approx}{F}^{T}\right) . \nabla + \rho \underset{\sim}{b} = \rho \ddot{\underset{\sim}{u}}$$

* This development follows Truesdell and Noll. Some tensors are the transpose of those defined by Malvern.

Recall from Eq. (2.3-8)

$$\left(\frac{1}{J}\underset{\approx}{F}^T\right)\cdot\nabla = \underset{\sim}{0}$$

and from Eq. (2.2-12)

$$(\ )\nabla\cdot\underset{\approx}{F} = (\ )\nabla_0$$

Then

$$\underset{\approx}{\sigma}\cdot\nabla = \left(\frac{1}{J}\,\hat{\underset{\approx}{\tau}}\cdot\underset{\approx}{F}^T\right)\cdot\nabla = \hat{\underset{\approx}{\tau}}\cdot\left(\frac{1}{J}\,\underset{\approx}{F}^T\right)\cdot\nabla + C_{23}\left[(\hat{\underset{\approx}{\tau}}\nabla)\cdot\frac{\underset{\approx}{F}}{J}\right]$$

$$= C_{23}\ (\hat{\underset{\approx}{\tau}}\nabla_0)\frac{1}{J}$$

Using Eq. (3.1-2) and the definition of the divergence we obtain

$$\underset{\approx}{\sigma}\cdot\nabla = (\hat{\underset{\approx}{\tau}}\cdot\nabla_0)\ \frac{\rho}{\rho_0}$$

Cauchy's first equation in terms of $\hat{\underset{\approx}{\tau}}$ then becomes

$$\hat{\underset{\approx}{\tau}}\cdot\nabla_0 + \rho_0\underset{\sim}{b} = \rho_0\ddot{u} \tag{3.7-5}$$

Observe that this equation is linear if all variables are expressed in terms of $\underset{\sim}{R}_0$ and t, i.e., the material reference frame is used. However, note that Eq. (3.7-4) is nonlinear since the product of the stress tensor and the deformation gradient is involved.

We introduce another stress tensor, $\underset{\approx}{\tau}$, called the Piola-Kirchhoff stress tensor of the second kind such that

$$\underset{\approx}{\tau} = \underset{\approx}{F}^{-1}\cdot\hat{\underset{\approx}{\tau}}$$

$$= J\underset{\approx}{F}^{-1}\cdot\underset{\approx}{\sigma}\cdot\underset{\approx}{F}^{-T}$$

or

$$\underset{\approx}{\sigma} = \frac{1}{J}\,\underset{\approx}{F}\cdot\underset{\approx}{\tau}\cdot\underset{\approx}{F}^T$$

$$\underset{\approx}{\sigma}^T = \frac{1}{J}\,\underset{\approx}{F}\cdot\underset{\approx}{\tau}^T\cdot\underset{\approx}{F}^T \tag{3.7-6}$$

Since $\underset{\approx}{\sigma}$ is symmetric then for the above equation to hold $\underset{\approx}{\tau}$ must be symmetric.

The Piola-Kirchhoff stress tensors of the first and second kind are related by

$$\hat{\underset{\approx}{\tau}} = \underset{\approx}{F} \cdot \underset{\approx}{\tau} \tag{3.7-7}$$

$$\underset{\approx}{\tau} = \underset{\approx}{F}^{-1} \cdot \hat{\underset{\approx}{\tau}}$$

Cauchy's first and second equations can be expressed in the form

$$(\underset{\approx}{F} \cdot \underset{\approx}{\tau}) \cdot \nabla_0 + \rho_0 \underset{\sim}{b} = \rho_0 \ddot{\underset{\sim}{u}}$$

$$\underset{\approx}{\tau} = \underset{\approx}{\tau}^T \tag{3.7-8}$$

As a final note observe that $\underset{\approx}{\tau}$ is symmetric but that there is still the nonlinear term $(\underset{\approx}{F} \cdot \underset{\approx}{\tau}) \cdot \nabla_0$ in the first equation of (3.7-8) and the boundary condition on stress becomes

$$\hat{\underset{\sim}{t}} = (\underset{\approx}{F} \cdot \underset{\approx}{\tau}) \cdot \underset{\sim}{N}$$

which is also nonlinear.

Section 8 Boundary Conditions

Boundary conditions can be expressed in the following forms:

1. Prescribed Displacement

$$\underset{\sim}{u} = \underset{\sim}{u}_0(\underset{\sim}{R},t) \quad \text{on } \partial R_{0_u}$$

or

$$\underset{\sim}{u} = \underset{\sim}{u}_0(\underset{\sim}{r},t) \quad \text{on } \partial R_u$$

The above relations state that the displacement specified on a portion of the boundary can be described in either the reference or spatial state.

2. Prescribed Traction

$$\underset{\sim}{\hat{t}} = \underset{\sim}{t}_0\ (\underset{\sim}{R},t) \quad \text{on } \partial R_{0_t}$$

or

$$\underset{\sim}{t} = \underset{\sim}{t}_0(\underset{\sim}{r},t) \quad \text{on } \partial R_t$$

Similar to the displacement case, the traction is specified on a portion of the body in either the reference or spatial configuration.

If the above two boundary conditions are sufficient to solve the specified problem then

$$\partial R_{ou} + \partial R_{ot} = \partial R_0$$

and

$$\partial R_u + \partial R_t = \partial R$$

3. One Component of Prescribed Displacement

Suppose $\underset{\sim}{u}_o \cdot \underset{\sim}{m} = u_o$ is specified where $\underset{\sim}{m}$ is a unit vector. Then the other two components of traction must be given to completely describe the problem. Hence

$$\underset{\sim}{t} - (\underset{\sim}{t} \cdot \underset{\sim}{m})\underset{\sim}{m} = \underset{\sim}{t}_o$$

must be specified, i.e., the stress vector in the plane perpendicular to $\underset{\sim}{m}$ must be specified.

4. One Component of Prescribed Traction

If one component of traction is specified, i.e.

$$\underset{\sim}{t}\cdot\underset{\sim}{m} = t_0$$

is prescribed, then

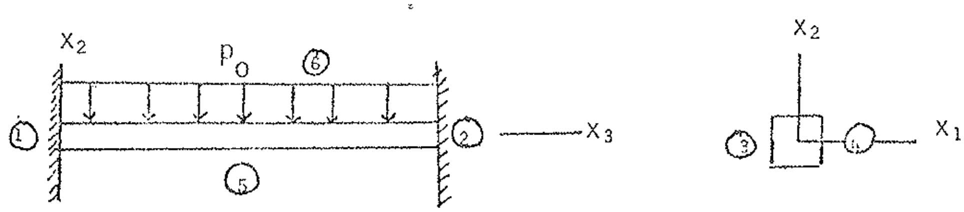
$$\underset{\sim}{u} - (\underset{\sim}{u}\cdot\underset{\sim}{m})\underset{\sim}{m} = u_o$$

must be specified.

Examples:

1. Fixed - Fixed Beam

Consider a fixed-fixed beam with the six sides of the beam shown on the figures.



The boundary conditions on the six sides are:

Side ①: $\underset{\sim}{u} = \underset{\sim}{o}$
" ②: $\underset{\sim}{u} = \underset{\sim}{o}$
" ③: $\underset{\sim}{t} = \underset{\sim}{o}$
" ④: $\underset{\sim}{t} = \underset{\sim}{o}$
" ⑤: $\underset{\sim}{t} = \underset{\sim}{o}$
" ⑥: $\underset{\sim}{t} = -p_o\underset{\sim}{e}_2$

For a traction prescribed boundary condition, sometimes the stress tensor components rather than the stress vector components are specified. For example, on side ⑥ where $\underset{\sim}{n} = \underset{\sim}{e}_2$

$$\underset{\sim}{t} = \underset{\approx}{\sigma}\cdot\underset{\sim}{n}$$

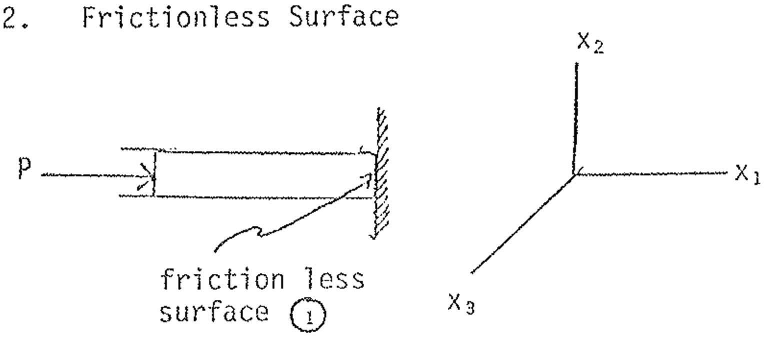
$$= \sigma_{12}\underset{\sim}{e}_1 + \sigma_{22}\underset{\sim}{e}_2 + \sigma_{32}\underset{\sim}{e}_3 = p_o\underset{\sim}{e}_2$$

Hence

$$\sigma_{12} = 0 \quad \sigma_{22} = -p_o \quad \sigma_{32} = 0$$

Thus for any point on this traction prescribed surface, three components of the stress tensor are known. However, the other three components are not known so the stress tensor is not completely defined at such a point by the boundary conditions.

2. Frictionless Surface



On Surface 1:

$u_1 = 0$

$t_2 = 0$

$t_3 = 0$

3.9 Thermodynamics of a Continuum

1. Mechanical Power

Consider a region R or any subregion R^* of a body. The power P applied to the body is defined to be

$$P_M = \int_{R^*} \underset{\sim}{b} \cdot \underset{\sim}{v} \rho \, dV + \int_{\partial R^*} \underset{\sim}{t} \cdot \underset{\sim}{v} \, dS \qquad (1.1)$$

$\underset{\sim}{b}$ - specific (per unit mass) body force field

$\underset{\sim}{t}$ - traction

Recall that

$$\underset{\sim}{t} = \underline{\underline{T}} \cdot \underset{\sim}{n} \qquad (1.2)$$

Then

$$\int_{\partial R^*} \underset{\sim}{t} \cdot \underset{\sim}{v} \, ds = \int_{\partial R^*} \underset{\sim}{v} \cdot \underline{\underline{T}} \cdot \underset{\sim}{n} \, dS$$

$$= \int_{R^*} (\underset{\sim}{v} \cdot \underline{\underline{T}}) \cdot \nabla \, dV$$

$$= \int_{R^*} \left[\underline{\underline{T}} : \underset{\sim}{v} \nabla + \underset{\sim}{v} \cdot (\underline{\underline{T}} \cdot \nabla) \right] dV \qquad (1.3)$$

Eqs. of Motion:

$$\underline{\underline{T}} = \underline{\underline{T}}^T \qquad (1.4)$$

$$\underline{\underline{T}} \cdot \nabla + \rho \underset{\sim}{b} = \rho \underset{\sim}{a}$$

The result is

$$P_M = \int_{R^*} \left[\underline{\underline{T}} : \underline{\underline{D}} + \rho \underset{\sim}{v} \cdot \underset{\sim}{a} \right] dV \qquad (1.5)$$

where

$$\underline{\underline{D}} = (\underset{\sim}{v} \nabla)_{sym} \qquad (1.6)$$

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Page

Consider the term

$$\int_{R^*} \rho \, \underset{\sim}{v} \cdot \underline{a} \, dV = \int_{R^*} \rho \, \underset{\sim}{v} \cdot \frac{d\underset{\sim}{v}}{dt} \, dV$$

$$= \int_{R^*} \rho \frac{d}{dt} \left(\frac{1}{2} \underset{\sim}{v} \cdot \underset{\sim}{v} \right) dV$$

$$= \frac{d}{dt} K \qquad (1.7)$$

where

$$K = \frac{1}{2} \int_{R^*} \rho \, \underset{\sim}{v} \cdot \underset{\sim}{v} \, dV \qquad (1.8)$$

is the kinetic energy of the body

Now

$$P_M = \int_{R^*} \underline{\underline{T}} : \underline{\underline{D}} \, dV + \frac{d}{dt} K \qquad (1.9)$$

Define the specific stress as

$$\underline{\underline{T}}^s = \frac{1}{\rho} \underline{\underline{T}} \qquad (1.10)$$

the specific stress power as

$$p^s = \underline{\underline{T}}^s : \underline{\underline{D}} = \frac{1}{\rho} \underline{\underline{T}} : \underline{\underline{D}} \qquad (1.11)$$

and the specific kinetic energy as

$$k^s = \frac{1}{2} \underset{\sim}{v} \cdot \underline{v} \qquad (1.12)$$

Note that for rigid body motion $\underline{\underline{D}} = \underline{\underline{0}}$
so p^s represents the "power of deformation"

Define the specific mechanical power:

$$\mathcal{P}_M^s = p^s + \overset{\circ}{k}^s$$

Then

$$P_M = \int_{R^*} \rho \, \mathcal{P}_M^s \, dV$$

<u>Conjugate Stresses & Deformation Rates</u> (Conjugate pairs)

The specific stress power is (recall: specific ⇒ per unit mass

$$p^s = \underline{\underline{\sigma}}^s : \underline{\underline{D}} = \frac{1}{\rho} \underline{\underline{\sigma}} : \underline{\underline{D}} \qquad \underline{\underline{D}} = (\underset{\sim}{\nabla} \underset{\sim}{v})_{sym}$$

$$\underline{\underline{\sigma}} : \underline{\underline{D}} = tr(\underline{\underline{\sigma}} \cdot \underline{\underline{D}}) = tr(\underline{\underline{\sigma}} \cdot \underline{\underline{L}}^T)$$

$$= tr\left(\frac{1}{J} \hat{\underline{\underline{\tau}}} \cdot \underline{\underline{F}}^T \cdot \underline{\underline{L}}^T\right)$$

$$= \frac{1}{J} tr(\hat{\underline{\underline{\tau}}} \cdot \dot{\underline{\underline{F}}}^T)$$

$$= tr\left[\frac{1}{J}(\underline{\underline{F}} \cdot \underline{\underline{\tau}} \cdot \underline{\underline{F}}^T) \cdot (\underline{\underline{F}}^{-T} \cdot \dot{\underline{\underline{E}}} \cdot \underline{\underline{F}}^{-1})\right]$$

$$= \frac{1}{J} tr(\underline{\underline{\tau}} \cdot \dot{\underline{\underline{E}}})$$

Also

$$\underline{\underline{\sigma}} : \underline{\underline{D}} = \underline{\underline{\Sigma}} : \underline{\underline{D}}^R \quad \text{where} \quad \underline{\underline{\Sigma}} = \underline{\underline{R}}^T \cdot \underline{\underline{\sigma}} \cdot \underline{\underline{R}} \quad \text{Rotated Cauchy stress}$$

$$\underline{\underline{D}}^R = \underline{\underline{R}}^T \cdot \underline{\underline{D}} \cdot \underline{\underline{R}} \quad \text{Rotated rate of deformation}$$

The following sets are called conjugate pairs:

$(\underline{\underline{\sigma}}, \underline{\underline{D}})$, $(\underline{\underline{\Sigma}}, \underline{\underline{D}}^R)$, $(\underline{\underline{\tau}}, \dot{\underline{\underline{E}}})$ and $(\hat{\underline{\underline{\tau}}}, \dot{\underline{\underline{F}}}^T)$

The "natural" pair is $(\underline{\underline{\tau}}, \dot{\underline{\underline{E}}})$ for then if $U = U(E)$, we have $\dot{U} = \partial U / \partial \underline{\underline{E}} : \dot{\underline{\underline{E}}}$ and $\frac{\partial U}{\partial \underline{\underline{E}}} = \underline{\underline{\tau}}$.

2. Thermal Powers

Let $\underset{\sim}{q}$ be the heat flux vector defined such that $\underset{\sim}{q} \cdot \underset{\sim}{n}$ is the rate at which heat is flowing out of a surface. Let Q be the specific (body) heat source. The thermal power is defined to be (applied to the body)

$$P_T = \int_{R^*} Q \rho \, dV - \int_{\partial R^*} \underset{\sim}{q} \cdot \underset{\sim}{n} \, dS$$

$$= \int_{R^*} \left[Q - \frac{1}{\rho} \underset{\sim}{q} \cdot \nabla \right] \rho \, dV \qquad (2.1)$$

$$= \int_{R^*} \rho \, P_T^s \, dV$$

$$P_T = Q - \frac{1}{\rho} \underset{\sim}{q} \cdot \nabla \quad \text{(specific thermal power)} \qquad (2.2)$$

3. Work

The mechanical work and thermal work associated with these two powers over a time span $t_2 - t_1$ is:

$$W_{M_{1-2}} = \int_{t_1}^{t_2} P_M \, dt \quad \text{or} \quad W_M(t) = \int_{t_1}^{t} P_M \, dt \qquad (3.1)$$

$$W_{T_{1-2}} = \int_{t_1}^{t_2} P_T \, dt \quad \text{or} \quad W_T(t) = \int_{t_1}^{t} P_T \, dt \qquad (3.2)$$

In general $\dot{W}_M \neq P_M$ $\quad \dot{W}_T \neq P_T$, i.e., the work down by power creating agencies is ... path dependent.

4. First Law of Thermodynamics

Even though the work done individually by the mechanical and thermal agencies is path dependent, it has been observed that the sum of the work done is not path dependent. This observation infers that there must exist an energy function E such that

$$\dot{E} = P_M + P_T \tag{4.1}$$

This observation is the <u>1st Law</u>.

Define a specific energy e $\ni$

$$E = \int_{R^t} \rho e \, dV \tag{4.2}$$

$\Rightarrow$

$$\dot{E} = \int_{R^t} \rho \dot{e} \, dV$$

Then alternative forms of the first law are:

$$\dot{e} = p_M^s + p_T^s \tag{4.3}$$

$$= p^s + \dot{k} + p_T^s$$

Define specific and global internal energy functions:

$$u = \dot{e} - \dot{k} \qquad U = \int_{R^t} \rho u \, dV \tag{4.4}$$

u is called a thermodynamic potential.

Then alternative forms of the 1st law are:

$$\dot{u} = \rho^s + \rho^s_T$$

$$= \underset{=}{\sigma}^s : \underset{=}{D} + Q - \frac{1}{\rho}(\underset{-}{q} \cdot \nabla)$$

$$= \frac{1}{\rho} \underset{=}{\sigma} : \underset{=}{D} + Q - \frac{1}{\rho}(\underset{-}{q} \cdot \nabla) \qquad (4.5)$$

Suppose u is a function of entropy, s, and strain $\underset{=}{e}$ (small strain). Then

$$\dot{u} = \frac{\partial u}{\partial s}\dot{s} + \frac{\partial u}{\partial \underset{=}{e}} : \dot{\underset{=}{e}} \qquad (4.6)$$

Define temperature T to be

$$T = \frac{\partial u}{\partial s} \qquad (4.7)$$

Then the 1st law becomes

$$\frac{\partial u}{\partial \underset{=}{e}} : \dot{\underset{=}{e}} + T\dot{s} = \frac{1}{\rho} \underset{=}{\sigma} : \underset{=}{D} + Q - \frac{1}{\rho}(\underset{-}{q} \cdot \nabla) \qquad (4.8)$$

But $\dot{\underset{=}{e}} = \underset{=}{D}$ (for small deformations). If this is to hold for all $\underset{=}{D}$, the 1st law becomes

$$\underset{=}{\sigma} = \rho \frac{\partial u}{\partial \underset{=}{e}}$$

$$T\dot{s} = Q - \frac{1}{\rho}(\underset{-}{q} \cdot \nabla) \qquad (4.9)$$

Now $\dot{u} = \frac{\partial u}{\partial s}\dot{s} + \frac{\partial u}{\partial \underset{=}{e}} : \dot{\underset{=}{e}} = T\dot{s} + \underset{=}{\sigma}^s : \dot{\underset{=}{e}}$ $\qquad \underset{=}{\sigma}^s = \underset{=}{\sigma}/\rho$

T and s ; $\underset{=}{\sigma}^s$ and $\underset{=}{e}$ are said to represent conjugate pairs with respect to the internal energy.

5. <u>Elementary Form for $\mathcal{U}$</u> :

Suppose

$$\mathcal{U}(s,\underline{\underline{e}}) = \frac{1}{2\rho}\, \underline{\underline{e}} : \underline{\underline{\underline{\underline{E}}}}_s : \underline{\underline{e}} + c_{\underline{e}} T_0 \left[e^{(s-s_0)/c_{\underline{e}}} - 1 \right] \qquad (5.1)$$

(Sometimes called the caloric equation of state

$c_{\underline{e}}$ – the specific heat at constant strain (sometimes denoted as c_v)

$\underline{\underline{\underline{\underline{E}}}}_s$ – elasticity tensor for constant s

Then, by defn:

$$T = T_0\, e^{(s-s_0)/c_{\underline{e}}} \qquad (5.2)$$

Constitutive eq. relating temperature & entropy

The 1st law yields

$$\underline{\underline{\nabla}} = \rho \frac{\partial U}{\partial \underline{\underline{e}}} = \underline{\underline{\underline{\underline{E}}}}_s : \underline{\underline{e}} \qquad (5.3)$$

Constit. eq. relating stress & strain

and

$$T \dot{s} = Q - \frac{1}{\rho}(\underline{q} \cdot \nabla) \qquad (5.4)$$

Suppose (5.2) is solved for s :

$$\frac{s - s_0}{c_e} = \ln \frac{T}{T_0} \qquad s = s_0 + c_e \ln \frac{T}{T_0} \qquad (5.5)$$

Then

$$\dot{s} = c_e \frac{\dot{T}}{T} \qquad (5.6)$$

and

$$T\dot{s} = c_e \dot{T}$$

The 1st law reduces to

$$c_e \dot{T} = -\frac{1}{\rho}(\underline{q} \cdot \nabla) + Q \qquad (5.7)$$

Suppose (5.5) is substituted in (5.1) to obtain the internal energy as a function of $\underset{=}{e}$ and T. The result is called a <u>thermal equation of state</u>

$$u^*(T, \underset{=}{e}) = \frac{1}{2\rho}\, \underset{=}{e} : \underset{\equiv}{E}_s : \underset{=}{e} + c_e (T - T_0) \qquad (5.8)$$

6. Fourier's Law & the 1st Law

The following particular constitutive relation between heat flux & temperature gradient is called Fourier's Law:

$$\underset{\sim}{q} = -\underline{\underline{K}}\cdot(T\nabla) \qquad \underline{\underline{K}} \text{ - conductivity} \qquad (6.1)$$

$\underline{\underline{K}}$ might be a general tensor (generally considered symmetric) for anisotropic materials. If $\underline{\underline{K}}$ is not a function of T, the constitutive ~~fn~~ relation is linear; if $\underline{\underline{K}}$ is not a function of $\underset{\sim}{x}$, the material is homogeneous. If

$$\underline{\underline{K}} = k\,\underline{\underline{I}} \qquad (6.2)$$

the material is isotropic. If k is a constant, the material is isotropic & homogeneous.

Suppose (6.1) is substituted in (5.7). Then the 1st law becomes:

$$\rho c_e \dot{T} = (\underline{\underline{K}}\cdot T\nabla)\cdot\nabla + \rho Q \qquad (6.3)$$

Similarly define <u>enthalpy</u> h by

$$h(s, \underline{\sigma}^s) = u - \underline{\sigma}^s : \underline{\underline{e}} \qquad (7.6)$$

$$\left(\frac{\partial h}{\partial s}\right)_{\sigma^s} = \left(\frac{\partial u}{\partial s}\right)_e = T \qquad \frac{\partial h}{\partial \underline{\sigma}^s} = -\underline{\underline{e}} \qquad (7.7)$$

Free <u>Enthalpy (Gibb's Fn)</u> g

$$g(T, \underline{\sigma}^s) = h - Ts \qquad (7.8)$$

$$s = -\frac{\partial g}{\partial T} \qquad \underline{\underline{e}} = -\frac{\partial g}{\partial \underline{\sigma}^s} \qquad (7.9)$$

To avoid the minus signs, Gibb's Free Energy is defined to be

$$G(T, \underline{\sigma}^s) = -g(T, \underline{\sigma}^s) \qquad (7.10)$$

Then

$$s = \frac{\partial G}{\partial T} \qquad \underline{\underline{e}} = \frac{\partial G}{\partial \underline{\sigma}^s} \qquad (7.11)$$

Note that $\quad U - A + g - h = 0 \qquad (7.12)$

For constant entropy problems, use $u(s, \underline{\underline{e}})$ and $h(s, \underline{\sigma}^s)$ for then these fns. are functions of single variables.

For constant temperature problems (isothermal) use $A(T, \underline{\underline{e}})$ or $G(T, \underline{\sigma}^s)$ as thermodynamic potentials.

8. Inferred Identities

Maxwellian Equations

– Involve X-derivatives of each potential function – can be interchanged.

$$\frac{\partial^2 u}{\partial \underline{\underline{e}}\, \partial s} \Rightarrow \left(\frac{\partial \underline{\underline{\sigma}}^s}{\partial s}\right)_{\underline{\underline{e}}} = \left(\frac{\partial T}{\partial \underline{\underline{e}}}\right)_s$$

$$\frac{\partial^2 A}{\partial \underline{\underline{e}}\, \partial T} \Rightarrow \left(\frac{\partial \underline{\underline{\sigma}}^s}{\partial T}\right)_{\underline{\underline{e}}} = -\left(\frac{\partial s}{\partial \underline{\underline{e}}}\right)_T$$

$$\frac{\partial^2 h}{\partial s\, \partial \underline{\underline{\sigma}}^s} \Rightarrow -\left(\frac{\partial \underline{\underline{e}}}{\partial s}\right)_{\underline{\underline{\sigma}}^s} = \left(\frac{\partial T}{\partial \underline{\underline{\sigma}}^s}\right)_s$$

$$\frac{\partial^2 G}{\partial T\, \partial \underline{\underline{\sigma}}^s} \Rightarrow \left(\frac{\partial s}{\partial \underline{\underline{\sigma}}^s}\right)_T = \left(\frac{\partial \underline{\underline{e}}}{\partial T}\right)_{\sigma^s}.$$

Gibbsian Eqs.

$$\dot{u} = \frac{\partial u}{\partial \underline{\underline{e}}} : \dot{\underline{\underline{e}}} + \frac{\partial u}{\partial s}\dot{s} = \underline{\underline{\sigma}}^s : \dot{\underline{\underline{e}}} + T\dot{s}$$

$$\dot{A} = \qquad = \underline{\underline{\sigma}}^s : \dot{\underline{\underline{e}}} - s\dot{T}$$

$$\dot{h} = \qquad = -\underline{\underline{e}} : \dot{\underline{\underline{\sigma}}}^s + T\dot{s}$$

$$\dot{G} = \qquad = \underline{\underline{e}} : \dot{\underline{\underline{\sigma}}}^s + s\dot{T}$$

9. Dissipation

Suppose the strain is decomposed into an elastic and plastic part:

$$\underline{\underline{e}} = \underline{\underline{e}}^e + \underline{\underline{e}}^p \qquad \dot{\underline{\underline{e}}} = \dot{\underline{\underline{e}}}^e + \dot{\underline{\underline{e}}}^p = \underline{\underline{D}} \tag{9.1}$$

Instead of 5.1, the internal energy is assumed to be

$$U(S, \underline{\underline{e}}^e) = \frac{1}{2\rho} \underline{\underline{e}}^e : \underline{\underline{\underline{\underline{E}}}}_s : \underline{\underline{e}}^e + c_e T_0 \left[e^{(S-S_0)/c_e} - 1 \right] \tag{9.2}$$

Then the def'n of temp. yields, as before,

$$T = T_0 \, e^{(S-S_0)/c_e} \tag{9.3}$$

and the 1st law implies

$$\underline{\underline{\sigma}} = \rho \frac{\partial U}{\partial \underline{\underline{e}}^e} = \underline{\underline{\underline{\underline{E}}}}_s : \underline{\underline{e}}^e$$

$$c_e \dot{T} = \frac{1}{\rho} \underline{\underline{\sigma}} : \dot{\underline{\underline{e}}}^p + Q - \frac{1}{\rho} (\underline{q} \cdot \nabla) \tag{9.4}$$

The term

$$D^p \equiv \underline{\underline{\sigma}} : \dot{\underline{\underline{e}}}^p \tag{9.5}$$

is called the plastic dissipation rate and is a source of heat in the energy equation.

For dynamic problems, (9.4) is often replaced with the approximation

$$c_e \dot{T} = (0.9) \frac{1}{\rho} \underline{\underline{\sigma}} : \dot{\underline{\underline{e}}}^p \tag{9.6}$$

10. Second Law of Thermodynamics

It is postulated, based on experimental observations, that the rate of increase of internal entropy must be greater than or equal to the rate at which entropy is added to the body by external heat sources and by heat flux over the boundary.

The rate at which entropy is added by the two heat sources is defined to be

$$\dot{\eta}_Q = \frac{Q}{T} \qquad (10.1)$$

$$\dot{\eta}_{\tilde{q}} = \tilde{q}/T$$

Then the second law for an arbitrary region is:

$$\int_{R^*} \rho \dot{s} \, dV \geq \int_{R^*} \rho \dot{\eta}_Q \, dV - \int_{\partial R^*} \dot{\eta}_{\tilde{q}} \cdot \tilde{n} \, dS \qquad (10.2)$$

Apply the divergence theorem, and define the specific total entropy rate $\dot{\eta}$ such that

$$\rho \dot{\eta} = \rho \dot{s} - \rho \dot{\eta}_Q + (\dot{\eta}_{\tilde{q}}) \cdot \nabla \qquad (10.3)$$

Then, because R^* is arbitrary, the second law reduces to

$$\dot{\eta} \geq 0 \qquad (10.4)$$

or

$$\rho \dot{s} \geq \rho \dot{\eta}_Q - (\dot{\eta}_q) \cdot \nabla \qquad 10.5$$

which is also known as the Clausius-Duhem Inequality.

11. Alternative Forms of the Second Law:

Observe that

$$\dot{\eta}_q \cdot \nabla = \frac{q \cdot \nabla}{T} - \frac{1}{T^2} q \cdot (T\nabla) \qquad (11.1)$$

Recall from (2.2) that the specific thermal power is

$$P_T^s = Q - \frac{1}{\rho} q \cdot \nabla \qquad (11.2)$$

so that

$$-\frac{1}{\rho}\frac{q \cdot \nabla}{T} = \frac{P_T^s - Q}{T} \qquad (11.3)$$

Let

$$\dot{\varsigma} = -\frac{1}{\rho T^2} q \cdot (T\nabla) \qquad (11.4)$$

Then the second law becomes

$$\dot{s} \geq \frac{P_T^s}{T} - \dot{\varsigma} \qquad (11.5)$$

Now use (4.5):

$$P_T^s = \dot{u} - P^s \qquad (11.6)$$

$$= \dot{u} - \frac{1}{\rho} \underline{\underline{\nabla}} : \underline{\underline{D}}$$

Another alternative form then is (Reduced Dissipation Inequality)

$$\dot{s} \geq \frac{1}{T}\left(\dot{u} - \frac{1}{\rho} \underline{\underline{\nabla}} : \underline{\underline{D}}\right) - \dot{\varsigma} \qquad (11.7)$$

a form that contains neither the thermal power nor the heat source explicitly.

The dissipation rate is $D_s = \underline{\underline{\nabla}} : \underline{\underline{D}} - \rho \dot{u}$

Then

$$\dot{s} \geq -\frac{D_s}{\rho T} - \dot{\varsigma}$$

Recall that the Helmholtz Free Energy is defined to be

$$A \doteq u - TS \tag{11.8}$$

so that

$$\dot{u} = \dot{A} + \dot{T}s + T\dot{s} \tag{11.9}$$

Then (11.7) becomes

$$T\dot{s} \geq \dot{A} + \dot{T}s + T\dot{s} - T\varrho - \frac{1}{\rho}\underline{\underline{\nabla}} : D$$

or

$$\frac{1}{T}\left(\frac{1}{\rho}\underline{\underline{\nabla}} : \underline{\underline{D}} - \dot{A} - s\dot{T}\right) + \varrho \geq 0 \tag{11.10}$$

which does not contain the entropy rate.

12. Implications of the Second Law

(12.1) Suppose $u = u(s, \underline{\underline{e}})$. (12.1)

Then $T = \frac{\partial u}{\partial s}$ by defn.

and the 1st law $\Rightarrow$

$$\frac{1}{\rho}\underline{\underline{\sigma}} = \frac{\partial u}{\partial \underline{\underline{e}}}$$

It follows that

$$\dot{u} = \frac{\partial u}{\partial s}\dot{s} + \frac{\partial u}{\partial \underline{\underline{e}}} : \overset{\circ}{\underline{\underline{e}}}$$

$$= T\dot{s} + \frac{1}{\rho}\underline{\underline{\sigma}} : \overset{\circ}{\underline{\underline{e}}} \qquad (12.2)$$

Substitute in (11.7)

$$T\dot{s} \geq T\dot{s} + \frac{1}{\rho}\underline{\underline{\sigma}} : \overset{\circ}{\underline{\underline{e}}} - \frac{1}{\rho}\underline{\underline{\sigma}} : \underline{\underline{D}} - T\varsigma \qquad (12.3)$$

or

$$T\varsigma \geq 0$$

or $$-\frac{1}{\rho T}\,\underset{\sim}{q} \cdot (T\nabla) \geq 0 \qquad (12.4)$$

i.e. heat must flow in the negative dir'n defined by the gradient of T.

Consider Fourier's Constitutive Eq. :

$$\underset{\sim}{q} = -\underline{\underline{K}} \cdot (T\nabla) \qquad (12.5)$$

Then the second law implies

$$(T\nabla) \cdot \underline{\underline{K}} \cdot (T\nabla) \geq 0 \qquad (12.6)$$

or $\underline{\underline{K}}$ must be positive, semi-definite.

(12.2) Suppose $u = u(S, \underline{\underline{e}}^e)$ and $\underline{\underline{D}} = \underline{\underline{D}}^e + \underline{\underline{D}}^p$

Then an analogous procedure yields

$$D + T\varsigma \geq 0 \qquad (12.7)$$

$$D = \frac{1}{\rho}\underline{\underline{\sigma}} : \dot{\underline{\underline{e}}}^p = \frac{1}{\rho}\underline{\underline{\sigma}} : \underline{\underline{D}}^p \quad \text{plastic dissipation}$$

Normally it is assumed that each term is positive or zero;

$$\varsigma \geq 0 \qquad (12.8)$$
$$D \geq 0$$

In plasticity theory, an evolution equation for $\underline{\underline{D}}^p$ must be postulated. One such form is that

$$\dot{\underline{\underline{e}}}^p = \dot{\lambda}\, \underline{\underline{m}}(\underline{\underline{\sigma}}, \underline{\underline{e}}^p) \qquad \dot{\lambda} \geq 0 \qquad (12.9)$$

The second law imposes the restriction that

$$\underline{\underline{\sigma}} : \underline{\underline{m}} \geq 0 \qquad \forall\ \underline{\underline{\sigma}} \text{ and } \underline{\underline{e}}^p \qquad (12.10)$$

Chapter 4 Constitutive Equations

Section 4.1 Basic Requirements

The equations of continuum mechanics derived up to now hold for all materials. Except in rather unusual circumstances (for ex., statically determinate problems) the stress field resulting from applied forces cannot be obtained. An extra set of equations, called the constitutive equations, that characterize the individual material and its reaction to applied loads is required to provide a complete set of field equations.

It is not feasible to use one set of equations to describe accurately a real material over its entire range of behavior. Instead, equations corresponding to ideal material response are formulated to describe, in an approximate manner, the response of a material over a suitably restricted range.

Constitutive equations will in general depend on the temperature or on the entropy of the medium. A purely mechanical theory is defined to be one in which neither of these variables is present. Such a theory is applicable for some materials and is a good approximation for others in which the response depends only slightly on termperature or entropy.

Three fundamental postulates are assumed to hold for purely mechanical constitutive equations of a continuous medium. They are:

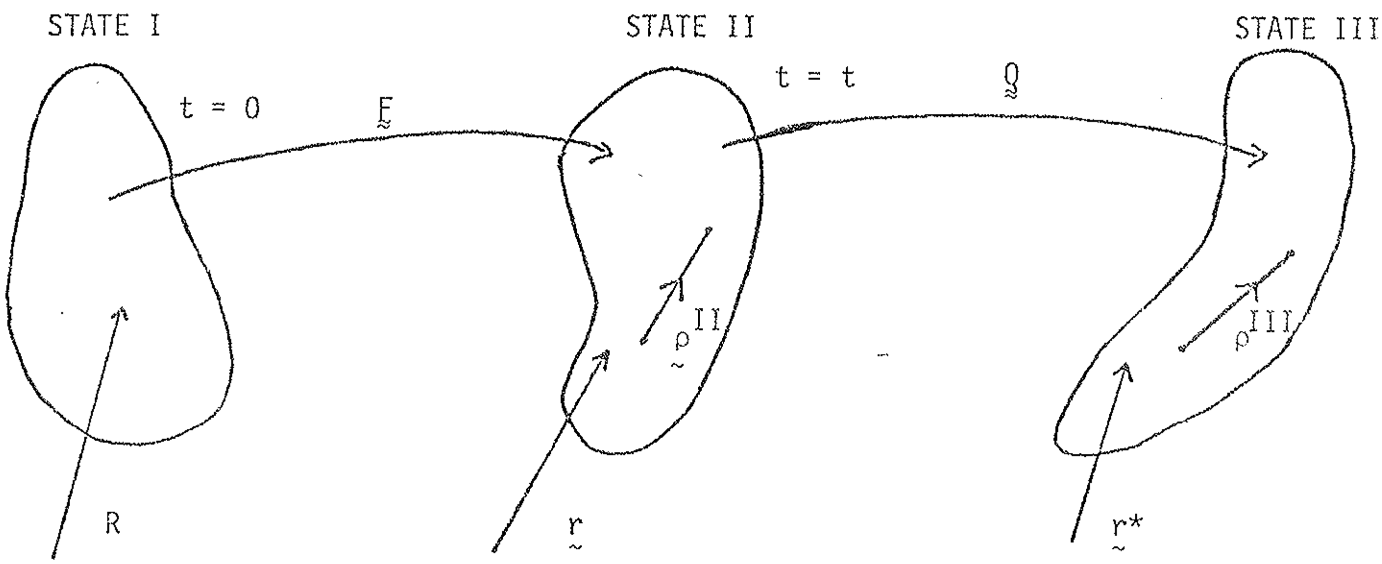
1. Principle of Determinism of Stress: The stress in a body is determined by the history of the motion of that body. We will also assume that the principle of determinism implies uniqueness, that is, if two motions of the same material differ by an arbitrarily small value as measured by a suitable norm, then the norms of the two resulting stress states will also differ by a small value.

2. Principle of Local Action: To determine the stress at a particle X, the motion outside an arbitrary neighborhood of X may be disregarded.

These first two postulates imply that the stress is determined by the history of the deformation gradient (and its time derivatives). The use of higher order deformation gradients is ruled out since by the use of Taylor's series, this would imply that information from more than an arbitrarily small neighborhood would be required. Such a material is often defined to be a "simple" material.

3. Principle of Material Frame - Indifference (PMI): Constitutive equations must be invariant under changes of the frame of reference. For practical applications, this principle is equivalent to saying that constitutive equations must not violate the following relations obtained by taking a deformed body and superimposing a rigid body displacement.

Suppose a body is in some deformed state at time t (State II), and that an arbitrary rigid body motion is added to obtain State III



From Eq. (2.2-6) relating States I and II

$$dr = F \cdot dR \tag{4.1-1}$$

and from Sec. 2.9 on rigid body motion, with Q, an orthogonal tensor, used instead of R

$r^* = c(t) + Q(t) \cdot r$

$$dr^* = Q \cdot dr \tag{4.1-2}$$

If $\underset{\sim}{\rho}^{II}$ denotes a vector joining any two particles in State II, then it follows that the vector joining the same two particles in State III is given by (Eq. 2.9-4))

$$\underset{\sim}{\rho}^{III} = \underset{\approx}{Q}\cdot\underset{\sim}{\rho}^{II} \tag{4.1-3}$$

Since $\underset{\sim}{n}$, the unit normal vector for State II and $\underset{\sim}{t}$, the applied traction for State II which is assumed to move "with" the body between States II and III, can be considered parallel to vectors fixed to the body, it follows that

$$\begin{aligned} \underset{\sim}{n}^* &= \underset{\approx}{Q}\cdot\underset{\sim}{n} \\ \underset{\sim}{t}^* &= \underset{\approx}{Q}\cdot\underset{\sim}{t} \end{aligned} \tag{4.1-4}$$

Since

$$\begin{aligned} \underset{\sim}{t} &= \underset{\approx}{\sigma}\cdot\underset{\sim}{n} \\ \underset{\sim}{t}^* &= \underset{\approx}{\sigma}^*\cdot\underset{\sim}{n}^* \end{aligned} \tag{4.1-5}$$

the use of Eq. (4.1-4) for the latter equation yields

$$\underset{\approx}{Q}\cdot\underset{\sim}{t} = \underset{\approx}{\sigma}^*\cdot\underset{\approx}{Q}\cdot\underset{\sim}{n} \tag{4.1-6}$$

The orthogonality relation $\underset{\approx}{Q}^T\cdot\underset{\approx}{Q} = \underset{\approx}{I}$ is used to obtain

$$\underset{\sim}{t} = \underset{\approx}{Q}^T\cdot\underset{\approx}{\sigma}^*\cdot\underset{\approx}{Q}\cdot\underset{\sim}{n} \tag{4.1-7}$$

which must hold for all $\underset{\sim}{t}$ and $\underset{\sim}{n}$. Thus

$$\begin{aligned} \underset{\approx}{\sigma} &= \underset{\approx}{Q}^T\cdot\underset{\approx}{\sigma}^*\cdot\underset{\approx}{Q} \\ \underset{\approx}{\sigma}^* &= \underset{\approx}{Q}\cdot\underset{\approx}{\sigma}\cdot\underset{\approx}{Q}^T \end{aligned} \tag{4.1-8}$$

The traction vector $\tilde{t}$ per unit undeformed state, and $\tilde{N}$, the unit normal vector for the undeformed state, do not change with time. Thus, the equation corresponding to Eq. (4.1-8) for the second Piola-Kirchhoff stress tensor is

$$\tau^* = \tau \tag{4.1-9}$$

If F^* is the deformation gradient between States I and III, then

$$dr^* = F^* \cdot dR \tag{4.1-10$_a$}$$

But from Eq. (4.1-1) and (4.1-2)

$$\begin{aligned} dr^* &= Q \cdot dr \\ &= Q \cdot F \cdot dR \end{aligned} \tag{4.1-10$_b$}$$

These equations must hold for all dR and dr. Hence

$$\begin{aligned} F^* &= Q \cdot F \\ F^{*-1} &= F^{-1} \cdot Q^T \\ F^{*-T} &= Q \cdot F^{-T} \\ F^{*T} &= F^T \cdot Q^T \end{aligned} \tag{4.1-11}$$

Refering to Eq. (2.6-2), the strain tensors in State III are

$$\begin{aligned} E^* &= \tfrac{1}{2}(F^{*T} \cdot F^* - I) \\ &= \tfrac{1}{2}(F^T \cdot Q^T \cdot Q \cdot F - I) \\ &= \tfrac{1}{2}(F^T \cdot F - I) \\ &= E \\ e^* &= \tfrac{1}{2}(I - F^{*-T} \cdot F^{*-1}) \\ &= \tfrac{1}{2}(I - Q \cdot F^{-T} \cdot F^{-1} \cdot Q^T) \\ &= Q \cdot \tfrac{1}{2}(Q^T \cdot Q - F^{-T} \cdot F^{-1}) \cdot Q^T \\ &= Q \cdot \tfrac{1}{2}(I - F^{-T} \cdot F^{-1}) \cdot Q^T \\ &= Q \cdot e \cdot Q^T \end{aligned}$$

In summary

$$\underset{\approx}{E}^* = \underset{\approx}{E} \qquad \underset{\approx}{e}^* = \underset{\approx}{Q}\cdot\underset{\approx}{e}\cdot\underset{\approx}{Q}^T \tag{4.1-12}$$

Recall from Eq. (2.5-1) that the velocity gradients are given by

$$\underset{\approx}{L} = \underset{\sim}{v}\nabla \qquad \underset{\approx}{L}^* = \underset{\sim}{v}^*\nabla^* \tag{4.1-13}$$

From Eq. (2.2-12) with $\underset{\approx}{F}$ replaced by $\underset{\approx}{Q}$,

$$(\ )\nabla = (\ )\nabla^*\cdot\underset{\approx}{Q} \qquad (\ )\nabla^* = (\ )\nabla\cdot\underset{\approx}{Q}^T \tag{4.1-14}$$

The velocity vector is obtained from (4.1-2):

$$\underset{\sim}{v}^* = \dot{\underset{\sim}{r}}^* = \dot{\underset{\sim}{c}}(t) + \dot{\underset{=}{Q}}(t)\cdot\underset{\sim}{r} + \underset{=}{Q}(t)\cdot\underset{\sim}{v}$$

or

$$\underset{\sim}{v}^* = \dot{\underset{\sim}{c}} + \underset{=}{Q}\cdot\underset{\sim}{v} + \dot{\underset{=}{Q}}\cdot\underset{=}{Q}^T\cdot(\underset{\sim}{r}^* - \underset{\sim}{c}) \tag{4.1-15}$$

Thus

$$\underset{=}{L}^* = \underset{\sim}{v}^*\nabla^* = \underset{=}{Q}\cdot\underset{\sim}{v}\nabla\cdot\underset{=}{Q}^T + \dot{\underset{=}{Q}}\cdot\underset{=}{Q}^T$$

since $\underset{=}{Q}\nabla^* = \underset{=}{0}$. Let the rigid body spin be

$$\underset{=}{\Omega}^Q = \dot{\underset{=}{Q}}\cdot\underset{=}{Q}^T$$

Then

$$\underset{=}{L}^* = \underset{=}{Q}\cdot\underset{=}{L}\cdot\underset{=}{Q}^T + \underset{=}{\Omega}^Q \tag{4.1-16}$$

It follows immediately that

$$\underset{=}{D}^{*} = \frac{1}{2}(\underset{=}{L}^{*} + \underset{=}{L}^{*T})$$

$$= \frac{1}{2}(\underset{=}{Q}\cdot\underset{=}{L}\cdot\underset{=}{Q}^{T} + \underset{=}{Q}\cdot\underset{=}{L}^{T}\cdot\underset{=}{Q}^{T}) + \frac{1}{2}(\underset{=}{\Omega}^{Q} + \underset{=}{\Omega}^{QT})$$

$$= \underset{=}{Q}\cdot\frac{1}{2}(\underset{=}{L} + \underset{=}{L}^{T})\cdot\underset{=}{Q}^{T}$$

$$= \underset{=}{Q}\cdot\underset{=}{D}\cdot\underset{=}{Q}^{T} \qquad (4.1\text{-}17)$$

since $\underset{=}{\Omega}^{Q}$ is skew-symmetric. Also

$$\underset{=}{W}^{*} = \frac{1}{2}(\underset{=}{L}^{*} - \underset{=}{L}^{*T}) = \underset{=}{Q}\cdot\underset{=}{W}\cdot\underset{=}{Q}^{T} + \underset{=}{\Omega}^{Q} \qquad (4.1\text{-}18)$$

From (4.1-12), the strain rates are

$$\overset{\circ}{\underset{=}{E}}{}^{*} = \overset{\circ}{\underset{=}{E}} \qquad (4.1\text{-}19)$$

$$\overset{\circ}{\underset{=}{e}}{}^{*} = \underset{=}{Q}\cdot\overset{\circ}{\underset{=}{e}}\cdot\underset{=}{Q}^{T} + \overset{\circ}{\underset{=}{Q}}\cdot\underset{=}{e}\cdot\underset{=}{Q}^{T} + \underset{=}{Q}\cdot\underset{=}{e}\cdot\overset{\circ}{\underset{=}{Q}}{}^{T}$$

Consider traction and the -Kirchhoff stress tensors:

$$\hat{\underset{\sim}{t}}\, dS_0 = \underset{\sim}{t}\, ds \qquad \underset{\sim}{t}^{*} = \underset{=}{Q}\cdot\underset{\sim}{t}$$

Therefore $\hat{\underset{\sim}{t}}^{*} = \underset{=}{Q}\cdot\hat{\underset{\sim}{t}}$ (4.1-20)

Recall that $\hat{\underset{=}{\tau}}\cdot\underset{=}{N} = \hat{\underset{\sim}{t}}$

Then $\hat{\underset{=}{\tau}}^{*}\cdot\underset{\sim}{N}^{*} = \hat{\underset{=}{\tau}}^{*}\cdot\underset{=}{N} = \hat{\underset{\sim}{t}}^{*} = \underset{=}{Q}\cdot\hat{\underset{\sim}{t}} = \underset{=}{Q}\cdot\underset{=}{\tau}\cdot\underset{\sim}{N}$

Thus $\hat{\underset{=}{\tau}}^{*} = \underset{=}{Q}\cdot\hat{\underset{\approx}{\tau}}$ (4.1-21)

Similarly, since $\underset{=}{\tau}^{*} = \underset{=}{F}^{*-1}\cdot\hat{\underset{=}{\tau}}^{*} = (\underset{=}{Q}\cdot\underset{=}{F})^{-1}\cdot\hat{\underset{=}{\tau}}^{*} = \underset{=}{F}^{-1}\cdot\underset{=}{Q}^{T}\cdot\underset{=}{Q}\cdot\hat{\underset{=}{\tau}}$

$$= \underset{=}{F}^{-1}\cdot\hat{\underset{=}{\tau}} = \underset{=}{F}^{-1}\cdot\underset{=}{F}\cdot\underset{=}{\tau}$$

it follows that

$$\underset{=}{\tau}^{*} = \underset{=}{\tau} \qquad (4.1\text{-}22)$$

Summary of Relations Based on Rigid Body Rotation

$$\underset{\sim}{n}^* = \underline{\underline{Q}} \cdot \underset{\sim}{n} \qquad \underset{\sim}{N}^* = \underset{\sim}{N}$$

$$\underset{\sim}{t}^* = \underline{\underline{Q}} \cdot \underset{\sim}{t}$$

$$\underset{\sim}{\hat{t}}^* = \underline{\underline{Q}} \cdot \underset{\sim}{\hat{t}}$$

$$\underline{\underline{\sigma}}^* = \underline{\underline{Q}} \cdot \underline{\underline{\sigma}} \cdot \underline{\underline{Q}}^T \qquad \underline{\underline{\tau}}^* = \underline{\underline{\tau}} \qquad \underset{\sim}{\hat{\tau}}^* = \underline{\underline{Q}} \cdot \underset{\approx}{\hat{\tau}}$$

$$\underline{\underline{e}}^* = \underline{\underline{Q}} \cdot \underline{\underline{e}} \cdot \underline{\underline{Q}}^T \qquad \underline{\underline{E}}^* = \underline{\underline{E}} \qquad \underline{\underline{F}}^* = \underline{\underline{Q}} \cdot \underline{\underline{F}}$$

$$\underline{\underline{D}}^* = \underline{\underline{Q}} \cdot \underline{\underline{D}} \cdot \underline{\underline{Q}}^T \qquad \dot{\underline{\underline{E}}}^* = \dot{\underline{\underline{E}}} \tag{4.1-23}$$

$$\underline{\underline{W}}^* = \underline{\underline{Q}} \cdot \underline{\underline{W}} \cdot \underline{\underline{Q}}^T + \underline{\underline{\Omega}}^Q \qquad \underline{\underline{L}}^* = \underline{\underline{Q}} \cdot \underline{\underline{L}} \cdot \underline{\underline{Q}}^T + \underline{\underline{\Omega}}^Q$$

$$\dot{\underline{\underline{e}}}^* = \underline{\underline{Q}} \cdot \dot{\underline{\underline{e}}} \cdot \underline{\underline{Q}}^T + \dot{\underline{\underline{Q}}} \cdot \underline{\underline{e}} \cdot \underline{\underline{Q}}^T + \underline{\underline{Q}} \cdot \underline{\underline{e}} \cdot \dot{\underline{\underline{Q}}}^T$$

$$\underset{\sim}{v}^* = \underline{\underline{Q}} \cdot \underset{\sim}{v} + \dot{\underset{\sim}{c}} + \underline{\underline{\Omega}}^Q \cdot (\underset{\sim}{r}^* - \underset{\sim}{c})$$

Under a rigid body rotation: (i) a tensor $\underline{\underline{M}}$ that satisfies $\underline{\underline{M}}^* = \underline{\underline{M}}$ is called a material tensor (examples are $\underline{\underline{\tau}}$, $\underline{\underline{E}}$, $\dot{\underline{\underline{E}}}$)
(ii) a tensor $\underline{\underline{s}}$ that satisfies $\underline{\underline{s}}^* = \underline{\underline{Q}} \cdot \underline{\underline{s}} \cdot \underline{\underline{Q}}^T$ is called a spatial tensor (examples are $\underline{\underline{\sigma}}$, $\underline{\underline{e}}$, $\underline{\underline{D}}$), and
(iii) a tensor $\underline{\underline{m}}$ that satisfies $\underline{\underline{m}}^* = \underline{\underline{Q}} \cdot \underline{\underline{m}}$ is called a mixed tensor (examples are $\hat{\underline{\underline{\tau}}}$, $\underline{\underline{F}}$)

The relations of Eq. (4.1-23) are said to be objective. The Principle of Material Frame Indifference states that constitutive relations should not yield a contradiction to any of these relations. Thus the principle is often referred to as the Principle of Objectivity. It follows that if a constitutive relation involves only material tensors, the principle of material frame indifference is automatically satisfied.

Section 4.2 Elasticity

A body formed of an ideally elastic material recovers its original form when the forces that caused the original deformation are completely removed, and there is a one-to-one relationship between the state of stress and the state of strain. Since thermal effects are small for a large class of problems, only purely mechnical theories will be considered.

Piola-Kirchhoff - Lagrangian Strain Relations

A possible elastic constitutive equation is one that relates the Lagrangian strain tensor and the Piola-Kirchhoff stress tensor of the second kind

$$\underset{\approx}{\tau} = \underset{\approx}{f}^{\tau}(\underset{\approx}{E}) \tag{4.2-1}$$

in which $\underset{\approx}{f}^{\tau}$ represents a tensor valued function of a tensor. The stress in State III (Sec. 4.1) introduced by an additional rigid body displacement is given by

$$\underset{\approx}{\tau}^* = \underset{\approx}{f}^{\tau}(\underset{\approx}{E}^*) \tag{4.2-2}$$

Since $\underset{\approx}{\tau}^* = \underset{\approx}{\tau}$ and $\underset{\approx}{E}^* = \underset{\approx}{E}$, this equation satisfies the principle of material frame indifference.

One special case of Eq. (4.2-1) is a power series expansion in $\underset{\approx}{E}$, which in component form becomes

$$\tau_{AB} = C_{ABCD}E_{CD} + \hat{C}_{ABCDEF}E_{CD}E_{EF} + \ldots \tag{4.2-3}$$

where $\underset{\approx\approx}{C}$, $\hat{\underset{\approx\approx\approx}{C}}$,... are called elasticity tensors. A subclass of this case is given by

$$\tau_{AB} = C_{ABCD}E_{CD} \tag{4.2-4}$$

where the deformation may or may not be infinitesimal, i.e., rigid body motion may occur. Since $\underset{\approx}{\tau}$ and $\underset{\approx}{E}$ are both symmetric, $\underset{\approx\approx}{C}$ must satisfy the restrictions

$$C_{ABCD} = C_{BACD} = C_{ABDC} \tag{4.2-5}$$

so that of the 81 components of $\underset{\approx\approx}{C}$, only 36 are actually independent.

If a strain energy function is postulated, then the material is said to be hyperelastic and for the subclass denoted by Eq. (4.2-4), the internal energy is

$$U = \tfrac{1}{2} C_{ABCD} E_{AB} E_{CD} \tag{4.2-6}$$

In order that U be a single-valued function of the strains, an additional condition is required, namely

$$C_{ABCD} = C_{CDAB} \tag{4.2-7}$$

Now, instead of 36 independent components, the linear stress-strain relation is characterized by 21 independent components for the most general anisotropic case.

Isotropic Elasticity

Suppose a coordinate transformation is applied to the elasticity tensor $\underset{\approx}{C}$. Then

$$\tilde{C}_{ijkl} = C_{ABCD} a_{Ai} a_{Bj} a_{Ck} a_{Dl} \tag{4.2-8}$$

where a_{Ai} represents the orthogonal transformation matrix (Sec. 1.8 and 1.10). The material is said to be isotropic if the components of $\underset{\approx}{C}$ are the same in any coordinate system, i.e.

$$\hat{C}_{ijkl} = C_{ijkl} \tag{4.2-9}$$

It can be shown that the most general fourth order isotropic tensor is

$$C_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + \nu(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) \tag{4.2-10}$$

where λ, μ and ν are independent constants.

However, since $C_{ijkl} = C_{jikl}$ for this application $\nu = 0$ and Eq. (4.2-10) reduces to

$$C_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \tag{4.2-11}$$

Hence, there are only two independent material constants. Note that in Eq. (4.2-11) $C_{ijkl} = C_{ijlk} = C_{klij}$, and hence an isotropic material is also hyperelastic. The substitution of Eq. (4.2-11) into (4.2-8) will show that the components of $\underset{\approx}{C}$ are the same in any coordinate system. (This constitutes the sufficient condition part of the proof of isotropy - the necessary condition has not been shown).

The substitution of Eq. (4.2-11) into (4.2-4) yields

$$\tau_{AB} = \lambda E_{DD}\delta_{AB} + 2\mu E_{AB} \tag{4.2-12}$$

in which λ and μ are called Lamè's elastic constants. It follows that

$$\tau_{AA} = (3\lambda + 2\mu)E_{AA} \tag{4.2-13}$$

and hence

$$E_{AB} = \frac{-\lambda\delta_{AB}}{2\mu(3\lambda+2\mu)}\tau_{DD} + \frac{1}{2\mu}\tau_{AB} \tag{4.2-14}$$

If the engineering constants, shear modulus G, Young's modulus E, and Poisson's ratio ν are introduced

$$\begin{aligned} \mu &= G = \frac{E}{2(1+\nu)} \\ \lambda &= \frac{\nu E}{(1+\nu)(1-2\nu)} \end{aligned} \tag{4.2-15}$$

Then

$$E_{AB} = \frac{(1+\nu)}{E} [\tau_{AB} - \frac{\nu}{1+\nu} \delta_{AB}\tau_{DD}] \tag{4.2-16}$$

or

$$\begin{aligned} E_{11} &= \frac{1}{E} [\tau_{11} - \nu(\tau_{22} + \tau_{33})] \\ &\vdots \\ E_{12} &= \frac{\tau_{12}}{2G} \end{aligned} \tag{4.2-17}$$

The inverse relations are:

$$\tau_{AB} = \frac{E}{(1+\nu)} [E_{AB} + \frac{\nu}{1-2\nu} \delta_{AB}E_{DD}] \tag{4.2-18}$$

or

$$\begin{aligned} \tau_{11} &= \frac{E}{(1+\nu)(1-2\nu)} [(1-\nu)E_{11} + \nu(E_{22} + E_{33})] \\ &\vdots \\ \tau_{12} &= 2G\, E_{12} \\ &\vdots \end{aligned} \tag{4.2-19}$$

The stress and strain deviators are defined as follows:

$$\begin{aligned} \underset{\approx}{\tau}^D &= \underset{\approx}{\tau} - \frac{1}{3} \underset{\approx}{I}\, tr(\underset{\approx}{\tau}) \\ \underset{\approx}{E}^D &= \underset{\approx}{E} - \frac{1}{3} \underset{\approx}{I}\, tr(\underset{\approx}{E}) \end{aligned} \tag{4.2-20}$$

Then the linear stress-strain relations become

$$\begin{aligned} \underset{\approx}{\tau}^D &= 2G\underset{\approx}{E}^D \\ P &= -Ke \end{aligned} \tag{4.2-21}$$

where

$$\begin{aligned} e &= tr(\underset{\approx}{E}) \\ p &= -\frac{1}{3} tr(\underset{\approx}{\tau}) \end{aligned} \tag{4.2-22}$$

where e is the volumetric strain, P is the mean normal pressure and K is the bulk modulus given by

$$K = \lambda + \frac{2}{3} G = \frac{E}{3(1-2\nu)} \tag{4.2-23}$$

The general linear stress-strain relation is theoretically correct even for large deformations. Its main area of applicability, however, is to metal structures that can buckle, in which case the strains can remain small but the rotations can be large.

Cauchy Stress - Eulerian Strain Relations

Now postulate an elastic constitutive equation relating the Cauchy stress tensor and the Eulerian strain tensor:

$$\underset{\approx}{\sigma} = \underset{\approx}{f}^{\sigma}(\underset{\approx}{e}) \tag{4.2-24}$$

If a rigid body motion is superimposed, then

$$\underset{\approx}{\sigma}^* = \underset{\approx}{f}^{\sigma}(\underset{\approx}{e}^*) \tag{4.2-25}$$

The use of Eqns. (4.1-8) and (4.1-12) yields

$$\underset{\approx}{Q}\cdot\underset{\approx}{\sigma}\cdot\underset{\approx}{Q}^T = \underset{\approx}{f}^{\sigma}(\underset{\approx}{Q}\cdot\underset{\approx}{e}\cdot\underset{\approx}{Q}^T) \tag{4.2-26}$$

for arbitrary $\underset{\approx}{Q}$. This relation is an application of the principle of material frame indifference. A function $\underset{\approx}{f}^{\sigma}$ that satisfies Eq. (4.2-26) is said to be an isotropic tensor function.

Now suppose the stress and strain are related linearly (which does not necessarily imply infinitesimal deformations). Then

$$\underset{\approx}{\sigma} = \underset{\approx}{C} : \underset{\approx}{e} \quad \text{or} \quad \sigma_{ij} = C_{ijkl}e_{kl} \tag{4.2-27}$$

The question is "What restrictions, if any, does objectivity place on 4.2-27?".

First note that for a rigid body rotation, the rigid body rotation can be represented by

$$\underset{\approx}{Q} = \underset{\sim}{e}_i^* \otimes \underset{\sim}{e}_i \qquad (4.2\text{-}28)$$

where $\underset{\sim}{e}_i^*$ is obtained from $\underset{\sim}{e}_i$ through the rigid body rotation. Postulate the constitutive relation

$$\underset{\approx}{\sigma} = \underset{\approx}{C} : \underset{\approx}{e}$$

which implies that

$$\underset{\approx}{\sigma}^* = \underset{\approx}{C} : \underset{\approx}{e}^* \qquad (4.2\text{-}29)$$

or

$$\underset{\approx}{Q} \cdot \underset{\approx}{\sigma} \cdot \underset{\approx}{Q}^T = \underset{\approx}{C} : (\underset{\approx}{Q} \cdot \underset{\approx}{e} \cdot \underset{\approx}{Q}^T)$$

$$\underset{\approx}{Q} \cdot (\underset{\approx}{C} : \underset{\approx}{e}) \cdot \underset{\approx}{Q}^T = \underset{\approx}{C} : (\underset{\approx}{Q} \cdot \underset{\approx}{e} \cdot \underset{\approx}{Q}^T)$$

Express $\underset{\approx}{C}$ in the $\underset{\sim}{e}_i$ system on the left and in the $\underset{\sim}{e}_i^*$ system on the right, i.e.

$$\underset{\approx}{C} = C_{ijkl}\, \underset{\sim}{e}_i \otimes \underset{\sim}{e}_j \otimes \underset{\sim}{e}_k \otimes \underset{\sim}{e}_l = C_{ijkl}\, \underset{\sim}{e}_i^* \otimes \underset{\sim}{e}_j^* \otimes \underset{\sim}{e}_k^* \otimes \underset{\sim}{e}_l^*$$

Also use

$$\underset{\approx}{e} = e_{lm}\, \underset{\sim}{e}_l \otimes \underset{\sim}{e}_m$$

Then

$$\underset{\approx}{Q} \cdot \underset{\approx}{e} \cdot \underset{\approx}{Q}^T = e_{lm}\, \underset{\sim}{e}_l^* \otimes \underset{\sim}{e}_m^*$$

The result is that

$$C_{jklm}\, e_{lm}\, \underset{\sim}{e}_j^* \otimes \underset{\sim}{e}_k^* = C_{jklm}\, e_{lm}\, \underset{\sim}{e}_j^* \otimes \underset{\sim}{e}_k^*$$

which must hold for all choices of e_{lm} and $\underset{\sim}{e}_i^*$

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The result is

$$C_{jk\ell m} = C^{*}_{jk\ell m} \qquad (4.2\text{-}30)$$

which implies that $\underset{\equiv}{C}$ is an isotropic tensor. This shows that the linear Cauchy stress - Eulerian strain relation with arbitrary deformations is valid only for isotropic materials (a similar statement holds for nonlinear relations). For non-isotropic materials and large deformations, an equation of the form (4.2-24) cannot be used.

For infinitesimal deformations, there is no need to differentiate between stress or strain tensors, large rotations are not allowed, and non-isotropic materials cause no problems. In other words, tensor relations automatically satisfy objectivity for infinitesimal deformations.

Section 4.3 Fluids

A fluid can be defined as a material which cannot sustain a shear stress when a body composed of such a material is at a state of rest (or equilibrium). For such a case, the Cauchy stress tensor must be of the form

$$\underset{\approx}{\sigma} = -p_s \underset{\approx}{I} \tag{4.3-1}$$

where p_s is called the static pressure. Such a condition is said to provide a hydrostatic state of stress. The static pressure is related to the density, ρ, and temperture, θ, by a kinetic equation of state

$$F(p_s, \rho, \theta) = 0 \tag{4.3-2}$$

If the fluid is not in a state of equilibrium, the thermodynamic pressure, p, is defined to be that pressure given by the same kinetic equation of state as for the static case, i.e.,

$$F\,(p, \rho, \theta) = 0 \tag{4.3-3}$$

For this situation there is the possibility that $\underset{\approx}{\sigma} \neq p\underset{\approx}{I}$ or even $tr(\underset{\approx}{\sigma}) \neq -3p$.

One equation of state is the perfect gas law

$$p = \rho R \theta \tag{4.3-4}$$

where R is the gas constant for the particular gas.

If θ is a constant, Eq. (4.3-3) describes isothermal flow. If the function F does not depend on θ at all, then Eq. (4.3-3) reduces to

$$f(p, \rho) = 0 \tag{4.3-5}$$

and such a fluid is said to be barotropic. Reversible adiabatic flow (isentropic flow with no heat transfer) is governed by the barotropic relation

$$\frac{p}{\rho^{\gamma}} = \text{constant} \tag{4.3-6}$$

where $\gamma = C_p/C_v = 1 + (R/C_v)$ and where C_p and C_v are the specific heat at constant pressure and constant volume, respectively.

An ideal incompressible fluid is given by

$$\rho = \text{constant} \quad \text{(for all flows)} \tag{4.3-7}$$

in which case p may be an indeterminable parameter, or determined by a boundary condition for the particular problem. There can be special problems for which ρ = constant even if the fluid is compressible.

An ideal nonviscous fluid is defined as a fluid which can sustain no shear stress even when the fluid is in motion. Then the Cauchy stress tesnor assumes the same form as the static case

$$\underset{\approx}{\sigma} = -p\underset{\approx}{I} \tag{4.3-8}$$

If the fluid is barotropic, Eq. (4.3-5) can be written $p = p(\rho)$ and the fluid is also called an elastic fluid.

Postulated constitutive equations for viscous fluids can assume several forms. One, attributed to Stokes, is

$$\left.\begin{aligned} \underset{\approx}{\sigma} &= -p\underset{\approx}{I} + \underset{\approx}{\sigma}_V \\ \underset{\approx}{\sigma}_V &= \underset{\approx}{F}(\underset{\approx}{D}) \end{aligned}\right\} \quad \text{(Stokesian Fluid)} \tag{4.3-9}$$

where $\underset{\approx}{\sigma}_V$ is called the viscous stress. The application of the principle of material frame indifference implies that $\underset{\approx}{F}$ is an isotropic tensor function of the rate of deformation tensor $\underset{\approx}{D}$.

If the function $\underset{\approx}{F}$ is linear in $\underset{\approx}{D}$, then the fluid is called Newtonian and Eq. (4.3-9) reduces to

$$\underset{\approx}{\sigma} = -p\underset{\approx}{I} + C_{35}C_{46}\ (\underset{\approx}{\underset{\approx}{C}} \otimes \underset{\approx}{D})$$
$$\sigma_{ij} = -p\delta_{ij} + C_{ijkl}D_{kl} \tag{4.3-10}$$

The requirement that $\underset{\approx}{\sigma}$ and $\underset{\approx}{D}$ satisfy the transformation relations for a rigid body superposition (principle of material frame indifference) of Sec. 4.1 leads to the same result as that for the linear Cauchy stress - Eulerian strain relation, namely, that the fluid must be isotropic. $\underset{\approx}{\underset{\approx}{C}}$ has at most two independent components and Eq. (4.3-10) becomes

$$\underset{\approx}{\sigma} = -p\underset{\approx}{I} + \lambda\ I_D\underset{\approx}{I} + 2\mu\underset{\approx}{D} \qquad \text{(Newtonian Fluid)}$$
$$\sigma_{ij} = -p\delta_{ij} + \lambda\ I_D\delta_{ij} + 2\mu D_{ij} \tag{4.3-11}$$
$$I_D = tr(\underset{\approx}{D}),\ \text{the first invariant of } \underset{\approx}{D}$$

The two parameters λ and μ characterize the viscosity of the fluid. Again, p, is the pressure obtained as a function of ρ (and perhaps Θ) under the assumption that the fluid is at rest.

Let $\bar{p}$ be the average pressure for the general case of fluid flow, i.e.

$$\bar{p} = -\frac{1}{3}\ I_\sigma = -\frac{1}{3}\ tr(\underset{\approx}{\sigma}) \tag{4.3-12}$$

Then the stress deviator is

$$\underset{\approx}{\sigma}^D = \underset{\approx}{\sigma} - \frac{1}{3}\ I_\sigma\underset{\approx}{I}$$
$$= \underset{\approx}{\sigma} + \bar{p}\underset{\approx}{I} \tag{4.3-13}$$

By taking the trace of each term in Eq. (4.3-11) we get

$$-3\bar{p} = -3p + (3\lambda + 2\mu)I_D$$

or

$$\bar{p} = p - K_v I_D$$
$$K_v = \frac{1}{3}\ (3\lambda + 2\mu) \tag{4.3-14}$$

in which K_v is called the bulk or volume viscosity. It follows that the mean pressure equals the static pressure if either K_v or I_D is zero. But from the equation of continuity (Eq. (3.1-2))

$$I_D = \mathrm{div}\ \underset{\sim}{v} = -\frac{1}{\rho}\frac{d\rho}{dt} \tag{4.3-15}$$

and I_D is zero for an incompressible fluid or a flow in which the density is constant. Thus

$$\bar{p} = p$$

$$\begin{aligned} &\text{if} \qquad \rho = \text{constant} \\ &\text{or} \qquad \lambda + \frac{2}{3}\mu = 0 \qquad\qquad \text{Stokes' Assumption} \end{aligned} \tag{4.3-16}$$

The rate-of-deformation deviator is

$$\underset{\approx}{D}^D = \underset{\approx}{D} - \frac{1}{3} I_D \underset{\approx}{I} \tag{4.3-17}$$

The substitution of Eqs. (4.3-13),(4.3-14), and (4.3-17) into Eq. (4.3-11) yields an alternate form of the constitutive equation for a Newtonian fluid:

$$\begin{aligned} \underset{\approx}{\sigma}^D &= 2\mu \underset{\approx}{D}^D \\ \bar{p} &= p - K_v I_D \end{aligned} \tag{4.3-18}$$

which, for either incompressible fluids or under Stokes' assumption, reduces to

$$\begin{aligned} \underset{\approx}{\sigma}^D &= 2\mu \underset{\approx}{D}^D \\ \bar{p} &= p \end{aligned}$$

A large number of problems are solved under Stokes' assumption in which case $\lambda = -\frac{2}{3}\mu$ and

$$\underset{\approx}{\sigma} = -p\underset{\approx}{I} + 2\mu\,(\underset{\approx}{D} - \frac{1}{3}\, I_D \underset{\approx}{I}) \qquad \text{(Newtonian Fluid with Stokes' Assumption)} \qquad (4.3\text{-}20)$$

$$= -p\underset{\approx}{I} + 2\mu \underset{\approx}{D}^{D}$$

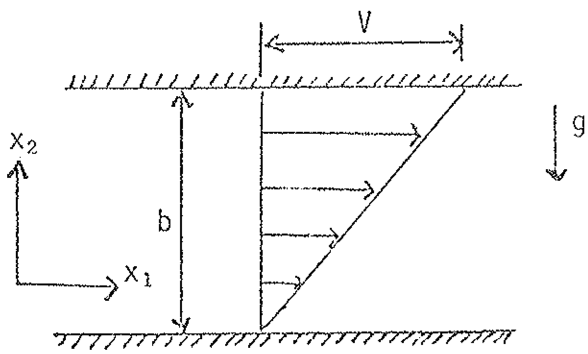
Hence

$$\bar{p} = p$$

even if the flow is not constant density or the fluid is incompressible.

Ex: Ref: Chia - Shun Yih, <u>Fluid Mechanics</u>, Mc Graw-Hill Book Co., N.Y.,N.Y., p.33.

Consider the following illustration



Velocity Distribution is the Fluid Between Two Parallel Plates the Upper of Which is Moving with Velocity V in its Own Plane.

Suppose the fluid is incompressible. From the figure

$$v_1 = \frac{V x_2}{b} \qquad v_2 = v_3 = 0$$

$$p = \text{constant} - \rho g x_2$$

The equation of continuity, boundary conditions and equations of motion are satisfied in this example. From Eq. (4.3-20), the shear stress is

$$\sigma_{12} = 2\mu D_{12} = 2\mu \cdot \tfrac{1}{2}\left(\frac{\partial v_1}{\partial x_2} + \frac{\partial v_2}{\partial x_1}\right)$$

or

$$\sigma_{12} = \frac{\mu V}{b}$$

The material parameter μ is called the viscosity of the fluid and relates the shear stress to the rate of shear deformation.

Note: The important point here is that, in general, a constitutive equation and an equation of state cannot be postulated independently, An equation of state and the first law yields a constitutive equation. The other approach is to ignore the first law and postulate a constitutive equation.

4.4 Derivatives of Spatial Tensors

Most constitutive models used in computer codes involve spatial tensors such as the Cauchy stress $\underset{\approx}{\sigma}$. Some constitutive relations require the stress rate $\dot{\underset{\approx}{\sigma}}$. Under the superposition of a rigid body rotation $\underset{\approx}{\sigma}$ changes to

$$\underset{\approx}{\sigma}^* = \underset{\approx}{Q} \cdot \underset{\approx}{\sigma} \cdot \underset{\approx}{Q}^T \qquad (4.4\text{-}1)$$

but the time derivative satisfies the more complex equation

$$\dot{\underset{\approx}{\sigma}}^* = \underset{\approx}{Q} \cdot \dot{\underset{\approx}{\sigma}} \cdot \underset{\approx}{Q}^T + \dot{\underset{\approx}{Q}} \cdot \underset{\approx}{\sigma} \cdot \underset{\approx}{Q}^T + \underset{\approx}{Q} \cdot \underset{\approx}{\sigma} \cdot \dot{\underset{\approx}{Q}}^T \qquad (4.4\text{-}2)$$

To avoid the dependence of a stress rate on the rate of rotation ($\dot{\underset{\approx}{Q}}$) several alternative rates have been proposed (a typical rate is denoted by $\overset{\nabla}{\underset{\approx}{\sigma}}$) such that

$$\overset{\nabla}{\underset{\approx}{\sigma}}{}^* = \underset{\approx}{Q} \cdot \overset{\nabla}{\underset{\approx}{\sigma}} \cdot \underset{\approx}{Q}^T \qquad (4.4\text{-}3)$$

i.e., $\overset{\nabla}{\underset{\approx}{\sigma}}$ satisfies the usual "objectivity" relation associated with spatial tensors.

It turns out that there are several definitions which satisfy (4.4-3) and the question becomes one of "Which is the most appropriate?". This question will not be considered. Instead we will show the derivation of some of these rates and illustrate some of their features.

For incremental constitutive models, the quantity $\overset{\nabla}{\underset{\approx}{\sigma}}{}^* \Delta t$ should be used in the constitutive equation whereas $\dot{\underset{\approx}{\sigma}} \Delta t$ should be used as the increment for the equation of motion.

Truesdell Rate

Since $\underline{\underline{\tau}}$ is a material tensor, $\underline{\underline{\tau}}^* = \underline{\underline{\tau}}$ and $\dot{\underline{\underline{\tau}}}^* = \dot{\underline{\underline{\tau}}}$. Recall that

$$\underline{\underline{\sigma}} = \frac{1}{J} \underline{\underline{F}} \cdot \underline{\underline{\tau}} \cdot \underline{\underline{F}}^T$$

Therefore

$$\dot{\underline{\underline{\sigma}}} = -\frac{\dot{J}}{J^2} \underline{\underline{F}} \cdot \underline{\underline{\tau}} \cdot \underline{\underline{F}}^T + \frac{1}{J} \dot{\underline{\underline{F}}} \cdot \underline{\underline{\tau}} \cdot \underline{\underline{F}}^T + \frac{1}{J} \underline{\underline{F}} \cdot \dot{\underline{\underline{\tau}}} \cdot \underline{\underline{F}}^T + \frac{1}{J} \underline{\underline{F}} \cdot \underline{\underline{\tau}} \cdot \dot{\underline{\underline{F}}}^T \quad (4.4\text{-}4)$$

Since $\dot{J} = J \operatorname{tr} \underline{\underline{L}} = J \operatorname{tr} \underline{\underline{D}}$ $\quad \dot{\underline{\underline{F}}} = \underline{\underline{L}} \cdot \underline{\underline{F}}$ $\quad \dot{\underline{\underline{F}}}^T = \underline{\underline{F}}^T \cdot \underline{\underline{L}}^T$

it follows that

$$\dot{\underline{\underline{\sigma}}} = -(\operatorname{tr} \underline{\underline{D}}) \underline{\underline{\sigma}} + \underline{\underline{L}} \cdot \underline{\underline{\sigma}} + \frac{1}{J} \underline{\underline{F}} \cdot \dot{\underline{\underline{\tau}}} \cdot \underline{\underline{F}}^T + \underline{\underline{\sigma}} \cdot \underline{\underline{L}}^T \quad (4.4\text{-}5)$$

The Truesdell rate is defined to be (Eringen, Pg. 112)

$$\begin{aligned} \underline{\underline{\sigma}}^{\nabla\textcircled{1}} &= \frac{1}{J} \underline{\underline{F}} \cdot \dot{\underline{\underline{\tau}}} \cdot \underline{\underline{F}}^T \qquad (4.4\text{-}6) \\ &= \dot{\underline{\underline{\sigma}}} + (\operatorname{tr} \underline{\underline{D}}) \underline{\underline{\sigma}} - \underline{\underline{L}} \cdot \underline{\underline{\sigma}} - \underline{\underline{\sigma}} \cdot \underline{\underline{L}}^T \\ &= \dot{\underline{\underline{\sigma}}} + (\underline{v} \cdot \nabla) \underline{\underline{\sigma}} - \underline{\underline{D}} \cdot \underline{\underline{\sigma}} - \underline{\underline{\sigma}} \cdot \underline{\underline{D}} - \underline{\underline{W}} \cdot \underline{\underline{\sigma}} + \underline{\underline{\sigma}} \cdot \underline{\underline{W}} \end{aligned}$$

where

$$\underline{\underline{L}} = \underline{\underline{D}} + \underline{\underline{W}} \quad (4.4\text{-}7)$$

To show that $\underline{\underline{\sigma}}^{\nabla\textcircled{1}}$ satisfies (4.4-3) consider

$$\begin{aligned} \underline{\underline{\sigma}}^{\nabla\textcircled{1}*} &= \frac{1}{J^*} \underline{\underline{F}}^* \cdot \dot{\underline{\underline{\tau}}}^* \cdot \underline{\underline{F}}^{*T} \\ &= \frac{1}{J} \underline{\underline{Q}} \cdot \underline{\underline{F}} \cdot \dot{\underline{\underline{\tau}}} \cdot \underline{\underline{F}}^T \cdot \underline{\underline{Q}}^T \qquad (4.4\text{-}8) \\ &= \frac{1}{J} \underline{\underline{Q}} \cdot \underline{\underline{\sigma}}^{\nabla\textcircled{1}} \cdot \underline{\underline{Q}}^T \end{aligned}$$

since $J = J^*$ (volume does not change under rigid body motion).

Zaremba - Jaumann - Noll (ZJN) Rate

Consider the set of terms (from 4.4-6)

$$\underset{=}{S} = (\underset{\sim}{v} \cdot \nabla)\, \underset{=}{\tau} - \underset{=}{D} \cdot \underset{=}{\tau} - \underset{=}{\tau} \cdot \underset{=}{D} \qquad (4.4\text{-}9)$$

Superimpose a rigid body rotation and note that

$$\underset{=}{D}^{*} = \underset{=}{Q} \cdot \underset{=}{D} \cdot \underset{=}{Q}^{T}$$

$$\operatorname{tr} \underset{=}{D}^{*} = \operatorname{tr} \underset{=}{D} \qquad (4.4\text{-}10)$$

$$\underset{=}{D}^{*} \cdot \underset{=}{\tau}^{*} = \underset{=}{Q} \cdot \underset{=}{D} \cdot \underset{=}{Q}^{T} \cdot \underset{=}{Q} \cdot \underset{=}{\tau} \cdot \underset{=}{Q}^{T} = \underset{=}{Q} \cdot \underset{=}{D} \cdot \underset{=}{\tau} \cdot \underset{=}{Q}^{T}$$

$$\underset{=}{\tau}^{*} \cdot \underset{=}{D}^{*} = \underset{=}{Q} \cdot \underset{=}{\tau} \cdot \underset{=}{D} \cdot \underset{=}{Q}^{T}$$

It follows that $\underset{=}{S}$ satisfies the objectivity relation

$$\underset{=}{S}^{*} = \underset{=}{Q} \cdot \underset{=}{S} \cdot \underset{=}{Q}^{T} \qquad (4.4\text{-}11)$$

Since the sum of terms in the Truesdell rate satisfies objectivity as does the partial set $\underset{=}{S}$, the remaining terms must also satisfy objectivity. These remaining terms are labelled the ZJN rate:

$$\underset{=}{\tau}^{\nabla(2)} = \dot{\underset{=}{\tau}} - \underset{=}{W} \cdot \underset{=}{\tau} + \underset{=}{\tau} \cdot \underset{=}{W} \qquad (4.4\text{-}12)$$

The Rate of Dienes and Dafalias

Many computer codes use the conjugate variables consisting of the rotated Cauchy stress and the material rate of deformation tensor $\underline{\underline{D}}^M$

$$\underline{\underline{\Sigma}} = \underline{\underline{R}}^T \cdot \underline{\underline{\sigma}} \cdot \underline{\underline{R}} = \frac{1}{J} \underline{\underline{U}} \cdot \underline{\underline{\tau}} \cdot \underline{\underline{U}} \qquad (4.4\text{-}13)$$

$$\underline{\underline{D}}^M = \underline{\underline{R}}^T \cdot \underline{\underline{D}} \cdot \underline{\underline{R}} = \frac{d}{dt} \ln \underline{\underline{U}} = \frac{1}{2}\left(\dot{\underline{\underline{U}}} \cdot \underline{\underline{U}}^{-1} + \underline{\underline{U}}^{-1} \cdot \dot{\underline{\underline{U}}}\right)$$

Then

$$\underline{\underline{\Sigma}}^* = \underline{\underline{\Sigma}} \qquad \underline{\underline{D}}^{M*} = \underline{\underline{D}}^M \qquad (4.4\text{-}14)$$

$$\dot{\underline{\underline{\Sigma}}}^* = \dot{\underline{\underline{\Sigma}}} \qquad \dot{\underline{\underline{D}}}^{M*} = \dot{\underline{\underline{D}}}^M$$

and

$$\dot{\underline{\underline{\sigma}}} = \underline{\underline{R}} \cdot \dot{\underline{\underline{\Sigma}}} \cdot \underline{\underline{R}}^T + \dot{\underline{\underline{R}}} \cdot \underline{\underline{\Sigma}} \cdot \underline{\underline{R}}^T + \underline{\underline{R}} \cdot \underline{\underline{\Sigma}} \cdot \dot{\underline{\underline{R}}}^T \qquad (4.4\text{-}15)$$

Define a third rate as

$$\overset{\nabla(3)}{\underline{\underline{\sigma}}} = \underline{\underline{R}} \cdot \dot{\underline{\underline{\Sigma}}} \cdot \underline{\underline{R}}^T \qquad (4.4\text{-}16)$$

and let

$$\underline{\underline{\Omega}} = \dot{\underline{\underline{R}}} \cdot \underline{\underline{R}}^T \qquad (4.4\text{-}17)$$

which implies

$$\dot{\underline{\underline{R}}} = \underline{\underline{\Omega}} \cdot \underline{\underline{R}} \qquad \dot{\underline{\underline{R}}}^T = \underline{\underline{R}}^T \cdot \underline{\underline{\Omega}}^T = -\underline{\underline{R}}^T \cdot \underline{\underline{\Omega}} \qquad (4.4\text{-}18)$$

and

$$\overset{\nabla(3)}{\underline{\underline{\sigma}}} = \dot{\underline{\underline{\sigma}}} - \underline{\underline{\Omega}} \cdot \underline{\underline{\sigma}} + \underline{\underline{\sigma}} \cdot \underline{\underline{\Omega}} \qquad (4.4\text{-}19)$$

To show objectivity:

$$\begin{aligned}\overset{\nabla(3)*}{\underline{\underline{\sigma}}} &= \underline{\underline{R}}^* \cdot \dot{\underline{\underline{\Sigma}}}^* \cdot \underline{\underline{R}}^{T*} \\ &= \underline{\underline{Q}} \cdot \underline{\underline{R}} \cdot \dot{\underline{\underline{\Sigma}}} \cdot \underline{\underline{R}}^T \cdot \underline{\underline{Q}}^T \\ &= \underline{\underline{Q}} \cdot \overset{\nabla(3)}{\underline{\underline{\sigma}}} \cdot \underline{\underline{Q}}^T\end{aligned} \qquad (4.4\text{-}20)$$

since $\underline{\underline{R}}^* = \underline{\underline{Q}} \cdot \underline{\underline{R}}$ $\qquad (4.4\text{-}21)$

The Use of Material Tensors

Recall that

$$\underset{=}{F} = \underset{=}{R}\,\underset{=}{U} \qquad \dot{\underset{=}{F}} = \dot{\underset{=}{R}}\cdot\underset{=}{U} + \underset{=}{R}\cdot\dot{\underset{=}{U}} = \underset{=}{L}\cdot\underset{=}{F}$$

$$\underset{=}{L} = \dot{\underset{=}{F}}\cdot\underset{=}{F}^{-1} = \dot{\underset{=}{R}}\cdot\underset{=}{R}^T + \underset{=}{R}\cdot\dot{\underset{=}{U}}\cdot\underset{=}{U}^{-1}\cdot\underset{=}{R}^T$$

$$\underset{=}{L}^T = \underset{=}{R}\cdot\dot{\underset{=}{R}}^T + \underset{=}{R}\cdot\underset{=}{U}^{-1}\cdot\dot{\underset{=}{U}}\cdot\underset{=}{R}^T$$

$$\underset{=}{D}^M = \frac{1}{2}(\dot{\underset{=}{U}}\cdot\underset{=}{U}^{-1} + \underset{=}{U}^{-1}\cdot\dot{\underset{=}{U}}) \qquad (4.4\text{-}22)$$

$$\underset{\approx}{\Omega} = \dot{\underset{=}{R}}\cdot\underset{=}{R}^T$$

Then

$$\underset{=}{D} = \frac{1}{2}(\underset{=}{L} + \underset{=}{L}^T) = \underset{=}{R}\cdot\underset{=}{D}^M\cdot\underset{=}{R}^T \qquad (4.4\text{-}23)$$

since $\dot{\underset{=}{R}}\cdot\underset{=}{R}^T + \underset{=}{R}\cdot\dot{\underset{=}{R}}^T = \underset{=}{0}$

The skew-symmetric part is

$$\underset{=}{W} = \frac{1}{2}(\underset{=}{L} - \underset{=}{L}^T) = \frac{1}{2}(\dot{\underset{=}{R}}\cdot\underset{=}{R}^T - \underset{=}{R}\cdot\dot{\underset{=}{R}}^T) + \frac{1}{2}\underset{=}{R}\cdot(\dot{\underset{=}{U}}\cdot\underset{=}{U}^{-1} - \underset{=}{U}^{-1}\cdot\dot{\underset{=}{U}})\cdot\underset{=}{R}^T$$

Let

$$\underset{=}{M} = \frac{1}{2}(\dot{\underset{=}{U}}\cdot\underset{=}{U}^{-1} - \underset{=}{U}^{-1}\cdot\dot{\underset{=}{U}}) \qquad (4.4\text{-}24)$$

Then

$$\underset{=}{W} = \underset{\approx}{\Omega} + \underset{=}{R}\cdot\underset{=}{M}\cdot\underset{=}{R}^T = \underset{=}{\Omega} + \underset{=}{m} \qquad (4.4\text{-}25)$$

where

$$\underset{=}{m} = \underset{=}{R}\cdot\underset{=}{M}\cdot\underset{=}{R}^T \qquad (4.4\text{-}26)$$

The important point to emphasize is that $\underset{=}{W}$ and $\underset{=}{\Omega}$ are not equal for general deformations. This point was not made clear in some early papers on the subject.

Define material tensors:

$$\underline{\underline{\Omega}}^M = \underline{\underline{R}}^T \cdot \underline{\underline{\Omega}} \cdot \underline{\underline{R}} = \underline{\underline{R}}^T \cdot \dot{\underline{\underline{R}}} \tag{4.4-27}$$

$$\underline{\underline{W}}^M = \underline{\underline{R}}^T \cdot \underline{\underline{W}} \cdot \underline{\underline{R}}$$

Then

$$\underline{\underline{W}}^M = \underline{\underline{\Omega}}^M + \underline{\underline{M}} \tag{4.4-28}$$

Also, from (4.4-27)

$$\dot{\underline{\underline{R}}} = \underline{\underline{R}} \cdot \underline{\underline{\Omega}}^M \qquad \dot{\underline{\underline{R}}}^T = \underline{\underline{\Omega}}^{M^T} \cdot \underline{\underline{R}}^T = -\underline{\underline{\Omega}}^M \cdot \underline{\underline{R}}^T \tag{4.4-29}$$

Then the use of (4.4-15) and (4.4-16) yields

$$\overset{\nabla(3)}{\underline{\underline{\sigma}}} = \dot{\underline{\underline{\sigma}}} - \underline{\underline{R}} \cdot \left(\underline{\underline{\Omega}}^M \cdot \underline{\underline{\Sigma}} - \underline{\underline{\Sigma}} \cdot \underline{\underline{\Omega}}^M \right) \cdot \underline{\underline{R}}^T \tag{4.4-30}$$

as an example of an alternative formulation for this particular rate.

Derivatives of Invariants

Many other rates that satisfy objectivity can be developed (e.g., Cotter & Rivlin, Oldroyd). Prager pointed out that of these rates only the ZJN rate provided the result that the rate of the second invariant of the Cauchy stress equaled the time derivative of the invariant. This relationship is particularly convenient for plasticity theory.

Consider the second invariant and use (4.4-12). Then

$$\mathrm{tr}\,\overset{\nabla\textcircled{2}}{(\underline{\underline{\sigma}}\cdot\underline{\underline{\sigma}})} = 2\,\mathrm{tr}(\overset{\nabla\textcircled{2}}{\underline{\underline{\sigma}}}\cdot\underline{\underline{\sigma}})$$

$$= 2\,\mathrm{tr}\left[(\dot{\underline{\underline{\sigma}}} - \underline{\underline{w}}\cdot\underline{\underline{\sigma}} + \underline{\underline{\sigma}}\cdot\underline{\underline{w}})\cdot\underline{\underline{\sigma}}\right] = 2\,\mathrm{tr}(\dot{\underline{\underline{\sigma}}}\cdot\underline{\underline{\sigma}})$$

$$= \frac{d}{dt}\,\mathrm{tr}(\underline{\underline{\sigma}}\cdot\underline{\underline{\sigma}})$$

The Truesdell rate does not satisfy this criterion but it can be shown that the Dienes-Dafalias rate (after Prager's paper) does.

References:

1. Dienes, J.K., "On the Analysis of Rotation and Stress Rate in Deforming Bodies," Acta Mechanica, Vol. 32, 1979, pp. 217-232.

2. Nagtegaal, J.C., and de Jong, J.E., "Some Aspects of Non-Isotropic Workhardening in Finite Strain Plasticity," Proc. of the Workshop on "Plasticity of Metals at Finite Strains: Theory, Experiment and Computation, pp. 65-102, Stanford Univ., 1981.

3. Dafalias, Y.F., "Corotational Rates for Kinematic Hardening at Large Plastic Deformations", J of App. Mech., Vol. 50, Sept., 1983, pp. 561-565.

4. Lee, E.H., Mallett, R.L. and Wertheimer, T.B., "Stress Analysis for Anisotropic Hardening in Finite-Deformation Plasticity," JAM, Vol. 50, Sept, 1983, pp. 554-560.

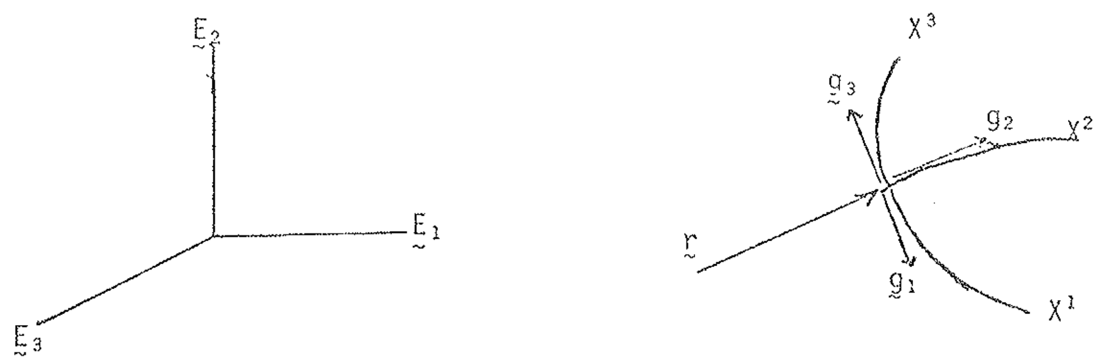
Chapter 5 Curvilinear Coordinates in a Euclidean Space

Section 1 Dual Bases, Contravariant, Covariant, and Mixed Components of Vectors

With curvilinear coordinates, the indicial notation assumes a slightly different form from that used previously. Summation can only occur if one dummy indice is at the top level and the other dummy indice is at the lower. A free indice in an equation must be at the same level.

Ex:
$$A^i = b^{i\cdot}{}_j V^j = C^{ijk} T_{jk} \tag{5.1-1}$$

In a Euclidian space a rectangular cartesian coordinate system that spans the space can always be constructed. We restrict ourselves to such a space and denote the rectangular cartesian coordinates by Z_A with base vectors $\underset{\sim}{E}_A$.



Denote a general curvilinear coordinate system by x^i (each point in the space is associated with a unique triplet of coordinate values). The base vectors associated with such a coordinate system are defined by

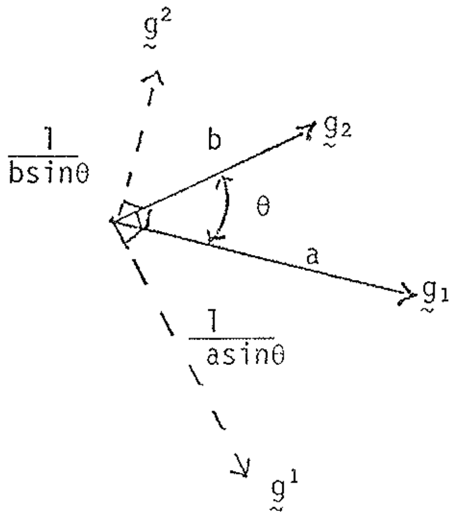
$$\underset{\sim}{g}_i = \frac{\partial \underset{\sim}{r}}{\partial x^i} \tag{5.1-2}$$

where $\underset{\sim}{r}$ is the position vector $\underset{\sim}{g}_i$ are tangent to the coordinate curves but not necessarily of unit length. The set $\underset{\sim}{g}_i$ is called the covariant basis. A contravariant basis $\underset{\sim}{g}^i$ is defined such that

$$\underset{\sim}{g}^i \cdot \underset{\sim}{g}_j = \delta^i_j \tag{5.1-3}$$

where δ^i_j is the Kronecker delta.

Example in 2-Space:



Let $|\underset{\sim}{g}_1| = a$

$|\underset{\sim}{g}_2| = b$

Then $\underset{\sim}{g}_1 \cdot \underset{\sim}{g}_2 = ab \cos\theta$

From (5.1-3) we have

$$\underset{\sim}{g}^1 \cdot \underset{\sim}{g}_1 = 1 = |\underset{\sim}{g}^1||\underset{\sim}{g}_1| \cos(90^0 - \theta)$$

Hence

$$|\underset{\sim}{g}^1| = \frac{1}{|\underset{\sim}{g}_{\cdot}|\sin\theta} = \frac{1}{a\sin\theta}$$

Similarly

$$|\underset{\sim}{g}^2| = \frac{1}{b\sin\theta}$$

The two bases taken together are called a dual bases. Either basis can be used and frequently it is convenient to use both. For example, a vector can be expressed in the forms

$$\underset{\sim}{v} = v^i \underset{\sim}{g}_i = v_i \underset{\sim}{g}^i \qquad (5.1\text{-}4)$$

where v^i are the contravariant components of the vector and v_i are the covariant components of the vector with respect to the two sets of base vectors. In general, these two sets of components are different.

For a second order tensor, more combinations are possible:

$$\underset{\approx}{T} = T^{ij}\underset{\sim}{g}_i \otimes \underset{\sim}{g}_j = T_{ij}\underset{\sim}{g}^i \otimes \underset{\sim}{g}^j$$
$$= T^{i}_{\cdot j}\underset{\sim}{g}_i \otimes \underset{\sim}{g}^j = T_{i}^{\cdot j}\underset{\sim}{g}^i \otimes \underset{\sim}{g}_j \tag{5.1-5}$$

where T^{ij} are the contravariant components

T_{ij} are the covariant components

and $T^{i}_{\cdot j}$, $T_{i}^{\cdot j}$ are called the mixed components.

Again each set of components may be different.

Section 2 Coordinate Transformations

Let

$$g_{ij} = \underset{\sim}{g}_i \cdot \underset{\sim}{g}_j$$

$$g^{ij} = \underset{\sim}{g}^i \cdot \underset{\sim}{g}^j \tag{5.2-1}$$

Since $\underset{\sim}{g}_i$ are linearly independent vectors, so are $\underset{\sim}{g}^i$. Thus let

$$\underset{\sim}{g}_i = A_{ik}\underset{\sim}{g}^k$$

$$\underset{\sim}{g}_i \cdot \underset{\sim}{g}_j = A_{ik}\delta^k_j = A_{ij}$$

Hence

$$\underset{\sim}{g}_i = g_{ij}\underset{\sim}{g}^j \tag{5.2-2}$$

and

$$\underset{\sim}{g}^i = g^{ij}\underset{\sim}{g}_i$$

Thus

$$\underset{\sim}{g}^i = g^{ij}g_{ij}\underset{\sim}{g}^i$$

so

$$g^{ij} = (g_{ij})^{-1}$$

i.e.

$$g^{ij}g_{jk} = \delta^i_k \tag{5.2-3}$$

Now consider

$$\underset{\sim}{g}_i \cdot \underset{\sim}{g}^k = g_{ij}\underset{\sim}{g}^j \cdot \underset{\sim}{g}^k$$

$$= g_{ij}g^{jk}$$

$$= \delta^k_i$$

Suppose a coordinate transformation exists between Z_A and x_i, i.e.,

$$Z_A = Z_A(x_i) \tag{5.2-4}$$

represents a one-to-one transformation, is continuous and has continuous first derivatives. Also

$$\left| \frac{\partial Z_A}{\partial x_i} \right| \neq 0,\infty \tag{5.2-5}$$

To obtain a relation between the dual base vectors and $\underset{\sim}{Z}_A$, consider

$$\underset{\sim}{g}_i = \frac{\partial r}{\partial x^i} = \frac{\partial (Z_A \underset{\sim}{E}_A)}{\partial x^i}$$

Thus

$$\underset{\sim}{g}_i = \frac{\partial Z_A}{\partial x^i} \underset{\sim}{E}_A + \frac{\partial \underset{\sim}{E}_A}{\partial x^i} Z_A$$

but the cartesian coordinate base vectors do not change from point to point so

$$\frac{\partial \underset{\sim}{E}_A}{\partial x^i} = 0$$

Hence

$$\underset{\sim}{g}_i = \frac{\partial Z_A}{\partial x^i} \underset{\sim}{E}_A$$

$$g_{ij} = \frac{\partial Z_A}{\partial x^i} \frac{\partial Z_A}{\partial x^j} \tag{5.2-6}$$

Since Eq. (5.2-5) holds, $\frac{\partial x^i}{\partial Z_A}$ exists and the use of $\frac{\partial x^i}{\partial Z_B} \frac{\partial Z_A}{\partial x^i} = \delta_{AB}$ yields

$$\frac{\partial x^i}{\partial Z_B} \frac{\partial Z_A}{\partial x^i} \underset{\sim}{E}_A = \frac{\partial x^i}{\partial Z_B} \underset{\sim}{g}_i$$

$$\delta_{AB} \underset{\sim}{E}_A = \frac{\partial x^i}{\partial Z_B} \underset{\sim}{g}_i$$

$$\underset{\sim}{E}_A = \frac{\partial x^i}{\partial Z_A} \underset{\sim}{g}_i \tag{5.2-7}$$

To obtain an expression for g^{ij} consider

$$\underset{\sim}{E}_A \cdot \underset{\sim}{E}_B = \left(\frac{\partial x^i}{\partial Z^A} \underset{\sim}{g}_i \right) \cdot \left(\frac{\partial x^j}{\partial Z^B} \underset{\sim}{g}_j \right) = \frac{\partial x^i}{\partial Z^A} \frac{\partial x^j}{\partial Z^B} g_{ij} = \delta_{AB} \tag{5.2-8}$$

Multiplying both sides of the above equation by $\frac{\partial Z_A}{\partial x^k}$ gives

$$\frac{\partial x^i}{\partial Z^A}\frac{\partial Z^A}{\partial x^k}\frac{\partial x^j}{\partial Z^B}g_{ij} = \frac{\partial Z_B}{\partial x^k}$$

Note that $\frac{\partial x_i}{\partial Z_A}\ \frac{\partial Z^A}{\partial x^k} = \frac{\partial x^i}{\partial x^k} = \delta_{ik}$

Hence

$$\delta_{ik}\frac{\partial\ ^j}{\partial Z^B}g_{ij} = \frac{\partial Z_B}{\partial x^k}$$

After multiplying both sides of the above equation by $\frac{\partial x^P}{\partial Z_B}$ we obtain

$$\frac{\partial x^P}{\partial Z_B}\frac{\partial x^j}{\partial Z_B}g_{jk} = \frac{\partial Z_B}{\partial x^k}\frac{\partial x^P}{\partial Z_B} = \frac{\partial x^P}{\partial x^k} = \delta^P_k$$

Since g_{jk} is arbitrary, the use of Eqn. (5.2-3) yields

$$g^{ij} = \frac{\partial x^i}{\partial Z_A}\frac{\partial x^j}{\partial Z_A} \tag{5.2-9}$$

The metric is defined to be

$$\begin{aligned} ds^2 &= d\underset{\sim}{r}\cdot d\underset{\sim}{r} \\ &= dZ_A dZ_A \qquad (5.2\text{-}10) \\ &= g_{ij}dx^i dx^j \end{aligned}$$

Note: If the space is not Euclidean, then the preceding formulas involving Z_A do not exist. Then the metric, and hence g_{ij}, must be known "a priori" - such a space is called Riemannian.

Consider the identity tensor

$$\underset{\approx}{I} = \underset{\sim}{E}_A \otimes \underset{\sim}{E}_A = \frac{\partial x^i}{\partial Z_A} \underset{\sim}{g}_i \otimes \frac{\partial x^j}{\partial Z_A} \underset{\sim}{g}_j$$

$$I = g^{ij} \underset{\sim}{g}_i \otimes \underset{\sim}{g}_j = \underset{\sim}{g}^j \otimes \underset{\sim}{g}_j = g_{ij}\, \underset{\sim}{g}^i \otimes \underset{\sim}{g}^j \tag{5.2-11}$$

$$= \delta^i_j \underset{\sim}{g}^j \otimes \underset{\sim}{g}_i$$

Since the covariant components of the identity tensor are the same as the components in the expression for the metric, $\underset{\approx}{I}$ is sometimes called the metric tensor.

Relationships Between Components

$$\underset{\sim}{v} = v^i \underset{\sim}{g}_i = v^i g_{ij} \underset{\sim}{g}^j = v_j \underset{\sim}{g}^j$$

Hence

$$v^i g_{ij} = v_j \tag{5.2-12}$$

$$v^i = g^{ij} v_j$$

Similarly

$$T^{ij} = g^{ik} g^{j\ell} T_{k\ell}$$

$$T_{ij} = g_{ik} g_{j\ell} T^{k\ell}$$

$$T_i^{\cdot j} = g_{ik} \delta^j_\ell T^{k\ell} \tag{5.2-13}$$

$$T_i^{\cdot j} = g_{ik} T^{kj}$$

$$T^i_{\cdot j} = g^{ik} T_{kj}$$

Contractions

$$\underset{\sim}{u}\cdot\underset{\sim}{v} = u^i v_i = u_i v^i \tag{5.2-14}$$

$$tr\underset{\approx}{T} = T^i{}_{\cdot i} = T_i{}^{\cdot i} \tag{5.2-15}$$

Tensor Operations

$$\begin{aligned}\underset{\approx}{T}\cdot\underset{\sim}{v} &= T^{ij} v_j \underset{\sim}{g}_i \\ &= T_{ij} v^j \underset{\sim}{g}^i\end{aligned} \tag{5.2-16}$$

$$\begin{aligned}\underset{\sim}{u} \otimes \underset{\approx}{T}\cdot\underset{\sim}{v} &= u^k T_{ij} v^j \underset{\sim}{g}_k \otimes \underset{\sim}{g}^i \\ &= u_k T^{ij} v_j \underset{\sim}{g}^k \otimes \underset{\sim}{g}_i\end{aligned} \tag{5.2-17}$$

Coordinate Transformations

Let y^α denote another curvilinear coordinate system with base vectors

$$\hat{\underset{\sim}{g}}_\alpha = \frac{\partial \underset{\sim}{r}}{\partial y^\alpha}$$

$$\hat{\underset{\sim}{g}}^\alpha \cdot \hat{\underset{\sim}{g}}_\beta = \delta^\alpha_\beta \tag{5.2-18}$$

If there is an allowable coordinate transformation, i.e., $y^\alpha = y^\alpha(x^i)$ is one-to-one, continuous, has continuous first derivative and $\left|\frac{\partial y^\alpha}{\partial x^i}\right| \neq 0, \infty$, then

$$\hat{\underset{\sim}{g}}_\alpha = \frac{\partial \underset{\sim}{r}}{\partial y^\alpha} = \frac{\partial \underset{\sim}{r}}{\partial x^i}\frac{\partial x^i}{\partial y^\alpha}$$

or

$$\begin{aligned}\hat{\underset{\sim}{g}}_\alpha &= \underset{\sim}{g}_i \frac{\partial x^i}{\partial y^\alpha} \\ \underset{\sim}{g}_i &= \frac{\partial y^\alpha}{\partial x^i}\hat{\underset{\sim}{g}}_\alpha\end{aligned} \tag{5.2-19}$$

Then

$$\hat{g}_{\alpha\beta} = \underset{\sim}{g}_i \frac{\partial x^i}{\partial y^\alpha}\cdot\underset{\sim}{g}_j \frac{\partial x^j}{\partial y^\beta}$$

$$= g_{ij} \frac{\partial x^i}{\partial y^\alpha} \frac{\partial x^j}{\partial y^\beta} \tag{5.2-20}$$

and

$$g_{ij} = \hat{g}_{\alpha\beta} \frac{\partial y^\alpha}{\partial x^i} \frac{\partial y^\beta}{\partial x^j}$$

represent the transformation relations for the metric component.

Consider

$$\underset{\sim}{g}_i \cdot \hat{\underset{\sim}{g}}^\beta = \frac{\partial y^\alpha}{\partial x^i} \hat{\underset{\sim}{g}}_\alpha \cdot \hat{\underset{\sim}{g}}^\beta = \frac{\partial y^\alpha}{\partial x^i} \delta^\beta_\alpha$$

$$= \frac{\partial y^\beta}{\partial x^i}$$

$$= \frac{\partial y^\beta}{\partial x^j} \underset{\sim}{g}^j \cdot \underset{\sim}{g}_i$$

which holds for arbitrary $\underset{\sim}{g}_i$. Thus

$$\hat{\underset{\sim}{g}}^\beta = \frac{\partial y^\beta}{\partial x^i} \underset{\sim}{g}^i$$

$$\underset{\sim}{g}^i = \frac{\partial x^i}{\partial y^\beta} \hat{\underset{\sim}{g}}^\beta \tag{5.2-21}$$

and

$$g^{ij} = \frac{\partial x^i}{\partial y^\beta} \hat{\underset{\sim}{g}}^\beta \cdot \frac{\partial x^j}{\partial y^\alpha} \hat{\underset{\sim}{g}}^\alpha$$

$$= \frac{\partial x^i}{\partial y^\beta} \frac{\partial x^j}{\partial y^\alpha} \hat{g}^{\alpha\beta} \tag{5.2-22}$$

and

$$\hat{g}^{\alpha\beta} = \frac{\partial y^\alpha}{\partial x^i} \frac{\partial y^\beta}{\partial x^j} g^{ij}$$

Let

$$g = |g_{ij}| \tag{5.2-23}$$

Then from Eq. (5.2-20)

$$g = \hat{g}\left|\frac{\partial y^\alpha}{\partial x^i}\right|^2$$

$$\hat{g} = g\left|\frac{\partial x^i}{\partial y^\alpha}\right|^2 \qquad (5.2\text{-}24)$$

Transformation of Components

Recall from Eqs. (5.1-4) and (5.1-5) the various ways in which a vector and a second order tensor can be expressed

$$\underset{\sim}{v} = v_i \underset{\sim}{g}^i = v^i \underset{\sim}{g}_i = \hat{v}_\alpha \hat{g}^\alpha = \hat{v}^\alpha \hat{g}_\alpha$$

$$(5.2\text{-}25)$$

$$\underset{\approx}{T} = T_{ij}\underset{\sim}{g}^i \otimes \underset{\sim}{g}^j = T^{ij}\underset{\sim}{g}_i \otimes \underset{\sim}{g}_j = T_i^{\cdot j}\underset{\sim}{g}^i \otimes \underset{\sim}{g}_j = T^i_{\cdot j}\underset{\sim}{g}_i \otimes \underset{\sim}{g}^j$$

$$= \hat{T}_{\alpha\beta}\hat{g}^\alpha \otimes \hat{g}^\beta = \hat{T}^{\alpha\beta}\hat{g}_\alpha \otimes \hat{g}_\beta = \hat{T}_\alpha^{\cdot\beta}\hat{g}^\alpha \otimes \hat{g}_\beta = \hat{T}^\alpha_{\cdot\beta}\hat{g}_\alpha \otimes \hat{g}^\beta$$

Then, with the use of Eqs. (5.2-19) and (5.2-21)

$$\hat{u}^\alpha = \frac{\partial y^\alpha}{\partial x^i} u^i \qquad u^i = \frac{\partial x^i}{\partial y^\alpha} \hat{u}^\alpha$$

$$(5.2\text{-}26)$$

$$u_i = \frac{\partial y^\alpha}{\partial x^i} \hat{u}_\alpha \qquad \hat{u}_\alpha = \frac{\partial x^i}{\partial y^\alpha} u_i$$

The first two equations in (5.2-26) represent a transformation of contravariant components while the other two equations represent a transformation of covariant components.

Also

$$\hat{T}^{\alpha\beta} = \frac{\partial y^\alpha}{\partial x^i}\frac{\partial y^\beta}{\partial x^j} T^{ij} \qquad \text{(contravariant components)}$$

$$(5.2\text{-}27)$$

$$\hat{T}_{\alpha\beta} = \frac{\partial x^i}{\partial y^\alpha}\frac{\partial x^j}{\partial y^\beta} T_{ij} \qquad \text{(covariant components)}$$

$$\hat{T}_{\alpha}^{\cdot\beta} = \frac{\partial x^i}{\partial y^\alpha}\frac{\partial y^\beta}{\partial x^j} T_i^{\cdot j} \qquad \text{(mixed components)}$$

Notes:

1. From Eqs. (5.2-12) and (5.2-13), (g^{ij}, g_{ij}) are the means of transforming from one set of components to another set within one coordinate system. Eqs. (5.2-26) and (5.2-27), $\left(\frac{\partial y^\alpha}{\partial x^i}, \frac{\partial x^i}{\partial y^\alpha}\right)$ are the means for transforming a particular set of components from one coordinate system to another.
2. Eqs. (5.2-20) and (5.2-22) are just particular cases of Eq. (5.2-27).
3. Eqs. (5.2-6) and (5.2-7) are just particular cases of Eq. (5.2-19).
4. The analogy involving coordinate transformations can be extended further. Given a coordinate system x^i with base vectors $\underset{\sim}{g}_i$, assume a dual coordinate system x_i can be obtained such that

$$\underset{\sim}{g}^i = \frac{\partial \underset{\sim}{r}}{\partial x_i}$$

Then

$$g_{ij} = \frac{\partial x_i}{\partial x^j}$$

$$g^{ij} = \frac{\partial x^j}{\partial x_i}$$

just represent the matrices for transforming from the x_i - coordinate system to the x^i - coordinate system.

Section 3 Relations Involving the Alternating Symbol

One motivating factor for introducing a contravariant basis is the desirability of a simple expression for the cross product. $\underset{\sim}{g}_1 \times \underset{\sim}{g}_2$ will be perpendicular to both $\underset{\sim}{g}_1$ and $\underset{\sim}{g}_2$ which is the direction denoted by $\underset{\sim}{g}^3$. Thus let

$$\begin{aligned} \underset{\sim}{g}_1 \times \underset{\sim}{g}_2 &= \alpha \underset{\sim}{g}^3 & \alpha &= \underset{\sim}{g}_3 \cdot (\underset{\sim}{g}_1 \times \underset{\sim}{g}_2) \\ \underset{\sim}{g}_2 \times \underset{\sim}{g}_3 &= \beta \underset{\sim}{g}^1 & \beta &= \underset{\sim}{g}_1 \cdot (\underset{\sim}{g}_2 \times \underset{\sim}{g}_3) \\ \underset{\sim}{g}_3 \times \underset{\sim}{g}_1 &= \gamma \underset{\sim}{g}^2 & \gamma &= \underset{\sim}{g}_2 \cdot (\underset{\sim}{g}_3 \times \underset{\sim}{g}_1) \end{aligned} \tag{5.3-1}$$

From the property $\underset{\sim}{a} \cdot (\underset{\sim}{b} \times \underset{\sim}{c}) = \underset{\sim}{b} \cdot (\underset{\sim}{c} \times \underset{\sim}{a}) = \underset{\sim}{c} \cdot (\underset{\sim}{a} \times \underset{\sim}{b})$ we have

$$\alpha = \beta = \gamma \tag{5.3-2}$$

Let

$$\begin{aligned} e^{ijk} = e_{ijk} &= 1 \text{ if ijk form a positive permutation} \\ &= -1 \text{ if ijk form a negative permutation} \\ &= 0 \text{ if otherwise} \end{aligned}$$

and

$$\varepsilon_{ijk} = \alpha e_{ijk} \tag{5.3-3}$$

which reduces to the previous definition of Eqn. (1.5-9) for a rectangular cartesian system for which $\alpha = 1$. Then

$$\underset{\sim}{g}_i \times \underset{\sim}{g}_j = \varepsilon_{ijk} \underset{\sim}{g}^k \tag{5.3-4}$$

In a similar manner let

$$\begin{aligned} \hat{\alpha} &= \underset{\sim}{g}^1 \cdot (\underset{\sim}{g}^2 \times \underset{\sim}{g}^3) \\ \varepsilon^{ijk} &= \hat{\alpha}\, e^{ijk} \end{aligned} \tag{5.3-5}$$

Then

$$\underset{\sim}{g}^i \times \underset{\sim}{g}^j = \varepsilon^{ijk} \underset{\sim}{g}_k \tag{5.3-6}$$

and

$$\underset{\sim}{u} \times \underset{\sim}{v} = \varepsilon^{ijk} u_i v_j \underset{\sim}{g}_k = \varepsilon_{ijk} u^i v^j \underset{\sim}{g}^k \tag{5.3-7}$$

Let $g = |g_{ij}|$ (5.3-8)

Then $$|g^{ij}| = \frac{1}{g}$$

since $$g^{ij}g_{j\cdot} = \delta^i_k$$

Theorem: $$\frac{\alpha}{\hat{\alpha}} = g \qquad (5.3\text{-}9)$$

Proof:

$$\underset{\sim}{g}_i \times \underset{\sim}{g}_j = \varepsilon_{ijk}\underset{\sim}{g}^k$$

$$g_{im}g_{jn}\underset{\sim}{g}^m \times \underset{\sim}{g}^n = \varepsilon_{ijk}\underset{\sim}{g}^k$$

$$(g_{im}g_{jn}\varepsilon^{mnp}\underset{\sim}{g}_p = \varepsilon_{ijk}\underset{\sim}{g}^k)\cdot\underset{\sim}{g}_\ell$$

$$g_{im}g_{jn}g_{\ell p}\hat{\alpha}e^{mnp} = \alpha e_{ijk}\delta^k_\ell$$

$$g_{im}g_{jn}g_{\ell p}\hat{\alpha}e^{mnp} = \alpha e_{ij\ell}$$

From Eqn. (1.5-19) which involves the determinant of a matrix,

$$\hat{\alpha}|g_{ij}|e_{ij\ell} = \alpha e_{ij\ell}$$

or $$g = \frac{\alpha}{\hat{\alpha}}$$ EOP

Theorem:

$$\alpha = \sqrt{g} \qquad \hat{\alpha} = \frac{1}{\sqrt{g}}$$

$$\varepsilon_{ijk} = \sqrt{g}\, e_{ijk} \quad \varepsilon^{ijk} = \frac{1}{\sqrt{g}}\cdot e^{ijk} \qquad (5.3\text{-}10)$$

Proof:

$$\alpha = \underset{\sim}{g}_1\cdot(\underset{\sim}{g}_2 \times \underset{\sim}{g}_3)$$

$$\underset{\sim}{g}_2 \times \underset{\sim}{g}_3 = |\underset{\sim}{g}_2||\underset{\sim}{g}_3|\sin\phi\underset{\sim}{n}$$

Let $$\underset{\sim}{g}_1 = a\underset{\sim}{g}_2 + b\underset{\sim}{g}_3 + c\underset{\sim}{n}$$

where $\underset{\sim}{n}.\underset{\sim}{g}_2 = 0$ and $\underset{\sim}{n}.\underset{\sim}{g}_3 = 0$ since $\underset{\sim}{n}$ is defined as a unit vector perpendicular to both $\underset{\sim}{g}_2$ and $\underset{\sim}{g}_3$

$$\alpha = c|\underset{\sim}{g}_2||\underset{\sim}{g}_3|\sin\phi$$

$$\alpha^2 = c^2 g_{22} g_{33} \sin^2\phi$$

$$\sin^2\phi = 1 - \cos^2\phi = 1 - \left(\frac{\underset{\sim}{g}_2 \cdot \underset{\sim}{g}_3}{|\underset{\sim}{g}_2||\underset{\sim}{g}_3|}\right)^2 = 1 - \frac{g_{23}^2}{g_{22}g_{33}}$$

Now $$\alpha^2 = c^2(g_{22}g_{33} - g_{23}^2)$$

$$\underset{\sim}{g}_1 \cdot \underset{\sim}{g}_1 = g_{11} = a^2 g_{22} + 2ab g_{23} + b^2 g_{33} + c^2$$
$$\underset{\sim}{g}_1 \cdot \underset{\sim}{g}_2 = g_{12} = a g_{22} + b g_{23}$$
$$\underset{\sim}{g}_1 \cdot \underset{\sim}{g}_3 = g_{13} = a g_{23} + b g_{33}$$

Solving the above equations for a and b we get

$$a = \frac{g_{12}g_{33} - g_{13}g_{23}}{g_{22}g_{33} - g_{23}^2} \qquad b = \frac{g_{22}g_{13} - g_{12}g_{23}}{g_{22}g_{33} - g_{23}^2}$$

Substituting for c^2 we obtain

$$\alpha^2 = (g_{11} - a^2 g_{22} - 2ab g_{23} - b^2 g_{33})(g_{22}g_{33} - g_{23}^2)$$

$$= [g_{11} - \left(\frac{g_{12}g_{33}-g_{13}g_{23}}{g_{22}g_{33}-g_{23}^2}\right)^2 g_{22} - 2\,\frac{(g_{12}g_{33}-g_{13}g_{23})(g_{22}g_{13}-g_{12}g_{23})g_{23}}{(g_{22}-g_{23}^2)^2}$$

$$- \left(\frac{g_{22}g_{13}-g_{12}g_{23}}{g_{22}g_{33}-g_{23}^2}\right)^2 g_{33}][g_{22}g_{33} - g_{23}^2]$$

$$= g_{11}g_{22}g_{33} - g_{11}g_{23}^2 + \frac{1}{g_{22}g_{33} - g_{23}^2}[(-g_{12}^2g_{33}^2 + 2g_{12}g_{33}g_{13}g_{23} - g_{13}^2g_{23}^2)g_{22}$$

$$- 2\,(g_{12}g_{33}g_{22}g_{13} - g_{12}^2g_{23}g_{33} - g_{13}^2g_{22}g_{23} + g_{12}g_{13}g_{23}^2)g_{23}$$

$$+ (-g_{22}^2g_{13}^2 + 2g_{12}g_{13}g_{22}g_{23} - g_{12}^2g_{23}^2)g_{33}]$$

$$= g_{11}g_{22}g_{33} - g_{11}g_{23}^2 + \frac{1}{g_{22}g_{33}-g_{23}^2}\,[g_{22}g_{33}(- g_{12}^2g_{33} + 2g_{12}g_{13}g_{23})$$

$$- g_{23}^2(g_{13}^2g_{22}) + g_{22}g_{33}(- 2g_{12}g_{13}g_{23}) - g_{23}^2(- 2g_{12}^2g_{33} - 2g_{13}^2g_{22} + 2g_{12}g_{13}g_{23})$$

$$+ g_{22}g_{33}(- g_{22}g_{13}^2 + 2g_{12}g_{13}g_{23}) - g_{23}^2(g_{12}^2g_{33})]$$

$$= g_{11}g_{22}g_{33} - g_{11}g_{23}^2 + \frac{1}{g_{22}g_{33}-g_{23}^2}\,[g_{22}g_{33}(- g_{12}^2g_{33} + 2g_{12}g_{13}g_{23} - g_{22}g_{13}^2)$$

$$- g_{23}^2(- g_{13}g_{22} + 2g_{12}g_{13}g_{23} - g_{12}^2g_{33})]$$

$$= g_{11}g_{22}g_{33} - g_{11}g_{23}^2 + \frac{g_{22}g_{33}-g_{23}^2}{g_{22}g_{33}-g_{23}^2}(- g_{12}^2g_{33} + 2g_{12}g_{13}g_{23} - g_{22}g_{13}^2)$$

$$= g_{11}g_{22}g_{33} - g_{11}g_{23}{}^2 - g_{12}^2g_{33} + 2g_{12}g_{13}g_{23} - g_{22}g_{13}^2$$

$$= \det|g_{ij}|$$

$$= |g_{ij}|$$

$$= g$$

which is positive for a right handed coordinate system.

Hence

$$\alpha = \sqrt{g} \qquad \text{EOP}$$

<u>Alternating Tensor</u>

Eqns. (5.3-4) and (5.3-6) can be summarized by

$$\varepsilon^{ijk} = \underset{\sim}{g}^i.(\underset{\sim}{g}^j \times \underset{\sim}{g}^k)$$

$$\varepsilon_{ijk} = \underset{\sim}{g}_i.(\underset{\sim}{g}_j \times \underset{\sim}{g}_k)$$

Substitute Eqn. (5.2-2) into the first of these relations

$$\varepsilon^{ijk} = g^{i\ell}g^{jm}g^{kn}\underset{\sim}{g}_\ell\cdot(\underset{\sim}{g}_m \times \underset{\sim}{g}_n)$$

The second relation yields

$$\varepsilon^{ijk} = g^{i\ell}g^{jm}g^{kn}\varepsilon_{\ell mn} \tag{5.3-11}$$

or

$$\varepsilon_{ijk} = g_{i\ell}g_{jm}g_{kn}\varepsilon^{\ell mn}$$

In a similar manner, we can use Eqns. (5.2-19) and (5.2-21) to obtain the transformation relations between two coordinate systems

$$\hat{\varepsilon}^{\alpha\beta\gamma} = \frac{\partial y^\alpha}{\partial x^i}\frac{\partial y^\beta}{\partial x^j}\frac{\partial y^\gamma}{\partial x^k}\varepsilon^{ijk}$$

$$\hat{\varepsilon}_{\alpha\beta\gamma} = \frac{\partial x^i}{\partial y^\alpha}\frac{\partial x^j}{\partial y^\beta}\frac{\partial x^k}{\partial y^\gamma}\varepsilon_{ijk} \tag{5.3-12}$$

Because these relationships between components for two arbitrary coordinate systems are the same relations for components of a third order tensor it is appropriate to define the tensor

$$\underset{\approx}{\varepsilon} = \varepsilon^{ijk}\underset{\sim}{g}_i \otimes \underset{\sim}{g}_j \otimes \underset{\sim}{g}_k$$

$$= \varepsilon_{ijk}\underset{\sim}{g}^i \otimes \underset{\sim}{g}^j \otimes \underset{\sim}{g}^k \tag{5.3-13}$$

However e_{ijk} are not the components of a tensor

Determinants of Matrices

Consider the contravariant components of a second order tensor $\underset{\approx}{A}$. From Eq. (1.5-19)

$$e_{ijk}A^{i\ell}A^{jm}A^{kn} = e^{\ell mn}|A^{pq}|$$

$$e_{ijk}e_{\ell mn}A^{i\ell}A^{jm}A^{kn} = 6|A^{pq}| \tag{5.3-14}$$

$$\varepsilon_{ijk}\varepsilon_{\ell mn}A^{i\ell}A^{jm}A^{kn} = 6g|A^{pq}|$$

Similarly for mixed and covariant components

$$\varepsilon^{ijk}\varepsilon_{\ell mn}A^{\ell}_{\cdot i}A^{m}_{\cdot j}A^{n}_{\cdot k} = 6|A^{p}_{\cdot q}|$$

$$= 6|A_{q}^{\cdot p}|$$

$$\varepsilon^{ijk}\varepsilon^{\ell mn}A_{i\ell}A_{jm}A_{kn} = \frac{6}{g}|A_{pq}|$$

Determinant of Tensors

Recall from Eqn. (1.7-4) that the definition of a tensor was obtained from

$$\det(\underset{\approx}{T})\ \underset{\sim}{u}\cdot(\underset{\sim}{v} \times \underset{\sim}{w}) = (\underset{\approx}{T}\cdot\underset{\sim}{u})\cdot[(\underset{\approx}{T}\cdot\underset{\sim}{v}) \times (\underset{\approx}{T}\cdot\underset{\sim}{w})]$$

or
$$\det(\underset{\approx}{T})\ \varepsilon^{ijk}u_i v_j w_k = \varepsilon^{\ell mn}T_{\ell}^{\cdot i}T_{m}^{\cdot j}T_{n}^{\cdot k}u_i v_j w_k$$

which must hold for all $\underset{\sim}{u}$, $\underset{\sim}{v}$ and $\underset{\sim}{w}$. Thus

$$6\det(\underset{\approx}{T}) = \varepsilon_{ijk}\varepsilon^{\ell mn}T_{\ell}^{\cdot i}T_{m}^{\cdot j}T_{n}^{\cdot k}$$

$$= 6\ |T_{\ell}^{\cdot i}|$$

or

$$\det(\underset{\approx}{T}) = |T_{\ell}^{\cdot i}| = g|T^{ij}| \qquad (5.3\text{-}15)$$

$$= |T^{i}_{\cdot\ell}| = \frac{1}{g}|T_{ij}|$$

The last three relations can be obtained by taking the determinants of

$$T_{\ell}^{\cdot i} = g_{\ell m}T^{mi} = T^{k}_{\cdot m}g_{k\ell}g^{mi} = g^{ik}T_{\ell k}$$

and using Eqn. (5.3-8) and the determinant rule for products of matrices.

ε-δ Identity

Using Eq. (1.5-15) we can show that

$$e^{ijk}e_{i\ell m} = \delta^{j}_{\ell}\delta^{k}_{m} - \delta^{j}_{m}\delta^{k}_{\ell}$$

and also

$$\varepsilon^{ijk}\varepsilon_{i\ell m} = \delta^j_\ell \delta^k_m - \delta^j_m \delta^k_\ell$$

$$\varepsilon^i_{.jk}\varepsilon_{i\ell m} = g_{j\ell} g_{km} - g_{jm} g_{k\ell} \tag{5.3-16}$$

Similarly,

$$e^{ijk} e_{ijm} = 2\delta^k_m$$

Hence

$$\varepsilon^{ijk}\varepsilon_{ijm} = 2\delta^k_m$$

$$\varepsilon^{ij}_{.k}\varepsilon_{ijm} = 2g_{km} \tag{5.3-17}$$

Some alternate approaches ~~+~~ summary

Let

$$e_{ijk} = e^{ijk} = \begin{cases} 1 & \text{positive permutation} \\ -1 & \text{negative permutation} \\ 0 & \text{otherwise} \end{cases}$$

$$\varepsilon_{ijk} = (\underset{\sim}{g}_i \times \underset{\sim}{g}_j) \cdot \underset{\sim}{g}_k$$

$$\underset{\sim}{g}_i = \frac{\partial \underset{\sim}{r}}{\partial x^i} = \frac{\partial \underset{\sim}{r}}{\partial z_A} \frac{\partial z_A}{\partial x^i} = \underset{\sim}{E}_A \frac{\partial z_A}{\partial x^i}$$

$$\varepsilon_{ijk} = \left(\underset{\sim}{E}_A \frac{\partial z_A}{\partial x^i} \times \underset{\sim}{E}_B \frac{\partial z_B}{\partial x^j}\right) \cdot \underset{\sim}{E}_C \frac{\partial z_C}{\partial x^k}$$

$$= e_{ABC} \frac{\partial z_A}{\partial x^i} \frac{\partial z_B}{\partial x^j} \frac{\partial z_C}{\partial x^k}$$

$$= e_{ijk} \left| \frac{\partial z_A}{\partial x^\ell} \right|$$

$$g_{ij} = \frac{\partial z_A}{\partial x^i} \frac{\partial z_A}{\partial x^j}$$

$$g = |g_{ij}| = \left| \frac{\partial z_A}{\partial x^i} \right|^2$$

Therefore

$$\varepsilon_{ijk} = \sqrt{g}\, e_{ijk}$$

Similarly

$$\varepsilon^{ijk} = \frac{1}{\sqrt{g}} e^{ijk}$$

Determinants of Matrices

$$e_{ijk} A^{il} A^{jm} A^{kn} = e^{lmn} |A^{il}|$$

$$e_{lmn} e_{ijk} A^{il} A^{jm} A^{kn} = 6 |A^{il}|$$

$$\varepsilon_{lmn} \varepsilon_{ijk} A^{il} A^{jm} A^{kn} = 6 g |A^{il}|$$

$$\varepsilon_{ijk} \; \varepsilon^{lmn} A^{i}_{\cdot l} A^{j}_{\cdot m} A^{k}_{\cdot n} = 6 \, |A^{i}_{\cdot l}|$$

Determinants of Tensors

$$(\underline{\underline{T}} \cdot \underset{\sim}{u}) \cdot [(\underline{\underline{T}} \cdot \underset{\sim}{v}) \times (\underline{\underline{T}} \cdot \underset{\sim}{w})] = \det \underline{\underline{T}} \;\; \underset{\sim}{u} \cdot (\underset{\sim}{v} \times \underset{\sim}{w})$$

$$\forall \; \underset{\sim}{u}, \underset{\sim}{v}, \underset{\sim}{w}$$

$$\varepsilon^{lmn} \, T_{l}^{\cdot i} \, T_{m}^{\cdot j} \, T_{n}^{\cdot k} u_i v_j w_k = (\det \underline{\underline{T}}) \; \varepsilon^{ijk} u_i v_j w_k$$

$$(\det \underline{\underline{T}}) = \frac{1}{6} \; \varepsilon_{ijk} \varepsilon^{lmn} T_{l}^{\cdot i} T_{m}^{\cdot j} T_{n}^{\cdot k}$$

$$= \frac{1}{6} \; |T_{l}^{\cdot i}| \quad = \frac{1}{6} \, |T^{i}_{\cdot l}|$$

$\varepsilon - \delta$ Identity

$$\varepsilon^{ijk} \varepsilon_{ilm} = \delta^{j}_{l} \delta^{k}_{m} - \delta^{j}_{m} \delta^{k}_{l}$$

Section 4 Gradient and Other Operators

The expression for the gradient operator provides a further justification for introducing contravariant basis. Recall that the gradient of scalars, vectors and tensors were defined such that

$$d\phi = \phi\nabla\cdot d\underset{\sim}{r}$$

$$d\underset{\sim}{v} = \underset{\sim}{v}\nabla\cdot d\underset{\sim}{r} \qquad (5.4\text{-}1)$$

$$d\underset{\approx}{T} = \underset{\approx}{T}\nabla\cdot d\underset{\sim}{r}$$

Now, in general, $d\underset{\sim}{r} = dx^i \underset{\sim}{g}_i$

and

$$d\phi = \frac{\partial\phi}{\partial x^i} dx^i$$

$$d\underset{\sim}{v} = \frac{\partial \underset{\sim}{v}}{\partial x^i} dx^i \qquad (5.4\text{-}2)$$

$$d\underset{\approx}{T} = \frac{\partial \underset{\approx}{T}}{\partial x^i} dx^i$$

Hence

$$\frac{\partial\phi}{\partial x^i} = \phi\nabla\cdot \underset{\sim}{g}_i$$

$$\frac{\partial \underset{\sim}{v}}{\partial x^i} = \underset{\sim}{v}\nabla\cdot \underset{\sim}{g}_i \qquad (5.4\text{-}3)$$

$$\frac{\partial \underset{\approx}{T}}{\partial x^i} = \underset{\approx}{T}\nabla\cdot \underset{\sim}{g}_i$$

Consider the first relation

$$\frac{\partial\phi}{\partial x^i} = \frac{\partial\phi}{\partial x^j}\delta^j_i = \frac{\partial\phi}{\partial x^j}\underset{\sim}{g}^j\cdot\underset{\sim}{g}_i$$

Similar equations hold for the derivatives of $\underset{\sim}{v}$ and $\underset{\approx}{T}$. Since these relations hold for any coordinate system (arbitrary $\underset{\sim}{g}_i$)

$$\phi\nabla = \frac{\partial\phi}{\partial x^i}\,\underset{\sim}{g}^i$$

$$\underset{\sim}{v}\nabla = \frac{\partial \underset{\sim}{v}}{\partial x^i} \otimes \underset{\sim}{g}^i \tag{5.4-4}$$

$$\underset{\approx}{T}\nabla = \frac{\partial \underset{\approx}{T}}{\partial x^i} \otimes \underset{\sim}{g}^i$$

Note that it is the contravariant and not the covariant base vectors that appear naturally in these expressions. Of course the relation $\underset{\sim}{g}^i = g^{ij}\underset{\sim}{g}_j$ could be used to obtain equivalent equations in terms of the covariant base vectors.

The divergence becomes

$$\underset{\sim}{v}\cdot\nabla = \frac{\partial \underset{\sim}{v}}{\partial x^i}\cdot \underset{\sim}{g}^i$$

$$\underset{\approx}{T}\cdot\nabla = \frac{\partial \underset{\approx}{T}}{\partial x^i}\cdot \underset{\sim}{g}^i \tag{5.4-5}$$

and the curl

$$\underset{\sim}{v} \times \nabla = \frac{\partial \underset{\sim}{v}}{\partial x^i} \times \underset{\sim}{g}^i$$

$$\underset{\approx}{T} \times \nabla = \frac{\partial \underset{\approx}{T}}{\partial x^i} \times \underset{\sim}{g}^i \tag{5.4-6}$$

Since the base vectors vary with position, the derivative of a vector becomes

$$\begin{aligned}\frac{\partial \underset{\sim}{v}}{\partial x^i} &= \underset{\sim}{v}_{,i} \\ &= (v^j\underset{\sim}{g}_j)_{,i} = (v_j\underset{\sim}{g}^j)_{,i} \\ &= v^j{}_{,i}\underset{\sim}{g}_j + v^j\underset{\sim}{g}_{j,i} \\ &= v_{j,i}\underset{\sim}{g}^j + v_j\underset{\sim}{g}^j{}_{,i}\end{aligned}$$

Let $\underset{\sim}{g}_{j,i}$ be expressed as a linear combination of the covariant base vectors, i.e.

$$\underset{\sim}{g}_{j,i} = \left\{ {k \atop ji} \right\} \underset{\sim}{g}_k \tag{5.4-7}$$

where $\left\{ {k \atop ji} \right\}$ are called the Christoffel symbols of the second kind. They are not components of third order tensors. The notation Γ^k_{ij} is also frequently used to represent Christoffel symbols.

If a RCC system exists,

$$\underset{\sim}{g}_j = \frac{\partial Z_A}{\partial x^j} \underset{\sim}{E}_A$$

$$\underset{\sim}{g}_{j,i} = \frac{\partial^2 Z_A}{\partial x^i \partial x^j} \underset{\sim}{E}_A = \frac{\partial^2 Z_A}{\partial x^j \partial x^i} \underset{\sim}{E}_A = \underset{\sim}{g}_{i,j}$$

if the second derivatives exist and are continuous.

Thus

$$\underset{\sim}{g}_{i,j} = \underset{\sim}{g}_{j,i}$$

or $$\left\{ {k \atop ij} \right\} = \left\{ {k \atop ji} \right\} \tag{5.4-8}$$

and there are at most 18 (rather than 27) independent components of the Christoffel symbol.

Note: Eqn. (5.4-8) is usually assumed even if an RCC system does not exist

Similarly, let the derivatives of the contravariant base vectors be expressed as the linear combination

$$\underset{\sim}{g}^k{}_{,i} = \left\langle {k \atop ij} \right\rangle \underset{\sim}{g}^j \tag{5.4-9}$$

and consider

$$(\delta^k_j)_{,i} = (\underset{\sim}{g}_j \cdot \underset{\sim}{g}^k)_{,i} = \underset{\sim}{g}_{j,i} \cdot \underset{\sim}{g}^k + \underset{\sim}{g}_j \cdot \underset{\sim}{g}^k{}_{,i}$$

$$= \{{\ell \atop ij}\}\, \underset{\sim}{g}_\ell \cdot \underset{\sim}{g}^k + \underset{\sim}{g}_j \cdot <{k \atop i\ell}> \underset{\sim}{g}^\ell$$

$$= \{{k \atop ij}\} + <{k \atop ij}> = 0$$

Thus

$$\underset{\sim}{g}^k{}_{,i} = <{k \atop ij}> \underset{\sim}{g}^j = - \{{k \atop ij}\} \underset{\sim}{g}^j \tag{5.4-10}$$

Now

$$\underset{\sim}{v}_{,i} = v^j{}_{,i} \underset{\sim}{g}_j + v^j \{{k \atop ij}\} \underset{\sim}{g}_k$$

$$= [v^j{}_{,i} + v^k \{{j \atop ik}\}] \underset{\sim}{g}_j$$

$$= v_{j,i} \underset{\sim}{g}^j + v_j \underset{\sim}{g}^j{}_{,i}$$

$$= v_{j,i} \underset{\sim}{g}^j + v_j <{j \atop ik}> \underset{\sim}{g}^k$$

$$= v_{j,i} \underset{\sim}{g}^j + v_k <{k \atop ij}> \underset{\sim}{g}^j$$

$$= [v_{j,i} - v_k \{{k \atop ij}\}] \underset{\sim}{g}^j$$

Let

$$\begin{aligned} v^j/_i &= v^j{}_{,i} + \{{j \atop ik}\} v^k \\ v_j/_i &= v_{j,i} - \{{k \atop ij}\} v_k \end{aligned} \tag{5.4-11}$$

These are the components of grad $\underset{\sim}{v}$ and they are also called the covariant derivatives of the components of $\underset{\sim}{v}$.

Then

$$v_{,i} = v^k/_i \underset{\sim}{g}_k = v_{k/i} \underset{\sim}{g}^k \tag{5.4-12}$$

and

$$\underset{\sim}{v}\nabla = v^k/_i \underset{\sim}{g}_k \otimes \underset{\sim}{g}^i = v_{k/i} \underset{\sim}{g}^k \otimes \underset{\sim}{g}^i \tag{5.4-13}$$

Similarly,

$$\begin{aligned} T^{ij}/_k &= T^{ij}{}_{,k} + T^{\ell j}\{{i \atop \ell k}\} + T^{i\ell}\{{j \atop \ell k}\} \\ T_{ij/k} &= T_{ij,k} - T_{\ell j}\{{\ell \atop ik}\} - T_{i\ell}\{{\ell \atop jk}\} \end{aligned} \tag{5.4-14}$$

$$T^{i}_{.j/k} = T^{i}_{.j,k} - T^{i}_{.\ell}\{{}^{\ell}_{jk}\} + T^{\ell}_{.j}\{{}^{i}_{\ell k}\}$$

$$T_{i}^{\cdot j}/_{k} = T_{i}^{\cdot j}{}_{,k} - T_{\ell}^{\cdot j}\{{}^{\ell}_{ik}\} + T_{i}^{\cdot \ell}\{{}^{j}_{\ell k}\}$$

so that

$$\underset{\approx}{T}\nabla = T^{ij}/_{k}\ \underset{\sim}{g}_i \otimes \underset{\sim}{g}_j \otimes \underset{\sim}{g}^k \tag{5.4-15}$$

$$= T_{ij/k}\underset{\sim}{g}^i \otimes \underset{\sim}{g}^j \otimes \underset{\sim}{g}^k$$

$$= T^{i}_{.j/k}\underset{\sim}{g}_i \otimes \underset{\sim}{g}^j \otimes \underset{\sim}{g}^k$$

$$= T_{i}^{\cdot j}/_{k}\ \underset{\sim}{g}^i \otimes \underset{\sim}{g}_j \otimes \underset{\sim}{g}^k$$

The divergence reduces to

$$v\cdot\nabla = v^{k}/_{k} \tag{5.4-16}$$

$$\underset{\approx}{T}.\nabla = T^{ij}/_{j}\ \underset{\sim}{g}_i = T_{i}^{\cdot j}/_{j}\ \underset{\sim}{g}^i$$

Also

$$\phi\nabla = \phi_{,i}\underset{\sim}{g}^i$$

$$(\phi\nabla).\nabla = \phi_{,i/j}\ \underset{\sim}{g}^i.\underset{\sim}{g}^j \tag{5.4-17}$$

or

$$\phi\nabla^2 = \phi/_{ij}\ g^{ij}$$

where

$$\phi/_{i} = \phi_{,i}$$

The gradient of the divergence of $\underset{\sim}{v}$ is

$$\text{grad (div } \underset{\sim}{v}) = (\underset{\sim}{v}.\nabla)\nabla \tag{5.4-18}$$

$$= v^{k}/_{k\ell}\underset{\sim}{g}^{\ell}$$

while the divergence of the gradient of $\underset{\sim}{v}$ is

$$\text{div (grad } \underset{\sim}{v}) = (\underset{\sim}{v}\nabla).\nabla = v\nabla^2$$

$$= v^i/_{k\ell} g^{k\ell} g_i \qquad (5.4\text{-}19)$$

$$= v_{i/k\ell} g^{k\ell} \underset{\sim}{g}^i$$

Note: The notation $v_i/^k$ could be used to mean $v_{i/\ell} g^{k\ell}$ but it is not a common representation so it will not be used.

The gradient of the identy tensor also assumes several forms:

$$\underset{\approx}{I}\nabla = \delta^i_{j/k} \underset{\sim}{g}_i \otimes \underset{\sim}{g}^j \otimes \underset{\sim}{g}^k$$

$$= g_{ij/k} \underset{\sim}{g}^i \otimes \underset{\sim}{g}^j \otimes \underset{\sim}{g}^k \qquad (5.4\text{-}20)$$

$$= g^{ij}/_k \underset{\sim}{g}_i \otimes \underset{\sim}{g}_j \otimes \underset{\sim}{g}^k$$

Since this third order tensor is a null tensor, it follows that the components must be zero, or

$$\delta^i_{j/k} = 0$$

$$g_{ij/k} = 0 \qquad (5.4\text{-}21)$$

$$g^{ij}/_k = 0$$

(These equations can also be shown by direct substitution with the use of the Christoffel symbols derived in the next section).

Equations for the Christoffel Symbols

It follows from Eqn. (5.4-7) that

$$\{\overset{k}{ij}\} = \underset{\sim}{g}^k \cdot \underset{\sim}{g}_{j,i} \qquad (5.4\text{-}22)$$

but this is usually not a convenient method for determining the Christoffel symobls of the second kind for a given coordinate system. Instead, the Christoffel symbols of the first kind are defined:

$$[ij,k] = [ji,k] = g_{\ell k}\{\overset{\ell}{ij}\} \qquad (5.4\text{-}23)$$

Then if these symbols are known, the Christoffel symbols of the second kind are derived from

$$\left\{ {\ell \atop ij} \right\} = g^{\ell k}[ij,k] \tag{5.4-24}$$

Suppose that for a given coordinate system the metric and hence g_{ij} are known. Then from Eqns. (5.4-22) and (5.4-23)

$$\begin{aligned}
[ij,k] &= g_{\ell k}\underset{\sim}{g}^{\ell}\cdot\underset{\sim}{g}_{j,i} \\
&= \underset{\sim}{g}_k\cdot\underset{\sim}{g}_{j,i} \\
&= (\underset{\sim}{g}_k\cdot\underset{\sim}{g}_j)_{,i} - \underset{\sim}{g}_j\cdot\underset{\sim}{g}_{k,i} \\
&= g_{kj,i} - \underset{\sim}{g}_j\cdot\underset{\sim}{g}_{i,k} && \text{(a)} \\
&= g_{kj,i} - (\underset{\sim}{g}_j\cdot\underset{\sim}{g}_i)_{,k} + \underset{\sim}{g}_i\cdot\underset{\sim}{g}_{j,k} \\
&= g_{kj,i} - g_{ij,k} + \underset{\sim}{g}_i\cdot\underset{\sim}{g}_{j,k} \\
&= g_{kj,i} - g_{ij,k} + \underset{\sim}{g}_i\cdot\underset{\sim}{g}_{k,j} && \text{(b)}
\end{aligned}$$

From (a) with i and j interchanged

$$[ji,k] = g_{ik,j} - \underset{\sim}{g}_i\cdot\underset{\sim}{g}_{k,j} \tag{c}$$

Add (b) and (c) and use the symmetry condition $[ij,k] = [ji,k]$ to get

$$[ij,k] = \tfrac{1}{2}(g_{ik,j} + g_{jk,i} - g_{ij,k}) \tag{5.4-25}$$

which is an expression that can be used directly to get the Christoffel symbols.

Since the Christoffel symbols are necessary to obtain gradients, they must be derived for each coordinate system. The procedure can be summarized as follows:

(1) Obtain the components g_{ij} from one of two possible sources:

(a) If the metric $ds^2 = g_{ij}dx^i dx^j$ is known, then g_{ij} can be determined by examining the coefficients of $dx^i dx^j$.

(b) If the relationship between the curvilinear coordinate system and the rectangular cortesian system is known, $Z_A = Z_A(x^i)$ then compute g_{ij} from

$$g_{ij} = \frac{\partial Z_A}{\partial x^i}\frac{\partial Z_A}{\partial x^j}$$

Note: Some information must be made available with regard to the coordinate system to allow the determination of g_{ij} and it is usually in the form of (a) or (b).

(2) Invert g_{ij} to get g^{ij} and g

(3) Determine the Christoffel symbols of the first kind from

$$[ij,k] = [ji,k] = \tfrac{1}{2}(g_{ik,j} + g_{jk,i} - g_{ij,k})$$

(4) Determine the Christoffel symbols of the second kind from

$$\left\{ {k \atop ij} \right\} = \left\{ {k \atop ji} \right\} = g^{k\ell}[ij,\ell]$$

Section 5 Orthogonal Curvilinear Coordinates and Physical Components

In the class of curvilinear coordinates, the most common are those that are orthogonal, i.e. $g_i \cdot g_j = g_{ij} = 0$ for $i \neq j$. Then

$$g_{ij} \Longrightarrow \begin{bmatrix} g_{11} & 0 & 0 \\ 0 & g_{22} & 0 \\ 0 & 0 & g_{33} \end{bmatrix}$$

$$g^{ij} \Longrightarrow \begin{bmatrix} \frac{1}{g_{11}} & 0 & 0 \\ 0 & \frac{1}{g_{22}} & 0 \\ 0 & 0 & \frac{1}{g_{33}} \end{bmatrix} \qquad (5.5\text{-}1)$$

$$g = |g_{ij}| = g_{11}g_{22}g_{33}$$

$$ds^2 = g_{ij}dx^i dx^j = g_{11}(dx^1)^2 + g_{22}(dx^2)^2 + g_{33}(dx^3)^2$$

Although frequently there is no good justification from an analysis or interpretive viewpoint, it has been common practice to convert to unit base vectors and to use components of tensors (physical components) in terms of these unit base vectors. In the following development, the unit base vectors and physical components will be denoted by underlining the indices. The summation convention for a pair of identical indices in parentheses is suspended. Indices outside the parentheses are considered as before.

The unit base vectors are obtained as follows: .

$$\underset{\sim}{g}_{\underline{i}} = \frac{\underset{\sim}{g}_i}{\sqrt{g_{(ii)}}}$$

$$\underset{\sim}{g}^{\underline{i}} = \frac{\underset{\sim}{g}^i}{\sqrt{g^{(ii)}}} \qquad (5.5\text{-}2)$$

Also

$$\underset{\sim}{g}_{\underline{i}} = \underset{\sim}{g}^{\underline{i}} \tag{5.5-3}$$

since

$$\underset{\sim}{g}^{\underline{i}} = \frac{\underset{\sim}{g}_k g^{ik}}{\sqrt{g^{(ii)}}}$$

$$= \frac{\underset{\sim}{g}_i g^{(ii)}}{\sqrt{g^{(ii)}}}$$

$$= \underset{\sim}{g}_i \sqrt{g^{(ii)}}$$

$$= \frac{\underset{\sim}{g}_i}{\sqrt{g_{(ii)}}}$$

$$= \underset{\sim}{g}_{\underline{i}}$$

The physical components of a vector are obtained by expressing the vector in several systems:

$$\begin{aligned} \underset{\sim}{v} &= v^i \underset{\sim}{g}_i = v_i \underset{\sim}{g}^i \\ &= v^{\underline{i}} \underset{\sim}{g}_{\underline{i}} = v_{\underline{i}} \underset{\sim}{g}^{\underline{i}} \end{aligned} \tag{5.5-4}$$

It follows from the use of Eqn. (5.5-2) and (5.5-4) that

$$v^{\underline{i}} \underset{\sim}{g}_{\underline{i}} = v^i \underset{\sim}{g}_i$$

$$= v^i \underset{\sim}{g}_{\underline{i}} \sqrt{g_{(ii)}}$$

Hence

$$
\begin{aligned}
v^{\underline{i}} &= v^i\sqrt{g_{(ii)}} \\
&= v_i\sqrt{g^{(ii)}} = v_i/\sqrt{g_{(ii)}} \qquad (5.5\text{-}5) \\
&= v_{\underline{i}}
\end{aligned}
$$

in which the last three equalities are obtained with the use of Eqns. (5.5-1), (5.5-2) and (5.5-4). In a similar fashion, the physical components of second order tensors can be shown to be

$$
\begin{aligned}
T^{\underline{ij}} &= T_{\underline{ij}} = T^{\underline{i}}{}_{\underline{.j}} = T_{\underline{i}}{}^{\underline{.j}} \\
&= T^{ij}\sqrt{g_{(ii)}}\sqrt{g_{(jj)}} = T_{ij}/\sqrt{g_{(ii)}}\sqrt{g_{(jj)}} \qquad (5.5\text{-}6) \\
&= T^{i}{}_{.j}\sqrt{g_{(ii)}}/\sqrt{g_{(jj)}} = T_i^{.j}\sqrt{g_{(jj)}}/\sqrt{g_{(ii)}}
\end{aligned}
$$

The physical components of the gradient of a vector can be derived as follows. Let

$$
\begin{aligned}
S &= \underset{\sim}{v}\nabla \\
&= v_i/_j\, \underset{\sim}{g}^i \otimes \underset{\sim}{g}^j
\end{aligned}
$$

or

$$
S_{ij} = v_i/_j
$$

Then

$$
S_{\underline{ij}} = (V_i/_j)/\sqrt{g_{(ii)}}\sqrt{g_{(jj)}}
$$

$$
= (v_{i,j} - \{{}^{\;k}_{ij}\}v_k)/\sqrt{g_{(ii)}}\sqrt{g_{(jj)}}
$$

To express the physical components of $S_{\underline{ij}}$ in terms of the physical components of $v_{\underline{i}}$, use Eqn. (5.5-5) to get

$$S_{\underline{ij}} = \{(\sqrt{g_{(ii)}}v_{\underline{i}})_{,j} - \{{}^{k}_{ij}\}\sqrt{g_{(kk)}}v_{\underline{k}}\}/\sqrt{g_{(ii)}}\sqrt{g_{(jj)}} \tag{5.5-7}$$

The physical components of the divergence of a tensor can be derived in a similar fashion. Let

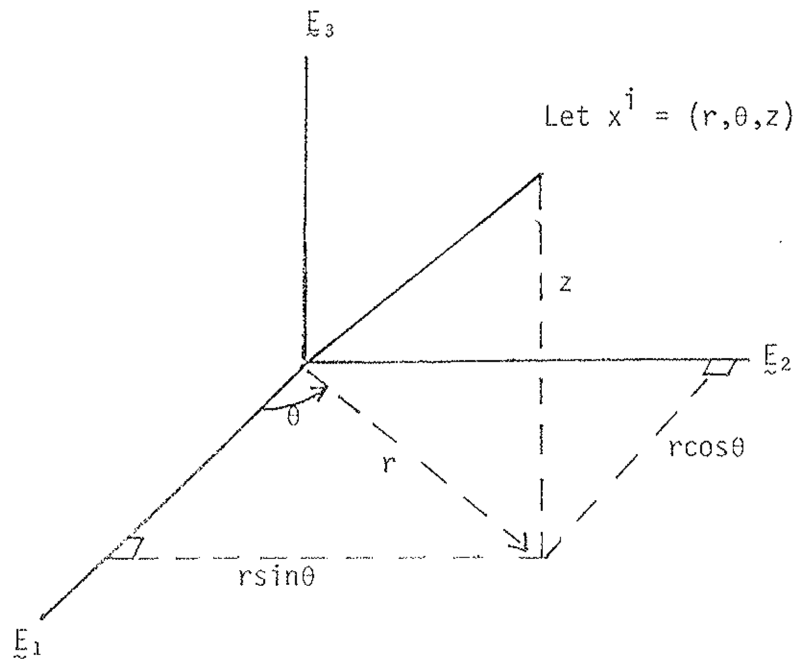
$$\underset{\sim}{b} = \underset{\approx}{T}.\nabla$$

Then

$$\begin{aligned} b_{\underline{i}} &= (T^{ij}/_{j})\sqrt{g_{(ii)}} \\ &= (T^{ij}{}_{,j} + T^{kj}\{{}^{i}_{kj}\} + T^{ik}\{{}^{j}_{kj}\})\sqrt{g_{(ii)}} \\ &= [(\frac{T_{\underline{ij}}}{\sqrt{g_{(ii)}g_{(jj)}}})_{,j} + \frac{T_{\underline{kj}}}{\sqrt{g_{(kk)}g_{(jj)}}}\{k^{i}j\} + \frac{T_{\underline{ik}}}{\sqrt{g_{(ii)}g_{(kk)}}}\{k^{j}j\}]\sqrt{g_{(ii)}} \end{aligned} \tag{5.5-8}$$

Section 6 Cylindrical and Spherical Coordinates

Cylindrical Coordinates



(i) $Z_A = Z_A(x^i)$

$$Z_1 = x^1 \cos x^2 = r\cos\theta$$
$$Z_2 = x^1 \sin x^2 = r\sin\theta$$
$$Z_3 = x^3 \qquad = z$$

(ii)

$$\frac{\partial Z_A}{\partial x^i} \Rightarrow \begin{bmatrix} \frac{\partial Z_1}{\partial x^1} & \frac{\partial Z_1}{\partial x^2} & \frac{\partial Z_1}{\partial x^3} \\ \frac{\partial Z_2}{\partial x^1} & \frac{\partial Z_2}{\partial x^2} & \frac{\partial Z_2}{\partial x^3} \\ \frac{\partial Z_3}{\partial x^1} & \frac{\partial Z_3}{\partial x^2} & \frac{\partial Z_3}{\partial x^3} \end{bmatrix} \Longrightarrow \begin{bmatrix} \cos x^2 & -x^1 \sin x^2 & 0 \\ \sin x^2 & x^1 \cos x^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Longrightarrow \begin{bmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(iii)

$$g_{ij} = \frac{\partial Z_A}{\partial x^i}\frac{\partial Z_A}{\partial x^j}$$

$$g_{11} = \frac{\partial Z_1}{\partial x^1}\frac{\partial Z_1}{\partial x^1} + \frac{\partial Z_2}{\partial x^1}\frac{\partial Z_2}{\partial x^1} + \frac{\partial Z_3}{\partial x^1}\frac{\partial Z_3}{\partial x^1} = \cos^2\theta + \sin^2\theta + 0 = 1$$

$$g_{12} = \frac{\partial Z_1}{\partial x^1}\frac{\partial Z_1}{\partial x^2} + \frac{\partial Z_2}{\partial x^1}\frac{\partial Z_2}{\partial x^2} + \frac{\partial Z_3}{\partial x^1}\frac{\partial Z_3}{\partial x^2} = - r\sin\theta\cos\theta + r\sin\theta\cos\theta + 0 = 0$$

$$g_{13} = \frac{\partial Z_1}{\partial x^1}\frac{\partial Z_1}{\partial x^3} + \frac{\partial Z_2}{\partial x^1}\frac{\partial Z_2}{\partial x^3} + \frac{\partial Z_3}{\partial x^1}\frac{\partial Z_3}{\partial x^3} = 0 + 0 + 0 = 0$$

$$g_{22} = \frac{\partial Z_1}{\partial x^2}\frac{\partial Z_1}{\partial x^2} + \frac{\partial Z_2}{\partial x^2}\frac{\partial Z_2}{\partial x^2} + \frac{\partial Z_3}{\partial x^2}\frac{\partial Z_3}{\partial x^2} = r^2\sin^2\theta + r^2\cos^2\theta = r^2$$

$$g_{23} = \frac{\partial Z_1}{\partial x^2}\frac{\partial Z_1}{\partial x^3} + \frac{\partial Z_2}{\partial x^2}\frac{\partial Z_2}{\partial x^3} + \frac{\partial Z_3}{\partial x^2}\frac{\partial Z_3}{\partial x^3} = 0 + 0 + 0 = 0$$

$$g_{33} = \frac{\partial Z_1}{\partial x^3}\frac{\partial Z_1}{\partial x^3} + \frac{\partial Z_2}{\partial x^3}\frac{\partial Z_2}{\partial x^3} + \frac{\partial Z_3}{\partial x^3}\frac{\partial Z_3}{\partial x^3} = 0 + 0 + 1 = 1$$

Since $g_{ij} = g_{ji}$ then $g_{21} = g_{31} = g_{32} = 0$.

$$g_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(iv) $g^{ij} = (g_{ij})^{-1}$

$$g^{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(v) $ds^2 = g_{ij}dx^i dx^j$

$$ds^2 = g_{11}(dx^1)^2 + g_{22}(dx^2)^2 + g_{33}(dx^3)^2$$

since all other components of g_{ij} are zero.

$$ds^2 = (dr)^2 + r^2(d\theta)^2 + (dz)^2$$

(vi) $\underset{\sim}{g}_i = \frac{\partial Z_A}{\partial x^i} \underset{\sim}{E}_A$

$$\underset{\sim}{g}_1 = \frac{\partial Z_1}{\partial x^1} \underset{\sim}{E}_1 + \frac{\partial Z_2}{\partial x^1} \underset{\sim}{E}_2 + \frac{\partial Z_3}{\partial x^1} \underset{\sim}{E}_3 = \cos\theta \underset{\sim}{E}_1 + \sin\theta \underset{\sim}{E}_2$$

$$\underset{\sim}{g}_2 = \frac{\partial Z_1}{\partial x^2} \underset{\sim}{E}_2 + \frac{\partial Z_2}{\partial x^2} \underset{\sim}{E}_2 + \frac{\partial Z_3}{\partial x^2} \underset{\sim}{E}_3 = -r\sin\theta E_1 + r\cos\theta \underset{\sim}{E}_2$$

$$\underset{\sim}{g}_3 = \frac{\partial Z_1}{\partial x^3} \underset{\sim}{E}_1 + \frac{\partial Z_2}{\partial x^3} \underset{\sim}{E}_2 + \frac{\partial Z_3}{\partial x^3} \underset{\sim}{E}_3 = \underset{\sim}{E}_3$$

$$g_{ij} = \underset{\sim}{g}_i \cdot \underset{\sim}{g}_j$$

$$\Longrightarrow \begin{aligned} g_{11} &= \cos^2\theta + \sin^2\theta = 1 \\ g_{12} &= -r\sin\theta\cos\theta + r\sin\theta\cos\theta = 0 \\ g_{13} &= 0 \\ g_{22} &= r^2\sin^2\theta + r^2\cos^2\theta = r^2 \\ g_{23} &= 0 \\ g_{33} &= 1.1 = 1 \end{aligned}$$

$$g_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(vii) $\underset{\sim}{g}^i = g^{ij} \underset{\sim}{g}_j$

$$\underset{\sim}{g}^1 = g^{11}\underset{\sim}{g}_1 + g^{12}\underset{\sim}{g}_2 + g^{13}\underset{\sim}{g}_3 = 1(\cos\theta \underset{\sim}{E}_1 + \sin\theta \underset{\sim}{E}_2) = \cos\theta \underset{\sim}{E}_1 + \sin\theta \underset{\sim}{E}_2$$

$$\underset{\sim}{g}^2 = g^{21}\underset{\sim}{g}_1 + \underset{\sim}{g}^{22}\underset{\sim}{g}_2 + \underset{\sim}{g}^{23}\underset{\sim}{g}_3 = \frac{1}{r^2}(-r\sin\theta \underset{\sim}{E}_1 + r\cos\theta \underset{\sim}{E}_2) = \frac{-1}{r}\sin\theta \underset{\sim}{E}_1 + \frac{1}{r}\cos\theta \underset{\sim}{E}_2$$

$$\underset{\sim}{g}^3 = g^{31}\underset{\sim}{g}_1 + g^{32}\underset{\sim}{g}_2 + g^{33}\underset{\sim}{g}_3 = 1(\underset{\sim}{E}_3) = \underset{\sim}{E}_3$$

(viii) $\underset{\sim}{g}_{\underline{i}} = \frac{\underset{\sim}{g}_i}{\sqrt{g_{(ii)}}}$

$$\underset{\sim}{g}_{\underline{1}} = \frac{\underset{\sim}{g}_1}{\sqrt{g_{(11)}}} = \frac{\underset{\sim}{g}_1}{1} = \cos\theta \underset{\sim}{E}_1 + \sin\theta \underset{\sim}{E}_2$$

$$\underset{\sim}{g}_{\underline{2}} = \frac{\underset{\sim}{g}_2}{\sqrt{g_{(22)}}} = \frac{\underset{\sim}{g}_2}{r} = -\sin\theta \underset{\sim}{E}_1 + \cos\theta \underset{\sim}{E}_2$$

$$\underset{\sim}{g}_{\underline{3}} = \frac{\underset{\sim}{g}_3}{\sqrt{g_{(33)}}} = \frac{\underset{\sim}{g}_3}{1} = \underset{\sim}{E}_3$$

If we denote the unit vectors $\underset{\sim}{g}_i$ by $\underset{\sim}{e}_r$, $\underset{\sim}{e}_\theta$, and $\underset{\sim}{e}_z$ we obtain

$$\underset{\sim}{e}_r = \cos\theta \underset{\sim}{E}_1 + \sin\theta \underset{\sim}{E}_2$$

$$\underset{\sim}{e}_\theta = -\sin\theta \underset{\sim}{E}_1 + \cos\theta \underset{\sim}{E}_2$$

$$\underset{\sim}{e}_z = \qquad \underset{\sim}{E}_3$$

(ix)

$$[ij,k] = \tfrac{1}{2}\,(g_{ik,j} + g_{kj,i} - g_{ij,k})$$

The only non-zero Christoffel symbols of the first kind are those that contain the term $g_{22,1}$. All other derivatives of the g_{ij} components are zero.

$$[12,2] = [21,2] = \tfrac{1}{2}\,(g_{12,2} + g_{22,1} - g_{12,2}) = \tfrac{1}{2}\, g_{22,1} = \tfrac{1}{2}(2r) = r$$

$$[22,1] = \tfrac{1}{2}\,(g_{21,2} + g_{12,2} - g_{22,1}) = -\tfrac{1}{2}\, g_{22,1} = -r$$

Hence

$$[ij,1] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -r & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad [ij,2] = \begin{bmatrix} 0 & r & 0 \\ r & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$[ij,3]$ = zero matrix

(x) $\quad \left\{{k \atop ij}\right\} = g^{k\ell}[ij,\ell]$

Similarly, the only non-zero Christoffel symbols of the second kind are

$$\left\{{2 \atop 1\;2}\right\} = \left\{{2 \atop 2\;1}\right\} = g^{2\ell}[12,\ell] = g^{21}[12,1] + g^{22}[12,2] + g^{23}[12,3]$$
$$= \frac{1}{r^2}(r) = \frac{1}{r}$$

$$\left\{{1 \atop 2\;2}\right\} = g^{1\ell}[22,\ell] = g^{11}[22,1] + g^{12}[22,2] + g^{13}[22,3] = 1(1-r) = -r$$

Hence

$$\{{}_{i}{}^{1}{}_{j}\} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -r & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \{{}_{i}{}^{2}{}_{j}\} = \begin{bmatrix} 0 & \frac{1}{r} & 0 \\ \frac{1}{r} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\{{}_{i}{}^{3}{}_{j}\} = \text{zero matrix}$$

(xi) Derive the physical components of $\underset{\approx}{e} = \frac{1}{2}(\nabla\underset{\sim}{u} + \underset{\sim}{u}\nabla)$ in terms of physical components of $\underset{\sim}{u}$ where $e_{\underline{11}} = e_{rr}$ etc. and $u_{\underline{1}} = u_r$, $u_{\underline{2}} = u_\theta$, and $u_{\underline{3}} = u_z$

$$e_{ij} = \tfrac{1}{2}(u_{i/j} + u_{j/i})$$

$$e_{\underline{ij}} = \frac{\frac{1}{2}\,[u_{i,j} - \{{}_{i}{}^{k}{}_{j}\}u_k + u_{j,i} - \{{}_{j}{}^{k}{}_{i}\}u_k]}{\sqrt{g_{(ii)}}\,\sqrt{g_{(kk)}}}$$

Noting that $\{{}_{i}{}^{k}{}_{j}\} = \{{}_{j}{}^{k}{}_{i}\}$ and substituting for the physical components of u

$$u_i = u_{\underline{i}}\,\sqrt{g_{(ii)}}$$

we have

$$e_{\underline{ij}} = \frac{\frac{1}{2}[(u_{\underline{i}}\sqrt{g_{(ii)}})_{,j} + (u_{\underline{j}}\sqrt{g_{(jj)}})_{,i} - 2\{{}_{i}{}^{k}{}_{j}\}u_{\underline{k}}\sqrt{g_{(kk)}}]}{\sqrt{g_{(ii)}}\sqrt{g_{(jj)}}}$$

$$e_{\underline{11}} = \frac{\frac{1}{2}[(u_r.1)_{,r} + (u_r.1)_{,r} - \cancel{2\{{}_{1}{}^{k}{}_{1}\}}^{\,0}\; u_{\underline{k}}\sqrt{g_{(kk)}}]}{(1)(1)}$$

$$e_{\underline{11}} = \tfrac{1}{2}(2u_{r,r}) = u_{r,r}$$

$$e_{\underline{12}} = \frac{\frac{1}{2}[(u_r.1)_{,\theta} + (u_\theta.r)_{,r} - 2\,\{{}_{1}{}^{k}{}_{2}\}\,u_{\underline{k}}\,\sqrt{g_{(kk)}}]}{(1)(r)}$$

$$= \frac{\frac{1}{2}\,(u_{r,\theta} + u_{\theta,r}.r + u_\theta - 2(\frac{1}{r})u_\theta(r)]}{r}$$

$$e_{\underline{12}} = \tfrac{1}{2}\left(\frac{u_{r,\theta}}{r} + u_{\theta,r} - \frac{u_\theta}{r}\right)$$

$$e_{\underline{13}} = \frac{\tfrac{1}{2}\left[(u_r.1)_{,z} + (u_z.1)_{,r} - 2\cancel{\left\{{k \atop 1\ 3}\right\}}^{0} u_{\underline{k}}\sqrt{g_{(kk)}}\right]}{(1)(1)}$$

$$= \tfrac{1}{2}(u_{r,z} + u_{z,r})$$

$$e_{\underline{22}} = \frac{\tfrac{1}{2}\left[(u_\theta.r)_{,\theta} + (u_\theta.r)_{,\theta} - 2\left\{{k \atop 2\ 2}\right\}u_{\underline{k}}\sqrt{g_{(kk)}}\right]}{(r)(r)}$$

$$= \tfrac{1}{2}\,\frac{(2\,u_{\theta,\theta}.r - 2(-r)u_r(1))}{r^2}$$

$$e_{\underline{22}} = \frac{u_{\theta,\theta}}{r} + \frac{u_r}{r}$$

$$e_{\underline{23}} = \frac{\tfrac{1}{2}\left[(u_\theta.r)_{,z} + (u_z.1)_{,\theta} - 2\cancel{\left\{{k \atop 2\ 3}\right\}}^{0} u_{\underline{k}}\sqrt{g_{(kk)}}\right]}{(r)(1)}$$

$$= \tfrac{1}{2}\,\frac{(u_{\theta,z}r + u_{z,\theta})}{r} \quad = \tfrac{1}{2}\left(u_{\theta,z} + \frac{u_{z,\theta}}{r}\right)$$

$$e_{\underline{33}} = \frac{\tfrac{1}{2}\left[(u_z.1)_{,z} + (u_z.1)_{,z} - 2\cancel{\left\{{k \atop 3\ 3}\right\}}^{0} u_{\underline{k}}\sqrt{g_{(kk)}}\right]}{(1)(1)}$$

$$= \tfrac{1}{2}(2u_{z,z})$$

$$e_{\underline{33}} = u_{z,z}$$

(xii) Derive the physical components of $\underset{\approx}{\sigma}.\nabla$ in terms of the physical components of $\underset{\approx}{\sigma}$, where $\sigma_{\underline{11}} = \sigma_{rr}$, etc.

Let $\underset{\sim}{S} = \underset{\approx}{\sigma}.\nabla$

$$S_{\underline{i}} = \sigma^{ij}/_j \sqrt{g_{(ii)}}$$

$$= [\sigma^{ij}{}_{,j} + \sigma^{kj}\left\{{i \atop k\ j}\right\} + \sigma^{ki}\left\{{j \atop k\ j}\right\}]\sqrt{g_{(ii)}}$$

After substituting to obtain the physical components we get

$$S_{\underline{i}} = [(\frac{\sigma_{\underline{ij}}}{\sqrt{g_{(ii)}g_{(jj)}}})_{,j} + \frac{\sigma_{\underline{kj}}}{\sqrt{g_{(kk)}g_{(jj)}}}\{{}_{k}{}^{i}{}_{j}\} + \frac{\sigma_{\underline{ki}}}{\sqrt{g_{(kk)}g_{(ii)}}}\{{}_{k}{}^{j}{}_{j}\}]\sqrt{g_{(ii)}}$$

$$S_{\underline{1}} = [(\frac{\sigma_{rr}}{(1)(1)})_{,r} + (\frac{\sigma_{r\theta}}{(1)(r)})_{,\theta} + (\frac{\sigma_{rz}}{(1)(1)})_{,z} + \frac{\sigma_{\theta\theta}}{(r)(r)}(-r) + \frac{\sigma_{rr}}{(1)(1)}(\frac{1}{r})](1)$$

$$= \sigma_{rr,r} + \frac{\sigma_{r\theta,\theta}}{r} + \sigma_{rz,z} - \frac{\sigma_{\theta\theta}}{r} + \frac{\sigma_{rr}}{r}$$

$$S_{\underline{2}} = [(\frac{\sigma_{\theta r}}{(r)(1)})_{,r} + (\frac{\sigma_{\theta\theta}}{(r)(r)})_{,\theta} + (\frac{\sigma_{\theta z}}{(r)(1)})_{,z} + 2\frac{\sigma_{r\theta}(\frac{1}{r})}{(1)(r)} + \frac{\sigma_{r\theta}}{(1)(r)}(\frac{1}{r})](r)$$

$$= (\frac{\sigma_{\theta r,r}}{r} - \frac{1}{r^2}\sigma_{\theta r} + \frac{1}{r^2}\sigma_{\theta\theta,\theta} + \frac{\sigma_{\theta z,z}}{r} + 2\frac{\sigma_{r\theta}}{r^2} + \frac{\sigma_{r\theta}}{r^2})r$$

$$= \sigma_{\theta r,r} + \frac{\sigma_{\theta\theta,\theta}}{r} + \sigma_{\theta z,z} + 2\frac{\sigma_{r\theta}}{r}$$

$$S_{\underline{3}} = [(\frac{\sigma_{zr}}{(1)(1)})_{,r} + (\frac{\sigma_{z\theta}}{(1)(r)})_{,\theta} + (\frac{\sigma_{zz}}{(1)(1)})_{,z} + \frac{\sigma_{rz}}{(1)(1)}(\frac{1}{r})](1)$$

$$= \sigma_{zr,r} + \frac{\sigma_{z\theta,\theta}}{r} + \sigma_{zz,z} + \frac{\sigma_{rz}}{r}$$

In summary, we have

$$e_{\underline{ij}} = \begin{bmatrix} e_{rr} & e_{r\theta} & e_{rz} \\ e_{\theta r} & e_{\theta\theta} & e_{\theta z} \\ e_{zr} & e_{z\theta} & e_{zz} \end{bmatrix} = \begin{bmatrix} u_{r,r} & \frac{1}{2}(\frac{u_{r,\theta}}{r} + u_{\theta,r} - \frac{u_\theta}{r}) & \frac{1}{2}(u_{r,z} + u_{z,r}) \\ & \frac{u_{\theta,\theta}}{r} + \frac{u_r}{r} & \frac{1}{2}(u_{\theta,z} + \frac{u_{z,\theta}}{r}) \\ \text{Symmetric} & & u_{z,z} \end{bmatrix}$$

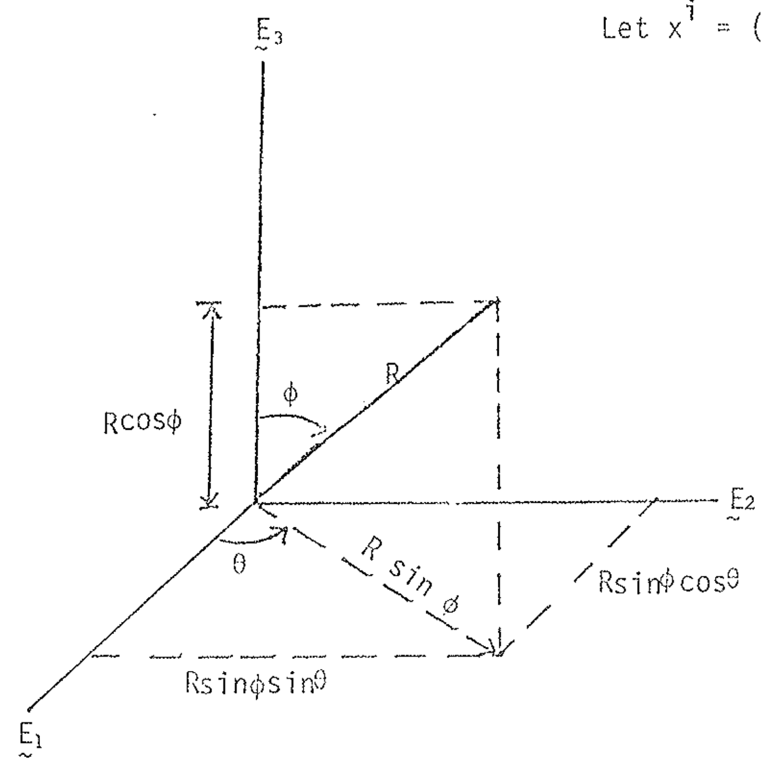
$$\underset{\approx}{\sigma}\cdot\nabla = [\frac{1}{r}(r\sigma_{rr})_{,r} + \frac{1}{r}\sigma_{r\theta,\theta} + \sigma_{rz,z} - \frac{1}{r}\sigma_{\theta\theta}]\,\underset{\sim}{e}_r$$

$$+ [\frac{1}{r}\sigma_{\theta\theta,\theta} + \sigma_{\theta z,z} + \frac{1}{r}(r\sigma_{r\theta})_{,r} + \frac{1}{r}\sigma_{r\theta}]\underset{\sim}{e}_\theta$$

$$+ [\sigma_{zz,z} + \frac{1}{r}(r\sigma_{rz})_{,r} + \frac{1}{r}\sigma_{\theta z,\theta}]\underset{\sim}{e}_z$$

Spherical Coordinates

Let $x^i = (R, \phi, \theta)$



(i) $Z_A = Z_A(x^i)$

$$
\begin{aligned}
Z_1 &= x^1 \sin x^2 \cos x^3 &&= R\sin\phi\cos\theta \\
Z_2 &= x^1 \sin x^2 \sin x^3 &&= R\sin\phi\sin\theta \\
Z_3 &= x^1 \cos x^2 &&= R\cos\phi
\end{aligned}
$$

(ii) $$\frac{\partial Z_A}{\partial x^i} \Rightarrow \begin{bmatrix} \sin x^2\cos x^3 & x^1\cos x^2\cos x^3 & -x^1\sin x^2\sin x^3 \\ \sin x^2\sin x^3 & x^1\cos x^2\sin x^3 & x^1\sin x^2\cos x^3 \\ \cos x^2 & -x^1\sin x^2 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \sin\phi\cos\theta & R\cos\phi\cos\theta & -R\sin\phi\sin\theta \\ \sin\phi\sin\theta & R\cos\phi\sin\theta & R\sin\phi\cos\theta \\ \cos\phi & -R\sin\phi & 0 \end{bmatrix}$$

(iii) $g_{ij} = \dfrac{\partial Z_A}{\partial x^i}\dfrac{\partial Z_A}{\partial x^j}$

$$g_{11} = (\sin\phi\cos\theta)^2 + (\sin\phi\sin\theta)^2 + \cos^2\phi$$
$$= \sin^2\phi(\cos^2\theta + \sin^2\theta) + \cos^2\phi$$
$$= \sin^2\phi + \cos^2\phi$$
$$= 1$$
$$g_{12} = R\sin\phi\cos\phi\cos^2\theta + R\cos\phi\sin\phi\sin^2\theta - R\sin\phi\cos\phi$$
$$= R\sin\phi\cos\phi(\cos^2\theta + \sin^2\theta) - R\sin\phi\cos\phi$$
$$= 0$$
$$g_{13} = -R\sin\theta\cos\theta\sin^2\phi + R\sin\theta\cos\theta\sin^2\phi + 0$$
$$= 0$$
$$g_{22} = R^2\cos^2\phi\cos^2\theta + R^2\cos^2\phi\sin^2\theta + R^2\sin^2\phi$$
$$= R^2\cos^2\phi(\cos^2\theta + \sin^2\theta) + R^2\sin^2\phi$$
$$= R^2$$
$$g_{23} = -R^2\sin\phi\cos\phi\sin\theta\cos\theta + R^2\sin\phi\cos\phi\sin\theta\cos\theta + 0$$
$$= 0$$
$$g_{33} = R^2\sin^2\phi\sin^2\theta + R^2\sin^2\phi\cos^2\theta + 0$$
$$= R^2\sin^2\phi(\sin^2\theta + \cos^2\theta)$$
$$= R^2\sin^2\phi \quad .$$
$$g_{21} = g_{31} = g_{32} = 0$$

$$g_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & R^2 & 0 \\ 0 & 0 & R^2\sin^2\phi \end{bmatrix}$$

(iv) $g^{ij} = (g_{ij})^{-1}$

$$g^{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{R^2} & 0 \\ 0 & 0 & \frac{1}{R^2\sin^2\phi} \end{bmatrix}$$

(v) $ds^2 = g_{ij}dx^i dx^j$

$$ds^2 = g_{11}(dx^1)^2 + g_{22}(dx^2)^2 + g_{33}(dx^3)^2$$

since all other components of g_{ij} are zero

$$ds^2 = (dR)^2 + R^2(d\phi)^2 + R^2\sin^2\phi(d\theta)^2$$

(vi) $$\underset{\sim}{g}_i = \frac{\partial Z_A}{\partial x^i} \underset{\sim}{E}_A$$

$$\underset{\sim}{g}_1 = \sin\phi\cos\theta\underset{\sim}{E}_1 + \sin\phi\sin\theta\underset{\sim}{E}_2 + \cos\phi\underset{\sim}{E}_3$$
$$\underset{\sim}{g}_2 = R\cos\phi\cos\theta\underset{\sim}{E}_1 + R\cos\phi\sin\theta\underset{\sim}{E}_2 - R\sin\phi\underset{\sim}{E}_3$$
$$\underset{\sim}{g}_3 = -R\sin\phi\sin\theta\underset{\sim}{E}_1 + R\sin\phi\cos\theta\underset{\sim}{E}_2$$

$$g_{ij} = \underset{\sim}{g}_i \cdot \underset{\sim}{g}_j$$
$$g_{11} = \sin^2\phi\cos^2\theta + \sin^2\phi\sin^2\theta + \cos^2\phi$$
$$= \sin^2\phi(\cos^2\theta + \sin^2\theta) + \cos^2\phi = \sin^2\phi + \cos^2\phi$$
$$= 1$$
$$g_{12} = R\sin\phi\cos\phi\cos^2\theta + R\sin\phi\cos\phi\sin^2\theta - R\sin\phi\cos\phi$$
$$= R\sin\phi\cos\phi(\cos^2\theta + \sin^2\theta) - R\sin\phi\cos\phi$$
$$= 0$$
$$g_{13} = -R\sin\theta\cos\theta\sin^2\phi + R\sin\theta\cos\theta\sin^2\phi = 0$$
$$g_{22} = R^2\cos^2\phi\cos^2\theta + R^2\cos^2\phi\sin^2\theta + R^2\sin^2\phi$$
$$= R^2\cos^2\phi(\cos^2\theta + \sin^2\theta) + R^2\sin^2\phi$$
$$= R^2$$
$$g_{23} = -R^2\sin\phi\cos\phi\sin\theta\cos\theta + R^2\sin\phi\cos\phi\sin\theta\cos\theta$$
$$= 0$$
$$g_{33} = R^2\sin^2\phi\sin^2\theta + R^2\sin^2\phi\cos^2\theta$$
$$= R^2\sin^2\phi(\sin^2\theta + \cos^2\theta)$$
$$= R^2\sin^2\phi$$

$$g_{21} = g_{31} = g_{32} = 0$$

$$g_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & R^2 & 0 \\ 0 & 0 & R^2\sin^2\phi \end{bmatrix}$$

(vii) $$\underset{\sim}{g}^i = g^{ij}\underset{\sim}{g}_j$$

$$\underset{\sim}{g}^1 = g^{11}\underset{\sim}{g}_1 + g^{12}\underset{\sim}{g}_2 + g^{13}\underset{\sim}{g}_3 = 1(\sin\phi\cos\theta\underset{\sim}{E}_1 + \sin\phi\sin\theta\underset{\sim}{E}_2 + \cos\phi\underset{\sim}{E}_3)$$
$$= \sin\phi\cos\theta\underset{\sim}{E}_1 + \sin\phi\sin\theta\underset{\sim}{E}_2 + \cos\phi\underset{\sim}{E}_3$$
$$\underset{\sim}{g}^2 = g^{21}\underset{\sim}{g}_1 + g^{22}\underset{\sim}{g}_2 + g^{23}\underset{\sim}{g}_3 = \frac{1}{R^2}(R\cos\phi\cos\theta\underset{\sim}{E}_1 + R\cos\phi\sin\theta\underset{\sim}{E}_2 - R\sin\phi\underset{\sim}{E}_3)$$
$$= \frac{1}{R}\cos\phi\cos\theta\underset{\sim}{E}_1 + \frac{1}{R}\cos\phi\sin\theta\underset{\sim}{E}_2 - \frac{1}{R}\sin\phi\underset{\sim}{E}_3$$
$$\underset{\sim}{g}^3 = g^{31}\underset{\sim}{g}_1 + g^{32}\underset{\sim}{g}_2 + g^{33}\underset{\sim}{g}_3 = \frac{1}{R^2\sin^2\phi}(-R\sin\phi\sin\theta\underset{\sim}{E}_1 + R\sin\phi\cos\theta\underset{\sim}{E}_2)$$

$$= \frac{-\sin\theta}{R\sin\phi}\underset{\sim}{E}_1 + \frac{\cos\theta}{R\sin\phi}\underset{\sim}{E}_2$$

(viii) $$\underset{\sim}{g}_{\underline{i}} = \frac{\underset{\sim}{g}_i}{\sqrt{g_{(ii)}}}$$

$$g_{\underline{1}} = \frac{\underset{\sim}{g}_1}{\sqrt{g_{11}}} = \underset{\sim}{g}_1 = \sin\phi\cos\theta\underset{\sim}{E}_1 + \sin\phi\sin\theta\underset{\sim}{E}_2 + \cos\phi\underset{\sim}{E}_3$$

$$g_{\underline{2}} = \frac{\underset{\sim}{g}_2}{\sqrt{g_{22}}} = \frac{1}{R}\underset{\sim}{g}_2 = \cos\phi\cos\theta\underset{\sim}{E}_1 + \cos\phi\sin\theta\underset{\sim}{E}_2 - \sin\phi E_3$$

$$g_{\underline{3}} = \frac{\underset{\sim}{g}_3}{\sqrt{g_{33}}} = \frac{1}{R\sin\phi}\underset{\sim}{g}_3 = -\sin\theta\underset{\sim}{E}_1 + \cos\theta\underset{\sim}{E}_2$$

Denoting $g_{\underline{i}}$ by $\underset{\sim}{e}_R$, $\underset{\sim}{e}_\phi$, and $\underset{\sim}{e}_\theta$ we have

$$\underset{\sim}{e}_R = \sin\phi\cos\theta\underset{\sim}{E}_1 + \sin\phi\sin\theta\underset{\sim}{E}_2 + \cos\phi\underset{\sim}{E}_3$$

$$\underset{\sim}{e}_\phi = \cos\phi\cos\theta\underset{\sim}{E}_1 + \cos\phi\sin\theta\underset{\sim}{E}_2 - \sin\phi\underset{\sim}{E}_3$$

$$\underset{\sim}{e}_\theta = -\sin\theta\underset{\sim}{E}_1 + \cos\theta\underset{\sim}{E}_2$$

(ix) $$[ij,k] = \tfrac{1}{2}(g_{ik,j} + g_{kj,i} - g_{ij,k})$$

The only non-zero Christoffel symbols of the first kind are those that contain the terms $g_{22,1}$ $g_{33,1}$ and $g_{33,2}$

$$g_{22,1} = 2R \qquad g_{33,1} = 2R\sin^2\phi \qquad g_{33,2} = 2R^2\sin\phi\cos\phi$$

$$[22,1] = \tfrac{1}{2}(g_{21,2} + g_{12,2} - g_{22,1}) = -\tfrac{1}{2}(2R) = -R$$

$$[33,1] = \tfrac{1}{2}(g_{31,3} + g_{13,3} - g_{33,1}) = -\tfrac{1}{2}(2R\sin^2\phi) = -R\sin^2\phi$$

$$[21,2] = [12,2] = \tfrac{1}{2}(g_{12,2} + g_{22,1} - g_{12,2}) = \tfrac{1}{2}(2R) = R$$

$$[33,2] = \tfrac{1}{2}(g_{32,3} + g_{23,3} - g_{33,2}) = -\tfrac{1}{2}(2R^2\sin\phi\cos\phi) = -R^2\sin\phi\cos\phi$$

$$[13,3] = [31,3] = \tfrac{1}{2}(g_{13,3} + g_{33,1} - g_{13,3}) = \tfrac{1}{2}(2R\sin^2\phi) = R\sin^2\phi$$

$$[23,3] = [32,3] = \tfrac{1}{2}(g_{23,3} + g_{33,2} - g_{23,3}) = \tfrac{1}{2}(2R^2\sin\phi\cos\phi) = R^2\sin\phi\cos\phi$$

Hence

$$[ij,1] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -R & 0 \\ 0 & 0 & -R\sin^2\phi \end{bmatrix} \qquad [ij,2] = \begin{bmatrix} 0 & R & 0 \\ R & 0 & 0 \\ 0 & 0 & -R^2\sin\phi\cos\phi \end{bmatrix}$$

$$[ij,3] = \begin{bmatrix} 0 & 0 & R\sin^2\phi \\ 0 & 0 & R^2\sin\phi\cos\phi \\ R\sin^2\phi & R^2\sin\phi\cos\phi & 0 \end{bmatrix}$$

(x) $\{{}_i{}^k{}_j\} = g^{k\ell}[ij,\ell]$

The only non-zero Christoffel symbols of the second kind are

$$\{{}_2{}^1{}_2\} = g^{1\ell}[22,\ell] = g^{11}[22,1] + g^{12}[22,2] + g^{13}[22,3]$$

$$= 1\ (-R) + 0 + 0 = -R$$

$$\{{}_3{}^1{}_3\} = g^{1\ell}[33,\ell] = g^{11}[33,1] + g^{12}[33,2] + g^{13}[33,3]$$

$$= 1(-R\sin^2\phi) + 0 + 0 = -R\sin^2\phi$$

$$\{{}_1{}^2{}_2\} = \{{}_2{}^2{}_1\} = g^{2\ell}[12,\ell] = g^{21}[12,1] + g^{22}[12,2] + g^{23}[12,3]$$

$$= 0 + \frac{1}{R^2}(R) + 0 = \frac{1}{R}$$

$$\{{}_3{}^2{}_3\} = g^{2\ell}[33,\ell] = g^{21}[33,1] + g^{22}[33,2] + g^{23}[33,3]$$

$$= 0 + \frac{1}{R^2}(-R^2\sin\phi\cos\phi) + 0 = -\sin\phi\cos\phi$$

$$\{{}_1{}^3{}_3\} = \{{}_3{}^3{}_1\} = g^{3\ell}[13,\ell] = g^{31}[13,1] + g^{32}[13,2] + g^{33}[13,3]$$

$$= 0 + 0 + \frac{1}{R^2\sin^2\phi}(R\sin^2\phi) = \frac{1}{R}$$

$$\left\{{}_{2}{}^{3}{}_{3}\right\} = \left\{{}_{3}{}^{3}{}_{2}\right\} = g^{3\ell}[23,\ell] = g^{31}[23,1] + g^{32}[23,2] + g^{33}[23,3]$$

$$= 0 + 0 + \frac{1}{R^2\sin^2\phi}\,(R^2\sin\phi\cos\phi) = \frac{\cos\phi}{\sin\phi} = \cot\phi$$

Hence

$$\left\{{}_{i}{}^{1}{}_{j}\right\} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -R & 0 \\ 0 & 0 & -R\sin^2\phi \end{bmatrix} \quad \left\{{}_{i}{}^{2}{}_{j}\right\} = \begin{bmatrix} 0 & \frac{1}{R} & 0 \\ \frac{1}{R} & 0 & 0 \\ 0 & 0 & -\sin\phi\cos\phi \end{bmatrix}$$

$$\left\{{}_{i}{}^{3}{}_{j}\right\} = \begin{bmatrix} 0 & 0 & \frac{1}{R} \\ 0 & 0 & \cot\phi \\ \frac{1}{R} & \cot\phi & 0 \end{bmatrix}$$

(xi)

$$e_{\underline{ij}} = \frac{\tfrac{1}{2}[(u_{\underline{i}}\sqrt{g_{(ii)}})_{,j} + (u_{\underline{j}}\sqrt{g_{(jj)}})_{,i} - 2\left\{{}_{i}{}^{k}{}_{j}\right\}u_{\underline{k}}\sqrt{g_{(kk)}}]}{\sqrt{g_{(ii)}}\ \sqrt{g_{(jj)}}}$$

$$e_{\underline{11}} = \frac{\tfrac{1}{2}[(u_R.1)_{,R} + (u_R.1)_{,R} - 2\cancelto{0}{\left\{{}_{1}{}^{k}{}_{1}\right\}}\ u_{\underline{k}}\sqrt{g_{(kk)}}]}{(1)(1)}$$

$$= \tfrac{1}{2}(2u_{R,R})$$

$$= u_{R,R}$$

$$e_{\underline{12}} = \frac{\tfrac{1}{2}\,[(u_R.1)_{,\phi} + (u_\phi.R)_{,R} - 2\left\{{}_{1}{}^{k}{}_{2}\right\}u_{\underline{k}}\sqrt{g_{(kk)}}]}{(1)(R)}$$

$$= \frac{\tfrac{1}{2}\,[u_{R,\phi} + u_{\phi,R}.R + u_\phi - 2(\frac{1}{R})u_\phi(R)]}{R}$$

$$= \tfrac{1}{2}\left(\frac{u_{R,\phi}}{R} + u_{\phi,R} - \frac{u_\phi}{R}\right)$$

$$e_{13} = \frac{\tfrac{1}{2}\,[(u_R.1)_{,\theta} + (u_\theta R\sin\phi)_{,R} - 2\{{}_{1}{}^{k}{}_{3}\}u_{\underline{k}}\sqrt{g_{(kk)}}]}{(1)(R\sin\phi)}$$

$$= \frac{\tfrac{1}{2}\,(u_{R,\theta} + u_{\theta,R}R\sin\phi + u_\theta\sin\phi - 2(\tfrac{1}{R})u_\theta(R\sin\phi))}{R\sin\phi}$$

$$= \tfrac{1}{2}\left(\frac{u_{R,\theta}}{R\sin\phi} + u_{\theta,R} - \frac{u_{\theta}}{R}\right)$$

$$e_{22} = \frac{\tfrac{1}{2}\,[(u_\phi.R)_{,\phi} + (u_\phi.R)_{,\phi} - 2\{{}_{2}{}^{k}{}_{2}\}u_{\underline{k}}\sqrt{g_{(kk)}}}{(R)(R)}$$

$$= \tfrac{1}{2}\,\frac{(2u_{\phi,\phi}R - 2(-R)u_R(1))}{R^2}$$

$$= \frac{u_{\phi,\phi}}{R} + \frac{u_R}{R}$$

$$e_{23} = \frac{\tfrac{1}{2}[(u_\phi.R)_{,\theta} + (u_\theta.R\sin\phi)_{,\phi} - 2\{{}_{2}{}^{k}{}_{3}\}u_{\underline{k}}\sqrt{g_{(kk)}}]}{(R)(R\sin\phi)}$$

$$= \frac{\tfrac{1}{2}[u_{\phi,\theta}R + u_{\theta,\phi}\,R\sin\phi + u_\theta R\cos\phi - 2(\cot\phi)u_\theta(R\sin\phi)]}{R^2\sin\phi}$$

$$e_{23} = \tfrac{1}{2}\left(\frac{u_{\phi,\theta}}{R\sin\phi} + \frac{u_{\theta,\phi}}{R} - \frac{u_\theta}{R}\cot\phi\right)$$

$$e_{33} = \frac{\tfrac{1}{2}\,[(u_\theta R\sin\phi)_{,\theta} + (u_\theta R\sin\phi)_{,\theta} - 2\{{}_{3}{}^{k}{}_{3}\}u_{\underline{k}}\sqrt{g_{(kk)}}]}{(R\sin\phi)(R\sin\phi)}$$

$$= \frac{\tfrac{1}{2}\,(2u_{\theta,\theta}R\sin\phi - 2(-R\sin^2\phi)u_R(1) - 2(-\sin\phi\cos\phi)u_\phi(R))}{R^2\sin^2\phi}$$

$$= \frac{u_{\theta,\theta}}{R\sin\phi} + \frac{u_R}{R} + \frac{u_\phi\cot\phi}{R}$$

(xii) Let $\underset{\sim}{S} = \underset{\approx}{\sigma}.\nabla$

$$S_i = \sigma^{ij}/_j\sqrt{g_{(ii)}}$$

$$= [\sigma^{ij}{}_{,j} + \sigma^{kj}\{{}^{\,i}_{k\,j}\} + \sigma^{ki}\{{}^{\,j}_{k\,j}\}]\sqrt{g_{(ii)}}$$

$$S_{\underline{i}} = [\left(\frac{\sigma_{\underline{ij}}}{\sqrt{g_{(ii)}g_{(jj)}}}\right)_{,j} + \frac{\sigma_{\underline{kj}}}{\sqrt{g_{(kk)}g_{(jj)}}}\{{}^{\,i}_{k\,j}\}$$

$$+ \frac{\sigma_{\underline{ki}}}{\sqrt{g_{(kk)}g_{(ii)}}}\{{}^{\,j}_{k\,j}\}]\sqrt{g_{(ii)}}$$

$$S_{\underline{1}} = [\left(\frac{\sigma_{RR}}{(1)(1)}\right)_{,R} + \left(\frac{\sigma_{R\phi}}{(1)(R)}\right)_{,\phi} + \left(\frac{\sigma_{R\theta}}{(1)(R\sin\phi)}\right)_{,\theta} + \frac{\sigma_{\phi\phi}}{(R)(R)}(-R)$$

$$+ \frac{\sigma_{\theta\theta}(-R\sin^2\phi)}{(R\sin\phi)(R\sin\phi)} + \frac{\sigma_{RR}}{(1)(1)}\left(\frac{1}{R} + \frac{1}{R}\right) + \frac{\sigma_{\phi R}}{(R)(1)}(\cot\phi)](1)$$

$$= \sigma_{RR,R} + \frac{\sigma_{R\phi,\phi}}{R} + \frac{\sigma_{R\theta,\theta}}{R\sin\phi} - \frac{\sigma_{\phi\phi}}{R} - \frac{\sigma_{\theta\theta}}{R} + \frac{2\sigma_{RR}}{R} + \frac{\sigma_{\phi R}\cot\phi}{R}$$

$$S_{\underline{2}} = [\left(\frac{\sigma_{\phi R}}{(R)(1)}\right)_{,R} + \left(\frac{\sigma_{\phi\phi}}{(R)(R)}\right)_{,\phi} + \left(\frac{\sigma_{\phi\theta}}{(R)(R\sin\phi)}\right)_{,\theta} + \frac{2\sigma_{R\phi}}{(1)(R)}\left(\frac{1}{R}\right)$$

$$+ \frac{\sigma_{\theta\theta}}{(R^2\sin^2\phi)}(-\sin\phi\cos\phi) + \frac{\sigma_{R\phi}}{(1)(R)}\left(\frac{1}{R} + \frac{1}{R}\right) + \frac{\sigma_{\phi\phi}}{(R)(R)}(\cot\phi)](R)$$

$$= \left(\frac{\sigma_{\phi R,R}}{R} - \frac{\sigma_{\phi R}}{R^2} + \frac{\sigma_{\phi\phi,\phi}}{R^2} + \frac{\sigma_{\phi\theta,\theta}}{R^2\sin\phi} + \frac{2\sigma_{R\phi}}{R^2} - \frac{\sigma_{\theta\theta}}{R^2}\cot\phi + \frac{2\sigma_{R\phi}}{R^2}\right.$$

$$\left. + \frac{\sigma_{\phi\phi}\cot\phi}{R^2}\right)(R)$$

$$= \sigma_{\phi R,R} + \frac{3\sigma_{\phi R}}{R} + \frac{\sigma_{\phi\phi,\phi}}{R} + \frac{\sigma_{\phi\theta,\theta}}{R\sin\phi} - \frac{\sigma_{\theta\theta}}{R}\cot\phi + \frac{\sigma_{\phi\phi}}{R}\cot\phi$$

$$S_{\underset{\sim}{3}} = [\left(\frac{\sigma_{\theta R}}{(R\sin\phi)(1)}\right)_{,R} + \left(\frac{\sigma_{\theta\phi}}{(R\sin\phi)(R)}\right)_{,\phi} + \left(\frac{\sigma_{\theta\theta}}{R^2\sin^2\phi}\right)_{,\theta} + \frac{2\sigma_{R\theta}}{(1)(R\sin\phi)}(\frac{1}{R})$$

$$+ \frac{2\sigma_{\phi\theta}}{(R)(R\sin\phi)}(\cot\phi) + \frac{\sigma_{R\theta}}{(1)(R\sin\phi)}(\frac{1}{R} + \frac{1}{R}) + \frac{\sigma_{\phi\theta}}{(R)(R\sin\phi)}(\cot\phi)]R\sin\phi$$

$$= \left(\frac{\sigma_{\theta R,R}}{R\sin\phi} - \frac{\sigma_{\theta R}}{R^2\sin\phi} + \frac{\sigma_{\theta\phi,\phi}}{R^2\sin\phi} - \frac{\sigma_{\theta\phi}}{R^2}\frac{\cot\phi}{\sin\phi} + \frac{\sigma_{\theta\theta,\theta}}{R^2\sin^2\phi} + \frac{2\sigma_{R\theta}}{R^2\sin\phi}\right.$$

$$\left. + \frac{2\sigma_{\phi\theta}}{R^2\sin\phi}\cot\phi + \frac{2\sigma_{R\theta}}{R^2\sin\phi} + \frac{\sigma_{\phi\theta}}{R^2\sin\phi}\cot\phi\right)(R\sin\phi)$$

$$= \sigma_{\theta R,R} + \frac{3\sigma_{R\theta}}{R} + \frac{\sigma_{\theta\phi,\phi}}{R} + \frac{2\sigma_{\phi\theta}}{R}\cot\phi + \frac{\sigma_{\theta\theta,\theta}}{R\sin\phi}$$

In summary we have

$$e_{\underline{ij}} \doteq \begin{bmatrix} e_{RR} & e_{R\phi} & e_{R\theta} \\ e_{\phi R} & e_{\phi\phi} & e_{\phi\theta} \\ e_{\theta R} & e_{\theta\phi} & e_{\theta\theta} \end{bmatrix} = \begin{bmatrix} u_{R,R} & \frac{1}{2}(\frac{u_{R,\phi}}{R} + u_{\phi,R} - \frac{u_\phi}{R}) & \frac{1}{2}(\frac{u_{R,\theta}}{R\sin\phi} + u_{\theta,R} - \frac{u_\theta}{R}) \\ & \frac{u_{\phi,\phi}}{R} + \frac{u_R}{R} & \frac{1}{2}(\frac{u_{\phi,\theta}}{R\sin\phi} + \frac{u_{\theta,\phi}}{R} - \frac{u_\theta}{R}\cot\phi) \\ \text{Symmetric} & & \frac{u_{\theta,\theta}}{R\sin\phi} + \frac{u_R}{R} + \frac{u_\phi\cot\phi}{R} \end{bmatrix}$$

$$\underset{\approx}{\sigma}.\nabla = [\frac{1}{R^2}(R^2\sigma_{RR})_{,R} + \frac{1}{R\sin\phi}(\sigma_{R\phi}\sin\phi)_{,\phi} + \frac{1}{R\sin\phi}\sigma_{R\theta,\theta} - \frac{1}{R}(\sigma_{\theta\theta} + \sigma_{\phi\phi})]\underset{\sim}{e}_R$$

$$+ [\frac{1}{R^2}(R^2\sigma_{R\phi})_{,R} + \frac{1}{R\sin\phi}(\sigma_{\phi\phi}\sin\phi)_{,\phi} + \frac{1}{R\sin\phi}\sigma_{\phi\theta,\theta} + \frac{1}{R}\sigma_{R\phi} - \frac{1}{R}\sigma_{\theta\theta}\cot\phi]\underset{\sim}{e}_\phi$$

$$+ [\frac{1}{R^2}(R^2\sigma_{R\theta})_{,R} + \frac{1}{R\sin\phi}(\sigma_{\phi\theta}\sin\phi)_{,\phi} + \frac{1}{R\sin\phi}\sigma_{\theta\theta,\theta} + \frac{1}{R}(\sigma_{R\theta} + \sigma_{\phi\theta}\cot\phi]\underset{\sim}{e}_\theta$$

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Points in brackets denote weight.

1. [20] Simplify the following expressions (if possible):

(i) $\delta_{ij}a_j$ (ii) $\delta_{ij}x_ix_j$ (iii) $a_{ij}x_ix_j$ with $a_{ij} = a_{ji}$

(iv) $a_{ij}x_ix_j$ with $a_{ij} = -a_{ji}$ (v) $(x_ix_j),_j$ (vi) $(x_ix_j),_i$

(vii) $(\delta_{ij} + c_{ij})(\delta_{ik} + c_{ik}) - \delta_{jk} - \delta_{mn}c_{mj}c_{nk}$

2. Let F be a scalar function of a vector field v.

(i) [8] Define what is meant by the gradient of F with respect to v.

(ii) [5] What is meant by an orthonormal basis?

(iii) [7] Use your definition in (i) to obtain an expression in indicial form for the gradient if an orthonormal basis is used.

(iv) [10] Use your result from (iii) to determine the gradient with respect to **v** of

(a) $F = \mathbf{v} \cdot \mathbf{v}$ (b) $F = \mathbf{v} \cdot \mathbf{T} \cdot \mathbf{v}$ for a constant tensor **T**.

Give your answers in both direct and indicial notation.

3. Suppose that $\int_R \phi\nabla dV = \int_{\partial R} \phi\mathbf{n}\, dS$

(i) [6] In this context what do each of the following terms denote: ∇, R, ∂R, dV, dS and **n**

(ii) [4] What conclusion can you draw by letting $\phi = 1$.

(iii) [7] Use the original equation and derive the corresponding equation where the function is a vector rather than a scalar.

(iv) [3] Use the result of (iii) to derive the divergence theorem for a vector field.

(v) 10] With either direct or indicial notation, use the divergence theorem to show that

$$\int_{\partial R} \phi(\psi\nabla) \cdot \mathbf{n}\, dS = \int_R \left[(\phi\nabla) \cdot (\psi\nabla) + \phi(\psi\nabla^2)\right] dV$$

4. Material elements in the undeformed and deformed configuration are related by $d\mathbf{r} = \mathbf{F} \cdot d\mathbf{R}$.

(i) [8] From this definition for **F**, give **F** in terms of a gradient.

(ii) [12] If L denotes a line segment in the deformed configuration, derive an equivalent expression for $\frac{d}{dt}\int_L \mathbf{v} \cdot d\mathbf{r}$ in which the time derivative is inside the integral.

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Numbers in brackets denote weights for parts of problems.

1. (a) [5] If a tensor is represented as $\mathbf{T} = a\mathbf{e}_1 \otimes \mathbf{e}_3 + b\mathbf{e}_2 \otimes \mathbf{e}_2$ give the matrix of its components in the basis $\mathbf{e}_i \otimes \mathbf{e}_j$.

Suppose $\mathbf{F} = \frac{\partial x_i}{\partial X_A}\mathbf{e}_i \otimes \mathbf{E}_A$ and $\mathbf{B} = \mathbf{E}_1 \otimes \mathbf{e}_1 + \mathbf{E}_1 \otimes \mathbf{e}_2$ where $\frac{\partial x_i}{\partial X_A} = \begin{array}{c} i=1 \\ 2 \end{array}\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

(b) [5] Determine $\mathbf{G} = \mathbf{F} \cdot \mathbf{B}$. What is the natural basis for $\mathbf{G}$ based on this expression.

(c) [10] Determine $\frac{\partial x_i}{\partial X_A}\frac{\partial x_i}{\partial X_B}$ and associate each free index with either a row or a column of the resulting matrix.

2. Suppose x^i represents a general curvilinear coordinate system.

(a) [5] How are covariant and contravariant bases defined?

(b) [5] Give general representations for vectors and tensors using all possible combinations of bases.

(c) [10] Define the transformation relations between the bases and derive the equation which relates covariant and contravariant components of a vector.

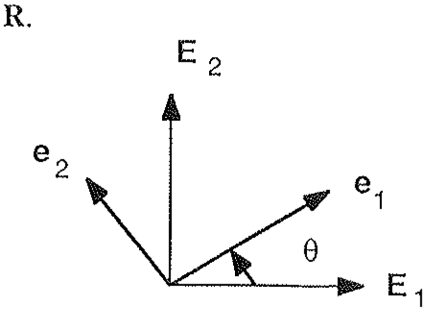
3. [30] State Euler's First and Second Laws. Outline the assumptions and steps necessary to arrive at Cauchy's First and Second Equations of Motion.

4. Consider the planar deformation $x_1 = a X_1$, $x_2 = X_2 + bX_1$ where the base vectors are related as shown in the sketch.

(a) [10] Obtain the displacement vector, $\mathbf{u}$. Express this vector two ways: (i) In terms of X_A and $\mathbf{E}_A$, and (ii) in terms of x_i and $\mathbf{e}_i$.

(b) [15] Obtain the components (specify the basis) of $\mathbf{F}$, $\mathbf{C} = \mathbf{F}^T \cdot \mathbf{F}$ and $\mathbf{E} = \frac{1}{2}(\mathbf{C} - \mathbf{I})$

(c) [5] If $\mathbf{F} = \mathbf{R} \cdot \mathbf{U}$ and $\mathbf{U}^2 = \mathbf{C}$ how would you obtain $\mathbf{R}$.



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Numbers in square brackets denote weights.

1. For each of the following expressions, give a definition and provide an illustrative example:

[18] (i) tensor product, (ii) second order tensor, (iii) direct notation, (iv) indicial notation, (v) trace, and (vi) invariant.

2. Suppose $\underset{\sim}{v} = F(r)\,\underset{\sim}{r}$, i.e., $\underset{\sim}{v}$ is a vector consisting of the product of the position vector, $\underset{\sim}{r}$, and a scalar function, F, which depends on the magnitude of $\underset{\sim}{r}$ [$r = (\underset{\sim}{r}\cdot\underset{\sim}{r})^{1/2}$]

(i) Determine (a) $\underset{\sim}{v}\nabla$, (b) curl $\underset{\sim}{v}$ and (c) $\underset{\sim}{v}\cdot\nabla$
[9] in terms of F and its derivative.

(ii) If a surface is defined by $F(r) = c$, a
[3] constant, obtain the expression for a unit vector normal to the surface.

3. Suppose the components of $\underset{=}{T}$ are given with respect to the basis $\underset{\sim}{e}_i \otimes \underset{\sim}{e}_j$.

(a) Outline the procedure for obtaining the eigenvalues
[10] and eigenvectors of $\underset{=}{T}$

[2] (b) What is a principal basis?

[2] (c) ~~How~~ What are the components of $\underset{=}{T}$ with respect to the principal basis?

(d) Under what conditions would $\underset{=}{T}^{1/3}$ exist
[6] and how would you obtain the components of $\underset{=}{T}^{1/3}$ with respect to the basis $\underset{\sim}{e}_i \otimes \underset{\sim}{e}_j$?

4\. The deformation gradient and the Eulerian strain tensor are defined by the equations

$$d\underset{\sim}{r} = \underset{=}{F} \cdot d\underset{\sim}{R} \qquad ds^2 - dS^2 = d\underset{\sim}{r} \cdot \underset{=}{e} \cdot d\underset{\sim}{r}$$

[4] (a) Define each of the terms in these equations.

(b) If the displacement vector is defined to be

[4] $\underset{\sim}{u} = \underset{\sim}{r} - \underset{\sim}{R}$, obtain an expression for $\underset{=}{F}$ in terms of $\underset{\sim}{u}$.

(c) Develop a physical interpretation of e_{33} in a

[4] rectangular Cartesian system.

[4] (d) Derive the relation between $\underset{=}{e}$ and $\underset{=}{F}$.

5\. ME 512 students do (a) only. ME 402 students do (b) only.

(a) Let x^i denote a curvilinear coordinate system.

[20] Start with the basic definition of a gradient and derive an expression for the gradient of a vector in the curvilinear system. Your answer should involve a "Christoffel" – type term.

(b) Derive the transformation equation which relates

[20] components of a tensor in two rectangular Cartesian systems.

6\. The First Law of Thermodynamics implies that for infinitesimal deformations, two of the constitutive relations must be

$$\underset{=}{\sigma} = \rho \frac{\partial U}{\partial \underset{=}{e}} \qquad \theta = \frac{\partial U}{\partial s}$$

where U - specific thermal energy, $\underset{=}{\sigma}$ - stress, $\underset{=}{e}$ - strain, θ - temp, s - entropy, ρ - mass density.

(a) If $U = U_1(\underset{=}{e}, \underset{=}{e}^p) + U_2(s)$ where $U_1 = \frac{1}{2\rho}(\underset{=}{e} - \underset{=}{e}^p) : \underset{\equiv}{E} : (\underset{=}{e} - \underset{=}{e}^p)$

[5] $U_2 = c\, e^{s/s^*}$ with ρ, $\underset{\equiv}{E}$, c, s^* constant

determine $\underset{=}{\sigma}$ and θ.

(b) Invert one of the equations to obtain s as a function

[4] of θ. Substitute for s in U to obtain $U^*(\underset{=}{e}, \underset{=}{e}^p, \theta)$.

(c) Perform a Legendre (or contact) transformation on U

[4] to obtain the Helmholtz Free Energy $A(\underset{=}{e}, \underset{=}{e}^p, \theta)$. Are U^* and A the same functions?

[1] (d) Determine $\dfrac{\partial U}{\partial \underset{=}{e}^p}$.

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Weights for problems are given in square brackets.

1. [20] For each of the following, give a definition and provide an illustrative example: (i) tensor product, (ii) second order tensor, (iii) indicial notation, (iv) gradient of a vector, and (v) invariant of a second order tensor.

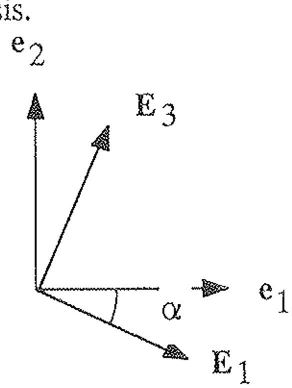
2. [20] Suppose $\mathbf{v} = F(r)\mathbf{r}$, i.e., $\mathbf{v}$ is a vector formed from the product of the position vector, $\mathbf{r}$, and a scalar function, F, which depends on the magnitude of $\mathbf{r}$ [$r = (\mathbf{r}\cdot\mathbf{r})^{1/2}$].

(i) Determine (a) $\mathbf{v}\nabla$, (b) curl $\mathbf{v}$, and (c) $\mathbf{v}\cdot\nabla$ in terms of F and its derivative.

(ii) If a surface is defined by $F(r) = c$, a constant, obtain a unit vector normal to the surface.

3. [20] Describe rather fully how you would obtain $\mathbf{T}^{1/3}$. Is it necessary to place any restrictions on the nature of the second order tensor, $\mathbf{T}$?

4. [20] Derive the transformation equations that relate components of vectors and tensors in two bases. Suppose two bases are related as shown in the sketch. A vector is given by $\mathbf{v} = \mathbf{e}_1 + 2\mathbf{e}_2 - \mathbf{e}_3$. Using your derived relation, determine the expression for the vector in the $\mathbf{E}_A$ basis.



$\mathbf{E}_2 = -\mathbf{e}_3$

5. [20] Given that ϕ is a scalar function, and that volume and surface integrals are related as follows: $\int_R \phi\nabla dV = \int_{\partial R} \phi \mathbf{n} dS$. Define all terms in the equation and use the equation to derive the divergence theorem for vectors.

Problem weights are given in square brackets.

1. [10] (i) What is the form that boundary conditions must take? Give an example of each possibility.

[10] (ii) What is the Principle of Material Frame Indifference? Give one example of an equation that satisfies the principle and an example that doesn't.

ME 512 Only (iii) In curvilinear coordinates, how are bases constructed? Give all the resulting representations for vectors and tensors. Describe the rules to be followed for the indicial notation.

[10]

ME 402 Only (iii) Describe the procedure for taking a time derivative inside the volume integral expressed in the current configuration.

2. [10] (i) Start with the definition of a gradient. Obtain expressions for $\mathbf{F}$ and $\dot{\mathbf{F}}$ where $d\mathbf{r} = \mathbf{F} \cdot d\mathbf{R}$.

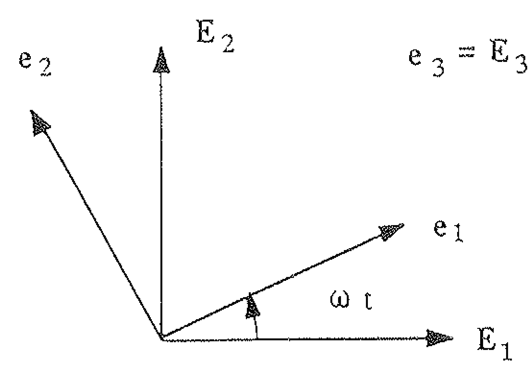
[10] (ii) Derive the relation between $\mathbf{e}$ and $\mathbf{F}$ where

$$ds^2 - dS^2 = 2 d\mathbf{r} \cdot \mathbf{e} \cdot d\mathbf{r} \qquad ds^2 = d\mathbf{r} \cdot d\mathbf{r} \qquad dS^2 = d\mathbf{R} \cdot d\mathbf{R}.$$

3. [25] Start with Euler's First and Second Laws. Outline the assumptions and steps required to arrive at Cauchy's First and Second Equations of Motion.

4. For constant ω, the base vectors $\mathbf{e}_i$ and $\mathbf{E}_A$ are related as shown. For constant α and β, a deformation is given by

$$x_1 = X_1 + \beta t X_2$$
$$x_2 = X_2$$
$$x_3 = X_3 e^{-\alpha t}$$



[15] (i) Obtain the components of $\mathbf{F}$ with respect to the bases: $\mathbf{E}_A \otimes \mathbf{E}_B$, $\mathbf{e}_i \otimes \mathbf{E}_B$, $\mathbf{e}_i \otimes \mathbf{e}_j$. Is the volume changing? Why?

[10] (ii) Determine $\mathbf{v}$, $\mathbf{L} = \mathbf{v}\nabla$, and $\dot{\mathbf{F}}$.

Problem weights are given in square brackets.

1. (A) [8] Many expressions can be written in direct or indicial. Give the equivalent other notation for each of the following:

$$\text{(i) } T_{ij}v_ju_i\,, \quad \text{(ii) } \mathbf{T}\cdot\mathbf{S}^T, \quad \text{(iii) } u_iv_j\,, \quad \text{(iv) } \mathbf{T}:\mathbf{S}$$

(B) [8] Indicate whether the result is a scalar, vector or second order tensor and give the corresponding expression in indicial form for each of the following (capital letters are second order tensors; lower case letters are vectors, $\boldsymbol{\varepsilon}$ is the third-order alternating tensor):

$$\text{(i) } C_{13}(\mathbf{R}\otimes\mathbf{T}^T) \qquad \text{(ii) } \mathbf{T}^T\cdot\nabla \qquad \text{(iii) } \mathbf{u}\cdot\boldsymbol{\varepsilon}\cdot\mathbf{v} \qquad \text{(iv) } \mathbf{B}\cdot\mathbf{C}^T\cdot\mathbf{D}$$

2. If the components of $\mathbf{T}$ in the $\mathbf{e}_i\otimes\mathbf{e}_j$ basis are $\begin{bmatrix} 3 & 0 & -\sqrt{2} \\ 0 & 5 & 0 \\ -\sqrt{2} & 0 & 2 \end{bmatrix}$.

(i) [12] Determine the principal basis and the components of $\mathbf{T}$ in that basis.

(ii) [12] What are the components of $\mathbf{T}^{1/2}$ in the $e_i\otimes e_j$ basis.

3. (**ME 512 only**) [20] Start with a basic definition and obtain a general expression for the gradient of a vector in terms of components and base vectors in a curvilinear coordinate system. Your final expression should have a "Christoffel" type symbol. Derive the corresponding expression for physical components of the gradient in terms of the physical components of the vector.

3. (**ME 402 only**) [20] Start with a basic definition for a gradient of a vector function with respect to a vector and derive the corresponding expression in indicial notation. What

is the gradient of $\dfrac{\mathbf{v}}{(\mathbf{v}\cdot\mathbf{v})^{1/2}}$ with respect to v.

4. [20] The first law of thermodynamics implies that $\boldsymbol{\sigma} = \rho \frac{\partial U}{\partial \mathbf{e}}$ and $\theta = \frac{\partial U}{\partial s}$ where U denotes the internal energy, θ - the temperature, s - the entropy, ρ - mass density, e - total strain, e^p - plastic strain and $\boldsymbol{\sigma}$ - stress. Suppose

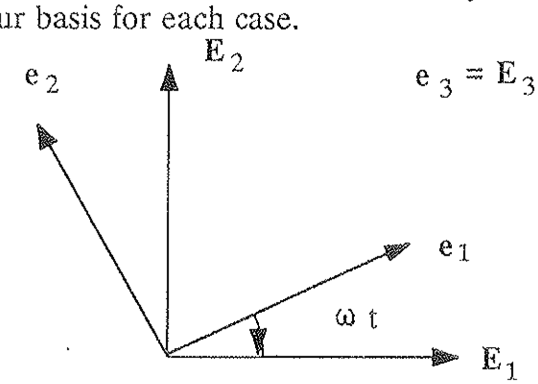
$U = \frac{1}{2\rho}(\mathbf{e} - \mathbf{e}^p):\mathbf{E}:(\mathbf{e} - \mathbf{e}^p) + c[1 + \frac{s}{s^*} + \frac{1}{2}\left(\frac{s}{s^*}\right)^2]$ where c and s^* are constants.

(i) Determine $\boldsymbol{\sigma}$, θ and $\frac{\partial U}{\partial \mathbf{e}^p}$.

(ii) Using your result from (i) express U in terms of strain and θ instead of strain and s to obtain a function U*.

(iii) Perform a contact (Legendre) transformation with respect to temperature and entropy to obtain the Helmholtz Free energy A in terms of strain and θ. What is an interpretation for $\frac{\partial A}{\partial \theta}$? Is there a similar interpretation for $\frac{\partial U^*}{\partial \theta}$? Do A and U* have the same independent variables? Are they the same functions?

5. [20] Consider the deformation described by $x_1 = X_1 + atX_2$, $x_2 = X_2 + btX_1$, $x_3 = X_3$ superposed on the rotation given by $\theta = \omega t$ where a and b are constants and t denotes time. Determine $\mathbf{F} = \mathbf{r}\nabla_0$, $\mathbf{v} = \dot{\mathbf{r}}$, $\mathbf{L} = \mathbf{v}\nabla$, and indicate how you would obtain the rotation **R**. Be sure to define your basis for each case.



ME 402/512 Continuum Mechanics Closed Book Exam 1 10/12/93

Weights for problems are given in square brackets. Allocate time accordingly.

1. [20] Use indicial notation to prove the following:

$$(\mathbf{u} \times \mathbf{v}) \times \nabla = (\mathbf{u} \cdot \nabla)\mathbf{v} - (\mathbf{v} \cdot \nabla)\mathbf{u} + (\mathbf{v}\nabla)\cdot \mathbf{u} - (\mathbf{u}\nabla)\cdot \mathbf{u}$$

2. [20] Provide the simplest possible expressions for each of the following:

(i) $\mathbf{r}\nabla$, (ii) $\varepsilon_{ijk}u_j u_k$, (iii) tr ($\mathbf{I}$), (iv) $C_{13}(\mathbf{u} \otimes \mathbf{T})$, (v) tr $(\mathbf{T}^s \cdot \mathbf{T}^a)$

in which $\mathbf{r}$ is the position vector, $\mathbf{I}$ is the identity tensor, $\mathbf{u}$ is a vector, $\mathbf{T}$ is any second order tensor, $\mathbf{T}^s$ is the symmetric part of $\mathbf{T}$, and $\mathbf{T}^a$ is the skew-symmetric part of $\mathbf{T}$.

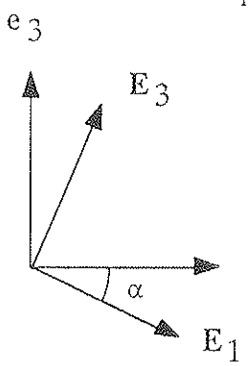
3. [15] What is meant by each of the following (give fairly complete descriptions):

(i) an invariant of a second-order tensor,

(ii) an eigenpair of a second-order tensor,

(iii) strain tensor.

4. [20] Derive the transformation equations that relate components of tensors in two bases. Suppose two bases are related as shown in the sketch. A tensor is given by $\mathbf{T} = 4\mathbf{e}_1 \otimes \mathbf{e}_1 + 3\mathbf{e}_1 \otimes \mathbf{e}_3 - 2\mathbf{e}_2 \otimes \mathbf{e}_1 + 2\mathbf{e}_3 \otimes \mathbf{e}_3$. Using your derived relation, determine the expression for the components of the tensor in the $\mathbf{E}_A$ basis.



$\mathbf{E}_2 = -\mathbf{e}_1$

5. (i) [5]Give the general definition of a gradient of a vector field with respect to a vector.

(ii)[5]Given that the deformation of a continuous medium is described by $d\mathbf{r} = \mathbf{F}\cdot d\mathbf{R}$ use your definition of a gradient to derive the relationship between the gradient based on $\mathbf{R}$ and the gradient based on $\mathbf{r}$.

(iii) [3] Use the definition of a gradient to obtain the explicit expression $\mathbf{F} = \mathbf{r}\nabla_o$.

(iv) [6] Suppose $x_1 = A(X_1)^2 + BX_2$, $x_2 = CX_1X_2$, $x_3 = X_3$. Determine the components of $\mathbf{F}$. What tensor basis is associated with these components?

(v) [6] Determine the components of the Lagrangian strain tensor

$\mathbf{E} = (\mathbf{F}^T\cdot\mathbf{F} - \mathbf{I})/2$. What tensor basis is associated with these components?

ME 402/512 Continuum Mechanics Closed Book Exam 2 11/30/93

Weights for problems are given in square brackets. Allocate time accordingly.

1. (i) [5] Use indicial notation to prove that $tr(\mathbf{A}\cdot\mathbf{B}\cdot\mathbf{C}) = tr\,(\mathbf{C}\cdot\mathbf{A}\cdot\mathbf{B})$.

(ii) [15] The deformation gradient $\mathbf{F}$ is defined such that $d\mathbf{r} = \mathbf{F}\cdot d\mathbf{R}$. What are $d\mathbf{r}$ and $d\mathbf{R}$? Use the definition to obtain an expression for $\mathbf{F}$ and to obtain the relationship between gradients with respect to $\mathbf{r}$ and $\mathbf{R}$.

2. [20] Give the assumptions and outline the derivation of the relationship which states that the traction is a linear transformation of the unit normal to a surface.

3. (i) [10] Describe the possible combinations of boundary conditions for a continuous body.

(ii) [5] A part of a rigid body, A, contains a frictionless surface defined by the arc of a circle, $r = c$, with c a constant. A portion of a body, B, is pressed against this surface. What is the boundary condition for body, B, for the portion of the boundary in contact with the frictionless surface. (Give a sketch).

(iii) [5] Another part of the surface of body, B, defined by the equation $f(x, y, z) = 0$ is acted on by a fluid which sustains a static pressure P. What is the boundary condition for this part of the surface?

4. [20] Give the principle of conservation of mass in integral form. Then take a time derivative to obtain the "continuity" equation.

ME 512 students do problem 5 and <u>not</u> 6. ME 402 students do problem 5 <u>or</u> 6.

5. Let x^i denote curvilinear coordinates.

(i) [5] How are covariant and contravariant base vectors constructed?

(ii) [5] Relate one set of these base vectors to an orthonormal set of base vectors associated with rectangular Cartesian coordinates.

(iii) [5] How are covariant and contravariant components of a vector defined.

(iv) [5] What is the relationship between the contravariant and covariant components of a vector?

6. (i) [5] What is the definition of the spin tensor?

(ii) [5] What is the definition of the vorticity tensor?

(iii) [10] Derive the relationship between the spin and vorticity tensors.

ME 402/512 Continuum Mechanics Final Exam Closed Book 12/15/93

Weights for problems are given in square brackets. Allocate time accordingly.

1. (i) [10] Derive the transformation relation between components of a second order tensor expressed with the use of two orthonormal bases. What does orthonormal mean?
(ii) [10] What is the definition of an invariant? Given an example and use your transformation relation to prove that your example is an invariant.

2. (i) [10] What is an eigenpair associated with a tensor, **T**? What is meant by the terms "principal values" and "principal directions" of a tensor? If **T** is symmetric and real what can you say about the principal values and principal directions?
(ii) [10] Obtain the eigensystem for the tensor, **T**, whose components are given as follows:

$$T_{ij} \Rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

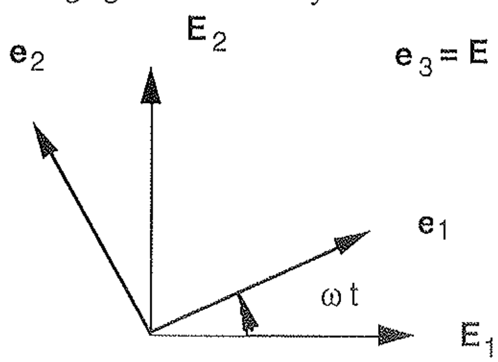
How would you obtain $\mathbf{T}^{1/4}$?

3. [20] Suppose only one component of strain, e, is nonzero so that the internal energy can be given as $u = Ee^2/2 + a + bs + cs^2/2$ in which E,a,b,c are constants and s is the entropy.
(i) Determine s and T, the functions conjugate to e and s, respectively.
(ii) Use the expression for T to determine the internal energy, u^*, as a function of e and T. What is $\partial u^* \;\; \partial T$?
(iii) Obtain the Helmholtz Free Energy, A, by performing a "contact transformation" on u with respect to T and s. What should you expect to get for $\partial A \;\; \partial T$?. What do you get for $\partial A \;\; \partial T$ for your function, A?

4. Consider the deformation described by $x_1 = X_1 + at(X_1)^2$, $x_2 = X_2 + btX_1$, $x_3 = X_3$ superposed on the rotation given by $q = wt$ where a, b and w are constants and t denotes time.

(i) [16] Determine $\mathbf{F} = \mathbf{r}\nabla_0$, $\mathbf{v} = \dot{\mathbf{r}}$, $\mathbf{L} = \mathbf{v}\nabla$, . Be sure to define your basis for each case.

(ii) [4] Sketch a segment of material in the undeformed and deformed configuration. Is the volume differential dV changing with time? Why?



Note: ME 512 students must do 5(a) and not 5(b). ME 402 students can do 5(a) or 5(b).

5(a) (i) [12] Start with a basic definition and obtain a general expression for the gradient of a tensor function of position in terms of components and base vectors in a curvilinear coordinate system. Your final expression should have "Christoffel" symbols. Obtain an expression for the divergence of a tensor.

(ii) [8] What is meant by "physical component?. How would you use the physical components of tensors in part (i)? How would you obtain the physical components of the divergence in part (i)?

5(b) (i) [8] Start with a basic definition for a gradient of a tensor function of position and derive the corresponding expression in indicial notation. Obtain an expression for the divergence of a tensor.

(ii) [12] Start with a basic definition for a gradient of a scalar function of a second-order tensor and derive the corresponding expression in indicial notation. What is the gradient of $(\mathbf{s}:\mathbf{s})^{1/2}$ with respect to $\mathbf{s}$, a second order tensor.

ME 402/512 Continuum Mechanics Closed Book Exam 1 10/10/95

Weights for problems are given in square brackets. Allocate time accordingly.

1. [20] Use indicial notation and an orthonormal basis for the following:

(a) For vectors **u** and **v**, show that

$$\text{(i)}\ (\mathbf{u}\cdot\mathbf{v})\nabla = \mathbf{u}\cdot(\mathbf{v}\nabla) + \mathbf{v}\cdot(\mathbf{u}\nabla))$$

$$\text{(ii)}\ (\mathbf{u}\cdot\mathbf{v})\nabla = (\mathbf{v}\nabla)\cdot\mathbf{u} + (\mathbf{u}\nabla)\cdot\mathbf{v} - \mathbf{u}\times(\mathbf{v}\times\nabla) - \mathbf{v}\times(\mathbf{u}\times\nabla)$$

(b) For tensors **A, B** and **C** verify that $\mathrm{tr}(\mathbf{A}\cdot\mathbf{B}\cdot\mathbf{C}) = \mathrm{tr}(\mathbf{B}\cdot\mathbf{C}\cdot\mathbf{A})$

2. [20] (i) Derive the transformation relation for components of a second-order tensor.

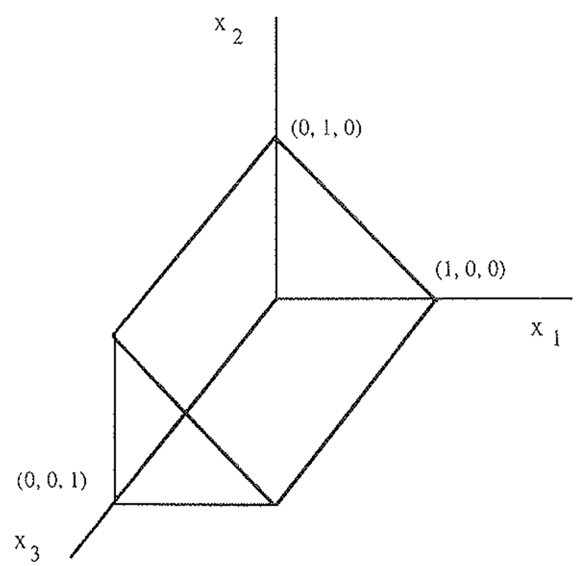
(ii) Give the result in both indical and matrix forms.

(iii) Construct your transformation matrix for two bases related as follows:

$$\mathbf{E}_1 = \frac{1}{\sqrt{14}}(\mathbf{e}_1 - 2\mathbf{e}_2 + 3\mathbf{e}_3) \qquad \mathbf{E}_2 = \frac{1}{\sqrt{5}}(2\mathbf{e}_1 + \mathbf{e}_2) \qquad \mathbf{E}_3 = \mathbf{E}_1 \times \mathbf{E}_2$$

3. [15] What is an invariant of a second-order tensor. In general, how many independent invariants are there. Give two examples of sets of independent invariants.

4. [25] A region is defined in the following sketch:



Suppose $\phi = x_1x_2$. Carry out the integrals to show that $\int_R \phi\nabla dV = \int_{\partial R} \phi\mathbf{n}dS$ holds.

5. [20] Suppose $\mathbf{r} = x_ie_i$, $\mathbf{R} = X_A\mathbf{E}_A$ and $\mathbf{E}_i = \mathbf{e}_i$. ∇_o denotes the gradient with respect to **R**, and ∇ the gradient with respect to **r**. Consider the deformation

$$x_1 = X_1 - X_2 \qquad x_2 = X_2 + X_1 \qquad x_3 = X_3$$

Let $\phi = ax_1^2 + bx_1x_2$ with a and b constants.

Determine $\phi\nabla_o$ and $\phi\nabla$.

ME 402/512 Continuum Mechanics Closed Book Exam 2 11/16/95

Weights for problems are given in square brackets. Allocate time accordingly.

1. (a) [5] If a tensor is represented as $\mathbf{T} = a\mathbf{e}_1 \otimes \mathbf{e}_3 + b\mathbf{e}_2 \otimes \mathbf{e}_2$ give the matrix of its components in the basis $\mathbf{e}_i \otimes \mathbf{e}_j$.

(b) [15] Suppose the eigenvalues of a tensor are 0, -2 and 3, and the eigenvectors corresponding to the first two eigenvalues are $\mathbf{p}_1 = \frac{1}{\sqrt{2}}(\mathbf{e}_2 - \mathbf{e}_3)$ and $\mathbf{p}_2 = \frac{1}{\sqrt{6}}(2\mathbf{e}_1 - \mathbf{e}_2 - \mathbf{e}_3)$. What is the third eigenvector? Give the components of the tensor with respect to the $\mathbf{e}_i \otimes \mathbf{e}_j$ basis.

2. The mapping between differentials of position vectors in the original and current configurations is given by $d\mathbf{r} = \mathbf{F} \cdot d\mathbf{R}$.

(a) [10] How is a differential volume element defined? If $J = \det(\mathbf{F})$ derive the mapping that relates differential volume elements in the two configurations.

(b) [10] Let ∇_0 denote a gradient with respect to $\mathbf{R}$ and ∇ the gradient with respect to $\mathbf{r}$. Use the basic definition of a gradient and the relation $d\mathbf{r} = \mathbf{F} \cdot d\mathbf{R}$ to derive the relationship that allows one to convert from one gradient to the other.

(c) [5] Derive an equivalent expression for $\frac{d}{dt}\int_R F(t)dV$ but with the time derivative inside the integral.

3. [15] Give the basic formulation that defines either the Eulerian or Lagrangian strain tensor. Then derive the equation that relates the strain tensor to $\mathbf{F}$.

4. [25] Outline the arguments that prove the existence of the Cauchy stress tensor $\boldsymbol{\sigma}$.

5. (i) [5] Describe the possible combinations of boundary conditions for a continuous body.

(ii) [5] A part of a rigid body, A, contains a frictionless surface defined by the arc of a circle, r = c, with c a constant. A portion of a second body, B, is pressed against this surface. What is the boundary condition for body, B, for the portion of the boundary in contact with the frictionless surface. (Give a sketch).

(iii) [5] Another part of the surface of body, B, defined by the equation $f(x, y, z) = 0$ is acted on by a fluid which sustains a static pressure P. What is the boundary condition for this part of the surface?

Weights for problems are given in square brackets. Allocate time accordingly.

Total number of points is 120. Work to maximize point count, not to finish the exam.

1. (i) $\delta_{ij}a_j$ (ii) $\delta_{ij}x_ix_j$ (iii) $(x_ix_j)_{,j}$ (iv) $(\delta_{ij} + a_{ij})(\delta_{ik} + a_{ik}) - \delta_{jk} - \delta_{mn}a_{mj}a_{nk}$

(b) [10] What is the Principle of Material Frame Indifference? Give one example of an equation that satisfies the principle and an example that doesn't.

(c) [5] Material elements in the undeformed and deformed configuration are related by $d\mathbf{r} = \mathbf{F} \cdot d\mathbf{R}$. If L denotes a line segment in the deformed configuration and $\mathbf{u}$ is a function of time t, derive an equivalent expression for $\frac{d}{dt}\int_L \mathbf{u} \cdot d\mathbf{r}$ in which the time derivative is inside

2. Let G be a scalar function of a vector field v.

(i) [5] Define what is meant by the gradient of G with respect to v.

(ii) [5] What is meant by an orthonormal basis?

(iii) [5] Use your definition in (i) to obtain an expression in indicial form for the gradient if an orthonormal basis is used.

(iv) [10] Use your result from (iii) to determine the gradient with respect to v of

(a) $G = \mathbf{v} \cdot \mathbf{v}$ (b) $G = \dfrac{\mathbf{v} \cdot \mathbf{T} \cdot \mathbf{v}}{(\mathbf{v} \cdot \mathbf{v})^{1/2}}$ for a constant tensor, $\mathbf{T}$.

Give your answers in both direct and indicial notation.

3. Suppose that the equation $\int_R \phi \nabla dV = \int_{\partial R} \phi \mathbf{n} dS$ is given.

(i) [6] In this context what do each of the following terms denote: ∇, R, ∂R, dV, dS and $\mathbf{n}$.

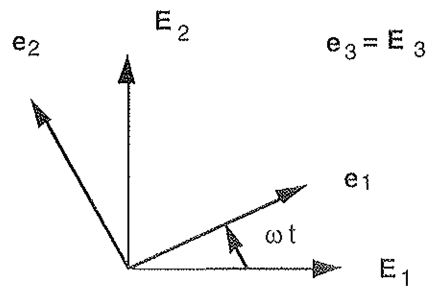
(ii) [10] Use the given equation to derive the corresponding equation where the function, ϕ, is a tensor rather than a scalar.

(iii) [4] Use the result of (ii) to derive the divergence theorem for a tensor field.

4. Consider the deformation described by $x_1 = X_1 + \alpha t X_2$, $x_2 = X_2(1 + \beta t X_1)$, $x_3 = X_3$ superposed on the rotation given by $\theta = \omega t$ where α and β are constants and t denotes time.

(i) [15] Determine $\mathbf{F} = \mathbf{r}\nabla_0$, $\mathbf{v} = \dot{\mathbf{r}}$, $\mathbf{L} = \mathbf{v}\nabla$, . Be sure to define your basis for each set of components.

(ii) [5] Describe rather thoroughly how you would obtain the rotation **R** and the right stretch **U**.



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5. Suppose x^i represents a general curvilinear coordinate system.

(i) [5] How are covariant and contravariant bases defined?

(ii) [5] How are covariant and contravariant components of a vector defined?

(iii) [10] Start with a basic definition and obtain a general expression for the gradient of a vector in terms of components and base vectors in a curvilinear coordinate system. Your final expression should include a "Christoffel" type symbol.

(iv) [4] What are physical components of vectors and tensors.

(v) [6] Indicate how you obtain the physical components of the gradient of a vector in terms of the physical components of the vector.

ME 402/512 Continuum Mechanics Closed Book Exam 1 10/9/97

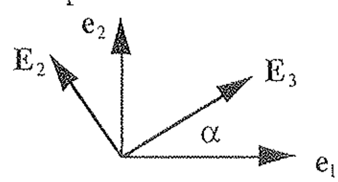
Weights for problems are given in square brackets. Allocate time accordingly.

1. The components of a tensor, **T**, and a vector, **v**, with respect to the orthonormal basis $\mathbf{e}_i$ are given as follows:

$$[T]_{e_i \otimes e_j} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \qquad \{v\}_{e_i} = \begin{Bmatrix} -3 \\ 2 \\ -1 \end{Bmatrix}$$

(i) [5] Determine the components of the symmetric and skew-symmetric parts of **T**.
(ii) [5] Obtain $\mathbf{T}\cdot\mathbf{v}$ and $\mathbf{v}\cdot\mathbf{T}$.
(iii) [5] Obtain w_i where $w_i = \varepsilon_{ijk}T_{jk}$.

(iv) [5] Obtain tr(**T**) and **I**:**T** where **I** is the identity tensor.
(v) [5] Determine det [**T**].
(v) [10] A new orthonormal basis, $\mathbf{E}_A$, is oriented with respect to $\mathbf{e}_i$ as shown in the sketch. Both pairs of base vectors lie in the plane and both bases form a right-handed system. Obtain the 2-3 component of **T** in this new basis.



2. The gradient theorem for a scalar function is given by $\int_R \phi\nabla dV = \int_{\partial R} \phi \mathbf{n} dS$.

(i) [5] What conclusions can you draw if the function is unity, i.e., $\phi = 1$.
(ii) [10] Derive the corresponding gradient and divergence theorems for vectors.

3. A definition of a gradient of a vector, $\mathbf{v}\nabla$, is given by $d\mathbf{v} = (\mathbf{v}\nabla)\cdot d\mathbf{r} \quad \forall\, d\mathbf{r}$.

(i) [5] What is meant by each of the terms **v**, d**v**, **r**, d**r**, and $\forall$.
(ii) [8] Use this definition to derive the expression for the gradient in a rectangular Cartesian coordinate system.
(iii) [7] Apply your result to find the gradient of the following vector:

$$\mathbf{v} = x_2^2\mathbf{e}_1 + x_1x_2\mathbf{e}_2 + \sin(x_1^2x_3)\mathbf{e}_3$$

(iv) [10] Evaluate the line integral, $\int_{P_1}^{P_2} \mathbf{v}\cdot d\mathbf{r}$, for the straight line between the points P_1 and P_2 which have coordinates (1, 2, 0) and (1, 4, 0), respectively.

4. [20] With respect to a symmetric second-order tensor, what is meant by each of the following terms: (i) eigenvector, (ii) eigenvalue, (iii) eigenpair, (iv) characteristic polynomial, (v) characteristic equation, (vi) principal directions, (vii) principal values, (viii) invariants, (ix) Cayley-Hamilton Theorem, and (x) a non-integer power.

ME 402/512 Continuum Mechanics Closed Book Exam 2 11/20/97

Weights for problems are given in square brackets. Allocate time accordingly. A total of 115 points are available so work on those problems you find easiest to maximize your point total.

1. The mapping between differentials of position vectors in the original and current configurations is given by $d\mathbf{r} = \mathbf{F} \cdot d\mathbf{R}$. Let ∇_0 denote a gradient with respect to $\mathbf{R}$ and ∇ the gradient with respect to $\mathbf{r}$.

(i) [5] Use the basic definition of a gradient to derive an expression for $\mathbf{F}$ involving one of the gradients.

(ii) [10] Derive the relationship that allows one to convert from one gradient to the other.

(iii) [10] Obtain two expressions for the time derivative of $\mathbf{F}$, one involving ∇_0 and one involving ∇.

(iv) [10] Obtain the time derivative of $\mathbf{F}^{-1}$.

3. [25] Give the basic formulation that defines either the Eulerian or Lagrangian strain tensor. Then derive the equation that relates the strain tensor to $\mathbf{F}$. Derive the relationship that gives a physical interpretation for one of the diagonal components of the strain tensor you defined.

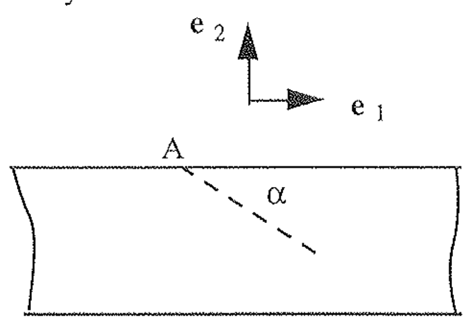
3. The stress tensor, σ, has been solved for a problem and the components at point A are as follows:

(i) [7] Determine the traction on the top surface of the body. Point A is located at the surface.

(ii) [7] Obtain the traction on a surface which forms an angle α with the top surface as indicated in the figure.

(iii) [6] If failure is governed by the maximum value of the normal component of the traction, at which angle is failure most likely to occur?

$$\sigma\big|_{e_i \otimes e_j} = \begin{bmatrix} 8 & 3 & 0 \\ 3 & 4 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$



4. Suppose a deformation in the X_1 - X_2 plane is given as follows for the case where $\mathbf{e}_i = \mathbf{E}_i$: $x_1 = X_1$ and $x_2 = X_2 - 2a(t)X_1$.

(i) [5] Provide a sketch which illustrates the deformation.

(ii) [5] Obtain the displacement field $\mathbf{u} = \mathbf{r} - \mathbf{R}$.

(iii) [5] Obtain $\mathbf{v}$ in terms of x_i and $\mathbf{e}_i$.

(iv) [5] Obtain $\mathbf{L} = \mathbf{v}\nabla$, and then $\mathbf{D}$ and $\mathbf{W}$.

(v) [5] Obtain $\mathbf{F}$.

(vi) [10] Obtain $\mathbf{e} = \frac{1}{2}[\mathbf{I} - \mathbf{F}^{-T} \cdot \mathbf{F}^{-1}]$.

Weights for problems are given in square brackets. Allocate time accordingly.

Note. Do only the appropriate version of Problem 5.

1. [20] With regard to the deformation of a continuous medium, express with words (and with sketches) what is meant by each of the following expressions. Also define each of the distinct tensors and symbols.

$$\text{(i) } d\mathbf{r} = \mathbf{F}\cdot d\mathbf{R}, \quad \text{(ii) } \mathbf{F} = \mathbf{R}\cdot\mathbf{U}, \quad \text{(iii) } \mathbf{D} = (\mathbf{v}\nabla)_{sym}, \quad \text{(iv) } \mathbf{L} = \mathbf{D} + \mathbf{W}$$

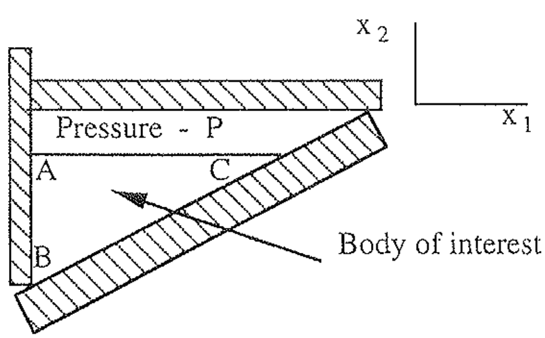
2. [20] Prove that the eigenvectors of a symmetric, real tensor, **T**, associated with two distinct eigenvalues are orthogonal.

3. (i) [5] What is the Cauchy stress tensor $\boldsymbol{\sigma}$?

(ii) [5] What are Cauchy's equations of motion?

(ii) [10] Outline the arguments which prove that such a tensor must exist and which provides a derivation of Cauchy's equations.

4. (i) [5] Suppose a body is confined by frictionless surfaces along the planes $x_3 = -a$ and $x_3 = a$. The body is glued along the surface A-B-C to a rigid base. On the top surface A-C, an air pressure of magnitude P is applied. Give the boundary conditions for the body.



(ii) [5] The boundary conditions constitute an essential part of the description of a completely formulated problem. In continuum mechanics what other equations must be specified?

5. (ME 512 only) Suppose x^i represents a general coordinate system.

The covariant components of the metric tensor are known to be $g_{ij} \Rightarrow \begin{bmatrix} (x^2)^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{(x^1)^2} \end{bmatrix}$

(i) [5] Is the coordinate system orthogonal? Why? Is the coordinate system curvilinear? Why? Will the contravariant components of the position vector be x^i? Why?

(ii) [5] What are the contravariant and mixed components of the metric or identity tensor? What are the components of the metric tensor in a rectangular Cartesian system?

(iii) [5] The covariant components of a vector $\mathbf{v}$ are $v_i \Rightarrow (1, 2, -1)$? What are the contravariant components? What is $\mathbf{v} \cdot \mathbf{v}$? Will the gradient of $\mathbf{v}$ be the zero tensor? Why?

(iv) [5] The contravariant components of a tensor $\mathbf{T}$ are $T^{ij} \Rightarrow \begin{bmatrix} 1 & 0 & 3 \\ 0 & 2 & 4 \\ 3 & 4 & 5 \end{bmatrix}$.

What are the covariant components? What is the determinant of $\mathbf{T}$?

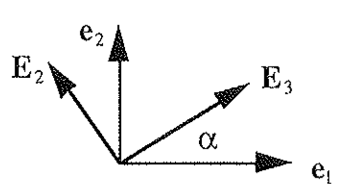
(v) [10] Start with a basic definition and obtain a general expression for the gradient of a vector in terms of components and base vectors in a curvilinear coordinate system. Your final expression should include a "Christoffel" type symbol.

5. (ME 404 only) Consider two orthonormal bases, $\mathbf{e}_i$ and $\mathbf{E}_A$.

(i) [10] Derive the transformation relations that relate components of a second-order tensor in these two systems.

(ii) [10] The components of $\mathbf{T}$ in the $\mathbf{e}_i$ system are $T_{ij} \Rightarrow \begin{bmatrix} 1 & 0 & 3 \\ 0 & 2 & 4 \\ 3 & 4 & 5 \end{bmatrix}$.

What are the components in the $\mathbf{E}$ system if the systems are oriented as follows:



What is the determinant of $\mathbf{T}$?

(iii) [10] From the basic definition, obtain the general expression for the gradient of a tensor.

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ME 402/512 **CONTINUUM MECHANICS** **ASSIGNMENT 1**

1. If $[A]=\begin{bmatrix} 4 & 0 & 2 \\ -1 & 1 & 1 \end{bmatrix}$ and $[B]=\begin{bmatrix} 2 & 1 \\ -2 & 0 \\ 3 & 5 \end{bmatrix}$ find [A] [B] and [B] [A].

2. Let

$$[A]=\begin{bmatrix} 4 & -4 & 2 \\ 7 & -2 & 2 \\ 21 & 0 & 4 \end{bmatrix} \qquad \{b\}=\begin{Bmatrix} 1 \\ -1 \\ 2 \end{Bmatrix} \qquad \{x\}=\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}$$

(a) Use the adjoint matrix formulation to obtain $[A]^{-1}$.

(b) Show that $[A]^{-1}[A] = [I]$.

(c) Find the solution {x} to the equation [A] {x} = {b}.

3. Obtain the inverse of $[A]=\begin{bmatrix} 1 & 0 & 0 \\ \lambda_1 & 1 & 0 \\ \lambda_2 & \lambda_3 & 1 \end{bmatrix}$.

4. Let $[A]=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$.

(4.1) Let [E] denote an "elementary" matrix. For each of the following cases determine [E][A] and [A][E] and state in words the effect on [A] of the matrix multiplication with [E].

(a) $[E]=\begin{bmatrix} 1 & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (b) $[E]=\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(4.2) For the following two cases, apply the expansion of $\sum_k a_{ik} C^A_{jk} = \delta_{ij}|A|$:

(a) Let i = j = 2 and obtain the expression for |A| .

(b) Let i = 1, j = 3 and show the equality holds.

5.Let $[A]=\begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix}$ and $[B]=\begin{bmatrix} -1 & 2 \\ 2 & -4 \end{bmatrix}$.Find $[A][B]$, $([A][B])^T$ and $[B]^T[A]^T$.

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ME 402/512 CONTINUUM MECHANICS ASSIGNMENT 2

1. Let $x = x_i e_i$ and $v = -e_1 + 3e_2 - 2e_3$. What restriction must be placed on the components of x so that x is perpendicular to v.

2. Given

$$T_{pq} = \begin{bmatrix} -1 & 2 & 3 \\ 2 & 3 & 1 \\ 4 & 3 & 4 \end{bmatrix} \qquad u_i = (0, 3, 2) \qquad v_i = (-2,4,-3)$$

Find the components of the following scalars, vectors, or tensors:

(a) $\mathbf{T} \cdot \mathbf{u}$ (b) $\mathbf{u} \cdot \mathbf{T}^T$ (c) $\mathbf{v} \cdot \mathbf{T} \cdot \mathbf{u}$ (d) $\mathbf{u} \otimes \mathbf{v}$

3. Show that the relation $\mathbf{v} = (\mathbf{v} \cdot \mathbf{n})\mathbf{n} + \mathbf{n} \times (\mathbf{v} \times \mathbf{n})$ holds $\forall\, \mathbf{n}$ and that this represents a resolution of **v** into vectors parallel and perpendicular to **n** (a unit vector).

4. What is wrong with the following indicial equations:

(a) $w_i = b_{ik} u_i v_k$ (b) $\phi = b_{ik} u_i$ (c) $\phi_{jp} = R_{ijkl} T_{kl} u_p$

Give forms which are correct.

5. Let $b_i = \varepsilon_{ijk} R_{jk}$. Give b_1, b_2 and b_3 in terms of explicit components of **R**.

6. Using indicial notation, prove that the determinant of the product of two matrices equals the product of the determinants of the matrices.

7. Prove the $\varepsilon - \delta$ identity $\varepsilon_{ijk}\, \varepsilon_{irs} = \delta_{jr}\, \delta_{ks} - \delta_{js}\, \delta_{kr}$. One way is to construct a table with various combinations of indices. Alternatively, Malvern (page 25) provides the following hint. Two of the four free indices must be equal. Show that if j = k or r = s, then both sides vanish. Then show that even with $j \neq k$ and $r \neq s$ both sides vanish when j = r unless also k = s. What other cases must be considered?

1. This problem is from Malvern, Pg. 46, No. 8. Leave answers in terms of the square root of 2. The angles between the respective base vectors in two systems are given in the table to the left. The components of **T** in the $\mathbf{e}_i$ system is given to the right:

	E_1	E_2	E_3
e_1	90^o	45^o	135^o
e_2	45^o	60^o	60^o
e_3	45^o	120^o	120^o

$$[T]_{e_i} = \begin{bmatrix} 2 & -2 & 0 \\ -2 & 6 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

(a) Express e_i in terms of E_A ; Express E_A in terms of e_i.

(b) Obtain the components of the transformation matrix and verify that the basis E_A is a right-handed orthonormal system. Show that the transpose equals the inverse

(c) Find the components of T in the E_A system.

2. (Malvern, Page 46, Nos. 11 and 12)

(a) Show that $\mathrm{tr}(T \cdot U) = \mathrm{tr}\left(T^T \cdot U\right)$ if either T or U is symmetric.

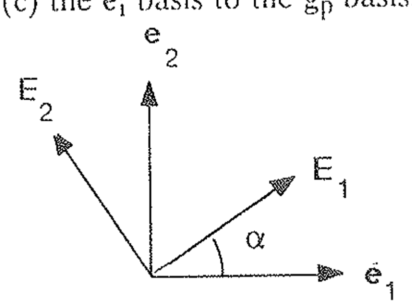
(b) Show that $\mathrm{tr}(T \cdot U) = 0$ if one one of the tensors is skew-symmetric and the other is symmetric.

3. E_A is related to e_i and g_p is related to E_A as shown. Obtain the transformation matrices for transforming components from:

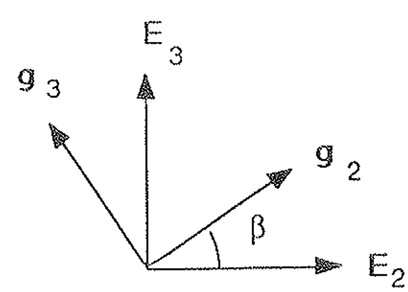
(a) the e_i basis to the E_A basis,

(b) the E_A basis to the g_p basis, and

(c) the e_i basis to the g_p basis.



$E_3 = e_3$



$E_1 = g_1$

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ME 402/512 CONTINUUM MECHANICS ASSIGNMENT 4

For the basis $\mathbf{e}_i$, the components of $\mathbb{T}$ are $\begin{bmatrix} 3 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 3 \end{bmatrix}$

1. Find the components of $\mathbb{T}^2$ and $\mathbb{T}^3$ for the same basis.

2. Find $\mathrm{I}_T = \operatorname{tr} \mathbb{T}$, $\mathrm{II}_T = \operatorname{tr} \mathbb{T}^2$, and $\mathrm{III}_T = \operatorname{tr} \mathbb{T}^3$.

3. Find the eigenvalues of $\mathbb{T}$ and the eigenvectors. Construct a principal basis ($\mathbb{P}_A$ say) expressed in terms of $\mathbf{e}_i$.

4. What are the components of $\mathbb{T}$, $\mathbb{T}^2$, and $\mathbb{T}^3$ with respect to the principal basis. Determine I_T, II_T, and III_T using these components.

5. Set up the transformation matrix between $\mathbb{P}_A$ and $\mathbf{e}_i$. Start with the components of $\mathbb{T}$ in the principal basis obtained in Prob. 4 and use the transformation relation to obtain the components with respect to the basis $\mathbf{e}_i$.

6. (a) Obtain the values of the invariants $\hat{\mathrm{I}}_T$, $\hat{\mathrm{II}}_T$, and $\hat{\mathrm{III}}_T$.
(b) Show that the Cayley-Hamilton theorem holds using components in the $\mathbf{e}_i$ system.
(c) With the use of components in either system, show that

$$\hat{\mathrm{III}}_T \equiv \frac{1}{6}\left[\mathrm{I}_T^3 - 3\,\mathrm{I}_T\,\mathrm{II}_T + 2\,\mathrm{III}_T\right] = \det(\mathbb{T}).$$

7. Find the components of the tensor $\mathbb{T}^{1/2}$ in the $\mathbf{e}_i$ system. Obtain $T_{ij}^{1/2}\,T_{jk}^{1/2}$.

8. In the $\mathbf{e}_i$ system find the components of $\mathbb{T}^{-1}$ from the equation

$$\mathbb{T}^{-1} = \left(\mathbb{T}^2 - \hat{\mathrm{I}}_T\,\mathbb{T} - \hat{\mathrm{II}}_T\,\mathbb{I}\right) / \left(\hat{\mathrm{III}}_T\right)$$

Transform these components to obtain the components of $\mathbb{T}^{-1}$ in the $\mathbb{P}_A$ system.

For the basis $\mathbf{e}_i$, the components of $\mathbf{T}$ are $\begin{bmatrix} 3 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 3 \end{bmatrix}$

1. Find the components of $\mathbf{T}^2$ and $\mathbf{T}^3$ for the same basis.

2. Find $\mathrm{I}_T = \mathrm{tr}\,\mathbf{T}$, $\mathrm{II}_T = \mathrm{tr}\,\mathbf{T}^2$, and $\mathrm{III}_T = \mathrm{tr}\,\mathbf{T}^3$.

3. Find the eigenvalues of $\mathbf{T}$ and the eigenvectors. Construct a principal basis ($\mathbf{P}_A$ say) expressed in terms of $\mathbf{e}_i$.

4. What are the components of $\mathbf{T}$, $\mathbf{T}^2$, and $\mathbf{T}^3$ with respect to the principal basis. Determine I_T, II_T, and III_T using these components.

5. Set up the transformation matrix between $\mathbf{P}_A$ and $\mathbf{e}_i$. Start with the components of $\mathbf{T}$ in the principal basis obtained in Prob. 4 and use the transformation relation to obtain the components with respect to the basis $\mathbf{e}_i$.

6. (a) Obtain the values of the invariants $\hat{\mathrm{I}}_T$, $\hat{\mathrm{II}}_T$, and $\hat{\mathrm{III}}_T$.
   (b) Show that the Cayley-Hamilton theorem holds using components in the $\mathbf{e}_i$ system.
   (c) With the use of components in either system, show that

$$\hat{\mathrm{III}}_T \equiv \frac{1}{6}\left[\mathrm{I}_T^3 - 3\,\mathrm{I}_T\,\mathrm{II}_T + 2\,\mathrm{III}_T\right] = \det(\mathbf{T}).$$

7. Find the components of the tensor $\mathbf{T}^{1/2}$ in the $\mathbf{e}_i$ system. Obtain $T_{ij}^{1/2}\,T_{jk}^{1/2}$.

8. In the $\mathbf{e}_i$ system find the components of $\mathbf{T}^{-1}$ from the equation

$$\mathbf{T}^{-1} = \left(\mathbf{T}^2 - \hat{\mathrm{I}}_T\,\mathbf{T} - \hat{\mathrm{II}}_T\,\mathbf{I}\right) / \left(\hat{\mathrm{III}}_T\right).$$

Transform these components to obtain the components of $\mathbf{T}^{-1}$ in the $\mathbf{P}_A$ system.

ME 402/512 CONTINUUM MECHANICS ASSIGNMENT 6

1. Use indicial notation to show that

$$
\begin{aligned}
&\text{(i) } (\mathbf{F}\mathbf{v})\cdot\nabla = \mathbf{v}\cdot(\mathbf{F}\nabla) + \mathbf{F}(\mathbf{v}\cdot\nabla) \\
&\text{(ii) } (\mathbf{F}\nabla)\times\nabla = 0 \\
&\text{(iii) } (\mathbf{v}\times\nabla)\cdot\nabla = 0 \\
&\text{(iv) } (\mathbf{v}\times\nabla)\times\nabla = (\mathbf{v}\cdot\nabla)\nabla - \mathbf{v}\nabla^2 \\
&\text{(v) } (\mathbf{u}\otimes\mathbf{v})\cdot\nabla = (\mathbf{u}\nabla)\cdot\mathbf{v} + \mathbf{u}(\mathbf{v}\cdot\nabla) \\
&\text{(vi) } \operatorname{curl}(\operatorname{grad}\phi) = 0
\end{aligned}
$$

2, If $\mathbf{u} = x_1x_2x_3\mathbf{e}_1 + x_1x_2\mathbf{e}_2 + x_1\mathbf{e}_3$, determine the divergence of $\mathbf{u}$, the curl of $\mathbf{u}$, and the gradient of $\mathbf{u}$. Verify that (iv) of Problem 1 is satisfied.

3. Find the components of (i) $\mathbf{T}\times\nabla$ and (ii) $C_{13}(\mathbf{I}\times\mathbf{T})$ where $\mathbf{T}$ is a second-order tensor.

4. A force of magnitude F acts in a direction radially away from the origin at a point $\left(\frac{2a}{3}, \frac{b}{3}, \frac{2c}{3}\right)$ on the surface of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$. Determine the component of the force in the direction of the normal to the surface.

5. (i) If $\mathbf{r}$ is the position vector, use the divergence theorem to express $\int_{\partial R} \mathbf{r}\cdot\mathbf{n}\,ds$ in terms of the volume of the region R.

(ii) Actually perform the surface integral for a unit cube with one corner at the reference point of an Euclidean point space.

6. A plane area in the x_1 - x_2 plane is bounded by the square with corners (0, 0), (b, 0), (b, b), (0, b). A vector $\mathbf{v}$ has components $v_1 = Ax_2, v_2 = Bx_2, v_3 = 0$ where A and B are constants.

(i) Verify that the divergence theorem holds for a cube with one surface coincident with the plane area.

(ii) Verify that Stokes' theorem holds.

1. The deformation of a continuous medium is defined by the equations

$$
\begin{aligned}
x_1 &= \frac{1}{2}(X_1 + X_2)\alpha + \frac{1}{2}(X_1 - X_2)\beta \\
x_2 &= \frac{1}{2}(X_1 + X_2)\alpha - \frac{1}{2}(X_1 - X_2)\beta \\
x_3 &= X_3
\end{aligned}
\tag{1}
$$

in which α and β are constants and the bases $\mathbf{e}_i$ and $\mathbf{E}_A$ coincide.

(a) Obtain the components of $\mathbf{F}$ as functions of material coordinates. Invert to obtain the components of $\mathbf{F}^{-1}$.

(b) Obtain the components of $\mathbf{E}$ and $\mathbf{e}$ as functions of material coordinates.

(c) Express the displacement components in terms of material coordinates and time.

(d) Obtain the components of $\mathbf{E}$ from the expression for $\mathbf{E}$ in terms of gradients of displacement.

(d) Express the small-strain components in terms of material coordinates and time.

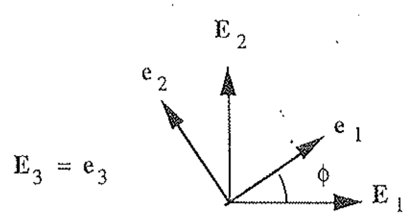
(e) Sketch the deformation provided by Eq. (1).

2. Invert the deformation relation of Eq. (1) to obtain X_A as functions of x_i.

(a) Obtain the components of $\mathbf{F}^{-1}$ as functions of spatial coordinates. Invert to obtain the components of $\mathbf{F}$.

(b) Obtain the components of $\mathbf{E}$ and $\mathbf{e}$ as functions of spatial coordinates. Show that these expressions agree with those obtained in part (b) of Problem 1.

3. Now suppose the bases are related as follows:



(a) With the use of Eq. (1), obtain the components of $\mathbf{F}$ in the mixed basis $\mathbf{e}_i \otimes \mathbf{E}_A$, in the material basis $\mathbf{E}_A \otimes \mathbf{E}_B$, and in the spatial basis $\mathbf{e}_i \otimes \mathbf{e}_j$.

(b) Obtain the components of $\mathbf{E}$ in the material basis two ways: (i) by using the mixed components of $\mathbf{F}$, and (ii) by using the material components of $\mathbf{F}$.

(c) Invert the mixed components, F_{iA} to obtain the mixed components of the inverse, $F^{-1}{}_{Ai}$. Use these latter components to obtain the components of $\mathbf{e}$ in the spatial basis.

4. Now revert to the form for $\mathbf{F}$ obtained in Problem 1. By a simple observation, obtain the components of the right stretch tensor $\mathbf{U}$ and the rotation tensor.

Ref. Malvern: Pages 212 & 213, Nos. 7, 8, 9, 11 and 12.

1. A velocity vector field $\mathbf{v}$ satisfying $\mathbf{v} \cdot \nabla = 0$ is called solenoidal. A volume-preserving motion is called isochoric. (The flow of an incompressible fluid is necessarily isochoric, but there may be isochoric flows of compressible fluids.)

(a) Show that for isochoric motion the velocity field is solenoidal, and conversely.

(b) Show that any velocity field $\mathbf{v}$ given in terms of a vector potential function $\mathbf{Q}$ by $\mathbf{v} = -\mathbf{Q} \times \nabla$ is solenoidal and the flow isochoric.

(c) For incompressible (or isochoric) plane flow in the x_1-x_2 plane, $\mathbf{Q} = Q\mathbf{e}_3$ where the component $Q(x_1, x_2)$ is called a stream function. Show that the volume flux $F_V \equiv \int_A^B \mathbf{v} \cdot \mathbf{n} dS$ across any plane curve joining points x^A and x^B equals $[Q(x^B) - Q(x^A)]$.

2. The circulation around a closed curve C is defined to be $\Gamma = \oint_c \mathbf{v} \cdot d\mathbf{x}$.

(i) Show that $\frac{d\Gamma}{dt} = \oint_c \mathbf{a} \cdot d\mathbf{x} + \oint_c \mathbf{v} \cdot \mathbb{L} \cdot d\mathbf{x}$

(ii) Show that $\oint_c \mathbf{v} \cdot \mathbb{L} \cdot d\mathbf{x} = \oint_c \mathbf{v} \cdot d\mathbf{v} = 0$

(iii) Show that if $\Gamma = 0$ for all curves (irrotational flow) then $\mathbf{v} = \phi\nabla$ where ϕ is a potential function.

(iv) Show that if $\mathbf{v} = \phi\nabla$ then $\dot{\mathbf{v}} = \dot{\phi}\nabla - \mathbf{v} \cdot \mathbb{L}$

3. Show that (i) $\frac{d}{dt}(\mathbf{n}dS) = (\mathbf{v} \cdot \nabla)\mathbf{n}dS - \mathbb{L}^T \cdot \mathbf{n}dS$

(ii) $\frac{d}{dt}\int f\mathbf{n}dS = \int \left[\frac{df}{dt}\mathbf{n} + f(\mathbf{v} \cdot \nabla)\mathbf{n} - f\mathbb{L}^T \cdot \mathbf{n}\right]dS$

ME 402/512 CONTINUUM MECHANICS ASSIGNMENT 9

1. If the intensity of illumination of a fluid particle at (x_1, x_2, x_3) at time t is given by $I = Ae^{-3t} / \left(x_1^2 + x_2^2 + x_3^2\right)$ and the fluid velocity field is given by $v_1 = B\left(x_2 + 2x_3\right)$, $v_2 = B\left(x_2 + 3x_3\right)$, $v_3 = B\left(2x_1 + 3x_2 + 2x_3\right)$ where A and B are known constants, determine the rate of change of the illumination experienced at time t by the fluid particle which is at point (1, 2, -2) at time t.

2. A velocity vector field v satisfying $\mathbf{v} \cdot \nabla = 0$ is called solenoidal. A volume-preserving motion is called isochoric. (The flow of an incompressible fluid is necessarily isochoric, but there may be isochoric flows of compressible fluids.)

(a) Show that for isochoric motion the velocity field is solenoidal, and conversely.

(b) Show that any velocity field **v** given in terms of a vector potential function Q by $\mathbf{v} = -\mathbf{Q} \times \nabla$ is solenoidal and the flow isochoric.

(c) For incompressible (or isochoric) plane flow in the x_1-x_2 plane, Q = Qe_3 where the component $Q(x_1, x_2)$ is called a stream function. Show that the volume flux $F_V \equiv \int_A^B \mathbf{v} \cdot \mathbf{n} dS$ across any plane curve joining points $(x_1, x_2)^A$ and $(x_1, x_2)^B$ equals

$[Q(x_1, x_2)^B - Q(x_1, x_2)^A]$.

2. The circulation around a closed curve C is defined to be $\Gamma = \oint_c \mathbf{v} \cdot d\mathbf{x}$.

(i) Show that $\dfrac{d\Gamma}{dt} = \oint_c \mathbf{a} \cdot d\mathbf{x} + \oint_c \mathbf{v} \cdot \mathbf{L} \cdot d\mathbf{x}$

(ii) Show that $\oint_c \mathbf{v} \cdot \mathbf{L} \cdot d\mathbf{x} = \oint_c \mathbf{v} \cdot d\mathbf{v} = 0$

(iii) Show that if Γ = 0 for all curves (irrotational flow) then $\mathbf{v} = \phi\nabla$ where ϕ is a potential function.

(iv) Show that if $\mathbf{v} = \phi\nabla$ then $\dot{\mathbf{v}} = \dot{\phi}\nabla - \mathbf{v} \cdot \mathbf{L}$

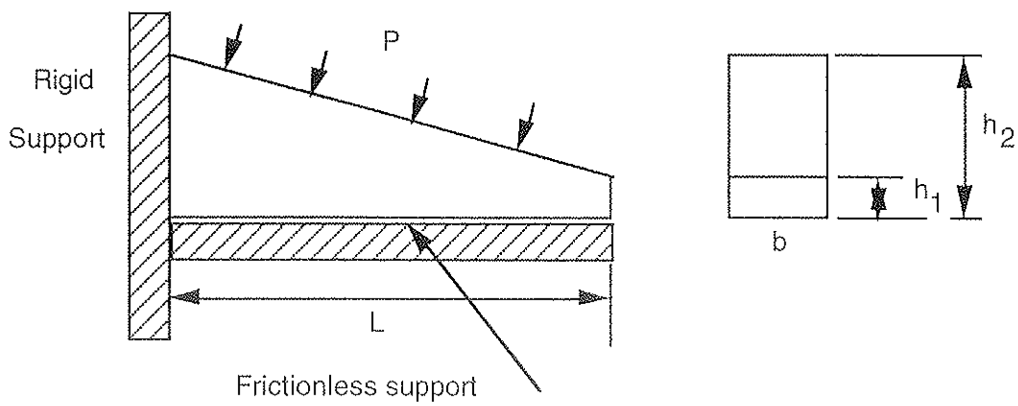
3. Show that (i) $\dfrac{d}{dt}(\mathbf{n}dS) = (\mathbf{v} \cdot \nabla)\mathbf{n}dS - \mathbf{L}^T \cdot \mathbf{n}dS$

(ii) $\dfrac{d}{dt}\int f\mathbf{n}dS = \int \left[\dfrac{df}{dt}\mathbf{n} + f(\mathbf{v} \cdot \nabla)\mathbf{n} - f\mathbf{L}^T \cdot \mathbf{n}\right]dS$

ME 402/512 CONTINUUM MECHANICS ASSIGNMENT 10

1. A strain gauge provides a measure of stretch in the direction of the strain gauge. A strain-gauge rosette is composed of three strain gauges oriented at angles α, β and γ with respect to the x_1-axis. Set up the equations that will provide components of the Lagrangian strain tensor in two dimensions with the readings of the strain gauges known.

2. A variable-thickness cantilevered beam supports a uniform pressure on the top surface. Give the exact (3-dimensional) boundary conditions.



3. A kinematically-admissible displacement field is one from which strains can be derived (i.e., sufficient continuity conditions) and which satisfies the displacement boundary consitions $u = u^*$ on ∂R_u. A variation $\delta\mathbf{u}$ is an imaginary variation from such a field with $\delta\mathbf{u} = 0$ on ∂R_u. Show that the principle of virtual displacements is equivalent to Cauchy's first equation of motion, i.e., if $\boldsymbol{\sigma} = \boldsymbol{\sigma}^T$ and if

$$\int_{\partial R_t} \mathbf{t}\cdot\delta\mathbf{u}\,dS + \int_R (\rho\mathbf{b} - \rho\mathbf{a})\cdot\delta\mathbf{u}\,dV = \int_R \mathrm{tr}(\boldsymbol{\sigma}\cdot\delta\mathbf{e})\,dV$$

for all kinematically admissible variations, then $\boldsymbol{\sigma}\cdot\nabla + \rho\mathbf{b} = \rho\mathbf{a}$. Show this for <u>infinitesimal</u> deformations, i.e., $\mathbf{e} = (\mathbf{u}\nabla)_{sym}$.

4. (i) Prove that $\mathrm{tr}(\mathbf{A}\cdot\mathbf{B}\cdot\mathbf{C}) = \mathrm{tr}(\mathbf{B}\cdot\mathbf{C}\cdot\mathbf{A})$ for second-order tensors $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$.

(ii) The stress power is defined to be $S_p = \int_R \mathrm{tr}(\boldsymbol{\sigma}\cdot\mathbf{D})dV$. Use the results of part (i) and the definitions $\boldsymbol{\Sigma} \equiv \mathbf{R}^T\cdot\boldsymbol{\sigma}\cdot\mathbf{R}$ and $\mathbf{D}^* \equiv \mathbf{R}^T\cdot\mathbf{D}\cdot\mathbf{R}$ to show that alternative expressions for the stress power are:

$$S_p = \int_R \mathrm{tr}\,(\boldsymbol{\Sigma}\cdot\mathbf{D}^*)dV = \int_{R_0} \mathrm{tr}\,(\hat{\boldsymbol{\tau}}\cdot\dot{\mathbf{F}}^T)dV_o = \int_{R_0} \mathrm{tr}\,(\boldsymbol{\tau}\cdot\dot{\mathbf{E}})dV_o$$

These combinations of stress and deformation rates are said to be "conjugate."

Recall that for a tensor, $\mathbf{T}$, two sets of invariants are defined by $I_T = \text{tr}\,\mathbf{T}$, $II_T = \text{tr}\,\mathbf{T}^2$, $III_T = \text{tr}\,\mathbf{T}^3$ and $\hat{I}_T = I_T$, $\hat{II}_T = \frac{1}{2}(II_T - I_T^2)$, $\hat{III}_T = \det \mathbf{T}$.

1. If $\boldsymbol{\sigma}$ denotes the Cauchy stress tensor, the stress deviator is defined to be $\boldsymbol{\sigma}^d = \boldsymbol{\sigma} - \frac{1}{3} I_\sigma \mathbf{I}$. (i) What are the relationships between the eigenvalues and the eigenvectors of $\boldsymbol{\sigma}$ and $\boldsymbol{\sigma}^d$.

(ii) Show that $\hat{I}_{\sigma^d} = 0$

2. The second invariant of the stress deviator is often used as a measure of when a metal will yield (Mises stress). The second invariant assumes several forms which, at first glance, appear to be different. The purpose of this problem is to show equivalence. [Hint: Some of the expressions are obtained by adding or subtracting zero in the form of the square of the first invariant.] Verify the following:

(i) $\hat{II}_{\sigma^d} = -(\sigma_{11}^d \sigma_{22}^d + \sigma_{22}^d \sigma_{33}^d + \sigma_{33}^d \sigma_{11}^d) + (\sigma_{12}^d)^2 + (\sigma_{23}^d)^2 + (\sigma_{31}^d)^2$

(ii) $\hat{II}_{\sigma^d} = \frac{1}{2}\left[(\sigma_{11}^d)^2 + (\sigma_{22}^d)^2 + (\sigma_{33}^d)^2\right] + (\sigma_{12}^d)^2 + (\sigma_{23}^d)^2 + (\sigma_{31}^d)^2$

(iii) $\hat{II}_{\sigma^d} = \frac{1}{6}\left[(\sigma_{11}^d - \sigma_{22}^d)^2 + (\sigma_{22}^d - \sigma_{33}^d)^2 + (\sigma_{33}^d - \sigma_{11}^d)^2\right] + (\sigma_{12}^d)^2 + (\sigma_{23}^d)^2 + (\sigma_{31}^d)^2$

(iv) $\hat{II}_{\sigma^d} = \frac{1}{6}\left[(\sigma_{11} - \sigma_{22})^2 + (\sigma_{22} - \sigma_{33})^2 + (\sigma_{33} - \sigma_{11})^2\right] + (\sigma_{12}^d)^2 + (\sigma_{23}^d)^2 + (\sigma_{31}^d)^2$

3. Let $\overline{\sigma} = \alpha\,[\,\hat{II}_{\sigma^d}]^{1/2}$ and $\overline{\sigma}_s = \beta\,[\,\hat{II}_{\sigma^d}]^{1/2}$ denote the "effective" stress and "equivalent" shear stress, respectively. Determine α so that $\overline{\sigma} = |\sigma_{11}|$ for uniaxial stress (σ_{11} the only nonzero component) and β so that $\overline{\sigma}_s = |\sigma_{12}|$ for pure shear (σ_{12} the only nonzero component).

4. Construct an orthonormal basis $\mathbf{E}_1$, $\mathbf{E}_2$, $\mathbf{E}_3$ such that $\mathbf{E}_3$ makes equal angles with the three principal directions $\mathbf{P}_1$, $\mathbf{P}_2$ and $\mathbf{P}_3$ of $\boldsymbol{\sigma}$. Other than being normal to $\mathbf{E}_3$, the choice for the other two, $\mathbf{E}_1$ and $\mathbf{E}_2$, is based on convenience. The plane $\mathbf{E}_1$ - $\mathbf{E}_2$ is called the octahedral plane.

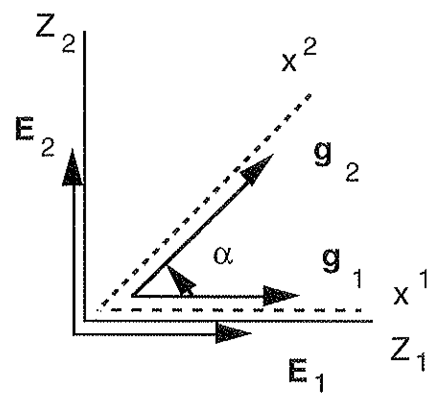
(i) Find an expression for the traction vector $\mathbf{t}$ on this plane in terms of the principal values, σ_i, of $\boldsymbol{\sigma}$, i.e., find t_i where $\mathbf{t} = t_i \mathbf{E}_i$.

(ii) Show that the octahedral normal stress, t_3, is $t_3 = \frac{1}{3} I_\sigma$.

(iii) The octahedral shear stress is defined to be $\tau_0 \equiv [(t_1)^2 + (t_2)^2]^{1/2}$.

Show that $\tau_0 = [\frac{2}{3}\,\hat{II}_{\sigma^d}]^{1/2}$.

1. In two dimensions a rectangular Cartesian system Z_1, Z_2 with base vectors $\mathbf{E}_1, \mathbf{E}_2$ is assumed to exist. Another Cartesian system x^i with base vectors $\mathbf{g}_i$ also exists as shown in the figure.



(i) If $|\mathbf{g}_1| = a$ and $|\mathbf{g}_2| = b$ construct a coordinate transformation $x^i = f^i(Z_1, Z_2)$ such that $\dfrac{\partial \mathbf{r}}{\partial x^i} = \mathbf{g}_i$; i.e., find f^1 and f^2.

(ii) Determine g_{ij}, g^{ij} and $\mathbf{g}^i$.

(iii) If the components of $\mathbf{v}$ in the $\mathbf{E}_A$ system are (2, 4), determine v^i and v_i.

2. For cylindrical coordinates where $x^i \Rightarrow (r, \theta, z)$

(i) Give the relations between x^i and Z_A, $\mathbf{g}_i$ and $\mathbf{E}_A$, $\mathbf{g}^i$ and $\mathbf{E}_A$.
(ii) Find r_i, r^i, $r_{\underline{i}}$ and $r^{\underline{i}}$ where $\mathbf{r} = r_A\mathbf{E}_A = r_i\mathbf{g}^i = r^i\mathbf{g}_i = r_{\underline{i}}\mathbf{g}^{\underline{i}} = r^{\underline{i}}\mathbf{g}_{\underline{i}}$.
(iii) Obtain $g_{ij} = \mathbf{g}_i \cdot \mathbf{g}_j$ and $g^{ij} = [g_{ij}]^{-1}$.

(iv) Verify that $g_{ij} = \dfrac{\partial Z_A}{\partial x^i}\dfrac{\partial Z_A}{\partial x^j}$ and $g^{ij} = \mathbf{g}^i \cdot \mathbf{g}^j$.

(v) Obtain the Christoffel symbols of the second kind using $\begin{Bmatrix} k \\ i\ j \end{Bmatrix} = \mathbf{g}_{i,j} \cdot \mathbf{g}^k$.

(vi) Obtain the Christoffel symbols of the first kind using $[ij, k] = \begin{Bmatrix} l \\ i\ j \end{Bmatrix} g_{kl}$.

(vii) Obtain the components of $\mathbf{H} = \mathbf{u}\nabla$
(viii) Obtain the physical components of $\mathbf{H}$ in terms of the physical components of $\mathbf{u}$.

ME 412/512 CONTINUUM MECHANICS (ALTERNATIVE) ASSIGNMENT 12

Note: This alternative is for ME 412. It can also serve as a review exercise for ME 512.

1. Express the following equations in terms of components and base vectors. Give the corresponding equations in indicial notation (capital letters are second-order tensors, lower-case letters are vectors):

$$\mathbf{A} = \mathbf{R} \cdot \mathbf{T} \qquad \mathbf{B} = C_{14}(\mathbf{R} \otimes \mathbf{T}) \qquad \mathbf{C} = C_{23}\left(\mathbf{R}^{\mathrm{T}} \otimes \mathbf{T}\right) \qquad \mathbf{d} = \mathbf{T} \cdot \nabla \qquad \mathbf{e} = \mathbf{T}^{\mathrm{T}} \cdot \nabla$$

2. Develop the transformation relations for the components of vectors and tensors expressed in terms of two bases. Illustrate your relations in terms of the transformation matrix corresponding to the bases shown in Fig. 1. Assume right-handed systems.

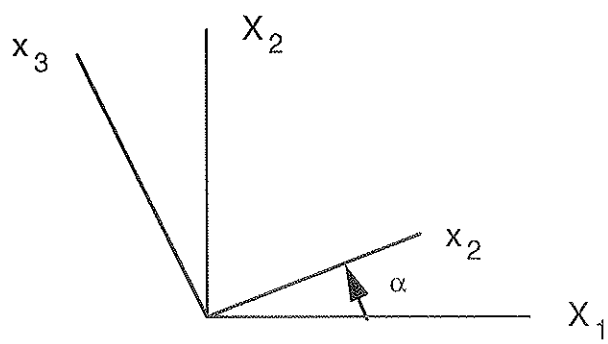


Figure 1.

3. Let a stress invariant be defined by $II^* = \frac{1}{2}\left[\mathrm{tr}\left(\boldsymbol{\sigma}^2\right) - (\mathrm{tr}\,\boldsymbol{\sigma})^2\right]$. Let II^{*d} be the corresponding invariant of the stress deviator $\boldsymbol{\sigma}^d$. Let ∇_σ denote the gradient with respect to $\boldsymbol{\sigma}$. Show that $II^{*d}\,\nabla_\sigma = \boldsymbol{\sigma}^d$.

4. If A, k are constants, x_i are spatial coordinates and

$$v_1 = -A\left(x_1^3 + x_1 x_2^2\right)e^{-kt} \qquad v_2 = -A\left(x_1^2 x_2 + x_2^3\right)e^{-kt} \qquad v_3 = 0$$

find the components of the acceleration vector at the point (1, 1, 0) at time t = 0.