3. VECTOR BASES, VECTORS AND TRANSFORMATION RELATIONS

3.1 Initial Comments

For some this will be the most difficult material because we are starting axiomatically with the concept of an orthonormal vector basis. Nothing in this chapter is related to coordinate systems or points in space; just the representation of a vector using several bases and the transformation relations between any two bases.

3.2 An Orthonormal Basis

We define an orthonormal basis, $\boldsymbol{e}_i$, as a set of three unit "physical" vectors that satisfies

$$\boldsymbol{e}_i \cdot \boldsymbol{e}_j = \delta_{ij} \tag{3-1}$$

We interpret this definition physically in the usual manner of a dot product, namely the product of the magnitudes of two vectors times the cosine of the angle between them. Each of these particular base vectors are chosen to have magnitude one, and each is orthogonal to the other two.

The particular orientation of the base vectors is implicitly understood. Each may be aligned with the side of a room, with the side of a solid object, or with the directions north or south, east or west, and up or down. Frequently we leave the directions arbitrary for the purpose of describing a theory but for a specific problem, enough information must be available so that a reader knows the orientation.

In addition we usually assume the basis is also right handed which implies that the following equation involving the cross product and the alternating symbol is satisfied:

$$\boldsymbol{e}_i \cdot (\boldsymbol{e}_j \times \boldsymbol{e}_k) = \varepsilon_{ijk} \tag{3-2}$$

Again we assume that the definition of a cross product is known implicitly. An alternative way to interpret (3-2) is that

$$\boldsymbol{e}_i \times \boldsymbol{e}_j = \varepsilon_{ijk} \boldsymbol{e}_k \tag{3-3}$$

In the future, we will simply sketch a right-handed orthonormal basis as shown in Fig. 3-1 with the understanding that (3-1) and (3-2) are satisfied.

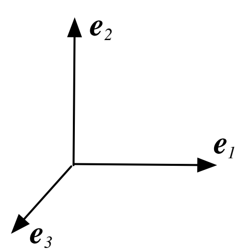

Fig. 3-1. The orthonormal basis $\boldsymbol{e}_i$.

3.3 Vectors

Any vector can be expressed as a product of components and base vectors:

$$\boldsymbol{v} = v_i \boldsymbol{e}_i \tag{3-4}$$

in which the set of numbers v_i are the components of $\boldsymbol{v}$ with respect to the basis $\boldsymbol{e}_i$. We say that both the single bold face letter $\boldsymbol{v}$ and the basis $\boldsymbol{e}_i$ are part of what we refer to as **direct notation**.

Now we proceed further and suppose we represent the components as either a row vector or a column vector, and similarly place the base vectors in a row or a column as follows:

$$\begin{aligned} v_i \quad &\Rightarrow \quad \{v\} = \begin{Bmatrix} v_1 \\ v_2 \\ v_3 \end{Bmatrix} \qquad & v_i \quad &\Rightarrow \quad \langle v \rangle = \langle \; v_1 \quad v_2 \quad v_3 \; \rangle \\ \boldsymbol{e}_i \quad &\Rightarrow \quad \{\boldsymbol{e}\} = \begin{Bmatrix} \boldsymbol{e}_1 \\ \boldsymbol{e}_2 \\ \boldsymbol{e}_3 \end{Bmatrix} \qquad & \boldsymbol{e}_i \quad &\Rightarrow \quad \langle \boldsymbol{e} \rangle = \langle \; \boldsymbol{e}_1 \quad \boldsymbol{e}_2 \quad \boldsymbol{e}_3 \; \rangle \end{aligned} \tag{3-5}$$

Then we have the following possible representations of a vector as follows:

$$\boldsymbol{v} = v_i \boldsymbol{e}_i = \langle v \rangle \{\boldsymbol{e}\} = \langle \boldsymbol{e} \rangle \{v\} \tag{3-6}$$

We note that we can use either indicial or matrix notation to obtain

$$\begin{aligned} &\boldsymbol{e}_i \cdot \boldsymbol{e}_j = \delta_{ij} & &\boldsymbol{e}_i \cdot \boldsymbol{e}_i = \delta_{ii} = 3 \\ &\{\boldsymbol{e}\} \cdot \langle \boldsymbol{e} \rangle = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} & &\langle \boldsymbol{e} \rangle \cdot \{\boldsymbol{e}\} = 3 \end{aligned} \tag{3-7}$$

We define the magnitude of $\boldsymbol{v}$ to be $|\boldsymbol{v}|$ as follows:

$$|\boldsymbol{v}| = \left(|\boldsymbol{v}|^2 \right)^{1/2} \qquad |\boldsymbol{v}|^2 = \boldsymbol{v} \cdot \boldsymbol{v} = v_i \boldsymbol{e}_i \cdot v_j \boldsymbol{e}_j = v_i v_j \delta_{ij} = v_i v_j \tag{3-8}$$

A unit vector in the direction of $\boldsymbol{v}$ is

$$\boldsymbol{n}^v = \frac{\boldsymbol{v}}{|\boldsymbol{v}|} \tag{3-9}$$

and these are simply the direction cosines of $\boldsymbol{v}$ with respect to the basis $\boldsymbol{e}_i$. The component of another vector $\boldsymbol{u}$ in the direction of $\boldsymbol{v}$ is simply

$$u_n = \boldsymbol{u} \cdot \boldsymbol{n}^v \tag{3-10}$$

The dot product of two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ is

$$\boldsymbol{u}\cdot\boldsymbol{v} = u_l \boldsymbol{e}_l \cdot v_m \boldsymbol{e}_m = u_l v_l = \langle u\rangle\{v\} = \langle v\rangle\{u\} \tag{3-11}$$

Note that there are no free indices.

The cross product of two vectors is given by

$$\begin{aligned} &\boldsymbol{w} = \boldsymbol{u}\times\boldsymbol{v} && \text{Direct notation} \\ &\boldsymbol{w} = u_r\boldsymbol{e}_r \times v_s\boldsymbol{e}_s = \varepsilon_{rst}u_r v_s \boldsymbol{e}_t = w_t\boldsymbol{e}_t \\ &w_t = \varepsilon_{rst}u_r v_s && \text{Indicial notation} \end{aligned} \tag{3-12}$$

3.4 Multiple Bases

It is not unusual to use three or four bases simultaneously in a real problem. Consider a complex structure. You set up one general basis, $\boldsymbol{e}_i$, that you call the global basis. One part of your structure is a thick rectangular plate with sides not parallel to your global basis. You set up a basis, $\boldsymbol{E}_A$, with base vectors parallel to the plate sides. The plate is composed of a fiber-reinforced material with fibers at an angle to the plate sides so for easy interpretation of the constitutive equation you set up a third basis, $\boldsymbol{g}_\alpha$. And then when the plate starts to fail it is convenient to have a basis with one base vector perpendicular to the failure surface and two base vectors tangent to the failure surface labeled $\boldsymbol{f}_r$.

Any vector is represented similarly in terms of components and any one of these bases. Therefore any one vector can have multiple components. To keep these components straight we use an identifier as follows:

$$\boldsymbol{v} = v_i^e\boldsymbol{e}_i = v_A^E\boldsymbol{E}_A = v_\alpha^g\boldsymbol{g}_\alpha = v_r^f\boldsymbol{f}_r \tag{3-13}$$

or

$$\boldsymbol{v} = \langle v^e\rangle\{\boldsymbol{e}\} = \langle v^E\rangle\{\boldsymbol{E}\} = \langle v^g\rangle\{\boldsymbol{g}\} = \langle v^f\rangle\{\boldsymbol{f}\} \tag{3-14}$$

We note that the single physical vector $\boldsymbol{v}$ has multiple representations of mathematical vectors; namely $\{v^e\}$, $\{v^E\}$, $\{v^g\}$, and $\{v^f\}$. This illustrates the power of using direct notation. Direct notation also forces us to think in terms of what the vector means, i.e., velocity. Then when we have to solve a specific problem we get into the details of components and base vectors.

The magnitude squared of the vector is

$$|\boldsymbol{v}|^2 = \boldsymbol{v}\cdot\boldsymbol{v} = v_j^e v_j^e = v_A^E v_A^E = v_\alpha^g v_\alpha^g = v_s^f v_s^f \tag{3-15}$$

This result is called an **invariant** of the vector because it is a unique scalar formed from the components of the vector irrespective of what basis is used. The fact that a scalar is obtained from the direct notation form $|\boldsymbol{v}|^2 = \boldsymbol{v}\cdot\boldsymbol{v}$ indicates immediately that the result is an invariant.

We note from (3-13) that

$$v_i^e = \boldsymbol{v}\cdot\boldsymbol{e}_i \qquad v_r^g = \boldsymbol{v}\cdot\boldsymbol{g}_r \qquad \text{etc.} \tag{3-16}$$

which is just an application of (3-10).

3.5 Transformation from One Basis to Another

Suppose for the moment that we wish to express one set of base vectors, $\boldsymbol{E}_i$, in terms of another set, $\boldsymbol{e}_i$. Each $\boldsymbol{E}_i$ is a vector that can be represented in terms of components in the $\boldsymbol{e}_i$ basis. We construct the relationship as follows:

$$\begin{aligned}
\boldsymbol{E}_1 &= {}^E a_{11}^e \boldsymbol{e}_1 + {}^E a_{12}^e \boldsymbol{e}_2 + {}^E a_{13}^e \boldsymbol{e}_3 = \left\langle {}^E a^e \right\rangle_1 \{\boldsymbol{e}\} \\
\boldsymbol{E}_2 &= {}^E a_{21}^e \boldsymbol{e}_1 + {}^E a_{22}^e \boldsymbol{e}_2 + {}^E a_{23}^e \boldsymbol{e}_3 = \left\langle {}^E a^e \right\rangle_2 \{\boldsymbol{e}\} \\
\boldsymbol{E}_3 &= {}^E a_{31}^e \boldsymbol{e}_1 + {}^E a_{32}^e \boldsymbol{e}_2 + {}^E a_{33}^e \boldsymbol{e}_3 = \left\langle {}^E a^e \right\rangle_3 \{\boldsymbol{e}\}
\end{aligned} \tag{3-17}$$

We combine the three equations into one matrix or indicial equation as follows:

$$\{\boldsymbol{E}\} = \left[{}^E a^e\right]\{\boldsymbol{e}\} \qquad \boldsymbol{E}_i = {}^E a_{ij}^e \boldsymbol{e}_j \tag{3-18}$$

where

$$\left[{}^E a^e\right] = \begin{bmatrix} \left\langle {}^E a^e \right\rangle_1 \\ \left\langle {}^E a^e \right\rangle_2 \\ \left\langle {}^E a^e \right\rangle_3 \end{bmatrix} \tag{3-19}$$

Take an outer dot product of the first of (3-18) to obtain

$$\{\boldsymbol{E}\}\cdot\langle\boldsymbol{e}\rangle = \left[{}^E a^e\right]\{\boldsymbol{e}\}\cdot\langle\boldsymbol{e}\rangle = \left[{}^E a^e\right][I] = \left[{}^E a^e\right] \tag{3-20}$$

or

$$\left[{}^E a^e\right] = \begin{bmatrix} \boldsymbol{E}_1\cdot\boldsymbol{e}_1 & \boldsymbol{E}_1\cdot\boldsymbol{e}_2 & \boldsymbol{E}_1\cdot\boldsymbol{e}_3 \\ \boldsymbol{E}_2\cdot\boldsymbol{e}_1 & \boldsymbol{E}_2\cdot\boldsymbol{e}_2 & \boldsymbol{E}_2\cdot\boldsymbol{e}_3 \\ \boldsymbol{E}_3\cdot\boldsymbol{e}_1 & \boldsymbol{E}_3\cdot\boldsymbol{e}_2 & \boldsymbol{E}_3\cdot\boldsymbol{e}_3 \end{bmatrix} \qquad {}^E a_{ij}^e = \boldsymbol{E}_i\cdot\boldsymbol{e}_j \tag{3-21}$$

The matrix $\left[{}^E a^e\right]$ is called the transformation matrix that expresses the base vectors $\boldsymbol{E}_i$ in terms of the base vectors $\boldsymbol{e}_j$. The prefix "E" and postscript "e" are used to specify the base vectors and the order of this transformation. The transformation the other way would be given as follows:

$$\{\boldsymbol{e}\} = \left[{}^e a^E\right]\{\boldsymbol{E}\} \qquad \boldsymbol{e}_i = {}^e a_{ij}^E \boldsymbol{E}_j \qquad {}^e a_{ij}^E = \boldsymbol{e}_i\cdot\boldsymbol{E}_j \tag{3-22}$$

By using the concept of a matrix inverse, we see that

$$\left[{}^{e}a^{E}\right]=\left[{}^{E}a^{e}\right]^{-1} \qquad \left[{}^{E}a^{e}\right]=\left[{}^{e}a^{E}\right]^{-1} \tag{3-23}$$

Therefore if we construct one of these matrices, the other one is obtained by taking an inverse, not a trivial task.

Next we show that when the two bases are orthonormal the transformation matrix is **orthogonal**. Consider the first two equations of (3-17) and use the identities $\boldsymbol{E}_1 \cdot \boldsymbol{E}_1 = 1$ and $\boldsymbol{E}_1 \cdot \boldsymbol{E}_2 = 0$ to show that

$$\begin{aligned}
\boldsymbol{E}_1 \cdot \boldsymbol{E}_1 &= \langle {}^{E}a^{e} \rangle_1 \{\boldsymbol{e}\} \cdot \langle \boldsymbol{e} \rangle \left\{ \langle {}^{E}a^{e} \rangle_1^T \right\} = \langle {}^{E}a^{e} \rangle_1 [I] \left\{ \langle {}^{E}a^{e} \rangle_1^T \right\} = \langle {}^{E}a^{e} \rangle_1 \left\{ \langle {}^{E}a^{e} \rangle_1^T \right\} = 1 \\
\boldsymbol{E}_1 \cdot \boldsymbol{E}_2 &= \langle {}^{E}a^{e} \rangle_1 \{\boldsymbol{e}\} \cdot \langle \boldsymbol{e} \rangle \left\{ \langle {}^{E}a^{e} \rangle_2^T \right\} = \langle {}^{E}a^{e} \rangle_1 [I] \left\{ \langle {}^{E}a^{e} \rangle_2^T \right\} = \langle {}^{E}a^{e} \rangle_1 \left\{ \langle {}^{E}a^{e} \rangle_2^T \right\} = 0
\end{aligned} \tag{3-24}$$

Similar results hold for the other combination of dot products for the basis $\boldsymbol{E}_i$. Now we take the product of $\left[{}^{E}a^{e}\right]$ with its transpose, and use the results of (3-24), and its extension for the remaining combinations of base vectors to obtain

$$\left[{}^{E}a^{e}\right]\left[{}^{E}a^{e}\right]^T = \begin{bmatrix} \langle {}^{E}a^{e} \rangle_1 \\ \langle {}^{E}a^{e} \rangle_2 \\ \langle {}^{E}a^{e} \rangle_3 \end{bmatrix} \begin{bmatrix} \left\{ \langle {}^{E}a^{e} \rangle_1^T \right\} & \left\{ \langle {}^{E}a^{e} \rangle_2^T \right\} & \left\{ \langle {}^{E}a^{e} \rangle_3^T \right\} \end{bmatrix} = [I] \tag{3-25}$$

This result, together with (3-23), shows that

$$\left[{}^{e}a^{E}\right]=\left[{}^{E}a^{e}\right]^{-1}=\left[{}^{E}a^{e}\right]^{T} \qquad \left[{}^{E}a^{e}\right]=\left[{}^{e}a^{E}\right]^{-1}=\left[{}^{e}a^{E}\right]^{T} \tag{3-26}$$

Once a matrix is obtained that transforms one orthonormal basis to another orthonormal basis, the reverse transformation is obtained by merely taking the transpose. Stated differently, when the two bases are orthonormal, the transformation matrix is orthogonal and this implies that the inverse of the matrix equals the transpose.

If we take the determinant of the first and last terms in (3-25), and use a result from the previous chapter, we obtain

$$\left| \left[{}^{E}a^{e}\right]\left[{}^{E}a^{e}\right]^T \right| = \left\| I \right\| \qquad \left\| {}^{E}a^{e} \right\|^2 = 1 \qquad \left\| {}^{E}a^{e} \right\| = \pm 1 \tag{3-27}$$

Without proof, if both bases are orthonormal, the determinant of the transformation matrix is positive one.

3.6 A Sequence of Transformations

Suppose we have three bases labeled $\boldsymbol{e}_i, \boldsymbol{E}_i$ and $\boldsymbol{g}_i$ and we have the transformation matrices $\left[{}^{e}a^{E}\right]$ and $\left[{}^{E}a^{g}\right]$, but we require $\left[{}^{e}a^{g}\right]$. We simply note that

$$\{\boldsymbol{e}\}=\left[{}^{e}a^{E}\right]\{\boldsymbol{E}\}=\left[{}^{e}a^{E}\right]\left[{}^{E}a^{g}\right]\{\boldsymbol{g}\}=\left[{}^{e}a^{g}\right]\{\boldsymbol{g}\} \tag{3-28}$$

or

$$\left[{}^{e}a^{g}\right]=\left[{}^{e}a^{E}\right]\left[{}^{E}a^{g}\right] \qquad (3\text{-}29)$$

Note that the first and last superscripts "e" and "g" are the same on both sides of the equation to indicate that the transformation provides $\{\boldsymbol{e}\}$ in terms of $\{\boldsymbol{g}\}$. The intermediate subscripts "E" and "E" are no longer needed for the final transformation matrix.

3.7 Examples of Transformation Matrices

Suppose the $\boldsymbol{E}_i$ basis is obtained by a rotation β about $\boldsymbol{g}_2$ with $\boldsymbol{E}_2=\boldsymbol{g}_2$, and the $\boldsymbol{e}_i$ basis is obtained by a rotation α about $\boldsymbol{E}_3$ with $\boldsymbol{e}_3=\boldsymbol{E}_3$ as sketched in Fig. 3-2. The result of using (3-21) is that

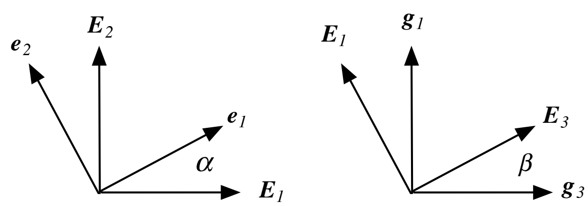

Fig. 3-2. Examples of transformations between base vectors

$$\left[{}^{e}a^{E}\right]=\begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \left[{}^{E}a^{g}\right]=\begin{bmatrix} \cos\beta & 0 & -\sin\beta \\ 0 & 1 & 0 \\ \sin\beta & 0 & \cos\beta \end{bmatrix} \qquad (3\text{-}30)$$

and

$$\begin{aligned}\left[{}^{e}a^{g}\right]&=\begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \cos\beta & 0 & -\sin\beta \\ 0 & 1 & 0 \\ \sin\beta & 0 & \cos\beta \end{bmatrix} \\ &=\begin{bmatrix} \cos\alpha\cos\beta & \sin\alpha & -\cos\alpha\sin\beta \\ -\sin\alpha\cos\beta & \cos\alpha & \sin\alpha\sin\beta \\ \sin\beta & 0 & \cos\beta \end{bmatrix}\end{aligned} \qquad (3\text{-}31)$$

Note that the final matrix is orthogonal as it should be and that the determinants of all transformation matrices are one.

3.8 Transformation of Vector Components

Recall from (3-13) that a vector is expressed in terms of components and base vectors as follows:

$$\boldsymbol{v} = v_i^e \boldsymbol{e}_i = v_A^E \boldsymbol{E}_A = v_\alpha^g \boldsymbol{g}_\alpha = v_r^f \boldsymbol{f}_r \tag{3-32}$$

Suppose the components v_i^e are known and we wish to obtain v_i^E. We simply use the transformation matrix relating the two bases

$$v_i^e \boldsymbol{e}_i = v_i^e \, {}^e a_{ij}^E \boldsymbol{E}_j = v_j^E \boldsymbol{E}_j \tag{3-33}$$

to obtain

$$v_j^E = v_i^e \, {}^e a_{ij}^E \qquad \langle v^E \rangle = \langle v^e \rangle \left[{}^e a^E \right] \tag{3-34}$$

Note that the repeated superscripts “e” are together and the superscript “E” appears singly on each side, in analogy with the summation index “i” and the free index “j”. Take the transpose to get

$$\left\{ v^E \right\} = \left[{}^e a^E \right]^T \left\{ v^e \right\} = \left[{}^E a^e \right] \left\{ v^e \right\} \qquad v_r^E = {}^E a_{rs}^e v_s^e \tag{3-35}$$

Note that the superscripts “e” are adjacent to each other again. This check is useful in determining if the transformation relations are being used in a consistent manner.

The transformation relations for components between any pair of orthonormal bases follow the same pattern.

3.9 The Use of Base Vectors that are not Orthonormal

Suppose $\boldsymbol{E}_i$ is a given orthonormal basis, and $\boldsymbol{e}_i^*$ is an orthogonal basis but with $\boldsymbol{e}_1^* \cdot \boldsymbol{e}_1^* = 1, \boldsymbol{e}_2^* \cdot \boldsymbol{e}_2^* = 4, \boldsymbol{e}_3^* \cdot \boldsymbol{e}_3^* = 1$ and $\boldsymbol{e}_3^* = \boldsymbol{E}_3$ as shown in Fig. 3-3. The transformation matrix is

$$\left[{}^{e*} a^E \right] = \begin{bmatrix} \cos\alpha & 2\sin\alpha & 0 \\ -2\sin\alpha & 2\cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{3-36}$$

We see immediately that the matrix is not orthogonal so that the matrix $\left[{}^E a^{e*} \right]$ cannot be obtained by taking the transpose. We also note that $v_2^{e*} \neq \boldsymbol{v} \cdot \boldsymbol{e}_2^*$.

Next we consider the case of normal but not orthogonal basis such as $\boldsymbol{e}_i^*$ defined by $\boldsymbol{e}_1^* = \boldsymbol{E}_1, \boldsymbol{e}_2^* = \boldsymbol{E}_1 \cos\gamma + \boldsymbol{E}_2 \sin\gamma$ and $\boldsymbol{e}_3^* = \boldsymbol{E}_3$ and shown in Fig. 3-4. The transformation matrix to transfer from $\boldsymbol{g}_i^*$ to $\boldsymbol{E}_i$ is

$$\left[{}^{e*} a^E \right] = \begin{bmatrix} 1 & 0 & 0 \\ \cos\gamma & \sin\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{3-37}$$

Again we see that the matrix is not orthogonal.

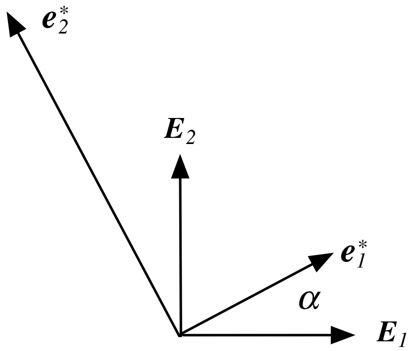

Fig. 3-3. Orthogonal but non-normal basis.

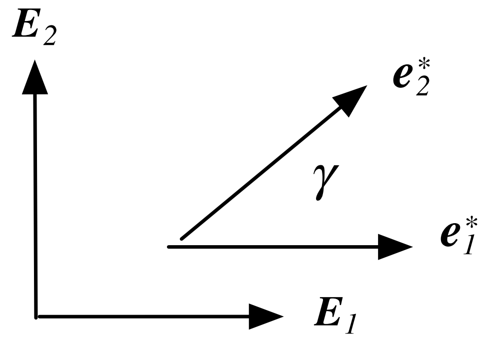

Fig. 3-4. Normal but non-orthogonal basis.

3.10 General Non-Orthonormal Bases

Here we look ahead to the situation and approach that is used with general curvilinear coordinates in which a basis that is neither orthogonal nor normal arises. Let this general basis be labeled $\boldsymbol{g}_i^*$ with the asterisk used here to emphasize that this not our conventional orthonormal basis. This basis is called covariant. It turns out to be especially convenient in connection with the gradient operator to introduce another general basis, called the contravariant (indices are now superscript) basis, or conjugate basis, $\boldsymbol{g}^{*i}$ with the property that

$$\boldsymbol{g}^{*i} \cdot \boldsymbol{g}_j^* = \delta_j^i \qquad \text{Kronecker Delta} \tag{3-38}$$

A two-dimensional representation is given in Fig. 3-5 in which it is assumed that $\boldsymbol{g}^{*3} = \boldsymbol{g}_3^*$, a unit vector perpendicular to the other four base vectors. Now we have two representations for a vector

$$\boldsymbol{v} = v^{*i}\boldsymbol{g}_i^* = v_i^*\boldsymbol{g}^{*i} \tag{3-39}$$

Here summation can only occur over indices when one is a subscript and one is a superscript, and free indices must be at the same level. It follows from (3-38) and (3-39) that we have the following convenient relations for extracting components from a given vector:

$$v^{*i} = \boldsymbol{v} \cdot \boldsymbol{g}^{*i} \qquad v_i^* = \boldsymbol{v} \cdot \boldsymbol{g}_i^* \tag{3-40}$$

If we define

$$g^{*ij} = \boldsymbol{g}^{*i} \cdot \boldsymbol{g}^{*j} \qquad g_{ij}^* = \boldsymbol{g}_i^* \cdot \boldsymbol{g}_j^* \tag{3-41}$$

then it follows that

$$v_i^* = g_{ij}^* v^{*j} \qquad v^{*i} = g^{*ij} v_j^* \qquad g_{ij}^* g^{*jk} = \delta_i^k \tag{3-42}$$

In addition there is the task of converting either one of these bases to an orthonormal basis if the problem requires the transformation.

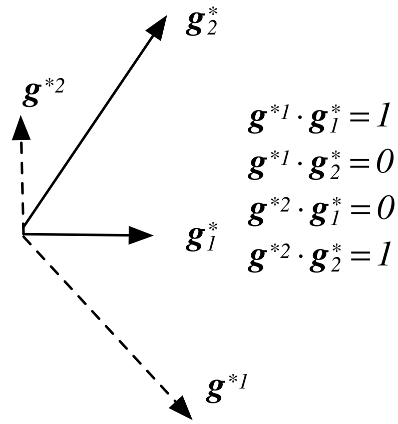

Fig. 3-5. Covariant and contravariant base vectors in two dimensions.

3.11 Closing Comments

The focus of this chapter has been on the concept of orthonormal bases and the transformation relations between these bases. Once the transformation matrices have been constructed, the equations for transforming components of a vector are obtained easily. The same will be true when we develop the analogous equations for transforming components of tensors. It is essential to be able to think in terms of having to use several bases at the same time.

The purpose of including Subsection 3.10 is to again indicate how special orthonormal bases are and to provide a small introduction to a notation that is widely used in general tensor analysis. However, if we prove a theorem holds in direct notation, we bypass the complications that can arise with the use of a general arbitrary basis.