9. RELATIONS BETWEEN VOLUME AND SURFACE INTEGRALS

9.1 Initial Comments

This chapter gathers a number of identities that relate surface and volume integrals. With a derivation of just a couple of basic theorems, a number of identities are easily shown.

9.2 Notation for Area and Volume Elements and Integrals

Let $d\boldsymbol{r}^{(1)}$ and $d\boldsymbol{r}^{(2)}$ denote two differentials of the position vector at one point. Then the differential of area, dA, at that point is defined such that

$$\boldsymbol{n}dA = d\boldsymbol{r}^{(1)} \times d\boldsymbol{r}^{(2)} \tag{9-1}$$

in which $\boldsymbol{n}$ is a unit vector with the direction defined by the cross product. Then

$$dA = \boldsymbol{n} \cdot (d\boldsymbol{r}^{(1)} \times d\boldsymbol{r}^{(2)}) \geq 0 \tag{9-2}$$

Now introduce a third differential of position vector, $d\boldsymbol{r}^{(3)}$, such that $d\boldsymbol{r}^{(3)} \cdot \boldsymbol{n} \geq 0$. Then a differential of volume, dV, is defined to be

$$dV = d\boldsymbol{r}^{(3)} \cdot (d\boldsymbol{r}^{(1)} \times d\boldsymbol{r}^{(2)}) = d\boldsymbol{r}^{(1)} \cdot (d\boldsymbol{r}^{(2)} \times d\boldsymbol{r}^{(3)}) = d\boldsymbol{r}^{(2)} \cdot (d\boldsymbol{r}^{(3)} \times d\boldsymbol{r}^{(1)}) \tag{9-3}$$

If a fixed orthonormal basis, or equivalently, a rectangular Cartesian system, is used then

$$dA = \varepsilon_{ijk} n_i dx_j^{(1)} dx_k^{(2)} \qquad dV = \varepsilon_{ijk} dx_i^{(1)} dx_j^{(2)} dx_k^{(3)} \tag{9-4}$$

Now suppose we pick the specific differentials of position of $d\boldsymbol{r}^{(1)} = dx_1\boldsymbol{e}_1$, $d\boldsymbol{r}^{(2)} = dx_2\boldsymbol{e}_2$ and $d\boldsymbol{r}^{(3)} = dx_3\boldsymbol{e}_3$ then $\boldsymbol{n} = \boldsymbol{e}_3$ and we arrive at the more familiar forms of

$$dA = dx_1 dx_2 \qquad dV = dx_1 dx_2 dx_3 \tag{9-5}$$

For an arbitrary surface, the area element depends on the unit normal and the integral over the complete surface of a body is actually a double integral, while a volume integral is actually a triple integral, also with variable limits of integration. These nontrivial integrations are represented by the following notation that are widely used for surface and volume integrals:

$$\int_{\partial R} f dA = \int_{g_L^1}^{g_U^1} \int_{g_L^2}^{g_U^2} (f) \varepsilon_{ijk} n_i dx_j^{(1)} dx_k^{(2)} \qquad \int_R f dV = \int_{g_L^1}^{g_U^1} \int_{g_L^2}^{g_U^2} \int_{g_L^3}^{g_U^3} (f) \varepsilon_{ijk}\, dx_i^{(1)}\, dx_j^{(2)}\, dx_k^{(3)} \tag{9-6}$$

The notations on the left will be used and it must be understood that the form masks a considerable amount of effort to actually evaluate the integral. The generic argument f may be a scalar, vector or tensor function, ∂R denotes the surface of a body that occupies a region R in three dimensions. The limits of integration for the integrals on the right are described as lower and upper limits for the respective variables, and these limits may be functions of the other variables.

9.3 Relationships Between Surface and Volume Integrals

Green's Theorem

We start with Green's theorem as given. The theorem will be derived later. Let R be a bounded region with piecewise smooth, orientable surface ∂R. Let $\phi(\boldsymbol{r})$ be a scalar function of position defined over $R+\partial R$ of class C^1 (the function and its gradient are continuous). Then

$$\int_{\partial R} \phi \boldsymbol{n} dA = \int_R \phi \bar{\bar{\nabla}} dV \tag{9-7}$$

where dV is a volume element, dA is an area element on the surface with $\boldsymbol{n}$ an outward directed unit normal vector. As an example, suppose $\phi = 1$. Then (9-7) yield the intuitive result that

$$\int_{\partial R} \boldsymbol{n} dA = \boldsymbol{0} \qquad \forall\ R \tag{9-8}$$

Suppose the dependent variable is a product of two scalar functions, or $\phi = fg$. Then (9-7) becomes

$$\int_{\partial R} fg\boldsymbol{n} dA = \int_R [(f\bar{\bar{\nabla}})g + (g\bar{\bar{\nabla}})f]\, dV \tag{9-9}$$

Rearranging terms yields

$$\int_R (f\bar{\bar{\nabla}})g\, dV = \int_{\partial R} fg\boldsymbol{n} dA - \int_R (g\bar{\bar{\nabla}})f\, dV \tag{9-10}$$

which is the three-dimensional version of integration by parts.

Gradient Theorem for a Vector Field

In (9-7), let $\phi(\boldsymbol{r}) = \boldsymbol{u} \cdot \boldsymbol{v}(\boldsymbol{r})$ where $\boldsymbol{u}$ is a constant but arbitrary vector. Then (9-7) becomes

$$\boldsymbol{u} \cdot \int_{\partial R} \boldsymbol{v} \otimes \boldsymbol{n} dA = \boldsymbol{u} \cdot \int_R \boldsymbol{v} \bar{\bar{\nabla}} dV \tag{9-11}$$

in which $\boldsymbol{u}$ can be taken outside the integrals because it is constant. Now, since $\boldsymbol{u}$ is arbitrary, it follows that

$$\int_{\partial R} \boldsymbol{v} \otimes \boldsymbol{n} dA = \int_R \boldsymbol{v} \bar{\bar{\nabla}} dV \tag{9-12}$$

which is the gradient theorem for vectors.

Divergence Theorem for a Vector Field

Take the trace of each of the left and right sides of (9-12) to obtain

$$\int_{\partial R} \boldsymbol{v} \cdot \boldsymbol{n} dA = \int_{R} \boldsymbol{v} \cdot \bar{\bar{\nabla}} dV \tag{9-13}$$

which is the divergence theorem for vectors.

<u>Gradient Theorem for a Tensor Field</u>

In (9-12), let $\boldsymbol{v} = \boldsymbol{u} \cdot \boldsymbol{T}(\boldsymbol{r})$ where $\boldsymbol{u}$ is a constant but arbitrary vector. Then (9-12) becomes

$$\boldsymbol{u} \cdot \int_{\partial R} \boldsymbol{T} \otimes \boldsymbol{n} dA = \boldsymbol{u} \cdot \int_{R} \boldsymbol{T} \bar{\bar{\nabla}} dV \tag{9-14}$$

with $\boldsymbol{u}$ factored out because it is constant. Since $\boldsymbol{u}$ is an arbitrary vector, (9-14) yields the gradient theorem for tensors:

$$\int_{\partial R} \boldsymbol{T} \otimes \boldsymbol{n} dA = \int_{R} \boldsymbol{T} \bar{\bar{\nabla}} dV \tag{9-15}$$

<u>Divergence Theorem for a Tensor Field</u>

Apply the contraction operator, C_{23}, to the left and right side of (9-15) to obtain the divergence theorem for tensors:

$$\int_{\partial R} \boldsymbol{T} \cdot \boldsymbol{n} dA = \int_{R} \boldsymbol{T} \cdot \bar{\bar{\nabla}} dV \tag{9-16}$$

<u>Curl Theorem for Vectors</u>

Apply the operator, $\boldsymbol{\varepsilon} \cdot\cdot$, (recall that $\boldsymbol{\varepsilon}$ is the third-order alternating tensor) to the left and right sides of (9-12) to obtain the curl theorem for vectors:

$$\int_{\partial R} \boldsymbol{v} \times \boldsymbol{n} dA = \int_{R} \boldsymbol{v} \times \bar{\bar{\nabla}} dV \tag{9-17}$$

<u>Curl Theorem for Tensors</u>

Now we apply a combination of contraction and multiplication of the alternating tensor to the left and right sides of (9-15):

$$C_{25} C_{36} \left[\boldsymbol{\varepsilon} \otimes \int_{\partial R} \boldsymbol{T} \otimes \boldsymbol{n} dA \right] = C_{25} C_{36} \left[\boldsymbol{\varepsilon} \otimes \int_{R} \boldsymbol{T} \bar{\bar{\nabla}} dV \right] \tag{9-18}$$

The result is the curl theorem for tensors:

$$\int_{\partial R} \boldsymbol{T} \times \boldsymbol{n} \, dA = \int_{R} \boldsymbol{T} \times \bar{\bar{\nabla}} dV \tag{9-19}$$

General Form for All Theorems in this Section

Let $\odot$ be a generic operator, and f a generic description of a scalar, vector and tensor function. Then all of the theorems in this section can be summarized as follows:

$$\int_{\partial R} f \odot \boldsymbol{n}\, dA = \int_{R} f \odot \bar{\nabla}\, dV \tag{9-20}$$

If f is a scalar, then $\odot$ denotes nothing. If f is a vector or a tensor, then $\odot$ denotes a tensor, dot or cross product.

9.4 Relationships Between Line and Surface Integrals

Green's Theorem in Two Dimensions

Suppose the region lies in the $x_1 - x_2$ plane. Then Green's theorem reduces to

$$\iint_{\partial R_2} (\phi,_1 \boldsymbol{e}_1 + \phi,_2 \boldsymbol{e}_2)\, dx_1 dx_2 = \oint_{\partial_L R_2} \phi(\boldsymbol{e}_1\, dx_2 - \boldsymbol{e}_2\, dx_1) \tag{9-21}$$

in which ∂R_2 denotes the planar region and $\partial_L R_2$ denotes the line that forms the boundary of the region.

Divergence Theorem in Two Dimensions

As before let $\phi = \boldsymbol{u} \cdot \boldsymbol{v}$ in which $\boldsymbol{u}$ is a constant but arbitrary vector. Then (9-21) yields

$$\iint_{\partial R_2} (\boldsymbol{v},_1 \otimes \boldsymbol{e}_1 + \boldsymbol{v},_2 \otimes \boldsymbol{e}_2)\, dx_1 dx_2 = \oint_{\partial_L R_2} \boldsymbol{v} \otimes (\boldsymbol{e}_1\, dx_2 - \boldsymbol{e}_2\, dx_1) \tag{9-22}$$

Now we take the trace or, equivalently, apply the contraction operator C_{12} to obtain the corresponding divergence theorem

$$\iint_{\partial R_2} (v_1,_1 + v_2,_2)\, dx_1 dx_2 = \oint_{\partial_L R_2} (v_1\, dx_2 - v_2\, dx_1) \tag{9-23}$$

The divergence theorem in one-dimension is the basic integration equation

$$\int_a^b v,_x\, dx = v\big|_a^b \tag{9-24}$$

Stokes' Theorem

For this case we consider a two-dimensional surface, $\partial \hat{R}_2$, that may not be planar, with a boundary line, $\partial_L \hat{R}_2$, that also may not be planar. We define a unit vector, $\boldsymbol{N}$, normal to the surface and a unit vector, $\boldsymbol{t}$, tangent to the boundary line. There are two choices for each vector, differing by a sign. Let $\boldsymbol{n}_{lat}$ be a unit vector tangent to the surface and perpendicular to the edge. The usual convention is to choose $\boldsymbol{t}$ such that $\boldsymbol{t} = \boldsymbol{N} \times \boldsymbol{n}_{lat}$. The sketch in Fig. 9-1 illustrates these three vectors. Another way of interpreting these vectors is to assume you are walking around the base of a mountain, the boundary line, in a counterclockwise direction. Then the unit normal, $\boldsymbol{N}$, is the normal to the surface of the mountain. Also shown in Fig. 9-1 is an arbitrary subregion, $\partial \hat{R}_2^*$, with boundary, $\partial_L \hat{R}_2^*$, that will be used later to prove Stokes' Theorem.

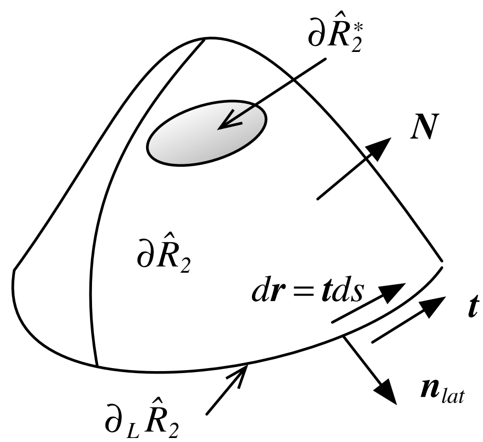

Fig. 9-1. Notation for use with Stokes' Theorem.

The theorem states that for a vector field, $\boldsymbol{v}$,

$$\oint_{\partial_L \hat{R}_2} \boldsymbol{v} \cdot d\boldsymbol{r} = - \int_{\partial \hat{R}_2} (\boldsymbol{v} \times \bar{\nabla}) \cdot \boldsymbol{N} \hat{d}A \tag{9-25}$$

If we define the vector, $\boldsymbol{v}$, as velocity, or the time derivative of the position, then

$$\oint_{\partial_L \hat{R}_2} \boldsymbol{v} \cdot d\boldsymbol{r} = \oint_{\partial_L \hat{R}_2} \boldsymbol{v} \cdot \boldsymbol{t}\, ds \tag{9-26}$$

and the right side of (9-25) is called the circulation.

9.5 Proof of Green's Theorem

Next we sketch the proof of Green's theorem since it is the basis of a number of other theorems relating integrals in a given dimension to integrals in a lower dimension. For the sake of convenience, we repeat (9-7) as

$$\int_{\partial R} \phi \boldsymbol{n} dA = \int_{R} \phi \tilde{\nabla} \, dV \tag{9-27}$$

For convenience we use the coordinate labels x, y and z. First we consider the z-component of (9-27), or

$$\int_{\partial R} \phi n_z dA = \int_{R} \phi,_z \, dV \tag{9-28}$$

If we prove that (9-28) holds, then the corresponding equations for the other two components hold by analogy.

First we "cut" a complex region, R, into a number of "simple" subregions formed by planar cuts with normal perpendicular to the z-axis, of which a typical one is denoted as R^* as indicated in the sketch of Fig. 9-2. We define the surfaces of the subregion R^* specifically as follows:

$$\begin{array}{ll} \text{Surface } \partial R_1^*: x = c_1 & \text{Surface } \partial R_2^*: x = c_2 \\ \text{Surface } \partial R_3^*: y = c_3 & \text{Surface } \partial R_4^*: y = c_4 \\ \text{Surface } \partial R_5^*: z = f_L(x,y) & \text{Surface } \partial R_6^*: z = f_U(x,y) \end{array} \tag{9-29}$$

where $c_1 \cdots c_4$ are constants. Sides 1, 2, 3 and 4 are cuts within the body to obtain the subregion, while Sides 5 and 6 are lower and upper surfaces of the subregion that also represent portions of the surface of the original region.

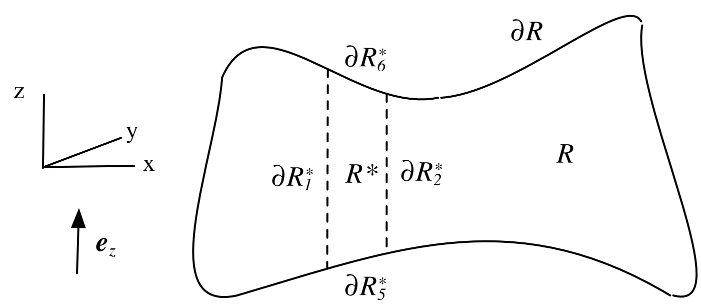

Fig. 9-2. Complex region with subregion with cut surfaces perpendicular to $\boldsymbol{e}_z$.

We compute the contributions to the integrals of (9-28) from each subregion and add them together to get the total contribution for each integral. First consider the surface integral of (9-28) for the subregion. There are six area contributions:

$$\int_{\partial R^*} \phi n_z dA = \sum_{i=1}^{4} \int_{\partial R_i^*} \phi n_z dA + \int_{\partial R_5^*} \phi n_z dA + \int_{\partial R_6^*} \phi n_z dA \tag{9-30}$$

The first four contributions over artificially cut surfaces are zero because $n_z = 0$. Next we are going to show that for each subregion, the following modified form of Green's theorem holds:

$$\int_{\partial R_5^*} \phi n_z dA + \int_{\partial R_6^*} \phi n_z dA = \int_{R^*} \phi,_z dV \tag{9-31}$$

First consider the volume integral of (9-31):

$$\begin{aligned}\int_{R^*} \phi,_z dV &= \int_{c_1}^{c_2}\int_{c_3}^{c_4}\int_{f_L}^{f_U} \frac{\partial \phi}{\partial z} dz dy dx = \int_{c_1}^{c_2}\int_{c_3}^{c_4} \phi\Big|_{z=f_L}^{z=f_U} dy dx \\ &= \int_{c_1}^{c_2}\int_{c_3}^{c_4} \left[\phi\{x,y,f_U(x,y)\} - \phi\{x,y,f_L(x,y)\}\right] dy dx\end{aligned} \tag{9-32}$$

Now we turn to the area integrals on the left side of (9-31) and obtain an expression for the differential of area expressed in terms of an area element in the x-y plane, dA_p:

$$dxdy \equiv dA_p = \cos\theta dA \tag{9-33}$$

But $\cos\theta = n_z$ on the upper surface, and $\cos\theta = -n_z$ on the lower surface as indicated in Fig. 9-3. Therefore the left side of (9-31) becomes

$$\begin{aligned}\int_{\partial R_5^*} \phi n_z dA + \int_{\partial R_6^*} \phi n_z dA &= \int_{\partial R_5^*} -\phi\big|_{z=f_L} \cos\theta dA + \int_{\partial R_6^*} \phi\big|_{z=f_L} \cos\theta dA \\ &= \int_{c_1}^{c_2}\int_{c_3}^{c_4} \left[\phi\{x,y,f_U(x,y)\} - \phi\{x,y,f_L(x,y)\}\right] dy dx\end{aligned} \tag{9-34}$$

which is the same as the right side of (9-32).

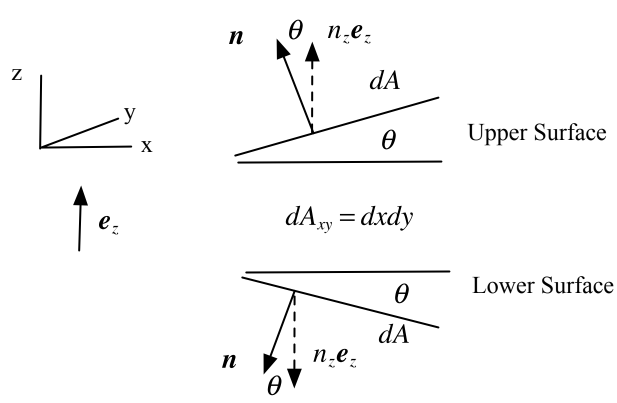

Fig. 9-3. Area elements on upper and lower surfaces with projections to the x-y plane.

Since (9-31) holds, when all contributions from the subregions are added, the result is that we have proven the z-component of Green's Theorem holds. In an analogous manner, by using cuts with normals perpendicular to the x-and y-directions, respectively, we obtain the same result for the other two components and the proof of Green's Theorem is complete.

9.6 Green's Theorem in Two Dimensions

Suppose the body is in the form of a pancake of small thickness, h, with center-plane of the pancake lying in the $x_1 - x_2$ plane. Suppose the dependent variable is the function $\phi(x_1, x_2)$. The contributions to the surface integral from the upper and lower surfaces $x_3 = \pm h/2$ will cancel because the component of the unit normal n_3 on the lower surface is +1 on the upper surface and -1 on the lower surface. If ∂R_{lat} denotes the lateral surface, Green's Theorem of (9-7) becomes

$$\int_R \phi \bar{\nabla} dV = \int_{\partial R_{lat}} \phi \boldsymbol{n} \, dA \tag{9-35}$$

But

$$dV = h dx_1 dx_2 \qquad \mathrm{dA}_{lat} = hds \tag{9-36}$$

in which ds is a differential of length of the boundary in the $x_1 - x_2$ plane as illustrated in Fig. 9-4. Since ϕ does not depend on x_3, Green's Theorem becomes

$$\int_{\partial \hat{R}} (\phi,_1 \boldsymbol{e}_1 + \phi,_2 \boldsymbol{e}_2)\, h \, dx_1 dx_2 = \int_{\partial_L \hat{R}} \phi(n_1 \boldsymbol{e}_1 + n_2 \boldsymbol{e}_2)\, h ds \tag{9-37}$$

in which $\partial \hat{R}$ denotes the two-dimensional domain with the line $\partial_L \hat{R}$ as boundary.

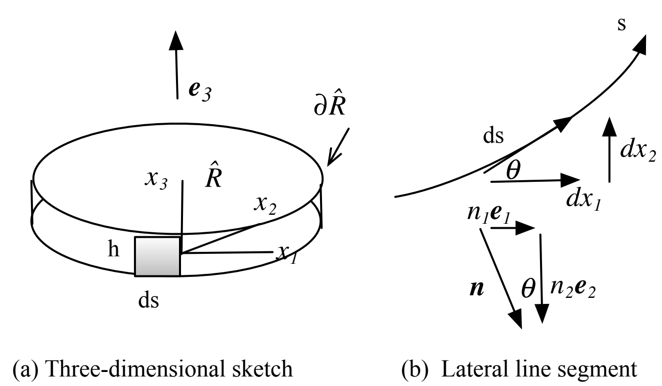

Fig. 9-4. Transition from 3-d to 2-d for Green's Theorem.

From Fig. 9-4 we make the following observations:

$$\begin{array}{ll} \cos\theta ds = dx_1 & \sin\theta ds = dx_2 \\ \cos\theta = -n_2 & \sin\theta = n_1 \end{array} \tag{9-38}$$

After a division by h Green's Theorem becomes the following two-dimensional form:

$$\iint_{\partial \hat{R}} (\phi,_1 \boldsymbol{e}_1 + \phi,_2 \boldsymbol{e}_2)\, dx_1 dx_2 = \oint_{\partial_L \hat{R}} \phi(\boldsymbol{e}_1 dx_2 - \boldsymbol{e}_2 dx_1) \tag{9-39}$$

9.7 Proof of Stokes' Theorem

In order to utilize our various versions of Green's Theorem consider the surface sketched in Fig. 9-3 to be the mid-surface of a shell of constant thickness h, which we choose to be infinitesimally small. Now the three-dimensional body has three surfaces: the "upper" surface, ∂R_U, with normal $\boldsymbol{N}$, the "lower" surface, ∂R_L, with normal $-\boldsymbol{N}$, and the lateral surface, ∂R_{lat}, with normal $\boldsymbol{n}_{lat}$.

We will use the curl and divergence theorems for vectors which we summarize here, for a vector field, $\boldsymbol{w}$, as

$$\int_R \boldsymbol{w} \times \bar{\bar{\nabla}}\, dV = \int_{\partial R} \boldsymbol{w} \times \boldsymbol{n}\, dA \qquad \int_R \boldsymbol{w} \cdot \bar{\bar{\nabla}}\, dV = \int_{\partial R} \boldsymbol{w} \cdot \boldsymbol{n}\, dA \tag{9-40}$$

For the body under consideration, the curl Theorem becomes

$$h \int_{\partial \hat{R}_2} \boldsymbol{w} \times \bar{\bar{\nabla}}\, dA = h \int_{\partial R_U} \boldsymbol{w} \times \boldsymbol{N}\, dA + h \int_{\partial R_U} \boldsymbol{w} \times (-\boldsymbol{N})\, dA + h \oint_{\partial_L \hat{R}_2} \boldsymbol{w} \times \boldsymbol{n}_{lat} ds \tag{9-41}$$

in which $\partial \hat{R}_2$ denotes the surface of interest with boundary defined by the line $\partial_L \hat{R}_2$ as indicated in Fig. 9-3. For small h the first two terms on the right side cancel. Then we divide by h to obtain

$$\int_{\partial \hat{R}_2} \boldsymbol{w} \times \bar{\bar{\nabla}}\, dA = \oint_{\partial_L \hat{R}_2} \boldsymbol{w} \times \boldsymbol{n}_{lat} ds \tag{9-42}$$

In a similar fashion, the divergence theorem yields

$$\int_{\partial \hat{R}_2} \boldsymbol{w} \cdot \bar{\bar{\nabla}}\, dA = \oint_{\partial_L \hat{R}_2} \boldsymbol{w} \cdot \boldsymbol{n}_{lat} ds \tag{9-43}$$

Now we substitute $\boldsymbol{w} = \boldsymbol{N}$ in (9-42) to obtain

$$\int_{\partial \hat{R}_2} \boldsymbol{N} \times \bar{\bar{\nabla}}\, dA = \oint_{\partial_L \hat{R}_2} \boldsymbol{N} \times \boldsymbol{n}_{lat} ds \tag{9-44}$$

All of the relations in this subsection and, in particular, (9-44) apply to any subregion:

$$\int_{\partial \hat{R}_2^*} \boldsymbol{N} \times \bar{\bar{\nabla}}\, dA = \oint_{\partial_L \hat{R}_2^*} \boldsymbol{N} \times \boldsymbol{n}_{lat} ds \tag{9-45}$$

But on the line boundary, $\boldsymbol{N} \times \boldsymbol{n}_{lat} = \boldsymbol{t}$, a unit vector tangent to the boundary. We use the identity

$$\oint_{\partial_L \hat{R}_2^*} \boldsymbol{t}\, ds = 0 \qquad \forall \ \partial \hat{R}_2^* \tag{9-46}$$

that is the line integral equals zero for all closed lines, not just a single particular one. The combination of (9-45) and (9-46) yields

$$\int_{\partial \hat{R}_2^*} \boldsymbol{N} \times \bar{\nabla}\, dA = \boldsymbol{0} \qquad \forall \ \hat{R}_2^* \tag{9-47}$$

The only way this equation can be satisfied for arbitrary choice of domain, $\partial \hat{R}_2^*$, is that

$$\boldsymbol{N} \times \bar{\nabla} = \boldsymbol{0} \tag{9-48}$$

Note: Another way to obtain (9-48) is to recall that the equation of the surface, $\partial \hat{R}_2$, can be written as $\phi(\boldsymbol{r}) = 0$. The gradient, $\phi \bar{\nabla}$, is proportional to the unit normal, $\boldsymbol{N}$. Then the curl of $\boldsymbol{N}$ is the curl of the gradient of a scalar function and the result is a null vector.

Next we choose $\boldsymbol{w} = \boldsymbol{v} \times \boldsymbol{N}$ and apply (9-43). With the use of (9-48) the argument in the surface integral becomes

$$\boldsymbol{w} \cdot \bar{\nabla} = (\boldsymbol{v} \times \boldsymbol{N}) \cdot \bar{\nabla} = \boldsymbol{v} \cdot (\boldsymbol{N} \times \bar{\nabla}) - (\boldsymbol{v} \times \bar{\nabla}) \cdot \boldsymbol{N} = -(\boldsymbol{v} \times \bar{\nabla}) \cdot \boldsymbol{N} \tag{9-49}$$

and the argument in the line integral is

$$\boldsymbol{w} \cdot \boldsymbol{n}_{lat} ds = (\boldsymbol{v} \times \boldsymbol{N}) \cdot \boldsymbol{n}_{lat} ds = \boldsymbol{v} \cdot (\boldsymbol{N} \times \boldsymbol{n}_{lat}) ds = \boldsymbol{v} \cdot \boldsymbol{t} ds = \boldsymbol{v} \cdot d\boldsymbol{r} \tag{9-50}$$

Then (9-43) becomes

$$\oint_{\boldsymbol{\partial_L \hat{R}_2}} \boldsymbol{v} \cdot d\boldsymbol{r} = -\int_{\partial \hat{R}_2} (\boldsymbol{v} \times \bar{\nabla}) \cdot \boldsymbol{N}\, dA \tag{9-51}$$

which is Stokes' Theorem.

9.8 Concluding Remarks

In this Chapter we have summarized many of the Theorems that relate volume integrals to surface integrals. The divergence theorems are used extensively in connection with the development of equations of motion and energy relations in continuum mechanics.

We have shown that all of these theorems can be considered variations of just one theorem, namely Green's Theorem. Also the essential aspects of the proof of Green's Theorem are not particularly difficult. The idea of breaking a region into subregions is useful for the development of the concept of a stress tensor. Looking at sub- surfaces is useful for specifying boundary conditions.

Next we move onto continuum mechanics for which the knowledge of this and the previous chapters is taken for granted.