Multifrontal methods

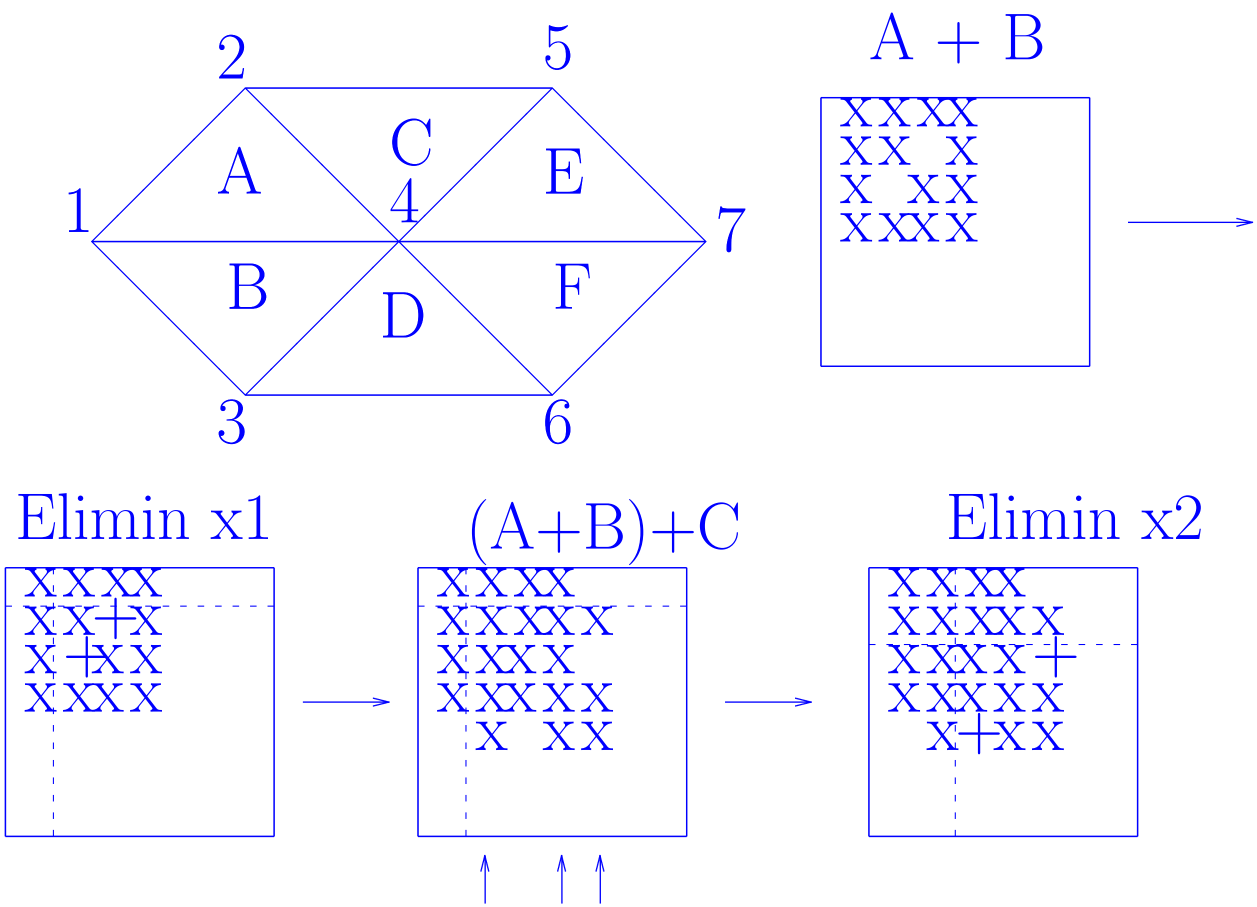
- Start with the frontal method.
- Recall: Finite element matrix:

$$\boldsymbol{A} = \sum \boldsymbol{A}^{[e]}$$

$\boldsymbol{A}^{[e]}$ = element matrix associated with element e.

- An old idea: Execute Gaussian elimination as the elements are being assembled
- This is called *the frontal method*
- Very popular among finite element users: saves storage

The origin: Frontal method



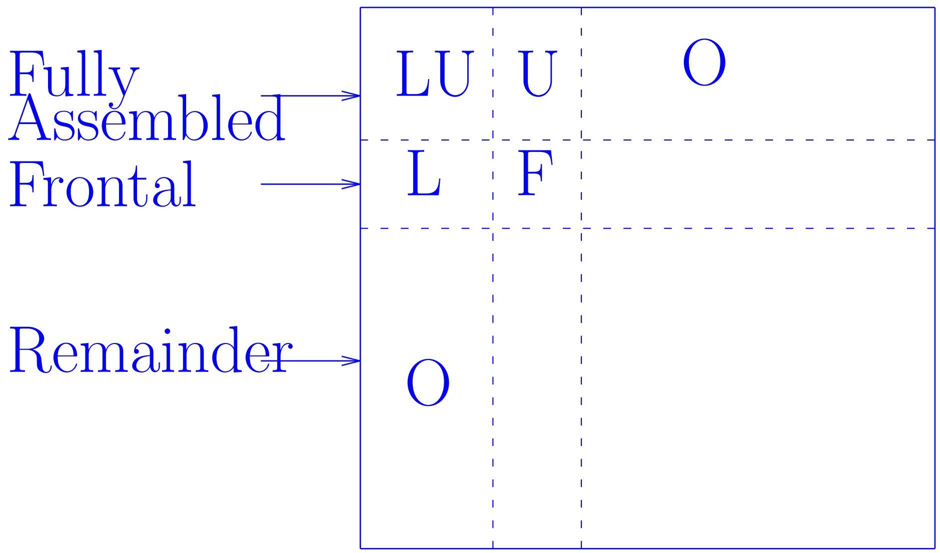
➤ Elimination of x_1 creates an update matrix

- Matrix has 3 parts:

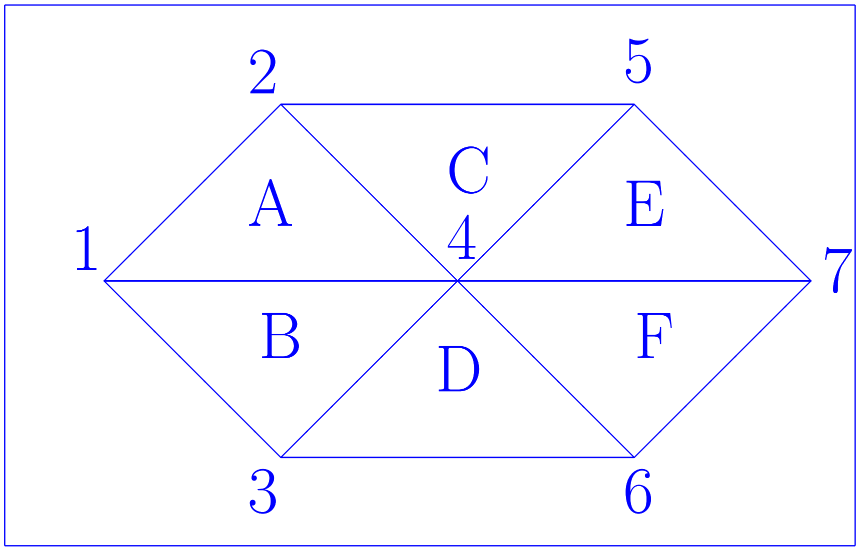
1) Fully assembled (no longer modified)

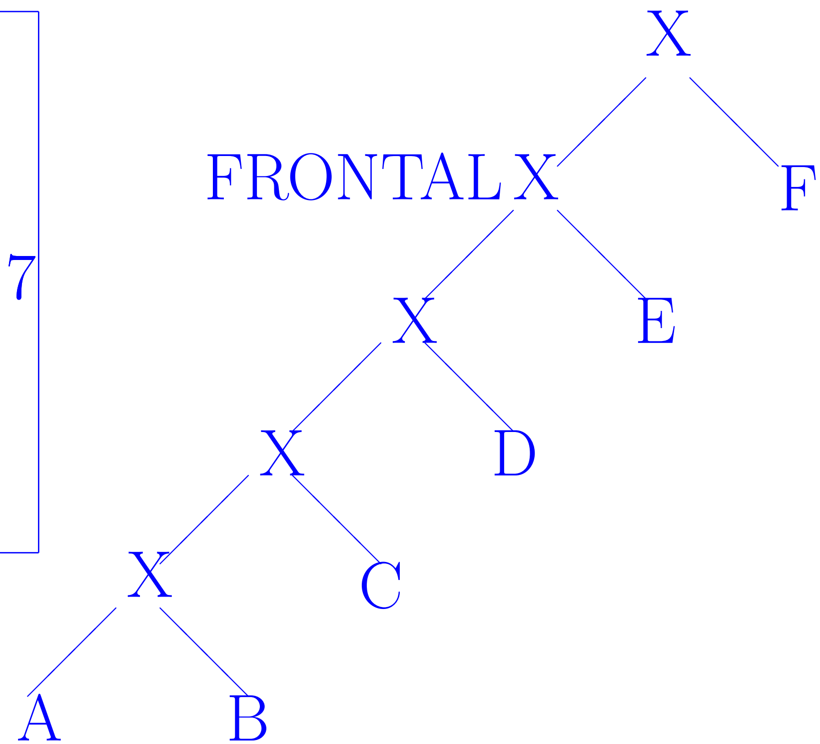
2) Frontal matrix: undergoes assembly + updates

3) Remainder: not accessed yet.



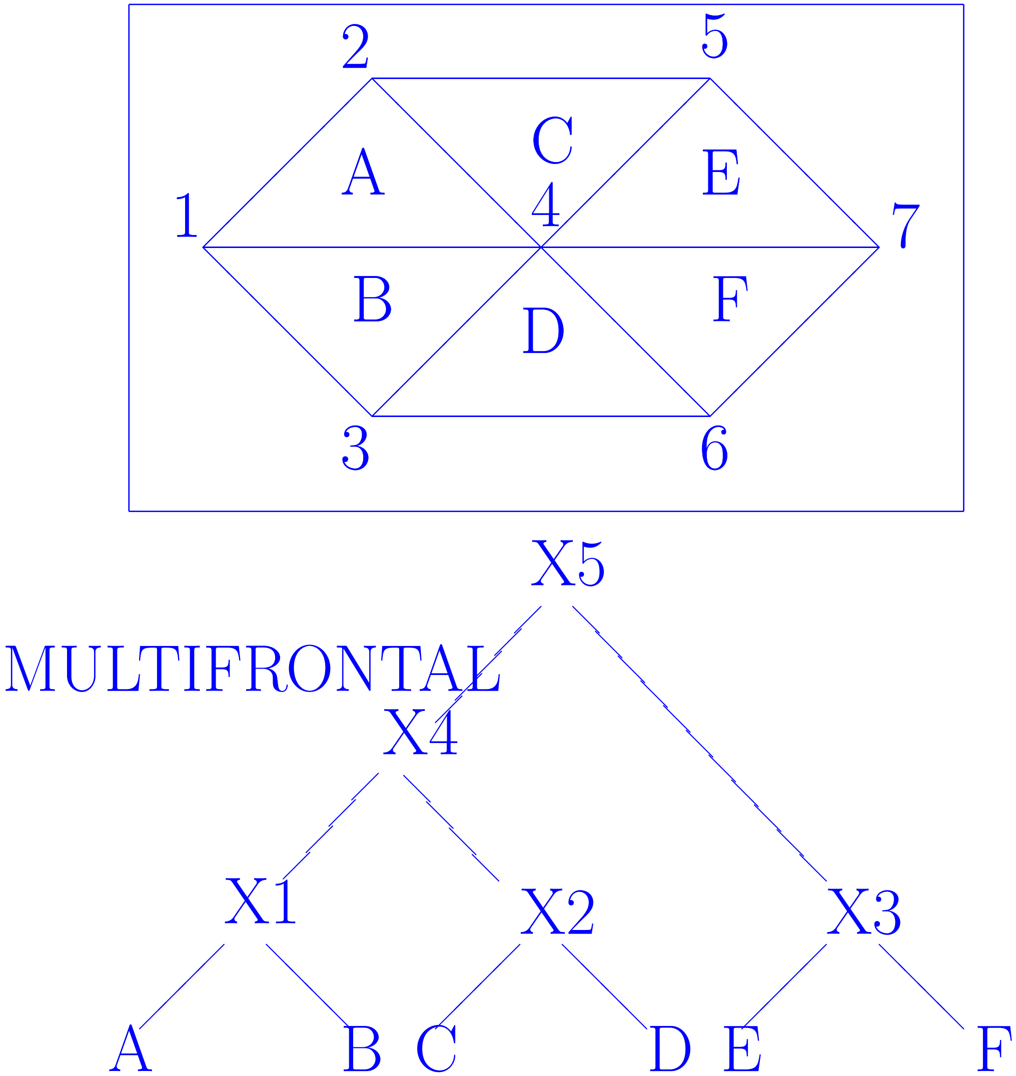
Assembly tree: - analogue to elimination tree





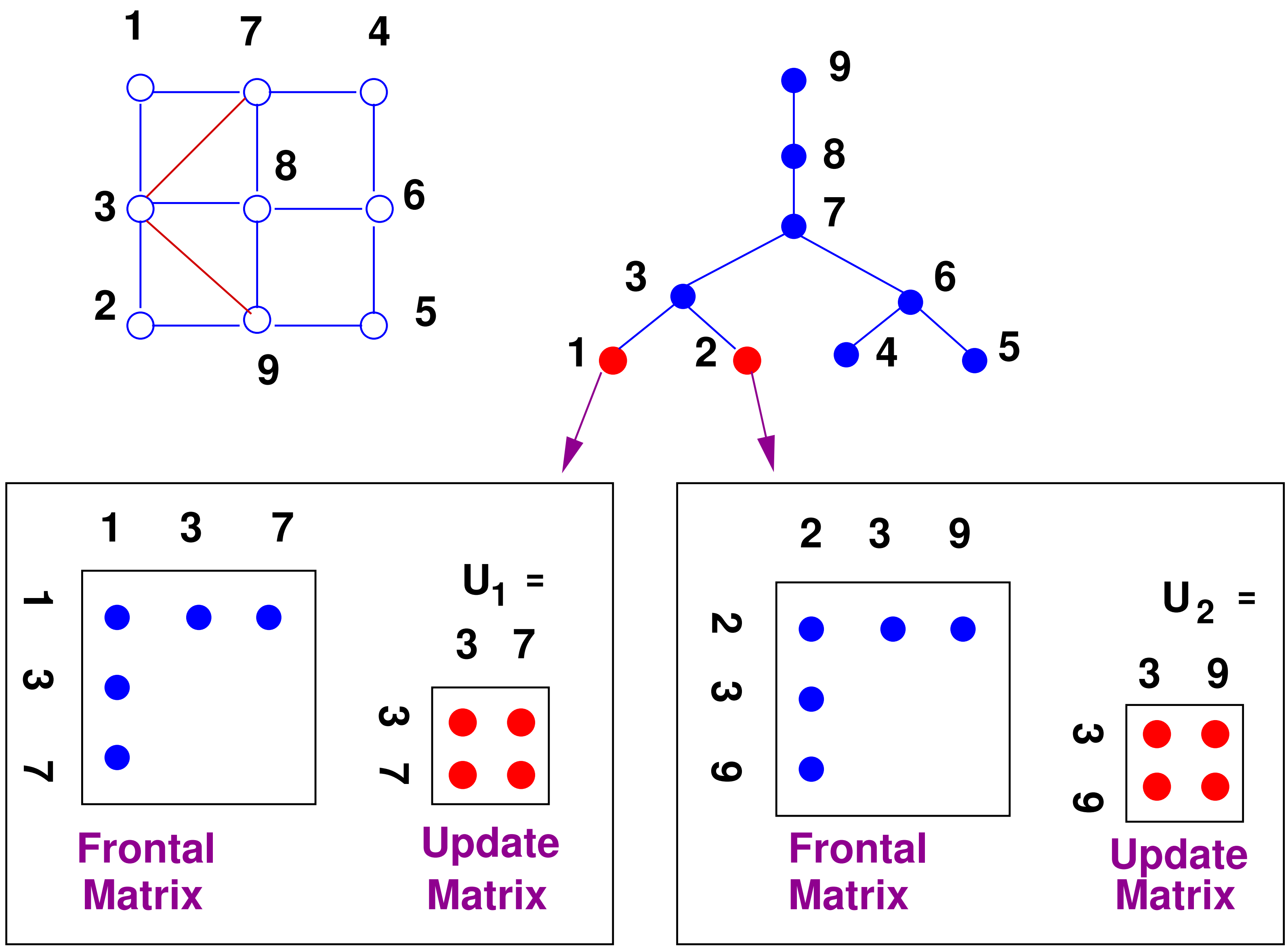
- Can proceed from several incoupled elements at the same time $\longrightarrow$ multifrontal technique [Duff & Reid, 1983]

Assembly tree for Multifrontal Method

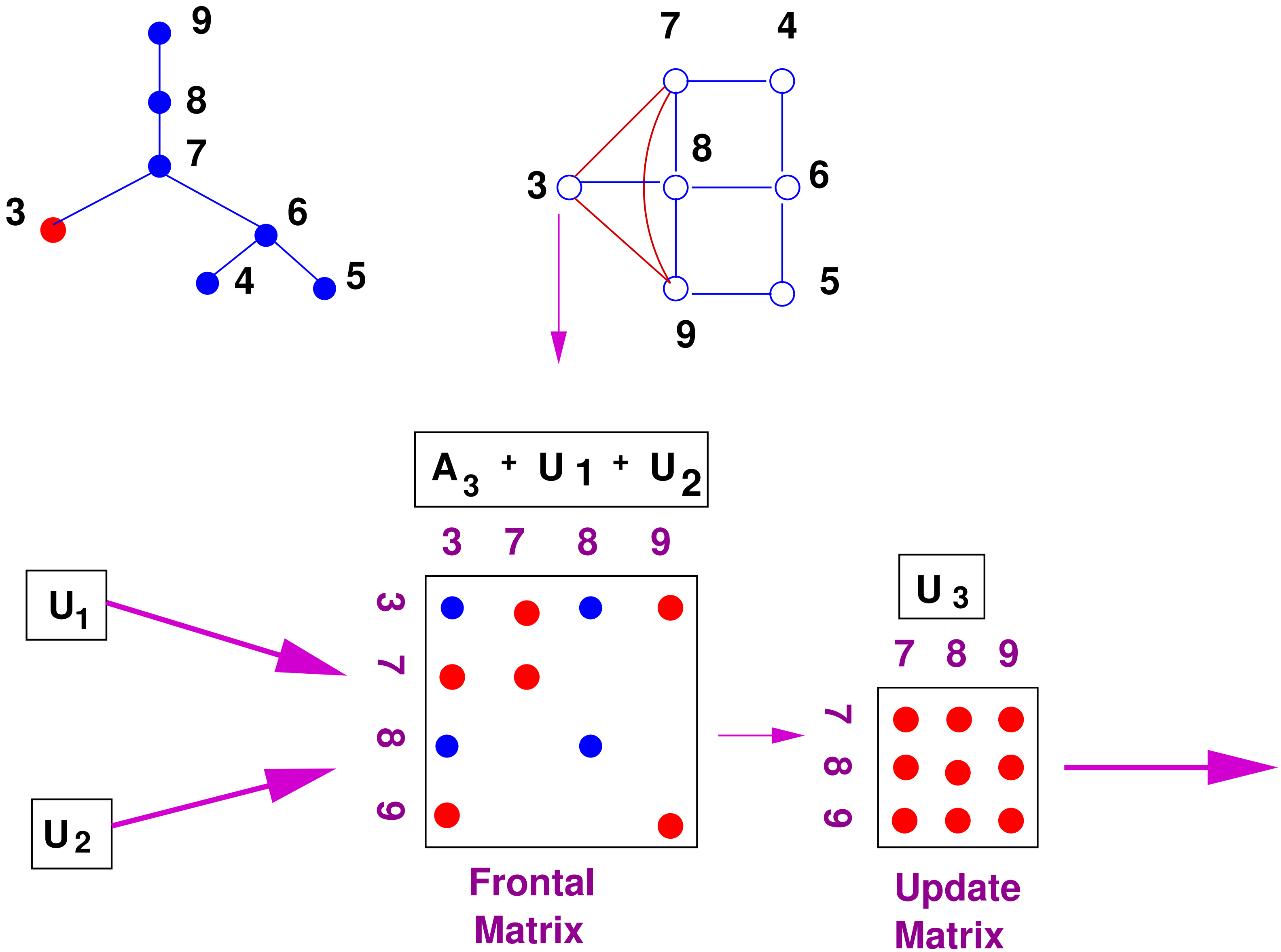


Multifrontal methods: extension to general matrices

- Elimination tree replaces assembly tree
- Proceed in post-order traversal of elimination tree in order not to violate task dependencies.
- When a node is eliminated an update matrix is created.
- This matrix is passed to the parent which adds it to its frontal matrix.
- Requires a stack of pending update matrices
- Update matrices popped out as they are needed
- Typically implemented with nested dissection ordering
- More complex than a left-looking algorithm



1
7
4
8
3
6
2
5
9
9
8
7
3
6
1
2
4
5
1 3 7
1
3
7
Frontal Matrix
U_1 =
3 7
3
7
Update Matrix
2 3 9
2
3
9
Frontal Matrix
U_2 =
3 9
3
9
Update Matrix



9
8
7
3
6
4
5
7
4
8
3
6
5
9
A3 + U1 + U2
3 7 8 9
U1
U2
U 3
7 8 9
Frontal Matrix
Update Matrix

Eliminating nodes 1 and 2: What happens on matrix

$$\left[\begin{array}{c|c|c|ccc|c|c|c}
1 & & \star & & & & \star & & \\ \hline
 & 2 & \star & & & & & & \star \\ \hline
\star & \star & 3 & & & & \blacksquare & \star & \blacksquare \\ \hline
 & & & 4 & & \star & \star & & \\
 & & & & 5 & \star & & & \star \\
 & & & \star & \star & 6 & & \star & \\ \hline
\star & & \blacksquare & \star & & & 7 & \star & \\ \hline
 & & \star & & & \star & \star & 8 & \star \\ \hline
 & \star & \blacksquare & & \star & & & \star & 9
\end{array}\right]
\begin{array}{l}
\\
\\
\leftarrow U_1(3,:) \leftarrow U_2(3,:) \\
\\
\\
\\
\leftarrow U_1(7,:) \\
\\
\qquad\qquad\quad \leftarrow U_2(9,:)
\end{array}$$

Supernodes

➤ In GE, contiguous columns tend to inherit the same pattern as the columns from they are updated $\longrightarrow$ Many columns will have same sparsity pattern.

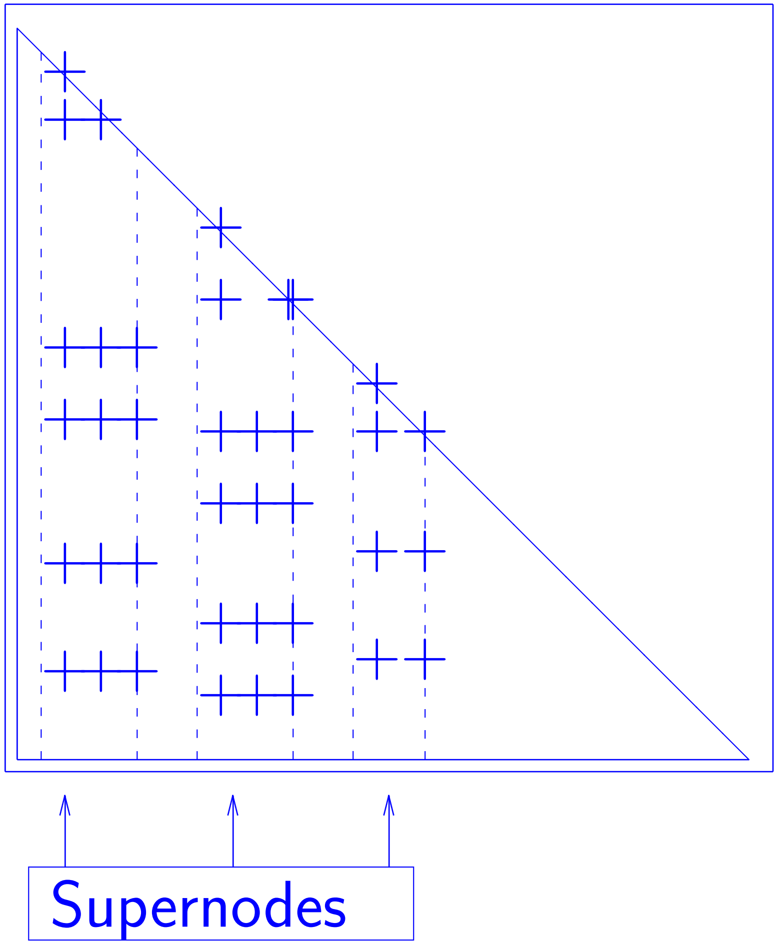
A supernode $=$ a set of contiguous columns in the Cholesky factor L which have the same sparsity pattern.

The set $\{j, j+1, ..., j+s\}$ is a supernode if

$$NZ(L_{*,k}) = NZ(L_{*,k+1}) \bigcup \{k+1\} \quad j \le k < j+s$$

where $NZ(L_{*,k})$ is nonzero set of column k of L.

Supernodes



Other terms used: Mass elimination, indistinguishible nodes, active variables in front, subscript compression,...

- Idea is old but first suggested by S. Eisenstat for speeding up sparse codes on vector machines.
- Beneficial on most machines
- Gains come in part from savings in Gather-Scatter operations.