1. The motion of a continuous medium is defined by the equations (You may want to use hyperbolic functions)

$$x_1 = \frac{1}{2}(X_1 + X_2)e^t + \frac{1}{2}(X_1 - X_2)e^{-t}$$

$$x_2 = \frac{1}{2}(X_1 + X_2)e^t - \frac{1}{2}(X_1 - X_2)e^{-t} \qquad x_3 = X_3$$

Let the bases $\mathbf{e}_i$ and $\mathbf{E}_A$ coincide for all time.
(a) Express the velocity components in terms of the material coordinates and time.
(b) Express the velocity components in terms of the spatial coordinates, i.e., show that $v_1 = x_2, v_2 = x_1, v_3 = 0$.
(c) Determine the components of $\boldsymbol{L}, \boldsymbol{d}$, and $\boldsymbol{W}$.
(d) Determine $\mathbf{F}$, J and $\mathbf{E}$. Take a time derivative to obtain the components of $\dot{\mathbf{F}}$, $\dot{J}$ and $\dot{\mathbf{E}}$.
(e) Show that the equations $\dot{\boldsymbol{F}} = \boldsymbol{L} \cdot \boldsymbol{F}$, $\dot{\boldsymbol{E}} = \boldsymbol{F}^T \cdot \boldsymbol{d} \cdot \boldsymbol{F}$ and $\dot{J}=J(\boldsymbol{v} \cdot \nabla)$ are satisfied.

2. If the intensity of illumination of a fluid particle at (x_1, x_2, x_3) at time t is given by $I = Ae^{-3t} / \left(x_1^2 + x_2^2 + x_3^2\right)$ and the fluid velocity field is given by $v_1 = B\left(x_2 + 2x_3\right)$, $v_2 = B\left(x_2 + 3x_3\right)$, $v_3 = B\left(2x_1 + 3x_2 + 2x_3\right)$ where A and B are known constants, determine the rate of change of the illumination experienced at time t by the fluid particle which is at point (1, 2, -2) at time t.

3. A velocity vector field v satisfying $\boldsymbol{v} \cdot \bar{\nabla} = 0$ is called solenoidal. A volume-preserving motion is called isochoric. (The flow of an incompressible fluid is necessarily isochoric, but there may be isochoric flows of compressible fluids.)
(a) Show that for isochoric motion the velocity field is solenoidal, and conversely.
(b) Show that any velocity field $\mathbf{v}$ given in terms of a vector potential function Q by $\boldsymbol{v} = -\boldsymbol{Q} \times \bar{\nabla}$ is solenoidal and the flow isochoric.
(c) For incompressible (or isochoric) plane flow in the x_1-x_2 plane, $\boldsymbol{Q} = Q\boldsymbol{e}_3$ where the component $Q(x_1,x_2)$ is called a stream function. Show that the volume flux $F_V \equiv \int_A^B \boldsymbol{v} \cdot \boldsymbol{n} \, da$ across any plane curve joining points $(x_1, x_2)^A$ and $(x_1, x_2)^B$ equals $[Q(x_1, x_2)^B - Q(x_1, x_2)^A]$.

4. The circulation around a closed curve C is defined to be $\Gamma = \oint_c \boldsymbol{v} \cdot d\boldsymbol{r}$.

(i) Show that $\dfrac{d\Gamma}{dt} = \oint_c \boldsymbol{a} \cdot d\boldsymbol{r} + \oint_c \boldsymbol{v} \cdot \boldsymbol{L} \cdot d\boldsymbol{r}$

(ii) Show that $\oint_c \boldsymbol{v} \cdot \boldsymbol{L} \cdot d\boldsymbol{r} = \oint_c \boldsymbol{v} \cdot d\boldsymbol{v} = 0$

(iii) Show that if $\Gamma = 0$ for all curves (irrotational flow) then $\boldsymbol{v} = \phi\bar{\bar{\nabla}}$ where ϕ is a potential function.

(iv) Show that if $\boldsymbol{v} = \phi\bar{\bar{\nabla}}$ then $\dot{\boldsymbol{v}} = \dot{\phi}\bar{\bar{\nabla}} - \boldsymbol{v} \cdot \boldsymbol{L}$

5. Show that (i) $\dfrac{d}{dt}(\boldsymbol{n}da) = \left(\boldsymbol{v} \cdot \bar{\bar{\nabla}}\right)\boldsymbol{n}da - \boldsymbol{L}^T \cdot \boldsymbol{n}da$

(ii) $\dfrac{d}{dt}\int f\boldsymbol{n}\,da = \int \left[\dfrac{df}{dt}\boldsymbol{n} + f\left(\boldsymbol{v} \cdot \bar{\bar{\nabla}}\right)\boldsymbol{n} - f\boldsymbol{L}^T \cdot \boldsymbol{n} \right] da$