13. Rates of Deformation and of Strain

13.1 Initial Comments

In this chapter we obtain expressions for velocity and acceleration of material points. There are several opportunities for confusion but with special emphasis on specifying the independent variables and the use of the chain rule, the equations are relatively straightforward. Again, the problem becomes one of learning an additional set of definitions. Then rates of various tensors related to deformation are defined and relationships among these rates are developed.

13.2 Velocity and Acceleration

Recall that time-independent deformation of a body is described if the current position is defined as a function of the material position vector, $\boldsymbol{r} = \boldsymbol{r}(\boldsymbol{R})$. Now we allow the possibility of time dependence by describing the deformation as follows:

$$\boldsymbol{r} = \boldsymbol{r}(\boldsymbol{R},t) \tag{13-1}$$

This equation explicitly describes the motion of material points so $\boldsymbol{R}$ and t are independent variables. The velocity, $\boldsymbol{v}$, and acceleration, $\boldsymbol{a}$, are defined to be

$$\boldsymbol{v}(\boldsymbol{R},t) = \left.\frac{\partial \boldsymbol{r}}{\partial t}\right|_{\boldsymbol{R}} \qquad \boldsymbol{a}(\boldsymbol{R},t) = \left.\frac{\partial \boldsymbol{v}}{\partial t}\right|_{\boldsymbol{R}} \tag{13-2}$$

i.e., the material vector is held fixed for the time derivative. The expressions for velocity and acceleration are material time derivatives of the position and velocity, respectively. Two other notations that are commonly used are

$$\boldsymbol{v}(\boldsymbol{R},t) = \frac{d\boldsymbol{r}}{dt} = \dot{\boldsymbol{r}} \qquad \boldsymbol{a}(\boldsymbol{R},t) = \frac{d\boldsymbol{v}}{dt} = \dot{\boldsymbol{v}} \tag{13-3}$$

At least theoretically, the inverse of (13-1) exists in which case

$$\boldsymbol{R} = \boldsymbol{R}\left\{\boldsymbol{r}(t),t\right\} \tag{13-4}$$

in which case the variable, t, appears explicitly as well as a parameter in the expression for the spatial position vector. If (13-4) is used in the expression for velocity, we obtain velocity as a function of the spatial position and time, or

$$\boldsymbol{v} = \boldsymbol{v}\left\{\boldsymbol{r}(t),t\right\} \tag{13-5}$$

Now the derivative with respect to time consists of two parts:

$$\boldsymbol{a} = \frac{d\boldsymbol{v}}{dt} = \left.\frac{\partial \boldsymbol{v}}{\partial t}\right|_{\boldsymbol{r}} + \boldsymbol{v}\bar{\bar{\nabla}} \cdot \frac{d\boldsymbol{r}}{dt} \tag{13-6}$$

or

$$\boldsymbol{a}(\boldsymbol{r},t) = \left.\frac{\partial \boldsymbol{v}}{\partial t}\right|_{\boldsymbol{r}} + \boldsymbol{v}\bar{\nabla} \cdot \boldsymbol{v} \tag{13-7}$$

Although the acceleration is just the time derivative of position following a material point, the phrase "substantial derivative" is often used in the literature related to fluid mechanics. The first term on the right is sometimes called the spatial acceleration.

If the origin for the spatial position vector is fixed, alterative forms for velocity and acceleration are obtained using the displacement vector $\boldsymbol{u}$:

$$\boldsymbol{v}(\boldsymbol{R},t) = \frac{d\boldsymbol{u}}{dt} = \dot{\boldsymbol{u}} \qquad \boldsymbol{a}(\boldsymbol{R},t) = \frac{d^2\boldsymbol{u}}{dt^2} = \dot{\boldsymbol{v}} \tag{13-8}$$

13.3 Rate of Deformation Gradient and Related Expressions

Recall that the deformation gradient is defined to be

$$\boldsymbol{F} = \left\{\boldsymbol{r}(\boldsymbol{R},\boldsymbol{t}\right\}\bar{\nabla}_0 \tag{13-9}$$

Since $\boldsymbol{R}$ and t are independent variables, the time derivative can be interchanged with the material gradient operator, or symbolically

$$\frac{d}{dt}\left[()\bar{\nabla}_0\right] = \left[\frac{d}{dt}()\right]\bar{\nabla}_0 \tag{13-10}$$

We apply (13-10) to (13-9) and obtain

$$\dot{\boldsymbol{F}} = \boldsymbol{v}\bar{\nabla}_0 \tag{13-11}$$

But we showed in (10-45) that the gradient operators are related by

$$()\bar{\nabla}_0 = ()\bar{\nabla} \cdot \boldsymbol{F} \tag{13-12}$$

It follows that an alternative form for the rate of the deformation gradient, and the corresponding expression for the transpose, are

$$\dot{\boldsymbol{F}} = \boldsymbol{L} \cdot \boldsymbol{F} \qquad \dot{\boldsymbol{F}}^T = \boldsymbol{F}^T \cdot \boldsymbol{L}^T \tag{13-13}$$

in which the spatial velocity gradient is denoted as

$$\boldsymbol{L} = \boldsymbol{v}\bar{\nabla} \tag{13-14}$$

If we decompose $\boldsymbol{L}$ into symmetric and skew-symmetric parts, then

$$\boldsymbol{L} = \boldsymbol{d} + \boldsymbol{W} \tag{13-15}$$

where **d** is called the "spatial" rate of deformation tensor, and $\boldsymbol{W}$ is the vorticity tensor:

$$\boldsymbol{d} = \frac{1}{2}\left(\boldsymbol{L} + \boldsymbol{L}^T\right) \qquad \boldsymbol{W} = \frac{1}{2}\left(\boldsymbol{L} - \boldsymbol{L}^T\right) \tag{13-16}$$

Since the vorticity tensor is skew symmetric, we define the vorticity vector to be

$$\boldsymbol{\omega} = \frac{1}{2}\boldsymbol{\varepsilon}\cdot\cdot\boldsymbol{W} \qquad \omega_i = \frac{1}{2}\varepsilon_{ijk}W_{jk} \tag{13-17}$$

and the inverse relation is

$$\boldsymbol{W} = \boldsymbol{\varepsilon}\cdot\boldsymbol{w} \qquad W_{ij} = \varepsilon_{ijk}w_k \tag{13-18}$$

Now we return to (13-13) and multiply on the right by $\boldsymbol{F}^{-1}$ to obtain an alternative form for $\boldsymbol{L}$, namely

$$\boldsymbol{L} = \dot{\boldsymbol{F}}\cdot\boldsymbol{F}^{-1} \tag{13-19}$$

Next we obtain the equation for the inverse of $\dot{\boldsymbol{F}}$ and its transpose as follows:

$$\frac{d}{dt}(\boldsymbol{I}) = \frac{d}{dt}\left(\boldsymbol{F}\cdot\boldsymbol{F}^{-1}\right) = \dot{\boldsymbol{F}}\cdot\boldsymbol{F}^{-1} + \boldsymbol{F}\cdot\dot{\boldsymbol{F}}^{-1} = \boldsymbol{0} \tag{13-20}$$

which, with the use of (13-13), yields the relations

$$\dot{\boldsymbol{F}}^{-1} = -\boldsymbol{F}^{-1}\cdot\boldsymbol{L} \qquad \dot{\boldsymbol{F}}^{-T} = -\boldsymbol{L}^T\cdot\boldsymbol{F}^{-T} \tag{13-21}$$

13.4 Rates of Strains

Recall that the Lagrangian strain is

$$\boldsymbol{E} = \tfrac{1}{2}\left[\boldsymbol{F}^T\cdot\boldsymbol{F} - \boldsymbol{I}\right] \tag{13-22}$$

The use of (13-13) yields

$$\dot{\boldsymbol{E}} = \tfrac{1}{2}\left[\dot{\boldsymbol{F}}^T\cdot\boldsymbol{F} + \boldsymbol{F}^T\cdot\dot{\boldsymbol{F}}\right] = \tfrac{1}{2}\left[\boldsymbol{F}^T\cdot\boldsymbol{L}^T\cdot\boldsymbol{F} + \boldsymbol{F}^T\cdot\boldsymbol{L}\cdot\boldsymbol{F}\right] \tag{13-23}$$

We define a "material" rate of deformation, $\boldsymbol{D}$, to be

$$\boldsymbol{D} = \boldsymbol{F}^T\cdot\boldsymbol{d}\cdot\boldsymbol{F} \qquad \boldsymbol{d} = \tfrac{1}{2}\left[\boldsymbol{L} + \boldsymbol{L}^T\right] \tag{13-24}$$

Then the Lagrangian strain rate is simply

$$\dot{\boldsymbol{E}} = \boldsymbol{D} \tag{13-25}$$

To obtain the Eulerian strain rate, we take the time derivative of

$$\boldsymbol{e} = \frac{1}{2}\left[\boldsymbol{I} - \boldsymbol{F}^{-T}\cdot\boldsymbol{F}^{-1}\right] \tag{13-26}$$

and use (13-21)

$$\dot{\boldsymbol{e}} = -\tfrac{1}{2}\left[\dot{\boldsymbol{F}}^{-T}\cdot\boldsymbol{F}^{-1} + \boldsymbol{F}^{-T}\cdot\dot{\boldsymbol{F}}^{-1}\right] = \tfrac{1}{2}\left[\boldsymbol{L}^T\cdot\boldsymbol{F}^{-T}\cdot\boldsymbol{F}^{-1} + \boldsymbol{F}^{-T}\cdot\boldsymbol{F}^{-1}\cdot\boldsymbol{L}\right] \tag{13-27}$$

We note from (13-26) that

$$\boldsymbol{F}^{-T}\cdot\boldsymbol{F}^{-1} = \boldsymbol{I} - 2\boldsymbol{e} \tag{13-28}$$

so that (13-27) becomes

$$\dot{\boldsymbol{e}} = \tfrac{1}{2}\left[\boldsymbol{L}^T \cdot \left(\boldsymbol{I} - 2\boldsymbol{e}\right) + \left(\boldsymbol{I} - 2\boldsymbol{e}\right) \cdot \boldsymbol{L}\right] \tag{13-29}$$

or

$$\dot{\boldsymbol{e}} = \boldsymbol{d} - \boldsymbol{L}^T \cdot \boldsymbol{e} - \boldsymbol{e} \cdot \boldsymbol{L} \tag{13-30}$$

Sometimes the logarithmic strain is proposed which in the spectral decomposition form is rather attractive:

$$\boldsymbol{L}_U = ln(\boldsymbol{U}) \equiv \sum_{i=1}^{3} (ln\, \Lambda_i) \boldsymbol{N}_i \otimes \boldsymbol{N}_i \tag{13-31}$$

However, its rate is

$$\boldsymbol{L}_U = ln(\boldsymbol{U}) \equiv \sum_{i=1}^{3} \left[\frac{\dot{\Lambda}_i}{\Lambda_i} \boldsymbol{N}_i \otimes \boldsymbol{N}_i + (ln\, \Lambda_i) \dot{\boldsymbol{N}}_i \otimes \boldsymbol{N}_i + (ln\, \Lambda_i) \boldsymbol{N}_i \otimes \dot{\boldsymbol{N}}_i \right] \tag{13-32}$$

a result that involves the rates of the eigenvectors which, in general, are not zero because the eigenvectors rotate.

Finally, we recall that the scalar J relates volume elements in the deformed and undeformed configuration. Therefore, any measure of volumetric stain involves J in one form or another, and the volumetric strain rate requires the rate of J. Next, we obtain the following relation for the rate of the determinant of $\boldsymbol{F}$:

$$\dot{J} = Jtr\left(\boldsymbol{L}\right) \qquad J = det\left(\boldsymbol{F}\right) \tag{13-33}$$

The use of (13-14) and (13-15) yields the following equivalent expressions:

$$\dot{J} = Jtr\left(\boldsymbol{d}\right) = J\left(\boldsymbol{v} \cdot \bar{\nabla}\right) \tag{13-34}$$

If $\boldsymbol{a}, \boldsymbol{b}$ and $\boldsymbol{c}$ are constant but arbitrary vectors, we start with the definition of a determinant in direct notation from (6-18):

$$J\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c}) = \left(\boldsymbol{F} \cdot \boldsymbol{a}\right) \cdot \left[\left(\boldsymbol{F} \cdot \boldsymbol{b}\right) \times \left(\boldsymbol{F} \cdot \boldsymbol{c}\right)\right] \qquad \forall \ \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \tag{13-35}$$

and the definition of the trace of a tensor in direct notation [This should have been developed in a previous chapter.]

$$\begin{aligned} tr\left(\boldsymbol{L}\right)\left[\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c})\right] &= \left(\boldsymbol{L} \cdot \boldsymbol{a}\right) \cdot \left[\left(\boldsymbol{b}\right) \times \left(\boldsymbol{c}\right)\right] \\ &+ \left(\boldsymbol{a}\right) \cdot \left[\left(\boldsymbol{L} \cdot \boldsymbol{b}\right) \times \left(\boldsymbol{c}\right)\right] + \left(\boldsymbol{a}\right) \cdot \left[\left(\boldsymbol{b}\right) \times \left(\boldsymbol{L}\boldsymbol{c}\right)\right] \end{aligned} \tag{13-36}$$

Take the time derivative of the terms in (13-35)

$$\begin{aligned} \dot{J}\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c}) &= \left(\dot{\boldsymbol{F}} \cdot \boldsymbol{a}\right) \cdot \left[\left(\boldsymbol{F} \cdot \boldsymbol{b}\right) \times \left(\boldsymbol{F} \cdot \boldsymbol{c}\right)\right] \\ &+ \left(\boldsymbol{F} \cdot \boldsymbol{a}\right) \cdot \left[\left(\dot{\boldsymbol{F}} \cdot \boldsymbol{b}\right) \times \left(\boldsymbol{F} \cdot \boldsymbol{c}\right)\right] \\ &+ \left(\boldsymbol{F} \cdot \boldsymbol{a}\right) \cdot \left[\left(\boldsymbol{F} \cdot \boldsymbol{b}\right) \times \left(\dot{\boldsymbol{F}} \cdot \boldsymbol{c}\right)\right] \end{aligned} \tag{13-37}$$

Now use (13-13) to obtain

$$
\begin{aligned}
\dot{J}\boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{c}) &= (\boldsymbol{L}\cdot\boldsymbol{F}\cdot\boldsymbol{a})\cdot\left[(\boldsymbol{F}\cdot\boldsymbol{b})\times(\boldsymbol{F}\cdot\boldsymbol{c})\right] \\
&+(\boldsymbol{F}\cdot\boldsymbol{a})\cdot\left[(\boldsymbol{L}\cdot\boldsymbol{F}\cdot\boldsymbol{b})\times(\boldsymbol{F}\cdot\boldsymbol{c})\right] \\
&+(\boldsymbol{F}\cdot\boldsymbol{a})\cdot\left[(\boldsymbol{F}\cdot\boldsymbol{b})\times(\boldsymbol{L}\cdot\boldsymbol{F}\cdot\boldsymbol{c})\right]
\end{aligned}
\tag{13-38}
$$

Then the expression for the trace of a tensor in direct notation, (13-36), yields

$$
\begin{aligned}
\dot{J}\boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{c}) &= tr(\boldsymbol{L})(\boldsymbol{F}\cdot\boldsymbol{a})\cdot\left[(\boldsymbol{F}\cdot\boldsymbol{b})\times(\boldsymbol{F}\cdot\boldsymbol{c})\right] \\
&= tr(\boldsymbol{L})det(\boldsymbol{F})\{\boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{c})\}
\end{aligned}
\tag{13-39}
$$

which results in (13-33).

13.5 Spin and Vorticity

Recall that $\boldsymbol{R}$ is orthogonal so that $\boldsymbol{R}\cdot\boldsymbol{R}^T=\boldsymbol{I}$. Take a time derivative to obtain

$$
\dot{\boldsymbol{R}}\cdot\boldsymbol{R}^T+\boldsymbol{R}\cdot\dot{\boldsymbol{R}}^T=\boldsymbol{0} \qquad \text{or} \qquad \dot{\boldsymbol{R}}\cdot\boldsymbol{R}^T=-\left(\dot{\boldsymbol{R}}\cdot\boldsymbol{R}^T\right)^T
\tag{13-40}
$$

The spin, $\boldsymbol{\Omega}$, is defined to be

$$
\boldsymbol{\Omega}=\dot{\boldsymbol{R}}\cdot\boldsymbol{R}^T
\tag{13-41}
$$

and (13-40) indicates that the spin is skew-symmetric. It follows that

$$
\dot{\boldsymbol{R}}=\boldsymbol{\Omega}\cdot\boldsymbol{R} \qquad \dot{\boldsymbol{R}}^T=\boldsymbol{R}^T\cdot\boldsymbol{\Omega}^T=-\boldsymbol{R}^T\cdot\boldsymbol{\Omega}
\tag{13-42}
$$

Also recall that

$$
\boldsymbol{R}=\boldsymbol{n}_i\otimes\boldsymbol{N}_i \qquad \boldsymbol{n}_i=\boldsymbol{R}\cdot\boldsymbol{N}_i \qquad \boldsymbol{N}_i=\boldsymbol{R}^T\cdot\boldsymbol{n}_i
\tag{13-43}
$$

By taking rates and using (13-43) we obtain

$$
\begin{aligned}
&\dot{\boldsymbol{n}}_i=\dot{\boldsymbol{R}}\cdot\boldsymbol{N}_i+\boldsymbol{R}\cdot\dot{\boldsymbol{N}}_i=\boldsymbol{\Omega}\cdot\boldsymbol{R}\cdot\boldsymbol{N}_i+\boldsymbol{R}\cdot\dot{\boldsymbol{N}}_i \\
&\dot{\boldsymbol{n}}_i=\boldsymbol{\Omega}\cdot\boldsymbol{n}_i+\boldsymbol{R}\cdot\dot{\boldsymbol{N}}_i \\
&\dot{\boldsymbol{N}}_i=\dot{\boldsymbol{R}}^T\cdot\boldsymbol{n}_i+\boldsymbol{R}^T\cdot\dot{\boldsymbol{n}}_i \\
&\dot{\boldsymbol{N}}_i=-\left(\boldsymbol{R}^T\cdot\boldsymbol{\Omega}\cdot\boldsymbol{R}\right)\cdot\boldsymbol{N}_i+\boldsymbol{R}^T\cdot\dot{\boldsymbol{n}}_i
\end{aligned}
\tag{13-44}
$$

To obtain the relation between spin and vorticity, $\boldsymbol{W}$, recall that

$$
\begin{aligned}
&\boldsymbol{F}=\boldsymbol{R}\cdot\boldsymbol{U} \qquad \boldsymbol{F}^{-1}=\boldsymbol{U}^{-1}\cdot\boldsymbol{R}^T \\
&\dot{\boldsymbol{F}}=\boldsymbol{L}\cdot\boldsymbol{F}=\dot{\boldsymbol{R}}\cdot\boldsymbol{U}+\boldsymbol{R}\cdot\dot{\boldsymbol{U}}
\end{aligned}
\tag{13-45}
$$

and

$$
\begin{aligned}
&\boldsymbol{L}=\boldsymbol{d}+\boldsymbol{W} \qquad \boldsymbol{L}^T=\boldsymbol{d}-\boldsymbol{W} \\
&\boldsymbol{L}=\dot{\boldsymbol{F}}\cdot\boldsymbol{F}^{-1}=\dot{\boldsymbol{R}}\cdot\boldsymbol{R}^T+\boldsymbol{R}\cdot\dot{\boldsymbol{U}}\cdot\boldsymbol{U}^{-1}\cdot\boldsymbol{R}^T \\
&\boldsymbol{L}^T=\boldsymbol{R}\cdot\dot{\boldsymbol{R}}^T+\boldsymbol{R}\cdot\boldsymbol{U}^{-1}\cdot\dot{\boldsymbol{U}}\cdot\boldsymbol{R}^T
\end{aligned}
\tag{13-46}
$$

It follows that

$$\boldsymbol{d} = \tfrac{1}{2}\left(\boldsymbol{L}+\boldsymbol{L}^T\right) = \tfrac{1}{2}\left(\boldsymbol{\Omega}+\boldsymbol{\Omega}^T\right) + \tfrac{1}{2}\boldsymbol{R}\cdot\left[\dot{\boldsymbol{U}}\cdot\boldsymbol{U}^{-1}+\boldsymbol{U}^{-1}\cdot\dot{\boldsymbol{U}}\right]\cdot\boldsymbol{R}^T \tag{13-47}$$

The sum of the first two terms on the right is zero. Let

$$\boldsymbol{D}^* = \tfrac{1}{2}\left(\dot{\boldsymbol{U}}\cdot\boldsymbol{U}^{-1}+\boldsymbol{U}^{-1}\cdot\dot{\boldsymbol{U}}\right) \tag{13-48}$$

Then

$$\boldsymbol{d} = \boldsymbol{R}\cdot\boldsymbol{D}^*\cdot\boldsymbol{R}^T \tag{13-49}$$

Note from (13-24) that $\boldsymbol{d}$ and $\boldsymbol{D}$ are related through the use of $\boldsymbol{F}$ rather than $\boldsymbol{R}$ and therefore

$$\boldsymbol{D} = \boldsymbol{F}^T\cdot\boldsymbol{d}\cdot\boldsymbol{F} = \boldsymbol{U}^T\cdot\boldsymbol{R}^T\cdot\boldsymbol{d}\cdot\boldsymbol{R}\cdot\boldsymbol{U} \tag{13-50}$$

We use (13-49) to obtain

$$\boldsymbol{D} = \boldsymbol{U}^T\cdot\boldsymbol{D}^*\cdot\boldsymbol{U} \tag{13-51}$$

and then (13-48) yields

$$\boldsymbol{D} = \tfrac{1}{2}\left(\boldsymbol{U}\cdot\dot{\boldsymbol{U}}+\dot{\boldsymbol{U}}\cdot\boldsymbol{U}\right) \tag{13-52}$$

The vorticity becomes

$$\boldsymbol{W} = \tfrac{1}{2}\left(\boldsymbol{L}-\boldsymbol{L}^T\right) = \tfrac{1}{2}\left(\boldsymbol{\Omega}-\boldsymbol{\Omega}^T\right) + \tfrac{1}{2}\boldsymbol{R}\cdot\left[\dot{\boldsymbol{U}}\cdot\boldsymbol{U}^{-1}-\boldsymbol{U}^{-1}\cdot\dot{\boldsymbol{U}}\right]\cdot\boldsymbol{R}^T \tag{13-53}$$

Define a material spin to be

$$\boldsymbol{\Omega}^* = \tfrac{1}{2}\left[\dot{\boldsymbol{U}}\cdot\boldsymbol{U}^{-1}-\boldsymbol{U}^{-1}\cdot\dot{\boldsymbol{U}}\right] \tag{13-54}$$

Then the vorticity becomes the sum of two spin contributions

$$\boldsymbol{W} = \boldsymbol{\Omega}+\boldsymbol{R}\cdot\boldsymbol{\Omega}^*\cdot\boldsymbol{R}^T \tag{13-55}$$

Next we provide another equation that relates vorticity and spin through the left stretch tensor $\boldsymbol{V}$. We start with

$$\boldsymbol{F} = \boldsymbol{V}\cdot\boldsymbol{R} \qquad \boldsymbol{F}^{-1} = \boldsymbol{R}^T\cdot\boldsymbol{V}^{-1} \tag{13-56}$$

Take a time derivative to obtain

$$\begin{aligned} \dot{\boldsymbol{F}} &= \dot{\boldsymbol{V}}\cdot\boldsymbol{R}+\boldsymbol{V}\cdot\dot{\boldsymbol{R}} \\ \boldsymbol{L} = \boldsymbol{d}+\boldsymbol{W} &= \dot{\boldsymbol{F}}\cdot\boldsymbol{F}^{-1} = \dot{\boldsymbol{V}}\cdot\boldsymbol{V}^{-1}+\boldsymbol{V}\cdot\dot{\boldsymbol{R}}\cdot\boldsymbol{R}^T\cdot\boldsymbol{V}^{-1} \end{aligned} \tag{13-57}$$

and

$$\left(\boldsymbol{d}+\boldsymbol{W}\right)\cdot\boldsymbol{V} = \dot{\boldsymbol{V}}+\boldsymbol{V}\cdot\boldsymbol{\Omega} \tag{13-58}$$

Take the transpose to obtain

$$\boldsymbol{V}\cdot\left(\boldsymbol{d}-\boldsymbol{W}\right) = \dot{\boldsymbol{V}}-\boldsymbol{\Omega}\cdot\boldsymbol{V} \tag{13-59}$$

Subtract the terms in (13-59) from corresponding terms in (13-58) to eliminate the rate of $\boldsymbol{V}$ and obtain

$$\boldsymbol{d}\cdot\boldsymbol{V}-\boldsymbol{V}\cdot\boldsymbol{d}+\boldsymbol{W}\cdot\boldsymbol{V}+\boldsymbol{V}\cdot\boldsymbol{W}=\boldsymbol{V}\cdot\boldsymbol{\Omega}+\boldsymbol{\Omega}\cdot\boldsymbol{V} \tag{13-60}$$

Rearrange terms to get

$$\boldsymbol{d}\cdot\boldsymbol{V}-\boldsymbol{V}\cdot\boldsymbol{d}=\boldsymbol{V}\cdot(\boldsymbol{\Omega}-\boldsymbol{W})+(\boldsymbol{\Omega}-\boldsymbol{W})\cdot\boldsymbol{V} \tag{13-61}$$

Let $\boldsymbol{w}, \boldsymbol{\omega}$ and $\boldsymbol{z}$ be the axial vectors of $\boldsymbol{W}, \boldsymbol{\Omega}$ and $(\boldsymbol{d}\cdot\boldsymbol{V}-\boldsymbol{V}\cdot\boldsymbol{d})$ respectively, i.e.,

$$\boldsymbol{W}=\boldsymbol{\varepsilon}\cdot\boldsymbol{w} \qquad \boldsymbol{\Omega}=\boldsymbol{\varepsilon}\cdot\boldsymbol{\omega} \qquad (\boldsymbol{d}\cdot\boldsymbol{V}-\boldsymbol{V}\cdot\boldsymbol{d})=\boldsymbol{\varepsilon}\cdot\boldsymbol{z} \tag{13-62}$$

so that (13-61) becomes

$$\boldsymbol{\varepsilon}\cdot\boldsymbol{z}=\boldsymbol{V}\cdot\left[\boldsymbol{\varepsilon}\cdot(\boldsymbol{\omega}-\boldsymbol{w})\right]+\left[\boldsymbol{\varepsilon}\cdot(\boldsymbol{\omega}-\boldsymbol{w})\right]\cdot\boldsymbol{V} \tag{13-63}$$

In indicial notation, (13-63) is

$$\varepsilon_{ijk}z_k=V_{il}\varepsilon_{ljk}(\omega_k-w_k)+\varepsilon_{ikl}(\omega_l-w_l)V_{kj} \tag{13-64}$$

Multiply by ε_{ijm} and use the two Kronecker delta-alternating system identities:

$$\begin{aligned}
&\varepsilon_{ijm}\varepsilon_{ijk}z_k=V_{il}\varepsilon_{ijm}\varepsilon_{ljk}(\omega_k-w_k)+\varepsilon_{ijm}\varepsilon_{ikl}(\omega_l-w_l)V_{kj}\\
&2\delta_{mk}z_k=(\delta_{km}\delta_{li}-\delta_{ki}\delta_{lm})V_{il}(\omega_k-w_k)+\left(\delta_{jk}\delta_{ml}-\delta_{jl}\delta_{mk}\right)(\omega_l-w_l)V_{kj}\\
&2z_m=V_{ii}(\omega_m-w_m)-V_{im}(\omega_i-w_i)+(\omega_m-w_m)V_{kk}-(\omega_j-w_j)V_{mj}
\end{aligned} \tag{13-65}$$

Now convert back to direct notation:

$$2\boldsymbol{z}=(tr\boldsymbol{V})(\boldsymbol{\omega}-\boldsymbol{w})-(\boldsymbol{\omega}-\boldsymbol{w})\cdot\boldsymbol{V}+(\boldsymbol{\omega}-\boldsymbol{w})(tr\boldsymbol{V})-\boldsymbol{V}\cdot(\boldsymbol{\omega}-\boldsymbol{w}) \tag{13-66}$$

Since $(\boldsymbol{\omega}-\boldsymbol{w})\cdot\boldsymbol{V}=\boldsymbol{V}^T\cdot(\boldsymbol{\omega}-\boldsymbol{w})=\boldsymbol{V}\cdot(\boldsymbol{\omega}-\boldsymbol{w})$, it follows that

$$\begin{aligned}
\boldsymbol{z}&=(tr\boldsymbol{V})(\boldsymbol{\omega}-\boldsymbol{w})-\boldsymbol{V}\cdot(\boldsymbol{\omega}-\boldsymbol{w})\\
\boldsymbol{z}&=\left[(tr\boldsymbol{V})\boldsymbol{I}-\boldsymbol{V}\right]\cdot(\boldsymbol{\omega}-\boldsymbol{w})
\end{aligned} \tag{13-67}$$

or

$$\boldsymbol{\omega}=\boldsymbol{w}+\left[(tr\boldsymbol{V})\boldsymbol{I}-\boldsymbol{V}\right]^{-1}\cdot\boldsymbol{z} \tag{13-68}$$

The spin is related to the vorticity, $\boldsymbol{V}$ and $\boldsymbol{d}$ without the need for determining the rate of either $\boldsymbol{U}$ or $\boldsymbol{V}$.

13.6 Summary of Important Relations

The deformation, velocity and acceleration vectors are

$$\begin{aligned}&\boldsymbol{r}=\boldsymbol{r}(\boldsymbol{R},t) \qquad \boldsymbol{v}(\boldsymbol{R},t)=\frac{d\boldsymbol{r}}{dt}=\dot{\boldsymbol{r}} \qquad \boldsymbol{a}(\boldsymbol{R},t)=\frac{d\boldsymbol{v}}{dt}=\dot{\boldsymbol{v}}\\ &\boldsymbol{R}=\boldsymbol{R}\{\boldsymbol{r}(t),t\} \quad \Rightarrow \quad \boldsymbol{v}=\boldsymbol{v}(\boldsymbol{r},t) \qquad \boldsymbol{a}(\boldsymbol{r},t)=\left.\frac{\partial \boldsymbol{v}}{\partial t}\right|_{\boldsymbol{r}}+\boldsymbol{v}\bar{\bar{\nabla}}\cdot\boldsymbol{v}\end{aligned} \tag{13-69}$$

The spatial velocity gradient and the rate of deformation gradients are

$$\boldsymbol{L}=\boldsymbol{v}\bar{\bar{\nabla}} \qquad \dot{\boldsymbol{F}}=\boldsymbol{L}\cdot\boldsymbol{F} \qquad \dot{\boldsymbol{F}}^T=\boldsymbol{F}^T\cdot\boldsymbol{L}^T \tag{13-70}$$

The rates of the inverse deformation gradients are

$$\dot{\boldsymbol{F}}^{-1}=-\boldsymbol{L}\cdot\boldsymbol{F}^{-1} \qquad \dot{\boldsymbol{F}}^{-T}=-\boldsymbol{F}^{-T}\cdot\boldsymbol{L}^T \tag{13-71}$$

The spatial rate of deformation, the vorticity, and the material rate of deformation tensors are

$$\begin{aligned}&\boldsymbol{d}=\tfrac{1}{2}\left(\boldsymbol{L}+\boldsymbol{L}^T\right) \qquad \boldsymbol{W}=\tfrac{1}{2}\left(\boldsymbol{L}-\boldsymbol{L}^T\right)\\ &\boldsymbol{D}=\boldsymbol{F}^T\cdot\boldsymbol{d}\cdot\boldsymbol{F}=\tfrac{1}{2}\left(\boldsymbol{U}\cdot\dot{\boldsymbol{U}}+\dot{\boldsymbol{U}}\cdot\boldsymbol{U}\right)\end{aligned} \tag{13-72}$$

The vorticity vector is

$$\boldsymbol{\omega}=\frac{1}{2}\boldsymbol{\varepsilon}\cdot\cdot\boldsymbol{W} \qquad \omega_i=\frac{1}{2}\varepsilon_{ijk}W_{jk} \tag{13-73}$$

The rates of the Lagrangian and Eulerian strain tensors are

$$\dot{\boldsymbol{E}}=\boldsymbol{D} \qquad \dot{\boldsymbol{e}}=\boldsymbol{d}-\boldsymbol{L}^T\cdot\boldsymbol{e}-\boldsymbol{e}\cdot\boldsymbol{L} \tag{13-74}$$

The rate of the determinant of the deformation gradient is

$$\dot{J}=Jtr\left(\boldsymbol{L}\right)=Jtr\left(\boldsymbol{d}\right)=J\left(\boldsymbol{v}\cdot\bar{\bar{\nabla}}\right) \tag{13-75}$$

The spin is

$$\boldsymbol{\Omega}=\dot{\boldsymbol{R}}\cdot\boldsymbol{R}^T \tag{13-76}$$

The spin and vorticity are related by

$$\boldsymbol{W}=\boldsymbol{\Omega}+\boldsymbol{R}\cdot\boldsymbol{\Omega}*\cdot\boldsymbol{R}^T \qquad \boldsymbol{\Omega}*=\tfrac{1}{2}\left[\dot{\boldsymbol{U}}\cdot\boldsymbol{U}^{-1}-\boldsymbol{U}^{-1}\cdot\dot{\boldsymbol{U}}\right] \tag{13-77}$$

The corresponding axial vectors satisfy

$$\boldsymbol{\omega}=\boldsymbol{w}+\left[\left(tr\boldsymbol{V}\right)\boldsymbol{I}-\boldsymbol{V}\right]^{-1}\cdot\boldsymbol{z} \qquad \boldsymbol{z}=\frac{1}{2}\boldsymbol{\varepsilon}\cdot\cdot\left(\boldsymbol{d}\cdot\boldsymbol{V}-\boldsymbol{V}\cdot\boldsymbol{d}\right) \tag{13-78}$$

This concludes the chapter providing rates of tensors related to deformation.