given: $x_1 = \frac{1}{2}(\alpha+\beta)X_1 + \frac{1}{2}(\alpha-\beta)X_2$

$x_1 = \frac{1}{2}(\alpha-\beta)X_1 + \frac{1}{2}(\alpha+\beta)X_2$

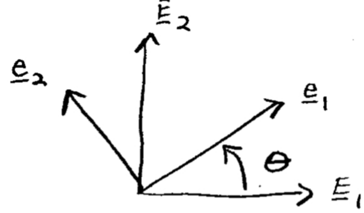
$x_3 = X_3$

Let: $C_1 = \frac{1}{2}(\alpha+\beta)$

$C_2 = \frac{1}{2}(\alpha-\beta)$

the deformation: $\underline{r} = \underline{r}(\underline{R}) = x_i \underline{e}_i$

1)



$\underline{E}_3 = \underline{e}_3$

$\underline{e}_1 = \cos(\theta)\underline{E}_1 + \cos(90° - \theta)\underline{E}_2$

$\underline{e}_2 = \cos(90° + \theta)\underline{E}_1 + \cos(\theta)\underline{E}_2$

$\underline{e}_3 = \underline{E}_3$

$$\overset{e\text{-}E}{[a]} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\overset{E\text{-}e}{[a]} = \overset{e\text{-}E}{[a]}^T$$

$\underline{E}_1 = \cos(\theta)\underline{e}_1 + \cos(90° + \theta)\underline{e}_2$

$\underline{E}_2 = \cos(90° - \theta)\underline{e}_1 + \cos(\theta)\underline{e}_2$

$\underline{E}_3 = \underline{e}_3$

$$\overset{E\text{-}e}{[a]} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- becaus $\underline{\underline{a}}$ is an orthogonal tensor ie. composed of orthogonal base vectors ($\underline{e}$ + $\underline{E}$), the inverse of $\overset{e\text{-}E}{a_{ij}}$ is equal to its transpose.
- Remember: when taking the transpose of a tensor, both the base vectors <u>and</u> components are transposed!
  - → this is obvious in the context of transformation tensors.

2) $$d\underline{r} = \underline{\underline{F}} \cdot d\underline{R}$$

$$\underline{\underline{F}} = \underline{r}\overleftarrow{\nabla}_0 = \frac{\partial \underline{r}}{\partial \underline{R}} = \frac{\partial x_i}{\partial X_j}\, \underline{e}_i \otimes \underline{E}_j$$

$$\overset{e\text{-}E}{[F]} = \begin{bmatrix} C_1 & C_2 & 0 \\ C_2 & C_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \text{recall: } \begin{aligned} C_1 &= \tfrac{1}{2}(\alpha+\beta) \\ C_2 &= \tfrac{1}{2}(\alpha-\beta) \end{aligned}$$

3) $$\underline{\underline{F}}^{-1} = \underline{R}\overleftarrow{\nabla} = \frac{\partial \underline{R}}{\partial \underline{r}} = \frac{\partial X_i}{\partial x_j}\, \underline{E}_i \otimes \underline{e}_j$$

=> But, instructions were to invert the matrix of components in the $\underline{e}$-$\underline{E}$ basis.

$$\overset{e\text{-}E}{[F]}^{cf} \Rightarrow \overset{e\text{-}E}{F}{}^{cf}_{ij} = (-1)^{(i+j)} \det(\overset{e\text{-}E}{F_{ij}})$$

→ Find components of cofactor tensor for $\overset{e\text{-}E}{\underline{\underline{F}}}$

$$(1,1) = \begin{bmatrix} C_1 & 0 \\ 0 & 1 \end{bmatrix} = C_1 \,;\quad (1,2) = -\begin{bmatrix} C_2 & 0 \\ 0 & 1 \end{bmatrix} = -C_2 \,;\quad (1,3) = \begin{bmatrix} C_2 & C_1 \\ 0 & 0 \end{bmatrix} = 0$$

$$(2,1) = -\begin{bmatrix} C_2 & 0 \\ 0 & 1 \end{bmatrix} = -C_2 \,;\quad (2,2) = \begin{bmatrix} C_1 & 0 \\ 0 & 1 \end{bmatrix} = C_1 \,;\quad (2,3) = -\begin{bmatrix} C_1 & C_2 \\ 0 & 0 \end{bmatrix} = 0$$

$$(3,1) = \begin{bmatrix} C_2 & 0 \\ C_1 & 0 \end{bmatrix} = 0 \,;\quad (3,2) = -\begin{bmatrix} C_1 & 0 \\ C_2 & 0 \end{bmatrix} = 0 \,;\quad (3,3) = \begin{bmatrix} C_1 & C_2 \\ C_2 & C_1 \end{bmatrix} = C_1^2 - C_2^2$$

$$\det(\overset{e\text{-}E}{[F]}) = C_1^2 - C_2^2$$

$$\underline{\underline{F}}^{-1} \Rightarrow \overset{E\text{-}e}{[F^{-1}]} = \frac{\overset{e\text{-}E}{[F^{cf}]}^T}{\left|\overset{e\text{-}E}{[F]}\right|} = \frac{\overset{E\text{-}e}{[F^{cfT}]}}{\left|\overset{E\text{-}e}{[F]}\right|} = \frac{\begin{bmatrix} C_1 & -C_2 & 0 \\ -C_2 & C_1 & 0 \\ 0 & 0 & C_1^2-C_2^2 \end{bmatrix}}{C_1^2 - C_2^2} = \begin{bmatrix} \frac{C_1}{C_1^2-C_2^2} & -\frac{C_2}{C_1^2-C_2^2} & 0 \\ \frac{-C_2}{C_1^2-C_2^2} & \frac{C_1}{C_1^2-C_2^2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Let: $D_1 = \dfrac{C_1}{C_1^2-C_2^2}$ + $D_2 = \dfrac{C_2}{C_1^2-C_2^2}$

then: $$\overset{E\text{-}e}{[F^{-1}]} = \begin{bmatrix} D_1 & -D_2 & 0 \\ -D_2 & D_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

4)

$$\overset{E-E}{F_{ij}} = \overset{e-E}{a_{ki}}\,\overset{e-E}{F_{kj}} = \overset{E-e}{a_{ik}}\,\overset{e-E}{F_{kj}} \Rightarrow \overset{E-e}{\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}} \overset{e-E}{\begin{bmatrix} C_1 & C_2 & 0 \\ C_2 & C_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

$$\overset{E-E}{[F]} = \begin{bmatrix} C_1\cos\theta - C_2\sin\theta & C_2\cos\theta - C_1\sin\theta & 0 \\ C_1\sin\theta + C_2\cos\theta & C_2\sin\theta + C_1\cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\overset{E-E}{F^{-1}_{ij}} = \overset{E-e}{F^{-1}_{ik}}\,\overset{e-E}{a_{kj}} \Rightarrow \overset{E-e}{\begin{bmatrix} D_1 & -D_2 & 0 \\ -D_2 & D_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \overset{e-E}{\begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}} = \overset{E-E}{\begin{bmatrix} D_1\cos\theta + D_2\sin\theta & D_1\sin\theta - D_2\cos\theta & 0 \\ -D_2\cos\theta - D_1\sin\theta & -D_2\sin\theta + D_1\cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

Let $a = \sin\theta$ & $b = \cos\theta$

$$\overset{E-E}{[F]}\,\overset{E-E}{[F^{-1}]} = \begin{bmatrix} (C_1 b - C_2 a)(D_1 b + D_2 a) + (C_2 b - C_1 a)(-D_2 b - D_1 a) & (C_1 b - C_2 a)(D_1 a - D_2 b) + (C_2 b - C_1 a)(-D_2 a + D_1 b) & 0 \\ (C_1 a + C_2 b)(D_1 b + D_2 a) + (C_2 a + C_1 b)(-D_2 b - D_1 a) & (C_1 a + C_2 b)(D_1 a - D_2 b) + (C_2 a + C_1 b)(-D_2 a + D_1 b) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

5)

$$\underline{\underline{F}}^{-1} = \underline{R}\overleftarrow{\nabla} = \frac{\partial X_i}{\partial x_j}\,\underline{E}_i \otimes \underline{e}_j$$

$$\begin{bmatrix} C_1 & C_2 \\ C_2 & C_1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

ues Cramer's rule for finding the inverse!

$$X_1 = \frac{\begin{vmatrix} x_1 & C_2 \\ x_2 & C_1 \end{vmatrix}}{\begin{vmatrix} C_1 & C_2 \\ C_2 & C_1 \end{vmatrix}} = \frac{x_1 C_1 - x_2 C_2}{C_1^2 - C_2^2} = \frac{x_1 C_1 - x_2 C_2}{\alpha\beta}$$

$$X_2 = \frac{\begin{vmatrix} C_1 & x_1 \\ C_2 & x_2 \end{vmatrix}}{\begin{vmatrix} C_1 & C_2 \\ C_2 & C_1 \end{vmatrix}} = \frac{x_2 C_1 - x_1 C_2}{C_1^2 - C_2^2} = \frac{x_2 C_1 - x_1 C_2}{\alpha\beta}$$

$$X_3 = x_3$$

$$\underline{R} = \underline{R}(\underline{r}) = X_i\,\underline{E}_i \quad \text{where } X_i = X_i(x_j)$$

$$= \left(\frac{x_1 C_1 - x_2 C_2}{C_1^2 - C_2^2}\right) \underline{E}_1 + \left(\frac{x_2 C_1 - x_1 C_2}{C_1^2 - C_2^2}\right) \underline{E}_2 + x_3\,\underline{E}_3$$

→ the determinant of $\underline{\underline{F}}^{-1}$ must $\neq 0$, i.e. $C_1^2 - C_2^2 \neq 0$

→ also stated as $\underline{\underline{F}}^{-1}$ must be nonsingular / cannot be singular

5) cont.

$$\underset{=}{F}^{-1} \Rightarrow \overset{E-e}{\begin{bmatrix} \frac{C_1}{C_1^2-C_2^2} & \frac{-C_2}{C_1^2-C_2^2} & 0 \\ \frac{-C_2}{C_1^2-C_2^2} & \frac{C_1}{C_1^2-C_2^2} & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

yes, this matches results from problem 3

6) $\overset{e-e}{[F]} = \overset{e-E}{[F]}\,\overset{e-E}{[a]}$

$\overset{e-e}{[F^{-1}]} = \overset{e-E}{[a]}\,\overset{E-e}{[F^{-1}]}$

→ use the transformation tensor to find components of a tensor in a different basis.

7) $d\underline{R} = 1\,\underline{E}_1 + 1\,\underline{E}_2 = \underline{t}_0\, dS_0$

$\underline{t}_0 = \frac{d\underline{R}}{dS_0} = \frac{d\underline{R}}{(d\underline{R}\cdot d\underline{R})^{1/2}} = \frac{1}{(1^2+1^2)^{1/2}}\underline{E}_1 + \frac{1}{(1^2+1^2)^{1/2}}\underline{E}_2$

i) $\underline{t}_0 = \frac{1}{\sqrt{2}}\underline{E}_1 + \frac{1}{\sqrt{2}}\underline{E}_2$ ≡ unit vector tangent to $d\underline{R}$

ii) $dS_0 = \sqrt{2}$ length of element

iii) $d\underline{r} = \underline{\underline{F}}\cdot d\underline{R}$

$$\{\overset{e}{d\underline{r}}\} = \overset{e-E}{[F]}\,\{\overset{E}{d\underline{R}}\} = \begin{bmatrix} C_1 & C_2 & 0 \\ C_2 & C_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{Bmatrix} 1 \\ 1 \\ 0 \end{Bmatrix} = \begin{Bmatrix} C_1+C_2 \\ C_2+C_1 \\ 0 \end{Bmatrix} = \begin{Bmatrix} \alpha \\ \alpha \\ 0 \end{Bmatrix}$$

iv) $\underline{t} = \frac{d\underline{r}}{ds} = \frac{d\underline{r}}{(d\underline{r}\cdot d\underline{r})^{1/2}} = \frac{d\underline{r}}{(\alpha^2+\alpha^2)^{1/2}} = \frac{\alpha}{\sqrt{2}\,\alpha}\underline{e}_1 + \frac{\alpha}{\sqrt{2}\,\alpha}\underline{e}_1$

$= \frac{1}{\sqrt{2}}\underline{e}_1 + \frac{1}{\sqrt{2}}\underline{e}_2$

v) $ds = (d\underline{r}\cdot d\underline{r})^{1/2} = (\alpha^2+\alpha^2)^{1/2} = \sqrt{2}\,\alpha$

↓

7. Cont.

$$ds = \{ \underline{t}_o \cdot \underline{\underline{F}}^T \cdot \underline{\underline{F}} \cdot \underline{t}_o \}^{1/2} dS_o$$

$$\overset{E-e}{[F^T]}\overset{e-E}{[F]} = \begin{bmatrix} c_1 & c_2 & 0 \\ c_2 & c_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 & c_2 & 0 \\ c_2 & c_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} c_1^2+c_2^2 & 2c_1c_2 & 0 \\ 2c_1c_2 & c_1^2+c_2^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\overset{E}{\langle t_o \rangle} \overset{E-e}{[F^T]}\overset{e-E}{[F]} = \langle \tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}, 0 \rangle \begin{bmatrix} c_1^2+c_2^2 & 2c_1c_2 & 0 \\ 2c_1c_2 & c_1^2+c_2^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\left\{ \overset{E}{\langle t_o \rangle} \overset{E-e}{[F^T]}\overset{e-E}{[F]} \overset{E}{\{t_o\}} \right\}^{1/2} = \left(\left\langle \frac{c_1^2+c_2^2}{\sqrt{2}} + \frac{2c_1c_2}{\sqrt{2}}, \frac{2c_1c_2}{\sqrt{2}} + \frac{c_1^2+c_2^2}{\sqrt{2}}, 0 \right\rangle \begin{Bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{Bmatrix} \right)^{1/2}$$

$$= \left(\frac{c_1^2+c_2^2}{2} + \frac{2c_1c_2}{2} + \frac{2c_1c_2}{2} + \frac{c_1^2+c_2^2}{2} \right)^{1/2}$$

$$= \left(c_1^2 + c_2^2 + 2c_1c_2\right)^{1/2} = \left(\tfrac{1}{4}(\alpha^2 + 2\alpha\beta + \beta^2) + \tfrac{1}{4}(\alpha^2 - 2\alpha\beta + \beta^2) + 2\left(\tfrac{1}{2}(\alpha+\beta)\ \tfrac{1}{2}(\alpha-\beta)\right)\right)^{1/2}$$

$$= \left(\frac{\alpha^2}{2} + \frac{\beta^2}{2} + \frac{1}{2}(\alpha^2 - \beta^2) \right)^{1/2} = (\alpha^2)^{1/2} = \alpha$$

$$\left\{ \overset{E}{\langle t_o \rangle} \overset{E-e}{[F^T]}\overset{e-E}{[F]} \overset{E}{\{t_o\}} \right\}^{1/2} dS_o = \alpha\, dS_o = \alpha \sqrt{2}$$

... yes the relation is satisfied

8) Nanson's: $\underline{\underline{F}}^T \cdot \underline{n}\, da = \underline{N}_0 \cdot J\, dA_0$

$$da = dx_1\, dx_2$$

$$dA_0 = dX_1\, dX_2$$

then: $\underline{\underline{F}}^T \cdot \underline{n}\, dx_1\, dx_2 = \underline{N}_0\, J\, dX_1\, dX_2$

$\underline{n}$ + $\underline{N}_0$ are $\perp$ to the $x_1 - x_2$ + $X_1 - X_2$ planes

$\therefore$ $\underline{\underline{F}}^T dx_1\, dx_2 = J\, dX_1\, dX_2$

9) $dV = J\, dV_0$

$$dx_1\, dx_2\, dx_3 = J\, dX_1\, dX_2\, dX_3$$

- recall $J = \det(\underline{\underline{F}})$, changes in area and volume are both related through J, the determinant of the deformation gradient.