6. MULTIPLE BASES AND TRANSFORMATION RELATIONS FOR COMPONENTS OF TENSORS

6.1 Initial Comments

As with vectors, any one vector can be expressed in terms of components and a number of bases. The same situation holds for tensors of any order. However, initially we restrict ourselves to second-order tensors and develop the transformation relations for components. At the same time we introduce the concept of "mixed" components followed by a brief treatment of components for higher-order tensors.

6.2 Bases, Components and Transformation Relations for Second-Order Tensors

In the previous Chapter, we indicated that an orthonormal vector basis, $\boldsymbol{e}_i$, could be used to generate an orthonormal tensor basis, $\boldsymbol{e}_i \otimes \boldsymbol{e}_j$. Now suppose we wish to work with two vector bases $\boldsymbol{e}_i$ and $\boldsymbol{E}_i$. These vector bases can generate four tensor bases, $\boldsymbol{e}_i \otimes \boldsymbol{e}_j, \boldsymbol{E}_i \otimes \boldsymbol{E}_j, \boldsymbol{e}_i \otimes \boldsymbol{E}_j$ and $\boldsymbol{E}_i \otimes \boldsymbol{e}_j$ with last two called mixed bases. Note that all four are orthonormal tensor bases. With these four tensor bases a single tensor, $\boldsymbol{T}$, can be represented four ways as follows:

$$\boldsymbol{T} = \overset{e-e}{T}_{ij}\, \boldsymbol{e}_i \otimes \boldsymbol{e}_j = \overset{E-E}{T}_{ij}\, \boldsymbol{E}_i \otimes \boldsymbol{E}_j = \overset{e-E}{T}_{ij}\, \boldsymbol{e}_i \otimes \boldsymbol{E}_j = \overset{E-e}{T}_{ij}\, \boldsymbol{E}_i \otimes \boldsymbol{e}_j \tag{6-1}$$

It is essential to specify the particular basis when identifying components. The first two representations are the ones most commonly used while the two mixed versions appear naturally in connection with deformation gradient introduced later. Using the alternative matrix notation, (6-1) becomes

$$\boldsymbol{T} = \langle \boldsymbol{e} \rangle \otimes \overset{e-e}{[T]} \{\boldsymbol{e}\} = \langle \boldsymbol{E} \rangle \otimes \overset{E-E}{[T]} \{\boldsymbol{E}\} = \langle \boldsymbol{e} \rangle \otimes \overset{e-E}{[T]} \{\boldsymbol{E}\} = \langle \boldsymbol{E} \rangle \otimes \overset{E-e}{[T]} \{\boldsymbol{e}\} \tag{6-2}$$

To derive the transformation relations for the components of a tensor recall that

$$\begin{aligned} \{\boldsymbol{E}\} &= \overset{E-e}{[a]} \{\boldsymbol{e}\} & \qquad \{\boldsymbol{e}\} &= \overset{e-E}{[a]} \{\boldsymbol{E}\} \\ \langle \boldsymbol{E} \rangle &= \langle \boldsymbol{e} \rangle \overset{e-E}{[a]} & \qquad \langle \boldsymbol{e} \rangle &= \langle \boldsymbol{E} \rangle \overset{E-e}{[a]} \end{aligned} \tag{6-3}$$

Insert the left two equations of (6-3) in the second representation of $\boldsymbol{T}$ in (6-2) to obtain

$$\langle \boldsymbol{e} \rangle \otimes \overset{e-e}{[T]} \{\boldsymbol{e}\} = \langle \boldsymbol{E} \rangle \otimes \overset{E-E}{[T]} \{\boldsymbol{E}\} = \langle \boldsymbol{e} \rangle \overset{e-E}{[a]} \otimes \overset{E-E}{[T]} \overset{E-e}{[a]} \{\boldsymbol{e}\} \tag{6-4}$$

The left and right representations have the same basis. Therefore the components must be the same, or

$$\overset{e-e}{[T]} = \overset{e-E}{[a]} \overset{E-E}{[T]} \overset{E-e}{[a]} \tag{6-5}$$

This is an equation for obtaining the components $\overset{e-e}{[T]}$ if the components $\overset{E-E}{[T]}$ are known. A similar procedure can be used to obtain the reverse transfomation

$$\overset{E-E}{[T]} = \overset{E-e}{[a]}\overset{e-e}{[T]}\overset{e-E}{[a]} \tag{6-6}$$

Transformations involving the mixed components satisfy relations of the following type:

$$\begin{aligned} \overset{e-E}{T} &= \overset{e-E}{[a]}\overset{E-E}{[T]} = \overset{e-E}{[T]}\overset{E-E}{[a]} \qquad & \overset{E-e}{T} &= \overset{E-e}{[a]}\overset{e-e}{[T]} = \overset{E-E}{[T]}\overset{E-e}{[a]} \\ \overset{E-e}{T} &= \overset{E-e}{[a]}\overset{e-E}{[T]}\overset{E-e}{[a]} \qquad & \overset{e-E}{T} &= \overset{e-E}{[a]}\overset{E-e}{[T]}\overset{e-E}{[a]} \end{aligned} \tag{6-7}$$

Note the structure displayed by the superscripts used to delineate the particular set of components provided in the respective matrices. The outer most superscripts are always the same, while the adjacent ones in the matrix multiplications must also be the same.

If indicial notation is preferred, then (6-3) becomes

$$\boldsymbol{E}_i = \overset{E-e}{a}_{ij}\boldsymbol{e}_j \qquad \boldsymbol{e}_i = \overset{E-e}{a}_{ji}\boldsymbol{E}_j = \overset{e-E}{a}_{ij}\boldsymbol{E}_j \tag{6-8}$$

The transformation relations given in (6-5) and (6-6) are

$$\overset{e-e}{T}_{ij} = \overset{e-E}{a}_{ik}\overset{E-E}{T}_{kl}\overset{E-e}{a}_{lj} \qquad \overset{E-E}{T}_{ij} = \overset{E-e}{a}_{ik}\overset{e-e}{T}_{kl}\overset{e-E}{a}_{lj} \tag{6-9}$$

Probably the matrix forms of the transformation relations are the most useful.

6.3 Bases, Components and Transformation Relations for Third- and Fourth-Order Tensors

Here we just briefly sketch the analogous representations and transformation relations for third and fourth-order tensors. We do not consider mixed representations although they can certainly be used. Generic third, $\overset{3}{\boldsymbol{C}}$, and fourth-order, $\overset{4}{\boldsymbol{E}}$, tensors are given as follows for two bases:

$$\begin{aligned} \overset{3}{\boldsymbol{C}} &= \overset{e-e-e}{C}_{ijk}\,\boldsymbol{e}_i\otimes\boldsymbol{e}_j\otimes\boldsymbol{e}_k = \overset{E-E-E}{C}_{ijk}\,\boldsymbol{E}_i\otimes\boldsymbol{E}_j\otimes\boldsymbol{E}_k \\ \overset{4}{\boldsymbol{F}} &= \overset{e-e-e-e}{F}_{ijkl}\,\boldsymbol{e}_i\otimes\boldsymbol{e}_j\otimes\boldsymbol{e}_k\otimes\boldsymbol{e}_l = \overset{E-E-E-E}{F}_{ijkl}\,\boldsymbol{E}_i\otimes\boldsymbol{E}_j\otimes\boldsymbol{E}_k\otimes\boldsymbol{E}_l \end{aligned} \tag{6-10}$$

Now the matrix form is not available so we use the indicial representation, Suppose the components in the $\boldsymbol{e}_i$ system are available and we want the component in the other system. Consider the third-order tensor and use the right-most equation of (6-8) to obtain

$$\overset{e-e-e}{C}_{ijk}\,\boldsymbol{e}_i \otimes \boldsymbol{e}_j \otimes \boldsymbol{e}_k = \overset{e-e-e}{C}_{ijk}\,\overset{e-E}{a}_{im}\,\boldsymbol{E}_m \otimes \overset{e-E}{a}_{in}\,\boldsymbol{E}_n \otimes \overset{e-E}{a}_{ip}\,\boldsymbol{E}_p$$
$$= \overset{E-E-E}{C}_{mnp}\,\boldsymbol{E}_m \otimes \boldsymbol{E}_n \otimes \boldsymbol{E}_p \tag{6-11}$$

where we have changed the summation indices on the right side so that the indices on the base vectors, $\boldsymbol{E}_i$, are identical. The components must be the same so the transformation relation becomes

$$\overset{E-E-E}{C}_{mnp} = \overset{e-e-e}{C}_{ijk}\,\overset{e-E}{a}_{im}\,\overset{e-E}{a}_{in}\,\overset{e-E}{a}_{ip} \tag{6-12}$$

The reverse transformation, and the corresponding transformation relations for fourth-order tensors are obtained in a similar manner.

6.5 Invariants

First we consider the identity tensor. Recall that

$$\boldsymbol{I} = \delta_{ij}\boldsymbol{e}_i \otimes \boldsymbol{e}_j = \langle \boldsymbol{e} \rangle \otimes [I]\{\boldsymbol{e}\} = \langle \boldsymbol{e} \rangle \otimes \{\boldsymbol{e}\} \tag{6-13}$$

Apply the transformation relations to obtain the form for a different basis:

$$\boldsymbol{I} = \langle \boldsymbol{E} \rangle \otimes \overset{E-e}{[a]}\overset{e-E}{[a]}\{\boldsymbol{E}\} = \langle \boldsymbol{E} \rangle \otimes [I]\{\boldsymbol{E}\} = \delta_{ij}\boldsymbol{E}_i \otimes \boldsymbol{E}_j \tag{6-14}$$

and the components also form the Kronecker delta in the new basis. Next consider the mixed components:

$$\boldsymbol{I} = \langle \boldsymbol{e} \rangle \otimes \{\boldsymbol{e}\} = \langle \boldsymbol{E} \rangle \otimes \overset{E-e}{[a]}\{\boldsymbol{e}\} = \langle \boldsymbol{e} \rangle \otimes \overset{e-E}{[a]}\{\boldsymbol{E}\} \tag{6-15}$$

Therefore we can express the various components of the identity tensor as

$$\overset{e-e}{[I]} = [I] \qquad \overset{E-E}{[I]} = [I] \qquad \overset{E-e}{[I]} = \overset{E-e}{[a]} \qquad \overset{e-E}{[I]} = \overset{e-E}{[a]} \tag{6-16}$$

Now consider the trace of $\boldsymbol{T}$ computed using the various components:

$$\begin{aligned}
tr\boldsymbol{T} &= \boldsymbol{I} \cdot\cdot \boldsymbol{T} = \delta_{ij}(\boldsymbol{e}_i \otimes \boldsymbol{e}_j)\cdot\cdot(\boldsymbol{e}_k \otimes \boldsymbol{e}_l)\overset{e-e}{T}_{kl} = \overset{e-e}{T}_{kk} = tr\overset{e-e}{[T]} \\
tr\boldsymbol{T} &= tr\overset{E-E}{[T]} \\
tr\boldsymbol{T} &= \boldsymbol{I} \cdot\cdot \boldsymbol{T} = \overset{e-E}{a}_{ij}(\boldsymbol{e}_i \otimes \boldsymbol{E}_j)\cdot\cdot(\boldsymbol{e}_k \otimes \boldsymbol{E}_l)\overset{e-E}{T}_{kl} = \overset{e-E}{a}_{kl}\overset{e-E}{T}_{kl} \\
&= \overset{E-e}{a}_{lk}\overset{e-E}{T}_{kl} = \delta_{lj}\overset{E-e}{a}_{lk}\overset{e-E}{T}_{kj} = \delta_{lj}\overset{E-E}{T}_{lj} = \overset{E-E}{T}_{jj} = tr\overset{E-E}{[T]}
\end{aligned} \tag{6-17}$$

A result similar to the last equation holds if the other mixed components are used. What we have shown is that the scalar $tr\boldsymbol{T}$ is the same no matter which components we use and is therefore called an invariant of the tensor $\boldsymbol{T}$. In a similar manner we can show that $tr(\boldsymbol{T}^2), tr(\boldsymbol{T}^3), \cdots$ are also invariants.

The determinant of $\boldsymbol{T}$ is the scalar $|\boldsymbol{T}|$ that satisfies the following equation:

$$(\boldsymbol{T}\cdot\boldsymbol{u})\cdot\left[(\boldsymbol{T}\cdot\boldsymbol{v})\times(\boldsymbol{T}\cdot\boldsymbol{w})\right]=\boldsymbol{u}\cdot[\boldsymbol{v}\times\boldsymbol{w}]|\boldsymbol{T}| \qquad \forall\ \boldsymbol{u}, \boldsymbol{v} \text{ and } \boldsymbol{w} \tag{6-18}$$

Since this equation is written in direct notation, the determinant cannot depend on any particular choice of a basis and is therefore an invariant. If the equation is expanded for any particular basis, the result is that

$$|\boldsymbol{T}|=|[T]| \qquad \varepsilon_{ijk}T_{il}T_{jm}T_{kn}=\varepsilon_{ijm}|T| \tag{6-19}$$

We will return to invariant when we discuss the eigenproblem in the next Chapter

6.5 Closing Comments

At this stage it is best to focus on the essential ideas and they are:

1. We start with orthonormal bases for vectors, $\boldsymbol{e}_i$ and $\boldsymbol{E}_i$, say.
2. We construct corresponding orthonormal bases for second-order tensors, $\boldsymbol{e}_i\otimes\boldsymbol{e}_j$ and $\boldsymbol{E}_i\otimes\boldsymbol{E}_j$.
3. We construct one of the transformation matrices and obtain the other by taking the transpose so that the two matrices $\overset{e-E}{[a]}$ and $\overset{E-e}{[a]}$ are available.
4. The two transformation relations of most use for second-order tensors are

$$\overset{e-e}{[T]}=\overset{e-E}{[a]}\overset{E-E}{[T]}\overset{E-e}{[a]} \qquad \overset{E-E}{[T]}=\overset{E-e}{[a]}\overset{e-e}{[T]}\overset{e-E}{[a]} \tag{6-20}$$

These last equations are simple enough that they are easier to derive when needed rather than looking them up in a reference source or memorizing them.

And finally, the components in any basis can be used to obtain an invariant.