Robert Malakhov

Dr. Tjiptowidjojo

CBE 521

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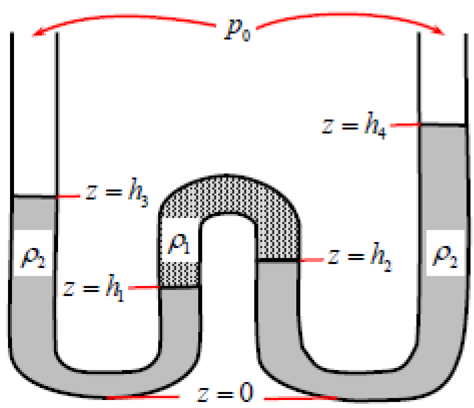
Homework 2

1. Fluid with unknown density ϱ_1 is measured with the system shown in the figure shown below. Derive an expression of $\gamma = \frac{\rho_1}{\rho_2}$ in terms of h_1, h_2, h_3, and h_4.



Taking the mechanical engineer's approach,

@ $z = h_4$ $\qquad P_{z=h_4} = P_4 = P_0$

@ $z = h_3$ $\qquad P_{z=h_3} = P_3 = P_0$

@ $z = h_2$ $\qquad P_{z=h_2} = P_2 = P_4 + \rho_2 g h_{4-2} \Rightarrow P_0 + \rho_2 g(h_4 - h_2)$

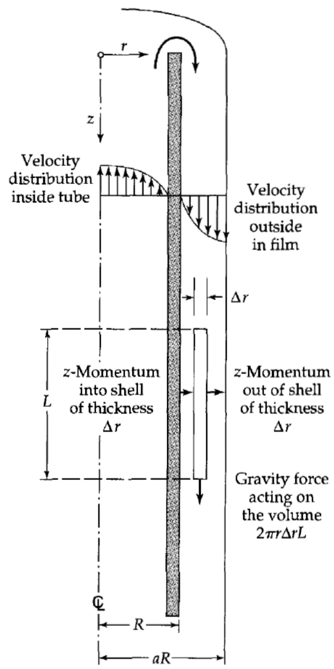
@ $z = h_{1-right}$ $\qquad P_{z=h_1} = P_1 = P_2 + \rho_1 g h_{2-1} \Rightarrow P_0 + \rho_2 g(h_4 - h_2) + \rho_1 g(h_2 - h_1)$

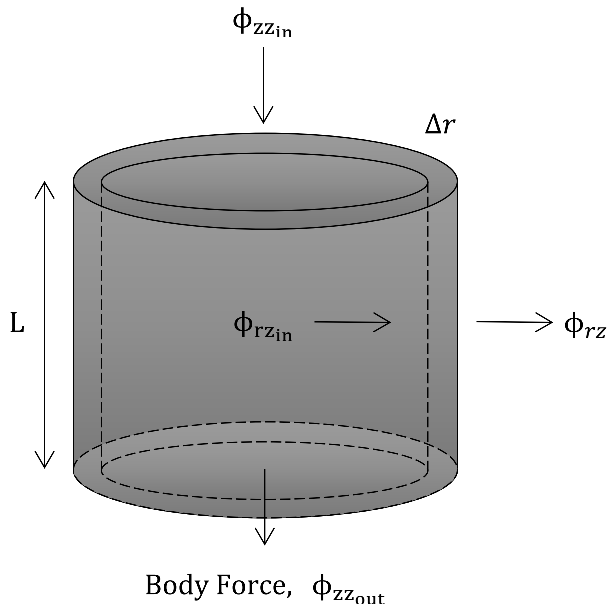
@ $z = h_{1-left}$ $\qquad P_{z=h_1} = P_1 = P_3 + \rho_2 g h_{3-1} \Rightarrow P_0 + \rho_2 g(h_3 - h_1)$

Setting left and right equal and solving for $\frac{\rho_1}{\rho_2}$,

$$P_0 + \rho_2 g(h_3 - h_1) = P_0 + \rho_2 g(h_4 - h_2) + \rho_1 g(h_2 - h_1)$$
$$P_0 - P_0 + \rho_2 g(h_3 - h_1) - \rho_2 g(h_4 - h_2) = \rho_1 g(h_2 - h_1)$$
$$\rho_2[(h_3 - h_1) - (h_4 - h_2)] = \rho_1[(h_2 - h_1)]$$
$$\frac{\rho_1}{\rho_2} = \frac{(h_3 - h_1) - (h_4 - h_2)}{(h_2 - h_1)}$$

2. Flow of a film on the outside of a circular tube. In a gas absorption experiment a viscous fluid flows upward through a small circular tube and then downward in laminar flow on the outside. Set up a momentum balance over a shell of thickness Δr in the film, as shown.





Note that the "momentum in" and "momentum out" arrows are always taken in the positive coordinate direction, even though in this problem the momentum is flowing through the cylindrical surfaces in the negative r direction.

a. Show that the velocity distribution in the falling film (neglecting end effects) is

$$v_z = \frac{\rho g R^2}{4\mu}\left[1 - \left(\frac{r}{R}\right)^2 + 2a^2 \ln\left(\frac{r}{R}\right)\right]$$

Setting up the momentum balance for a shell shown above yields the following,

$$2\pi r L(\phi_{rz}|_r - \phi_{rz}|_{r+\Delta r}) + 2\pi r \Delta r(\phi_{zz}|_{z=0} - \phi_{zz}|_{z=L}) + 2\pi r \Delta r L(\rho g) = 0$$

Simplifying and taking the limit,

$$2\pi r L(\phi_{rz}|_{r+\Delta r} - \phi_{rz}|_r) - 2\pi r \Delta r(\phi_{zz}|_{z=0} - \phi_{zz}|_{z=L}) = 2\pi r \Delta r L(\rho g)$$

$$r\frac{(\phi_{rz}|_{r+\Delta r} - \phi_{rz}|_r)}{\Delta r} - r\frac{(\phi_{zz}|_{z=0} - \phi_{zz}|_{z=L})}{L} = \rho g r$$

$$\lim_{\Delta r \to \infty} r\frac{(\phi_{rz}|_{r+\Delta r} - \phi_{rz}|_r)}{\Delta r} - r\frac{(\phi_{zz}|_{z=0} - \phi_{zz}|_{z=L})}{L} = \rho g r$$

$$\frac{\partial r\phi_{rz}}{\partial r} - r\frac{(\phi_{zz}|_{z=0} - \phi_{zz}|_{z=L})}{L} = \rho g r$$

Writing the components out explicitly using Appendix B.1,

$$\phi_{rz} = \tau_{rz} + \rho v_r v_z = -\mu \frac{\partial v_z}{\partial r} + \rho v_r v_z$$

$$\phi_{zz} = p + \tau_{zz} + \rho v_z v_z = p - 2\mu \frac{\partial v_z}{\partial z} + \rho v_z v_z$$

Applying the following conditions,

$$v_r = 0$$

$$v_z = v_z(r)$$

$$p = p(r)$$

The components and momentum balance becomes,

$$\phi_{rz} = \tau_{rz} = -\mu \frac{\partial v_z}{\partial r}$$

$$\phi_{zz} = 0$$

$$\frac{\partial r\tau_{rz}}{\partial r} = \rho g r$$

$$-\mu \frac{d}{dr}\left(r \frac{dv_z}{dr}\right) = \rho g r$$

Integrating the momentum balance twice and applying boundary conditions,

$$-\mu r \frac{dv_z}{dr} = \frac{\rho g r^2}{2} + c_1$$

$$\frac{dv_z}{dr} = -\frac{\rho g r}{2\mu} + c_1 \frac{1}{r}$$

$$v_z = -\frac{\rho g r^2}{4\mu} + c_1 \ln r + c_2$$

$$At\ r = R, v_z = 0$$

$$c_2 = \frac{\rho g R^2}{4\mu} - c_1 \ln R$$

$$At\ r = aR, dv_z/dr = 0$$

$$c_1 = \frac{\rho g a R^2}{2\mu}$$

$$v_z = \frac{\rho g R^2}{4\mu}\left[1 - \left(\frac{r}{R}\right)^2 + 2a^2 \ln \frac{r}{R}\right]$$

b. Obtain an expression for the mass rate of flow in the film.

The mass rate of flow is the velocity integrated over the entire cross section,

$$\omega = \int_0^{2\pi} \int_R^{aR} \rho v_z r dr d\theta$$

First, the angular integration is performed because it's easy,

$$\omega = 2\pi \int_R^{aR} \rho v_z r dr$$

Next, the radial component is non-dimensionalized with $\zeta = r/R$ and integrated,

$$\omega = 2\pi\rho g R^2 \int_1^a v_z \zeta d\zeta$$

$$\omega = \frac{\pi\rho^2 g R^4}{2\mu} \int_1^a [1 - \zeta^2 + 2a^2 \ln\zeta]\zeta d\zeta$$

$$\omega = \frac{\pi\rho^2 g R^4}{2\mu} \int_1^a \zeta - \zeta^3 + 2a^2 \zeta \ln\zeta \, d\zeta$$

$$\omega = \frac{\pi\rho^2 g R^4}{2\mu} \left[\frac{1}{2}\zeta^2 - \frac{1}{4}\zeta^4 + 2a^2\left(-\frac{1}{4}\zeta^2 + \frac{1}{2}\zeta^2 \ln\zeta\right)\right]\Bigg|_1^a$$

$$\omega = \frac{\pi\rho^2 g R^4}{2\mu} \left\{\left[\frac{1}{2}a^2 - \frac{1}{4}a^4 + 2a^2\left(-\frac{1}{4}a^2 + \frac{1}{2}a^2 \ln a\right)\right] - \left[\frac{1}{2}1^2 - \frac{1}{4}1^4 + 2a^2\left(-\frac{1}{4}1^2 + \frac{1}{2}1^2 \ln 1\right)\right]\right\}$$

$$\omega = \frac{\pi\rho^2 g R^4}{8\mu}(-1 + 4a^2 - 3a^4 + 4a^4 \ln a)$$

c. Show that the result (b) simplifies to $\omega = \frac{\rho^2 g W \delta^3 \cos\beta}{3\mu}$ if the film thickness is very small.

If film thickness is very small, the value is set such that $a = 1 + \varepsilon$,

$$\omega = \frac{\pi\rho^2 g R^4}{8\mu}\left(-1 + 4(1+\varepsilon)^2 - 3(1+\varepsilon)^4 + 4(1+\varepsilon)^4 \ln(1+\varepsilon)\right)$$

$$\omega = \frac{\pi\rho^2 g R^4}{8\mu}\Bigg(-3\varepsilon^4 - 12\varepsilon^3 - 14\varepsilon^2 - 4\varepsilon$$
$$+ (4\varepsilon^4 + 16\varepsilon^3 + 24\varepsilon^2 + 16\varepsilon + 4)\left(\varepsilon - \frac{\varepsilon^2}{2} + \frac{\varepsilon^3}{3} - \frac{\varepsilon^4}{4} + \cdots\right)\Bigg)$$

For the case of a vertically aligned cylindrical shell,

$$W = 2\pi R$$

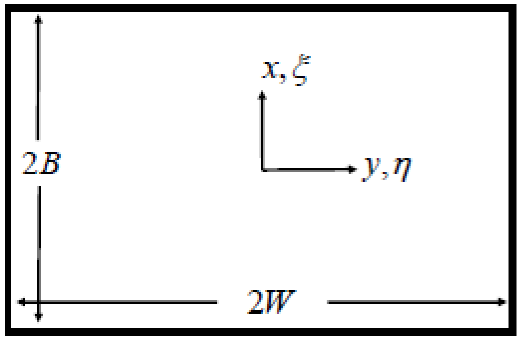
$$\delta = \varepsilon R$$

$$\cos\beta = 1$$

Thus, the mass rate of flow can be approximated,

$$\omega = \frac{2\pi\rho^2 g R^4}{16\mu}\left(\frac{16}{3}\varepsilon^3\right) \approx \frac{\rho^2 g W \delta^3 \cos\beta}{3\mu}$$

3. Steady parallel rectilinear flow in a duct of rectangular cross-section.



a. Show that dimensionless v, i.e. v_z (x,y) scaled with B^2(-dP/dz)/μ, satisfies a Poisson equation and boundary conditions:

$$\frac{\partial^2 v}{\partial \xi^2} + \frac{\partial^2 v}{\partial \eta^2} = -1, \quad v(\pm 1, \eta) = 0, \quad v(\xi, \pm r) = 0.$$

Dimensionless lengths are defined as follows: $\xi \equiv x/B\,, \eta \equiv y/B\,, and\ r \equiv W/B.$

Starting with the continuity and momentum equations in z,

$$\frac{\partial v_z}{\partial z} = 0$$

$$\rho v_z \frac{\partial v_z}{\partial z} = -\frac{\partial P}{\partial z} + \mu \left[\frac{\partial^2 v_z}{\partial x^2} + \frac{\partial^2 v_z}{\partial y^2} + \frac{\partial^2 v_z}{\partial z^2}\right]$$

$$\frac{\partial P}{\partial z}/\mu = \frac{\partial^2 v_z}{\partial x^2} + \frac{\partial^2 v_z}{\partial y^2}, \quad v(\pm B, y) = 0, \quad v(x, \pm W) = 0$$

The dimensionless parameters are introduced and substituted,

$$v \equiv \frac{v_z}{B^2\left(-\frac{dP}{dz}\right)/\mu}, \quad \xi \equiv x/B\,, \quad \eta \equiv y/B\,, \quad r \equiv W/B$$

$$\frac{\partial P}{\partial z}/\mu = \frac{\partial^2\left(\left[B^2\left(-\frac{dP}{dz}\right)/\mu\right]v\right)}{\partial (B\xi)^2} + \frac{\partial^2\left(\left[B^2\left(-\frac{dP}{dz}\right)/\mu\right]v\right)}{\partial (B\eta)^2}$$

$$\frac{\partial P}{\partial z}/\mu = \frac{B^2\left(-\frac{dP}{dz}\right)/\mu}{B^2}\frac{\partial^2 v}{\partial \xi^2} + \frac{B^2\left(-\frac{dP}{dz}\right)/\mu}{B^2}\frac{\partial^2 v}{\partial \eta^2}$$

$$\frac{\partial P}{\partial z}/\mu = \left(-\frac{dP}{dz}\right)/\mu\left[\frac{\partial^2 v}{\partial \xi^2} + \frac{\partial^2 v}{\partial \eta^2}\right]$$

$$\frac{\partial^2 v}{\partial \xi^2} + \frac{\partial^2 v}{\partial \eta^2} = -1, \quad v(\pm 1, \eta) = 0, \quad v(\xi, \pm r) = 0$$

b. Show that the solution can be written as the sum of $v = v_p + v_c$ where v_p is a particular solution which satisfies the equation system except for the boundary conditions at $\eta = \pm r$ and v_c is a complimentary solution which satisfies

$$\frac{\partial^2 v_c}{\partial \xi^2} + \frac{\partial^2 v_c}{\partial \eta^2} = 0, \quad v_c(\pm 1, \eta) = 0, \quad v_c(\xi, \pm r) = -v_p(\xi, \pm r).$$

If the full solution is the following,

$$\frac{\partial^2 v}{\partial \xi^2} + \frac{\partial^2 v}{\partial \eta^2} = -1, \quad v(\pm 1, \eta) = 0, \quad v(\xi, \pm r) = 0$$

It can be written as the sum of two solutions,

$$v = v_c + v_p$$

Where v_p bears the non-homogenous solution,

$$\frac{\partial^2 v_p}{\partial \xi^2} = -1, \quad v_p(\pm 1, \eta) = 0$$

In order to complete the solution, homogeneous solution, v_c, must be such that the sum of the complimentary and particular solutions and their boundary conditions is equal to the full solution,

$$v - v_p = v_c$$

$$\frac{\partial^2 v}{\partial \xi^2} + \frac{\partial^2 v}{\partial \eta^2} - \frac{\partial^2 v_p}{\partial \xi^2} = \frac{\partial^2 v_c}{\partial \xi^2} + \frac{\partial^2 v_c}{\partial \eta^2} \Rightarrow -1 - (-1) = 0$$

$$v(\pm 1, \eta) - v_p(\pm 1, \eta) = v_c(\pm 1, \eta) \Rightarrow 0 - 0 = 0$$

$$v(\xi, \pm r) - v_p(\xi, \pm r) = v_c(\xi, \pm r) \Rightarrow 0 - \alpha = \beta$$

The complimentary solution, v_c, then becomes,

$$\frac{\partial^2 v_c}{\partial \xi^2} + \frac{\partial^2 v_c}{\partial \eta^2} = 0, \quad v_c(\pm 1, \eta) = 0, \quad v_c(\xi, \pm r) = -v_p(\xi, \pm r)$$

c. Show that the particular solution is

$$v_p = \frac{1}{2}(1 - \xi^2).$$

Starting with the differential equation and its boundary condition,

$$\frac{\partial^2 v_p}{\partial \xi^2} = -1, \quad v_p(\pm 1, \eta) = 0$$

Integrating twice and solving for boundary conditions,

$$\frac{\partial v_p}{\partial \xi} = -\xi + c_1$$

$$v_p = -\frac{1}{2}\xi^2 + c_1\xi + c_2$$

$$At\ \xi = +1, v_p = 0 \Rightarrow c_2 = \frac{1}{2} - c_1$$

$$At\ \xi = -1, v_p = 0 \Rightarrow c_1 = 0$$

$$v_p = \frac{1}{2}(1 - \xi^2)$$

d. Obtain the complementary solution v_c.

First, the variables are separated,

$$\frac{\partial^2 v}{\partial \xi^2} + \frac{\partial^2 v}{\partial \eta^2} = 0$$

$$v(\xi, \eta) = \Xi_{(\xi)} \mathrm{H}_{(\eta)}$$

$$\frac{\partial^2 v}{\partial \xi^2} = \mathrm{H}\frac{d^2\Xi}{d\xi^2} = \mathrm{H}\Xi''$$

$$\frac{\partial^2 v}{\partial \mathrm{\eta}^2} = \Xi\frac{d^2\mathrm{H}}{\mathrm{d\eta}^2} = \Xi\mathrm{H}''$$

$$\mathrm{H}\Xi'' + \Xi\mathrm{H}'' = 0 \Rightarrow \mathrm{H}\Xi'' = -\Xi\mathrm{H}'' \Rightarrow \frac{\Xi''}{\Xi} = -\frac{\mathrm{H}''}{\mathrm{H}} \Leftrightarrow -\lambda^2$$

$$\Xi'' + \lambda^2\Xi = 0$$

$$\mathrm{H}'' - \lambda^2\mathrm{H} = 0$$

Second, the boundary conditions are translated,

$$\Xi(-1) = 0$$

$$\Xi(1) = 0$$

$$\mathrm{H}(-1) = 0$$

$$\mathrm{H}(1) = 0$$

Third, both of the equations are solved with BSL Table-C.1,

$$\Xi = c_1 \cos \lambda\xi + c_2 \sin \lambda\xi$$

$$At\ \xi = -1 \rightarrow \Xi = 0 \Rightarrow c_1 = -c_2 \tan \lambda$$

$$At\ \xi = 1 \rightarrow \Xi = 0 \Rightarrow 0 = -c_2 \sin \lambda + c_2 \sin \lambda$$

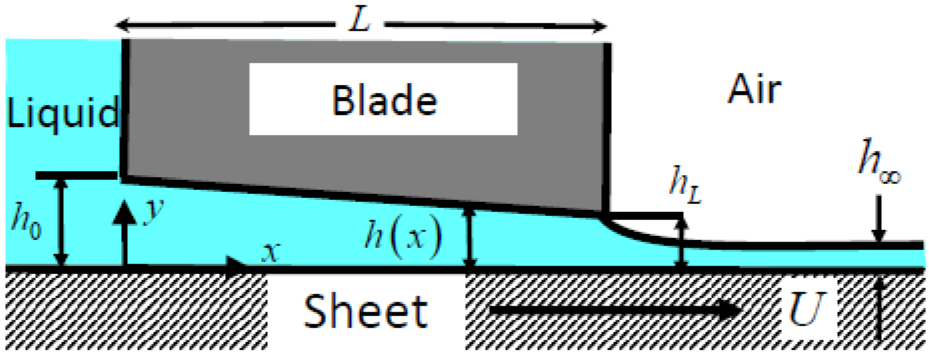
$$\mathrm{H} = c_1 \cosh \lambda\eta + c_2 \sinh \lambda\eta$$

$$At\ \eta = -1 \rightarrow \mathrm{H} = 0 \Rightarrow$$

$$At\ \eta = 1 \rightarrow \mathrm{H} = 0 \Rightarrow$$

e. Show that as the aspect ratio r grows large, the velocity distribution approaches that of the particular solution v_p, i.e. planar Poiseuille flow, except in the vicinity of the side walls at $\eta = \pm r$.

4. Blade coating is a continuous coating process in which liquid is entrained by a moving sheet and metered out by a blade, as shown in the figure below. In general, the final coating thickness h_∞ depends on the sheet speed U, blade dimensions, pressure difference across the blade $\Delta P = P_0 - P_L$, and the liquid properties. The gap thickness $h(x)$ is very thin compared to the blade's length L such that the lubrication approximation applies.



a. Relate h_∞ to ΔP, U, and the blade geometry. Obtain a general result first, and then one for the special case where $h(x)$ is linear.

Starting with the continuity equation and lubrication approximation,

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0$$

$$\frac{\partial^2 v_x}{\partial y^2} = \frac{1}{\mu}\frac{dP}{dx}$$

Integrating twice and solving for the boundary conditions,

$$\frac{\partial v_x}{\partial y} = \frac{1}{\mu}\frac{dP}{dx}y + c_1$$

$$v_x = \frac{1}{2\mu}\frac{dP}{dx}y^2 + c_1 y + c_2$$

$$At\ y = h(x), v_x = 0 \rightarrow \partial v_x/\partial y = 0 \Rightarrow c_1 = -\frac{1}{2\mu}\frac{dP}{dx}h(x)$$

$$At\ y = 0, v_x = U \Rightarrow c_2 = U\left(1 - \frac{y}{h(x)}\right)$$

$$v_x = \frac{1}{2\mu}\frac{dP}{dx}y^2 - \frac{1}{2\mu}\frac{dP}{dx}h(x)y + U\left(1 - \frac{y}{h(x)}\right)$$

$$v_x = \frac{h^2(x)}{2\mu}\frac{dP}{dx}\left[\left(\frac{y}{h(x)}\right)^2 - \frac{y}{h(x)}\right] + U\left(1 - \frac{y}{h(x)}\right)$$

The flow rate is found by integrating with respect to dy,

$$q = \int_0^{h(x)} v_x dy$$

$$q = \int_0^{h(x)} \left\{ \frac{h^2(x)}{2\mu} \frac{dP}{dx} \left[\left(\frac{y}{h(x)} \right)^2 - \frac{y}{h(x)} \right] + U \left(1 - \frac{y}{h(x)} \right) \right\} dy$$

$$q = \frac{h^2(x)}{2\mu} \frac{dP}{dx} \left[\frac{y^3}{3h^2(x)} - \frac{y^2}{2h(x)} \right] + U \left(y - \frac{y^2}{2h(x)} \right) \Bigg|_0^{h(x)}$$

$$q = \frac{Uh(x)}{2} - \frac{h^3(x)}{12\mu} \frac{dP}{dx}$$

Solving for the pressure gradient and integrating,

$$\frac{dP}{dx} = \frac{6\mu U}{h^2(x)} - \frac{12\mu q}{h^3(x)}$$

$$\int_{P_0}^{P_L} dP = \int_0^L \left(\frac{6\mu U}{h^2(x)} - \frac{12\mu q}{h^3(x)} \right) dx$$

$$\Delta P = P_L - P_0 = 6\mu U \int_0^L \frac{1}{h^2(x)} dx - 12\mu q \int_0^L \frac{1}{h^3(x)} dx$$

$$\Delta P = 6\mu U \int_{h_0}^{h_L} \frac{1}{h^2(x)} dH - 12\mu q \int_{h_0}^{h_L} \frac{1}{h^3(x)} dh$$

$$\Delta P = -6\mu U \frac{1}{h(x)} \Bigg|_{h_0}^{h_L} + 12\mu q \frac{1}{2h^2(x)} \Bigg|_{h_0}^{h_L}$$

$$\Delta P = -6\mu U \left(\frac{1}{h_L} - \frac{1}{h_0} \right) + 6\mu q \left(\frac{1}{h_L^2} - \frac{1}{h_0^2} \right)$$

Solving for q, then equating to the flow rate at h_∞ and solving for it,

$$q = \frac{\Delta P}{6\mu \left(\frac{1}{h_L^2} - \frac{1}{h_0^2} \right)} + U \frac{\left(\frac{1}{h_L} - \frac{1}{h_0} \right)}{\left(\frac{1}{h_L^2} - \frac{1}{h_0^2} \right)} == q_{h_\infty} = h_\infty U$$

$$h_\infty = \frac{\Delta P}{6\mu U \left(\frac{1}{h_L^2} - \frac{1}{h_0^2} \right)} + \frac{\left(\frac{1}{h_L} - \frac{1}{h_0} \right)}{\left(\frac{1}{h_L^2} - \frac{1}{h_0^2} \right)}$$

b. If h_0 is too large, some of the liquid will recirculate instead of being drawn out with the sheet, an undesirable situation. Assuming h_∞ is given, determine the maximum opening $h_0 = h_{max}$ for which there will be no recirculation.

Substituting dP/dx in v_x,

$$v_x = \frac{h^2(x)}{2\mu}\left[\frac{6\mu U}{h^2(x)} - \frac{12\mu q}{h^3(x)}\right]\left[\left(\frac{y}{h(x)}\right)^2 - \frac{y}{h(x)}\right] + U\left(1 - \frac{y}{h(x)}\right)$$

$$v_x = U\left[1 - 4\frac{y}{h(x)} + 3\left(\frac{y}{h(x)}\right)^2\right] + \frac{6q}{h(x)}\left[\frac{y}{h(x)} - \left(\frac{y}{h(x)}\right)^2\right]$$

Deriving the velocity with respect to y,

$$\frac{\partial v_x}{\partial y} = \frac{U}{h(x)}\left[-4 + 6\frac{y}{h(x)}\right] + \frac{6q}{h^2(x)}\left[1 - 2\frac{y}{h(x)}\right]$$

Applying the boundary condition that the velocity is zero just at the top of h(x),

$$\left.\frac{\partial v_x}{v_y}\right|_{y=h(x)} = 0$$

$$0 = \frac{U}{h(x)}\left[-4 + 6\frac{h(x)}{h(x)}\right] + \frac{6q}{h^2(x)}\left[1 - 2\frac{h(x)}{h(x)}\right]$$

$$h_{max} = \frac{6q}{2U}$$

5. Flow near a wall suddenly set in motion (approximate solution). Apply a procedure like that of example 4.4-1 to get an approximate solution for Example 4.1.1

    a. Integrate Eq. 4.4-1 over y to get

    $$\int_0^\infty \frac{\partial v_x}{\partial t} dy = v \frac{\partial v_x}{\partial y}\bigg|_0^\infty$$

    Make use of the boundary conditions and the Leibniz rule for differentiating an integral (Eq. C.3-2) to rewrite Eq. 4B.2-1 in the form

    $$\frac{d}{dt}\int_0^\infty \rho v_x dy = \tau_{yx}\big|_{y=0}$$

    Interpret this result physically.

    b. We know roughly what the velocity profiles look like. We can make the following reasonable postulate for the profiles:

    $$\frac{v_x}{v_\infty} = 1 - \frac{3}{2}\frac{y}{\delta(t)} + \frac{1}{2}\left(\frac{y}{\delta(t)}\right)^3 \qquad for\ 0 \le y \le \delta(t)$$

    $$\frac{v_x}{v_\infty} = 1 \qquad for\ y \ge \delta(t)$$

    Here $\delta(t)$ is a time-dependent boundary-layer thickness. Insert this approximate expression into Eq. 4B.2-2 to obtain

    $$\delta \frac{d\delta}{dt} = 4v$$

    c. Integrate Eq. 4B.2-5 with a suitable initial value of $\delta(t)$, and insert the result into Eq. 4B2-3 to get the approximate velocity profiles.

    d. Compare the values of v_x/v_∞ obtained from (c) with those from Eq. 4.1-15 at $y/\sqrt{4vt}$ = 0.2, 0.5, and 1.0. Express the results as the ratio of the approximate value to the exact value.

    answer (d) 1.015, 1.026, 0.738