Topic:Heat equation/Solution to the 2-D Heat Equation

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Definition

The solution to the 2-dimensional heat equation (in rectangular coordinates) deals with two spatial and a time dimension, $u(x,y,t)$. The heat equation, the variable limits, the Robin boundary conditions, and the initial condition are defined as:

$$\begin{aligned}
u_t = k\left[u_{xx}+u_{yy}\right]+h(x,y,t)\\
(x,y,t)\in[0,L]\times[0,M]\times[0,\infty)\\
\alpha_1 u(0,y,t)-\beta_1 u_x(0,y,t)=b_1(y,t)\\
\alpha_2 u(L,y,t)+\beta_2 u_x(L,y,t)=b_2(y,t)\\
\alpha_3 u(x,0,t)-\beta_3 u_y(x,0,t)=b_3(x,t)\\
\alpha_4 u(x,M,t)+\beta_4 u_y(x,M,t)=b_4(x,t)\\
u(x,y,0)=f(x,y)
\end{aligned}$$

Solution

The solution is just an advanced version of the solution in 1 dimension. If you have questions about the steps shown here, review the 1-D solution.

Step 1: Partition Solution

Just as in the 1-D solution, we partition the solution into a "steady-state" and a "variable" portion:

$$u(x,y,t)=\underbrace{s(x,y,t)}_{\text{steady-state}}+\underbrace{v(x,y,t)}_{\text{variable}}$$

We substitute this equation into the initial boundary value problem (IBVP):

$$\begin{cases}
s_t+v_t=k\left[s_{xx}+v_{xx}+s_{yy}+v_{yy}\right]+h(x,y,t)\\
\alpha_1 s(0,y,t)+\alpha_1 v(0,y,t)-\beta_1 s_x(0,y,t)-\beta_1 v_x(0,y,t)=b_1(y,t)\\
\alpha_2 s(L,y,t)+\alpha_2 v(L,y,t)+\beta_2 s_x(L,y,t)+\beta_2 v_x(L,y,t)=b_2(y,t)\\
\alpha_3 s(x,0,t)+\alpha_3 v(x,0,t)-\beta_3 s_y(x,0,t)-\beta_3 v_y(x,0,t)=b_3(x,t)\\
\alpha_4 s(x,M,t)+\alpha_4 v(x,M,t)+\beta_4 s_y(x,M,t)+\beta_4 v_y(x,M,t)=b_4(x,t)\\
s(x,y,0)+v(x,y,0)=f(x,y)
\end{cases}$$

We want to set some conditions on s and v:

1. Let s satisfy the Laplace equation: $s_{xx}+s_{yy}=0$.
2. Let s satisfy the non-homogeneous boundary conditions.
3. Let v satisfy the non-homogeneous equation and homogeneous boundary conditions.

We end up with 2 separate IBVPs:

$$\begin{cases} s_{xx} + s_{yy} = 0 \\ \alpha_1 s(0,y,t) - \beta_1 s_x(0,y,t) = b_1(y,t) \\ \alpha_2 s(L,y,t) + \beta_2 s_x(L,y,t) = b_2(y,t) \\ \alpha_3 s(x,0,t) - \beta_3 s_y(x,0,t) = b_3(x,t) \\ \alpha_4 s(x,M,t) + \beta_4 s_y(x,M,t) = b_4(x,t) \end{cases}$$

$$\begin{cases} v_t = k\left[v_{xx} + v_{yy}\right] + h(x,y,t) - s_t(x,y,t) \\ \alpha_1 v(0,y,t) - \beta_1 v_x(0,y,t) = 0 \\ \alpha_2 v(L,y,t) + \beta_2 v_x(L,y,t) = 0 \\ \alpha_3 v(x,0,t) - \beta_3 v_y(x,0,t) = 0 \\ \alpha_4 v(x,M,t) + \beta_4 v_y(x,M,t) = 0 \\ v(x,y,0) = f(x,y) - s(x,y,0) \end{cases}$$

Step 2: Solve Steady-State Portion

Solving for the steady-state portion is exactly like solving the Laplace equation with 4 non-homogeneous boundary conditions. Using that technique, a solution can be found for all types of boundary conditions.

Step 3: Solve Variable Portion

Step 3.1: Solve Associated Homogeneous BVP

The associated homogeneous BVP equation is:

$$v_t = k\left[v_{xx} + v_{yy}\right]$$

The boundary conditions for v are the ones in the IBVP above.

Separate Variables

$$v(x,y,t) = X(x)Y(y)T(t)$$

$$\Rightarrow XYT' = k[X''YT + XY''T]$$

$$\Rightarrow \frac{T'}{kT} = \frac{X''}{X} + \frac{Y''}{Y} = \mu$$

By similar methods, you obtain the following ODEs:

$$\begin{cases} T' - \mu k T = 0 \\ X'' - \rho X = 0 \\ Y'' - \delta Y = 0 \\ \mu = \rho + \delta \quad \text{(coupling equation)} \end{cases}$$

Translate Boundary Conditions

$$\left.\begin{aligned} \left[\alpha_1 X(0) - \beta_1 X'(0)\right] Y(y)T(t) = 0 \\ \left[\alpha_2 X(L) + \beta_2 X'(L)\right] Y(y)T(t) = 0 \\ \left[\alpha_3 Y(0) - \beta_3 Y'(0)\right] X(x)T(t) = 0 \\ \left[\alpha_4 Y(M) + \beta_4 Y'(M)\right] X(x)T(t) = 0 \end{aligned}\right\} \Rightarrow \quad \begin{aligned} \alpha_1 X(0) - \beta_1 X'(0) = 0 \\ \alpha_2 X(L) + \beta_2 X'(L) = 0 \\ \alpha_3 Y(0) - \beta_3 Y'(0) = 0 \\ \alpha_4 Y(M) + \beta_4 Y'(M) = 0 \end{aligned}$$

Solve SLPs

$$\left.\begin{aligned} X''-\rho X=0 \\ \alpha_1 X(0)-\beta_1 X'(0)=0 \\ \alpha_2 X(L)+\beta_2 X'(L)=0 \end{aligned}\right\} \begin{array}{l} -\rho=\lambda^2 \\ \text{Eigenvalues } \lambda_n\text{: solutions to equation } (\alpha_1\alpha_2-\beta_1\beta_2\lambda^2)\sin(\lambda L)+(\alpha_1\beta_2+\alpha_2\beta_1)\lambda\cos(\lambda L)=0 \\ X_n(x)=\beta_1\lambda_n\cos(\lambda_n x)+\alpha_1\sin(\lambda_n x), n=0,1,2,\cdots \end{array}$$

$$\left.\begin{aligned} Y''-\delta Y=0 \\ \alpha_3 Y(0)-\beta_3 Y'(0)=0 \\ \alpha_4 Y(M)+\beta_4 Y'(M)=0 \end{aligned}\right\} \begin{array}{l} -\delta=\hat{\lambda}^2 \\ \text{Eigenvalues } \hat{\lambda}_m\text{: solutions to equation } (\alpha_3\alpha_4-\beta_3\beta_4\hat{\lambda}^2)\sin(\hat{\lambda} M)+(\alpha_3\beta_4+\alpha_4\beta_3)\hat{\lambda}\cos(\hat{\lambda} M)=0 \\ Y_m(x)=\beta_3\hat{\lambda}_m\cos(\hat{\lambda}_m y)+\alpha_3\sin(\hat{\lambda}_m y), m=0,1,2,\cdots \end{array}$$

We have obtained eigenfunctions that we can use to solve the nonhomogeneous IBVP.

Step 3.2: Solve Non-homogeneous IBVP

Setup Problem

Just like in the 1-D case, we define v(x,y,t) and q(x,y,t) as infinite sums:

$$v(x,y,t):=\sum_{m,n=0}^{\infty} T_{mn}(t)X_n(x)Y_m(y)$$

$$q(x,y,t):=\sum_{m,n=0}^{\infty} Q_{mn}(t)X_n(x)Y_m(y)\ ,\ Q_{mn}(t)=\frac{\int\limits_0^L\int\limits_0^M q(x,y,t)X_n(x)Y_m(y)dydx}{\int\limits_0^L X_n^2(x)dx\int\limits_0^M Y_m^2(y)dy}$$

Determine Coefficients

We then substitute expansion into the PDE:

$$\frac{\partial}{\partial t}\left[\sum T_{mn}(t)X_n(x)Y_m(y)\right]=k\left\{\frac{\partial}{\partial x^2}\left[\sum T_{mn}(t)X_n(x)Y_m(y)\right]+\frac{\partial}{\partial y^2}\left[\sum T_{mn}(t)X_n(x)Y_m(y)\right]\right\}+\sum Q_{mn}(t)X_n(x)Y_m(y)$$

$$\Rightarrow \sum T'_{mn}(t)X_n(x)Y_m(y)=k\left\{\sum T_{mn}(t)X''_n(x)Y_m(y)+\sum T_{mn}(t)X_n(x)Y''_m(y)\right\}+\sum Q_{mn}(t)X_n(x)Y_m(y)$$

$$\Rightarrow \sum T'_{mn}(t)X_n(x)Y_m(y)=\sum k\left\{T_{mn}(t)[-\lambda_n^2X_n(x)]Y_m(y)+\sum T_{mn}(t)X_n(x)[-\hat{\lambda}_m^2Y_m(y)]\right\}+\sum Q_{mn}(t)X_n(x)Y_m(y)$$

$$\Rightarrow \sum\left[T'_{mn}(t)+k(\lambda_n^2+\hat{\lambda}_m^2)\right]X_n(x)Y_m(y)=\sum Q_{mn}(t)X_n(x)Y_m(y)$$

This implies that $X_n(x)\otimes Y_m(y)$ forms an orthogonal basis. This means that we can write the following:

$$\Rightarrow T'_{mn}(t)+k(\lambda_n^2+\hat{\lambda}_m^2)=Q_{mn}(t)$$

This is a first-order ODE which can be solved using the integration factor:

$$\mu(t)=e^{\int k(\lambda_n^2+\hat{\lambda}_m^2)dt}=e^{k(\lambda_n^2+\hat{\lambda}_m^2)t}$$

Solving for our coefficient we get:

$$T_{mn}(t)=e^{-k(\lambda_n^2+\hat{\lambda}_m^2)t}\int\limits_0^t e^{k(\lambda_n^2+\hat{\lambda}_m^2)s}Q_{mn}(s)ds+C_{mn}e^{-k(\lambda_n^2+\hat{\lambda}_m^2)t}$$

Satisfy Initial Condition

We apply the initial condition to our equation above:

$$\begin{aligned} v(x,y,0) &= f(x,y) - s(x,y,0) \\ &= \sum T_{mn}(0)X_n(x)Y_m(y) \\ &= \sum C_{mn}X_n(x)Y_m(y) \end{aligned}$$

The Fourier coefficients can be solved using the inner product definition:

$$C_{mn} = \frac{\int_0^L \int_0^M \left[f(x,y) - s(x,y,0)\right] X_n(x)Y_m(y)dydx}{\int_0^L X_n^2(x)dx \int_0^M Y_m^2(y)dy}$$

We have all the necessary information about the variable portion of the function.

Step 4: Combine Solutions

We now have solved for the "steady-state" and "variable" portions, so we just add them together to get the complete solution to the 2-D heat equation.