Low Reynolds Number “Creeping” Flow in Micro and Nanofluidic Systems

CBE/NE/BME 525

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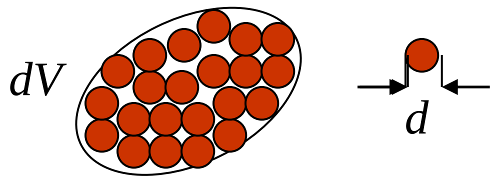
Outline

1. Continuum Hypothesis
2. Concept of low Reynolds number and its implications for the fluid flow.
3. Flows in narrow slits and capillaries. Simplification of the Navier-Stokes equations using low Re number and/or symmetry arguments. Hagen-Poiseuille flow.
4. Flow in porous media: Carman-Kozeny empirical equation. Cell hydrodynamic model of Happel.
5. Movement of a spherical particle in an unbound viscous fluid at low Re (Stokes problem). Forces on a moving particle
6. Approach of surfaces in viscous fluids. Reynolds problem.

Continuum Hypothesis

Hydrodynamics: Ignores the structure of the fluid, solvent, dissolved species



$$dV \gg \frac{\pi d^3}{6}$$

Typical Dimensions:

- Ionic Range → 0.1 ÷ 1.0 nm
- Molecular Range → 0.2 ÷ 10 nm
- Macro Molecular and Colloidal Range → 1.0 ÷ 1000 nm
- Micro Particle Range → 1.0 ÷ 50 μm
- Macro Particle Range → 50 ÷ > 1000 μm

Flow of Incompressible Fluid

$$\rho\left(\frac{\partial \mathbf{v}}{\partial \mathbf{t}}+\mathbf{v}\nabla\mathbf{v}\right)=-\nabla p+\eta\nabla^2\mathbf{v}+\text{force terms}\leftarrow\ \text{balance of linear momentum}$$

$$\nabla\cdot\mathbf{v}=0\leftarrow\ \text{continuity equation}$$

Dimensionless Variables

$$\tilde{\mathbf{r}}=\mathbf{r}/l,\ \tilde{\mathbf{v}}=\mathbf{v}/U,\ \tilde{t}=tU/l$$

$$\tilde{p}=\ \ p-p_\infty\ \ l/\eta U,\ \tilde{\nabla}=l\nabla$$

l – characteristic length

U – characteristic velocity

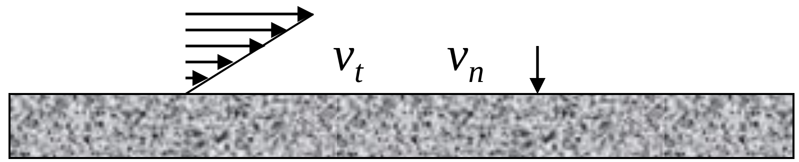
$$\mathrm{Re}\left(\frac{\partial \tilde{\mathbf{v}}}{\partial \mathbf{t}}+\tilde{\mathbf{v}}\tilde{\nabla}\tilde{\mathbf{v}}\right)=-\tilde{\nabla}\tilde{p}+\eta\tilde{\nabla}^2\tilde{\mathbf{v}}$$

$$\tilde{\nabla}\cdot\tilde{\mathbf{v}}=0,\quad \mathrm{Re}=\frac{\rho U l}{\eta}$$

Boundary Conditions

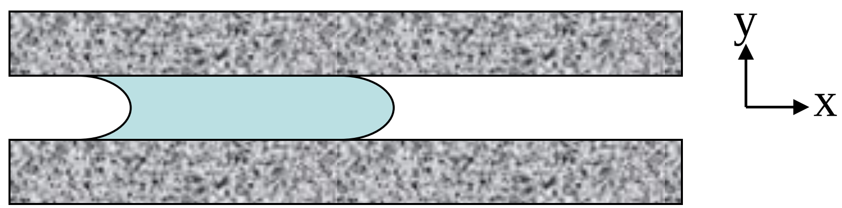
No-slip

Hard surfaces



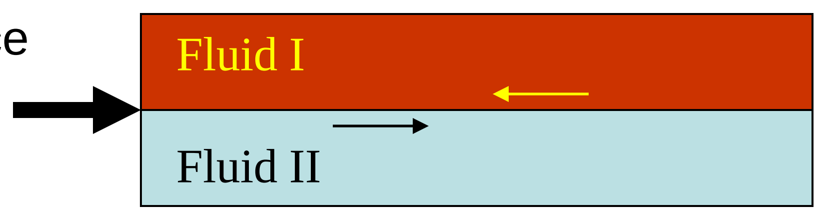
$$v_t = v_n = 0, \text{ or } \mathbf{v} = \mathbf{0}$$

Symmetry



$$\left.\frac{\partial v_x}{\partial y}\right|_{\text{center}} = 0$$

Stress balance



$$\boldsymbol{\tau}_{\mathrm{I}} = \boldsymbol{\tau}_{\mathrm{II}}$$

Low Reynolds Number Flow

Steady Flow

$$\eta\tilde{\nabla}^2\tilde{\mathbf{v}} - \tilde{\nabla}\tilde{p} = 0, \quad \tilde{\nabla}\cdot\tilde{\mathbf{v}} = 0$$

Unsteady Flow

$$v = U\cos\ \omega t + \alpha$$

$$\mathrm{Re}^{r}\ \frac{\partial\tilde{\mathbf{v}}}{\partial\mathbf{t}} + \mathrm{Re}^{t}\ \tilde{\mathbf{v}}\tilde{\nabla}\tilde{\mathbf{v}} = -\tilde{\nabla}\tilde{p} + \eta\tilde{\nabla}^2\tilde{\mathbf{v}}$$

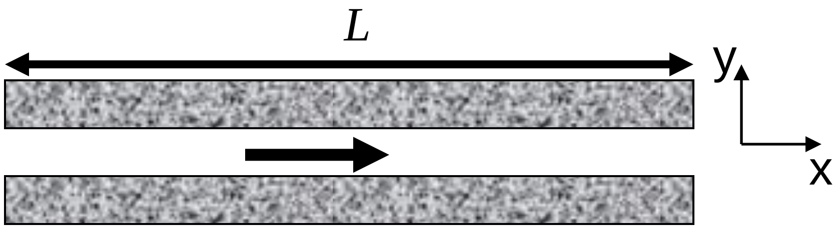
$$\tilde{\nabla}\cdot\tilde{\mathbf{v}} = 0, \quad \mathrm{Re}^{t} = \frac{\rho U l}{\eta}, \mathrm{Re}^{r} = \frac{\rho\omega l^2}{\eta}$$

$$\Rightarrow$$

$$\mathrm{Re}^{r}\ \frac{\partial\tilde{\mathbf{v}}}{\partial\tilde{\mathbf{t}}} = -\tilde{\nabla}\tilde{p} + \eta\tilde{\nabla}^2\tilde{\mathbf{v}}, \mathrm{Re}^{t} \ll \mathrm{Re}^{r}$$

$$\frac{\partial\mathbf{v}}{\partial\mathbf{t}} = -\nabla p + \eta\nabla^2\mathbf{v}, \quad \nabla\cdot\mathbf{v} = 0$$

Equations for Steady Flow in Narrow Slits



$$\rho\left(\frac{\partial v_x}{\partial t}+v_x\frac{\partial v_x}{\partial x}+v_y\frac{\partial v_x}{\partial y}\right)=-\frac{\partial p}{\partial x}+\eta\left(\frac{\partial^2 v_x}{\partial x^2}+\frac{\partial^2 v_x}{\partial y^2}\right)$$

$$\rho\left(\frac{\partial v_y}{\partial t}+v_x\frac{\partial v_y}{\partial x}+v_y\frac{\partial v_y}{\partial y}\right)=-\frac{\partial p}{\partial y}+\eta\left(\frac{\partial^2 v_y}{\partial x^2}+\frac{\partial^2 v_y}{\partial y^2}\right)$$

$$\frac{\partial v_x}{\partial x}+\frac{\partial v_y}{\partial y}=0$$

$$\Rightarrow \quad \eta\frac{\partial^2 v_x}{\partial y^2}=\frac{\partial p}{\partial x}$$

Boundary Conditions

$\mathbf{v}=\mathbf{0}$ at the walls

$$\frac{\partial v_x}{\partial y}=0 \text{ in the center}$$

$$-\frac{\partial p}{\partial x}=\frac{p_{in}-p_{out}}{L}$$

Flow Velocity and Shear Stress in Narrow Slits

Velocity Profile

$$v_x = \frac{1}{\eta}\frac{dp}{dx}\; y^2 - R^2 \;= -\frac{1}{\eta}\frac{p_{in} - p_{out}}{L}\; y^2 - R^2$$

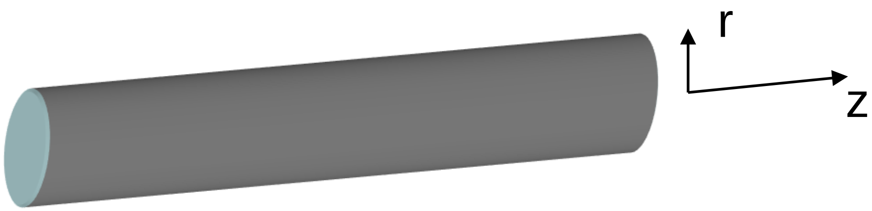
Shear Stress

$$\tau_{xy} = \tau_{yx} = 2\frac{dp}{dx}y = -2\frac{p_{in} - p_{out}}{L}y$$

Friction Force
per unit area
at the wall

$$\frac{F}{A} = \tau_{yx} = 2\frac{dp}{dx}R$$

Equations for Steady Flow in Narrow Capillaries



$$\rho\left(\frac{\partial v_r}{\partial t}+v_r\frac{\partial v_r}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta}+v_z\frac{\partial v_z}{\partial z}-\frac{v_\theta^2}{r}\right)=-\frac{\partial p}{\partial r}+\eta\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial r v_r}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2}+\frac{\partial^2 v_r}{\partial z^2}-\frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}\right]$$

$$\rho\left(\frac{\partial v_\theta}{\partial t}+v_r\frac{\partial v_\theta}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta}+v_z\frac{\partial v_z}{\partial z}-\frac{v_r v_\theta}{r}\right)=-\frac{1}{r}\frac{\partial p}{\partial \theta}+\eta\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial r v_\theta}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2}+\frac{\partial^2 v_\theta}{\partial z^2}-\frac{2}{r^2}\frac{\partial v_r}{\partial \theta}\right]$$

$$\rho\left(\frac{\partial v_z}{\partial t}+v_r\frac{\partial v_z}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta}+v_z\frac{\partial v_z}{\partial z}\right)=-\frac{\partial p}{\partial z}+\eta\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2}+\frac{\partial^2 v_z}{\partial z^2}\right]$$

$$\Rightarrow \quad \eta\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right)=\frac{\partial p}{\partial z}$$

$$\frac{1}{r}\frac{\partial\ r v_r}{\partial r}+\frac{1}{r}\frac{\partial v_\theta}{\partial \theta}+\frac{\partial v_z}{\partial z}=0$$

Boundary Conditions

$\mathbf{v}=\mathbf{0}$ at the wall, $\frac{\partial v_z}{\partial r}=0$ in the center

$$-\frac{\partial p}{\partial z}=\frac{p_{in}-p_{out}}{L}$$

<u>Example:</u> Flow in a Circular Capillary

(Hagen-Poiseuille Solution)

$$\eta \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) = \frac{\partial p}{\partial z} \quad , \mathbf{v} = \mathbf{0} \text{ at the wall, } \frac{\partial v_z}{\partial r} = 0 \text{ in the center, } -\frac{\partial p}{\partial z} = \frac{p_{in} - p_{out}}{L}$$

$$\Rightarrow \quad \left(r\frac{\partial v_z}{\partial r}\right) = \frac{1}{2\eta}\frac{\partial p}{\partial z} r^2 + C_1 \quad \text{first integration, } C_1 \equiv 0$$

$$\Rightarrow v_z = \frac{1}{4\eta}\frac{\partial p}{\partial z} r^2 + C_2 \quad \text{second integration, } C_2 = -\frac{1}{4\eta}\frac{\partial p}{\partial z} R^2$$

$$\Rightarrow v_z = -\frac{1}{4\eta}\frac{\partial p}{\partial z}\left(R^2 - r^2\right) = \frac{1}{4\eta}\frac{p_{in} - p_{out}}{L}\left(R^2 - r^2\right) \text{ parabolic profile!}$$

Example: Flow in a Circular Capillary

(Hagen-Poiseuille Solution)

Maximum velocity $$v_{z,\max} = \frac{p_{in} - p_{out}}{4\eta L} R^2$$

Average velocity $$\langle v_z \rangle = \frac{\int_0^{2\pi} \int_0^R v_z r dr d\theta}{\int_0^{2\pi} \int_0^R r dr d\theta} = \frac{p_{in} - p_{out}}{8\eta L} R^2 = \frac{1}{2} v_{z,\max}$$

Bulk flow rate $$Q = \pi R^2 \langle v_z \rangle = \int_0^{2\pi} \int_0^R v_z r dr d\theta = \frac{\pi R^4 \quad p_{in} - p_{out}}{8\eta L}$$

Flow Velocity and Shear Stress in a Capillary

(Pressure Driven Flow)

Velocity Profile

$$v_z = \frac{1}{4\eta}\frac{dp}{dz}\ r^2 - R^2\ = -\frac{1}{4\eta}\frac{p_{in} - p_{out}}{L}\ r^2 - R^2$$

Shear Stress

$$\tau_{rz} = \tau_{zr} = \frac{1}{2}\frac{dp}{dz}r = -\frac{1}{2}\frac{p_{in} - p_{out}}{L}r$$

Friction Force per unit area at the wall

$$\frac{F}{A} = \tau_{zr} = \frac{1}{2}\frac{dp}{dz}R$$

Porous Media

D'Arcy Law: the bulk fluid flow rate is linearly proportional to the driving force (pressure difference)

$$U = \frac{k}{\mu}\frac{\Delta p}{l} = k'\Delta p$$

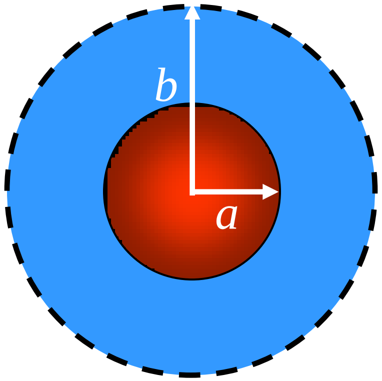
The main problem is to find k

Empirical Approach: Kozeny-Carman Equation

Capillaries $\quad k = \frac{\varepsilon^3 d_C^2}{4k_k}, \qquad k_k = 2\frac{l_e}{l} \quad \leftarrow \quad \text{Kozeny constant}$

Packed spheres $\quad k = \frac{\varepsilon^3 d_p^2}{36(1-\varepsilon)^2 k_k}, \quad k_k \simeq 5 \quad \leftarrow \quad \text{Kozeny constant}$

Cell Model



$$\frac{a}{b} = \gamma; \quad \left(\frac{a}{b}\right)^3 = \gamma^3 = \Phi \quad \rightarrow \text{ volume fraction}$$

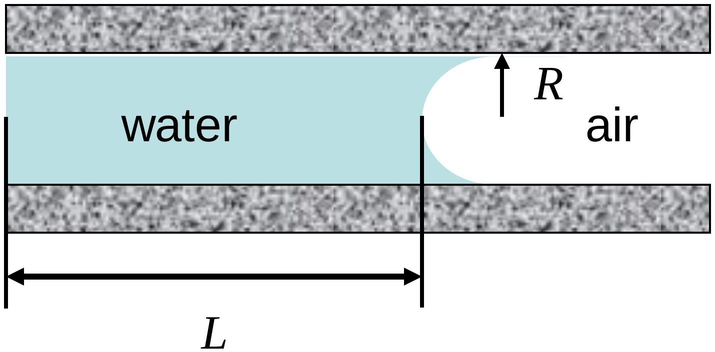
$$\varepsilon = 1 - \Phi = 1 - \gamma^3 \quad \rightarrow \quad \text{porosity}$$

$$k = \frac{3 - \ 9/2 \ \gamma + \ 9/2 \ \gamma^5 - 3\gamma^6}{3 + 2\gamma^5} \frac{2a^2}{9\gamma^3}$$

$$U = \left\{ \left[\frac{3 - \ 9/2 \ \gamma + \ 9/2 \ \gamma^5 - 3\gamma^6}{3 + 2\gamma^5} \right] \frac{2a^2}{9\gamma^3} \right\} \frac{\Delta p}{\eta l}$$

The cell model can be easily generalized to include electrokinetic transport!

Capillary Driven Flows. Perfectly Wettable Walls

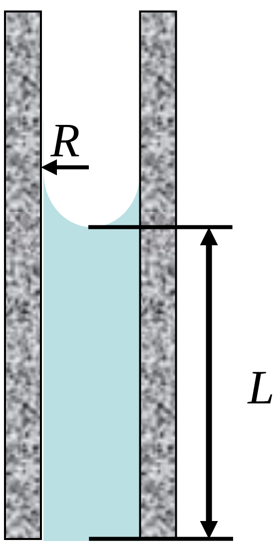


$$\langle v_z \rangle = \frac{\int_0^{2\pi}\int_0^{R} v_z r dr d\theta}{\int_0^{2\pi}\int_0^{R} r dr d\theta} = \frac{\Delta p}{8\eta l} R^2$$

$$\Delta p = p_C = \gamma\left(\frac{1}{R_1} + \frac{1}{R_2}\right) = \frac{2\gamma}{R}, \quad \text{for } R_1 = R_2 = R$$

Washburn Equation

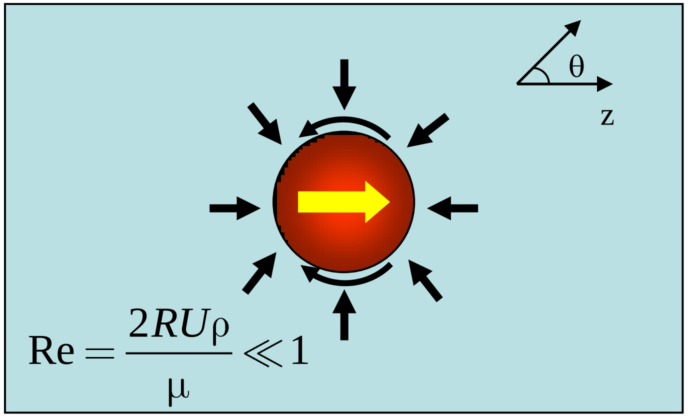
$$\langle v_z \rangle = \frac{p_C R^2}{8\eta l} = \frac{\gamma R}{4\eta l}$$



$$\langle v_z \rangle = \frac{p_C - \rho g l \;\; R^2}{8\eta l} = \frac{\frac{2\gamma}{R} - \rho g l \;\; R^2}{8\eta l} = \frac{\gamma R}{4\eta l} - \frac{\rho g R^2}{8\eta}$$

Movement of a Spherical Nanoparticle in Viscous Fluid

(Stokes Problem)



Navier-Stokes Equations

$$\eta\left[\frac{1}{r^2}\frac{\partial^2 \; r v_r}{\partial r^2} + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial v_r}{\partial\theta}\right)\right] = \frac{\partial p}{\partial r}$$

$$\eta\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r\frac{\partial v_\theta}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial v_\theta \sin\theta}{\partial\theta}\right) + \frac{2}{r^2}\frac{\partial v_r}{\partial\theta}\right] = \frac{1}{r}\frac{\partial p}{\partial\theta}$$

Continuity

$$\frac{1}{r^2}\frac{\partial \; r^2 v_r}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial v_\theta \sin\theta}{\partial\theta} = 0$$

Viscous Stresses

$$\tau_{rr} = -2\eta\frac{\partial v_r}{\partial r}, \quad \tau_{r\theta} = \tau_{\theta r} = -\eta\left[r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right) + \frac{1}{r}\frac{\partial v_r}{\partial\theta}\right]$$

Far from the sphere

$$v_r = U\cos\theta, \quad v_\theta = -U\sin\theta, \quad r \to \infty$$

At the surface

$$v_r = v_\theta = 0, \quad r = R$$

Movement of a Spherical Nanoparticle in Viscous Fluid: Solution Procedure

Assuming the solutions have the forms

$$v_r = f \;\; r \;\; \cos\theta, \quad v_\theta = \varphi \;\; r \;\; \sin\theta, \quad p = \mu\psi \;\; r \;\; \cos\theta$$

Navier Stokes Equations become

$$\frac{d^2 f}{dr^2} + \frac{2}{r}\frac{df}{dr} - \frac{4 \;\; f + \varphi}{r^2} = \frac{d\psi}{dr}$$

$$\frac{d^2\varphi}{dr^2} + \frac{2}{r}\frac{d\varphi}{dr} - \frac{f + \varphi}{r^2} = -\frac{\psi}{r}$$

$$\frac{df}{dr} + \frac{2 \;\; f + \varphi}{r} = 0$$

Solving eqs 2 and 3 for φ and ψ and replacing in eq 1

$$\varphi = -\frac{r}{2}\frac{df}{dr} - f, \quad \psi = \frac{r^2}{2}\frac{d^3 f}{dr^3} + 3r\frac{d^2 f}{dr^2} + 2\frac{df}{dr}$$

$$r^3\frac{d^4 f}{dr^4} + 8r^2\frac{d^3 f}{dr^3} + 8r\frac{d^2 f}{dr^2} - 8\frac{df}{dr} = 0$$

Euler equation

$$f \sim r^n$$

Movement of a Spherical Nanoparticle in Viscous Fluid: Solution Procedure

$$n \;\; n-1 \;\; n-2 \;\; n-3 \; +8n \;\; n-1 \;\; n-2 \; +8n \;\; n-1 \; -8n=0$$

$$\Rightarrow n \;\; n-1 \;\; n+2 \;\; n+3 \; =0$$

$$n_1=0, \quad n_2=2, \quad n_3=-1, \quad n_4=-3$$

Then, f, φ, and ψ are

$$f=\frac{b_1}{r^3}+\frac{b_2}{r}+b_3+a_1r^2$$

$$\varphi=\frac{b_1}{2r^3}-\frac{b_2}{2r}-b_3-2a_1r^2$$

$$\psi=\frac{b_2}{r^2}+10a_1r^2$$

The solutions are

$$v_r=\left(\frac{b_1}{r^3}+\frac{b_2}{r}+b_3+a_1r^2\right)\cos\theta$$

$$v_\theta=\left(\frac{b_1}{2r^3}-\frac{b_2}{2r}-b_3-2a_1r^2\right)\sin\theta$$

$$p=\eta\left(\frac{b_2}{r^2}+10a_1r^2\right)\cos\theta$$

Application of the Boundary Conditions

At infinity, $r \to \infty$

$$a_1 = 0, \quad b_3 = -U$$

$$v_r = \left(\frac{b_1}{r^3} + \frac{b_2}{r} - U\right)\cos\theta, \quad v_\theta = \left(\frac{b_1}{2r^3} - \frac{b_2}{2r} + U\right)\sin\theta$$

$$p = \eta\frac{b_2}{r^2}\cos\theta$$

At the surface, $r = R$

$$\frac{b_1}{r^3} + \frac{b_2}{r} - U = 0, \quad \frac{b_1}{2r^3} - \frac{b_2}{2r} + U = 0$$

Solution

$$v_r \;\; r,\theta \;\; = -U\left(1 - \frac{3}{2}\frac{R}{r} + \frac{1}{2}\frac{R^3}{r^3}\right)\cos\theta$$

$$v_\theta \;\; r,\theta \;\; = U\left(1 - \frac{3}{4}\frac{R}{r} - \frac{1}{4}\frac{R^3}{r^3}\right)\sin\theta$$

$$p \;\; r,\theta \;\; = U\left(\frac{3}{2}\eta\frac{R}{r^2}\right)\cos\theta$$

Stresses and Forces

$$\tau_{rr} = -2\eta \frac{\partial v_r}{\partial r}, \quad \tau_{r\theta} = \tau_{\theta r} = -\eta \left[r \frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right) + \frac{1}{r}\frac{\partial v_r}{\partial \theta} \right]$$

$$\tau_{rr} = -\frac{3\eta U}{R}\left[\left(\frac{R}{r}\right) - \left(\frac{R}{r}\right)^4\right]\cos\theta \quad \Rightarrow \text{ at the surface } r = R, \quad \tau_{rr}\big|_{r=R} \equiv 0$$

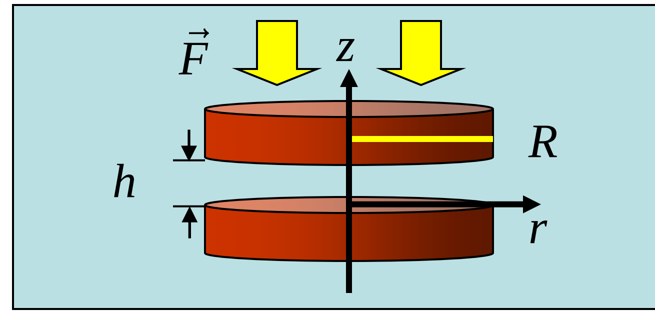
$$\tau_{r\theta} = \tau_{\theta r} = \frac{3\eta U}{2R}\left(\frac{R}{r}\right)^4 \sin\theta \quad \Rightarrow \text{ at the surface } r = R, \quad \tau_{r\theta} = \tau_{\theta r} = \frac{3\eta U}{2R}\sin\theta$$

Normal Force $$F_n = \int_0^{2\pi}\int_0^{\pi}\left[- p + \tau_{rr}\big|_{r=R}\cos\theta\right] R^2 \sin\theta \, d\theta \, d\phi = 2\pi\eta R U$$

Tangential Force $$F_t = \int_0^{2\pi}\int_0^{\pi} \tau_{r\theta}\big|_{r=R} \sin\theta \; R^2 \sin\theta \, d\theta \, d\phi = 4\pi\eta R U$$

Total Force $$F = F_n + F_t = 6\pi\eta R U$$

Approach of Disks in Viscous Fluid (O. Reynolds). Lubrication Approximation



Additional simplifications due to symmetry and geometry: $v_\theta = 0,\ h \ll R$

$$\eta\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial r v_r}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2}+\frac{\partial^2 v_r}{\partial z^2}-\frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}\right]=\frac{\partial p}{\partial r}$$

$$\eta\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial r v_\theta}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2}+\frac{\partial^2 v_\theta}{\partial z^2}-\frac{2}{r^2}\frac{\partial v_r}{\partial \theta}\right]=\frac{1}{r}\frac{\partial p}{\partial \theta}$$

$$\eta\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2}+\frac{\partial^2 v_z}{\partial z^2}\right]=\frac{\partial p}{\partial z}$$

$$\frac{1}{r}\frac{\partial\ r v_r}{\partial r}+\frac{1}{r}\frac{\partial v_\theta}{\partial \theta}+\frac{\partial v_z}{\partial z}=0$$

$$\eta\frac{\partial^2 v_r}{\partial z^2}=\frac{\partial p}{\partial r}$$

$$v_\theta \equiv 0$$

$$\frac{\partial p}{\partial z}=0$$

$$\frac{1}{r}\frac{\partial r v_r}{\partial r}+\frac{\partial v_z}{\partial z}=0$$

Approach of Disks in Viscous Fluid

Boundary Conditions

$$z = 0: \quad v_r = v_z = 0$$
$$z = h: \quad v_r = 0, v_z = -U$$
$$r = R: \quad p = p_0$$

Integrating twice $\quad \eta \dfrac{\partial^2 v_r}{\partial z^2} = \dfrac{\partial p}{\partial r} \quad \Rightarrow \quad v_r = \dfrac{1}{2\eta}\dfrac{dp}{dr} z \;\; z - h$ $\quad$ Radial velocity

From the continuity $\quad \dfrac{1}{r}\dfrac{\partial r v_r}{\partial r} + \dfrac{\partial v_z}{\partial z} = 0 \quad \Rightarrow \quad U = \dfrac{1}{r}\dfrac{d}{dr}\displaystyle\int_0^h r v_r dz = -\dfrac{h^3}{12\eta r}\dfrac{d}{dr}\left(r \dfrac{dp}{dr} \right)$

The pressure is $\quad p = p_0 + \dfrac{3\eta U}{h^3} \;\; R^2 - r^2$

The force bringing the two disks together with rate of approach U is

$$F = 2\pi \int_0^R \;\; p - p_0 \;\; r dr = \frac{6\pi\eta U}{h^3} \int_0^R \;\; R^2 - r^2 \;\; r dr = \frac{3\pi\eta U R^4}{2h^3}$$

Summary

1. The above considerations are valid for systems that are unaffected by long range molecular and surface forces (e.g. electrostatic, van der Waals).
2. Flows in micro and nanochannels are usually characterized by low Reynolds numbers because of the small lengthscales.
3. The forces in channels or on moving particles are predominantly viscous. Inertial forces are unimportant.

References

- J. Happel and H. Brenner, “Low Reynolds Number Hydrodynamics”, Martinius Nijhof, 1983.
- J. O. Wilkes, “Fluid Mechanics for Chemical Engineers with Microfluidics and CFD”, Prentice Hall, 2006.