# 4. THE EIGENPROBLEM AND RELATED TOPICS

## 4.1 INTRODUCTION

The eigenproblem is the name of a matrix equation whose solution consists of a set of scalars, called eigenvalues, and a set of vectors, known as eigenvectors. The engineer is often introduced to the eigenproblem in an indirect manner through such topics as principal values and directions of the moment of inertia tensor from rigid-body dynamics, principal values and directions of stress and strain from mechanics of materials, and as natural frequencies and modes of vibration from vibration theory. This disjointed method of exposure to the underlying theory is a disservice to the inherent beauty of the structure implicit in the eigenproblem. For a finite-dimensional space, the theory can be covered as a natural part of linear algebra, and the topics mentioned previously are, then, just special cases. Viewed strictly from a mathematical viewpoint, the eigenproblem provides an important method for viewing the structure of a matrix and gives an insight to the background for many of the algorithms used to solve the linear algebraic problem. Because the topic is so important, this chapter is devoted to the eigenproblem including basic definitions, some of the more elementary theorems, and some applications in the analysis of numerical algorithms. No general algorithms are given because they are readily available from other sources such as Numerical Recipes [1992]. Example problems are given near the end of the chapter so that several concepts can be illustrated at one time for a given matrix.

The eigenproblem is associated with a rich history that includes matrices of arbitrary dimension with complex components. With the assumption that this is the first general exposure to the topic by the reader, the restriction is made to real, symmetric matrices that are obtained as a part of a procedure for finding approximate solutions to boundary value problems. The complications that can arise from nonsymmetric matrices are discussed briefly. No attempt is made to include matrices with complex numbers as components, although complex eigenvalues do arise from nonsymmetric matrices. The topics that are presented in this chapter have been selected because of their importance to computational mechanics and to provide a reasonable, although selective, exposure to the eigenproblem. With the introduction provided here, the reader should be able to profitably read more advanced texts on the subject.

## 4.2 STANDARD EIGENPROBLEM FOR SYMMETRIC MATRICES

### 4.2.1 Basic Definitions

Suppose a scalar, $\lambda$, and a vector, $\{e\}$, can be found such that

$$[A]\{e\} = \lambda\{e\} \tag{4.2-1}$$

Then $\lambda$ is said to be an **eigenvalue** and $\{e\}$ an **eigenvector** of $[A]$. Taken together, they represent a solution to the **standard eigenproblem**. Note that any scalar times $\{e\}$ is still an eigenvector or, alternatively, that the magnitude of an eigenvector is not defined unless prescribed by an independent criterion.

An equivalent formulation of (4.2-1) is

$$[[A] - \lambda[I]]\{e\} = \{0\} \tag{4.2-2}$$

where $[I]$ is the identity matrix. Since the algebraic equation is homogeneous, the use of Cramer's rule shows that a nontrivial solution for $\{e\}$ exists only if the following equation is satisfied:

$$\det [[A] - \lambda[I]] = 0 \tag{4.2-3}$$

which is a polynomial equation for $\lambda$. Equation (4.2-3) is called the **characteristic equation** and is also written symbolically as

$$P(\lambda) = 0 \tag{4.2-4}$$

in which P is the **characteristic polynomial**. For an n-dimensional matrix, the characteristic polynomial will be of order n, so there will be n zeros or roots (or eigenvalues), $\lambda_i$, i = 1, ..., n. These eigenvalues are often ordered sequentially:

$$\lambda_1 \le \lambda_2 \le \cdots \le \lambda_n \tag{4.2-5}$$

For each eigenvalue, $\lambda_i$, there is an associated eigenvector, $\{e\}^i$. The n eigenvalues and eigenvectors form the **eigensystem** of $[A]$. Each set $(\lambda_i, \{e\}^i)$ is called an **eigenpair**. The set of all eigenvalues is called the **spectrum** of $[A]$. As an example, for some solid systems, $\lambda_i$ corresponds to the square of the natural frequency and the eigenvector is called a mode of vibration.

Because real, symmetric matrices arise so often in practice, the following theorems and proofs are included.

**Theorem**: The eigenvalues of real, symmetric matrices are real.
**Proof**: Assume that the eigenvalue $\lambda_i$ and eigenvector $\{e\}^i$ are complex. Let an asterisk denote the conjugate of any quantity immediately to the left. By definition of an eigenpair

$$[A]\{e\}^i = \lambda_i \{e\}^i \tag{i}$$

Take the conjugate:

$$\overline{[A]\{e\}^i} = \overline{\lambda_i \{e\}^i} \tag{ii}$$

The conjugate of a product equals the product of conjugates so (ii) becomes

$$[\overline{A}]\{\overline{e}\}^i = \overline{\lambda}_i \{\overline{e}\}^i \tag{iii}$$

Use the assumption that [A] is real so that $[\overline{A}] = [A]$. Then, we get

$$[A]\{\overline{e}\}^i = \overline{\lambda}_i \{\overline{e}\}^i \tag{iv}$$

Multiply (i) on the left by $< \overline{e} >^i$ and (iv) on the left by $<e>^i$ to yield

$$< \overline{e} >^i [A]\{e\}^i = \overline{\lambda}_i < \overline{e} >^i \{e\}^i \tag{v}$$
$$< e >^i [A]\{\overline{e}\}^i = \overline{\lambda}_i < e >^i \{\overline{e}\}^i \tag{vi}$$

Take the transpose of (v):

$$< e >^i [A]^T\{\overline{e}\}^i = \lambda_i < e >^i \{\overline{e}\}^i \tag{vii}$$

By symmetry $[A] = [A]^T$ so that

$$< e >^i [A]\{\overline{e}\}^i = \lambda_i < e >^i \{\overline{e}\}^i \tag{viii}$$

Subtract corresponding terms in (vi) and (viii) to obtain

$$0 = (\lambda_i - \overline{\lambda}_i) < e >^i \{\overline{e}\}^i \tag{ix}$$

But the product of a complex number and its conjugate is positive. Therefore the product of a vector and its conjugate is the sum of positive terms, or

$$< e >^i \{\overline{e}\}^i > 0 \tag{x}$$

Then, (ix) implies that

$$\lambda_i - \overline{\lambda}_i = 0 \tag{xi}$$

Thus, the imaginary part of the eigenvalue is zero, i.e., $\lambda_i$ is real. **QED**

Nothing is said about the imaginary part of the eigenvector. However, the real and imaginary parts of the eigenvector satisfy the same eigenproblem so there is no loss of generality if $\{e\}^i$ is assumed to be real as well.

**Theorem**: Eigenvectors of real, symmetric matrices corresponding to distinct eigenvalues are unique (to within a scalar multiplication) and orthogonal.
**Proof**: Assume that the eigenvalues $\lambda_i$ and $\lambda_j$ are distinct for i not equal to j:

$$[A]\{e\}^i = \lambda_i\{e\}^i \quad \text{(i)}$$
$$[A]\{e\}^j = \lambda_j\{e\}^j \quad \text{(ii)}$$

Multiply the transpose of (i) on the right by $\{e\}^j$ and multiply (ii) on the left by $\langle e\rangle^i$ to obtain the following:

$$\langle e\rangle^i[A]^T\{e\}^j = \lambda_i\langle e\rangle^i\{e\}^j \quad \text{(iii)}$$
$$\langle e\rangle^i[A]\{e\}^j = \lambda_j\langle e\rangle^i\{e\}^j \quad \text{(iv)}$$

With the assumption of symmetry, $[A] = [A]^T$ and the left side of (iii) is identical to the left side of (iv). Subtract corresponding terms in (iii) and (iv) to obtain

$$0 = (\lambda_i - \lambda_j)\langle e\rangle^i\{e\}^j \quad \text{(v)}$$

But, $(\lambda_i - \lambda_j)$ is not zero by assumption. Therefore,

$$\langle e\rangle^i\{e\}^j = 0 \qquad \text{for i not equal to j} \quad \text{(vi)}$$

which indicates the eigenvectors are orthogonal. This result also indicates the eigenvectors are unique to within a scalar factor. **QED**

Repeated Eigenvalues

Suppose the first two eigenvalues are equal so that $\lambda_1 = \lambda_2 < \lambda_3 < \lambda_4 < \ldots < \lambda_n$ with eigenvectors $\{e\}^1, \{e\}^3, \ldots, \{e\}^n$. Recall that each eigenvector can only be defined to within a scale factor. Because of the repeated root, there is an additional degree of indeterminacy in the equation used for obtaining $\{e\}^1$. The implication is that one of the nonzero components of $\{e\}^1$ can be chosen arbitrarily. An independent eigenvector corresponding to $\lambda_2$ can then be constructed using the equations

$$[A]\{e\}^2 = \lambda_1\{e\}^2 \quad \text{and} \quad \langle e\rangle^2\{e\}^1 = 0 \quad \text{(4.2-6)}$$

Actually, $\{e\}^1$ and $\{e\}^2$ define a vector subspace. Any two orthogonal vectors in this subspace can be interpreted as an eigenvector corresponding to the repeated roots.

If an eigenvalue represents three roots, then the eigenvectors represent a three-dimensional subspace in which any three orthogonal vectors in that subspace can be used as

independent eigenvectors. Two additional degrees of indeterminacy will exist in the equation used for obtaining the first eigenvector in the subspace. Orthogonality relations are then used as additional restrictions for subsequent eigenvectors in the subspace.

### 4.2.2 Modal Matrices and Spectral Decomposition

Modal Matrices

Suppose n eigenvectors have been obtained and each eigenvector is scaled (**normalized**) such that

$$\langle e \rangle^i \{e\}^i = 1 \tag{4.2-7}$$

Then the eigenvectors form an **orthonormal** set:

$$\langle e \rangle^i \{e\}^j = \delta_{ij} \tag{4.2-8}$$

Construct the **modal matrix**, $[M^o]$, defined as the matrix whose columns are the normalized eigenvectors:

$$[M^o] = [\{e\}^1, \{e\}^2, ..., \{e\}^n] \tag{4.2-9}$$

Then

$$[M^o]^T[M^o] = [I] \tag{4.2-10}$$

i.e., $[M^o]$ is an orthogonal matrix. The set of vectors $\{e\}^i$, i = 1,..., n can be considered as both the contravariant and covariant basis of $[M^o]$.

Let $[\lambda]$ be a diagonal matrix with diagonal terms consisting of the ordered eigenvalues:

$$[\lambda] = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix} \tag{4.2-11}$$

Consider the expansion of the product of the matrices [A] and $[M^o]$ as follows:

$$\begin{aligned} [A][M^o] &= [A][\{e\}^1, \{e\}^2, ..., \{e\}^n] \\ &= [[A]\{e\}^1, [A]\{e\}^2, ..., [A]\{e\}^n] \\ &= [\lambda_1\{e\}^1, \lambda_2\{e\}^2, ..., \lambda_n\{e\}^n] \end{aligned}$$

The product of the matrices $[M^o]$ and $[\lambda]$ is also

$$[M^o][\lambda] = [\lambda_1\{e\}^1, \lambda_2\{e\}^2, ..., \lambda_n\{e\}^n$$

so we have shown that the complete set of eigen equations are reflected in the single matrix equation

$$[A][M^o] = [M^0][\Lambda] \tag{4.2-12}$$

This equation is used extensively in the derivation of various properties related to the eigenproblem of which an example is given next.

Spectral Decomposition of a Matrix

Since $[A][M^o] = [M^o][\lambda]$, it follows by multiplying on the right with $[M^o]^T$ and with the use of (4.2-10) that

$$[A] = [M^o][\lambda][M^o]^T \tag{4.2-13}$$

which is called the **spectral decomposition** of [A]. An alternative form that is perhaps more illustrative is obtained by investigating the right side. The product of the first two matrices is

$$[M^o][\lambda] = \begin{bmatrix} \{e\}^1 & \{e\}^2 & \cdots & \{e\}^n \end{bmatrix} \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}$$

$$= \begin{bmatrix} \lambda_1\{e\}^1 & \lambda_2\{e\}^2 & \cdots & \lambda_n\{e\}^n \end{bmatrix}$$

Next, we multiply on the right with the transpose of $[M^o]$ to obtain

$$[M^o][\lambda][M^o]^T = \begin{bmatrix} \lambda_1\{e\}^1 & \lambda_2\{e\}^2 & \cdots & \lambda_n\{e\}^n \end{bmatrix} \begin{bmatrix} < e >^1 \\ < e >^2 \\ \vdots \\ < e >^n \end{bmatrix}$$

$$= \lambda_1\{e\}^1 < e >^1 + \lambda_2\{e\}^2 < e >^2 + \cdots + \lambda_n\{e\}^n < e >^n$$

which denotes a sum of matrices. Each matrix consists of an eigenvalue times the outer product of each eigenvector with itself. The result is that the spectral decomposition of [A] given in (4.2-13) can also be written as follows:

$$[A] = \sum_{i=1}^{n} \lambda_i \{e\}^i < e >^i \tag{4.2-14}$$

In a similar fashion, the inverse of $[A] = [M^o][\lambda][M^o]^T$ yields the spectral decomposition of $[A]^{-1}$ in the two corresponding forms:

$$[A]^{-1} = [M^o][\lambda]^{-1}[M^o]^T = \sum_{i=1}^{n} \frac{1}{\lambda_i} \{e\}^i < e >^i \tag{4.2-15}$$

This representation shows that the inverse cannot exist if one of the eigenvalues is zero.

Take the determinant of $[A] = [M^o][\lambda][M^o]^T$ with the observation that $[M]$ is orthogonal which implies $\det[M]^T = \det[M] = \pm 1$ to obtain

$$\det [A] = \det [\Lambda] = \lambda_1 \lambda_2 \cdots \lambda_n \tag{4.2-16}$$

The determinant of a matrix is merely the product of its eigenvalues. We see immediately from (4.2-15) and (4.2-16) that the determinant of [A] cannot be zero for the inverse to exist.

### 4.2.3 Eigenvector Basis

Now we return to the linear algebraic problem $[A]\{x\} = \{b\}$. With the use of (4.2-13), an alternative but equivalent form of the equation is

$$[M^o][\lambda][M^o]^T\{x\} = \{b\} \tag{4.2-17}$$

By multiplying on the left by $[M^o]^T$ and then by $[\lambda]^{-1}$, the result is

$$[M^0]^T\{x\} = \{y\} \tag{4.2-18}$$

in which the vector $\{y\}$ is given by

$$\{y\} = [\lambda]^{-1}[M^o]^T\{b\} \tag{4.2-19}$$

If $\{y\}$ is determined from (4.2-19), then a multiplication by $[M^o]$ on the left of the terms in (4.2-18) provides the solution for $\{x\}$ as a linear combination of the eigenvectors, i.e., the **eigenvector basis**:

$$\{x\} = [M^o]\{y\} = \sum_{i=1}^{n} y_i \{e\}^i \tag{4.2-20}$$

This equation also implies that the eigenvectors form a vector space that spans the column space of [A]. Of course, an equivalent relationship to (4.2-20) is

$$\{x\} = [M^o][\lambda]^{-1}[M^o]^T\{b\} \tag{4.2-21}$$

Suppose s eigenvalues are zero and r = n - s eigenvalues are not zero as follows:

$$\begin{aligned} &\lambda_1 \le \lambda_2 \le \cdots \le \lambda_m < 0 \\ &\lambda_{m+1} = \lambda_{m+2} = \cdots = \lambda_{m+s} = 0 \\ &0 < \lambda_{m+s+1} \le \cdots \le \lambda_n \end{aligned} \tag{4.2-22}$$

Then the eigenvectors $\{e\}^{m+1}$, $\{e\}^{m+2}$, ..., $\{e\}^{m+s}$ span the **null space** of [A], denoted as $N([A])$. The dimension of $N([A])$ is $\dim N([A]) = s$. The vector space formed by the remaining eigenvectors is called the **range** of [A], denoted as $R([A])$. The dimension of $R([A])$ is $\dim R([A]) = r$, the number of nonzero eigenvalues. The number, r, is also called the **rank** of [A].

### 4.2.4 Positive Definiteness

Recall that [A] is positive definite if $<x>[A]\{x\} > 0$ for all $\{x\} \neq \{0\}$. Let $\{x\}$ be a linear combination of the eigenvectors of [A]:

$$\{x\} = \sum_{i=1}^{n} \alpha_i \{e\}^i \tag{4.2-23}$$

By taking the inner product of $\{x\}$ with $[A]\{x\}$, we get

$$< x > [A]\{x\} = < x > \sum_{i=1}^{n} \alpha_i [A]\{e\}^i = < x > \sum_{i=1}^{n} \alpha_i \lambda_i \{e\}^i \tag{4.2-24}$$

where we have used the fact that $\{e\}^i$ is an eigenvector of [A]. Now, utilize (4.2-23) again to obtain

$$< x > [A]\{x\} = \sum_{j=1}^{n} \alpha_j < e >^j \sum_{i=1}^{n} \alpha_i \lambda_i \{e\}^i = \sum_{i,j=1}^{n} \alpha_i \alpha_j \lambda_i \delta_{ij} \tag{4.2-25}$$

in which the orthonormality of the eigenvectors has been used. The last result indicates that nonzero contributions are made to the sum only if i = j, so an alternative form for the inner product of $\{x\}$ with $[A]\{x\}$ is

$$< x > [A]\{x\} = \sum_{i=1}^{n} (\alpha_i)^2 \lambda_i \tag{4.2-26}$$

Since the choice of $\{x\}$, and consequently of $\alpha_i$, is arbitrary, one consequence of this equation is that a matrix is **positive definite** if and only if all its eigenvalues are positive. If all eigenvalues satisfy the inequality $\lambda_i \ge 0$, the matrix is **positive, semi-definite**; if the eigenvalues are positive and negative, the matrix is **indefinite**.

### 4.2.5 Similarity

Suppose [S] is a matrix whose inverse exists. Suppose further that [A] is a given matrix. If another matrix $[A^s]$ is defined by the equation

$$[A^s] = [S][A][S]^{-1} \tag{4.2-27}$$

then $[A^s]$ is said to be a matrix **similar** to [A] and [S] is called the **similarity transformation** matrix (see Subsection 3.4.5). Now, suppose the eigensystem of [A] is known and consider

$$[A^s][S] = [S][A] \tag{4.2-28}$$

Multiply on the right by $\{e\}^i$ to obtain:

$$[A^s][S]\{e\}^i = [S][A]\{e\}^i = [S]\,\lambda_i\{e\}^i \tag{4.2-29}$$

Define a set of vectors $\{e^s\}^i$ from the equation

$$\{e^s\}^i = [S]\{e\}^i \qquad i = 1, ..., n \tag{4.2-30}$$

Then, (4.2-29) and (4.2-30) yield

$$[A^s]\{e^s\}^i = \lambda_i\{e^s\}^i \tag{4.2-31}$$

In words, the last equation implies that similar matrices, $[A^s]$, have the same eigenvalues as [A]. The eigenvectors of $[A^s]$ and [A] are different although they are related through the similarity transformation of (4.2-30).

As a trivial example, suppose $[S] = [M^o]^T$, the modal matrix of [A]. Then, (4.2-27) becomes

$$[A^s] = [M^o]^T[A][M^o]^{-T} = [M^o]^T[A][M^o] = [\lambda] \tag{4.2-32}$$

in which $[M^o]^{-T}$ denotes $([M^o]^T)^{-1}$ and $[M^o]^{-T} = [M^o]$ because $[M^o]$ is an orthogonal matrix. The result is that [A] and $[\lambda]$ are similar matrices. Because the similarity transformation matrix is orthogonal, [A] and $[\lambda]$ are also congruent matrices. Their eigenvalues are the same and the eigenvectors of $[\lambda]$ are

$$\{e^s\}^i = [M^o]^T\{e\}^i = \begin{bmatrix} < e >^1 \\ < e >^2 \\ \cdots \\ < e >^n \end{bmatrix}\{e\}^i = \{I\}^i \tag{4.2-33}$$

The eigenvectors of $[\lambda]$ are the coordinate basis, $\{I\}^i$, a result which could have been determined by inspection since $[\lambda]$ is a diagonal matrix.

### 4.2.6 Power of a Matrix

Consider the spectral decomposition of a matrix:

$$[A] = [M^o][\lambda][M^o]^T$$

and its square:

$$\begin{aligned}[A]^2 &= [M^o][\lambda][M^o]^T[M^o][\lambda][M^o]^T \\ &= [M^o][\lambda][\lambda][M^o]^T \quad (4.2\text{-}34) \\ &= [M^o][\lambda]^2[M^o]^T\end{aligned}$$

The process can be repeated indefinitely to obtain, for any integer, k:

$$[A]^k = [M^o][\lambda]^k[M^o]^T \tag{4.2-35}$$

Because $[\lambda]$ is a diagonal matrix, $[\lambda]^\kappa$, is also a diagonal matrix with nonzero components $(\lambda_i)^k$ which are well defined even when k is not an integer. Therefore, (4.2-35) is used to define the k'th **power of a matrix** for arbitrary k. Note that the process of obtaining a power of a matrix is not trivial because the eigensystem of [A] must be completely determined.

A function of a matrix can be defined in a similar manner. Let f(x) be a scalar function of the scalar variable x. Then $[f([A])]$ is a matrix defined in a manner suggested by (4.2-35) as follows:

$$[f([A])] = [M^o][f([\lambda])][M^o]^T \tag{4.2-36}$$

in which $[f([\lambda])]$ is the diagonal matrix

$$[f([\lambda])] = \begin{bmatrix} f(\lambda_1) & 0 & \cdots & 0 \\ 0 & f(\lambda_2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & f(\lambda_n) \end{bmatrix} \tag{4.2-37}$$

In this manner, a function of a matrix can be defined provided the scalar function of each eigenvalue is defined. A typical example is to define the exponential of a matrix in which case the i'th diagonal component in (4.2-37) is $e^{\lambda_i}$ .

### 4.2.7 Orthogonal Projections

A set of **orthogonal projections** is a set of matrices $[P]_i$, where the index ranges from 1 to m, with the following special properties:

(i) the product of one matrix with any other matrix from the same set is zero,
(ii) the product of any matrix with itself is the same matrix, and
(iii) the sum of all projection matrices in the set is the identity matrix.

These properties are reflected in the equations

$$[P]_i[P]_j = \delta_{ij}[P]_i \tag{4.2-38}$$

and

$$\sum_{i=1}^{m} [P]_i = [I] \tag{4.2-39}$$

Note that [I] itself is an orthogonal projection since [I][I] = [I]. Suppose the eigenvalues of [A] are distinct. Then the projections formed from the outer products of the normalized eigenvectors form such a set. To show this let

$$[P^e]_i = \{e\}_i <e>_i , \qquad i = 1, ..., n \tag{4.2-40}$$

Then, the product of the projection matrix with another matrix from the same set is

$$[P^e]_i[P^e]_j = \{e\}_i <e>_i \{e\}_j <e>_j = (<e>_i \{e\}_i)\{e\}_i <e>_j \tag{4.2-41}$$

If it is assumed that the eigenvectors have been normalized, or, $<e>_j\{e\}_i = \delta_{ij}$. Then

$$[P^e]_i[P^e]_j = \delta_{ij}\{e\}_i <e>_j = \delta_{ij}[P]_i \tag{4.2-42}$$

so that satisfaction of (4.2-38) is shown. Now, we need to show that (4.2-39) is satisfied. To this end, consider

$$\sum_{i=1}^{n} [P^e]_i\{x\} = \sum_{i=1}^{n} \{e\}^i < e >^i \{x\} = \sum_{i=1}^{n} \{e\}^i (< e >^i \{x\}) \tag{4.2-43}$$

But, the right side is just an expression for {x} in terms of components and the basis, $\{e\}^i$. Therefore, (4.2-43) is an alternative expression for [I]{x} = {x}, and (4.2-39) is shown.

With the use of the projection matrices, the spectral decomposition of [A] from (4.2-14) becomes

$$[A] = \sum_{i=1}^{n} \lambda_i [P^e]_i \tag{4.2-44}$$

If an eigenvalue is repeated for symmetric, real matrices, then more than one eigenvector is associated with that eigenvalue; the number of eigenvectors equals the number of repeated roots, *q*. The projection operator associated with the repeated eigenvalue, $\lambda_i$, is

$$[P^e]_i = \sum_{j=1}^{q} \{e\}_{i,j} < e >_{i,j} \tag{4.2-45}$$

in which $\{e\}_{i,j}$ denotes the j'th eigenvector associated with $\lambda_i$. The spectral decomposition now becomes

$$[A] = \sum_{i=1}^{m} \lambda_i [P^e]_i \tag{4.2-46}$$

in which m denotes the number of distinct eigenvalues.

It is not necessary to obtain the eigenvectors to obtain the eigenprojection matrices $[P^e]$. Luehr and Rubin (1990) provide a direct method for obtaining these matrices:

$$[P^e]_i = \frac{\prod_{j=1,\ldots,m(j\neq i)} ([A] - \lambda_j [I])}{\prod_{j=1,\ldots,m(j\neq i)} (\lambda_i - \lambda_j)}, \qquad i = 1,\ldots, m \tag{4.2-47}$$

in which $\prod()$ denotes the product symbol.

If the eigenvalues and eigenprojections are available, and f(x) denotes a function of a scalar variable x, then the corresponding function of a matrix is defined as follows:

$$[f([A])] = \sum_{i=1}^{m} f(\lambda_i)[P^e]_i \tag{4.2-48}$$

which is merely an alternative form of (4.2-36). This equation holds even if the eigenvalues are not distinct. The power of a matrix is a special case. However the **function of a matrix** can also be, as an example, the sine of [A].

### 4.2.8 The Cayley-Hamilton Theorem

**Cayley-Hamilton Theorem**: Suppose the characteristic equation associated with a matrix, [A], is

$$\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \cdots + a_n = 0 \tag{4.2-49}$$

Then, the matrix satisfies its own characteristic equation, i.e.,

$$[A]^n + a_1[A]^{n-1} + a_2[A]^{n-2} + \cdots + a_n[I] = [0] \tag{4.2-50}$$

**Proof**: Recall that

$$\begin{aligned} [A] &= [M^o][\lambda][M^o]^T \\ [A]^2 &= [M^o][\lambda]^2[M^o]^T \\ &\cdots\cdots \\ [A]^n &= [M^o][\lambda]^n[M^o]^T \end{aligned} \tag{4.2-51}$$

Substitute (4.2-51) in (4.2-50) to obtain

$$[M^o]\left[\ [\lambda]^n + a_1[\lambda]^{n-1} + a_2[\lambda]^{n-2} + \ldots + a_n[I]\ \right][M^o]^T = [0] \tag{4.2-52}$$

In the large square bracket, the only nonzero terms in the matrices are on the diagonal. The first nonzero term is $\lambda_1^{\,n} + a_1\lambda_1^{\,n-1} + a_2\lambda_1^{\,n-2} + \cdots + a_n$. But $\lambda_1$ is a root of the Characteristic Equation given in (4.2-47). Therefore this first diagonal term is zero. Similarly, all the other diagonal terms are zero since the other eigenvalues also satisfy the same characteristic equation. The result is that (4.2-50) is an identity.

**QED**

### 4.2.9 Rayleigh Quotient

Let [A] be symmetric and real. For any nonzero vector, {x}, the **Rayleigh Quotient** is defined to be

$$R(\{x\}) = \frac{< x > [A]\{x\}}{< x > \{x\}} \tag{4.2-53}$$

**Theorem**: If [A] is positive definite, then $R(\{x\}) > 0$ and

$$\lambda_1 \le R(\{x\}) \le \lambda_n \tag{4.2-54}$$

**Proof**: Let $\{e\}_i$, i = 1, ..., n be the eigenvectors of [A]. Then the vector {x} and the denominator of Rayleigh's Quotient can be represented as follows:

$$\{x\} = \sum_{i=1}^{n} \alpha_i \{e\}_i \tag{4.2-55}$$

and

$$< x > \{x\} = \sum_{j=1}^{n}\sum_{i=1}^{n} \alpha_i < e >_i \alpha_j \{e\}_j = \sum_{j=1}^{n}\sum_{i=1}^{n} \alpha_i\alpha_j\delta_{ij} = \sum_{i=1}^{n} (\alpha_i)^2 \tag{4.2-56}$$

Similarly, the numerator is expanded to yield

$$\begin{aligned} < x > [A]\{x\} &= \sum_{i=1}^{n} \alpha_i < e >_i \sum_{j=1}^{n} \alpha_j [A]\{e\}_j \\ &= \sum_{i=1}^{n} \alpha_i < e >_i \sum_{j=1}^{n} \alpha_j \lambda_j \{e\}_j = \sum_{i=1}^{n} \lambda_i (\alpha_i)^2 \end{aligned} \tag{4.2-57}$$

Rayleigh's Quotient becomes

$$\begin{aligned} R(\{x\}) &= \frac{\lambda_1 \alpha_1^2 + \lambda_2 \alpha_2^2 + \cdots + \lambda_n \alpha_n^2}{\alpha_1^2 + \alpha_2^2 + \cdots + \alpha_n^2} \\ &= \frac{\lambda_1 [\alpha_1^2 + \frac{\lambda_2}{\lambda_1} \alpha_2^2 + \cdots + \frac{\lambda_n}{\lambda_1} \alpha_n^2]}{\alpha_1^2 + \alpha_2^2 + \cdots + \alpha_n^2} \geq \lambda_1 \end{aligned} \tag{4.2-58}$$

because $\frac{\lambda_i}{\lambda_1} \geq 1$ for all i. In an analogous manner

$$R(\{x\}) = \frac{\lambda_n [\frac{\lambda_1}{\lambda_n} \alpha_1^2 + \frac{\lambda_2}{\lambda_n} \alpha_2^2 + \cdots + \alpha_n^2]}{\alpha_1^2 + \alpha_2^2 + \cdots + \alpha_n^2} \leq \lambda_n \tag{4.2-59}$$

because $\frac{\lambda_i}{\lambda_n} \leq 1$ for all i. The combination of (4.2-58) and (4.2-59) yields the result (4.2-54), the statement which was to be proved. **QED**

Next, we show that if if an eigenvector is chosen for {x}, then Rayleigh's quotient is simply the associated eigenvalue.

**Theorem**: $$R(\{e\}_i) = \lambda_i \tag{4.2-60}$$

**Proof**: When the substitution of the eigenvector is made, the result is as follows:

$$R(\{e\}_i) = \frac{< e >_i [A]\{e\}_i}{< e >_i \{e\}_i} = \frac{< e >_i \lambda_i \{e\}_i}{< e >_i \{e\}_i} = \lambda_i \tag{4.2-61}$$

which is just (4.2-60). **QED**

Particular consequences of the theorem are $R(\{e\}_1) = \lambda_1$ and, similarly, $R(\{e\}_n) = \lambda_n$. Therefore $\lambda_1$ and $\lambda_n$ can be obtained as the minimum and maximum, respectively, of $R(\{x\})$ with respect to {x}:

$$\lambda_1 = \min R(\{x\}) , \qquad \lambda_n = \max R(\{x\}) \tag{4.2-61}$$

In vibration theory, it is well known that if a mode is assumed, the corresponding approximation for the natural frequency is often quite accurate. This is one example of the application of the following theorem.

**Theorem**: If $\{e\}$ is an approximation to the eigenvector $\{e\}_i$ of order $\varepsilon$, then $R(\{e\})$ is an approximation to the eigenvalue $\lambda_i$ of order $\varepsilon^2$.

**Proof**: Normalize the eigenvectors so that $<e>_i\{e\}_j = \delta_{ij}$. Suppose $\{x\}$ is a unit vector perpendicular to $\{e\}_i$, i.e.,

$$<x>\{e\}_i = 0 , \qquad <x>\{x\} = 1 \tag{4.2-62}$$

Let the approximation to $\{e\}_i$ be $\{e\}$, which is represented as follows:

$$\{e\} = \{e\}_i + \varepsilon\{x\} \tag{4.2-63}$$

Then

$$<e>\{e\} = <<e>_i + \varepsilon<x>>\{\{e\}_i + \varepsilon\{x\}\} = 1 + \varepsilon^2 \tag{4.2-64}$$

Since $\{x\}$ is orthogonal to $\{e\}_i$, the following representations hold:

$$\{x\} = \sum_{j\neq 1}^{n} \alpha_j\{e\}_j \qquad \Rightarrow \qquad < x >\{x\} = \sum_{j\neq 1}^{n} (\alpha_j)^2 = 1$$

$$[A]\{x\} = \sum_{j\neq i}^{n} \alpha_j\lambda_j\{e\}_j \quad \Rightarrow \quad [A]\{e\} = \lambda_i\{e\}_i + \varepsilon\sum_{j\neq 1}^{n} \alpha_j\lambda_j\{e\}_j \tag{4.2-65}$$

$$< e > [A]\{e\} = \lambda_i + \varepsilon^2\sum_{j\neq i}^{n} (\alpha_j)^2\lambda_j$$

Now, the Rayleigh Quotient becomes

$$R(\{e\}) = \frac{< e > [A]\{e\}}{< e > \{e\}} = \frac{\lambda_i + \varepsilon^2\sum_{j\neq i}^{n} (\alpha_j)^2\lambda_j}{1 + \varepsilon^2} \tag{4.2-66}$$

$$= [\lambda_i + \varepsilon^2\sum_{j\neq i}^{n} (\alpha_j)^2\lambda_j][1 - \varepsilon^2 + \cdots]$$

Use the relation from (4.2-65) that $\sum_{j\neq i}^{n}(\alpha_j)^2 = 1$ to obtain

$$R(\{e\}) = \lambda_i + \varepsilon^2 \sum_{j=1}^{n}(\alpha_j)^2(\lambda_j - \lambda_i) + \cdots \tag{4.2-67}$$

in which $\alpha_i$ is arbitrary because its coefficient in the summation is zero. The result is that the Rayleigh Quotient provides an estimate to the eigenvalue of order $\varepsilon^2$ when the vector $\{x\}$ is an estimate to the eigenvector of order $\varepsilon$. In practice, relatively poor choices for $\{e\}$ based on engineering judgment provides a reasonably good estimate for $\lambda$. **QED**

### 4.2.10 Eigenvalue Separation Property

Suppose the standard eigenproblem $[A]\{e\} = \lambda\{e\}$ is to be solved subject to a constraint $\langle c\rangle\{e\} = 0$. The constraint can be invoked by deleting a variable from $\{e\}$ with the result that the modified problem becomes one of solving $[A]^{(1)}\{e\}^{(1)} = \lambda^{(1)}\{e\}^{(1)}$ in which $[A]^{(1)} \in R^{(n-1)x(n-1)}$. The **eigenvalue separation property** states that the eigenvalues of the restricted problem are related to the original eigenvalues through the inequalities

$$\lambda_1 \le \lambda_1^{(1)} \le \lambda_2 \le \lambda_2^{(1)} \le \cdots \le \lambda_{n-1}^{(1)} \le \lambda_n \tag{4.2-69}$$

If an additional constraint is invoked, the eigenvalues $\lambda_i^{(2)}$ are related to $\lambda_i^{(1)}$ through a second application of the eigenvalue separation property, and so on.

## 4.3 STANDARD EIGENPROBLEM FOR UNSYMMETRIC MATRICES

### 4.3.1 Properties of the Eigensystem

Suppose [A] is real but $[A] \neq [A]^T$. For some matrices that are not symmetric, the eigenvalues will be real, but there is now the possibility that the eigenvalues will not be real. Also, both a **left eigensystem** and a **right eigensystem** may exist. With superscripts L and R denoting the left and right systems, respectively, the definitions are given by the following equations:

$$\begin{gathered} [A]\{e\}^R = \lambda^R \{e\}^R \\ < e >^L [A] = \lambda^L < e >^L \qquad \text{or} \qquad [A]^T \{e\}^L = \lambda^L \{e\}^L \end{gathered} \tag{4.3-1}$$

The left and right eigenvalues are obtained from the respective characteristic equations:

$$\begin{gathered} \det\left[[A] - \lambda^R [I]\right] = 0 \\ \det\left[[A]^T - \lambda^L [I]\right] = 0 \qquad \text{or} \qquad \det\left[[A] - \lambda^L [I]\right]^T = 0 \end{gathered} \tag{4.3-2}$$

However, the determinant of a transposed matrix equals the determinant of the matrix itself so there is actually only one characteristic equation. Therefore, the eigenvalues of the left and right eigensystems are identical, i.e.,

$$\lambda_i^R = \lambda_i^L = \lambda_i \tag{4.3-3}$$

Eigenvectors associated with symmetric matrices and distinct eigenvalues are orthogonal. For unsymmetric matrices the corresponding relationship involves both left and right eigenvectors. To derive the relationship, consider the two governing equations for the right and left eigenvectors:

$$[A]\{e\}_i^R = \lambda_i \{e\}_i^R \qquad \text{and} \qquad < e >_j^L [A] = \lambda_j < e >_j^L \tag{4.3-4}$$

Multiply the first equation on the left by $< e >_j^L$ and the second on the right by $\{e\}_i^R$. The result is

$$\begin{aligned} < e >_j^L [A]\{e\}_i^R &= \lambda_i < e >_j^L \{e\}_i^R \\ < e >_j^L [A]\{e\}_i^R &= \lambda_j < e >_j^L \{e\}_i^R \end{aligned} \tag{4.3-5}$$

The left sides of the equation are identical so a subtraction of terms in the two equations yields $(\lambda_i - \lambda_j) < e >_j^L \{e\}_i^R = 0$. If the eigenvalues are distinct, the left and right eigenvectors corresponding to different eigenvalues are orthogonal, i.e.,

$$< e >_j^L \{e\}_i^R = 0\,, \qquad i \neq j \tag{4.3-6}$$

Suppose the length of each eigenvector in one set is chosen arbitrarily but the second set is chosen such that each eigenvector is normalized by the equation $< e >_i^L \{e\}_i^R = 1$. Then the two sets of eigenvectors satisfy the relation

$$< e >_i^L \{e\}_j^R = \delta_{ij} \tag{4.3-7}$$

that is, the two sets form a dual basis. One set can be chosen as a contravariant basis, and the other would form a covariant basis. The eigenvectors may be complex.

Let $[M^{oR}]$ be a matrix whose columns consist of the right eigenvectors normalized in a suitable manner. Let the rows of $[M^{oL}]^T$ be the left eigenvectors normalized according to the relation implied by (4.3-7) when i = j:

$$[M^{oR}] = \begin{bmatrix} \{e\}_1^R & \{e\}_2^R & \cdots & \{e\}_n^R \end{bmatrix}$$

$$[M^{oL}]^T = \begin{bmatrix} < e >_1^L \\ < e >_2^L \\ \vdots \\ < e >_n^L \end{bmatrix} \tag{4.3-8}$$

An alternative form of (4.3-7) is the matrix equation $[M^{oL}]\,[M^{oR}] = [I]$ or

$$[M^{oL}]^T = [M^{oR}]^{-1} \tag{4.3-9}$$

If $[\lambda]$ denotes the diagonal matrix with the eigenvalues of [A] on the diagonal, then (4.3-5) provides the following form for the spectral decomposition of [A]:

$$[M^{oR}]^{-1}[A][M^{oR}] = [\lambda] \qquad [A] = [M^{oR}][\lambda][M^{oR}]^{-1} \tag{4.3-10}$$

or, that [A] and $[\lambda]$ are similar matrices. However, they are not congruent matrices (see Subsection 3.4.5). If [A] is symmetric, then $[M^{oR}]$ is the modal matrix, $[M^o]$, and $[M^o]^{-1} = [M^o]^T$. The results of the previous section are recovered.

### 4.3.2 Gerschgorin's Theorem

**Gerschgorin's theorem** is often useful for obtaining bounds to eigenvalues. The theorem states that each eigenvalue, $\lambda$, (real or complex) of [A] satisfies at least one of the inequalities

$$|\lambda - A_{ii}| \le r_i , \qquad i = 1, ..., n \tag{4.3-11}$$

where

$$r_i = \sum_{\substack{j=1 \\ j \ne i}}^{n} |A_{ij}|, \qquad i = 1, ..., n \tag{4.3-12}$$

In words, each eigenvalue lies in at least one of the discs with center $A_{ii}$ and radius $r_i$ in the complex plane. Moreover, if the union of p of the discs is disjoint from the remainder, then there are precisely p eigenvalues of [A] in the union of the p discs.

As an example, consider the following matrix:

$$[A] = \begin{bmatrix} 10 & 2 & -2 \\ 2 & 0 & -1 \\ -2 & -1 & 1 \end{bmatrix}$$

As shown in Fig. 4.3-1, the centers of the discs are located at $A_{11} = 10$, $A_{22} = 0$, and $A_{33} = 1$, with radii of $r_1 = |2| + |-2| = 4$, $r_2 = |2| + |-1| = 3$ and $r_3 = |-2| + |-1| = 3$, corresponding, respectively, with the centers of the discs. Because the matrix is symmetric, the eigenvalues are real and, therefore, they must lie on the real axis. Two eigenvalues lie in the union of the first two discs while another eigenvalue lies in the second disc. Necessarily, we have the following bounds on the eigenvalues: $-3 \le \lambda_1, \lambda_2 \le 4$ and $6 \le \lambda_3 \le 14$.

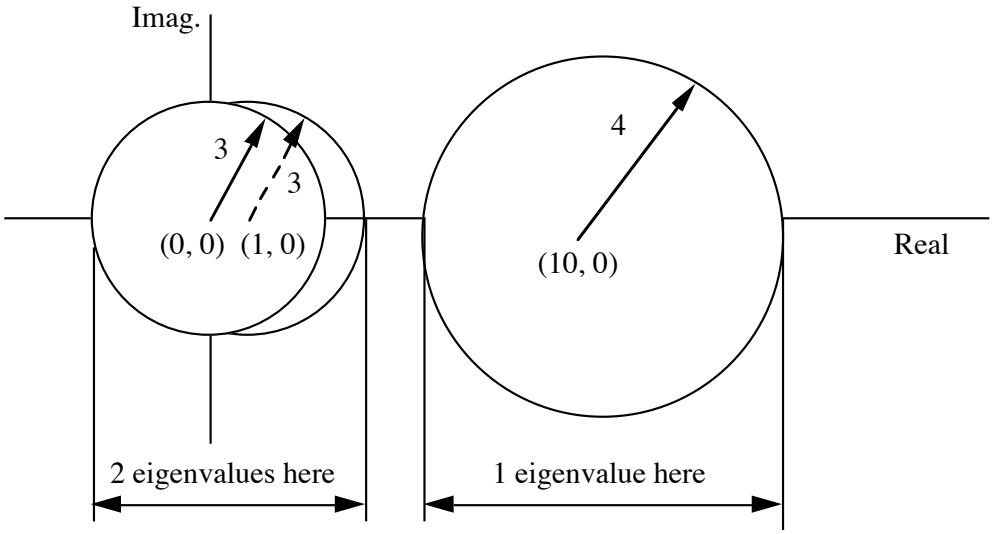


Fig. 4.3-1. Gerschgorin discs for the given matrix.

One example of an application is to relate diagonally dominant matrices to positive definiteness. Recall that a matrix is **diagonally dominant** if $|A_{ii}| \ge \sum_{j \ne i} |A_{ij}|$ for all i, with an inequality for at least one i. Suppose [A] is real and the diagonal terms are all positive. It follows from Gerschgorin's Theorem that the eigenvalues are positive or zero. Therefore,

**real, diagonally dominant** matrices with positive diagonal components are at least **positive, semi-definite**.

Sometimes in applications, a large matrix $[C] \in R^{2n \times 2n}$ is partitioned and formed from two matrices $[A] \in R^{n \times n}$ and $[B] \in R^{n \times n}$ as follows:

$$[C] = \begin{bmatrix} [A] & [B] \\ [B] & [A] \end{bmatrix} \tag{4.3-13}$$

Then, the set of eigenvalues of [C] is the set of 2n numbers consisting of the eigenvalues of [A] + [B] and the eigenvalues of [A] - [B] [Marcus, 1964; pg. 12]. Gerschgorin's theorem could be applied to both the sum and the difference to obtain bounds on [C].

## 4.4 GENERAL EIGENPROBLEM FOR SYMMETRIC MATRICES

### 4.4.1 Basic Relationships for Symmetric Matrices

Although the standard eigenproblem provides considerable insight into the structure of matrices, and is itself an important problem, there is a large class of problems governed by the general eigenproblem. Consider two matrices [A] and [B], both symmetric and real with [B] positive definite. For most engineering applications [A] is positive, semi-definite but this requirement need not hold for the derivation of the following equations. If

$$[A]\{e\} = \lambda[B]\{e\} \tag{4.4-1}$$

then $(\lambda, \{e\})$ is an eigenpair of the **general eigenproblem**. If [B] = [I], the problem reduces to the standard eigenproblem. The characteristic equation, from which the eigenvalues, $\lambda_i$, are obtained as roots, is the equation

$$\det\left[[A] - \lambda[B]\right] = 0 \tag{4.4-2}$$

It is customary to normalize each eigenvector with respect to the matrix [B], i.e., to choose the length of each eigenvector so that

$$\langle e \rangle_i [B]\{e\}_i = 1 , \qquad i = 1, ..., n \tag{4.4-3}$$

This normalization cannot be performed if [B] is not positive definite. Then we have the following result which is analogous to the orthonormal relationship of eigenvectors obtained from the standard eigenproblem.

**Theorem**: For distinct eigenvalues, the eigenvectors are **B-orthogonal**.
**Proof**: By definition of each eigenpair

$$[A]\{e\}_i = \lambda_i[B]\{e\}_i \qquad \text{and} \qquad [A]\{e\}_j = \lambda_j[B]\{e\}_j \tag{4.4-4}$$

Multiply the first equation on the left by $<e>_j$ and the second by $<e>_i$:

$$\begin{aligned} < e >_j [A]\{e\}_i &= \lambda_i < e >_j [B]\{e\}_i \\ < e >_i [A]\{e\}_j &= \lambda_j < e >_i [B]\{e\}_j \end{aligned} \tag{4.4-5}$$

Subtract and use the symmetry of [A] and [B] to obtain

$$(\lambda_i - \lambda_j) < e >_i [B]\{e\}_j = 0 \tag{4.4-6}$$

But the eigenvalues were assumed to be distinct so $\lambda_i \neq \lambda_j$. When use is made of (4.4-3) the result is

$$< e >_i [B]\{e\}_j = \delta_{ij} \tag{4.4-7}$$

which is a combined orthogonality and normality relation for the eigenvectors with respect to [B]. **QED**

If some of the eigenvalues are not distinct, eigenvectors can be constructed to span the subspace associated with the repeated eigenvalues in a manner similar to that for the standard eigenproblem. An approach similar to that used for the standard eigenproblem is used to show that the eigenvalues must be real. The modal matrix is also constructed in a similar fashion:

$$[M^o] = \left[\{e\}_1 \quad \{e\}_2 \quad \cdots \quad \{e\}_n\right] \tag{4.4-8}$$

Thus, there follows

$$[M^o]^T[B][M^o] = [I] \quad \text{and} \quad [M^0]^T[A][M^0] = [\lambda] \tag{4.4-9}$$

### 4.4.2 Relationships between General and Standard Eigenproblems

The majority of numerical algorithms hold for the standard and not for the general eigenproblem. Therefore, it is useful to consider various methods for converting the general eigenproblem to the standard eigenproblem. The purpose of this subsection is to illustrate how this is done.

Since [B] is positive definite and symmetric, its inverse exists so the general eigenproblem

$$[A]\{e\} = \lambda[B]\{e] \tag{4.4-10}$$

can be translated to the following equivalent problem:

$$[B]^{-1}[A]\{e\} = \lambda\{e\} \tag{4.4-11}$$

However, $[B]^{-1}[A]$ is not symmetric, so a standard eigenproblem algorithm based on symmetry could not be used even after the work has been performed to invert [B]. Nevertheless, (4.4-11) must yield the real, actual eigenvalues and eigenvectors of (4.4-10).

An alternative approach is to first obtain the eigensystem of [B] and to then determine $[B]^{1/2}$. Rewrite the general eigenproblem of (4.4-10) by inserting $[I] = [B]^{-1/2}[B]^{1/2}$ on the left and using $[B] = [B]^{1/2}[B]^{1/2}$ on the right:

$$[A][B]^{-1/2}[B]^{1/2}\{e\} = \lambda[B]^{1/2}[B]^{1/2}\{e] \tag{4.4-12}$$

Define a scaled vector $\{e^*\}$ by

$$\{e^*\} = [B]^{1/2}\{e\} \tag{4.4-13}$$

Multiply (4.4-12) on the left by $[B]^{-1/2}$ to obtain

$$[A^*]\{e^*\} = \lambda\{e^*\} \qquad \text{where} \qquad [A^*] = [B]^{-1/2}[A][B]^{-1/2} \tag{4.4-14}$$

which is a standard eigenproblem, and [A*] is symmetric. We have obtained the result that if an eigenpair of (4.4-14) is $(\lambda_i, \{e^*\}_i)$, then the corresponding eigenpair of the general eigenproblem is $(\lambda_i, [B]^{-1/2}\{e^*\}_i)$.

An approach that doesn't require obtaining the matrix $[B]^{-1/2}$ is to perform a Cholesky factorization of [B]:

$$[B] = [G][G]^T \tag{4.4-15}$$

in which [G] is lower triangular. With the insertion of $[I] = [G]^{-T}[G]^T$ on the left side of (4.4-10), the general eigenproblem becomes

$$[A][G]^{-T}[G]^T\{e\} = \lambda[G][G]^T\{e\} \tag{4.4-16}$$

Now, multiply on the left by $[G]^{-1}$. Then (4.4-16) becomes

$$[A^{**}]\{e^{**}\} = \lambda\{e^{**}\} \tag{4.4-17}$$

where

$$\{e^{**}\} = [G]^T\{e\} \quad \text{and} \quad [A^{**}] = [G]^{-1}[A][G]^{-T} \tag{4.4-18}$$

and [A**] is symmetric. With the solution to the standard eigenproblem of (4.4-17) denoted by the eigenpair $(\lambda_i, \{e^{**}\}_i)$, the eigensystem for the general eigenproblem of (4.4-10) displays the same eigenvalues. However, the eigenvectors, $\{e\}_i$, are obtained by solving the first of (4.4-18), i.e.,

$$\{e^{**}\}_i = [G]^T\{e\}_i \tag{4.4-19}$$

This process requires only back substitution because $[G]^T$ is upper triangular.

A procedure which is numerically superior to both of the above approaches is that of the QZ algorithm in which [A] and [B] are reduced simultaneously to upper triangular form [Heath, 1997; p. 140]. If the triangular forms of [A] and [B] are $[T]^A$ and $[T]^B$, respectively, then the eigenvalues are simply $T_{ii}^A / T_{ii}^B$, i = 1, ..., n provided $T_{ii}^B \neq 0$.

## 4.5 NORMS

### 4.5.1 Vector Norms

For many engineering problems, several approximate solution vectors to linear algebraic problems are often available. These solutions might be obtained as a result of using various mesh sizes for a finite difference or finite element discretization of a problem. Normally one wishes to know if one solution is better than another, or how the solution improves with mesh refinement. This requirement leads to a need to compare vectors. Because there may be hundreds or even thousands of components, a comparison of all components is impossible. It is particularly convenient if a single positive number can be defined as the magnitude of the vector. With an appropriate theorem, a "best" vector can be selected based on a comparison of magnitudes. Definitions of magnitudes, which are part of a well-established mathematical structure, are called **norms**. A **vector norm** of $\{x\}$ is denoted by $\|\{x\}\|$ and satisfies the following conditions:

(a) $\|\{x\}\| > 0$ for $\{x\} \neq \{0\}$; $\|\{x\}\| = 0$ implies $\{x\} = \{0\}$.
(b) $\|k\{x\}\| = |k|\,\|\{x\}\|$ for any scalar k.
(c) $\|\{x\} + \{y\}\| \leq \|\{x\}\| + \|\{y\}\|$ (Triangle Inequality).

A class of norms that is widely used is the **p-norm**:

$$\|\{x\}\|_p = (|x_1|^p + |x_2|^p + \cdots + |x_n|^p)^{1/p}, \qquad p \geq 1 \tag{4.5-1}$$

Special cases of the p-norm are:

p = 1 (one-norm):

$$\|\{x\}\|_1 = |x_1| + |x_2| + \cdots + |x_n| \tag{4.5-2}$$

p = 2 (two-norm):

$$\|\{x\}\|_2 = [(x_1)^2 + (x_2)^2 + \cdots + (x_n)^2]^{1/2} = [< x > \{x\}]^{1/2} \tag{4.5-3}$$

p = $\infty$ (infinity-norm):

$$\|\{x\}\|_\infty = \max_i |x_i| \tag{4.5-4}$$

The **Hölder Inequality** always holds:

$$|< x > \{y\}| \leq \|\{x\}\|_p \|\{y\}\|_q, \qquad \frac{1}{p} + \frac{1}{q} = 1 \tag{4.5-5}$$

of which a special case is the **Cauchy-Schwartz Inequality**:

$$|\langle x \rangle \{y\}| \le \|\{x\}\|_2 \|\{y\}\|_2 \tag{4.5-6}$$

All norms are equivalent in the sense that there exist positive constants $c_1$ and $c_2$ such that

$$c_1 \|\{x\}\|_\alpha \le \|\{x\}\|_\beta \le c_2 \|\{x\}\|_\alpha \tag{4.5-7}$$

Specific examples are the following:

$$\begin{aligned}
&\|\{x\}\|_\infty \le \|\{x\}\|_1 \le n\|\{x\}\|_\infty , && \frac{1}{\sqrt{n}}\|\{x\}\|_2 \le \|\{x\}\|_\infty \le \|\{x\}\|_2 \\
&\|\{x\}\|_\infty \le \|\{x\}\|_2 \le \sqrt{n}\|\{x\}\|_\infty , && \frac{1}{n}\|\{x\}\|_1 \le \|\{x\}\|_\infty \le \|\{x\}\|_1 \\
&\frac{1}{\sqrt{n}}\|\{x\}\|_\alpha \le \|\{x\}\|_\beta \le \|\{x\}\|_1 , && \|\{x\}\|_2 \le \|\{x\}\|_1 \le \sqrt{n}\|\{x\}\|_2
\end{aligned} \tag{4.5-8}$$

where n is the size of the vector.

### 4.5.2 Matrix Norms

A **matrix norm** of [A] is denoted by $\|[A]\|$ and satisfies the following relations:

(a) $\|[A]\| > 0$ for $\|[A]\| \neq [0]$; $\|[A]\| = 0$ implies that [A] = [0].
(b) $\|k[A]\| = |k| \|[A]\|$ for any scalar k.
(c) $\|[A] + [B]\| \le \|[A]\| + \|[B]\|$. (Triangle Inequality)
(d) $\|[A][B]\| \le \|[A]\| \|[B]\|$.

Condition (d) is the analogue to the Cauchy-Schwartz Inequality. A matrix norm is **compatible** (**subordinate**) with a vector norm if, in addition,

(e) $\|[A]\{x\}\| \le \|[A]\| \|\{x\}\|$.

A **natural norm** (**p - norm**) is one that satisfies the condition

$$\|[A]\|_p = \sup_{\{x\} \neq \{0\}} \frac{\|[A]\{x\}\|_p}{\|\{x\}\|_p} = \sup_{\|\{x\}\|_p = 1} \|[A]\{x\}\|_p \tag{4.5-9}$$

where **sup** denotes the **least upper bound**. Special cases are the following:
p = 1 (one - norm):

$$\|[A]\|_1 = \max_j \sum_{i=1}^{n} |A_{ij}| = \text{maximum absolute column sum.} \tag{4.5-10}$$

p = 2 (two - norm):

$$\|[A]\|_2 = (\text{maximum eigenvalue of } [A]^T[A])^{1/2}. \tag{4.5-11}$$

$p = \infty$ (infinity norm):

$$\|[A]\|_\infty = \max_i \sum_{j=1}^{n} |A_{ij}| = \text{maximum absolute row sum.} \tag{4.5-12}$$

The matrix p-norms are compatible with the vector p-norms. The two-norm is also called the **spectral norm**. On the other hand, the **spectral radius** $\rho([A])$ is defined to be

$$\rho([A]) = \max_i |\lambda_i| \tag{4.5-13}$$

which may differ from $\|[A]\|_2$ if [A] is not symmetric. In numerical analysis the **Frobenius norm** is frequently used:

$$\|[A]\|_F = \left(\sum_{i,j=1}^{n} |A_{ij}|^2\right)^{1/2} \tag{4.5-14}$$

A bound on eigenvalues is easily obtained using compatible norms. For example, recall the eigenproblem

$$[A]\{e\} = \lambda\{e\}$$

Then

$$\|[A]\{e\}\| = \|\lambda\{e\}\| = |\lambda| \|\{e\}\| \le \|[A]\| \|\{e\}\|$$

Therefore, if the eigenvector is normalized to unit length, we have

$$|\lambda| \le \|[A]\| \tag{4.5-15}$$

If [A] is positive, semi-definite, (4.5-15) states that any p - norm of [A] is greater than the maximum eigenvalue.

Similarly to the relationships among vector norms given by (4.5-8), there are specific relationships among the various matrix norms. Some examples are:

$$\left.\begin{array}{c} \|[A]\|_2 \le \|[A]\|_F \le \sqrt{n}\,\|[A]\|_2 \\ \|[A]\|_2 \le (\|[A]\|_1 \|[A]\|_\infty)^{1/2} \\ \frac{1}{\sqrt{n}}\|[A]\|_\infty \le \|[A]\|_2 \le \sqrt{n}\,\|[A]\|_\infty \\ \frac{1}{\sqrt{n}}\|[A]\|_1 \le \|[A]\|_2 \le \sqrt{n}\,\|[A]\|_1 \\ \max_{i,j} |A_{ij}| \le \|[A]\|_2 \le n \max_{i,j} |A_{ij}| \end{array}\right\} \tag{4.5-16}$$

Often, a rough estimate of a norm is sufficient for a particular application. If a bound is obtained by using norms that are easier to calculate, then these inequalities can be used to great advantage.

### 4.5.3 Weighted Norms

A norm used in connection with the conjugate gradient iterative method involves the use of a symmetric, positive definite matrix [W]. Suppose $\{y\} = [W]^{-1/2}\{x\}$. Then the p-norm of {y} is a weighted norm of {x}:

$$\|\{x\}\|_{p[w]^{-1}} = \|\{y\}\|_p = \|\{x\}[W]^{-1/2}\|_p \tag{4.5-17}$$

In particular, the weighted 2-norm is

$$\|\{x\}\|_{2[w]^{-1}} = (< x > [W]^{-1}\{x\})^{1/2} \tag{4.5-18}$$

The weighted matrix norm compatible with the weighted vector norm is

$$\|[A]\|_{p[w]^{-1}} = \|[A][W]^{1/2}\|_p \tag{4.5-19}$$

and as an example, the 2-norm is

$$\|[A]\|_{2[w]^{-1}} = \text{max. eigenvalue of } ([W]^{1/2}[A]^T[A][W]^{1/2})^{1/2} \tag{4.5-19}$$

This is the particular norm used most often.

### 4.5.4 Scaled P-Norms

In connection with error analysis involving the finite difference method, it is useful to utilize a scaled p-norm of vectors defined as follows:

$$\|\{v\}\|_{sp} = \frac{1}{n^{1/p}}\|\{v\}\|_p = \frac{1}{n^{1/p}}\left[\sum_{i=1}^{n} |v_i|^p\right]^{1/p}, \qquad p \geq 1 \tag{4.5-20}$$

The scaled p-norm satisfies the rules for a norm given in Subsection 4.4. Special cases are the scaled one, two and infinity norms:

$$\left.\begin{aligned} \|\{v\}\|_{s1} &= \frac{1}{n}\sum_{i=1}^{n}|v_i| = \frac{1}{n}\|\{v\}\|_1 \\ \|\{v\}\|_{s2} &= \frac{1}{\sqrt{n}}\left[\sum_{i=1}^{n}|v_i|^2\right]^{1/2} = \frac{1}{\sqrt{n}}\|\{v\}\|_2 \\ \|\{v\}\|_{s\infty} &= \max_i|v_i| = \|\{v\}\|_\infty \end{aligned}\right\} \tag{4.5-21}$$

One nice property of the scaled p-norm follows from (4.5-8); namely, the norm increases with p so that, for example,

$$\|\{v\}\|_{s1} \le \|\{v\}\|_{s2} \le \|\{v\}\|_{s\infty} \tag{4.5-22}$$

Recall that a compatible matrix norm is defined such that

$$\|[A]\| = \sup_{\{v\}\neq 0}\frac{\|[A]\{v\}\|}{\{v\}} \tag{4.5-23}$$

The scaling factor drops out so the p-norm of a matrix is also compatible with the scaled p-norm of a vector.

## 4.6 CONDITION OF LINEAR EQUATIONS

Suppose $\{x\}$ is the solution to $[A]\{x\} = \{b\}$. $[A]$ and $\{b\}$ are often based on experimental data for a given problem and, therefore, may be subject to measurement errors so that a more accurate description of the problem is

$$[[A] + [\delta A]]\{\{x\} + \{\delta x\}\} = \{b\} + \{\delta b\} \tag{4.6-1}$$

In numerical analysis, whenever small changes in the data can lead to large changes in the solution, the problem is said to be **ill-conditioned**. Otherwise, it is **well-conditioned**.

**Theorem** (without proof): Suppose

$$[A]\{x\} = \{b\} \quad \text{and} \quad [[A] + [\delta A]]\{\{x\} + \{\delta x\}\} = \{b\} + \{\delta b\}$$

Then

$$\frac{\|\{\delta x\}\|}{\|\{x\}\|} \le c([A])M\left(\frac{\|\{\delta b\}\|}{\|\{b\}\|} + \frac{\|[\delta A]\|}{\|[A]\|}\right) \tag{4.6-2}$$

where

$$M = \frac{1}{1 - \|[\delta A][A]^{-1}\|} \tag{4.6-3}$$

and

$$c([A]) = \|[A]\|\|[A]^{-1}\| \tag{4.6-4}$$

The positive number, c([A]), is called the **condition number** of [A].

The implication of this theorem is as follows. If [δA] is small enough, M will be close to unity. Then, if c is also small, minute changes {δb} and [δA] will limit the size of {x}. However, if c is large, small changes in {b} and [A] may cause large changes in {x}. Note the implication of the inequality. A large condition number, c, does not indicate that the solution to [A]{x} = {b} is ill-conditioned for every {b}; rather it implies that the problem is ill-conditioned for some {b}.

Suppose [A] is symmetric, positive definite. If the two-norm, or spectral norm, is used, then

$$\|[A]\|_2 = \lambda_n \qquad \text{and} \qquad \|[A]^{-1}\|_2 = \frac{1}{\lambda_1} \tag{4.6-5}$$

and (4.6-4) yields

$$c([A]) = \frac{\lambda_n}{\lambda_1} \tag{4.6-6}$$

i.e., c([A]) is the ratio of the largest to the smallest eigenvalues. Systems with large ratios are sometimes said to be **stiff**. The addition of more points in a finite element or finite difference mesh will generally increase the condition number.

This theorem also provides insight into the manner in which boundary conditions can affect the stiffness and condition number of a linear algebraic problem. Boundary conditions can be viewed as constraints on a system. Adding constraints decreases the condition number. Without the use of boundary conditions, a matrix is often ill-conditioned ($\lambda_1 = 0$). A constraint can only raise (or leave unchanged) the lowest eigenvalue and lower (or leave unchanged) the highest eigenvalue based on the eigenvalue separation property. As will be discussed in detail later, one method for applying a constraint is through the **penalty method**. In effect, the penalty method for adding a constraint removes a singularity ($\lambda_1 = 0$) by adding a mode with a large eigenvalue, $p_T$, which is called the "penalty" term. This procedure changes the condition number to $c = p_T/\lambda_2$. It is now easy to see that there is a dilemma. A large value for $p_T$ is desired in order to accurately satisfy the constraint; however, too large a value produces a large condition number, which increases the possibility of numerical error.

## 4.7 DECOMPOSITIONS

### 4.7.1 Singular Value Decomposition

**Theorem** (without proof): Suppose $[A] \in R^{m \times n}$. Then there exist orthogonal matrices, [U] and [V], defined by

$$\left.\begin{aligned} [U] &= [\{U\}_1 \{U\}_2 \cdots \{U\}_m]_{m \times m} \\ [V] &= [\{V\}_1 \{V\}_2 \cdots \{V\}_n]_{n \times n} \end{aligned}\right\} \tag{4.7-1}$$

such that

$$[U]^T[A][V] = \begin{bmatrix} [\Sigma] & [0] \\ [0] & [0] \end{bmatrix}, \qquad [\Sigma] = \begin{bmatrix} \Sigma_{11} & & & 0 \\ & \Sigma_{22} & & \\ & & \ddots & \\ 0 & & & \Sigma_{pp} \end{bmatrix} \tag{4.7-2}$$

where $[\Sigma]$ is a diagonal matrix with

$$0 \le \Sigma_{11} \le \Sigma_{22} \le \cdots \le \Sigma_{pp}\ , \qquad p = \min(m, n) \tag{4.7-3}$$

The **singular value decomposition** of [A] is the relationship given in the equation (4.7-2). By definition, the $\Sigma_{ii}$ are the **singular values** of [A]. $\{U\}_i$ and $\{V\}_i$ are the i'th **left singular vector** and **right singular vector**, respectively, of [A], which satisfy the following equations:

$$\left.\begin{aligned} [A]\{V\}_i &= \Sigma_{ii}\{U\}_i \\ < U >_i [A] &= \Sigma_{ii} < V >_i \end{aligned}\right\} \tag{4.7-4}$$

If there are r nonzero $\Sigma_{ii}$'s, then r is the rank of [A]. The range of [A] becomes $R$[A] = span $\{\{U\}_1 \cdots \{U\}_r\}$, and $N$[A] = span $\{\{V\}_{r+1} \cdots \{V\}_n\}$ is the null space. Let

$$[U]_r = [\{U\}_1 \cdots \{U\}_r], \qquad [V]_r = [\{V\}_1 \cdots \{V\}_r]$$
$$[\Sigma]_r = \begin{bmatrix} \Sigma_{11} & & 0 \\ & \ddots & \\ 0 & & \Sigma_{rr} \end{bmatrix} \tag{4.7-5}$$

Then the spectral decomposition of [A] becomes

$$[A] = [U]_r[\Sigma]_r[V]^T = \sum_{i=1}^{r} \Sigma_{ii}\{U\}_i < V >_i \tag{4.7-6}$$

An equivalent development by Noble [1969; page 336] is given next. If [A] is a general m x n matrix, $\Sigma$ is a nonzero number, and {U} and {V} are vectors such that

$$[A]\{U\} = \Sigma\{V\} \quad \text{and} \quad [A]^T\{V\} = \Sigma\{U\} \tag{4.7-7}$$

then $\Sigma$ is called a singular value and ($\{U\}$, $\{V\}$) are a pair of singular vectors of [A]. An alternative form for (4.7-7) is

$$\begin{bmatrix} [0] & [A] \\ [A]^T & [0] \end{bmatrix} \begin{Bmatrix} \{V\} \\ \{U\} \end{Bmatrix} = \Sigma \begin{Bmatrix} \{V\} \\ \{U\} \end{Bmatrix} \tag{4.7-8}$$

so that $\Sigma$ is an eigenvalue of a symmetric matrix and, therefore, is real. Multiply the first of (4.7-7) with $[A]^T$, and use the second of (4.7-7) to obtain

$$[A]^T[A]\{U\} = \Sigma[A]^T\{V\} = \Sigma^2\{U\} \tag{4.7-9}$$

Therefore, $\{U\}$ is an eigenvector of $[A]^T[A]$ with eigenvalue $\Sigma^2$. Similarly, by multiplying the second of (4.7-7) with [A] and using the first of (4.7-7) it follows that $\{V\}$ is an eigenvector of $[A][A]^T$, also with eigenvalue $\Sigma^2$.

### 4.7.2 Polar Decomposition

Suppose $[A] \in R^{m\,x\,n}$. Then the **right polar decomposition** and **left polar decomposition** of [A] are

$$[A] = [Q][R] = [L][Q] \tag{4.7-10}$$

in which $[L] \in R^{m\,x\,m}$ and $[R] \in R^{n\,x\,n}$ are symmetric, positive semi-definite matrices, and $[Q] \in R^{m\,x\,n}$ is orthogonal. Since [Q] is not square, orthogonality in this context implies

$$[Q][Q]^T = [I]_{mxm}, \qquad [Q]^T[Q] = [I]_{nxn} \tag{4.7-11}$$

The decomposition can be constructed by considering

$$[A]^T[A] = [R]^T[Q]^T[Q][R] = [R]^T[R] = [R]^2 \tag{4.7-12}$$

in which the last relation holds because [R] is symmetric. But $[A]^T[A]$ is symmetric. It is also positive semi-definite because the inner product $\langle x\rangle[A]^T[A]\{x\} = \langle y\rangle\{y\} > 0$ unless $\{y\} = \{0\}$ where $\{y\} = [A]\{x\}$. Therefore,

$$[R] = \left[[A]^T[A]\right]^{1/2} \quad \text{and} \quad [Q] = [A][R]^{-1} \tag{4.7-13}$$

where the second equation holds because $[A]^T[A]$ is positive definite and, therefore, invertible. A similar construction holds for the left decomposition. These decompositions are widely used in continuum mechanics.

## 4.8 THE MODAL METHOD FOR SOLVING THE ALGEBRAIC PROBLEM

### 4.8.1 The Complete Modal Method

Consider the problem $[A]\{x\} = \{b\}$ and suppose the eigensystem of [A] has been determined, i. e., the modal matrix, [M], and the diagonal eigenvalue matrix, $[\Lambda]$, exist. Then, the spectral decomposition of [A] is

$$[A] = [M][\Lambda][M]^T \tag{4.8-1}$$

Since [A] is orthogonal, it follows immediately that the solution to the algebraic problem is

$$\{x\} = [M][\Lambda]^{-1}[M]^T\{b\} \tag{4.8-2}$$

An alternative way of viewing (4.8-2) is to utilize the fact that the eigenvectors of [A] span the column space of [A] so that {x} and {b} can be represented as follows:

$$\{x\} = \sum_{i=1}^{n} \alpha_i \{e\}_i \qquad \text{and} \qquad \{b\} = \sum_{i=1}^{n} \beta_i \{e\}_i \tag{4.8-3}$$

The vector {b} is known so the coefficients, $\beta_i$, can be determined by premultiplying the second equation of (4.8-3) by $\langle e \rangle_j$ to obtain

$$\langle e \rangle_j \{b\} = \langle e \rangle_j \sum_{i=1}^{n} \beta_i \{e\}_i \tag{4.8-4}$$

Use the orthormal relationship $\langle e \rangle_j \{e\}_i = \delta_{ij}$ to obtain

$$\beta_i = \langle e \rangle_i \{b\} \tag{4.8-5}$$

Now, with the coefficients $\beta_i$ considered known, return to the algebraic problem, $[A]\{x\} = \{b\}$, which becomes

$$\sum_{i=1}^{n} \alpha_i \lambda_i \{e\}_i = \sum_{i=1}^{n} \beta_i \{e\}_i \tag{4.8-6}$$

As before, premultiply by $\langle e \rangle_j$ and use $\langle e \rangle_j \{e\}_i = \delta_{ij}$ to obtain

$$\alpha_i = \frac{\beta_i}{\lambda_i} \tag{4.8-7}$$

Again, no solution exists if $\lambda_i = 0$. With all components, $\alpha_i$, determined in this manner (**modal method**), the solution {x} is obtained from the first of (4.8-3).

### 4.8.2 The Approximate Modal Method

Suppose there is reason to believe that the force vector, {b}, consists primarily of contributions from the first m modes, i. e., $\beta_i = 0$ for $i > m$. Then $\alpha_i = 0$ for $i > m$ and there is no need to compute the eigenpairs for $i > m$. In practice, m is estimated based on prior experience and only the first m eigenpairs are evaluted. For a model consisting of thousands of degrees of freedom, $m < 100$ may be sufficient. With the first m eigenpairs considered known, construct the truncated modal matrix, $[M]^{(m)}$, and truncated diagonal matrix of eigenvalues, $[\Lambda]^{(m)}$, as follows:

$$[M]^{(m)} = [\{e\}_1 \{e\}_2 \cdots \{e\}_m]_{n \times m}, \qquad [\Lambda]^{(m)} = \begin{bmatrix} \Lambda_1 & & 0 \\ & \ddots & \\ 0 & & \Lambda_m \end{bmatrix}_{m \times m} \tag{4.8-8}$$

Then, in anology to (4.8-2), an approximate solution is defined to be

$$\{x\}^{(m)} = [M]^{(m)} [\Lambda]^{(m)^{-1}} [M]^{(m)^T} \{b\} \tag{4.8-9}$$

which is the equation generally associated with what is called the "modal method" but is more appropriately called the **approximate modal method**. Again, the superscript m merely denotes the number of modes or eigenpairs evaluated from the original matrix. Any eigenpair can be included, not just those identified with the lowest eigenvalues.

The approach is used in several general-purpose elastic finite-element codes, primarily in connection with forcing functions associated with seismic disturbances. The method is particularly efficient if $m << n$ and the solution to a large number of forcing vectors is required.

The fact is that m is never known with certainty ahead of time so there is a need for a criterion to determine when enough modes have been utilized in the approximate modal method. One criterion that is often used is to increase m until

$$\frac{\left\| \{x\}^{(m)} - \{x\}^{(m-1)} \right\|}{\left\| \{x\}^{(m)} \right\|} < \varepsilon \tag{4.8-10}$$

where $\varepsilon$ is a small positive number. This measure of error can result in considerable solution error whenever the force vector contains a significant contribution in a higher mode. For example, $\beta_m = 0$ will allow (4.8-10) to be satisfied, but if $\beta_{m+1}$ is large then this part will not be reflected in the solution. A preferable approach is to ensure that the most significant part of {b} is reflected in the solution by performing the check:

$$\frac{\|\{b\} - \{b\}^{(m)}\|}{\|\{b\}\|} < \varepsilon \tag{4.8-11}$$

where

$$\{b\}^{(m)} = \sum_{i=1}^{m} \beta_i \{e\}_i , \qquad \beta_i = < e >_i \{b\} \tag{4.8-12}$$

With the use of the truncated modal matrix, $[M]^{(m)}$, (4.8-12) can be written directly as

$$\{b\}^{(m)} = [M]^{(m)} [M]^{(m)^T} \{b\} \tag{4.8-13}$$

Now, modes can be added in a systematic manner until (4.8-11) is satisfied for a user-prescribed error parameter, $\varepsilon$.

## 4.9 ITERATIVE METHODS FOR SOLVING THE ALGEBRAIC PROBLEM

The previous chapter is primarily concerned with the linear algebraic problem and this chapter presents various aspects of the eigenproblem. Therefore, a subsection on iterative methods for solving the algebraic problem may, at first glance, seem out of place. However, any discussion on why iterative methods work relies heavily on a knowledge of the eigenproblem. With the background of the previous part of this chapter, a presentation of basic iterative procedures can be done in a much more logical manner than if the method was presented in the previous chapter.

Suppose [A] is split (rather than decomposed) in the form

$$[A] = [E] - [F] \tag{4.9-1}$$

Then $[A]\{x\} = \{b\}$ can be written

$$[E]\{x\} = [F]\{x\} + \{b\} \tag{4.9-2}$$

An iterative procedure is defined as follows:

$$[E]\{x\}^{\{r+1\}} = [F]\{x\}^{\{r\}} + \{b\} \tag{4.9-3}$$

where $\{x\}^{(r)}$ is the approximation to $\{x\}$ at the r'th stage. The initial approximation, $\{x\}^{(0)}$, is chosen arbitrarily. The matrix, [E], must be nonsingular and chosen to be of a form amenable to a straightforward solution of

$$[E]\{x\} = \{g\} \tag{4.9-4}$$

Instead of working in terms of $\{x\}^{(r)}$, it is often more convenient to work in terms of the corrections

$$\{\delta\}^{(r+1)} = \{x\}^{(r+1)} - \{x\}^{(r)} \tag{4.9-5}$$

To illustrate how the procedure works, consider two consecutive iterates:

$$[E]\{x\}^{(r+1)} = [F]\{x\}^{(r)} + \{b\}\ , \qquad [E]\{x\}^{(r)} = [F]\{x\}^{(r-1)} + \{b\} \tag{4.9-6}$$

Subtract corresponding terms to obtain

$$[E]\{\delta\}^{(r+1)} = [F]\{\delta\}^{(r)} \tag{4.9-7}$$

To obtain approximate solutions, $\{x\}^{(0)}$ is assumed and $\{x\}^{(1)}$ is obtained from (4.9-3). Then,

$$\{\delta\}^{(0)} = \{x\}^{(0)}\ , \qquad \{\delta\}^{(1)} = \{x\}^{(1)} - \{x\}^{(0)} \tag{4.9-8}$$

Now, (4.9-7) is used sequentially together with the following summation:

$$\{x\}^{(r+1)} = \{x\}^{(0)} + \{\delta\}^{(1)} + \cdots + \{\delta\}^{(r+1)} = \sum_{i=0}^{r+1} \{\delta\}^{(i)} \tag{4.9-9}$$

If the procedure is to work, the difference in consecutive iterates $\{\delta\}^{(r)}$ must approach zero for large r. The specific condition for a satisfactory algorithm is given in the following theorem.

**Theorem**: The iterative procedure $[E]\{x\}^{(r+1)} = [F]\{x\}^{(r)} + \{b\}$ yields an approximate sequence $\{x\}^{(r)}$ that converges to the solution if and only if the eigenvalues $\lambda$ of

$$[H] = [E]^{-1}[F] \tag{4.9-10}$$

satisfy $|\lambda| < 1$.

**Proof**: The correction at the r'th step is

$$\{\delta\}^{(r)} = [H]\{\delta\}^{(r-1)} = [H][H]\{\delta\}^{(r-2)} = \cdots = [H]^r\{\delta\}^{(0)} \tag{4.9-11}$$

Use the spectral decomposition formula of (4.3-10) to obtain

$$\{\delta\}^{(r)} = [M][\Lambda]^r[M]^{-1}\{\delta\}^{(0)} \tag{4.9-12}$$

in which [M] is the modal matrix whose columns are the right eigenvectors of [H], which may not be a symmetric matrix. $[\Lambda]$ is a diagonal matrix containing the eigenvalues of [H]. An alternative form of (4.9-12) is

$$[M]^{-1}\{\delta\}^{(r)} = [\Lambda]^r[M]^{-1}\{\delta\}^{(0)} \tag{4.9-13}$$

Let

$$\{\delta^*\} = [M]^{-1}\{\delta\} \tag{4.9-14}$$

Then the uncoupled form of the iterative procedure given in (4.9-12) is

$$\{\delta^*\}^{(r)} = [\Lambda]^r\{\delta^*\}^{(0)} \tag{4.9-15}$$

It follows that $\{\delta^*\}^{(r)} \rightarrow 0$ if and only if $|\lambda| < 1$. **QED**

**Note:** In general, [H] is not a symmetric matrix, even if the original matrix [A] is symmetric so the eigenvalues of [H] may be complex. Then the absolute values of eigenvalues in the theorem must be interpreted as the magnitudes of complex numbers, if necessary.

The proof shows that the smaller the value of $\max|\lambda|$, the faster the procedure converges. The matrix, [H], is called the **iterative matrix**. If the spectral radius of [H], defined to be the maximum absolute value of the eigenvalue, is less than one, then [H] is said to be **convergent**.

For the rare situation when both [E] and [F] are symmetric, and [E] is positive definite, an alternative approach can be used to prove the theorem with results from the following general eigenproblem:

$$[\,[F] - \lambda[E]\,]\{e\} = \{0\} \tag{4.9-16}$$

Let the eigenpairs be $(\lambda_i, \{e\}_i)$, the modal matrix be [M], and the diagonal matrix of eigenvalues be [Λ]. Then

$$[M]^T[E][M] = [I] \quad \text{and} \quad [M]^T[F][M] = [\Lambda] \tag{4.9-17}$$

so that the iterative form of (4.9-7) becomes

$$[M][M]^T\{\delta\}^{(r+1)} = [M][\Lambda][M]^T\{\delta\}^{(r)} \tag{4.9-18}$$

Let $\{\delta^*\}^{(r)} = [M]^T\{\delta\}^{(r)}$ and multiply on the left by $[M]^T$. The result is

$$\{\delta^*\}^{(r+1)} = [\Lambda]\{\delta^*\}^{(r)} = \cdots = [\Lambda]^r\{\delta^*\}^{(0)} \tag{4.9-19}$$

and, as before, the iterates approach zero if $|\lambda_i| < 1$ for all i.

This general eigenproblem approach for the analysis might be more convenient if [E] and [F] are symmetric matrices. This would be the case if [E] is chosen to be the diagonal or tridiagonal part of an original symmetric matrix [A]. For example, let

$$[E] = [A]_D \quad \text{and} \quad [F] = -[A] + [A]_D \tag{4.9-20}$$

in which $[A]_D$ is the diagonal part of [A]. Then the iterative procedure is **Jacobi's Method**. On the other hand, if

$$[E] = [A]_{LT} \quad \text{and} \quad [F] = -[A] + [A]_{LT} \tag{4.9-21}$$

in which $[A]_{LT}$ is the lower triangular part of [A] including the diagonal terms, then the result is called the **Gauss-Seidel Method**. [E] is not symmetric and there is the possibility that the eigenvalues will be complex.

An alternative way to describe these two methods is to let

$$[A] = [L] + [D] + [U] \tag{4.9-22}$$

where [D] is now the diagonal part of [A], and [L] and [U] are the lower and upper triangular parts, respectively, of [A] with zeros on the diagonal. Then the two iterative procedures become

$$\left.\begin{aligned} [D]\{x\}^{(r+1)} &= -\big[[L]+[U]\big]\{x\}^{\{r\}} + \{b\} \qquad \text{(Jacobi)} \\ \big[[L]+[D]\big]\{x\}^{(r+1)} &= -[U]\{x\}^{\{r\}} + \{b\} \qquad \text{(Gauss - Seidel)} \end{aligned}\right\} \tag{4.9-23}$$

Introduce a parameter, p, which is a prescribed constant. Then the **Accelerated Gauss-Seidel** or **Successive Over-Relaxation (SOR)** method is

$$\big[[L] + p[D]\big]\{x\}^{(r+1)} = -\big[(1 - p)[D] + [U]\big]\{x\}^{(r)} + \{b\} \tag{4.9-24}$$

An alternative way of writing (4.9-24) is to introduce an **over-relaxation factor** $\omega = 1/p$:

$$\{x\}^{(r+1)} = \{x\}^{(r)} + \omega[D]^{-1}\Big\{-[L]\{x\}^{(r+1)} - \big[[D] + [U]\big]\{x\}^{(r)} + \{b\}\Big\} \tag{4.9-25}$$

The interpretation obtained by writing the equation in this form is that $\{x\}^{(r+1)}$ is obtained by adding to $\{x\}^{(r)}$ a multiple of a scaled residual or error. The optimal value of $\omega$ for the most rapid convergence is greater than one and this is the reason for the adjective "over-relaxation." At first it seems unusual to write the unknown vector, $\{x\}^{(r+1)}$, on the right side. However, the coefficient of $\{x\}^{(r+1)}$ is the lower triangular matrix, [L], so that only updated components of $\{x\}^{(r+1)}$ are used.

There are also different ways to arrive at the SOR Method. One alternative approach is to start with the Gauss--Seidel procedure, which is given by $\big[[L]+[D]\big]\{x\}^{(r+1)} = -[U]\{x\}^{(r)}+\{b\}$, and to rewrite the equation as follows:

$$[D]\{x\}^{(r+1)} = -[L]\{x\}^{(r+1)} - [U]\{x\}^{(r)} + \{b\}$$

By inverting the matrix, [D], we obtain

$$\begin{aligned} \{x\}^{(r+1)} &= [D]^{-1}\{-[L]\{x\}^{(r+1)} - [U]\{x\}^{(r)} + \{b\}\} \\ &= \{x\}^{(r)} + [D]^{-1}\Big\{-[L]\{x\}^{(r+1)} - \big[[D] + [U]\big]\{x\}^{(r)} + \{b\}\Big\} \end{aligned}$$

The SOR method is obtained as the following modification:

$$\{x\}^{(r+1)} = \{x\}^{(r)} + \omega[D]^{-1}\left\{-[L]\{x\}^{(r+1)} - \left[[D] + [U]\right]\{x\}^{(r)} + \{b\}\right\} \tag{4.9-26}$$

Yet another way is to decompose [A] as follows:

$$[A] = [L] + p[D] + (1-p)[D] + [U] \tag{4.9-27}$$

and let

$$[E] = [L] + p[D] \tag{4.9-28}$$

Then (4.9-3) becomes

$$\left[[L] + p[D]\right]\{x\}^{(r+1)} = -\left[(1-p)[D] + [U]\right]\{x\}^{(r)} + \{b\} \tag{4.9-29}$$

Move the term involving [L] to the right side to obtain

$$p[D]\{x\}^{(r+1)} = -[L]\{x\}^{(r+1)} - \left[(1-p)[D] + [U]\right]\{x\}^{(r)} + \{b\} \tag{4.9-30}$$

Again, inverting [D] yields

$$\{x\}^{(r+1)} = \omega[D]^{-1}\left\{-[L]\{x\}^{(r+1)} - \left[(1-p)[D] + [U]\right]\{x\}^{(r)} + \{b\}\right\}$$

The following form is more customary:

$$\{x\}^{(r+1)} = \{x\}^{(r)} - \omega[D]^{-1}\left\{[L]\{x\}^{(r+1)} + \left[[D] + [U]\right]\{x\}^{(r)} - \{b\}\right\} \tag{4.9-31}$$

Although the notation of (4.9-31) provides the matrix description of the SOR method, the essential simplicity is somewhat obscure. Suppose [D] is chosen to be the diagonal components of [A]. An iteration is performed by computing the components of {x} in which a typical component is merely replaced by the right side as indicated in the following expression:

$$x_i \leftarrow x_i - \frac{\omega}{A_{ii}}\left[\sum_{j=1}^{n} A_{ij}x_j - b_i\right] \tag{4.9-32}$$

The current values of $x_i$ are used as they become available. The Gauss-Seidel algorithm is obtained when $\omega = 1$.

Pertinent theorems are the following:

1. The Jacobi matrix and the Gauss-Seidel matrix are either both convergent, or both divergent [Varga, 1962; page 70] .
2. The SOR method is convergent if and only if [A] is positive definite and $0 < \omega < 2$ [Varga, 1962; page 77] .

## 4.10 ITERATIVE METHODS FOR OBTAINING EIGENPAIRS

### 4.10.1 Introductory Comments

Although the theory associated with the eigenproblem provides great insight into the structure of matrices, nothing has been said how eigenpairs are obtained. In most applications, of which the approximate modal method is one example, not all eigenpairs are required. Normally the eigenpairs associated with the lowest eigenvalues or the largest eigenvalues are the most useful. We are at the stage now where classical techniques for obtaining such pairs can be introduced. A procedure called **backward iteration** (or **inverse iteration**) together with Gram-Schmidt orthogonalization can be used to obtain the eigenpairs with the lowest eigenvalues. Alternatively, **forward iteration** with Gram-Schmidt orthogonalization is used to obtain the eigenpairs with the highest eigenvalues. These methods provide valuable insight to the general eigenproblem and are often useful for problems with a small number of degrees of freedom but, in general practice, more sophisticated techniques are more efficient [Hughes, 1987].

### 4.10.2 Backward or Inverse Iteration

Consider the general eigenproblem

$$[A]\{e\} = \lambda[B]\{e\} \tag{4.10-1}$$

Then the Rayleigh Quotient is

$$R(\{x\}) = \frac{< x > [A]\{x\}}{< x > [B]\{x\}} \tag{4.10-2}$$

Inverse iteration consists of assuming a starting vector, $\{x\}^1$, and performing the following iterations starting with k = 1. First, solve

$$[A]\{x^*\}^{(k+1)} = [B]\{x\}^{(k)} \tag{4.10-3}$$

for $\{x^*\}^{(k+1)}$, and then obtain

$$\{x\}^{(k+1)} = \frac{\{x^*\}^{(k+1)}}{\left(< x^* >^{(k+1)} [B]\{x^*\}^{(k+1)}\right)^{1/2}} \tag{4.10-4}$$

Then, $\{x\}^{(k+1)} \rightarrow \{e\}_1$ as $k \rightarrow \infty$ provided the initial guess, $\{x\}^{(1)}$, is not B-orthogonal to the first eigenvector, $\{e\}_1$, i.e., $< x >^{(1)} [B]\{e\}_1 \neq 0$. It is highly unlikely that an arbitrary initial guess will happen to meet the B-orthogonality condition. The approximation to the first eigenvalue, $\lambda_1$, is obtained by using the Rayleigh Quotient on $\{x\}^{(k+1)}$.

Numerically, the calculations are performed by assigning a small positive value to $\varepsilon$, a convergence parameter, assuming a vector $\{x\}^{(0)}$, obtaining $\{y\}^{(0)} = [B]\{x\}^{(0)}$, setting R = 0, and assigning the value k = 1 for a counter. The procedure follows:

1. Solve $[A]\{x\}^{(k)} = \{y\}^{(k-1)}$ for $\{x\}^{(k)}$.

2. Obtain $\{y^*\}^{(k)} = [B]\{x\}^{(k)}$.

3. Determine $D = \langle x \rangle^{(k)} \{y^*\}^{(k)}$.

4. Save the previous Rayleigh Quotient, $R_p = R$ and obtain the current Rayleigh Quotient $R = \left( \langle x \rangle^{(k)} \{y\}^{(k-1)} \right) / D$.

5. Set $\{y\}^{(k)} = \{y^*\}^{(k)} / \sqrt{D}$.

6. Evaluate the convergence parameter $\varepsilon^* = \left| R - R_p \right| / R$.

7. Check the convergence criterion $\varepsilon^* \leq \varepsilon$. If the criterion is not met, increment the counter, $k = k + 1$, and repeat the process starting with Step 1. If the criterion is met, then R and $\{x\}^{(k)}$ are the approximations to the lowest eigenvalue and the associated eigenvector, respectively.

The result is that $\{x\}^{(k)} \rightarrow \{e\}_1$ and $R \rightarrow \lambda_1$ as $k \rightarrow \infty$ provided $\langle y \rangle^{(0)} \{e\}_1 \neq 0$. Convergence for this method can be shown to be linear and the rate of convergence is $\lambda_1/\lambda_2$. Of course, many of the steps are much simpler for the standard eigenproblem for which [B] = [I].

### 4.10.3 Forward Iteration

Here the roles of [A] and [B] are interchanged from those of backward iteration to obtain an estimate for the largest eigenvalue and the corresponding eigenvector. Similarly to the previous case, we assign a small positive value to $\varepsilon$, a convergence parameter, choose a vector $\{y\}^{(0)}$ with the assumption that $\langle e \rangle_n \{y\}^{(0)} \neq 0$, set R = 0, and assign the value for the counter at k = 1. The procedure follows:

1. Solve $[B]\{x\}^{(k)} = \{y\}^{(k-1)}$ for $\{x\}^{(k)}$.

2. Obtain $\{y^*\}^{(k)} = [A]\{x\}^{(k)}$.

3. Determine $D = \langle x \rangle^{(k)} \{y\}^{(k-1)}$.

4. Save the previous Rayleigh Quotient, $R_p = R$ and obtain the current Rayleigh Quotient $R = \left( \langle x \rangle^{(k)} \{y^*\}^{(k)} \right) / D$.

5. Set $\{y\}^{(k)} = \left( \{y^*\}^{(k)} \right) / \sqrt{D}$.

6. Evaluate the convergence parameter $\varepsilon^* = \left| R - R_p \right| / R$.

7. Check the convergence criterion $\varepsilon^* \le \varepsilon$. If the criterion is not met, increment the counter, k = k + 1, and repeat the process starting with Step 1. If the criterion is met, then R and $\{x\}^{(k)}$ are the approximations to the largest eigenvalue and the associated eigenvector, respectively.

With this iterative sequence, $\{x\}^{(k)} \rightarrow \{e\}_n$ and $R \rightarrow \lambda_n$ as $k \rightarrow \infty$.

### 4.10.4 Gram-Schmidt Method for Other Eigenpairs

Suppose the first m eigenvalues and eigenvectors have been obtained by inverse iteration and the next eigenpair, $(\lambda_{m+1}, \{e\}_{m+1})$, is desired. Let $\{\hat{x}\}^{(1)}$ be the first guess for $\{e\}_{m+1}$. Define

$$\{x\}^{(1)} = \{\hat{x}\}^{(1)} - \sum_{i=1}^{m} \alpha_i \{e\}_i \tag{4.10-5}$$

By construction, $\{x\}^{(1)}$ is made orthogonal to the first m eigenvalues. Premultiply (4.10-5) by $<e>_j[B]$ to obtain

$$< e >_j [B]\{x\}^{(1)} = < e >_j [B]\{\hat{x}\}^{(1)} - \sum_{i=1}^{m} \alpha_i < e >_j [B]\{e\}_i \tag{4.10-6}$$

However, the eigenvalues are B-orthonormal, i.e.,

$$\sum_{i=1}^{m} \alpha_i < e >_j [B]\{e\}_i = \sum_{i=1}^{m} \alpha_i \delta_{ij} = \alpha_j \tag{4.10-7}$$

The left hand side of (4.10-6) should be zero. With j replaced by i, the result is

$$\alpha_i = < e >_i [B]\{\hat{x}\}^{(1)} \tag{4.10-8}$$

The use of (4.10-8) in (4.10-5) is an enforcement of orthogonality. Now, invoke the inverse iteration procedure together with orthogonality to the preceding modes with $\{y\}^{(0)}$ chosen arbitrarily and begin with k = 1:

1. Solve $[A]\{x\}^{(k)} = \{y\}^{(k-1)}$ for $\{x\}^{(k)}$.

2. Enforce orthogonality (perhaps not for every k).

3. Determine $\{y^*\}^{(k)} = [B]\{x\}^{(k)}$.

4. Obtain $D = <x>^{(k)}\{y^*\}^{(k)}$.

5. Evaluate $R = (< x >^{(k)} \{y\}^{(k-1)})/D$.

6. Obtain $\{y\}^k = \{y^*\}^{(k)}/\sqrt{D}$.

7. Check for convergence and either stop or return to 1 with k incremented.

Now $\{x\}^{(k)} \rightarrow \{e\}_{m+1}$ and $R \rightarrow \lambda_{m+1}$ as $k \rightarrow \infty$. A similar procedure using a combination of forward iteration and enforcing orthogonality can be invoked for obtaining eigenpairs associated with the largest eigenvalues.

## 4.11 EXAMPLE PROBLEMS

Here, we illustrate some of the concepts of the chapter with the use of small matrices that allow elementary solutions.

Example 1: Consider the following symmetric matrix:

$$[A] = \begin{bmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

(i) Determine the eigenpairs for the matrix.

Solution: First we develop the characteristic equation and solve for the eigenvalues. The characteristic equation is

$$\det\,[[A] - \lambda[I]] = \begin{vmatrix} 3-\lambda & -1 & 0 \\ -1 & 2-\lambda & -1 \\ 0 & -1 & 1-\lambda \end{vmatrix} = (\lambda - 2)(\lambda^2 - 4\lambda + 1) = 0$$

Solve the quadratic equation. The ordered roots are the eigenvalues: $\lambda_1 = 2 - \sqrt{3}$, $\lambda_2 = 2$, $\lambda_3 = 2 + \sqrt{3}$. To obtain the first eigenvector, solve the equation $[[A] - \lambda_1[I]]\{e^*\} = \{0\}$ for $\{e^*\}$. The algebraic equations become

$$\begin{aligned} (3-\lambda_1)e_1^* - e_2^* &= 0 \\ -e_1^* + (2-\lambda_1)e_2^* - e_3^* &= 0 \\ -e_2^* + (1-\lambda_1)e_3^* &= 0 \end{aligned}$$

If the second equation is multiplied by $(3 - \lambda_1)$ and the new equation added to the first, the result is identical to the third equation (show), i.e., there are only two independent equations. Because of this dependence, arbitrarily set $e_1^* = 1$. (If a contradiction should arise later in the process of obtaining a solution, the appropriate interpretation is that $e_1^* = 0$ and one should then try choosing $e_2^* = 1$, etc.) With $e_1^* = 1$, the first equation yields $e_2^* = (3 - \lambda_1)$ and the third equation is $e_3^* = (3 - \lambda_1)/(1 - \lambda_1)$. The result is $< e^* >=< 1,\ 1 + \sqrt{3},\ 1 - \sqrt{3} >$. Divide by the length of the vector to obtain the first normalized eigenvector. The first eigenpair is

$$\lambda_1 = 2 - \sqrt{3} \quad \text{and} \quad < e >_1 = \frac{1}{\alpha_1} < 1,\ 1 + \sqrt{3},\ 2 + \sqrt{3} > \quad \text{with} \quad \alpha_1^2 = 12 + 6\sqrt{3}$$

Now, we move on to obtain the second eigenvector in a similar manner. With $\lambda$ equal to the second eigenvalue, the algebraic equations are

$$\begin{aligned} (3 - \lambda_2)e_1^* - e_2^* &= 0 \\ -e_1^* + (2 - \lambda_2)e_2^* - e_3^* &= 0 \\ -e_2^* + (1 - \lambda_2)e_3^* &= 0 \end{aligned}$$

The solution yields the second eigenpair:

$$\lambda_2 = 2 \quad \text{and} \quad < e >_2 = \frac{1}{\sqrt{3}} < 1,\ 1,\ -1 >$$

When the process is repeated, the third eigenpair is obtained:

$$\lambda_3 = 2 + \sqrt{3} \quad \text{and} \quad < e >_3 = \frac{1}{\alpha_3} < 1,\ 1 - \sqrt{3},\ 2 - \sqrt{3} > \quad \text{with} \quad \alpha_3^2 = 12 - 6\sqrt{3}$$

As a check, we note that the eigenvectors are orthogonal and the sum of the eigenvalues equals the sum of the diagonal components of [A].

(ii) Construct the modal matrix [M and obtain $[M]^T[M]$ and $[M]^T[A][M]$.

Solution: The columns of the modal matrix are just the normalized eigenvectors:

$$[M] = \left[ \frac{1}{\alpha_1} \begin{Bmatrix} 1 \\ 1 + \sqrt{3} \\ 2 + \sqrt{3} \end{Bmatrix} \quad \frac{1}{\sqrt{3}} \begin{Bmatrix} 1 \\ 1 \\ -1 \end{Bmatrix} \quad \frac{1}{\alpha_2} \begin{Bmatrix} 1 \\ 1 - \sqrt{3} \\ 2 - \sqrt{3} \end{Bmatrix} \right]$$

The rows of $[M]^T$ are these same vectors:

$$[M]^T = \begin{bmatrix} \frac{1}{\alpha_1} < 1,\ 1+\sqrt{3},\ 2+\sqrt{3} > \\ \frac{1}{\sqrt{3}} < 1,\ 1,\ -1 > \\ \frac{1}{\alpha_3} < 1,\ 1-\sqrt{3},\ 2-\sqrt{3} > \end{bmatrix}$$

Matrix multiplication shows that $[M]^T[M] = [I]$, a result to be expected because of the orthonormal properties of the eigenvectors. To obtain the next result first perform the following multiplication:

$$[M]^T[A] = \begin{bmatrix} \frac{1}{\alpha_1}(2-\sqrt{3}) & \frac{-1}{\alpha_1}(1-\sqrt{3}) & \frac{1}{\alpha_1} \\ \frac{2}{\sqrt{3}} & \frac{2}{\sqrt{3}} & \frac{-2}{\sqrt{3}} \\ \frac{1}{\alpha_3}(2+\sqrt{3}) & \frac{-1}{\alpha_3}(1+\sqrt{3}) & \frac{1}{\alpha_3} \end{bmatrix}$$

Then, post multiplying by [M] yields

$$\begin{bmatrix} \frac{1}{\alpha_1}(2-\sqrt{3}) & \frac{-1}{\alpha_1}(1-\sqrt{3}) & \frac{1}{\alpha_1} \\ \frac{2}{\sqrt{3}} & \frac{2}{\sqrt{3}} & \frac{-2}{\sqrt{3}} \\ \frac{1}{\alpha_3}(2+\sqrt{3}) & \frac{-1}{\alpha_3}(1+\sqrt{3}) & \frac{1}{\alpha_3} \end{bmatrix} \begin{bmatrix} \frac{1}{\alpha_1}\begin{Bmatrix} 1 \\ 1+\sqrt{3} \\ 2+\sqrt{3} \end{Bmatrix} & \frac{1}{\sqrt{3}}\begin{Bmatrix} 1 \\ 1 \\ -1 \end{Bmatrix} & \frac{1}{\alpha_2}\begin{Bmatrix} 1 \\ 1-\sqrt{3} \\ 2-\sqrt{3} \end{Bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} 2-\sqrt{3} & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2+\sqrt{3} \end{bmatrix}$$

in which the only nonzero components are the eigenvalues, as expected.

(iii) Obtain the Rayleigh Quotient for the vector $\langle x \rangle = \langle 1, 2, 3 \rangle$.

Solution: Recall that the Rayleigh Quotient is $R = \langle x \rangle [A]\{x\}/(\langle x \rangle \{x\})$. Carry out the indicated multiplications to obtain

$$[A]\{x\} = \begin{Bmatrix} 1 \\ 0 \\ 1 \end{Bmatrix}, \qquad \langle x \rangle [A]\{x\} = 4, \qquad \langle x \rangle \{x\} = 14, \qquad R = 0.286$$

Note that $\lambda_1 = 0.227 \le R \le \lambda_3 = 3.763$.

(iv) Construct the projection matrices $[P]_i = \{e\}_i \langle e \rangle_i$ and evaluate $\sum_{i=1}^{3} \lambda_i [P]_i$.

Solution: The projection matrices become

$$[P]_1 = \frac{1}{\alpha_1^2}\begin{bmatrix} 1 & 1+\sqrt{3} & 2+\sqrt{3} \\ 1+\sqrt{3} & 4+2\sqrt{3} & 5+3\sqrt{3} \\ 2+\sqrt{3} & 5+3\sqrt{3} & 7+4\sqrt{3} \end{bmatrix}, \qquad [P]_2 = \frac{2}{3}\begin{bmatrix} 1 & 1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix}$$

$$[P]_3 = \frac{1}{\alpha_3^2}\begin{bmatrix} 1 & 1-\sqrt{3} & 2-\sqrt{3} \\ 1-\sqrt{3} & 4-2\sqrt{3} & 5-3\sqrt{3} \\ 2-\sqrt{3} & 5-3\sqrt{3} & 7-4\sqrt{3} \end{bmatrix}$$

When the algebra is carried out, the sum becomes

$$\sum_{i=1}^{3} \lambda_i [P]_i = (2-\sqrt{3})[P]_1 + 2[P]_2 + (2-\sqrt{3})[P]_3 = [A]$$

(v) We leave as an exercise the following verification:

If $[B] = \sum_{i=1}^{3} \frac{1}{\lambda_i}[P]_i$, then [B][A] = [I], i.e., [B] is the inverse of [A].

Example 2: Consider the following matrix:

$$[A] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

(i) Determine the eigenpairs for the matrix.

Solution: Whenever a symmetric matrix is given with the components of a particular row and column zero except for the diagonal term, then the diagonal term is an eigenvalue and the corresponding eigenvector is the coordinate basis with the unit component in the location that agrees with the location of the eigenvalue. For this matrix the first row and column fit the description so, $\lambda = 1$ and $\langle e \rangle = \langle 1, 0, 0 \rangle$ form an eigenpair. This result is easily verified by showing that $[A]\{e\} = \lambda\{e\}$.

Independent of this result, we develop the characteristic equation and solve for the eigenvalues. The characteristic equation is

$$\det\,[[A] - \lambda[I]] = \begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 2-\lambda & -1 \\ 0 & -1 & 2-\lambda \end{vmatrix} = (\lambda-1)^2(\lambda-3) = 0$$

and we see that not only is $\lambda = 1$ a root, but for this case it is a repeated root. The ordered roots are the eigenvalues: $\lambda_1 = 1$, $\lambda_2 = 1$, $\lambda_3 = 3$. To obtain the first eigenvector, solve the equation $[[A] - \lambda_1[I]]\{e^*\} = \{0\}$ for $\{e^*\}$. The algebraic equations become

$$\begin{aligned} (1-\lambda_1)e_1^* &= 0 \\ (2-\lambda_1)e_2^* - e_3^* &= 0 \\ -e_2^* + (2-\lambda_1)e_3^* &= 0 \end{aligned}$$

After the substitution of $\lambda_1 = 1$, we see that there is only one independent equation. One solution is the choice $\langle e \rangle_1 = \langle 1, 0, 0 \rangle$ which is the vector mentioned above obtained by inspection. Now, if we seek a solution for the second eigenvalue, the algebraic equations are identical to those associated with the first eigenvalue because the eigenvalues are equal. The second and third equations becomes $e^*_2 - e^*_3 = 0$. Choose $e_2^* = 1$ and solve to get $e_3^* = 1$. Normalize and we have the eigenpair $\lambda_2 = 1$ and $\langle e \rangle_2 = \langle 0, 1, 1 \rangle / \sqrt{2}$. Any combination of $\{e\}_1$ and $\{e\}_2$ will also be an eigenvector associated with $\lambda = 1$. For example, if the sum and difference of these two vectors are used, an alternative set of eigenvectors is $\langle e^a \rangle_1 = \langle 1, 1/\sqrt{2}, 1/\sqrt{2} \rangle / \sqrt{2}$ for the one and $\langle e^a \rangle_2 = \langle 1, -1/\sqrt{2}, -1/\sqrt{2} \rangle / \sqrt{2}$ for the other. Note that each pair of vectors is mutually orthogonal. Now for the third eigenpair, set $\lambda_3 = 3$ in the algebraic equations and solve. The eigenvector is $\langle e \rangle_3 = \langle 0, 1, -1 \rangle / \sqrt{2}$. For the matrix, the eigenpairs are summarized as follows:

$$\lambda_1 = 1,\ \{e\}_1 = \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix}, \qquad \lambda_2 = 1,\ \{e\}_2 = \frac{1}{\sqrt{2}} \begin{Bmatrix} 0 \\ 1 \\ 1 \end{Bmatrix}, \qquad \lambda_3 = 3,\ \{e\}_3 = \frac{1}{\sqrt{2}} \begin{Bmatrix} 0 \\ 1 \\ -1 \end{Bmatrix}$$

An equivalent alternative set is

$$\lambda_1 = 1,\ \{e^a\}_1 = \frac{1}{\sqrt{2}} \begin{Bmatrix} 1 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{Bmatrix}, \quad \lambda_2 = 1,\ \{e^a\}_2 = \frac{1}{\sqrt{2}} \begin{Bmatrix} 1 \\ -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{Bmatrix}, \quad \lambda_3 = 3,\ \{e\}_3 = \frac{1}{\sqrt{2}} \begin{Bmatrix} 0 \\ 1 \\ -1 \end{Bmatrix}$$

There are actually an infinite number of choices for the first two eigenvectors; any two vectors which span the space defined by $\{e\}_1$ and $\{e\}_2$ are eigenvectors.

(iv) Construct the projection matrices $[P]_i = \{e\}_i \langle e \rangle_i$ and evaluate $\sum_{i=1}^{3} \lambda_i [P]_i$ .

Solution: Because the eigenvectors are not unique for $\lambda = 1$, the definition of projection matrices changes to include the contributions from both eigenvectors. The result is

$$[P]_1 = \{e\}_1 \langle e \rangle_1 + \{e\}_2 \langle e \rangle_2 = \{e^a\}_1 \langle e^a \rangle_1 + \{e^a\}_2 \langle e^a \rangle_2$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$[P]_3 = \{e\}_1 \langle e \rangle_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

A quick calculation shows that $\lambda_1[P]_1 + \lambda_3[P]_3 = [A]$.

Example 3: Determine the eigensystem for the following unsymmetric matrix in which the skew-symmetrical component, $\varepsilon$, is left as an arbitrary parameter:

$$[A] = \begin{bmatrix} 1 & \varepsilon \\ -\varepsilon & 2 \end{bmatrix}$$

Solution: When $\varepsilon = 0$, the eigenvalues are 1 and 2, and the corresponding eigenvectors are just the coordinate vectors, $\{I\}_1$ and $\{I\}_2$, respectively. For arbitrary $\varepsilon$, the characteristic polynomial is

$$\begin{vmatrix} 1-\lambda & \varepsilon \\ -\varepsilon & 2-\lambda \end{vmatrix} = \lambda^2 - 3\lambda + 2 + \varepsilon^2 = 0$$

The roots are

$$\lambda_{1,2} = (3 \mp \alpha)/2 , \qquad \alpha = \sqrt{1 - 4\varepsilon^2}$$

We see that even though the matrix is not symmetric for any nonzero $\varepsilon$, the eigenvalues remain real provided $-\frac{1}{2} < \varepsilon < \frac{1}{2}$. When $\varepsilon = \pm\frac{1}{2}$, the system degenerates to the case of a repeated eigenvalue, 3/2. For $|\varepsilon| > \frac{1}{2}$, the eigenvalues are complex.

A right eigenvector satisfies the set of equations

$$\begin{bmatrix} 1-\lambda & \varepsilon \\ -\varepsilon & 2-\lambda \end{bmatrix} \begin{Bmatrix} \hat{e}_1 \\ \hat{e}_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

or

$$(1-\lambda)\hat{e}_1 + \varepsilon\hat{e}_2 = 0$$
$$-\varepsilon\hat{e}_1 + (2-\lambda)\hat{e}_2 = 0$$

A solution exists, only if $\lambda$ satisfies the characteristic equation, in which case, there is only one independent equation for $\varepsilon \neq \pm(1/2)$. For nonzero $\varepsilon$, choose $\hat{e}_1 = 2\varepsilon$. The first equation yields $\hat{e}_2 = 2(\lambda - 1)$. After the two eigenvalues are substituted, in turn, in the expression for the second component, the resulting expressions for the right eigenvectors are

$$\{e\}_1^R = \begin{Bmatrix} 2\varepsilon \\ 1-\alpha \end{Bmatrix}, \qquad \{e\}_2^R = \begin{Bmatrix} 2\varepsilon \\ 1+\alpha \end{Bmatrix}$$

Note that the eigenvectors become complex simultaneously (through $\alpha$) with the eigenvalues as the parameter, $\varepsilon$, is allowed to vary.

A left eigenvector satisfies the set of equations

$$\langle \tilde{e}_1 \quad \tilde{e}_2 \rangle \begin{bmatrix} 1-\lambda & \varepsilon \\ -\varepsilon & 2-\lambda \end{bmatrix} = \langle 0 \quad 0 \rangle$$

or

$$(1-\lambda)\tilde{e}_1 - \varepsilon\tilde{e}_2 = 0$$
$$\varepsilon\tilde{e}_1 + (2-\lambda)\tilde{e}_2 = 0$$

As before, select $\tilde{e}_1 = 2\varepsilon$ and solve for $\tilde{e}_2$. The resulting expressions for the left eigenvectors are

$$\{e\}_1^L = s_1 \begin{Bmatrix} 2\varepsilon \\ -1+\alpha \end{Bmatrix}, \qquad \{e\}_2^L = s_2 \begin{Bmatrix} 2\varepsilon \\ -1-\alpha \end{Bmatrix}$$

in which $s_1$ and $s_2$ are scale factors determined from the following normalizing equations:

$$\langle e \rangle_1^L \{e\}_1^R = 1 \qquad \text{and} \qquad \langle e \rangle_2^L \{e\}_2^R = 1$$

The result of satisfying these equations is

$$s_1 = 1/[2\alpha(1-\alpha)] \qquad \text{and} \qquad s_2 = -1/[2\alpha(1+\alpha)]$$

Next we note that $\langle e \rangle_1^L \{e\}_2^R = 0$ and $\langle e \rangle_2^L \{e\}_1^R = 0$ so that, after normalization, the left and right eigenvectors satisfy the orthogonality relation $\langle e \rangle_i^L \{e\}_j^R = \delta_{ij}$ given in (4.3-7).

## 4.12 CONCLUDING REMARKS

In this chapter, we have introduced basic concepts associated with the standard eigenproblem and the general eigenproblem. When the matrices are real and symmetric, then the eigenvalues must be real. Furthermore, the eigenvectors must be orthogonal for the standard eigenproblem, and B-orthogonal for the general eigenproblem. The modal matrix is constructed with columns consisting of eigenvectors and the concept of a spectral decomposition shows how a matrix is related to a diagonal matrix whose components are eigenvalues. The spectral decomposition provides the means for defining a function of a matrix. The Cayley-Hamilton Theorem shows that a matrix satisfies its own characteristic equation. The Rayleigh Quotient provides an upper bound to the lowest eigenvalue and a lower bound to the largest eigenvalue.

The concept of compatible vector and matrix norms provides the tools for measuring error and, hence, the accuracy of a solution. The approximate modal method is one approach for obtaining solutions to the algebraic problem for which a good error parameter is required . Finally, an elementary introduction to iterative methods for solving the algebraic problem and for finding eigenpairs is given.

Taken as a whole, this chapter is probably the most important one in the book. We will see that concepts associated with the eigenproblem appear repeatedly in connection with the finite difference method and the finite element method.

## 4.13 EXERCISES

1. If the inverse of [A] exists, show that the eigenvalues of $[A]^{-1}$ consist of the inverse of the eigenvalues of [A], and the eigenvectors of $[A]^{-1}$ are identical to those of [A].

2. Two matrices are given as follows:

$$[A] = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \qquad [A] = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

Do the following for each matrix [A]:

(a) Obtain the eigenpairs.

(b) Construct the modal matrix, $[M] = [\{e\}_1, \{e\}_2, \{e\}_3]$.

(c) Determine $[M]^T[A][M]$, $[M]^T[M]$ and $\sum_i \lambda_i \{e\}^i \{e\}^{iT}$

(d) Determine the range and the rank.

3. Given

$$[A] = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 2 \end{bmatrix} \qquad \{b\}_1 = \begin{Bmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 0 \end{Bmatrix} \qquad \{b\}_2 = \begin{Bmatrix} 2 \\ -1 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

(a) Solve $[A]\{x\} = \{b\}_1$ and $[A]\{x\} = \{b\}_2$ using an iterative procedure developed by you. Show how accuracy improves with the number of iterations.

(b) Find a subroutine that provides eigenvalues and eigenvectors. Use the subroutine to obtain the eigensystem for [A]. Construct the matrices

(i) [M], (ii) $[M]^T[M]$, (iii) $[A^*] = \sum_i \lambda_i \{e\}^i \{e\}^{iT}$,

(iv) $[A^*]^{-1} = \sum_i \frac{1}{\lambda_i} \{e\}^i \{e\}^{iT}$, and (v) $[A^*][A^*]^{-1}$

(c) Obtain one-mode, three-mode and five-mode solutions to the algebraic equation $[A]\{x\} = \{b\}_1$.
(d) Use Gershgorin's theorem to obtain bounds on the eigenvalues of [A].
(e) Obtain the Rayleigh quotient $R(\{v\})$ for $\langle v \rangle = \langle 1, 2, 3, 2, 1 \rangle$. Is the inequality $\lambda_1 \leq R \leq \lambda_n$ satisfied?
(f) Determine $\|A\|_1$, $\|A\|_2$, and $\|A\|_\infty$. Do these norms form an upper bound to the maximum eigenvalue?
(g) What is the condition number of [A]?

4. Apply the results of this chapter to Hilbert matrixes, $[H]_{n \times n}$ which are defined in Problem 6 of Chapter 3. Consider a range for n, (2 to 6, say), and do the following:

(a) Obtain the eigenpairs.
(b) Determine how the condition number changes with n.
(c) Use an iteration solver to obtain solutions to the algebraic problem for Hilbert matrices of dimension n = 2 up to one higher than the dimension for which you can obtain reasonable answers with your machine. Discuss your results.