8. Two-Dimensional Laplace and Poisson Equations

In the previous chapter we saw that when solving a wave or heat equation it may be necessary to first compute the solution to the steady state equation. In the case of one-dimensional equations this steady state equation is a second order ordinary differential equation. In this chapter will look at two-dimensional steady state equations, that is equations of the form

$$\Delta u(x,y) \equiv u_{xx}(x,y) + u_{yy}(x,y) = f(x,y),\ (x,y) \in \Omega,$$

where Ω is some region in the plane. If f is not identically zero this equation is called the *Poisson Equation*. If $f \equiv 0$ the equation is homogeneous and is called the *Laplace Equation*. Although the methods for solving these equations is different from those used to solve the heat and wave equations, there is a great deal of similarity. The method for solving these problems again depends on eigenfunction expansions. The eigenfunctions may be functions of one variable, x or y, as they were in the previous chapter, or they might be functions of both variables. Indeed, the eigenvalue problem for elliptic partial differential operators is of fundamental importance and the Sturm-Liouville problem is merely the one dimensional version of this more general concept. Therefore, besides the Laplace and Poisson equations we will also have to look at eigenvalue problems of the type

$$-\Delta u(x,y) = \lambda u(x,y), \quad (x,y) \in \Omega,$$

together with certain prescribed homogeneous boundary conditions.

This chapter consists of four sections. In the first section we will look at the Laplace equation with boundary conditions of various types. These boundary conditions will be linear and of the general form

$$\mathcal{B}\, u(x,y) \equiv \alpha(x,y)\tfrac{\partial u}{\partial n}(x,y) + \beta(x,y)\, u(x,y) = g(x,y), \quad (x,y) \in \partial\Omega,$$

where $\partial/\partial n$ denotes the outward normal derivative. We always assume that $\alpha^2 + \beta^2 > 0$. If $\alpha \equiv 0$ the boundary condition is a Dirichlet boundary condition, if $\beta \equiv 0$ it is a Neumann boundary condition, and if $\alpha(x,y)$ and $\beta(x,y)$ are both nonvanishing on the boundary then it is a Robin boundary condition. If either α or β has the property that it is zero on only part of the boundary then the boundary condition is sometimes referred to as *mixed.* In other words, the boundary condition is mixed if it is of Dirichlet type on part of the boundary, of Neumann type on another part of the boundary, and of Robin type on a third part of the boundary (where at most one of these parts is the empty set). If f is not identically zero this equation is called the *Poisson Equation*. In the second section we study the two-dimensional eigenvalue problem. In the third section we will use the results on eigenfunctions that were obtained in section 2 to solve the Poisson problem with homogeneous boundary conditions (the caveat about eigenvalue problems only making sense for problems with homogeneous boundary conditions is still in effect). Finally, in the last section, we will look at several examples, including problems of the form

$$\Delta u(x,y) = f(x,y),\ (x,y) \in \Omega,$$

$$\mathcal{B}u(x,y) = g(x,y),\ (x,y) \in \partial\Omega.)$$

Problems such as these are solved by using superposition (i.e. the linearity of the operators $\mathcal{B}$ and Δ); we let $u = v + w$ where

$$\Delta v(x,y) = 0, \quad (x,y) \in \Omega,$$

$$\mathcal{B}v(x,y) = g(x,y),\ (x,y) \in \partial\Omega,$$

and

$$\Delta w(x,y) = f(x,y),\ (x,y) \in \Omega,$$

$$\mathcal{B}w(x,y) = 0,\ (x,y) \in \partial\Omega.$$

In other words we do not really need to treat the most general problems; when confronted with a “monster" problem we just “divide and conquer". In the first example that we shall look at we see that if the strategy of “divide and conquer" is not good enough then we resort to “further divide and then conquer"

8.1 THE LAPLACE EQUATION.

We begin with the Laplace equation on a rectangle with homogeneous Dirichlet boundary conditions on three sides and a nonhomogeneous Dirichlet boundary condition on the fourth side.

Example 1. Solve the problem

$$\Delta u \equiv u_{xx} + u_{yy} = 0$$

on the rectangle $\mathcal{Q} : 0 < x < a,\ \ 0 < y < b,$ with boundary conditions

$$u(0, y) = u(x, 0) = u(a, y) = 0,$$

$$u(x, b) = f(x).$$

8.1 *Dirichlet boundary conditions along the rectangular plate.*

Solution. We separate variables again. Let us look for solutions of the form $X(x)\,Y(y)$. Substituting this into the equation we get

$$X''(x)\,Y(y) = \,-\,X(x)\,Y''(y).$$

Upon dividing this equation through by $X\,Y$ we obtain

$$\tfrac{X''(x)}{X(x)} \;=\; -\;\tfrac{Y''(y)}{Y(y)} \;=\; -\,\lambda.$$

The homogeneous boundary conditions at $x = 0$ and $x = a$ indicate that we should demand that

$$X(0) = X(a) = 0.$$

But this leads us to a familiar eigenvalue problem in the variable x. We know that the eigenpairs are

$$\lambda_n = (n\pi/a)^2 \quad \text{and} \quad X_n(x) = \sin\,(n\pi x/a).$$

Since the equation is homogeneous the corresponding functions $Y(y)$ are useful and so we compute them.

$$Y_n''(y) = (n\pi/a)^2 Y(y).$$

Taking into account the homogeneous boundary condition at $y = 0$ we see that $Y_n(y) = \sinh\,(n\pi y/a)$. Therefore we consider a solution of the form

$$u(x, y) = \sum_{n=0}^{\infty} c_n \sin(n\pi x/a)\,\sinh(n\pi y/a). \tag{1}$$

If we substitute this into the boundary condition at $y = b$ we see that we must have

$$f(x) = \sum_{n=0}^{\infty} c_n \sin(n\pi x/a)\sinh(n\pi \mathrm{b}/a).$$

This means that the quantities $c_n \sinh(n\pi \mathrm{b}/a)$ are the Fourier coefficients of f, and so

$$c_n = \frac{2}{a\sinh(n\pi b/a)} \int_0^a f(x)\sin(n\pi x/a)\,dx. \tag{2}$$

Equations (1)-(2) give us the solution to the problem.

Example 2. Solve the problem $\Delta v = 0$ in the rectangle $0 < x < a,\ \ 0 < y < b$, with boundary conditions

$$v(0,y) = v(a,y) = v(x,b) = 0,$$

$$v(x,0) = g(x).$$

Solution. We proceed as before by separating variables and finding that our solutions must be of the form $X_n(x)Y_n(y)$ where $X_n(x) = \sin(n\pi x/a)$ and where

$$Y_n''(y) = (n\pi/a)^2 Y_n(y),$$

but where Y must now satisfy the homogeneous boundary condition

$$Y_n(b) = 0.$$

The most convenient form for the solution is

$$Y_n(y) = \sinh\left(n\pi(b-y)/a\right).$$

This means that our solution has the form

$$v(x,y) = \sum_{n=1}^{\infty} b_n \sinh\left(n\pi(b-y)/a\right)\sin(n\pi x/a).$$

Setting $y = 0$ we have

$$g(x) = \sum_{n=1}^{\infty} b_n \sinh\left(n\pi b/a\right)\sin(n\pi x/a).$$

This implies that we must have

$$b_n = \frac{2}{a\sinh(n\pi b/a)} \int_0^a g(x)\sin(n\pi x/a)dx.$$

Remark. We observe that the superposition $\widetilde{u} = u + v$ satisfies the Laplace equation $\Delta\widetilde{u} = 0$ with the boundary conditions

$$\widetilde{u}\,(x,0) = g(x),\ \ \widetilde{u}\,(x,b) = f(x),$$

$$\widetilde{u}\,(0,y) = \widetilde{u}\,(a,y) = 0;$$

$$\widetilde{u}\,(x,y) = \sum_{n=0}^{\infty}[c_n \sinh\left(n\pi y/a\right) + b_n \sinh(n\pi(b-y)/a)]\sin(n\pi x/a).$$

This form could also have been obtained directly without superposition of solutions to two separate problems. This will be illustrated in the next example. However if we have nonhomogeneous boundary conditions on two adjacent sides then superposition usually cannot be avoided. This can be seen by doing the next exercise.

Exercise 1. Solve the problem $\Delta w = 0$ on the rectangle $0 < x < a,\ \ 0 < y < b,$ with boundary conditions

$$w(x,0) = g(x),\ \ w(x,b) = f(x),$$

$$w(0,y) = \phi(y),\ \ w(a,y) = \psi(y).$$

Example 3. Solve the problem $\Delta w = 0$ on the rectangle $0 < x < a,\ \ 0 < y < b,$ with boundary conditions

$$w_x(0,y) = w_x(a,y) = 0,$$

$$w(x,0) = f(x),\ \ w_y(x,b) = h(x).$$

Solution. We may, as indicated in the previous remark, split this problem into two problems, each with homogeneous boundary conditions on three of the four sides. However, here we will show that this not really necessary. Again we separate variables. This leads to the equations

$$\tfrac{Y''(y)}{Y(y)} = -\tfrac{X''(x)}{X(x)} = \lambda,$$

with the boundary conditions $X'(0) = X'(a) = 0$. This leads to the eigenfunctions

$$X_0(x) \equiv 1,\ \ X_n(x) = \cos(n\pi x/a),\ n = 1,\ 2,\ ,\ \ldots$$

For $n \geq 0$ we see that the corresponding functions Y_n are given , for example, by

$$Y_n(y) = a_n \exp(n\pi y/a) + b_n \exp(-n\pi y/a).$$

However we can take any other linear combination of two linearly independent solutions. Any two of the following set of functions are linearly independent solutions of the differential equation: $\exp(n\pi y/a)$, $\exp(-n\pi y/a)$, $\sinh(n\pi y/a)$, $\cosh(n\pi y/a)$, $\sinh(n\pi(y-b)/a)$, $\cosh(n\pi(y-b)/a)$. We will choose the form

$$Y_n(y) = a_n \sinh(n\pi y/a) + b_n \cosh(n\pi(y-b)/a).$$

The reason for this choice will soon become obvious. For the case $n = 0$ we will choose

$$Y_0(y) = a_0 y + b_0.$$

The solution to the problem will be given by

$$w(x,y) = \tfrac{1}{2}(a_0 y + b_0) + \sum_{n=1}^{\infty}(a_n \sinh(n\pi y/a) + b_n \cosh(n\pi(y-b)/a))\cos(n\pi x/a).$$

The boundary conditions require that

$$h(x) = \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n(n\pi/a)\cosh(n\pi b/a)\cos(n\pi x/a),$$

$$f(x) = \tfrac{1}{2}b_0 + \sum_{n=1}^{\infty} b_n \cosh(n\pi b/a)\cos(n\pi x/a).$$

This means that the coefficients are

$$a_n = \tfrac{2}{n\pi\cosh(n\pi b/a)}\int_0^a h(x)\cos(n\pi x/a)dx\ ,\ \ n > 0,$$

$$b_n = \tfrac{2}{a\cosh(n\pi b/a)}\int_0^a g(x)\cos(n\pi x/a)dx\ ,\ \ n \geq 0.$$

If we integrate the series expansion we obtained for $h(x)$ from 0 to a we obtain

$$a_0 = \frac{2}{a}\int_0^a h(x)\,dx\ ,\ \ n > 0,$$

Exercise 2. Solve the problem $\Delta u = 0$ on the rectangle $0 < x < a,\ 0 < y < b$, with boundary conditions

$$u_x(0,y) = \phi(y),\ \ u_x(a,y) = \psi(y),$$

$$u(x,0) = f(x),\ \ u_y(x,b) = \mathrm{g}(x).$$

Remark. The problem $\Delta u = 0$ with nonhomogeneous Neumann boundary conditions on the entire boundary ($\frac{\partial u}{\partial n} = \phi(x,y)$ on the boundary) usually does not have a solution (for example, look at the steady state equation for exercise 11 of the previous chapter). If it does happen to have a solution u, then that solution is not unique since $u + c$ is another solution for any constant c.

Next we look at the Laplace equation on a disk. The appropriate independent variables to use in this case are of course the polar coordinates (r,θ). The Laplacian operator in polar coordinates is

$$\Delta u = \frac{1}{r}(ru_r)_r + \frac{1}{r^2}u_{\theta\theta}.$$

Exercise 3. Prove the above formula by using the chain rule.

Example 4. Solve the problem

$$\Delta u = 0,\ \ 0 < r < R,$$

$$u(R,\theta) = f(\theta).$$

We again try separation of variables and substitute a solution of the form $\mathcal{R}(r)\Theta(\theta)$. This yields

$$-\frac{r(r\mathcal{R}'(r))'}{\mathcal{R}(r)} = \frac{\Theta''(\theta)}{\Theta(\theta)} = -\lambda.$$

The functions $\Theta(\theta)$ must be periodic of period 2π:

$$\Theta(\pi) = \Theta(-\pi),\ \ \Theta'(\pi) = \Theta'(-\pi).$$

These are implicit boundary conditions. They are homogeneous boundary conditions and therefore we can solve the eigenvalue problem

$$-\Theta''(\theta) = \lambda\Theta(\theta),$$

with the above boundary conditions. But we already solved this problem at the beginning of chapter 5. The eigenfunctions are $e^{in\theta}$ with corresponding eigenvalues n^2. Let us now solve the equation for $\mathcal{R}$ which we expect to give useful information since the partial differential equation is homogeneous.

$$\frac{r(r\mathcal{R}'(r))'}{\mathcal{R}(r)} = n^2.$$

This is a Cauchy-Euler equation,

$$r^2\mathcal{R}'' + r\mathcal{R}' - n^2\mathcal{R} = 0.$$

The solutions of the Cauchy-Euler equation are known to be of the form r^p. Substituting this into the equation we obtain the characteristic equation $p^2 - n^2 = 0$. If $n \neq 0$ then the general solution is of the form $cr^n + dr^{-n}$. If

$n = 0$ then the general solution is of the form $c + d\,\ell n\, r$ (if the characteristic equation for the Cauchy-Euler equation has a repeated root ν then, besides the solution r^ν there is the solution $x^\nu \ell n\, r$). This means that we have found the following solutions to the Laplace equation:

$$r^{|n|}e^{in\theta},\ r^{-|n|}e^{in\theta},\quad n = \pm 1,\ \pm 2,\ \ldots,$$

$$1,\ \ell n\, r.$$

Consequently we want to try to obtain a solution of the form

$$u(r,\theta) = \sum_{n=-\infty}^{\infty} c_n\, r^{|n|} e^{in\theta} + d_0\, \ell n\, r + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} d_n\, r^{-|n|} e^{in\theta}\,. \tag{3}$$

However the terms which are not defined at $r = 0$ are unacceptable since we expect a perfectly well- defined solution at the origin. Therefore we eliminate these terms and arrive at the form

$$u(r,\theta) = \sum_{n=-\infty}^{\infty} c_n\, r^{|n|} e^{in\theta}\,. \tag{4}$$

Next we impose the boundary condition:

$$f(\theta) = \sum_{n=-\infty}^{\infty} c_n\, R^{|n|} e^{in\theta}\,.$$

This is recognized as the complex Fourier series for the function f. This means that

$$c_n = \tfrac{1}{2\pi R^{|n|}} \int_{-\pi}^{\pi} f(\phi) e^{-in\phi}\, d\phi. \tag{5}$$

Although equations (4)-(5) provide the solution to the problem, we will proceed to simplify our answer. First substitute the integral in equation (5) for the coefficient c_n in equation (4) and interchange the integral and the sum. It is easy to see that by the Weierstrass M-test this interchange is justified for $0 \leq r < R$. We obtain

$$u(r,\theta) = \tfrac{1}{2\pi} \int_{-\pi}^{\pi} f(\phi) \sum_{n=-\infty}^{\infty} \left(\tfrac{r}{R}\right)^{|n|} e^{in(\theta-\phi)} d\phi\,.$$

The sum inside the integral can be broken up into two geometric series which can be summed

$$\sum_{n=-\infty}^{\infty} \left(\tfrac{r}{R}\right)^{|n|} e^{in(\theta-\phi)} = \sum_{n=0}^{\infty} \left(\tfrac{r}{R}\right)^{|n|} e^{in(\theta-\phi)} + \sum_{n=1}^{\infty} \left(\tfrac{r}{R}\right)^{|n|} e^{-in(\theta-\phi)}$$

$$= \tfrac{1}{1-(r/R)e^{i(\theta-\phi)}} + \tfrac{(r/R)e^{-i(\theta-\phi)}}{1-(r/R)e^{-i(\theta-\phi)}}$$

$$= \tfrac{R^2-r^2}{R^2-2rR\cos(\theta-\phi)+r^2}\ ,$$

where we used the fact that $e^{i(\theta-\phi)} + e^{-i(\theta-\phi)} = 2\cos(\theta-\phi)$. The solution can therefore be simply written as

$$u(r,\theta) = \tfrac{1}{2\pi} \int_{-\pi}^{\pi} \tfrac{(R^2-r^2)\, f(\phi)}{R^2-2rR\cos(\theta-\phi)+r^2}\, d\phi.$$

This is the *Poisson Integral Formula* for the Dirichlet problem on the disk. The factor

$$\frac{R^2 - r^2}{2\pi(R^2 - 2rR\cos(\theta - \phi) + r^2)}$$

which appears in this integral is called the *Poisson kernel* and it is closely related to the concepts of *Green's functions* and *fundamental solutions,* topics that will be investigated in chapter 12.

Remark on the exterior problem. An interesting situation arises if we attempt to solve the Dirichlet problem in the region $r > R$ with the boundary condition $u(R, \theta) = f(\theta)$. In this case we can use all the terms in the series (3). But that will mean that we cannot determine all the coefficients. This indicates that if we want a well-posed problem then we must impose some additional auxiliary conditions. These are sometimes loosely called *boundary conditions at infinity.* Such conditions should, ideally, come from the application that is being modeled. For example, if we are looking at a heat equation then the total heat energy of the steady state should be finite. Hence we might require that the $u(r, \theta)$ be absolutely integrable over the region $r > R$. This might lead us to require that each of the terms in the series be absolutely integrable. However this leads to the series

$$u(r, \theta) = \sum_{\substack{n=\infty \\ |n|>1}}^{\infty} d_n \, r^{-|n|} e^{in\theta}$$

which would in general not allow us to fit the boundary condition since the Fourier series for f will be missing three terms:

$$f(\theta) = \sum_{\substack{n=-\infty \\ |n| > 1}}^{\infty} c_n \, R^{|n|} e^{in\theta} \; .$$

This equality cannot be satisfied if we pick boundary values $f(\theta)$ whose complex Fourier series contains nonzero coefficients for $n = 0, \; \pm 1$. Hence we should make our requirement on the solution less stringent. In fact, it is clear that if we only eliminate terms that are unbounded then we arrive at a well-posed problem whose solution is of the form

$$u(r, \theta) = c_0 + \sum_{\substack{n=\infty \\ n \neq 0}}^{\infty} d_n \, r^{-|n|} e^{in\theta} \; .$$

This kind of solution seems unacceptable on the basis of physical considerations since it requires an infinite amount of heat energy. The conclusion that we must draw is that the heat equation in the exterior region will never attain a steady state. A little reflection upon the problem may lead us to conclude that this is reasonable. Suppose for example that the initial temperature is 0 and that $f(\theta) \equiv 1$. Since the region is unbounded it will never receive enough heat energy to achieve the steady state $u \equiv 1$. At any time, no matter how far in the future, the temperature far away from the origin will still be nearly zero.

Exercise 4. Solve the Laplace equation in the anulus $\rho < r < R$ with boundary conditions $u(\rho, \theta) = g(\theta)$ and $u(R, \theta) = f(\theta)$.

Exercise 5. Solve the Laplace equation in the region $0 < r < R, \; 0 < \theta < \pi/3$, with boundary conditions $u(R, \theta) = f(\theta)$, $u(r, 0) = 0$, $u_\theta(r, \pi/3) = 0$.

We can also use the methods of separation of variables and eigenfunction expansion to solve certain boundary value problems on semi-bounded domains. Let us illustrate this with an example.

Example 5. Solve the Laplace equation on the region $0 < y < b, \; 0 < x$, with boundary conditions $u(x, 0) = 0$, $u(x, b) + \gamma u_y(x, b) = 0$, $u(0, y) = f(y)$, where $\gamma > 0$.

Solution. We again separate variables. This leads to the equations

$$-Y''(y) = \lambda Y(y),$$

$$X''(x) = \lambda X(x),$$

together with the homogeneous boundary conditions

$$Y(0) = 0, \;\; Y(b) + \gamma Y'(b) = 0.$$

We can solve the eigenvalue problem for Y. There are no negative eigenvalues. One also easily verifies that 0 cannot be an eigenvalue. Hence we are left with eigenvalues $\lambda = \mu^2$, $\mu > 0$. In view of the boundary condition at $y = 0$ we see that $Y = \sin \mu y$. The second boundary condition yields the equation

$$\tan \mu b = \; - \gamma\mu$$

which has positive solutions $\mu_1 < \mu_2 < \ldots$. The corresponding functions X_n must be of the form

$$X_n(x) = a_n e^{-\mu_n x} + b_n e^{\mu_n x}.$$

We now argue that an appropriate solution should not tend to infinity as $x \to \infty$ and that we must therefore demand that the coefficients b_n be zero. The solution should therefore be of the form

$$u(x, y) = \sum_{n=1}^{\infty} a_n e^{-\mu_n x} \sin \mu_n y.$$

The boundary condition at $x = 0$ requires that

$$f(y) = \sum_{n=1}^{\infty} a_n \sin \mu_n y.$$

This means that we must have

$$a_n = \frac{\int_0^b f(y) \sin \mu_n y \, dy}{\int_0^b \sin^2 \mu_n y \, dy}.$$

Remark. If the boundary conditions in the above problem are changed to $u(0, y) = 0$, $u(x, 0) = 0$, $u(x, b) = g(x)$ then we have a problem which cannot be solved with the tools we have available at this point. In this case there is a nonhomogeneous boundary condition at $y = b$ and hence we must search for an eigenvalue problem for X. If we look at the equation for X, that is $- X'' = \lambda X$, we see that the solutions are of the form $Ae^{(\alpha + i\beta)x} + Be^{-(\alpha + i\beta)x}$, where $(\alpha + i\beta)^2 = \lambda$. If we impose the requirements that $X(0) = 0$ and that $X(x)$ be bounded we are left with $X(x) = \sin \beta x$. There is no further boundary condition that will lead to a discrete set (i.e. a countable set) of allowable values β_n. If $X(x) = \sin \beta x$ then we easily see that we must have $Y = \sinh \beta y$. This means that we must consider the solution of the boundary value problem to have the form

$$u(x, y) = \int_0^{\infty} U(\beta) \sinh \beta y \; \sin \beta x \, d\beta.$$

This is an example of a Fourier transform and it will be studied in the next chapter.

8.2a. THE EIGENVALUE PROBLEM IN THE RECTANGLE.

In this section we will look at an eigenvalue problem

$$- (u_{xx} + u_{yy}) = \lambda u$$

on the rectangle $\mathcal{Q} = \{(x, y)\colon 0 < x < a, \; 0 < y < b\}$. This problem makes sense with homogeneous boundary conditions. We will choose the boundary conditions

$$u(x, 0) = u(x, b) = u_x(0, y) = u_x(a, y) = 0.$$

Since all boundary conditions are homogeneous, it makes sense to separate variables and look at one-dimensional eigenvalue problems in both x and y. If we try separation of variables we obtain

$$-\left(X''Y + XY''\right) = \lambda XY,$$

or

$$\frac{X''}{X} = -\lambda - \frac{Y''}{Y}.$$

This implies that both of the ratios in the above equation must be constant, leading us to <u>two</u> one-dimensional eigenvalue problems:

$$\begin{gathered} -X'' = \lambda_1 X \\ X'(0) = X'(a) = 0 \end{gathered}$$

$$\begin{gathered} -Y'' = \lambda_2 Y \\ Y(0) = Y(b) = 0. \end{gathered}$$

These are familiar eigenvalue problems and we know that the corresponding eigenfunctions are

$$\begin{gathered} X_0(x) \equiv 1,\ \ X_n(x) = \cos\frac{n\pi x}{a},\ n = 1,\ 2,\ ,\ldots, \\ Y_m(y) = \sin\frac{m\pi y}{b}\ ,\ m = 1,\ 2,\ \ldots. \end{gathered}$$

The eigenfunctions to the two-dimensional eigenvalue problem are therefore

$$\Phi_{nm}(x,y) = X_n(x)Y_m(y),$$

and the corresponding eigenvalues are

$$\lambda_{nm} = \left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2.$$

The eigenfunctions $\Phi_{nm}(x,y)$ can be shown to form a complete orthogonal family, or basis, on the linear space $L^2(\mathcal{Q})$. This space is endowed with the inner product

$$\left\langle f, g\right\rangle = \int_0^b \int_0^a f(x,y)\,\overline{g(x,y)}\,dxdy.$$

A function $f(x,y)$ in this space may therefore be expanded as an eigenfunction expansion

$$f(x,y) \sim \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \widetilde{F}_{nm}\Phi_{nm}(x,y) =$$

$$\sum_{n=0}^{\infty} \frac{1}{2} F_{0m} \sin\left(\frac{m\pi y}{b}\right) + \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} F_{nm} \sin\frac{m\pi y}{b} \cos\frac{n\pi x}{a}, \tag{6}$$

where the coefficients $\widetilde{F}_{nm} = F_{nm}$ if $n > 0$ and $2\widetilde{F}_{0m} = F_{0m}$. The coefficients are computed in the usual manner by taking advantage of the orthogonality. They are

$$F_{nm} = \frac{4}{ab}\int_0^b \int_0^a f(x,y) \sin\frac{m\pi y}{b} \cos\frac{n\pi x}{a}\,dxdy. \tag{7}$$

Series of the form (6) are called *double Fourier series.* In this case it is a Fourier sine series in y and a Fourier cosine series in x. In general such Fourier series can occur in a multitude of other forms: let $\{\phi_n(y)\}$ be an orthogonal basis, derived perhaps from some self-adjoint regular Sturm-Liouville problem on $0 \le y \le b$, and similarly let $\{\psi_m(x)\}$ be such an orthogonal basis defined on the interval $0 \le x \le a$. Then the products $\{\psi_m(x)\phi_n(y)\}$ form an orthogonal basis for $L^2(\mathcal{Q})$ and can be used to form generalized double Fourier series for square integrable functions defined on the rectangle $\mathcal{Q}$. Indeed, the concept of generalized Fourier series on two-dimensional regions is not constrained to regions and boundary conditions for which we may obtain eigenfunctions for the Laplacian operator by means of separation of variables. It can be shown that for bounded domains $\mathcal{D}$ whose boundaries are not too bizarre the eigenfunctions $\Psi_k(x,y)$ which satisfy $\Delta\Psi_k = \lambda_k\Psi_k$ in $\mathcal{D}$ and $\Psi_k(x,y) = 0$ on $\partial\mathcal{D}$ form an orthogonal basis for the space of square integrable functions on $\mathcal{D}$. This is also true in higher dimensions as well as for other boundary conditions and more general elliptic equations.

Exercise 6. Find the eigenfunctions and eigenvalues for the problem

$$-\Delta u = \lambda u \text{ in the rectangle } 0 < x < a,\ 0 < y < b,$$

$$u(x,0) = u(0,y) = u_y(x,b) = u(a,y) = 0.$$

Also find the double Fourier series of the function

$$f(x,y) = \begin{cases} 1 & \text{if } 0<x<a/2, 0<y<b/2 \\ 0 & \text{elsewhere} \end{cases}$$

8.2b. THE EIGENVALUE PROBLEM ON THE DISK.

Before we study the eigenvalue problem on the disk we need to look at some special functions that will be needed. The first of these is the gamma function $\Gamma(\tau)$. This function allows us to extend the definition of the *factorial* of a number to numbers other than nonnegative integers. The gamma function is defined by

$$\Gamma(\tau) = \int_0^\infty e^{-x}\, x^{\tau-1} dx. \qquad (\tau > 0)$$

One can, without too much difficulty, prove the following properties of the Gamma function:

$\Gamma(1) = 1$
$\Gamma(\tau + 1) = \tau\Gamma(\tau)$ [1]
$\Gamma(n + 1) = n!$ if n is a nonnegative integer
$\Gamma(\frac{1}{2}) = \sqrt{\pi}.$

Fig. 8.2 *Graph of the gamma function* $\Gamma(\tau)$ [2].

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When k is a positive integer we will sometimes use the notation

$$(\tau)_k = \tau \cdot (\tau+1) \cdot (\tau+2) \cdot \ \cdots \ \cdot (\tau+k-1) = \tfrac{\Gamma(\tau+k)}{\Gamma(\tau)}.$$

$$(\tau)_0 = 1.$$

Next we we take a brief look at Bessel's equation:

$$y''(x) + \tfrac{1}{x}\, y'(x) + \left(1 - \tfrac{\nu^2}{x^2}\right) y(x) = 0,$$

where we take $\nu \geq 0$. It can be shown that this equation has a solution of the form

$$\sum_{k=0}^{\infty} c_k x^{\nu+k}.$$

The first coefficient c_0 is arbitrary. Once it has been assigned, the value of the others may be determined by substituting the above expression into the differential equation and balancing coefficients of like powers of x. The *Bessel function of the first kind of order* ν is obtained by choosing the first coefficient to be $c_0 = [2^\nu \Gamma(\nu+1)]^{-1}$. This function is denoted by J_ν and is given by the series

$$J_\nu(x) = \tfrac{x^\nu}{2^\nu \Gamma(\nu+1)} \sum_{k=0}^{\infty} \tfrac{\left(-\frac{1}{4}x^2\right)^k}{k!\,(\nu+1)_k}.$$

Fig 8.3 *The graphs for the Bessel functions of the first kind of orders 0, 1, and 2.*

We note that the function $J_\nu(x) \approx c_0 x^\nu$ for small values of x. For large values of x it can be shown that this function approaches the damped sinusoid $\sqrt{\frac{2}{\pi x}} \cos(x - \frac{1}{2}n\pi - \frac{1}{4})$. The graphs for the functions J_0, J_1, and J_2 are shown below.

Since Bessel's equation is of second order we know that there must be a second, linearly independent solution. If ν is not an integer then a second solution may be found which is of the form

$$\sum_{k=0}^{\infty} c_k x^{-\nu+k}.$$

If ν is not an integer and the first coefficient is chosen as $c_0 = [2^\nu \Gamma(-\nu+1)]^{-1}$ then we obtain the function $J_{-\nu}$. This function behaves like $c_0 x^{-\nu}$ for small values of x. If ν is an integer then, since the gamma function is not defined for nonpositive integers, $J_{-\nu}$ makes no sense and we must find a second solution by some other means. For example a second solution may be obtained by the technique of reduction of order (look for a solution of the form $u(x)J_\nu(x)$). Once we have found two linearly independent solutions the other solutions may be formed by linear combinations. A standard second solution in case the order is an integer is either Weber's function or Neumann's function. We will not supply a formula, however it should be pointed out that these solutions tend to $-\infty$ as $x \downarrow 0$. Hence we see that in all cases the second solution becomes undefined (*singular*) at $x = 0$. These singular solutions also approach damped sinusoids as x gets large. The graphs of Weber functions of orders 0, 1, and 2 are shown below.

Fig. 8.4 *The graphs of Weber functions of orders 0, 1, and 2.*

We are now ready to look at the eigenvalue problem on the disk:

$$-\,r(ru_r)_r - u_{\theta\theta} = \lambda r^2 u, \quad (r,\theta) \in \mathcal{D} := \{(r,\theta)\colon 0 \le r < R\},$$

$$u(R,\theta) = 0.$$

Separation of variables, $u(r,\theta) = \mathcal{R}(r)\Theta(\theta)$, leads to

$$\lambda r^2 + \tfrac{r(r\mathcal{R}'(r))'}{\mathcal{R}(r)} = -\,\tfrac{\Theta''(\theta)}{\Theta(\theta)}.$$

As before, since the function $\Theta(\theta)$ must be periodic with period 2π so that it must be a linear combination of the functions

$$\Theta_n(\theta) = e^{in\theta}, \quad n = 0,\ \pm 1,\ \pm 2,\ \ldots.$$

This leads us to an equation for $\mathcal{R}$:

$$\mathcal{R}'' + r^{-1}\mathcal{R}' + (\lambda - n^2/r^2)\mathcal{R} = 0. \tag{8}$$

This is nearly Bessel's equation. First we will show that λ must be positive. To do this we note that the above equation may be written as

$$-\,(r\mathcal{R}')' + n^2\mathcal{R} = \lambda r\mathcal{R}. \tag{9}$$

We see that this is a Sturm-Liouville problem. Although it is not regular and only has a boundary condition at $r = R$, namely $\mathcal{R}(R) = 0$ we can still use the same proofs as were used in Chapter 6 to prove that the eigenvalues are real and positive, and that eigenfunctions corresponding to different eigenvalues are orthogonal with respect to the weight function r. However, in order to carry out the various manipulations we need to adjoin the auxiliary condition requiring that the eigenfunctions are bounded.

Exercise 7. Show that the solutions to (9) which are bounded and satisfy $\mathcal{R}(R) = 0$ must correspond to positive eigenvalues. Show that eigenfunctions corresponding to different eigenvalues are orthogonal with respect to the weight function $w(r) \equiv r$.

Next we make a change of variables in equation (8). Let $s = \sqrt{\lambda}\,r$ and let $\mathcal{S}$ be the function of s defined by $\mathcal{S}(s) = \mathcal{R}(s/\sqrt{\lambda})$. With this change of variables equation (8) turns into a Bessel's equation:

$$\mathcal{S}'' + \tfrac{1}{s}\mathcal{S}' + \left(1 - \tfrac{n^2}{s^2}\right)\mathcal{S} = 0.$$

The only bounded solutions of this equation are multiples of $J_n(s)$. The required boundary condition at $r = R$ translates into the requirement that $J_n(\sqrt{\lambda}\, R) = 0$. This means that $\sqrt{\lambda}\, R$ must correspond to a root of J_n. Let us use the notation that these roots be labeled as β_{nm}. We assume that they are indexed consecutively in increasing order:

$$J_n(\beta_{nm}) = 0, \quad \beta_{n1} < \beta_{n2} < \beta_{n3} < \cdot \ \cdot \ \cdot \ .$$

Therefore we see that $\sqrt{\lambda}\, R = \beta_{nm}$ for some index m so that the eigenvalues must be

$$\lambda_{nm} = (\beta_{nm}/R)^2$$

and the corresponding eigenfunctions are

$$\Phi_{nm}(r,\theta) = J_n(\beta_{nm} r/R)\, e^{\pm in\theta}.$$

In order to simplify notation we will define J_{-n} to be the same as J_n whenever n is an integer:

$$J_{-n}(s) \overset{def}{=} J_n(s), \ \beta_{-nm} = \beta_{nm}$$

so that the eigenfunctions may be labeled as

$$\Phi_{nm}(r,\theta) = J_n(\beta_{nm} r/R)\, e^{in\theta}, \ n = 0, \ \pm 1, \ \pm 2, \ \ldots \text{ and } m = 1, \ 2, \ \ldots.$$

It can be shown that **for each fixed nonnegative integer n the functions $\{J_n(\beta_{nm} r/R)\}_{m=1}^{\infty}$ form an orthogonal basis for the space $L^2_w(0,r)$ with weight function $w(r) \equiv r$.** What this means is that the functions $\Phi_{nm}(r,\theta) = J_n(\beta_{nm} r/R)\, e^{in\theta}$ form an orthogonal basis for $L^2(\mathcal{D})$. We will not prove the completeness of this family, and the orthogonality is left as an exercise for the reader.

Exercise 8. Show that

$$\int_{\mathcal{D}} \Phi_{nm} \overline{\Phi_{jk}}\, dA = 0$$

unless $n = j$ and $m = k$. Note that the area differential dA is $r dr d\theta$ in polar coordinates, thus providing exactly the correct weight function.

Although it is possible to consider eigenvalue problems on the disk with other kinds of boundary conditions, this will take us too far afield. However it is possible, with the information given above to solve eigenvalue problems of the form

$$\begin{gathered} -\Delta u(r,\theta) = \lambda u(r,\theta), \ 0 < r < R, \ 0 < \theta < \alpha, \\ u(R,\theta) = 0, \\ a u_\theta(r,0) + b u(r,0) = 0, \\ c u_\theta(r,\alpha) + d u(r,\alpha) = 0, \end{gathered}$$

with $ab \leq 0, \ cd \geq 0$.

Exercise 9. Solve the above problem with $\alpha = \pi/3, \ a = 0, \ d = 0$.

8.3 THE POISSON PROBLEM.

The Poisson problem may be solved by looking for a solution which is expressed as a generalized Fourier series in terms of the eigenfunctions of the Laplacian, The method is easily described in general. Suppose that we have obtained all the eigenfunctions for

$$-\Delta u = \lambda u \text{ in } \Omega,$$

$$\mathcal{B}u = 0 \text{ on } \partial\Omega.$$

Let us denote the eigenpairs by (λ_k, Φ_k) -- we are using a single index although generally these arise with double indices for planar regions and triple indices for three-dimensional regions. Suppose we need to solve the Poisson problem

$$\Delta u(x,y) = f(x,y) \text{ in } \Omega,$$

$$u(x,y) = 0 \text{ on } \partial\Omega.$$

We express f as an eigenfunction expansion

$$f = \sum_{n=1}^{\infty} F_n \Phi_n$$

where the coefficients are computed, as usual, by the formula

$$F_n = \frac{\int_\Omega f\overline{\Phi_n}\,dA}{\int_\Omega |\Phi_n|^2\,dA}. \tag{10}$$

Next we suppose that u may also be expressed as a series in terms of the eigenfunctions

$$u = \sum_{n=1}^{\infty} U_n \Phi_n. \tag{11}$$

If we substitute the series expressions for u and f into the Poisson equation and use the fact that $\Delta\Phi_n = -\lambda_n \Phi_n$, we see that

$$\sum_{n=1}^{\infty} -\lambda_n U_n \Phi_n = \sum_{n=1}^{\infty} F_n \Phi_n,$$

and hence the coefficients for the series expansion of the solution $u$are given by

$$U_n = -\frac{F_n}{\lambda_n}. \tag{12}$$

Equations (10)-(12) provide a formal solution to the Poisson problem. In some cases the solution of the Poisson problem may be given in a form much like the Poisson Integral formula:

$$u(x,y) = \int\int_\Omega G(x,y;\xi,\eta)\, f(\xi,\eta)\, d\xi d\eta.$$

The function G is called Green's function. A formal calculation seems to indicate that

$$G(x,y;\xi,\eta) = \sum_{n=1}^{\infty} -\frac{\Phi_n(x,y)\Phi_n(\xi,\eta)}{\lambda_n}.$$

The question that arises is in what sense does this series converge? To answer that would take us far outside the scope of this chapter. For certain problems such Green's functions can be cleverly constructed and can be used to give the solution of the Poisson problem in a much more convenient form than the above series representation. One such construction is the so-called *method of images* that is often seen in the study of electrostatics.

We conclude this chapter with an easy example.

Example 6. Solve the Poisson problem

$$\Delta u = 1,\ \ 0 < x < \pi,\ 0 < y < \pi,$$

$$u(x,0) = u(\pi,y) = u(x,\pi) = u(0,y) = 0.$$

This problem represents the vertical displacement of a membrane due to a uniform downward force, such as gravity.

Solution. It is easy to see that the eigenfunctions are $\sin mx \sin ny$. Applying the Laplacian to them we see that $\Delta \sin mx \sin ny = -(m^2+n^2)\sin mx \sin ny$. The function $f(x,y) \equiv 1$ can be expanded as a double Fourier series with coefficients

$$F_{mn} = \frac{4}{\pi^2}\int_0^\pi \int_0^\pi \sin mx \sin ny dx dy$$

$$= \frac{4}{mn\pi^2}[1-(-1)^m][1-(-1)^n].$$

Therefore

$$U_{mn} = \frac{-4[1-(-1)^m][1-(-1)^n]}{(m^2+n^2)mn\pi^2}.$$

We have

$$u(x,y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} U_{mn} \sin mx \sin ny$$

$$= -\frac{8}{\pi^2}\ \left\{\sin x \sin y + \frac{1}{15}\sin x \sin 3y + \frac{1}{15}\sin 3x \sin y + \frac{1}{81}\sin 3x \sin 3y + \ldots\right\}.$$

Just a few terms give a good approximation since the series converges rapidly.

Exercise 10. Solve the problem

$$\Delta u = 1,\ \ 0 < x < \pi,\ 0 < y < \pi,$$

$$u_y(x,0) = u(\pi,y) = u(x,\pi) = u(0,y) = 0.$$

Exercise 11. Solve the problem $\Delta u = -c$, c a constant, on $0 < x < \pi$, $0 < y < \pi$, with boundary conditions $u(x,0) = 0$, $u(x,\pi) = 0$, $u(0,y) = 0$, $u(\pi,y) = y(\pi - y)$.

Exercise 12. Solve the problem $\Delta u = 0$ in the semi-infinite strip $y > 0$, $0 < x < L$, with boundary conditions $u(0,y) = u(L,y) = 0$, and $u_y(x,0) = g(x)$.

Exercise 13. Obtain a solution to the problem in example 6 as follows: note that $v(x,y) := x^2/2$ is a solution to the PDE, so that $w := u - v$ satisfies the Laplace equation. Solve for w and obtain v. In chapter 9 we will obtain a method for finding particular solutions to the Poisson problem so that w3e may reduce Poisson equations to Laplace equations.