2. TIME-DEPENDENT PROBLEMS IN ONE DIMENSION

2.1 INTRODUCTION

When a new computer program is written, or even if an existing program is being used for the first time, it is useful to have solutions available for model problems. In this chapter such a set of solutions is given for transient problems in heat conduction, axial motion of a bar and transverse motion of a beam. In the process, classical solution procedures are summarized to provide the background that will prove useful for a good understanding of the finite element method. Also, the approach is available for obtaining solutions to other model problems that can be part of the verification process for codes designed to solve these classes of problems.

2.2 TRANSIENT HEAT CONDUCTION

2.2.1 Formulation

Recall from (1.2-9) and (1.2-10) that the governing equations for transient heat conduction in a bar of unit cross section are

$$-q_{,x} + Q = \rho c T_{,t} \ , \qquad q = -KT_{,x} \tag{2.2-1}$$

which can be combined to read

$$(KT_{,x})_{,x} + Q = \rho c T_{,t} \tag{2.2-2}$$

The terms in (2.2-1) and (2.2-2) are defined as follows (typical units in the SI system are included in parentheses):

q - x-component of the heat flux (W/m^2)
T - temperature (K)
ρ - mass per unit volume (kg/m^3)
c - specific heat ($J/kg{\cdot}K$) ($J = W\,s$)

x - spatial coordinate (m)
t - temporal coordinate or time (s)
K - thermal conductivity (W/m·K)
Q - heat source per unit volume (W/m^3)

where W denotes Watts, m is meters, K is Kelvins, J is Joules, kg is kilograms and s is seconds.

The boundary condition imposed at each end (x = 0 and x = L) consists of either a prescribed temperature

$$T = T^*(t) \tag{2.2-3}$$

or a prescribed flux into the body

$$qn = -q^*(t) \tag{2.2-4}$$

In addition, the initial value of T must be given:

$$T(0,x) = \hat{T}(x) \tag{2.2-5}$$

2.2.2 Analytical Solution

Suppose ρ, C and K are constant and let α denote the thermal diffusivity, i.e.,

$$\alpha = \frac{K}{\rho C} \tag{2.2-6}$$

Then for Q = 0, (2.2-2) becomes

$$T,_t = \alpha T,_{xx} \tag{2.2-7}$$

with the initial condition

$$T(0,x) = \hat{T}(x) \tag{2.2-8}$$

and the boundary condition of (2.2-3) or (2.2-4). To obtain a solution, the classical approach is to seek a similarity solution. Define the similarity variable, η, as

$$\eta = \frac{x}{\sqrt{\alpha t}} \tag{2.2-9}$$

If a similarity solution exists for this problem, the temperature depends only upon η, i.e., we consider $T(\eta)$ as a possible solution. With the use of the chain rule,

$$\begin{gathered} T_{,t} = T_{,\eta}\,\eta_{,t} = -\frac{1}{2}\frac{\alpha x}{(\alpha t)^{3/2}}\,T_{,\eta} \\ T_{,x} = T_{,\eta}\,\eta_{,x} = \frac{1}{\sqrt{\alpha t}}\,T_{,\eta} \quad \text{and} \quad T_{,xx} = \frac{1}{\alpha t}\,T_{,\eta\eta} \end{gathered} \tag{2.2-10}$$

The homogeneous differential equation becomes

$$T_{,\eta\eta} + \tfrac{1}{2}\eta T_{,\eta} = 0 \tag{2.2-11}$$

One solution is $T = c_1$, a constant. Let $\Psi = T_{,\eta}$. Then the governing differential equation can be rewritten as

$$\frac{\Psi_{,\eta}}{\Psi} = -\tfrac{1}{2}\eta \qquad \text{or} \qquad \frac{d}{d\eta}(\ln\Psi) = -\tfrac{1}{2}\eta \tag{2.2-12}$$

Integrating once yields

$$\ln\Psi = -\tfrac{1}{4}\eta^2 + c^* \tag{2.2-13}$$

or

$$\Psi = c_2^* e^{-\eta^{*2}}, \qquad \eta^* = \tfrac{\eta}{2} \tag{2.2-14}$$

A second integration to obtain T yields

$$T = c_1 + c_2 \mathrm{erf}(\eta^*) \tag{2.2-15}$$

where erf(η^*) is called the error function. It is normally tabulated as a probability integral defined by

$$\mathrm{erf}(\eta^*) = \frac{2}{\sqrt{\pi}} \int_0^{\eta^*} e^{-\phi^2} d\phi \tag{2.2-16}$$

The coefficients c^* and c^*_2 have been absorbed into the integration constant c_2.

The analytical solution given by (2.2-15) is a natural one for considering semi-infinite problems ($0 < x < \infty$) in which T is a prescribed constant, c_1, at the boundary x = 0 ($\eta^* = 0$). The initial condition at t = 0 corresponds to the solution at $\eta^* \to \infty$ for which the error function is one. Therefore c_1+c_2 can be interpreted as the initial (constant) temperature.

The use of the analytical solution for general forms of initial and boundary conditions requires further manipulation. Instead of presenting examples the approach based on separation of variables is given next.

2.2.3 Procedure for Obtaining Series Solutions

Again, suppose ρ, C and K are constant and let α denote the thermal diffusivity so that the governing equation is

$$T_{,t} = \alpha T_{,xx} \tag{2.2-17}$$

with the initial condition

$$T(0,x) = \hat{T}(x) \tag{2.2-18}$$

and the boundary condition of (2.2-3) or (2.2-4). Suppose further that neither the boundary condition functions nor the internal heat source is a function of time, that is, T* is constant, q* is constant and

$$Q(x,t) = Q(x) \tag{2.2-19}$$

Let the temperature be represented as the sum of a steady-state contribution, T_s, and a time-varying part, $\overline{T}$, viz.,

$$T(t,x) = T_s(x) + \overline{T}(t,x) \qquad (2.2\text{-}20)$$

where $T_S(x)$ satisfies the steady state equation, i.e.,

$$T_{s},_{xx} + \frac{Q}{K} = 0 \qquad (2.2\text{-}21)$$

and a suitable combination of boundary conditions consisting of either a prescribed constant temperature at each boundary, or a prescribed constant temperature at one boundary and a prescribed constant flux at the other boundary. Then the time-dependent part of the solution satisfies the equation

$$\frac{1}{\alpha}\overline{T},_t = \overline{T},_{xx} \qquad (2.2\text{-}22)$$

which must be solved subject to homogeneous (zero) boundary conditions. Also, the initial condition becomes

$$\overline{T}(0,x) = T(0,x) - T_s(x) = \hat{T}(x) - T_s(x) \qquad (2.2\text{-}23)$$

Let $T_d(x)$ denote the difference between the initial value of T and the steady-state value, i.e.,

$$T_d(x) = \hat{T}(x) - T_s(x) \qquad (2.2\text{-}24)$$

which is known once the steady-state solution is found. Then the initial condition on $\overline{T}$ is

$$\overline{T}(0,x) = T^*(x) \qquad (2.2\text{-}25)$$

In summary, suppose the restrictions on C, K, Q and the boundary functions are met and that $T_S(x)$ can be found using elementary integrations. The remaining problem is to solve (2.2-22) for $\overline{T}$ subject to homogenous boundary conditions in place of the original boundary conditions and the initial condition given by (2.2-25). Then the complete solution is the sum $T_s(x) + \overline{T}(t,x)$. The approach is to assume $\overline{T}$ can be represented as a product of a function of t and a function of x, i.e., let

$$\overline{T}(t,x) = \Gamma(t)\overline{X}(x) \tag{2.2-26}$$

The **separation-of-variables** solution technique requires a homogeneous differential equation and homogeneous boundary conditions, which is why the steady-state part has been subtracted out. More complex boundary conditions and forcing functions can be handled in conjunction with the separation-of-variables approach but the more general cases are not considered here for the sake of simplicity.

It follows from (2.2-26) that

$$\overline{T}_{,t} = \Gamma_{,t}\,\overline{X} \qquad \text{and} \qquad \overline{T}_{,xx} = \Gamma\overline{X}_{,xx} \tag{2.2-27}$$

so that (2.2-22) becomes

$$\frac{1}{\alpha}\Gamma_{,t}\,\overline{X} = \Gamma\overline{X}_{,xx} \tag{2.2-28}$$

Now, divide (2.2-28) through by $\Gamma\overline{X}$ with the result

$$\frac{\Gamma_{,t}}{\Gamma} = \alpha\frac{\overline{X}_{,xx}}{\overline{X}} \tag{2.2-29}$$

The left side is a function of t only, while the right side is a function only of x. Two functions of different variables can be equal only if they equal the same constant value. Choose the constant to be $-\alpha\overline{k}^2$, wherefore

$$\frac{\Gamma_{,t}}{\Gamma} = -\alpha\overline{k}^2 \qquad \text{and} \qquad \alpha\frac{\overline{X}_{,xx}}{\overline{X}} = -\alpha\overline{k}^2 \tag{2.2-30}$$

which can be rewritten as

$$\Gamma_{,t} + \alpha\overline{k}^2\Gamma = 0 \qquad \text{and} \qquad \overline{X}_{,xx} + \overline{k}^2\overline{X} = 0 \tag{2.2-31}$$

These two ordinary differential equations have the solutions

$$\Gamma = e^{-\alpha \bar{k}^2 t} \qquad \text{and} \qquad \bar{X} = a\cos\bar{k}x + b\sin\bar{k}x \tag{2.2-32}$$

Since $\bar{T} = \Gamma\bar{X}$, the constant of integration associated with the first solution has been absorbed in a and b with no loss in generality. The quantity $\bar{k}$ is called the wave number. The choice of $\bar{k}^2$ implies that $-\alpha\bar{k}^2$ is always negative. With this choice of sign, Γ decays with time; a choice of a positive sign would have allowed the solution to become infinite which is unacceptable based on a physical interpretation of the problem.

Now, the function $\bar{X}(x)$ must satisfy the homogenous boundary conditions. This yields, in turn, two conditions:

(i) the coefficients a and b are not independent, and
(ii) $\bar{k}$ must equal one of an infinite number of **eigenvalues** $\bar{k}_n$, $n = 1,2,..\infty$.

Consequently, there is actually an infinite number of solutions represented symbolically as follows:

$$\bar{X}_n = c_n \chi_n(x)\,, \qquad \bar{k} = \bar{k}_n\,, \qquad n = 1,\ 2,\ ...,\ \infty \tag{2.2-33}$$

where c_n represents a sequence of constants and the functions $\chi_n(x)$ are known as **eigenfunctions**. As a consequence of the structure of the differential equation, eigenfunctions satisfy an important property called orthogonality. To show orthogonality, consider any two eigenfunctions, χ_n and χ_m, with $m \neq n$. Each eigenfunction must satisfy the govering differential equation given as the second of (2.2-31):

$$\chi_{n,xx} + \bar{k}_n^2 \chi_n = 0 \qquad\qquad \chi_{m,xx} + \bar{k}_m^2 \chi_m = 0 \tag{2.2-34}$$

Multiply the first equation by χ_m, the second by χ_n and integrate to obtain

$$\begin{aligned}\chi_{n,x}\chi_m\Big|_0^L-\int_0^L\chi_{n,x}\chi_{m,x}dx+\bar{k}_n^2\int_0^L\chi_n\chi_m dx=0\\ \chi_{m,x}\chi_n\Big|_0^L-\int_0^L\chi_{m,x}\chi_{n,x}dx+\bar{k}_m^2\int_0^L\chi_m\chi_n dx=0\end{aligned} \tag{2.2-35}$$

Since the boundary conditions are homogeneous, either the function itself is zero, $\chi_m = 0$, or the gradient is zero, $\chi_{m,x} = 0$, at $x = 0$ and $x = L$ with the identical conditions for χ_n. If the corresponding terms in the two equations of (2.2-35) are subtracted, and since the boundary terms are zero, the result is

$$\left(\bar{k}_n^2-\bar{k}_m^2\right)\int_0^L\chi_n\chi_m dx=0 \tag{2.2-36}$$

For $m \neq n$ and $k_n^2 \neq k_m^2$, we obtain the following **orthogonality condition**:

$$\int_0^L\chi_n\chi_m dx=0\ ,\qquad n\neq m \tag{2.2-37}$$

As a result of solving (2.2-31), the **eigenfunctions** χ_n are known. For convenience, let

$$\psi_n=\int_0^L\chi_n^2 dx \tag{2.2-38}$$

From (2.2-26) the solution assumes the form

$$\bar{T}(t,x)=\sum_{n=1}^{\infty}c_n\chi_n(x)e^{-\alpha\bar{k}_n^2 t} \tag{2.2-39}$$

To evaluate the unknown coefficients, c_n, apply the initial condition given previously in (2.2-25), i.e., $\bar{T}(0,x)=T_d(x)$, to (2.2-39) to obtain

$$\sum_{n=1}^{\infty}c_n\chi_n(x)=T_d(x) \tag{2.2-40}$$

Multiply by $\chi_m(x)$, integrate over the domain and use (2.2-37) and (2.2-38). The result is

$$c_n = \frac{1}{\psi_n}\int_0^L \chi_n(x)T_d(x)dx \tag{2.2-41}$$

which can be evaluated. Thus $\overline{T}(t,x)$ and, then, the complete solution $T(t,x) = T_s(x) + \overline{T}(t,x)$ are obtained explicitly.

2.2.4 Series Solutions

To illustrate the use of the equations associated with the approach based on the assumption of separation of variables, two example problems are given. The first problem is one with mixed boundary conditions while the second has Dirichlet boundary conditions.

<u>Example One:</u>

Consider a problem described by the following data:

(i) the domain, $0 < x < L$;

(ii) the coefficient functions ρ, C, and K, are considered constants;

(iii) the forcing function $Q = 0$;

(iv) the initial condition $\hat{T}(x) = 0$; and

(v) the boundary conditions of $T(t,0) = T_0$, a constant, and $q_L^*(t) = nq(t,L) = -kT_{,x}(t,L) = 0$.

First consider the steady-state part governed by (2.2-21) and the boundary conditions of (v):

$$T_{s,xx} = 0\ , \qquad T_s(0) = T_0\ , \qquad -kT_{s,x}(L) = 0 \tag{2.2-42}$$

The solution is $T_s(x) = T_0$.

Now we seek the transient part, $\overline{T}$. From (2.2-24) the initial condition for $\overline{T}$ is

$$T_d(x) = \hat{T}(x) - T_s(x) = -T_o \tag{2.2-43}$$

From (2.2-26), recall that $\overline{T}(t,x) = \Gamma(t)\,\overline{X}(x)$ with

$$\Gamma = e^{-\alpha \bar{k}^2 t} \qquad \text{and} \qquad \bar{X} = a\cos\bar{k}x + b\sin\bar{k}x \tag{2.2-44}$$

The homogeneous boundary conditions are:

$$\bar{X}(0) = 0 \ , \qquad \bar{X},_x(L) = 0 \tag{2.2-45}$$

which results in

$$\bar{X}(0) = a = 0 \ , \qquad \bar{X},_x(L) = b\bar{k}\cos\bar{k}L = 0 \tag{2.2-46}$$

To obtain a nontrivial solution to the second equation, the argument of the cosine function must be a multiple of $\pi/2$, wherefore

$$\bar{k} = \bar{k}_n = \frac{(2n-1)\pi/2}{L} \tag{2.2-47}$$

Thus, the solution for a typical component $\bar{X}_n$ is

$$\bar{X}_n = c_n \chi_n \ , \qquad \chi_n = \sin\bar{k}_n x \ , \qquad \Gamma_n = e^{-\alpha \bar{k}_n^2 t} \tag{2.2-48}$$

and the solution for the transient part is the sum

$$\bar{T}(t,x) = \sum_{n=1}^{\infty} c_n e^{-\alpha \bar{k}_n^2 t} \sin\bar{k}_n x \tag{2.2-49}$$

For later use, the parameter, ψ_n, defined in (2.2-38) is required. Since the length of the bar, L, times $\bar{k}_n$ is a multiple of $\pi/2$, the equation

$$\int_0^L \sin^2\bar{k}_n x dx = \int_0^L \cos^2\bar{k}_n x dx \tag{2.2-50}$$

is valid and the following result is obtained:

$$\psi_n = \int_0^L \chi_n^2 dx = \int_0^L \sin^2\bar{k}_n x \, dx = \frac{1}{2}\int_0^L (\sin^2\bar{k}_n x + \cos^2\bar{k}_n x) dx = \frac{L}{2} \tag{2.2-51}$$

Here, ψ_n is independent of the index n. Such a result is unusual so, in a sense, this example is a pathological case. Now, we apply (2.2-41) to obtain

$$c_n = \frac{1}{\psi_n}\int_0^L \sin\bar{k}_n x(-\bar{T}_o)dx = \frac{-\bar{T}_0}{\psi_n}\left(\frac{-1}{\bar{k}_n}\right)\cos\bar{k}_n x\Big|_0^L = \frac{T_0}{\psi_n \bar{k}_n}\left[\cos\bar{k}_n L - 1\right] \quad (2.2\text{-}52)$$

However, we know that $\cos\bar{k}_n(2m-1)\pi/2 = 0$. Therefore

$$c_n = -T_0 \cdot \frac{2}{L}\cdot\frac{L}{(2n-1)\pi/2} = \frac{-4T_0}{(2n-1)\pi} \quad (2.2\text{-}53)$$

When all of these results are combined, the complete solution is

$$\begin{aligned} T(t,x) &= T_s(x) + \bar{T}(t,x) \\ &= T_0 - \frac{4T_0}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)}\sin\left\{(2n-1)\frac{\pi x}{2L}\right\}\exp\left\{\frac{-\alpha(2n-1)^2\pi^2 t}{4L^2}\right\} \end{aligned} \quad (2.2\text{-}54)$$

where "exp" denotes the exponential function.

End of Example 1.

<u>Example Two:</u>

Consider a problem described by the following data:

(i) the domain is $0 < x < L$;
(ii) the coefficient functions ρ, C, and K are constants;
(iii) the forcing function $Q = 0$;
(iv) the initial condition $\hat{T}(x) = 0$; and
(v) the boundary conditions are $T(t,0) = T_0$, a constant, and $T(t,L) = 0$.

Since the procedure has already been illustrated with the first example, only the key steps will be highlighted for this case. The steady state solution is readily shown to be

$$T_s(x) = T_0\left(1 - \frac{x}{L}\right) \quad (2.2\text{-}55)$$

so that

$$T_d(x) = -T_0(1 - \tfrac{x}{L}) \tag{2.2-56}$$

The homogeneous form of the boundary conditions imply that

$$\overline{X}(0) = 0 \quad \text{and} \quad \overline{X}(L) = 0 \tag{2.2-57}$$

which yields

$$\sin \overline{k}L = 0 \quad \text{or} \quad \overline{k}L = n\pi\,, \qquad n = 1,2,... \tag{2.2-58}$$

Thus, we conclude that the eigenvalues and eigenfunctions are

$$\overline{k}_n = \frac{n\pi}{L} \qquad \text{and} \qquad \chi_n = \sin n\pi\frac{x}{L} \tag{2.2-59}$$

From (2.2-38) and (2.2-41), the parameter, ψ_n, and the coefficient, c_n, become

$$\begin{aligned} \psi_n &= \int_0^L \sin^2 \overline{k}_n x\, dx = \tfrac{L}{2} \\ c_n &= \frac{1}{\psi_n}\int_0^L \chi_n(x) T^*(x) dx = \tfrac{2}{L}(-T_0)\int_0^L (1 - \tfrac{x}{L}) \sin \overline{k}_n x\, dx \end{aligned} \tag{2.2-60}$$

The integral for c_n involves two terms. The first is

$$\int_0^L \sin \overline{k}_n x\, dx = \frac{-1}{\overline{k}_n} \cos \overline{k}_n x\,\Big|_0^L = \frac{-1}{\overline{k}_n}[\cos n\pi - 1] = \frac{-L}{n\pi}\left[(-1)^n - 1\right] \tag{2.2-61}$$

The second integral, which involves the term (x/L), requires an integration by parts. The result of performing the integral is

$$\int_0^L \frac{x}{L}\sin\bar{k}_n x\,dx = \int_0^L \frac{x}{L}\,d\left(\frac{-1}{\bar{k}_n}\cos\bar{k}_n x\right)$$

$$\begin{aligned}
&= -\frac{1}{\bar{k}_n}\left[\frac{x}{L}\cos\bar{k}_n x\right]_0^L + \frac{1}{\bar{k}_n}\int_0^L \cos\bar{k}_n x\,\frac{dx}{L} \\
&= -\frac{L}{L}\frac{1}{\bar{k}_n}\cos k_n L + \frac{1}{\bar{k}_n L}\left[\frac{1}{\bar{k}_n}\sin\bar{k}_n x\right]_0^L \\
&= -\frac{1}{\bar{k}_n}\cos n\pi + \frac{1}{(\bar{k}_n)^2 L}\left[\sin n\pi - 0\right] = -\frac{L}{n\pi}(-1)^n
\end{aligned} \tag{2.2-62}$$

Therefore, the series coefficient, c_n, is

$$c_n = -\frac{-2T_0}{L}\left\{\frac{-L}{n\pi}\left[(-1)^n - 1\right] - \frac{(-L)}{n\pi}(-1)^n\right\} = \frac{-2T_0}{n\pi} \tag{2.2-63}$$

and the complete solution is

$$\begin{aligned}
T(t,x) &= T_s(x) + \bar{T}(t,x) \\
&= T_0\left(1 - \frac{x}{L}\right) - \frac{2T_0}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\left(n\pi\frac{x}{L}\right)\exp\left\{-\alpha\frac{n^2\pi^2}{L^2}t\right\}
\end{aligned} \tag{2.2-64}$$

End of Example 2.

The basic formulation and two examples have been provided to illustrate the approach for obtaining a series solution to the transient heat conduction equation. This material is important for two reasons: first the techniques and terminology associated with the theoretical development will prove to be of great value in connection with the development of the finite element method and, second, analytical solutions are necessary to verify any numerical procedure that is used to obtain approximate solutions.

2.3 WAVE MOTION IN A BAR

2.3.1 Formulation

Recall from Subsection 1.3.1 that the governing equations for the longitudinal motion of a bar are:

$$(\sigma A),_x + f = \rho A\ddot{u} \ , \qquad \sigma = Ee \ , \qquad e = u,_x \qquad (2.3\text{-}1)$$

or

$$(ku,_x),_x + f = \bar{\rho}\ddot{u} \ , \qquad k = AE \qquad (2.3\text{-}2)$$

where the terms are defined as follows (typical units in the SI system are included in parentheses):

f	- distributed force/unit length	(N/m)
ρ	- mass density	(kg/m^3)
A	- cross-sectional area	(m^2)
u	- displacement	(m)
$\ddot{u}$	- acceleration	(m/s^2)
s	- stress	(N/m^2) or (Pa)
x	- longitudinal coordinate	(m)
e	- strain	dimensionless
E	- Young's modulus	(N/m^2) or (Pa)
k=AE	- stiffness	(N)
$\bar{\rho} = \rho A$	- mass/unit length	(kg/m)

where N denotes Newtons, m is meters, kg is kilograms, s is seconds and Pa is Pascals.

The boundary condition at each end consists of either a prescribed displacement, $u = u^*(t)$, or a prescribed traction, $\tau = \tau^*(t)$, at each end, $x = 0$ and $x = L$. In addition, the initial distribution for displacement and velocity must be given: $u(0,x) = \hat{u}(x)$ and $\dot{u}(0,x) = \hat{v}(x)$, respectively.

2.3.2 Wave Solutions

In (2.3-2), suppose f=0, and assume k and $\bar{\rho}$ are constant. To simplify the formulation, let

$$c^2 = \frac{k}{\bar{\rho}} = \frac{E}{\rho} \qquad (2.3\text{-}3)$$

The quantity, c, is readily shown to have the dimensions of velocity. Then (2.3-2) reduces to a particular partial differential equation called the **wave equation**

$$u_{,tt} = c^2 u_{,xx} \qquad (2.3\text{-}4)$$

which has two general solutions. Because the equation is linear, the complete solution is the sum:

$$u = f(t - \tfrac{x}{c}) + g(t + \tfrac{x}{c}) \qquad (2.3\text{-}5)$$

As we will show later, the functions f and g, which must be determined from initial and boundary conditions, correspond to waves propagating to the right and left, respectively. To verify that (2.3-5) satisfies (2.3-4), we note that

$$\begin{aligned}
u_{,x} &= -\tfrac{1}{c} f'(t - \tfrac{x}{c}) + \tfrac{1}{c} g'(t + \tfrac{x}{c}) \\
u_{,xx} &= \tfrac{1}{c^2} f''(t - \tfrac{x}{c}) + \tfrac{1}{c^2} g''(t + \tfrac{x}{c}) \\
u_{,t} &= f'(t - \tfrac{x}{c}) + g'(t + \tfrac{x}{c}) \\
u_{,tt} &= f''(t - \tfrac{x}{c}) + g''(t + \tfrac{x}{c})
\end{aligned} \qquad (2.3\text{-}6)$$

where primes denote derivatives with respect to the particular similarity arguments, t + x/c and t - x/c, i.e.,

$$f'(t - \tfrac{x}{c}) = \frac{\partial f}{\partial(t - \tfrac{x}{c})}\ , \qquad g'(t + \tfrac{x}{c}) = \frac{\partial g}{\partial(t + \tfrac{x}{c})} \qquad (2.3\text{-}7)$$

Consider a semi-infinite bar aligned with the positive x axis $(x \geq 0)$ and suppose that

$$u = f(t - \tfrac{x}{c}) \qquad (2.3\text{-}8)$$

At x = 0, u = f(t). If u is prescribed at x = 0 in the form

$$u\big|_{x=0} = F(t) \tag{2.3-9}$$

then the solution to (2.3-4) that meets this boundary condition is

$$u = F(t - \tfrac{x}{c}) \tag{2.3-10}$$

that is, a general solution is obtained by replacing the argument "t" in the boundary function with the argument “t - x/c”.

Pick a particular value of t, say t*, and let F*=F(t*). Then the general solution will exhibit the same value when t - x/c = t* or

$$t = t^* + \tfrac{x}{c} \tag{2.3-11}$$

In other words, for some value, x, the same value for u will appear at a time x/c later than when the feature was initiated at x = 0. The argument can be repeated for all choices of t*. That is, any feature is propagated unchanged to the right with a velocity c, as indicated in Fig. 2.3-1 by the first three sketches which show the displacement as a function of time for a point at the origin and for two other points, x_1 and x_2. The last sketch shows the same information but plotted as “snapshots” of displacement versus position for different times.

The parameter c is called the wave velocity. The functions f(t - x/c) and g(t + x/c) represent general disturbances propagating to the right (positive x direction) and to the left (negative x direction), respectively, since the arguments associated with g(t + x/c) are similar to those for f(t - x/c) but with a change of sign for x.

In general, the various functions of interest are as follows:

$$\begin{aligned}
u &= f(t - \tfrac{x}{c}) + g(t + \tfrac{x}{c}) && \text{Displacement} \\
v &= u_{,t} = f' + g' && \text{Particle velocity} \\
e &= u_{,x} = -\frac{f'}{c} + \frac{g'}{c} && \text{Strain} \\
\sigma &= Ee = \frac{E}{c}(-f' + g') && \text{Stress} \\
&= \rho c(-f' + g')
\end{aligned} \tag{2.3-12}$$

in which the relation $E = \rho c^2$ from (2.3-3) is used to obtain the last equation for stress. Note that the particle velocity v is not the same as the wave velocity c. It also follows from (2.3-12) that for a wave traveling strictly in the positive x direction $\sigma = -\rho c v$ ($g = 0$) while for a wave traveling strictly in the negative x direction $\sigma = \rho c v$ ($f = 0$).

Before proceeding to the examples, there are several observations worthy of mention:

1. The wave solution of (2.3-12) is applied normally to an infinite or semi-infinite body. However, the following examples show that solutions can be obtained for regions in a finite domain as well. The problems considered are assumed to have zero initial conditions with a non-zero boundary condition.
2. The case of wave propagation in a bounded domain in which the boundary conditions are homogeneous but an initial condition is prescribed is called one of **free vibration**. Such problems are normally handled by a separation-of-variables technique rather than through the use of (2.3-12).
3. The separation of the solution into the two types of wave propagation is a natural consequence of seeking a similarity solution. When the same problem is solved numerically, the two classes are automatically taken into consideration with no special consideration needed on the part of the analyst.
4. With the following examples, we show how the wave form of the solutions to the wave equation are used for problems with a finite domain. We see that the key aspect is to introduce "fictional" forcing functions as a source of additional waves so as to satisfy prescribed boundary conditions.

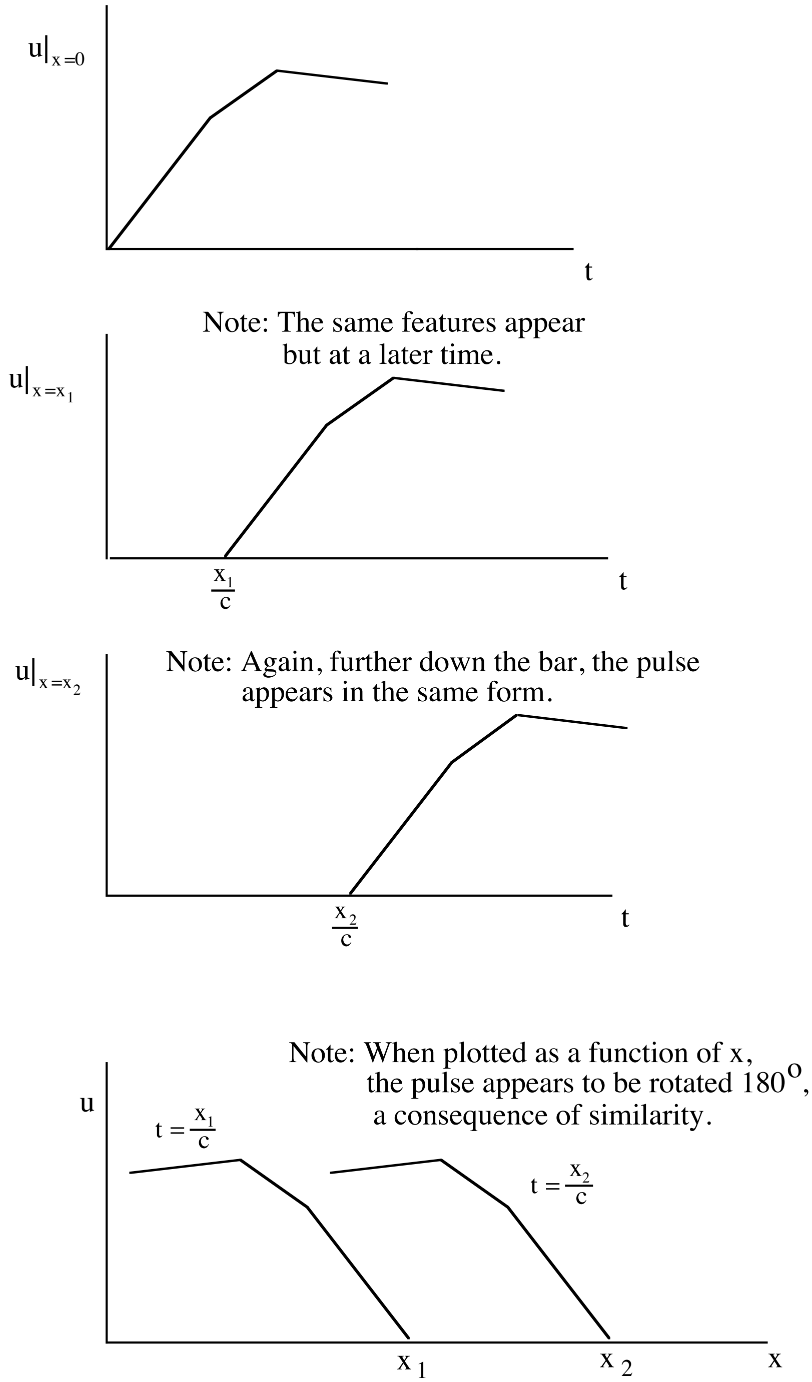

Fig. 2.3-1. Sketches of wave profiles as functions of t and x.

<u>Example 1:</u>

Consider a bar of unit cross-sectional area (A = 1) and of length L with the x coordinate introduced such that the domain of the bar is $0 \leq x \leq L$. Suppose a step force of unit magnitude, i.e., F(t) = H[t], is applied at x = 0. Further, assume the right end is fixed, so that u(L,t) = 0. Since the traction, τ, is given by $\tau = F/A$ and the traction is related to the stress by $\tau = n\sigma$ it follows that $\sigma(0,t) = -H[t]$. For an initial segment of time, postulate a solution characterized as a wave traveling to the right. Physically, we think of a disturbance initiated at the left end and propagated to the right. If the postulate turns out to be wrong, one of the boundary conditions will be violated. From (2.3-12) it follows that

$$v = \frac{-\sigma}{\rho c} = \frac{1}{\rho c}H[t] \quad \text{at} \quad x = 0 \tag{2.3-13}$$

If the argument is replaced with t - x/c, then a possible solution to the wave equation that satisfies the boundary condition at x = 0 is

$$v(x,t) = \frac{1}{\rho c}H\left[t - \frac{x}{c}\right] \tag{2.3-14}$$

Now, we integrate to obtain the displacement which is assumed to be zero when t = 0:

$$u(x,t) = \frac{1}{\rho c} < t - \frac{x}{c} >^{1} \tag{2.3-15}$$

The solution for u and v at two different times is shown in Fig. 2.3-2.

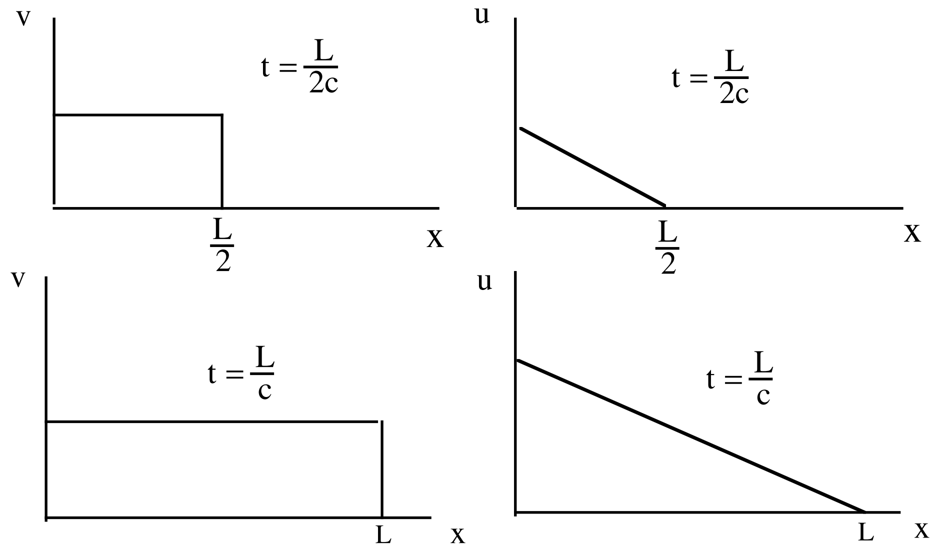

Fig. 2.3-2. Solution for Problem 1 at two different times.

The stress wave is a step function whose front propagates to the right at the wave velocity, c. To the right of the wave front, nothing is happening. In particular, the prescribed boundary condition of zero displacement at the right end is satisfied until t = L/c at which time the displacement begins to increase, i.e.,

$$u(L,t) = \frac{1}{\rho c} < t - \frac{L}{c} >^1 \tag{2.3-16}$$

and there is a contradiction with the boundary condition u(L,t) = 0 once t > L/c. One method for removing the contradiction is to suppose that a disturbance in the form of a force acting to the left is introduced at x = 2L (an imaginary point) and that a wave that results from this force propagates to the left. The sign and magnitude of the second part is chosen so that the sum of the two parts is as follows:

$$u = \frac{1}{\rho c} < t - \frac{x}{c} >^1 - \frac{1}{\rho c} < t - \frac{2L}{c} + \frac{x}{c} >^1 \tag{2.3-17}$$

With this choice we see that the boundary condition at x = L is satisfied for both t < L/c and t ≥ L/c as shown when x = L is substituted in (2.3-17):

$$u(t,L) = \frac{1}{\rho c} < t - \frac{L}{c} >^1 - \frac{1}{\rho c} < t - \frac{L}{c} >^1 = 0 \tag{2.3-18}$$

The boundary x = L can be perceived as a generator of a wave traveling to the left with a shift in time scale of 2L/c. Now the solution is valid until the front of the wave propagates back to x = 0 at which time another wave traveling to the right is introduced. The result when this third wave is incorporated is

$$\begin{aligned} u &= \frac{1}{\rho c} < t - \frac{x}{c} >^1 - \frac{1}{\rho c} < t - \frac{2L}{c} + \frac{x}{c} >^1 - \frac{1}{\rho c} < t - \frac{2L}{c} - \frac{x}{c} >^1 \\ v &= \frac{1}{\rho c} H\left[t - \frac{x}{c}\right] - \frac{1}{\rho c} H\left[t - \frac{2L}{c} + \frac{x}{c}\right] - \frac{1}{\rho c} H\left[t - \frac{2L}{c} - \frac{x}{c}\right] \\ \sigma &= -H\left[t - \frac{x}{c}\right] - H\left[t - \frac{2L}{c} + \frac{x}{c}\right] + H\left[t - \frac{2L}{c} - \frac{x}{c}\right] \end{aligned} \tag{2.3-19}$$

which holds for $0 \le t \le \frac{3L}{c}$. As a check, consider the left and right boundary conditions. The stress and displacement are

$\sigma(0,t) = -H[t] - H\left[t - \frac{2L}{c}\right] + H\left[t - \frac{2L}{c}\right] = -H[t]$, the prescribed function

$$u(L,t) = \frac{1}{\rho c} < t - \frac{L}{c} >^1 - \frac{1}{\rho c} < t - \frac{L}{c} >^1 - \frac{1}{\rho c} < t - \frac{3L}{c} >^1$$
$$= -\frac{1}{\rho c} < t - \frac{3L}{c} >^1 = 0 \quad \text{for} \quad 0 \le t < \frac{3L}{c} \qquad (2.3\text{-}20)$$

The addition of contributions due to reflections at the boundaries continues indefinitely.

Example 2:

Consider a bar of unit cross-sectional area and of length L. The x coordinate is introduced such that the domain of the bar is $0 < x < L$. On the end at $x = 0$ a force $F(t) = H[t]$ is introduced, and assume the end $x = L$ is free, i.e., the traction at $x = L$ is zero which implies that $\sigma(t,L) = 0$. For $0 \le t \le L/c$, the solution is identical to that of Example 1. However, now a wave that propagates to the left must be introduced to cancel the stress at the boundary, instead of the displacement. A similar contribution must be added later at the left end so that for the period of time $0 \le t < 3L/c$ the solution is the following:

$$u(x,t) = \frac{1}{\rho c} < t - \frac{x}{c} >^1 + \frac{1}{\rho c} < t - \frac{2L}{c} + \frac{x}{c} >^1 + \frac{1}{\rho c} < t - \frac{2L}{c} - \frac{x}{c} >^1$$
$$v(x,t) = \frac{1}{\rho c} H\left[t - \frac{x}{c}\right] + \frac{1}{\rho c} H\left[t - \frac{2L}{c} + \frac{x}{c}\right] + \frac{1}{\rho c} H\left[t - \frac{2L}{c} - \frac{x}{c}\right] \qquad (2.3\text{-}21)$$
$$\sigma(x,t) = -H\left[t - \frac{x}{c}\right] + H\left[t - \frac{2L}{c} + \frac{x}{c}\right] - H\left[t - \frac{2L}{c} - \frac{x}{c}\right]$$

To verify the solution is valid, evaluate the stress at $x = 0$ and $x = L$. The results are

$$\sigma(0,t) = -H[t] + H\left[t - \frac{2L}{c}\right] - H\left[t - \frac{2L}{c}\right] = -H[t]$$
$$\sigma(L,t) = -H\left[t - \frac{L}{c}\right] + H\left[t - \frac{3L}{c}\right] = -H\left[t - \frac{3L}{c}\right] = 0 \qquad (2.3\text{-}22)$$

provided $0 \le t < \frac{3L}{c}$.

Note that this problem is a case where the bar is free to translate. There is no problem with regard to ill-posedness as long as it is understood that the bar can move as a body and the analysis or numerical procedure can accommodate the motion. Most numerical procedures can accommodate such motions, at least for small time (infinitesimal displacements). If the bar has unit cross-sectional area and if the bar is rigid, then the acceleration of the center of mass from Newton's Law is (dotted line in Fig. 2.3-3)

$$a^c = \frac{1}{\rho L} \qquad \text{Rigid bar} \tag{2.3-23}$$

for a unit force and a mass of ρL. The velocity of the centroid of a deformable bar based on the wave solution (indicated by the solid line in Fig. 2.3-4) is

$$v\left(\tfrac{L}{2},t\right) = \tfrac{1}{\rho c} H\left[t - \tfrac{L}{2c}\right] + \tfrac{1}{\rho c} H\left[t - \tfrac{3L}{2c}\right] + \tfrac{1}{\rho c} H\left[t - \tfrac{5L}{2c}\right] \tag{2.3-24}$$

which is valid for $0 \le t < \frac{3L}{c}$.

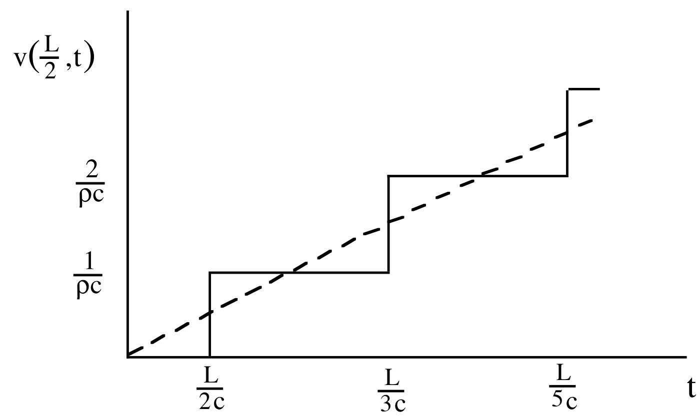

Fig. 2.3-3. Motion of a rigid bar (dotted line) and the center of a deformable bar (solid line) that is free to translate.

<u>Example 3:</u>

The forcing function used for the previous two examples, a step function, is a particularly simple one. Here we show that the same ideas can be used to develop a solution when a more complicated forcing function is prescribed. Consider the bar described as in Examples 1 and 2. Let the boundary condition at x = L be unspecified for the moment. Assume a force is applied to the left end as described by the following function:

$$F(t) = \begin{cases} A \sin \Omega t, & 0 < t < \frac{\pi}{\Omega} \\ 0, & t < 0 \text{ and } t > \frac{\pi}{\Omega} \end{cases} \tag{2.3-25}$$

where the period of the forcing function is chosen to be L/c. This choice is equivalent to specifying the forcing frequency as follows:

$$T = \frac{2\pi}{\Omega} = \frac{L}{c} \quad \Rightarrow \quad \Omega = \frac{2\pi c}{L} \tag{2.3-26}$$

An alternative form of the boundary condition is to consider the stress to be prescribed with Heaviside functions used to conveniently express the duration of time for which the stress is nonzero. The result is

$$\sigma(0,t) = -\left\{H[t] - H[t - \tfrac{\pi}{\Omega}]\right\} \sin \Omega t \tag{2.3-27}$$

With the assumption of a wave traveling to the right, t is replaced by the similarity argument to obtain a general solution

$$\sigma(x,t) = -\left\{H[t - \tfrac{x}{c}] - H[t - \tfrac{\pi}{\Omega} - \tfrac{x}{c}]\right\} \sin \Omega(t - \tfrac{x}{c}) \tag{2.3-28}$$

Also, we know that for a wave traveling to the right, $v = -\frac{\sigma}{\rho c}$ so that the velocity is immediately obtained and an integration yields the displacement. The results are

$$\begin{aligned} v(x,t) &= \frac{1}{\rho c}\left\{H[t - \tfrac{x}{c}] - H[t - \tfrac{\pi}{\Omega} - \tfrac{x}{c}]\right\} \sin \Omega(t - \tfrac{x}{c}) \\ u(x,t) &= \frac{1}{\Omega \rho c}\left\{H[t - \tfrac{x}{c}] - H[t - \tfrac{\pi}{\Omega} - \tfrac{x}{c}]\right\}\left\{1 - \cos \Omega(t - \tfrac{x}{c})\right\} \end{aligned} \tag{2.3-29}$$

when we make use of the assumption that u(0, 0) = 0. The solution holds for $0 \le t < L/c$. As in Examples 1 and 2, appropriate adjustments must be made to extend the time of solution according to the specific boundary condition at the right end, x = L.

2.3.3 Procedure for Obtaining Series Solutions

An alternative method for obtaining an analytical solution to the wave equation is to postulate that the solution can be expressed as the product of a function of the spatial variable and a function of the temporal variable. Not all problems can be solved with this approach but, nevertheless, the procedure is a classical one for obtaining solutions to an important class of problems. Since the procedure has already been discussed in the section on transient heat conduction, we provide the material here in a more condensed form.

Recall from (2.3-2) that for constant k and $\bar{\rho}$, the equation of motion can be written as:

$$u_{,tt} = c^2 u_{,xx} + \frac{f(x,t)}{\bar{\rho}} \tag{2.3-40}$$

To limit the complexity of the problem, suppose the boundary conditions are independent of time. To proceed, let the complete solution be represented as a sum of two functions, one of which is independent of time, i.e.,

$$u(x,t) = u_s(x) + \bar{u}(x,t) \tag{2.3-41}$$

where $u_S(x)$ satisfies the static condition

$$u_{s,xx} = 0 \tag{2.3-42}$$

and the prescribed boundary conditions. Then $\bar{u}(x,t)$ must satisfy corresponding homogenous boundary conditions. (It is possible to handle more general boundary conditions but these approaches will not be discussed.)

If the initial conditions are

$$u(x,0) = u^0(x)\ , \qquad \dot{u}(x,0) = v^0(x) \qquad \text{(Prescribed)} \tag{2.3-43}$$

then from (2.3-41), the function $\bar{u}$ must satisfy the initial conditions

$$\bar{u}(x,0) = u^0(x) - u_s(x)\ , \qquad \dot{\bar{u}}(x,0) = v^0(x) \tag{2.3-44}$$

since $\dot{u}_s(x) = 0$ by definition. Substitute (2.3-41) in (2.3-40) and use (2.3-42). Then $\bar{u}(x,t)$ must satisfy

$$\bar{u}_{,tt} = c^2 \bar{u}_{,xx} + \frac{f(x,t)}{\bar{\rho}} \tag{2.3-45}$$

as well as the homogeneous boundary conditions and the initial conditions of (2.3-44). First consider the homogeneous form of (2.3-45):

$$\bar{u}_{,tt} = c^2 \bar{u}_{,xx} \tag{2.3-46}$$

Assume a separation-of-variables solution:

$$\bar{u} = \bar{X}(x)\bar{T}(t) \tag{2.3-47}$$

Then, substitute (2.3-47) in (2.3-46), and divide by $\bar{u}$ to obtain

$$\frac{\ddot{\bar{T}}}{\bar{T}} = c^2 \frac{\bar{X}_{,xx}}{\bar{X}} = -\omega^2 \tag{2.3-48}$$

where the minus sign has been chosen to ensure finite values of $\bar{T}$ and $\bar{X}$ for all time. Define

$$\phi^2 = \frac{\omega^2}{c^2} \tag{2.3-49}$$

Then (2.3-48) yields the following pair of ordinary differential equations:

$$\ddot{\bar{T}} + \omega^2 \bar{T} = 0 \ , \qquad \bar{X}_{,xx} + \phi^2 \bar{X} = 0 \tag{2.3-50}$$

with the solutions

$$\bar{T} = a \sin \omega t + b \cos \omega t \ , \qquad \bar{X} = c^* \sin \phi x + d^* \cos \phi x \tag{2.3-51}$$

We note that if, alternatively, a positive sign had been chosen as the coefficient of ω^2 in (2.3-48), then the coefficient of ω^2 in (2.3-50) would be negative and the solution would contain an exponential term in time. Such a choice of sign is ruled out based on the physical argument that the solution should remain bounded.

At the boundaries, $\overline{X}$ must satisfy homogeneous boundary conditions. Solutions will exist only if c* and d* are linearly related and $\phi = \phi_n$, a particular value for $n = 1, ..., \infty$. If the coefficients c* and d* are incorporated with a and b, the general solution is the superposition

$$\bar{u} = \sum_{n=1}^{\infty} \overline{X}_n(x, \phi_n)(a_n \sin \omega_n t + b_n \cos \omega_n t) \tag{2.3-52}$$

where the following terminology is used in this context:

(i) $\omega_n = c\ \phi_n$ are **eigenvalues** (**natural frequencies**, **principal values**), and

(ii) $\overline{X}_n$ are **eigenfunctions**, **principal modes** or **modes of vibration**.

The form of (2.3-52) is said to be one of modal superpostion. Also, $\overline{X}_1$ is called the **fundamental mode** and ω_1 is called the **fundamental frequency**.

Orthogonality

Recall from (2.3-50) that any eigenfunction must satisfy

$$\overline{X}_{n,xx} + \phi_n^2 \overline{X}_n = 0 \qquad \text{or} \qquad \overline{X}_{m,xx} + \phi_m^2 \overline{X}_m = 0 \tag{2.3-53}$$

Multiply the first by $\overline{X}_m$, the second by $\overline{X}_n$ and integrate by parts

$$\left.\begin{aligned} \overline{X}_m \overline{X}_{n,x}\Big|_0^L - \int_0^L \overline{X}_{m,x} \overline{X}_{n,x} dx + \phi_n^2 \int_0^L \overline{X}_m \overline{X}_n dx = 0 \\ \overline{X}_n \overline{X}_{m,x}\Big|_0^L - \int_0^L \overline{X}_{n,x} \overline{X}_{m,x} dx + \phi_m^2 \int_0^L \overline{X}_n \overline{X}_m dx = 0 \end{aligned}\right\} \tag{2.3-54}$$

The function, $\overline{X}$, is required to satisfy homogeneous boundary conditions, which implies that either $\overline{X}$ or $\overline{X}_{,x}$ is zero at each boundary. Therefore, the boundary terms in (2.3-54) are zero. By subtracting terms in the two equations we obtain

$$\left(\phi_n^2 - \phi_m^2\right)\int_0^L \overline{X}_m \overline{X}_n dx = 0 \tag{2.3-55}$$

In general, similar to our analysis in Subsection 2.2.4, $\phi_n^2 \neq \phi_m^2$ when $n \neq m$. Thus (2.3-55) and (2.3-54) yield the following conditions, either of which is called the **orthogonality condition** satisfied by the eigenfunctions:

$$\int_0^L \overline{X}_n \overline{X}_m dx = 0\ , \qquad \int_0^L \overline{X}_{n,x} \overline{X}_{m,x} dx = 0 \qquad m \neq n \tag{2.3-56}$$

The function defined in (2.3-52) is the solution to the homogeneous problem. For the inhomogeneous problem

$$\overline{u}_{,tt} = c^2 \overline{u}_{,xx} + \frac{f(x,t)}{\overline{\rho}} \tag{2.3-57}$$

assume a solution of the form

$$\overline{u}(x,t) = \sum_{n=1}^{\infty} \tau_n(t) \overline{X}_n(x) \tag{2.3-58}$$

Substitute (2.3-58) in (2.3-57) to obtain

$$\sum_{n=1}^{\infty} \ddot{\tau}_n(t) \overline{X}_n(x) = \sum_{n=1}^{\infty} c^2 \tau_n \overline{X}_{n,xx} + \frac{f(x,t)}{\overline{\rho}} \tag{2.3-59}$$

With the use of (2.3-53) it follows that

$$c^2 \sum_{n=1}^{\infty} \tau_n \overline{X}_{n,xx} = -c^2 \sum_{n=1}^{\infty} \tau_n \phi_n^2 \overline{X}_n = -\sum_{n=1}^{\infty} \tau_n \omega_n^2 \overline{X}_n \tag{2.3-60}$$

Thus an alternative form for (2.3-59) is

$$\sum_{n=1}^{\infty} (\ddot{\tau}_n + \omega_n^2 \tau_n) \overline{X}_n = \frac{f(x,t)}{\overline{\rho}} \tag{2.3-61}$$

The problem now is to obtain separate equations for τ_n, n = 1 to ∞. Multiply each side of (2.3-61) by $\overline{X}_m$, and integrate over the domain. Because of orthogonality, all terms in the sum are zero except for the case of m = n. Let

$$A_n = \int_0^L \overline{X}_n^2(x)dx \qquad \text{and} \qquad F_n(t) = \int_0^L \frac{f(x,t)}{\overline{\rho}} \overline{X}_n(x)dx \tag{2.3-62}$$

The result is the following governing equation for $\tau_n(t)$:

$$\ddot{\tau}_n + \omega_n^2 \tau_n = F_n(t)/A_n \qquad \text{for} \qquad n = 1,2,... \tag{2.3-63}$$

The solution is

$$\tau_n = a_n \sin\omega_n t + b_n \cos\omega_n t + \tau_n^p(t) \tag{2.3-64}$$

where τ_n^p is a particular solution for $F_n(t)/A_n$. Now consider the initial condition of (2.3-40), i.e.,

$$\overline{u}(x,0) = \sum_{n=1}^{\infty} \tau_n(0)\overline{X}_n(x) = u^0(x) - u_s(x) \tag{2.3-65}$$

Again, multiply by $\overline{X}_m(x)$, integrate over the domain and use orthogonality to obtain

$$\tau_n(0) = \frac{1}{A_n}\int_0^L \overline{X}_n(x)[u^0(x) - u_s(x)]dx \tag{2.3-66}$$

A similar operation on the initial velocity

$$\dot{\overline{u}}(x,0) = \sum_{n=1}^{\infty} \dot{\tau}_n(0)\overline{X}_n(x) = v^0(x) \tag{2.3-67}$$

yields the following initial condition for the time derivative of τ_n:

$$\dot{\tau}_n(0) = \frac{1}{A_n}\int_0^L \overline{X}_n(x)v^0(x)\,dx \tag{2.3-68}$$

The inital conditions of (2.3-66) and (2.3-67) are used to evaluate a_n and b_n in (2.3-64). Then the complete solution is obtained by noting that $u(x,t) = \overline{u}(x,t) + u_s(x)$.

$$\tau_n = a_n \sin\omega_n t + b_n \cos\omega_n t + \tau_n^p(t) \tag{2.3-64}$$

where τ_n^p is a particular solution for $F_n(t)/A_n$. Now consider the initial condition of (2.3-40), i.e.,

$$\overline{u}(x,0) = \sum_{n=1}^{\infty}\tau_n(0)\overline{X}_n(x) = u^0(x) - u_s(x) \tag{2.3-65}$$

Again, multiply by $\overline{X}_m(x)$, integrate over the domain and use orthogonality to obtain

$$\tau_n(0) = \frac{1}{A_n}\int_0^L \overline{X}_n(x)[u^0(x) - u_s(x)]\,dx \tag{2.3-66}$$

A similar operation on the initial velocity

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The inital conditions of (2.3-66) and (2.3-67) are used to evaluate a_n and b_n in (2.3-64). Then the complete solution is obtained by noting that $u(x,t) = \overline{u}(x,t) + u_s(x)$.

2.3.4 An Example of a Series Solution

Consider the problem of a bar of unit cross-sectional area with an applied force F = H[t] at x = 0 and a fixed boundary at x = L. We state this mathematically as follows:

$$\sigma(0,t) = Eu_{,x}(0,t) = -H[t] \qquad \text{and} \qquad u(t,L) = 0 \tag{2.3-69}$$

Incorporate the boundary condition function, a constant, into the steady state part u_s, which is the solution to the following problem:

$$u_{s,xx} = 0 \quad \text{subject to} \quad u_{s,x}(0) = -\frac{1}{E} \quad \text{and} \quad u_s(L) = 0 \tag{2.3-70}$$

The solution to this problem is

$$u_s = \frac{L}{E}\left(1 - \frac{x}{L}\right) \tag{2.3-71}$$

To obtain the complete solution we must solve for the time-varying part $\bar{u}(x,t)$, which we assume to be the following series:

$$\bar{u}(x,t) = \sum_{n=1}^{\infty} \tau_n(t)\bar{X}_n(x) \tag{2.3-72}$$

From the development of the previous subsection, we know that

$$\begin{aligned} \bar{X}_n &= c^* \sin\phi x + d^* \cos\phi x \\ \bar{X}_{n,x} &= c^* \phi\cos\phi x - d^* \phi\sin\phi x \end{aligned} \tag{2.3-73}$$

The boundary condition at x = 0, which is $\bar{X}_{n,x}(0) = 0$, yields c* = 0. Then the boundary condition at x = L becomes

$$\bar{X}_n(L) = d^* \cos\phi L = 0 \qquad \Rightarrow \qquad \phi L = \frac{\pi}{2}(2n-1) \tag{2.3-74}$$

for n = 1, 2, ... Therefore, we conclude that

$$\phi_n = \frac{\pi}{2L}(2n-1) \tag{2.3-75}$$

so that the natural frequencies, ω_n, and modes of vibration, $\overline{X}_n$, are

$$\omega_n = \frac{\pi c}{2L}(2n-1) \qquad \text{and} \qquad \overline{X}_n = \cos\phi_n x \tag{2.3-76}$$

The fundamental frequency is $\omega_1 = \pi c/(2L)$ and it follows that the fundamental period is $T_1 = 2\pi/\omega_1 = 4L/c$. Note that a wave would travel for a time of 4L/c before the solution repeats.

Since f(x,t) is zero, the forcing function $F_n(t) = 0$. Thus, the particular solution is zero, and from (2.3-64) we obtain

$$\tau_n = a_n \sin\omega_n t + b_n \cos\omega_n t \tag{2.3-77}$$

Since $v^o(x) = 0$, it follows that $\dot{\tau}_n(0) = a_n\omega_n = 0$ and, therefore, $a_n = 0$. The other initial condition requires the following expression:

$$u^0(x) - u_s(x) = -u_s(x) = -\frac{L}{E}(1-\frac{x}{L}) \tag{2.3-78}$$

Then, (2.3-66) yields

$$\tau_n(0) = b_n = \frac{1}{A_n}\int_0^L \overline{X}_n(x)(-\frac{L}{E})(1-\frac{x}{L})dx \tag{2.3-79}$$

where the coefficient, A_n, is

$$A_n = \int_0^L \cos^2\phi_n x dx = \frac{L}{2} \tag{2.3-80}$$

The integrals of (2.3-79) become

$$\int_0^L \cos\phi_n x\,dx = \frac{1}{\phi_n}\sin\phi_n x\Big|_0^L = \frac{1}{\phi_n}\sin\phi_n L$$

$$\begin{aligned}\int_0^L x\cos\phi_n x\,dx &= \frac{1}{\phi_n}[x\sin\phi_n x]_0^L - \frac{1}{\phi_n}\int_0^L \sin\phi_n x\,dx \\ &= \frac{L}{\phi_n}\sin\phi_n L + \frac{1}{\phi_n^2}\cos\phi_n L - \frac{1}{\phi_n^2}\end{aligned} \tag{2.3-81}$$

However, ϕ_n is determined from the condition $\cos\phi_n L = 0$ so that, when the terms are combined, we obtain

$$\int_0^L (1-\tfrac{x}{L})\cos\phi_n x\,dx = \frac{1}{\phi_n^2 L} \tag{2.3-82}$$

Therefore, the coefficients, b_n, are

$$b_n = \frac{1}{L/2}\cdot\frac{1}{\phi_n^2 L}(-\tfrac{L}{E}) = -\frac{2}{L^2\phi_n^2}\frac{L}{E} \tag{2.3-83}$$

and, the complete solution is

$$u(x,t) = u_s(x) + \bar{u}(x,t) = \frac{L}{E}(1-\tfrac{x}{L}) - \frac{2}{LE}\sum_{n=1}^{\infty}\frac{1}{\phi_n^2}\cos\phi_n x\cos\omega_n t \tag{2.3-84}$$

As a check, consider the strain, and then the stress at $x = 0$:

$$\begin{aligned}u,_x(x,t) &= \frac{L}{E}\left[-\frac{1}{L} + \frac{8}{\pi^2}\frac{\pi}{2L}\cdot\sum_{n=1}^{\infty}\frac{1}{(2n-1)}\cos\left\{\frac{\pi c}{2L}(2n-1)t\right\}\sin\left\{\frac{\pi}{2L}(2n-1)x\right\}\right] \\ &= -\frac{1}{E}\left[1 - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)}\cos\left\{\frac{\pi c}{2L}(2n-1)t\right\}\sin\left\{\frac{\pi}{2L}(2n-1)x\right\}\right]\end{aligned} \tag{2.3-85}$$

$$\sigma(0,t) = Ee(0,t) = Eu,_x(0,t) = -1$$

The stress is the prescribed value at the left end of the bar.

2.3.5 Approximate Solutions

Frequently, it is useful to have an approximate solution available for design purposes or to ensure that there are no obvious errors in numerical solutions because of an incompatible set of units for prescribed data. Here we mean an approximate solution of the order $\pm 50\%$ of the exact peak value, say, rather than an approximate solution that is accurate to within $\pm 5\%$ that one might expect from a finite element or a finite difference program. Results of the previous subsections can be used to suggest very simple approximate solutions that require an insignificant amount of work to obtain.

(1) Solution using the Fundamental Mode

If the natural frequencies and fundamental modes are known, the fundamental (first) mode often provides an excellent approximation to the complete solution. That is,

$$u^a(x,t) = u_s(x) + \tau_1(t)\overline{X}_1(x) \tag{2.3-86}$$

For the example of the previous section

$$u^a(x,t) = \frac{L}{E}\left(1 - \frac{x}{L} - \frac{8}{\pi^2}\cos\frac{\pi ct}{2L}\cos\frac{\pi x}{2L}\right) \tag{2.3-87}$$

(2) Assumed Mode Solution

If the principal modes are not known, use a simple algebraic form that satisfies the homogeneous boundary conditions, i.e., let

$$u^a(x,t) = u_s(x) + \tau^a(t)\overline{X}^a(x) \tag{2.3-88}$$

where $\overline{X}^a(x)$ satisfies the homogeneous boundary conditions. Substitute the part $\tau^a(t)\overline{X}^a(x)$ in (2.3-45) which is

$$\overline{u}_{,tt} = c^2\overline{u}_{,xx} + \frac{f(x,t)}{\overline{\rho}} \tag{2.3-89}$$

to obtain

$$\tau^a{}_{,tt}\,\overline{X}^a - c^2\tau^a\overline{X}^a{}_{,xx} = \frac{f(x,t)}{\overline{\rho}} \qquad (2.3\text{-}90)$$

Then, multiply by $\overline{X}^a$ and integrate over the domain to obtain

$$\tau^a{}_{,tt}D_1 + c^2\tau^a D_2 = F^a(t) \qquad (2.3\text{-}91)$$

where

$$D_1 = \int_0^L (\overline{X}^a)^2 dx\ , \quad D_2 = \int_0^L (\overline{X}^a{}_{,x})^2 dx\ , \quad F^a(t) = \int_0^L \overline{X}^a(x)\frac{f(x,t)}{\rho}dx \qquad (2.3\text{-}92)$$

The expression for D_2 arises from an integration by parts and the use of $\overline{X}^a{}_{,x}\overline{X}^a\,|_0^L = 0$ since either $\overline{X}^a = 0$ or $\overline{X}^a{}_{,x} = 0$ at each boundary.

To illustrate the procedure, consider the example of the previous section where

$$u_s = \frac{L}{E}(1-\frac{x}{L})\ , \qquad \overline{X}^a{}_{,x}(0) = 0\ , \qquad \overline{X}^a(L) = 0 \qquad (2.3\text{-}93)$$

For the approximation function $\overline{X}^a$ choose a quadratic function of x, viz.,

$$\overline{X}^a = a + bx + cx^2 \qquad (2.3\text{-}94)$$

The application of the two homogeneous boundary conditions yields

$$\overline{X}^a{}_{,x}(0) = b = 0 \qquad \text{and} \qquad \overline{X}^a(L) = a + cL^2 = 0 \Rightarrow c = -\frac{a}{L^2} \qquad (2.3\text{-}95)$$

Incorporate the constant a with τ^a in which case

$$\overline{X}^a = 1 - \frac{x^2}{L^2}\ , \quad D_1 = \int_0^L \left(1-\frac{x^2}{L^2}\right)^2 dx = \frac{8}{15}L\ , \quad D_2 = \int_0^L \left(-\frac{2x}{L^2}\right)^2 dx = \frac{4}{3L} \qquad (2.3\text{-}96)$$

But, $F^a(t) = 0$ since $f(x,t) = 0$ so the governing equation for τ^a becomes

$$\frac{8}{15}L\ddot{\tau}^a + c^2\frac{4}{3L}\tau^a = 0 \qquad (2.3\text{-}97)$$

which is just the governing equation for a single-degree-of-freedom spring-mass system. The "approximate" natural frequency, ω^a, is obtained by considering the ratio of the coefficient of τ^a to the coefficient of $\ddot{\tau}^a$ as follows:

$$(\omega^a)^2 = c^2 \cdot \frac{4}{3L} \cdot \frac{15}{8L} = \frac{5c^2}{2L^2} \tag{2.3-98}$$

Wherefore the approximate frequency is

$$\omega^a = \sqrt{\frac{5}{2}}\frac{c}{L} \tag{2.3-99}$$

Note that this approximation is within 0.6% of the exact fundamental frequency of $\omega_1 = \frac{\pi}{2}\frac{c}{L}$.

Now, collecting the results obtained so far results in the approximate solution

$$u^a = u_s = \tau^a(t)x^a(x) = \frac{L}{E}(1-\frac{x}{L}) + \left(1-\frac{x^2}{L^2}\right)(a_1 \sin\omega^a t + b_1 \cos\omega^a t) \tag{2.3-100}$$

in which a_1 and b_1 are integration constants to be used in association with the solution to (2.3-97). To evaluate these constants, we use the initial conditions

$$\begin{aligned} u^a(x,0) &= \frac{L}{E}(1-\frac{x}{L}) + b_1\left(1-\frac{x^2}{L^2}\right) = u^*(x) \\ \dot{u}^a(x,0) &= a_1\omega^a\left(1-\frac{x^2}{L^2}\right) = \dot{u}^*(x) \end{aligned} \tag{2.3-101}$$

in which $u^*(x)$ and $\dot{u}^*(x)$ are prescribed functions. For the homogenous initial conditions considered here, $u^*(x) = 0$ and $\dot{u}^*(x) = 0$. The relations of (2.3-101) cannot hold for all values of x, so approximate relations must be used. The variable, x, must be eliminated. To accomplish this end, multiply the first initial condition by $\overline{X}^a$ and integrate over the domain. The result is

$$\int_0^L \bar{X}^a u^a(x,0)dx = \frac{L}{E}\int_0^L \bar{X}^a(1-\frac{x}{L})dx + b_1\int_0^L \bar{X}^a\left(1-\frac{x^2}{L^2}\right)dx = 0$$

$$\frac{L}{E}\int_0^L\left(1-\frac{x^2}{L^2}\right)(1-\frac{x}{L})dx + \frac{8}{15}Lb_1 = 0 \tag{2.3-102}$$

$$\frac{L^2}{E}\cdot\frac{5}{12}+\frac{8}{15}L\cdot b_1 = 0 \qquad \Rightarrow \qquad b_1 = -\frac{25L}{32E}$$

For the initial velocity the same approach yields

$$\int_0^L \bar{X}^a \dot{u}^a(x,0)dx = \int_0^L \bar{X}^a\left(1-\frac{x^2}{L^2}\right)(a_1\omega^a)dx = 0 \qquad \Rightarrow \quad a_1 = 0 \tag{2.3-103}$$

In summary, the approximate solution is

$$u^a(x,t) = \frac{L}{E}(1-\frac{x}{L}) - \frac{25}{32}\frac{L}{E}\left(1-\frac{x^2}{L^2}\right)\cos\left(\sqrt{\frac{5}{2}}\frac{c}{L}\right)t \tag{2.3-104}$$

The approach outlined above presupposes very little knowledge of partial differential equations. Nevertheless, an approximate solution is obtained which satisfies the boundary and initial conditions, and with a frequency that is very close to the fundamental frequency of the system. A similar approach can be used to great advantage with more complicated problems such as beams with variable cross section for which an analytical solution may not be available, and for plates as well.

2.4 TRANSIENT MOTION OF A BEAM

2.4.1 Formulation

Recall from Subsection 1.4.1 that the equation of motion for an Euler-Bernouilli beam with no applied moment is

$$(EIw,_{xx}),_{xx} + \bar{\rho}\ddot{w} = F , \qquad 0 < x < L \tag{2.4-1}$$

in which w(x,t) is the transverse displacement, E is Young's modulus, I is the second-area moment, $\bar{\rho}$ is the mass per unit length and F(x,t) is the applied lateral force. The bending moment and shear functions are

$$M = EIw,_{xx} \ , \qquad V = -M,_{x} \tag{2.4-2}$$

The boundary conditions are of the form

$$\left.\begin{array}{ll} nM = M^* & \text{or} \quad w,_x = \theta^* \\ nV = V^* & \text{or} \quad w = w^* \end{array}\right\} \quad \text{at } \ x = 0 \ \text{ and } \ x = L \tag{2.4-3}$$

All quantities with an asterisk are given. In addition, the initial functions for displacement and velocity must be prescribed, i.e.,

$$w(x,0) = w_o(x) \qquad \text{and} \qquad \dot{w}(x,0) = \dot{w}_o(x) \tag{2.4-4}$$

The next subsection describes a method for obtaining a solution for the motion of beams.

2.4.2 Procedure for Obtaining Series Solutions

To limit the complexity of the development, suppose the boundary conditions are not functions of time and that $\bar{\rho}$ and EI are constants. Then the solution to the dynamic equation

$$(EIw,_{xx}),_{xx} + \bar{\rho}\ddot{w} = F \tag{2.4-5}$$

can be separated into two parts:

$$w(x,t) = w_s(x) + \overline{w}(x,t) \tag{2.4-6}$$

where w_S satisfies the static condition

$$(EIw_{s,xx})_{,xx} = 0 \tag{2.4-7}$$

and the prescribed boundary conditions. Then $\overline{w}(x,t)$ satisfies the dynamic equation and homogenous boundary conditions. If the given initial conditions are

$$w(x,0) = w_o(x) \ , \qquad \dot{w}(x,0) = \dot{w}_o(x) \tag{2.4-8}$$

then, from (2.4-6), the function $\overline{w}$ must satisfy the initial conditions

$$\overline{w}(x,0) = w_o(x) - w_s(x) \ , \qquad \dot{\overline{w}}(x,0) = \dot{w}_o(x) \tag{2.4-9}$$

since $\dot{w}_s = 0$ by definition.

The substitution of (2.4-6) in (2.4-5) and the use (2.4-7) shows that $\overline{w}(x,t)$ must satisfy

$$(EI\overline{w}_{,xx})_{,xx} + \overline{\rho}\ddot{\overline{w}} = F \tag{2.4-10}$$

and homogeneous boundary conditions. The initial conditions are given by (2.4-9). To obtain the homogenous part, consider the equation

$$(EI\overline{w}_{,xx})_{,xx} + \overline{\rho}\ddot{\overline{w}} = 0 \tag{2.4-11}$$

Assume a separation-of-variables solution, so that $\overline{w}$ is given by

$$\overline{w} = \overline{X}(x)\overline{T}(t) \tag{2.4-12}$$

Substitute (2.4-12) in (2.4-11), and divide by $\overline{w}$. The result is

$$EI\frac{\overline{X}^{iv}}{\overline{X}} = -\overline{\rho}\frac{\ddot{\overline{T}}}{\overline{T}} \tag{2.4-13}$$

where the superscript "iv" indicates the fourth derivative. The function on the left side of the equality depends only on x; the function on the right depends only on t. The only way for this equation to be satisfied is to have each side equal to the same constant. If the solution is to remain bounded in time, then the sign of the constant ω^2 is chosen to ensure sinusoidal rather than exponential terms in the solution for $\overline{T}$, i.e.,

$$\ddot{\overline{T}} + \omega^2 \overline{T} = 0 \tag{2.4-14}$$

Then the second equation becomes

$$\overline{X}^{iv} - \lambda^4 \overline{X} = 0 \tag{2.4-15}$$

where

$$\lambda^4 = \frac{\omega^2 \overline{\rho}}{EI} = \frac{\omega^2 \rho A}{EI} = \frac{\omega^2}{c^2 r_g^2} \tag{2.4-16}$$

with

$$c^2 = E/\rho\,, \qquad r_g^2 = I/A \tag{2.4-17}$$

The solution to (2.4-15) is

$$\overline{X} = C_1 \cosh \lambda x + C_2 \sinh \lambda x + C_3 \cos \lambda x + C_4 \sin \lambda x \tag{2.4-18}$$

in which the C_i's are constants of integration. The result of invoking the homogeneous boundary conditions is that $\lambda = \lambda_n$, a set of discrete numbers with $n = 1, \ldots, \infty$ and that three of the coefficients C_i can be related to the fourth. If the amplitude parameter is incorporated in $\overline{T}$, then the eigenpair $(\lambda_n, \overline{X}_n)$ are obtained where λ_n is an **eigenvalue** and $\overline{X}_n$ is an **eigenfunction, principal mode**, or **mode of vibration**. From (2.4-16), the **natural frequencies** are

$$\omega_n = \lambda_n^2 c r_g \tag{2.4-19}$$

Orthogonality

As with the heat conduction and bar problems, orthogonality is an inherent property of the differential equation. This property is used repeatedly to solve for the unkown coefficients in the series solution. Since the proof of orthogonality has been given for these previous cases, the development here is somewhat briefer and is done only to show the particular features that arise in connection with the fourth-order differential equation. From (2.4-15) each eigenpair must satisfy

$$\overline{X}_m^{iv} - \lambda_m^4 \overline{X}_m = 0\ , \qquad \overline{X}_n^{iv} - \lambda_n^4 \overline{X}_n = 0 \tag{2.4-20}$$

Multiply the first equation by $\overline{X}_n$ and integrate by parts twice. The result is

$$\overline{X}_n \overline{X}_m^{iii}\Big|_0^L - \overline{X}_{n,x}\, \overline{X}_{m,xx}\Big|_0^L + \int_0^L \overline{X}_{n,xx}\, \overline{X}_{m,xx}\, dx - \lambda_m^4 \int_0^L \overline{X}_n \overline{X}_m dx = 0 \tag{2.4-21}$$

A similar operation obtained by taking the product of $\overline{X}_m$ with terms in the second of (2.4-20) yields

$$\overline{X}_m \overline{X}_n^{iii}\Big|_0^L - \overline{X}_{m,x}\, \overline{X}_{n,xx}\Big|_0^L + \int_0^L \overline{X}_{m,xx}\, \overline{X}_{n,xx}\, dx - \lambda_n^4 \int_0^L \overline{X}_m \overline{X}_n dx = 0 \tag{2.4-22}$$

By construction, $\overline{X}_m$ and $\overline{X}_n$ satisfy homogenous boundary conditions. Therefore all boundary terms are zero. If the remaining terms in (2.4-22) are subtracted from (2.4-21) and with the assumption that $\lambda_m \neq \lambda_n$, then

$$\left.\begin{aligned} &\int_0^L \overline{X}_m \overline{X}_n dx = 0 \qquad && m \neq n \\ &\int_0^L \overline{X}_{m,xx}\, \overline{X}_{n,xx}\, dx = 0 \qquad && m \neq n \end{aligned}\right\} \tag{2.4-23}$$

either of which can be interpreted as the orthogonality condition for beam modes.

<u>Solution to the Nonhomogeneous Problem</u>

For the nonhomogeneous differential equation

$$(EI\overline{w}_{,xx})_{,xx} + \overline{\rho}\ddot{\overline{w}} = F \tag{2.4-24}$$

assume a solution of the form

$$\overline{w}(x,t) = \sum_{n=1}^{\infty} \tau_n(t)\overline{X}_n(x) \tag{2.4-25}$$

Substitute (2.4-25) in (2.4-24) to obtain

$$\sum_{n=1}^{\infty} \tau_n(t)(EI\overline{X}_{n},_{xx}),_{xx} + \overline{\rho}\sum_{n=1}^{\infty} \ddot{\tau}_n(t)\overline{X}_n(x) = F(x,t) \tag{2.4-26}$$

But from (2.4-20) with EI constant we have

$$(EI\overline{X}_{n},_{xx}),_{xx} = EI\lambda_n^4\overline{X}_n = \overline{\rho}\,\omega_n^2\overline{X}_n \tag{2.4-27}$$

in which (2.4-16) has been used. Then (2.4-26) yields

$$\sum_{n=1}^{\infty} (\ddot{\tau}_n + \omega_n^2\tau_n)\overline{X}_n = \frac{F(x,t)}{\overline{\rho}} \tag{2.4-28}$$

To obtain the differential equation for τ_n, we multiply by $\overline{X}_m$, and integrate over the domain. Because of orthogonality, the only term in the sum that is nonzero is the product when m = n. With our attention now focused on this case, define

$$A_n = \int_0^L \overline{X}_n^2(x)dx\ , \qquad F_n(t) = \int_0^L \frac{F(x,t)}{\overline{\rho}}\overline{X}_n(x)dx \tag{2.4-29}$$

The result is that τ_n must satisfy the ordinary differential equation

$$\ddot{\tau}_n + \omega^2\tau_n = \frac{F_n(t)}{A_n}\ , \qquad n = 1,2,... \tag{2.4-30}$$

The solution is

$$\tau_n = a_n \sin\omega_n t + b_n \cos\omega_n t + \tau_n^p(t) \tag{2.4-31}$$

where $\tau_n^p(t)$ is a particular solution for the forcing term $F_n(t)/A_n$. The initial value of $\overline{w}$ is

$$\overline{w}(x,0) = \sum_{n=1}^{\infty} \tau_n(0)\overline{X}_n(x) = w_o(x) - w_s(x) \tag{2.4-32}$$

in which (2.4-6) has been used. By multiplying by $\overline{X}_m$, integrating over the domain, and again using orthogonality, we obtain

$$\tau_n(0) = \frac{1}{A_n}\int_0^L \overline{X}_n(x)\left[w_o(x) - w_s(x)\right]dx \tag{2.4-33}$$

in which A_n is defined in (2.4-29). In a similar procedure for the initial velocity, we have

$$\dot{\overline{w}}(x,0) = \sum_{n=1}^{\infty} \dot{\tau}_n(0)\overline{X}_n(x) = \dot{w}_o(x) \tag{2.4-34}$$

Hence, the result is

$$\dot{\tau}_n(0) = \frac{1}{A_n}\int_0^L \overline{X}_n(x)\dot{w}_o(x)dx \tag{2.4-35}$$

The initial values given by (2.4-33) and (2.4-35) are used in (2.4-31) to determine a_n and b_n. Then, the complete solution is

$$w(x,t) = w_s(x) + \sum_{n=1}^{\infty} \tau_n(t)\overline{X}_n(x) \tag{2.4-36}$$

In this subsection, the procedure for obtaining a series solution to a dynamic beam problem has been outlined. Because the process was given in a brief format, we now illustrate the formulation with a sample problem whose solution can be put to good use in verifying numerical algorithms designed for obtaining approximate solutions to dynamic loads on beams.

2.4.3 An Example Solution

Consider a cantilevered beam with the right end fixed and a point force applied in the upward direction at the left end as shown in Fig. 2.4-1. The data for the problem consist of the following:

(i) the domain, $0 < x < L$;

(ii) the material properties consisting of the beam stiffness, EI, and the mass per unit length, $\bar{\rho}$, both of which are taken to be constants;

(iii) the forcing function at the left end is prescribed to be a unit step function, $F(t) = H[t]$;

(iv) the initial conditions for w and $\dot{w}$ are prescribed to be zero; and

(v) the boundary conditions, which are given as follows:

$$\begin{aligned} V^*(0,t) &= F(t)\,, \qquad & w^*(L,T) &= 0 \\ M^*(0,t) &= 0\,, \qquad & \theta^*(L,t) &= 0 \end{aligned} \tag{2.4-37}$$

Since n = -1 at the left end, an alternative form for the transverse shear boundary condition is $V(0,t) = -F(t)$.

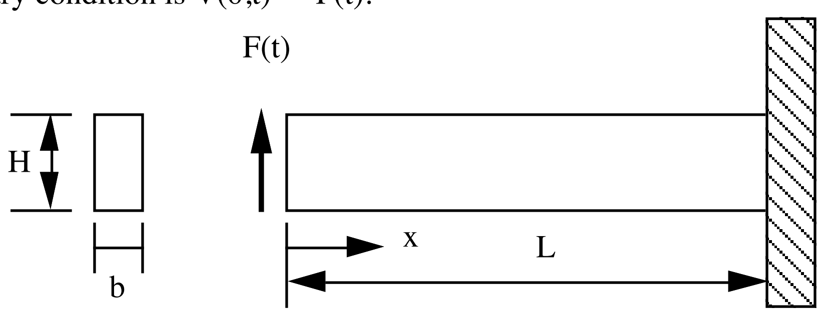

Fig. 2.4-1. Details of the model problem for illustrating the dynamic response of a beam.

The static problem can be considered as one in which a unit load is applied to the left end. The governing equation for the static part is

$$(EIw_{s,xx})_{,xx} = 0 \tag{2.4-38}$$

Integrate once to obtain

$$(EIw_{s,xx})_{,x} = c_1 = -V \tag{2.4-39}$$

But, the boundary condition on V at x = 0 is V(0) = -1 which yields c_1 = 1. Integrate repeatedly and evaluate the remaining constants based on boundary conditions as follows:

$$\begin{aligned}
&EIw_{s,xx} = x + c_2 \\
&M(0) = 0 \qquad => \qquad c_2 = 0 \\
&w_{s,xx} = \frac{x}{EI} \\
&w_{s,x} = \frac{1}{EI}\left(\frac{x^2}{2} + c_3\right) \\
&w_{s,x}(L) = 0 \qquad => \qquad c_3 = -L^2/2 \\
&w_s = \frac{1}{EI}\left(\frac{x^3}{6} - \frac{L^2 x}{2} + c_4\right) \\
&w_s(L) = 0 \qquad => \qquad c_4 = -\frac{L^3}{6} + \frac{L^3}{2} = \frac{L^3}{3}
\end{aligned} \tag{2.4-40}$$

Therefore, the static part of the solution is

$$w_s(x) = \frac{1}{EI}\left(\frac{x^3}{6} - \frac{L^2 x}{2} + \frac{L^3}{3}\right) \tag{2.4-41}$$

The modal solution (mode) is given by (2.4-18). The homogenous boundary conditions for this problem are:

$$\begin{aligned}
\overline{X}_{,xxx}(0) = 0\ , \qquad\qquad & \overline{X}(L) = 0 \\
\overline{X}_{,xx}(0) = 0\ , \qquad\qquad & \overline{X}_{,x}(L) = 0
\end{aligned} \tag{2.4-42}$$

The use of these conditions in (2.4-18) yields the characteristic equation

$$\cos\lambda L \cosh\lambda L = -1 \tag{2.4-43}$$

The solutions are the eigenvalues

$$\lambda_n L = 1.875,\ 4.694,\ 7.855,\ 10.966,\ 14.137,\ 17.279.\ .... \tag{2.4-44}$$

The corresponding eigenfunctions (modes of vibration) are:

$$\overline{X}_n = \cosh\lambda_n x + \cos\lambda_n x + B_n(\sin\lambda_n x + \sinh\lambda_n x)\,, \qquad n = 1,2,... \qquad (2.4\text{-}45)$$

where

$$B_n = \frac{(\sin\lambda_n L - \sinh\lambda_n L)}{(\cos\lambda_n L + \cosh\lambda_n L)} \qquad (2.4\text{-}46)$$

For later use, we note that

$$\begin{aligned} A_n &= \int_0^L \overline{X}_n^2(x)dx \\ &= \frac{1}{2\lambda_n}[\,2\lambda_n L + \sin\lambda_n L\cos\lambda_n L + \sinh\lambda_n L\cosh\lambda_n L \\ &\quad + 2\cos\lambda_n L\sinh\lambda_n L + 2\sin\lambda_n L\cosh\lambda_n L \\ &\quad + 2B_n(\sin^2\lambda_n L + \sinh^2\lambda_n L + 2\sin\lambda_n L\sinh\lambda_n L) \\ &\quad + B_n^2(\sinh\lambda_n L\cosh\lambda_n L - \sin\lambda_n L\cos\lambda_n L \\ &\quad + 2\sin\lambda_n L\cosh\lambda_n L - 2\cos\lambda_n L\sinh\lambda_n L)\big] \end{aligned} \qquad (2.4\text{-}47)$$

Since $F(x,t) = 0$ it follows that $F_n(t) = 0$. Assume that the initial displacement and velocity are zero. Then, from (2.4-33) and (2.4-41), it follows that

$$\begin{aligned} \tau_n(0) = b_n &= \frac{-1}{A_n}\int_0^L \overline{X}_n(x)w_s(x)dx = \frac{-1}{EIA_n}\int_0^L \overline{X}_n(x)\Big(\frac{x^3}{6} - \frac{L^2x}{2} + \frac{L^3}{3}\Big)dx \\ &= \frac{-1}{EIA_n}\frac{1}{\lambda_n^4}\big[(\lambda_n L - B_n)\sinh\lambda_n L + (B_n\lambda_n L - 1)\cosh\lambda_n L \\ &\qquad - (\lambda_n L + B_n)\sin\lambda_n L + (B_n\lambda_n L - 1)\cos\lambda_n L + 2\big] \end{aligned} \qquad (2.4\text{-}48)$$

Then, with the use of (2.4-35) the initial condition of $\dot{\tau}_n$ at $t = 0$ yields $\dot{\tau}_n(0) = 0$ which implies $a_n = 0$.

In summary, the solutions for the displacement and the bending moment are:

$$
\begin{aligned}
w(x,t) &= w_s(x) + \sum_{n=1}^{\infty} \tau_n(0)\overline{X}_n(x)\cos\omega_n t \\
M(x,t) &= x + EI\sum_{n=1}^{\infty} \tau_n(0)\overline{X}_{n,xx}(x)\cos\omega_n t \qquad (2.4\text{-}49) \\
\overline{X}_{n,xx}(x) &= \lambda_n^2[\cosh\lambda_n x - \cos\lambda_n x + B_n(\sinh\lambda_n x - \sin\lambda_n x)]
\end{aligned}
$$

Suppose the displacement is normalized by the static displacement at x = 0 and the time normalized by the fundamental period T_1. A plot of the dimensionless displacement at x = 0 as a function of dimensionless time is shown in Fig. 2.4-2
in which 1, 2 and 4 terms have been retained in the series. The two and four-term solutions are practically indistinguishable and not too different from the one-term solution which indicates that the beam is responding primarily in the fundamental mode. In a similar fashion, suppose the moment is normalized with the moment at x = L for the static problem. One, two, four and eight-term solutions for the moment as a function of dimensionless time are given similarly in Fig. 2.4-3 with the latter two almost overlaying each other. Because of the spatial derivatives, it is to be expected that more terms must be retained to obtain an accurate solution for the moment in comparison with the number of terms required for the displacement.

PLOT MISSING

Fig. 2.4-2. Plot of displacement at x = 0 as a function of time.

PLOT MISSING

Fig. 2.4-3. Plot of moment at x = L as a function of time.

2.5 CONCLUDING REMARKS

Analytical solutions to time-dependent one-dimensional problems have been given for some special cases. The solutions are valuable for indicating qualitative features and for providing cases with which to compare numerical solutions. Of greatest importance, the process of obtaining numerical solutions requires knowledge of the concepts of characteristic equations, eigenvalues, eigenfunctions, orthogonality and integration by parts, all of which will arise in different contexts later on. For example, the finite element method can be viewed as an approach where separation of variables is used locally (element by element) rather than globally. Over all, knowledge of the terminology and solution features that should be expected are invaluable when obtaining and interpreting numerical results.

2.6 EXERCISES

1. Obtain a series solution for the case of heat conduction in a bar with lateral convection for prescribed temperatures of T_0 and T_L at $x = 0$ and $x = L$, respectively. For this case, what are the orthogonality conditions satisfied by the eigenfunctions? Plot the solution using consecutively more terms. How many terms is enough?

2. For one of the problems involving wave propagation caused by a step pulse, overlay plots of the wave solution and the series solution. How good can the series solution represent the jump in stress? How many terms is enough?

3. Obtain a series solution for the case of a simply supported beam with a point load applied suddenly at the center. Plot displacement and moment at the center as functions of time. Include sequentially more terms until the change from one representation to the next becomes insignificant.