ASSIGNMENT 5

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ME 512 - Continuum Mechanics

September 25, 2014

1 Find the derivatives with respect to components of the independent variable with the bases assumed constant.

(i) Show that for $\phi = (\boldsymbol{v}\cdot\boldsymbol{v})^{3/2}$; $\quad (\phi)\overleftarrow{\nabla}_{\boldsymbol{v}} = \frac{3}{2}(\boldsymbol{v}\cdot\boldsymbol{v})^{1/2}\boldsymbol{v}$

- $\phi = (v_j v_j)^{3/2}$
- $(\phi)\overleftarrow{\nabla}_{\boldsymbol{v}} = \frac{\partial\phi}{\partial\boldsymbol{v}} \Rightarrow \frac{d\phi}{dv_i} = \frac{3}{2}(v_j v_j)^{1/2}\frac{d(v_k v_k)}{dv_i}$
- $\frac{d(v_k v_k)}{dv_i} = 2v_l\frac{\partial v_l}{\partial v_i} = 2v_l\delta_{il} = 2v_l$
- $(\phi)\overleftarrow{\nabla}_{\boldsymbol{v}} = \frac{3}{2}(v_j v_j)^{1/2}2v_l = 3(v_j v_j)^{1/2}v_l = 3(\boldsymbol{v}\cdot\boldsymbol{v})^{1/2}\boldsymbol{v}$

(ii) Show that for $\phi = tr(\boldsymbol{T}) + tr(\boldsymbol{T}^2) + (\boldsymbol{T}\cdot\cdot\boldsymbol{T})^{1/2}$; $\quad (\phi)\overleftarrow{\nabla}_{\boldsymbol{T}} = \boldsymbol{I} + 2\boldsymbol{I}^T + \frac{\boldsymbol{T}}{(\boldsymbol{T}\cdot\cdot\boldsymbol{T})^{1/2}}$

- $\phi = T_{pp} + T_{pq}T_{qp} + (T_{pq}T_{qp})^{1/2}$
- $$\begin{aligned}(\phi)\overleftarrow{\nabla}_{\boldsymbol{T}} &= \frac{\partial\phi}{\partial\boldsymbol{T}}\\ &= \frac{\partial T_{pp}}{\partial T_{ij}} + \left(T_{pq}\frac{T_{qp}}{\partial T_{ij}} + T_{qp}\frac{T_{pq}}{\partial T_{ij}}\right) + \frac{1}{2}\left(T_{pq}T_{qp}\right)^{-1/2}\frac{\partial(T_{rs}T_{sr})}{\partial T_{ij}}\\ &= \delta_{pi}\delta_{pj} + 2T_{pq}\delta_{pi}\delta_{qj} + \frac{1}{2(T_{pq}T_{qp})^{1/2}}2T_{rs}\delta_{ri}\delta_{sj}\\ &= \delta_{ij} + 2T_{ij} + \frac{T_{ij}}{(T_{pq}T_{pq})^{1/2}} \Rightarrow \boldsymbol{I} + 2\boldsymbol{T}^T + \frac{\boldsymbol{T}}{(\boldsymbol{T}\cdot\cdot\boldsymbol{T})^{1/2}}\end{aligned}$$

(iii) Show that for $\phi = \boldsymbol{v}(\boldsymbol{v}\cdot\boldsymbol{v})^{3/2}$; $\quad (\boldsymbol{\phi})\overleftarrow{\nabla}_{\boldsymbol{v}} = \left[\boldsymbol{I}(\boldsymbol{v}\cdot\boldsymbol{v})^{3/2} + 3(\boldsymbol{v}\cdot\boldsymbol{v})^{1/2}(\boldsymbol{v}\otimes\boldsymbol{v})\right]$

- Postulate: $(\boldsymbol{\phi})\overleftarrow{\nabla}_{\boldsymbol{v}} = \frac{\partial\boldsymbol{\phi}}{\partial v_i}\otimes\boldsymbol{e}_i$
- $\boldsymbol{\phi} = v_k\boldsymbol{e}_k(v_l v_l)^{3/2}$
- $$\begin{aligned}\frac{\partial\boldsymbol{\phi}}{\partial v_i} &= \frac{\partial v_k}{\partial v_i}\boldsymbol{e}_k(v_l v_l)^{3/2} + v_k\boldsymbol{e}_k\frac{\partial(v_l v_l)}{\partial v_i}\\ &= \delta_{ki}\boldsymbol{e}_k(v_l v_l)^{3/2} + v_k\boldsymbol{e}_k(3/2)(v_l v_l)^{1/2}\left(v_m\frac{\partial v_m}{\partial v_i} + \frac{\partial v_m}{\partial v_i}v_m\right)\\ &= \delta_{ki}\boldsymbol{e}_k(v_l v_l)^{3/2} + v_k\boldsymbol{e}_k(3/2)(v_l v_l)^{1/2}(2v_m\delta_{mi})\\ &= (\boldsymbol{v}\cdot\boldsymbol{v})^{3/2}\boldsymbol{e}_i + 3(\boldsymbol{v}\cdot\boldsymbol{v})^{1/2}v_k\boldsymbol{e}_k v_i\end{aligned}$$

- $$\frac{\partial \boldsymbol{\phi}}{\partial v_i} \otimes \boldsymbol{e}_i = (\boldsymbol{v}\cdot\boldsymbol{v})^{3/2}\boldsymbol{e}_i \otimes \boldsymbol{e}_i + 3(\boldsymbol{v}\cdot\boldsymbol{v})^{1/2} v_k \boldsymbol{e}_k v_i \otimes \boldsymbol{e}_i$$
$$= (\boldsymbol{v}\cdot\boldsymbol{v})^{3/2}\boldsymbol{e}_i \otimes \boldsymbol{e}_i + 3(\boldsymbol{v}\cdot\boldsymbol{v})^{1/2}(\boldsymbol{v}\otimes\boldsymbol{v})$$
$$= \boldsymbol{I}(\boldsymbol{v}\cdot\boldsymbol{v})^{3/2} + 3(\boldsymbol{v}\cdot\boldsymbol{v})^{1/2}(\boldsymbol{v}\otimes\boldsymbol{v})$$

2 if F and ϕ are scalars, use indicial notation to show:

(i) $(F\boldsymbol{v})\cdot\overleftarrow{\nabla} = \boldsymbol{v}\cdot(F\overleftarrow{\nabla}) + F(\boldsymbol{v}\cdot\overleftarrow{\nabla}) \Rightarrow \frac{\partial(Fv_i)}{\partial x_i} = \left(\frac{\partial F}{\partial x_i}\right)v_i + F\left(\frac{\partial v_i}{\partial x_i}\right) = F_{,i}v_i + Fv_{i,i}$

(ii) $(F\overleftarrow{\nabla})\times\overleftarrow{\nabla} = \mathcal{E}_{ijk}\frac{\partial(\partial F/\partial x_k)}{\partial x_j} = \mathcal{E}_{ijk}\frac{\partial^2 F}{\partial x_j \partial x_k} = \mathcal{E}_{ijk}F_{,jk} = \boldsymbol{0}$

(iii) $(\boldsymbol{v}\times\overleftarrow{\nabla})\cdot\overleftarrow{\nabla} = \frac{\partial}{\partial x_i}\left(\mathcal{E}_{ijk}\frac{\partial v_k}{\partial x_j}\right) = \mathcal{E}_{ijk}\frac{\partial^2 v_k}{\partial x_i \partial x_j} = \mathcal{E}_{ijk}v_{k,ij}$

(iv) $(\boldsymbol{v}\times\overleftarrow{\nabla})\times\overleftarrow{\nabla} = \mathcal{E}_{mni}\mathcal{E}_{ijk}v_{k,jn} = (\delta_{mj}\delta_{nk} - \delta_{mk}\delta_{jn})v_{k,jn} = \delta_{mj}\delta_{nk}v_{k,jn} - \delta_{mk}\delta_{jn}v_{k,jn} = v_{k,mk} - v_{m,nn}$
$(\boldsymbol{v}\cdot\overleftarrow{\nabla})\overleftarrow{\nabla} = v_{k,mk}$
$\boldsymbol{v}\overleftarrow{\nabla}^2 = Laplacian = v_{m,nn}$
$(\boldsymbol{v}\times\overleftarrow{\nabla})\times\overleftarrow{\nabla} = (\boldsymbol{v}\cdot\overleftarrow{\nabla})\overleftarrow{\nabla} - \boldsymbol{v}\overleftarrow{\nabla}^2$

(v) $(\boldsymbol{u}\otimes\boldsymbol{v})\cdot\overleftarrow{\nabla} = (\boldsymbol{u}\overleftarrow{\nabla})\cdot\boldsymbol{v} + \boldsymbol{u}(\boldsymbol{v}\cdot\overleftarrow{\nabla})$
$Product\ rule: \frac{\partial(\boldsymbol{uv})}{\partial x_i} = \frac{\partial(\boldsymbol{u})}{\partial x_i}\boldsymbol{v} + \frac{\partial(\boldsymbol{v})}{\partial x_i}\boldsymbol{u}$
$\boldsymbol{uv} \Rightarrow T_{ij}$ then $T_{ij,j} = u_{j,j}v_i + v_{j,j}u_i$

(vi) curl(gradϕ)) $\Rightarrow \mathcal{E}_{ijk}\phi_{,jk} = \boldsymbol{0}$

3 Determine the gradient, divergence, and curl of $\boldsymbol{u}$

- $\boldsymbol{u} = x_1x_2x_3\boldsymbol{e}_1 + x_1x_2\boldsymbol{e}_2 + x_1\boldsymbol{e}_3$
- Gradient: $\boldsymbol{u}\overleftarrow{\nabla} = u_{i,j}\boldsymbol{e}_i\otimes\boldsymbol{e}_j \Rightarrow \begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} \\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & \frac{\partial u_3}{\partial x_3} \end{bmatrix} = \begin{bmatrix} x_2x_3 & x_2 & 1 \\ x_1x_3 & x_1 & 0 \\ x_1x_2 & 0 & 0 \end{bmatrix}$
- Divergence: $\boldsymbol{u}\cdot\overleftarrow{\nabla} = u_{i,i} = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} = x_2x_3 + x_1 + 0$
- Curl: $\boldsymbol{u}\times\overleftarrow{\nabla} = u_{i,i} = \mathcal{E}_{ijk}u_{j,k}\boldsymbol{e}_i \Rightarrow \begin{Bmatrix} \frac{\partial u_2}{\partial x_3} - \frac{\partial u_3}{\partial x_2} \\ \frac{\partial u_2}{\partial x_3} - \frac{\partial u_3}{\partial x_2} \\ \frac{\partial u_2}{\partial x_3} - \frac{\partial u_3}{\partial x_2} \end{Bmatrix} = \begin{Bmatrix} 0-0 \\ 1 - x_1x_2 \\ x_1x_2 - x_2 \end{Bmatrix}$
- Verify that (iv) of Problem 2 is satisfied:

4 Express in terms of components and base vectors.

(i) $\boldsymbol{T}\times\overleftarrow{\nabla} \Rightarrow \mathcal{E}_{jkl}T_{ij,k}\boldsymbol{e}_i\otimes\boldsymbol{e}_l \Rightarrow \begin{bmatrix} \left(\frac{\partial T_{12}}{\partial x_3} - \frac{\partial T_{13}}{\partial x_2}\right)\boldsymbol{e}_1\otimes\boldsymbol{e}_1 & \left(\frac{\partial T_{13}}{\partial x_1} - \frac{\partial T_{11}}{\partial x_3}\right)\boldsymbol{e}_1\otimes\boldsymbol{e}_2 & \left(\frac{\partial T_{11}}{\partial x_2} - \frac{\partial T_{12}}{\partial x_1}\right)\boldsymbol{e}_1\otimes\boldsymbol{e}_3 \\ \left(\frac{\partial T_{22}}{\partial x_3} - \frac{\partial T_{23}}{\partial x_2}\right)\boldsymbol{e}_2\otimes\boldsymbol{e}_1 & \left(\frac{\partial T_{23}}{\partial x_1} - \frac{\partial T_{21}}{\partial x_3}\right)\boldsymbol{e}_2\otimes\boldsymbol{e}_2 & \left(\frac{\partial T_{21}}{\partial x_2} - \frac{\partial T_{22}}{\partial x_1}\right)\boldsymbol{e}_2\otimes\boldsymbol{e}_3 \\ \left(\frac{\partial T_{32}}{\partial x_3} - \frac{\partial T_{33}}{\partial x_2}\right)\boldsymbol{e}_3\otimes\boldsymbol{e}_1 & \left(\frac{\partial T_{33}}{\partial x_1} - \frac{\partial T_{31}}{\partial x_3}\right)\boldsymbol{e}_3\otimes\boldsymbol{e}_2 & \left(\frac{\partial T_{31}}{\partial x_2} - \frac{\partial T_{32}}{\partial x_1}\right)\boldsymbol{e}_3\otimes\boldsymbol{e}_3 \end{bmatrix}$

(ii) $C_{13}(\boldsymbol{I}\times\boldsymbol{T})$???

5 Express in indicial form and longhand.

- $\boldsymbol{T} \cdot \overleftarrow{\nabla} + \boldsymbol{f} = \boldsymbol{0} \Rightarrow T_{ij,j}\boldsymbol{e}_i + f_i\boldsymbol{e}_i$
- $\boldsymbol{T} \cdot \overleftarrow{\nabla} + \boldsymbol{f} = \boldsymbol{0} \Rightarrow \begin{Bmatrix} \frac{\partial T_{11}}{\partial x_1} + \frac{\partial T_{12}}{\partial x_2} + \frac{\partial T_{13}}{\partial x_3} \\ \frac{\partial T_{21}}{\partial x_1} + \frac{\partial T_{22}}{\partial x_2} + \frac{\partial T_{23}}{\partial x_3} \\ \frac{\partial T_{31}}{\partial x_1} + \frac{\partial T_{32}}{\partial x_2} + \frac{\partial T_{33}}{\partial x_3} \end{Bmatrix} + \begin{Bmatrix} f_1 \\ f_2 \\ f_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$