Poisson Equation

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Poisson's Equation

Definition

$$\nabla^2 u = f \Rightarrow \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \frac{\partial^2 u}{\partial x_3^2} = f \ .$$

Description

Appears in almost every field of physics.

Solution to Case with 4 Homogeneous Boundary Conditions

Let's consider the following example, where $u_{xx} + u_{yy} = F(x,y), (x,y) \in [0,L] \times [0,M]$. and the Dirichlet boundary conditions are as follows:

$$\begin{aligned} u(0,y) &= 0 \\ u(L,y) &= 0 \\ u(x,0) &= 0 \\ u(x,M) &= 0 \end{aligned}$$

In order to solve this equation, let's consider that the solution to the homogeneous equation will allow us to obtain a system of basis functions that satisfy the given boundary conditions. We start with the Laplace equation: $u_{xx} + u_{yy} = 0$.

Step 1: Separate Variables

Consider the solution to the Poisson equation as $u(x,y) = X(x)Y(y)$. Separating variables as in the solution to the Laplace equation yields:

$$X'' - \mu X = 0$$
$$Y'' + \mu Y = 0$$

Step 2: Translate Boundary Conditions

As in the solution to the Laplace equation, translation of the boundary conditions yields:

$$\begin{aligned} X(0) &= 0 \\ X(L) &= 0 \\ Y(0) &= 0 \\ Y(M) &= 0 \end{aligned}$$

Step 3: Solve Both SLPs

Because all of the boundary conditions are homogeneous, we can solve both SLPs separately.

$$\left.\begin{aligned} X''-\mu X &= 0 \\ X(0) &= 0 \\ X(L) &= 0 \end{aligned}\right\} X_m(x)=\sin\frac{(m+1)\pi x}{L}, m=0,1,2,\cdots$$

$$\left.\begin{aligned} Y''-\mu Y &= 0 \\ Y(0) &= 0 \\ Y(M) &= 0 \end{aligned}\right\} Y_n(y)=\sin\frac{(n+1)\pi y}{M}, n=0,1,2,\cdots$$

Step 4: Solve Non-homogeneous Equation

Consider the solution to the non-homogeneous equation as follows:

$$\begin{aligned} u(x,y) &:= \sum_{m,n=0}^{\infty} a_{mn} X_m(x) Y_n(y) \\ &= \sum_{m,n=0}^{\infty} a_{mn}\sin\frac{(m+1)\pi x}{L}\sin\frac{(n+1)\pi y}{M} \end{aligned}$$

We substitute this into the Poisson equation and solve:

$$\begin{aligned} F(x,y) &= u_{xx}+u_{yy} \\ &= \sum_{m,n=0}^{\infty}\left\{a_{mn}\left[-\frac{(m+1)^2\pi^2}{L^2}\right]\sin\frac{(m+1)\pi x}{L}\sin\frac{(n+1)\pi y}{M}\right\}+\left\{a_{mn}\left[-\frac{(n+1)^2\pi^2}{M^2}\right]\sin\frac{(m+1)\pi x}{L}\sin\frac{(n+1)\pi y}{M}\right\} \\ &= \sum_{m,n=0}^{\infty}\underbrace{\left[-a_{mn}\left(\frac{(m+1)^2\pi^2}{L^2}+\frac{(n+1)^2\pi^2}{M^2}\right)\right]}_{A_{mn}}\sin\frac{(m+1)\pi x}{L}\sin\frac{(n+1)\pi y}{M} \end{aligned}$$

$$\begin{aligned} A_{mn} &= \frac{\int_0^M\int_0^L F(x,y)\sin\frac{(m+1)\pi x}{L}\sin\frac{(n+1)\pi y}{M}dxdy}{\int_0^M\sin^2\frac{(n+1)\pi y}{M}dy\int_0^L\sin^2\frac{(m+1)\pi x}{L}dx} \\ &= \frac{4}{LM}\int_0^M\int_0^L F(x,y)\sin\frac{(m+1)\pi x}{L}\sin\frac{(n+1)\pi y}{M}dxdy \end{aligned}$$

$$a_{mn}=-\frac{4}{LM\left[\frac{(m+1)^2\pi^2}{L^2}+\frac{(n+1)^2\pi^2}{M^2}\right]}\int_0^M\int_0^L F(x,y)\sin\frac{(m+1)\pi x}{L}\sin\frac{(n+1)\pi y}{M}dxdy; m,n=0,1,2,\cdots$$

Solution to General Case with 4 Non-homogeneous Boundary Conditions

Let's consider the following example, where $u_{xx}+u_{yy}=F(x,y),(x,y)\in[0,L]\times[0,M]$. and the boundary conditions are as follows:

$$\begin{aligned} u(x,0) &= f_1 \\ u(x,M) &= f_2 \\ u(0,y) &= f_3 \\ u(L,y) &= f_4 \end{aligned}$$

The boundary conditions can be Dirichlet, Neumann or Robin type.

Step 1: Decompose Problem

For the Poisson equation, we must decompose the problem into 2 sub-problems and use superposition to combine the separate solutions into one complete solution.

1. The first sub-problem is the homogeneous Laplace equation with the non-homogeneous boundary conditions. The individual conditions must retain their type (Dirichlet, Neumann or Robin type) in the sub-problem:

$$\begin{cases} u_{xx} + u_{yy} = 0 \\ u(x,0) = f_1 \\ u(x,M) = f_2 \\ u(0,y) = f_3 \\ u(L,y) = f_4 \end{cases}$$

2. The second sub-problem is the non-homogeneous Poisson equation with all homogeneous boundary conditions. The individual conditions must retain their type (Dirichlet, Neumann or Robin type) in the sub-problem:

$$\begin{cases} u_{xx} + u_{yy} = F(x,y) \\ u(x,0) = 0 \\ u(x,M) = 0 \\ u(0,y) = 0 \\ u(L,y) = 0 \end{cases}$$

Step 2: Solve Subproblems

Depending on how many boundary conditions are non-homogeneous, the Laplace equation problem will have to be subdivided into as many sub-problems. The Poisson sub-problem can be solved just as described above.

Step 3: Combine Solutions

The complete solution to the Poisson equation is the sum of the solution from the Laplace sub-problem $u_1(x,y)$ and the homogeneous Poisson sub-problem $u_2(x,y)$:
$u(x,y) = u_1(x,y) + u_2(x,y)$