7. THE EIGENPROBLEM FOR SYMMETRIC, SECOND-ORDER TENSORS

7.1 Initial Comments

The eigenproblem is one of the most important topics in continuum and computational mechanics. Here we focus on the development in connection with second-order tensors, such the stress, strain and moment-of-inertia tensors, and the terminology, which is applicable to the general eigenproblem.

7.2 The Eigenproblem

When a second-order tensor, $\boldsymbol{T}$, operates on a vector, $\boldsymbol{u}$, the result is another vector, $\boldsymbol{v}$. There is the possibility that when $\boldsymbol{T}$ operates on a particular vector, $\boldsymbol{p}$, the result is one of simply a scalar times the vector itself. This possibility is called the eigenproblem and is stated as follows. If a vector $\boldsymbol{p}$ exists such that

$$\boldsymbol{T} \cdot \boldsymbol{p} = \lambda \boldsymbol{p} \tag{7-1}$$

then λ and $\boldsymbol{p}$ are said to be an eigenvalue and eigenvector, respectively, of the tensor $\boldsymbol{T}$. Taken together, the set $(\lambda, \boldsymbol{p})$ is an eigenpair of $\boldsymbol{T}$. An alternative description is to call λ a principal value and $\boldsymbol{p}$ a principal direction of $\boldsymbol{T}$. The indicial form of (7-1) is

$$T_{ij} p_j = \lambda p_i \tag{7-2}$$

and two equivalent matrix forms are

$$[T]\{p\} = \lambda\{p\} \qquad \big[[T] - \lambda[I]\big]\{p\} = \{0\} \tag{7-3}$$

We will work with the second form of (7-3) under the assumption that $\boldsymbol{T}$ is symmetric, or that $\boldsymbol{T} = \boldsymbol{T}^T$.

7.3 Obtaining Eigenpairs of Symmetric Tensors

Suppose we are given the components, $\overset{e-e}{[T]}$ of a symmetric tensor with respect to the basis $\boldsymbol{e}_i \otimes \boldsymbol{e}_j$ and we wish to find a solution $\overset{e}{\{p\}}$ of (7-3). One solution is the null vector but this is of no interest. Is it possible that a nontrivial solution exists? Suppose we use Cramer's rule which states that the solution for the first component of the eigenvector is

$$\overset{e}{p}_1 = \frac{\begin{vmatrix} 0 & \overset{e}{T}_{12} & \overset{e}{T}_{13} \\ 0 & \overset{e}{T}_{22}-\lambda & \overset{e}{T}_{23} \\ 0 & \overset{e}{T}_{32} & \overset{e}{T}_{33}-\lambda \end{vmatrix}}{\left\| \left[\overset{e-e}{T} \right] - \lambda [I] \right\|} \tag{7-4}$$

If we expand the determinant in the denominator the result is a third-order polynomial in λ which we write as follows:

$$\left\| \left[\overset{e-e}{T} \right] - \lambda [I] \right\| = \begin{vmatrix} \overset{e}{T}_{11}-\lambda & \overset{e}{T}_{12} & \overset{e}{T}_{13} \\ \overset{e}{T}_{21} & \overset{e}{T}_{22}-\lambda & \overset{e}{T}_{23} \\ \overset{e}{T}_{31} & \overset{e}{T}_{32} & \overset{e}{T}_{33}-\lambda \end{vmatrix} = -P(\lambda) \tag{7-5}$$

The polynomial $P(\lambda)$ is called the characteristic polynomial. Since the numerator of (7-4) is zero the only possibility of a nontrivial solution is that the denominator is zero as well so we impose the condition, called the characteristic equation, that

$$P(\lambda) = 0 \tag{7-6}$$

The three values of λ labeled λ_1, λ_2 and λ_3, that are the zeroes of $P(\lambda)$ or, equivalently, that are the solutions to (7-6), are the eigenvalues of $\boldsymbol{T}$.

Suppose for the moment that the eigenvalues are distinct, i.e., $\lambda_1 \neq \lambda_2 \neq \lambda_3$. Now we obtain the components of the eigenvector associated with the eigenvalue λ_1 by obtaining a solution to the equation

$$\left[\left[\overset{e-e}{T} \right] - \lambda_1 [I] \right] \begin{Bmatrix} p^e_{1,1} \\ p^e_{1,2} \\ p^e_{1,3} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \tag{7-7}$$

The first subscript for the components of the eigenvector denotes that these components are associated with the first eigenvalue. Only two of the three scalar equations in (7-7) are independent because of the characteristic equation. Therefore we can choose one of the components arbitrarily. Suppose we choose the first component to be unity, i.e., $p^e_{1,1} = 1$. Then we use any two of the equations to solve for $p^e_{1,2}$ and $p^e_{1,3}$. The third equation should automatically be satisfied. If a contradiction in the solution arises, this often indicates that you should choose $p^e_{1,1} = 0$ and $p^e_{1,2} = 1$ and then use any one of the equations to solve for the other component.

For the second and third eigenvectors just set $\lambda = \lambda_2$ and $\lambda = \lambda_3$, respectively, and repeat the process.

As a check we note the following properties (proven later) of eigenvectors and eigenvalues must be satisfied for symmetric, real tensors:

1. The eigenvalues are real.
2. The eigenvectors may be complex but the imaginary part is proportional to the real part for each eigenvector. Therefore, there is no loss in generality if just the real part is retained.
3. The eigenvectors are orthogonal.

Important consequences of Item 3 are: (i) the third eigenvector can be obtained by taking the cross product of the first two, and (ii) after normalization, the eigenvectors form an orthonormal basis, $\boldsymbol{p}_i$. Since any scalar times an eigenvector is still an eigenvector, a sign can be changed if necessary to form a right-handed orthonormal principal basis.

Recall that

$$\boldsymbol{T} \cdot \boldsymbol{p}_1 = \lambda_1 \boldsymbol{p}_1 \tag{7-8}$$

Dot with respect to the other two eigenvectors to obtain

$$\begin{aligned} &\boldsymbol{p}_2 \cdot \boldsymbol{T} \cdot \boldsymbol{p}_1 = \lambda_1 \boldsymbol{p}_2 \cdot \boldsymbol{p}_1 = 0 \quad \text{or} \quad \overset{p-p}{T}_{21} = 0 \quad \overset{p-p}{T}_{12} = 0 \\ &\boldsymbol{p}_3 \cdot \boldsymbol{T} \cdot \boldsymbol{p}_1 = \lambda_1 \boldsymbol{p}_3 \cdot \boldsymbol{p}_1 = 0 \quad \text{or} \quad \overset{p-p}{T}_{31} = 0 \quad \overset{p-p}{T}_{13} = 0 \end{aligned} \tag{7-9}$$

where the equations on the right are obtained by symmetry. The process is repeated starting with the other two eigenvectors to show that all the off-diagonal components of $\boldsymbol{T}$ with respect to the principal basis $\boldsymbol{p}_i \otimes \boldsymbol{p}_j$ are zero. The result is that we have what is called the **spectral decomposition** of $\boldsymbol{T}$ given in the following equivalent forms:

$$\begin{aligned} &\boldsymbol{T} = \overset{p-p}{T}_{ij}\, \boldsymbol{p}_i \otimes \boldsymbol{p}_j = \sum_{i=1}^{3} \lambda_i \boldsymbol{p}_i \otimes \boldsymbol{p}_i = \lambda_1 \boldsymbol{p}_1 \otimes \boldsymbol{p}_1 + \lambda_2 \boldsymbol{p}_2 \otimes \boldsymbol{p}_2 + \lambda_3 \boldsymbol{p}_3 \otimes \boldsymbol{p}_3 \\ &\boldsymbol{T} = \langle \boldsymbol{p} \rangle \otimes \overset{p-p}{[T]}_d \{\boldsymbol{p}\} \qquad \overset{p-p}{[T]}_d = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \end{aligned} \tag{7-10}$$

where the subscript "d" on the matrix is sometimes used to emphasize the point that the matrix is diagonal.

Recall that we started with the assumption that the components of $\boldsymbol{T}$ were given with respect to the $\boldsymbol{e}_i \otimes \boldsymbol{e}_j$ basis. When the eigenvectors are computed, the starting point implies that the principal basis is described in terms of the basis $\boldsymbol{e}_i$ or

$$\begin{aligned}\boldsymbol{p}_1 &= p^e_{1,1}\boldsymbol{e}_1 + p^e_{1,2}\boldsymbol{e}_2 + p^e_{1,3}\boldsymbol{e}_3\\ \boldsymbol{p}_2 &= p^e_{2,1}\boldsymbol{e}_1 + p^e_{2,2}\boldsymbol{e}_2 + p^e_{2,3}\boldsymbol{e}_3\\ \boldsymbol{p}_3 &= p^e_{3,1}\boldsymbol{e}_1 + p^e_{3,2}\boldsymbol{e}_2 + p^e_{3,3}\boldsymbol{e}_3\end{aligned} \tag{7-11}$$

With the use of (7-11), the transformation matrices between the two bases are simply

$$\overset{p-e}{[a]} = \begin{bmatrix} p^e_{1,1} & p^e_{1,2} & p^e_{1,3} \\ p^e_{2,1} & p^e_{2,2} & p^e_{2,3} \\ p^e_{3,1} & p^e_{3,2} & p^e_{3,3} \end{bmatrix} \qquad \overset{e-p}{[a]} = \overset{p-e}{[a]}{}^T \tag{7-12}$$

The equation on the right holds because both bases are orthonormal. It follows that the components with respect to the two bases must satisfy the following transformation relations:

$$\overset{e-e}{[T]} = \overset{e-p}{[a]}\overset{p-p}{[T]}_d\overset{p-e}{[a]} \qquad \overset{p-p}{[T]}_d = \overset{p-e}{[a]}\overset{e-e}{[T]}\overset{e-p}{[a]} \tag{7-13}$$

These equations can often be used as a check as to whether or not the eigenpairs have been obtained correctly.

Suppose the matrix of the components in the $\boldsymbol{e}_i \otimes \boldsymbol{e}_j$ contain zeroes in a row (or column) other than a diagonal term. As an example suppose

$$\overset{e-e}{[T]} = \begin{bmatrix} \overset{e-e}{T}_{11} & \overset{e-e}{T}_{12} & 0 \\ \overset{e-e}{T}_{21} & \overset{e-e}{T}_{22} & 0 \\ 0 & 0 & \overset{e-e}{T}_{33} \end{bmatrix} \tag{7-14}$$

Then it follows immediately that $\boldsymbol{e}_3$ is an eigenvector with an associated eigenvalue of $\overset{e-e}{T}_{33}$.

7.4 Repeated Eigenvalues

Now suppose the first two eigenvalues are repeated bust distinct from the third, i.e., $\lambda_1 = \lambda_2 = \lambda^* \neq \lambda_3$. Now the equations of (7-7) will have two degrees of redundancy or, expressed differently, two components of the eigenvector can be picked arbitrarily unless a contradiction arises that suggests a zero should have been chosen for at least one of the components.

Another way to approach the problem of finding the set of eigenvectors is to first set $\lambda = \lambda_3$ and find the eigenvector $\boldsymbol{p}_3$. Now choose any vector perpendicular to $\boldsymbol{p}_3$ and label this vector $\boldsymbol{p}_2$. This eigenvector will be a solution to the eigenproblem for $\lambda = \lambda^*$.

Then choose $\boldsymbol{p}_1 = \boldsymbol{p}_2 \times \boldsymbol{p}_3$, normalize all three vectors to obtain the principal basis. Then the spectral decomposition of $\boldsymbol{T}$ becomes

$$\boldsymbol{T} = \lambda^* (\boldsymbol{p}_1 \otimes \boldsymbol{p}_1 + \boldsymbol{p}_2 \otimes \boldsymbol{p}_2) + \lambda_3 \boldsymbol{p}_3 \otimes \boldsymbol{p}_3 \tag{7-15}$$

There is not a unique choice for a vector perpendicular to $\boldsymbol{p}_3$. Suppose an alternative choice is $\hat{\boldsymbol{p}}_2$ with a corresponding derivation for the third eigenvector $\hat{\boldsymbol{p}}_1$. The spectral decomposition of $\boldsymbol{T}$ is

$$\boldsymbol{T} = \lambda^* (\hat{\boldsymbol{p}}_1 \otimes \hat{\boldsymbol{p}}_1 + \hat{\boldsymbol{p}}_2 \otimes \hat{\boldsymbol{p}}_2) + \lambda_3 \boldsymbol{p}_3 \otimes \boldsymbol{p}_3 \tag{7-16}$$

The two forms (7-15) and (7-16) are the same if and only if

$$(\boldsymbol{p}_1 \otimes \boldsymbol{p}_1 + \boldsymbol{p}_2 \otimes \boldsymbol{p}_2) = (\hat{\boldsymbol{p}}_1 \otimes \hat{\boldsymbol{p}}_1 + \hat{\boldsymbol{p}}_2 \otimes \hat{\boldsymbol{p}}_2) \tag{7-17}$$

The use of the transformation relations can be used to show that (7-17) holds for any pair of orthogonal bases in a plane. Either side is just the two-dimensional identity tensor.

If all three eigenvalues are the same, i.e., $\lambda_1 = \lambda_2 = \lambda_3 = \lambda^*$, then any vector is an eigenvector and the tensor is said to be isotropic (no preferred direction) and must have the form

$$\boldsymbol{T} = \lambda^* \boldsymbol{I} \qquad \text{(isotropic)} \tag{7-18}$$

7.5 The Characteristic Equation

Recall that the characteristic equation is $P(\lambda) = 0$ in which the characteristic polynomial is given by (7-5). Here we provide a form for the polynomial without the need for taking the determinant. First we define three primary invariants of $\boldsymbol{T}$ as follows:

$$I_T = tr\boldsymbol{T} \qquad II_T = tr\boldsymbol{T}^2 \qquad III_T = tr\boldsymbol{T}^3 \tag{7-19}$$

or, in matrix form if the components are given in the $\boldsymbol{e}_i \otimes \boldsymbol{e}_j$ system

$$I_T = tr\overset{e-e}{[T]} \qquad II_T = tr[\overset{e-e}{[T]}\overset{e-e}{[T]}] \qquad III_T = tr[\overset{e-e}{[T]}\overset{e-e}{[T]}\overset{e-e}{[T]}] \tag{7-20}$$

It can be shown (see the extended notes) that the characteristic equation can be given in the form

$$\lambda^3 - I^* \lambda^2 - II^* \lambda - III^* = 0 \tag{7-21}$$

in which the coefficients are called the characteristic invariants and are related to the primary invariants as follows:

$$I_T^* = I_T \qquad II_T^* = \frac{1}{2}(II_T - I_T^2) \qquad \mathrm{III}_T^* = det\boldsymbol{T} = \frac{1}{6}(I_T^3 - 3I_T II_T + 2III_T) \tag{7-22}$$

Note: Some use minus one times the above definition for II_T^* in which case the third term in the characteristic equation becomes positive, i.e., then the signs in the characteristic equation alternate. Since the primary and characteristic invariants can be evaluated in any basis, suppose we use the components in the principal basis to obtain

$$
\begin{aligned}
&I_T = \lambda_1 + \lambda_2 + \lambda_3 \qquad \mathrm{II}_T = \lambda_1^2 + \lambda_2^2 + \lambda_3^2 \qquad \mathrm{III}_T = \lambda_1^3 + \lambda_2^3 + \lambda_3^3 \\
&I_T^* = \lambda_1 + \lambda_2 + \lambda_3 \qquad \mathrm{II}_T^* = -(\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1) \qquad \mathrm{III}_T^* = \lambda_1\lambda_2\lambda_3
\end{aligned} \tag{7-23}
$$

Since the eigenvalues are obtained from the characteristic equation where the coefficients are invariant, it follows that the eigenvalues are also invariants. Stated differently, it doesn't matter which set of components of a tensor $\boldsymbol{T}$ that we use, the characteristic equation will be the same. For convenience we summarize the three sets of invariants that we have defined so far:

<u>Sets of Invariants</u>

Primary: I_T, II_T, III_T Characteristic: I_T^*, II_T^*, III_T^* Eigenvalues: $\lambda_1, \lambda_2, \lambda_3$ (7-24)

We will show later that there are at most, three independent invariants.

Sometimes we see functions of a tensor that seem odd at first such as $sin(\boldsymbol{T})$, or $\boldsymbol{T}^{1/3}$. We use the generic description $f(\boldsymbol{T})$ for such cases. The definition is based on the spectral form of the tensor as follows:

$$
f(\boldsymbol{T}) = f(\lambda_1)\boldsymbol{p}_1 \otimes \boldsymbol{p}_1 + f(\lambda_2)\boldsymbol{p}_2 \otimes \boldsymbol{p}_2 + f(\lambda_3)\boldsymbol{p}_3 \otimes \boldsymbol{p}_3 \tag{7-25}
$$

We note the following: (i) the result is a second-order tensor, (ii) if the tensor is expressed in terms of the basis $\boldsymbol{e}_i \otimes \boldsymbol{e}_j$, then the complete eigenproblem must be solved to get the eigenvalues and eigenvectors, and (iii) the principal components of the resulting tensor given in (7-25) must be transformed back to the $\boldsymbol{e}_i \otimes \boldsymbol{e}_j$ basis.

7.6 The Cayley-Hamilton Theorem

The Cayley-Hamilton theorem states that a tensor (matrix) satisfies its own characteristic equation. In the next equation, we first give the characteristic equation followed by the Cayley-Hamilton theorem for a tensor and then the same theorem expressed in terms of the matrix of components for any basis;

$$
\begin{aligned}
&\lambda^3 - I^*\lambda^2 - II^*\lambda - III^* = 0 \\
&\boldsymbol{T}^3 - I^*\boldsymbol{T}^2 - II^*\boldsymbol{T} - III^*\boldsymbol{I} = \boldsymbol{0} \\
&[T]^3 - I^*[T]^2 - II^*[T] - III^*[I] = [0]
\end{aligned} \tag{7-26}
$$

in which $\boldsymbol{I}$ and $\boldsymbol{0}$ denote the identity and null tensors, respectively. To prove the theorem we choose to use the last form in which components in the principal basis are used. The only nonzero terms are on the diagonal. If we look at any one equation associated with a diagonal term we merely get the first equation with ne of the eigenvalues in place of λ. But each eigenvalue satisfies the characteristic equation. If an equation holds for components in one basis it also holds for any other basis. Therefore it is appropriate to write the Cayley-Hamilton theorem in the form involving direct notation.

Next we look at certain implications that follow from the Cayley-Hamilton theorem. Suppose we multiply the second of (7-26) by $\boldsymbol{T}$ and take the trace of each term. The result is that

$$tr(\boldsymbol{T}^4) = I^* tr(\boldsymbol{T}^3) + II^* tr(\boldsymbol{T}^2) + III^* tr(\boldsymbol{T}) \qquad (7\text{-}27)$$

This result states that the fourth primary invariant is a function of the first three invariants and is, therefore, not an independent invariant. The same argument holds for higher primary invariants. There are, at most, three independent invariants of a tensor.

Now we multiply the second of (7-26) by $\boldsymbol{T}^{-1}$ to obtain

$$\boldsymbol{T}^{-1} = \frac{1}{det(\boldsymbol{T})}\left(\boldsymbol{T}^2 - I^*\boldsymbol{T} - II^*\boldsymbol{I}\right) \qquad III^* = det(\boldsymbol{T}) \qquad (7\text{-}28)$$

which is an expression for $\boldsymbol{T}^{-1}$. We see that the inverse does not exist if the determinant is zero, a result known from linear algebra for the inverse of a matrix.

7.6 Theorems Related to the Eigenpairs

Theorem: If $\boldsymbol{T}$ is real and symmetric, then the eigenvalues are real.
Proof: Suppose an eigenvalue is complex, i.e., $\lambda = \lambda_R + i\lambda_I$ in which $i = \sqrt{-1}$. The complex conjugate is $\bar{\lambda} = \lambda_R - i\lambda_I$ and we note that $\bar{\lambda}\lambda = \lambda_R^2 + \lambda_I^2 > 0$. By definition of an eigenpair

$$\boldsymbol{T}\cdot\boldsymbol{v} = \lambda\boldsymbol{v} \qquad \text{(i)}$$

Take the complex conjugate

$$\overline{\boldsymbol{T}\cdot\boldsymbol{v}} = \overline{\lambda\boldsymbol{v}} \qquad \text{(ii)}$$

The conjugate of a product equals the product of the conjugates, and $\boldsymbol{T} = \bar{\boldsymbol{T}}$ because $\boldsymbol{T}$ is real. Therefore (ii) becomes

$$\boldsymbol{T}\cdot\bar{\boldsymbol{v}} = \bar{\lambda}\bar{\boldsymbol{v}} \qquad \text{(iii)}$$

Dot the terms in (i) with $\bar{\boldsymbol{v}}$ and the terms in (iii) with $\boldsymbol{v}$ and subtract to obtain

$$\bar{\boldsymbol{v}}\cdot\boldsymbol{T}\cdot\boldsymbol{v} - \boldsymbol{v}\cdot\boldsymbol{T}\cdot\bar{\boldsymbol{v}} = (\lambda - \bar{\lambda})\boldsymbol{v}\cdot\bar{\boldsymbol{v}} \qquad \text{(iv)}$$

But $\bar{\boldsymbol{v}}\cdot\boldsymbol{T}\cdot\boldsymbol{v} = \boldsymbol{v}\cdot\boldsymbol{T}\cdot\bar{\boldsymbol{v}}$ because $\boldsymbol{T}$ is symmetric and $\boldsymbol{v}\cdot\bar{\boldsymbol{v}} > 0$ using the same argument that yields $\bar{\lambda}\lambda > 0$. Therefore (iv) implies that $\lambda_I = 0$ and, therefore, λ is real. **EOP**

We note from (i) and (iii) that $\boldsymbol{T}\cdot\overline{\boldsymbol{v}}=\lambda\overline{\boldsymbol{v}}$ and $\boldsymbol{T}\cdot\boldsymbol{v}=\lambda\boldsymbol{v}$. By an addition and a subtraction we have $\boldsymbol{T}\cdot\boldsymbol{v}_R=\lambda\boldsymbol{v}_R$ and $\boldsymbol{T}\cdot\boldsymbol{v}_I=\lambda\boldsymbol{v}_I$ so that the imaginary part of the eigenvector is proportional to the real part. Therefore there is no loss of generality if we just ignore the imaginary part and assume the eigenvector is real.

Theorem: If two eigenvalues λ_1 and λ_2 are distinct, then the corresponding eigenvalues are orthogonal.

Proof: From the definition of eigenpairs, we have

$$\boldsymbol{T}\cdot\boldsymbol{p}_1=\lambda_1\boldsymbol{p}_1 \qquad \text{(i)}$$

$$\boldsymbol{T}\cdot\boldsymbol{p}_2=\lambda_2\boldsymbol{p}_2 \qquad \text{(ii)}$$

Dot the terms in (i) with $\boldsymbol{p}_2$ and the terms in (ii) with $\boldsymbol{p}_1$ and subtract:

$$\boldsymbol{p}_2\cdot\boldsymbol{T}\cdot\boldsymbol{p}_1-\boldsymbol{p}_1\cdot\boldsymbol{T}\cdot\boldsymbol{p}_2=(\lambda_1-\lambda_2)\boldsymbol{p}_1\cdot\boldsymbol{p}_2 \qquad \text{(iii)}$$

But $\boldsymbol{p}_2\cdot\boldsymbol{T}\cdot\boldsymbol{p}_1=\boldsymbol{p}_1\cdot\boldsymbol{T}\cdot\boldsymbol{p}_2$ because $\boldsymbol{T}$ is symmetric and $\lambda_1-\lambda_2\neq 0$ because the eigenvalues are distinct. Therefore $\boldsymbol{p}_1\cdot\boldsymbol{p}_2=0$. **EOP**

7.7 Mohr's Circle

Suppose we know one principal direction. Call this direction $\boldsymbol{p}_3=\boldsymbol{e}_3$. Now consider two bases $\boldsymbol{E}_i$ and $\boldsymbol{e}_i$ with $\boldsymbol{E}_1=\cos\theta\boldsymbol{e}_1+\sin\theta\boldsymbol{e}_2$ and $\boldsymbol{E}_2=-\sin\theta\boldsymbol{e}_1+\cos\theta\boldsymbol{e}_2$. The corresponding transformation matrices for the plane perpendicular to $\boldsymbol{p}_3$ are

$$\overset{E-e}{[a]}=\begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix} \qquad \overset{e-E}{[a]}=\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix} \tag{7-29}$$

Consider the transformation of tensor components from the $\boldsymbol{e}_i\otimes\boldsymbol{e}_j$ system to the $\boldsymbol{E}_i\otimes\boldsymbol{E}_j$ basis:

$$\overset{E-E}{[T]}=\overset{E-e}{[a]}\overset{e-e}{[T]}\overset{e-E}{[a]} \tag{7-30}$$

The intermediate steps are

$$\overset{e-e}{[T]}\overset{e-E}{[a]}=\begin{bmatrix}\overset{e-e}{T}_{11} & \overset{e-e}{T}_{12}\\ \overset{e-e}{T}_{21} & \overset{e-e}{T}_{22}\end{bmatrix}\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}$$

$$=\begin{bmatrix}\cos\theta\overset{e-e}{T}_{11}+\sin\theta\overset{e-e}{T}_{12} & -\sin\theta\overset{e-e}{T}_{11}+\cos\theta\overset{e-e}{T}_{12}\\ \cos\theta\overset{e-e}{T}_{21}+\sin\theta\overset{e-e}{T}_{22} & -\sin\theta\overset{e-e}{T}_{21}+\cos\theta\overset{e-e}{T}_{22}\end{bmatrix}$$

Multiplying on the left by $\overset{E-e}{[a]}$ results in

$$\overset{E-E}{T}_{11} = \cos\theta\left(\cos\theta \overset{e-e}{T}_{11} + \sin\theta \overset{e-e}{T}_{12}\right) + \sin\theta\left(\cos\theta \overset{e-e}{T}_{21} + \sin\theta \overset{e-e}{T}_{22}\right)$$

$$\overset{E-E}{T}_{12} = \cos\theta\left(-\sin\theta \overset{e-e}{T}_{11} + \cos\theta \overset{e-e}{T}_{12}\right) + \sin\theta\left(-\sin\theta \overset{e-e}{T}_{21} + \cos\theta \overset{e-e}{T}_{22}\right) \tag{7-31}$$

$$\overset{E-E}{T}_{22} = -\sin\theta\left(-\sin\theta \overset{e-e}{T}_{11} + \cos\theta \overset{e-e}{T}_{12}\right) + \cos\theta\left(-\sin\theta \overset{e-e}{T}_{21} + \cos\theta \overset{e-e}{T}_{22}\right)$$

and, with the use of symmetry,

$$\overset{E-E}{T}_{11} = \cos^2\theta \overset{e-e}{T}_{11} + \sin^2\theta \overset{e-e}{T}_{22} + 2\sin\theta\cos\theta \overset{e-e}{T}_{21}$$

$$\overset{E-E}{T}_{12} = \cos\theta\sin\theta(-\overset{e-e}{T}_{11} + \overset{e-e}{T}_{22}) + (\cos^2\theta - \sin^2\theta)\overset{e-e}{T}_{12} \tag{7-32}$$

$$\overset{E-E}{T}_{22} = \sin^2\theta \overset{e-e}{T}_{11} + \cos^2\theta \overset{e-e}{T}_{22} - 2\sin\theta\cos\theta \overset{e-e}{T}_{21}$$

Now use the trigonometric relations

$$2\sin\theta\cos\theta = \sin 2\theta \qquad \cos^2\theta = \frac{1+\cos 2\theta}{2} \qquad \sin^2\theta = \frac{1-\cos 2\theta}{2} \tag{7-33}$$

so that (7-32) becomes

$$\overset{E-E}{T}_{11} = \frac{\overset{e-e}{T}_{11} + \overset{e-e}{T}_{22}}{2} + \frac{\overset{e-e}{T}_{11} - \overset{e-e}{T}_{22}}{2}\cos 2\theta + \overset{e-e}{T}_{21}\sin 2\theta$$

$$\overset{E-E}{T}_{12} = -\frac{\overset{e-e}{T}_{11} - \overset{e-e}{T}_{22}}{2}\sin 2\theta + \overset{e-e}{T}_{12}\cos 2\theta \tag{7-34}$$

$$\overset{E-E}{T}_{22} = \frac{\overset{e-e}{T}_{11} + \overset{e-e}{T}_{22}}{2} - \frac{\overset{e-e}{T}_{11} - \overset{e-e}{T}_{22}}{2}\cos 2\theta - \overset{e-e}{T}_{21}\sin 2\theta$$

First let us find the principal values and directions by letting $\boldsymbol{E}_i = \boldsymbol{p}_i$ which then implies that

$$\lambda_1 = \overset{E-E}{T}_{11} \qquad \lambda_2 = \overset{E-E}{T}_{22} \qquad \overset{E-E}{T}_{12} = 0 \tag{7-35}$$

The last equation of (7-35) yields the value $\theta = \theta_p$ that provides the orientation of $\boldsymbol{p}_1$ with respect to $\boldsymbol{e}_1$. The use of the second equation in (7-34) yields

$$\tan 2\theta_p = \frac{\overset{e-e}{T}_{11} - \overset{e-e}{T}_{22}}{2\overset{e-e}{T}_{12}} \tag{7-36}$$

Now with the location of the principal axes determined, we now re-label the axes in (7-34) as follows: $\boldsymbol{e}_i \to \boldsymbol{p}_i$ and $\boldsymbol{E}_i \to \boldsymbol{e}_i$ which also means that

$$\overset{e-e}{T}_{11} = \frac{\overset{p-p}{T}_{11} + \overset{p-p}{T}_{22}}{2} + \frac{\overset{p-p}{T}_{11} - \overset{p-p}{T}_{22}}{2} \cos 2\theta + \overset{p-p}{T}_{21} \sin 2\theta$$

$$\overset{e-e}{T}_{12} = -\frac{\overset{p-p}{T}_{11} - \overset{p-p}{T}_{22}}{2} \sin 2\theta + \overset{p-p}{T}_{12} \cos 2\theta \tag{7-37}$$

$$\overset{e-e}{T}_{22} = \frac{\overset{p-p}{T}_{11} + \overset{p-p}{T}_{22}}{2} - \frac{\overset{p-p}{T}_{11} - \overset{p-p}{T}_{22}}{2} \cos 2\theta - \overset{p-p}{T}_{21} \sin 2\theta$$

But

$$\overset{p-p}{T}_{12} = 0 \qquad \overset{p-p}{T}_{11} = \lambda_1 \qquad \overset{p-p}{T}_{22} = \lambda_2 \tag{7-38}$$

Let

$$T_c = \frac{\overset{p-p}{T}_{11} + \overset{p-p}{T}_{22}}{2} \qquad R = \frac{\overset{p-p}{T}_{11} - \overset{p-p}{T}_{22}}{2} \tag{7-39}$$

Then (7-37) reduces to

$$\overset{e-e}{T}_{11} = T_c + R \cos 2\theta$$

$$\overset{e-e}{T}_{12} = -R \sin 2\theta \tag{7-40}$$

$$\overset{e-e}{T}_{22} = T_c - R \cos 2\theta$$

We note that alternative expressions for $\boldsymbol{T}_c$ and R are

$$T_c = \frac{\overset{e-e}{T}_{11} + \overset{e-e}{T}_{22}}{2} \qquad R = \left(\overset{e-e}{T}_{11} - T_c \right)^2 + \left(\overset{e-e}{T}_{12} \right)^2 = \left(\frac{\overset{e-e}{T}_{11} - \overset{e-e}{T}_{22}}{2} \right)^2 + \left(\overset{e-e}{T}_{12} \right)^2 \tag{7-41}$$

These equations are the same as those inferred by the geometrical construction shown in Fig. 7-1 known as Mohr' circle. A rotation of 2θ in Mohr's circle corresponds to a rotation of θ in the same direction in the physical plane.

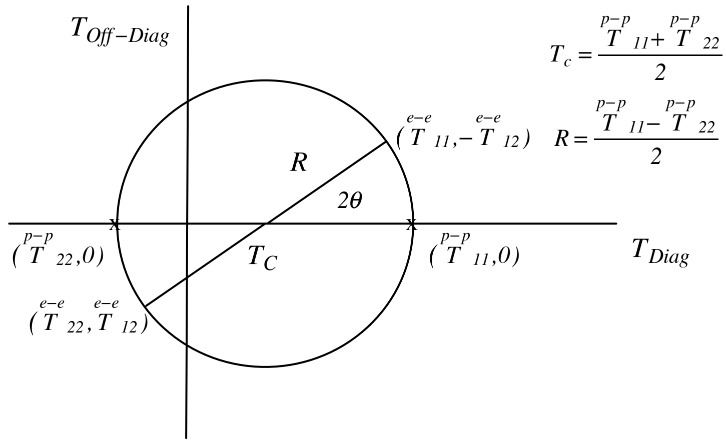

Fig. 7-1. Geometrical interpretation of Mohr's circle.

7.8 Summary

Suppose we have a second-order symmetric tensor with a representation in two bases:

$$\boldsymbol{T} = \overset{e-e}{T} \boldsymbol{e}_i \otimes \boldsymbol{e}_j = \overset{E-E}{T} \boldsymbol{E}_i \otimes \boldsymbol{E}_j \tag{7-42}$$

To find the eigenpairs, we work with one set of components or the others. If we use the $\boldsymbol{e}_i \otimes \boldsymbol{e}_j$ system we use the characteristic equation to obtain the eigenvalues, and the eigenvectors are implicitly described using the $\boldsymbol{e}_i$ basis. If we choose to use the $\boldsymbol{E}_i \otimes \boldsymbol{E}_j$ basis, the characteristic equation will be same because all coefficients are invariants. However the eigenvectors will be expressed in terms of the $\boldsymbol{E}_i$ basis. To show that the eigenvectors are the same the components of one basis have to be transformed to components in the other basis.

A function of a tensor, denoted symbolically as $f(\boldsymbol{T})$, is defined to be the following form based on the spectral decomposition of $\boldsymbol{T}$:

$$f(\boldsymbol{T}) = f(\lambda_1)\boldsymbol{p}_1 \otimes \boldsymbol{p}_1 + f(\lambda_2)\boldsymbol{p}_2 \otimes \boldsymbol{p}_2 + f(\lambda_3)\boldsymbol{p}_3 \otimes \boldsymbol{p}_3 \tag{7-43}$$

The components of this tensor in a given basis are obtained using the transformation relation.

The elementary form of Mohr's circle is simply a geometrical representation of the transformation relation in a plane perpendicular to a principal direction.