ME 500
NUMERICAL METHODS IN MECHANICAL ENGINEERING
TEST REVIEW

BRANDON LAMPE

1. Definitions

1.1. **Differential Equations**

. Terms are related to a **second-order linear equation** in one independent variable (**ordinary**) and two independent variables (**partial**) is of the form

$$Au_{xx} + Bu_x + Cu = D \tag{1}$$

$$Au_{xx} + Bu_{xy} + Cu_{xx} + Du_x + Eu_y + Fu = G \tag{2}$$

where $A, B, C, D, E, F,$ and G are constants (often some are zero valued) or functions of the independent variable(s)

$$u = \text{dependent variable} \implies \text{temp., disp., head, etc.}$$
$$x, y = \text{independent variable} \implies \text{spatial reference or time}$$

1.2. **Linear Algebra**

. The following definitions pertain to terminology associated with the field of linear algebra.

1.2.1. *Mathematical Vectors*

. Following terms are specific to mathematical vectors, which implies these vectors are not related to a physical basis (unlike physical vectors).

column vector $\{v\}$, a column of ordered terms with components $v_1, v_2, ..., v_n$.

row vector $< v >$, a row of ordered terms with components $v_1, v_2, ..., v_n$.

size n, the number of components in the vector; also referred to as the dimension of the vector

transpose ^{T}, an operation that swaps column and row components:

$$\{v\}^T = < v > \quad \text{and} \quad < v >^T = \{v\}$$

inner product results in a scalar and is only defined between vectors (or vector spaces) of the same dimension n, and is only defined if the vectors are of the same size

$$< v > \{x\} = \langle v, x \rangle = \sum_{i=1}^{n} v_i x_i$$

analogous is:

$$\mathbf{v} \cdot \mathbf{x} = v_i x_j \mathbf{e}_i \cdot \mathbf{e}_j = v_i x_i \implies \text{physical vectors}$$

magnitude $|v|$, a nonnegative scalar value of a vector defined as the square root of the inner product of a vector with itself; analogous to the L_2 norm

$$|v| = \sqrt{< v > \{v\}}$$

Date: October 3, 2015.

norm $\|x\|$, a nonnegative scalar measure of a vector that can be zero only if every component of the vector is zero

$$\|x\| = |x| \implies 1D \quad \text{Absolute Value norm}$$

$$\|x\|_{L_1} = \sum_{i=1}^{n} |x_i| \implies \text{sum of positive values, Taxicab or Manhattan norms}$$

$$\|x\|_{L_2} = \sqrt{< x > \{x\}} \implies \text{square root of the sum of squares, magnitude or Euclidean norm}$$

unit vector $\{\hat{v}\}$, a vector having magnitude of unity, which is the vector divided by its magnitude

$$\{\hat{v}\} = \frac{\{v\}}{|v|}$$

angle between vectors θ, defined as

$$cos(\theta_{uv}) = < \hat{u} > \{\hat{v}\} \implies \theta_{uv} = cos^{-1}(< \hat{u} > \{\hat{v}\})$$

real vector space a set of vectors together with eight rules for vector addition and multiplication by real numbers. The vector space is denoted as R^n where n indicates the number of components, and the sets of vectors that make up the space must be given; rules include

(1) association of addition: $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$
(2) commutativity of addition: $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
(3) identity element of addition: $\mathbf{v} + \mathbf{0} = \mathbf{v} \quad \forall \quad \mathbf{v}$
(4) inverse elements of addition: $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$
(5) compatibility of scalar multiplication with field multiplication: $a(b\mathbf{v}) = (ab)\mathbf{v}$
(6) identity element: $1\mathbf{v} = \mathbf{v}$
(7) distributivity of scalar multiplication with respect to vector addition: $a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$
(8) distributivity of scalar multiplication with respect to field addition: $(a + b)\mathbf{v} = a\mathbf{v} + b\mathbf{v}$

Within all vector spaces, two operations are possible that allow us to take *linear combinations* of the vectors. These operations result in vectors that are in the same vector space:

- we can add any two vectors
- we can multiply all vectors by scalars

notation for a vector space R^n, consists of *all column vectors with n components.* R is used to denote the space because the components are real numbers. The number of components in the vector space is denoted by n. If the dimension of R^n is m, then the set of given vectors also define a subspace R^n_m, where m is the number of linearly independent vectors in the vector space and $m \leq n$.

Example: say we have a 3×4 matrix that defines a vector space that is R^3. A column $\{c\}$ of the matrix is a linear combination of two other columns e.g., $\{c\} = \{a\} + \{b\}$. $\{c\}$ defines a subspace that is R^1, and that subspace is a one-dimensional line that lies on the two-dimensional plane defined by the columns $\{a\}$ and $\{b\}$.

span if the vector subspace $\{c\}$ can be expressed in terms of a linear combination of vectors in the vector space $\{a\}^i$, then $\{a\}^i$ is said to span the subspace $\{c\}$. This concept holds in higher dimensions.

dimension of a vector space is the number of linearly independent vectors given in the definition of the vector space, this is different than the size or dimension of a single vector. The dimension of a vector space may be determined via the Gram-Schmidt procedure. Vectors in a vector space are linearly independent if

$$\sum_{i=1}^{m} \alpha_i \{v\}^i = \{0\} \implies \alpha = 0 \text{ for } i = 1...m$$

linear independence of a vector a set of vectors is not linearly independent one of the vectors in the set can be defined as a linear combination of other vectors in the set. If no vector in the set can be written in this way, then the vectors are linearly independent.

basis of a vector space unless otherwise state: $\mathbf{e_i}$ or $[I]$, is the frame of reference for the vector space. That is, the components of vector $\{x\}$ are implicitly the components with respect to the coordinate basis.

$$[I] = \delta_{ij}$$

projection the vector projection of $\{a\}$ onto $\{b\}$ results in vector $\{c\}$ having the components of $\{a\}$ that are parallel to $\{b\}$. $\{c\}$ is formed by multiplying the inner product between $\{a\}$ and the unit vector $\{\hat{b}\}$ by $\{\hat{b}\}$

$$\{c\} = (< a > \{\hat{b}\})\{\hat{b}\}$$

orthonormal basis the set of vectors defining the basis are orthonormal if

(1) vectors are normal (magnitude of unity): $|\{v\}| = 1$ or $< v >^i \{v\}^i = 1$
(2) vectors in the space are orthogonal: $< v >^i \{v\}^j = 0 \quad \forall \quad i \neq j$
(3) vectors are orthonormal if: $< v >^i \{v\}^j = \delta_{ij} \quad \forall \quad i$ and j

An orthonormal basis ($< e >$) is particularly convenient if the components of a vector with respect to that basis $\{x\}^e$ are desired, the components of that vector are obtained via the inner product with each of the base vectors

$$x_i^e = < e >^i \{x\}$$

Gram-Schmidt procedure a method of obtaining an orthonormal set of vectors $\{Q\}^i$ that span the same vector space of a given set of vectors $\{A\}^i$. In essence, the $\{Q\}^1$ calculated from the unit vector of $\{A\}^1$ and each additional $\{Q\}^k$ results from subtracting out the sum of previously calculated values of $\{Q\}^k$

$$\{Q\}^{*k} = \{A\}^k - \sum_{j=1}^{k-1} < \{Q\}^{jT} > \{A\}^k$$
$$\{Q\}^k = \frac{\{Q\}^{*k}}{|\{Q\}^{*k}|}$$

1.2.2. *Matrices*
. Following terms are specific to matrices.

matrix $[A]_{m\times n}$, is an ordered array of column or row vectors with m rows and n columns. Additionally, a matrix can be an ordered array of components (scalars). $R^{m\times n}$ denotes the vector space of all real $m \times n$ matrices, and the dimension of the vector space is the number of independent matrices used to fine the space. Note: dimensions of the matrix and dimensions of the vector space are different items.

ordered array of column vectors

$$[A] = \begin{bmatrix} \{A\}^1 & \{A\}^2 & \{A\}^3 & \dots & \{A\}^n \end{bmatrix}$$

ordered array of row vectors

$$[A] = \begin{bmatrix} < A >^1 \\ < A >^2 \\ < A >^3 \\ \vdots \\ < A >^m \end{bmatrix}$$

ordered array of scalars

$$[A] = \begin{bmatrix} A_{11} & A_{12} & \dots & A_{1n} \\ A_{21} & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ A_{m1} & \dots\dots\dots & & A_{mn} \end{bmatrix}$$

transpose $[A]^T$, rows and columns are exchanged. In indicinal form: $A_{ij}^T = A_{ji}$

matrix product $[A]_{m\times n}[B]_{p\times q} = [C]_{m\times q}$, is only defined if $n = p$

$$C_{ij} = \sum_{k=1}^{n} A_{ik}B_{kj} = A_{i1}B_{1j} + A_{i2}B_{2j} + \ldots + A_{in}B_{nj}$$

transpose of a matrix product transpose of the product of two matrices equals the product of the transpose of the two matrices in the reverse order

$$\big[[A][B]\big]^T = [B]^T[A]^T$$

multiplication with a vector $< x > [A]$ or $[A]\{x\}$, results in a vector having the same length $\{x\}$, analogous to a dot product between a vector and a tensor.

$$< x > [A] = [A]\{x\} = \sum_{j=1}^{n} a_{ij}x_j$$

outer product of vectors Suppose $\{u\} \in R^m$ and $\{v\} \in R^n$, then the outer product of this two vectors is the matrix $[A] \in R^{m\times n}$

$$[A] = \{u\} < v > = \begin{bmatrix} u_1v_1 & u_1v_2 & \ldots & u_1v_n \\ u_2v_1 & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ u_mv_1 & \ldots\ldots\ldots & & u_mv_n \end{bmatrix}$$

partitioned matrix results from partitioning a general matrix into an array of sub matrices and is useful because sub matrices follow the same rules as general matrices

$$[A] = \begin{bmatrix} [A]_{11} & \vdots & [A]_{12} \\ \ldots & \ldots & \ldots \\ [A]_{21} & \vdots & [A]_{22} \end{bmatrix}$$

diagonal

lower triangular

upper triangular

range vector space formed by the columns of a matrix, also known as the column space, which is a subspace of R^m (the whole space)

rank r, is the number of independent vectors in a given vector space, which may be obtained via the Gram-Schmidt procedure.

$$\text{for } [A]_{m\times n} \; r \leq n$$

nullspace a vector space of $[A]$ formed from the solution of $[A]\{x\} = 0$. The *nullspace* of a matrix consists of all vectors $\{x\}$

inverse of a product of matrices the inverse of a product of matrices equals the product of the inverse of the two matrices in reverse order

$$\big[[A][B]\big]^{-1} = [B]^{-1}[A]^{-1}$$

transpose of the inverse transpose of the inverse of a matrix is equal to the inverse of the transpose of the matrix

$$\big[[A]^T\big]^{-1} = \big[[A]^{-1}\big]^T$$

$$\big[[A][B]\big]^{-T} = [A]^{-T}[B]^{-T}$$

orthogonal matrix composed of orthonormal columns and rows

$$[Q][Q]^T = [I] \implies [Q]^T = [Q]^{-1}$$
$$det[Q] = \pm 1$$

positive definite $< x > [A]\{x\} > 0 \ \forall \ \{x\}$, additionally, a matrix is said to be positive definite if

- its eigenvalues are all positive.
- matrix is nonsingular (an inverse exists)
- its determinant is positive
- all diagonal components must be positive, $A_{ii} > 0 \ \forall \ i$

if a matris is symmetric-positive definite, then

$$|A_{ij}| \leq \frac{1}{2}\left(A_{ii} + A_{jj}\right)$$
$$|A_{ij}| \leq \sqrt{A_{ii} + A_{jj}}$$

elementary $[E]$, is a matrix that differs from the identity matrix by one single elementary row operation. By pre or post multiplying a matrix by $[E]$ you can affect either a rows or columns

$$[E][A] \rightarrow \text{elementary row operation}$$
$$[A][E] \rightarrow \text{elementary column operation}$$

determinant

minor M_{ij}, the determinant obtained by deleting the i^{th} row and j^{th} column of a matrix.

cofactor the number obtained by $M^C_{ij} = (-1)^{i+j} M_{ij}$

Inverse via cofactor $[A]^{-1} = \frac{[M]^a}{det[A]}$ where $[M]^a$ (the adjoint matrix) is the transpose of $[M]^c$

determinant of a matrix product determinant of a product of matrices equals the product of the determinants of the matrices

$$det\left([A][B]\right) = det[A]det[B]$$

determinant of the identity matrix $det[I] = 1$

singular a square matrix is not invertible, and a square matrix is not invertible iff its determinant is zero

trace sum of diagonal terms

magnitude Frobenius norm, which is a scalar measure of a matrix

$$|[A]| = \left(tr\left[[A][A]^T\right]\right)$$
$$|[I]| = \sqrt{n}$$

1.2.3. *the Linear Algebraic Problem*
. Following terms relate specifically to: $[A]\{x\} = \{b\} \implies$ the Linear Algebraic Problem

Cramer's method

Iterative method

Direct methods LU, QR, SE

direct method procedures forward, backward

manufacture a solution

error analysis

error measure

exact error

residual error
normalized error
residual error of a matrix solution