ME-504 Take-Home Exam: Problem 4 (Finite Element Theory)

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Strong Form of the Problem:

$$-(sin(x)\theta')' + cos(x)\theta = x, \quad \Omega \rightarrow 0 < x < 1$$
$$\theta(0) = 1$$
$$\theta(1) = -1$$

Weak Form of the Problem:

$$\int_0^1 [v'\theta' sin(x) + v\theta cos(x)]dx = 0$$

Examine Convergence:

Approximate solutions over the problem domain are shown in Figure 1 and first derivatives of the approximate solutions are shown in Figure 2. Convergence was analyzed at $x = 0.5$, as shown in Figure 3.

Evaluate the Error:

Figure 4 shows the calculated values of the Energy and L2 error norms with respect to the inverse of the element width. The errors were calculated between elements of width h and width $\frac{h}{2}$, not with respect to the analytical solution. The error norms scale with the order of the basis function such that the energy error norm scales with h^{k+1} and the L2 error norm scales with h^k, where h is the element width and k is the order of the polynomial used for the basis function (order 1 polynomials were used for this assignment).

Energy Error Norm

The work function is:

$$\prod[\theta] = \frac{1}{2}\int_0^1 [sin(x)\theta'^2 + cos(x)\theta^2]dx - \int_0^1 [x\theta]dx = 0$$

Therefore, the relative energy norm, which takes into account only the stiffness matrix (i.e., not the forcing function) is defined as:

$$||e||_{Energy} = \Big(\frac{1}{2}\int_0^1 [sin(x)(\theta'_N - \theta'_{2N})^2 + cos(x)(\theta_N - \theta_{2N})^2]dx\Big)^{\frac{1}{2}}$$

L2 Error Norm:

$$||e||_{L2} = \Big(\int_0^1 [(\theta_N - \theta_{2N})^2]dx\Big)^{\frac{1}{2}}$$

0.1 Discussion of Results

Convergence rates vary dramatically across the domain. The approximate solution appears to have converged on the right side of the domain, near x = 1, when as few as four elements were used. This quick convergence rate is the result of the solution being approximately linear on the right side of the domain and the implementation of linear basis functions. However, the left side of the domain, near x = 0, had a solution that was highly nonlinear and required a very fine mesh to obtain convergence (more than 128 elements). Convergence would have been reached much quicker over the nonlinear portion of the mesh had higher order (nonlinear) basis functions been used. Difference in convergence rates across the domain is especially evident when comparing the spatial derivatives of the approximate solution across the domain. The approximate derivatives all overly on the right side of the domain while there are substantial difference as more elements are used in to results on the right side of the domain. The approximate solution at $x = 0.5$ shows convergence as the number of elements are increased, and both error norms decreased when an increasing number of elements were used.

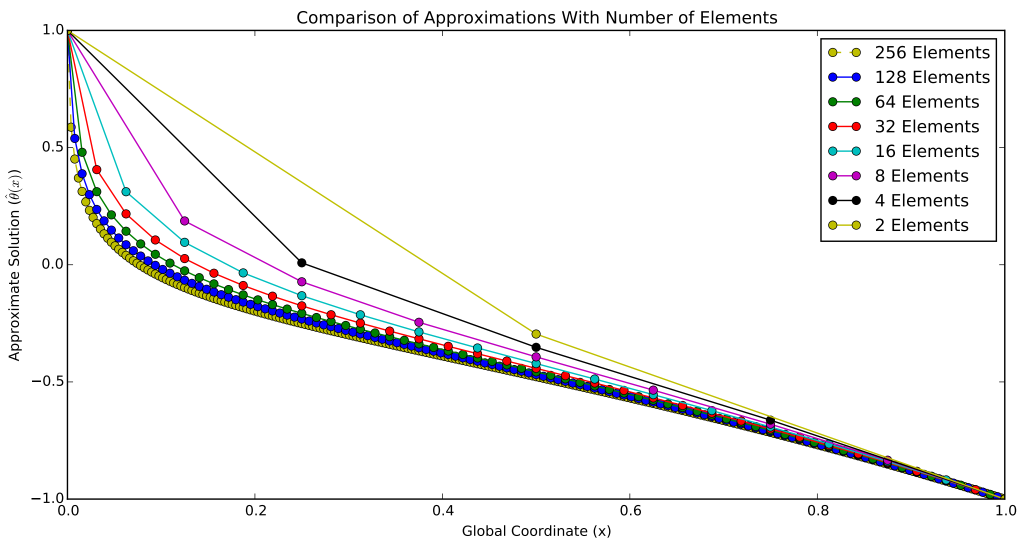

Figure 1: Approximate Solutions

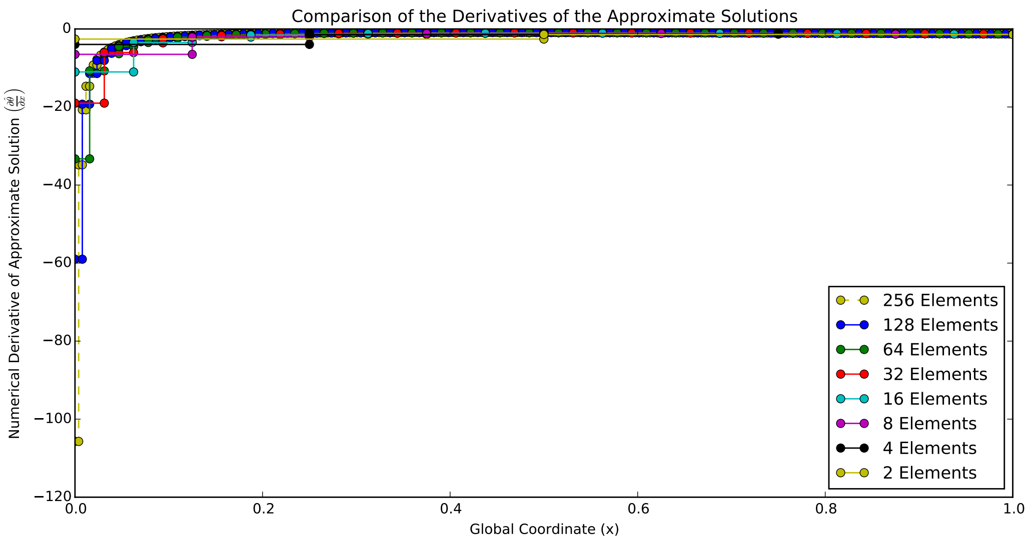

Figure 2: Derivatives of the Approximate Solutions

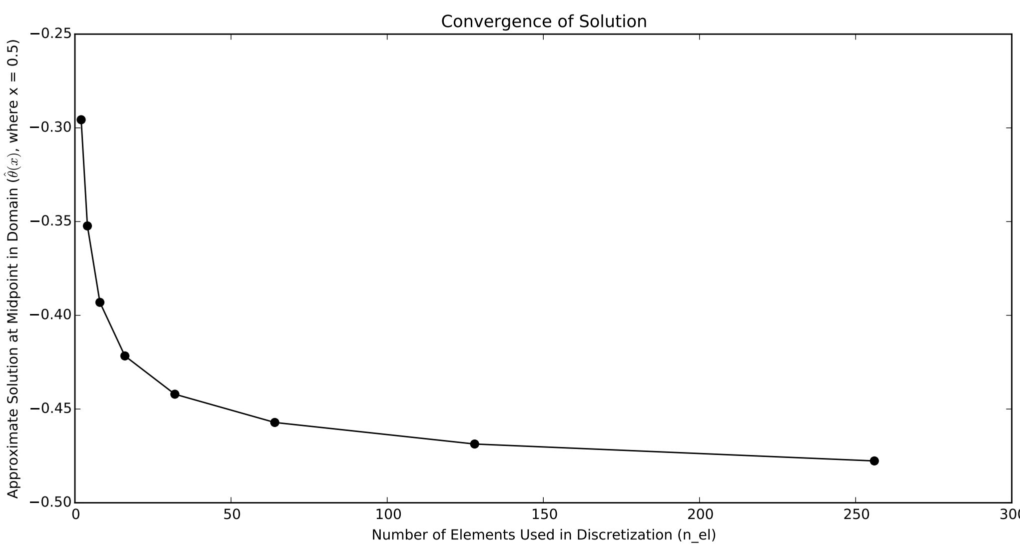

Figure 3: Approximate solutions at x = 0.5 for differing numbers of elements.

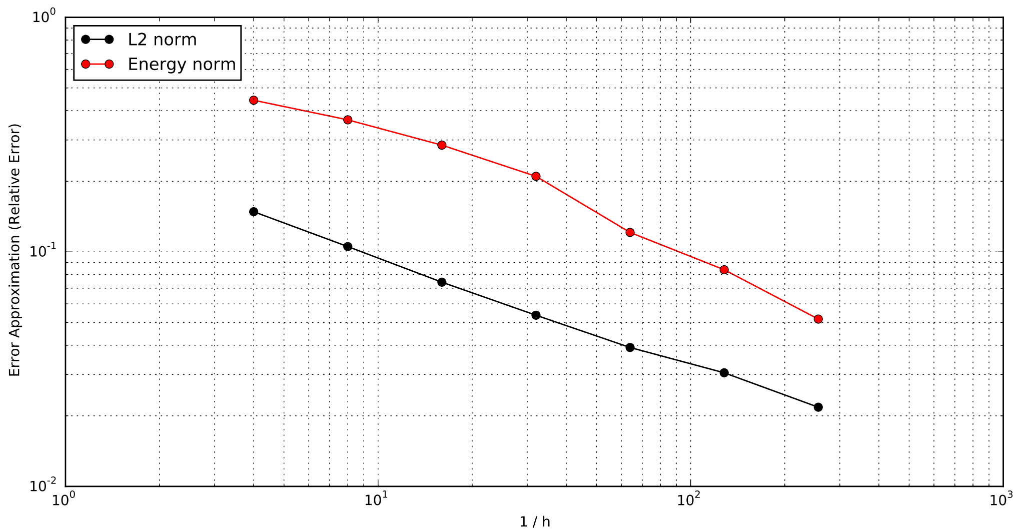

Figure 4: L2 and Energy error norms with respect to the inverse of element width.