CBE 521
Advanced Transport Phenomena
Examination 1

Brandon Lampe

October 14, 2015

1 Temperature distribution in wire

Assumptions

- wire has negligible resistance to radial conduction $\implies$ temperature fluctuates in axial direction only ($T = T(z)$)
- $\rho, \hat{C}_p, k, v_z$ are all constant
- no viscous heating ($\Phi_v = 0$)
- as z $\to \infty$, $\frac{dT}{dz} \to 0$

General PDE

$$\rho\hat{C}_p\left(\frac{\partial T}{\partial t} + v_r\frac{\partial T}{\partial r} + v_\theta\frac{\partial T}{\partial \theta} + v_z\frac{\partial T}{\partial z}\right) = k\left[\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{r\partial T}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2} + \frac{\partial^2 T}{\partial z^2}\right] + \mu\Phi_v$$

after assumptions are applied:

$$\rho\hat{C}_p v_z\frac{dT}{dz} = k\frac{d^2T}{dz^2} \tag{1}$$

from the problem description:

$$v_z = -v \tag{2}$$

$$\text{Let } C = -\frac{\rho\hat{C}_p v}{k} \tag{3}$$

$$\Theta(z) = \frac{T(z) - T_\infty}{T_0 - T_\infty} \implies T(z) = \Theta(T_0 - T_\infty) + T_\infty \quad \text{(dimensionless } T\text{)} \tag{4}$$

Problem Data

- problem domain: $0 \le z \le \infty$
- coefficient function: C is constant
- forcing function: $Q = 0$, no external heat
- boundary conditions:

$$T\Big|_{z=0} = T_0, \quad T\Big|_{z=\infty} = T_\infty, \quad \frac{dT}{dz}\Big|_{z=\infty} = 0$$

- governing equation:

$$C\frac{dT}{dz} = \frac{d^2T}{dz^2}$$

Solve the ODE

$$C\int \frac{dT}{dz}dz = \int \frac{d^2T}{dz^2}dz$$
$$CT = \frac{dT}{dz} + C_1$$

Evaluate at $z = \infty$

$$CT\Big|_{z=\infty} = \frac{dT}{dz}\Big|_{z=\infty} + C_1 \implies C_1 = CT_\infty$$

substitute in terms from equation 4 and separate the variables

$$\frac{dT}{dz} = -C(T - T_\infty)$$
$$(T_0 - T_\infty)\frac{d\Theta}{dz} + \frac{dT_\infty}{dz} = -C\Big(\Theta(T_0 - T_\infty)\Big)$$
$$\frac{d\Theta}{\Theta} = -Cdz$$

integrate both sides

$$\int \frac{d\Theta}{\Theta} = -C\int dz$$
$$ln(\Theta) = -Cz + C_2$$

evaluate $\Theta\Big|_{z=0} = 1$ and solve for C_2

$$ln(1) = 0 + C_2 \implies C_2 = 0$$

$$\Theta = exp(-Cz)$$

substitute back in terms to obtained $T(z)$

$$\boxed{T(z) = exp\left(\frac{\rho\hat{C}_p vz}{k}\right)(T_0 - T_\infty) + T_\infty} \tag{5}$$

2 Derive 2-D temperature profile $T = T(x, y)$

Problem Data

- problem domain: $0 \leq x \leq 1,\ 0 \leq y \leq 1$
- coefficient function: k = 1
- forcing function: Q = 0
- boundary conditions:

$$T\Big|_{x=0} = 0, \quad T\Big|_{x=1} = g(y) \quad \Big|_{y=0} = 0, \quad T\Big|_{y=1} = f(x)$$

- governing equation:

$$\nabla^2 T = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$$

Solution Method Because the governing equation is linear in in T, the method of superposition may be used to obtain a solution. The solution will be obtained separating the problem into two subproblems, solving each with only one of the inhomogeneous boundary conditions, and summing the solutions of each subproblem to obtain the full solution:

$$T = T_1 + T_2$$

Solve for T_1

define the boundary conditions for the subproblem:

$$T_1(x,0) = 0$$
$$T_1(x,1) = f(x)$$
$$T_1(0,y) = 0$$
$$T_1(1,y) = 0$$

assume a solution takes the form:

$$T_1(x,y) = X(x)Y(y)$$

calculate the second derivatives of the new variables, and substitute these derivatives into the governing equation:

$$\frac{\partial^2 T_1}{\partial X^2} = X''(x)Y(y)$$
$$\frac{\partial^2 T_1}{\partial Y^2} = X(x)Y''(y)$$

$$\frac{\partial^2 T_1}{\partial X^2} + \frac{\partial^2 T_1}{\partial Y^2} = X''(x)Y(y) + X(x)Y''(y) = 0$$

rearrange the new equation, because X is independent of y and Y is independent of x, then each side of the equation below is independent and therefore each side must equal a fixed constant (μ):

$$-\frac{X''}{X} = \frac{Y''}{Y} = \mu^2$$

therefore, we now have two homogeneous ODEs:

$$X'' + \mu^2 X = 0 \tag{6}$$
$$Y'' - \mu^2 Y = 0 \tag{7}$$

define the boundary conditions in terms of X and Y:

$$T_1(x,0) = X(x)Y(0) = 0 \ \forall\ x \in [0,1] \implies Y(0) = 0$$
$$T_1(x,1) = X(x)Y(1) = f(x) \ \forall\ x \in [0,1] \implies Y(1) = f(x)$$
$$T_1(0,y) = X(0)Y(y) = 0 \ \forall\ y \in [0,1] \implies X(0) = 0$$
$$T_1(1,y) = X(1)Y(y) = 0 \ \forall\ y \in [0,1] \implies X(1) = 0$$

Equation 6 has homogeneous boundary conditions and is of the form of a Sturm-Liouville problem. The characteristic equation for Equation 6 is $m^2+\mu^2 = 0$. The roots of this characteristic equation are complex, resulting in solutions of $m = \pm i\sqrt{\mu^2}$ and the general solution to the differential equation is $X(x) = e^{\alpha x}\left[B\ sin(\sqrt{\mu^2}x) + C\ cos(\sqrt{\mu^2}x)\right]$. Recognizing that $\alpha = 0$ and letting $\mu = \lambda$ results in:

$$X(x) = B\ sin(\lambda x) + C\ cos(\lambda x)$$

which for any $\lambda, B,$ or C will satisfy the differential equation. Now, obtain a solution that satisfies the particular boundary conditions:

$$X(0) = 0$$
$$X(1) = 0$$

to satisfy the boundary conditions, $X(0) \implies C = 0$. If $B = 0$, the solution reduces to the trivial solution $X(x) = 0$. A non-trivial solution to $sin(\lambda(1)) = 0$ is obtained by letting $\lambda(1) = n\pi$ for $n = 1, 2, ...$ a solution for λ may be obtained:

$$\lambda_n = n\pi \text{ for } n = 1, 2, 3, \ldots \tag{8}$$
$$X_n(x) = sin\,(n\pi x) \ \text{ for } n = 1, 2, 3, \ldots \tag{9}$$

where $X_n(x)$ are eigenfunctions and each λ_n are corresponding eigenvalues. The solution is now in terms of an infinite number of functions (fundamental solutions).

The solution to the remaining ODE must now be obtained, where characteristic equation for Equation 7 is $m^2 - \mu^2 = 0$. Solving this quadratic equation results in the solutions of $m = 1$. Again, Let $\mu = \lambda$ and the general solution to the differential equation is

$$Y(y) = D\ cosh(\lambda y) + E\ sinh(\lambda y)$$

which for any $\lambda, D,$ or E will satisfy the differential equation. Now, obtain a solution that satisfies the homogeneous boundary condition only:

$$Y(0) = 0$$

$Y(0) \implies D = 0$. Let $E = 1$, then a nontrivial solution to $sinh(\lambda y) = 0$ is obtained by letting $\lambda = n\pi$ for $n = 1, 2, ...$ a solution for λ may be obtained:

$$\lambda_n = n\pi \text{ for } n = 1, 2, 3, \ldots \tag{10}$$

$$Y_n(y) = sinh\,(n\pi y) \text{ for } n = 1, 2, 3, \ldots \tag{11}$$

combine the pair of homogeneous soltions:

$$T_{1n}(x, y) = sinh\,(n\pi y)\, sin\,(n\pi x)$$

this function satisfies the three homogeneous boundary conditions. To solve for the solution including the non-homogeneous boundary conditions, let the complete solution consist of the following terms:

$$\begin{aligned} T_1(x, y) &\equiv \sum_{n=1}^{\infty} a_n T_{1n} \\ &= \sum_{n=1}^{\infty} a_n sinh\,(n\pi y)\, sin\,(n\pi x) \end{aligned}$$

at the non-homogeneous boundary conditions:

$$\begin{aligned} f(x) = T_1(x, 1) &= \sum_{n=1}^{\infty} a_n sinh\,(n\pi)\, sin\,(n\pi x) \\ &= \sum_{n=1}^{\infty} A_n sin\,(n\pi x) \end{aligned}$$

this is an orthogonal expansion of $f(x)$ relative to the orthogonal basis of the sine function. A_n is a Fourier coefficient which is defined as the inner product:

$$A_n = \frac{\int_0^1 f(x) sin(n\pi x) dx}{\int_0^1 sin^2(n\pi x)} = 2\int_0^1 f(x) sin(n\pi x) dx \text{ for } n = 1, 2, 3, \ldots$$

therefore, a_n must be:

$$a_n = -\frac{2}{sinh(n\pi)} \int_0^1 f(x) sin(n\pi x) dx \text{ for } n = 1, 2, 3, \ldots$$

and the solution to T_1 is:

$$T_1 = \sum_{n=1}^{\infty} \left[-\frac{2}{sinh(n\pi)} \int_0^1 f(x) sin(n\pi x) dx \right] sinh\,(n\pi y)\, sin\,(n\pi x) \tag{12}$$

Solve for T_2

define the boundary conditions for the subproblem:

$$\begin{aligned} T_2(x, 0) &= 0 \\ T_2(x, 1) &= 0 \\ T_2(0, y) &= 0 \\ T_2(1, y) &= g(y) \end{aligned}$$

assume a solution takes the form:

$$T_2(x, y) = X(x)Y(y)$$

calculate the second derivatives of the new variables, and substitute these derivatives into the governing equation:

$$\frac{\partial^2 T_2}{\partial X^2} = X''(x)Y(y)$$
$$\frac{\partial^2 T_2}{\partial Y^2} = X(x)Y''(y)$$

$$\frac{\partial^2 T_2}{\partial X^2} + \frac{\partial^2 T_2}{\partial Y^2} = X''(x)Y(y) + X(x)Y''(y) = 0$$

rearrange the new equation, because X is independent of y and Y is independent of x, then each side of the equation below is independent and therefore each side must equal a fixed constant (μ):

$$-\frac{X''}{X} = \frac{Y''}{Y} = \mu^2$$

therefore, we now have two homogeneous ODEs:

$$X'' + \mu^2 X = 0 \tag{13}$$
$$Y'' - \mu^2 Y = 0 \tag{14}$$

define the boundary conditions in terms of X and Y:

$$T_2(x, 0) = X(x)Y(0) = 0 \ \forall \ x \in [0, 1] \implies Y(0) = 0$$
$$T_2(x, 1) = X(x)Y(1) = 0 \ \forall \ x \in [0, 1] \implies Y(1) = 0$$
$$T_2(0, y) = X(0)Y(y) = 0 \ \forall \ y \in [0, 1] \implies X(0) = 0$$
$$T_2(1, y) = X(1)Y(y) = g(y) \ \forall \ y \in [0, 1] \implies X(1) = g(y)$$

Equation 14 has homogeneous boundary conditions and is of the form of a Sturm-Liouville problem. The characteristic equation for Equation 14 is $m^2 - \mu^2 = 0$ which as roots of $m = 1$ and the general solution to the differential equation is:

$$Y(y) = B \ sinh(\lambda y) + C \ cosh(\lambda y)$$

which for any $\lambda, B,$ or C will satisfy the differential equation. Now, obtain a solution that satisfies the particular boundary conditions:

$$Y(0) = 0$$
$$Y(1) = 0$$

to satisfy the boundary conditions, $Y(0) \implies C = 0$. If $B = 0$, the solution reduces to the trivial solution $Y(y) = 0$. A non-trivial solution to $sinh(\lambda(1)) = 0$ is obtained by letting $\lambda(1) = n\pi$ for $n = 1, 2, ...$ a solution for λ may be obtained:

$$\lambda_m = m\pi \text{ for } m = 1, 2, 3, \ldots \tag{15}$$
$$Y_m(y) = sinh\left(m\pi y\right) \text{ for } m = 1, 2, 3, \ldots \tag{16}$$

The solution to the remaining ODE must now be obtained, where characteristic equation for Equation 13 is $m^2 + \mu^2 = 0$. The roots of this characteristic equation are complex, resulting in solutions of $m = \pm i\sqrt{\mu^2}$ and the general solution to the differential equation is $X(x) = e^{\alpha x}\left[D \ sin(\sqrt{\mu^2}x) + E \ cos(\sqrt{\mu^2}x)\right]$. Recognizing that $\alpha = 0$ and letting $\mu = \lambda$ results in:

$$X(x) = D \ sin(\lambda x) + E \ cos(\lambda x)$$

which for any $\lambda, D,$ or E will satisfy the differential equation. Now, obtain a solution that satisfies the homogeneous boundary condition only:

$$X(0) = 0$$

$X(0) \implies E = 0$. Let $D = 1$, then a nontrivial solution to $sin(\lambda x) = 0$ is obtained by letting $\lambda = m\pi$ for $m = 1, 2, ...$ a solution for λ may be obtained:

$$\lambda_m = m\pi \text{ for } m = 1, 2, 3, \ldots \tag{17}$$

$$X_m(x) = sin\,(m\pi x) \;\text{ for } m = 1, 2, 3, \ldots \tag{18}$$

combine the pair of homogeneous solutions:

$$T_{2m}(x, y) = sinh\,(m\pi y)\,sin\,(m\pi x)$$

this function satisfies the three homogeneous boundary conditions. To solve for the solution including the non-homogeneous boundary conditions, let the complete solution consist of the following terms:

$$\begin{aligned} T_2(x, y) &\equiv \sum_{m=1}^{\infty} a_m T_{2m} \\ &= \sum_{m=1}^{\infty} a_m sinh\,(m\pi y)\,sin\,(m\pi x) \end{aligned}$$

at the non-homogeneous boundary conditions:

$$\begin{aligned} g(y) = T_2(y, 1) &= \sum_{m=1}^{\infty} a_m sinh\,(m\pi y)\,sin\,(m\pi) \\ &= \sum_{m=1}^{\infty} A_m sinh\,(n\pi y) \end{aligned}$$

this is an orthogonal expansion of $g(y)$ relative to the orthogonal basis of the hyperbolic sine function. A_m is a Fourier coefficient which is defined as the inner product:

$$A_m = \frac{\int_0^1 g(y) sinh(m\pi y) dy}{\int_0^1 sinh^2(m\pi y)} = 2\int_0^1 g(y) sinh(m\pi y) dy \text{ for } m = 1, 2, 3, \ldots$$

therefore, a_m must be:

$$a_m = -\frac{2}{sin(m\pi)} \int_0^1 g(y) sinh(m\pi y) dy \text{ for } m = 1, 2, 3, \ldots$$

and the solution to T_2 is:

$$T_2 = \sum_{m=1}^{\infty} \left[-\frac{2}{sin(m\pi)} \int_0^1 g(y) sinh(m\pi y) dy \right] sinh\,(m\pi y)\,sin\,(m\pi x) \tag{19}$$

Combine solutions to the subproblems

$$T = T_1 + T_2 \tag{20}$$

$$= \sum_{n=1}^{\infty} \left[-\frac{2}{sinh(n\pi)} \int_0^1 f(x) sin(n\pi x) dx \right] sinh\,(n\pi y)\,sin\,(n\pi x) + \tag{21}$$

$$\left[-\frac{2}{sin(n\pi)} \int_0^1 g(y) sinh(n\pi y) dy \right] sinh\,(n\pi y)\,sin\,(n\pi x) \tag{22}$$

3 Slot coating analysis

Assumptions

- steady-state flow
- gravitation influence is negligible
- flow occurs in x-direction only
- external pressures (p_0 and p_1) are constant

- total velocity profile is the sum of Couette (v^c) and Poiseuille (v^p) types
- no-slip conditions exist at wall surfaces

Governing Equations

- continuity:

$$\frac{\partial v_x}{\partial x} = 0$$

- Navier-Stokes EOM:

$$0 = -\frac{\partial p}{\partial x} + \mu \frac{\partial^2 v_x}{\partial y^2}$$

Problem Data

- problem domain: $-L_2 \leq x \leq L_2, \quad 0 \leq y \leq h$
- coefficient function: μ is constant
- forcing function: p_0 is constant
- boundary conditions:

$$v_x^c\Big|_{y=0} = U, \quad v_x^c\Big|_{y=h} = 0, \quad v_x^p\Big|_{y=0} = 0, \quad v_x^p\Big|_{y=h} = 0$$

- governing equations:

$$\text{Couette flow} \quad 0 = \mu \frac{\partial^2 v_x^c}{\partial y^2} \tag{23}$$

$$\text{Poiseuille flow} \quad 0 = -\frac{\partial p}{\partial x} + \mu \frac{\partial^2 v_x^p}{\partial y^2} \tag{24}$$

Analyze Couette (viscous) contribution resulting form wall velocity

integrate Equation 23 twice to obtain:

$$\int \frac{d^2 v_x^c}{dy^2} dy = \frac{dv_x^c}{dy} + C_1$$

$$\int \frac{dv_x^c}{dy} = \int C_1 dy$$

$$v_x^c = C_1 y + C_2$$

apply boundary conditions to solve for constants:

$$C_2 = U$$

$$C_1 = -\frac{U}{h}$$

$$v_x^c = -\frac{U}{h} y + U$$

Analyze Poiseuille (pressure driven) contribution

integrate Equation 24:

$$\int \frac{d^2 v_x^p}{dy^2} dy = \int \left(\frac{1}{\mu} \frac{dp}{dx} \right) dy$$

$$\int \frac{dv_x^p}{dy} dy = \int \left(\frac{1}{\mu} \frac{dp}{dx} y + C_3 \right) dy$$

$$v_x^p = \frac{1}{2\mu} \frac{dp}{dx} y^2 + C_3 y + C_4$$

apply boundary conditions to solve for constants:

$$C_4 = 0$$

$$C_3 = -\frac{1}{2\mu} \frac{dp}{dx} h$$

$$v_x^p = \frac{1}{2\mu} \frac{dp}{dx} y^2 - \frac{1}{2\mu} \frac{dp_0}{dx} hy$$

Combine Couette and Poiseuille velocity terms

the total flow velocity is then:

$$v_x = v_x^c + v_x^p \tag{25}$$
$$= -\frac{U}{h}y + U + \frac{1}{2\mu}\frac{dp}{dx}y^2 - \frac{1}{2\mu}\frac{dp}{dx}hy \tag{26}$$
$$= U\left[1 - \frac{y}{h}\right] + \frac{1}{2\mu}\frac{dp}{dx}\left[y^2 - yh\right] \tag{27}$$

Integrate over cross-sectional area to obtain the volumetric flow rate

assume a unit width and integrate Equation 27 over the slot height (h); additionally, assume the total flow rate is the sum of :

$$q = \int_0^h v_x dy \tag{28}$$
$$= U\left[y - \frac{y^2}{2h}\right] + \frac{1}{2\mu}\frac{dp}{dx}\left[\frac{y^3}{3} - \frac{y^2 h}{2}\right]\Bigg|_{y=0}^{y=h} \tag{29}$$
$$= \frac{Uh}{2} - \frac{h^3}{12\mu}\frac{dp}{dx} \tag{30}$$

Use flow rate to calculate change in pressure over L_1

rearrange Equation 30 and solve for the external pressure by integrating over L_1, then rearrange to express q without the pressure differential:

$$\frac{dp}{dx} = \frac{6\mu U}{h^2} - \frac{q_1 12\mu}{h^3} \tag{31}$$
$$\int_{p_1}^{p_0} dp = \int_0^{L_1}\left(\frac{6\mu U}{h^2} - \frac{q_1 12\mu}{h^3}\right)dx \tag{32}$$
$$p_0 - p_1 = \left(\frac{6\mu U}{h^2} - \frac{q_1 12\mu}{h^3}\right)L_1 \tag{33}$$
$$q_1 = \frac{6\mu U L_1 h + (p_1 - p_0)h^3}{12\mu L_1} \tag{34}$$

3(a) Solve for $v_x(y)$ in the forward-flow region (over L_1)

substitute in Equation 31 for $\frac{dp}{dx}$

$$\begin{aligned} v_x &= U\left[1 - \frac{y}{h}\right] + \frac{1}{2\mu}\frac{dp}{dx}\left[y^2 - yh\right] \\ &= U\left[1 - \frac{y}{h}\right] + \frac{1}{2\mu}\left[\frac{6\mu U}{h^2} - \frac{q_1 12\mu}{h^3}\right]\left[y^2 - yh\right] \\ &= U\left[1 - \frac{y}{h}\right] + \frac{1}{2\mu}\left[\frac{6\mu U}{h^2} - \left(\frac{6\mu U L_1 h + (p_1 - p_0)h^3}{12\mu L_1}\right)\frac{12\mu}{h^3}\right]\left[y^2 - yh\right] \end{aligned}$$

therefore, in the forward-flow region:

$$\boxed{v_{x1}(y) = U\left[1 - \frac{y}{h}\right] + \frac{1}{2\mu}\left[\frac{6\mu U}{h^2} - \left(\frac{6\mu U L_1 h + (p_1 - p_0)h^3}{12\mu L_1}\right)\frac{12\mu}{h^3}\right]\left[y^2 - yh\right]} \tag{35}$$

Use flow rate to calculate change in pressure over L_2

assume that no fluid is stored in region L_2; therefore, the net flow rate is zero in this region ($q_2 = 0$)
rearrange Equation 30, let $q_2 = 0$, and solve for the external pressure by integrating over L_2:

$$\frac{dp}{dx} = \frac{6\mu U}{h^2} - \frac{q_2 12\mu}{h^3} \tag{36}$$

$$= \frac{6\mu U}{h^2} \tag{37}$$

$$\int_{p_0}^{p_1} dp = \int_{-L_2}^{0} \left(\frac{6\mu U}{h^2}\right) dx \tag{38}$$

$$p_1 - p_0 = -\left(\frac{6\mu U}{h^2}\right)(-L_2) \tag{39}$$

$$L_2 = \frac{(p_0 - p_1)h^2}{6\mu U} \tag{40}$$

3(b) Solve for $v_x(y)$ in the back-flow region (over L_2)

because $q_2 = 0$, Equation 36 may be substituted into Equation 27; therefore, in the back-flow region:

$$v_x = U\left[1 - \frac{y}{h}\right] + \frac{1}{2\mu}\frac{dp}{dx}\left[y^2 - yh\right]$$
$$= U\left[1 - \frac{y}{h}\right] + \frac{1}{2\mu}\frac{6\mu U}{h^2}\left[y^2 - yh\right]$$

$$\boxed{v_{x2}(y) = U\left[1 - \frac{y}{h}\right] + \frac{1}{2\mu}\frac{6\mu U}{h^2}\left[y^2 - yh\right]} \tag{41}$$

and L_2, solved for in Equation 40:

$$\boxed{L_2 = \frac{(p_0 - p_1)h^2}{6\mu U}} \tag{42}$$

3(c) Evaluate the final coating thickness h_∞

for mass to be conserved the flow rate q_1 must be constant; therefore by setting Equation 34 equal to Uh_∞:

$$q_1 = \frac{6\mu U L_1 h + (p_1 - p_0)h^3}{12\mu L_1}$$
$$= Uh_\infty$$

$$\boxed{h_\infty = \frac{6\mu U L_1 h + (p_1 - p_0)h^3}{12\mu L_1 U}} \tag{43}$$

3(d) Determine the shear force (F) per unit width applied by the fluid on the substrate

$$F = \int_{-L_2}^{L_1} \left(\mu \frac{dv_x}{dy}\right) dx$$
$$= \int_{-L_2}^{0} \left(\mu \frac{dv_{x2}}{dy}\right) dx + \int_{0}^{L_1} \left(\mu \frac{dv_{x1}}{dy}\right) dx$$
$$= \int_{-L_2}^{0} \mu\left[-\frac{U}{h} + \frac{3U}{h^2}[2y - h]\right] dx + \int_{0}^{L_1} \mu\left(-\frac{U}{h} + \frac{1}{2\mu}\left[\frac{6\mu U}{h^2} - \left(\frac{6\mu U L_1 h + (p_1 - p_0)h^3}{12\mu L_1}\right)\frac{12\mu}{h^3}\right][2y - h]\right) dx$$
$$= \mu\left[-\frac{U}{h} + \frac{3U}{h^2}[2y - h]\right] L_2 + \mu\left(-\frac{U}{h} + \frac{1}{2\mu}\left[\frac{6\mu U}{h^2} - \left(\frac{6\mu U L_1 h + (p_1 - p_0)h^3}{12\mu L_1}\right)\frac{12\mu}{h^3}\right][2y - h]\right) L_1$$

evaluate $F\Big|_{y=0}$

$$F(0) = \mu\left[-\frac{U}{h} + \frac{3U}{h^2}[-h]\right]L_2 + \mu\left(-\frac{U}{h} + \frac{1}{2\mu}\left[\frac{6\mu U}{h^2} - \left(\frac{6\mu U L_1 h + (p_1 - p_0)h^3}{12\mu L_1}\right)\frac{12\mu}{h^3}\right][-h]\right)L_1 \tag{44}$$

$$= \frac{\mu U}{h}(-4L_2 - L_1) + \frac{(p_0 - p_1)h}{2} \tag{45}$$

assuming $p_1 > p_0$, the net shear force is in the negative x direction

4 Axisymmetric flow between parallel plates

Assumptions

- $v = v(r, z), \quad v_z = v_\theta = 0$
- creeping (Stoke's) flow $\implies R_e = \frac{\text{inertial forces}}{\text{viscous forces}} << 1$ i.e., viscous forces are dominant and fluid inertia is negligible ($\rho v_r \frac{\partial v_r}{\partial r} = 0$)

Governing Equations by applying these assumptions, the conservation equations (mass and momentum) simplify to:

- continuity (BSL B.4-2):

$$\frac{1}{r}\frac{\partial}{\partial r}(r v_r) = 0$$

- Equation of Motion in the radial direction (BSL B.6-4):

$$0 = -\frac{d\mathscr{P}}{dr} + \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r v_r)\right) + \frac{\partial^2 v_r}{\partial z^2}\right]$$

4.1 Relate mean radial velocity $U(r)$, i.e. v_r averaged along the gap thickness of $2H$, to a volumetric flow rate Q and the geometric quantities.

define a flow rate Q, where U is the mean radial velocity:

$$Q = Q(r) = U(r)a(r)$$

where a is perpendicular to the direction of flow:

$$a = 4\pi H r$$

therefore:

$$\boxed{U(r) = \frac{Q}{4\pi H r}} \tag{46}$$

4.2 Show the continuity and momentum equations have the form given

Obtain an expression for rv_r as a function of only z

$$\frac{1}{r}\frac{\partial}{\partial r}(r v_r) = 0 \implies r v_r(r) \text{ is a constant, i.e. } r v_r \text{ with respect to the radial direction } (r) \text{ is constant}$$

therefore, rv_r can be a function of z only:

$$r v_r = f(z) \rightarrow v_r = \frac{f(z)}{r}$$

$$\boxed{v_r = \frac{f(z)}{r}} \tag{47}$$

obtain an expresion for $\mathscr{P}$

substitute this expression and that of continuity into the conservation of momentum to obtain:

$$0 = -\frac{d\mathscr{P}}{dr} + \mu\frac{1}{r}\frac{d^2 f(z)}{dz^2}$$

seperate the variables to obtain two ODEs

$$r\frac{d\mathscr{P}}{dr} = \mu\frac{d^2 f(z)}{dz^2}$$

the left side is now a function of only r and the right side is now only a function of z; therefore, both sides must be equal to a constant:

$$r\frac{d\mathscr{P}}{dr} = \mu\frac{d^2 f(z)}{dz^2} = C$$

Therefore, the following must hold:

$$\boxed{\mathscr{P} = \mathscr{P}(r)} \tag{48}$$

4.3 Determine $v_r(r, z)$

Problem Data

- problem domain: $0 \leq z \leq 2H$, $r_1 \leq r \leq r_2$, and $0 \leq \theta \leq 2\pi$
- coefficient functions: C and μ are constant
- forcing function: $Q = 0$, no external heat
- no-slip boundary conditions:

$$v_r\Big|_{z=0} = 0, \quad v_r\Big|_{z=2H} = 0$$

- governing equations:

$$r\frac{d\mathscr{P}}{dr} = C; \quad \mu\frac{d^2 f(z)}{dz^2} = C$$

solve for the seperation constant C

integrate the pressure term over the radial domain:

$$\int_{\mathscr{P}_1}^{\mathscr{P}_2} d\mathscr{P} = C\int_{r_1}^{r_2}\frac{dr}{r}$$

$$\mathscr{P}_2 - \mathscr{P}_1 = C ln\left(\frac{r_2}{r_1}\right) \rightarrow C = \frac{\mathscr{P}_2 - \mathscr{P}_1}{ln\left(\frac{r_2}{r_1}\right)}$$

substitute this expression into the remaining ODE to solve for $f(z)$

$$\int d^2 f(z) = \frac{\mathscr{P}_2 - \mathscr{P}_1}{\mu\, ln\left(\frac{r_2}{r_1}\right)}\int dz$$

$$\int df(z) = \frac{\mathscr{P}_2 - \mathscr{P}_1}{\mu\, ln\left(\frac{r_2}{r_1}\right)} z + C_1$$

$$f(z) = \frac{\mathscr{P}_2 - \mathscr{P}_1}{\mu\, ln\left(\frac{r_2}{r_1}\right)}\frac{z^2}{2} + C_1 z + C_2$$

apply boundary conditions and solve for constants of integration

substitute back in $rv_r = f(z)$

$$rv_r(z) = \frac{\mathscr{P}_2 - \mathscr{P}_1}{\mu \, ln\left(\frac{r_2}{r_1}\right)} \frac{z^2}{2} + C_1 z + C_2$$

Let

$$A = \frac{\mathscr{P}_2 - \mathscr{P}_1}{\mu \, ln\left(\frac{r_2}{r_1}\right)}$$

$$v_r\Big|_{z=0} = 0 \quad \rightarrow \quad 0 = C_2$$

$$v_r\Big|_{z=2H} = 0 \quad \rightarrow \quad 0 = A\frac{(2H)^2}{2} + C_2$$

$$C_2 = -A\frac{(2H)^2}{2}$$

$$rv_r(z) = A\frac{z^2}{2} - A\frac{(2H)^2}{2}$$

$$v_r(r,z) = A\frac{z^2}{2r} - A\frac{(2H)^2}{2r}$$

calculate the volumetric flow rate

the flow rate must be constant through any cylindrical surface at a distance $r_1 \leq r \leq r_2$, therefore, integrate over the circumference of the domain (2π) at a distance (r_1) and over the domain height $(2H)$:

$$Q = \int_0^{2\pi} \int_0^{2H} v_r\Big|_{r=r_1} dz d\theta$$

$$Q = \int_0^{2\pi} \int_0^{2H} \left(A\frac{z^2}{2r_1} - A\frac{(2H)^2}{2r_1} \right) dz d\theta$$

$$Q = \int_0^{2\pi} \left(A\frac{z^3}{6r_1} - A\frac{(2H)^2 z}{2r_1} \right)\Big|_{z=0}^{z=2H} d\theta$$

$$Q = \int_0^{2\pi} \left(A\frac{(2H)^3}{6r_1} - A\frac{(2H)^2 2H}{2r_1} \right) d\theta$$

$$Q = A\frac{(2H)^3 2\pi}{6r_1} - A\frac{(2H)^2 2H 2\pi}{2r_1}$$

express $v_r(r,z)$ in terms of Q

because Q is constant:

$$\boxed{v_r(r) = \frac{Q}{a} = \frac{Q}{4\pi H r}} \tag{49}$$