9. TRANSIENT HEAT CONDUCTION

9.1 INTRODUCTION

In this chapter the transient heat conduction equation is spatially discretized with the use of finite elements to form a coupled set of ordinary differential equations in time. These ordinary differential equations are discretized in time, and the resulting equations are combined with the trapezoidal rule to obtain a system time integrator. The numerical algorithm provides approximate solutions to the transient heat conduction equation. Now the formulation involves capacitance and conductivity matrices rather than scalars. Stability of the system integrator is analyzed from two points of view; the first approach involves the equations themselves, while the second approach utilizes a modal decomposition of the system matrices. It is shown that conclusions obtained from the analysis of the single-degree-of-freedom problem carry over to the general formulation of this chapter.

9.2 GOVERNING EQUATIONS

Recall from (1.2.1-9) and (1.2.1-10) that the governing equations for transient heat conduction in a bar of unit cross section are

$$-q_{,x} + Q = \rho c T_{,t} \qquad q = -KT_{,x} \tag{9.2-1}$$

or

$$(KT_{,x})_{,x} + Q = \rho c T_{,t} \tag{9.2-2}$$

where the terms are defined as follows and typical units in the SI system are given:

q - x-component of the heat flux (W/m^2)
T - temperature (^oK)
ρ - mass per unit volume (kg/m^3)
c - specific heat ($J/(kg^oK)$) (J = Ws)
x - spatial coordinate (m)
t - temporal coordinate or time (s)
K - thermal conductivity (W/m^oK)
Q - heat source per unit volume (W/m^3)

The boundary condition consists of either a prescribed temperature

$$T = T\ (t) \tag{9.2-3a}$$

or a prescribed flux into the body

$$qn = -q\ (t) \tag{9.2-3b}$$

at each end (x = 0 or x = L). In addition the initial value of T must be given:

$$T(0,x) = \hat{T}(x) \tag{9.2-4}$$

For a weight function w(x), the weak form of (9.2-2) is

$$\int_0^L [CT_{,t} - (KT_{,x})_{,x} - Q]w(x)dx = 0 \qquad w(x) \tag{9.2-5}$$

where C=ρc is called the capacitance (with units, $\frac{J}{^0 Km^3}$).

Consider the second term in (9.2-5). An integration by parts yields

$$-\int_0^L (KT_{,x})_{,x} w dx = (-KT_{,x} w)\Big|_0^L + \int_0^L KT_{,x} w_{,x} dx \tag{9.2-6}$$

The boundary term is

$$(-KT_{,x} w)\Big|_0^L = (qw)\Big|_0^L = (nqw)\,|_0 + (nqw)|_L \tag{9.2-7}$$

If T is prescribed on a boundary then nq is not known and w must be chosen zero on that boundary. If T is not prescribed then $nq = -q$ must be given. Thus, with the understanding that if T is prescribed then w is zero, (9.2-7) can be expressed as

$$(qw)\Big|_0^L = -q_0 w_0 - q_L^* w_L \tag{9.2-8}$$

The forcing function Q denotes a heat source per unit length whereas q_0 and q_L denote fluxes into the body through the boundary. Define an augmented forcing function by

$$Q^a = Q + q_0(t)\delta[x] + q_L(t)\delta[x - L] \tag{9.2-9}$$

Then the substitution of (9.2-9) and (9.2-6) in (9.2-5) yields the symmetric weak formulation:

$$\int_0^L [CT_{,t} w + Kw_{,x} T_{,x} - Q^a w]dx = 0 \qquad w \tag{9.2-10}$$

9.3 THE GALERKIN FORMULATION FOR NODAL BASIS FUNCTIONS

The Galerkin approximation is obtained by representing the function T and w as a finite sum involving basis functions. If nodal basis functions $N_i(x)$ are used, then

$$w^N = \sum_{i=1}^{N} w_i N_i(x) \qquad T^N = \sum_{i=1}^{N} T_i(t) N_i(x) \tag{9.3-1}$$

where now, the nodal parameters T_i are functions of t, rather than being constants as is the case for static problems. The representation for T can be viewed as a separation of variables solutionbased on basis functions with compact support rather than global support. Let

$$\begin{aligned} C_{ij} &= \int_0^L C N_i(x) N_j(x)\,dx && \text{Capacitance matrix} \\ K_{ij} &= \int_0^l K N_{i,x}(x) N_{j,x}(x)\,dx && \text{Conductance matrix} \\ Q_i &= \int_0^L Q^a N_i(x)\,dx && \text{Thermal force vector} \end{aligned} \tag{9.3-2}$$

Then the substitution of (9.3-1) in (9.2-10) yields

$$\sum_{i=1}^{N} w_i \left[\sum_{j=1}^{N} \left(C_{ij} \dot{T}_j + K_{ij} T_j \right) - Q_i \right] = 0 \tag{9.3-3}$$

which must hold for arbitrary w_i except for those nodes where w must be zero. Suppose all w_i's are arbitrary. Then (9.3-3) implies

$$\sum_{j=1}^{N} (C_{ij} \dot{T}_j + K_{ij} T_j) = Q_i \qquad i = 1,..., N \tag{9.3-4}$$

or

$$[C]\{\dot{T}\} + [K]\{T\} = \{Q\} \tag{9.3-5}$$

9.4 MATRICES USING FINITE ELEMENTS

The finite element method provides a systematic means for developing local nodal basis functions and consequently for obtaining the matrices of (9.3-2) which are used in the Galerkin form of the symmetric weak formulation. From the development of Section 7.3 the element shape functions for a two-node element are:

$$N_1^e = 1 - \eta \qquad N_2^e = \eta \qquad \eta = \frac{x - x_1^e}{h^e} \tag{9.4-1}$$

where x_1^e is the coordinate of the first node and h^e is the element length. Components of element matrices are defined as follows:

$$C_{ij}^e = \int_{\Omega^e} C N_i^e N_j^e dx \qquad K_{ij}^e = \int_{\Omega^e} K N_{i,x}^e N_{j,x}^e dx \qquad Q_i^e = \int_{\Omega^e} Q^{sa} N_i^e dx \tag{9.4-2}$$

in which Ω^e denotes the domain of the element and Q^{sa} is the smooth part of the augmented thermal force function, i.e.,

$$Q^a(x,t) = Q^{sa}(x,t) + Q^{pa}(x,t) \tag{9.4-3}$$

in which Q^{pa} represents point sources including contributions from the boundaries.

Suppose C, K and Q^{sa} are piecewise constant and, in particular, constant over each element. Then

$$C_{ij}^e = C^e h^e \int_0^1 N_i^e N_j^e d\eta \qquad K_{ij}^e = \frac{K^e}{h^e} \int_o^1 N_{i,\eta}^e N_{j,\eta}^e d\eta \qquad Q_i^e = Q^e h^e \int_o^1 N_i^e dn \tag{9.4-4}$$

For the shape functions of (9.4-1), the matrices of (9.4-4) become

$$[C]^e = C^e h^e \begin{bmatrix} \frac{1}{3} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{3} \end{bmatrix} \qquad [K]^e = \frac{K^e}{h^e} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \qquad \{Q\}^e = \frac{Q^e h^e}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \tag{9.4-5}$$

As was also discussed in Subsection 7.6 the global matrices can be assembled in a systematic manner. For example, if all terms remain constant the global matrices would look like the following for a uniform mesh:

$$[C]^G = C^e h \begin{bmatrix} \frac{1}{3} & \frac{1}{6} & 0 & 0 & 0 & \cdots \\ \frac{1}{6} & \frac{2}{3} & \frac{1}{6} & 0 & 0 & \cdots \\ 0 & \frac{1}{6} & \frac{2}{3} & \frac{1}{6} & 0 & \cdots \\ 0 & 0 & \frac{1}{6} & \frac{2}{3} & \frac{1}{6} & \cdots \\ 0 & 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \cdots \end{bmatrix}$$

$$[K]^G = \frac{K^e}{h} \begin{bmatrix} 1 & -1 & 0 & - & - & \cdots \\ -1 & 2 & -1 & 0 & - & \cdots \\ 0 & -1 & 2 & -1 & 0 & \cdots \\ & 0 & -1 & 2 & - & \cdots \\ \vdots & 0 & 0 & 0 & - & \cdots \\ \vdots & \vdots & \vdots & \vdots & - & \cdots \end{bmatrix} \qquad \{Q^s\}^G = \frac{Q^e h}{2} \begin{Bmatrix} 1 \\ 2 \\ 2 \\ - \\ 2 \\ 1 \end{Bmatrix} \tag{9.4-6}$$

in which only the smooth part of the thermal force vector is included in the force vector. Point sources have to be added to the smooth thermal force vector to obtain the complete global version of the thermal force vector.

9.5 TIME INTEGRATION OF THE GALERKIN EQUATIONS

Derivation of the Matrix Equation

Suppose all matrices have been determined via the finite element or even some other method. Then the governing set of coupled ordinary differential equations is

$$[C]\{\dot{T}\} + [K]\{T\} = \{Q\} \tag{9.5-1}$$

Let $\{\dot{T}\}^k$ and $\{T\}^k$ represent the approximations to the vectors $\{\dot{T}(t^k)\}$ and $\{T(t^k)\}$, respectively, where $t^k = ks$ and $k = 0, 1, 2, \ldots$. Suppose (9.5-1) is discretized as follows

$$[C]\{\dot{T}\}^k + [K]\{T\}^k = \{Q\}^k \tag{9.5-2}$$

where

$$\{Q\}^k = \{Q(t^k)\} \qquad t^k = ks \qquad k = 0,1,... \tag{9.5-3}$$

represents the thermal force vector evaluated at discrete times. As given in Subsection 8.3 for a single component, introduce the trapezoidal rule for a time integrator:

$$\{T\}^{k+1} = \{T\}^k + s\alpha\{\dot{T}\}^{k+1} + s(1-\alpha)\{\dot{T}\}^k \qquad 0 \le \alpha \le 1 \tag{9.5-4}$$

Increment (9.5-2) one time step and multiple by $s\alpha$

$$s\alpha[C]\{\dot{T}\}^{k+1} = -s\alpha[K]\{T\}^{k+1} + s\alpha\{Q\}^{k+1} \tag{9.5-5}$$

Multiply the terms in (9.5-2) by $s(1-\alpha)$

$$s(1-\alpha)[C]\{\dot{T}\}^{k} = -s(1-\alpha)[K]\{T\}^{k} + s(1-\alpha)\{Q\}^{k} \tag{9.5-6}$$

Add (9.5.1-5) and (9.5.1-6) and use (9.5.1-4) in the form

$$s\alpha\{\dot{T}\}^{k+1} + s(1-\alpha)\{\dot{T}\}^{k} = \{T\}^{k+1} - \{T\}^{k} \tag{9.5-7}$$

to yield

$$[A]\{T\}^{k+1} = [B]\{T\}^{k} + \{F\}^{k} \qquad k = 0, 1, 2,... \tag{9.5-8}$$

where

$$\begin{aligned} [A] &= [C] + s\alpha[K] \\ [B] &= [C] - s(1-\alpha)[K] \\ \{F\}^{k} &= s\alpha\{Q\}^{k+1} + s(1-\alpha)\{Q\}^{k} \end{aligned} \tag{9.5-9}$$

Special cases are given as follows:

$\alpha = 1$:

$[A]=[C]+s[K]$ $\quad [B]=[C]$ $\quad \{F\}^{k} = s\{Q\}^{k+1}$

$\alpha = 0$:

$[A]=[C]$ $\quad [B]=[C]-s[K]$ $\quad \{F\}^{k} = s\{Q\}^{k}$ (9.5-10)

$\alpha = \frac{1}{2}$:

$$[A] = [C] + \frac{s}{2}[K] \qquad [B]=[C]- \frac{s}{2}[K] \qquad \{F\}^{k} = \frac{s}{2}\left\{\{Q\}^{n} + \{Q\}^{k+1}\right\}$$

The Basic Algorithm

The force vector $\{F\}^{k}$ can be evaluated explicitly at each time step. The previous temperature vector $\{T\}^{k}$ is presumed known. Let the right-hand-side vector be

$$\{R\}^{k} = [B]\{T\}^{k} + \{F\}^{k} \tag{9.5-11}$$

Then at each time step, and subject to boundary conditions, the equation to be solved is

$$[A]\{T\}^{k+1} = \{R\}^{k} \tag{9.5-12}$$

If an LU decomposition is performed on [A], (9.5-12) provides the means for updating the temperature for each discrete time.

Initial Conditions

Suppose the inital condition $T(0,x) = \hat{T}(x)$ is given. Then the approximate solution governed by (9.5-12) is prescribed to have the initial values:

$$T_i^0 = \hat{T}(x_i) \qquad i = 1,\ldots,N \tag{9.5.-13}$$

which form the components of the vector $\{T\}^0$ to be used in (9.5.-11) for k = 0.

Boundary Conditions

If a boundary condition is a flux condition, the boundary term is handled "naturally" through the augmented forcing function Q^a and consequently through {Q}. The problem is with temperature prescribed boundaries. To be specific suppose the temperature at x = 0 is prescribed to be

$$T(0,t) = T_o(t) \tag{9.5.-14}$$

which implies the temperature of node 1 is

$$T_1(t) = T_o(t) \tag{9.5.-15}$$

Suppose the method of Subsection 6.9.2 is used to apply boundary conditions. Then, the first row and column of [A] must be replaced with zeros except for the diagonal which is replaced with unity. The right-hand-side vector must be modified by subtracting a vector consisting of the product of $T_1(t)$ and the first column of [A]. Then the first component of the new right-hand-side vector must be replaced with $T_1(t)$. A similar procedure must be followed if the temperature at the right boundary is prescribed.

Summary of Algorithm

Here, the algorithm is summarized with the assumption that the matrices [A] and [B], and the vector $\{Q\}^k$ are available. Let l_1 and l_2 be logical switches to indicate if the boundary conditions are temperature prescribed at the left and right ends, respectively. A value of zero for a switch indicates a flux-prescribed boundary condition; a value of one denotes a temperature prescribed boundary condition. The procedure follows:

(i) If the boundary condition at the left end is prescribed ($l_1 = 1$) save the first column of [A], denoted by $\{A\}_1$, zero the first row and column of [A] and set $A_{11} = 1$; if the boundary condition at the right end is prescribed ($l_2 = 1$), save the last column of [A] denoted by $\{A\}_N$, zero the last row and column of [A] and set $A_{NN} = 1$.
(ii) Perform an LU decomposition of [A].
(iii) Set the initial values of T in the vector $\{T\}^0$.
(iv) For each value of k starting with k = 0 perform the following operations:

(a) Determine $\{F\}^k = s\alpha\{Q\}^{k+1} + s(1-\alpha)\{Q\}^k$

(b) Determine $\{R\}^k = [B]\{T\}^k + \{F\}^k$

(c) Determine $\{R^*\}^k = \{R\}^k - l_1 T(0, t^{k+1})\{A\}_1 - l_2 T(L, t^{k+1})\{A\}_N$

(d) Replace $R^*_1{}^k$ with $l_1 T(0, t^{k+1})$ and $R^*_N{}^k$ with $l_2 T(L, t^{k+1})$. Denote this new vector as $\{R^{**}\}^k$.

(e) Solve $[A]\{T\}^{k+1} = \{R^{**}\}^k$ for $\{T\}^{k+1}$.

9.6 IMPLICIT AND EXPLICIT TIME INTEGRATION

For a system of transient (or time dependent) ordinary differential equations, if a matrix decomposition is required to obtain the updated solutions the system integrator is said to be implicit. Therefore, the scheme given above is implicit no matter what value of α is used. Since the analysis of the single-degree-of-freedom suggests that $\alpha = 1/2$ provides higher order accuracy than the integrator for any other value of α, most use implicit time integration with $\alpha = 1/2$.

The matrix that needs to be decomposed is [A]. If [A] is replaced with a diagonal matrix, $[A^d]$ say, with diagonal components A^d_{ii}, then the inverse of [A] denoted by $[A]^{-1}$ is also diagonal with components $1/A^d_{ii}$. The solution to (9.5-12) becomes

$$\{T\}^{k+1} = [A^d]^{-1}\{R\}^k \tag{9.6-1}$$

A decomposition is not required. An alternative form for this equation is obtained when the component equations are written instead. In fact, the component form is the equation usually programmed as a sweep over all nodes. The result is the explicit system time integrator:

$$T_i^{k+1} = \frac{1}{A^d_{ii}} R_i^k \qquad i = 1, 2, \ldots, N \tag{9.6-2}$$

Generally speaking, such an integrator is usually conditionally stable, i.e., the time step must be below a critical value. However, because no matrices are required and the algorithm is so simple many users tend to use the algorithm even though more computer timemay be required in comparison with an implicit scheme.

The problem is that a diagonal matrix must be defined as an approximation to [A]. A possible approach is to use the sum of each row of [A] as the corresponding diagonal term in $[A^d]$. Frequently, the matrix [K] is such that each row sums to zero and then $[A^d]$ is just the diagonal matrix $[C^d]$ obtained by summing the rows of [C].

If the rows of [K] do not sum to zero, then usually α is set to zero and again $[A^d] = [C^d]$. This latter approach of choosing $\alpha = 0$ and using $[C^d]$ is usually what is inferred in the literature if reference is made to explicit time integration. There are two common approaches for obtaining $[C^d]$. The first is to sum rows as mentioned previously in connection with [A]. The second approach based on engineering intuition is to just lump capacitances at nodes. For example, the total capacitance for an element can be defined as the integral of C over the element. Then this integral is often allocated equally to all nodes of the element to obtain a diagonal element capacitance matrix which is then accumulated into the global capacitance matrix.

The matrix [C] obtained by a rigorous application of the weak formulation is said to be the consistent capacitance matrix, to differentiate [C] from the diagonal capacitance matrix $[C^d]$. The diagonal form infers an approximation which must produce an error that is not easy to quantify so some authors simply refuse to use the diagonal form.

As an example of a diagonal capacitance matrix consider a two-node element. Suppose one-half the total capacitance Ch is allocated equally to each node. For this case the result is identical to that of summing rows. The corresponding consistent and diagonal element capacitance matrices are:

$$[C]^e = C^e h^e \begin{bmatrix} \frac{1}{3} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{3} \end{bmatrix} \qquad [C^d]^e = \frac{C^e h^e}{2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{9.6-3}$$

For a uniform mesh, and with the capacitance a constant, $C^e = C$, the corresponding global matrices assume the form:

$$[C]^G = C^e h \begin{bmatrix} \frac{1}{3} & \frac{1}{6} & - & 0 \\ \frac{1}{6} & \frac{2}{3} & - & - \\ - & - & \frac{2}{3} & \frac{1}{6} \\ 0 & - & \frac{1}{6} & \frac{1}{3} \end{bmatrix} \qquad [C^d]^G = C^e h \begin{bmatrix} \frac{1}{2} & 0 & - & 0 \\ 0 & 1 & - & - \\ - & - & 1 & - \\ 0 & - & 0 & \frac{1}{2} \end{bmatrix} \tag{9.6-4}$$

Another advantage of the explicit method is that with $\alpha = 0$ the matrix [B] does not even have to be computed. Recall from (9.5-8), (9.5-9) and (9.5-10) that the algorithm reduces to

$$[C^d]\{T\}^{k+1} = [C^d]\{T\}^k - s[K]\{T\}^k + s\{Q\}^k \tag{9.6-5}$$

Define an "internal" heat vector:

$$\{Q^{int}\}^k = [K]\{T\}^k \tag{9.6-6}$$

Then the component form of (9.6-2) is simply

$$T_i^{k+1} = T_i^k + \frac{s}{C_{ii}^d}\,(Q_i - Q_i^{int})^k \tag{9.6-7}$$

Now steady-state heat conduction is simply described by the equation that the internal heat vector equals the external heat vector:

$$\{Q^{int}\}^k = \{Q\}^k \tag{9.6-8}$$

The internal heat vector can be assembled on an element-by-element basis in the same fashion as the external heat source vector by defining an element internal heat vector to be

$$\{Q^{int}\}^e = [K]^e\{T\}^e \tag{9.6-9}$$

in which $\{T\}^e$ denotes the vector of current temperatures at the nodes of the element. The great advantage of this equation is that the element conductivity K^e used as the coefficient for the element conductivity matix

$$[K]^e = \frac{K^e}{h^e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \tag{9.6-10}$$

can be easily taken as a function of the average temperature of the element. Therefore, nonlinear problems in which K is a function of T, and even if C is a function of T, are handled in an elementary way using explicit time integration with the finite element method.

Although the computational advantage of an explicit integrator (which implies a diagonal capacitance matrix) is significant, the mathematical rigor of the Galerkin formulation is lost. For this reason, some choose to not even discuss the explicit approach. However, because of the simplicity of the method, explicit time integration is widely used for unusually large problems, for problems which are nonlinear, and for unusually large nonlinear problems.

9.7 DIRECT STABILITY ANALYSIS

A determination of stability with regard to the time integrator is now considerably more complex because the system has been discretized in both space and time. The single-degree-of-freedom problem involved discretization with respect to time only. General results concerning stability are obtained most easily using theorems associated with eigenvalues and eigenvectors of matrices. For this subsection, it is assumed that these theorems are not available so a direct approach is used. The idea remains the same, and that is to attempt to obtain an exact solution to the time-stepping equation that is used for a numerical solution. An exact solution can normally be obtained only for special cases, but the results provide good quidelines for use in a more general environment.

The homogeneous form of (9.5-8) is considered with the assumptions that the capacitance C and conductance K are constant and that the mesh is uniform, i.e., h is contant

For these assumptions the global form of the capacitance and conductance matrices will be

$$[C]^G = Ch\begin{bmatrix} - & - & - & - & - & - \\ 0 & \frac{1}{6} & \frac{2}{3} & \frac{1}{6} & 0 & - \\ - & 0 & \frac{1}{6} & \frac{2}{3} & \frac{1}{6} & 0 \\ - & - & - & - & - & - \end{bmatrix} \quad [K]^G = \frac{k}{h}\begin{bmatrix} - & - & - & - & - & - \\ 0 & -1 & 2 & -1 & 0 & - \\ 0 & 0 & -1 & 2 & -1 & 0 \\ - & - & - & - & - & - \end{bmatrix} \tag{9.7-1}$$

so that the matrices [A] and [B] will have the form

$$[A]^G = [C]^G + s\alpha[K]^G = \begin{bmatrix} - & - & - & - & - & - \\ 0 & a^* & a & a^* & 0 & 0 \\ 0 & 0 & a^* & a & a^* & 0 \\ - & - & - & - & - & - \end{bmatrix} \tag{9.7-2}$$

$$[B]^G = [C]^G - s(1-\alpha)[K]^G = \begin{bmatrix} - & - & - & - & - & - \\ 0 & b^* & b & b^* & 0 & 0 \\ 0 & 0 & b^* & b & b^* & 0 \\ - & - & - & - & - & - \end{bmatrix} \tag{9.7-3}$$

that is, the tridiagonal structure is maintained with

$$\begin{aligned} a &= \frac{2}{3}Ch + s\alpha\frac{2K}{h} & a^* &= \frac{1}{6}Ch - s\alpha\frac{k}{h} \\ b &= \frac{2}{3}Ch - s(1-\alpha)\frac{2K}{h} & b^* &= \frac{1}{6}Ch + s(1-\alpha)\frac{K}{h} \end{aligned} \tag{9.7-4}$$

Consider the homogeneous form of (9.5-8)

$$[A]\{T\}^{k+1} - [B]\{T\}^{k} = \{0\} \tag{9.7-5}$$

A typical equation for an interior node (labeled i) is the following:

$$a * T_{i+1}^{k+1} + aT_i^{k+1} + a * T_{i-1}^{k+1} - b * T_{i+l}^{k} - bT_i^{k} - b * T_{i-l}^{k} = 0 \tag{9.7-6}$$

Postulate a solution of the form:

$$T_i^k = T^0 \lambda^k e^{i\beta\sqrt{-1}} \tag{9.7-7}$$

in which the initial value, T^o, is assumed to be the same for all nodes i. Then

$$T_{i+1}^k = T^0 \lambda^k e^{i\beta\sqrt{-1}} e^{\beta\sqrt{-1}} = T_i^k e^{\beta\sqrt{-1}} \qquad T_{i-1}^k = T_i^k e^{-\beta\sqrt{-1}} \tag{9.7-8}$$

in which the identities $e^{(i+l)x} = e^{ix}\, e^{x}$ and $e^{(i-l)x} = e^{ix}\, e^{-x}$ have been used. Similarly

$$T_i^{k+l} = T_i^k \lambda \qquad T_{i+l}^{k+l} = T_i^k \lambda e^{\beta\sqrt{-1}} \qquad T_{i-l}^{k+l} = T_i^k \lambda\, e^{-\beta\sqrt{-1}} \tag{9.7-9}$$

Substitute (9.7-7), (9.7-8) and (9.7-9) in (9.7-6), factor out T_i^k and set the coefficient to zero. The result is

$$\lambda\left(a * e^{\beta\sqrt{-1}} + a + a * e^{-\beta\sqrt{-1}}\right) - \left(b * e^{\beta\sqrt{-1}} + b + b * e^{-\beta\sqrt{-1}}\right) = 0 \tag{9.7-10}$$

With the use of $e^{\beta\sqrt{-1}} + e^{-\beta\sqrt{-1}} = 2\cos\beta$, (9.7-10) yields

$$\lambda = \frac{b + 2b * \cos\beta}{a + 2a * \cos\beta} \tag{9.7-11}$$

The integrator is said to be stable if $|\lambda| \le 1$ for all values of β (or for all values of cos β between -1 and 1). With the use of (9.7-4)

$$\lambda = \frac{\frac{2}{3}\mathrm{Ch} - s(1-\alpha)\frac{2K}{h} + 2\cos\beta\{\frac{1}{6}\mathrm{Ch} + s(1-\alpha)\frac{K}{h}\}}{\frac{2}{3}\mathrm{Ch} + s\alpha\frac{2K}{h} + 2\cos\beta\{\frac{1}{6}\mathrm{Ch} - s\alpha\frac{K}{h}\}} \tag{9.7-12}$$

Let

$$\bar{s} = \frac{sK}{\mathrm{Ch}^2} \qquad \text{(dimensionless time step)} \tag{9.7-13}$$

Then

$$\lambda = \frac{1 - 3(1-\alpha)\bar{s} + \left\{\frac{1}{2} + 3(1-\alpha)\bar{s}\right\}\cos\beta}{1 + 3\alpha\bar{s} + \left\{\frac{1}{2} - 3\alpha\bar{s}\right\}\cos\beta} \tag{9.7-14}$$

Consider limiting cases of λ in terms of the parameters α, β and $\bar{s}$. Note that $\lambda|_{\bar{s}=0} = 1$ for all values of α and $\cos\beta$. Let

$$\begin{aligned} \lambda_1 &= \lambda|_{\cos\beta=1} = \frac{3/2}{3/2} = 1 \\ \lambda_0 &= \lambda|_{\cos\beta=0} = \frac{1-3(1-\alpha)\bar{s}}{1+3\alpha\bar{s}} \\ \lambda_{-1} &= \lambda|_{\cos\beta=-1} = \frac{1-12(1-\alpha)\bar{s}}{1+12\alpha\bar{s}} \end{aligned} \tag{9.7-15}$$

For $\bar{s} \to \infty$

$$\lim_{\bar{s}\to\infty} \lambda_0 = \frac{\alpha-1}{\alpha} \equiv \lambda_\infty \qquad \lim_{\bar{s}\to\infty} \lambda_{-1} = \frac{\alpha-1}{\alpha} = \lambda_\infty \tag{9.7-16}$$

and

$$\lambda_\infty|_{\alpha=1} = 0 \qquad \lambda_\infty|_{\alpha=\frac{1}{2}} = -1 \tag{9.7-17}$$

Thus, the time integrator is unconditionally stable for $\frac{1}{2} \le \alpha \le 1$ which is the same conslusion obtained for the single-degree-of-freedom system.

Set $\alpha = 0$. Then

$$\lambda_0|_{\alpha=0} = 1 - \bar{s} \qquad \lambda_{-1}|_{\alpha=0} = 1 - 12\bar{s} \tag{9.7-18}$$

This suggests that λ_{-1} is the worst case. Consider $0 < \alpha < 1/2$ and set $\lambda_1 = -1$ to determine the critical time step $\bar{s}_c$, i.e.,

$$-1 = \frac{1-12(1-\alpha)\bar{s}_c}{1+12\alpha\bar{s}_c} \tag{9.7-19}$$

which yields

$$\bar{s}_c = \frac{1}{6(1-2\alpha)} \tag{9.7-20}$$

This implies that for $0 < \alpha < 1/2$, the algorithm is conditionally stable. A time step must be used such that $\bar{s} \le \bar{s}_c$, or from (9.7-13),

$$s \le \frac{Ch^2}{K}\bar{s}_c \tag{9.7-21}$$

which indicates that the time step is limited by the mesh size h.

Consider the special case of an explicit integrator involving a diagonal capacitance matrix. A typical diagonal term on the capacitance matrix is Ch so with $\alpha = 0$ in (9.7-4)

$$\begin{aligned} a &= Ch & a^* &= 0 \\ b &= Ch - s\frac{2K}{h} & b^* &= \frac{sK}{h} \end{aligned} \tag{9.7-22}$$

After a substitution in (9.7-12)

$$\lambda = \frac{Ch - 2\frac{sK}{h} + 2\frac{sK}{h}\cos\beta}{Ch} = 1 - 2\bar{s}(1 - \cos\beta) \tag{9.7-23}$$

The worst situation is $\cos\beta = -1$. The critical value of $\bar{s}$ is obtained by setting $\lambda = -1$ with the result that $-1 = 1 - 4\bar{s}_c$ which yields

$$\bar{s}_c = \frac{1}{2} \tag{9.7-24}$$

Therefore, for explicit time integration with $\alpha = 0$, the time step must meet the inequality

$$s \le \frac{Ch^2}{2K} \tag{9.7-25}$$

9.8 STABILITY ANALYSIS USING MATRIX THEORY

In this subsection, a stability analysis of the discretized equations is performed with the use of results based on Subsection 4.4. Consider the general eigenproblem

$$[K]\{e\} = \lambda[C]\{e\} \tag{9.8-1}$$

in which [K] is the global conductance matrix and [C] the global capacitance matrix. [C] is assumed to be positive definite and both matrices are symmetric and real. Let $[M^o]$ denote the modal matrix formed from the eigenvectors $\{e\}_i$:

$$[M^o] = [\{e\}_1\ \{e\}_2\ ...] \tag{9.8-2}$$

Then

$$[M^o]^T[C][M^o] = [I] \qquad [M^o]^T[K][M^o] = [\lambda] \tag{9.8-3}$$

in which $[\lambda]$ is a diagonal matrix with diagonal components λ_i, the eigenvalues of the system.

Consider the discretized set of equations from (8.5-8) and (8.5-9):

$$\begin{aligned} &[A]\{T\}^{k+1} = [B]\{T\}^k + \{F\}^k \qquad k = 0, 1, 2,... \\ &[A] = [C] + s\alpha[K] \\ &[B] = [C] - s(1-\alpha)[K] \\ &\{F\}^k = s\alpha\{Q\}^{k+1} + s(1-\alpha)\{Q\}^k \end{aligned} \tag{9.8-4}$$

Introduce a vector of transformed temperatures $\{T^{Tr}\}$ such that

$$\{T\} = [M^o]\{T^{Tr}\} \tag{9.8-5}$$

Substitute (9.8-5) in (9.8-4) and multiply on the left by $[M^o]^T$:

$$[M^o]^T[A][M^o]\{T^{Tr}\}^{k+1} = [M^o]^T[B][M^o]\{T^{Tr}\}^k + [M^o]^T\{F\}^k \tag{9.8-6}$$

Define a transformed force vector such that

$$\{F^{Tr}\} = [M^o]^T\{F\} \tag{9.8-7}$$

Use (9.8-3) to obtain the set of equations in **modal form**:

$$\big[[I] + s\alpha[\lambda]\big]\{T^{Tr}\}^{k+1} = \big[[I] - s(1-\alpha)[\lambda]\big]\{T^{Tr}\}^k + \{F^{Tr}\}^k \tag{9.8-8}$$

which represents a set of **uncoupled** ordinary differential equations in time. A typical component equation is:

$$(1 + s\alpha\lambda_i)T^{Tr}{}_i^{k+1} = [1 - s(1-\alpha)\lambda_i]T^{Tr}{}_i^k + F^{Tr}{}_i^k \tag{9.8-9}$$

which is exactly the same form considered for the one-degree-of-freedom case given by (8.3-4) and (8.3-5). Therefore, conclusions obtained for the one-degree-of-freedom analysis carry over to the general case.

Some numerical programs actually use the modal form as a general technique. The primary expense is that of obtaining the modal matrix $[M^o]$ which is unrealistic for large problems. However, many problems can be handled adequately if only the modes with the lowest eigenvalues are retained. This simplification results in an approximate modal method that is quite efficient.

The one-degree-of-freedom analysis indicated that the algorithm of (9.8-9) is unconditionally stable for $\frac{1}{2} \le \alpha \le 1$. The modal form shows that the algorithm is stable for the multi-degree-of-freedom case as well, a result consistent with that of the last subsection. For $0 \le \alpha < \frac{1}{2}$, the one-degree-of-freedom analysis also showed from (8.5-2) and (8.5-7) that the critical time step is

$$s_c = \frac{2}{(1-2\alpha)\Psi} \qquad \Psi = \frac{K}{C} \tag{9.8-10}$$

The same analysis can be applied to (9.8-9) with the result that there is a critical time step for each mode:

$$s_{c(i)} = \frac{2}{(1-2\alpha)\lambda_i} \tag{9.8-11}$$

However, a general integrator must be stable for all modes. The worst case is the mode with the maximum eigenvalue, λ_N, for an N - degree-of-freedom system. Therefore, the critical time step is

$$s_c = \frac{2}{(1-2\alpha)\lambda_N} \qquad 0 \le \alpha < \tfrac{1}{2} \tag{9.8-12}$$

Usually, it is not a trivial matter to obtain λ_N but relatively easy to obtain an upper bound λ_U. Then the following inequality holds:

$$s_c \ge \frac{2}{(1-2\alpha)\lambda_U} \qquad 0 \le \alpha < \tfrac{1}{2} \tag{9.8-13}$$

The results of Subsection 7.14 can be used to obtain an upper bound using element matrices. To illustrate the use of (9.8-13) to estimate the critical time step, two typical element matrices are considered next.

Consistent Capacitance Matrix

Suppose the capacitance and conductance matrices for the smallest element in a discretized system is given as follows:

$$[C]^e = Ch\begin{bmatrix} \frac{1}{3} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{3} \end{bmatrix} \qquad [K]^e = \frac{K}{h}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \tag{9.8-14}$$

The general eigenproblem

$$\left[[K]^e - \lambda[C]^e\right]\{e\} = \{0\} \tag{9.8-15}$$

yields the characteristic equation

$$\bar{\lambda}\left(1 - \frac{\bar{\lambda}}{12}\right) = 0 \qquad \bar{\lambda} = \lambda\frac{Ch^2}{K} \tag{9.8-16}$$

Therefore, an upper bound to the system eigenvalue and the corresponding inequality for the critical time step is

$$\lambda_U = 12\frac{K}{Ch^2} \qquad s_c \geq \frac{Ch^2}{6(1-2\alpha)K} \tag{9.8-17}$$

Diagonal Capacitance Matrix

If a diagonal capacitance matrix is used to obtain an explicit time integrator, then

$$[C]^e = Ch\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \qquad [K]^e = \frac{K}{h}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \tag{9.8-18}$$

The eigenvalues are $\bar{\lambda} = 0$ and $\bar{\lambda} = 4$ so the upper bounds on the system eigenvalues and critical time step are:

$$\lambda_U = 4\frac{K}{Ch^2} \qquad s_c \geq \frac{Ch^2}{2(1-2\alpha)K} \tag{9.8-19}$$

If explicit time integration is chosen, typically $\alpha = 0$ is used. Then the bound on the critical time step becomes

$$s_c\big|_{\alpha=0} \geq \frac{Ch^2}{2K} \tag{9.8-20}$$

which coincides with the value obtained in the previous subsection and given in (9.7-25).

9.9 CONVERGENCE

An error analysis is not provided. However any numerical solution should be accompanied by results showing convergence with refinement of the mesh in both space and time. A result such as (9.8-20) suggests that the element size and the mesh size should be refined simultaneously. For a given mesh, the time step should be refined until no significant improvement in error is detected. If an analytical solution is not available to provide a measure of error, then the reduction in time step should be made until no change from the previous solution is observed. From then on, each time the mesh size is reduced by a factor of two, the time step should be reduced simultaneously by a factor of four. If the time step is held fixed, and only the element size is reduced, eventually the error due to time discretization will dominate and mesh refinement will provide no reduction in error.

9.10 CONCLUDING REMARKS

In this section, the general trapezoidal rule was applied to the spatially discretized equations obtained from the transient heat conduction equation using the finite element method. The derivation is a rather elementary generalization from the single-degree-of-freedom problem of the previous section. The concepts of implicit and explicit time integration weree introduced. A stability analysis was performed using two approaches; one involving the discretized equations directly and the other using concepts associated with the general eigenproblem. For explicit time integration, the idea of using the smallest element to provide an estimate of the critical time step was introduced. These concepts carry over directly to other differential equations and to higher dimensions.

9.11 EXERCISES

1. Extend the capability of your finite element program to incorporate (i) the development of consistent capacitance matrices, and (ii) a time integrator for transient heat conduction. Provide a partial verification of your program by giving a comparison of analytical and approximate solutions that involve parts (i) and (ii).

2. Add the capability for handling a temperature prescribed boundary condition where the temperature is a function of time. With a simple model problem of your choice demonstrate this capability of your program.

3. Write a version of a finite element program utilizing explicit time integration and internal heat "force" vectors. Your program should not involve the development of global matrices. Verify your program by comparison with analytical solutions such as those used in problem 1. Use elements of unequal length and show that (9.8-20) provides an adequate bound on the critical time step. Illustrate the power of the program by obtaining a solution to a nonlinear problem in which the conductivity is a given function of temperature.

4. The containment building for some nuclear reactors consists of a prestressed concrete cylinder topped with a spherical cap. An accident can result in internal pressure and heat generation. The properties of concrete deteriorate significantly if temperatures exceed 75^{o}C so it has been proposed that part of the concrete wall be a refractory liner. How effective such a liner would be is an important technical question.

(i) Suppose the initial temperature of the concrete is 25^{o}C and the temperature of the outer surface is maintained at 25^{o}C. If the temperature at the inner surface is suddenly raised to 200^{o}C and maintained at that temperature how long is it before a point 0.3 m from the inner surface reaches a temperature of 75^{o}C if the wall is plain concrete 1 meter thick?

(ii) How thick should the refractory concrete part of the wall be if this time is to be increased by a factor of two?

Plain Concrete

$\rho = 2000\ kg/m^3$

$c = 880 \frac{Ws}{kg\ ^oK}$

$K = 1.0 \frac{W}{m\ ^oK}$

Re fractory Concrete

$\rho = 2000\ kg/m^3$

$c = 880 \frac{Ws}{kg\ ^oK}$

$K = 0.25 \frac{W}{m\ ^oK}$