# ME 562 - Assignment 2

Brandon Lampe

February 26, 2015

**Abstract**

Overview of assignment

## 1 Write a program that provides the matrix of elasticity coefficients with respect to a given basis

The function "Elastic()" was written to calculate the coefficients of the stiffness matrix for a material having at least orthotropic symmetry (i.e., nine independent elastic parameters). If a material has greater symmetry (e.g., transverse isotropy (five in independent parameters), cubic symmetry (3 independent parameters), complete isotropy (2 independent parameters), etc), the input parameters into Elastic() must be modified accordingly.

Restrictions have been made on the input values to maintain a nonsingular (invertible) flexibility matrix, those restrictions include:

1. Poisson's ratios ($\nu$) have been constrained to values greater than -1.0 and less than 0.5 $\Rightarrow -1.0 < \nu < 0.5$
2. Young's Moduli ($M$) have been constrained to positive values greater than zero $\Rightarrow Y > 0$
3. Shear Moduli ($G$) have been constrained to positive values greater than zero $\Rightarrow G > 0$

As verification of the Elastic() routine, parameters for a completely isotropic material were used as input and the output was checked against hand calculations. Only three matrix components were needed to verify the isotropic parameters because for such a material $E_{11} = E_{22} = E_{33}$, $E_{12} = E_{23} = E_{31}$, $E_{44} = E_{55} = E_{66}$ and the matrix is symmetric, all remaining components take values of zero. Only two isotropic parameters are independent ($E_{11}$ and $E_{12}$), while the third ($E_{44}$) parameter is dependent upon the previous two. Results from this verification are shown below and the numerical output from Elastic() is shown in the attached code under Problem 1.

$$E_{11} = \frac{Y(1-\nu)}{(1+\nu)(1-2\nu)} = 12$$

$$E_{12} = E_{11}\frac{\nu}{(1-\nu)} = \frac{Y\nu}{(1+\nu)(1-2\nu)} = 4$$

$$E_{44} = 2G = E_{11} - E_{12} = \frac{Y}{(1+\nu)} = 16$$

## 2 Orthotropic Material

Material planes of symmetry may be defined as planes on which only normal stresses are present (i.e., no shear stress). Therefore, a plane of symmetry may be viewed as a principal direction of the material, this problem was approached with concept at hand.

An elastic-orthotropic material has three orthogonal planes of elastic symmetry. If a cube is cut with two orthogonal edges aligned with two planes of symmetry, the third edge must also be aligned with a plane of symmetry. The linear-elastic constitutive equation for an elastic-orthotropic material is shown below in V-M notation, where this representation implies that material planes of symmetry are aligned with the chosen basis. Regarding the constitutive relationship shown below, $\left[E^{nn}\right]$ represents a full-symmetric 3x3 matrix and $\left[E^{ss}\right]_D$ represents a diagonal 3x3 matrix and the $[E^{ns}]$ and $[E^{sn}]$ matrices represent 3x3 null matrices (only when the bases are aligned with planes of orthotropic symmetry). Additionally, $\{\sigma^{nn}\}$ and $\{\sigma^{ss}\}$ are 3x1 mathematical vectors of the normal and shear components, respectively, of the Cauchy stress tensor and $\{e^{nn}\}$and $\{e^{ss}\}$ are their corresponding strain components. When the constitutive equation below is not represented in a bases aligned with the material planes of symmetry, the constitutive relation $[E]_{6x6}$ will likely be a full rank matrix (e.g., the matrix will likely contain all nonzero values because $[E^{sn}] = [E^{ns}]^T \neq [0]$ ).

$$\begin{Bmatrix} \{\sigma^{nn}\} \\ \{\sigma^{ss}\} \end{Bmatrix} = \begin{bmatrix} [E^{nn}] & [E^{ns}] \\ [E^{sn}] & [E^{ss}]_D \end{bmatrix} \begin{Bmatrix} \{e^{nn}\} \\ \{e^{ss}\} \end{Bmatrix}$$

For the cube cut with edges aligned with the planes of symmetry ($[E^{sn}] = [E^{ns}]^T = [0]$) :

a. Yes, the normal components of the strain are the same because the normal components stress are all the same and $[E^{nn}]$is symmetric.

b. Yes, the shear components of strain are zero.

For the cube with edges not aligned with a plan of symmetry($[E^{sn}] = [E^{ns}]^T \neq [0]$):

a. No, the normal components of the strain will not be the same (although the normal stress components will still be) because $[E]_{6x6}$ is a full matrix $[E^{ns}]$ is not symmetric.

b. No, the shear components of strain are nonzero, again this results from $[E]_{6x6}$ being a full matrix.

# 3 Material of Choice

I chose rock salt (halite) as my anisotropic material. Halite has a cubic crystalline structure; therefore, it has cubic symmetry (three independent elastic parameters). The following values for engineering moduli were obtained:

- $Y = 30.8$ GPa
- $\nu = 0.347$
- $G = 11.3$ GPa
- Reference: Robertson, E., R. Robie, K. Books. 1958. "Physical Properties of salt, Anhydrite, and Gypsum – Preliminary Results," Trace Elements Memorandum Report 1048, US Dept. of the Interior Geological Survey, 40 pages.

I also found published values for the elastic constants directly. The author used Lamés parameters for the stiffness matrix; therefore, I converted these values to be in terms consistent with those used in class (i.e., Young's Modulus, Poission's Ratio, and Shear Modulus), both sets of values are provided below. Additionally, to verify the

- $E_{11} = 49$ GPa
- $E_{12} = 13$ GPa
- $E_{44} = 13$ GPa
- $Y = 32.5$ GPa
- $\nu = 0.25$

- $G = 13$ GPa
- Reference: Meyers, M. and K. Chawla. 2009. "Mechanical Behavior of Materials, Second Edition," Table 2.6 on page 114 titled: Elastic Coefficients of Ceramics, published by University Cambridge Press, 856 pages.

### 3.1 Construct Corresponding Matrix

Shear and Young's moduli values from both authors are within 15 percent of each other, although the reported values for Poission's ratio vary substantially. The components of the stiffness matrix obtained using engineering moduli as described by Robertson et al. are shown below. Calculations for the stiffness and flexibility ($[F] = [E]^{-1}$) matrices are shown in the attached code under Problem 3.

$$[E] \text{ in the E-E basis: } \begin{bmatrix} 48.8 & 25.9 & 25.9 & 0 & 0 & 0 \\ 25.9 & 48.8 & 25.9 & 0 & 0 & 0 \\ 25.9 & 25.9 & 48.8 & 0 & 0 & 0 \\ 0 & 0 & 0 & 22.6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 22.6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 22.6 \end{bmatrix} GPa$$

### 3.2 Best Isotropic Approximation

The best possible choices for the bulk (B) and shear (G) moduli are shown below:

- $Y_{iso} = 18.3$ Gpa
- $\nu_{iso} = 0.409$

The "goodness" of the isotropic approximation was based on a residual (R) scaled from 0 to 1, where 1 indicates a very poor approximation and 0 indicates the elasticity components are perfectly represented by the isotropic approximation. The R value for the isotropic approximation was 0.32.

## 4 Elasticity Components in an Alternate Basis

The components of the fourth order stiffness tensor, $\underset{\equiv}{E}$, were transformed from a basis with axes aligned with the orthotropic planes of symmetry$\left(\overset{E-E}{[E]}\right)$ to a different orthonormal basis with an arbitrary orientation $\left(\overset{e-e}{[E]}\right)$ in V-M notation. This was accomplished by performing the operation shown below, where $\overset{e-E}{[A]}$ is the transformation matrix for fourth order tensors (represented in V-M notation) to represent components in the $e$ and $E$ bases.

$$\overset{e-e}{[E]} = \overset{e-E}{[A]}\overset{E-E}{[E]}\overset{E-e}{[A]}$$

The results of this operation are shown in the attached code under problem 4, and the stiffness matrix $\overset{e-e}{[E]}$ takes the variable "E_ee" in the numerical calculations. The matrix of components resulting from this transformation contained no zero values, where the matrix of components aligned with the orthotropic planes of symmetry had upper right and lower left quarters of all zero values and the lower right quarter contained only diagonal nonzero components.

An isotropic approximation of the $\overset{e-e}{[E]}$ components was again made. This approximation resulted in the same values of engineering moduli $Y_{iso}$ and $\nu_{iso}$ as were obtained from the $\overset{E-E}{[E]}$ components. This is to be expected as both fits were to the same tensor $\underset{\equiv}{E}$, which is constant regardless of basis.

# 5 Find Material Axes

## 5.1 Find Eigenvectors of Elasticity Matrix

The eigenvalues and eigenvectors of $\underline{\underline{\underline{\underline{E}}}}$ were obtained with respect to the $e-e-e-e$ basis, and these values are shown under Problem 5 in the attached code. Of the six eigenvalues obtained, only 3 are unique. All six eigenvectors appear unique when viewed in V-M notation as a 6x1 vector and converted to the components of second-order eigentensors. The components of these eigentensors provide a transformation relationship between the the principal basis of $\underline{\underline{\underline{\underline{E}}}}$ and the $e-e-e-e$ basis $\left(\overset{P-e}{[A]}\right)$. Although I am unsure of this reasoning as typically the transformation matrices have been orthonormal, but my calculations show that $\overset{P-e}{[A]}$ is not orthonormal.

## 5.2 Convert Eigenvectors from V-M Notation to Second Order Eigentensors

Each eigenvector of $\underline{\underline{\underline{\underline{E}}}}$, or row of $\overset{P-e}{[A]}$, was converted from a to a second-order eigentensor. The components of each of these six second-order eigentensors relate the specific eigenvector to the original arbitrarily orientated basis (i.e., the transformation matrices $\overset{P_1-e}{[a]}$, $\overset{P_2-e}{[a]}$, etc.).

## 5.3 Determine Eigenvectors for Each of the 6 Second Order Eigentensors

The eigenvectors for each of the six second-order eigentensors were determined, where these eigenvectors are defined to be the "material vectors". I believe this implies that each of these material vectors lies within a plan of material symmetry. These material vectors are associated with the basis as defined by the six eigentensors, and the six eigentensors were associated with the arbitrarily orientated $\mathbf{e}$ basis; therefore, the material vectors are also associated with the $\mathbf{e}$ basis. Therefore, components of each respective material vector set relate a material basis to the arbitrary basis $\mathbf{e}$:

$$\overset{M-e}{[a]} \Rightarrow \begin{bmatrix} M_1^1\mathbf{e}_1 & M_2^1\mathbf{e}_2 & M_3^1\mathbf{e}_3 \\ M_1^2\mathbf{e}_1 & M_2^2\mathbf{e}_2 & M_3^2\mathbf{e}_3 \\ M_1^3\mathbf{e}_1 & M_2^3\mathbf{e}_2 & M_3^3\mathbf{e}_3 \end{bmatrix}$$

## 5.4 Obtain V-M Components of the Elasticity Matrix With Respect to Two Orthonormal Material Bases

Results from this are shown in the attached code. Elasticity matrices with respect to material bases 4 and 5 (i.e., those calculated from the fourth and fifth eigenvectors of the elasticity matrix) reproduce the matrix exactly. However, components of the elasticity matrix in the remaining four material bases do not reproduce the original components. Additionally, the material basis from the first elasticity eigenvector is not orthonormal.

## 5.5 Obtain Components of the Material Vectors

Componenets of the material vectors are shown in the attached code, and the figures below show their relationship with both the $\mathbf{E}$ and $\mathbf{e}$bases.

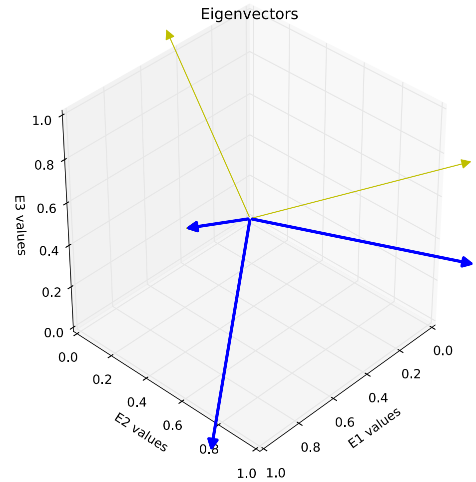


Figure 1: Orientations with respect to the **E** basis, where yellow is the first material basis (not orthonormal) and blue represents the **e** basis.

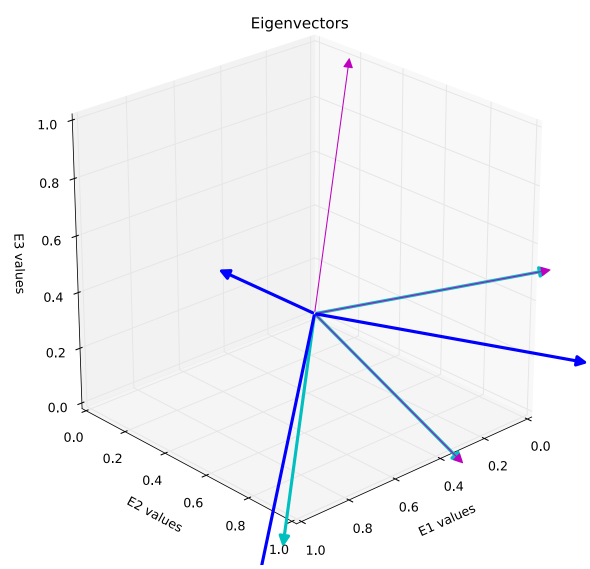


Figure 2: Orientations with respect to the **E** basis, where cyan and magenta bases are the fourth and fifth material bases, and the blue represents the **e** basis. The fourth and fifth bases are mirrored images of each other about a plane of symmetry.

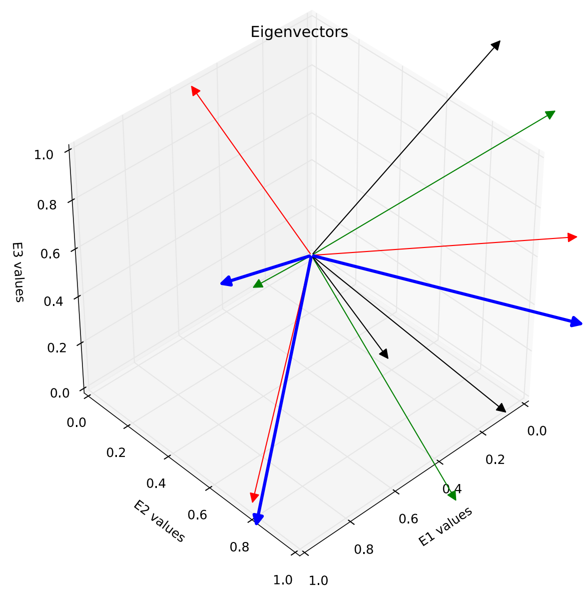


Figure 3: Orientations with respect to the **E** basis, where the black, red, and green vectors represent the remaining material bases, and the blue represents the **e** basis. I don't recognize the relationships here?