2) for $[A]$ find:

a) eigenpairs

b) Modal matrix , c) calc. $[M^o]^T[A][M^o]$, $[M^o][M^o]^T$, $\sum \lambda_i \underline{e}^i \underline{e}^i$

d) rank and range of null space

i) $[A] = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$

eigenproblem : $[[A] - \lambda [I]]\{e\} = \{0\}$

Cayley-Hamilton: $\det([A] - \lambda [I]) = 0$

$$\det\left(\begin{bmatrix} 1-\lambda & -1 & 0 \\ -1 & 2-\lambda & -1 \\ 0 & -1 & 1-\lambda \end{bmatrix}\right) = (1-\lambda)(2-\lambda)(1-\lambda) - (1-\lambda) - (1-\lambda)$$

$$= (1-2\lambda+\lambda^2)(2-\lambda) - 2 + 2\lambda$$

$$= \cancel{2} - 4\lambda + \cancel{2\lambda^2} - \lambda + \cancel{2\lambda^2} - \lambda^3 - \cancel{2} + 2\lambda$$

$$= -\lambda^3 + 4\lambda^2 - 3\lambda$$

$$= \lambda(-\lambda^2 + 4\lambda - 3)$$

- characteristic eqn $\rightarrow$ $\lambda(-\lambda^2 + 4\lambda - 3) = 0$
- solve for the roots:

$\lambda_1 = 0$

$$\frac{-4 \pm \sqrt{16 - 12}}{-2} \rightarrow \frac{-4 \pm 2}{-2} = \lambda_2, \lambda_3$$

$\lambda_2 = 1$, $\lambda_3 = 3$

- solve for the eigenvectors $\Rightarrow$ sub. λ back into eigenproblem:

for λ_1; solve for $\underline{e}^1$

$$\begin{bmatrix} 1-\lambda & -1 & 0 \\ -1 & 2-\lambda & -1 \\ 0 & -1 & 1-\lambda \end{bmatrix} \begin{Bmatrix} e_1^1 \\ e_2^1 \\ e_3^1 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}, \quad \text{Let } \underline{e}_1^1 = 1 \text{ and } \lambda = 0$$

$\underline{e}_2^1 = 1$, $\underline{e}_3^1 = 1$

cont. $\underline{e}^1$:

normalize $\underline{e}^1$:
$$\frac{\{\underline{e}^1\}}{\|\{\underline{e}^1\}\|_2} = \frac{\langle 1, 1, 1\rangle}{\sqrt{1^2+1^2+1^2}}$$
$$= \left\langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right\rangle$$

solve for $\underline{e}^2$:
$$\begin{bmatrix} 1-\lambda_2 & -1 & 0 \\ -1 & 2-\lambda_2 & -1 \\ 0 & -1 & 1-\lambda_2 \end{bmatrix} \begin{Bmatrix} e^2_1 \\ e^2_2 \\ e^2_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}, \quad \lambda_2 = 1$$

$$\left.\begin{aligned} &\text{Let } e^2_1 = 1 \\ &-e^2_2 = 0 \\ &-1 + 0 - 1e^2_3 = 0 \\ &\qquad e^2_3 = -1 \end{aligned}\right\} \text{ normalize: } \frac{\underline{e}^2}{\sqrt{1^2+0^2+1^2}}$$

$$\underline{e}^2 = \left\langle \frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}} \right\rangle$$

- solve for $\underline{e}^3$:
$$\begin{bmatrix} 1-\lambda_3 & -1 & 0 \\ -1 & 2-\lambda_3 & -1 \\ 0 & -1 & 1-\lambda_3 \end{bmatrix} \begin{Bmatrix} e^3_1 \\ e^3_2 \\ e^3_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \qquad \begin{aligned} &\lambda_3 = 3 \\ &\text{Let } e^3_1 = 1 \end{aligned}$$

$$-2 \; -1e^3_2 = 0 \quad \longrightarrow \quad e^3_2 = -2$$
$$2 - 2e^3_3 = 0 \quad \longrightarrow \quad e^3_3 = 1$$

normalize: $\dfrac{\underline{e}^3}{\sqrt{1^2+2^2+1^2}} \Rightarrow \underline{e}^3 = \left\langle \frac{1}{\sqrt{6}}, \frac{-2}{\sqrt{6}}, \frac{1}{\sqrt{6}} \right\rangle$

$\therefore$ a) the eigenpairs are $(\lambda_i, \underline{e}^i)$:

$0, \left\langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right\rangle$

$1, \left\langle \frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right\rangle$

$3, \left\langle \frac{1}{\sqrt{6}}, \frac{-2}{\sqrt{6}}, \frac{1}{\sqrt{6}} \right\rangle$

b) the modal matrix:

$$\overset{a-p}{[M^0]} = \left[\frac{1}{\sqrt{3}}\begin{Bmatrix}1\\1\\1\end{Bmatrix} \quad \frac{1}{\sqrt{2}}\begin{Bmatrix}1\\0\\-1\end{Bmatrix} \quad \frac{1}{\sqrt{6}}\begin{Bmatrix}1\\-2\\1\end{Bmatrix}\right]$$

c)

$$\overset{p-a}{[M^0]}^T \overset{a-p}{[M^0]} = \begin{bmatrix} \frac{1}{\sqrt{3}}\langle 1 & 1 & 1\rangle \\ 1/\sqrt{2}\,\langle 1 & 0 & -1\rangle \\ 1/\sqrt{6}\,\langle 1 & -2 & 1\rangle \end{bmatrix} \left[\frac{1}{\sqrt{3}}\begin{Bmatrix}1\\1\\1\end{Bmatrix} \; \frac{1}{\sqrt{2}}\begin{Bmatrix}1\\0\\-1\end{Bmatrix} \; \frac{1}{\sqrt{6}}\begin{Bmatrix}1\\-2\\1\end{Bmatrix}\right]$$

$$= \begin{bmatrix} (1/3 + 1/3 + 1/3) & 0 & 0 \\ 0 & (\frac12 + 0 + \frac12) & 0 \\ 0 & 0 & (\frac16 + \frac46 + \frac16) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Bigg\} \text{ because eigenvectors are orthonormal} \Rightarrow [M^0]^T = [M^0]^{-T}$$

$$\overset{p-a}{[M^0]}^T \overset{a-a}{[A]} \overset{a-p}{[M^0]} = \overset{p-p}{[A]} \quad \Longrightarrow \overset{p-p}{[A]} \text{ is in the principal basis}$$

$$[M^0]^T[A] = \begin{bmatrix} (\frac{1}{\sqrt3} + -\frac{1}{\sqrt3}) & (-\frac{1}{\sqrt3} + 2/\sqrt3 - 1/\sqrt3) & (0 + 1/\sqrt3 - 1/\sqrt3) \\ (+1/\sqrt2 + 0 + 0) & (-1/\sqrt2 + 0 + 1/\sqrt2) & (0 + 0 - 1/\sqrt2) \\ (1/\sqrt6 + 2/\sqrt6 + 0) & (-1/\sqrt6 - 4/\sqrt6 - 1/\sqrt6) & (0 + 2/\sqrt6 + \frac{1}{\sqrt6}) \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ +1/\sqrt2 & 0 & -1/\sqrt2 \\ 3/\sqrt6 & -6/\sqrt6 & 3/\sqrt6 \end{bmatrix}$$

$$[M^0]^T[A][M^0] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac12 + \frac12 & 0 \\ 0 & 0 & (\frac36 + \frac{12}{6} + \frac36) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$= \lambda [I]$$

$$\sum_{i=1}^{n} \lambda_i \{e^i\}\langle e^i\rangle = [0] + 1\begin{bmatrix} 1/2 & 0 & -1/2 \\ 0 & 0 & 0 \\ -1/2 & 0 & 1/2 \end{bmatrix} + 3\begin{bmatrix} 1/6 & -2/6 & +1/6 \\ -2/6 & 4/6 & -2/6 \\ 1/6 & -2/6 & 1/6 \end{bmatrix}$$

$$\therefore \sum_{i=1}^{3} \lambda_i \{e^i\}\langle e^i\rangle = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} = [A]$$

d) rank => number of nonzero eigenvalues

$\therefore r = 2$

=> only 2 of the 3 rows are linearly independent

i.e. $-1 \cdot \langle A^1\rangle + -1 \cdot \langle A^3\rangle = \langle A^2\rangle$

range => the column space of [A]

this is the plane $\underline{e}^2$ + $\underline{e}^3$ lie in and $\underline{e}^1$ is orthogonal to $\therefore$ it may be described by a vector $\perp$ to it:

range = $\underline{e}^2 \times \underline{e}^3$

nullspace of [A] => the vector space of [A] that is not indepentent

$\therefore$ $\underline{e}^1$ is the nullspace of [A]

2. b)

$$[A] = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

eigenproblem: $\big[[A] - \lambda[I]\big]\{e\} = \{0\}$

-> because the (2,2) component of [A] is the only nonzero component on the row, 2 is an eigenvalue

$\lambda_1 = 2$

$$\rightarrow \begin{bmatrix} 1-\lambda & 0 & -1 \\ 0 & 2-\lambda & 0 \\ -1 & 0 & 1-\lambda \end{bmatrix} \{e\} = \{0\}$$

$$\det\Big(\big[[A]-\lambda[I]\big]\Big) = (1-\lambda)(2-\lambda)(1-\lambda) - (2-\lambda) \Rightarrow \text{Char. Poly.}$$

$$\underbrace{(2-\lambda)}_{\lambda_1 = 2}\underbrace{\big[(1-\lambda)(1-\lambda) - 1\big]}_{\text{solve for roots}} = 0 \Rightarrow \text{char. Eqn.}$$

$$1 - 2\lambda + \lambda^2 - 1 = 0$$

$$\lambda^2 - 2\lambda = 0$$

$$\frac{2 \pm \sqrt{4-0}}{2}$$

$$\frac{2 \pm 2}{2} = \lambda_2, \lambda_3 \rightarrow \begin{array}{l} \lambda_2 = 2 \\ \lambda_3 = 0 \end{array}$$

-> <u>find eigenvectors</u>: solve for $\underline{e}^1$ ->

by observation, $\underline{e}^1 = \begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix}$ will satisfy the char. Eqn.

solve for $\underline{e}^2$ ->

$$\begin{bmatrix} 1-\lambda & 0 & -1 \\ 0 & 2-\lambda & 0 \\ -1 & 0 & 1-\lambda \end{bmatrix} \begin{Bmatrix} e^2_1 \\ e^2_2 \\ e^2_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \qquad \text{for } \lambda_2 = 2, \text{ Let } e^2_1 = 1$$

$$-1 - 1e^2_3 = 0 \rightarrow e^2_3 = -1$$

$$0e^2_2 = 0 \rightarrow e^2_2 = 0 \quad \therefore \underline{e}^2 = \begin{Bmatrix} 1 \\ 0 \\ -1 \end{Bmatrix} = \begin{Bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{Bmatrix}$$

solve for $\underline{e}^3 \rightarrow$

$$\begin{bmatrix} 1-\lambda & 0 & -1 \\ 0 & 2-\lambda & 0 \\ -1 & 0 & 1-\lambda \end{bmatrix} \begin{Bmatrix} e^3_1 \\ e^3_2 \\ e^3_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \qquad \text{for } \lambda_3 = 0 \text{, let } e^3_1 = 1$$

$$1 - 1 e^3_3 = 0 \quad \rightarrow \quad e^3_3 = 1$$

$$2 e^3_2 = 0 \quad \rightarrow \quad e^3_2 = 0 \qquad \therefore \quad \underline{e}^3 = \begin{Bmatrix} 1 \\ 0 \\ 1 \end{Bmatrix} = \begin{Bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{Bmatrix}$$

a) the eigenpairs are then:

$2, \langle 0, 1, 0 \rangle$

$2, \langle 1/\sqrt{2}, 0, -1/\sqrt{2} \rangle$

$0, \langle 1/\sqrt{2}, 0, 1/\sqrt{2} \rangle$

b) the modal matrix:

$$\overset{a\cdot p}{[M^0]} = \left[\begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix} \quad 1/\sqrt{2} \begin{Bmatrix} 1 \\ 0 \\ -1 \end{Bmatrix} \quad 1/\sqrt{2} \begin{Bmatrix} 1 \\ 0 \\ 1 \end{Bmatrix} \right]$$

c) $\overset{p-a}{[M^0]}^T \overset{a-a}{[A]} \overset{a-p}{[M^0]} = \overset{p-p}{[A]} \Rightarrow \overset{p-p}{\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}}$, in the principal basis

$$[M^0]^T [M^0] = [I] \quad \Rightarrow \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\sum_{i=1}^{n} \lambda_i \{e^i\}\{e^i\}^T = 2\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + 2 \begin{bmatrix} 1/2 & 0 & -1/2 \\ 0 & 0 & 0 \\ -1/2 & 0 & 1/2 \end{bmatrix} + [0]$$

$$= \begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

d) rank $= 2$

range = plane defined by $\underline{e}^1 + \underline{e}^2$; the vector normal to this plane is $\underline{e}^3 = \underline{e}^1 \times \underline{e}^2$

null space $= \underline{e}^3$