Heat Transfer Problems

Introductory Course on Multiphysics Modelling

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1 Introduction

1.1 Mechanisms of heat transfer

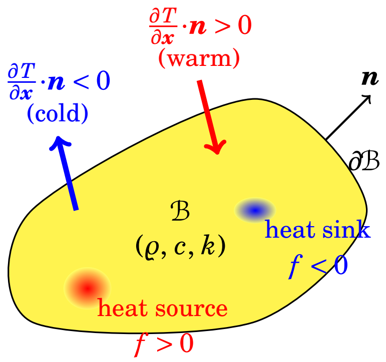
Motivation

- In many engineering systems and devices there is often a need for **optimal thermal performance**.
- Most **material properties are temperature-dependent** so the effects of heat enter many other disciplines and drive the requirement for **multiphysics modeling**.

Heat transfer: the movement of energy due to a temperature difference and according to the following **three mechanisms**

- **Conduction** – heat transfer by diffusion in a stationary medium due to a temperature gradient. The medium can be a solid or a liquid.
- **Convection** – heat transfer between either a hot surface and a cold moving fluid or a hot moving fluid and a cold surface. Convection occurs in fluids (liquids and gases).
- **Radiation** – heat transfer via electromagnetic waves between two surfaces with different temperatures.

1.2 Heat conduction and the energy conservation principle



- *Problem*: to find the **temperature** in a solid, $T = T(\boldsymbol{x}, t) = ?$ [K].
- **Temperature** is related to **heat** which is a form of **energy**.
- The **principle of conservation of energy** should be used to determine the temperature.
- Thermal energy can be: **stored**, **generated** (or absorbed), and **supplied** (transferred).

The law of conservation of thermal energy
The rate of change of internal thermal energy with respect to time in $\mathcal{B}$ is equal to the net flow of energy across the surface of $\mathcal{B}$ plus the rate at which the heat is generated within $\mathcal{B}$.

2 Heat transfer equation

2.1 Balance of thermal energy

- The **rate of change of thermal energy**, $\dot{E} = \frac{\mathrm{d}E}{\mathrm{d}t}$ [W]:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{B}} \varrho\, e \,\mathrm{d}V = \int_{\mathcal{B}} \varrho \frac{\partial e}{\partial t}\,\mathrm{d}V$$

$\varrho = \varrho(\boldsymbol{x})$ – the mass density $\left[\frac{\mathrm{kg}}{\mathrm{m}^3}\right]$

$e = e(\boldsymbol{x}, t)$ – the specific internal energy $\left[\frac{\mathrm{J}}{\mathrm{kg}}\right]$

- The **flow of heat**, Q [W] (the amount of heat per unit time flowing-in across the boundary $\partial\mathcal{B}$):

$$-\int_{\partial\mathcal{B}} \boldsymbol{q} \cdot \boldsymbol{n}\,\mathrm{d}S$$

$\boldsymbol{q} = \boldsymbol{q}(\boldsymbol{x}, t)$ – the heat flux vector $\left[\frac{\mathrm{W}}{\mathrm{m}^2}\right]$

$\boldsymbol{n}$ – the outward normal vector

- The total **rate of heat production**, $\boldsymbol{F}$ [W] (the amount of heat per unit time produced in $\mathcal{B}$ by the volumetric heat sources):

$$\int_{\mathcal{B}} f \, \mathrm{d}V \qquad\qquad f = f(\boldsymbol{x},t) \text{ – the rate of heat production per unit volume } \left[\tfrac{\mathrm{W}}{\mathrm{m}^3}\right]$$

The thermal energy conservation law, $\dot{E} = Q + F$, leads to the following equation.

The global form of thermal energy balance

$$\int_{\mathcal{B}} \varrho \frac{\partial e}{\partial t} \, \mathrm{d}V = -\int_{\partial\mathcal{B}} \boldsymbol{q} \cdot \boldsymbol{n} \, \mathrm{d}S + \int_{\mathcal{B}} f \, \mathrm{d}V \tag{1}$$

or, after using the divergence theorem, $\int_{\partial\mathcal{B}} \boldsymbol{q} \cdot \boldsymbol{n} \, \mathrm{d}S = \int_{\mathcal{B}} \nabla \cdot \boldsymbol{q} \, \mathrm{d}V$,

$$\int_{\mathcal{B}} \left(\varrho \frac{\partial e}{\partial t} + \nabla \cdot \boldsymbol{q} - f \right) \mathrm{d}V = 0 \, . \tag{2}$$

Assuming the continuity of the above integral and using the fact that this equality holds not only for the whole domain $\mathcal{B}$, but also for its every single subdomain the following PDE is obtained.

The local form of thermal energy balance

$$\varrho \frac{\partial e}{\partial t} + \nabla \cdot \boldsymbol{q} = f \quad \text{in } \mathcal{B} \, . \tag{3}$$

- In every case, the unknown fields are: $e = e(\boldsymbol{x},t) =?$, $\boldsymbol{q} = \boldsymbol{q}(\boldsymbol{x},t) =?$.
- These fields are related to the unknown temperature $T = T(\boldsymbol{x},t) =?$.
- To determine an equation for the temperature additional information of an empirical nature is needed.
- In other words, relations $e = e(T)$ and $\boldsymbol{q} = \boldsymbol{q}(T)$ are to be established and applied.

2.2 Specific thermal energy

Observation:

- for many materials, over fairly wide (but not too large) temperature ranges, the specific thermal energy depends **linearly** on the temperature.

Specific thermal energy vs. temperature

$$\frac{\partial e}{\partial t} = c \, \frac{\partial T}{\partial t} \tag{4}$$

where $c = c(\boldsymbol{x},t)$ is the **thermal capacity** $\left[\frac{\mathrm{J}}{\mathrm{kg \cdot K}}\right]$.

- The thermal capacity is also called *specific heat capacity* (or simply, *specific heat*).
- It describes the ability of a material to store the heat and refers to the quantity that represents the amount of heat required to change the temperature of one unit of mass by one degree.

2.3 Fourier's law

Observations:

- the **heat flows** from regions of **high temperature to** regions of **low temperature**,
- the rate of heat **flow is bigger** if the **temperature differences** (between neighboring regions) are **larger**.

Postulate: linear relationship between the rate of **heat flow** and the rate of **temperature change**.

Fourier's law of heat conduction

$$\boldsymbol{q} = -k\,\nabla T \tag{5}$$

where $k = k(\boldsymbol{x})$ is the **thermal conductivity** $\left[\frac{\mathrm{W}}{\mathrm{m\cdot K}}\right]$.

The minus in equation (5) means that the conduction of heat proceeds from regions of high temperature to regions of low temperature.

- The thermal conductivity is a material constant that describes the ability of a material to conduct the heat.
- If the thermal conductivity is anisotropic, k becomes the (second order) *thermal conductivity tensor*.

2.4 Heat equation

$$\begin{array}{ccccc} \text{Energy vs. temp.} & & \text{Energy conservation law} & & \text{Fourier's law} \\ \boxed{\dfrac{\partial e}{\partial t} = c\,\dfrac{\partial T}{\partial t}} & \longrightarrow & \boxed{\varrho\,\dfrac{\partial e}{\partial t} + \nabla\cdot\boldsymbol{q} = f} & \longleftarrow & \boxed{\boldsymbol{q} = -k\,\nabla T} \\ & & \downarrow & & \end{array}$$

Heat conduction equation

$$\varrho\,c\,\frac{\partial T}{\partial t} - \nabla\cdot(k\,\nabla T) = f \tag{6}$$

where the only unknown is the temperature: $T(\boldsymbol{x},t) = ?$

Thermally-homogeneous material: For $k(\boldsymbol{x})$ = const. the heat PDE can be presented as follows

$$\frac{\partial T}{\partial t} = \alpha^2\,\triangle T + \tilde{f} \quad \text{where} \quad \alpha^2 = \frac{k}{\varrho\,c} \quad \text{and} \quad \tilde{f} = \frac{f}{\varrho\,c}. \tag{7}$$

Here: $\alpha^2 = \alpha^2(\boldsymbol{x})$ is the thermal diffusivity $\left[\frac{\mathrm{m}^2}{\mathrm{s}}\right]$,

$\tilde{f} = \tilde{f}(\boldsymbol{x},t)$ is the rate of change of temperature $\left[\frac{\mathrm{K}}{\mathrm{s}}\right]$ due to internal heat sources.

3 Boundary and initial conditions

3.1 Mathematical point of view

From the point of view of mathematics there are **three kinds** of boundary conditions:

1. the **first kind** or **Dirichlet** b.c. – to set a *temperature*, $\hat{T}$ [K], on a boundary:
$$T = \hat{T} \quad \text{on } \partial\mathcal{B}_T, \tag{8}$$
2. the **second kind** or **Neumann** b.c. – to set an *inward heat flux*, $\hat{q}$ [W], normal to the boundary:
$$-\boldsymbol{q}(T)\cdot\boldsymbol{n} = \hat{q} \quad \text{on } \partial\mathcal{B}_q, \tag{9}$$
3. the **third kind** or **Robin** (or **generalized Neumann**) b.c. – to specify the heat flux in terms of an explicit *heat flux*, $\hat{q}$, and a *convective heat transfer coefficient*, h $\left[\frac{\mathrm{W}}{\mathrm{m^2 \cdot K}}\right]$, relative to a *reference temperature*, $\hat{T}$:
$$-\boldsymbol{q}(T)\cdot\boldsymbol{n} = \hat{q} + h(\hat{T} - T) \quad \text{on } \partial\mathcal{B}_h. \tag{10}$$

Here, $\partial\mathcal{B}_T$, $\partial\mathcal{B}_q$, and $\partial\mathcal{B}_h$ are mutually disjoint, complementary parts of the boundary $\partial\mathcal{B}$.

3.2 Physical interpretations

Prescribed temperature: $T = \hat{T}$

Along a boundary the specified temperature, $\hat{T}$, is maintained (the surrounding medium is thermostatic).

*Use the **Dirichlet** b.c. Specify: $\hat{T}$.*

Insulation or symmetry: $-\boldsymbol{q}(T)\cdot\boldsymbol{n} = 0$

To specify where a domain is well insulated, or to reduce model size by taking advantage of symmetry. The condition means that the temperature gradient across the boundary must equal zero. For this to be true, the temperature on one side of the boundary must equal the temperature on the other side (heat cannot transfer across the boundary if there is no temperature difference).

*Use the (homogeneous) **Neumann** b.c. with $\hat{q} = 0$.*

Conductive heat flux: $-\boldsymbol{q}(T)\cdot\boldsymbol{n} = \hat{q}$

To specify a heat flux, $\hat{q}$, that enters a domain. This condition is well suited to represent, for example, any electric heater (neglecting its geometry).

*Use the **Neumann** b.c. Specify: $\hat{q}$.*

Convective heat flux: $-\boldsymbol{q}(T)\cdot\boldsymbol{n} = h(\hat{T} - T)$

To model convective heat transfer with the surrounding environment, where the heat transfer coefficient, h, depends on the geometry and the ambient flow conditions; $\hat{T}$ is the external bulk temperature.

*Use the **Robin** b.c. with $\hat{q} = 0$. Specify: h and $\hat{T}$.*

Heat flux from convection and conduction: $-\boldsymbol{q}(T)\cdot\boldsymbol{n} = \hat{q} + h\,(\hat{T} - T)$

Heat is transferred by convection and conduction. Both contributions are significant and none of them can be neglected. Notice that the conduction heat flux, $\hat{q}$, is in the direction of the inward normal whereas the convection term, $h\,(\hat{T} - T)$, in the direction of the outward normal.

*Use the **Robin** b.c. Specify: h, $\hat{T}$, and $\hat{q}$.*

3.3 Initial-Boundary-Value Problem

IBVP of the heat transfer

Find $T = T(\boldsymbol{x}, t)$ for $\boldsymbol{x} \in \mathcal{B}$ and $t \in [t_0, t_1]$ satisfying the **heat equation**:

$$\varrho\, c\, \dot{T} + \nabla\cdot\boldsymbol{q} - f = 0 \quad \text{where} \quad \boldsymbol{q} = \boldsymbol{q}(T) = -k\,\nabla T\,, \tag{11}$$

with the **initial condition** (at $t = t_0$):

$$T(\boldsymbol{x}, t_0) = T_0(\boldsymbol{x}) \quad \text{in } \mathcal{B}\,, \tag{12}$$

and subject to the **boundary conditions**:

$$T(\boldsymbol{x}, t) = \hat{T}(\boldsymbol{x}, t) \;\text{ on } \partial\mathcal{B}_T\,, \tag{13}$$

$$-\boldsymbol{q}(T)\,\boldsymbol{n} = \hat{q}(\boldsymbol{x}, t) \;\text{ on } \partial\mathcal{B}_q\,, \tag{14}$$

$$-\boldsymbol{q}(T)\cdot\boldsymbol{n} = \hat{q} + h\,(\hat{T} - T) \;\text{ on } \partial\mathcal{B}_h\,, \tag{15}$$

where $\partial\mathcal{B}_T \cup \partial\mathcal{B}_q \cup \partial\mathcal{B}_h = \partial\mathcal{B}$, and $\partial\mathcal{B}_T \cap \partial\mathcal{B}_q = \emptyset$, $\partial\mathcal{B}_T \cap \partial\mathcal{B}_h = \emptyset$, $\partial\mathcal{B}_q \cap \partial\mathcal{B}_h = \emptyset$.

4 Convective heat transfer

4.1 Heat transfer by convection (and conduction)

An important mechanism of heat transfer in fluids is **convection**.

- Heat can be transferred with fluid in motion.
- In such case, a **convective term** containing the convective velocity vector, $\boldsymbol{u}\,\left[\frac{\mathrm{m}}{\mathrm{s}}\right]$, must be added to the Fourier's law of heat conduction:

$$\boldsymbol{q} = -k\,\nabla T + \varrho\, c\, \boldsymbol{u}\, T\,. \tag{16}$$

This modified constitutive law of heat conduction and convection leads to a new form of heat equation.

(Conservative) heat transfer equation with convection

$$c\,\frac{\partial(\varrho T)}{\partial t} + \nabla\cdot(-k\,\nabla T + \varrho\, c\, \boldsymbol{u}\, T) = f\,. \tag{17}$$

Note that here the density is allowed to be time-dependent, $\varrho = \varrho(\boldsymbol{x}, t)$, since it can change in time and space due to the fluid motion causing local compressions and decompressions.

4.2 Nonconservative convective heat transfer

For homogeneous, **incompressible** fluid:

$$\nabla \cdot \boldsymbol{u} = 0 \quad \rightarrow \quad \varrho(\boldsymbol{x}, t) = \text{const}. \tag{18}$$

This assumption produces the following result

$$\nabla \cdot (\varrho\, c\, \boldsymbol{u}\, T) = \varrho\, c\, \nabla T \cdot \boldsymbol{u} + \underbrace{T\, \nabla \cdot (\varrho\, c\, \boldsymbol{u})}_{0} = \varrho\, c\, \nabla T \cdot \boldsymbol{u}\,. \tag{19}$$

By using this result for (17) the following equation is obtained.

Nonconservative heat transfer equation with convection

$$\varrho\, c\, \frac{\partial T}{\partial t} - \nabla \cdot (k\, \nabla T) + \varrho\, c\, \nabla T \cdot \boldsymbol{u} = f \tag{20}$$

$$\text{or, for } k(\boldsymbol{x}) = \text{const}.: \quad \frac{\partial T}{\partial t} = \tilde{f} + \alpha^2\, \triangle T - \nabla T \cdot \boldsymbol{u}\,. \tag{21}$$