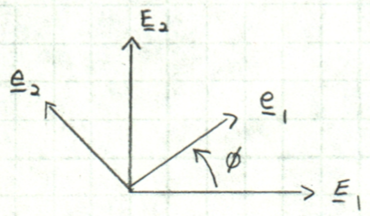


$\underline{E}_3 = \underline{e}_3$

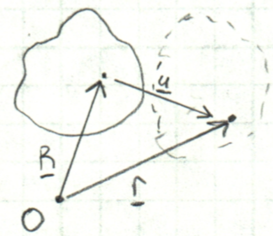
$$\underline{e}_1 = \cos\phi\,\underline{E}_1 + \cos(90^\circ-\phi)\,\underline{E}_2 + 0\,\underline{E}_3$$
$$\underline{e}_2 = \cos(90^\circ+\phi)\,\underline{E}_1 + \cos(\phi)\,\underline{E}_2 + 0\,\underline{E}_3$$
$$\underline{e}_3 = 0\,\underline{E}_1 + 0\,\underline{E}_2 + \underline{E}_3$$

$$\overset{e-E}{[a]} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\underline{R} = X_i \underline{E}_i$

$\underline{r} = x_i \underline{e}_i$

$\underline{u} = \underline{r} - \underline{R}$



given: the deformation is defined by

$$x_1 = \alpha X_1 + \gamma X_2\,,\quad x_2 = \beta X_2\,,\quad x_3 = X_3$$

1) $\underline{r} = \underline{r}(\underline{R})$

$$\overset{e}{\underline{r}} = (\alpha X_1 + \gamma X_2)\,\underline{e}_1 + \beta X_2\,\underline{e}_2 + X_3\,\underline{e}_3$$

2) $\underline{R} = \underline{R}(\underline{r})$

- must invert the deformation to obtain the original position vector in terms of the deformed components

$$X_3 = x_3\,,\quad X_2 = x_2/\beta\,,\quad x_1 = \alpha X_1 + \gamma(x_2/\beta)$$
$$X_1 = x_1/\alpha - \frac{\gamma x_2}{\alpha\beta}$$

$$\overset{E}{\underline{R}} = \left(\frac{x_1}{\alpha} - \frac{\gamma x_2}{\alpha\beta}\right)\underline{E}_1 + \frac{x_2}{\beta}\,\underline{E}_2 + x_3\,\underline{E}_3$$

3) $\underline{u} = \underline{u}(\underline{R})$

$$\langle \overset{E}{r} \rangle = \langle \overset{e}{r} \rangle \overset{e-E}{[a]} \Rightarrow \left[\cos\phi(\alpha X_1 + \gamma X_2) - \sin\phi(\beta X_2)\right]\underline{E}_1 + \left[\sin\phi(\alpha X_1 + \gamma X_2) + \cos\phi(\beta X_2)\right]\underline{E}_2 + X_3\,\underline{E}_3$$

=> the deformed position vect in the undeformed basis

→ the original position of a material point is $(X\ X\ X)$ given by $\underline{R} = X_1\underline{E}_1 + X_2\underline{E}_2 + X_3\underline{E}_3$; therefore, the displacement ($\underline{u}$) can be obtained from $\underline{u} = \underline{r} - \underline{R}$

↘

in the undeformed basis:

$$\overset{E}{r}_i E_i = \left[X_1 \alpha\cos\phi + X_2(\gamma\cos\phi - \beta\sin\phi)\right] \underline{E}_1 + \left[X_1 \alpha \sin\phi + X_2(\gamma\sin\phi + \beta\cos\phi)\right] \underline{E}_2 + X_3 \underline{E}_3$$

$$\langle \overset{E}{u} \rangle = \langle \overset{E}{r} \rangle - \langle \overset{E}{R} \rangle$$

$$\underline{u} = \left(X_1(\alpha\cos\phi - 1) + X_2(\gamma\cos\phi - \beta\sin\phi)\right) \underline{E}_1 + \left(X_1 \alpha\sin\phi + X_2\left[(\gamma\sin\phi + \beta\cos\phi) - 1\right]\right) \underline{E}_2 + 0\, \underline{E}_3$$

4) => this presents the displacement as a function of the position in the undefored basis.

$$\underline{R} = X_i \underline{E}_i \Rightarrow \underline{u} = \underline{u}(\underline{R})$$

4) Express $\underline{u}$ as a function of $\underline{r}$:

$$\langle \overset{e}{R} \rangle = \langle \overset{E}{R} \rangle \overset{E-e}{[a]} \Rightarrow \left[\left(\frac{x_1}{\alpha} - \frac{\gamma x_2}{\alpha\beta}\right)\cos\phi + \frac{x_2}{\beta}\sin\phi\right] \underline{e}_1 + \left[\left(\frac{x_1}{\alpha} - \frac{\gamma x_2}{\alpha\beta}\right)(-\sin\phi) + \frac{x_2}{\beta}\cos\phi\right] \underline{e}_2 + x_3 \underline{e}_3$$

$$\overset{e}{R}_i \underline{e}_i = \left[\frac{x_1}{\alpha}\cos\phi - x_2\left(\frac{\gamma}{\alpha\beta}\cos\phi - \frac{1}{\beta}\sin\phi\right)\right] \underline{e}_1 + \left[-\frac{x_1}{\alpha}\sin\phi + x_2\left(\frac{\gamma}{\alpha\beta}\sin\phi + \frac{1}{\beta}\cos\phi\right)\right] \underline{e}_2 + x_3 \underline{e}_3$$

$\underline{r} = x_i \underline{e}_i$, $\underline{u} = \underline{r} - \underline{R}$, $\underline{u} = \underline{u}(\underline{r})$ is then:

$$\underline{u} = \left(x_1\left(1 - \frac{1}{\alpha}\cos\phi\right) + x_2\left(\frac{\gamma}{\alpha\beta}\cos\phi - \frac{1}{\beta}\sin\phi\right)\right) \underline{e}_1 + \left(+\frac{x_1}{\alpha}\sin\phi + x_2\left(1 - \left[\frac{\alpha}{\alpha\beta}\sin\phi + \frac{1}{\beta}\cos\phi\right]\right)\right) \underline{e}_2 + 0\, \underline{e}_3$$

=> this is the displacement as a function of the position vector in the deformed basis.

- obtain strains as a function of the deformation gradient

5) $$\underline{\underline{F}} = \underline{r}\overleftarrow{\nabla}_0 = \frac{\partial x_i}{\partial X_j}\, \underline{e}_i \otimes \underline{E}_j = \overset{e\text{-}E}{F_{ij}}\, \underline{e}_i \otimes \underline{E}_j$$

$$\overset{e\text{-}E}{[F]} \Rightarrow \begin{bmatrix} \alpha & \gamma & 0 \\ 0 & \beta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\underline{\underline{F}}^T = \langle \underline{E} \rangle \overset{E\text{-}e}{[F^T]} \otimes \{\underline{e}\}, \quad \text{where } \overset{E\text{-}e}{[F^T]} = \overset{e\text{-}E}{[F]}^T$$

$$\overset{E\text{-}e}{[F^T]} \Rightarrow \begin{bmatrix} \alpha & 0 & 0 \\ \gamma & \beta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\underline{\underline{E}} = \frac{1}{2}\left[\underline{\underline{F}}^T \cdot \underline{\underline{F}} - \underline{\underline{I}}\right]$$

$$\overset{E\text{-}E}{[E]} = \frac{1}{2}\left[\overset{E\text{-}e}{[F^T]}\,\overset{e\text{-}E}{[F]} - \overset{E\text{-}E}{[I]}\right] = \frac{1}{2}\begin{bmatrix} \alpha^2+\beta^2-1 & \gamma\beta & 0 \\ \gamma\beta & \beta^2-1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\underline{\underline{e}} = \frac{1}{2}\left[\underline{\underline{I}} - F^{-T} \cdot F^{-1}\right]$$

$$\overset{e\text{-}e}{[e]} = \frac{1}{2}\left[\overset{e\text{-}e}{[I]} - \overset{e\text{-}E}{[F^{-T}]}\,\overset{E\text{-}e}{[F^{-1}]}\right]$$

where: $$\underline{\underline{F}}^{-1} = \underline{R}\overleftarrow{\nabla} = \frac{\partial X_i}{\partial x_j}\, \underline{E}_i \otimes \underline{e}_j = \overset{E\text{-}e}{F}^{-1}\, \underline{E}_i \otimes \underline{e}_j$$

$$\overset{E\text{-}e}{[F^{-1}]} = \begin{bmatrix} 1/\alpha & -\frac{\gamma}{\alpha\beta} & 0 \\ 0 & 1/\beta & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad ; \text{ check: } \overset{E\text{-}e}{[F^{-1}]}\,\overset{e\text{-}E}{[F]} = \overset{E\text{-}E}{[I]}$$

$$\underline{\underline{F}}^{-T} = \overset{E\text{-}e}{F_{ji}}\, \underline{e}_i \otimes \underline{E}_j$$

$$\overset{E\text{-}e}{[F^{-T}]} = \begin{bmatrix} 1/\alpha & 0 & 0 \\ \frac{-\gamma}{\alpha\beta} & 1/\beta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\overset{e\text{-}e}{[e]} = \frac{1}{2}\begin{bmatrix} 1-\frac{1}{\alpha^2} & \frac{\gamma}{\alpha^2\beta} & 0 \\ \frac{\gamma}{\alpha^2\beta} & 1-\left(\frac{\gamma}{\alpha\beta}\right)^2-\frac{1}{\beta^2} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Check: $$\overset{E\text{-}E}{[E]} = \overset{E\text{-}e}{[F^T]}\,\overset{e\text{-}e}{[e]}\,\overset{e\text{-}E}{[F]}$$

6) calculate $\underline{\underline{H}}$ using alt. method to check, via $\underline{\underline{F}}$, the deformation gradient

$$\underline{\underline{H}} = \underline{\underline{F}} - \underline{\underline{I}}$$

$$\overset{E-E}{[H]} = \overset{E-e}{[a]}\,\overset{e-E}{[F]} - \overset{E-E}{[I]}$$, Let $c = \cos\phi$ + $s = \sin\phi$

$$= \begin{bmatrix} c & -s & 0 \\ s & c & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \alpha & \gamma & 0 \\ 0 & \beta & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \alpha c - 1 & \gamma c - \beta s & 0 \\ \alpha s & \gamma s + \beta c - 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

calculat $\underline{\underline{H}}$ using displacement gradient, $\underline{u}$

$$\underline{\underline{H}} = \underline{u}\overleftarrow{\nabla}_0 = \frac{\partial u_i}{\partial X_j}\,\underline{E}_i \otimes \underline{E}_j = \overset{E-E}{H_{ij}}\,\underline{E}_i \otimes \underline{E}_j$$

$$\overset{E-E}{[H]} = \begin{bmatrix} \alpha c - 1 & \gamma c - \beta s & 0 \\ \alpha s & \gamma s - \beta c - 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$; same values for both approaches

- calculate the Lagrangian strain tensor wrt the undeformed basis

$$\underline{\underline{E}} = \tfrac{1}{2}\left[\underline{\underline{H}} + \underline{\underline{H}}^T + \underline{\underline{H}}^T \cdot \underline{\underline{H}}\right] = \frac{1}{2}\left[\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i} + \frac{\partial u_k}{\partial X_i}\frac{\partial u_k}{\partial X_j}\right]\underline{E}_i \otimes \underline{E}_j$$

$$\overset{E-E}{[H]}\,\overset{E-E}{[H^T]} = \begin{bmatrix} \alpha c - 1 & \alpha c - \beta s & 0 \\ \alpha s & \gamma s + \beta c - 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \alpha c - 1 & \alpha s & 0 \\ \alpha c - \beta s & \gamma s + \beta c - 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} (\alpha c - 1)^2 + (\alpha c - \beta s)^2 & \alpha s(\alpha c - 1) + (\alpha c - \beta s)(\gamma s + \beta c - 1) & 0 \\ \alpha s(\alpha c - 1) + (\alpha c - \beta s)(\gamma s + \beta c - 1) & (\alpha s)^2 + (\gamma s + \beta c - 1)^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\overset{E-E}{[H]} + \overset{E-E}{[H^T]} = \begin{bmatrix} 2(\alpha c - 1) & \alpha s + \alpha c - \beta s & 0 \\ \alpha s + \alpha c - \beta s & 2(\gamma s + \beta c - 1) & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\overset{E-E}{[E]} = \tfrac{1}{2}\left[\overset{E-E}{[H]} + \overset{E-E}{[H^T]} + \overset{E-E}{[H]}\,\overset{E-E}{[H^T]}\right] \Rightarrow$$ See next page

6) cont.

$$\overset{E\text{-}E}{[E]} = \frac{1}{2}\begin{bmatrix} 2(\alpha c-1)+(\alpha c-1)^2+(\alpha c-\beta s)^2 & \alpha s+\alpha c-\beta s+\alpha s(\alpha c-1)+(\alpha c-\beta s)(\gamma s+\beta c-1) & 0\\ \alpha s+\alpha c-\beta s+\alpha s(\alpha c-1)+(\alpha c-\beta s)(\gamma s+\beta c-1) & 2(\gamma s+\beta c-1)+(\alpha s)^2+(\gamma s+\beta c-1)^2 & 0 \\ 0 & 0 & 0\end{bmatrix}$$

- Check:
for $\alpha = 2$, $\beta = 2$, + $\gamma = 1$; $\phi = \pi/2$ ... results for

$\underline{\underline{E}}$ calculated via $\underline{u}$ + $\underline{\underline{F}}$ do not agree; maybe a mistake with algebra?

- calculate the eulerian strain using the deformation gradient $\underline{u}$

$$\underline{\underline{h}} = \underline{\underline{I}} - \underline{\underline{F}}^{-1}$$

$$\overset{e\text{-}e}{[h]} = \overset{e\text{-}e}{[I]} - \overset{e\text{-}E}{[a]}\overset{E\text{-}e}{[F^{-1}]} = \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} - \begin{bmatrix}c&s&0\\-s&c&0\\0&0&1\end{bmatrix}\begin{bmatrix}1/\alpha & -\gamma/\alpha\beta & 0\\ 0 & 1/\beta & 0\\ 0&0&1\end{bmatrix}$$

$$= \begin{bmatrix}1-c/\alpha & +(\gamma/\alpha\beta)c - s/\beta & 0\\ +s/\alpha & 1-(\gamma/\alpha\beta)s - c/\beta & 0\\ 0&0&0\end{bmatrix}$$

calculate $\underline{\underline{h}}$ using the displacement gradient:

$$\underline{\underline{h}} = \underline{u}\overleftarrow{\nabla} = \frac{\partial u_i}{\partial x_j}\,\underline{e}_i\otimes\underline{e}_j = \overset{e\text{-}e}{h_{ij}}\,\underline{e}_i\otimes\underline{e}_j$$

$$\overset{e\text{-}e}{[h]} = \begin{bmatrix}1-c/\alpha & (\gamma/\alpha\beta)c - s/\beta & 0\\ +s/\alpha & 1-(\gamma/\alpha\beta)s - c/\beta & 0\\ 0&0&0\end{bmatrix}$$; same results regardless of approach

$$\underline{\underline{e}} = \frac{1}{2}\left[\underline{\underline{h}} + \underline{\underline{h}}^T + \underline{\underline{h}}^T\cdot\underline{\underline{h}}\right] = \frac{1}{2}\left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \frac{\partial u_k}{\partial x_i}\frac{\partial u_k}{\partial x_j}\right]\underline{e}_i\otimes\underline{e}_j$$

$$\overset{e\text{-}e}{[h]} + \overset{e\text{-}e}{[h^T]} = \begin{bmatrix}2(1-c/\alpha) & \left(\frac{\gamma}{\alpha\beta}\right)c & 0\\ \left(\frac{\gamma}{\alpha\beta}\right)c & 2\left(1-\left(\frac{\gamma}{\alpha\beta}\right)-c/\beta\right) & 0\\ 0&0&0\end{bmatrix}$$

$$\overset{e-e}{[h^T]}\overset{e-e}{[h]} = \begin{bmatrix} 1-\frac{c}{\alpha} & \left(\frac{\gamma}{\alpha\beta}\right)c - \frac{s}{\beta} & 0 \\ \frac{s}{\alpha} & 1-\left(\frac{\gamma}{\alpha\beta}\right)s - \frac{s}{\beta} & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1-\frac{c}{\alpha} & \frac{s}{\alpha} & 0 \\ \left(\frac{\gamma}{\alpha\beta}\right)c - \frac{s}{\beta} & 1-\left(\frac{\gamma}{\alpha\beta}\right)s - \frac{s}{\beta} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

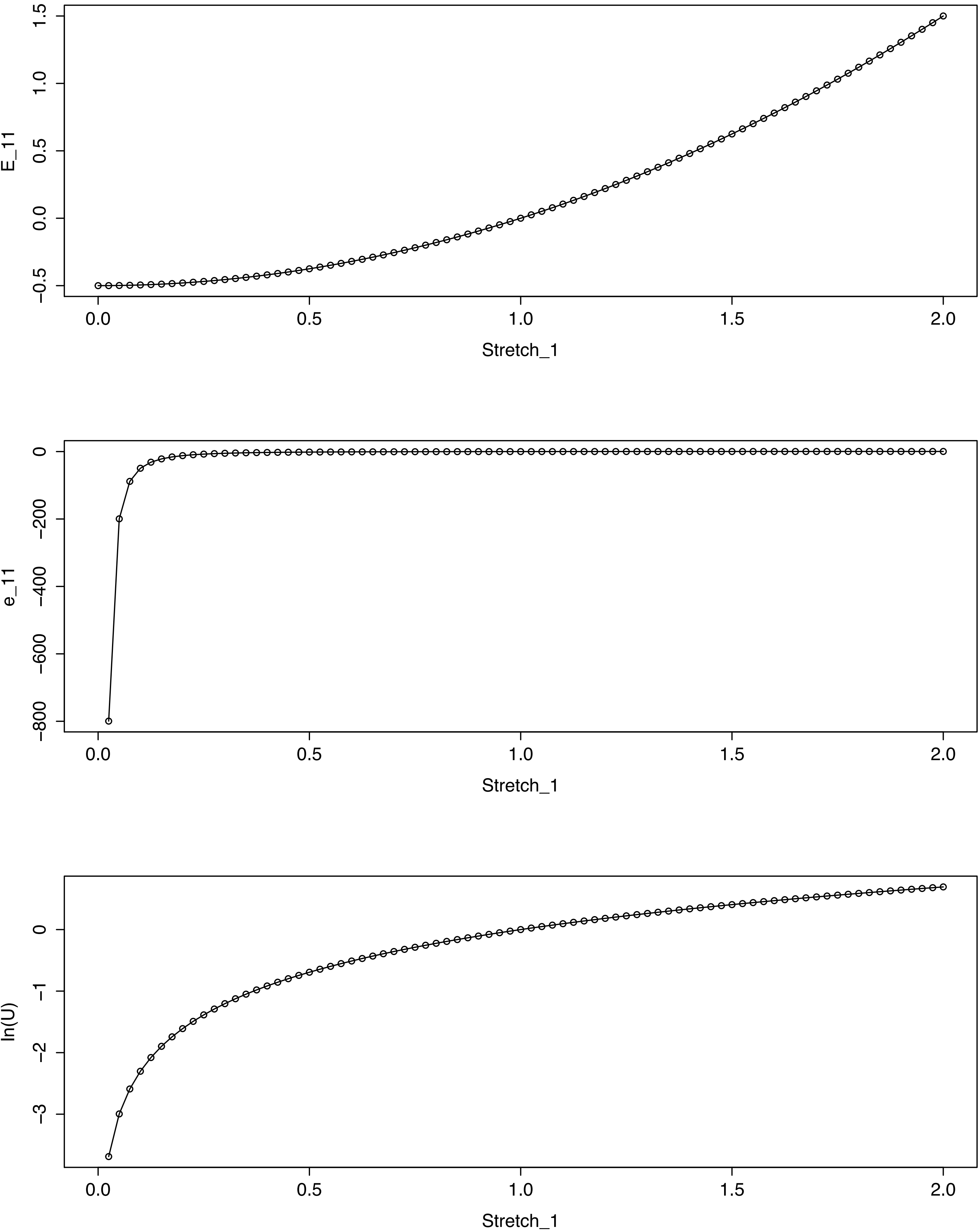
Let: $m = \frac{\gamma}{\alpha\beta}$

$$= \begin{bmatrix} \left(1-\frac{c}{\alpha}\right)^2 + \left(mc - \frac{s}{\beta}\right)^2 & \left(1-\frac{c}{\alpha}\right)\left(\frac{s}{\alpha}\right) + \left(mc-\frac{s}{\beta}\right)\left(1-ms-\frac{s}{\beta}\right) & 0 \\ \left(1-\frac{c}{\alpha}\right)\frac{s}{\alpha} + \left(1-ms-\frac{s}{\beta}\right)\left(mc-\frac{s}{\beta}\right) & \left(\frac{s}{\alpha}\right)^2 + \left(1-ms-\frac{s}{\beta}\right)^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

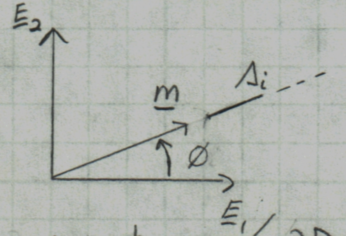
$$\overset{e-e}{[e]} = \frac{1}{2}\begin{bmatrix} 2\left(1-\frac{c}{\alpha}\right) + \left(1-\frac{c}{\alpha}\right)^2 + \left(mc-\frac{s}{\beta}\right)^2 & mc + \left(1-\frac{c}{\alpha}\right)\left(\frac{s}{\alpha}\right) + \left(mc-\frac{s}{\beta}\right)\left(1-ms-\frac{s}{\beta}\right) & 0 \\ mc + \left(1-\frac{c}{\alpha}\right)\frac{s}{\alpha} + \left(1-ms-\frac{s}{\beta}\right)\left(mc-\frac{s}{\beta}\right) & 2\left(1-m-\frac{c}{\beta}\right) + \left(\frac{s}{\alpha}\right)^2 + \left(1-ms-\frac{s}{\beta}\right)^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

- again, disagreement between $\underline{e}$ when calculated with deformation grad. and disp. grad.

Problem 7



8)



- on a plane / 2D
- Say the Lagrangian strain components in the undeformed basis (E_{EE}) are desired, but the strain rosette provides this strain measure along $\underline{m}$ instead. The components of $\underline{\underline{E}}$ in the $\underline{E} \otimes \underline{E}$ basis may be obtained as follows.

Lagrangian strain in the $\underline{m}$ basis

$$\overset{m-m}{[E]} = \begin{bmatrix} E_{11} & E_{12} & 0 \\ E_{21} & E_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad E = \tfrac{1}{2}\left[\Lambda^2 - 1\right]$$

Let $E_{12} = E_{21}$

- We obtain E from measurements aligned w/ $\underline{m}$; therefore, must solve for the components of $\underline{\underline{E}}$ in the $\underline{E}$ basis.

$$\begin{aligned} E_{mm} &= \underline{m} \cdot \underline{\underline{E}} \cdot \underline{m} \\ &= \underline{m} \cdot \left(E_{11}\, \underline{E}_1 \otimes \underline{E}_1 + E_{12}\, \underline{E}_1 \otimes \underline{E}_2 + E_{21}\, \underline{E}_2 \otimes \underline{E}_1 + E_{22}\, \underline{E}_2 \otimes \underline{E}_2\right) \cdot \underline{m} \\ &= E_{11}(\underline{m}\,\underline{E}_1)^2 + 2E_{12}(\underline{m}\,\underline{E}_1 + \underline{m}\,\underline{E}_2) + E_{22}(\underline{m}\,\underline{E}_2)^2 \\ &= E_{11}(\cos\phi)^2 + 2E_{12}(\cos\phi + \sin\phi) + E_{22}(\sin\phi)^2 \end{aligned}$$

→ for 3 unique values of ϕ, say α, β, + γ, correlating to 3 unique vectors (m, m', + m''), the E_{11}, E_{12}, E_{22} components may be determined by solving the following system.

$$\begin{aligned} E_{11}\cos^2\alpha + 2E_{12}(\cos\alpha + \sin\alpha) + E_{22}\sin^2\alpha &= E_{mm} \\ E_{11}\cos^2\beta + 2E_{12}(\cos\beta + \sin\beta) + E_{22}\sin^2\beta &= E_{m'm'} \\ E_{11}\cos^2\gamma + 2E_{12}(\cos\gamma + \sin\gamma) + E_{22}\sin^2\gamma &= E_{m''m''} \end{aligned}$$

- where α, β, + γ are the angles between $\underline{E}_1$ and m, m', + m'' respectively. The scalar values E_{mm}, $E_{m'm'}$, + $E_{m''m''}$ are those calculated from data obtained directly from the strain rosette.