APPENDIX

A.1 FUNDAMENTAL THEOREM OF CONTINUUM MECHANICS

Theorem: If f is piecewise continuous and

$$\int_{x_1}^{x_2} f\,dx = 0 \qquad \forall\ x_1, x_2 \in [0,1] \tag{A.1-1}$$

then f = 0. Here $\forall\ x_1, x_2 \in [0,1]$ means all x_1 and x_2 such that $0 \le x_1 \le 1$ and $0 \le x_2 \le 1$.

Proof: Suppose $f > 0$ for $a < x < b$ and $\int_{x_1}^{x_2} f\,dx = 0$. Choose $x_1 = a$ and $x_2 = b$. Then $\int_a^b f\,dx > 0$ which is a contradiction. Therefore f = 0.

A.2 FUNDAMENTAL LEMMA OF THE CALCULUS OF VARIATIONS

Theorem: Suppose f and v are piecewise constant and

$$\int_0^1 fv\,dx = 0 \qquad \forall\ v \tag{A.2-1}$$

where. again, $\forall$ v means for arbitrary v or for all x. Then f = 0.

Proof: Suppose $\int_0^1 fv\,dx = 0$ and $f \neq 0$. Since v is arbitrary, choose v = f. Then $\int_0^1 f^2\,dx > 0$ which is a contradiction. Therefore, f must be zero.

A.3 TAYLOR SERIES

Suppose f(x) is continuous and has continuous derivatives in the domain of interest. The Taylor series for the value of the function at x = s in terms of the values of the function and its derivatives at x = a is

$$f(s) = f(a) + \frac{f'(a)}{1!}(s-a) + \frac{f''(a)}{2!}(s-a)^2 + \ldots + \frac{f^{(n)}(a)}{n!}(s-a)^n + R_{n+1}(s) \tag{A.3-1}$$

where primes denote derivatives with respect to x and $R_{n+1}(s)$ is the remainder given by

$$R_{n+1}(s) = \frac{(s-a)^{n+1}}{(n+1)!} f^{(n+1)}(\zeta) \qquad \zeta \in [a,s] \tag{A.3-2}$$

A.4 NUMERICAL QUADRATURE

There are numerous applications (the evaluation of components of finite element matrices is one notable example) in which definite integrals are performed numerically. The general problem is to obtain a numerical approximation for the integral

$$I = \int_a^b F(x)dx \qquad b - a = L \tag{A.4-1}$$

for a given function F(x). The numerical approximation is typically expressed as a quadrature formula that requires the domain of integration to be either [-1, 1] or [0,1]. Suppose we use the latter, and let $\eta = (x - a)/L$. To emphasize the transformation, we let $f(\eta)$ represent the function of the transformed variable. Then the integral becomes

$$I = hI_\eta \qquad I_\eta = \int_0^1 f(\eta)d\eta \tag{A.4-2}$$

Quadrature formulas provide weights, W_i, and assigned quadrature points (sampling points), η_i, so that the numerical approximation is of the form

$$I_\eta^{num} = \sum_{j=1}^{n_q} W_j f(\eta_j) \tag{A.4-3}$$

The number of sampling points is n_q. The locations of the sampling points are assigned according to a predefined rule. The weights are determined to provide an exact integration of a complete polynomial of highest possible order, n_p, a number which is also the order of accuracy. Some conventional quadrature rules are summarized next.

Newton-Cotes Quadrature

With this procedure, the end points of the domain are always chosen as sampling points. Any additional interior sampling points are chosen so that the intervals between the points are equal. The coefficients for values of n_q from 2 to 4 are given as follows:

$\underline{n}_q$	$\underline{n}_p$	$\underline{\eta}_j$	$\underline{W}_j^{NC}$
2	1	$0, 1$	$\frac{1}{2}, \frac{1}{2}$
3	3	$0, \frac{1}{2}, 1$	$\frac{1}{6}, \frac{2}{3}, \frac{1}{6}$
4	3	$0, \frac{1}{3}, \frac{2}{3}, 2$	$\frac{1}{8}, \frac{3}{8}, \frac{3}{8}, \frac{1}{8}$

(A.4-4)

The Trapezoidal Rule is identified with $n_q = 2$, and Simpson's Rule with $n_q = 3$. Since the order of accuracy for $n_q = 4$ is no better than the accuracy for $n_q = 3$, the rule identified as $n_q = 4$ is rarely used.

The Newton-Cotes formula is a natural one to use with the Finite Difference Method because function values are given at nodes and equally-spaced intervals are often used.

Gauss Quadrature

With Gauss Quadrature the locations of the sampling points are optimized with the result that higher accuracy is obtained for a given number of sampling points in comparison with the Newton-Cotes procedure. However, the intervals on the subdomain are not equal. With Gauss Quadrature, n_q sampling points results in exact integration for polynomials up to order $2n_q - 1$. The sampling points and corresponding weights for values of n_q from 1 to 3 is given as follows with the degree symbol used to denote repeating numerals:

$\underline{n}_q$	$\underline{n}_p$	$\underline{\eta}_j$	$\underline{W}_j^{G}$
1	1	0.5	1
2	3	$0.5 \pm \frac{1}{2}(0.57735\cdots)$	$\frac{1}{2}, \frac{1}{2}$
3	5	0.5 and	0.4444^0
		$0.5 \pm \frac{1}{2}(0.774596\cdots)$	0.2777^0

(A.4-5)

Gauss quadrature is extensively used to obtain the element matrices associated with the finite element method.

If both end points are specified to be sampling points and the remaining points are optimized, the resulting procedure is called Lobatto Quadrature. When one point is arbitrarily specified, and the remainder optimized, the result is Radau Quadrature.

Trapezoidal Rule

With this formula, only the end points of the interval are used. The approximation for the integral is given by

$$I_\eta^{num} = \alpha f(1) + (1-\alpha) f(0) \qquad 0 \le \alpha \le 1 \qquad \text{(A.4-6)}$$

The weights depend on the parameter, α. The **general trapezoidal rule** is identified with $\alpha \neq 1/2$ and provides an exact integral only for functions that are constants ($n_p = 0$). The **trapezoidal rule** is the case $\alpha = 1/2$ which is identical with the lowest order Newton-Cotes quadrature ($n_p = 1$).

Error Analysis

Now we return to the original problem which is to obtain a numerical approximation for the integral defined in (A.4-1). To limit the error one can divide the original domain into subdomains, use a higher-order algorithm, or a combination of both. Suppose we introduce n_s subdomains of equal length, h, i.e., $n_s h = L$. Then

$$I = \sum_{i=1}^{n_s} I_i \qquad I_i = \int_{x_i}^{x_{i+1}} F(x)\,dx = h\int_0^1 f_i(\eta)\,d\eta \qquad x_{i+1} - x_i = h \qquad \text{(A.4-7)}$$

in which the function f_I is defined over the i'th segment as follows:

$$f_i(\eta) = F(x_i + \eta h) \qquad 0 \le \eta \le 1 \qquad \text{(A.4-8)}$$

The corresponding numerical evaluations are

$$I^{num} = h\sum_{i=1}^{n_s} I_i^{num} \qquad I_i^{num} = \sum_{j=1}^{n_q} W_j f_i(\eta_j) \qquad \text{(A.4-9)}$$

with numerical quadrature applied to each segment.

Quadrature formulas are derived to provide exact integrals of polynomials for a specified order, n_p. This simple relationship provides an elementary way of estimating the error when the function is not a polynomial, or a polynomial of higher order. To obtain a theoretical bound on the error for the integral of a typical segment, perform a Taylor expansion about point x_i for a typical segment:

$$F(x) = F_i + F_i'(x - x_i) + F_i'' \frac{(x - x_i)^2}{2!} + \cdots + F_i^{(n_p)} \frac{(x - x_i)^{n_p}}{n_p!} + R_{n_p+1}(x)$$

$$R_{n_p+1}(x) = \frac{(x - x_i)^{n_p+1}}{(n_p+1)!} F^{(n_p+1)}(\zeta) \qquad x_i \le \zeta \le x_{i+1} \tag{A.4-10}$$

where the expansion has been carried out to order n_p and R denotes the remainder. The function and its derivatives evaluated at x_i are denoted by F_i, F_i', F_i''.... Note that $x - x_i = \eta$ so that the first n_p terms of the expansion is a polynomial in η of order n_p. The quadrature formula provides an exact integration for all terms except the remainder. Therefore, the error for a typical segment becomes

$$e_i = I_i - hI_i^{num} = \int_{x_i}^{x_{i+1}} R_{n_p+1}(x)dx \tag{A.4-11}$$

Let

$$F_{i,\,max}^{(n_p+1)} \equiv \max_{x \in [x_i, x_{i+1}]} \left| F^{(n_p+1)}(x) \right| \tag{A.4-12}$$

denote the upper bound of the absolute value of $F^{(n_p+1)}(x)$ on the segment. Then an upper bound on the absolute value of the error over the segment is

$$\left|e_i\right| \le E_i \equiv \frac{F_{i,\,max}^{(n_p+1)}}{(n_p+1)!} \int_{x_i}^{x_{i+1}} (x - x_i)^{n_p+1} dx = \frac{h^{(n_p+2)}}{(n_p+2)!} F_{i,\,max}^{(n_p+1)} \tag{A.4-13}$$

Instead of using the maximum value of the (n_p+1)'th derivative for each segment, a looser bound is obtained by using the maximum value over the complete domain:

$$E_i \le E_h \equiv \frac{h^{(n_p+2)}}{(n_p+2)!} F_{L,\,max}^{(n_p+1)} \qquad F_{L,\,max}^{(n_p+1)} = F_{L,\,max}^{(n_p+1)} \equiv \max_{x \in [a,b]} \left| F^{(n_p+1)}(x) \right| \tag{A.4-14}$$

Now, an error bound for the complete domain is obtained by summing the errors from all segments

$$E_L \le n_s E_h = \frac{L}{h} E_h \qquad \text{or} \qquad E_L \le \frac{L F_{L,\,max}^{(n_p+1)}}{(n_p+2)!} h^{n_p+1} \tag{A.4-15}$$

The result is that in the expression for a bound on the error for the complete integral, the order of the highest derivative of the integrand as well as the exponent of the interval length, h, is one order higher than the order of the polynomial that is integrated exactly.

Finally, we emphasize that there are two methods for reducing quadrature error; either use a higher-order quadrature formula (increase n_p) or use more segments (reduce h).

A.5 INTEGRATION BY PARTS

Theorem: Let f and g represent two functions defined over $0 \le x \le L$. Then

$$\int_0^L f \frac{dg}{dx} dx = fg\Big|_0^L - \int_0^L g \frac{df}{dx} dx \tag{A.5-1}$$

Proof: Use the product rule and consider the following integral:

$$\int_0^L \frac{d}{dx}(fg) dx = \int_0^L \left(f \frac{dg}{dx} + g \frac{df}{dx} \right) dx = fg\Big|_0^L$$

A rearrangement of terms yields (A.5-1).

A.6 ONE-DIMENSIONAL DIVERGENCE THEOREM

Consider a three-dimensional space spanned by the rectangular cartesian coordinates x, y and z. The gradient operator involves partial derivatives with respect to each of the coordinates and unit base vectors $\mathbf{e}_x$, $\mathbf{e}_y$ and $\mathbf{e}_z$ associated with the coordinate system:

$$\nabla(\ ) = \frac{\partial(\ )}{\partial x}\mathbf{e}_x + \frac{\partial(\ )}{\partial y}\mathbf{e}_y + \frac{\partial(\ )}{\partial z}\mathbf{e}_z \tag{A.6-1}$$

If n denotes the unit vector normal to the surface, ∂R, of the three-dimensional body mapped onto a region, R, then the divergence theorem states that

$$\iiint_R \nabla \cdot \mathbf{v}\, dx\, dy\, dz = \iint_{\partial R} \mathbf{n} \cdot \mathbf{v}\, da \tag{A.6-2}$$

in which $\mathbf{v}$ is a vector field and da represents an area element on ∂R. In one-dimension $\mathbf{v} = v\mathbf{e}_x$, $\mathbf{n} = n\mathbf{e}_x$ and the vector representation can be dropped. Suppose the region R is defined by $0 < x < L$ and the boundary by the points $x = 0$ and $x = L$ as shown in Fig. (A.6-1). If the cross-section is uniform in x, then $\mathbf{n} \cdot \mathbf{v}$ is zero on the lateral surface and (A.6-2) reduces to

$$\int_0^L \frac{\partial v}{\partial x} dx = nv\Big|_0 + nv\Big|_L \tag{A.6-3}$$

if v is a function of x. However, n = -1 at x = 0 and n = 1 at x = L so an alternative expression is the conventional integral relation

$$\int_0^L \frac{\partial v}{\partial x} dx = v\Big|_o^L \tag{A.6-4}$$

The notations of both (A.6-3) and (A.6-4) will be used.

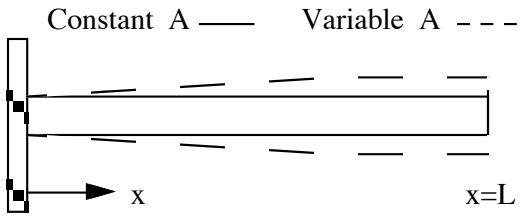

Fig. A.6-1. Domain for one-dimensional problem.

In (A.6-3) suppose v is replaced by Av. Then

$$\int_0^L \frac{\partial}{\partial x}(Av)dx = nAv\Big|_0 + nAv\Big|_L \tag{A.6-5}$$

which is the divergence theorem for a bar of cross-sectional area A which can vary with x. On the other hand if (A.6-2) is used directly, the result is

$$\int_0^L \left(\frac{\partial v}{\partial x}\iint dydz\right)dx = \int_{\partial R^*} n * vda + nvA\Big|_0 + nvA\Big|_L \tag{A.6-6}$$

where ∂R^* denotes the lateral surface of the bar and n* is the x-component of the unit normal to the lateral surface. Since

$$\int_0^L \left(\frac{\partial v}{\partial x}\iint dydz\right)dx = \int_0^L A\frac{\partial v}{\partial x}dx \tag{A.6-7}$$

A substitution of (A.6-5) and (A.6-7) in (A.6-6) yields

$$\int_{\partial R^*} n * vda = -\int_0^L v\frac{\partial A}{\partial x}dx \tag{A.6-8}$$

which is an additional relation that must be used in the derivation for equations governing bars of arbitrary cross-section.

A.7 GENERALIZED FUNCTIONS

For describing loading functions or variable material properties, it is convenient to utilize generalized functions which are defined as follows:

$$\langle x-a\rangle^n = \begin{cases} 0 & x \le a \\ (x-a)^n & x > a \end{cases} \qquad n \ge 0 \qquad n \text{ an integer} \qquad \text{(A.7-1)}$$

Sketches of such functions are shown in Fig. A.7-1. It follows that the integrals satisfy the relations

$$\int \langle x-a\rangle^n dx = \frac{\langle x-a\rangle^{n+1}}{n+1} \qquad n \ge 0 \qquad \text{(A.7-2)}$$

and for all positive integers except n = 0, the derivatives are

$$\frac{d}{dx}\langle x-a\rangle^n = n\langle x-a\rangle^{n-1} \qquad n \ge 1 \qquad \text{(A.7-3)}$$

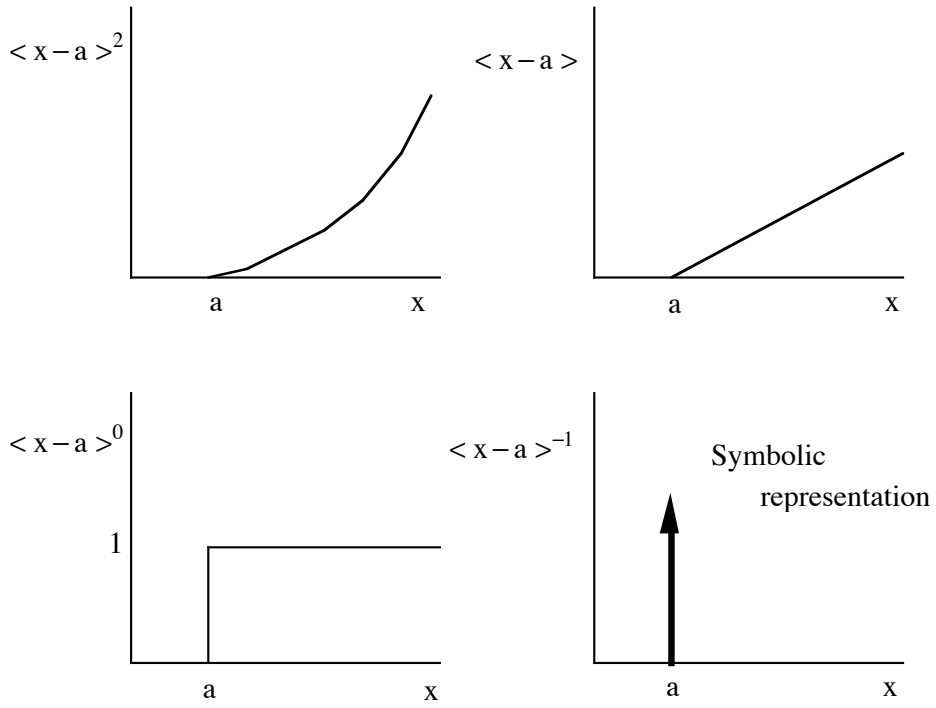

Fig. A.7-1. Sketches of generalized functions.

The special case of n = 0 is often called the Heaviside, or step, function with a separate notation:

$$H[x-a] = \begin{cases} 0 & x \le a \\ 1 & x > a \end{cases} \tag{A.7-4}$$

With the use of the Heaviside function, an alternative way to express (A.7-1) is as follows:

$$\langle x-a \rangle^n = (x-a)^n H[x-a] \tag{A.7-5}$$

The definitions of generalized functions can be extended to n = -1 (Dirac delta function) which is defined such that

$$\begin{aligned} &\delta[x-a] = 0 \qquad x \neq a \\ &\int_{a-\varepsilon}^{a+\varepsilon} \delta[x-a]dx = \int_{-\infty}^{\infty} \delta[x-a]dx = 1 \end{aligned} \tag{A.7-6}$$

for any positive parameter ε. Even though the function is not defined at x = a, the integral representation holds. If y is a variable of integration then (A.7-6) can also be written as

$$\int_{-\infty}^{x} \delta[y-a]dy = H[x-a] \tag{A.7-7}$$

It also follows that if f(x) is a continuous function, then

$$\int_{-\infty}^{x} f(y)\delta[y-a]dy = \begin{cases} 0 & \text{if } x \le a \\ f(a) & \text{if } x > a \end{cases} \tag{A.7-8}$$

In the more mathematical literature, (A.7-8) is used as the definition of the Dirac delta function. If use is made of the Heaviside function, (A.7-8) is expressed more succinctly in the form

$$\int_{-\infty}^{x} f(y)\delta[y-a]dy = H[x-a]f(a) \tag{A.7-9}$$

Care must be taken with the use of the Dirac delta function if the independent variables are scaled, e.g., if dimensionless variables are introduced. Let

$$x^* = \frac{x}{L} \qquad y^* = \frac{y}{L} \qquad a^* = \frac{a}{L} \tag{A.7-10}$$

where L denotes a characteristic length associated with a problem of interest. Suppose (A.7-7) is applied in each coordinate system to obtain

$$\int_{-\infty}^{x^*} \delta[y^* - a^*]\,dy^* = H[x^* - a^*]$$
$$\int_{-\infty}^{x} \delta[y - a]\,dy = H[x - a] \qquad \text{(A.7-11)}$$

Based on the definition of the Heaviside function, the coordinate transformation leaves its values unaltered, i.e., H[x* - a*] = H[x - a] so that (A.7-11) yields

$$\int_{-\infty}^{x^*} \delta[y^* - a^*]\,dy^* = \int_{-\infty}^{x} \delta[y - a]\,dy \qquad \text{(A.7-12)}$$

With the use of the transformation of variables defined in (A.7-10), it follows that

$$\frac{1}{L}\delta[y^* - a^*] = \delta[y - a] \qquad \text{(A.7-13)}$$

In other words, a change in the argument of the Dirac delta function results in a scale factor. The same result can be derived using (A.7-9).

The definition of a generalized function can be extended even further to include n = -2 (unit couple) but this case will not be discussed here.

Although the derivative of the Heaviside function does not exist, (A.7-7) is often used to provide the symbolic representation for the derivative:

$$\frac{d}{dx}H[x - a] = \delta[x - a]. \qquad \text{(A.7-14)}$$

The integrals of the generalized functions exist for $n \geq -1$ and, in fact, for most cases the integrals of the square of the generalized functions exist:

$$\int_0^L \langle x - a \rangle^{2n}\,dx < \infty \quad \text{for} \quad n \geq 0 \qquad \text{(A.7-15)}$$

In particular, for $0 \leq a \leq L$

$$\int_0^L (H[x - a])^2\,dx = \int_0^L H[x - a]\,dx = (L - a) \qquad \text{(A.7-16)}$$

However, $\int_0^L (\delta[x - a])^2\,dx$ does not exist, i.e., the Dirac function is not square integrable.

A.8 CLASSIFICATION OF FUNCTIONS

Classification by Continuity

If a function, f(x), and its first n derivatives are continuous then the notation C^n is used to describe these features of the function. For example, the linear ramp function is of class C^0, and described as follows:

$$< x - a >^1 \in C^0 \tag{A.8-1}$$

since its derivative, the step function, is not continuous. In the same manner, the generalized functions with n greater than zero are described by

$$< x - a >^n \in C^{n-1} \qquad n \geq 1 \tag{A.8-2}$$

With this classification, the Heaviside and Dirac delta functions are sometimes said to be of class C^{-1} and C^{-2}, respectively. However, for functions of this type, it is more meaningful to use a different classification scheme, namely, one based on integrability.

Classification by Square Integrability

If the integral of the square of f and of the square of all its derivatives up to and including n exists, then the function is said to belong to a class denoted by H^n, i.e.,

$$f(x) \in H^n \quad \text{if} \quad \int_D \{f^{(m)}(x)\}^2 \, dx < \infty \quad \text{for} \quad m = 0,1,...,n \tag{A.8-3}$$

where D denotes the domain. Examples of this classification for the generalized functions are the following:

$$< x - a >^1 \in H^1 , \qquad < x - a >^0 = H[x - a] \in H^0 \tag{A.8-4}$$

These results are derived by noting, again, that the integral of the square of the Dirac delta function is not defined. The examples of (A.8-4) can be considered as special cases of the following result:

$$< x - a >^n \in H^n \qquad \text{for} \qquad n \geq 0 \tag{A.8-5}$$

A special case is that for which the function is square integrable. Then the notation L_2 is often used, i.e., $f \in L_2$ if $f \in H^0$.

The classification by square integrability is the natural one to use in connection with the finite element method although the classification based on continuity is often used when any numerical method is analysed.