CE 598 – Peridynamics September 3, 2015

SOLUTION: Assignment #3 – Do four problems of your choosing from Chapter 3. Due in one week.

3.1 Show that, according to Definition 3.2, the set of all polynomials $\boldsymbol{P} = \sum a_n x^n$, $n = 1,2,3,$ on the interval $0 \le x \le 1$, is a vector space P_n over the scalar field of real numbers. Find two different sets of basis vectors that can be used to describe all vectors in P_n.

Solution: Show the eight axioms are satisfied; Let $= \sum u_n x^n$; $\boldsymbol{v} = \sum v_n x^n$; $\boldsymbol{w} = \sum w_n x^n$:
Definition 3.2 A *vector space* over a field F is a set, V, together with the operations of vector addition and scalar multiplication, that satisfy the eight axioms listed below. We call the elements of vector space V *vectors*. We call the elements of field F *scalars*. Let $\boldsymbol{u}$, $\boldsymbol{v}$ and $\boldsymbol{w}$ be arbitrary vectors in V, and a and b scalars in *F*. The eight axioms are:

1) Associativity of addition: $\boldsymbol{u} + (\boldsymbol{v} + \boldsymbol{w}) = (\boldsymbol{u} + \boldsymbol{v}) + \boldsymbol{w}$
The order in which polynomial functions are added together does not affect the result, because $\boldsymbol{u} + (\boldsymbol{v} + \boldsymbol{w}) = \sum u_n x^n + (\sum v_n x^n + \sum w_n x^n) = \sum([u_n + (v_n + w_n)]x^n) = \sum([(u_n + v_n) + w_n]x^n) = (\sum u_n x^n + \sum v_n x^n) + \sum w_n x^n = (\boldsymbol{u} + \boldsymbol{v}) + \boldsymbol{w}$

2) Commutativity of addition: $\boldsymbol{u} + \boldsymbol{v} = \boldsymbol{v} + \boldsymbol{u}$
Again, the order in which polynomial functions are added together does not affect the result.

$$\boldsymbol{u} + \boldsymbol{v} = \sum u_n x^n + \sum v_n x^n = \sum v_n x^n + \sum u_n x^n = \boldsymbol{v} + \boldsymbol{u}$$

3) Identity element of addition: $\exists \mathbf{0} \in V \ : \ \boldsymbol{v} + \mathbf{0} = \boldsymbol{v} \quad \forall \boldsymbol{v} \in V$
Define the zero element as $\mathbf{0} = \sum 0x^n$. Clearly then $\boldsymbol{v} + \mathbf{0} = \boldsymbol{v} \qquad \forall \boldsymbol{v} \in V$.

4) Inverse element of addition: $\forall \boldsymbol{v} \in V, \exists(-\boldsymbol{v}) \in V : \quad \boldsymbol{v} + (-\boldsymbol{v}) = \mathbf{0} \qquad \forall \boldsymbol{v} \in V$
Define the inverse element as $-\boldsymbol{v} = \sum -v_n x^n$. Clearly then $\boldsymbol{v} + (-\boldsymbol{v}) = \mathbf{0} \qquad \forall \boldsymbol{v} \in V$.

5) Associativity of multiplication: $a(b\boldsymbol{v}) = (ab)\boldsymbol{v}$
For any scalars *a* and *b*, $a(b \sum v_n x^n) = ab(\sum v_n x^n) = (ab)\boldsymbol{v}$

6) Scalar identity element of multiplication: $1\boldsymbol{v} = \boldsymbol{v}$
For the scalars *1*, $1\boldsymbol{v} = 1(\sum v_n x^n) = \sum 1 v_n x^n = \boldsymbol{v}$

7) First distributivity scalar/vector property: $a(\boldsymbol{u} + \boldsymbol{v}) = a\boldsymbol{u} + a\boldsymbol{v}$
For the scalar *a*, $a(\boldsymbol{u} + \boldsymbol{v}) = a(\sum u_n x^n + \sum v_n x^n) = a \sum u_n x^n + a \sum v_n x^n = a\boldsymbol{u} + a\boldsymbol{v}$

8) Second distributivity scalar/vector property: $(a + b)\boldsymbol{v} = a\boldsymbol{v} + b\boldsymbol{v}$.
For the scalars *a* and *b*, $(a + b)\boldsymbol{v} = a(\sum v_n x^n) + b(\sum v_n x^n) = a\boldsymbol{v} + b\boldsymbol{v}$.

Thus the set of all polynomials $\boldsymbol{P} = \sum a_n x^n$, $n = 1,2,3,$ on the interval $0 \le x \le 1$, is a vector space P_n over the scalar field of real numbers.

Two different sets of basis vectors are:

$$\boldsymbol{e_1} = 1x^1 + 0x^2 + 0x^3 \ ; \boldsymbol{e_2} = 0x^1 + 1x^2 + 0x^3; \boldsymbol{e_3} = 0x^1 + 0x^2 + 1x^3;$$

and

$$\boldsymbol{e_1} = 1x^1 + 0x^2 + 0x^3 \ ; \boldsymbol{e_2} = 1x^1 + 1x^2 + 0x^3; \boldsymbol{e_3} = 1x^1 + 1x^2 + 1x^3 \ .$$

3.2 Show that the vector space defined in Problem 3.1 is a valid inner product space if the inner product is defined as $\boldsymbol{P}\cdot\boldsymbol{Q} = \int_0^1 (p_n x^n)(q_m x^m)dx$, and the norm is defined as $\|\boldsymbol{P}\| = \boldsymbol{P}\cdot\ \boldsymbol{P}$.

Solution:

Definition 3.3 A Euclidean vector space is a vector space, defined over the field of real numbers, together with a norm and a dot product. A Euclidean vector is an element of a Euclidean vector space.

The inner product is $\cdot\ \boldsymbol{Q} = \int_0^1 (p_n x^n)(q_m x^m)dx$, which produces a scalar real number for any pair of vectors. This definition of inner product is valid, because it has conjugate symmetry: $\boldsymbol{P}\cdot\ \boldsymbol{Q} = \boldsymbol{Q}\cdot\ \boldsymbol{P}$, linearity in the first argument: $(a\boldsymbol{P})\cdot\ \boldsymbol{Q} = a(\boldsymbol{Q}\cdot\ \boldsymbol{P})$, positive definiteness: $\boldsymbol{P}\cdot\ \boldsymbol{P} \geq 0$ for all $\boldsymbol{P}$, and if $\boldsymbol{P}\cdot\ \boldsymbol{P} = 0$, then $\boldsymbol{P} = 0$.

Also, the norm $\|\boldsymbol{P}\| = \boldsymbol{P}\cdot\ \boldsymbol{P} = \int_0^1 (p_n x^n)(p_m x^m)dx$ is easily shown to be non-negative:

$$\int_0^1 (p_n x^n)(p_m x^m)dx \ = \int_0^1 (p_1 x^1 + p_2 x^2 + p_3 x^3)(p_1 x^1 + p_2 x^2 + p_3 x^3)dx$$

$$= \int_0^1 (p_1 x^1 + p_2 x^2 + p_3 x^3)^2 dx \ ;$$

and because the integrand is non-negative for all values of x, therefore the integral must be non-negative.

The only way that the norm $\|\boldsymbol{P}\| = \boldsymbol{P}\cdot\boldsymbol{P}$ can be zero is if $= 0$.

3.3 Using a Euclidian space with the definition of the dot product being $\boldsymbol{a} \cdot \boldsymbol{b} = \|\boldsymbol{a}\|\|\boldsymbol{b}\| cos(\theta)$, prove that $\boldsymbol{a} \cdot \boldsymbol{b} = a_i b_i$, where a_i and b_i are the Cartesian components of $\boldsymbol{a}$ and $\boldsymbol{b}$.

Solution: Let $\boldsymbol{a} = a_i \hat{\boldsymbol{e}}_{\boldsymbol{i}}$ and $\boldsymbol{b} = b_i \hat{\boldsymbol{e}}_{\boldsymbol{i}}$, where the unit vectors $\hat{\boldsymbol{e}}_{\boldsymbol{i}}$ form an orthonormal basis for the Euclidian space. Thus, $\hat{\boldsymbol{e}}_{\boldsymbol{i}} \cdot \hat{\boldsymbol{e}}_{\boldsymbol{j}} = \boldsymbol{a} \cdot \boldsymbol{b} = \|\hat{\boldsymbol{e}}_{\boldsymbol{i}}\|\|\hat{\boldsymbol{e}}_{\boldsymbol{j}}\| cos(\theta) = 1 \times 1 \times cos(\theta) = \delta_{ij}$ (the Kronecker delta).

If $\boldsymbol{a} \cdot \boldsymbol{b} = (a_i \hat{\boldsymbol{e}}_{\boldsymbol{i}}) \cdot (b_j \hat{\boldsymbol{e}}_{\boldsymbol{j}}) =$

$$
\begin{aligned}
&= (a_1 \hat{\boldsymbol{e}}_{\boldsymbol{1}} + a_2 \hat{\boldsymbol{e}}_{\boldsymbol{2}} + a_3 \hat{\boldsymbol{e}}_{\boldsymbol{3}}) \cdot (b_1 \hat{\boldsymbol{e}}_{\boldsymbol{1}} + b_2 \hat{\boldsymbol{e}}_{\boldsymbol{2}} + b_3 \hat{\boldsymbol{e}}_{\boldsymbol{3}}) \\
&= a_1 \hat{\boldsymbol{e}}_{\boldsymbol{1}} \cdot (b_1 \hat{\boldsymbol{e}}_{\boldsymbol{1}} + b_2 \hat{\boldsymbol{e}}_{\boldsymbol{2}} + b_3 \hat{\boldsymbol{e}}_{\boldsymbol{3}}) + a_2 \hat{\boldsymbol{e}}_{\boldsymbol{2}} \cdot (b_1 \hat{\boldsymbol{e}}_{\boldsymbol{1}} + b_2 \hat{\boldsymbol{e}}_{\boldsymbol{2}} + b_3 \hat{\boldsymbol{e}}_{\boldsymbol{3}}) + a_3 \hat{\boldsymbol{e}}_{\boldsymbol{3}} \cdot (b_1 \hat{\boldsymbol{e}}_{\boldsymbol{1}} + b_2 \hat{\boldsymbol{e}}_{\boldsymbol{2}} + b_3 \hat{\boldsymbol{e}}_{\boldsymbol{3}}) \\
&= (a_1 b_1 \hat{\boldsymbol{e}}_{\boldsymbol{1}} \cdot \hat{\boldsymbol{e}}_{\boldsymbol{1}} + a_1 b_2 \hat{\boldsymbol{e}}_{\boldsymbol{1}} \cdot \hat{\boldsymbol{e}}_{\boldsymbol{2}} + a_1 b_3 \hat{\boldsymbol{e}}_{\boldsymbol{1}} \cdot \hat{\boldsymbol{e}}_{\boldsymbol{3}}) + (a_2 b_1 \hat{\boldsymbol{e}}_{\boldsymbol{2}} \cdot \hat{\boldsymbol{e}}_{\boldsymbol{1}} + a_2 b_2 \hat{\boldsymbol{e}}_{\boldsymbol{2}} \cdot \hat{\boldsymbol{e}}_{\boldsymbol{2}} + a_2 b_3 \hat{\boldsymbol{e}}_{\boldsymbol{2}} \cdot \hat{\boldsymbol{e}}_{\boldsymbol{3}}) \\
&\quad + (a_3 b_1 \hat{\boldsymbol{e}}_{\boldsymbol{3}} \cdot \hat{\boldsymbol{e}}_{\boldsymbol{1}} + a_3 b_2 \hat{\boldsymbol{e}}_{\boldsymbol{3}} \cdot \hat{\boldsymbol{e}}_{\boldsymbol{2}} + a_3 b_3 \hat{\boldsymbol{e}}_{\boldsymbol{3}} \cdot \hat{\boldsymbol{e}}_{\boldsymbol{3}}) \\
&= \big(a_1 b_1 (1) + a_1 b_2 (0) + a_1 b_3 (0)\big) + \big(a_2 b_1 (0) + a_2 b_2 (1) + a_2 b_3 (0)\big) \\
&\quad + \big(a_3 b_1 (0) + a_3 b_2 (0) + a_3 b_3 (1)\big) \\
&= a_1 b_1 + a_2 b_2 + a_3 b_3 \text{ . QED}
\end{aligned}
$$

3.4 Expand the following indicial expressions to as many terms as possible, and then express them as matrices. Indicate the rank of each matrix. The indices all vary from 1 to 3.

a. $A_i x_i$
b. $A_i x_j$
c. $A_{ij} x_j$
d. $A_{ijk} x_k$
e. $A_{ij} x_i x_j$

Solution:

a. $A_i x_i = A_1 x_1 + A_2 x_2 + A_3 x_3 = \lfloor A_i \rfloor \{x_i\}$. (A scalar expression - a 1x1 matrix)

b. $A_i x_j = A_1 x_1, A_1 x_2,\ A_1 x_3,\ A_2 x_1, A_2 x_2,\ A_2 x_3,\ A_3 x_1, A_3 x_2,\ A_3 x_3$ (Nine independent expressions.) In matrix form: $A_i x_j = \{A_i\} \lfloor x_j \rfloor = \begin{bmatrix} A_1 x_1 & A_1 x_2 & A_1 x_3 \\ A_2 x_1 & A_2 x_2 & A_2 x_3 \\ A_3 x_1 & A_3 x_2 & A_3 x_3 \end{bmatrix}$ (A 3x3 matrix.)

c. Expanding repeated index j: $A_{ij} x_j = A_{i1} x_1 + A_{i2} x_2 + A_{i3} x_3$

Expanding non-repeated index : $\begin{matrix} A_{1j} x_j = A_{11} x_1 + A_{12} x_2 + A_{13} x_3 \\ A_{2j} x_j = A_{21} x_1 + A_{22} x_2 + A_{23} x_3 \\ A_{3j} x_j = A_{31} x_1 + A_{32} x_2 + A_{33} x_3 \end{matrix} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}$

(A 3x1 matrix.)

d. Expanding repeated index k: $A_{ijk} x_k = A_{ij1} x_1 + A_{ij2} x_2 + A_{ij3} x_3$

Expanding non-repeated indices k: $A_{ijk} x_k = A_{ij1} x_1 + A_{ij2} x_2 + A_{ij3} x_3$

(This represents nine equations)

$$A_{ijk} x_k = \begin{bmatrix} A_{111} & A_{112} & A_{113} \\ A_{121} & A_{122} & A_{123} \\ A_{131} & A_{132} & A_{133} \\ A_{211} & A_{212} & A_{213} \\ A_{221} & A_{222} & A_{223} \\ A_{231} & A_{232} & A_{233} \\ A_{311} & A_{312} & A_{313} \\ A_{321} & A_{322} & A_{323} \\ A_{331} & A_{332} & A_{333} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} \text{ (A 9x1 matrix.)}$$

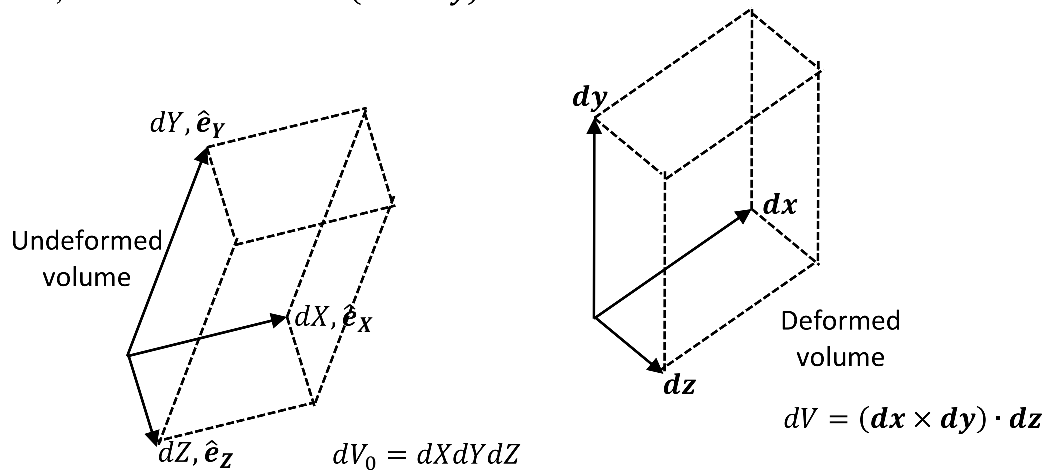
e. $A_{ij} x_i x_j = (A_{i1} x_1 + A_{i2} x_2 + A_{i3} x_3) x_i = \big((A_{11} x_1 + A_{12} x_2 + A_{13} x_3) x_1 + (A_{21} x_1 + A_{22} x_2 + A_{23} x_3) x_2 + (A_{31} x_1 + A_{32} x_2 + A_{33} x_3) x_3 \big) = (A_{11} x_1 x_1 + A_{12} x_2 x_1 + A_{13} x_3 x_1) + (A_{21} x_1 x_2 + A_{22} x_2 x_2 + A_{23} x_3 x_2) + (A_{31} x_1 x_3 + A_{32} x_2 x_3 + A_{33} x_3 x_3) =$
$\lfloor x_1 \quad x_2 \quad x_3 \rfloor \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}$ (A 1x1 matrix.)

3.5 Show that the deformation gradient $\boldsymbol{F}$ is a second-order tensor.

Solution: $\boldsymbol{F} \equiv \frac{\partial \boldsymbol{x}_i}{\partial \boldsymbol{X}_j}$. By the chain rule, $dx_i = \frac{\partial \boldsymbol{x}_i}{\partial \boldsymbol{X}_j} dX_j$, regardless of coordinate system. In other terms, $\boldsymbol{dx} = \boldsymbol{F}\boldsymbol{dX}$. Because both $\boldsymbol{dx}$ and $\boldsymbol{dX}$ are vectors (first-order tensors), and thus independent of coordinate system, therefore $\boldsymbol{F}$ must also be independent of coordinate system as well. In addition, any linear transformation from a vector to another vector is a second-order tensor. Therefore, the deformation gradient $\boldsymbol{F}$ is a second-order tensor.

3.6 Prove that $\frac{dV}{dV_0} = det\boldsymbol{F}$.

Solution: Choose the differential element whose edges, of lengths dX, dY, and dZ are aligned with the $(\hat{\boldsymbol{e}}_{\boldsymbol{X}}, \hat{\boldsymbol{e}}_{\boldsymbol{Y}}, \hat{\boldsymbol{e}}_{\boldsymbol{Z}})$ reference orthonormal Cartesian coordinate system shown below. The volume of this differential element is $dV_0 = dXdYdZ$. This differential element deforms into the element shown, whose volume is $dV = (\boldsymbol{dx} \times \boldsymbol{dy}) \cdot \boldsymbol{dz}$:



$$\boldsymbol{dx} = \boldsymbol{FdX}, \text{ or } x_i = F_{ij}dX_j \; ;$$

thus $\boldsymbol{dx} = F_{11}dX\hat{\boldsymbol{e}}_{\boldsymbol{X}} + F_{12}dY\hat{\boldsymbol{e}}_{\boldsymbol{Y}} + F_{13}dZ\hat{\boldsymbol{e}}_{\boldsymbol{Z}}$ and $\boldsymbol{dy} = F_{21}dX\hat{\boldsymbol{e}}_{\boldsymbol{X}} + F_{22}dY\hat{\boldsymbol{e}}_{\boldsymbol{Y}} + F_{23}dZ\hat{\boldsymbol{e}}_{\boldsymbol{Z}}$

$$\begin{aligned}\boldsymbol{dx} \times \boldsymbol{dy} = det &\begin{vmatrix} \hat{\boldsymbol{e}}_{\boldsymbol{X}} & \hat{\boldsymbol{e}}_{\boldsymbol{Y}} & \hat{\boldsymbol{e}}_{\boldsymbol{Z}} \\ F_{11}dX & F_{12}dY & F_{13}dZ \\ F_{21}dX & F_{22}dY & F_{23}dZ \end{vmatrix} \\ &= \hat{\boldsymbol{e}}_{\boldsymbol{X}}(F_{12}dYF_{23}dZ - F_{13}dZF_{22}dY) - \hat{\boldsymbol{e}}_{\boldsymbol{Y}}(F_{11}dXF_{23}dZ - F_{13}dZF_{21}dX) \\ &+ \hat{\boldsymbol{e}}_{\boldsymbol{Z}}(F_{11}dXF_{22}dY - F_{12}dYF_{21}dX)\end{aligned}$$

and $\boldsymbol{dz} = F_{31}dX\hat{\boldsymbol{e}}_{\boldsymbol{X}} + F_{32}dY\hat{\boldsymbol{e}}_{\boldsymbol{Y}} + F_{33}dZ\hat{\boldsymbol{e}}_{\boldsymbol{Z}}$,

$$\begin{aligned}dV &= (\boldsymbol{dx} \times \boldsymbol{dy}) \cdot \boldsymbol{dz} \\ &= [\hat{\boldsymbol{e}}_{\boldsymbol{X}}(F_{12}dYF_{23}dZ - F_{13}dZF_{22}dY) - \hat{\boldsymbol{e}}_{\boldsymbol{Y}}(F_{11}dXF_{23}dZ - F_{13}dZF_{21}dX) \\ &+ \hat{\boldsymbol{e}}_{\boldsymbol{Z}}(F_{11}dXF_{22}dY - F_{12}dYF_{21}dX)] \cdot (F_{31}dX\hat{\boldsymbol{e}}_{\boldsymbol{X}} + F_{32}dY\hat{\boldsymbol{e}}_{\boldsymbol{Y}} + F_{33}dZ\hat{\boldsymbol{e}}_{\boldsymbol{Z}})\end{aligned}$$

$$\begin{aligned}dV &= (\boldsymbol{dx} \times \boldsymbol{dy}) \cdot \boldsymbol{dz} \\ &= (F_{12}dYF_{23}dZ - F_{13}dZF_{22}dY)F_{31}dX - (F_{11}dXF_{23}dZ - F_{13}dZF_{21}dX)F_{32}dY \\ &+ (F_{11}dXF_{22}dY - F_{12}dYF_{21}dX)F_{33}dZ\end{aligned}$$

$$\begin{aligned}dV &= (\boldsymbol{dx} \times \boldsymbol{dy}) \cdot \boldsymbol{dz} \\ &= [(F_{12}F_{23} - F_{13}F_{22})F_{31} - (F_{11}F_{23} - F_{13}F_{21})F_{32} + (F_{11}F_{22} - F_{12}F_{21})F_{33}]dXdYdZ\end{aligned}$$

$$dV = det\begin{vmatrix} F_{11} & F_{12} & F_{13} \\ F_{21} & F_{22} & F_{23} \\ F_{31} & F_{32} & F_{33} \end{vmatrix} dV_0 \quad \text{or } \frac{dV}{dV_0} = det\boldsymbol{F} \;\; \text{QED.}$$

3.7 A body is rotated as a rigid body about the $\hat{\boldsymbol{e}}_3$ axis by the angle θ.

a) Determine the deformation function $\boldsymbol{x} = \boldsymbol{\varphi}(\boldsymbol{X})$ in terms of θ.
b) Determine the deformation gradient, $\boldsymbol{F}$.
c) Determine the Cauchy-Green deformation tensor $\boldsymbol{C}$.
d) Determine the Lagrangian strain tensor $\boldsymbol{E}$.
e) Determine the small-strain tensor $\boldsymbol{\epsilon}$.
f) Show that $\boldsymbol{\epsilon}$ and $\boldsymbol{E}$ are approximately the same for small θ.

Solution:

a) Deformation function $= \boldsymbol{\varphi}(\boldsymbol{X})$:

$$\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{bmatrix} cos(\theta) & -sin(\theta) & 0 \\ sin(\theta) & cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix}, \text{ so } \quad \begin{aligned} x_1 &= X_1 cos(\theta) - X_2 sin(\theta) + 0X_3 \\ x_2 &= X_1 sin(\theta) + X_2 cos(\theta) + 0X_3 \\ x_3 &= 0X_1 - 0X_2 + 1X_3 \end{aligned}$$

b) Deformation gradient, $\boldsymbol{F} = \frac{\partial \boldsymbol{\varphi}}{\partial \boldsymbol{X}} = \begin{bmatrix} \frac{\partial x_1}{\partial X_1} & \frac{\partial x_1}{\partial X_2} & \frac{\partial x_1}{\partial X_3} \\ \frac{\partial x_2}{\partial X_1} & \frac{\partial x_2}{\partial X_2} & \frac{\partial x_2}{\partial X_3} \\ \frac{\partial x_3}{\partial X_1} & \frac{\partial x_3}{\partial X_2} & \frac{\partial x_3}{\partial X_3} \end{bmatrix} = \begin{bmatrix} cos(\theta) & -sin(\theta) & 0 \\ sin(\theta) & cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

c) Cauchy-Green deformation tensor $\boldsymbol{C} = \boldsymbol{F}^T \boldsymbol{F} =$

$$\begin{bmatrix} cos(\theta) & -sin(\theta) & 0 \\ sin(\theta) & cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} cos(\theta) & sin(\theta) & 0 \\ -sin(\theta) & cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} =$$

$$\begin{bmatrix} [cos^2(\theta) + sin^2(\theta)] & [sin(\theta)cos(\theta) - sin(\theta)cos(\theta)] & 0 \\ [sin(\theta)cos(\theta) - sin(\theta)cos(\theta)] & [sin^2(\theta) + cos^2(\theta)] & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

d) Lagrangian strain tensor: $\boldsymbol{E} = \frac{1}{2}(\boldsymbol{C} - \boldsymbol{I}) = \frac{1}{2}\left[\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

e) $\boldsymbol{\epsilon}$: $\epsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$

$$\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} - \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix} = \begin{bmatrix} cos(\theta) & -sin(\theta) & 0 \\ sin(\theta) & cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix}$$

$$= \begin{bmatrix} cos(\theta) - 1 & -sin(\theta) & 0 \\ sin(\theta) & cos(\theta) - 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix}$$

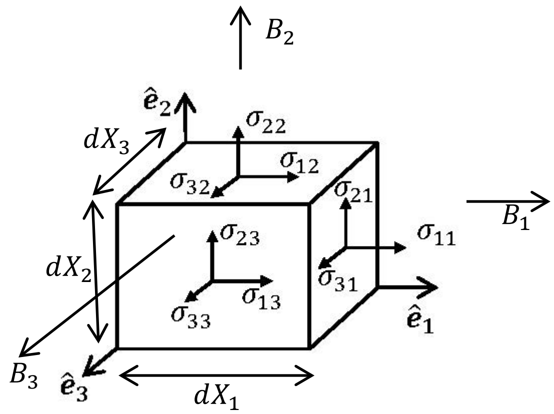
$$[u_{i,j}] = \begin{bmatrix} cos(\theta) - 1 & -sin(\theta) & 0 \\ sin(\theta) & cos(\theta) - 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\epsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) = \begin{bmatrix} cos(\theta) - 1 & 0 & 0 \\ 0 & cos(\theta) - 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

f) If θ is small then $\boldsymbol{\epsilon} \approx \boldsymbol{0} = \boldsymbol{E}$. (However, if θ is *not* small, then the small-strain tensor $\epsilon_{ij} \neq [0]$.)

3.8 Show that the Cauchy stress tensor is symmetric if body moment is absent: $\boldsymbol{\sigma} = \boldsymbol{\sigma}^{\boldsymbol{T}}$.

Solution: Consider the following free body diagram, with stresses and body forces acting upon it:



A symmetric tensor $\boldsymbol{\sigma}$ is a tensor that is invariant under a permutation of its vector arguments, so that $\boldsymbol{\sigma}(\boldsymbol{v_1}, \boldsymbol{v_2}) = \boldsymbol{\sigma}(\boldsymbol{v_2}, \boldsymbol{v_1})$. If the tensor is expressed in terms of its tensor components, then $\sigma_{ij} = \sigma_{ji}$.

Summing moments about the $\hat{\boldsymbol{e}}_3$ axis, the normal stresses produce no net moment about the $\hat{\boldsymbol{e}}_3$ axis because they produce equal and opposite moments on opposing faces. Only the shear stresses σ_{ij} and body forces B_i cause a moment about the $\hat{\boldsymbol{e}}_3$ axis:

$$\sigma_{21}dX_2dX_3(dX_1) - \sigma_{12}dX_1dX_3(dX_2) - B_1dX_1dX_2dX_3\left(\frac{dX_2}{2}\right) + B_2dX_1dX_2dX_3\left(\frac{dX_1}{2}\right) = 0 .$$

As the moments due to applied body forces are higher-order terms, they disappear in the limit as $dX_1dX_2dX_3$ goes to zero. Therefore $\sigma_{12} = \sigma_{21}$.

Similarly, by summing moments about the other two axes: $\sigma_{23} = \sigma_{32}$ and $\sigma_{13} = \sigma_{31}$.

Therefore, the stress tensor is symmetric: $\boldsymbol{\sigma} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{bmatrix} = \boldsymbol{\sigma}^{\boldsymbol{T}}$.

3.9 In the reference configuration, the mass density of a body is given by the function $\rho_0(\boldsymbol{X})$. Show that the conservation of mass equation, Eq. 3.41, is satisfied for all deformations of the form $x_i = \alpha_i(t)X_i$, (no summation implied) where functions $\alpha_i(t)$ are differentiable functions of time. (Hint: $div\,\boldsymbol{v} = \frac{\frac{\partial}{\partial t}(det\boldsymbol{F})}{det\boldsymbol{F}}$).

Solution:

$$\frac{D\rho}{Dt} + \rho(div\,\boldsymbol{v}) = 0\,, \qquad (3.1)$$

Consider the fact that $\frac{dV}{dV_0} = det|\boldsymbol{F}|$, so $dV_0 det|\boldsymbol{F}| = dV$, and $\frac{dm}{dV} det|\boldsymbol{F}| = \frac{dm}{dV_0}$ or $\rho det|\boldsymbol{F}| = \rho_0$

so $= \frac{\rho_0}{det|\boldsymbol{F}|}$.

Substituting this formula for the density ρ into Eq. 3.41:

$$\frac{D\left(\frac{\rho_0}{det|\boldsymbol{F}|}\right)}{Dt} + \frac{\rho_0}{det|\boldsymbol{F}|}(div\,\boldsymbol{v}) = 0 \text{ or } \rho_0 \frac{D\left(\frac{1}{det|\boldsymbol{F}|}\right)}{Dt} + \rho_0 \frac{1}{det|\boldsymbol{F}|}(div\,\boldsymbol{v}) =?\,0\,.$$

But $div\,\boldsymbol{v} = \frac{\frac{\partial}{\partial t}(det\boldsymbol{F})}{det\boldsymbol{F}}$ so

$$\frac{D\left(\frac{1}{det|\boldsymbol{F}|}\right)}{Dt} + \frac{1}{det|\boldsymbol{F}|}\left(\frac{\frac{\partial}{\partial t}(det\boldsymbol{F})}{det\boldsymbol{F}}\right) =?\,0 \text{ or } \frac{D\left(\frac{1}{det|\boldsymbol{F}|}\right)}{Dt} + \frac{1}{(det|\boldsymbol{F}|)^2}\left(\frac{\partial}{\partial t}(det\boldsymbol{F})\right) =?\,0\,.$$

$$\boldsymbol{F} = \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{X}} = \left\{\frac{\partial}{\partial X_i}\right\}[\alpha_1(t)X_1 \quad \alpha_2(t)X_2 \quad \alpha_3(t)X_3] = \begin{bmatrix} \alpha_1(t) & 0 & 0 \\ 0 & \alpha_2(t) & 0 \\ 0 & 0 & \alpha_3(t) \end{bmatrix}.$$

So $= \alpha_1(t)\alpha_2(t)\alpha_3(t) = \alpha_1\alpha_2\alpha_3$. Thus

$$\frac{D\left(\frac{1}{\alpha_1\alpha_2\alpha_3}\right)}{Dt} + \frac{1}{(\alpha_1\alpha_2\alpha_3)^2}\left(\frac{\partial}{\partial t}(\alpha_1\alpha_2\alpha_3)\right) = \frac{d({\alpha_1}^{-1}{\alpha_2}^{-1}{\alpha_3}^{-1})}{dt} + \frac{1}{(\alpha_1\alpha_2\alpha_3)^2}\left(\frac{d}{dt}(\alpha_1\alpha_2\alpha_3)\right)$$
$$=$$
$$\left(-\frac{\dot{\alpha_1}}{{\alpha_1}^2}{\alpha_2}^{-1}{\alpha_3}^{-1} - \frac{\dot{\alpha_2}}{{\alpha_2}^2}{\alpha_1}^{-1}{\alpha_3}^{-1} - \frac{\dot{\alpha_3}}{{\alpha_3}^2}{\alpha_1}^{-1}{\alpha_2}^{-1}\right) + \frac{1}{(\alpha_1\alpha_2\alpha_3)^2}(\dot{\alpha_1}\alpha_2\alpha_3 + \dot{\alpha_2}\alpha_1\alpha_3 + \dot{\alpha_3}\alpha_1\alpha_2) = 0.$$

QED.

3.10 Solve the problem specified in Fig. 3.10 in the time domain, assuming a uniform density $\rho_0 = 1.0\ kg/m^3$, $E = 1.0\ N/m^2$, $L = 1.0\ m$, $A = 1.0\ m^2$, $k = 0.001$, and that the applied tip displacement $\Delta = v_0 t$ is a ramp function, where $v_0 = 2kL/t_{max}\ m/s$ is the rate of end-displacement application, and $t_{max} = 25\ s$ is the total time of the simulation. Use small-displacement theory, as the strain ϵ is much smaller than unity. Use MatLab to solve the problem numerically using 10, 15 and 25 particles.

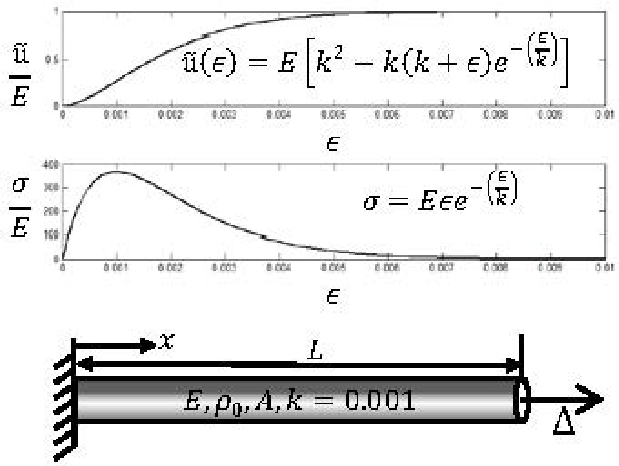


Figure 0.1. Uniaxial bar with nonlinear stress-strain relationship.

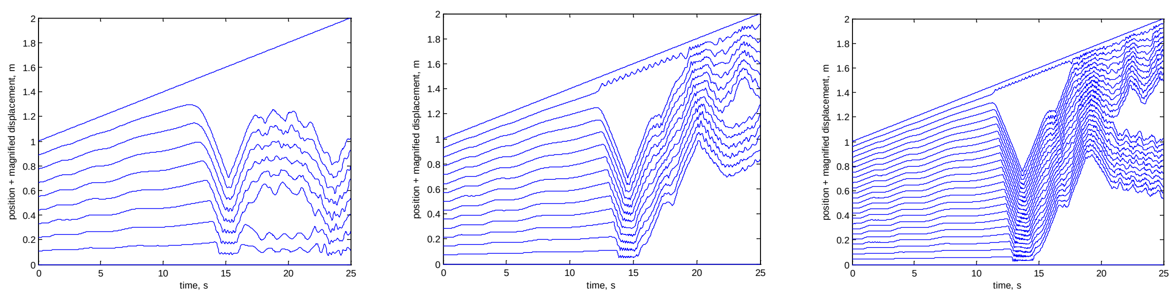
Solution: The MatLab code is:

```
function Prob3_10
%Problem 3.10 Solution
rho0 = 1.0; % kg/m^3;
E = 1.0; % N/m^2
L = 1.0; %m
A = 1.0; %m^2
k = 0.001; % unitless constant
EA = E*A;
tMax = 25.; % total simulation time, in seconds
numPtcls = 15; % number of particles,
               % including leftmost (fixed) and rightmost (applied velocity)
dL = L/(numPtcls - 1);
m = dL*A*rho0; % mass of a particle
xRef = [0 : dL : L];   %reference positions of all particles
x = xRef;   %current positions of all particles
time = 0.;
K2 = 2*EA/dL;
omegaMax = sqrt(K2/m);
PeriodMin = 2*pi/omegaMax;
dtCrit = PeriodMin/(2.*pi);
dt = dtCrit/2.
numSteps = floor(tMax/dt);
force = zeros(1, numPtcls);
accel = zeros(1, numPtcls);
vel = zeros(1, numPtcls);
xPlot = zeros(numSteps, numPtcls);
uPlot = zeros(numSteps, numPtcls);
timePlot = zeros(numSteps, 1);
velTip = 2.*k*L/tMax %m/s tip x-velocity
```

```
for iStep = 1 : numSteps
    time = time + dt;
    for iPtcl = 2 : numPtcls - 1
        stretchLeft = ((x(iPtcl) - x(iPtcl - 1)) - dL)/dL;
        forceLeft = EA*stretchLeft*exp(-(stretchLeft/k));
        stretchRight = ((x(iPtcl + 1) - x(iPtcl)) - dL)/dL;
        forceRight = EA*stretchRight*exp(-(stretchRight/k));
        force(iPtcl) = forceRight - forceLeft;
    end
    x(numPtcls) = xRef(numPtcls) + velTip*time;
    accel(2 : (numPtcls - 1) ) = force(2 : (numPtcls - 1) )/m;
    vel(2 : (numPtcls - 1) ) = vel(2 : (numPtcls - 1) ) ...
                                + accel(2 : (numPtcls - 1) )*dt;
    x(2 : (numPtcls - 1) ) = x(2 : (numPtcls - 1) ) ...
                                + vel(2 : (numPtcls - 1) )*dt;
    xPlot(iStep,:) = x;
    xRefPlot(iStep,:) = xRef;
    uPlot(iStep,:) = x-xRef;
    timePlot(iStep) = time;
end
magFactor = 500.;
for iPtcl = 1 : numPtcls
    plot(timePlot, xRefPlot(:,iPtcl) + magFactor*uPlot(:, iPtcl));
    hold on
end
xlabel 'time, s'
ylabel 'position + magnified displacement, m'
return
end
```

Giving the graphical output:

10 particles: 15 particles: 25 particles:



One sees that "cracks" between particles form and heal, finally resulting in one large discontinuity (crack). The strain field is clearly discontinuous across the cracks. The results are very sensitive to the number of particles in the simulation. The solution is very nonlinear.

3.11 The cantilever beam shown in Fig. 3.11 has four corners at which the stress field is problematic. Explain why the stress field is problematic. Assume small deformation linear elastic theory.

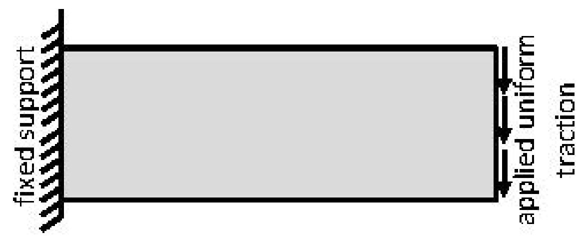
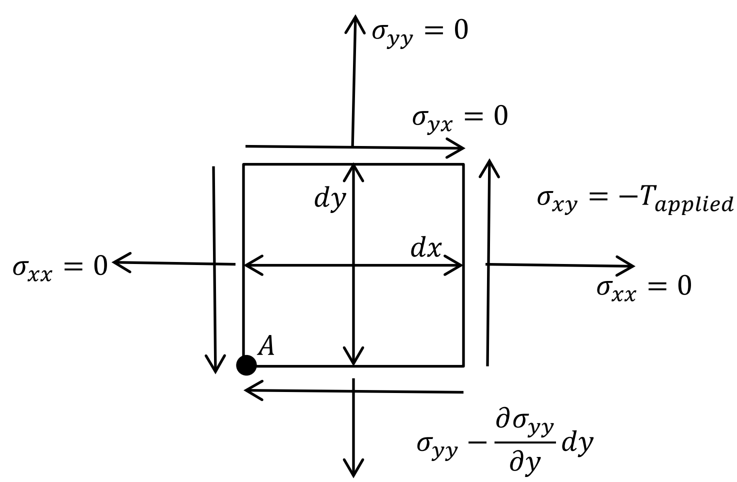


Figure 0.2. Cantilever beam for Problem 3.11.

Solution: At the right end of the beam, the applied vertical traction would not be permitted at the top and bottom corners of the beam, as this would imply that $\sigma_{xy} \neq \sigma_{yx}$. But, considering the top right infinitesimal element, it cannot be in moment equilibrium. Consider taking moments about point A:



$$\sum M_A = 0 : \quad -T_{applied} \times dx - \left(\sigma_{yy} - \frac{\partial \sigma_{yy}}{\partial y} dy\right) \times \frac{dx}{2} = 0\ .$$

But $\sigma_{yy} = 0$ on the top surface of the beam, and $\frac{\partial \sigma_{yy}}{\partial y}$ must be finite. Therefore, because $dx \times dy$ is a higher-order differential term, this implies that $T_{applied} = 0$, but this contradicts the initial assumption of applied traction. The only way to satisfy moment equilibrium of the differential element is if $\frac{\partial \sigma_{yy}}{\partial y} = \infty$. The stress field at the top right corner would need to have an infinite gradient.

One can make similar arguments about the other three corners of the beam. Because the boundary of the domain is not smooth, there is no *finite* continuum mechanics solution at the corners of the beam.