# 11

# Variational Formulation of Bar Element

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## §11.1. A New Beginning

This Chapter begins Part II of the course. This Part focuses on the construction of structural and continuum finite elements using a *variational formulation* based on the Total Potential Energy. Why only elements? Because the other synthesis steps of the DSM: globalization, merge, BC application and solution, remain the same as in Part I. Those operations are not element dependent.

*Individual elements* are constructed in this Part beginning with the simplest ones and progressing to more complicated ones. The formulation of 2D finite elements from a variational standpoint is discussed in Chapters 14 and following. Although the scope of that formulation is broad, exceeding structural mechanics, it is better understood by going through specific elements first.

From a geometrical standpoint the simplest finite elements are one-dimensional or *line elements*. This means that the *intrinsic dimensionality* is one, although these elements may be used in one, two or three space dimensions upon transformation to global coordinates as appropriate. The simplest one-dimensional structural element is the *two-node bar element*, which we have already encountered in Chapters 2, 3 and 5 as the truss member.

In this Chapter the bar stiffness equations are rederived using the variational formulation. For uniform properties the resulting equations are the same as those found previously using the physical or Mechanics of Materials approach. The variational method has the advantage of being readily extendible to more complicated situations, such as variable cross section or more than two nodes.

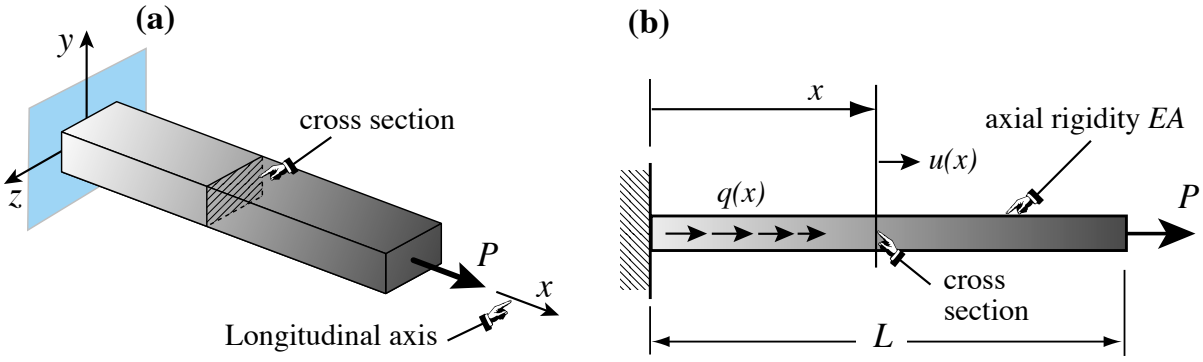


FIGURE 11.1. A fixed-free bar member: (a) 3D view showing reference frame; (b) 2D view on $\{x, y\}$ plane highlighting some quantities that are important in bar analysis.

## §11.2. Definition of Bar Member

In structural mechanics a *bar* is a structural component characterized by two properties:

(1) One preferred dimension: the *longitudinal dimension* or *axial dimension* is much larger that the other two dimensions, which are collectively known as *transverse dimensions*. The intersection of a plane normal to the longitudinal dimension and the bar defines the *cross sections*. The longitudinal dimension defines the *longitudinal axis*. See Figure 11.1(a).

(2) The bar resists an internal axial force along its longitudinal dimension.

**Table 11.1 Nomenclature for Mathematical Model of Axially Loaded Bar**

| *Quantity* | *Meaning* |
|---:|---|
| $x$ | Longitudinal bar axis* |
| $(.)'$ | $d(.)/dx$ |
| $u(x)$ | Axial displacement |
| $q(x)$ | Distributed axial force, given per unit of bar length |
| $L$ | Total bar length |
| $E$ | Elastic modulus |
| $A$ | Cross section area; may vary with $x$ |
| $EA$ | Axial rigidity |
| $e = du/dx = u'$ | Infinitesimal axial strain |
| $\sigma = Ee = Eu'$ | Axial stress |
| $F = A\sigma = EA\,e = EAu'$ | Internal axial force |
| $P$ | Prescribed end load |

* $x$ is used in this Chapter instead of $\bar{x}$ (as in Chapters 2–3) to simplify the notation.

In addition to trusses, bar elements are used to model cables, chains and ropes. They are also used as fictitious elements in penalty function methods, as discussed in Chapter 9.

We will consider here only *straight bars*, although their cross section may vary. Our one-dimensional mathematical model assumes that the bar material is linearly elastic obeying Hooke's law, and that displacements and strains are infinitesimal. Figure 11.1(b) pictures some relevant quantities for a fixed-free bar. Table 11.1 collects the necessary terminology for the governing equations.

Figure 11.2 displays the governing equations of the bar in a graphic format called a *Tonti diagram*. The formal similarity with the diagrams used in Chapter 5 to explain MoM elements should be noted, although the diagram of Figure 11.2 pertains to the continuum bar model rather than to the discrete one. (The qualifier "strong form" is explained in the next Chapter.)

## §11.3. Variational Formulation

To illustrate the variational formulation, the finite element equations of the bar will be derived from the Minimum Potential Energy principle.

### §11.3.1. The Total Potential Energy Functional

In Mechanics of Materials it is shown that the *internal energy density* at a point of a linear-elastic material subjected to a one-dimensional state of stress $\sigma$ and strain $e$ is $\mathcal{U} = \frac{1}{2}\sigma(x)e(x)$, where $\sigma$ is to be regarded as linked to the displacement $u$ through Hooke's law $\sigma = Ee$ and the strain-displacement relation $e = u' = du/dx$. This $\mathcal{U}$ is also called the *strain energy density*. Integration over the volume of the bar gives the total internal energy

$$U = \tfrac{1}{2}\int_V \sigma\, e\, dV = \tfrac{1}{2}\int_0^L F e\, dx = \tfrac{1}{2}\int_0^L (EAu')u'\, dx = \tfrac{1}{2}\int_0^L u' EA\, u'\, dx. \tag{11.1}$$

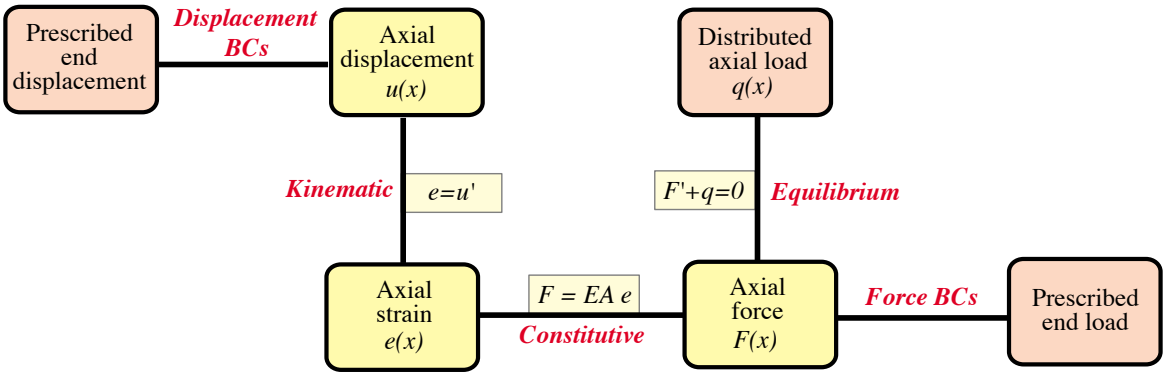


FIGURE 11.2. Strong-form Tonti diagram for the continuum model of a bar member. Field equations and BCs are represented as lines connecting the boxes. Yellow (brown) boxes contain unknown (given) quantities.

All integrand quantities in (11.1) may depend on $x$.

The *external work potential* is the work performed by applied mechanical loads working on the bar displacements. This potential is denoted by $W$. (The *external energy* $V$ is the negative of the work potential: $V = -W$. In the ensuing derivations $W$ will be used instead of $V$.) It collects contributions from two sources:

1. The distributed load $q(x)$. This contributes a cross-section density of $q(x)u(x)$ because $q$ is assumed to be already integrated over the section.
2. Any specified axial point load(s). For the fixed-free example of Figure 11.1 the end load $P$ would contribute $P\,u(L)$.

The second source may be folded into the first by conventionally writing any point load $P$ acting at a cross section $x = a$ as a contribution $P\,\delta(a)$ to $q(x)$, in which $\delta(a)$ denotes the one-dimensional Dirac delta function at $x = a$. If this is done the external energy can be concisely expressed as

$$W = \int_0^L q\,u\,dx. \tag{11.2}$$

The total potential energy of the bar is given by

$$\boxed{\Pi = U - W} \tag{11.3}$$

Mathematically $\Pi$ is a *functional*, called the *Total Potential Energy* functional or TPE. It depends only on the axial displacement $u(x)$. In Variational Calculus $u(x)$ is called the *primary variable* of the functional. When the dependence of $\Pi$ on $u$ needs to be emphasized we shall write $\Pi[u] = U[u]-W[u]$, with brackets enclosing the primary variable. To display both primary and independent variables we write, for example, $\Pi[u(x)] = U[u(x)] - W[u(x)]$.

**Remark 11.1**. According to the rules of Variational Calculus, the Euler-Lagrange equation for $\Pi$ is

$$\mathcal{E} = \frac{\partial \Pi}{\partial u} - \frac{d}{dx}\frac{\partial \Pi}{\partial u'} = -q - (E\,A\,u')' \tag{11.4}$$

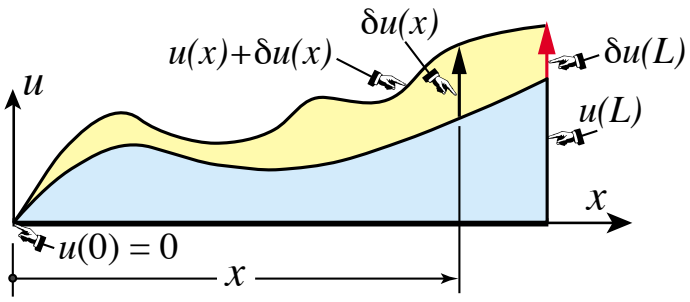


FIGURE 11.3. Concept of admissible variation of the axial displacement function $u(x)$. For convenience $u(x)$ is plotted normal to the longitudinal axis. Both $u(x)$ and $u(x) + \delta u(x)$ shown above are kinematically admissible, and so is the variation $\delta u(x)$. Note that the variation $\delta u(L)$ is not zero because the BC at $x = L$ is natural.

The stationary condition for $\Pi$ is $\mathcal{E} = 0$, or

$$(E A u')' + q = 0 \tag{11.5}$$

This is the strong (pointwise) equation of equilibrium in terms of the axial displacement, which reduces to $E A u'' + q = 0$ if $E A$ is constant. This equation is not explicitly used in the FEM development. It is instead replaced by $\delta\Pi = 0$, with the variation restricted over the class of finite element interpolation functions.

### §11.3.2. Admissible Variations

The concept of *admissible variation* is fundamental in both variational calculus and the variationally formulated FEM. *Only the primary variable(s) of a functional may be varied*. For the TPE functional (11.3) this is the axial displacement $u(x)$. Suppose that $u(x)$ is changed to $u(x) + \delta\, u(x)$.[1] This is illustrated in Figure 11.3, where for convenience $u(x)$ is plotted normal to $x$. The TPE functional changes accordingly as

$$\Pi = \Pi[u] \;\Rightarrow\; \Pi + \delta\Pi = \Pi[u + \delta u]. \tag{11.6}$$

The function $\delta\, u(x)$ and the scalar $\delta\Pi$ are called the *variations* of $u(x)$ and $\Pi$, respectively. The variation $\delta\, u(x)$ should not be confused with the ordinary differential $du(x) = u'(x)\, dx$ since on taking the variation the independent variable $x$ is *frozen*; that is, $\delta x = 0$.

A displacement variation $\delta u(x)$ is said to be *admissible* when both $u(x)$ and $u(x) + \delta\, u(x)$ are *kinematically admissible* in the sense of the Principle of Virtual Work (PVW). This agrees with the conditions of classical variational calculus, and are restated next.

A *kinematically admissible* axial displacement $u(x)$ obeys two conditions:

(i) It is continuous over the bar length, that is, $u(x) \in \mathcal{C}^0$ in $x \in [0, L]$.

(ii) It satisfies exactly any displacement boundary condition, such as the fixed-end specification $u(0) = 0$ of Figure 11.1. See of Figure 11.3.

The variation $\delta\, u(x)$ pictured in Figure 11.3 is kinematically admissible because both $u(x)$ and $u(x) + \delta\, u(x)$ satisfy the foregoing conditions. Note that the variation $\delta u(L)$ at the free end $x = L$ is not necessarily zero because that boundary condition is natural; that is, not specified directly in terms of the displacement $u(L)$. On the other hand, $\delta(0) = 0$.

The physical meaning of conditions (i)–(ii) is the subject of Exercise 11.1.

[1] The symbol $\delta$ not immediately followed by a parenthesis is not a delta function but instead denotes variation with respect to the variable that follows.

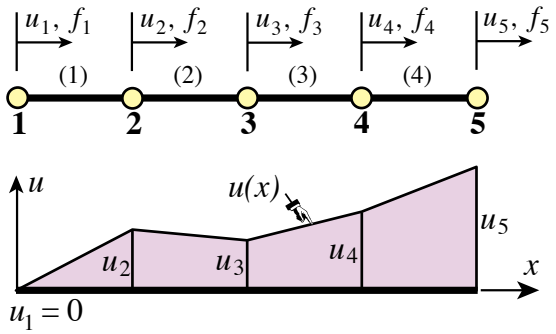


FIGURE 11.4. FEM discretization of bar member. A piecewise- linear admissible displacement trial function $u(x)$ is drawn underneath the mesh. It is assumed that the left end is fixed; thus $u_1 = 0$.

### §11.3.3. The Minimum Total Potential Energy Principle

The Minimum Total Potential Energy (MTPE) principle states that the actual displacement solution $u^*(x)$ that satisfies the governing equations is that which renders $\Pi$ stationary:[2]

$$\boxed{\delta\Pi = \delta U - \delta W = 0 \quad \text{iff} \quad u = u^*} \tag{11.7}$$

with respect to *admissible* variations $u = u^* + \delta u$ of the exact displacement field $u^*(x)$.

**Remark 11.2**. Using standard techniques of variational calculus[3] it can be shown that if $EA > 0$ and kinematic boundary conditions weed out any rigid motions, the solution $u^*(x)$ of (11.7) exists, is unique, and renders $\Pi[u]$ a minimum over the class of kinematically admissible displacements. The last attribute explains the "minimum" in the name of the principle.

### §11.3.4. TPE Discretization

To apply the TPE functional (11.3) to the derivation of FEM equations we replace the continuum mathematical model by a discrete one consisting of a union of bar elements. For example, Figure 11.4 illustrates the subdivision of a fixed-free bar member into four two-node elements.

Functionals are scalars. Therefore, for a discretization such as that shown in Figure 11.4, the TPE functional (11.3) may be decomposed into a sum of contributions of individual elements:

$$\boxed{\Pi = \Pi^{(1)} + \Pi^{(2)} + \ldots + \Pi^{(N_e)}} \tag{11.8}$$

in which $N_e$ denotes the number of elements. The same decomposition applies to both its internal energy and external work potential components:

$$\boxed{\delta U = \delta U^{(1)} + \ldots + \delta U^{(N_e)} = 0, \qquad \delta W = \delta W^{(1)} + \ldots + \delta W^{(N_e)} = 0,} \tag{11.9}$$

as well as to the stationarity condition (11.7):

$$\boxed{\delta\Pi = \delta\Pi^{(1)} + \delta\Pi^{(2)} + \ldots + \delta\Pi^{(N_e)} = 0.} \tag{11.10}$$

[2] The symbol "iff" in (11.7) is an abbreviation for "if and only if".

[3] See references in **Notes and Bibliography** at the end of Chapter.

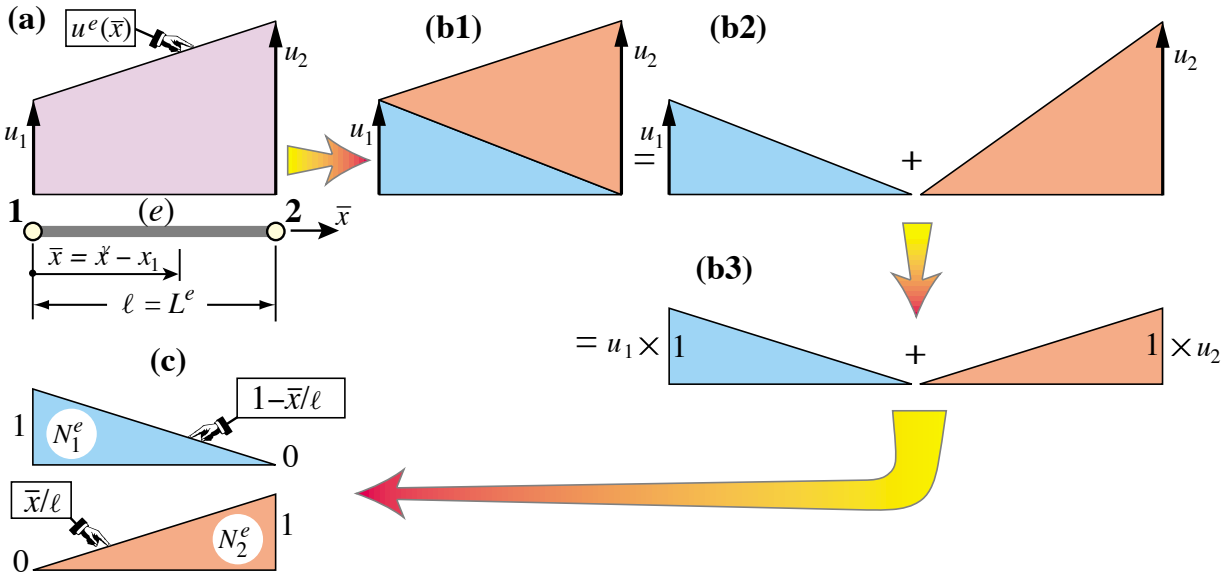


FIGURE 11.5. A two-node, TPE-based bar element: (a) element configuration and axial displacement variation (plotted normal to element axis for better visibility); (b1,b2,b3) displacement interpolation expressed in terms of linear shape functions; (c) element shape functions.

Using the fundamental lemma of variational calculus,[4] it can be shown that (11.10) implies that for a generic element $e$ we may write

$$\boxed{\delta\Pi^e = \delta U^e - \delta W^e = 0.} \tag{11.11}$$

This *variational equation* is the basis for the derivation of element stiffness equations once the displacement field has been discretized over the element, as described next.

**Remark 11.3**. In mathematics (11.11) is called a *first variation form*. It is a special case of a more general expression called a *weak form*, which is covered in more detail later. In mechanics it states the *Principle of Virtual Work* or PVW for each element: $\delta U^e = \delta W^e$, which says that the virtual work of internal and external forces on admissible displacement variations is equal if the element is in equilibrium [588].

### §11.3.5. Bar Element Discretization

Figure 11.5(a) depicts a generic bar element $e$. It has two nodes, which are labeled 1 and 2. These are called the *local node numbers*.[5] The element is referred to its local axis $\bar{x} = x - x_1$, which measures the distance from its left end. The two degrees of freedom are $u_1^e$ and $u_2^e$. (Bars are not necessary since the directions of $\bar{x}$ and $x$ are the same.) The element length is $\ell = L^e$.

The mathematical concept of bar finite elements is based on *approximating* axial displacement $u(x)$ over the element. The exact displacement $u^*$ is replaced by an approximate displacement

$$u^*(x) \approx u^e(x) \tag{11.12}$$

[4] See, e.g., Chapter II of Gelfand and Fomin [297].

[5] Note the notational change from the labels $i$ and $j$ of Part I. This will facilitate transition to multidimensional elements in Chapters 14ff.

over the finite element mesh. This approximate displacement, $u^e(x)$, taken over all elements $e = 1, 2, \ldots N^e$, is called the *finite element trial expansion* or simply *trial expansion*. See Figure 11.4. This FE trial expansion must belong to the class of kinematically admissible displacements defined in **?**. Consequently, it must be $C^0$ continuous over and between elements. The most common choices fpr $u^e$ are polynomials in $x$, as in the development that follows.

### §11.3.6. Interpolation by Shape Functions

In a two-node bar element the only possible polynomial choice of the displacement $u^e$ that satisfies the interelement continuity requirement is *linear*. It can be expressed by the following interpolation formula, which is graphically developed in Figure 11.5(b1,b2,b3):

$$\boxed{u^e(x) = N_1^e\, u_1^e + N_2^e\, u_2^e = [\, N_1^e \quad N_2^e\, ] \begin{bmatrix} u_1^e \\ u_2^e \end{bmatrix} = \mathbf{N}^e\, \mathbf{u}^e.} \tag{11.13}$$

The functions $N_1^e$ and $N_2^e$ that multiply the node displacements $u_1$ and $u_2$ are called *shape functions*, while $\mathbf{N}$ is called the *shape function matrix*. In this case $\mathbf{N}^e$ reduces to a row vector.

The shape functions *interpolate* the internal displacement $u^e$ directly from the node values. They are pictured in Figure 11.5(c). For this element, with $\bar{x} = x - x_1$ measuring the axial distance from the left node $i$, the shape functions are

$$N_1^e = 1 - \frac{\bar{x}}{\ell} = 1 - \zeta, \quad N_2^e = \frac{\bar{x}}{\ell} = \zeta. \tag{11.14}$$

Here

$$\zeta = \frac{x - x_1}{\ell} = \frac{\bar{x}}{\ell}, \tag{11.15}$$

is a dimensionless coordinate, also known as a *natural coordinate*, that varies from 0 through 1 over the element. Note that $dx = \ell\, d\zeta$ and $d\zeta = dx/\ell$. The shape function $N_1^e$ has the value 1 at node 1 and 0 at node 2. Conversely, shape function $N_2^e$ has the value 0 at node 1 and 1 at node 2. This is a general property of shape functions. It follows from the fact that element displacement interpolations such as (11.13) are based on physical node values.

### §11.3.7. The Strain-Displacement Matrix

The axial strain associated with the trial function $u^e$ is

$$e = \frac{du^e}{dx} = (u^e)' = \left[\, \frac{dN_1^e}{dx} \quad \frac{dN_2^e}{dx}\, \right] \begin{bmatrix} u_1^e \\ u_2^e \end{bmatrix} = \frac{1}{\ell}\, [\, -1 \quad 1\, ] \begin{bmatrix} u_1^e \\ u_2^e \end{bmatrix} = \mathbf{B}\, \mathbf{u}^e, \tag{11.16}$$

in which

$$\mathbf{B} = \frac{1}{\ell}\, [\, -1 \quad 1\, ], \tag{11.17}$$

is called the *strain-displacement* matrix. Note that $\mathbf{B}$ is constant over the element.

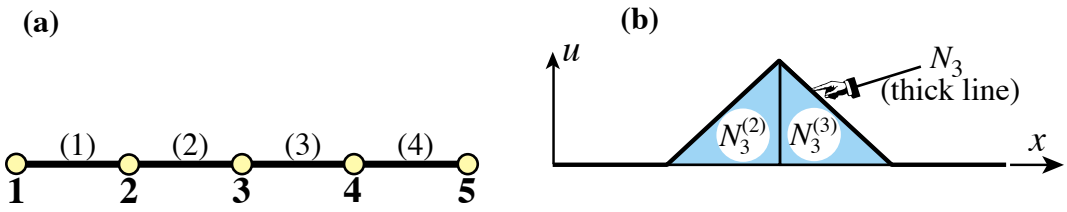


FIGURE 11.6. Trial basis function (a.k.a. hat function) for node 3 of a four-element bar discretization.

### §11.3.8. *Trial Basis Functions

Shape functions are associated with elements. A *trial basis function*, or simply *basis function*, is associated with a node. Suppose node $i$ of a bar discretization connects elements $(e1)$ and $(e2)$. The trial basis function $N_i$ is defined as

$$N_i(x) = \begin{cases} N_i^{(e1)} & \text{if } x \in \text{element } (e1) \\ N_i^{(e2)} & \text{if } x \in \text{element } (e2) \\ 0 & \text{otherwise} \end{cases} \tag{11.18}$$

For a piecewise linear discretization, such as that used in the two-node bar, this function has the shape of a hat. Thus it is also called a *hat function* or *chapeau function*. See Figure 11.6, in which $i = 3, e1 = 2$, and $e2 = 3$. The concept is important in the variational interpretation of FEM as a Rayleigh-Ritz method.

**Remark 11.4**. In addition to continuity, shape and trial functions must satisfy a *completeness* requirement with respect to the governing variational principle. This condition is stated and discussed in later Chapters. Suffices for now to say that the shape functions (11.14), as well as the associated trial functions, do satisfy this requirement.

## §11.4. The Finite Element Equations

In linear FEM the discretization process based on the TPE functional leads to the following algebraic form in the node displacements

$$\Pi^e = U^e - W^e, \quad \text{in which} \quad U^e \stackrel{\text{def}}{=} \tfrac{1}{2}(\mathbf{u}^e)^T \mathbf{K}^e \mathbf{u}^e \quad \text{and} \quad W^e \stackrel{\text{def}}{=} (\mathbf{u}^e)^T \mathbf{f}^e. \tag{11.19}$$

Here $\mathbf{K}^e$ and $\mathbf{f}^e$ are called the *element stiffness matrix* and the *element consistent nodal force vector*, respectively. The three scalars $\Pi^e, U^e$ and $W^e$ are only function of the node displacements $\mathbf{u}^e$. (This is a consequence of displacements being the only primary variable of the TPE functional.) Note that $U^e$ and $W^e$ depend *quadratically* and *linearly*, respectively, on those displacements. Taking the variation of $\Pi^e$ with respect to the node displacements gives[6]

$$\delta\Pi^e = \left(\delta\mathbf{u}^e\right)^T \frac{\partial \Pi^e}{\partial \mathbf{u}^e} = \left(\delta\mathbf{u}^e\right)^T \left[\mathbf{K}^e \mathbf{u}^e - \mathbf{f}^e\right] = 0. \tag{11.20}$$

Because the variations $\delta\mathbf{u}^e$ can be arbitrary, the bracketed expression must vanish, which yields

$$\boxed{\mathbf{K}^e \mathbf{u}^e = \mathbf{f}^e.} \tag{11.21}$$

These are the familiar element stiffness equations. Hence the foregoing names given to $\mathbf{K}^e$ and $\mathbf{f}^e$ are justified *a posteriori*.

[6] The $\frac{1}{2}$ factor disappears on taking the variation because $U^e$ is quadratic in the node displacements. For a review on the calculus of discrete quadratic forms, see Appendix D.

### §11.4.1. The Stiffness Matrix

We now apply the foregoing expressions to the two-node bar element. Its internal energy $U^e$ is

$$U^e = \tfrac{1}{2}\int_{x_1}^{x_2} e\,EA\,e\,dx = \tfrac{1}{2}\int_0^1 e\,EA\,e\,\ell\,d\zeta. \tag{11.22}$$

Note that the integration variable $x$ has been changed to the natural coordinate $\zeta$ defined in (11.15) that varies from 0 through 1, whence $dx = \ell\,d\zeta$. This form is symmetrically expanded using the strain-displacement matrix relation (11.16), by inserting $e = e^T = (\mathbf{u}^e)^T\mathbf{B}^T$ and $e = \mathbf{B}\,\mathbf{u}^e$ into the first and second $e$ of (11.22), respectively, to get

$$U^e = \tfrac{1}{2}\int_0^1 (\mathbf{u}^e)^T\mathbf{B}^T EA\,\mathbf{B}\,\mathbf{u}^e\,\ell\,d\zeta = \tfrac{1}{2}\int_0^1 [\,u_1^e \quad u_2^e\,]\,\frac{1}{\ell}\begin{bmatrix}-1\\1\end{bmatrix} EA\,\frac{1}{\ell}[\,-1 \quad 1\,]\begin{bmatrix}u_1^e\\u_2^e\end{bmatrix}\ell\,d\zeta. \tag{11.23}$$

The nodal displacements do not depend on position and can be moved out of the integral. Also $\mathbf{B}^T EA\,\mathbf{B} = EA\,\mathbf{B}^T\mathbf{B}$ since $EA$ is a scalar:

$$U^e = \tfrac{1}{2}(\mathbf{u}^e)^T\int_0^1 EA\,\mathbf{B}^T\mathbf{B}\,\ell\,d\zeta\,\mathbf{u}^e = \tfrac{1}{2}[\,u_1^e \quad u_2^e\,]\int_0^1 \frac{EA}{\ell^2}\begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix}\ell\,d\zeta\,\begin{bmatrix}u_1^e\\u_2^e\end{bmatrix}. \tag{11.24}$$

By (11.19) this is expressible as $\frac{1}{2}\left(\mathbf{u}^e\right)^T\mathbf{K}^e\,\mathbf{u}^e$. Since $\mathbf{u}^e$ is arbitrary, $\mathbf{K}^e$ is extracted as

$$\boxed{\mathbf{K}^e = \int_0^1 EA\,\mathbf{B}^T\mathbf{B}\,\ell\,d\zeta = \int_0^1 \frac{EA}{\ell^2}\begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix}\ell\,d\zeta = \frac{1}{\ell}\begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix}\int_0^1 EA\,d\zeta.} \tag{11.25}$$

This is the bar element stiffness matrix. For a homogeneous and prismatic bar of constant rigidity, $EA$ can be moved outside the integral, $\int_0^1 d\zeta = 1$ and (11.25) collapses to

$$\boxed{\mathbf{K}^e = \frac{EA}{\ell}\begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix}.} \tag{11.26}$$

This is the same element stiffness matrix of the prismatic truss member derived in Chapters 2 and 5 by a Mechanics of Materials approach, but now obtained through a variational argument.

### §11.4.2. The Consistent Node Force Vector

The *consistent node force vector* $\mathbf{f}^e$ defined in (11.19) comes from the element contribution to the external work potential $W$:

$$W^e = \int_{x_1}^{x_2} q\,u\,dx = \int_0^1 q\,\mathbf{N}^T\mathbf{u}^e\,\ell\,d\zeta = \left(\mathbf{u}^e\right)^T\int_0^1 q\begin{bmatrix}1-\zeta\\ \zeta\end{bmatrix}\ell\,d\zeta \stackrel{\text{def}}{=} \left(\mathbf{u}^e\right)^T\mathbf{f}^e, \tag{11.27}$$

Since $\mathbf{u}^e$ is arbitrary,

$$\boxed{\mathbf{f}^e = \int_{x_1}^{x_2} q\begin{bmatrix}1-\zeta\\ \zeta\end{bmatrix}dx = \int_0^1 q\begin{bmatrix}1-\zeta\\ \zeta\end{bmatrix}\ell\,d\zeta.} \tag{11.28}$$

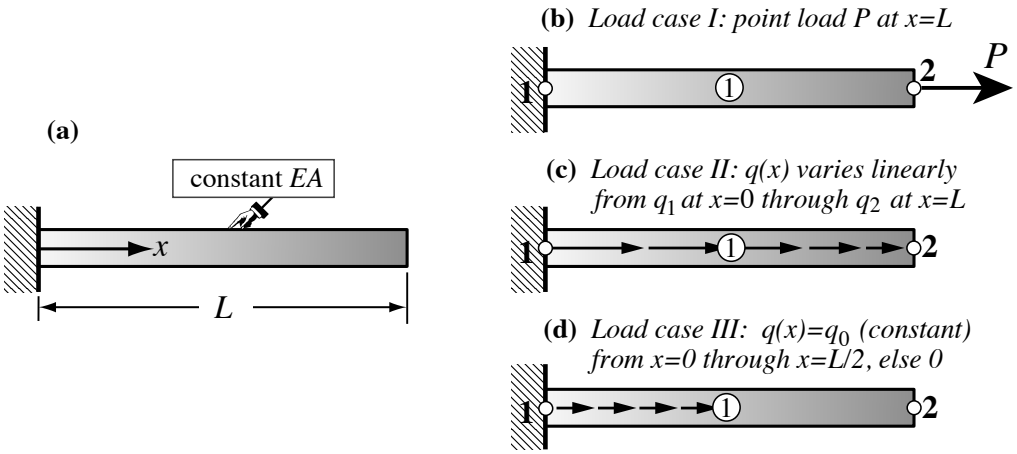


FIGURE 11.7. Fixed-free, prismatic bar example: (a) configuration; (b,c,d) FEM discretization and load cases.

in which $\zeta$ is defined by (11.15). If $q$ is constant over the element, it may be taken out of the integral:

$$\boxed{\mathbf{f}^e = q \int_0^1 \begin{bmatrix} 1-\zeta \\ \zeta \end{bmatrix} \ell \, d\zeta.} \tag{11.29}$$

This gives the same results as with the EbE lumping method of Chapter 7. See Exercise 11.3.

**Example 11.1**. The two-node bar element is tested on the benchmark problem defined in Figure 11.7. A fixed-free, homogeneous, prismatic bar of length $L$, elastic modulus $E$ and cross section area $A$ has the configuration illustrated in Figure 11.7(a). It is discretized with a *single* element as shown in Figure 11.7(b,c,d), and subjected to the three load cases pictured there. Case I involves a point load $P$ at the free end, which may be formally represented as

$$q^I(x) = P \ \delta(L) \tag{11.30}$$

where $\delta()$ denotes the delta function with argument $x$.

Case II involves a distributed axial load that varies linearly from $q_1 = q(0)$ at the fixed end through $q_2 = q(L)$ at the free end:

$$q^{II}(x) = q_1(1-\zeta) + q_2\zeta, \tag{11.31}$$

in which $\zeta = 1 - x/L$. Case III involves a "box" distributed load $q(x)$ that is constant and equal to $q_0$ from the fixed end $x = 0$ through midspan $x = L/2$, and zero otherwise:

$$q^{III}(x) = q_0 \left(H(x) - H(x - \tfrac{1}{2}L)\right), \tag{11.32}$$

in which $H()$ denotes the Heaviside unit step function with argument $x$. The master stiffness equations constructed using the prismatic stiffness matrix (11.26) with $\ell = L$ and $\bar{x} \to x$ are

$$\frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} u_1^m \\ u_2^m \end{bmatrix} = \begin{bmatrix} f_1^m \\ f_2^m \end{bmatrix} = \mathbf{f}^m. \tag{11.33}$$

Here supercript $m$ identifies the load case. The consistent node forces computed from (11.28) with $\ell = L$ and $\bar{x} \to x$ are

$$\mathbf{f}^I = \begin{bmatrix} 0 \\ P \end{bmatrix}, \qquad \mathbf{f}^{II} = \frac{L}{6}\begin{bmatrix} 2q_1 + q_2 \\ q_1 + 2q_2 \end{bmatrix}, \qquad \mathbf{f}^{III} = \frac{q_0 L}{8}\begin{bmatrix} 3 \\ 1 \end{bmatrix}. \tag{11.34}$$

On applying the fixed end support condition $u_1^m = 0$ and solving for $u_2^m$, the free end deflections are

$$u_2^I = \frac{PL}{EA}, \qquad u_2^{II} = \frac{(q_1 + 2q_2)L^2}{6EA}, \qquad u_2^{III} = \frac{q_0 L^2}{8EA}. \tag{11.35}$$

The analytical solutions for $u(x)$, obtained on integrating the ODE $EAu'' + q = 0$ with boundary conditions $u(0) = 0$, $F(L) = EA\,u'(L) = P$ for case I and $F(L) = EA\,u'(L) = 0$ for cases II and III, are

$$u^I(x) = \frac{Px}{EA}, \quad u^{II}(x) = \frac{x\,[3(q_1+q_2)L^2 - 3q_1Lx + (q_1-q_2)x^2]}{6EA}, \quad u^{III}(x) = \frac{q_0}{2EA}\big(Lx - x^2 + \langle x - \tfrac{1}{2}L\rangle^2\big). \tag{11.36}$$

In the expression of $u^{III}(x)$, $\langle x - \frac{1}{2}L\rangle^2$ means $(x - \frac{1}{2}L)^2$ if $x \ge \frac{1}{2}L$, else zero (Macauley's bracket notation for discontinuity functions). Evaluating (11.36) at $x = L$ and comparing to (11.35), one finds that the three computed end deflections are *exact*.

For case I this agreement is no surprise: the exact $u^I(x)$ is linear in $x$, which is contained in the span of the linear shape functions. But for II and III this is far from obvious since the exact solutions are cubic and piecewise quadratic, respectively, in $x$. The fact that the exact solution is verified at the free end node is an instance of the *nodal exactness* property discussed in §11.6.1.

Note that in cases II and III the FEM displacement solutions *inside* the element, which vary linearly, will not agree pointwise with the exact solutions, which do not. For example the exact midspan displacement is $u^{III}(\frac{1}{2}L) = q_0L^2/(8EA) = u^{III}(L)$, whereas the FEM interpolation would give $q_0L^2/(16EA)$ there, in error by 100%. To reduce such internal discrepancies the member may be divided into more elements.

## §11.5. Weak Forms

Weak forms are expressions notoriously difficult to explain to newcomers. They occupy an intermediate position between differential equations and functionals. There are so many variants and procedural kinks, however, that their position in the mathematical food chain is fuzzy. Confusion is compounded by the use of disparate terminology, some generic, some application oriented. To shed some sunlight into this murky swamp, we go through a specific example: the bar member.

### §11.5.1. From Strong to Weak

The governing differential equation for a bar member in terms of axial displacements is $\big(EA\,u'(x)\big)' + q(x) = 0$, or $EA\,u''(x) + q(x) = 0$ if the rigidity $EA$ is constant. Replace the zero by $r(x)$, which stands for *residual*, and move it to the left-hand side:

$$r(x) = \big(EA\,u'(x)\big)' + q(x), \quad \text{or if } EA \text{ is constant:} \quad r(x) = EA\,u''(x) + q. \tag{11.37}$$

The governing ODE may be compactly stated as $r(x) = 0$. This must hold *at each point* over the member span, say $x \in [0, L]$. Hence the term *strong form* (SF) used for this kind of mathematical model. No ambiguity so far. But suppose that insisting on $r(x) = 0$ everywhere is too demanding. We would like to relax that condition so it is satisfied only in an *average sense*. To accomplish that, multiply the residual by a function $v(x)$, integrate over the problem domain, and set the result to zero:

$$J = \int_0^L r(x)\,v(x)\,dx = 0. \tag{11.38}$$

Here $v(x)$ is supposed to be sufficiently well behaved for the integral to exist. Ignoring boundary conditions for now, (11.38) is called a *weak form*, which is often abbreviated to WF in the sequel.

Function $v(x)$ receives two names in the literature: *test function* in a general mathematical context, and *weight function* (also *weighting function*) in the context of approximation methods based on weak forms. In what follows both terms will be used within the appropriate context.

### §11.5.2. Weak Form Based Approximation Example

To show how weak forms can be used to generate approximate solutions, consider again a fixed-free, prismatic, homogeneous bar member (that is, $EA$ is constant), under uniform load $q(x) = q_0$ along its length and zero load at the free end. The WF (11.38) becomes

$$J = \int_0^L \big(EA\,u''(x) + q_0\big)\,v(x)\,dx = 0. \tag{11.39}$$

subject to the end conditions

$$u(0) = 0, \quad F(L) = EA\,u'(L) = 0. \tag{11.40}$$

We will restrict both $u(x)$ and $v(x)$ to be *quadratic* polynomials:

$$u(x) = a_0 + a_1\,x + a_2\,x^2, \quad v(x) = b_0 + b_1\,x + b_2\,x^2. \tag{11.41}$$

in which $a_i$ and $b_i$ are numerical coefficients, real in this case. Once assumptions such as those in (11.41) are made, more terminology kicks in. The assumed $u(x)$ is now called a *trial function*, which is spanned by the linear-space basis $\{1, x, x^2\}$ of dimension 3. The assumed $v(x)$ is called a *weight function*, which is spanned by exactly the same basis. There is a special name for the scenario when the trial and weight function bases coalesce: the *Galerkin method*.[7]. We will call the end result a *Galerkin solution*. Replacing (11.41) into (11.39) we get

$$J = \frac{L}{6}(6b_0 + 3b_1\,L + 2b_2\,L^2)\,(2EA\,a_2 + q_0). \tag{11.42}$$

Now $J$ must vanish for any arbitrary value of $\{b_0, b_1, b_2\}$. On extracting the expressions that multiply those coefficients we obtain the same equation thrice: $2EA\,a_2 + q_0 = 0$. Thus $a_2 = -q_0/(2EA)$, whereas $a_0$ and $a_1$ remain arbitrary. Consequently the Galerkin solution *before* BC is

$$u(x) = a_0 + a_1\,x - \frac{q_0}{2EA}x^2. \tag{11.43}$$

ODE aficionados would recognize this as the general solution of $EAu'' + q_0 = 0$ so Uncle Boris has done the job. Applying the end conditions (11.40) gives $a_0 = 0$ and $a_1 = q_0/(EA)$ whence the final solution is

$$u(x) = \frac{q_0}{2EA}\,x(2L - x). \tag{11.44}$$

Replacing into (11.37) and (11.40) it may be verified that this is the exact analytical solution.

Instead of applying the end conditions *a posteriori* we may try to incorporate them *a priori* into the trial function assumption. On enforcing (11.40) into the assumed $u(x)$ of (11.41) we find that $a_0 = 0$ and $a_1 = -2a_2\,L$. The trial function becomes

$$u(x) = a_2\,x\,(x - 2L), \tag{11.45}$$

[7] Introduced by Boris Galerkin in 1912. For a brief account of the general methodology, see **Notes and Bibliography**

and only one free coefficient remains. Accordingly only one weight basis function is needed: either $1, x$ or $x^2$ does the job, and the exact solution (11.44) is obtained again.[8]

What happens if the load $q(x)$ varies, say, linearly and the same quadratic polynomial assumptions (11.41) are used? Then Galerkin goes gaga. See Exercise 11.8.

Even for this trivial example, several procedural choices are apparent. If we allow the trial and weight function spaces to differ, volatility zooms up. Furthermore, we can apply transformations to the residual integral as done in the next subsection. Compared to the well ordered world of variational-based FEM, confusion reigns supreme.

### §11.5.3. Balanced Weak Forms

Some method in the madness can be injected by balancing. A look at (11.39) reveals an unpleasant asymmetry. Second derivatives of $u(x)$ appear, but none of $v(x)$. This places unequal restrictions on smoothness of the trial and test function spaces. Integration by parts restores derivative order balance. Replacing $\int_0^L E\,A\,u''\,v\,dx = -\int_0^L E\,A\,u'\,v'\,dx + (E\,A u')v\big|_0^L$ and rearranging terms yields

$$J = \int_0^L E\,A\,u'(x)\,v'(x)\,dx - \int_0^L q(x)\,v(x)\,dx - \big(E\,A\,u'(x)\big)\,v(x)\big|_0^L\,. \tag{11.46}$$

This will be called a *balanced-derivative weak form*, or simply a *balanced weak form* (BWF). It displays obvious advantages: (i) same smoothness requirements for assumed $u$ and $v$, and (ii) end BC appear explicitly in the non-integral term, neatly factored into essential and natural. A minor flaw is that the original residual is no longer clearly visible.

For a bar with variable axial rigidity replace $E\,A\,u''$ by $(E\,A\,u')'$ in the first integrand.

On repeating the Galerkin procedure of the previous subsection with the assumptions (11.41) one finds an identical $J$, as may be expected, and the same final solution. Again one has the choice of pre- or post-imposing the end conditions (11.40). Generally the latter choice is far more convenient in a computer implementation.

### §11.5.4. Principle of Virtual Work as Balanced Weak Form

There is a close relationship between the BWF (11.46) and one of the fundamental tools of Analytical Mechanics: the Principle of Virtual Work (PVW). To exhibit it, set the test function to be an admissible variation of $u(x)$: $v(x) = \delta\,u(x)$, in which $\delta u(x)$ strongly satisfies all essential BC. Then assume that $J$ is the first variation of a functional $\Pi$:

$$J = \int_0^L E\,A\,u'(x)\,\delta u'(x)\,dx - \int_0^L q(x)\,\delta u(x)\,dx - \big(E\,A\,u'(x)\big)\,\delta u(x)\big|_0^L \stackrel{\text{def}}{=} \delta\Pi. \tag{11.47}$$

Indeed this is the first variation of the TPE functional:

$$\Pi = U - W = \tfrac{1}{2}\int_0^L u'(x)\,E\,A\,u'(x)\,dx - \int_0^L q(x)\,u(x)\,dx \tag{11.48}$$

[8] Some early works covering weighted residual methods, for example Crandall [159], proclaim that the trial function must satisfy *all* BC *ab initio*. Later ones, e.g., [260,261], relax that rule to BC of essential type (in Galerkin methods, this rule applies to both trial and test functions since the spaces coalesce). In practice this rule can be often relaxed further, as in the example of §11.5.2, applying essential BCs at the last moment.

Hence $J = 0$ is the same as $\delta\Pi = 0$ or $\delta U = \delta W$, which is the PVW for an elastic bar member. This relationship can be used to prove an important property: *Galerkin method is equivalent to a variational formulation* if the residual is the Euler-Lagrange equation of a functional.

**Remark 11.5**. Where does the boundary term $\big(E A\, u'(x)\big)\, \delta u(x)\big|_0^L$ in (11.47) go? Actually, into $\delta W$. This immersion is a bit tricky, and depends on redefining $q(x)$ to include prescribed end point forces such as $N(L) = E A\, u'(L) = P$ through delta functions. This is the subject of Exercise 11.9.

### §11.5.5. *Weighted Residual Methods

Galerkin method is widely used in computational mechanics, but does not exhaust all possibilities of using a weak form as source for obtaining numerical solutions. The main generalization consist of allowing trial and test (weight) functions to be different. This leads to a rich class of approximation methods unified under the name *Method of Weighted Residuals* or MWR.

The key idea is as follows. Both $u(x)$ and $v(x)$ are restricted to belong to linear function spaces of *finite dimension* $N_u$ and $N_v$. These are the *trial function space* and the *test function space*, respectively. which are spanned by basis functions $\phi_i(x)$ and $\psi(x)$, respectively:

$$u(x) = \text{span}\big\{\phi_i(x),\, 1 \le i \le N_u\big\}, \qquad v(x) = \text{span}\big\{\psi_i(x),\, 1 \le i \le N_v\big\} \tag{11.49}$$

in which usually $N_u = N_v$. Since the spaces are linear, any $u(x)$ and $v(x)$ can be represented as linear combination of the basis functions:

$$u(x) = \sum_{i=1}^{N_u} a_i\, \phi_i(x), \qquad v(x) = \sum_{i=1}^{N_v} b_i\, \psi_i(x). \tag{11.50}$$

Here $a_i$ and $b_i$ are scalar coefficients, which may be real or complex depending on the nature of the problem. Insert these into the weak form, perform the necessary integrations, and extract the $N_v$ expressions that are coefficients of the $b_i$. Solve these equations for the coefficients $a_i$, and replace in the first of (11.50) to get the approximate solution $u(x)$.

The MWR methodology is of course not restricted to one space dimension. It also extends to time-dependent problems. It can be merged smoothly with the FEM concept of piecewise approximation using shape functions. Some references are provided under **Notes and Bibliography**.

## §11.6. *Accuracy Analysis

Low order 1D elements may give surprisingly high accuracy. In particular the lowly two-node bar element can display infinite accuracy under some conditions. This phenomenon is studied in this advanced section as it provides an introduction to modified equation methods and Fourier analysis along the way.

### §11.6.1. *Nodal Exactness and Superconvergence

Suppose that the following two conditions are satisfied:

1. The bar properties are constant along the length (prismatic member).
2. The distributed load $q(x)$ is zero between nodes. The only applied loads are point forces at the nodes.

If so, a linear axial displacement $u(x)$ as defined by (11.13) and (11.14) is the exact solution over each element since constant strain and stress satisfy, element by element, all of the governing equations listed in Figure 11.2.[9]

[9] The internal equilibrium equation $p' + q = E A\, u'' + q = 0$ is trivially verified because $q = 0$ from the second assumption, and $u'' = 0$ because of shape function linearity.

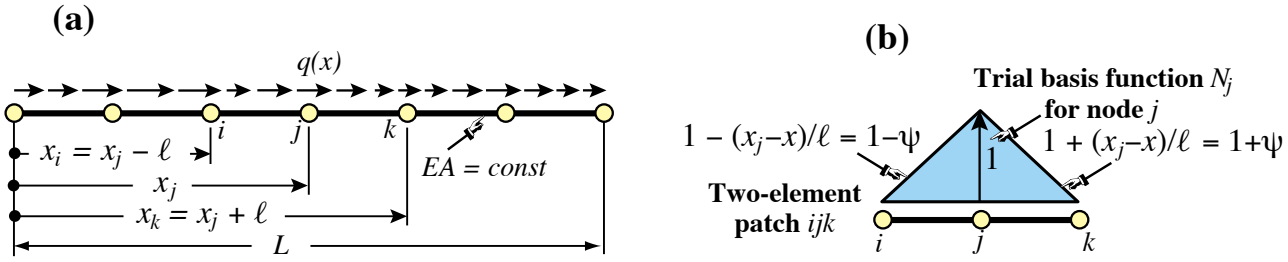


FIGURE 11.8. Superconvergence patch analysis: (a) lattice of bar elements; (b) two element patch.

It follows that if the foregoing conditions are verified the FEM solution is *exact*; that is, it agrees with the analytical solution of the mathematical model.[10] Adding extra elements and nodes would not change the solution. That is the reason behind the truss discretizations used in Chapters 2–3: *one element per member is enough* if they are prismatic and loads are applied to joints. Such models are called *nodally exact*.

What happens if the foregoing assumptions are not met? Exactness is then generally lost, and several elements per member may be beneficial if spurious mechanisms are avoided.[11] For a 1D lattice of equal-length, prismatic two-node bar elements, an interesting and more difficult result is: *the solution is nodally exact for any loading if consistent node forces are used*. This is proven in the subsection below. This result underlies the importance of computing node forces correctly.

If conditions such as equal-length are relaxed, the solution is no longer nodally exact but convergence at the nodes is extremely rapid (faster than could be expected by standard error analysis) as long as consistent node forces are used. This phenomenon is called *superconvergence* in the FEM literature.

### §11.6.2. *Fourier Patch Analysis

The following analysis is based on the modified differential equation (MoDE) method of Warming and Hyett [**?**] combined with the Fourier patch analysis approach of Park and Flaggs [553,554]. Consider a lattice of two-node prismatic bar elements of constant rigidity $EA$ and equal length $\ell$, as illustrated in Figure 11.8. The total length of the lattice is $L$. The system is subject to an arbitrary axial load $q(x)$. The only requirement on $q(x)$ is that it has a convergent Fourier series in the space direction.

From the lattice extract a patch[12] of two elements connecting nodes $x_i$, $x_j$ and $x_k$ as shown in Figure 11.8. The FEM patch equations at node $j$ are

$$\frac{EA}{\ell}\,[\,-1 \quad 2 \quad -1\,]\begin{bmatrix} u_i \\ u_j \\ u_k \end{bmatrix} = f_j, \tag{11.51}$$

in which the node force $f_j$ is obtained by consistent lumping:

$$f_j = \int_{x_i}^{x_k} q(x) N_j(x)\,dx = \int_{-1}^{0} q(x_j + \psi\ell)(1+\psi)\,\ell\,d\psi + \int_{0}^{1} q(x_j + \psi\ell)(1-\psi)\,\ell\,d\psi. \tag{11.52}$$

Here $N_j(x)$ is the "hat" trial basis function for node $j$, depicted in Figure 11.8, and $\psi = (x - x_j)/\ell$ is a dimensionless coordinate that takes the values $-1, 0$ and 1 at nodes $i$, $j$ and $k$, respectively. If $q(x)$ is expanded

[10] In variational language: the Green function of the $u'' = 0$ problem is included in the FEM trial space.

[11] These can happen when transforming such elements for 2D and 3D trusses. See Exercise E11.7.

[12] A patch is the set of all elements connected to a node; in this case $j$.

in Fourier series

$$q(x) = \sum_{m=1}^{M} q_m e^{i\beta_m x}, \quad \beta_m = m\pi/L, \tag{11.53}$$

(the term $m = 0$ requires special handling) the exact solution of the continuum equation $EA\,u'' + q = 0$ is

$$u^*(x) = \sum_{m=1}^{M} u^*_m e^{i\beta_m x}, \quad u^*_m = \frac{q_m e^{i\beta_m x}}{EA\beta_m^2}. \tag{11.54}$$

Evaluation of the consistent force using (11.52) gives

$$f_j = \sum_{m=1}^{M} f_{jm}, \quad f_{jm} = q_m \ell \frac{\sin^2(\frac{1}{2}\beta_m \ell)}{\frac{1}{4}\beta_m^2 \ell^2} e^{i\beta_m x_2}. \tag{11.55}$$

To construct a modified differential equation (MoDE), expand the displacement by Taylor series centered at node $j$. Evaluate at $i$ and $k$: $u_i = u_j - \ell u'_j + \ell^2 u''_j/2! - \ell^3 u'''_j/3! + \ell^4 u^{iv}_j/4! + \ldots$ and $u_k = u_j + \ell u'_j + \ell^2 u''_j/2 + \ell^3 u'''_j/3! + \ell^4 u^{iv}_j/4! + \ldots$. Replace these series into (11.51) to get

$$-2EA\ell \left( \frac{1}{2!} u''_j + \frac{\ell^2}{4!} u^{iv}_j + \frac{\ell^4}{6!} u^{vi}_j + \ldots \right) = f_j. \tag{11.56}$$

This is an ODE of infinite order. It can be reduced to an algebraic equation by assuming that the response of (11.56) to $q_m\, e^{i\beta_m x}$ is harmonic: $u_{jm} e^{i\beta_m x}$. If so $u''_{jm} = -\beta_m^2 u_{jm}$, $u^{iv}_{jm} = \beta_m^4 u_{jm}$, etc, and the MoDE becomes

$$2EA\ell\beta_m^2 \left( \frac{1}{2!} - \frac{\beta_m^2 \ell^2}{4!} + \frac{\beta_m^4 \ell^4}{6!} - \ldots \right) u_{jm} = 4EA\ell \sin^2(\tfrac{1}{2}\beta_m \ell)\, u_{jm} = f_{jm} = q_m \ell \frac{\sin^2(\frac{1}{2}\beta_m \ell)}{\frac{1}{4}\beta_m^2 \ell^2} e^{i\beta_m x_j}. \tag{11.57}$$

Solving gives $u_{jm} = q_m e^{i\beta_m x_j}/(EA\beta_m^2)$, which compared with (11.54) shows that $u_{jm} = u^*_m$ for any $m > 0$. Consequently $u_j = u^*_j$. In other words, the MoDE (11.56) and the original ODE: $EAu'' + q = 0$ have the same value at $x = x_j$ for any load $q(x)$ developable as (11.53). This proves nodal exactness. In between nodes the two solutions will not agree.[13]

The case $m = 0$ has to be treated separately since the foregoing expressions become $0/0$. The response to a uniform $q = q_0$ is a quadratic in $x$, and it is not difficult to prove nodal exactness.

### §11.6.3. *Robin Boundary Conditions

Suppose that for a bar of length $L$ one has the following end conditions: $u'(0) = au(0) + b$ at $x = 0$ and $u'(L) = au(L) + b$ at $x = L$, in which $a$ and $b$ are given coefficients. Those are called Robin BCs in the literature. Adjoining them as Courant penalty terms gives the functional

$$F(u) = \int_0^L [\tfrac{1}{2} EA\,(u')^2 - q\,u]\,dx + \tfrac{1}{2}[u'(0) - au(0) - b]^2 + \tfrac{1}{2}[u'(L) + au(L) + b]^2. \tag{11.58}$$

Divide [0,L] into $N_e$ elements and $N = N_e + 1$ nodes. Do $C^0$ linear interpolation over each element, insert into $F(u)$ to get $F_d(u) = \frac{1}{2}\mathbf{u}^T\mathbf{K}\mathbf{u} - \mathbf{f}^T\mathbf{v}$, in which $\mathbf{u}$ is the vector of node values, $\mathbf{K}$ the master stiffness matrix and $\mathbf{f}$ the master force vector. Coefficients $a$ and $b$ will affect both $\mathbf{K}$ and $\mathbf{f}$.

[13] The FEM solution varies linearly between nodes whereas the exact one is generally trigonometric.

Vanishing of the first variation: $\delta F_d = 0$ yields the FEM equations $\mathbf{Ku} = \mathbf{f}$ to be solved for $\mathbf{u}$. The Robin BCs at $x = 0$ and $x = L$ will affect the stiffness and force contributions of the first and last elements, but not those of interior elements.

This kind of boundary value problem (i.e., with Robin BCs) is common in heat conduction and heat transfer with convection given over cooling surfaces. In that case the heat flux is proportional to the difference of the (unknown) surface temperature and that of the cooling fluid. Elements that "touch" the convention boundary are affected.

**Notes and Bibliography**

The foregoing account pertains to the simplest structural finite element: the two-node bar element. For bar members these developments may be generalized in several directions, three of which are mentioned next.

*Refined bar models*. Adding internal nodes we can pass from linear to quadratic and cubic shape functions. These elements are rarely useful on their own right, but as accessories to 2D and 3D high order continuum elements (for example, to model edge reinforcements.) For that reason they are not considered here. The 3-node bar element is developed in exercises assigned in Chapter 16.

*Use in 2D and 3D truss structures*. The only additional ingredients are the local-to-global transformations discussed in Chapters 3 and 6.

*Curved bar elements*. These can be derived using isoparametric mapping, a device introduced later.

Matrices for straight bar elements are available in any finite element book; for example Przemieniecki [596].

Tonti diagrams were introduced in the 1970s in papers now difficult to access, for example [749]. Scanned images are available, howewer, from `http://www.dic.units.it/perspage/discretephysics`

The fundamentals of Variational Calculus may be studied in the excellent textbook [297], which is now available in an inexpensive Dover edition. The proof of the MPE principle can be found in texts on variational methods in mechanics. For example: Langhaar [435], which is the most readable "old fashioned" treatment of the energy principles of structural mechanics, with a clear treatment of virtual work. (Out of print but used copies may be found via the web engines cited in §1.5.2.) The elegant treatment by Lanczos [434] is recommended as reading material although it is more oriented to physics than structural mechanics.

It was noted that weak forms occupy an intermediate position between two older classical areas: differential equations (introduced in the XVII Century by the Calculus founders) and variational forms (introduced by Euler in the XVIII Century). Some weak forms in disguise are also ancient; e.g., the PVW was placed on firm mathematical grounds by Lagrange in the late XVIII Century [430]. But their rapid development as tools for producing approximate solutions of ODEs and PDEs took place in the early XIX Century. Five important variants are: Galerkin (1915), subdomain (1923), least squares (1928), moments (1932), and collocation (1937). These, as well as a few others of less importance, were unified in 1956 under the label Method of Weighted Residuals or MWR, by Crandall [159]. Other attempts at unification during this period may be found in [19,147]. The use of MWR methods, especially Galerkin's, as enabling devices to generate finite element equations developed rapidly following the pioneer paper [820]. The chief motivation was to accommodate application problems where a classical variational formulation does not exist, or is inconvenient to use.

The first accuracy study of FEM discretizations using modified equation methods is by Waltz et. al. [780]; however their procedures were faulty, which led to incorrect conclusions. The first correct derivation of modified equations appeared in [783]. The topic has recently attracted interest from applied mathematicians because modified equations provide a systematic tool for *backward error analysis* of differential equations: the discrete solution is the exact solution of the modified problem. This is particularly important for the study of long term behavior of discrete dynamical systems, whether deterministic or chaotic. Recommended references along these lines are [318,327,709].

Nodal exactness of bar models for point node loads is a particular case of a theorem by Tong [746]. For arbitrary loads it was proven by Park and Flaggs [553,554], who followed a variant of the scheme of §11.6.2.

A different technique is used in Exercise 11.10. The budding concept of superconvergence, which emerged in the late 1960s, is outlined in the book of Strang and Fix [698]. There is a monograph [781] devoted to the subject; it covers only Poisson problems but provides a comprehensive reference list until 1995.

**References**

Referenced items moved to Appendix R.

## Homework Exercises for Chapter 11
## Variational Formulation of Bar Element

**EXERCISE 11.1** [D:10] Explain the kinematic admissibility requirements stated in **?** in terms of physics, namely ruling out the possibility of gaps or interpenetration as the bar material deforms.

**EXERCISE 11.2** [A/C:15] Using (11.25), derive the stiffness matrix for a *tapered* bar element in which the cross section area varies linearly along the element length:

$$A = A_i(1-\zeta) + A_j\,\zeta, \tag{E11.1}$$

where $A_i$ and $A_j$ are the areas at the end nodes, and $\zeta = x^e/\ell$ is the dimensionless coordinate defined in §11.3.6. Show that this yields the same answer as that of a stiffness of a constant-area bar with cross section $\frac{1}{2}(A_i + A_j)$. Note: the following *Mathematica* script may be used to solve this exercise:[14]

```
ClearAll[Le,x,Em,A,Ai,Aj];
Be={{-1,1}}/Le; ζ=x/Le; A=Ai*(1-ζ)+Aj*ζ;
Ke=Integrate[Em*A*Transpose[Be].Be,{x,0,Le}];
Ke=Simplify[Ke];
Print["Ke for varying cross section bar: ",Ke//MatrixForm];
```

In this and following scripts `Le` stands for $\ell$.

**EXERCISE 11.3** [A:10] Find the consistent load vector $\mathbf{f}^e$ for a bar of *constant* area $A$ subject to a *uniform* axial force $q = \rho g A$ per unit length along the element. Show that this vector is the same as that obtained with the element-by-element (EbE) "lumping" method of §8.4, which simply assigns half of the total load: $\frac{1}{2}\rho g A\ell$, to each node. Hint: use (11.29) and $\int_0^1 \zeta\, d\zeta = 1/2$.

**EXERCISE 11.4** [A/C:15] Repeat the previous calculation for the tapered bar element subject to a force $q = \rho g A$ per unit length, in which $A$ varies according to (E11.1) whereas $\rho$ and $g$ are constant. Check that if $A_i = A_j$ one recovers $f_i = f_j = \frac{1}{2}\rho g A\ell$. Note: the following *Mathematica* script may be used to solve this exercise:[15]

```
ClearAll[q,A,Ai,Aj,ρ,g,Le,x];
ζ=x/Le; Ne={{1-ζ,ζ}}; A=Ai*(1-ζ)+Aj*ζ; q=ρ*g*A;
fe=Integrate[q*Ne,{x,0,Le}];
fe=Simplify[fe];
Print["fe for uniform load q: ",fe//MatrixForm];
ClearAll[A];
Print["fe check: ",Simplify[fe/.{Ai->A,Aj->A}]//MatrixForm];
```

**EXERCISE 11.5** [A/C:20] A tapered bar element of length $\ell$, end areas $A_i$ and $A_j$ with $A$ interpolated as per (E11.1), and constant density $\rho$, rotates on a plane at uniform angular velocity $\omega$ (rad/sec) about node $i$. Taking axis $x$ along the rotating bar with origin at node $i$, the centrifugal axial force is $q(x) = \rho A\omega^2 x$ along the length, in which $x \equiv x^e$. Find the consistent node forces as functions of $\rho$, $A_i$, $A_j$, $\omega$ and $\ell$, and specialize the result to the prismatic bar $A = A_i = A_j$. Partial result check: $f_j = \frac{1}{3}\rho\omega^2 A\ell^2$ for $A = A_i = A_j$.

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[14] The `ClearAll[...]` at the start of the script is recommended programming practice to initialize variables and avoid "cell crosstalk." In a `Module` this is done by listing the local variables after the `Module` keyword.

[15] The `ClearAll[A]` before the last statement is essential; else `A` would retain the previous assignation.

**EXERCISE 11.6** [A:15] (Requires knowledge of Dirac's delta function properties.) Find the consistent load vector $\mathbf{f}^e$ if the bar is subjected to a concentrated axial force $Q$ at a distance $x = a$ from its left end. Use (11.28), with $q(x) = Q\,\delta(a)$, in which $\delta(a)$ is the one-dimensional Dirac's delta function at $x = a$. Note: the following script does it by *Mathematica*, but it is overkill:

```
ClearAll[Le,q,Q,a,x];
ζ=x/Le; Ne={{1-ζ,ζ}}; q=Q*DiracDelta[x-a];
fe=Simplify[ Integrate[q*Ne,{x,-Infinity,Infinity}] ];
Print["fe for point load Q at x=a: ",fe//MatrixForm];
```

**EXERCISE 11.7** [C+D:20] In a learned paper, Dr. I. M. Clueless proposes "improving" the result for the example truss by putting three extra nodes, 4, 5 and 6, at the midpoint of members 1–2, 2–3 and 1–3, respectively. His "reasoning" is that more is better. Try Dr. C.'s suggestion using the *Mathematica* implementation of Chapter 4 and verify that the solution "blows up" because the modified master stiffness is singular. Explain physically what happens.

**EXERCISE 11.8** [C+D:15] This exercise illustrates "Galerkin surprises." Take up again the example of §11.5.2, but suppose now that the axial load varies linearly, as in (11.31). The trial and weight function assumptions are the quadratic polynomials (11.41). Show that the integral (11.39) is given by

$$12\,J/L = b_0\left(24EA\,a_2 + 6(q_1{+}q_2)\right) + b_1\left(12EA\,a_2 + 2(q_1{+}2q_2)\right) + b_2\left(8EA\,a_2 + (q_1{+}3q_2)\right), \quad \text{(E11.2)}$$

and that the resulting 3 equations for $a_2$ are *inconsistent* unless $q_1 = q_2$. Only one weight function gives the correct solution at $x = L$; which one? Note that the Galerkin method is generally viewed as the "most reliable" member of the MWR tribe. But unforeseen surprises have a silver lining: more papers can be written to explain them. Here is a partial fix: make the test function satisfy the essential BC *a priori*.

**EXERCISE 11.9** [A:20]. Prove that (11.47) is the first variation of (11.48), thus linking the PVW with the TPE functional. See Remark 11.5 for a hint on how to treat the boundary term in (11.47).

**EXERCISE 11.10** [A:35, close to research paper level]. Prove nodal exactness of the two-node bar element for arbitrary but Taylor expandable loading without using the Fourier series approach. Hints: expand $q(x) = q(x_j) + (\ell\psi)q'(x_j) + (\ell\psi)^2 q''(x_j)/2! + \ldots$, where $\ell\psi = x - x_j$ is the distance to node $j$, compute the consistent force $f_j(x)$ from (11.52), and differentiate the MoDE (11.56) repeatedly in $x$ while truncating all derivatives to a maximum order $n \geq 2$. Show that the original ODE: $EAu'' + q = 0$, emerges as an identity regardless of how many derivatives are kept.