$$\nabla(\phi+\psi) = \nabla\phi+\nabla\psi \qquad \frac{\partial(\phi+\psi)}{\partial x_n} = \frac{\partial\phi}{\partial x_n}+\frac{\partial\psi}{\partial x_n} \tag{21.54}$$

$$\nabla(\underset{\sim}{\boldsymbol{u}}+\underset{\sim}{\boldsymbol{v}}) = \nabla\underset{\sim}{\boldsymbol{u}}+\nabla\underset{\sim}{\boldsymbol{v}} \qquad \frac{\partial(u_i+v_i)}{\partial x_n} = \frac{\partial u_i}{\partial x_n}+\frac{\partial v_i}{\partial x_n} \tag{21.55}$$

$$\nabla(\phi\underset{\sim}{\boldsymbol{u}}) = (\nabla\phi)\underset{\sim}{\boldsymbol{u}}+\phi(\nabla\underset{\sim}{\boldsymbol{u}}) \qquad \frac{\partial(\phi u_i)}{\partial x_n} = \left(\frac{\partial\phi}{\partial x_n}\right)u_i+\phi\left(\frac{\partial u_i}{\partial x_n}\right) \tag{21.56}$$

$$\nabla(\underset{\sim}{\boldsymbol{u}}\underset{\sim}{\boldsymbol{v}}) = (\nabla\underset{\sim}{\boldsymbol{u}})\underset{\sim}{\boldsymbol{v}}+X_1^2[\underset{\sim}{\boldsymbol{u}}(\nabla\underset{\sim}{\boldsymbol{v}})] \qquad \frac{\partial(u_i v_j)}{\partial x_n} = \frac{\partial u_i}{\partial x_n}v_j+u_i\frac{\partial v_j}{\partial x_n} \tag{21.57}$$

$$\nabla\bullet(\nabla\times\underset{\sim}{\boldsymbol{v}}) = 0 \quad \textit{(see below)} \qquad \frac{\partial}{\partial x_i}\varepsilon_{ijk}\frac{\partial u_k}{\partial x_j} = \varepsilon_{ijk}\frac{\partial^2 u_k}{\partial x_i\partial x_j} = 0 \tag{21.58}$$

$$\nabla\times(\nabla\phi) = \underset{\sim}{\boldsymbol{0}} \quad \textit{(see below)} \qquad \varepsilon_{ijk}\frac{\partial(\partial\phi/\partial x_k)}{\partial x_j} = \varepsilon_{ijk}\frac{\partial^2\phi}{\partial x_j\partial x_k} = 0 \tag{21.59}$$

$$\nabla\bullet(\phi\underset{\sim}{\boldsymbol{v}}) = \underset{\sim}{\boldsymbol{v}}\bullet\nabla\phi+\phi\nabla\bullet\underset{\sim}{\boldsymbol{v}} \qquad \frac{\partial(\phi v_i)}{\partial x_i} = \left(\frac{\partial\phi}{\partial x_i}\right)v_i+\phi\left(\frac{\partial v_i}{\partial x_i}\right) \tag{21.60}$$

$$\nabla\times(\phi\underset{\sim}{\boldsymbol{v}}) = \phi\nabla\times\underset{\sim}{\boldsymbol{v}}+(\nabla\phi)\times\underset{\sim}{\boldsymbol{v}} \tag{21.61}$$

$$\nabla\bullet(\underset{\sim}{\boldsymbol{u}}\times\underset{\sim}{\boldsymbol{v}}) = \underset{\sim}{\boldsymbol{v}}\bullet(\nabla\times\underset{\sim}{\boldsymbol{u}})-\underset{\sim}{\boldsymbol{u}}\bullet(\nabla\times\underset{\sim}{\boldsymbol{v}}) \tag{21.62}$$

$$\nabla\times(\underset{\sim}{\boldsymbol{u}}\times\underset{\sim}{\boldsymbol{v}}) = \underset{\sim}{\boldsymbol{u}}(\nabla\bullet\underset{\sim}{v})-\underset{\sim}{\boldsymbol{u}}\bullet(\nabla\underset{\sim}{\boldsymbol{v}})+\underset{\sim}{v}\bullet(\nabla\underset{\sim}{\boldsymbol{u}})-\underset{\sim}{\boldsymbol{v}}(\nabla\bullet\underset{\sim}{\boldsymbol{u}}) \tag{21.63}$$

$$\nabla\bullet(\nabla\phi\times\nabla\psi) = 0 \quad \textit{(see below)} \tag{21.64}$$

$$\nabla\times(\nabla\times\underset{\sim}{\boldsymbol{v}}) = \nabla(\nabla\bullet\underset{\sim}{\boldsymbol{v}})-\nabla^2\underset{\sim}{\boldsymbol{v}} \tag{21.65}$$

$$\underset{\sim}{\boldsymbol{v}}\times(\nabla\times\underset{\sim}{\boldsymbol{u}}) = 2[\mathrm{skw}(\nabla\underset{\sim}{\boldsymbol{u}})]\bullet\underset{\sim}{\boldsymbol{v}} \tag{21.66}$$

The identity (21.58) follows from noting that, for any sufficiently smooth function $f(x, y)$,

$$\frac{\partial f}{\partial x\partial y} = \frac{\partial f}{\partial y\partial x} \tag{21.67}$$

When applied to a double gradient, this calculus identity may be written

$$\frac{\partial(\quad)}{\partial x_i\partial x_j} = \frac{\partial(\quad)}{\partial x_j\partial x_i} \tag{21.68}$$

In other words, there is symmetry with respect to the indices i and j. Consequently, the second partial derivative in Eq. (21.58) is symmetric with respect to the i and j indices. When contracted with ε_{ijk}, which is *skew-symmetric* in its i and j indices, the result must be zero. Equations (21.59) and (21.64) are zero for similar reasons.