A Brief Introduction to Partial Differential Equations

Introductory Course on Multiphysics Modelling

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(after: S.J. FARLOW's "*Partial Differential Equations for Scientists and Engineers*")

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Contents

1 Introduction

1.1 Basic notions and notations

Motivation: most physical phenomena, whether in the domain of fluid dynamics, electricity, magnetism, mechanics, optics or heat flow, can be in general (and actually are) described by *partial differential equations*.

Definition 1 (Partial Differential Equation)**.** A **partial differential equation (PDE)** is an equation that

1. has an *unknown function* depending on *at least two variables*,
2. contains some *partial derivatives* of the unknown function.

A **solution to PDE** is, generally speaking, any function (in the independent variables) that satisfies the PDE. However, from this family of functions one may be uniquely selected by imposing adequate **initial** and/or **boundary conditions**. A PDE with initial and boundary conditions constitutes the so-called **initial-boundary-value problem (IBVP)**. Such problems are mathematical models of most physical phenomena.

The following notation will be used throughout this lecture:

- t, x, y, z (or, e.g., r, θ, ϕ) – the **independent variables** (here, t represents time while the other variables are space coordinates),
- $u = u(t, x, \ldots)$ – the **dependent variable** (the unknown function),
- the *partial derivatives* will be denoted as follows (e.g.)

$$u_t = \frac{\partial u}{\partial t}, \qquad u_{tt} = \frac{\partial^2 u}{\partial t^2}, \qquad u_{xy} = \frac{\partial^2 u}{\partial x \partial y}, \qquad \text{etc.}$$

1.2 Methods and techniques for solving PDEs

A very brief review of most methods and techniques for solving PDEs is presented below.

Separation of variables. A PDE in n independent variables is reduced to n ODEs.

Integral transforms. A PDE in n independent variables is reduced to one in $(n-1)$ independent variables. Hence, a PDE in two variables could be changed to an ODE.

Change of coordinates. A PDE can be changed to an ODE or to an easier PDE by changing the coordinates of the problem (rotating the axes, etc.).

Transformation of the dependent variable. The unknown of a PDE is transformed into a new unknown that is easier to find.

Numerical methods. A PDE is changed to a system of *difference equations* that can be solved by means of iterative techniques (*Finite Difference Methods*). These methods can be divided into two main groups, namely: **explicit** and **implicit** methods. There are also other methods that attempt to approximate solutions by polynomial functions (eg., *Finite Element Method*).

Perturbation methods. A nonlinear problem (a nonlinear PDE) is changed into a sequence of linear problems that approximates the nonlinear one.

Impulse-response technique. Initial and boundary conditions of a problem are decomposed into simple impulses and the response is found for each impulse. The overall response is then obtained by adding these simple responses.

Integral equations. A PDE is changed to an integral equation (that is, an equation where the unknown is inside the integral). The integral equations is then solved by various techniques.

Variational methods. The solution to a PDE is found by reformulating the equation as a minimization problem. It turns out that the minimum of a certain expression (very likely the expression will stand for total energy) is also the solution to the PDE.

Eigenfunction expansion. The solution of a PDE is as an infinite sum of eigenfunctions. These eigenfunctions are found by solving the so-called eigenvalue problem corresponding to the original problem.

1.3 Well-posed and ill-posed problems

Definition 2 (A well-posed problem)**.** An initial-boundary-value problem is **well-posed** if:

1. it has a **unique solution**,
2. the **solution vary continuously** with the given inhomogeneous data, that is,

   small changes in the data (i.e., in the PDE coefficients as well as in the boundary and initial conditions) should cause only *small changes in the solution*.

Importance of well-posedness The following statements explain why well-posedness is a very desirable attribute for any physical problem and its mathematical model.

- In practice, the initial and boundary data are measured and so small errors occur.
- Very often the problem must be solved numerically which involves truncation and round-off errors.
- If the problem is well-posed then these unavoidable small errors produce only slight errors in the computed solution, and, hence, useful results are obtained.

2 Classifications

2.1 Basic classifications of PDEs

Partial differential equations are classified according to many things. Classification is an important concept because the general theory and methods of solution usually apply to a given class of equations. Six basic classifications of PDEs are presented below.

Order of the PDE. The order of a PDE is the order of the highest partial derivative in the equation.

Example 3.

$$\begin{aligned} \text{first order:} &\quad u_t = u_x\,, \\ \text{second order:} &\quad u_t = u_{xx}\,, \quad u_{xy} = 0\,, \\ \text{third order:} &\quad u_t + u\,u_{xxx} = \sin(x) \\ \text{fourth order:} &\quad u_{xxxx} = u_{tt}\,. \end{aligned}$$

Number of variables. PDEs may be classified by the number of their independent variables, that is, the number of variables the unknown function depends on.

Example 4.

$$\begin{aligned} \text{PDE in two variables:} \quad & u_t = u_{xx}\,, \quad \big(u = u(t,x)\big) \\ \text{PDE in three variables:} \quad & u_t = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} \quad \big(u = u(t,r,\theta)\big) \\ \text{PDE in four variables:} \quad & u_t = u_{xx} + u_{yy} + u_{zz}\,, \quad \big(u = u(t,x,y,z)\big)\,. \end{aligned}$$

Linearity. PDE is linear if the dependent variable and all its derivatives appear in a linear fashion.

Example 5.

$$\begin{aligned} \text{linear:} \quad & u_{tt} + \exp(-t)\,u_{xx} = \sin(t)\,, \\ \text{nonlinear:} \quad & u\,u_{xx} + u_t = 0\,, \\ \text{linear:} \quad & x\,u_{xx} + y\,u_{yy} = 0\,, \\ \text{nonlinear:} \quad & u_x + u_y + u^2 = 0\,. \end{aligned}$$

Kinds of coefficients. PDE can be with constant or variable coefficients (if at least one of the coefficients is a function of (some of) independent variables).

Example 6.

$$\begin{aligned} \text{constant coefficients:} \quad & u_{tt} + 5u_{xx} - 3u_{xy} = \cos(x)\,, \\ \text{variable coefficients:} \quad & u_t + \exp(-t)\,u_{xx} = 0\,. \end{aligned}$$

Homogeneity. PDE is homogeneous if the free term (the right-hand side term) is zero.

Example 7.

$$\begin{aligned} \text{homogeneous:} \quad & u_{tt} - u_{xx} = 0\,, \\ \text{nonhomogeneous:} \quad & u_{tt} - u_{xx} = x^2 \sin(t)\,. \end{aligned}$$

Kind of PDE. All linear second-order PDEs are either:

- hyperbolic (e.g., $u_{tt} - u_{xx} = f(t,x,u,u_t,u_x)$),
- parabolic (e.g., $u_{xx} = f(t,x,u,u_t,u_x)$),
- elliptic (e.g., $u_{xx} + u_{yy} = f(x,y,u,u_x,u_y)$).

Here, f is an an arbitrary function of independent variables, dependent variable and its first derivatives. The matter of hyperbolic, parabolic, and elliptic equations will be explained at length in next Sections.

2.2 Kinds of nonlinearity

Definition 8 (Semi-linearity, quasi-linearity, and full nonlinearity)**.** A partial differential equation is:

semi-linear – if the highest derivatives appear in a linear fashion and their coefficients do not depend on the unknown function or its derivatives;

Example 9 (here and below: $u = u(\boldsymbol{x})$ and $\boldsymbol{x} = (x, y)$).

$$C_1(\boldsymbol{x})\,u_{xx} + C_2(\boldsymbol{x})\,u_{xy} + C_3(\boldsymbol{x})\,u_{yy} + C_0(\boldsymbol{x}, u, u_x, u_y) = 0$$

quasi-linear – if the highest derivatives appear in a linear fashion;

Example 10.

$$C_1(\boldsymbol{x}, u, u_x, u_y)\,u_{xx} + C_2(\boldsymbol{x}, u, u_x, u_y)\,u_{xy} + C_0(\boldsymbol{x}, u, u_x, u_y) = 0$$

fully nonlinear – if the highest derivatives appear in a nonlinear fashion.

Example 11.

$$u_{xx}\,u_{xy} = 0$$

2.3 Types of second-order linear PDEs

A **second-order linear PDE in two variables** can be in general written in the following form

$$A\,u_{xx} + B\,u_{xy} + C\,u_{yy} + D\,u_x + E\,u_y + F\,u = G$$

where A, B, C, D, E, and F are coefficients, and G is a non-homogeneous (or right-hand side) term. All these quantities are constants or functions of x and y.

The second-order linear PDE is either

hyperbolic: if $B^2 - 4AC > 0$

Example 12.

$$\begin{aligned} u_{tt} - u_{xx} = 0 \quad &\to \quad B^2 - 4AC = 0^2 - 4\cdot(-1)\cdot 1 = 4 > 0\,, \\ u_{tx} = 0 \quad &\to \quad B^2 - 4AC = 1^2 - 4\cdot 0\cdot 0 = 1 > 0\,. \end{aligned}$$

parabolic: if $B^2 - 4AC = 0$

Example 13.

$$u_t - u_{xx} = 0 \quad \to \quad B^2 - 4AC = 0^2 - 4\cdot(-1)\cdot 0 = 0\,.$$

elliptic: if $B^2 - 4AC < 0$

Example 14.

$$u_{xx} + u_{yy} = 0 \quad \to \quad B^2 - 4AC = 0^2 - 4\cdot 1\cdot 1 = -4 < 0\,.$$

Note that whether PDE is hyperbolic, parabolic, or elliptic depends only on the coefficients of the second derivatives (i.e., A, B, and C); it has nothing to do with the first-derivative terms, the termi in u, or the non-homogeneous term. The mathematical solutions to these three types of equations are quite different.

The three major classifications of linear PDEs essentially classify physical problems into three basic types:

- vibrating systems and **wave** propagation (**hyperbolic** case),

- heat flow and **diffusion** processes (**parabolic** case),
- **steady-state** phenomena (**elliptic** case).

In general, $B^2 - 4AC$ is a function of the independent variables. Hence, an equation can change from one basic type to another throughout the domain of the equation.

Example 15.

$$y\,u_{xx} + u_{yy} = 0 \quad \rightarrow \quad B^2 - 4AC = -4y \begin{cases} > 0 & \text{for } y < 0 \text{ (hyperbolic)}, \\ = 0 & \text{for } y = 0 \text{ (parabolic)}, \\ < 0 & \text{for } y > 0 \text{ (elliptic)}. \end{cases}$$

Second-order linear equations in *three or more variables* can also be classified except that matrix analysis must be used.

Example 16.

$$\begin{aligned} u_t = u_{xx} + u_{yy} \quad &\leftarrow \quad \text{parabolic equation,} \\ u_{tt} = u_{xx} + u_{yy} + u_{zz} \quad &\leftarrow \quad \text{hyperbolic equation.} \end{aligned}$$

2.4 Classic linear PDEs

Below we present classical second-order linear PDEs and a couple of classic higher-order PDEs.

Hyperbolic PDEs:

- Vibrating string (1D wave equation): $u_{tt} - c^2 u_{xx} = 0$
- Wave equation with damping (if $h \neq 0$): $u_{tt} - c^2 \nabla^2 u + h\,u_t = 0$
- Transmission line equation: $u_{tt} - c^2 \nabla^2 u + h\,u_t + k\,u = 0$

Parabolic PDEs:

- Diffusion-convection equation: $u_t - \alpha^2 u_{xx} + h\,u_x = 0$
- Diffusion with lateral heat-concentration loss: $u_t - \alpha^2 u_{xx} + k\,u = 0$

Elliptic PDEs:

- Laplace's equation: $\nabla^2 u = 0$
- Poisson's equation: $\nabla^2 u = k$
- Helmholtz's equation: $\nabla^2 u + \lambda^2 u = 0$
- Shrödinger's equation: $\nabla^2 u + k\,(E - V)\,u = 0$

Higher-order PDEs:

- Airy's equation (third order): $u_t + u_{xxx} = 0$
- Bernouli's beam equation (fourth order): $\alpha^2 u_{tt} + u_{xxxx} = 0$
- Kirchhoff's plate equation (fourth order): $\alpha^2 u_{tt} + \nabla^4 u = 0$

(Here: ∇^2 is the Laplace operator, $\nabla^4 = \nabla^2\nabla^2$ is the biharmonic operator.)

3 Canonical forms

3.1 Canonical forms of second order PDEs

Any second-order linear PDE (in two variables)

$$A\,u_{xx} + B\,u_{xy} + C\,u_{yy} + D\,u_x + E\,u_y + F\,u = G \tag{1}$$

(where A, B, C, D, E, F, and G are constants or functions of x and y) can be transformed into the so-called **canonical form**. This can be achieved by **introducing new coordinates**

$$\xi = \xi(x,y) \quad \text{and} \quad \eta = \eta(x,y) \tag{2}$$

(in place of x and y) that simplify the equation to its canonical form.

The type of PDE determines the canonical form:

▶ **for hyperbolic PDE** (that is, if $B^2 - 4A\,C > 0$) there are, in fact, two possibilities:

$$u_{\xi\xi} - u_{\eta\eta} = f(\xi,\eta,u,u_\xi,u_\eta) \quad \left(\tilde{B}^2 - 4\tilde{A}\,\tilde{C} = 0^2 - 4\cdot 1\cdot(-1) = 4 > 0\right), \tag{3}$$

$$\text{or} \quad u_{\xi\eta} = f(\xi,\eta,u,u_\xi,u_\eta) \quad \left(\tilde{B}^2 - 4\tilde{A}\,\tilde{C} = 1^2 - 4\cdot 0\cdot 0 = 1 > 0\right), \tag{4}$$

▶ **for parabolic PDE** (that is, if $B^2 - 4A\,C = 0$):

$$u_{\xi\xi} = f(\xi,\eta,u,u_\xi,u_\eta) \quad \left(\tilde{B}^2 - 4\tilde{A}\,\tilde{C} = 0^2 - 4\cdot 1\cdot 0 = 0\right), \tag{5}$$

▶ **for elliptic PDE** (that is, if $B^2 - 4A\,C < 0$):

$$u_{\xi\xi} + u_{\eta\eta} = f(\xi,\eta,u,u_\xi,u_\eta) \quad \left(\tilde{B}^2 - 4\tilde{A}\,\tilde{C} = 0^2 - 4\cdot 1\cdot 1 = -4 < 0\right). \tag{6}$$

Here, f states for an arbitrary function of the new independent variables ξ and η, the dependent variable u, and the first derivatives u_ξ and u_η.

3.2 Reduction to a canonical form

Steps and calculations to simplify a second-order linear PDE to canonical form:

1. **Introduce new coordinates** $\xi = \xi(x,y)$ and $\eta = \eta(x,y)$.

   - Compute the partial derivatives

$$\begin{aligned}
u_x &= u_\xi\,\xi_x + u_\eta\,\eta_x\,, \qquad u_y = u_\xi\,\xi_y + u_\eta\,\eta_y\,,\\
u_{xx} &= u_{\xi\xi}\,\xi_x^2 + 2u_{\xi\eta}\,\xi_x\,\eta_x + u_{\eta\eta}\,\eta_x^2 + u_\xi\,\xi_{xx} + u_\eta\,\eta_{xx}\,,\\
u_{yy} &= u_{\xi\xi}\,\xi_y^2 + 2u_{\xi\eta}\,\xi_y\,\eta_y + u_{\eta\eta}\,\eta_y^2 + u_\xi\,\xi_{yy} + u_\eta\,\eta_{yy}\,,\\
u_{xy} &= u_{\xi\xi}\,\xi_x\,\xi_y + u_{\xi\eta}\left(\xi_x\,\eta_y + \xi_y\,\eta_x\right) + u_{\eta\eta}\,\eta_x\,\eta_y + u_\xi\,\xi_{xy} + u_\eta\,\eta_{xy}\,.
\end{aligned} \tag{7}$$

- Substitute these values into the original equation to obtain a new form

$$\widetilde{A}\, u_{\xi\xi} + \widetilde{B}\, u_{\xi\eta} + \widetilde{C}\, u_{\eta\eta} + \widetilde{D}\, u_{\xi} + \widetilde{E}\, u_{\eta} + F\, u = G \tag{8}$$

  where the new coefficients are as follows

$$\begin{gathered}\widetilde{A} = A\,\xi_x^2 + B\,\xi_x\,\xi_y + C\,\xi_y^2, \quad \widetilde{B} = 2A\,\xi_x\,\eta_x + B\big(\xi_x\,\eta_y + \xi_y\,\eta_x\big) + 2C\,\xi_y\,\eta_y\,,\\ \widetilde{C} = A\,\eta_x^2 + B\,\eta_x\,\eta_y + C\,\eta_y^2, \quad \widetilde{D} = A\,\xi_{xx} + B\,\xi_{xy} + C\,\xi_{yy} + D\,\xi_x + E\,\xi_y\,,\\ \widetilde{E} = A\,\eta_{xx} + B\,\eta_{xy} + C\,\eta_{yy} + D\,\eta_x + E\,\eta_y\,.\end{gathered} \tag{9}$$

2. **Impose the requirements onto coefficients** $\widetilde{A}$, $\widetilde{B}$, $\widetilde{C}$, and solve for ξ and η. The requirements depend on the type of the PDE:
   - set $\widetilde{A} = \widetilde{C} = 0$ for the **hyperbolic** PDE (when $B^2 - 4A\,C > 0$);
   - set either $\widetilde{A} = 0$ or $\widetilde{C} = 0$ for the **parabolic** PDE; in this case another necessary requirement $\widetilde{B} = 0$ will follow automatically (since $B^2 - 4A\,C = 0$);
   - for the **elliptic** PDE (when $B^2 - 4A\,C < 0$) first, proceed as in the hyperbolic case, setting $\widetilde{A} = \widetilde{C} = 0$ to find the *complex conjugate coordinates* ξ, η (which would lead to a form of *complex hyperbolic equation* $u_{\xi\eta} = f(\xi,\eta,u,u_\xi,u_\eta)$), and then transform them as follows

$$\alpha \leftarrow \frac{\xi+\eta}{2} \text{ (real part of } \xi \text{ and } \eta), \quad \beta \leftarrow \frac{\xi-\eta}{2\mathrm{i}} \text{ (imaginary part of } \xi \text{ and } \eta), \tag{10}$$

     which gives the final canonical elliptic form: $u_{\alpha\alpha} + u_{\beta\beta} = f(\alpha,\beta,u,u_\alpha,u_\beta)$.

3. **Use the new coordinates** for the coefficients and homogeneous term (to replace $x = x(\xi,\eta)$ and $y = y(\xi,\eta)$) in the new canonical form.

3.3 Transforming the hyperbolic equation

For **hyperbolic equation** the canonical form

$$u_{\xi\eta} = f(\xi,\eta,u,u_\xi,u_\eta) \tag{11}$$

is achieved by setting $\boxed{\widetilde{A} = \widetilde{C} = 0}$, that is,

$$\widetilde{A} = A\,\xi_x^2 + B\,\xi_x\,\xi_y + C\,\xi_y^2 = 0\,, \qquad \widetilde{C} = A\,\eta_x^2 + B\,\eta_x\,\eta_y + C\,\eta_y^2 = 0\,, \tag{12}$$

which can be rewritten as

$$A\left(\frac{\xi_x}{\xi_y}\right)^2 + B\,\frac{\xi_x}{\xi_y} + C = 0\,, \qquad A\left(\frac{\eta_x}{\eta_y}\right)^2 + B\,\frac{\eta_x}{\eta_y} + C = 0\,. \tag{13}$$

Solving these equations for $\frac{\xi_x}{\xi_y}$ and $\frac{\eta_x}{\eta_y}$ one finds the so-called **characteristic equations**:

$$\frac{\xi_x}{\xi_y} = \frac{-B + \sqrt{B^2 - 4A\,C}}{2A}\,, \qquad \frac{\eta_x}{\eta_y} = \frac{-B - \sqrt{B^2 - 4A\,C}}{2A}\,. \tag{14}$$

Note that $\frac{\xi_x}{\xi_y}$ and $\frac{\eta_x}{\eta_y}$ each have two solutions to their quadratic equations, but only one solution must be found for each in order for $\widetilde{A}$ and $\widetilde{C}$ to be zero. The only restriction is that one cannot pick up the same roots, or else will end up with the two coordinates the same.

The new coordinates equated to constant values define the parametric lines of the new system of coordinates. That means that the total derivatives are zero, i.e.,

$$\xi(x,y) = \text{const.} \quad \rightarrow \quad \mathrm{d}\xi = \xi_x\,\mathrm{d}x + \xi_y\,\mathrm{d}y = 0 \quad \rightarrow \quad \frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{\xi_x}{\xi_y}, \tag{15}$$

$$\eta(x,y) = \text{const.} \quad \rightarrow \quad \mathrm{d}\eta = \eta_x\,\mathrm{d}x + \eta_y\,\mathrm{d}y = 0 \quad \rightarrow \quad \frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{\eta_x}{\eta_y}, \tag{16}$$

Therefore, the **characteristic equations** are

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{\xi_x}{\xi_y} = \frac{B - \sqrt{B^2 - 4AC}}{2A}, \qquad \frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{\eta_x}{\eta_y} = \frac{B + \sqrt{B^2 - 4AC}}{2A}, \tag{17}$$

and can be easily integrated to find the implicit solutions, $\xi(x,y) = \text{const.}$ and $\eta(x,y) = \text{const.}$, that is, the new coordinates ensuring the simple canonical form of the PDE.

To present the hyperbolic PDE in the other another canonical form, i.e.,

$$u_{\alpha\alpha} - u_{\beta\beta} = f(\alpha, \beta, u, u_\alpha, u_\beta) \tag{18}$$

the new coordinates need yet to be transformed as follows

$$\alpha \leftarrow \xi + \eta, \qquad \beta \leftarrow \xi - \eta. \tag{19}$$

Example

Rewrite in canonical form the following equation

$$y^2 u_{xx} - x^2 u_{yy} = 0 \qquad x \in (0, +\infty), \quad y \in (0, +\infty). \tag{20}$$

(In the first quadrant this is a hyperbolic equation since $B^2 - 4AC = 4y^2x^2 > 0$ for $x \neq 0$ and $y \neq 0$.)

- Writing the two **characteristic equations**

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{B - \sqrt{B^2 - 4AC}}{2A} = -\frac{x}{y}, \qquad \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{B + \sqrt{B^2 - 4AC}}{2A} = \frac{x}{y}. \tag{21}$$

- **Solving**: separating the variables

$$y\,\mathrm{d}y = -x\,\mathrm{d}x, \qquad y\,\mathrm{d}y = x\,\mathrm{d}x, \tag{22}$$

  and integrating

$$\xi(x,y) = y^2 + x^2 = \text{const.}, \qquad \eta(x,y) = y^2 - x^2 = \text{const.} \tag{23}$$

  Parametric lines of the new coordinates are presented in Fig. 1.

- **Using the new coordinates** for the (non-zero) coefficients

$$\widetilde{B} = -16x^2y^2 = 4(\eta^2 - \xi^2), \quad \widetilde{D} = -2(y^2 + x^2) = -2\xi, \quad \widetilde{E} = 2(y^2 - x^2) = 2\eta, \tag{24}$$

  to **present the PDE in the canonical form**:

$$u_{\xi\eta} = \frac{\widetilde{D}\,u_\xi + \widetilde{E}\,u_\eta}{\widetilde{B}} = \frac{\xi\,u_\xi - \eta\,u_\eta}{2(\xi^2 - \eta^2)}. \tag{25}$$

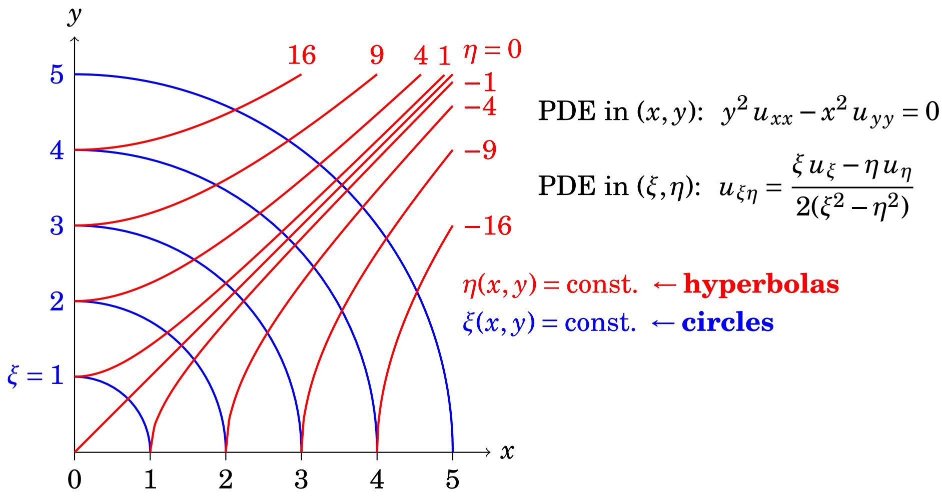

Figure 1: New coordinates, $\xi \in (0, +\infty)$ and $\eta \in (-\infty, +\infty)$, for the canonical form (25) of the hyperbolic PDE (20).

4 Separation of variables

4.1 Necessary assumptions

This technique applies to problems where

- **the PDE is *linear* and *homogeneous*** (not necessary constant coefficients)

  A second-order PDE in two variables (x and t) is *linear* and *homogeneous* if it can be written in the following form

  $$A u_{xx} + B u_{xt} + C u_{tt} + D u_x + E u_t + F u = 0 \tag{26}$$

  where the coefficients A, B, C, D, E, and F do not depend on the dependent variable $u = u(x, t)$ or any of its derivatives though can be functions of independent variables (x, t).

- **the boundary conditions are *linear* and *homogeneous***

  In the case of the second-order PDE a general form of such boundary conditions is

  $$G_1 u_x(x_1, t) + H_1 u(x_1, t) = 0, \tag{27}$$

  $$G_2 u_x(x_2, t) + H_2 u(x_2, t) = 0, \tag{28}$$

  where G_1, G_2, H_1, H_2 are constants.

4.2 Explanation of the method

Basic idea of the separation of variables technique can be expressed as follows:

1. break down the initial conditions of the problem into simple components,
2. find the response to each component,

3. add up these individual responses to obtain the result for the original problem.

The separation of variables technique looks first for the so-called **fundamental solutions**. They are simple-type solutions to the PDE of the form

$$u_i(x,t) = X_i(x)\,T_i(t)\,, \tag{29}$$

where $X_i(x)$ is a sort of "shape" of the solution i whereas $T_i(t)$ scales this "shape" for different values of time t.

Thus, the **fundamental solution will**:

- always **retain its basic "shape"**,
- at the same time, **satisfy the BCs** which puts a requirement only on the "shape" function $X_i(x)$ since the BCs are linear and homogeneous.

The general idea is that it is possible to find an infinite number of these fundamental solutions (everyone corresponding to an adequate simple component of initial conditions).

The **solution of the problem** is found by adding the simple fundamental solutions in such a way that the resulting sum

$$u(x,t) = \sum_{i=1}^{n} a_i\,u_i(x,t) = \sum_{i=1}^{n} a_i\,X_i(x)\,T_i(t) \tag{30}$$

satisfies the initial conditions which is attained by a proper selection of the coefficients a_i. The properties of linearity and homogeneity assure that this sum still satisfies the PDE and the BCs, thus being the solution of the problem.

Example

Consider the following initial-boundary-value problem (IBVP) for heat flow (or diffusion process):

Find $u = u(x,t) = ?$ satisfying

PDE: for $x \in (0,1)$ and $t \in (0,\infty)$

$$u_t = \alpha^2\,u_{xx}\,, \tag{31}$$

BCs: for $t \in (0,\infty)$ and

$$x = 0: \quad u(0,t) = 0\,, \tag{32}$$

$$x = 1: \quad u_x(1,t) + h\,u(1,t) = 0\,, \tag{33}$$

IC: for $x \in [0,1]$ and

$$t = 0: \quad u(x,0) = f(x)\,. \tag{34}$$

Step 1. Separating the PDE into two ODEs.

► Substituting the separated form (of the fundamental solution),

$$u(x,t) = u_i(x,t) = X_i(x)\,T_i(t)\,, \tag{35}$$

into the PDE gives (after division by $\alpha^2 X_i(x) T_i(t)$)

$$\frac{T_i'(t)}{\alpha^2 T_i(t)} = \frac{X_i''(x)}{X_i(x)}. \tag{36}$$

► Both sides of this equation must be constant (since they depend only on x or t which are *independent*). Setting them both equal to μ_i gives the two ODEs:

$$T_i'(t) - \mu_i \alpha^2 T_i(t) = 0, \qquad X_i''(x) - \mu_i X_i(x) = 0. \tag{37}$$

Step 2. Finding the separation constant and fundamental solutions.

- If $\mu_i = 0$ then: (after using the BCs) a trivial solution $u(x,t) \equiv 0$ is obtained.
- For $\mu_i > 0$: $T(t)$ (and so $u(x,t) = X(x)T(t)$) will grow exponentially to infinity which can be rejected on physical grounds.
- Therefore: $\mu_i = -\lambda_i^2 < 0$.

► Now, the two ODEs can be written as

$$T_i'(t) + \lambda_i^2 \alpha^2 T_i(t) = 0, \qquad X_i''(x) + \lambda_i^2 X_i(x) = 0, \tag{38}$$

and solutions to them are

$$T_i(t) = \tilde{C}_0 \exp\big(-\lambda_i^2 \alpha^2 t\big), \qquad X_i(x) = \tilde{C}_1 \sin(\lambda_i x) + \tilde{C}_2 \cos(\lambda_i x), \tag{39}$$

where $\tilde{C}_0$, $\tilde{C}_1$, and $\tilde{C}_2$ are constants.

► That leads to the following fundamental solution (with new constants C_1, C_2)

$$u_i(x,t) = X_i(x) T_i(t) = \big[C_1 \sin(\lambda_i x) + C_2 \cos(\lambda_i x)\big] \exp(-\lambda_i^2 \alpha^2 t). \tag{40}$$

► Applying the boundary conditions:

- $x = 0$: $\quad C_2 \exp(-\lambda_i^2 \alpha^2 t) = 0 \quad \to \quad C_2 = 0,$ $\qquad$ (41)
- $x = 1$: $\quad C_1 \exp(-\lambda_i^2 \alpha^2 t)\big[\lambda_i \cos(\lambda_i) + h \sin(\lambda_i)\big] = 0 \quad \to \quad \tan\lambda_i = -\dfrac{\lambda_i}{h}.$ $\qquad$ (42)

gives a desired condition on λ_i (they are **eigenvalues** for which there exists a nonzero solution). A graphical representation of eigenvalues solution (for $h = 3$) is presented in Fig. 2.

► The **fundamental solutions** are as follows

$$u_i(x,t) = \sin(\lambda_i x) \exp(-\lambda_i^2 \alpha^2 t). \tag{43}$$

Figure 3[left] presents initial shapes (i.e., for $t = 0$) of four fundamental solutions (for $h = 3$).

Step 3. Expansion of the IC as a sum of eigenfunctions.

▶ The final solution is such linear combination (with coefficients a_i) of infinite number of fundamental solutions,

$$u(x,t) = \sum_{i=1}^{\infty} a_i\, u_i(x,t) = \sum_{i=1}^{\infty} a_i\, \sin(\lambda_i\, x)\, \exp(-\lambda_i^2\, \alpha^2\, t)\,, \tag{44}$$

that satisfies the initial condition:

$$f(x) \equiv u(x,0) = \sum_{i=1}^{\infty} a_i\, \sin(\lambda_i\, x)\,. \tag{45}$$

Note that each fundamental solution satisfies the PDE and the BCs, and their linear combination will still satisfy the PDE and BCs, since both the PDE and BCs are linear and homogeneous.

▶ The coefficients a_i in the eigenfunction expansion (44) are found by multiplying both sides of the IC equation (45) by $\sin(\lambda_j x)$ and integrating using the orthogonality property, i.e.,

$$\begin{aligned}\int_0^1 f(x)\, \sin(\lambda_j\, x)\, \mathrm{d}x &= \sum_{i=1}^{\infty} a_i \int_0^1 \sin(\lambda_i\, x)\, \sin(\lambda_j\, x)\, \mathrm{d}x \\ &= a_j \int_0^1 \sin^2(\lambda_j\, x)\, \mathrm{d}x \\ &= a_j\, \frac{\lambda_j - \sin(\lambda_j)\, \cos(\lambda_j)}{2\lambda_j}\end{aligned} \tag{46}$$

which gives the following formula for the coefficients

$$a_i = \frac{2\lambda_i}{\lambda_i - \sin(\lambda_i)\, \cos(\lambda_i)} \int_0^1 f(x)\, \sin(\lambda_i\, x)\, \mathrm{d}x\,. \tag{47}$$

Figure 3[right] presents the initial shapes of four fundamental solutions scaled by the coefficients a_i (for $h = 3$ and $\alpha = 1$). The final solution for: $\alpha = 1$, $h = 3$, and $f(x) = x^2$, is presented in Fig. 4.

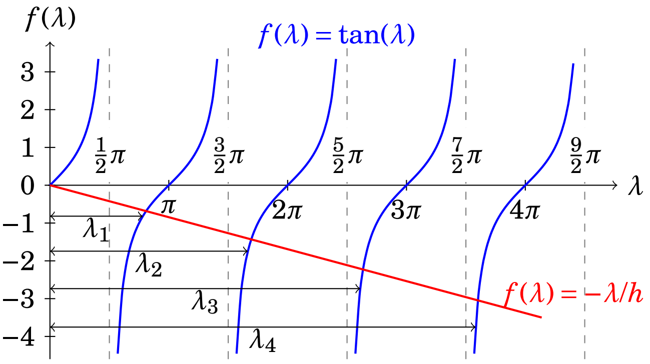

Figure 2: Eigenvalues solution.

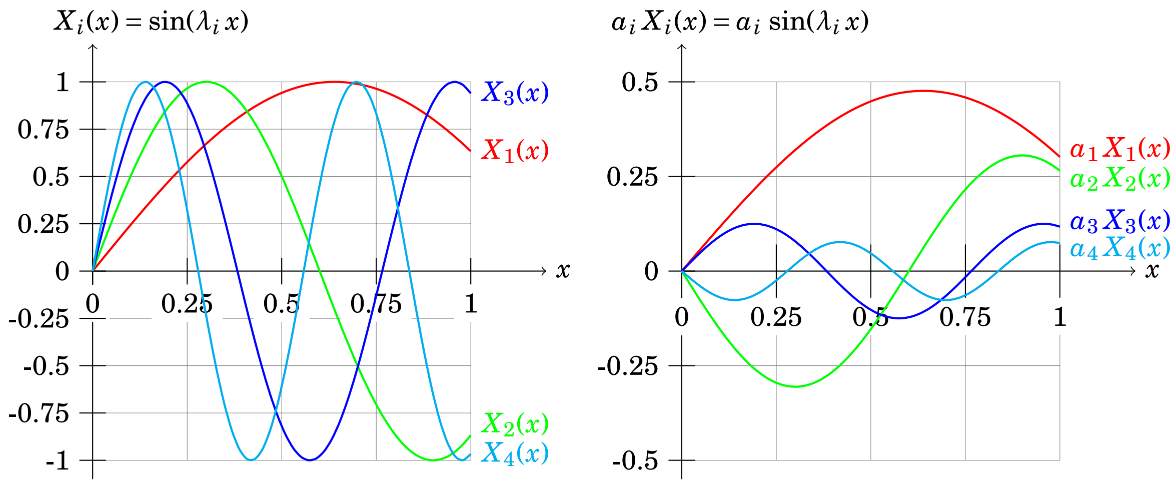

Figure 3: Initial shapes (i.e., $t = 0$) of four fundamental solutions (for $h = 3$) [left], and the shapes scaled by the coefficients a_i (for $\alpha = 1$, $h = 3$, and $f(x) = x^2$) [right].

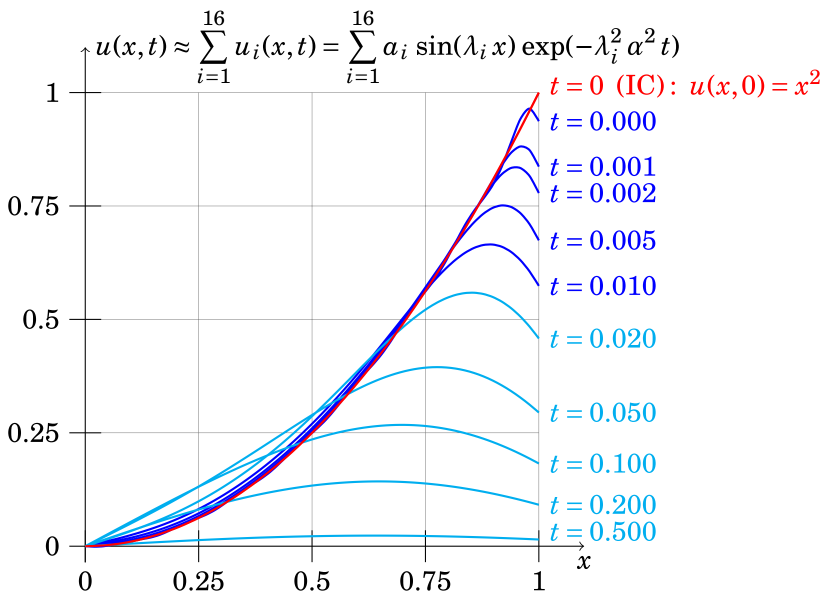

Figure 4: The final solution. (Notice that $f(x) = x^2$ does not satisfy the BC at $x = 1$.)