11. POLAR DECOMPOSITION OF F

11.1 Initial Comments

Now we initiate the formal study of continuum mechanics, which we define to be the study of the equations that describe the deformation of a continuous body subjected to static or dynamic physical and thermal loads. By continuous we mean that no new free surfaces (cracks or voids) are created from the loading process. The implication is that gradients are assumed to exist and are unique.

This Chapter provides the definitions for tensors that are commonly used to describe the deformation of a body that include the deformation gradient, strain tensors, stretch tensors and the rotation tensor

11.2 Polar Decomposition of F

Recall that it is assumed that a positive volume element in the reference configuration maps into a positive volume element in the current configuration. This assumption has the consequence that the determinant of $\boldsymbol{F}$ is positive $(J > 0)$ or that $\boldsymbol{F}$ is not singular. Then $\boldsymbol{F}$ admits the decompositions of

$$\boldsymbol{F} = \hat{\boldsymbol{R}} \cdot \boldsymbol{U} \qquad \boldsymbol{F} = \boldsymbol{V} \cdot \hat{\boldsymbol{R}} \tag{11-1}$$

The right and left stretch tensors, $\boldsymbol{U}$ and $\boldsymbol{V}$ respectively, are positive definite and symmetric. The tensor $\hat{\boldsymbol{R}}$ is orthogonal, or

$$\hat{\boldsymbol{R}} \cdot \hat{\boldsymbol{R}}^T = \boldsymbol{I} \qquad \hat{\boldsymbol{R}}^T \cdot \hat{\boldsymbol{R}} = \boldsymbol{I} \tag{11-2}$$

Suppose $\boldsymbol{u}$ and $\boldsymbol{v}$ are two. If

$$\boldsymbol{a} = \hat{\boldsymbol{R}} \cdot \boldsymbol{u} \qquad \boldsymbol{b} = \hat{\boldsymbol{R}} \cdot \boldsymbol{v} \tag{11-3}$$

then

$$\boldsymbol{a} \cdot \boldsymbol{a} = \boldsymbol{u} \cdot \hat{\boldsymbol{R}}^T \cdot \hat{\boldsymbol{R}} \cdot \boldsymbol{u} = \boldsymbol{u} \cdot \boldsymbol{u} \qquad \boldsymbol{a} \cdot \boldsymbol{b} = \boldsymbol{u} \cdot \hat{\boldsymbol{R}}^T \cdot \hat{\boldsymbol{R}} \cdot \boldsymbol{v} = \boldsymbol{u} \cdot \boldsymbol{v} \tag{11-4}$$

so that the mapping associated with the rotation tensor preserves the magnitudes of and angles between vectors, i.e., the tensor $\hat{\boldsymbol{R}}$ merely rotates vectors.

With the assumption that (11-1) holds, it follows that

$$\boldsymbol{F}^T \cdot \boldsymbol{F} = \boldsymbol{U}^T \cdot \hat{\boldsymbol{R}}^T \cdot \hat{\boldsymbol{R}} \cdot \boldsymbol{U} = \boldsymbol{U}^2 \qquad \boldsymbol{F} \cdot \boldsymbol{F}^T = \boldsymbol{V} \cdot \hat{\boldsymbol{R}} \cdot \hat{\boldsymbol{R}}^T \cdot \boldsymbol{V}^T = \boldsymbol{V}^2 \tag{11-5}$$

in which the properties of orthogonality and symmetry have been used. A common notation is to let

$$\boldsymbol{C} = \boldsymbol{F}^T \cdot \boldsymbol{F} = \boldsymbol{U}^2 \qquad \boldsymbol{B} = \boldsymbol{F} \cdot \boldsymbol{F}^T = \boldsymbol{V}^2 \tag{11-6}$$

where $\boldsymbol{C}$ and $\boldsymbol{B}$ are the right and left Cauchy-Green tensors, respectively.

Now we obtain explicit expressions for $\boldsymbol{U}$ and $\hat{\boldsymbol{R}}$ with $\boldsymbol{F}$ given. Recall that $d\boldsymbol{r} = \boldsymbol{F} \cdot d\boldsymbol{R}$ and for any $d\boldsymbol{R}$ the resulting vector $d\boldsymbol{r}$ is never zero based on the assumption that the stretch, Λ_{st}, is always positive. Therefore

$$d\boldsymbol{R} \cdot \boldsymbol{C} \cdot d\boldsymbol{R} = ds^2 > 0 \quad \forall \quad \mathrm{d}\boldsymbol{R} \neq \boldsymbol{0} \tag{11-7}$$

or $\boldsymbol{C}$ is positive definite. It follows that if the positive square root is taken, $\boldsymbol{U}$ and $\hat{\boldsymbol{R}}$ are uniquely defined to be

$$\boldsymbol{U} = (\boldsymbol{F}^T \cdot \boldsymbol{F})^{1/2} \qquad \hat{\boldsymbol{R}} = \boldsymbol{F} \cdot \boldsymbol{U}^{-1} \tag{11-8}$$

With the assumption that $\boldsymbol{V} = \boldsymbol{V} \cdot \hat{\boldsymbol{R}}^*$, we obtain explicit expressions for $\boldsymbol{V}$ and $\hat{\boldsymbol{R}}^*$ in a similar manner:

$$\boldsymbol{V} = \boldsymbol{B}^{1/2} \qquad \hat{\boldsymbol{R}}^* = \boldsymbol{V}^{-1} \cdot \boldsymbol{F} \tag{11-9}$$

To show that $\hat{\boldsymbol{R}}^* = \hat{\boldsymbol{R}}$, we perform the following operations

$$\boldsymbol{F} = \boldsymbol{V} \cdot \hat{\boldsymbol{R}}^* = (\boldsymbol{I}) \cdot \boldsymbol{V} \cdot \hat{\boldsymbol{R}}^* = (\hat{\boldsymbol{R}}^* \cdot \hat{\boldsymbol{R}}^*) \cdot \boldsymbol{V} \cdot \hat{\boldsymbol{R}}^* = \hat{\boldsymbol{R}}^* \cdot (\hat{\boldsymbol{R}}^* \cdot \boldsymbol{V} \cdot \hat{\boldsymbol{R}}^*) = \hat{\boldsymbol{R}} \cdot \boldsymbol{U} \tag{11-10}$$

Since $\boldsymbol{U}$ and $\hat{\boldsymbol{R}}$ are unique it follows that

$$\hat{\boldsymbol{R}}^* = \hat{\boldsymbol{R}} \qquad \boldsymbol{U} = \hat{\boldsymbol{R}}^T \cdot \boldsymbol{V} \cdot \hat{\boldsymbol{R}} \tag{11-11}$$

Recall that the deformation gradient maps a line element in the undeformed configuration to the corresponding material line element in the deformed configuration, or

$$d\boldsymbol{r} = \boldsymbol{F} \cdot d\boldsymbol{R} = \hat{\boldsymbol{R}} \cdot \boldsymbol{U} \cdot d\boldsymbol{R} = \boldsymbol{V} \cdot \hat{\boldsymbol{R}} \cdot d\boldsymbol{R} \tag{11-12}$$

Let

$$d\boldsymbol{R}^* = \boldsymbol{U} \cdot d\boldsymbol{R} \qquad d\boldsymbol{r}^* = \hat{\boldsymbol{R}} \cdot d\boldsymbol{r} \tag{11-13}$$

where $d\boldsymbol{R}^*$ is obtained from $d\boldsymbol{R}$ by a pure stretch, and $d\boldsymbol{r}^*$ is obtained from $d\boldsymbol{r}$ by a pure rotation. Then (11-12) becomes

$$d\boldsymbol{r} = \hat{\boldsymbol{R}} \cdot d\boldsymbol{R}^* \qquad d\boldsymbol{r} = \boldsymbol{V} \cdot d\boldsymbol{r}^* \tag{11-14}$$

Thus, the decomposition $\boldsymbol{F} = \hat{\boldsymbol{R}} \cdot \boldsymbol{U}$ can be considered as producing a stretch followed by a rotation; while $\boldsymbol{F} = \boldsymbol{V} \cdot \hat{\boldsymbol{R}}$ produces a rotation followed by a stretch.

11.3 Principal Stretches and Logarithmic Strains

It follows from (11-1) that

$$\hat{\boldsymbol{R}} \cdot \boldsymbol{U} = \boldsymbol{V} \cdot \hat{\boldsymbol{R}} \qquad \boldsymbol{U} = \hat{\boldsymbol{R}}^T \cdot \boldsymbol{V} \cdot \hat{\boldsymbol{R}} \qquad \boldsymbol{V} = \hat{\boldsymbol{R}} \cdot \boldsymbol{U} \cdot \hat{\boldsymbol{R}}^T \tag{11-15}$$

Let $(\Lambda, \boldsymbol{N})$ and $(\lambda, \boldsymbol{n})$ be eigenpairs of $\boldsymbol{U}$ and $\boldsymbol{V}$, respectively. Then the spectral decompositions of these two tensors are

$$\boldsymbol{U} = \sum_{i=1}^{3} \Lambda_i \boldsymbol{N}_i \otimes \boldsymbol{N}_i \qquad \boldsymbol{V} = \sum_{i=1}^{3} \lambda_i \boldsymbol{n}_i \otimes \boldsymbol{n}_i \tag{11-16}$$

We use (11-15) to obtain

$$\boldsymbol{V} = \sum_{i=1}^{3} \Lambda_i (\hat{\boldsymbol{R}} \cdot \boldsymbol{N}_i) \otimes (\boldsymbol{N}_i \cdot \hat{\boldsymbol{R}}^T) = \sum_{i=1}^{3} \lambda_i \boldsymbol{n}_i \otimes \boldsymbol{n}_i \tag{11-17}$$

When an orthogonal tensor operates on an orthonormal basis, the result is simply a rotation of the base vectors with lengths and angles between the vectors unchanged. It follows that the eigenvalues and eigenvector of the right and left stretch tensors are related as follows:

$$\lambda_i = \Lambda_i \qquad \boldsymbol{n}_i = \hat{\boldsymbol{R}} \cdot \boldsymbol{N}_i \qquad \boldsymbol{N}_i = \hat{\boldsymbol{R}}^T \cdot \boldsymbol{n}_i \tag{11-18}$$

Also, we note that an alternative expression for the rotation tensor is

$$\hat{\boldsymbol{R}} = \boldsymbol{n}_i \otimes \boldsymbol{N}_i \tag{11-19}$$

The use of (11-16), (11-18) and (11-19) allow us to perform a spectral decomposition of the deformation gradient as follows:

$$\boldsymbol{F} = \hat{\boldsymbol{R}} \cdot \boldsymbol{U} = \left(\boldsymbol{n}_j \otimes \boldsymbol{N}_j\right) \cdot \sum_{i=1}^{3} \Lambda_i \boldsymbol{N}_i \otimes \boldsymbol{N}_i = \sum_{i=1}^{3} \Lambda_i \boldsymbol{n}_i \otimes \boldsymbol{N}_i \tag{11-20}$$

Since $\boldsymbol{F}$ is not symmetric, it has left and right eigenvectors that are $\boldsymbol{n}_i$ and $\boldsymbol{N}_i$, respectively, with eigenvalues Λ_i. At this time we will not pursue the topic of the eigenproblem for tensors that are not symmetric.

As a closing note, we emphasize the point that both the eigenvalues and the eigenvectors of $\boldsymbol{U}$ and $\boldsymbol{V}$ will change whenever the deformation changes.

Recall that the stretch is defined to be

$$\Lambda_{st} = \frac{ds}{dS_0} = \{\boldsymbol{t}_0 \cdot \boldsymbol{F}^T \cdot \boldsymbol{F} \cdot \boldsymbol{t}_0\}^{1/2} \tag{11-21}$$

With the use of (11-6) this relation becomes

$$\Lambda_{st} = \{\boldsymbol{t}_0 \cdot \boldsymbol{U}^2 \cdot \boldsymbol{t}_0\}^{1/2} \tag{11-22}$$

Now choose the unit tangent vector, $\boldsymbol{t}_0$, to be $\boldsymbol{N}_i$, an eigenvector of $\boldsymbol{U}$. It follows from (11-16) that $\Lambda_{st} = \Lambda_{(i)}$, the associated eigenvalue of $\boldsymbol{U}$. The set of eigenvalues, Λ_i, are called the principal stretches of the deformation because they are the stretches for fibers that are originally in the principal directions of $\boldsymbol{U}$. Recall that $\boldsymbol{U}$ is positive definite so all of the eigenvalues are positive which satisfies the requirement that all stretches be positive.

Recall that ligaments do not change length if the stretch is unity. Typically measures of deformation called strains are defined such that the strain is zero if the stretch is one. Logarithmic strains satisfy this property and are based on the use of the spectral decompositions of the stretch tensors as follows:

$$\boldsymbol{L}_U = ln(\boldsymbol{U}) \equiv \sum_{i=1}^{3} (ln\,\Lambda_i)\boldsymbol{N}_i \otimes \boldsymbol{N}_i \qquad \boldsymbol{L}_V = ln(\boldsymbol{V}) \equiv \sum_{i=1}^{3} (ln\,\lambda_i)\boldsymbol{n}_i \otimes \boldsymbol{n}_i \tag{11-23}$$

If a principal stretch is one, the corresponding logarithmic term is zero. For stretches less (larger) than one, the logarithm is negative (positive).

11.4 Spatial, Material and Mixed Tensors

The deformation gradient will vary, in general from one material point to another, or $\boldsymbol{F} = \boldsymbol{F}(\boldsymbol{R})$. Suppose we superpose a constant rotation, defined by the orthogonal tensor, $\boldsymbol{Q}$, on all material points in the body, or stated otherwise, apply an additional rigid-body rotation to the body. How do our various tensors that describe the deformation of a body change due to this added rotation?

The new differential of the position vector due to the rotation is simply

$$d\boldsymbol{r}^* = \boldsymbol{Q} \cdot d\boldsymbol{r} \tag{11-24}$$

so that the new deformation gradient, $\boldsymbol{F}^*$, is defined to be

$$d\boldsymbol{r}^* = \boldsymbol{F}^* \cdot d\boldsymbol{R} = \boldsymbol{Q} \cdot \boldsymbol{F} \cdot d\boldsymbol{R} \tag{11-25}$$

It follows that $\boldsymbol{F}^*$, its inverse, and their transposes must satisfy the relations

$$\boldsymbol{F}^* = \boldsymbol{Q} \cdot \boldsymbol{F} \qquad \boldsymbol{F}^{*T} = \boldsymbol{F}^T \cdot \boldsymbol{Q}^T \qquad \boldsymbol{F}^{*-1} = \boldsymbol{F}^{-1} \cdot \boldsymbol{Q}^T \qquad \boldsymbol{F}^{*-T} = \boldsymbol{Q} \cdot \boldsymbol{F}^{-T} \tag{11-26}$$

in which we have used the orthogonality condition that $\boldsymbol{Q}^{-1} = \boldsymbol{Q}^T$. The polar decomposition involving the right stretch becomes

$$\boldsymbol{F}^* = \boldsymbol{R}^* \cdot \boldsymbol{U}^* = \boldsymbol{Q} \cdot \boldsymbol{F} = \boldsymbol{Q} \cdot \hat{\boldsymbol{R}} \cdot \boldsymbol{U} \tag{11-27}$$

We note that the product $\boldsymbol{Q} \cdot \hat{\boldsymbol{R}}$ is orthogonal and recall that the rotation is unique. Therefore, we draw the conclusions that

$$\hat{\boldsymbol{R}}^* = \boldsymbol{Q} \cdot \hat{\boldsymbol{R}} \qquad \boldsymbol{U}^* = \boldsymbol{U} \tag{11-28}$$

Now consider the decomposition involving the left stretch:

$$\boldsymbol{F}^* = \boldsymbol{V}^* \cdot \hat{\boldsymbol{R}}^* = \boldsymbol{V}^* \cdot \boldsymbol{Q} \cdot \hat{\boldsymbol{R}} = \boldsymbol{Q} \cdot \boldsymbol{F} = \boldsymbol{Q} \cdot \boldsymbol{V} \cdot \hat{\boldsymbol{R}} = \left(\boldsymbol{Q} \cdot \boldsymbol{V} \cdot \boldsymbol{Q}^T\right) \cdot \boldsymbol{Q} \cdot \hat{\boldsymbol{R}} \tag{11-29}$$

from which we draw the conclusion that

$$\boldsymbol{V}^* = \boldsymbol{Q} \cdot \boldsymbol{V} \cdot \boldsymbol{Q}^T \tag{11-30}$$

From the above we see that there are four general types of transformations that occur when a rigid-body rotation is superimposed on an existing deformation. Tensors such as $\boldsymbol{U}$ that remain unchanged are called “material” (or material-material) tensors.

Tensors that change such as $\boldsymbol{V}$ in (11-30) are called spatial (or spatial-spatial) tensors. Tensors that are modified with just a $\boldsymbol{Q}$ on the left are called spatial-material mixed. Those that are modified by a $\boldsymbol{Q}^T$ on the right are material-spatial mixed. We summarize these cases as follows:

Type	Label	Transformation	Examples	
Material-Material	$\overset{m-m}{\boldsymbol{T}}$	$\overset{m-m}{\boldsymbol{T}}{}^* = \overset{m-m}{\boldsymbol{T}}$	$\boldsymbol{U}$	
Spatial-Spatial	$\overset{s-s}{\boldsymbol{T}}$	$\overset{s-s}{\boldsymbol{T}}{}^* = \boldsymbol{Q}\cdot\overset{s-s}{\boldsymbol{T}}\cdot\boldsymbol{Q}^T$	$\boldsymbol{V}$	(11-31)
Material-Spatial	$\overset{m-s}{\boldsymbol{T}}$	$\overset{m-s}{\boldsymbol{T}}{}^* = \overset{m-s}{\boldsymbol{T}}\cdot\boldsymbol{Q}^T$	$\boldsymbol{F}^T, \hat{\boldsymbol{R}}, \boldsymbol{F}^{-1}$	
Spatial-Material	$\overset{s-m}{\boldsymbol{T}}$	$\overset{s-m}{\boldsymbol{T}}{}^* = \boldsymbol{Q}\cdot\overset{m-m}{\boldsymbol{T}}$	$\boldsymbol{F}, \hat{\boldsymbol{R}}, \boldsymbol{F}^{-T}$	

Note that the labels over the tensors describe the type of tensor and have nothing to do with components for which a similar label was used previously. Now we have the task of identifying both the type of tensor and the set of components being used.

The use of material tensors is of particular significance when formulating constitutive equations that must satisfy the principle of material frame indifference.

11.5 Euler-Rodriguez Formula

Let the unit vector $\boldsymbol{a}$ denote an axis of rotation. Then the Euler-Rodriguez formula states that the rotation tensor $\hat{\boldsymbol{R}}$ is represented as follows:

$$\hat{\boldsymbol{R}} = cos\theta\left(\boldsymbol{I}-\boldsymbol{a}\otimes\boldsymbol{a}\right)+\boldsymbol{a}\otimes\boldsymbol{a}+sin\theta\left(\boldsymbol{\varepsilon}\cdot\boldsymbol{a}\right) \tag{11-32}$$

in which θ denotes the angle of rotation. In words, any rotation tensor can be described as a physical rotation, θ, following the right-hand rule about a particular oriented line identified by the unit vector $\boldsymbol{a}$. [Exercise: Show that $\hat{\boldsymbol{R}}\cdot\hat{\boldsymbol{R}}^T = \boldsymbol{I}$.]

Now we obtain the eigenpairs of $\hat{\boldsymbol{R}}$. First we note that

$$\boldsymbol{a}\cdot\hat{\boldsymbol{R}} = \hat{\boldsymbol{R}}\cdot\boldsymbol{a} = \boldsymbol{a} \tag{11-33}$$

It follows that $\boldsymbol{a}$ is both a left and right eigenvector of $\boldsymbol{R}$ with eigenvalue one. Consider an orthonormal basis $\boldsymbol{e}_i'$ and choose $\boldsymbol{e}_3' = \boldsymbol{a}$. Then $\boldsymbol{I}-\boldsymbol{e}_3'\otimes\boldsymbol{e}_3' = \boldsymbol{e}_1'\otimes\boldsymbol{e}_1'+\boldsymbol{e}_2'\otimes\boldsymbol{e}_2'$. Let $c = cos\theta$ and $s = sin\theta$. Then (11-32) becomes

$$\begin{aligned}\hat{\mathbf{R}} &= \mathrm{c}(\mathbf{e}_1'\otimes\mathbf{e}_1'+\mathbf{e}_2'\otimes\mathbf{e}_2')+\mathrm{s}\varepsilon_{ij3}\mathbf{e}_i'\otimes\mathbf{e}_j'+\mathbf{e}_3'\otimes\mathbf{e}_3' \\ &= \mathrm{c}(\mathbf{e}_1'\otimes\mathbf{e}_1'+\mathbf{e}_2'\otimes\mathbf{e}_2')+\mathrm{s}(\mathbf{e}_1'\otimes\mathbf{e}_2'-\mathbf{e}_2'\otimes\mathbf{e}_1')+\mathbf{e}_3'\otimes\mathbf{e}_3'\end{aligned} \tag{11-34}$$

and the matrix of components is

$$\left[\hat{\mathrm{R}}\right]^{e'-e'} = \begin{bmatrix} c & s & 0 \\ -s & c & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{11-35}$$

It follows that the eigenvalues are $\lambda_{1,2} = c \pm is = e^{\pm i\theta}$ and $\lambda_3 = 1$ with $i = \sqrt{-1}$. The first two eigenvectors are complex conjugate and perpendicular to **a**. The complete set of eigenvectors is

$$\begin{aligned} &\mathbf{e}_1^R = s(\mathbf{e}_1' + i\mathbf{e}_2') \qquad \mathbf{e}_2^R = s(\mathbf{e}_1' - i\mathbf{e}_2') \\ &\mathbf{e}_1^L = s(\mathbf{e}_1' - i\mathbf{e}_2') \qquad \mathbf{e}_2^L = s(\mathbf{e}_1' + i\mathbf{e}_2') \\ &\mathbf{e}_3^R = \mathbf{e}_3^L = \mathbf{e}_3' \end{aligned} \tag{11-36}$$

Note that $\mathbf{e}_1^R = \mathbf{e}_2^L$ and $\mathbf{e}_2^R = \mathbf{e}_1^L$.

For a given $\hat{\boldsymbol{R}}$, the terms in the Euler-Rodrigues formula are obtained as follows. First, $\boldsymbol{a}$ is obtained by determining the eigenvector of $\hat{\boldsymbol{R}}$ with unit eigenvalue. Then we note that $2\cos\theta = tr\left(\hat{\boldsymbol{R}}\right)$ from which θ is obtained.

11.6 Summary

Here we summarize the key relations developed in this Chapter. First we have the polar decomposition of the deformation gradient and equations for obtaining the stretch and rotation tensors:

$$\begin{aligned} &\boldsymbol{F} = \hat{\boldsymbol{R}} \cdot \boldsymbol{U} \qquad\qquad \boldsymbol{F} = \boldsymbol{V} \cdot \hat{\boldsymbol{R}} \\ &\boldsymbol{U} = (\boldsymbol{F}^T \cdot \boldsymbol{F})^{1/2} \qquad \boldsymbol{V} = (\boldsymbol{F} \cdot \boldsymbol{F}^T)^{1/2} \qquad \hat{\boldsymbol{R}} = \boldsymbol{F} \cdot \boldsymbol{U}^{-1} = \boldsymbol{V}^{-1} \cdot \boldsymbol{F} \end{aligned} \tag{11-37}$$

The spectral decompositions of the stretch tensors resulted in the relations

$$\boldsymbol{U} = \sum_{i=1}^{3} \Lambda_i \boldsymbol{N}_i \otimes \boldsymbol{N}_i \qquad \boldsymbol{V} = \sum_{i=1}^{3} \Lambda_i \boldsymbol{n}_i \otimes \boldsymbol{n}_i \qquad \boldsymbol{n}_i = \hat{\boldsymbol{R}} \cdot \boldsymbol{N}_i \tag{11-38}$$

The eigenvalues are called the principal stretches. An alternative form for the rotation tensor is

$$\hat{\boldsymbol{R}} = \boldsymbol{n}_i \otimes \boldsymbol{N}_i \tag{11-39}$$

Logarithmic strains are defined to be

$$\boldsymbol{L}_U = ln(\boldsymbol{U}) \equiv \sum_{i=1}^{3} (ln\,\Lambda_i)\boldsymbol{N}_i \otimes \boldsymbol{N}_i \qquad \boldsymbol{L}_V = ln(\boldsymbol{V}) \equiv \sum_{i=1}^{3} (ln\,\lambda_i)\boldsymbol{n}_i \otimes \boldsymbol{n}_i \tag{11-40}$$

The Euler-Rodriguez formula is

$$\hat{\boldsymbol{R}} = cos\theta\left(\boldsymbol{I} - \boldsymbol{a} \otimes \boldsymbol{a}\right) + \boldsymbol{a} \otimes \boldsymbol{a} + sin\theta\left(\boldsymbol{\varepsilon} \cdot \boldsymbol{a}\right) \tag{11-41}$$

The equations that characterize material, spatial and mixed tensors are given in (11-31).