MANE 4240 & CIVL 4240
Introduction to Finite Elements

Prof. Suvranu De

Principles of minimum potential energy and Rayleigh-Ritz

Reading assignment:

Section 2.6 + Lecture notes

Summary:
- Potential energy of a system
  - Elastic bar
  - String in tension
- Principle of Minimum Potential Energy
- Rayleigh-Ritz Principle

A generic problem in 1D

$$\frac{d^2u}{dx^2} + x = 0; \qquad 0 < x < 1$$
$$u = 0 \quad at\ x = 0$$
$$u = 1 \quad at\ x = 1$$

Approximate solution strategy:
Guess $u(x) = a_0\varphi_o(x) + a_1\varphi_1(x) + a_2\varphi_2(x) + ...$
Where $\varphi_o(x)$, $\varphi_1(x)$,… are "known" functions and a_o, a_1, etc are constants chosen such that the approximate solution
1. Satisfies the boundary conditions
2. Satisfies the differential equation

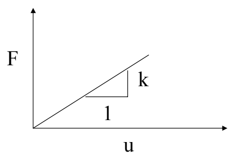
Too difficult to satisfy for general problems!!

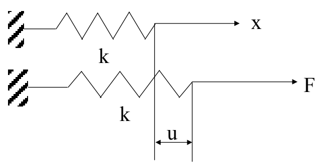
Potential energy

The **potential energy** of an elastic body is defined as

$$\boxed{\Pi = \text{Strain energy (U)} - \text{potential energy of loading (W)}}$$

Strain energy of a linear spring

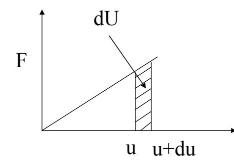




F = Force in the spring
u = deflection of the spring
k = "stiffness" of the spring

Hooke's Law
F = ku

Strain energy of a linear spring



Differential strain energy of the spring for a small change in displacement (du) of the spring

$$dU = Fdu$$

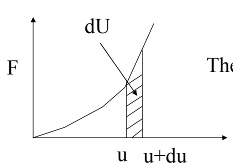
For a linear spring

$$dU = kudu$$

The total strain energy of the spring

$$U = \int_0^u k\,u\,du = \frac{1}{2}k\,u^2$$

Strain energy of a nonlinear spring



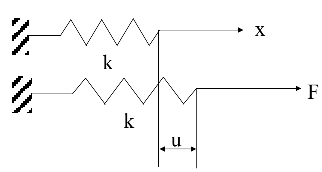
$$dU = Fdu$$

The total strain energy of the spring

$$U = \int_0^u F\,du = \text{Area under the force} - \text{dispalcement curve}$$

Potential energy of the loading (for a single spring as in the figure)

$$W = Fu$$



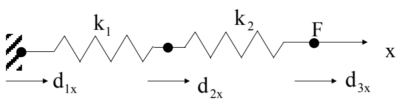
Potential energy of a linear spring

$$\boxed{\Pi = \text{Strain energy (U)} - \text{potential energy of loading (W)}}$$

$$\boxed{\Pi = \frac{1}{2}ku^2 - Fu}$$

Example of how to obtain the equlibr

Principle of <u>minimum potential energy</u> for a system of springs



For this system of spring, first write down the total potential energy of the system as:

$$\Pi_{system} = \left[\frac{1}{2}k_1(d_{2x})^2 + \frac{1}{2}k_2(d_{3x} - d_{2x})^2\right] - Fd_{3x}$$

Obtain the equilibrium equations by minimizing the potential energy

$$\frac{\partial \Pi_{system}}{\partial d_{2x}} = k_1 d_{2x} - k_2(d_{3x} - d_{2x}) = 0 \qquad \textit{Equation (1)}$$

$$\frac{\partial \Pi_{system}}{\partial d_{3x}} = k_2(d_{3x} - d_{2x}) - F = 0 \qquad \textit{Equation (2)}$$

Principle of <u>minimum potential energy</u> for a system of springs

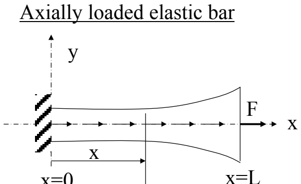
In matrix form, equations 1 and 2 look like

$$\begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}\begin{bmatrix} d_{2x} \\ d_{3x} \end{bmatrix} = \begin{bmatrix} 0 \\ F \end{bmatrix}$$

Does this equation look familiar?

Also look at example problem **<u>worked out in class</u>**

<u>Axially loaded elastic bar</u>



A(x) = cross section at x
b(x) = body force distribution (force per unit length)
E(x) = Young's modulus
u(x) = displacement of the bar at x

Axial strain $\varepsilon = \frac{du}{dx}$

Axial stress $\sigma = E\varepsilon = E\frac{du}{dx}$

Strain energy per unit volume of the bar $dU = \frac{1}{2}\sigma\varepsilon = \frac{1}{2}E\left(\frac{du}{dx}\right)^2$

Strain energy of the bar

$$U = \int dU = \int \frac{1}{2}\sigma\varepsilon\, dV = \int_{x=0}^{L} \frac{1}{2}\sigma\varepsilon\, A dx \qquad \text{since } dV = Adx$$

<u>Axially loaded elastic bar</u>

Strain energy of the bar

$$\boxed{U = \int_0^L \frac{1}{2}\sigma\varepsilon A\, dx = \frac{1}{2}\int_0^L EA\left(\frac{du}{dx}\right)^2 dx}$$

Potential energy of the loading

$$\boxed{W = \int_0^L bu\, dx + Fu(x = L)}$$

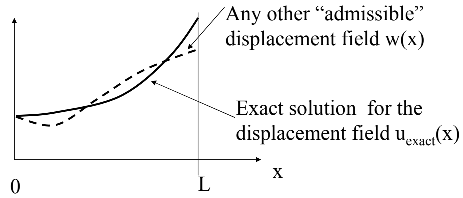
Potential energy of the axially loaded bar

$$\boxed{\Pi = \frac{1}{2}\int_0^L EA\left(\frac{du}{dx}\right)^2 dx - \int_0^L bu\, dx - Fu(x = L)}$$

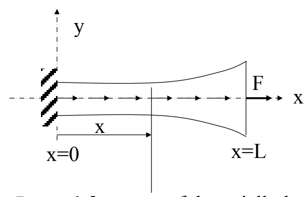
Principle of Minimum Potential Energy

Among all admissible displacements that a body can have, the one that minimizes the total potential energy of the body satisfies the strong formulation

Admissible displacements: these are any reasonable displacement that you can think of that satisfy the ***displacement boundary conditions of the original problem*** (and of course certain minimum ***continuity requirements***). Example:



Lets see what this means for an axially loaded elastic bar



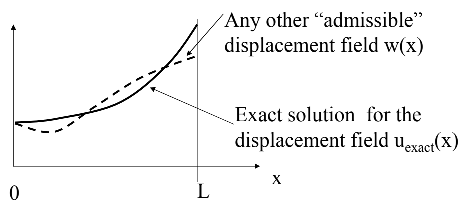
A(x) = cross section at x
b(x) = body force distribution (force per unit length)
E(x) = Young's modulus

Potential energy of the axially loaded bar corresponding to the exact solution $u_{exact}(x)$

$$\Pi(u_{exact}) = \frac{1}{2}\int_0^L EA\left(\frac{du_{exact}}{dx}\right)^2 dx - \int_0^L bu_{exact}\, dx - Fu_{exact}(x = L)$$

Potential energy of the axially loaded bar corresponding to the "**admissible**" displacement w(x)

$$\Pi(w) = \frac{1}{2}\int_0^L EA\left(\frac{dw}{dx}\right)^2 dx - \int_0^L bw\, dx - Fw(x = L)$$



Example:

$$AE\frac{d^2u}{dx^2} + b = 0; \qquad 0 < x < L$$
$$u = 0 \quad at \;\; x = 0$$
$$EA\frac{du}{dx} = F \;\; at \;\; x = L$$

Assume EA=1; b=1; L=1; F=1

Analytical solution is

$$u_{exact} = 2x - \frac{x^2}{2}$$

Potential energy corresponding to this analytical solution

$$\Pi(u_{exact}) = \frac{1}{2}\int_0^1 \left(\frac{du_{exact}}{dx}\right)^2 dx - \int_0^1 u_{exact}\, dx - u_{exact}(x = 1) = -\frac{7}{6}$$

Now assume an admissible displacement

$$w = x$$

Why is this an "**admissible**" displacement? This displacement is quite arbitrary. But, it satisfies the given **displacement boundary condition** w(x=0)=0. Also, its first derivate does not blow up.

Potential energy corresponding to this admissible displacement

$$\Pi(w) = \frac{1}{2}\int_0^1 \left(\frac{dw}{dx}\right)^2 dx - \int_0^1 w\ dx - w(x=1) = -1$$

Notice

since $-\frac{7}{6} < -1$

$$\boxed{\Pi(u_{exact}) < \Pi(w)}$$

Principle of Minimum Potential Energy

Among all admissible displacements that a body can have, the one that minimizes the total potential energy of the body satisfies the strong formulation

Mathematical statement: If 'u_{exact}' is the exact solution (which satisfies the differential equation together with the boundary conditions), and 'w' is an admissible displacement (that is quite arbitrary except for the fact that it **satisfies the displacement boundary conditions** and its **first derivative does not blow up**), then

$$\Pi(u_{exact}) < \Pi(w)$$

unless $w=u_{exact}$ (i.e. **the exact solution minimizes the potential energy**)

The Principle of Minimum Potential Energy and the strong formulation are exactly equivalent statements of the same problem.

The exact solution (u_{exact}) that satisfies the strong form, renders the potential energy of the system a minimum.

So, why use the Principle of Minimum Potential Energy?

The short answer is that it is much less demanding than the strong formulation. The long answer is, it

1. requires only the first derivative to be finite
2. incorporates the force boundary condition automatically. The admissible displacement (which is the function that you need to choose) needs to satisfy only the displacement boundary condition

Finite element formulation, takes as its starting point, not the strong formulation, but the **Principle of Minimum Potential Energy**.

Task is to find the function 'w' that minimizes the potential energy of the system

$$\boxed{\Pi(w) = \frac{1}{2}\int_0^L EA\left(\frac{dw}{dx}\right)^2 dx - \int_0^L bw\ dx - Fw(x=L)}$$

From the Principle of Minimum Potential Energy, that function 'w' is the exact solution.

<u>Rayleigh-Ritz Principle</u>

The minimization of the potential energy is difficult to perform exactly.
The Rayleigh-Ritz principle is an **<u>approximate</u>** way of doing this.

Step 1. Assume a solution

$$w(x) = a_0\varphi_o(x) + a_1\varphi_1(x) + a_2\varphi_2(x) + \ldots$$

Where $\varphi_o(x)$, $\varphi_1(x)$,… are "***admissible***" functions and a_o, a_1, etc are constants to be determined from the solution.

<u>Rayleigh-Ritz Principle</u>

Step 2. Plug the approximate solution into the potential energy

$$\Pi(\mathrm{w}) = \frac{1}{2}\int_0^L \mathrm{EA}\left(\frac{\mathrm{dw}}{\mathrm{dx}}\right)^2 \mathrm{dx} - \int_0^L \mathrm{bw\ dx} - \mathrm{Fw(x = L)}$$

$$\Rightarrow \Pi(\mathrm{a}_0, \mathrm{a}_1, \ldots) = \frac{1}{2}\int_0^L \mathrm{EA}\left(a_0\frac{\mathrm{d}\varphi_0}{\mathrm{dx}} + a_1\frac{\mathrm{d}\varphi_1}{\mathrm{dx}} + \ldots\right)^2 \mathrm{dx}$$
$$- \int_0^L \mathrm{b}\left(a_0\varphi_0 + a_1\varphi_1 + \ldots\right)\mathrm{dx}$$
$$- \mathrm{F}\left(a_0\varphi_0(x = L) + a_1\varphi_1(x = L) + \ldots\right)$$

<u>Rayleigh-Ritz Principle</u>

Step 3. Obtain the coefficients a_o, a_1, etc by setting

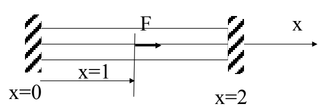
$$\frac{\partial \Pi(\mathrm{w})}{\partial a_i} = 0, \quad i = 0,1,2,\ldots$$

The approximate solution is

$$u(x) = a_0\varphi_o(x) + a_1\varphi_1(x) + a_2\varphi_2(x) + \ldots$$

Where the coefficients have been obtained from step 3

Example of application of Rayleigh Ritz Principle



E=A=1
F=2

The potential energy of this bar (of length 2) is

$$\Pi(\mathrm{u}) = \underbrace{\frac{1}{2}\int_0^2 \left(\frac{\mathrm{du}}{\mathrm{dx}}\right)^2 \mathrm{dx}}_{\textit{Strain Energy}} - \underbrace{\mathrm{Fu(x = 1)}}_{\substack{\textit{Potential Energy} \\ \textit{of load F applied} \\ \textit{at x=1}}}$$

Let us assume a polynomial "admissible" displacement field

$$\mathrm{u} = a_0 + a_1 x + a_2 x^2$$

Note that this is NOT the analytical solution for this problem.

Example of application of Rayleigh Ritz Principle

For this "admissible" displacement to satisfy the **displacement boundary conditions** the following conditions must be satisfied:

$$\mathrm{u}(\mathrm{x}=0)=a_0=0$$
$$\mathrm{u}(\mathrm{x}=2)=a_0+2a_1+4a_2=0$$

Hence, we obtain

$$a_0=0$$
$$a_1=-2a_2$$

Hence, the "admissible" displacement simplifies to

$$\mathrm{u}=a_0+a_1x+a_2x^2$$
$$=a_2\left(-2x+x^2\right)$$

Now we apply **Rayleigh Ritz principle**, which says that if I plug this approximation into the expression for the potential energy Π, I can obtain the unknown (in this case a_2) by minimizing Π

$$\Pi(\mathrm{u})=\frac{1}{2}\int_0^2\left(\frac{\mathrm{du}}{\mathrm{dx}}\right)^2\mathrm{dx}-\mathrm{Fu}(\mathrm{x}=1)$$
$$=\frac{1}{2}\int_0^2\left(\frac{\mathrm{d}}{\mathrm{dx}}\left\{a_2\left(-2x+x^2\right)\right\}\right)^2\mathrm{dx}-\mathrm{F}\left\{a_2\left(-2x+x^2\right)\right\}_{\substack{evaluated\\ at\ x=1}}$$
$$=\frac{4}{3}a_2^{\;2}+2a_2$$
$$\frac{\partial\Pi}{\partial a_2}=0$$
$$\Rightarrow\frac{8}{3}a_2+2=0$$
$$\Rightarrow a_2=-\frac{3}{4}$$

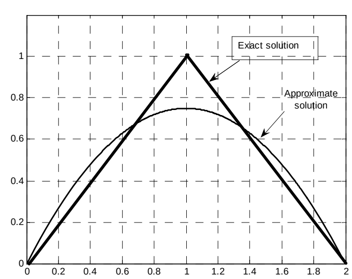
Hence the approximate solution to this problem, using the Rayleigh-Ritz principle is

$$\mathrm{u}=a_0+a_1x+a_2x^2$$
$$=a_2\left(-2x+x^2\right)$$
$$=-\frac{3}{4}\left(-2x+x^2\right)$$

Notice that the exact answer to this problem (can you prove this?) is

$$\mathrm{u}_{\mathrm{exact}}=\begin{cases} x & for\ \ 0\le x<1\\ 2-x & for\ \ 1\le x\le 2\end{cases}$$

The **displacement** solution :



How can you improve the approximation?

The **stress** within the bar: