16. Euler's Equations, Cauchy Stress and Equations of Motion

16.1 Initial Comments

The development of the equations of motion required a series of remarkably intelligent assumptions to arrive at a logical set of equations that culminate in the equations of motion that are currently used. The primary contributors to the foundations of mechanics and the years of their lives are Newton (1643-1705), Euler (1707-1783) and Cauchy (1789-1857). The starting point is the formulation of Newton for the motion of point masses, but Euler provided the essential idea of how to treat a continuum. Then Cauchy provided the final break through that led to the concepts of traction and a stress tensor, and the relation between the two. We highlight the major new equations by listing them as steps.

16.2 Euler's First and Second Laws

Consider a body currently occupying a region R with boundary ∂R. The reference configuration, $\boldsymbol{E}_i$, is considered to be Galilean, i.e., fixed relative to the stars although this is now a nebulous concept. Let $\boldsymbol{r}(\boldsymbol{R},t)$ denote the position of a material point in the current configuration where the position at time $t=0$ is $\boldsymbol{R}$. For simplicity, assume these two position vectors have a common origin, O, as indicated in Fig. 16-1.

Let $\boldsymbol{F}$ be the total force acting on the body, and let $\boldsymbol{L}$ be the total moment with respect to point O acting on the body. The total moment is the sum of moments of forces acting on the body.

We define the total momentum of the body to be

$$\boldsymbol{P} = \int_R \boldsymbol{v}\rho\, dV \tag{16-1}$$

and the moment of momentum, or angular momentum, to be

$$\boldsymbol{H} = \int_R (\boldsymbol{r}\times\boldsymbol{v})\rho\, dV \tag{16-2}$$

Euler's first and second laws are the postulates that

$$\boldsymbol{F} = \dot{\boldsymbol{P}} \qquad\qquad \boldsymbol{L} = \dot{\boldsymbol{H}} \tag{16-3}$$

The first can be considered a generalization of Newton's Law while the second can be interpreted as the dynamic version of the "law of the lever". With the use of (15.8), it follows that

$$\begin{aligned}\dot{\boldsymbol{P}} &= \int_R \dot{\boldsymbol{v}}\rho\, dV = \int_R \boldsymbol{a}\rho\, dV\\ \dot{\boldsymbol{H}} &= \int_R \frac{d}{dt}(\boldsymbol{r}\times\boldsymbol{v})\rho\, dV = \int_R \left[(\boldsymbol{v}\times\boldsymbol{v})+(\boldsymbol{r}\times\boldsymbol{a})\right]\rho\, dV = \int_R (\boldsymbol{r}\times\boldsymbol{a})\rho\, dV\end{aligned} \tag{16-4}$$

Step 1. Euler's first and second laws become

$$\boldsymbol{F} = \int_R \boldsymbol{a}\rho\, dV \qquad \boldsymbol{L} = \int_R (\boldsymbol{r}\times\boldsymbol{a})\rho\, dV \tag{16-5}$$

The rigid-body equations of motion follow from these two equations.

Any subregion, R^*, with boundary ∂R^* can also be considered a body. Let $\boldsymbol{F}^*$ and $\boldsymbol{L}^*$ be the resultant force and moment, respectively, acting on the subregion.

Step 2. Euler's first and second laws for the subregion, in analogy with (16-5), become

$$\boldsymbol{F}^* = \int_{R^*} \boldsymbol{a}\rho\, dV \qquad \boldsymbol{L}^* = \int_{R^*} (\boldsymbol{r}\times\boldsymbol{a})\rho\, dV \tag{16-6}$$

The second equation is what led to the Euler-Bernouilli formulation for beams that is used to this day. The problem is "What are the general definitions of $\boldsymbol{F}^*$ and $\boldsymbol{L}^*$ for a segment of a body separated or cut from a larger body"?

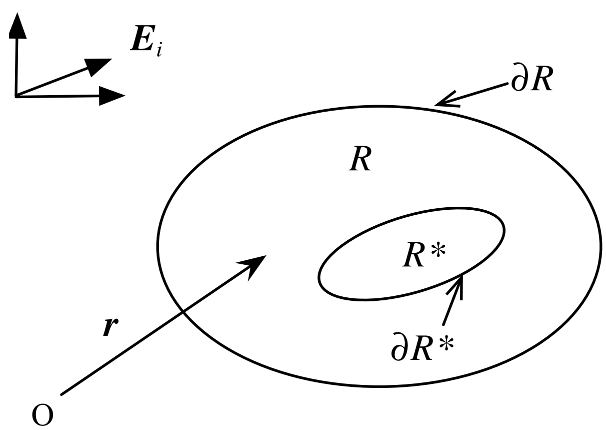


Fig. 16-1. Sketch showing notation for complete body, R, and a subregion, R^*.

16.3 Traction Principle of Euler and Cauchy

Assume that the total force, $\boldsymbol{F}^*$, acting on the subregion is composed of body force, $\boldsymbol{F}_b^*$, associated with the points in R^* and a contact force, $\boldsymbol{F}_c^*$, due to the action of the material external to R^* acting on the surface ∂R^*, i.e.,

$$\boldsymbol{F}^* = \boldsymbol{F}_c^* + \boldsymbol{F}_b^* \tag{16-7}$$

The formulation for the body force goes one step further by making the assumption that there is a specific body force function $\boldsymbol{b}(\boldsymbol{r},t)$ such that

$$\boldsymbol{F}_b^* = \int_{R^*} \boldsymbol{b}\rho \, dV \tag{16-8}$$

Recall that in this context the adjective "specific" means "per unit mass". However, the big advancement was the assumption that the effect of the external material on the subregion R^* could be reflected simply by identifying a traction vector, $\boldsymbol{\tau}$, on the "cut" surface ∂R^* that represents a force per unit area.

Step 3. The contact force can be described as follows:

$$\boldsymbol{F}_c^* = \int_{\partial R^*} \boldsymbol{\tau} \, da \tag{16-9}$$

Now the problem becomes "What variables should be used to describe the function $\boldsymbol{\tau}$"?

Consider the sketch of Fig. 16-2 that shows two subregions, ∂R_1^* and ∂R_2^*, that have a common tangent at the point P of interest. One possible hypothesis is that the traction at this point depends on the subregion, or that the respective tractions are $\boldsymbol{\tau}_1 = \boldsymbol{\tau}(\partial R_1^*)$ and $\boldsymbol{\tau}_2 = \boldsymbol{\tau}(\partial R_2^*)$. Although this might seem to be a reasonable hypothesis, it did not lead to a set of useful equations. Instead the hypothesis that served as the breakthrough and the one that is used now is that the traction depends on the normal vector, $\boldsymbol{n}$, at the point on the surface of interest.

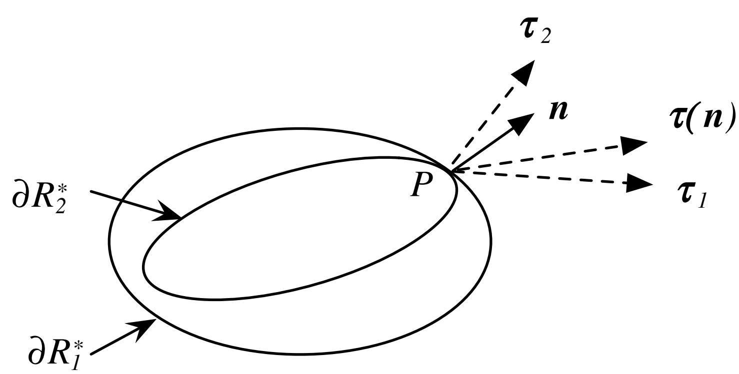


Fig. 16-2. Various interpretations of traction at two cut surfaces with a common normal.

Step 4. The traction at any point on any "cut" surface is given as follows:

$$\boldsymbol{\tau} = \boldsymbol{\tau}(\boldsymbol{n}, \boldsymbol{r}, t) \tag{16-10}$$

where the point P is identified by the position vector, $\boldsymbol{r}$. Now the traction for point P in Fig. 16-2 is a unique vector for the two subregions because they have the same normal vector.

16.4 Cauchy's Fundamental Theorem on Stress

The object of this subsection is to show that the traction is a linear transformation of the unit normal, and this transformation is represented by the Cauchy stress tensor.

If we combine (16-6), (16-7), (16-8) and (16-9), then Euler's first law becomes

$$\int_{\partial R^*} \boldsymbol{\tau}\, da + \int_{R^*} \boldsymbol{b}\rho\, dV = \int_{R^*} \boldsymbol{a}\rho\, dV \qquad (16\text{-}11)$$

Suppose the region R^* is circumscribed by a sphere of diameter "d". Then each of the terms in (16-11) is the order of magnitude that follows:

$$\int_{\partial R^*} \boldsymbol{\tau}\, da = \boldsymbol{O}(d^2) \qquad \int_{R^*} \boldsymbol{b}\rho\, dV = \boldsymbol{O}(d^3) \qquad \int_{R^*} \boldsymbol{a}\rho\, dV = \boldsymbol{O}(d^3) \qquad (16\text{-}12)$$

As $d \to 0$ the first term in (16-11) dominates so we have the equation

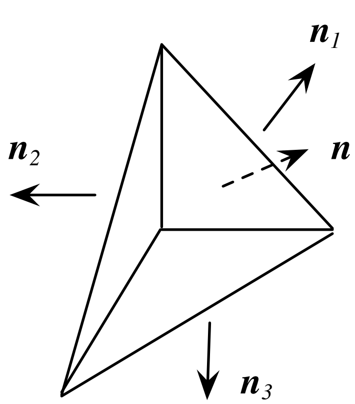
$$\lim_{d\to 0} \int_{\partial R^*} \boldsymbol{\tau}\, da = \boldsymbol{0} \qquad (16\text{-}13)$$

Now represent the region as a tetrahedron (See Fig. 16-3) whose sides have areas A_1, A_2, A_3 and A . Since the region is infinitesimal, the position of each point on the surface can be considered the same as the position of the centroid of the region, and the traction is constant over each surface with a constant normal. The traction vectors for the four surfaces are $\boldsymbol{\tau}_1, \boldsymbol{\tau}_2, \boldsymbol{\tau}_3$ and $\boldsymbol{\tau}$. The result is that (16-13) becomes

$$\boldsymbol{\tau}_1 A_1 + \boldsymbol{\tau}_2 A_2 + \boldsymbol{\tau}_3 A_3 + \boldsymbol{\tau} A = \boldsymbol{0} \qquad (16\text{-}14)$$

or

$$\boldsymbol{\tau} = \boldsymbol{\tau}_1 \hat{A}_1 + \boldsymbol{\tau}_2 \hat{A}_2 + \boldsymbol{\tau}_3 \hat{A}_3 \qquad \hat{A}_i = -\frac{A_i}{A} \qquad i = 1,2,3 \qquad (16\text{-}15)$$



(a) Normals to surfaces of tetrahedron

(b) Normals and tractions on opposite sides of a cut

Fig. 16-3.

With the use of (16-10), we know that

$$\boldsymbol{\tau} = \boldsymbol{\tau}(\boldsymbol{n}) \qquad \boldsymbol{\tau}_1 = \boldsymbol{\tau}(\boldsymbol{n}_1) \qquad \boldsymbol{\tau}_2 = \boldsymbol{\tau}(\boldsymbol{n}_2) \qquad \boldsymbol{\tau}_3 = \boldsymbol{\tau}(\boldsymbol{n}_3) \tag{16-16}$$

Recall that

$$\int_{R^*} \boldsymbol{n}\, da = \boldsymbol{0} \tag{16-17}$$

for any region. For the infinitesimal region under consideration, this equation implies that

$$\boldsymbol{n} = \hat{A}_1 \boldsymbol{n}_1 + \hat{A}_2 \boldsymbol{n}_2 + \hat{A}_3 \boldsymbol{n}_3 \tag{16-18}$$

Now we combine (16-15), (16-16) and (16-18) to obtain

$$\begin{aligned} \boldsymbol{\tau}\left(\hat{A}_1 \boldsymbol{n}_1 + \hat{A}_2 \boldsymbol{n}_2 + \hat{A}_3 \boldsymbol{n}_3\right) &= \boldsymbol{\tau}_1 \hat{A}_1 + \boldsymbol{\tau}_2 \hat{A}_2 + \boldsymbol{\tau}_3 \hat{A}_3 \\ &= \hat{A}_1 \boldsymbol{\tau}(\boldsymbol{n}_1) + \hat{A}_2 \boldsymbol{\tau}(\boldsymbol{n}_2) + \hat{A}_3 \boldsymbol{\tau}(\boldsymbol{n}_3) \end{aligned} \tag{16-19}$$

Since the choice of the orientations of the sides of the tetrahedron is arbitrary, (16-19) holds for all values of $\hat{A}_i$ and the implication is that $\boldsymbol{\tau}$ varies linearly with $\boldsymbol{n}$.

Step 4. The linear transformation implies the existence of a tensor, $\boldsymbol{\sigma}(\boldsymbol{r},t)$, now called the Cauchy stress tensor, such that

$$\boldsymbol{\tau} = \boldsymbol{\sigma} \cdot \boldsymbol{n} \tag{16-20}$$

Cauchy's Lemma states that

$$\boldsymbol{\tau}(\boldsymbol{r},\boldsymbol{n}) = -\boldsymbol{\tau}(\boldsymbol{r},-\boldsymbol{n}) \tag{16-21}$$

or one traction vector is the negative of the other for tractions defined at a point of the two cut surfaces for which one normal is the negative of the other.

16.5 Cauchy's First and Second Equations of Motion

If we substitute (16-20) in Euler's first equation (16-11), then

$$\int_{\partial R^*} \boldsymbol{\sigma} \cdot \boldsymbol{n}\, da + \int_{R^*} \boldsymbol{b} \rho\, dV = \int_{R^*} \boldsymbol{a} \rho\, dV \tag{16-22}$$

Apply the divergence theorem to obtain

$$\int_{R^*} \boldsymbol{\sigma} \cdot \bar{\nabla}\, da + \int_{R^*} \boldsymbol{b} \rho\, dV = \int_{R^*} \boldsymbol{a} \rho\, dV \tag{16-23}$$

Since R^* is arbitrary the result is that the integral signs can be removed.

Step 4. Cauchy's first equation of motion is

$$\boldsymbol{\sigma} \cdot \bar{\nabla} + \rho \boldsymbol{b} = \rho \boldsymbol{a} \tag{16-24}$$

Now we return to Euler's second equation in (16-6). In a manner similar to that used for the resultant force, the resultant moment for the subregion is

$$\boldsymbol{L}^* = \boldsymbol{L}_c^* + L_b^* \qquad \boldsymbol{L}_c^* = \int_{\partial R^*} (\boldsymbol{r}\times\boldsymbol{\tau})\,da \qquad \boldsymbol{F}_b^* = \int_{R^*} (\boldsymbol{r}\times\boldsymbol{\tau})\,\rho\,dV \tag{16-25}$$

so that Euler's second equation becomes

$$\int_{\partial R^*} (\boldsymbol{r}\times\boldsymbol{\tau})da + \int_{R^*} (\boldsymbol{r}\times\boldsymbol{b})\rho dV = \int_{R^*} (\boldsymbol{r}\times\boldsymbol{a})\,\rho\,dV \tag{16-26}$$

Now substitute (16-20) into the first term and apply the divergence theorem:

$$\begin{aligned}\int_{\partial R^*} (\boldsymbol{r}\times\boldsymbol{\tau})da &= \int_{\partial R^*} (\boldsymbol{r}\times\boldsymbol{\sigma})\cdot\boldsymbol{n}da \\ &= \int_{R^*} (\boldsymbol{r}\times\boldsymbol{\sigma})\cdot\bar{\bar{\nabla}} dV\end{aligned} \tag{16-27}$$

Since (16-26) holds for arbitrary R^*, Euler's second law implies that

$$(\boldsymbol{r}\times\boldsymbol{\sigma})\cdot\bar{\bar{\nabla}} + \rho(\boldsymbol{r}\times\boldsymbol{b}) = \rho(\boldsymbol{r}\times\boldsymbol{b}) \tag{16-28}$$

We note that the first term becomes

$$\begin{aligned}(\boldsymbol{r}\times\boldsymbol{\sigma})\cdot\bar{\bar{\nabla}} &= \boldsymbol{\varepsilon}\cdot\cdot\left\{(\boldsymbol{r}\otimes\boldsymbol{\sigma})\cdot\bar{\bar{\nabla}}\right\} \\ &= \boldsymbol{\varepsilon}\cdot\cdot\left\{\boldsymbol{r}\otimes\left(\boldsymbol{\sigma}\cdot\bar{\bar{\nabla}}\right)\right\} + \boldsymbol{\varepsilon}\cdot\cdot\left\{\boldsymbol{r}\bar{\bar{\nabla}}\cdot\boldsymbol{\sigma}^T\right\} \\ &= \boldsymbol{r}\times\left(\boldsymbol{\sigma}\cdot\bar{\bar{\nabla}}\right) + \boldsymbol{\varepsilon}\cdot\cdot\left\{\boldsymbol{I}\cdot\boldsymbol{\sigma}^T\right\} \\ &= \boldsymbol{r}\times\left(\boldsymbol{\sigma}\cdot\bar{\bar{\nabla}}\right) + \boldsymbol{\varepsilon}\cdot\cdot\boldsymbol{\sigma}^T\end{aligned} \tag{16-29}$$

Now (16-28) becomes

$$\boldsymbol{\varepsilon}\cdot\cdot\boldsymbol{\sigma}^T + \boldsymbol{r}\times\left[\boldsymbol{\sigma}\cdot\bar{\bar{\nabla}} + \rho\boldsymbol{b} - \rho\boldsymbol{a}\right] = \boldsymbol{0} \tag{16-30}$$

If Cauchy's first equation of motion (16-24) is invoked, then (16-30) reduces to

$$\boldsymbol{\varepsilon}\cdot\cdot\boldsymbol{\sigma}^T = \boldsymbol{0} \tag{16-31}$$

Invoking the skew-symmetric properties of the alternating tensor, the result is that the Cauchy stress tensor is symmetric.

Step 5. Cauchy's second equation of motion is

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^T \tag{16-32}$$

16.6 Equations of Motion in Reference Configuration

Suppose we define a traction vector, $\hat{\boldsymbol{\tau}}$, referenced to an area element, dA_0, in the original configuration as follows:

$$\hat{\boldsymbol{\tau}} dA_0 = \boldsymbol{\tau} da \tag{16-33}$$

The Piola-Kirchhoff stress tensor of the first kind, $\hat{\boldsymbol{P}}$, is defined such that

$$\hat{\boldsymbol{P}} \cdot \boldsymbol{N} = \hat{\boldsymbol{\tau}} \tag{16-34}$$

Use Nanson's relation that relates area elements

$$\boldsymbol{n} da = J\boldsymbol{F}^{-T} \cdot \boldsymbol{N} dA_0 \tag{16-35}$$

Cauchy's relation $\boldsymbol{\tau} = \boldsymbol{\sigma} \cdot \boldsymbol{n}$, and (16-34) to obtain

$$\begin{aligned} \hat{\boldsymbol{P}} \cdot \boldsymbol{N} dA_0 &= \boldsymbol{\sigma} \cdot \boldsymbol{n} da \\ &= \boldsymbol{\sigma} \cdot J\boldsymbol{F}^{-T} \cdot \boldsymbol{N} dA_0 \end{aligned} \tag{16-36}$$

This equation must hold for arbitrary $\boldsymbol{N}$. The result is an equation that relates the two stress tensors:

$$\hat{\boldsymbol{P}} = J\boldsymbol{\sigma} \cdot \boldsymbol{F}^{-T} \qquad \text{or} \qquad \boldsymbol{\sigma} = \tfrac{1}{J} \hat{\boldsymbol{P}} \cdot \boldsymbol{F}^T \tag{16-37}$$

Cauchy's second equation of motion $\boldsymbol{\sigma} = \boldsymbol{\sigma}^T$ becomes

$$\hat{\boldsymbol{P}} \cdot \boldsymbol{F}^T = \boldsymbol{F} \cdot \hat{\boldsymbol{P}}^T \tag{16-38}$$

a relation that indicates $\hat{\boldsymbol{P}}$ is not symmetric.

Now we substitute the second of (16-37) into Cauchy's first equation of motion, (16-24) to obtain

$$\left(\tfrac{1}{J} \hat{\boldsymbol{P}} \cdot \boldsymbol{F}^T\right) \cdot \bar{\bar{\nabla}} + \rho \boldsymbol{b} = \rho \boldsymbol{a} \tag{16-39}$$

Now we recall from (10-54) that

$$\left(\tfrac{1}{J} \boldsymbol{F}^T\right) \cdot \bar{\bar{\nabla}} = \boldsymbol{0} \tag{16-40}$$

Then with the use of the contraction operator, it follows that

$$\left(\tfrac{1}{J} \hat{\boldsymbol{P}} \cdot \boldsymbol{F}^T\right) \cdot \bar{\bar{\nabla}} = \hat{\boldsymbol{P}} \cdot \left[\left(\tfrac{1}{J} \boldsymbol{F}^T\right) \cdot \bar{\bar{\nabla}}\right] + C_{23}\left[\left(\hat{\boldsymbol{P}} \otimes \bar{\bar{\nabla}}\right) \cdot \left(\tfrac{1}{J} \boldsymbol{F}\right)\right] \tag{16-41}$$

Now recall that $()\bar{\bar{\nabla}} \cdot \boldsymbol{F} = ()\bar{\bar{\nabla}}_0$. The last term in (16-41) becomes

$$C_{23}\left[\left(\hat{\boldsymbol{P}} \otimes \bar{\bar{\nabla}}\right) \cdot \left(\tfrac{1}{J} \boldsymbol{F}\right)\right] = C_{23}\left[\left(\hat{\boldsymbol{P}} \otimes \bar{\bar{\nabla}}_0\right) \tfrac{1}{J}\right] = \left(\hat{\boldsymbol{P}} \cdot \bar{\bar{\nabla}}_0\right) \tfrac{1}{J} \tag{16-42}$$

Recall that conservation of mass implies $J = \rho_0 / \rho$. If we multiply through by J and use (16-42), then (16-39) becomes

$$\hat{\boldsymbol{P}} \cdot \bar{\bar{\nabla}} + \rho_0 \boldsymbol{b} = \rho_0 \boldsymbol{a} \tag{16-43}$$

Cauchy's first equation of motion in the reference configuration (16-43) is analogous to to the equation in the current configuration (16-24) with the appropriate changes in stress tensor, gradient and density. However, remember that the Piola-Kirchhoff stress tensor of he first kind is not symmetric, a feature that can be inconvenient in formulating constitutive equations. To overcome this perceived drawback, we introduce the Piola-Kirchhoff stress tensor of the second kind, $\boldsymbol{P}$, as follows:

$$\begin{aligned} &\boldsymbol{P} = \boldsymbol{F}^{-1} \cdot \hat{\boldsymbol{P}} = J\boldsymbol{F}^{-1} \cdot \boldsymbol{\sigma} \cdot \boldsymbol{F}^{-T} \\ &\boldsymbol{\sigma} = \tfrac{1}{J}\boldsymbol{F} \cdot \boldsymbol{\sigma} \cdot \boldsymbol{F}^{T} \end{aligned} \tag{16-44}$$

In terms of this stress tensor, Cauchy's first and second equations of motion become

$$\left(\boldsymbol{F} \cdot \boldsymbol{P}\right) \cdot \bar{\nabla}_0 + \rho_0 \boldsymbol{b} = \rho_0 \boldsymbol{a} \qquad \boldsymbol{P} = \boldsymbol{P}^{T} \tag{16-45}$$

An added complication is that the relation between $\boldsymbol{P}$ and the traction $\hat{\boldsymbol{\tau}}$ becomes

$$\hat{\boldsymbol{\tau}} = \left(\boldsymbol{F} \cdot \boldsymbol{P}\right) \cdot \boldsymbol{N} \tag{16-46}$$

This concludes the section that indicates how conversions are made to the equations of motion so that they can be applied in either the reference or the current configuration.

16.7 Summary

It took approximately two centuries after Newton that the appropriate assumptions were made to arrive at the equations that we currently use. Of course indicial and direct notation had not been developed so it would be difficult for us to even understand these fundamental contributions as written originally.

The key developments and the results are the following:

Euler's equations for subregions:

$$\boldsymbol{F}^* = \int_{R^*} \boldsymbol{a}\rho\, dV \qquad \boldsymbol{L}^* = \int_{R^*} (\boldsymbol{r}\times\boldsymbol{a})\rho\, dV \tag{16-47}$$

The existence of a specific distributed body force function and a surface traction vector provides specific forms for the applied force and moment

$$\begin{aligned} &\boldsymbol{F}^* = \boldsymbol{F}_c^* + \boldsymbol{F}_b^* \qquad \boldsymbol{F}_c^* = \int_{\partial R^*} \boldsymbol{\tau}\, da \qquad \boldsymbol{F}_b^* = \int_{R^*} \boldsymbol{b}\rho\, dV \\ &\boldsymbol{L}^* = \boldsymbol{L}_c^* + L_b^* \qquad \boldsymbol{L}_c^* = \int_{\partial R^*} (\boldsymbol{r}\times\boldsymbol{\tau})\, da \qquad \boldsymbol{F}_b^* = \int_{R^*} (\boldsymbol{r}\times\boldsymbol{\tau})\rho\, dV \end{aligned} \tag{16-48}$$

The traction depends linearly on the surface normal and this implies the existence of a tensor called Cauchy's stress tensor:

$$\boldsymbol{\tau} = \boldsymbol{\sigma}\cdot\boldsymbol{n} \tag{16-49}$$

If Euler's equations hold for arbitrary subregions, R^*, the result is Cauchy's two equations of motion:

$$\boldsymbol{\sigma}\cdot\bar{\bar{\nabla}} + \rho\boldsymbol{b} = \rho\boldsymbol{a} \qquad \boldsymbol{\sigma} = \boldsymbol{\sigma}^T \tag{16-50}$$

If distributed couples over the volume or the surface of the body are allowed, then the stress tensor is no longer symmetric and the resulting formulation is called couple-stress theory.

The definition of the first Piola-Kirchoff stress tensor and the corresponding relation to traction and the equations of motion are

$$\begin{aligned} &\hat{\boldsymbol{P}} = J\boldsymbol{\sigma}\cdot\boldsymbol{F}^{-T} \qquad \hat{\boldsymbol{P}}\cdot\boldsymbol{N} = \hat{\boldsymbol{\tau}} = \boldsymbol{\tau}\tfrac{da}{dA_0} \\ &\hat{\boldsymbol{P}}\cdot\bar{\bar{\nabla}} + \rho_0\boldsymbol{b} = \rho_0\boldsymbol{a} \qquad \hat{\boldsymbol{P}}\cdot\boldsymbol{F}^T = \boldsymbol{F}\cdot\hat{\boldsymbol{P}}^T \end{aligned} \tag{16-51}$$

The definition of the second Piola-Kirchoff stress tensor and the corresponding relation to traction and the equations of motion are

$$\begin{aligned} &\boldsymbol{P} = \boldsymbol{F}^{-1}\cdot\hat{\boldsymbol{P}} = J\boldsymbol{F}^{-1}\cdot\boldsymbol{\sigma}\cdot\boldsymbol{F}^{-T} \qquad (\boldsymbol{F}\cdot\boldsymbol{P})\cdot\boldsymbol{N} = \hat{\boldsymbol{\tau}} \\ &(\boldsymbol{F}\cdot\boldsymbol{P})\cdot\bar{\bar{\nabla}}_0 + \rho_0\boldsymbol{b} = \rho_0\boldsymbol{a} \qquad \boldsymbol{P} = \boldsymbol{P}^T \end{aligned} \tag{16-52}$$

Other stress tensors can be introduced but this chapter has provided the foundation for the equations of motion in either the current or the reference configuration.