2.1 Rubber specimens, having an initial length of 5 cm, are tested, one in compression and the other in tension. If the engineering displacements are –1.5 cm and 1.5 cm, respectively, what will be the final lengths of the specimens? What are true strains, and why are they numerically different?

$$\ell_0 = 5 \text{ cm}$$

Engineering strain, $e = \Delta\ell/\ell_0 = (\ell_f - \ell_0)/\ \ell_0 = \ell_f/\ell_0 - 1$
Or

$$\ell_f = \ell_0\,(1+ e)$$

$e_{tension} = 1.5/5 = 0.3$
$\ell_f = 5\ (1 + 0.3) = 0.262$ cm

$e_{comp} = -1.5/5 = -0.3$
$\ell_f = 5\ (1 - 0.3) = 3.5$ cm

True strains

$$\varepsilon = \ln(\ell_f/\ell_0) = \ln(e+1)$$

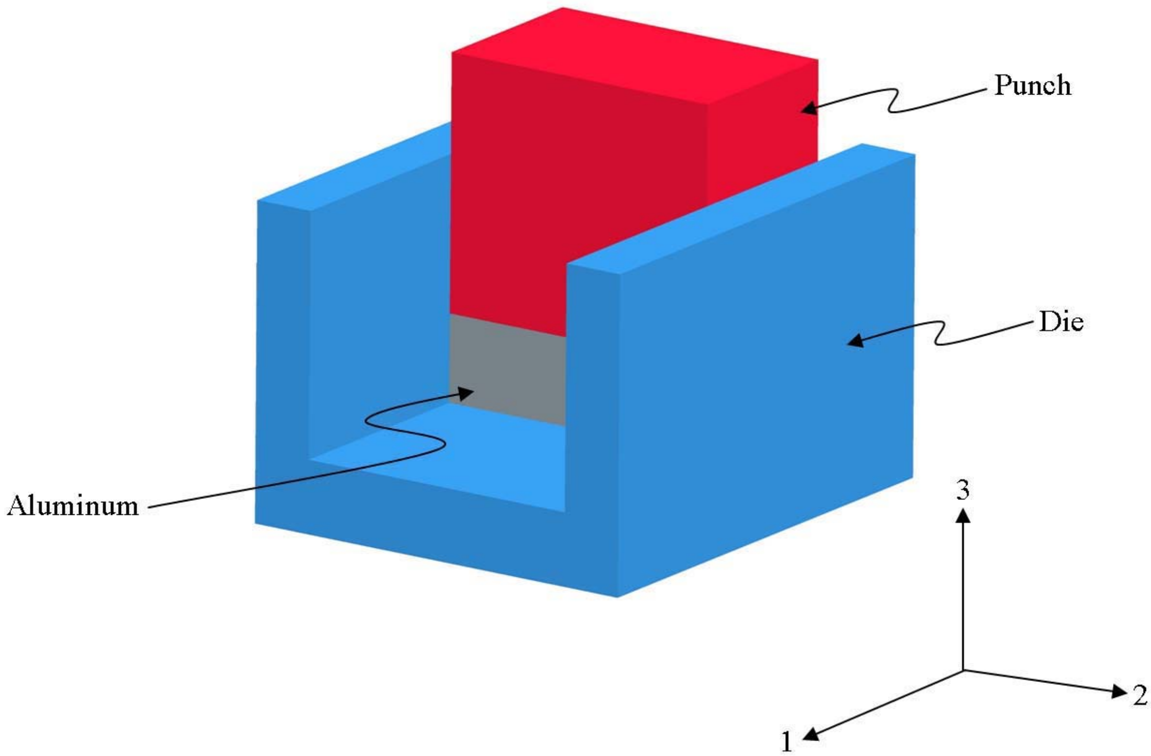
$$\varepsilon_{ten} = \ln\ (0.3 + 1) = \ln 1.3 = 0.262$$

$$\varepsilon_{comp} = \ln\ (-0.3 + 1) = \ln 0.7 = -0.357$$

The true strains are numerically different from engineering strains because in the former, the instantaneous gage length is used while in the latter; the original gage length is used.

2.2 An aluminum polycrystalline specimen is being elastically compressed in plane strain. If the true strain along the compression direction is –2 x 10^{-4}, what are the other two longitudinal strains?

The figure below shows the conditions of the test.



Strains

$\varepsilon_{22} = 0$ (rigid die wall provides the constraint)

$$\varepsilon_{33} = -2\ x\ 10^{-4}\ (given)$$

$$\varepsilon_{11} = ?$$

Stresses

$\sigma_{11} = 0$ because the material is free to flow in direction 1

$$\sigma_{22}, \sigma_{33} \neq 0$$

Hooke's Law

$$\varepsilon_{11} = \frac{1}{E}\left[0 - \nu\left(\sigma_{22} + \sigma_{33}\right)\right]$$

$$\varepsilon_{22} = 0 = \frac{1}{E}\left[\sigma_{22} - \nu\sigma_{33}\right]$$

$$\varepsilon_{33} = -2\ x\ 10^{-4} = \frac{1}{E}\left[\sigma_{33} - \nu\sigma_{12}\right]$$

$$\varepsilon_{22} = \frac{1}{E}\left[\sigma_{22} - \nu\sigma_{33}\right] = 0$$

$$\therefore \sigma_{22} = \nu\sigma_{33}$$

$$\varepsilon_{33} = \frac{1}{E}\left[\sigma_{33} - \nu\left(\sigma_{11} + \sigma_{22}\right)\right]$$

$$= \frac{1}{E}\left[\sigma_{33} - \nu\sigma_{22}\right] = \frac{\sigma_{33}}{E}\left[1 - \nu^2\right]$$

$$\varepsilon_{11} = \frac{1}{E}\left[\sigma_{11} - \nu\left(\sigma_{22} + \sigma_{33}\right)\right] = -\frac{\nu}{E}\left(\sigma_{22} + \sigma_{33}\right)$$

$$= -\frac{\nu}{E}\left(\sigma_{33} + \nu\sigma_{33}\right) = \frac{-\nu\sigma_{33}\left(1+\nu\right)}{E}$$

$$= \frac{-\nu\left(1+\nu\right)}{E} \cdot \frac{E\varepsilon_{33}}{\left(1-\nu^2\right)} = -\left(\frac{\nu}{1-\nu}\right)\varepsilon_{33}$$

$$\varepsilon_{11} = -\left(\frac{0.345}{1-0.345}\right)\left(-2\times 10^{-4}\right) = 1.05\times 10^{-4}$$

2.3 Determine K, λ, and G for polycrystalline niobium, titanium, and iron, from E and γ.

$$K = \frac{E}{3(1-2\nu)}$$

$$G = \frac{E}{2(1+\nu)}$$

$$\lambda = \frac{E\upsilon}{(1+\nu)(1-2\nu)}$$

For polycrystalline niobium

$$E = 104.9\ GPa \quad \upsilon = 0.397$$

$$K = \frac{104.9GPa}{3(1-2\times 0.397)} = 169.74\ GPa$$

$$G = \frac{104.9GPa}{2(1+0.397)} = 37.54\,GPa$$

$$\lambda = \frac{104.9GPa(0.397)}{(1+0.397)(1-2(0.397))} = 144.71\ GPa$$

For titanium

$$E = \frac{115.7}{2(1-2(0.321))} = 107.73\ GPa$$

$$K = \frac{115.7}{2(1+0.321)} = 43.78\ GPa$$

$$\lambda = \frac{115.7(0.321)}{(1+0.321)(1-2\times 0.321)} = 78.53\ GPa$$

For Iron

$$K = \frac{211.4}{3(1-2(0293))} = 170.2\ GPa$$

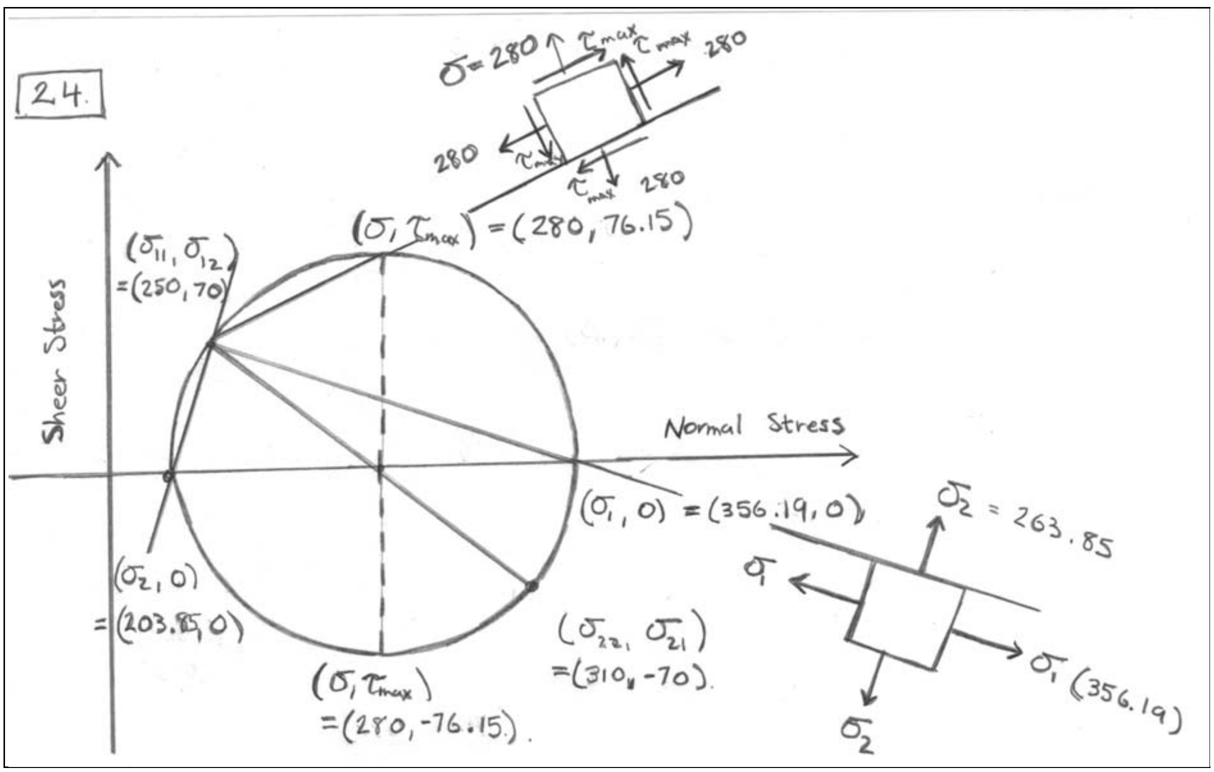
$$G = \frac{211.4}{2(1+0.393)} = 81.75\ GPa$$

$$\lambda = \frac{211.4(0.293)}{(1.293)(0.414)} 115.71\ GPa$$

2.4 A state of stress is given by

σ_{11} = 250 MPa,
σ_{12} = 70 MPa,
σ_{22} = 310 MPa.

Determine the principal stresses and the maximum shear stress, as well as their angle with the given direction.



$A = (\sigma_{11}, \sigma_{12}) = (250{,}70)$

$B = (\sigma_{22}, -\sigma_{21}) = (310{,}-70)$

Need to find a line through the origin

Slope from A-B = $\frac{\Delta y}{\Delta x} = \frac{-70-(70)}{310-250} = \frac{-7}{3}$

For Pt. A: $Y = \frac{-7}{3}x + B$

$$70 = \frac{-7}{3}(250) + B$$

653 = B

Find X intercept when Y = 0

$$0 = \frac{-7}{3}(X) + 653.33$$

280 = X

Center = (280, 0)

Radius = $\sqrt{(310-280)^2 + (-70-0)^2} = 76.15$

Principal Stresses:

$$280 + 76.15 = \sigma_1$$

$$280 - 76.15 = \sigma_2$$

$$\boxed{\begin{aligned} \sigma_1 &= 356.15 MPa \\ \sigma_2 &= 203.85 MPa \end{aligned}}$$

$$\tan 2\theta = \frac{(-70-0)}{310-280} =$$

$$2\theta = \tan^{-1}\left(\frac{7}{3}\right) = -66.80$$

$$\theta = 33.4\,^\circ$$

Maximum shear = $\tau = 76.15 MPa$

2.5 Calculate the anisotropy ratio for the cubic metals in Table 2.3.

$$Ag \Rightarrow A = \frac{2(46.1)}{124 - 93.4} = 3.01$$

$$A\ell \Rightarrow A = \frac{2(28.5)}{108.2 - 61.3} = 1.22$$

$$Cu \Rightarrow A = \frac{2(75.4)}{168.4 - 126.4} = 1.25$$

$$Ni \Rightarrow A = \frac{(124.7)^2}{246.5 - 147.3} = 2.51$$

$$Fe \Rightarrow A = \frac{2(116.5)}{228.132} = 2.43$$

$$Ta \Rightarrow A = \frac{(82.5)2}{267 - 161} = 1.56$$

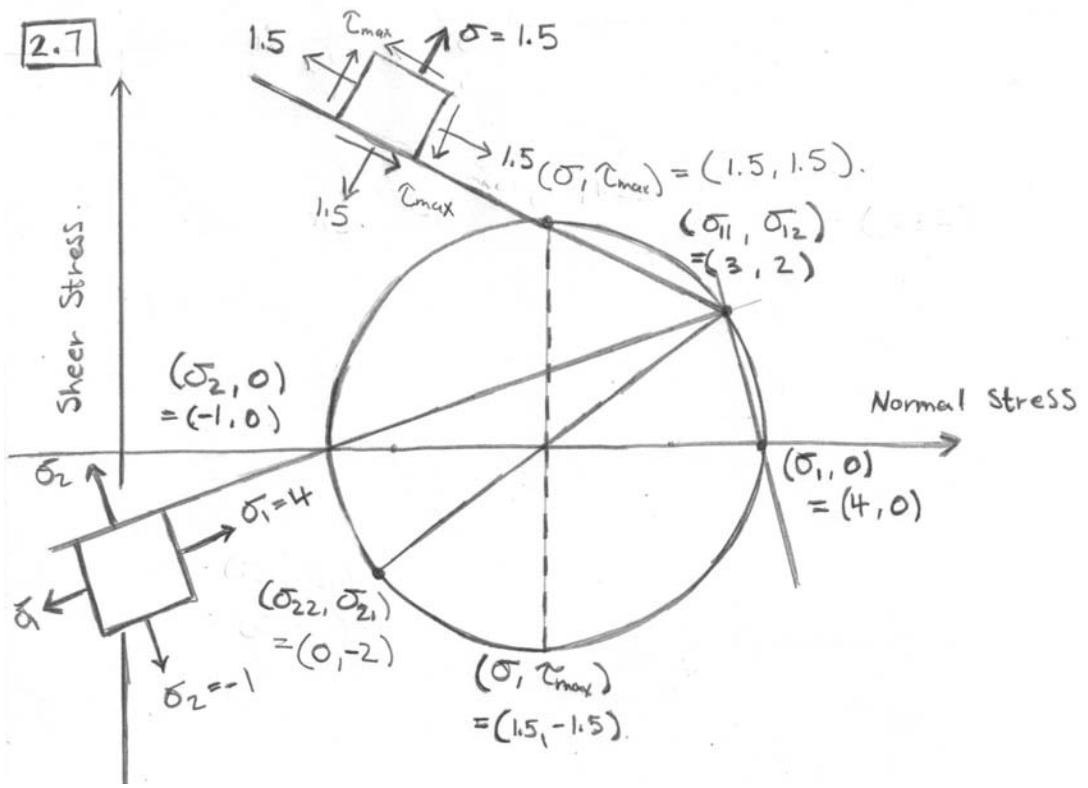
$$W \Rightarrow A = \frac{2(151.4)}{501.0 - 198} = 1.00$$

$$Au \Rightarrow A = \frac{2(42)}{186.0 - 157} = 2.90$$

$$Pb \Rightarrow A = \frac{2(14.9)}{49.5 - 42.3} = 4.14$$

2.7 Determine the principal stresses and the maximum shear stress, as well as their angles with the system of reference given by the following stresses:

$$\sigma_{ij} = \begin{pmatrix} 3 & 2 \\ 2 & 0 \end{pmatrix} MPa$$



$\sigma_{11} = 3$ $\quad$ $A = (\sigma_{11}, \sigma_{12}) = (3,2)$

$\sigma_{12} = 2$ $\quad$ $B = (\sigma_{22}, -\sigma_{12}) = (0,-2)$

$\sigma_{22} = 0$

Slope from A-B = $\frac{\Delta Y}{\Delta X} = \frac{2-(-2)}{3-(0)} = \frac{4}{3}$

$Y = \frac{4}{3}x + B$

For Pt. A: $2 = \frac{4}{3}(3) + b$

$b = -2$

$Y = \frac{4}{3}x - 2$

Find X intercept when Y = 0

$$0 = \frac{4}{3}x - 2$$

$$X = \frac{3}{2}$$

Center = $\left(\frac{3}{2}, 0\right)$

Radius = $\sqrt{\left(3 - \frac{3}{2}\right)^2 + (2 - 0)^2} = 2.5$

$$\tan 2\theta = \left(\frac{2 - 0}{3 - \frac{3}{2}}\right)$$

$$\sigma_1 = \frac{3}{2} + 2.5 = 4$$

$$2\theta = \tan^{-1}\left(\frac{4}{3}\right) = 53$$

$$\sigma_2 = \frac{3}{2} - 2.5 = -1$$

Maximum Shear = $\tau = 1.5$

$$\theta = 26.56^\circ$$

2.8 Extensometers attached to the external surface of a steel pressure vessel indicate that $\varepsilon_l = 0.002$ and $\varepsilon_t = 0.005$ along the longitudinal and transverse directions, respectively. Determine the corresponding stresses. What would be the error if Poisson's ratio were not considered?

$$E_{steel} = 210GPa$$
$$\varepsilon_l = .002$$
$$\varepsilon_t = .005$$

$$\nu = \frac{\varepsilon_l}{\varepsilon_t} = \frac{.002}{.005} = .4$$

Generalized Hooke's Law with consideration to Poisson's ratio

$$\varepsilon_l = \frac{1}{E}[\sigma_l - \nu(\sigma_t)] \qquad (1)$$

$$\varepsilon_t = \frac{1}{E}[\sigma_t - \nu(\sigma_l)] \qquad (2)$$

Solve equation (2) for σ_l and substitute into (1)

$$\sigma_l = E\varepsilon_t + \nu\sigma_t$$

$$\varepsilon_l = \frac{1}{E}[\sigma_l - \nu(E\varepsilon_t + \nu\sigma_t)]$$

$$\varepsilon_l = \frac{1}{E}[\sigma_l - \nu E\varepsilon_t - \nu^2\sigma_l)]$$

Solve for σ_l

$$\sigma_l = \frac{E(\varepsilon_l + \nu\varepsilon_t)}{1-\nu^2} = \frac{210\times10^9(.002+.4(.005))}{1-.4^2} = 1GPa$$

$$\sigma_t = E\varepsilon_t + \nu\sigma_l) = 210\times10^9(.005) + .4(1\times10^9) = 1.45GPa$$

Without consideration to Poisson's ratio

$$\sigma_l = E\varepsilon_l = 210\times10^9(.002) = 0.42GPa$$
$$\sigma_t = E\varepsilon_t = 210\times10^9(.005) = 1.05GPa$$

Error in stress calculation if Poisson's ratio is not taken into consideration:

$\sigma_t = 1.45\times10^9 - 1.05\times10^9 = 400MPa$

$\sigma_l = 1\times10^9 - .42\times10^9 = 580MPa$

2.9 Calculate Young's and shear moduli for monocrystalline iron along [100], [110], and [111].

Young's modulus: $\frac{1}{E_{ijk}} = S_{11} - 2(S_{11} - S_{12} - \frac{1}{2}S_{44})(\ell_{i1}{}^2\ell_{j2}{}^2 + \ell_{j2}{}^2\ell_{k3}{}^2 + \ell_{i1}{}^2\ell_{k3}{}^2)$

The direction cosines are

	ℓ_{11}	ℓ_{j2}	ℓ_{k3}	$\left(\ell_{i1}{}^2\ell_{j2}{}^2 + \ell_{j2}{}^2\ell_{k3}{}^2 + \ell_{i1}{}^2\ell_{k3}{}^2\right)$
[100]	1	0	0	0
[110]	$\sqrt{2}/2$	$\sqrt{2}/2$	0	$1/4$
[111]	$1/\sqrt{3}$	$1/\sqrt{3}$	$1/\sqrt{3}$	$1/3$

From table 2.4 (p. 112)

$S_{11} = 0.762 \text{ x } 10^{-2}$ GPa^{-1}
$S_{12} = -0.279 \text{ x } 10^{-2}$ GPa^{-1}
$S_{44} = 0.858 \text{ x } 10^{-2}$ GPa^{-1}

$$E_{100} = \frac{1}{0.762\times10^{-2}} = 131.23GPa$$

$$E_{110} = \frac{1}{0.762\times10^{-2} - 2\left[\left(0.76\times10^{-2} - \left(-0.279\times10^{-2}\right) - \left(0.5\times0.858\times10^{-2}\right)\right)\right](0.25)}$$

$$= \frac{1}{4.56\times10^{-3}} = 219.3\ GPa$$

$$E_{111} = \frac{1}{0.762\times10^{-2} - 2[\left(0.762\times10^{-2}\right) - \left(-0.279\times10^{-2}\right) - \left(0.5\times0.858\times10^{-2}\right)\left(1/3\right)}$$

$$= \frac{1}{3.54\times10^{-3}} = 282.49\ GPa$$

Shear moduli: $\frac{1}{Gijk} = S_{44} + 4(S_{11} - S_{12} - \frac{1}{2S_{44}})\left(\ell_{i1}{}^{2}\ell_{j2}{}^{2} + \ell_{j2}{}^{2}\ell_{k3}{}^{2} + \ell_{i1}{}^{2}\ell_{k3}{}^{2}\right)$

$$G_{100} = \frac{1}{0.858 \times 10^{-2}} = 116.55\ GPa$$

$$G_{110} = \frac{1}{0.858 \times 10^{-2} + 4\left[0.762 \times 10^{-2} - \left(-0.297 \times 10^{-2}\right) - \left(0.5 \times 0.858 \times 10^{-2}\right)\right](0.25)}$$

$$= \frac{1}{0.01488} = 67.20\ GPa$$

$$G_{111} = \frac{1}{0.858 \times 10^{-2} + 4\left[0.762 \times 10^{-2} - \left(0.297 \times 10^{-2}\right) - \left(0.5 \times 0.858 \times 10^{-2}\right)\right]\left(\frac{1}{3}\right)}$$

$$= \frac{1}{0.04522} = 22.11\ GPa$$

2.10 Calculate Young's and shear moduli for monocrystalline iron along [100], [110], and [111].

In GPa

$E_{100} = 131.23$ $\qquad$ $E_{110} = 219.3$ $\qquad$ $E_{111} = 282.3$

$$\overline{E} = \frac{E_{100} + E_{110} + E_{111}}{3} = \frac{1}{3}(131.23 + 219.3 + 282.3)$$

$$\overline{E} = 210.94$$

Refer to Table 2.3 and Table 2.4 for stiffness and compliance values for Fe, respectively, For Cubic Structure:

$$\begin{aligned} C_{11} &= C_{22} = C_{33} \\ C_{44} &= C_{55} = C_{66} \\ C_{12} &= C_{23} = C_{13} \end{aligned} \qquad \begin{aligned} S_{11} &= S_{22} = S_{33} \\ S_{44} &= S_{55} = S_{66} \\ S_{12} &= S_{23} = S_{13} \end{aligned}$$

Voigt Average

$$E = \frac{1}{5}\left(3F + 2G^{*} + H\right)$$

$$F = \frac{1}{3}\left(C_{11} + C_{11} + C_{11}\right) = \frac{1}{3}\left(3C_{11}\right) = C_{11}$$

$$G^{*} = \frac{1}{3}\left(3C_{12}\right) = C_{12}$$

$$H = \frac{1}{3}\left(3C_{44}\right) = C_{44}$$

$$\overline{E} = \frac{1}{5}\left(3\left(C_{11}\right) + 2\left(C_{12}\right) + C_{44}\right) = \frac{1}{5}\left(3(228) + 2(132.0) + (116.5)\right)$$

$$\overline{E} = 212.9GPa$$

Reuss Average

$$\frac{1}{E} = \frac{1}{5}\left(3F' + 2G' + H'\right)$$

$$F' = \frac{1}{3}\left(S_{11} + S_{11} + S_{11}\right) = \frac{1}{3}\left(3S_{11}\right) = S_{11}$$

$$G' = \frac{1}{3}\left(3S_{12}\right) = S_{12}$$

$$H' = \frac{1}{3}\left(3S_{44}\right) = S_{44}$$

$$\frac{1}{E} = \frac{1}{5}\left(3S_{11} + 2S_{12} + S_{44}\right)$$

$$\frac{1}{E} = \frac{1}{5}\left(3(.762) + 2(-.279) + (.858)\right) \times 10^{-2}\, GPa^{-1}$$

$$\frac{1}{E} = .5172 \times 10^{-2}\, GPa^{-1}$$

$$E = 193.3 GPa$$

2.11 A silver monocrystal is extended along [100]. Obtain the values for the Young's and shear moduli, as well as Poisson's ratio.

$$\frac{1}{E_{ijk}} = S_{11} - 2\left(S_{11} - S_{12} - \frac{1}{2}S_{44}\right)\left(l_{i1}^{\,2} l_{j2}^{\,2} + l_{i1}^{\,2} l_{k3}^{\,2} + l_{j2}^{\,2} l_{k3}^{\,2}\right)$$

	l_{i1}	l_{j2}	l_{k3}	B
For 100	1	0	0	0

$$\frac{1}{E_{100}} = S_{11}$$

$$S_{11} = 2.29 \times 10^{-2}\, GPa^{-1}$$
$$S_{12} = -0.983 \times 10^{-2}\, GPa^{-1}$$

$$S_{44} = 2.17 \times 10^{-2}\, GPa^{-1}$$

$$E_{100} = \frac{1}{S_{11}} = \frac{1}{2.29 \times 10^{-2}\, GPa^{-1}} = 43.668 GPa$$

$$G = \frac{1}{2\left(S_{11} - S_{22}\right)} = \frac{1}{2(2.29 + .983) \times 10^{-2}\, GPa^{-1}} = 15.27 GPa$$

$$\nu = -1 + \frac{E_{100}}{2G_{100}} = -1 + \frac{43.668}{2(15.27)} = .429 \approx .43$$

2.13 A steel specimen is subjected to elastic stresses represented by matrix

$$\sigma_{ij} = \begin{bmatrix} 2 & -3 & 1 \\ -3 & 4 & 5 \\ 1 & 5 & -1 \end{bmatrix} MPa$$

Calculate the corresponding strains.

For steel,

E = 210 GPa, ν = 0.3

$$G = \frac{E}{2(1+\nu)} = \frac{210}{2(1+0\times 3)} = 80.77\ GPa$$

$$\varepsilon_1 = \frac{1}{E}\left[\sigma_1 - \nu(\sigma_2 + \sigma_3)\right] = 5.24\times 10^{-6}$$

$$\varepsilon_2 = \frac{1}{E}\left[\sigma_2 - \nu(\sigma_1 + \sigma_3)\right] = 17.6\times 10^{-6}$$

$$\varepsilon_3 = \frac{1}{E}\left[\sigma_3 - \nu(\sigma_1 + \sigma_2)\right] = -13.3\times 10^{-6}$$

$$\varepsilon_4 = \gamma_{23} = \frac{\sigma_4}{G} = 61.8\times 10^{-6}$$

$$\varepsilon_5 = \gamma_{13} = \frac{\sigma_5}{G} = 12.4\times 10^{-6}$$

$$\varepsilon_6 = \gamma_{12} = \frac{\sigma_6}{G} = -37.1\times 10^{-6}$$

Recalling that the value of engineering shear strain, γ is twice that of tensorial shear strains, ε, we can write the strain tensor as

$$\varepsilon_{ij} = \begin{bmatrix} 5.24 & -18.5 & 6.2 \\ -9.25 & 17.6 & 30.9 \\ 3.75 & 15.42 & -13.3 \end{bmatrix} \times 10^{-6}$$

2.14 Ultrasonic equipment was used to determine the longitudinal and shear sound velocities of a metallic specimen having a density of 7.8 g/cm^3. The values obtained are:

$V_l = 5{,}300$m/s
$V_s = 3{,}000$m/s

Determine the Young's and shear moduli and Poisson's ratio for this material. What is the material?

$$V_\ell = \sqrt{\frac{E}{\rho}} \qquad V_s = \sqrt{\frac{G}{\rho}} \qquad G = \frac{E}{(1+\nu)}$$

$$E = V_l^2 \ell = (5300 m/s)^2 \times (7.8 g/cm^2)\left(\frac{1\times 10^6 cm^3}{1m^3} \times \frac{1kg}{1000g}\right)$$

$$= 2.19 \times 10^{11} Pa = 219 GPa$$

$$G = V_s^2 \ell = (3330)^2 \times (7.8)\left(\frac{1\times 10^6}{1000}\right)$$

$$= 8.49 \times 10^{10} Pa$$

$$= 84.9 GPa$$

$$\nu = \frac{E}{2G} - 1 = 0.29$$

From Table 2.3, we infer that this material could be iron!

2.15 A tubular specimen is being subjected to torsional moment T = 600 Nm. If the shear modulus of the material (Al) is equal to 26.1 GPa, what is the total angular deflection if the length is 1 m? The tube has a diameter of 5 cm and a wall thickness of 0.5 cm. Assume the process to be elastic.

$$T = \frac{\tau J}{r}$$

$$J = \frac{\pi\left(0.05^4 - 0.045^4\right)}{32} = 36.2 \times 10^{-8} m^4$$

T= 600 Nm, G = 26.1 GPa, $\ell = 1$ m

$$\theta = \frac{T\ell}{JG}$$

$$= \frac{600 \times 1}{36.2 \times 10^{-8} \quad 26.1 \times 10^{9}} = 0.0635\, rad = 3.64 \deg$$

2.16 Using the Mohr circle construction, calculate the principal stresses and the maximum shear stresses, as well as their orientation, for the sheet subjected to the stresses shown in figure Ex 2.16.

The center of Mohr circle is at

$$\frac{300+(-200)}{2} = 50\ MPa$$

$$radius = \sqrt{(450)^2 + (300-500)^2}$$

$$= 514.78\ MPa$$

Principal stresses are given by points A′ and B′:

$$\sigma_1 = 50 + 514.78 = 564.78\ MPa$$

$$\sigma_2 = 50 - 514.78 = -464.78\ MPa$$

The angle between the principal planes and the maximum shear stress are given as:

$$\sin 2\theta = \frac{450}{514.78}$$

$$2\theta = 60.945$$

$$\theta = 30.47°$$

$$Max.\ shear\ stress$$

$$\tau_{\max} = 514.78\ MPa$$

2.17 A state of stress is given by

$\sigma_{11} = -500$ MPa

$\sigma_{22} = 300$ MPa

$\sigma_{12} = 150$ MPa

Determine the principal stresses and the maximum shear stress, as well as their orientation, using the Mohr circle construction.

center at –400 + 300/2 = -50 MPa

$$radius = \sqrt{150^2 + (-350)^2} = 380.79\ MPa$$

Principal stresses are:

$$\sigma_1 = -50 + 380.79 = 330.79\ MPa$$

$$\sigma_2 = -50 - 380.79 = 430.79\ MPa$$

$$\sin\ 2\theta = \frac{150}{380.79}$$

$$2\theta = 23.2°$$

$$\theta = 11.6°$$

$$Max.\ shear\ stress$$

$$\tau_{\max} = 380.79\ MPa$$

2.18 From the elastic stiffness for copper (see Table 2.3), determine the elastic compliances.

C_{11} = 168.4 GPa
C_{44} = 75.4 GPa
C_{12} = 121.4 GPa

$S_{44} = 1/C_{44} = 1/75.4 = 0.0136\ \text{GPa}^{-1}$
$= 1.326 \times 10^{-2}\ \text{GPa}^{-1}$

$$S_{12} = \frac{-C_{12}}{(C_{11}+2C_{12})(C_{11}-C_{12})} = \frac{-124.4}{(168.4+2(121.4))(168.4-121.4)}$$
$$= -0.628\times 10^{-2}\ GPa^{-1}$$
$$S_{11} = \frac{1}{C_{11}} + \frac{2C_{12}^2}{C_{11}(C_{11}+2C_{12})(C_{11}-C_{12})}$$
$$= \frac{1}{168.4} + \frac{2(121.4)^2}{168.4(168.4+2(121.4))(168.4-121.4)} = 0.015\ GPa^{-1}$$

2.19 From the elastic compliances S_{11}, S_{12}, and S_{44} for iron and tungsten, determine the Young's moduli along [111], [110], and [100].

Table 2.3	S_{11}	S_{44}	S_{12}
Fe BCC	0.762	0.858	– 0.279
W BCC	0.257	0.66	– 0.0783

* units are $10^{-2}\ GPa^{-1}$

Equation 2.20

$$\frac{1}{E_{ijk}} = S_{11} - 2\left(S_{11} - S_{12} - \frac{1}{2}S_{44}\right) \times \left(l_{i1}^2 l_{j2}^2 + l_{j2}^2 l_{k3}^2 + l_{i1}^2 l_{k3}^2\right)$$

	l_{i1}	l_{j2}	l_{k3}	B
$[100]$	1	0	0	0
$[110]$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	0	$\frac{1}{4}$
$[111]$	$\frac{\sqrt{3}}{3}$	$\frac{\sqrt{3}}{3}$	$\frac{\sqrt{3}}{3}$	$\frac{1}{3}$

Fe:

$$\frac{1}{E_{ijk}} = 0.762 - 2\left(0.762 + 0.279 - \frac{1}{2}(0.858)\right)(B)$$

$$\frac{1}{E_{ijk}} = 0.762 - 1.224(B)$$

[100]

$$\frac{1}{E_{100}} = 0.762 - 1.224(0) = 0.762 \times 10^{-2}\, GPa^{-1}$$

$$E_{100} = 131.2 GPa$$

[110]

$$B_{[110]} = \frac{1}{4}$$

$$\frac{1}{E_{110}} = 0.762 - 1.224\left(\frac{1}{4}\right) = 0.456 \times 10^{-2}\, GPa^{-1}$$

$$E_{110} = 219.3 GPa$$

[111]

$$B_{[111]} = \frac{1}{3}$$

$$\frac{1}{E_{111}} = 0.762 - 1.224\left(\frac{1}{3}\right) = 0.354 \times 10^{-2}\, GPa^{-1}$$

$$E_{111} = 282.5 GPa$$

W:

$$\frac{1}{E_{ijk}} = 0.257 - 2\left(0.257 + 0.073 - \frac{1}{2}(66)\right)(B)$$

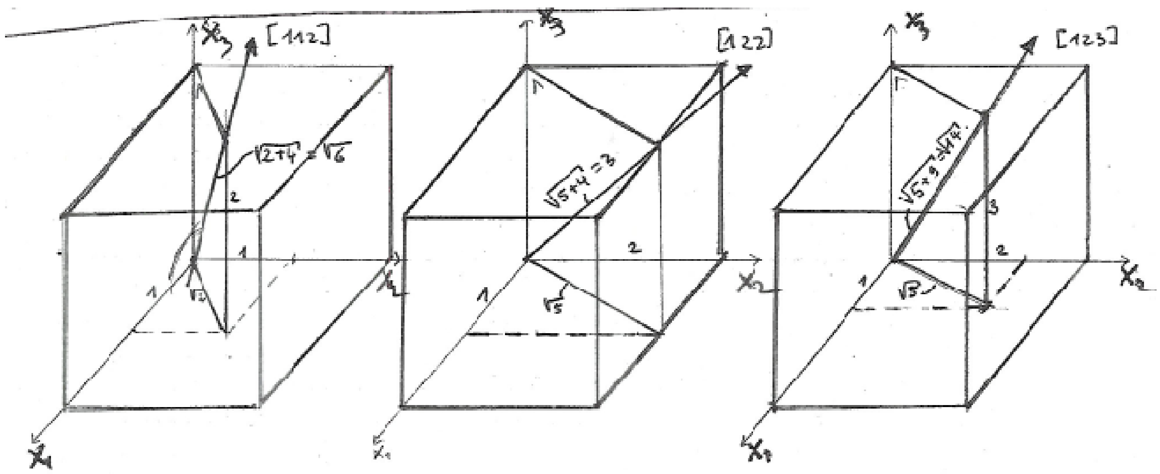
$$\frac{1}{E_{ijk}} = 0.257 - 0\left(B_{ijk}\right)$$

$$\frac{1}{E_{ijk}} = 0.257 \times 10^{-2}\, GPa^{-1}$$

$$E_{100} = E_{110} = E_{111} = 389.1 GPa$$

2.20 Determine the elastic Young's moduli for tungsten and ZrO_2 along [112], [122], and [123].

Solution:



$$\ell_1 = \frac{1}{\sqrt{6}} \qquad \ell_1 = \frac{1}{\sqrt{9}} = \frac{1}{3} \qquad \ell_1 = \frac{1}{\sqrt{14}}$$

$$\ell_2 = \frac{1}{\sqrt{6}} \qquad \ell_2 = \frac{2}{\sqrt{9}} = \frac{2}{3} \qquad \ell_2 = \frac{2}{\sqrt{14}}$$

$$\ell_3 = \frac{2}{\sqrt{6}} \qquad \ell_3 = \frac{2}{3} \qquad \ell_3 = \frac{3}{\sqrt{14}}$$

We first calculate the $(\ell_1^2\ell_2^2 + \ell_2^2\ell_3^2 + \ell_1^2\ell_3^2)$ - factor for each direction, because this calculation returns several times in the rest of the exercise.

$$[112] = \ell_1^2\ell_2^2 + \ell_2^2\ell_3^2 + \ell_1^2\ell_3^2 = \frac{1}{6}\cdot\frac{1}{6} + \frac{1}{6}\cdot\frac{4}{6} + \frac{1}{6}\cdot\frac{4}{6} = 0{,}25$$

$$[122] = \ell_1^2\ell_2^2 + \ell_2^2\ell_3^2 + \ell_1^2\ell_3^2 = \frac{1}{9}\cdot\frac{4}{9} + \frac{4}{9}\cdot\frac{4}{9} + \frac{1}{9}\cdot\frac{4}{9} = 0{,}2963$$

$$[123] = \ell_1^2\ell_2^2 + \ell_2^2\ell_3^2 + \ell_1^2\ell_3^2 = \frac{1}{14}\cdot\frac{4}{14} + \frac{4}{14}\cdot\frac{9}{14} + \frac{1}{14}\cdot\frac{9}{14} = 0{,}25$$

To calculate the Young's modulus in a certain direction, we use the formula:

$$\frac{1}{E_{ijk}} = S_{11} - 2\left(S_{11} - S_{12} - \frac{1}{2} S_{44}\right)\left(l_1^2 l_2^2 + l_2^2 l_3^2 + l_1^2 l_3^2\right)$$

From Table 2.4, we find the compliances for Tungsten:

$S_{11} = 0.257 \times 10^{-2}\ GPa^{-1}$

$S_{44} = 0.66 \times 10^{-2}\ GPa^{-1}$

$S_{12} = -0.073 \times 10^{-2}\ GPa^{-1}$

1) E_{112}?

$$\frac{1}{E_{112}} = 0.257 \times 10^{-2} - 2.\left(0.257 \times 10^{-2} + 0.073 \times 10^{-2} - \frac{1}{2} \times 0.66 \times 10^{-2}\right) \times (0.25)$$

$$\frac{1}{E_{112}} = 0.257 \times 10^{-2} \Rightarrow E_{112} = 389\ GPa$$

2) E_{122}?

$$\frac{1}{E_{122}} = 0.257 \times 10^{-2} - 2\left(0.257 \times 10^{-2} + 0.073 \times 10^{-2} - \frac{1}{2} \times 0.66 \times 10^{-2}\right) \times (0.2963)$$

$$\frac{1}{E_{122}} = 0.257 \times 10^{-2} \Rightarrow E_{122} = 389\ GPa$$

3) E_{123}?

$$\frac{1}{E_{123}} = 0.257 \times 10^{-2} - 2\left(0.257 \times 10^{-2} + 0.073 \times 10^{-2} - \frac{1}{2} \times 0.66 \times 10^{-2}\right) \times (0.25)$$

$$\frac{1}{E_{123}} = 0.257 \times 10^{-2} \Rightarrow E_{123} = 389\ GPa$$

$$\boxed{E_{112} = E_{122} = E_{123} = 389\ GPa}$$

This was of course expected since $A_{Tungsten} = 1$. Tungsten is isotropic and hence has the same properties in all directions.

* From table 2.6, we find the stiffnesses for ZrO_2

$C_{11} = 410$ GPa $\quad C_{44} = 60$ GPa

$C_{12} = 110$ GPa

Now we have to recalculate the stiffnesses to compliances, therefore we need to derive the necessary formula's.

$$\Rightarrow \quad S_{44} = \frac{1}{C_{44}}$$

$$S_{12} = \frac{-C_{12}}{(C_{11} + 2C_{12})(C_{11} - C_{12})}$$

$$S_{11} = \frac{C_{11} + C_{12}}{(C_{11} + 2C_{12})(C_{11} - C_{12})}$$

$$\Rightarrow S_{44} = \frac{1}{60} = 0.016667 \ GPa^{-1}$$

$$\Rightarrow S_{12} = \frac{-110}{(410 + 2.110)(410 - 110)} = -0{,}000582 \ GPa^{-1}$$

$$\Rightarrow S_{11} = \frac{410 + 110}{(410 + 2.110)(410 - 110)} = 0.002751 \ GPa^{-1}$$

1) E_{112} ?

$$\frac{1}{E_{112}} = 0.0027513 - 2 \times (0.0027513 + 5{,}82.10^{-4} - \frac{1}{2} \times 0.016667) \times (0.25)$$

$$= 0{,}0052514$$

$$\Rightarrow E_{112} = 190{,}42 \ GPa$$

2) E_{122}?

$$\frac{1}{E_{122}} = 0.002751 - 2 \times (0.0027513 + 5{,}82.10^{-4} - \frac{1}{2} \times 0.016667) \times (0.2963)$$

$$= 0.005714$$

$$\Rightarrow E_{122} = 175{,}00 \ GPa$$

3) E_{123}?

$$\frac{1}{E_{123}} = 0.002751 - 2 \times (0.0027513 + 5.82 \cdot 10^{-4} - \frac{1}{2} \times 0.016667) \times (0.025)$$

$$= 0.0052514$$

$\Rightarrow E_{123} = 190.42$ GPa

$$\boxed{\begin{array}{l} E_{112} = 190.42 \text{ GPa} \\ E_{122} = 175.00 \text{ GPa} \\ E_{123} = 190.42 \text{ GPa} \end{array}}$$

2.21 Determine the polycrystalline Young's modulus for molybdenum using Reuss' and Voigt's averages. Use elastic stiffnesses and compliances from Tables 2.3 and 2.4.

Solution:

From table 2.3 and 2.4 we find that

$C_{11} = 460$ GPa $\qquad S_{11} = 0.28 \cdot 10^{-2}$ GPa^{-1}

$C_{44} = 110$ GPa $\qquad S_{44} = 0.91 \cdot 10^{-2}$ GPa^{-1}

$C_{12} = 176$ GPa $\qquad S_{12} = -0.078 \cdot 10^{-2}$ GPa^{-1}

Molybdenum has a Body Centered Cubic (BCC) crystallographic structure, hence the compliance C_{ij} an stiffness S_{ij} matrix looks as follows:

$$C_{ij} = \begin{bmatrix} C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\ & C_{11} & C_{12} & 0 & 0 & 0 \\ & & C_{11} & 0 & 0 & 0 \\ & & & C_{44} & 0 & 0 \\ & & & & C_{44} & 0 \\ & & & & & C_{44} \end{bmatrix}$$

$$S_{ij} = \begin{bmatrix} S_{11} & S_{12} & S_{12} & 0 & 0 & 0 \\ & S_{11} & S_{12} & 0 & 0 & 0 \\ & & S_{11} & 0 & 0 & 0 \\ & & & S_{44} & 0 & 0 \\ & & & & S_{44} & 0 \\ & & & & & S_{44} \end{bmatrix}$$

1) Voigt average

$$F = \frac{1}{3}(C_{11} + C_{22} + C_{33}) = \frac{3}{3} C_{11} = 460 \text{ GPa}$$

$$G = \frac{1}{3}(C_{12} + C_{23} + C_{13}) = \frac{3}{3} C_{12} = 176 \text{ GPa}$$

$$H = \frac{1}{3}(C_{44} + C_{55} + C_{66}) = \frac{3}{3} C_{44} = 110 \text{ GPa}$$

$$\Rightarrow E = \frac{1}{5}(3F + 2G + H) = \frac{1}{5}(3 \cdot 460 + 2 \cdot 176 + 110)$$

$$\Rightarrow \boxed{E = 368.4 \text{ GPa}}$$

2) Reuss average

$$F' = \frac{1}{3}(S_{11} + S_{22} + S_{33}) = \frac{3}{3} S_{11} = 0.28 \cdot 10^{-2} \text{ GPa}^{-1}$$

$$G' = \frac{1}{3}(S_{12} + S_{23} + S_{13}) = \frac{3}{3} S_{12} = -0.078 \cdot 10^{-2} \text{ GPa}^{-1}$$

$$H' = \frac{1}{3}(S_{44} + S_{55} + S_{66}) = \frac{3}{3} S_{44} = 0.91 \cdot 10^{-2} \text{ GPa}^{-1}$$

$$\Rightarrow (E)^{-1} = \frac{1}{5}(3F' + 2G' + H') = \frac{1}{5}(3 \cdot 0.28 \cdot 10^{-2} + 2 \cdot -0.078 \cdot 10^{-2} + 0.91 \cdot 10^{-2})$$

$$\Rightarrow \frac{1}{E} = 0{,}003188$$

$$\Rightarrow \boxed{E = 313.67 \text{ GPa}}$$

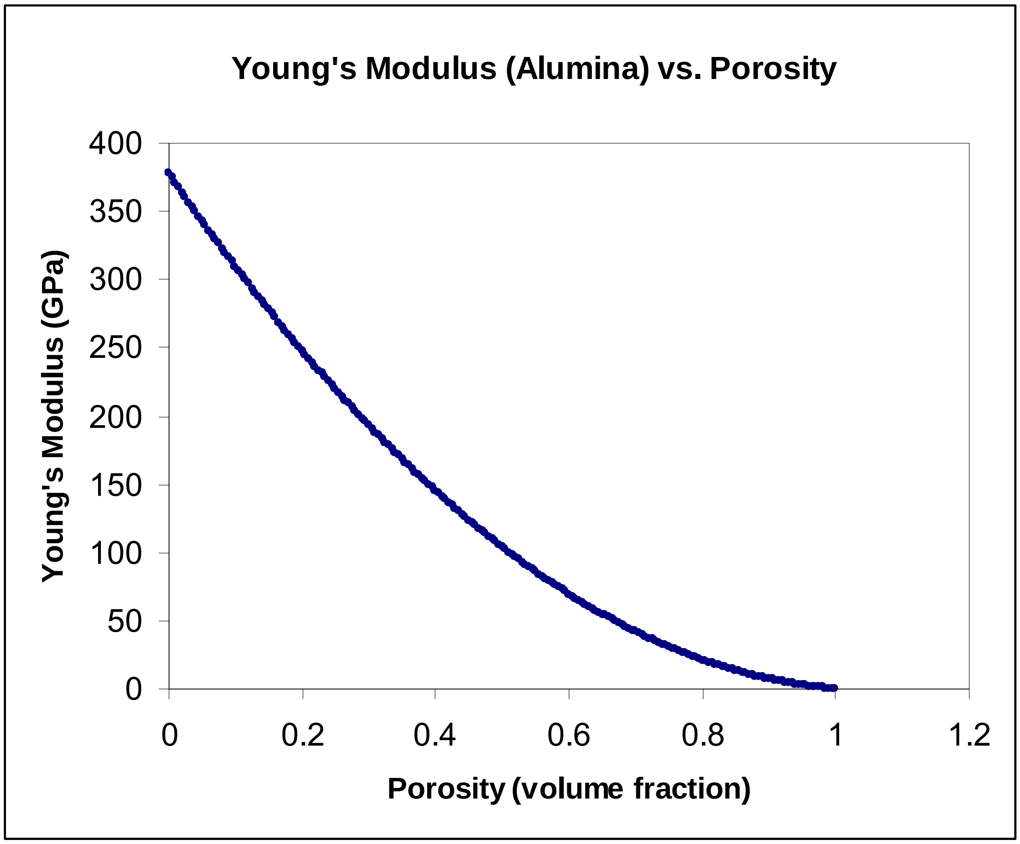
2.24 Plot Young's modulus as a function of porosity for alumina, and show what the value should be for a specimen having 5% porosity (E_{Al2O3}= 378 GPa).

$E = E_o(1 - 1.9\ p + 0.9\ p^2)$
$E_o = 378$ GPa

For $p = 0.05$
$E = 378\ \text{GPa}\ [1 - 1.9(0.05) + 0.9\ (0.05)^2\ ]$
$= 342.94$ GPa

Porosity, p	$E = E_o\ (1 - 1.9\ p + 0.9\ p^2)$
0	378
0.1	309.582
0.2	247.968
0.3	193.158
0.4	145.152
0.5	103.95
0.6	69.552
0.7	41.958
0.8	21.168
0.9	7.182
1	$4.19664\ 10^{-14}$

The plot is shown on the next page.


Young's Modulus (Alumina) vs. Porosity
400
350
300
250
200
150
100
50
0
Young's Modulus (GPa)
0
0.2
0.4
0.6
0.8
1
1.2
Porosity (volume fraction)

2.25 A specimen of Al_2O_3 contains microcracks that are approximately equal to its grain size (20 μm). One grain in each 10 grains contains cracks. If the uncracked materials has E_o = 378 GPa, determine Young's modulus for the cracked material using Budiansky and O'Connell's and Salganik's eequations.

<u>Budiansky and O'Connell</u>:

$$\frac{E}{E_0} = 1 - \frac{16(10-3\nu)(1-\nu^2)}{45(2-\nu)} f_s$$

$$\nu = \nu_o\left(1 - \frac{16 f_s}{9}\right)$$

$$E_o = 378\ GPa$$

$$f_s = N^3$$

$$N = \frac{number\ of\ cracks}{unit\ volume}$$

$$= \frac{1}{10(20x10^{-6}\,m)^3} = 1.25\times10^{13} 1/m^3$$

$$a = mean\ crack\ radius = \frac{20\mu\, m}{2} = 10\mu\, m$$

$$f_s = Na^3 = \left(1.25\times10^{13}\frac{1}{m^3}\right)(10\times10^{-6}\,m)^3 = 0.0125$$

$$\nu = \nu_o\left(1 - \frac{16 f_s}{9}\right) = 0.32\left(1 - \frac{16}{9}(0.0125)\right) = 0.31$$

$$E = E_o\left[1 - \frac{16(10-3(0.31)(1-(0.31)^2))}{45(2-0.31)}(0.0125)\right]$$

$$= 378\ GPa\left[1 - \frac{16(9.07)(0.9039)}{76.05}(0.0125)\right]$$

$$= 378\ GPa[1 - 1.725(0.0125)]$$

$$E = 370\ GPa$$

Salganik:

$$\frac{E}{E_o} = \left[1 + \frac{16(10 - 3\nu_o)(1 - \nu_o^2)}{45(2 - \upsilon_o)} Na^3\right]^{-1}$$

$$E_o = 378\ GPa \quad \nu_o = 0.32$$

$$Na^3 = 0.0125 \qquad (same\ as\ first\ method)$$

$$\therefore E = E_o\left[1 + \frac{16(10 - 3(0.32))(1 - (0.32))^2}{45(2 - 0.32)}(0.0125)\right]^{-1}$$

$$= 378\ GPa\left[1 + \frac{16(9.09)(0.5996)}{75.6}(0.0125)\right]^{-1}$$

$$E = 370\ GPa$$

2.26 Young's modulus (E) of a cubic single crystal as a function of orientation is given by

$$\frac{1}{E_{hkl}}=\frac{1}{E_{100}}-3\left(\frac{1}{E_{100}}-\frac{1}{E_{111}}\right)\left(\ell_1^2\ \ell_2^2+\ell_2^2\ \ell_3^2+\ell_3^2\ \ell_1^2\right)$$

where ℓ_1, ℓ_2, and ℓ_3 are the direction cosines between the direction $hk\ell$ and [100], [010], and [001], respectively. This is another version of the expression given in Example 2.10. For copper, E_{111} = 191 GPa and E_{100} = 66 GPa. Calculate Young's modulus for a copper single crystal in the {110} direction, and check your answer against the one in Example 2.10.

$$\frac{1}{E_{hkl}}=\frac{1}{E_{100}}-3\left(\frac{1}{E_{100}}-\frac{1}{E_{111}}\right)\left(\ell_1^2\ \ell_2^2+\ell_2^2\ \ell_3^2+\ell_1^2\ \ell_3^2\right)$$

$$Given\ E_{100}=66\ GPa$$

$$E_{111}=191\ GPa$$

The sum of the direction cosines term in the parenthesis for [110] is

$$\left(\ell_1^2\ \ell_2^2+\ell_2^2\ \ell_3^2+\ell_3^2\ \ell_3^2\right)=1/4$$

$$\therefore \frac{1}{E_{110}}=\frac{1}{66\ GPa}-3\left(\frac{1}{66\ GPa}-\frac{1}{191\ GPa}\right)(1/4)$$

$$E=129.6\ GPa$$

2.27 A polymer has a viscosity of 10^{12} Pa s at 150°C. If this polymer is subjected to a tensile stress of 100 MPa at that temperature, compute the deformation after 10 h. Assume the polymer to behave as a Maxwell solid. Take E = 5 GPa, and use the equation

$$\varepsilon_t = \frac{\sigma}{E} + \frac{1}{3\eta}\sigma\, t$$

$$\varepsilon_t = \frac{\sigma}{E} + \frac{1}{3\eta}\sigma\, t$$
$$\varepsilon_t = \frac{100\times10^6}{5\times10^9} + \frac{1}{3\times10^{12}}\left(100\times10^6\right)\left(10h\right)\left(\frac{3600\ s}{1h}\right)$$
$$\varepsilon_t = 1.22\ Pa$$

2.29 For an elastomeric material, we have the constitutive equation

$$\sigma = G\left(\lambda - \frac{1}{\lambda^2}\right) = \frac{E}{3}\left[\lambda - \frac{1}{\lambda^2}\right]$$

where E is the elastic modulus at zero elongation. Show that, for very small strains, this equation reduces to σ = Eε.

$$\sigma = G\left(\lambda - \frac{1}{\lambda 2}\right) = \frac{E}{3}\left[\lambda - \frac{1}{\lambda^2}\right] \qquad (1)$$

By definition

$$\varepsilon = \frac{\Delta\ell}{\ell_0} = \frac{\ell_i - \ell_0}{\ell_0} = \frac{\ell_i}{\ell_0} - 1 = \lambda - 1 \quad (2)$$
$$\lambda = \varepsilon + 1$$

Substituting (2) into (1), we get

$$\sigma = \frac{E}{3}\left[(\varepsilon+1) - \frac{1}{(\varepsilon+1)^2}\right]$$
$$= \frac{E}{3}\left[\frac{\varepsilon^3 + 2\varepsilon^2 + \varepsilon + \varepsilon^2 + 2\varepsilon + 1 - 1}{\varepsilon^2 + 2\varepsilon + 1}\right]$$
$$= \frac{E}{3}\left[\frac{\varepsilon^3 + 3\varepsilon^2 + 3\varepsilon}{\varepsilon^2 + 2\varepsilon + 1}\right]$$

For very small strains, we can neglect the higher order terms, i.e., ε^3 and ε^2 can be neglected. Thus,

$$\sigma = \frac{E}{3} \cdot \frac{3\varepsilon}{2\varepsilon + 1} = \frac{E\varepsilon}{2\varepsilon + 1}$$

At small strain, the term 2ε in the denominator is negligible compared to 1 and can be neglected. Thus, at small strains

$$\sigma = E\varepsilon$$

2.30 A cylindrical steel specimen (length = 200 mm, diameter = 5 mm) is subjected to a torque equal to 40 Nm. If one end of the specimen is fixed what is the deflection of the other end? Take E = 210 GPa and ν =0.3.

Given:

$$E = 210\ GPa \quad \upsilon = 0.3$$
$$\ell = 200\ mm = 0.2\ m$$
$$d = 5\ mm = 0.005\ m$$
$$T = 40\ Nm$$
$$c = \frac{d}{2} = 2.5\ mm$$

$$J = \frac{\pi\ c^4}{2}$$
$$\tau_{\max} = \frac{Tc}{\frac{\tau\ c^4}{2}} = \frac{2T}{\pi\ c^3} = \frac{2 \times 40\ Nm}{\pi \times 2.5^3\ mm}$$
$$= 1.63\ GPa$$
$$\tau = G\gamma$$
$$G = \frac{E}{2(1+\upsilon)} = \frac{210}{2(1+0.3)} \approx 81\ GPa$$
$$\gamma = \frac{\tau_{\max}}{G} = \frac{1.63}{81} = 0.02$$
$$deflection = \frac{r}{c} = \frac{0.02}{2.5}$$
$$\therefore deflection = 0.008\ rad / mm$$

$\boxed{2.33}$ Describe dilation that occurs in the elastic deformation of a solid. Give a mathematical expression in terms of strain components.

$$V_0 = 1$$

$$V_f = (1+\varepsilon_{11})(1+\varepsilon_{22})(1+\varepsilon_{33})$$

Neglect the cross product of the strains

$$V_f = 1 + \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}$$

Dilation $= \dfrac{\Delta V}{V_i} = \dfrac{V_f - V_i}{V_i} = \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}$

The sum of the diagonal terms in the strain matrix represents the dilation.

2.34 Consider a solid subjected to hydrostatic pressure, p, that results in a dilation or volumetric strain given by
$V/V = \varepsilon_p$.

The bulk modulus, K, is defined by the ratio p/ε_p. Using the generalized Hooke's law, show that

$K = E/3(1 - 2v)$,

where E is the Young's modulus and v is the Poisson's ratio.

Assume initial volume $V_i = 1$

$$V_f = (1 + \varepsilon_{11})(1 + \varepsilon_{22})(1 + \varepsilon_{33})$$

$$V_f = 1 + \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}$$

$$V_f - V_i = \Delta V = 1 + \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33} - 1 = \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}$$

Hydrostatic pressure implies $\varepsilon_{11} = \varepsilon_{22} = \varepsilon_{33}$

$$\Delta V = 3\varepsilon_{11}$$

$$K = \frac{p}{3\varepsilon_{11}}$$

Equation 2.11: $\varepsilon_{11} = \dfrac{1}{E}\left(\sigma_{11} - v(\sigma_{22} + \sigma_{33})\right)$

Hydrostatic pressure implies $= \sigma_{11} = \sigma_{22} = \sigma_{33} = p$

$$\varepsilon_{11} = \frac{1}{E}\left(p - 2\nu p\right) = \frac{p}{E}(1 - 2\nu) \quad (1)$$

$$K = \frac{p}{\varepsilon_p} = \frac{p}{\frac{\Delta V}{V}} = \frac{p}{\Delta V} = \frac{p}{3\varepsilon_{11}}$$

Substitute in Equation (1) for ε_{11}

$$K = \frac{p}{\frac{3p}{E}(1 - 2\nu)} = \boxed{\frac{E}{3(1 - 2\nu)}}$$

2.35 (a) Compute the Poisson's ratio for a material that is undergoing a uniaxial tensile test with zero dilation.

(b) A student was given three different unidentified materials to determine their Poisson's ratio. She determined the Poisson's ratios to be 0.5, 0.3, and 0. She needs your help in identifying the class of material for each of these ν values.

(a) Zero dilation means $\varepsilon_1 + \varepsilon_2 + \varepsilon_3 = 0$. For an isotropic material, $\varepsilon_2 = \varepsilon_3$,

$\varepsilon_1 + \varepsilon_2 + \varepsilon_2 = 0$
$\varepsilon_1 = -2\,\varepsilon_2$

Poisson's ratio is given by

$\nu = -\,\varepsilon_2/\varepsilon_1 = 0.5$

(b)

$\nu = 0.5$ corresponds to zero dilation, which would imply a rubbery or elastomeric material. Such a material deforms elastically at a constant volume.

$\nu = 0.3$ indicates a metallic material.

$\nu = 0$ indicates a material that does not deform in the transverse direction when compressed or stretched. A cellular material would show such a behavior. Cork is natural material that comes close to such material. That is the reason cork is used for bottle caps.

$\boxed{\textbf{2.37}}$ From Table 2.5, estimate the theoretical shear and cleavage strength for nickel and titanium.

From Table 2.5

	E (GPa)	G (GPa)
Nickel	199.5	76.0
Titanium	115.7	43.8

From Section 4.1 p. 251

$$\sigma_{th} = \sqrt{\frac{E\gamma}{a}} \approx \frac{E}{\pi} \text{ and } \tau_{th} = \frac{Gb}{2\pi G} \approx \frac{G}{2\pi}$$

Nickel: $\sigma_{th} = \dfrac{199.5}{\pi} = 63.5GPa$ $\qquad \tau_{th} = \dfrac{76.0}{2\pi} = 12.1GPa$

Titanium: $\sigma_{th} = \dfrac{115.7}{\pi} = 36.8GPa$ $\qquad \tau_{th} = \dfrac{43.8}{2\pi} = 6.9GPa$

$\boxed{\textbf{2.38}}$ From Table 2.5, estimate the theoretical shear and cleavage strength for magnesium and niobium.

From Table 2.5

	E (GPa)	G (GPa)
Niobium	104.9	37.5
Magnesium	44.7	17.3

From Section 4.1 p. 251

$$\sigma_{th} = \sqrt{\frac{E\gamma}{a}} \approx \frac{E}{\pi} \text{ and } \tau_{th} = \frac{Gb}{2\pi G} \approx \frac{G}{2\pi}$$

Niobium: $\sigma_{th} = \dfrac{104.9}{\pi} = 33.40GPa$ $\qquad \tau_{th} = \dfrac{37.5}{2\pi} = 5.97GPa$

Magnesium: $\sigma_{th} = \dfrac{447}{\pi} = 14.230GPa$ $\qquad \tau_{th} = \dfrac{17.3}{2\pi} = 2.75GPa$

$\boxed{\textbf{2.39}}$ From Table 2.3 find the elastic compliances for nickel.

From Table 2.3

		C_{11}*	C_{44}*	C_{12}*
Nickel	FCC	246.5	124.7	147.3

*[GPa]

$$S_{44} = \frac{1}{C_{44}}$$

$$S_{12} = \frac{-C_{12}}{(C_{11} + 2C_{12})(C_{11} - C_{12})}$$

$$S_{11} = \frac{C_{11} + C_{12}}{(C_{11} + 2C_{12})(C_{11} - C_{12})}$$

$$S_{44} = \frac{1}{124.7} = 0.801 \times 10^{-2}\, GPa^{-1}$$

$$S_{12} = \frac{-147.3}{(246.5 + 2(147.3)(246.5 - 147.3))} = -207 \times 10^{-2}\, GPa^{-1}$$

$$S_{11} = \frac{246.5 + 147.3}{(246.5 + 2(147.3)(246.5 - 147.3))} = 0.733 \times 10^{-2}\, GPa^{-1}$$

2.40 From Table 2.3 find the elastic compliances for aluminum.

From Table 2.3

		C_{11} *	C_{44} *	C_{12} *
Aluminum	FCC	108.2	28.5	61.3

*[GPa]

$$S_{44} = \frac{1}{C_{44}}$$

$$S_{12} = \frac{-C_{12}}{(C_{11} + 2C_{12})(C_{11} - C_{12})}$$

$$S_{11} = \frac{C_{11} + C_{12}}{(C_{11} + 2C_{12})(C_{11} - C_{12})}$$

$$S_{44} = \frac{1}{28.5} = 3.51 \times 10^{-2}\, GPa^{-1}$$

$$S_{12} = \frac{-61.3}{(108.2 + 2(61.3))(108.2 - 61.3)} = -.506 \times 10^{-2}\, GPa^{-1}$$

$$S_{11} = \frac{108.2 + 61.3}{(108.2 + 2(61.3))(108.2 - 61.3)} = 1.5 \times 10^{-2}\, GPa^{-1}$$

2.43 The following values are given for niobium:
E=105 GPa,
$v = 0.4$.
Calculate the values of *G*, *B*, *K*, and *λ*.

$$G = \frac{E}{2(1+v)} = \frac{105}{2(1+0.4)} = 37.5\ GPa$$

$$K = \frac{E}{3(1-\nu)} = \frac{105}{3(1-0.4)} = 58.3\ GPa$$

$$B = \frac{1}{K} = \frac{1}{58.3} = 0.0172\ GPa^{-1}$$

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)} = \frac{105\ x\ 0.4}{(1+0.4)(1-2\,x\,0.4)} = 150\ GPa$$

2.42 The following are given for tantalum:
$C_{11} = 267$ GPa,
$C_{44} = 82.5$ GPa,
$C_{12} = 161$ GPa. E

Determine the Young's moduli in the directions [100], [110], and [111] after calculating the elastic compliances.

Solution:

$$\frac{1}{E_{ijk}} = S_{11} - 2\left(S_{11} - S_{12} - \frac{1}{2}S_{44}\right)\left(l_{i1}^2 l_{j2}^2 + l_{j2}^2 l_{k3}^2 + l_{i1}^2 l_{k3}^2\right)$$

	l_{i1}	l_{j2}	l_{k3}	$l_{i1}^2 l_{j2}^2 + l_{j2}^2 l_{k3}^2 + l_{i1}^2 l_{k3}^2$
[100]	1	0	0	0
[110]	$\sqrt{2}/2$	$\sqrt{2}/2$	0	1/4
[111]	$1/\sqrt{3}$	$1/\sqrt{3}$	$1/\sqrt{3}$	1/3

$$S_{44} = \frac{1}{C_{44}} = \frac{1}{82.5} = 1.212 \times 10^{-2}\ \text{GPa}^{-1}$$

$$S_{12} = \frac{-C_{12}}{(C_{11} + 2C_{12})(C_{11} - C_{12})} = -2.579 \times 10^{-3}\ \text{GPa}^{-1}$$

$$S_{11} = \frac{1}{C_{11}} + \frac{2C_{12}^2}{C_{11}(C_{11} + 2C_{12})(C_{11} - C_{12})} = 6.855 \times 10^{-3}\ \text{GPa}^{-1}$$

$$\frac{1}{E_{100}} = 6.855 \times 10^{-3}\ \text{GPa}^{-1} \Rightarrow E_{100} = 149.879\ \text{GPa}$$

$$\frac{1}{E_{110}} = 5.168 \times 10^{-3}\ \text{GPa}^{-1} \Rightarrow E_{110} = 193.498\ \text{GPa}$$

$$\frac{1}{E_{111}} = 4.606 \times 10^{-3}\ \text{GPa}^{-1} \Rightarrow E_{111} = 217.124\ \text{GPa}$$

2.43 The following values are given for niobium:
E = 105 GPa,
$\nu =$ 0.4.

Calculate the values of G, B, K, and λ.

Solution:

$$E = 105 GPa$$
$$\nu = 0.4$$

Calculate the values of G, B, K, and λ.
Knowing that

$$G = \frac{E}{2(1+\nu)}$$

$$K = \frac{E}{3(1-\nu)}$$

$$B = \frac{1}{K}$$

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}$$

So,

$$G = \frac{105GPa}{2(1+0.4)} = 37.5GPa$$

$$K = \frac{105GPa}{3(1-0.4)} = 58.3GPa$$

$$B = \frac{1}{58.3} 0.0172GPa$$

$$\lambda = \frac{105GPa * 0.4}{(1+0.4)(1-2*0.4)} = 150GPa$$

2.44 Plot the engineering stress--engineering strain curve for a rubber at ambient temperature and liquid nitrogen temperature, up to a strain of 10, using the Equation 2.48. The number of chain segments per unit volume (m^3) is 2×10^{25}.

$$\sigma = nkT[\lambda_1^2 + \lambda_2^{-1}]$$

$$k = 13.81 * 10^{-24} J/K$$

$$\varepsilon \approx 10$$

$$\lambda = 1 + \varepsilon$$

$$n = 2 * 10^{25} m^{-3}$$

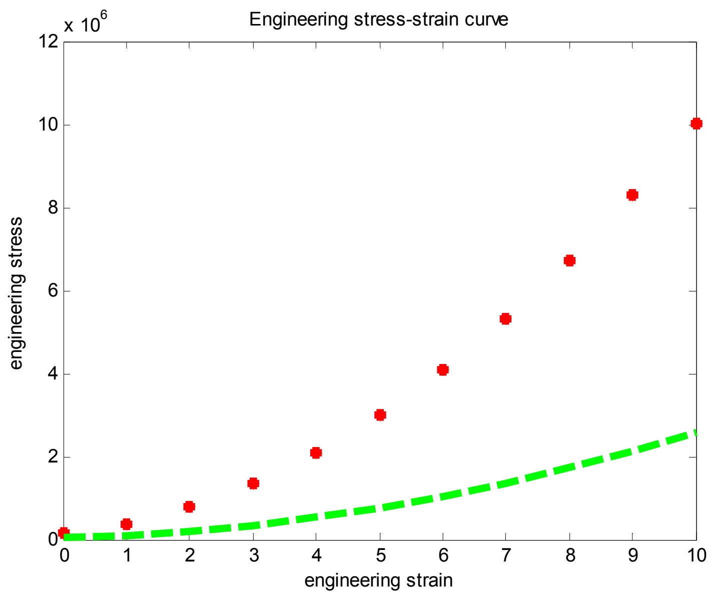
(1) At ambient temperature Ta=300K,

$$\sigma_1 = 82860[(1+\varepsilon)^2 + (1+\varepsilon)^{-1}]$$

(2) At liquid Nitrogen temperature TN=77K

$$\sigma_2 = 21267.4[(1+\varepsilon)^2 + (1+\varepsilon)^{-1}]$$

```
e=0:1:10;
s1=82860*((1+e).^2+(1+e).^(-1));
s2=21267.4*((1+e).^2+(1+e).^(-1))
figure(2)
hand2=plot(e,s1,'r*',e,s2,'g--')
xlabel('engineering strain')
ylabel('engineering stress')
title('Engineering stress-strain curve')
set(hand2,'Linewidth',4)
```



2.45 From the elastic stiffness for a cubic material, Nb (C_{11} = 242 GPa, C_{12} = 129 GPa, C_{44} = 286 GPa), find the elastic compliances.

$S_{44} = 1/C_{44} = 1/286 = 0.3497\ 10^{-2}$ GPa $^{-1}$

$$S_{12} = \frac{-C_{12}}{(C_{11}+2C_{12})(C_{11}-C_{12})} = \frac{-129}{(242+2(129))(242-129)}$$

$$= -0.2283\times 10^{-2}\, GPa^{-1}$$

$$S_{11} = \frac{1}{C_{11}} + \frac{2C_{12}^2}{C_{11}(C_{11}+2C_{12})(C_{11}-C_{12})}$$

$$= \frac{1}{242} + \frac{2(129)^2}{242(242+2(129))(242-129)} = 0.657\, 10^{-2}\, GPa^{-1}$$

$\boxed{\mathbf{2.46}}$ The potential energy of a Na^+ Cl^- ion pair at the distance r is given by:

$$U = U_i - \frac{q^2}{4\pi\varepsilon_0 r} + \frac{B}{r^9},$$

where q =1.6 × 10−19 C is the electronic charge, ε_0 =8.85 × 10^{-12} $C^2/(N \cdot m^2)$ is the permittivity of vacuum, and U_i=1.12 eV is the reference energy of two infinitely separated ions. If the equilibrium distance between the ions is r_0 =0.276 nm, calculate:
(a) the value of the constant B;
(b) the total force between ions, and its attractive and repulsive portions, when r = 0.25 nm;
(c) the total force between ions, and its attractive and repulsive portions, when r = 0.3 nm;
(d) the potential energy between two ions when they are at a distance r = 1 nm.

a) at equilibrium $r = r_0$ and $\frac{dU}{dr} = 0$

$$\frac{dU}{dr} = 0 = \frac{q^2}{4\pi\varepsilon_0}\frac{1}{r_0^2} - \frac{9B}{r_0^{10}}$$

$$\frac{q^2}{4\pi\varepsilon_0}\frac{1}{r_0^2} = \frac{9B}{r_0^{10}} \Rightarrow B = \frac{r_0^8 q^2}{9 \cdot 4\pi\varepsilon_0}$$

$$\boxed{B = 8.61 \times 10^{-106}\, N \cdot m^{10}}$$

b) $$F = \frac{dU}{dr} = \underbrace{\frac{q^2}{4\pi\varepsilon_0}\frac{1}{r^2}}_{\text{Attractive}} - \underbrace{\frac{9B}{r^{10}}}_{\text{Repulsive}}$$

Attractive force = $3.68 \times 10^{-9}\, N$

Repulsive force = $-\,8.13 \times 10^{-9}\, N$

Total Force = $\boxed{4.44 \times 10^{-9}\ N}$

c) Attractive Force = $2.56 \times 10^{-9}\ N$
Repulsive Force = $-\ 1.31 \times 10^{-9}\ N$

$\boxed{\text{Total Force} = 1.25 \times 10^{-9}\ N}$

d) $U = U_i - \dfrac{q^2}{4\pi\varepsilon_0 r} + \dfrac{B}{r^9}$

r = .1nm

$U = 1.12eV - 2.30 \times 10^{-19}\ J + 8.61 \times 10^{-25}\ J$

<u>Conversion:</u> $1eV = 1.602 \times 10^{-19}\ J$

$U = 1.12\left(1.602 \times 10^{-19}\right) - 2.30 \times 10^{-19} + 8.01 \times 10^{-23}$

$\boxed{U = -5.07 \times 10^{-20}\ J}$

2.47 The potential energy of two atoms, a distance r apart, is

$$U = -\frac{A}{r^m} + \frac{B}{r^n}.$$

Given that the atoms form a stable molecule at a separation $r = r_0$, with a binding energy $U = U_0$, derive:
(a) the expressions for the constants A and B in terms of m, n, r_0, and U_0;
(b) the expressions for the stiffness S of the bond at arbitrary r and at r_0;
(c) the expression for the distance r^* of the maximum tensile force (needed to break the bond between atoms), and the expression for that force ($F*$).
(d) Given that m=2, n=10, and that the atoms form a stable molecule at a separation r_0 =0.3 nm, with a binding energy U_0=−4 eV, evaluate A, B, $r*$, $F*$, and the stiffness $S0$ of the bond at $r = r_0$.

$$U = \frac{-A}{r^m} + \frac{B}{r^n}$$

Atoms form a stable molecule at a separation $r = r_0$ with a binding energy $U = U_0$

a) $U = \dfrac{-A}{r^m} + \dfrac{B}{r^n}$ at equilibrium $r = r_0$, $\dfrac{dU}{dr} = 0$

$$\frac{dU}{dr} = \frac{mA}{r^{m+1}} - \frac{nB}{r^{n+1}} = 0 \qquad \text{Equation (1)}$$

Stable molecule $r = r_0$, $U = U_0$

$$U_0 = \frac{-A}{r_0^{\;m}} + \frac{B}{r^n} \qquad \text{Equation (2)}$$

There are two equations and two unknowns, so we can solve them

$$\frac{A}{r^{m+1}} = \frac{nB}{r^{n+1}} \qquad (1)$$

$$\frac{A}{r^m \cdot r} = \frac{nB}{r^n \cdot r} \Rightarrow A = r_0^{\;m-n} nB$$

$\Uparrow$

Plug into Equation (2)

$$U_0 = \frac{-r_0^{\;m-n} nB}{r_0^{\;m}} + \frac{B}{r_0^{\;n}}$$

$$U_0 = \frac{-nB}{r_0^{\;n}} + \frac{B}{r_0^{\;n}} \Rightarrow B = \frac{U_0 r_0^{\;n}}{1-n}$$

$$A = r_0^{\;m-n} n\left(\frac{U_0 r_0^{\;n}}{1-n}\right) = \frac{n r_0^{\;m} U_0}{1-n}$$

b)

Equation 2.52: $E = \dfrac{A(n-1)}{r_0^{\;4}}$, $\qquad A = \dfrac{e^2}{4\pi\varepsilon_0}$

$e = 1.6 \times 10^{-19} eV$

$\varepsilon_0 = 8.8 \times 10^{-12} \frac{c^2}{nm^2}$

$$\boxed{E = \frac{e^2(n-1)}{4\pi\varepsilon_0 r_0^{\ 4}}}$$

c) $\frac{dU}{dr} = F$ At Maximum in F, $\frac{d^2U}{dr^2} = 0$

$$\frac{dU}{dr} = \frac{mA}{r^{m+1}} - \frac{nB}{r^{n+1}}$$

$$\frac{d^2U}{dr^2} = \frac{m(m+1)A}{r^{m+2}} - \frac{n(n+1)B}{r^{n+2}} = 0$$

$$\frac{m(m+1)A}{n(n+1)B} = \frac{r^{m+2}}{r^{n+2}} = r_*^{\ m-n}$$

$$\frac{m(m+1)nr_0^{\ m}U_0(1-n)}{n(n+1)(1-n)r_0^{\ n}U_0} = r_*^{\ m-n}$$

$$\frac{m(m+1)r_0^{\ m-n}}{(n+1)} = r_*^{\ m-n}$$

$$\left(\frac{m(m+1)}{n+1}\right)^{\frac{1}{m-n}} r_0 = r_*$$

Expression for Force:

$r = r_*$

$$F_* = \frac{mA}{r^{m+1}} - \frac{nB}{r^{n+1}}$$

$$F_* = \frac{mnr_0{}^m U_0}{1-n\left(\left(\frac{m(m+1)}{(n+1)}\right)^{\frac{1}{m-n}} r_0\right)^{m+1}} - \frac{nU_0 r_0{}^n}{(1-n)\left(\left(\frac{m(m+1)}{(n+1)}\right)^{\frac{1}{m-n}} r_0\right)^{n+1}}$$

d) m = 2, n = 10, $r_0 = .3$nm, $U_0 = -4eV$

$A = 4 \times 10^{-19} m^2 eV$

$B = 2.62 \times 10^{-96} m^{10} eV$

$E_0 = S_0 = 2.55 \times 10^{11} = 255GPa$

$r^* = .323 \times 10^{-9} m$

$F^* = 2.37 \times 10^{10} - 1.47 \times 10^{10} = 8.9GPa$

2.48 Plot the stress-strain curve for alumina in tension, knowing that the density of microcracks increases linearly with stress ($N = k\sigma$). The grain size is 20 μm. The failure stress is 1 GPa; given: $k = 5.45 \times 104\ m^3/Pa$, E_0 =420GPa.

Given : Grain size 20 μm
failure stress = 1 GPa
$k = 5.45 \times 10^4\ m^{-3}\ Pa^{-1}$
$E_0 = 420$ GPa
$\nu = 0.17$

Using Budiansky & O'Connell

$$\frac{E}{E_0} = 1 - \underbrace{\frac{16(10-3\nu)(1-\nu^2)}{45(2-\nu)}}_{k_1 = 1.79} f_s$$

$f_s = N a^3$

$N = k\sigma$

$$N = \frac{\#\text{ of cracks}}{\text{unit volume}}$$

$$\frac{E}{E_0} = 1 - k_1 k \sigma a^3$$

$$E = \frac{d\sigma}{d\varepsilon} = E_0 (1 - k_1 k \sigma a^3)$$

$$\Rightarrow \frac{d\sigma}{1 - k_1 k a^3} = E_0\, d\varepsilon$$

let $b = -k_1 k a^3$

$$\Rightarrow \frac{d\sigma}{1 + b\sigma} = E_0\, d\varepsilon \quad \text{(Substitution)}$$

let $u = 1 + b\sigma$

$du = b\, d\sigma$

$$\frac{1}{b}\int \frac{du}{u} = E_0 \int d\varepsilon$$

$$\frac{1}{b} \ln u = E_0 \varepsilon$$

$$\frac{1}{b} \ln(1 + b\sigma) = E_0 \varepsilon$$

2.48 (cont'd)

$$\frac{1}{-k_1 k a^3} \ln(1 - k_1 k a^3 \sigma) = E_0 \varepsilon$$

$$b = -k_1 k a^3 = -1.79 \times 5.45 \times 10^4 \ m^{-3}\ Pa^{-1} \times (20 \times 10^{-6}/2)^3$$

$$b = -9.76 \times 10^{-11}$$

$$\ln(1 + b\sigma) = E_0 b \varepsilon$$

$$1 + b\sigma = e^{E_0 b \varepsilon}$$

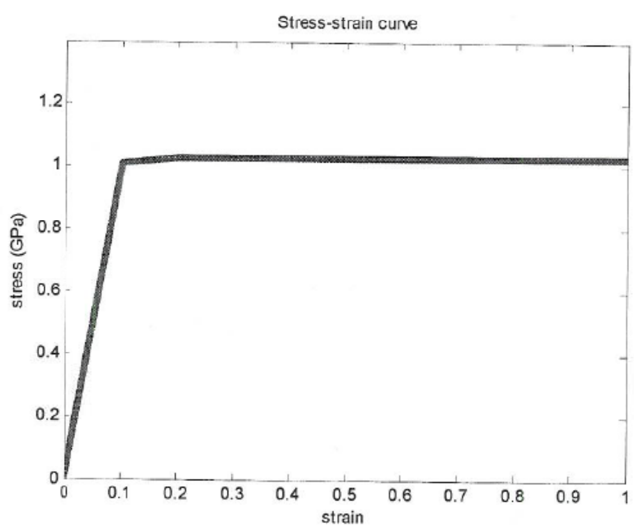
$$\sigma = \frac{e^{E_0 b \varepsilon} - 1}{b}$$

$$E_0 = 420 \times 10^9 \ Pa$$

$$E_0 b = -40.99$$

$$\sigma = \frac{e^{-40.99\varepsilon} - 1}{-9.76 \times 10^{-11}} = -9.76 \times 10^{4} \ (e^{-40.99\varepsilon} - 1)$$

```
x=0:0.1:1;
b=-9.76*10E-11;
y2=1E-9*(exp(-40.99*x)/b-1/b);
figure(1)
hand=plot(x,y2)
xlabel('strain')
ylabel('stress (GPa)')
title('Stress-strain curve')
axis([0,1,0,1.4])
set(hand,'LineWidth',4)
```



$\boxed{\textbf{2.49}}$ Derive the expression:

$G = E/2(1 + \nu)$.

The symbols have their usual significance.

Done in text, Sec 2.8, for bi-dimensional case for which:

$\sigma_{22} = -\sigma_{11}$

(1) $\varepsilon_{11} = \frac{1}{E}\left(\sigma_1 - \nu\sigma_2\right) = \frac{\sigma_1}{E}(1+\nu)$

(2) $\tau = -\sigma_1$

(3) $\tau = G\gamma$

Plug in equations (2) and (3) into (1)

$$\varepsilon_{11} = \frac{1}{E}\left(-G\gamma - \nu G\gamma\right)$$

$$\varepsilon_{11} = \frac{-G\gamma}{E}(1+\nu)$$

$$G = \frac{E}{1+\nu}\left(\frac{-\varepsilon_{11}}{\gamma}\right)$$

<u>From figure 2.8c</u>

$\tan\frac{-\gamma}{2} = \varepsilon_{11}$

small angle $\tan\frac{-\gamma}{2} \approx \frac{\gamma}{2}$

$\frac{-\gamma}{2} = \varepsilon_{11}$

$$\frac{1}{2} = \frac{-\varepsilon_{11}}{\gamma}$$

Small angle approximation

$$\boxed{G = \frac{E}{2(1+\nu)}}$$

2.51 From the data on elastic stiffness and compliances for HCP zirconium (Tables 2.3 and 2.4), determine the elastic stiffness, $C13$, missing in Table 2.3.

Template for HCP Structure

$$\begin{bmatrix} 11 & 12 & 13 & 0 & 0 & 0 \\ 0 & 11 & 13 & 0 & 0 & 0 \\ 0 & 0 & 33 & 0 & 0 & 0 \\ 0 & 0 & 0 & 44 & 0 & 0 \\ 0 & 0 & 0 & 0 & 44 & 0 \\ 0 & 0 & 0 & 0 & 0 & X \end{bmatrix}$$

$$X = 2\left(S_{11} - S_{12}\right) \qquad \text{or} \qquad X = 0.5\left(C_{11} - C_{12}\right)$$

In GPa

$C_{11} = 143.4$

$C_{44} = 32.0$

$C_{12} = 72.8$

$C_{33} = 164.8$

$C_{66} = 65.3$

In GPa^{-1}

$S_{11} = 1.013 \times 10^{-2}$

$S_{44} = 3.13 \times 10^{-2}$

$S_{12} = -0.404 \times 10^{-2}$

$S_{23} = 0.799 \times 10^{-2}$

$S_{13} = -0.241 \times 10^{-2}$

When we multiply stiffness and compliance matrices, we get the identity matrix.

Row 2 of stiffness matrix x Column 2 of compliance matrix = 1

$C_{11}S_{12} + C_{13}S_{11} = 1$

$$C_{13} = \frac{1 - C_{11}S_{12}}{S_{11}}$$

$$C_{13} = \frac{1 - (143.4GPa)\left(-.404 \times 10^{-2}\,GPa^{-1}\right)}{1.013 \times 10^{-2}\,GPa^{-1}} = \boxed{155.9GPa}$$

2.52

<u>From Table 2.3</u>

Sn [GPa]	$C_{11} = 73.5$	$C_{44} = 22.0$	$C_{12} = 23.4$
	$C_{33} = 87.0$	$C_{66} = 22.6$	$C_{13} = 28.0$

Compliance matrix with values

S_{ij}

$$\begin{bmatrix} 11 & 12 & 13 & 0 & 0 & 16 \\ 0 & 11 & 13 & 0 & 0 & -16 \\ 0 & 0 & 33 & 0 & 0 & 0 \\ 0 & 0 & 0 & 44 & 0 & 0 \\ 0 & 0 & 0 & 0 & 44 & 0 \\ 0 & 0 & 0 & 0 & 0 & 66 \end{bmatrix}$$

Stiffness matrix with values

C_{ij}

$$\begin{bmatrix} 73.5 & 23.4 & 28.0 & 0 & 0 & C_{16} \\ 0 & 73.5 & 28.0 & 0 & 0 & -C_{16} \\ 0 & 0 & 87 & 0 & 0 & 0 \\ 0 & 0 & 0 & 22.0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 22.0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 22.6 \end{bmatrix}$$

Stiffness Row 4 x Compliance column 4 = 1

$$C_{44} \cdot S_{44} = 1$$

$$S_{44} = 4.54 \times 10^{-2} \text{ GPa}^{-1}$$

Stiffness Row 6 x Compliance column 6 = 1

$$0\left(S_{16}\right) + 0\left(-S_{16}\right) + 22.6\left(S_{66}\right) = 1$$

$$S_{66} = 4.42 \times 10^{-2} \text{ GPa}^{-1}$$

Stiffness Row 2 x Compliance column 6 = 0

$$0\left(S_{16}\right) + C_{11}\left(-S_{16}\right) + C_{13}(0) - C_{16}\left(S_{66}\right) = 0$$

$$-C_{11}S_{16} = C_{16}S_{66} \qquad \Leftarrow \text{Equation (1)}$$

Stiffness Row 3 x Compliance column 3 = 1

$$0\left(S_{13}\right) + 0\left(S_{13}\right) + C_{33}\left(S_{33}\right) = 1$$

$$S_{33} = \frac{1}{C_{33}}$$

$$S_{33} = 1.15 \times 10^{-2} \text{ GPa}^{-1}$$

Stiffness Row 1 x Compliance column 1 = 1

$$C_{11}S_{11} + C_{12}(0) + C_{13}(0) + C_{16}(0) = 1$$

$S_{11} = 1.36 \times 10^{-2}$ GPa^{-1}

Stiffness Row 1 x Compliance column 2 = 0

$$C_{11}S_{12} + C_{12}S_{11} + C_{13}(0) + C_{16}(0) = 0$$

$$C_{11}S_{12} + C_{12}S_{11} = 0$$

$$S_{12} = \frac{-C_{12}S_{11}}{C_{11}} = -.433 \times 10^{-2} \text{ GPa}^{-1}$$

Stiffness Row 1 x Compliance column 3 = 0

$$C_{11}S_{13} + C_{12}S_{13} + C_{13}S_{33} + C_{16}(0) = 0$$

$$S_{13} = \frac{-C_{13}S_{33}}{C_{11} + C_{12}} = .332 \times 10^{-2} \text{ GPa}^{-1}$$

Stiffness Row 1 x Compliance column 6 = 0

$$C_{11}S_{16} + C_{12}\left(-S_{16}\right) + C_{13}(0) + C_{16}S_{66} = 0$$

$$-\left(C_{11} - C_{12}\right)S_{16} = C_{16}S_{66} \Leftarrow \text{ Equation (2)}$$

$$-C_{11}S_{16} = -\left(C_{11} - C_{12}\right)S_{16}$$

$$0 = -C_{12}S_{16}$$

$$C_{12} \neq 0 \therefore \boxed{S_{16} = 0}$$

$$(1)\ \frac{-C_{11}S_{16}}{S_{60}} = \boxed{C_{16} = 0}$$