Calculus II (part 4): Differential Equations (by Evan Dummit, 2012, v. 1.10)

Contents

1 Differential Equations

- Differential equations crop in every area of anything that can be described mathematically. In every branch of science, from physics to chemistry to biology, as well as other fields such as engineering, economics, and demography, virtually any interesting kind of process is modeled by a differential equation or a system of differential equations.
- Morally, the reason for this is that most anything interesting involves change of some kind, and thus rates of change – in the guise of a growth rate for a population, or the velocity and acceleration of a physical object, or the diffusion rates of molecules involved in a reaction, or rates of quantities in economic processes.
- In general, a differential equation is merely an equation involving a derivative (or several derivatives) of a function or functions.
  - Examples: $y' + y = 0$, or $y'' + 2y' + y = 3x^2$, or $f'' \cdot f = (f')^2$, or $f' + g' = x^3$, or $\dfrac{df}{ds} + \dfrac{df}{dt} = s + t$.
  - "Most" differential equations are difficult if not impossible to find exact solutions to, in the same way that 'most' random integrals or infinite series are hard to evaluate exactly.
  - In this course we will only cover how to solve a few basic types of equations: (first-order) separable equations, first-order linear equations, and second-order linear equations with constant coefficients.
- Sometimes we will be looking for every function which satisfies some particular equation: e.g., for all functions such that $y' + y = 0$. Other times we will be looking for a particular function, subject to some additional conditions – e.g., a function y such that $y' + y = 0$ and $y(0) = 10$. We will discuss both types of problems.

1.1 Terminology

- If a differential equation involves functions of only a single variable (i.e., if y is a function only of x) then it is called an ordinary differential equation (or ODE).
  - We will only talk about ODEs in this course, since we don't know how to differentiate functions of more than one variable.
  - For completeness, differential equations involving functions of several variables are called partial differential equations, or PDEs. (Derivatives of functions of more than one variable are called partial derivatives, hence the name.)

- The standard form of a differential equation is when it is written with all terms involving y or higher derivatives on one side, and functions of the variable on the other side.
  - Example: The equation $y'' + y' + y = 0$ is in standard form.
  - Example: The equation $y' = 3x^2 - xy$ is not in standard form.
- An equation is homogeneous if, when it is put into standard form, the x-side is zero. An equation is nonhomogeneous otherwise.
  - Example: The equation $y'' + y' + y = 0$ is homogeneous.
  - Example: The equation $y' + xy = 3x^2$ is nonhomogeneous.
- An nth order differential equation is an equation in which the highest derivative is the nth derivative.
  - Example: The equations $y' + xy = 3x^2$ and $y' \cdot y = 2$ are first-order.
  - Example: The equation $y'' + y' + y = 0$ is second-order.
- A differential equation is linear if it is linear in the y terms. In other words, if there are no terms like y^2, or $(y')^3$, or $y \cdot y'$.
  - Example: The equations $y' + xy = 3x^2$ and $y'' + y' + y = 0$ are linear.
  - Example: The equation $y' \cdot y = 3x^2$ is not linear.
- We say a linear differential equation has constant coefficients if the coefficients of y, y', y'', ... are all constants.
  - Example: The equation $y'' + y' + y = 0$ has constant coefficients.
  - Example: The equation $y' + xy = 3x^2$ does not have constant coefficients.
- Theorem: If y_0 and y_1 both satisfy the same linear homogeneous differential equation (of any order), then $C_0 \cdot y_0 + C_1 \cdot y_1$ will also satisfy the equation, for any constants C_0 and C_1.
  - This is just a formal check. For example, if the differential equation is $y'' + y = 0$, then $[C_0 y_0 + C_1 y_1]'' + [C_0 y_0 + C_1 y_1] = C_0[y_0'' + y_0] + C_1[y_1'' + y_1] = C_0[0] + C_1[0] = 0$.

1.2 Some Motivating Applications

- Simple motivating example: A population (unrestricted by space or resources) tends to grow at a rate proportional to its size. [Reason: imagine each male pairing off with a female and having a fixed number of offspring each year.]
  - In symbols, this means that $\frac{dP}{dt} = k \cdot P$, where $P(t)$ is the population at time t and k is the growth rate. This is a homogeneous first-order linear differential equation with constant coefficients.
  - It's not hard to see that one population model that works is $P(t) = e^{k \cdot t}$ – hence, "exponential growth".
- More complicated example: The Happy Sunshine Valley is home to Cute Mice and Adorable Kittens. The Cute Mice grow at a rate proportional to their population, minus the number of Mice that are eaten by their predators, the Kittens. The population of Adorable Kittens grows proportional to the number of mice (since they have to catch Mice to survive and reproduce).
  - In symbols this means $\frac{dM}{dt} = k_1 \cdot M - k_2 \cdot K$, and $\frac{dK}{dt} = k_3 \cdot M$, where $M(t)$ and $K(t)$ are the populations of Mice and Kittens, and k_1, k_2, k_3 are some constants.
  - Now it's a lot harder to see what a solution to this system could be. (We won't explicitly learn how to solve a system like this, but it can be converted to a single second-order linear equation, which can then be solved using the methods of this course.)

- The conditions here are not particularly unnatural for a simple predator-prey system. But in general, there could be non-linear terms too – perhaps when two Kittens meet, they fight with each other and cause injury, which might change the equation to $\frac{dK}{dt} = k_3 \cdot M - k_4 \cdot K^2$.
- Now imagine trying to model even a 'small' ecosystem with 5 species, each of which interacts with all of the others.

- Higher-order example: A simple pendulum consists of a weight suspended on a string, with gravity the only force acting on the weight. If θ is the angle the pendulum's string makes with a vertical line, then horizontal force on the weight toward the vertical is proportional to $\sin(\theta)$.
  - In symbols, this means that $\frac{d^2\theta}{dt^2} = -k \cdot \sin(\theta)$. This is a non-linear second-order differential equation.
  - This equation cannot be solved exactly for the function $\theta(t)$. However, a reasonably good approximation can be found by using $\sin(\theta) \approx \theta$.

1.3 First-Order: Separable

- A separable equation is of the form $y' = f(x) \cdot g(y)$ for some functions $f(x)$ and $g(y)$, or an equation equivalent to something of this form.
- Here is the method for solving such equations:
  - Step 1: Replace y' with $\frac{dy}{dx}$, and then cross-multiply as necessary to get all the y-stuff on one side (including dy) and all of the x-stuff on the other (including dx).
  - Step 2: Stick an indefinite integral sign on both sides.
  - Step 3: Evaluate both integrals. Don't forget to put the $+C$ on the side with the x-terms.
  - Step 4: If given, plug in the initial condition to solve for the constant C. (Otherwise, just leave it where it is.)
  - Step 5: Solve for y as a function of x, if possible.
- Example: Solve $y' = k \cdot y$, where k is some constant.
  - Step 1: Rewrite as $\frac{dy}{y} = k\,dx$.
  - Step 2: Stick integral signs to get $\int \frac{dy}{y} = \int k\,dx$.
  - Step 3: Evaluate to get $\ln(y) = kx + C$.
  - Step 5: Exponentiate to get $y = \boxed{e^{kx+C} = D \cdot e^{kx}}$.
- Example: Solve the differential equation $\frac{dy}{dx} = xy$ given the initial condition $y(2) = 4$.
  - Step 1: Rewrite as $\frac{dy}{y} = x\,dx$.
  - Step 2: Stick integral signs to get $\int \frac{dy}{y} = \int x\,dx$.
  - Step 3: Evaluate to get $\ln(y) = \frac{1}{2}x^2 + C$.
  - Step 4: Plug in initial condition to get $\ln(4) = \frac{1}{2} \cdot 2^2 + C$ hence $C = \ln(4) - 4$.
  - Step 5: So $\ln(y) = \frac{1}{2}x^2 + \ln(4) - 4$. Exponentiating gives $y = \boxed{e^{\frac{1}{2}x^2 - 2 + \ln(4)} = 4 \cdot e^{\frac{1}{2}x^2 - 2}}$.
- Example: Solve the differential equation $y' = e^{x-y}$.

- Step 1: Rewrite as $e^y\,dy = e^x\,dx$.
- Step 2: Stick integral signs to get $\int e^y\,dy = \int e^x\,dx$.
- Step 3: Evaluate to get $e^y = e^x + C$.
- Step 5: Take the natural logarithm to get $y = \boxed{\ln(e^x + C)}$.

- Example: Find y given that $y' = x + xy^2$ and $y(0) = 1$.
  - Step 1: Rewrite as $\dfrac{dy}{1+y^2} = x\,dx$.
  - Step 2: Stick integral signs to get $\int \dfrac{dy}{1+y^2} = \int x\,dx$.
  - Step 3: Evaluate to get $\tan^{-1}(y) = \dfrac{1}{2}x^2 + C$.
  - Step 4: Plug in the initial condition to get $\tan^{-1}(1) = C$ hence $C = \pi/4$.
  - Step 5: Take the natural logarithm to get $y = \boxed{\tan\left(\dfrac{1}{2}x^2 + \dfrac{\pi}{4}\right)}$.

1.4 First-Order: Linear

- The general form for a first-order linear differential equation is (upon dividing by the coefficient of y') given by $y' + P(x) \cdot y = Q(x)$, where $P(x)$ and $Q(x)$ are some functions of x.
- To solve this equation we use an "integrating factor": we multiply by something which will turn the left-hand side into the derivative of a single function. Here is the general method:
  - Step 1: Put the equation into the form $y' + P(x) \cdot y = Q(x)$.
  - Step 2: Multiply both sides by the integrating factor $e^{\int P(x)\,dx}$ to get $e^{\int P(x)\,dx}y' + e^{\int P(x)\,dx}P(x) \cdot y = e^{\int P(x)\,dx}Q(x)$.
  - Step 3: Observe that the right-hand side is $\dfrac{d}{dx}\left[e^{\int P(x)\,dx} \cdot y\right]$, and take the antiderivative on both sides. Don't forget the constant of integration C.
  - Step 4: If given, plug in the initial condition to solve for the constant C. (Otherwise, just leave it where it is.)
  - Step 5: Solve for y as a function of x, if possible.
- Example: Find y given that $y' + 2xy = x$ and $y(0) = 1$.
  - Step 1: We have $P(x) = 2x$ and $Q(x) = x$.
  - Step 2: Multiply both sides by $e^{\int P(x)\,dx} = e^{x^2}$ to get $e^{x^2}y' + e^{x^2} \cdot 2x \cdot y = x \cdot e^{x^2}$.
  - Step 3: Taking the antiderivative on both sides yields $e^{x^2}y = \dfrac{1}{2}e^{x^2} + C$.
  - Step 4: Plugging in yields $e^0 \cdot 1 = \dfrac{1}{2}e^0 + C$ hence $C = \dfrac{1}{2}$.
  - Step 5: Solving for y gives $y = \boxed{\dfrac{1}{2} + \dfrac{1}{2}e^{-x^2}}$.
- Example: Find all functions y for which $xy' = x^4 - 4y$.
  - Step 1: We have $y' + \dfrac{4}{x}y = x^3$, so $P(x) = \dfrac{4}{x}$ and $Q(x) = x^3$.
  - Step 2: Multiply both sides by $e^{\int P(x)\,dx} = e^{4\ln(x)} = x^4$ to get $x^4y' + 4x^3y = x^7$,
  - Step 3: Taking the antiderivative on both sides yields $x^4y = \dfrac{1}{8}x^8 + C$.

- Step 5: Solving for y gives $y = \boxed{\frac{1}{8}x^4 + C \cdot x^{-4}}$.

- Example: Find y given that $y' \cdot \cot(x) = y + 2\cos(x)$ and $y(0) = -\frac{1}{2}$.
  - Step 1: We have $y' - y\,\tan(x) = 2\sin(x)$, with $P(x) = -\tan(x)$ and $Q(x) = 2\sin(x)$.
  - Step 2: Multiply both sides by $e^{\int P(x)\,dx} = e^{\ln(\cos(x))} = \cos(x)$ to get $y' \cdot \cos(x) - y \cdot \sin(x) = 2\sin(x)\cos(x)$.
  - Step 3: Taking the antiderivative on both sides yields $[y \cdot \cos(x)] = -\frac{1}{2}\cos(2x) + C$.
  - Step 4: Plugging in yields $-\frac{1}{2} = -\frac{1}{2} \cdot 1 + C$ hence $C = 0$.
  - Step 5: Solving for y gives $y = \boxed{-\frac{\cos(2x)}{2\cos(x)}}$.

1.5 Second-Order: Linear, Constant Coefficients, Homogeneous

- The general second-order linear homogeneous differential equation with constant coefficients is of the form $ay'' + by' + cy = 0$, where a, b, and c are some constants.
- Theorem: There exist two linearly-independent functions $y_1(x)$ and $y_2(x)$ such that every solution to the equation $ay'' + by' + cy = 0$ is of the form $C_1 y_1 + C_2 y_2$ for some constants C_1 and C_2.
  - The term "linearly-independent" means that $y_1(x)$ is not a constant multiple of $y_2(x)$, as a function of x. For example, 1 and x are linearly independent, as are x^3 and e^x, but e^x and $\pi \cdot e^x$ are not.
  - In essence this theorem says that there are "two" different solutions to this second-order equation, and all other solutions are just a simple combination of these two.
  - The proof of this theorem is more advanced than the things we will cover in this course. (But you'll learn about it if you take a course on differential equations or linear algebra.)
- Based on solving first-order linear homogeneous equations (like $y' = ky$), we expect the solutions to involve exponentials. If we try setting $y = e^{rx}$ then after some arithmetic we end up with $ar^2e^{rx} + bre^{rx} + ce^{rx} = 0$. Multiplying both sides by e^{-rx} and cancelling yields the characteristic equation $\boxed{ar^2 + br + c = 0}$. So if we can find two values of r satisfying this much easier quadratic equation – i.e., by using the quadratic formula which says we get $r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ – we will get solutions to the original differential equation.
- There are three kinds of behavior to the values of r we get, based on the discriminant $D = b^2 - 4ac$ of the quadratic:
  - Case $D > 0$. In this case we get the two unequal real numbers $r_1 = \frac{-b - \sqrt{D}}{2a}$ and $r_2 = \frac{-b + \sqrt{D}}{2a}$, and the general solution is $\boxed{y = C_1 e^{r_1 x} + C_2 e^{r_2 x}}$.
  - Case $D = 0$. In this case we're a little bit sad because both roots are equal – so we only get one value $r = \frac{-b}{2a}$ yielding the solution $y = e^{rx}$. We know there must be another solution, and (based on what we see occurs in the simple example of $y'' = 0$) we can go off and check that $y = xe^{rx}$ also works. Therefore we have a general solution of $\boxed{y = C_1 e^{rx} + C_2 x e^{rx}}$.
  - Case $D < 0$. In this case we get two complex conjugate values of r, namely $r_1 = \alpha + \beta i$ and $r_2 = \alpha - \beta i$ with $\alpha = -\frac{b}{2a}$ and $\beta = \frac{\sqrt{|D|}}{2a}$. As in case 1, we could just write down $e^{r_1 x}$ and $e^{r_2 x}$ as our solutions, but we really want real-valued solutions, and $e^{r_1 x}$ and $e^{r_2 x}$ have complex numbers in the exponents. However we can just write out the real and imaginary parts and verify that we can write $e^{\alpha x}\sin(\beta x) = \frac{1}{2i}\left[e^{r_1 x} - e^{r_2 x}\right]$ and $e^{\alpha x}\cos(\beta x) = \frac{1}{2}\left[e^{r_1 x} + e^{r_2 x}\right]$. Therefore the general solution in this case is $\boxed{y = C_1 e^{\alpha x}\sin(\beta x) + C_2 e^{\alpha x}\cos(\beta x)}$.

- Therefore, to solve the general second-order linear homogeneous differential equation with constant coefficients, follow these steps:
  - Step 1: Rewrite the differential equation in the form $ay'' + by' + cy = 0$.
  - Step 2: Solve the characteristic equation $ar^2 + br + c = 0$.
  - Step 3: Determine which of the 3 cases applies, and write down the general solution.
  - Step 4: If given additional conditions, solve for the constants C_1 and C_2 using the conditions.

- Example: Find all functions y such that $y'' + y' - 6 = 0$.
  - Step 2: The characteristic equation is $r^2 + r - 6 = 0$ which has roots $r = 2$ and $r = -3$.
  - Step 3: We are in Case $D > 0$ since there are unequal real roots. So the general solution is $\boxed{y = C_1 e^{2x} + C_2 e^{-3x}}$.

- Example: Find all functions y such that $y'' - 2y' + 1 = 0$, with $y(0) = 1$ and $y'(0) = 2$.
  - Step 2: The characteristic equation is $r^2 - 2r + 1 = 0$ which has the single root $r = 1$.
  - Step 3: We are in Case $D = 0$ since there is a repeated real root. So the general solution is $y = C_1 e^x + C_2 x e^x$.
  - Step 4: Plugging in the two conditions gives $1 = C_1 \cdot e^0 + C_2 \cdot 0$, and $2 = C_1 e^0 + C_2 \left[(0+1)e^0\right]$ from which $C_1 = 1$ and $C_2 = 1$. Hence the particular solution requested is $\boxed{y = e^x + xe^x}$.

- Example: Find all real-valued functions y such that $y'' = -4y$.
  - Step 1: The standard form here is $y'' + 4y = 0$.
  - Step 2: The characteristic equation is $r^2 + 4 = 0$ which has roots $r = 2i$ and $r = -2i$.
  - Step 3: We are in Case $D < 0$ since there are two nonreal roots. Since the problem asks for real-valued functions we write $e^{r_1 x} = \cos(2x) + i\sin(2x)$ and $e^{r_2 x} = \cos(2x) - i\sin(2x)$ to see that the general solution is $\boxed{y = C_1 \cos(2x) + C_2 \sin(2x)}$.

1.6 Second-Order: Linear, Constant Coefficients, Non-homogeneous

- The general second-order linear nonhomogeneous differential equation with constant coefficients is of the form $ay'' + by' + cy = R(x)$, where a, b, and c are some constants, and $R(x)$ is some function of x.

- Theorem: If $y_p(x)$ is any solution to $ay'' + by' + cy = R(x)$, then every other solution to $ay'' + by' + cy = R(x)$ is of the form $y_p(x) + y_h(x)$, where y_h is a solution to the homogeneous differential equation $ay'' + by' + cy = 0$.
  - This theorem allows us to find the general solution of $ay'' + by' + cy = R(x)$, simply by finding one solution y_p – the rest are merely this one solution, plus a solution y_h to the homogeneous equation $ay'' + by' + cy = 0$, which we know how to solve already.
  - The proof of the theorem is simply to observe that if y_0 and y_1 are solutions to $ay'' + by' + cy = R(x)$, then $y_0 - y_1$ is a solution to $ay'' + by' + cy = 0$, which follows from simple derivatives rules.

- Thus, by the theorem, all we really need to do is find a single solution to $ay'' + by' + cy = R(x)$.

- There are essentially two ways of doing this – the Method of Undetermined Coefficients (which is really just a fancy way to say "guessing what we think the form of the solution will be and then checking if it works"), and Variation of Parameters.

1.6.1 Undetermined Coefficients

- The idea behind the method of undetermined coefficients is that we can 'guess' what our solution should look like (up to some coefficients we have to solve for), if $R(x)$ involves sums and products of polynomials, exponentials, and trigonometric functions. Specifically, we try a solution $y = $ [stuff], where the 'stuff' is a sum of things similar to the terms in $R(x)$.
- Note that the method of undetermined coefficients really does not care whether the equation is second-order or not, as long as it has constant coefficients. It's possible to use the same ideas to solve differential equations of higher order (as long as they have constant coefficients), too.
- Here is the procedure for generating the trial solution:
  - Step 1: Generate the "first guess" for the trial solution as follows:
    - Replace all numerical coefficients of terms in $R(x)$ with variable coefficients. If there is a sine (or cosine) term, add in the companion cosine (or sine) terms, if they are missing. Then group terms of $R(x)$ into "blocks" of terms which are the same up to a power of x, and add in any missing lower-degree terms in each "block".
    - Thus, if a term of the form $x^n e^{rx}$ appears in $R(x)$, fill in the terms of the form $e^{rx} \cdot [A_0 + A_1 x + \cdots + A_n x^n]$, and if a term of the form $x^n e^{\alpha x} \sin(\beta x)$ or $x^n e^{\alpha x} \cos(\beta x)$ appears in $R(x)$, fill in the terms of the form $e^{ax} \cos(bx) \cdot [D_0 + D_1 x + \cdots + D_n x^n] + e^{ax} \sin(bx)\,[E_0 + E_1 x + \cdots + E_n x^n]$.
  - Step 2: Solve the homogeneous equation, and write down the general solution.
  - Step 3: Compare the "first guess" for the trial solution with the solutions to the homogeneous equation. If any terms overlap, multiply all terms in the overlapping "block" by the appropriate power of x which will remove the duplication.
- Here is a series of examples demonstrating the procedure for generating the trial solution:
  - Example: $y'' - y = x$.
    - Step 1: We fill in the missing constant term in $Q(x)$ to get $D_0 + D_1 x$.
    - Step 2: General solution is $A_1 e^x + A_2 e^{-x}$.
    - Step 3: There is no overlap, so the trial solution is $\boxed{D_0 + D_1 x}$.
  - Example: $y'' + y' = x - 2$.
    - Step 1: We have $D_0 + D_1 x$.
    - Step 2: General homogeneous solution is $A + Be^{-x}$.
    - Step 3: There is an overlap (the solution D_0) so we multiply the corresponding trial solution terms by x, to get $D_0 x + D_1 x^2$. Now there is no overlap, so $\boxed{D_0 x + D_1 x^2}$ is the trial solution.
  - Example: $y'' - y = e^x$.
    - Step 1: We have $D_0 e^x$.
    - Step 2: General homogeneous solution is $Ae^x + Be^{-x}$.
    - Step 3: There is an overlap (the solution $D_0 e^x$) so we multiply the trial solution term by x, to get $D_0 x e^x$. Now there is no overlap, so $\boxed{D_0 x e^x}$ is the trial solution.
  - Example: $y'' - 2y' + y = 3e^x$.
    - Step 1: We have $D_0 e^x$.
    - Step 2: General homogeneous solution is $Ae^x + Bxe^x$.
    - Step 3: There is an overlap (the solution $D_0 e^x$) so we multiply the trial solution term by x^2, to get rid of the overlap, giving us the trial solution $\boxed{D_0 x^2 e^x}$.
  - Example: $y'' - 2y' + y = x^3 e^x$.
    - Step 1: We fill in the lower-degree terms to get $D_0 e^x + D_1 x e^x + D_2 x^2 e^x + D_3 x^3 e^x$.
    - Step 2: The general homogeneous solution is $A_0 e^x + A_1 x e^x$.

    * Step 3: There is an overlap (namely $D_0 e^x + D_1 x e^x$) so we multiply the trial solution terms by x^2 to get $\boxed{D_0 x^2 e^x + D_1 x^3 e^x + D_2 x^4 e^x + D_3 x^5 e^x}$ as the trial solution.

  - Example: $y'' + y = \sin(x)$.
    * Step 1: We fill in the missing cosine term to get $D_0 \cos(x) + E_0 \sin(x)$.
    * Step 2: The general homogeneous solution is $A\cos(x) + B\sin(x)$.
    * Step 3: There is an overlap (all of $D_0 \cos(x) + E_0 \sin(x)$) so we multiply the trial solution terms by x to get $D_0 x \cos(x) + E_0 x \sin(x)$. There is now no overlap so $\boxed{D_0 x \cos(x) + E_0 x \sin(x)}$ is the trial solution.

  - Example: $y'' + y = x\sin(x)$.
    * Step 1: We fill in the missing cosine term and then all the lower-degree terms to get $D_0 \cos(x) + E_0 \sin(x) + D_1 x \cos(x) + E_1 x \sin(x)$.
    * Step 2: The general homogeneous solution is $A\cos(x) + B\sin(x)$.
    * Step 3: There is an overlap (all of $D_0 \cos(x) + E_0 \sin(x)$) so we multiply the trial solution terms in that group by x to get $\boxed{D_0 x \cos(x) + E_0 x \sin(x) + D_1 x^2 \cos(x) + E_1 x^2 \sin(x)}$, which is the trial solution since now there is no overlap.

- Here is a series of examples finding the general trial solution and then solving for the coefficients:
  - Example: Find a function y such that $y'' + y' + y = x$.
    * The procedure produces our trial solution as $y = D_0 + D_1 x$, because there is no overlap with the solutions to the homogeneous equation.
    * We plug in and get $0 + (D_1) + (D_1 x + D_0) = x$, so that $D_1 = 1$ and $D_0 = -1$.
    * So our solution is $\boxed{y = x - 1}$.
  - Example: Find a function y such that $y'' - y = 2e^x$.
    * The procedure gives the trial solution as $y = D_0 x e^x$, since $D_0 e^x$ overlaps with the solution to the homogeneous equation.
    * If $y = D_0 x e^x$ then $y'' = D_0(x+2)e^x$ so plugging in yields $y'' - y = [D_0(x+2)e^x] - [D_1 x\, e^x] = 2e^x$. Solving yields $D_0 = 1$, so our solution is $\boxed{y = xe^x}$.
  - Example: Find a function y such that $y'' - 2y' + y = x + \sin(x)$.
    * The procedure gives the trial solution as $y = (D_0 + D_1 x) + (D_2 \cos(x) + D_3 \sin(x))$, by filling in the missing constant term and cosine term, and because there is no overlap with the solutions to the homogeneous equation.
    * Then we have $y'' = -D_2 \cos(x) - D_3 \sin(x)$ and $y' = D_1 - D_2 \sin(x) + D_3 \cos(x)$ so plugging in yields
    $$y'' - 2y' + y = [-D_2 \cos(x) - D_3 \sin(x)] - 2\,[D_1 - D_2 \sin(x) + D_3 \cos(x)] + [D_0 + D_1 x + D_2 \cos(x) + D_3 \sin(x)]$$
    and setting this equal to $x + \sin(x)$ then requires $D_0 - 2D_1 = 0$, $D_1 = 1$, $D_2 + 2D_3 - D_2 = 1$, $D_3 - 2D_2 - D_3 = 0$, so our solution is $\boxed{y = x + 2 + \frac{1}{2}\cos(x)}$.
  - Example: Find all functions y such that $y'' + y = \sin(x)$.
    * The solutions to the homogeneous system $y'' + y = 0$ are $y = C_1 \cos(x) + C_2 \sin(x)$.
    * Then the procedure gives the trial solution for the non-homogeneous equation as $y = D_0 x \cos(x) + D_1 x \sin(x)$, by filling in the missing cosine term and then multiplying both by x due to the overlap with the solutions to the homogeneous equation.
    * We can compute (eventually) that $y'' = -D_0 x \cos(x) - 2D_0 \sin(x) - D_1 x \sin(x) + 2D_1 \cos(x)$.
    * Plugging in yields $y'' + y = (-D_0 x \cos(x) - 2D_0 \sin(x) - D_1 x \sin(x) + 2D_1 \cos(x)) + (D_0 x \sin(x) + D_1 x \cos(x))$, and so setting this equal to $\sin(x)$, we obtain $D_0 = 0$ and $D_1 = -\frac{1}{2}$.
    * Therefore the set of solutions is $\boxed{y = -\frac{1}{2} x \cos(x) + C_1 \cos(x) + C_2 \sin(x)}$, for constants C_1 and C_2.

1.6.2 Variation of Parameters

- This method requires more thought, but less computation, than the method of undetermined coefficients. However, it will work for a general function $R(x)$. The derivation is not terribly enlightening, so I will just give the steps to follow to solve $ay'' + by' + cy = R(x)$.

  - Step 1: Solve the corresponding homogeneous equation $ay'' + by' + cy = 0$ and find two (linearly independent) solutions y_1 and y_2. Also calculate y_1' and y_2'.
  - Step 2: Look for functions v_1 and v_2 making $y_p = v_1 \cdot y_1 + v_2 \cdot y_2$ a solution to the original equation: do this by requiring v_1' and v_2' to satisfy the two equations

$$\begin{aligned} v_1' \cdot y_1 + v_2' \cdot y_2 &= 0 \\ v_1' \cdot y_1' + v_2' \cdot y_2' &= R(x)/a \end{aligned}$$

    Solve the relations for v_1' and v_2' in any way you would normally solve a system of two linear equations in two variables. (Cramer's Rule will work, or you can just multiply the first equation by y_1', the second by y_1, and subtract.) Or, even easier, plug in to these explicit formulas:

    * $\boxed{v_1' = \begin{vmatrix} 0 & y_2 \\ R(x)/a & y_2' \end{vmatrix} \Big/ \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \dfrac{-y_2 \cdot R(x)/a}{y_1 y_2' - y_1' y_2}}$ and
    * $\boxed{v_2' = \begin{vmatrix} y_1 & 0 \\ y_1' & R(x)/a \end{vmatrix} \Big/ \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \dfrac{+y_1 \cdot R(x)/a}{y_1 y_2' - y_1' y_2}}$.

  - Step 3: Integrate to find v_1 and v_2. (Ignore constants of integration.)
  - Step 4: Write down the particular solution to the nonhomogeneous equation, $\boxed{y_p = v_1 \cdot y_1 + v_2 \cdot y_2}$.
  - Step 5: If asked, add the particular solution to the general solution to the homogeneous equation, to find all solutions of the nonhomogeneous equation. You will end up with $\boxed{y = y_p + C_1 y_1 + C_2 y_2}$. Plug in any extra conditions given to solve for coefficients.

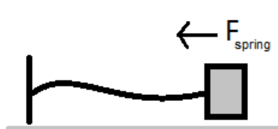
- Example: Find all functions y for which $y'' + y = \sec(x)$.

  - Step 1: The homogeneous equation is $y'' + y = 0$ which has two solutions of $y_1 = \cos(x)$ and $y_2 = \sin(x)$. Observe $y_1' = -\sin(x)$ and $y_2' = \cos(x)$.
  - Step 2: We have $G(x)/a = \sec(x)$. Also we have $y_1 y_2' - y_1' y_2 = \cos(x) \cdot \cos(x) - (-\sin(x)) \cdot \sin(x) = 1$. Thus plugging in to the formulas gives $v_1' = -\sin(x) \cdot \sec(x) = -\tan(x)$ and $v_2' = \cos(x) \cdot \sec(x) = 1$.
  - Step 3: Integrating yields $v_1 = \ln(\cos(x))$ and $v_2 = x$.
  - Step 4: We obtain the particular solution of $y_p = \ln(\cos(x)) \cdot \cos(x) + x \cdot \sin(x)$.
  - Step 5: The general solution is, therefore, given by $\boxed{y = [\ln(\cos(x)) \cdot \cos(x) + x \cdot \sin(x)] + C_1 \sin(x) + C_2 \cos(x)}$.

- Example: Find a function y for which $y'' - y = e^x$.

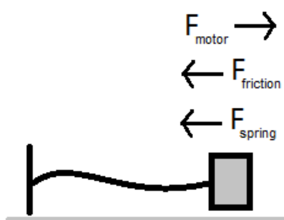
  - We could use undetermined coefficients to solve this – we would end up with $\frac{1}{2} x\, e^x$ – but let's use variation of parameters instead.
  - Step 1: The homogeneous equation is $y'' - y = 0$ which has two solutions of $y_1 = e^{-x}$ and $y_2 = e^x$; then $y_1' = -e^{-x}$ and $y_2' = e^x$.
  - Step 2: We have $G(x)/a = e^x$. Also we have $y_1 y_2' - y_1' y_2 = e^{-x} \cdot (e^x) - (-e^{-x}) e^x = 2$. Thus plugging in to the formulas gives $v_1' = -e^x \cdot e^x/2 = -e^{2x}/2$ and $v_2' = e^{-x} \cdot e^x/2 = 1/2$.
  - Step 3: Integrating yields $v_1 = -e^{2x}/4$ and $v_2 = x/2$.
  - Step 4: We obtain the particular solution of $y_p = e^{-x}(-e^{2x}/4) + e^x(x/2) = \boxed{-\frac{1}{4} e^x + \frac{1}{2} x\, e^x}$.

1.7 Applications to Newtonian Mechanics

- One of the applications we care about in this class is the use of second-order differential equations to solve certain physics problems. Most of the examples involve springs.
- Basic setup: An object is attached to one end of a spring whose other end is fixed. The mass is displaced some amount from the equilibrium position, and the problem is to find the object's position as a function of time.
  - Various modifications to this basic setup include any or all of (i) the object slides across a surface thus adding a force (friction) depending on the object's velocity or position, (ii) the object hangs vertically thus adding a constant gravitational force, (iii) a motor or other device imparts some additional nonconstant force (varying with time) to the object.
- In order to solve problems like this one, follow these steps:
  - Step 1: Draw a diagram and label the quantity or quantities of interest (typically, it is the position of a moving object) and identify and label all forces acting on those quantities.
  - Step 2: Find the values of the forces involved, and then use Newton's Second Law ($F = ma$) to write down a differential equation modeling the problem. Also use any information given to write down initial conditions.
    - In the above, F is the net force on the object – i.e., the sum of each of the individual forces acting on the mass (with the proper sign) – while m is the mass of the object, and a is the object's acceleration.
    - Remember that acceleration is the second derivative of position with respect to time – thus, if $y(t)$ is the object's position, acceleration is $y''(t)$.
    - You may need to do additional work to solve for unknown constants – e.g., for a spring constant, if it is not explicitly given to you – before you can fully set up the problem.
  - Step 3: Solve the differential equation and find its general solution.
  - Step 4: Plug in any initial conditions to find the specific solution.
  - Step 5: Check that the answer obtained makes sense in the physical context of the problem.
    - In other words, if you have an object attached to a fixed spring sliding on a frictionless surface, you should expect the position to be sinusoidal, something like $C_1 \sin(\omega t) + C_2 \cos(\omega t) + D$ for some constants C_1, C_2, ω, D.
    - If you have an object on a spring sliding on a surface imparting friction, you should expect the position to tend to some equilibrium value as t grows to ∞, since the object should be 'slowing down' as time goes on.
- Basic Example: An object, mass m, is attached to a spring of spring constant k whose other end is fixed. The object is displaced a distance d from the equilibrium position of the spring, and is let go with velocity v_0 at time $t = 0$. If the object is restricted to sliding horizontally on a frictionless surface, find the position of the object as a function of time.
  - Step 1: 

Take $y(t)$ to be the displacement of the object from the equilibrium position. The only force acting on the object is from the spring, F_{spring}.
  - Step 2: We know that $F_{spring} = -k \cdot y$ from Hooke's Law (aka, the only thing we know about springs). Therefore we have the differential equation $-k \cdot y = m \cdot y''$. We are also given the initial conditions $y(0) = d$ and $y'(0) = v_0$.
  - Step 3: We can rewrite the differential equation as $m \cdot y'' + k \cdot y = 0$, or as $y'' + \frac{k}{m} \cdot y = 0$. The characteristic equation is then $r^2 + \frac{k}{m} = 0$ with roots $r = \pm\sqrt{\frac{k}{m}}i$. Hence the general solution is $\boxed{y = C_1 \cos(\omega t) + C_2 \sin(\omega t)}$, where $\omega = \sqrt{\frac{k}{m}}$.

- Step 4: The initial conditions give $d = y(0) = C_1$ and $v_0 = y'(0) = \omega C_2$ hence $C_1 = d$ and $C_2 = v_0/\omega$. Hence the solution we want is $\boxed{y = d \cdot \cos(\omega t) + \frac{v_0}{\omega} \cdot \sin(\omega t)}$.
- Step 5: The solution we have obtained makes sense in the context of this problem, since on a frictionless surface we should expect that the object's motion would be purely oscillatory – it should just bounce back and forth along the spring forever since there is nothing to slow its motion. We can even see that the form of the solution agrees with our intuition: the fact that the frequency $\omega = \sqrt{\frac{k}{m}}$ increases with bigger spring constant but decreases with bigger mass makes sense – a stronger spring with larger k should pull back harder on the object and cause it to oscillate more quickly, while a heavier object should resist the spring's force and oscillate more slowly.

- Most General Example: An object, mass m, is attached to a spring of spring constant k whose other end is fixed. The object is displaced a distance d from the equilibrium position of the spring, and is let go with velocity v_0 at time $t = 0$. A motor attached to the object imparts a force along its direction of motion given by $R(t)$. If the object is restricted to sliding horizontally on a surface which imparts a frictional force of μ times the velocity of the object (opposite to the object's motion), set up a differential equation modeling the problem.
  - Here is the diagram: 

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  - As before we take $y(t)$ to be the displacement of the object from the equilibrium position. The forces acting on the object are from the spring, F_{spring}, from friction, $F_{friction}$, and from the motor, F_{motor}.
  - We know that $F_{spring} = -k \cdot y$ from Hooke's Law (aka, the only thing we know about springs). We are also given that $F_{fric} = -\mu \cdot y'$, since the force acts opposite to the direction of motion and velocity is given by y'. And we are just given $F_{motor} = R(t)$.
  - Plugging in gives us the differential equation $-k \cdot y - \mu \cdot y' + R(t) = m \cdot y''$, which in standard form is $\boxed{m \cdot y'' + \mu \cdot y' + k \cdot y = R(t)}$. We are also given the initial conditions $y(0) = d$ and $y'(0) = v_0$.

- Some Terminology: If we were to solve the differential equation $m \cdot y'' + \mu \cdot y' + k \cdot y = 0$ (here we assume that there is no outside force acting on the object, other than the spring and friction), we would observe a few different kinds of behavior depending on the parameters m, μ, and k.
  - Overdamped Case: If $\mu^2 - 4mk > 0$ and $R(t) = 0$, we would end up with general solutions of the form $C_1 e^{-r_1 t} + C_2 e^{-r_2 t}$, which when graphed is just a sum of two exponentially-decaying functions. Physically, as we can see from the condition $\mu^2 - 4mk > 0$, this means we have 'too much' friction, since we can just see from the form of the solution function that the position of the object will just slide back towards its equilibrium at $y = 0$ without oscillating at all. This is the "overdamped" case. ["Overdamped" because there is 'too much' damping.]
  - Critically Damped Case: If $\mu^2 - 4mk = 0$ and $R(t) = 0$, we would end up with general solutions of the form $(C_1 + C_2 t)e^{-rt}$, which when graphed is a slightly-slower-decaying exponential function that still does not oscillate, but could possibly cross the position $y = 0$ once, depending on the values of C_1 and C_2. This is the "critically damped" case. ["Critically" because we give the name 'critical' to values where some kind of behavior transitions from one thing to another.]
  - Underdamped Case: If $\mu^2 - 4mk < 0$ and $R(t) = 0$, we end up with general solutions of the form $e^{-\alpha t} \cdot [C_1 \cos(\omega t) + C_2 \sin(\omega t)]$, where $\alpha = -\frac{\mu}{2m}$ and $\omega^2 = \frac{4mk - \mu^2}{4m^2}$. When graphed this is a sine curve times an exponentially-decaying function. Physically, this means that there is some friction (the exponential), but 'not enough' friction to eliminate the oscillations entirely – the position of the object

will still tend toward $y = 0$, but the sine and cosine terms will ensure that it continues oscillating. This is the "underdamped" case. ["Underdamped" because there's not enough damping.]

- Undamped Case: If there is no friction (i.e., $\mu = 0$), we saw earlier that the solutions are of the form $y = C_1 \cos(\omega t) + C_2 \sin(\omega t)$ where $\omega^2 = k/m$. Since there is no friction, it is not a surprise that this is referred to as the "undamped" case.

- Example (Resonance): Suppose an object of mass m (sliding on a frictionless surface) oscillating on a spring is such that its oscillations have frequency ω. Examine what would happen to the object's motion if an external force were applied which oscillated at the same frequency ω.
  - From the solution to the "Basic Example" above, we know that $\omega = \sqrt{k/m}$ so we must have $k = m \cdot \omega^2$.
  - Then if $y(t)$ is the position of the object once we add in this new force $R(t) = A_1 \cos(\omega t) + A_2 \sin(\omega t)$, Newton's Second Law now gives $-k \cdot y + R(t) = m \cdot y''$, or $m \cdot y'' + k \cdot y = R(t)$.
  - If we divide through by m and put in $k = m \cdot \omega^2$ we get $y'' + \omega^2 y = \frac{A_1}{m} \cos(\omega t) + \frac{A_2}{m} \sin(\omega t)$.
  - Now we can think a little and realize that we can the method of undetermined coefficients to find a solution to this differential equation.
  - We would like to try something of the form $y = D_1 \cos(\omega t) + D_2 \sin(\omega t)$, but this will not work because functions of that form are already solutions to the homogeneous equation $y'' + \omega^2 y = 0$.
  - Instead the method instructs that the appropriate solution will be of the form $y = D_1 t \cdot \cos(\omega t) + D_2 t \cdot \sin(\omega t)$.
  - We can see from this formula that as t grows, so does the amplitude of the oscillation: in other words, as time goes on, the object will continue oscillating around its equilibrium point, but the swings back and forth will get larger and larger.
  - You can observe this phenomenon for yourself if you sit in a rocking chair, or swing an object back and forth – you will quickly find that the most effective way to rock the chair or swing the object is to move back and forth at the same frequency that the object is already moving at.

Well, you're at the end of my handout. Hope it was helpful.