1. The stress power is defined to be $S_p = \int_R tr(\boldsymbol{\sigma} \cdot \boldsymbol{d}) dV$ where $\boldsymbol{\sigma}$ is the Cauchy stress and $\boldsymbol{d} = \boldsymbol{L}_{sym}$.

Other measures of stress are

$\boldsymbol{\Sigma} = \boldsymbol{R}^T \cdot \boldsymbol{\sigma} \cdot \boldsymbol{R}$ Rotated Cauchy stress

$\hat{\boldsymbol{P}} = J\boldsymbol{\sigma} \cdot \boldsymbol{F}^{-T}$ Piola-Kirchoff stress of the first kind

$\boldsymbol{P} = J\boldsymbol{F}^{-1} \cdot \boldsymbol{\sigma} \cdot \boldsymbol{F}^{-T} = \boldsymbol{F}^{-1} \cdot \hat{\boldsymbol{P}}$ Piola-Kirchoff stress of the second kind

and other rates of deformation are

$$\boldsymbol{D} = \boldsymbol{F}^T \cdot \boldsymbol{d} \cdot \boldsymbol{F} = \dot{\boldsymbol{E}} \qquad \boldsymbol{D}^* = \boldsymbol{R}^T \cdot \boldsymbol{d} \cdot \boldsymbol{R}$$

Show that alternative expressions for the stress power are:

$$S_p = \int_R tr\,(\boldsymbol{\Sigma} \cdot \boldsymbol{D}^*) dV = \int_{R_o} tr\,(\hat{\boldsymbol{P}} \cdot \dot{\boldsymbol{F}}^T) dV_o = \int_{R_o} tr\,(\boldsymbol{P} \cdot \dot{\boldsymbol{E}}) dV_o$$

These combinations of stress and deformation rates are said to be "conjugate."

2. Recall that in connection with the study of a continuum, tensors could be defined as one of four possibilities: m-m, s-s, s-m, m-s where “m” denotes “material” and “s” denotes “spatial”. Recall the classifications of $\boldsymbol{F}$ and $\boldsymbol{R}$ from the notes. Assume $\boldsymbol{\sigma}$ and $\boldsymbol{d}$ are both s-s. Use the relations given in Prob. 1 to classify the tensors $\boldsymbol{\Sigma}, \hat{\boldsymbol{P}}, \boldsymbol{P}, \boldsymbol{D}$ and $\boldsymbol{D}^*$.

3. A bar of original length L that is initially horizontal (Fig. 1) deforms in a plane as the result of a simultaneous stretch and rotation as indicated in Fig. 2. The end O is fixed in space. The rotation is defined by $\theta = \omega t$ and the elongation of the end of the bar is $\delta_A = \varepsilon L t$ with both ω and ε constant.

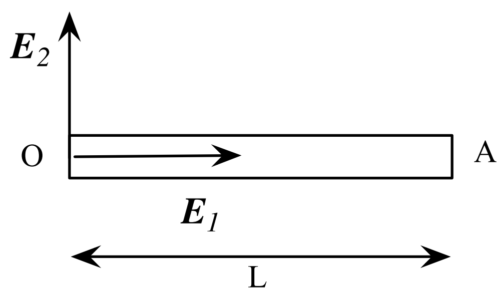

Fig. 1. Initial Position

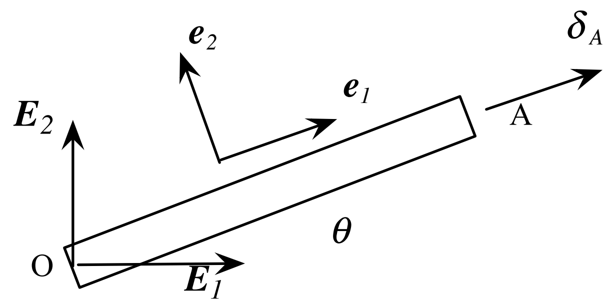

Fig. 2. Deformed position

The easiest way to describe the deformation is to use the two bases. Then the deformation is defined by

$$\boldsymbol{r} = x_i \boldsymbol{e}_i \quad \text{and} \quad \boldsymbol{R} = X_i \boldsymbol{E}_i$$
$$x_1 = X_1(1+\varepsilon t) \qquad x_2 = X_2 \qquad x_3 = X_3 \tag{0-1}$$

The elongation at the end of the bar is $\delta_A = \left(x_1 - X_1\right)\Big|_{X_1=L} = \left(X_1 \varepsilon t\right)\Big|_{X_1=L} = \varepsilon L t$.

3.1 Perform the transformation so that the components of $\boldsymbol{r}$ are given with respect to the basis $\boldsymbol{E}_i$. Let $\boldsymbol{r} = x_i^E \boldsymbol{E}_i$. Use this form to obtain $\boldsymbol{F}, \boldsymbol{R}, \boldsymbol{U}, \dot{\boldsymbol{F}}, \dot{\boldsymbol{R}}, \dot{\boldsymbol{U}}$ and $\boldsymbol{\Omega}$ all with the tensor basis $\boldsymbol{E}_i \otimes \boldsymbol{E}_j$.

3.2 Now use the two bases and the deformation as given by (0-1).
(i) Determine $\boldsymbol{F}, \boldsymbol{R}$ and $\boldsymbol{U}$

(ii) The basis $\boldsymbol{e}_i$ is a function of time. Determine $\dot{\boldsymbol{e}}_i$ and determine the tensor $\boldsymbol{\Omega}^*$ such that $\dot{\boldsymbol{e}}_i = \boldsymbol{\Omega}^* \cdot \boldsymbol{e}_i$.

(iii) Determine $\dot{\boldsymbol{F}}, \dot{\boldsymbol{U}}, \dot{\boldsymbol{R}}$ and $\boldsymbol{\Omega}$

Verify that your results for parts (ii) and (iii) agree with the results of 3.1.

3.3 Obtain $\boldsymbol{v} = \boldsymbol{v}(\boldsymbol{r}, t)$ and $\boldsymbol{L}$, and show that $\dot{\boldsymbol{F}} = \boldsymbol{L} \cdot \boldsymbol{F}$

3.4 Consider an element $\mathrm{d}\mathbf{X} = \mathrm{d}X_1 \mathbf{E}_1$. Determine the vector $\mathbf{U} \cdot \mathrm{d}\mathbf{X}$ and then the vector $\mathbf{R} \cdot (\mathbf{U} \cdot \mathrm{d}\mathbf{X})$.