17. Constitutive Equations

17.1 Initial Comments

If we consider Cauchy's first equation of motion under the assumption of equilibrium (acceleration is zero), there are three scalar equations and six unknown components of the symmetric Cauchy stress tensor. Only under very special conditions (statically determinant) can these equations be used to obtain the components of stress. In general, the details of the deformation must be taken into account and, if we work with the displacement vector, we have three additional variables. If strain is used to define the deformation, there are six components of strain expressed in terms of the three displacement components. Overall, we have nine scalar equations (three equilibrium and six strain-displacement relations) and fifteen variables consisting of six independent strain components, six independent stress components and three displacement components. The missing link that provides a complete set of equations is the set of constitutive equations that describe the nature of the material.

A constitutive equation provides stress as a function of parameters related to deformation. As we have seen, there are several choices for stress, and the choice of deformation parameters may be the deformation gradient, a strain tensor, or a rate of a suitable deformation tensor. The hope is that a given constitutive equation will describe a family of materials. However, it is often the case that any one constitutive equation may be suitable for only a certain range of deformation for a particular material.

There are two assumptions that are generally invoked, usually implicitly, to arrive at a suitable constitutive equation. They are:

<u>1. The Assumption of Determinism</u>

The basic idea here is that it is assumed that the stress at a material point is uniquely defined by the history of the deformation of the body.

<u>2. The Assumption of Local Action</u>

We go one (big) step further and assume the stress at a material point is uniquely determined by the deformation at that material point. The implication is that the stress is not affected by the deformation of neighboring material points.

There are constitutive equations that do not satisfy Assumption 2 and these are called nonlocal constitutive equations. An example is the assumption that stress depends on the gradient of strain as well as strain.

Any constitutive equation must satisfy the following restriction:

<u>The Principle of Material Frame Indifference</u>

Suppose a boundary-value problem, called Problem A, is solved so that the stress and strain tensors are known everywhere. Now suppose that the body and boundary conditions are subject to an arbitrary rotation and this defines Problem B. One solution to Problem B is to merely take the solutions to Problem A and determine the stress and strain by applying the appropriate rotation conditions for the respective tensors. A second solution to Problem B is obtained by applying the constitutive equation to the deformation field obtained by rotation from Problem A. The Principle of Material Frame Indifference states that the two solutions to Problem B should be identical. We will provide a simple example of the restriction that this principle places on a constitutive equation.

In the remainder of this chapter, we describe in a very cursory manner two elementary constitutive equations that are surprisingly broad in their application; namely, linear elasticity and constitutive equations for fluids.

17.2 Linear Elasticity

As examples, we choose two forms for linear elasticity. One relates the material tensors of the second Piola-Kirchhoff stress to the Lagrangian strain, and the second connects the spatial tensors of Cauchy stress to Eulerian strain:

$$\boldsymbol{P} = {}^{(4)}\boldsymbol{C} \cdot\cdot \boldsymbol{E} \qquad \boldsymbol{\sigma} = {}^{(4)}\boldsymbol{c} \cdot\cdot \boldsymbol{e} \tag{17-1}$$

where $\boldsymbol{C}$ and $\boldsymbol{c}$ are constant tensors called elasticity constitutive tensors. The prefix *(4)* is used in (17-1) to emphasize that these are fourth-order tensors. Even though these relations are linear, they may be chosen to describe a material undergoing large deformations as exemplified by large strains, or large rotations or both. Even if $\boldsymbol{C} = \boldsymbol{c}$, the two equations in (17-1) represent different constitutive equations unless the deformation is infinitesimal. Recall that the values of the components of the strain tensors may be different for given values of principal stretches.

Because the stress tensors are symmetric, these elasticity tensors must satisfy the minor symmetry condition that

$$C_{ijkl} = C_{jikl} \qquad \mathrm{c}_{ijkl} = c_{jikl} \tag{17-2}$$

Because of the symmetry of the strain tensors, any skew-symmetric part of the elasticity tensors will sum out so there is no loss in generality if we assume a second minor symmetry condition that

$$C_{ijkl} = C_{ijlk} \qquad \mathrm{c}_{ijkl} = c_{ijlk} \tag{17-3}$$

If we invoke a thermodynamic argument that a strain energy function must exist, then in addition we have a major symmetry condition that

$$C_{ijkl} = C_{klij} \qquad \mathrm{c}_{ijkl} = c_{klij} \tag{17-4}$$

The consequence of the two minor symmetry and the major symmetry conditions is that each elasticity tensor contains, at most, 21 independent material parameters that are used to construct the components for a given basis. Note that in general, there are more than 21 nonzero components.

Most materials exhibit planes of symmetry. For each independent plane of symmetry, the number of independent material parameters is reduced. The limiting case is that of isotropy for which every plane is a plane of symmetry and there are two independent material constants. Stated yet another way, the components of the elasticity tensors are the same for all bases.

Now we apply the Principle of Material Frame-Indifference to the two constitutive equations of (17-1). Consider a material point for which the solutions for Problem 1 are $(\boldsymbol{P},\boldsymbol{E})$ and $(\boldsymbol{\sigma},\boldsymbol{e})$. Apply a rotation $\boldsymbol{Q}$. First we consider the traction in terms of the Cauchy stress in the two configurations

$$\boldsymbol{\tau} = \boldsymbol{\sigma}\cdot\boldsymbol{n} \qquad \boldsymbol{\tau}^* = \boldsymbol{\sigma}^*\cdot\boldsymbol{n}^* \tag{17-5}$$

where the asterisk denotes variables in the rotated configuration. If we consider a surface on which the vectors $\boldsymbol{\tau}$ and $\boldsymbol{n}$ are defined, the corresponding vectors in the rotated configuration are

$$\boldsymbol{\tau}^* = \boldsymbol{Q}\cdot\boldsymbol{\tau} \qquad \boldsymbol{n}^* = \boldsymbol{Q}\cdot\boldsymbol{n} \tag{17-6}$$

The second and first equations of (17-5) yield

$$\boldsymbol{Q}\cdot\boldsymbol{\tau} = \boldsymbol{\sigma}^*\cdot\boldsymbol{Q}\cdot\boldsymbol{n} \quad \text{or} \quad \boldsymbol{Q}\cdot\boldsymbol{\sigma}\cdot\boldsymbol{n} = \boldsymbol{\sigma}^*\cdot\boldsymbol{Q}\cdot\boldsymbol{n} \tag{17-7}$$

with the result that

$$\boldsymbol{\sigma}^* = \boldsymbol{Q}\cdot\boldsymbol{\sigma}\cdot\boldsymbol{Q}^T \tag{17-8}$$

Recall that $\boldsymbol{F}^* = \boldsymbol{Q}\cdot\boldsymbol{F}$ with the consequence that $J^* = J$. Now we use (16-44) and (17-8) to obtain

$$\boldsymbol{P}^* = \boldsymbol{P} \tag{17-9}$$

We have shown previously that

$$\boldsymbol{E}^* = \boldsymbol{E} \quad \text{and} \quad \boldsymbol{e}^* = \boldsymbol{Q}\cdot\boldsymbol{e}\cdot\boldsymbol{Q}^* \tag{17-10}$$

Now suppose we apply the constitutive equation in configuration B with the rotated versions of the strains considered known. The resulting stress tensors are

$$\hat{\boldsymbol{P}} = \boldsymbol{C}\cdot\cdot\boldsymbol{E}^* \qquad \hat{\boldsymbol{\sigma}} = \boldsymbol{c}\cdot\cdot\boldsymbol{e}^* \tag{17-11}$$

The Principle of Material Frame-Indifference implies that the constitutive equation must be constructed such that

$$\hat{\boldsymbol{P}} = \boldsymbol{P}^* \qquad \hat{\boldsymbol{\sigma}} = \boldsymbol{\sigma}^* \tag{17-12}$$

The use of (17-9) and (17-10) implies that the first of (17-12) is automatically satisfied. However the second of (17-12) requires that the elasticity constitutive tensor $\boldsymbol{c}$ must be constructed so that the following equation is satisfied:

$$\boldsymbol{c} \cdot\cdot \boldsymbol{e}^* = Q \cdot \boldsymbol{\sigma} \cdot \boldsymbol{Q}^T \quad \text{or} \quad \boldsymbol{c} \cdot\cdot \left(\boldsymbol{Q} \cdot \boldsymbol{e} \cdot \boldsymbol{Q}^T\right) = \boldsymbol{Q} \cdot \left(\boldsymbol{c} \cdot\cdot \boldsymbol{e}\right) \cdot \boldsymbol{Q}^T \quad \forall \quad \boldsymbol{e} \text{ and } \boldsymbol{Q} \tag{17-13}$$

We rewrite this equation in indicial form as follows:

$$c_{ijkl} Q_{kp} e_{pq} Q^T_{ql} = Q_{il} c_{lmpq} e_{pq} Q^T_{mj} \tag{17-14}$$

Since the strain is arbitrary and using the trick of interchanging indices to express the transpose, we have

$$c_{ijkl} Q_{kp} Q_{lq} = Q_{il} Q_{jm} c_{lmpq} \tag{17-15}$$

Since $\boldsymbol{Q}$ is orthogonal, $Q^T_{ri} Q_{il} = Q_{ir} Q_{il} = \delta_{rl}$ and (17-15) becomes

$$c_{rspq} = c_{ijkl} Q_{ir} Q_{js} Q_{kp} Q_{lq} \qquad \forall \qquad \boldsymbol{Q} \tag{17-16}$$

The implication of (17-16) is that the components of $\boldsymbol{c}$ must be identical for all bases or, stated differently, the Principle of Material Frame Indifference implies that the relationship between spatial stress and strain tensors in a constitutive equation must be isotropic.

Remember that this result is based on the assumption of large deformations. If the deformations are small, the various strain tensors are identical, as are the stress tensors and there is no restriction that the elasticity tensor be isotropic. However, because the isotropic assumption is so widespread and useful, we devote the next section to the topic.

17.3 Linear Elastic Isotropy

The only isotropic second-order tensor that is symmetric is the identity tensor $\boldsymbol{I}$. Similarly, and without proof, there are only two independent, isotropic fourth-order tensors that satisfy the two minor symmetries and the major symmetry. One such tensor is the fourth-order identity, ${}^{(4)}\boldsymbol{I}$, that satisfies the symmetry conditions which may be expressed in direct notation as

$${}^{(4)}\boldsymbol{I} \cdot\cdot \boldsymbol{\sigma} = \boldsymbol{\sigma} \qquad \boldsymbol{\sigma} \cdot\cdot {}^{(4)}\boldsymbol{I} = \boldsymbol{\sigma} \qquad \boldsymbol{\sigma} \cdot\cdot {}^{(4)}\boldsymbol{I} \cdot\cdot \boldsymbol{e} = \boldsymbol{e} \cdot\cdot {}^{(4)}\boldsymbol{I} \cdot\cdot \boldsymbol{\sigma} \tag{17-17}$$

for arbitrary symmetric tensors $\boldsymbol{\sigma}$ and $\boldsymbol{e}$.

There are two other isotropic tensors that we call spherical, ${}^{(4)}\boldsymbol{P}^{sp}$, and deviatoric, ${}^{(4)}\boldsymbol{P}^{dev}$, projectors. These tensors have the properties that

$$\begin{aligned} &{}^{(4)}\boldsymbol{P}^{sp} + {}^{(4)}\boldsymbol{P}^{dev} = {}^{(4)}\boldsymbol{I} \\ &{}^{(4)}\boldsymbol{P}^{sp} \cdot\cdot {}^{(4)}\boldsymbol{P}^{sp} = {}^{(4)}\boldsymbol{P}^{sp} \qquad {}^{(4)}\boldsymbol{P}^{dev} \cdot\cdot {}^{(4)}\boldsymbol{P}^{dev} = {}^{(4)}\boldsymbol{P}^{dev} \\ &{}^{(4)}\boldsymbol{P}^{sp} \cdot\cdot {}^{(4)}\boldsymbol{P}^{dev} = {}^{(4)}\boldsymbol{0} \qquad {}^{(4)}\boldsymbol{P}^{dev} \cdot\cdot {}^{(4)}\boldsymbol{P}^{sp} = {}^{(4)}\boldsymbol{0} \end{aligned} \tag{17-18}$$

We define the projectors as follows:

$$ {}^{(4)}\boldsymbol{P}^{sp} = \frac{1}{3}\boldsymbol{I}\otimes\boldsymbol{I} \qquad {}^{(4)}\boldsymbol{P}^{dev} = {}^{(4)}\boldsymbol{I} - {}^{(4)}\boldsymbol{P}^{sp} \tag{17-19} $$

in which $\boldsymbol{I}$ is the second-order identity tensor. In indicial form these tensors are

$$ \begin{aligned} &{}^{(4)}I_{ijkl} = \tfrac{1}{2}\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right) \qquad {}^{(4)}P^{sp}_{ijkl} = \tfrac{1}{3}\delta_{ij}\delta_{kl} \\ &{}^{(4)}P^{dev}_{ijkl} = \tfrac{1}{2}\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right) - \tfrac{1}{3}\delta_{ij}\delta_{kl} \end{aligned} \tag{17-20} $$

Since only two of these tensors are independent, there are three combinations that can be chosen to represent an isotopic elastic constitutive tensor as follows:

$$ \begin{aligned} {}^{(4)}\boldsymbol{c} &= 3B\,{}^{(4)}\boldsymbol{P}^{sp} + 2G\,{}^{(4)}\boldsymbol{P}^{dev} \\ &= 3\lambda\,{}^{(4)}\boldsymbol{P}^{sp} + 2\mu\,{}^{(4)}\boldsymbol{I} \\ &= \tilde{C}_1\,{}^{(4)}\boldsymbol{I} + \tilde{C}_2\boldsymbol{P}^{dev} \end{aligned} \tag{17-21} $$

in which B is the bulk modulus, G is the shear modulus, λ and μ are the Lame parameters and the combination involving $\tilde{C}_1$ and $\tilde{C}_2$, a form that has not been used historically. A comparison of the first two forms indicates that

$$ 3\lambda + 2\mu = 3B \qquad \mu = G \tag{17-22} $$

For convenience, we will use the first of (17-21), an equation that is actually a spectral decomposition of the elasticity constitutive tensor. The tensor and its inverse are

$$ \begin{aligned} {}^{(4)}\boldsymbol{c} &= 3B\,{}^{(4)}\boldsymbol{P}^{sp} + 2G\,{}^{(4)}\boldsymbol{P}^{dev} \\ {}^{(4)}\boldsymbol{c}^{-1} &= \frac{1}{3B}\,{}^{(4)}\boldsymbol{P}^{sp} + \frac{1}{2G}\,{}^{(4)}\boldsymbol{P}^{dev} \end{aligned} \tag{17-23} $$

With the use of (17-18), it is easy to show that

$$ {}^{(4)}\boldsymbol{c}\cdot\cdot{}^{(4)}\boldsymbol{c}^{-1} = {}^{(4)}\boldsymbol{I} \tag{17-24} $$

With the Cauchy stress and Eulerian strain tensors, the linear elastic constitutive equation becomes

$$ \boldsymbol{\sigma} = \left\{3B\,{}^{(4)}\boldsymbol{P}^{sp} + 2G\,{}^{(4)}\boldsymbol{P}^{dev}\right\}\cdot\cdot\boldsymbol{e} \tag{17-25} $$

By operating on the left with the spherical and deviatoric projectors and using (17-18) we obtain the decoupled form

$$ {}^{(4)}\boldsymbol{P}^{sp}\cdot\cdot\boldsymbol{\sigma} = 3B\,{}^{(4)}\boldsymbol{P}^{sp}\cdot\cdot\boldsymbol{e} \qquad {}^{(4)}\boldsymbol{P}^{dev}\cdot\cdot\boldsymbol{\sigma} = 2G\,{}^{(4)}\boldsymbol{P}^{dev}\cdot\cdot\boldsymbol{e} \tag{17-26} $$

We define the spherical and deviatoric parts of the tress and strain tensors as follows:

$$ \begin{aligned} &\boldsymbol{\sigma}^{sp} = {}^{(4)}\boldsymbol{P}^{sp}\cdot\cdot\boldsymbol{\sigma} \qquad &\boldsymbol{\sigma}^{dev} = {}^{(4)}\boldsymbol{P}^{dev}\cdot\cdot\boldsymbol{\sigma} \\ &\boldsymbol{e}^{sp} = 3B\,{}^{(4)}\boldsymbol{P}^{sp}\cdot\cdot\boldsymbol{e} \qquad &\boldsymbol{e}^{dev} = 2G\,{}^{(4)}\boldsymbol{P}^{dev}\cdot\cdot\boldsymbol{e} \end{aligned} \tag{17-27} $$

It follows from the definitions of the spherical and deviatoric projectors that

$$\boldsymbol{\sigma}^{sp} = \tfrac{1}{3}\boldsymbol{I}\left(\boldsymbol{I}\cdot\cdot\boldsymbol{\sigma}\right) = P_T\boldsymbol{I} \qquad \boldsymbol{\sigma}^{dev} = \boldsymbol{\sigma} - \boldsymbol{\sigma}^{sp}$$
$$\boldsymbol{e}^{sp} = \tfrac{1}{3}\boldsymbol{I}\left(\boldsymbol{I}\cdot\cdot\boldsymbol{e}\right) = \tfrac{1}{3}e_V\boldsymbol{I} \qquad \boldsymbol{e}^{dev} = \boldsymbol{e} - \boldsymbol{e}^{sp} \tag{17-28}$$

where the "tensile" mean pressure, P_T, and volumetric strain for small strains, e_V, are given by

$$P_T = \tfrac{1}{3}\sigma_{ii} \qquad e_V = e_{ii} \tag{17-29}$$

If the mean pressure, $P = -P_T$, is used then the minus sign must be carried in the constitutive equation. Now the constitutive equation, as given in (17-25), becomes

$$P_T = Be_v \qquad \boldsymbol{\sigma}^{dev} = 2G\boldsymbol{e}^{dev} \tag{17-30}$$

A point is said to be in a state of pure shear if $P_T = 0$. An "effective" measure of shear, $\bar{\sigma}$, is often chosen as a scalar formed from the second invariant of the stress deviator:

$$\bar{\sigma} = \alpha\left(\boldsymbol{\sigma}^{dev}\cdot\cdot\boldsymbol{\sigma}^{dev}\right) \tag{17-31}$$

in which α is a positive scalar chosen for convenient comparison of experimental data with predictions based on plasticity models of material under either uniaxial stress, or pure shear.

17.4 Basic Constitutive Equations for Fluids

Isotropy

Constitutive equations for fluids often involve the Cauchy stress, $\boldsymbol{\sigma}$, and the rate of deformation, $\boldsymbol{d}$. Since both are spatial tensors, such constitutive equations must reflect isotropic behavior, at least for the forms considered in this section.

Incompressible-Inviscid (Ideal) Fluid

The assumptions on which this model is based are the following:

$$\boldsymbol{v}\cdot\bar{\nabla} = 0 \qquad \boldsymbol{\sigma}^{dev} = \boldsymbol{0} \tag{17-32}$$

The first equation is a constitutive constraint on the velocity field that is the consequence of incompressibility while the second states that the material cannot support shear. As a consequence of the constraint, the stress tensor is of the form

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^{sp} = -\tfrac{1}{3}P\boldsymbol{I} \tag{17-33}$$

in which P is a function that is conjugate to the constraint on velocity and is interpreted as the mean pressure.

The set of governing equations consists of the three scalar equations of motion and the incompressibility constraint and the unknowns are the three components of velocity and the pressure.

Incompressible-Viscous Fluid

The constitutive assumptions for this model are

$$\boldsymbol{v} \cdot \tilde{\nabla} = 0 \qquad \boldsymbol{\sigma}^{dev} = 2\mu \boldsymbol{d} \tag{17-34}$$

Now the material can support shear stress. An alternative form for the constraint is $tr(\boldsymbol{d}) = 0$ so (17-34) can also be expressed as

$$tr(\boldsymbol{d}) = 0 \qquad \boldsymbol{\sigma}^{dev} = 2\mu \boldsymbol{d}^{dev} \tag{17-35}$$

The stress tensor becomes

$$\boldsymbol{\sigma} = -\tfrac{1}{3} P\boldsymbol{I} + \boldsymbol{\sigma}^{dev} \tag{17-36}$$

in which P is, again, an unknown function that must be obtained by solving the set of governing equations with boundary and initial conditions.

Completely Viscous Fluid

Here we remove the assumption of incompressibility and assume the volumetric part behaves viscously as well to obtain

$$-P = B^v \left\{ tr(\boldsymbol{d}) \right\} \qquad \boldsymbol{\sigma}^{dev} = 2\mu \boldsymbol{d}^{dev} \tag{17-37}$$

in which B^v is the bulk viscosity. Note that this form is completely analogous to isotropic elasticity with the elastic moduli, B and G, replaced by the viscous parameters B^v and μ, respectively, and $\boldsymbol{e}$ replaced by $\boldsymbol{d}$. However, the volumetric part of this constitutive equation is not representative for most fluids.

Elastic-Viscous Fluid

If a compressible viscous fluid is being considered, then the constitutive equation is likely to be the following:

$$P = P(\theta, \rho) \qquad \boldsymbol{\sigma}^{dev} = 2\mu \boldsymbol{d}^{dev} \tag{17-38}$$

in which the pressure P is expressed as a function of temperature θ and density ρ, and called an "equation of state". However, recall that the function J relates volume elements and conservation of mass becomes $J = \rho_0 / \rho$. Therefore an equation of state can be interpreted as an equation that relates pressure to volumetric strain, which is an elastic constitutive equation. The equation can be highly nonlinear, especially for gases.

Elastic-Totally Viscous Fluid

A fairly comprehensive model is the case where a bulk viscous term is added to the elastic term to obtain

$$P = P^*(\theta,\rho) - B^v\{tr(\boldsymbol{d})\} \qquad \boldsymbol{\sigma}^{dev} = 2\mu\boldsymbol{d}^{dev} \tag{17-39}$$

More advanced models might include nonlinear terms for the deviatoric parts but still within the isotropic framework.

17.5 Summary

In this chapter, we indicate that constitutive equations are necessary to provide a complete set of governing equations to describe the response of a particular body. The set of constitutive equations is the part that differentiates one material from another.

We started with the basic constitutive equations used for solids, namely elasticity with an emphasis on the principle of Material Frame Indifference. If the stress and strain tensors are material tensors, this principle is automatically satisfied. If spatial tensors are used, then the restriction of isotropy must be imposed on the constitutive equation. Fortunately, many materials fall in this category so the elasticity equation reduces to the isotropic form of

$$-P = P_T = Be_v \qquad \boldsymbol{\sigma}^{dev} = 2G\boldsymbol{e}^{dev} \tag{17-40}$$

Rubber elasticity is often characterized as being incompressible and a shear part that is nonlinear. Other common classes of constitutive equations for solids are viscoelastic, plasticity and visco-plasticity.

For compressible fluids, the pressure part is nonlinearly elastic and the shear part depends on the rate of deformation instead of strain:

$$P = P(\theta,\rho) \qquad \boldsymbol{\sigma}^{dev} = 2\mu\boldsymbol{d}^{dev} \tag{17-41}$$

For some fluids, the shear part may also be nonlinear.

In order to obtain solutions to the complete set of equations, initial conditions and boundary conditions must be specified and these are considered in the next chapter.