Galerkin FEM for Heat Transfer Problem

Introductory Course on Multiphysics Modelling

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Contents

1 Notation preliminaries

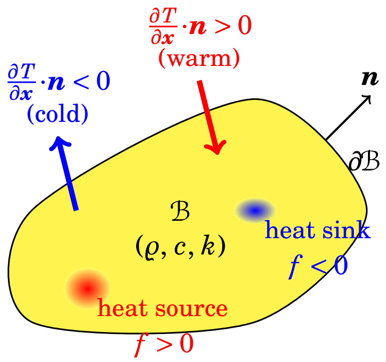
- The index notation is used with the summation over the index i. Obviously, $i = 1$ in 1D, $i = 1,2$ in 2D, and $i = 1,2,3$ in 3D.
- Consequently, the summation rule is also used for the approximation expressions, that is, over the indices $r,s = 1,\dots N$ (N is the number of degrees of freedom).
- The symbol $(\dots)_{|i}$ means a (generalized) invariant partial differentiation over the i-th coordinate:
$$(\dots)_{|i} = \frac{\partial(\dots)}{\partial x_i}.$$
The invariancy involves the so-called Christoffel symbols (in the case of curvilinear systems of reference).

- Symbols dV and dS are completely omitted in all the integrals presented below since it is obvious that one integrates over the specified domain or boundary. Therefore, one should understand that:

$$\int_{\mathcal{B}} (\dots) = \int_{\mathcal{B}} (\dots)\,\mathrm{d}V(\boldsymbol{x}), \qquad \int_{\partial\mathcal{B}} (\dots) = \int_{\partial\mathcal{B}} (\dots)\,\mathrm{d}S(\boldsymbol{x}).$$

2 Local differential formulation

2.1 Partial Differential Equation



- **Material data**
  $\varrho = \varrho(\boldsymbol{x})$ – the density $\left[\frac{\mathrm{kg}}{\mathrm{m}^3}\right]$
  $c = c(\boldsymbol{x})$ – the thermal capacity $\left[\frac{\mathrm{J}}{\mathrm{kg\cdot K}}\right]$
  $k = k(\boldsymbol{x})$ – the thermal conductivity $\left[\frac{\mathrm{W}}{\mathrm{m\cdot K}}\right]$
- **Known fields**
  $f = f(\boldsymbol{x}, t)$ – the heat production rate $\left[\frac{\mathrm{W}}{\mathrm{m}^3}\right]$
  $u_i = u_i(\boldsymbol{x}, t)$ – the convective velocity $\left[\frac{\mathrm{m}}{\mathrm{s}}\right]$
- **The unknown field**
  $T = T(\boldsymbol{x}, t) = ?$ – the temperature $[\mathrm{K}]$

Heat transfer equation

$$\varrho\, c\, \dot{T} + q_{i|i} - f = 0 \quad \text{where the heat flux vector } \left[\frac{\mathrm{W}}{\mathrm{K}}\right]:$$

$$q_i = q_i(T) = \begin{cases} -k\, T_{|i} & \text{– for conduction (only),} \\ -k\, T_{|i} + \varrho\, c\, u_i\, T & \text{– for conduction and convection,} \end{cases}$$

and $\dot{T} = \frac{\partial T}{\partial t}$ is the time rate of change of temperature $\left[\frac{\mathrm{K}}{\mathrm{s}}\right]$.

2.2 Initial and boundary conditions

The initial condition (at $t = t_0$)

- $T(\boldsymbol{x}, t_0) = T_0(\boldsymbol{x})$ in $\mathcal{B}$

  Prescribed field:

$T_0 = T_0(\boldsymbol{x})$ – the initial temperature $[\mathrm{K}]$

The boundary conditions (on $\partial\mathcal{B} = \partial\mathcal{B}_T \cup \partial\mathcal{B}_q$)

- the Dirichlet type:
  $T(\boldsymbol{x}, t) = \hat{T}(\boldsymbol{x}, t)$ on $\partial\mathcal{B}_T$

- the Neumann type:
  $-q_i\big(T(\boldsymbol{x},t)\big)\,n_i = \hat{q}(\boldsymbol{x},t)$ on $\partial\mathcal{B}_q$

Prescribed fields:
$\hat{T} = \hat{T}(\boldsymbol{x},t)$ – the temperature on the boundary $[\mathrm{K}]$
$\hat{q} = \hat{q}(\boldsymbol{x},t)$ – the inward heat flux $\left[\frac{\mathrm{W}}{\mathrm{m}^2}\right]$

For the sake of brevity, the Robin boundary condition is omitted. Usually, this condition is applied (in FEM) by means of the Lagrange multipliers.

2.3 Initial-Boundary-Value Problem

IBVP of the heat transfer
Find $T = T(\boldsymbol{x},t)$ for $\boldsymbol{x} \in \mathcal{B}$ and $t \in [t_0,t_1]$ satisfying the **equation of heat transfer** by conduction (a), or by conduction and convection (b):

$$\varrho\,c\,\dot{T} + q_{i|i} - f = 0 \quad \text{where} \quad q_i = q_i(T) = \begin{cases} -k\,T_{|i} & \leftarrow \text{(a)} \\ -k\,T_{|i} + \varrho\,c\,u_i\,T & \leftarrow \text{(b)} \end{cases} \tag{1}$$

with the **initial condition** (at $t = t_0$):

$$T(\boldsymbol{x},t_0) = T_0(\boldsymbol{x}) \quad \text{in } \mathcal{B}, \tag{2}$$

and subject to the **boundary conditions**:

$$T(\boldsymbol{x},t) = \hat{T}(\boldsymbol{x},t) \;\text{ on } \partial\mathcal{B}_T, \qquad -q_i\big(T(\boldsymbol{x},t)\big)\,n_i = \hat{q}(\boldsymbol{x},t) \;\text{ on } \partial\mathcal{B}_q \tag{3}$$

where $\partial\mathcal{B}_T \cup \partial\mathcal{B}_q = \partial\mathcal{B}$ and $\partial\mathcal{B}_T \cap \partial\mathcal{B}_q = \emptyset$.

3 Global integral formulations

3.1 Test functions

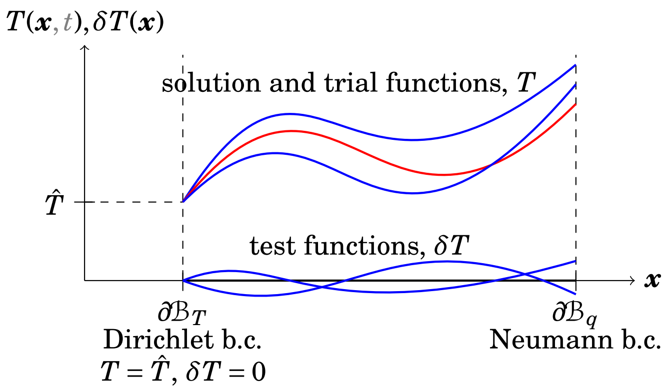


Figure 1: Test and trial functions.

Test function (see Fig. 1) – an arbitrary (but sufficiently regular) function $\delta T(\boldsymbol{x})$, defined in $\mathcal{B}$, which meets the *admissibility condition*:

$$\delta T = 0 \quad \text{on } \partial\mathcal{B}_T\,. \tag{4}$$

Notice that **test functions are always time-independent** (even for time-dependent problems since this approximation concerns only space).

3.2 Weighted formulation and weak variational form

The heat transfer PDE is multiplied by a test function δT and integrated over the domain $\mathcal{B}$ to obtain the **weighted integral formulation**

$$\boxed{\int_{\mathcal{B}} \varrho\, c\, \dot{T}\, \delta T + \int_{\mathcal{B}} q_{i|i}\, \delta T - \int_{\mathcal{B}} f\, \delta T = 0 \quad (\text{for } \textbf{every } \delta T)} \tag{5}$$

The term $q_{i|i}$ introduces the second derivative of T: $q_{i|i} = -k\, T_{|ii} + \ldots$. However, the heat PDE needs to be satisfied in the integral sense. Therefore, the requirements for T can be **weaken** as follows.

- Integrating by parts (using the divergence theorem)

$$\int_{\mathcal{B}} q_{i|i}\, \delta T = \int_{\mathcal{B}} (q_i\, \delta T)_{|i} - \int_{\mathcal{B}} q_i\, \delta T_{|i} = \int_{\partial\mathcal{B}} q_i\, \delta T\, n_i - \int_{\mathcal{B}} q_i\, \delta T_{|i} \tag{6}$$

- Using the Neumann boundary condition $(3)_2$ and the property of test function $(4)_2$

$$\int_{\partial\mathcal{B}} q_i\, n_i\, \delta T = \int_{\partial\mathcal{B}_q} \underbrace{q_i\, n_i}_{-\hat{q}}\, \delta T + \int_{\partial\mathcal{B}_T} q_i\, n_i\, \underbrace{\delta T}_{0} = - \int_{\partial\mathcal{B}_q} \hat{q}\, \delta T \tag{7}$$

Weak variational form is obtained

$$\boxed{\int_{\mathcal{B}} \varrho\, c\, \dot{T}\, \delta T - \int_{\mathcal{B}} q_i\, \delta T_{|i} - \int_{\partial\mathcal{B}_q} \hat{q}\, \delta T - \int_{\mathcal{B}} f\, \delta T = 0 \quad (\text{for } \textbf{every } \delta T)} \tag{8}$$

Now, only the first order spatial-differentiability of T is required.

In this formulation the Neumann boundary condition is already met (it has been used in a *natural* way). Therefore, the only additional requirements are the Dirichlet boundary condition $(3)_1$ and the initial condition (2).

4 Matrix formulations

4.1 Approximation

The spatial approximation of solution in the domain $\mathcal{B}$ is accomplished by a linear combination of (global) **shape functions**, $\phi_s = \phi_s(\boldsymbol{x})$,

$$T(\boldsymbol{x}, t) = \theta_s(t)\, \phi_s(\boldsymbol{x}) \qquad (s = 1, \ldots N) \tag{9}$$

where $\theta_s(t)$ $[\mathrm{K}]$ are (time-dependent) coefficients – the **degrees of freedom** (N is the total number of degrees of freedom). Consistent result is obtained now for the time rate of temperature

$$\dot{T}(\boldsymbol{x},t) = \dot{\theta}_s(t)\,\phi_s(\boldsymbol{x}) \quad \text{where} \quad \dot{\theta}_s(t) = \frac{\mathrm{d}\theta_s(t)}{\mathrm{d}t} \quad \left[\tfrac{\mathrm{K}}{\mathrm{s}}\right]. \tag{10}$$

Distinctive feature of the Galerkin method:
the same shape functions are used to approximate the solution as well as the test function

$$\delta T(\boldsymbol{x}) = \delta\theta_r\,\phi_r(\boldsymbol{x}) \qquad (r = 1,\dots N). \tag{11}$$

However, the test functions are time-independent (even for non-stationary problems) so the coefficients $\delta\theta_s$ will be always numbers. Moreover, the approximation (11) must satisfy the admissibility condition which is achieved easily in case of FEM, where the degrees of freedom are values in nodes some of which lie on the boundary.

4.2 Transient heat transfer (ordinary differential equations)

To reduce the "regularity" requirements for solution the approximations

$$T = \theta_s\,\phi_s \quad \left(\dot{T} = \dot{\theta}_s\,\phi_s\right), \qquad \delta T = \delta\theta_r\,\phi_r \tag{12}$$

are used for the **weak variational form** of the heat transfer problem

$$\int_{\mathcal{B}} \varrho\, c\, \dot{T}\,\delta T - \int_{\mathcal{B}} q_i\,\delta T_{|i} - \int_{\partial\mathcal{B}_q} \hat{q}\,\delta T - \int_{\mathcal{B}} f\,\delta T = 0\,. \tag{13}$$

This substitution is carried out below successively for all four integral terms of the weak form.

1. $\displaystyle\int_{\mathcal{B}} \varrho\, c\, \dot{T}\,\delta T = \dot{\theta}_s\,\delta\theta_r \int_{\mathcal{B}} \varrho\, c\, \phi_s\,\phi_r = \dot{\theta}_s\,\delta\theta_r\,\boldsymbol{M}_{rs}$
2. $\displaystyle\int_{\mathcal{B}} q_i\,\delta T_{|i} = \int_{\mathcal{B}} \big(-k\,T_{|i} + \varrho\, c\, u_i\,T\big)\,\delta T_{|i} = \theta_s\,\delta\theta_r \int_{\mathcal{B}} \big(-k\,\phi_{s|i} + \varrho\, c\, u_i\,\phi_s\big)\,\phi_{r|i} = \theta_s\,\delta\theta_r\,\boldsymbol{K}_{rs}$
3. $\displaystyle\int_{\partial\mathcal{B}_q} \hat{q}\,\delta T = \delta\theta_r \int_{\partial\mathcal{B}_q} \hat{q}\,\phi_r = \delta\theta_r\,\boldsymbol{Q}_r$
4. $\displaystyle\int_{\mathcal{B}} f\,\delta T = \delta\theta_r \int_{\mathcal{B}} f\,\phi_r = \delta\theta_r\,\boldsymbol{F}_r$

During the procedure, coefficient matrices and right-hand-side vectors are defined so the final result can be presented in the compact form as follows.

Matrix formulation of the heat transfer problem

$$\big[M_{rs}\,\dot{\theta}_s - K_{rs}\,\theta_s - (Q_r + F_r)\big]\,\delta\theta_r = 0 \quad \text{for } \textbf{every} \text{ admissible } \delta\theta_r. \tag{14}$$

This produces the following system of first-order **ordinary differential equations** (for $\theta_s = \theta_s(t) = ?$):

$$\boxed{M_{rs}\,\dot{\theta}_s - K_{rs}\,\theta_s = (Q_r + F_r)} \qquad (r, s = 1, \dots N). \tag{15}$$

This system can now be solved numerically using implicit or explicit time integration procedures where initial condition (2) will be required. Moreover, the Dirichlet boundary condition $(3)_1$ must be used (in a standard way, for FEM) rendering the system uniquely solvable.

The coefficient matrices and right-hand-side vectors can be assigned physical meanings:

- $M_{rs} = \int\limits_{\mathcal{B}} \varrho\, c\, \phi_s\, \phi_r$ – the thermal capacity matrix $\left[\frac{\mathrm{J}}{\mathrm{K}}\right]$,
- $K_{rs} = \int\limits_{\mathcal{B}} \big(-k\, \phi_{s|i} + \varrho\, c\, u_i\, \phi_s\big)\, \phi_{r|i}$ – the heat transfer matrix $\left[\frac{\mathrm{W}}{\mathrm{K}}\right]$,
- $Q_r = \int\limits_{\partial\mathcal{B}_q} \hat{q}\, \phi_r$ – the inward flow of heat vector $[\mathrm{W}]$,
- $F_r = \int\limits_{\mathcal{B}} f\, \phi_r$ – the heat production vector $[\mathrm{W}]$.

Note that in the case of heat transfer by convection a *time-variant* convective velocity, $u_i = u_i(t)$, results in a *time-dependent* heat transfer matrix, $K_{rs} = K_{rs}(t)$. Convection with a constant velocity renders this matrix time-independent.

4.3 Stationary heat transfer (algebraic equations)

For stationary heat transfer all fields do not depend on time but only on the spatial independent variables

$$T = T(\boldsymbol{x}), \qquad f = f(\boldsymbol{x}), \qquad u_i = u_i(\boldsymbol{x}) \qquad (\text{for } \boldsymbol{x} \in \mathcal{B}). \tag{16}$$

This means that heat flow by convection must be with a *constant* convective velocity.

These assumptions lead to the following

BVP of stationary heat flow: Find $T = T(\boldsymbol{x})$ satisfying (in $\mathcal{B}$)

$$q_{i|i} - f = 0 \quad \text{where} \quad q_i = q_i(T) = \begin{cases} -k\, T_{|i} & \text{(no convection)} \\ -k\, T_{|i} + \varrho\, c\, u_i\, T & \text{(with convection)} \end{cases} \tag{17}$$

with boundary conditions: $T = \hat{T}$ on $\partial\mathcal{B}_T$ (Dirichlet), and $-q_i(T)\, n_i = \hat{q}$ on $\partial\mathcal{B}_q$ (Neumann).

Consequently:

- The weak variational form lacks the rate integrand

$$-\int_{\mathcal{B}} q_i\, \delta T_{|i} - \int_{\partial\mathcal{B}_q} \hat{q}\, \delta T - \int_{\mathcal{B}} f\, \delta T = 0\,. \tag{18}$$

- The approximations $T(\boldsymbol{x}) = \theta_s\, \phi_s(\boldsymbol{x})$, $\delta T(\boldsymbol{x}) = \delta\theta_r\, \phi_r(\boldsymbol{x})$ (where both θ_s and $\delta\theta_r$ are number values) lead to the following **system of linear algebraic equations** (for $\theta_s = ?$):

$$\boxed{-K_{rs}\,\theta_s = (Q_r + F_r)} \qquad (r, s = 1, \ldots N). \tag{19}$$

  This result can be obtained directly from (15) by applying $\dot{\theta}_s = 0$. To solve that system the Dirichlet boundary condition will also be used.