5. TENSORS

5.1 Initial Comments

In this chapter we introduce the concept of a tensor product, which is different than both dot and cross products. The tensor product allows us to define a tensor basis and, consequently, tensor components with the accompanying transformation relations for both bases and components. Again, the individual ideas are not complex but the terms pile up and it is essential to keep the definitions clearly in mind. Scalars and vectors are sometimes referred to as zero and first-order tensors respectively. In this chapter we will be describing properties of second and higher-order tensors.

5.2 Tensor Products and the Connection with Dot Products

In this subsection all boldface letters denote vectors. The tensor product of two vectors $\boldsymbol{b} \otimes \boldsymbol{c}$ is called a second-order tensor that exhibits the following properties when dotted on the left and the right by a vector:

$$\boldsymbol{a} \cdot (\boldsymbol{b} \otimes \boldsymbol{c}) = (\boldsymbol{a} \cdot \boldsymbol{b})\boldsymbol{c} \qquad (\boldsymbol{b} \otimes \boldsymbol{c}) \cdot \boldsymbol{d} = (\boldsymbol{c} \cdot \boldsymbol{d})\boldsymbol{b} \tag{5-1}$$

The transpose is obtained by simply interchanging the order of the tensor product

$$(\boldsymbol{b} \otimes \boldsymbol{c})^T = (\boldsymbol{c} \otimes \boldsymbol{b}) \tag{5-2}$$

The composition of two second-order tensors is a second-order tensor

$$(\boldsymbol{a} \otimes \boldsymbol{b}) \cdot (\boldsymbol{c} \otimes \boldsymbol{d}) = (\boldsymbol{b} \cdot \boldsymbol{c})(\boldsymbol{a} \otimes \boldsymbol{d}) \tag{5-3}$$

The inner product of two tensors is the scalar

$$(\boldsymbol{a} \otimes \boldsymbol{b}) \cdot\cdot (\boldsymbol{c} \otimes \boldsymbol{d}) = (\boldsymbol{a} \cdot \boldsymbol{c})(\boldsymbol{b} \cdot \boldsymbol{d}) \tag{5-4}$$

The reason for calling this an inner product is that the inner product of a tensor with itself is positive semi-definite, and zero if and only if the tensor is zero:

$$(\boldsymbol{a} \otimes \boldsymbol{b}) \cdot\cdot (\boldsymbol{a} \otimes \boldsymbol{b}) = (\boldsymbol{a} \cdot \boldsymbol{a})(\boldsymbol{b} \cdot \boldsymbol{b}) \geq 0 \tag{5-5}$$

It is sometimes useful to define another product where the adjacent vectors are dotted first:

$$(\boldsymbol{a} \otimes \boldsymbol{b}) : (\boldsymbol{c} \otimes \boldsymbol{d}) = (\boldsymbol{b} \cdot \boldsymbol{c})(\boldsymbol{a} \cdot \boldsymbol{d}) \tag{5-6}$$

The reason that this is not an inner product is that when the product of a second-order tensor is taken with itself, the result

$$(\boldsymbol{a} \otimes \boldsymbol{b}) : (\boldsymbol{a} \otimes \boldsymbol{b}) = (\boldsymbol{a} \cdot \boldsymbol{b})^2 \tag{5-7}$$

may be zero even though both vectors are not null vectors. We also note that the two products are related through the transpose as follows:

$$(\boldsymbol{a} \otimes \boldsymbol{b}) \cdot\cdot (\boldsymbol{c} \otimes \boldsymbol{d}) = (\boldsymbol{a} \otimes \boldsymbol{b}) : (\boldsymbol{c} \otimes \boldsymbol{d})^T \tag{5-8}$$

A note of caution; some authors interchange the definitions for $\cdot\cdot$ and $:$ that are given here.

Now we proceed to extend these products to higher order tensors. A third-order tensor $\boldsymbol{a}\otimes\boldsymbol{b}\otimes\boldsymbol{c}$ involves two tensor products. Dotting on either side by a vector results in a second-order tensor:

$$(\boldsymbol{a}\otimes\boldsymbol{b}\otimes\boldsymbol{c})\cdot\boldsymbol{d}=(\boldsymbol{c}\cdot\boldsymbol{d})(\boldsymbol{a}\otimes\boldsymbol{b}) \qquad \boldsymbol{d}\cdot(\boldsymbol{a}\otimes\boldsymbol{b}\otimes\boldsymbol{c})=(\boldsymbol{a}\cdot\boldsymbol{d})(\boldsymbol{b}\otimes\boldsymbol{c}) \tag{5-9}$$

Double dotting on either side with a second-order tensor results in a vector:

$$(\boldsymbol{a}\otimes\boldsymbol{b}\otimes\boldsymbol{c})\cdot\cdot(\boldsymbol{d}\otimes\boldsymbol{e})=(\boldsymbol{b}\cdot\boldsymbol{d})(\boldsymbol{c}\cdot\boldsymbol{e})\boldsymbol{a} \qquad (\boldsymbol{d}\otimes\boldsymbol{e})\cdot\cdot(\boldsymbol{a}\otimes\boldsymbol{b}\otimes\boldsymbol{c})=(\boldsymbol{d}\cdot\boldsymbol{a})(\boldsymbol{e}\cdot\boldsymbol{b})\boldsymbol{c} \tag{5-10}$$

Sometimes a dot product with an "interior vector is required in which case a "contraction" operator is used. The contraction $C_{(A,B)}$ means dot the A'th vector from the left tensor with the B'th vector of the right tensor. Two examples are

$$\begin{aligned}(\boldsymbol{a}\otimes\boldsymbol{b}\otimes\boldsymbol{c})C_{(2,1)}(\boldsymbol{d}\otimes\boldsymbol{e})&=(\boldsymbol{b}\cdot\boldsymbol{d})(\boldsymbol{a}\otimes\boldsymbol{c}\otimes\boldsymbol{e})\\(\boldsymbol{d}\otimes\boldsymbol{e})C_{(1,3)}(\boldsymbol{a}\otimes\boldsymbol{b}\otimes\boldsymbol{c})&=(\boldsymbol{d}\cdot\boldsymbol{c})(\boldsymbol{e}\otimes\boldsymbol{a}\otimes\boldsymbol{b})\end{aligned} \tag{5-11}$$

Next, we extend the idea to a fourth-order tensor $\boldsymbol{a}\otimes\boldsymbol{b}\otimes\boldsymbol{c}\otimes\boldsymbol{d}$, which is normally contracted with a second-order tensor, with the results

$$\begin{aligned}(\boldsymbol{a}\otimes\boldsymbol{b}\otimes\boldsymbol{c}\otimes\boldsymbol{d})\cdot\cdot(\boldsymbol{e}\otimes\boldsymbol{f})&=(\boldsymbol{c}\cdot\boldsymbol{e})(\boldsymbol{d}\cdot\boldsymbol{f})(\boldsymbol{a}\otimes\boldsymbol{b})\\(\boldsymbol{e}\otimes\boldsymbol{f})\cdot\cdot(\boldsymbol{a}\otimes\boldsymbol{b}\otimes\boldsymbol{c}\otimes\boldsymbol{d})&=(\boldsymbol{a}\cdot\boldsymbol{e})(\boldsymbol{b}\cdot\boldsymbol{f})(\boldsymbol{c}\otimes\boldsymbol{d})\end{aligned} \tag{5-12}$$

In this subsection, we have introduced the various tensor and dot products through the use of very simple tensors. In the next subsection we develop similar relations for general forms of tensors.

5.3 Second-Order Tensors, Tensor Basis and Components

Recall that for vectors, we assume an orthonormal basis $\boldsymbol{e}_i$ with the property that $\boldsymbol{e}_i\cdot\boldsymbol{e}_j=\delta_{ij}$ and represent a vector through the use of components v_i with $\boldsymbol{v}=v_i\boldsymbol{e}_i$. In an analogous manner, we assume an orthormal basis for second-order tensors that is represented as $\boldsymbol{e}_i\otimes\boldsymbol{e}_j$. The notation indicates that there are nine base tensors with the property that

$$(\boldsymbol{e}_i\otimes\boldsymbol{e}_j)\cdot\cdot(\boldsymbol{e}_k\otimes\boldsymbol{e}_l)=\delta_{ik}\delta_{jl} \tag{5-13}$$

The product is zero if $i\neq k$ or $\mathrm{j}\neq 1$, i.e., the base tensors are orthogonal and the product is one if $i=k$ and $j=l$, which implies that the base tensors are of magnitude one. Taken together, these properties are implied by the word "orthonormal".

An arbitrary second-order tensor, $\boldsymbol{T}$, is represented as the product of base tensors and components, T_{ij}, as a sum of nine terms as follows:

$$\boldsymbol{T}=\boldsymbol{T}_{ij}\boldsymbol{e}_i\otimes\boldsymbol{e}_j \tag{5-14}$$

The transpose is obtained by interchanging the indices on either the base tensors, or the components, but not both:

$$\boldsymbol{T}^T = T_{ij}\boldsymbol{e}_j \otimes \boldsymbol{e}_i = T_{ji}\boldsymbol{e}_i \otimes \boldsymbol{e}_j \tag{5-15}$$

Any second-order tensor can be decomposed into the sum of symmetric and skew-symmetric parts:

$$\begin{gathered}
\boldsymbol{T} = \boldsymbol{T}^{sym} + \boldsymbol{T}^{sk} \qquad \boldsymbol{T}^{sym} = \tfrac{1}{2}\left(\boldsymbol{T} + \boldsymbol{T}^T\right) \qquad \boldsymbol{T}^{sym} = \tfrac{1}{2}\left(\boldsymbol{T} - \boldsymbol{T}^T\right) \\
\left[T\right]^{sym} = \begin{bmatrix} T_{11} & T_{12}^{sym} & T_{31}^{sym} \\ T_{12}^{sym} & T_{22} & T_{23}^{sym} \\ T_{31}^{sym} & T_{23}^{sym} & T_{33} \end{bmatrix} \qquad \begin{aligned} T_{12}^{sym} &= \tfrac{1}{2}(T_{12} + T_{21}) \\ T_{23}^{sym} &= \tfrac{1}{2}(T_{23} + T_{32}) \\ T_{31}^{sym} &= \tfrac{1}{2}(T_{31} + T_{13}) \end{aligned} \\
\left[T\right]^{sk} = \begin{bmatrix} 0 & T_{12}^{sk} & -T_{31}^{sk} \\ -T_{12}^{sk} & 0 & T_{23}^{sk} \\ T_{31}^{sk} & -T_{23}^{sk} & 0 \end{bmatrix} \qquad \begin{aligned} T_{12}^{sk} &= \tfrac{1}{2}(T_{12} - T_{21}) \\ T_{23}^{sk} &= \tfrac{1}{2}(T_{23} - T_{32}) \\ T_{31}^{sk} &= \tfrac{1}{2}(T_{31} - T_{13}) \end{aligned}
\end{gathered} \tag{5-16}$$

If a second-order tensor "operates" on a vector, we interpret this to mean dotting on the right, and the result is a vector

$$\boldsymbol{T} \cdot \boldsymbol{v} = T_{ij}(\boldsymbol{e}_i \otimes \boldsymbol{e}_j) \cdot v_k \boldsymbol{e}_k = T_{ij} v_k \delta_{jk} \boldsymbol{e}_i = T_{ij} v_j \boldsymbol{e}_i = w_i \boldsymbol{e}_i \tag{5-17}$$

This equation is rewritten in direct, indicial and matrix notation as follows:

$$\boldsymbol{w} = \boldsymbol{T} \cdot \boldsymbol{v} \qquad w_i = T_{ij} v_j \qquad \{v\} = [T]\{v\} \tag{5-18}$$

If we dot on the left with a vector the result is

$$\boldsymbol{v} \cdot \boldsymbol{T} = v_r \boldsymbol{e}_r \cdot T_{st}(\boldsymbol{e}_s \otimes \boldsymbol{e}_t) = T_{st} v_r \delta_{rs} \boldsymbol{e}_t = T_{rt} v_r \boldsymbol{e}_t = u_t \boldsymbol{e}_t \tag{5-19}$$

This equation is rewritten in direct, indicial and matrix notation as follows:

$$\boldsymbol{u} = \boldsymbol{v} \cdot \boldsymbol{T} \qquad \mathrm{u}_s = T_{rs} v_s \qquad \langle u \rangle = \langle v \rangle [T] \tag{5-20}$$

A dot product with two vectors becomes a scalar

$$w = \boldsymbol{u} \cdot \boldsymbol{T} \cdot \boldsymbol{v} = \boldsymbol{T} \cdot\cdot (\boldsymbol{u} \otimes \boldsymbol{v}) = T_{ij} u_i v_j = \langle u \rangle [T] \{v\} \tag{5-21}$$

Because the result is a scalar the equal sign holds for terms involving direct notation, indicial notation and matrix notation.

We note that

$$\boldsymbol{T} \cdot \boldsymbol{v} = \boldsymbol{v} \cdot \boldsymbol{T}^T \tag{5-22}$$

The composition of two second-order tensors, $\boldsymbol{R}$ and $\boldsymbol{S}$ is a second-order tensor $\boldsymbol{T}$ where

$$\boldsymbol{T} = \boldsymbol{R} \cdot \boldsymbol{S} \qquad T_{ij} = R_{ik} S_{kj} \qquad [T] = [R][S] \tag{5-23}$$

The composition of a tensor with itself is

$$\boldsymbol{T}^2 = \boldsymbol{T}\cdot\boldsymbol{T} \qquad T_{ik}^2 = T_{ij}T_{jk} \qquad [T]^2 = [T][T] \tag{5-24}$$

A special second-order tensor is the identity tensor

$$\boldsymbol{I} = \delta_{ij}\boldsymbol{e}_i \otimes \boldsymbol{e}_j = \boldsymbol{e}_i \otimes \boldsymbol{e}_i \tag{5-25}$$

It follows that for a vector $\boldsymbol{u}$ and a second-order tensor $\boldsymbol{T}$

$$\boldsymbol{I}\cdot\boldsymbol{u} = \boldsymbol{u} \qquad \boldsymbol{u}\cdot\boldsymbol{I} = \boldsymbol{u} \qquad \boldsymbol{T}\cdot\boldsymbol{I} = \boldsymbol{T} \qquad \boldsymbol{T}\cdot\boldsymbol{I} = \boldsymbol{T} \tag{5-26}$$

The inner product of a second-order tensor with itself is

$$\boldsymbol{T}\cdot\cdot\boldsymbol{T} = T_{ij}T_{ij} = tr\Big[[T][T]^T\Big] \tag{5-27}$$

The inner product of a second-order tensor and the identity is the trace:

$$\boldsymbol{T}^{tr} = \boldsymbol{T}\cdot\cdot\boldsymbol{I} = \boldsymbol{I}\cdot\cdot\boldsymbol{T} = T_{ii} = tr[T] \tag{5-28}$$

Next we provide a brief development for higher-order tensors

5.4 An Alternative Notation

Recall that for vectors, we let $\{\boldsymbol{e}\}$ be an alternative form for representing the set of base vectors $\boldsymbol{e}_i$. We perform an analogous representation here where we represent the second-order tensor basis, a set of nine terms, as

$$\boldsymbol{e}_i \otimes \boldsymbol{e}_j \quad \Rightarrow \quad \{\boldsymbol{e}\}\otimes\langle\boldsymbol{e}\rangle = \begin{bmatrix} \boldsymbol{e}_1\otimes\boldsymbol{e}_1 & \boldsymbol{e}_1\otimes\boldsymbol{e}_2 & \boldsymbol{e}_1\otimes\boldsymbol{e}_3 \\ \boldsymbol{e}_2\otimes\boldsymbol{e}_1 & \boldsymbol{e}_2\otimes\boldsymbol{e}_2 & \boldsymbol{e}_2\otimes\boldsymbol{e}_3 \\ \boldsymbol{e}_3\otimes\boldsymbol{e}_1 & \boldsymbol{e}_3\otimes\boldsymbol{e}_2 & \boldsymbol{e}_3\otimes\boldsymbol{e}_3 \end{bmatrix} \tag{5-29}$$

Any second-order tensor, a sum of nine terms, can be represented as follows:

$$\boldsymbol{T} = T_{ij}\boldsymbol{e}_i \otimes \boldsymbol{e}_j = \langle\boldsymbol{e}\rangle\otimes[T]\{\boldsymbol{e}\} \tag{5-30}$$

In particular, the identity tensor is

$$\boldsymbol{I} = \delta_{ij}\boldsymbol{e}_i \otimes \boldsymbol{e}_j = \boldsymbol{e}_i \otimes \boldsymbol{e}_i = \langle\boldsymbol{e}\rangle\otimes[I]\{\boldsymbol{e}\} = \langle\boldsymbol{e}\rangle\otimes\{\boldsymbol{e}\} \tag{5-31}$$

The product of two tensors is

$$\boldsymbol{T} = \boldsymbol{R}\cdot\boldsymbol{S} = \langle\boldsymbol{e}\rangle\otimes[T]\{\boldsymbol{e}\}\cdot\langle\boldsymbol{e}\rangle\otimes[T]\{\boldsymbol{e}\} = \langle\boldsymbol{e}\rangle\otimes[T][I][T]\{\boldsymbol{e}\} = \langle\boldsymbol{e}\rangle\otimes[T]^2\{\boldsymbol{e}\} \tag{5-32}$$

We note that

$$\{\boldsymbol{e}\}\cdot\langle\boldsymbol{e}\rangle = [I] \tag{5-33}$$

Alternative expressions can be constructed for a dot product with a vector, taking a transpose, etc.

5.4 Third-order Tensors with an Example

In a manner similar to that for second-order tensors, we construct a basis $\boldsymbol{e}_i \otimes \boldsymbol{e}_j \otimes \boldsymbol{e}_k$ and represent a general third-order tensor, $\overset{3}{\boldsymbol{M}}$, as follows:

$$\overset{3}{\boldsymbol{M}} = M_{ijk}\boldsymbol{e}_i \otimes \boldsymbol{e}_j \otimes \boldsymbol{e}_k \tag{5-34}$$

Contractions with a vector and a second order tensor result in a second-order tensor and a vector as follows:

$$\begin{aligned} \boldsymbol{T} &= \overset{3}{\boldsymbol{M}}\cdot\boldsymbol{u} \qquad & T_{ij} &= M_{ijk}u_k \\ \boldsymbol{v} &= \overset{3}{\boldsymbol{M}}\cdot\cdot\boldsymbol{R} \qquad & v_i &= M_{ijk}R_{jk} \end{aligned} \tag{5-35}$$

Now, constructing a corresponding matrix equation is not trivial.

A third-order tensor that is widely employed is the alternating tensor for which the components in any orthonormal basis are the terms associated with the alternating symbol:

$$\overset{3}{\boldsymbol{\varepsilon}} = \varepsilon_{ijk}\boldsymbol{e}_i \otimes \boldsymbol{e}_j \otimes \boldsymbol{e}_k \tag{5-36}$$

Suppose $\boldsymbol{T}^{sk}$ is a skew-symmetric second-order tensor as defined in (5-16). Then the "axial vector", $\boldsymbol{a}$, associated with $\boldsymbol{T}^{sk}$, and the inverse relation are

$$\boldsymbol{a} = \frac{1}{2}\overset{3}{\boldsymbol{\varepsilon}}\cdot\cdot\boldsymbol{T}^{sk} \qquad \boldsymbol{T}^{sk} = \overset{3}{\boldsymbol{\varepsilon}}\cdot\boldsymbol{a} \quad \text{or} \quad a_1 = T^{sk}_{23} \qquad a_2 = T^{sk}_{31} \qquad a_3 = T^{sk}_{12} \tag{5-37}$$

This relationship will be used when we discuss spin and vorticity.

5.5 Fourth-order Tensors

We will only mention that fourth-order tensors typically appear in connection with the elasticity tensor and with a tangent tensor that relate stress to strain for elastic materials, and stress rate to strain rate, for general materials, respectively. Following the general structure given previously, let us denote the elasticity tensor as

$$\overset{4}{\boldsymbol{E}} = E_{ijkl}\boldsymbol{e}_i \otimes \boldsymbol{e}_j \otimes \boldsymbol{e}_k \otimes \boldsymbol{e}_l \tag{5-38}$$

Suppose $\boldsymbol{e}$ and $\boldsymbol{\sigma}$ denote strain and stress, respectively, where both are second-order tensors. Then the elasticity relation is

$$\boldsymbol{\sigma} = \overset{4}{\boldsymbol{E}}\cdot\cdot\boldsymbol{e} \qquad \sigma_{ij} = E_{ijkl}e_{kl} \tag{5-39}$$

Other operations can be performed but these relations are more appropriate for a study of constitutive equations.

5.6 Closing Comments

Since the orthonormal basis is arbitrary, the operations defined in this Chapter hold for all orthonormal bases. Stated differently, the use of direct notation for an equation indicates that the result must hold for any basis, and in particular for any orthonormal basis.