SOLVING LINEAR SYSTEMS OF EQUATIONS

- See Chapter 3 of text
- Background on linear systems
- Gaussian elimination and the Gauss-Jordan algorithms
- The LU factorization
- Gaussian Elimination with pivoting

Background: Linear systems

The Problem: A is an $n \times n$ matrix, and b a vector of $\mathbb{R}^n$. Find x such that:

$$Ax = b$$

➤ x is the **unknown** vector, b the **right-hand side**, and A is the **coefficient matrix**

Example:

$$\begin{cases} 2x_1 + 4x_2 + 4x_3 = 6 \\ x_1 + 5x_2 + 6x_3 = 4 \\ x_1 + 3x_2 + x_3 = 8 \end{cases} \text{ or } \begin{pmatrix} 2 & 4 & 4 \\ 1 & 5 & 6 \\ 1 & 3 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 6 \\ 4 \\ 8 \end{pmatrix}$$

✍ Solution of above system ?

➤ Standard mathematical solution by Cramer's rule:

$$x_i = \det(A_i)/\det(A)$$

A_i = matrix obtained by replacing i-th column by b.

➤ Note: This formula is useless in practice beyond $n = 3$ or $n = 4$.

Three situations:

1. The matrix A is nonsingular. There is a unique solution given by $x = A^{-1}b$.
2. The matrix A is singular and $b \in \text{Ran}(A)$. There are infinitely many solutions.
3. The matrix A is singular and $b \notin \text{Ran}(A)$. There are no solutions.

Example: (1) Let $A = \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}$ $b = \begin{pmatrix} 1 \\ 8 \end{pmatrix}$. A is nonsingular ➤ a unique solution $x = \begin{pmatrix} 0.5 \\ 2 \end{pmatrix}$.

Example: (2) Case where A is singular & $b \in \text{Ran}(A)$:

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}, \quad b = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

➤ infinitely many solutions: $x(\alpha) = \begin{pmatrix} 0.5 \\ \alpha \end{pmatrix} \quad \forall \alpha$.

Example: (3) Let A same as above, but $b = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

➤ No solutions since 2nd equation cannot be satisfied

Triangular linear systems

Example:

$$\begin{pmatrix} 2 & 4 & 4 \\ 0 & 5 & -2 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix}$$

➤ One equation can be trivially solved: the last one.

$$x_3 = 2$$

➤ x_3 is known we can now solve the 2nd equation:

$$5x_2 - 2x_3 = 1 \;\rightarrow\; 5x_2 - 2 \times 4 = 1 \;\rightarrow\; x_2 = 1$$

➤ Finally x_1 can be determined similarly:

$$2x_1 + 4x_2 + 4x_3 = 2 \rightarrow \ldots \rightarrow x_1 = -5$$

ALGORITHM : 1. **Back-Substitution algorithm**

```
For i = n : -1 : 1 do:
    t := b_i
    For j = i + 1 : n do
        t := t - a_ij x_j
    End
    x_i = t/a_ii
End
```

➤ We must require that each $a_{ii} \neq 0$

➤ Operation count?

Backward error analysis for the triangular solve

The computed solution $\hat{x}$ of the triangular system $Ux = b$ computed by the previous algorithm satisfies:

$$(U + E)\hat{x} = b$$

with

$$|E| \leq n\,\underline{\mathbf{u}}\;|U| + O(\underline{\mathbf{u}}^2)$$

➤ Backward error analysis. Computed x solves a slightly perturbed system.

➤ Backward error not large in general. It is said that triangular solve is "backward stable".

➤ Column version of back-substitution:

Back-Substitution algorithm. Column version

```
For j = n : -1 : 1 do:
    x_j = b_j/a_jj
    For i = 1 : j - 1 do
        b_i := b_i - x_j * a_ij
    End
End
```

✍ Justify the above algorithm [Show that it does indeed compute the solution]

➤ See text for analogous algorithms for lower triangular systems.

Linear Systems of Equations: Gaussian Elimination

- Back to arbitrary linear systems.

Principle of the method: Since triangular systems are easy to solve, we will transform a linear system into one that is triangular. Main operation: combine rows so that zeros appear in the required locations to make the system triangular.

Notation.

$$\begin{cases} 2x_1 + 4x_2 + 4x_3 = 2 \\ x_1 + 3x_2 + 1x_3 = 1 \\ x_1 + 5x_2 + 6x_3 = -6 \end{cases} \quad \text{notation:} \quad \left[\begin{array}{ccc|c} 2 & 4 & 4 & 2 \\ 1 & 3 & 1 & 1 \\ 1 & 5 & 6 & -6 \end{array}\right]$$

- Main operation used: scaling and adding rows.

Example: Replace row2 by: row2 - $\frac{1}{2}$*row1:

$$\left[\begin{array}{ccc|c} 2 & 4 & 4 & 2 \\ 1 & 3 & 1 & 1 \\ 1 & 5 & 6 & -6 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 2 & 4 & 4 & 2 \\ 0 & 1 & -1 & 0 \\ 1 & 5 & 6 & -6 \end{array}\right]$$

- This is equivalent to:

$$\left[\begin{array}{ccc} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \times \left[\begin{array}{ccc|c} 2 & 4 & 4 & 2 \\ 1 & 3 & 1 & 1 \\ 1 & 5 & 6 & -6 \end{array}\right] = \left[\begin{array}{ccc|c} 2 & 4 & 4 & 2 \\ 0 & 1 & -1 & 0 \\ 1 & 5 & 6 & -6 \end{array}\right]$$

- The left-hand matrix is of the form

$$M = I - ve_1^T \text{ with } v = \begin{pmatrix} 0 \\ \frac{1}{2} \\ 0 \end{pmatrix}$$

Linear Systems of Equations: Gaussian Elimination

Go back to original system. Step 1 must transform:

$$\left[\begin{array}{ccc|c} 2 & 4 & 4 & 2 \\ 1 & 3 & 1 & 1 \\ 1 & 5 & 6 & -6 \end{array}\right] \text{ into: } \left[\begin{array}{ccc|c} x & x & x & x \\ 0 & x & x & x \\ 0 & x & x & x \end{array}\right]$$

$row_2 := row_2 - \frac{1}{2} \times row_1$:

$$\left[\begin{array}{ccc|c} 2 & 4 & 4 & 2 \\ 0 & 1 & -1 & 0 \\ 1 & 5 & 6 & -6 \end{array}\right]$$

$row_3 := row_3 - \frac{1}{2} \times row_1$:

$$\left[\begin{array}{ccc|c} 2 & 4 & 4 & 2 \\ 0 & 1 & -1 & 0 \\ 0 & 3 & 4 & -7 \end{array}\right]$$

- Equivalent to

$$\left[\begin{array}{ccc} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ -\frac{1}{2} & 0 & 1 \end{array}\right] \times \left[\begin{array}{ccc|c} 2 & 4 & 4 & 2 \\ 1 & 3 & 1 & 1 \\ 1 & 5 & 6 & -6 \end{array}\right] = \left[\begin{array}{ccc|c} 2 & 4 & 4 & 2 \\ 0 & 1 & -1 & 0 \\ 0 & 3 & 4 & -7 \end{array}\right]$$

$$[A, b] \rightarrow [M_1 A, M_1 b] \;\; M_1 = I - v^{(1)} e_1^T \;\; v^{(1)} = \begin{pmatrix} 0 \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$$

- New system $A_1 x = b_1$. Step 2 must now transform:

$$\left[\begin{array}{ccc|c} 2 & 4 & 4 & 2 \\ 0 & 1 & -1 & 0 \\ 0 & 3 & 4 & -7 \end{array}\right] \text{ into: } \left[\begin{array}{ccc|c} x & x & x & x \\ 0 & x & x & x \\ 0 & 0 & x & x \end{array}\right]$$

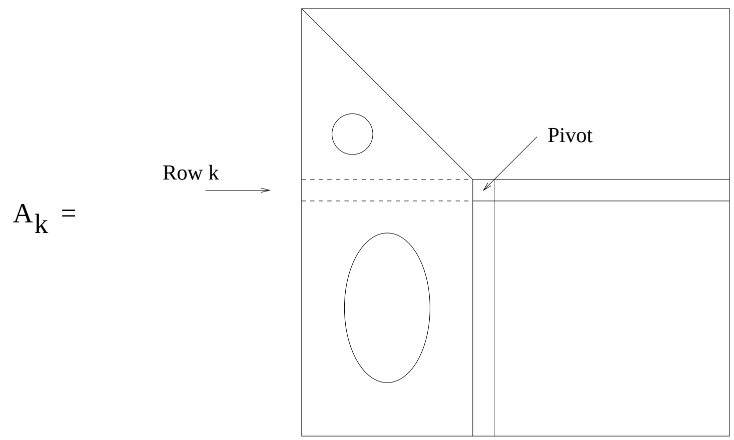
$$row_3 := row_3 - 3 \times row_2 : \rightarrow \left[\begin{array}{ccc|c} 2 & 4 & 4 & 2 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 7 & -7 \end{array}\right]$$

- Equivalent to

$$\begin{array}{|ccc|}\hline 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \\ \hline \end{array} \times \begin{array}{|ccc|c|}\hline 2 & 4 & 4 & 2 \\ 0 & 1 & -1 & 0 \\ 0 & 3 & 4 & -7 \\ \hline \end{array} = \begin{array}{|ccc|c|}\hline 2 & 4 & 4 & 2 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 7 & -7 \\ \hline \end{array}$$

- Triangular system ➤ Solve.
- Second transformation is as follows:

$$[A_1, b_1] \to [M_2A_1, M_2b_1] \;\; M_2 = I - v^{(2)}e_2^T \;\; v^{(2)} = \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix}$$



ALGORITHM : 2. **Gaussian Elimination**

1. For $k = 1 : n-1$ Do:
2.   For $i = k+1 : n$ Do:
3. $piv := a_{ik}/a_{kk}$
4.   For $j := k+1 : n+1$ Do :
5. $a_{ij} := a_{ij} - piv * a_{kj}$
6.   End
6.   End
7. End

- Operation count:

$$T = \sum_{k=1}^{n-1} \sum_{i=k+1}^{n} [1 + \sum_{j=k+1}^{n+1} 2] = \sum_{k=1}^{n-1} \sum_{i=k+1}^{n} (2(n-k)+3) = \ldots$$

✍ Complete the above calculation. Order of the cost?

The LU factorization

- Now ignore the right-hand side from the transformations.

Observation: Gaussian elimination is equivalent to $n-1$ successive Gaussian transformations, i.e., multiplications with matrices of the form $M_k = I - v^{(k)}e_k^T$, where the first k components of $v^{(k)}$ equal zero.

- Set $A_0 \equiv A$

$$A \to M_1A_0 = A_1 \to M_2A_1 = A_2 \to M_3A_2 = A_3 \cdots \\ \to M_{n-1}A_{n-2} = A_{n-1} \equiv U$$

- Last $A_k \equiv U$ is an upper triangular matrix.

- At each step we have: $A_k = M_{k+1}^{-1} A_{k+1}$. Therefore:

$$\begin{aligned} A_0 &= M_1^{-1} A_1 \\ &= M_1^{-1} M_2^{-1} A_2 \\ &= M_1^{-1} M_2^{-1} M_3^{-1} A_3 \\ &= \dots \\ &= M_1^{-1} M_2^{-1} M_3^{-1} \cdots M_{n-1}^{-1} A_{n-1} \end{aligned}$$

- $$L = M_1^{-1} M_2^{-1} M_3^{-1} \cdots M_{n-1}^{-1}$$
- Note: L is Lower triangular, A_{n-1} is upper triangular
- LU decomposition : $A = LU$

How to get L?

$$L = M_1^{-1} M_2^{-1} M_3^{-1} \cdots M_{n-1}^{-1}$$

- Consider only the first 2 matrices in this product.
- Note $M_k^{-1} = (I - v^{(k)} e_k^T)^{-1} = (I + v^{(k)} e_k^T)$. So:

$$M_1^{-1} M_2^{-1} = (I + v^{(1)} e_1^T)(I + v^{(2)} e_2^T) = I + v^{(1)} e_1^T + v^{(2)} e_2^T$$

- Generally,

$$M_1^{-1} M_2^{-1} \cdots M_k^{-1} = I + v^{(1)} e_1^T + v^{(2)} e_2^T + \cdots v^{(k)} e_k^T$$

The L factor is a lower triangular matrix with ones on the diagonal. Column k of L, contains the multipliers l_{ik} used used in the k-th step of Gaussian elimination.

A matrix A has an LU decomposition if

$$\det(A(1:k, 1:k)) \neq 0 \quad \text{for} \quad k = 1, \cdots, n-1.$$

In this case, the determinant of A satisfies:

$$\det A = \det(U) = \prod_{i=1}^{n} u_{ii}$$

If, in addition, A is nonsingular, then the LU factorization is unique.

- Show how to obtain L directly from the "multipliers"
- Practical use: Show how to use the LU factorization to solve linear systems with the same matrix A and different b's.
- LU factorization of the matrix $A = \begin{pmatrix} 2 & 4 & 4 \\ 1 & 5 & 6 \\ 1 & 3 & 1 \end{pmatrix}$?
- Determinant of A?
- True or false: "Computing the LU factorization of matrix A involves more arithmetic operations than solving a linear system $Ax = b$ by Gaussian elimination".

Gauss-Jordan Elimination

Principle of the method: We will now transform the system into one that is even easier to solve than triangular systems, namely a **diagonal** system. The method is very similar to Gaussian Elimination. It is just a bit more expensive.

Back to original system. Step 1 must transform:

$$\left[\begin{array}{ccc|c} 2 & 4 & 4 & 2 \\ 1 & 3 & 1 & 1 \\ 1 & 5 & 6 & -6 \end{array}\right] \text{ into: } \left[\begin{array}{ccc|c} x & x & x & x \\ 0 & x & x & x \\ 0 & x & x & x \end{array}\right]$$

$row_2 := row_2 - 0.5 \times row_1$: $\quad$ $row_3 := row_3 - 0.5 \times row_1$:

$$\left[\begin{array}{ccc|c} 2 & 4 & 4 & 2 \\ 0 & 1 & -1 & 0 \\ 1 & 5 & 6 & -6 \end{array}\right] \qquad \left[\begin{array}{ccc|c} 2 & 4 & 4 & 2 \\ 0 & 1 & -1 & 0 \\ 0 & 3 & 4 & -7 \end{array}\right]$$

Step 2: $\left[\begin{array}{ccc|c} 2 & 4 & 4 & 2 \\ 0 & 1 & -1 & 0 \\ 0 & 3 & 4 & -7 \end{array}\right]$ into: $\left[\begin{array}{ccc|c} x & 0 & x & x \\ 0 & x & x & x \\ 0 & 0 & x & x \end{array}\right]$

$row_1 := row_1 - 4 \times row_2$: $\quad$ $row_3 := row_3 - 3 \times row_2$:

$$\left[\begin{array}{ccc|c} 2 & 0 & 8 & 2 \\ 0 & 1 & -1 & 0 \\ 0 & 3 & 4 & -7 \end{array}\right] \qquad \left[\begin{array}{ccc|c} 2 & 0 & 8 & 2 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 7 & -7 \end{array}\right]$$

There is now a third step:

To transform: $\left[\begin{array}{ccc|c} 2 & 0 & 8 & 2 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 7 & -7 \end{array}\right]$ into: $\left[\begin{array}{ccc|c} x & 0 & 0 & x \\ 0 & x & 0 & x \\ 0 & 0 & x & x \end{array}\right]$

$row_1 := row_1 - \frac{8}{7} \times row_3$: $\quad$ $row_2 := row_2 - \frac{-1}{7} \times row_3$:

$$\left[\begin{array}{ccc|c} 2 & 0 & 0 & 10 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 7 & -7 \end{array}\right] \qquad \left[\begin{array}{ccc|c} 2 & 0 & 0 & 10 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 7 & -7 \end{array}\right]$$

Solution: $x_3 = -1; \ x_2 = -1; \ x_1 = 5$

ALGORITHM : 3. Gauss-Jordan elimination

1. For $k = 1 : n$ Do:
2. For $i = 1 : n$ and if $i! = k$ Do :
3. $piv := a_{ik}/a_{kk}$
4. For $j := k + 1 : n + 1$ Do :
5. $a_{ij} := a_{ij} - piv * a_{kj}$
6. End
6. End
7. End

➤ Operation count:

$$T = \sum_{k=1}^{n} \sum_{i=1}^{n-1} [1 + \sum_{j=k+1}^{n+1} 2] = \sum_{k=1}^{n-1} \sum_{i=1}^{n-1} (2(n-k) + 3) = \cdots$$

Complete the above calculation. Order of the cost? How does it compare with Gaussian Elimination?

```
function x = gaussj (A, b)
%---------------------------------------------------
%  function x = gaussj (A, b)
%  solves A x  = b by Gauss-Jordan elimination
%---------------------------------------------------
 n = size(A,1) ;
 A = [A,b];
 for k=1:n
   for i=1:n
     if (i ~= k)
         piv = A(i,k) / A(k,k) ;
         A(i,k+1:n+1) = A(i,k+1:n+1) - piv*A(k,k+1:n+1);
      end
   end
 end
 x = A(:,n+1) ./ diag(A) ;
```


Gaussian Elimination: Partial Pivoting

Consider again Gaussian Elimination for the linear system

$$\begin{cases} 2x_1 + 2x_2 + 4x_3 = 2 \\ x_1 + x_2 + x_3 = 1 \\ x_1 + 4x_2 + 6x_3 = -5 \end{cases} \quad \textbf{Or:} \quad \begin{array}{|ccc|c|} \hline 2 & 2 & 4 & 2 \\ 1 & 1 & 1 & 1 \\ 1 & 4 & 6 & -5 \\ \hline \end{array}$$

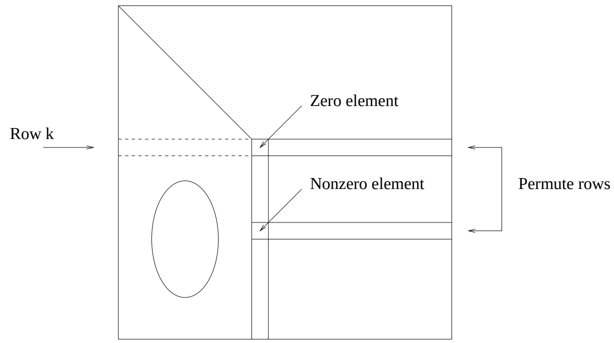
$row_2 := row_2 - \frac{1}{2} \times row_1$: $\quad$ $row_3 := row_3 - \frac{1}{2} \times row_1$:

$$\begin{array}{|ccc|c|} \hline 2 & 2 & 4 & 2 \\ 0 & 0 & -1 & 0 \\ 1 & 4 & 6 & -5 \\ \hline \end{array} \qquad \begin{array}{|ccc|c|} \hline 2 & 2 & 4 & 2 \\ 0 & 0 & -1 & 0 \\ 0 & 3 & 4 & -6 \\ \hline \end{array}$$

➤ **Pivot a_{22} is zero. Solution : permute rows 2 and 3:**

$$\begin{array}{|ccc|c|} \hline 2 & 2 & 4 & 2 \\ 0 & 3 & 4 & -6 \\ 0 & 0 & -1 & 0 \\ \hline \end{array}$$

Gaussian Elimination with Partial Pivoting



General situation

➤ **Partial Pivoting: Permute row k with row l such that**

$$|a_{lk}| = \max_{i=k,\ldots,n} |a_{ik}|$$

➤ **More 'stable' algorithm.**


```
function x = gaussp (A, b)
%---------------------------------------------------
%  function x = guassp (A, b)
%  solves A x  = b by Gaussian elimination with
%  partial pivoting/
%---------------------------------------------------
 n = size(A,1) ;
 A = [A,b]
 for k=1:n-1
    [t, ip] = max(abs(A(k:n,k)));
    ip = ip+k-1 ;
%% swap
     temp = A(k,k:n+1) ;
     A(k,k:n+1) =  A(ip,k:n+1);
     A(ip,k:n+1) = temp;
%%
      for i=k+1:n
      piv = A(i,k) / A(k,k) ;
      A(i,k+1:n+1) = A(i,k+1:n+1) - piv*A(k,k+1:n+1);
    end
 end
 x = backsolv(A,A(:,n+1));
```

Pivoting and permutation matrices

➤ A permutation matrix is a matrix obtained from the identity matrix by permuting its rows

➤ For example for the permutation $\pi = \{3, 1, 4, 2\}$ we obtain

$$P = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

➤ Important observation: the matrix PA is obtained from A by permuting its rows with the permutation π

$$(PA)_{i,:} = A_{\pi(i),:}$$

✍ What is the matrix PA when

$$P = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix} \quad A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 0 & -1 & 2 \\ -3 & 4 & -5 & 6 \end{pmatrix} ?$$

➤ Any permutation matrix is the product of interchange permutations, which only swap two rows of I.

➤ Notation: E_{ij} = Identity with rows i and j swapped

Example: To obtain $\pi = \{3, 1, 4, 2\}$ from $\pi = \{1, 2, 3, 4\}$ – we need to swap $\pi(2) \leftrightarrow \pi(3)$ then $\pi(3) \leftrightarrow \pi(4)$ and finally $\pi(1) \leftrightarrow \pi(2)$. Hence:

$$P = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix} = E_{1,2} \times E_{3,4} \times E_{2,3}$$

✍ In the previous example where

```
>> A = [ 1 2 3 4; 5 6 7 8; 9  0 -1 2 ; -3 4 -5 6]
```

Matlab gives $det(A) = -896$. What is $det(PA)$?

➤ At each step of G.E. with partial pivoting:

$$M_{k+1} E_{k+1} A_k = A_{k+1}$$

➤ Notes: (1) $E_i^{-1} = E_i$ and (2) $M_j^{-1} \times E_{k+1} = E_{k+1} \times \tilde{M}_j^{-1}$ for $k \geq j$. Where $\tilde{M}_j$ has a permuted Gauss vector:

$$\begin{aligned} (I + v^{(j)} e_j^T) E_{k+1} &= E_{k+1} (I + E_{k+1} v^{(j)} e_j^T) \\ &\equiv E_{k+1} (I + \tilde{v}^{(j)} e_j^T) \\ &\equiv E_{k+1} \tilde{M}_j \end{aligned}$$

Result:

$$\begin{aligned} A_0 &= E_1 M_1^{-1} A_1 \\ &= E_1 M_1^{-1} E_2 M_2^{-1} A_2 = E_1 E_2 \tilde{M}_1^{-1} M_2^{-1} A_2 \\ &= E_1 E_2 \tilde{M}_1^{-1} M_2^{-1} E_3 M_3^{-1} A_3 \\ &= E_1 E_2 E_3 \tilde{M}_1^{-1} \tilde{M}_2^{-1} M_3^{-1} A_3 \\ &= \dots \\ &= E_1 \cdots E_{n-1} \ \times \ \tilde{M}_1^{-1} \tilde{M}_2^{-1} \tilde{M}_3^{-1} \cdots \tilde{M}_{n-1}^{-1} \ \times \ A_{n-1} \end{aligned}$$

- In the end

$$PA = LU \text{ with } P = E_{n-1} \cdots E_1$$

Error Analysis

If no zero pivots are encountered during Gaussian elimination (no pivoting) then the computed factors $\hat{L}$ and $\hat{U}$ satisfy

$$\hat{L}\hat{U} = A + H$$

with

$$|H| \le 3(n-1) \ \times \ \underline{\mathbf{u}} \ \left(|A| + |\hat{L}| \ |\hat{U}|\right) + O(\underline{\mathbf{u}}^2)$$

Solution $\hat{x}$ computed via $\hat{L}\hat{y} = b$ and $\hat{U}\hat{x} = \hat{y}$ is s. t.

$$(A + E)\hat{x} = b \text{ with}$$

$$|E| \le n\underline{\mathbf{u}} \left(3|A| \ + 5 \ |\hat{L}| \ |\hat{U}|\right) + O(\underline{\mathbf{u}}^2)$$

- "Backward" error estimate.
- $|\hat{L}|$ and $|\hat{U}|$ are not known in advance – they can be large.
- What if partial pivoting is used?
- Permutations introduce no errors. Equivalent to standard LU factorization on matrix PA.
- $|\hat{L}|$ is small since $l_{ij} \le 1$. Therefore, only U is "uncertain"
- In practice partial pivoting is "stable" – i.e., it is highly unlikely to have a very large U.