B

LET $\hat{u}_2 = u_0 + T_1 \cdot x + \sum_{j=1}^{2} \alpha_j \phi_j$

ϕ_j IS THUS SUBJECT TO $\phi_1(0)=0$ & $\phi_1'(1)=0$

IN THIS CASE, $u_0 = 0$.

SUBSTITUTION OF $\hat{u}_{c2}$ INTO THE SYSTEM EQUATION YIELDS

$$-\left(x \sum_{j=1}^{2} \alpha_j \phi_j'\right)' + 2\sum_{j=1}^{2} \alpha_j \phi_j = f - \underbrace{\left[-T_1 + 2u_0 + 2T_1 x\right]}_{L[u_0+T_1x] = -(x(u_0+T_1x)')' + 2(u_0+T_1x)}$$

DEFINE $f^* = f - [-T_1 + 2u_0 + 2T_1 x]$, THE WORK FUNCTION THEN BECOMES, WITH $V = \sum_{k=1}^{2} \beta_k \psi_k$,

$$\sum_{k=1}^{2} \beta_k \left[\sum_{j=1}^{2} \alpha_j \int_0^1 \psi_k L[\phi_j] - \psi_k f^* \, dx\right] = 0$$

LET $\phi_1 = x(1-x)^2$ & $\phi_2 = (x-1)^2 - 1 \Rightarrow \phi_1' = 3x^2 - 4x + 1$, $\phi_2' = 2x - 2$

$\int \phi_1 dx = \frac{1}{4}x^4 - \frac{2}{3}x^3 + \frac{1}{2}x^2$, $\int \phi_2 dx = \frac{1}{3}x^3 - x^2$

i. THE COLLOCATION METHOD

$$\psi_k = \delta(x - x_k)$$

$$K_{jk} = \int_0^1 \delta(x-x_k)\left(-(x\phi_j')' + 2\phi_j\right) dx$$

$$= -(x_k \phi_j'(x_k))' + 2\phi_j(x_k) = \begin{cases} 2x_k^3 - 13x_k^2 + 10x_k - 1 & j=1 \\ 2x_k^2 - 8x_k + 2 & j=2 \end{cases}$$

$$f_k = \int_0^1 \delta(x - x_k) f^* dx$$

$$= f^*(x_k)$$

LET $x_1 = \frac{1}{3}$ & $x_2 = \frac{2}{3}$, $K = \begin{bmatrix} \frac{26}{27} & -\frac{4}{9} \\ \frac{13}{27} & -\frac{22}{9} \end{bmatrix}$, $\underset{\sim}{f} = \begin{Bmatrix} 15.312 \\ -10.6563 \end{Bmatrix} \Rightarrow K^{-1}\underset{\sim}{f} = \underset{\sim}{\alpha} = \begin{Bmatrix} 19.7042 \\ 8.2405 \end{Bmatrix}$

$$\hat{U}_N = \underbrace{1.8\pi\cos(1.8\pi)x}_{u_b = T_1 x} + \underbrace{19.7042}_{\alpha_1}\underbrace{x(1-x)^2}_{\phi_1} + \underbrace{8.24105}_{\alpha_2}\underbrace{((x-1)^2-1)}_{\phi_2}$$

ii THE SUB-DOMAIN METHOD

$$\psi_1 = \begin{cases} 1 & 0 \leq x \leq 1/2 \\ 0 & 1/2 < x \leq 1 \end{cases} \qquad \psi_2 = \begin{cases} 0 & 0 \leq x < 1/2 \\ 1 & 1/2 \leq x \leq 1 \end{cases}$$

WE CAN DEFINE THE SUBDOMAINS

$$\Omega_1 : \quad x \in [0, 1/2]$$

$$\Omega_2 : \quad x \in [1/2, 1]$$

$$\mathbb{K}_{jk} = \int_{\Omega_k} (-x\phi_j')' + 2\phi_j \, d\Omega_k = (-x\phi_j')\Big|_{\Omega_k} + \int_{\Omega_k} 2\phi_j \, d\Omega_k$$

$$f_k = \int_{\Omega_k} f^* \, d\Omega_k = -\frac{10}{9\pi}\cos\left(\frac{9}{5}\pi x\right) - \frac{9\pi}{5}x\cos\left(\frac{9}{5}\pi x\right) + \frac{9\pi}{5}x\cos\left(\frac{9}{5}\pi\right) - \frac{9\pi}{5}x^2\cos\left(\frac{9}{5}\pi\right)$$

NOTE, THE SAME BASIS FUNCTIONS FROM PART i ARE USED. ONLY ψ CHANGES

$$\mathbb{K} = \begin{bmatrix} \frac{23}{96} & \frac{1}{12} \\ \frac{-7}{96} & \frac{-17}{12} \end{bmatrix}, \quad \underset{\sim}{f} = \begin{Bmatrix} 4.5228 \\ -9.0302 \end{Bmatrix} \Rightarrow \mathbb{K}^{-1}\underset{\sim}{f} = \underset{\sim}{\alpha} = \begin{Bmatrix} 16.9644 \\ 5.5011 \end{Bmatrix}$$

$$\hat{u}_N = 1.8\pi\cos(1.8\pi)x + 16.9644\left(x(1-x)^2\right) + 5.5011\left((x-1)^2 - 1\right)$$

iii The Least Squares Method

$$\psi_k = \frac{\partial r}{\partial \alpha_k} = \mathcal{L}(\phi_k) = -(x\phi_k')' + 2\phi_k$$

$$K_{jk} = \int_0^1 \left(-(x\phi_k')' + 2\phi_k\right)\left(-(x\phi_j')' + 2\phi_j\right) dx$$

$$f_k = \int_0^1 \left(-(x\phi_k')' + 2\phi_k\right) f^* \, dx$$

$$K = \begin{bmatrix} \frac{74}{105} & \frac{4}{15} \\ \frac{4}{15} & \frac{24}{5} \end{bmatrix}, \quad \underset{\sim}{f} = \begin{Bmatrix} 11.247 \\ 28.778 \end{Bmatrix} \Rightarrow K^{-1}\underset{\sim}{f} = \underset{\sim}{\alpha} = \begin{Bmatrix} 13.984 \\ 5.218 \end{Bmatrix}$$

$$\hat{U}_N = 1.8\pi\cos(1.8\pi)x + 13.984\left(x(1-x)^2\right) + 5.218\left((x-1)^2 - 1\right)$$

iv THE GALERKIN METHOD

$$\psi_k = \phi_k$$

$$\mathbb{K}_{jk} = \int_0^1 \phi_k \left(-(x\phi_j')' + 2\phi_j \right) dx$$

$$f_k = \int_0^1 \phi_k f^* \, dx$$

$$\mathbb{K} = \begin{bmatrix} \frac{11}{210} & \frac{-1}{15} \\ \frac{-1}{15} & \frac{7}{5} \end{bmatrix}, \quad \underset{\sim}{f} = \begin{Bmatrix} 6.0457 \\ 6.333 \end{Bmatrix} \Rightarrow \mathbb{K}^{-1}\underset{\sim}{f} = \underset{\sim}{\alpha} = \begin{Bmatrix} 15.424 \\ 5.258 \end{Bmatrix}$$

$$\hat{u}_N = 1.8\pi \cos(1.8\pi)\, x + 15.424 \left(x(1-x)^2 \right) + 5.258 \left((x-1)^2 - 1 \right)$$

HOMEWORK #2 SOLUTION

COMPUTATIONAL MECHANICS
ME 404-504

① A DEFINE THE WEAK FORM

$$-(xu')' + 2u = -1.8\pi\cos(1.8\pi x) + \sin(1.8\pi x)(2 + (1.8\pi)^2 x)$$

SUBJECT TO

$$u(0)=0, \quad u'(1) = 1.8\pi\cos(1.8\pi)$$

FIRST, DEFINE THE RESIDUAL

$$r = -(xu')' + 2u - f,$$

WITH $f = -1.8\pi\cos(1.8\pi x) + \sin(1.8\pi x)(2 + (1.8\pi)^2 x)$

NEXT,

$$\int_0^1 r v \, dx = 0,$$

$$\int_0^1 \left(-(xu')' + 2u - f\right) v \, dx$$

INTEGRATION BY PARTS YIELDS

$$\Big[-(xu')v\Big]_0^1 + \int_0^1 xu'v' + 2uv - fv \, dx = 0$$

GIVEN $u(0)=0$, THIS REQUIRES THAT $v(0)=0$ IN ORDER FOR v TO BE AN ADMISSIBLE TEST FUNCTION.

$$\int_0^1 xu'v' + 2uv - fv \, dx - v(1)\, T_1 = 0$$

WHERE $T_1 = 1.8\pi\cos(1.8\pi) = 1 \cdot u'(1)$.

C THE GALERKIN METHOD NOT SATISFYING THE NATURAL B.C.'S

$$\hat{U}_N = \sum_{j=1}^{N} \alpha_j \phi_j , \quad \phi_j(0) = 0.$$

FROM THE WEAK FORM, PART 4,

$$\mathbb{K}_{jk} = \int_0^1 (x\phi_j'\phi_k' + 2\phi_j\phi_k)\,dx$$

$$f_k = \int_0^1 \phi_k f \, dx + \phi_k(1) T_1$$

FOR CONSISTENCY, CHOOSE

$$\phi_1 = x(1-x)^2, \quad \phi_2 = (x-1)^2 - 1, \quad \phi_3 = x$$

THESE ARE ADMISSIBLE SINCE THE ESSENTIAL B.C.'S ARE SATISFIED. THE NATURAL B.C.'S ARE AUTOMATICALLY SATISFIED BY SOLUTIONS TO THE ORIGINAL PROBLEM.

1 TERM:

$$\mathbb{K} = \frac{11}{210}, \quad f = 0.3811, \quad \alpha_1 = \frac{f}{\mathbb{K}} = 7.2765$$

2 TERM:

NOTE, THESE ARE THE SAME AS PART b.iv. THE STRONG & WEAK FORMS YIELD THE SAME $\mathbb{K}$!

$$\mathbb{K} = \begin{bmatrix} \frac{11}{210} & -\frac{1}{15} \\ -\frac{1}{15} & \frac{7}{5} \end{bmatrix}, \quad \underset{\sim}{f} = \begin{Bmatrix} 0.3811 \\ 0.9952 \end{Bmatrix}, \quad \mathbb{K}^{-1}\underset{\sim}{f} = \underset{\sim}{\alpha} = \begin{Bmatrix} 8.7091 \\ 1.1256 \end{Bmatrix}$$

THE FORCE TERMS, HOWEVER, DIFFER!

3 TERM

$$\mathbb{K} = \begin{bmatrix} \frac{11}{210} & -\frac{1}{15} & -\frac{1}{60} \\ -\frac{1}{15} & \frac{7}{5} & -\frac{7}{6} \\ -\frac{1}{60} & -\frac{7}{6} & \frac{7}{6} \end{bmatrix}, \quad \underset{\sim}{f} = \begin{Bmatrix} 0.3811 \\ 0.9952 \\ -0.9445 \end{Bmatrix}, \quad \mathbb{K}^{-1}\underset{\sim}{f} = \underset{\sim}{\alpha} = \begin{Bmatrix} 17.2377 \\ 6.374 \\ 5.8107 \end{Bmatrix}$$

$$\hat{U}_N = \sum_{j=1}^{N} \alpha_j \phi_j .$$

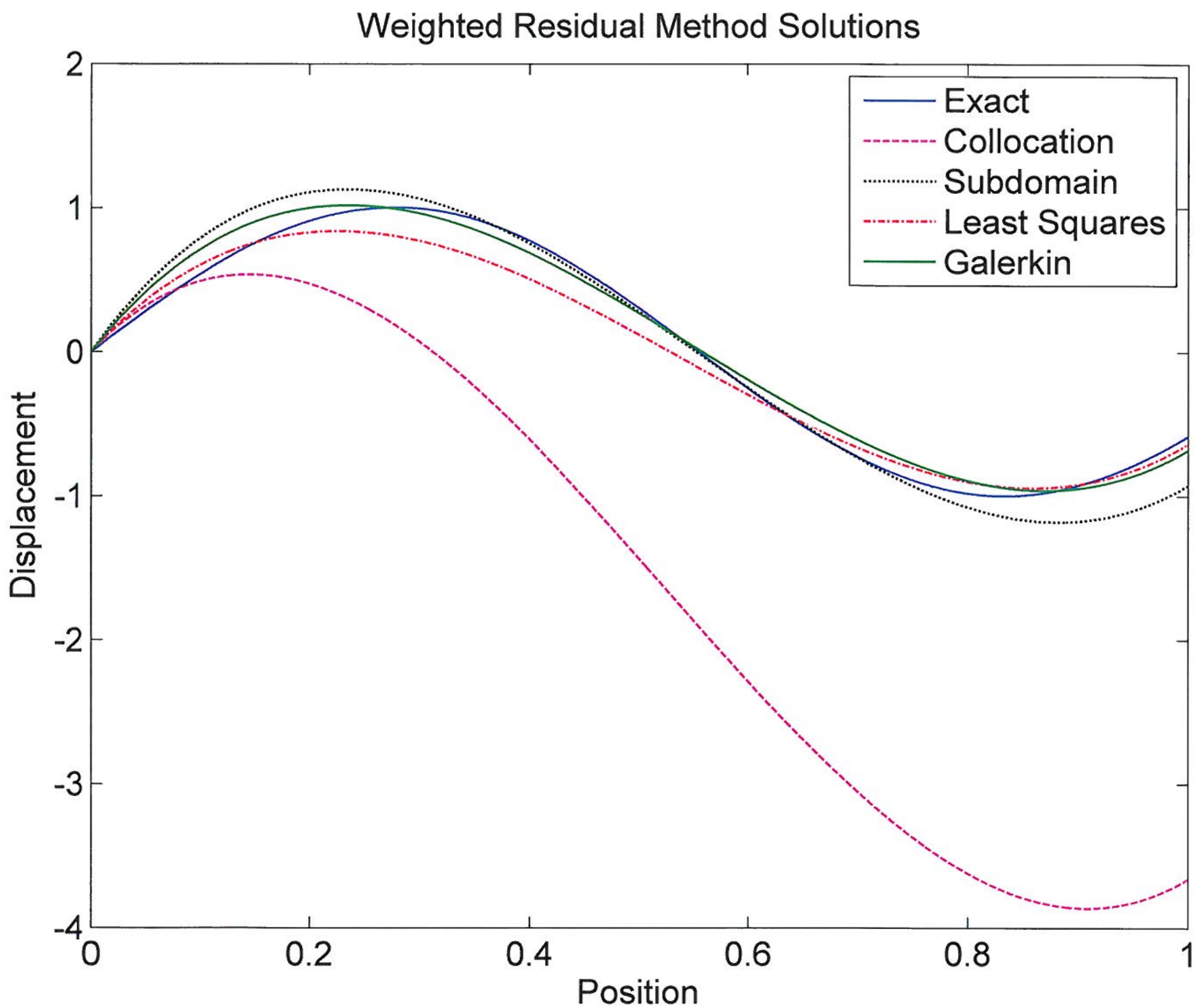
THE ERROR NORMS ARE GIVEN AS

$$\|e(x)\|_{L_2} = \sqrt{\int_0^1 (e(x))^2 dx}$$

$$\|e(x)\|_E = \sqrt{\frac{1}{2}\int_0^1 x\,(e'(x))^2 dx}$$

METHOD	$\hat{u}_N(1)$	$x\hat{u}_N'(1)$	$\|e(x)\|_{L_2}$	$\|e(x)\|_E$
COLLOCATION	1.705	4.575	1.023	1.477
SUBDOMAIN	-0.976	4.575	0.154	0.483
LEAST SQUARES	-0.644	4.575	0.135	0.46
GALERKIN (b)	-0.683	4.575	0.089	0.413
GALERKIN (c)				
1 TERM	0	0	0.738	1.376
2 TERM	-1.126	0	0.203	1.25
3 TERM	-0.563	5.811	0.1	0.318
EXACT	-6.588	4.575	—	—

THE ERROR IN THE FLUX AT $x=1$ FOR PART C IS NOT SURPRISING DUE TO THE CHOICE IN BASIS FUNCTIONS. ALSO, NOTE THAT THE GALERKIN METHOD MINIMIZES THE ERROR AS EXPECTED, AND THAT INCREASING N LEADS TO LOWER ERRORS.


Weighted Residual Method Solutions
Exact
Collocation
Subdomain
Least Squares
Galerkin
Displacement
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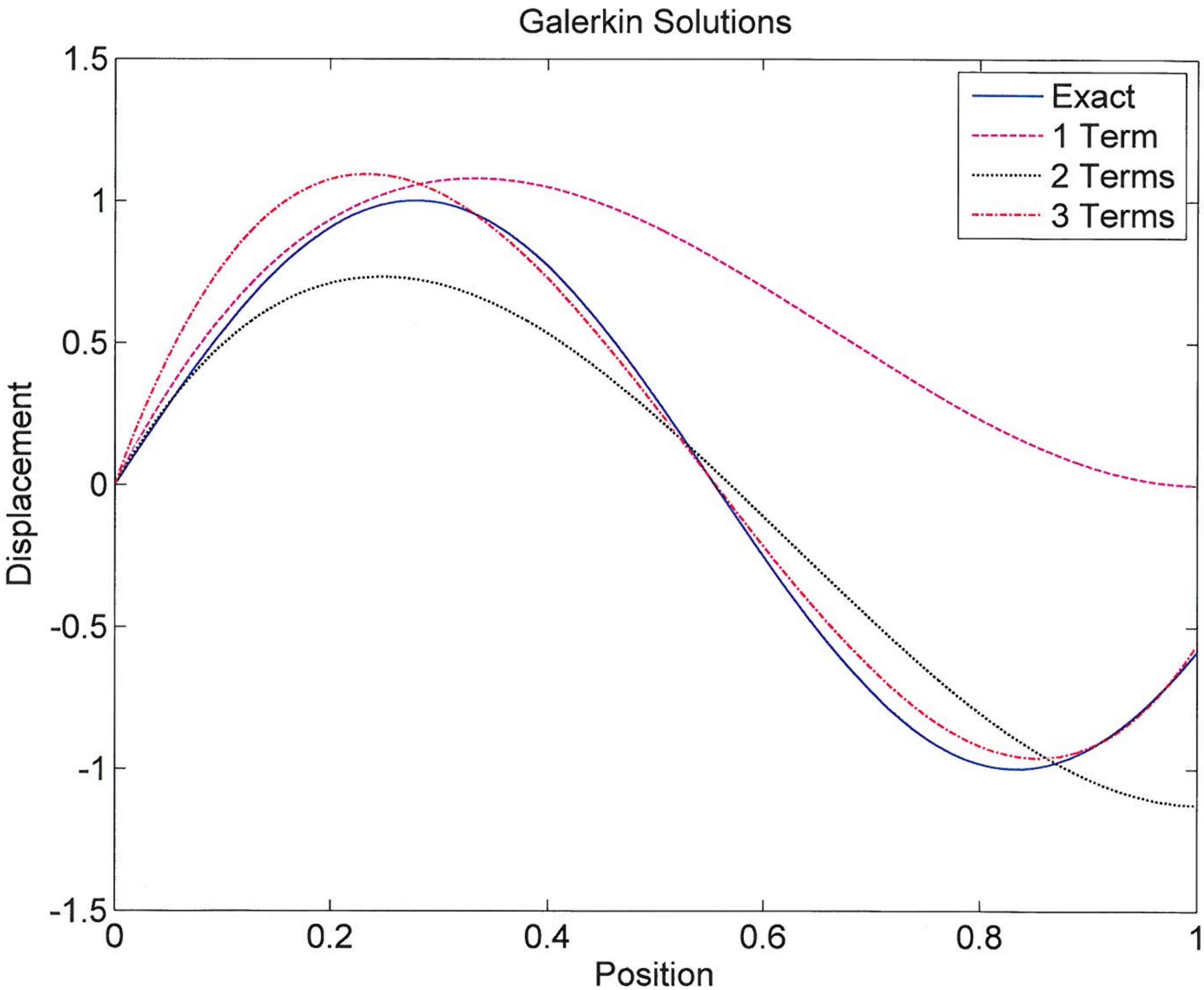
Galerkin Solutions
Exact
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2 Terms
3 Terms
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