12B.3 Heating of a wall (constant heat flux)

a. The equation to be solved for $q_y(y,t)$ is

$$\frac{\partial q_y}{\partial t} = \alpha \frac{\partial^2 q_y}{\partial y^2} \qquad \text{with } q_y(y,0)=0,\ q_y(0,t)=q_0, \text{ and } q_y(\infty,t)=0$$

This is the same mathematical problem that is solved in Example 12.1-1, so that we can write down at once

$$\frac{q_y}{q_0} = \frac{2}{\sqrt{\pi}} \int_{y/\sqrt{4\alpha t}}^{\infty} e^{-u^2} du$$

To get the temperature we use Eq. 12.1-39

$$\begin{aligned}
T - T_0 &= \frac{q_0}{k} \int_y^{\infty} \frac{q_y(\bar{y},t)}{q_0} d\bar{y} = \frac{q_0}{k} \frac{2}{\sqrt{\pi}} \int_y^{\infty} \int_{\bar{y}/\sqrt{4\alpha t}}^{\infty} e^{-u^2} du d\bar{y} \\
&= \frac{q_0}{k} \frac{2}{\sqrt{\pi}} \sqrt{4\alpha t} \int_{y/\sqrt{4\alpha t}}^{\infty} \int_Y^{\infty} e^{-u^2} du dY = \frac{q_0}{k} \frac{2}{\sqrt{\pi}} \sqrt{4\alpha t} \int_{y/\sqrt{4\alpha t}}^{\infty} \int_{y/\sqrt{4\alpha t}}^{u} e^{-u^2} dY du \\
&= \frac{q_0}{k} \frac{2}{\sqrt{\pi}} \sqrt{4\alpha t} \int_{y/\sqrt{4\alpha t}}^{\infty} e^{-u^2} \left[u - \frac{y}{\sqrt{4\alpha t}} \right] du \\
&= \frac{q_0}{k} \left(\sqrt{\frac{4\alpha t}{\pi}} \int_{y/\sqrt{4\alpha t}}^{\infty} e^{-u^2} u du - \frac{2y}{\sqrt{\pi}} \int_{y/\sqrt{4\alpha t}}^{\infty} e^{-u^2} du \right)
\end{aligned}$$

When the first integral is evaluated, we get Eq. 12B.3-1. In the above, to get the second line, we made a change of variables and then interchanged the order of integration; in the third line, we performed the inner integration.

b. Some intermediate steps in showing that the partial differential equation is satisfied: We write $k(T-T_0)/q_0 = F - G$. Then

$$\frac{\partial F}{\partial t} = \left(\sqrt{\frac{\alpha}{\pi t}} + \sqrt{\frac{1}{4\pi\alpha t}} \frac{y^2}{t} \right) e^{-y^2/4\alpha t}; \quad \alpha \frac{\partial^2 F}{\partial y^2} = \left(\sqrt{\frac{1}{4\pi\alpha t}} \frac{y^2}{t} - \sqrt{\frac{\alpha}{\pi t}} \right) e^{-y^2/4\alpha t}$$

$$\frac{\partial G}{\partial t} = \left(\sqrt{\frac{1}{4\pi\alpha t}} \frac{y^2}{t} \right) e^{-y^2/4\alpha t}; \quad \alpha \frac{\partial^2 G}{\partial y^2} = \left(-2\sqrt{\frac{\alpha}{\pi t}} + \sqrt{\frac{1}{4\pi\alpha t}} \frac{y^2}{t} \right) e^{-y^2/4\alpha t}$$