ME 512 CONTINUUM MECHANICS Fall 2014 Soln: ASSIGNMENT 11

1. The stress power is defined to be $S_p = \int_R tr(\boldsymbol{\sigma} \cdot \boldsymbol{d}) dV$ where $\boldsymbol{\sigma}$ is the Cauchy stress and $\boldsymbol{d} = \boldsymbol{L}_{sym}$.

Other measures of stress are

$\boldsymbol{\Sigma} = \boldsymbol{R}^T \cdot \boldsymbol{\sigma} \cdot \boldsymbol{R}$ Rotated Cauchy stress

$\hat{\boldsymbol{P}} = J\boldsymbol{\sigma} \cdot \boldsymbol{F}^{-T}$ Piola-Kirchoff stress of the first kind

$\boldsymbol{P} = J\boldsymbol{F}^{-1} \cdot \boldsymbol{\sigma} \cdot \boldsymbol{F}^{-T} = \boldsymbol{F}^{-1} \cdot \hat{\boldsymbol{P}}$ Piola-Kirchoff stress of the second kind

and other rates of deformation are

$$\boldsymbol{D} = \boldsymbol{F}^T \cdot \boldsymbol{d} \cdot \boldsymbol{F} = \dot{\boldsymbol{E}} \qquad \boldsymbol{D}^* = \boldsymbol{R}^T \cdot \boldsymbol{d} \cdot \boldsymbol{R}$$

Show that alternative expressions for the stress power are:

$$S_p = \int_R tr\,(\boldsymbol{\Sigma} \cdot \boldsymbol{D}^*) dV = \int_{R_o} tr\,(\hat{\boldsymbol{P}} \cdot \dot{\boldsymbol{F}}^T) dV_o = \int_{R_o} tr\,(\boldsymbol{P} \cdot \dot{\boldsymbol{E}}) dV_o$$

These combinations of stress and deformation rates are said to be "conjugate."

Soln:

We use $tr(\boldsymbol{A} \cdot \boldsymbol{B} \cdot \boldsymbol{C}) = tr(\boldsymbol{B} \cdot \boldsymbol{C} \cdot \boldsymbol{A})$, $dV = JdV_0$ and $tr(\boldsymbol{A} \cdot \boldsymbol{B}) = tr(\boldsymbol{A} \cdot \boldsymbol{B}^T)$ if $\boldsymbol{A} = \boldsymbol{A}^T$.

$$\begin{aligned} tr(\boldsymbol{\sigma} \cdot \boldsymbol{d}) &= tr(\boldsymbol{R} \cdot \boldsymbol{\Sigma} \cdot \boldsymbol{R}^T \cdot \boldsymbol{R} \cdot \boldsymbol{D}^* \cdot \boldsymbol{R}^T) = tr(\boldsymbol{R} \cdot \boldsymbol{\Sigma} \cdot \boldsymbol{D}^* \cdot \boldsymbol{R}^T) \\ &= tr(\boldsymbol{\Sigma} \cdot \boldsymbol{D}^* \cdot \boldsymbol{R}^T \cdot \boldsymbol{R}) = tr(\boldsymbol{\Sigma} \cdot \boldsymbol{D}^*) \end{aligned}$$

$$tr(\boldsymbol{\sigma} \cdot \boldsymbol{d}) = tr(\boldsymbol{\sigma} \cdot \boldsymbol{L}) = tr(\boldsymbol{\sigma} \cdot \boldsymbol{L}^T) = \tfrac{1}{J} tr(\hat{\boldsymbol{P}} \cdot \boldsymbol{F}^T \cdot \boldsymbol{L}^T) = \tfrac{1}{J} tr(\hat{\boldsymbol{P}} \cdot \dot{\boldsymbol{F}}^T)$$

$$\begin{aligned} tr(\boldsymbol{\sigma} \cdot \boldsymbol{d}) &= \tfrac{1}{J} tr(\boldsymbol{F} \cdot \boldsymbol{P} \cdot \boldsymbol{F}^T \cdot \boldsymbol{F}^{-T} \cdot \boldsymbol{D} \cdot \boldsymbol{F}^{-1}) = \tfrac{1}{J} tr(\boldsymbol{F} \cdot \boldsymbol{P} \cdot \boldsymbol{D} \cdot \boldsymbol{F}^{-1}) \\ &= \tfrac{1}{J} tr(\boldsymbol{P} \cdot \boldsymbol{D} \cdot \boldsymbol{F}^{-1} \cdot \boldsymbol{F}) = \tfrac{1}{J} tr(\boldsymbol{P} \cdot \boldsymbol{D}) \end{aligned}$$

2. Recall that in connection with the study of a continuum, tensors could be defined as one of four possibilities: m-m, s-s, s-m, m-s where "m" denotes "material" and "s" denotes "spatial". Recall the classifications of $\boldsymbol{F}$ and $\boldsymbol{R}$ from the notes. Assume $\boldsymbol{\sigma}$ and $\boldsymbol{d}$ are both s-s. Use the relations given in Prob. 1 to classify the tensors $\boldsymbol{\Sigma}, \hat{\boldsymbol{P}}, \boldsymbol{P}, \boldsymbol{D}$ and $\boldsymbol{D}^*$.

Soln:

Start with

$$\boldsymbol{\sigma} = \overset{s-s}{\boldsymbol{\sigma}}, \boldsymbol{d} = \overset{s-s}{\boldsymbol{d}}, \boldsymbol{F} = \overset{s-m}{\boldsymbol{F}}, \boldsymbol{F}^T = \overset{m-s}{\boldsymbol{F}^T}, \boldsymbol{F}^{-1} = \overset{m-s}{\boldsymbol{F}^{-1}}, \boldsymbol{F}^{-T} = \overset{s-m}{\boldsymbol{F}^{-1}}, \boldsymbol{R} = \overset{s-m}{\boldsymbol{R}} \text{ and } \boldsymbol{R}^T = \overset{m-s}{\boldsymbol{R}^T}$$

We look for the two outermost superscripts to identify the following tensors:

$$\boldsymbol{\Sigma} = \overset{m-s}{\boldsymbol{R}^T} \cdot \overset{s-s}{\boldsymbol{\sigma}} \cdot \overset{s-m}{\boldsymbol{R}} \quad \Rightarrow \quad \boldsymbol{\Sigma} = \overset{m-m}{\boldsymbol{\Sigma}} \qquad \text{material-material}$$

$$\hat{\boldsymbol{P}} = J\, \overset{s-s}{\boldsymbol{\sigma}} \cdot \overset{s-m}{\boldsymbol{F}^{-T}} \quad \Rightarrow \quad \hat{\boldsymbol{P}} = \overset{s-m}{\hat{\boldsymbol{P}}} \qquad \text{spatial-material}$$

$$\boldsymbol{P} = J\, \overset{m-s}{\boldsymbol{F}^{-1}} \cdot \overset{s-s}{\boldsymbol{\sigma}} \cdot \overset{s-m}{\boldsymbol{F}^{-T}} \quad \Rightarrow \quad \boldsymbol{P} = \overset{m-m}{\boldsymbol{P}} \qquad \text{material-material}$$

$$\boldsymbol{D} = \overset{m-s}{\boldsymbol{F}^T} \cdot \overset{s-s}{\boldsymbol{d}} \cdot \overset{s-m}{\boldsymbol{F}} \quad \Rightarrow \quad \boldsymbol{D} = \overset{m-m}{\boldsymbol{D}} \qquad \text{material-material}$$

$$\boldsymbol{D}^* = \overset{m-s}{\boldsymbol{R}^T} \cdot \overset{s-s}{\boldsymbol{d}} \cdot \overset{s-m}{R} \quad \Rightarrow \quad \boldsymbol{D}^* = \overset{m-m}{\boldsymbol{D}^*} \qquad \text{material-material}$$

3. A bar of original length L that is initially horizontal (Fig. 1) deforms in a plane as the result of a simultaneous stretch and rotation as indicated in Fig. 2. The end O is fixed in space. The rotation is defined by $\theta = \omega t$ and the elongation of the end of the bar is $\delta_A = \varepsilon L t$ with both ω and ε constant.

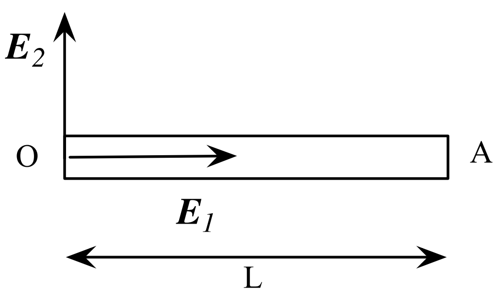

Fig. 1. Initial Position

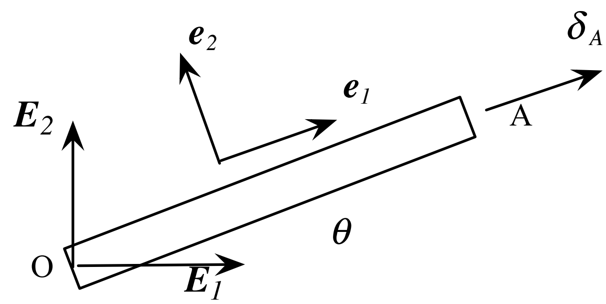

Fig. 2. Deformed position

The easiest way to describe the deformation is to use the two bases. Then the deformation is defined by

$$\begin{aligned} &\boldsymbol{r} = x_i \boldsymbol{e}_i \quad \text{and} \quad \boldsymbol{R} = X_i \boldsymbol{E}_i \\ &x_1 = X_1(1+\varepsilon t) \qquad x_2 = X_2 \qquad x_3 = X_3 \end{aligned} \tag{0-1}$$

The elongation at the end of the bar is $\delta_A = \left(x_1 - X_1\right)\big|_{X_1=L} = \left(X_1 \varepsilon t\right)\big|_{X_1=L} = \varepsilon L t$.

3.1 Perform the transformation so that the components of $\boldsymbol{r}$ are given with respect to the basis $\boldsymbol{E}_i$. Let $\boldsymbol{r} = x_i^E \boldsymbol{E}_i$. Use this form to obtain $\boldsymbol{F}, \boldsymbol{R}, \boldsymbol{U}, \dot{\boldsymbol{F}}, \dot{\boldsymbol{R}}, \dot{\boldsymbol{U}}$ and $\boldsymbol{\Omega}$ all with the tensor basis $\boldsymbol{E}_i \otimes \boldsymbol{E}_j$.

Soln:

The deformed and undeformed position vectors are

$$\boldsymbol{r} = x_i^e \boldsymbol{e}_i \quad \text{and} \quad \boldsymbol{R} = X_i \boldsymbol{E}_i$$

For simplicity, we drop the superscript notation and let $x_i \equiv x_i^e$. The deformation is

$$x_1 = X_1(1+\varepsilon t) \qquad x_2 = X_2 \qquad x_3 = X_3$$
$$\boldsymbol{r} = X_1(1+\varepsilon t)\boldsymbol{e}_1 + X_2\boldsymbol{e}_2 + X_3\boldsymbol{e}_3$$

and the two bases are related by

$$\{\boldsymbol{e}\} = \overset{e-E}{[a]}\{\boldsymbol{E}\} \qquad c = \cos\theta \quad s = \sin\theta$$
$$\overset{e-E}{[a]} = \begin{bmatrix} c & s & 0 \\ -s & c & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

To use a fixed basis for both position vectors, we transform $\boldsymbol{e}_i$ to $\boldsymbol{E}_i$. Then

$$\boldsymbol{r} = \langle x \rangle \{\boldsymbol{e}\} = \langle x \rangle \overset{e-E}{[a]}\{\boldsymbol{E}\} = \langle x^E \rangle \{\boldsymbol{E}\}$$
$$\langle x^E \rangle = \langle x \rangle \overset{e-E}{[a]}$$
$$x_1^E = cX_1(1+\varepsilon t) - sX_2 \qquad x_2^E = sX_1(1+\varepsilon t) + cX_2 \qquad x_3^E = X_3$$
$$\boldsymbol{r} = x_1^E\boldsymbol{E}_1 + x_2^E\boldsymbol{E}_1 + x_3^E\boldsymbol{E}_3$$

Now we have expressions for the deformed position vector where the components with respect to $\boldsymbol{E}_i$ are explicitly denoted with a superscript “E” as x_i^E.

Error Statement: In the original problem statement I had stated the second component was

$$x_2^E = X_1(1+\varepsilon t)\sin\theta + X_2 \sin\theta$$

and the above shows that it should have been

$$x_2^E = X_1(1+\varepsilon t)\sin\theta + X_2 \cos\theta$$

End of Error Statement

Now we obtain $\boldsymbol{F}, \boldsymbol{R}, \boldsymbol{U}, \dot{\boldsymbol{F}}, \dot{\boldsymbol{R}}, \dot{\boldsymbol{U}}$ and $\boldsymbol{\Omega}$.
Start with

$$x_1^E = cX_1(1+\varepsilon t) - sX_2 \qquad x_2^E = sX_1(1+\varepsilon t) + cX_2 \qquad x_3^E = X_3$$
$$\boldsymbol{F} = \overset{E-E}{F}_{ij}\, \boldsymbol{E}_i \otimes \boldsymbol{E}_j \qquad \overset{E-E}{F}_{ij} = \partial x_i^E / \partial X_j$$

Then

$$\overset{E-E}{\left[F\right]}=\begin{bmatrix}(1+\varepsilon t)c & -s & 0\\ (1+\varepsilon t)s & c & 0\\ 0 & 0 & 1\end{bmatrix}\qquad \overset{E-E}{\left[F^T\right]}=\begin{bmatrix}(1+\varepsilon t)c & (1+\varepsilon t)s & 0\\ -s & c & 0\\ 0 & 0 & 1\end{bmatrix}$$

and $\boldsymbol{U}^2=\boldsymbol{F}^T\cdot\boldsymbol{F}$ or

$$\overset{E-E}{\left[U^2\right]}=\begin{bmatrix}(1+\varepsilon t)c & (1+\varepsilon t)s & 0\\ -s & c & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}(1+\varepsilon t)c & -s & 0\\ (1+\varepsilon t)s & c & 0\\ 0 & 0 & 1\end{bmatrix}=\begin{bmatrix}(1+\varepsilon t)^2 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

We have the principal basis $\boldsymbol{N}_i=\boldsymbol{E}_i$ so that

$$\overset{E-E}{\left[U\right]}=\begin{bmatrix}(1+\varepsilon t) & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\qquad \overset{E-E}{\left[U^{-1}\right]}=\begin{bmatrix}\frac{1}{(1+\varepsilon t)} & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

It follows that

$$\boldsymbol{R}=\boldsymbol{F}\cdot\boldsymbol{U}^{-1}\qquad \overset{E-E}{\left[R\right]}=\overset{E-E}{\left[F\right]}\overset{E-E}{\left[U^{-1}\right]}$$

and

$$\overset{E-E}{\left[R\right]}=\begin{bmatrix}(1+\varepsilon t)c & -s & 0\\ (1+\varepsilon t)s & c & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\frac{1}{(1+\varepsilon t)} & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}=\begin{bmatrix}c & -s & 0\\ s & c & 0\\ 0 & 0 & 1\end{bmatrix}$$

Because $\boldsymbol{E}_i$ are time independent, it follows that

$$\dot{\boldsymbol{F}}=\overset{E-E}{\dot{F}}_{ij}\,\boldsymbol{E}_i\otimes\boldsymbol{E}_j\qquad \dot{\boldsymbol{U}}=\overset{E-E}{\dot{U}}_{ij}\,\boldsymbol{E}_i\otimes\boldsymbol{E}_j\qquad \dot{\boldsymbol{R}}=\overset{E-E}{\dot{R}}_{ij}\,\boldsymbol{E}_i\otimes\boldsymbol{E}_j$$

Note that $\dot{c}=-\omega s$ and $\dot{s}=\omega c$. Then it follows that

$$\overset{E-E}{\left[\dot{F}\right]}=\begin{bmatrix}\varepsilon c-\omega(1+\varepsilon t)s & -\omega c & 0\\ \varepsilon s+\omega(1+\varepsilon t)c & -\omega s & 0\\ 0 & 0 & 0\end{bmatrix}$$

$$\overset{E-E}{\left[\dot{U}\right]}=\begin{bmatrix}\varepsilon & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}\qquad \overset{E-E}{\left[\dot{R}\right]}=\omega\begin{bmatrix}-s & -c & 0\\ c & -s & 0\\ 0 & 0 & 0\end{bmatrix}$$

Recall that $\boldsymbol{\Omega}=\dot{\boldsymbol{R}}\cdot\boldsymbol{R}^T$ so that

$$\overset{E-E}{[\Omega]} = \overset{E-E}{[\dot{R}]}\overset{E-E}{[R^T]} = \omega \begin{bmatrix} -s & -c & 0 \\ c & -s & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} c & s & 0 \\ -s & c & 0 \\ 0 & 0 & 1 \end{bmatrix} = \omega \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

End of 3.1

3.2 Now use the two bases and the deformation as given by (0-1).
(i) Determine $\boldsymbol{F}, \boldsymbol{R}$ and $\boldsymbol{U}$

Soln: (i) .
Start with

$$c = \cos\theta \quad s = \sin\theta \quad \dot{\theta} = \omega \quad \dot{c} = -\omega s \quad \dot{s} = \omega c$$
$$x_1 = X_1(1+\varepsilon t) \quad x_2 = X_2 \quad x_3 = X_3$$

and

$$\boldsymbol{F} = \overset{e-E}{F}_{ij}\, \boldsymbol{e}_i \otimes \boldsymbol{E}_j \qquad \overset{e-E}{F}_{ij} = \partial x_i / \partial X_j$$

Then

$$\overset{e-E}{[F]} = \begin{bmatrix} (1+\varepsilon t) & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and

$$\overset{E-E}{[U^2]} = \overset{E-e}{[F^T]}\overset{e-E}{[F^T]} = \begin{bmatrix} (1+\varepsilon t)^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \overset{E-E}{[U]} = \begin{bmatrix} (1+\varepsilon t) & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

with $\boldsymbol{N}_i = \boldsymbol{E}_i$. Now $\boldsymbol{R} = \boldsymbol{F} \cdot \boldsymbol{U}^{-1}$ so that

$$\boldsymbol{R} = \overset{e-E}{R}_{ij}\, \boldsymbol{e}_i \otimes \boldsymbol{E}_j \qquad \overset{e-E}{[R]} = \overset{e-E}{[F]}\overset{E-E}{[U^{-1}]} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

or $\boldsymbol{R} = \boldsymbol{e}_i \otimes \boldsymbol{E}_i$ and $\boldsymbol{R}^T = \boldsymbol{E}_i \otimes \boldsymbol{e}_i$.

End of (i).

(ii) The basis $\boldsymbol{e}_i$ is a function of time. Determine $\dot{\boldsymbol{e}}_i$ and determine the tensor $\boldsymbol{\Omega}^*$ such that $\dot{\boldsymbol{e}}_i = \boldsymbol{\Omega}^* \cdot \boldsymbol{e}_i$.

Soln: (ii)
Recall that

$$\{\boldsymbol{e}\} = \overset{e-E}{[a]}\{\boldsymbol{E}\} \qquad \overset{e-E}{[a]} = \begin{bmatrix} c & s & 0 \\ -s & c & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Then

$$\{\dot{\boldsymbol{e}}\} = \overset{e-E}{[\dot{a}]}\{\boldsymbol{E}\} \qquad \overset{e-E}{[\dot{a}]} = \omega\begin{bmatrix} -s & c & 0 \\ -c & -s & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Now transform to the basis $\boldsymbol{e}_i$:

$$\{\dot{\boldsymbol{e}}\} = \overset{e-E}{[\dot{a}]}\overset{E-e}{[a]}\{\boldsymbol{e}\} \qquad \overset{e-E}{[\dot{a}]} = \omega\begin{bmatrix} -s & c & 0 \\ -c & -s & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \overset{E-e}{[a]} = \begin{bmatrix} c & -s & 0 \\ s & c & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\overset{e-E}{[\dot{a}]}\overset{E-e}{[a]} = \omega\begin{bmatrix} -s & c & 0 \\ -c & -s & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} c & -s & 0 \\ s & c & 0 \\ 0 & 0 & 1 \end{bmatrix} = \omega\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

It follows that

$$\dot{\boldsymbol{e}}_1 = \omega \boldsymbol{e}_2 \qquad \dot{\boldsymbol{e}}_2 = -\omega \boldsymbol{e}_1$$

Let $\boldsymbol{\Omega}^* = \overset{e-e}{\Omega^*_{ij}}\, \boldsymbol{e}_i \otimes \boldsymbol{e}_j$. Then $\boldsymbol{\Omega}^* \cdot \boldsymbol{e}_k = \overset{e-e}{\Omega^*_{ij}}\, \boldsymbol{e}_i \delta_{jk} = \overset{e-e}{\Omega^*_{ik}}\, \boldsymbol{e}_i$ and

$$\begin{aligned} &\dot{\boldsymbol{e}}_i = \boldsymbol{\Omega}^* \cdot \boldsymbol{e}_i \quad \Rightarrow \quad \{\dot{\boldsymbol{e}}\} = \{\boldsymbol{e}\} \cdot \boldsymbol{\Omega}^* = \{\boldsymbol{e}\} \cdot \langle \boldsymbol{e} \rangle \overset{e-e}{[\Omega^*]}\{\boldsymbol{e}\} = \overset{e-e}{[\Omega^*]}\{\boldsymbol{e}\} \\ &\dot{\boldsymbol{e}}_1 = \Omega^*_{11}\boldsymbol{e}_1 + \Omega^*_{12}\boldsymbol{e}_2 + \Omega^*_{13}\boldsymbol{e}_3 = \omega \boldsymbol{e}_2 \\ &\dot{\boldsymbol{e}}_2 = \Omega^*_{21}\boldsymbol{e}_1 + \Omega^*_{22}\boldsymbol{e}_2 + \Omega^*_{23}\boldsymbol{e}_3 = -\omega \boldsymbol{e}_1 \end{aligned}$$

It follows that

$$\overset{e-e}{[\Omega^*]} = \omega\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \boldsymbol{\Omega}^* = \omega\left(\boldsymbol{e}_1 \otimes \boldsymbol{e}_2 - \boldsymbol{e}_2 \otimes \boldsymbol{e}_1\right)$$

End of (ii).

(iii) Determine $\dot{\boldsymbol{F}}, \dot{\boldsymbol{U}}, \dot{\boldsymbol{R}}$ and $\boldsymbol{\Omega}$

Soln: (iii)

Start with

$$\dot{\boldsymbol{F}} = \overset{e-E}{\dot{F}}_{ij}\, \boldsymbol{e}_i \otimes \boldsymbol{E}_j + \overset{e-E}{F}_{ij}\, \dot{\boldsymbol{e}}_i \otimes \boldsymbol{E}_j = \langle \boldsymbol{e} \rangle \otimes \overset{e-E}{\left[\dot{F}\right]} \{\boldsymbol{E}\} + \langle \boldsymbol{e} \rangle \overset{e-e}{\left[\Omega^{*T}\right]} \otimes \overset{e-E}{\left[F\right]} \{\boldsymbol{E}\}$$

where

$$\overset{e-E}{\left[\dot{F}\right]} = \begin{bmatrix} \varepsilon & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\overset{e-e}{\left[\Omega^{*T}\right]} \overset{e-E}{\left[F\right]} = \omega \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} (1+\varepsilon t) & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \omega \begin{bmatrix} 0 & -1 & 0 \\ (1+\varepsilon t) & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Define $\overset{e-E}{\dot{F}_{ij}^{Tot}}$ such that $\dot{\boldsymbol{F}} = \overset{e-E}{\dot{F}_{ij}^{Tot}}\, \boldsymbol{e}_i \otimes \boldsymbol{E}_j = \langle \boldsymbol{e} \rangle \otimes \overset{e-E}{\left[\dot{F}^{Tot}\right]} \{\boldsymbol{E}\}$. Then

$$\dot{\boldsymbol{F}} = \langle \boldsymbol{e} \rangle \otimes \overset{e-E}{\left[\dot{F}^{Tot}\right]} \{\boldsymbol{E}\} \qquad \overset{e-E}{\left[\dot{F}^{Tot}\right]} = \begin{bmatrix} \varepsilon & -\omega & 0 \\ \omega(1+\varepsilon t) & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Convert components:

$$\overset{E-E}{\left[\dot{F}^{Tot}\right]} = \overset{E-e}{\left[a\right]} \overset{e-E}{\left[\dot{F}^{Tot}\right]} = \begin{bmatrix} c & -s & 0 \\ s & c & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \varepsilon & -\omega & 0 \\ \omega(1+\varepsilon t) & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} c\varepsilon - s\omega(1+\varepsilon t) & -c\omega & 0 \\ s\varepsilon + c\omega(1+\varepsilon t) & -s\omega & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

which agrees with 3.1.

The right stretch is already expresed in terms of the fixed basis so that the result is the same as 3.1.

$$\dot{\boldsymbol{U}} = \langle \boldsymbol{E} \rangle \overset{E-E}{\left[\dot{U}\right]} \otimes \{\boldsymbol{E}\} \qquad \overset{E-E}{\left[\dot{U}\right]} = \begin{bmatrix} \varepsilon & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Recall that $\boldsymbol{R} = \boldsymbol{e}_i \otimes \boldsymbol{E}_i$ and $\boldsymbol{R}^T = \boldsymbol{E}_i \otimes \boldsymbol{e}_i$. Then the spin is

$$\Omega = \dot{\boldsymbol{R}} \cdot \boldsymbol{R}^T = \left(\dot{\boldsymbol{e}}_i \otimes \boldsymbol{E}_i\right) \cdot \left(\boldsymbol{E}_j \otimes \boldsymbol{e}_j\right) = \dot{\boldsymbol{e}}_i \otimes \boldsymbol{e}_i = \langle \dot{\boldsymbol{e}} \rangle \otimes \{\boldsymbol{e}\}$$

Recall that we have derived

$$\{\dot{\boldsymbol{e}}\} = \overset{e-e}{\left[\Omega^*\right]} \{\boldsymbol{e}\} \quad \Rightarrow \quad \langle \dot{\boldsymbol{e}} \rangle = \langle \boldsymbol{e} \rangle \overset{e-e}{\left[\Omega^{*T}\right]}$$

Therefore, for this simple problem $\boldsymbol{\Omega} = \boldsymbol{\Omega}^{*T}$ and $\boldsymbol{n}_i = \boldsymbol{e}_i$.

As a check, do the transformation

$$\overset{E-E}{[\Omega]} = \overset{E-e}{[a]} \overset{e-e}{[\Omega]} \overset{e-E}{[a]} = \omega \begin{bmatrix} c & -s & 0 \\ s & c & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} c & s & 0 \\ -s & c & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \omega \begin{bmatrix} -s & -c & 0 \\ c & -s & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} c & s & 0 \\ -s & c & 0 \\ 0 & 0 & 1 \end{bmatrix} = \omega \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

which agrees with 3.1.

End of solution to 3.2

3.3 Obtain $\boldsymbol{v} = \boldsymbol{v}(\boldsymbol{r},t)$ and $\boldsymbol{L}$, and show that $\dot{\boldsymbol{F}} = \boldsymbol{L}\cdot\boldsymbol{F}$

Soln. Start with

$$\boldsymbol{r} = x_i\boldsymbol{e}_i \qquad \boldsymbol{v} = \dot{x}_i\boldsymbol{e}_i + x_i\dot{\boldsymbol{e}}_i \qquad \{\dot{\boldsymbol{e}}\} = \overset{e-e}{[\Omega^*]}\{\boldsymbol{e}\}$$

$$x_1 = X_1(1+\varepsilon t) \qquad x_2 = X_2 \qquad x_3 = X_3$$

$$\boldsymbol{v} = \langle\ \varepsilon X_1 \quad 0 \quad 0\ \rangle\{\boldsymbol{e}\} + \omega\langle\ x_1 \quad x_2 \quad x_3\ \rangle\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\{\boldsymbol{e}\}$$

Now substitute $X_1 = \dfrac{x_1}{(1+\varepsilon t)}$ to obtain

$$\boldsymbol{v} = v_i^e\boldsymbol{e}_i = \left\langle\ \frac{\varepsilon x_1}{(1+\varepsilon t)} - \omega x_2 \quad \omega x_1 \quad 0\ \right\rangle\{\boldsymbol{e}\}$$

Then

$$\boldsymbol{L} = \overset{e-e}{L_{ij}}\boldsymbol{e}_i\otimes\boldsymbol{e}_j \qquad \overset{e-e}{L_{ij}} = \frac{\partial v_i^e}{\partial x_j} \qquad \overset{e-e}{[L]} = \begin{bmatrix} \dfrac{\varepsilon}{(1+\varepsilon t)} & -\omega & 0 \\ \omega & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and

$$\dot{\boldsymbol{F}} = \boldsymbol{L}\cdot\boldsymbol{F} \qquad \dot{\boldsymbol{F}} = \overset{e-E}{\dot{F}_{ij}^{Tot}} \qquad \overset{e-E}{[\dot{F}^{Tot}]} = \overset{e-e}{[L]}\overset{e-E}{[F]}$$

$$= \begin{bmatrix} \dfrac{\varepsilon}{(1+\varepsilon t)} & -\omega & 0 \\ \omega & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} (1+\varepsilon t) & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \varepsilon & -\omega & 0 \\ \omega(1+\varepsilon t) & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

which agrees with the result in 3.2(iii).

3.4 Consider an element $\mathrm{d}\mathbf{X} = \mathrm{dX}_1\mathbf{E}_1$. Determine the vector $\mathbf{U}\cdot\mathrm{d}\mathbf{X}$ and then the vector $\mathbf{R}\cdot(\mathbf{U}\cdot\mathrm{d}\mathbf{X})$.

Soln: Use the approach of 3.2.

$$\boldsymbol{U}\cdot dX_1\boldsymbol{E}_1 = \{(1+\varepsilon t)\boldsymbol{E}_1\otimes\boldsymbol{E}_1 + \boldsymbol{E}_2\otimes\boldsymbol{E}_2 + \boldsymbol{E}_3\otimes\boldsymbol{E}_3\}\cdot dX_1\boldsymbol{E}_1 = (1+\varepsilon t)dX_1\boldsymbol{E}_1$$

which represents a stretch of the element with no rotation. Then

$$\boldsymbol{R}\cdot(\boldsymbol{U}\cdot d\boldsymbol{X}) = [\boldsymbol{e}_1\otimes\boldsymbol{E}_1 + \boldsymbol{e}_2\otimes\boldsymbol{E}_2 + \boldsymbol{e}_3\otimes\boldsymbol{E}_3]\cdot(1+\varepsilon t)dX_1\boldsymbol{E}_1 = (1+\varepsilon t)dX_1\boldsymbol{e}_1$$

which represents a pure rotation of the stretched bar.