# 8. GRADIENTS AND RELATED OPERATORS

## 8.1 Initial Comments

We start this Chapter with a general definition of a gradient that holds for functions and independent variables that can be any combination of scalars, vectors and tensors. Then we focus more closely on the conventional gradient with respect to position vector and related definitions involving divergence and curl.

## 8.2 General Gradients

Generic Forms of a Gradient

Gradients are defined in the context of a function defined in terms of a specific variable. The function and the variable may each be scalars, vectors or tensors. A gradient relates the total differential of the variable to the total differential of the function. Let $f$ denote a function of the variable $\Sigma$. Then the gradient of $f$ with respect to $\Sigma$ is denoted symbolically by $G_\Sigma(f)$ is defined such that

$$df = \left[G_\Sigma(f)\right] \odot d\Sigma \tag{8-1}$$

in which $\odot$ depends on the type of the variable. If $\Sigma$ is a scalar, then $\odot$ is just a regular multiplication. If $\Sigma$ is a vector, then $\odot \Rightarrow \cdot$, and if $\Sigma$ is a second-order tensor, then $\odot \Rightarrow \cdot\cdot$. For the case where $\Sigma$ is a vector, or a tensor, and alternative definition to (8-1) is possible, namely

$$df = d\Sigma \odot \left[G^*_\Sigma(f)\right] \tag{8-2}$$

To emphasize the point that (8-1) will always be used, we switch to the following notation of placing the del operator on the right side of the function as follows:

$$\left[G_\Sigma(f)\right] \equiv (f)\bar{\nabla}_\Sigma \tag{8-3}$$

The superscript arrow indicates the placement of the argument, and the subscript indicates the variable and we replace (8-1) with the general form for a gradient:

$$df = (f)\bar{\nabla}_\Sigma \otimes d\Sigma \tag{8-4}$$

Function is a Scalar

Now suppose we restrict ourselves to the case where $f$ is a scalar function, $f = \phi$. If $\phi$ depends on a single scalar variable $\alpha$ then

$$d\phi = \frac{d\phi}{d\alpha} d\alpha \tag{8-5}$$

The use of (8-4)implies that for this case

$$(\phi)\bar{\bar{\nabla}}_{\alpha} = \frac{d\phi}{d\alpha} \tag{8-6}$$

Now suppose $\phi$ is a function of a vector, $\boldsymbol{v}$. Then

$$d\phi = (\phi)\bar{\bar{\nabla}}_{\boldsymbol{v}} \cdot d\boldsymbol{v} \tag{8-7}$$

which suggests that the gradient of a scalar with respect to a vector is a vector.

**Example (1):** Consider the following:

$$\phi = (\boldsymbol{v}\cdot\boldsymbol{v})^{3/2} \qquad d\phi = \frac{3}{2}(\boldsymbol{v}\cdot\boldsymbol{v})^{1/2}(2\boldsymbol{v}\cdot d\boldsymbol{v})\cdot d\boldsymbol{v} = 3(\boldsymbol{v}\cdot\boldsymbol{v})^{1/2}\boldsymbol{v}\cdot d\boldsymbol{v} \tag{8-8}$$

It follows from (8-7) that

$$(\phi)\bar{\bar{\nabla}}_{\boldsymbol{v}} = 3(\boldsymbol{v}\cdot\boldsymbol{v})^{1/2}\boldsymbol{v} \tag{8-9}$$

which is a vector.

Next suppose $\phi$ is a function of a tensor $\boldsymbol{T}$. Then

$$d\phi = (\phi)\bar{\bar{\nabla}}_{\boldsymbol{T}} \cdot\cdot d\boldsymbol{T} \tag{8-10}$$

which indicates the gradient of a scalar with respect to a second-order tensor is a second-order tensor.

**Example (2):** Suppose

$$\begin{aligned} \phi &= tr(\boldsymbol{T}) + tr(\boldsymbol{T}^2) + (\boldsymbol{T}\cdot\cdot\boldsymbol{T})^{1/2} = \boldsymbol{I}\cdot\cdot\boldsymbol{T} + \boldsymbol{T}\cdot\cdot\boldsymbol{T}^T + (\boldsymbol{T}\cdot\cdot\boldsymbol{T})^{1/2} \\ d\phi &= \boldsymbol{I}\cdot\cdot d\boldsymbol{T} + d\boldsymbol{T}\cdot\cdot\boldsymbol{T}^T + \boldsymbol{T}\cdot\cdot d\boldsymbol{T}^T + \frac{1}{2}\frac{(d\boldsymbol{T}\cdot\cdot\boldsymbol{T} + \boldsymbol{T}\cdot\cdot d\boldsymbol{T})}{(\boldsymbol{T}\cdot\cdot\boldsymbol{T})^{1/2}} \\ &= \boldsymbol{I}\cdot\cdot d\boldsymbol{T} + \boldsymbol{T}^T\cdot\cdot d\boldsymbol{T} + \boldsymbol{T}^T\cdot\cdot d\boldsymbol{T} + \frac{(\boldsymbol{T}\cdot\cdot d\boldsymbol{T})}{(\boldsymbol{T}\cdot\cdot\boldsymbol{T})^{1/2}} \end{aligned} \tag{8-11}$$

It follows that

$$(\phi)\bar{\bar{\nabla}}_{\boldsymbol{T}} = \boldsymbol{I} + 2\boldsymbol{T}^T + \frac{\boldsymbol{T}}{(\boldsymbol{T}\cdot\cdot\boldsymbol{T})^{1/2}} \tag{8-12}$$

which satisfies the condition that the gradient of a scalar function with respect to a second-order tensor is a second-order tensor.

<u>Function is a vector</u>

Now we consider a vector function of a vector, $\boldsymbol{\phi}(\boldsymbol{v})$. Then

$$d\boldsymbol{\phi} = (\boldsymbol{\phi})\bar{\bar{\nabla}}_{\boldsymbol{v}} \cdot d\boldsymbol{v} \tag{8-13}$$

For this equation to be self-consistent the gradient must be a second-order tensor, i.e., a second-order tensor maps a vector to a vector.

**Example (3):**

Suppose

$$\boldsymbol{\phi} = \boldsymbol{v}(\boldsymbol{v}\cdot\boldsymbol{v})^{3/2}$$
$$d\boldsymbol{\phi} = d\boldsymbol{v}(\boldsymbol{v}\cdot\boldsymbol{v})^{3/2} + \boldsymbol{v}\frac{3}{2}(\boldsymbol{v}\cdot\boldsymbol{v})^{1/2}(2\boldsymbol{v}\cdot d\boldsymbol{v}) = \left[\boldsymbol{I}(\boldsymbol{v}\cdot\boldsymbol{v})^{3/2} + 3(\boldsymbol{v}\cdot\boldsymbol{v})^{1/2}(\boldsymbol{v}\otimes\boldsymbol{v})\right]\cdot d\boldsymbol{v} \tag{8-14}$$

It follows that the gradient is the second-order tensor

$$(\boldsymbol{\phi})\bar{\nabla}_{\boldsymbol{v}} = \left[\boldsymbol{I}(\boldsymbol{v}\cdot\boldsymbol{v})^{3/2} + 3(\boldsymbol{v}\cdot\boldsymbol{v})^{1/2}(\boldsymbol{v}\otimes\boldsymbol{v})\right] \tag{8-15}$$

We can go on to the case where a second-order tensor function depends on a variable that, itself, is a second-order tensor. The resulting gradient is a fourth-order tensor, a situation that arises in connection of tangent tensors for constitutive equations, and will not be presented here.

<u>Expressions for Gradients for Constant Bases</u>

The previous forms have been expressed in direct notation and, hence, always hold. Now we move on to the case where the vector basis, $\boldsymbol{e}_i$, and the tensor basis, $\boldsymbol{e}_i\otimes\boldsymbol{e}_j$ are constants. It follows that for primary variables that are vectors or second-order tensors the total differentials are

$$d\boldsymbol{v} = dv_i\boldsymbol{e}_i \qquad d\boldsymbol{T} = dT_{ij}\boldsymbol{e}_i\otimes\boldsymbol{e}_j \tag{8-16}$$

It follows that the dependent variable functions can be written in the alternative forms

$$\phi(\boldsymbol{v}) = \phi(v_{(k)}) \qquad \phi(\boldsymbol{T}) = \phi(T_{(k,l)}) \qquad \boldsymbol{\phi}(\boldsymbol{v}) = \boldsymbol{\phi}(v_{(k)}) \tag{8-17}$$

The brackets around the indices indicate that the usual indicial notation that free indices must be the same on the left and right side of an equation is not being used. Using the chain rule, the total differentials are

$$d\phi(\boldsymbol{v}) = \frac{\partial\phi}{\partial v_i}dv_i \qquad d\phi(\boldsymbol{T}) = \frac{\partial\phi}{\partial T_{ij}}dT_{ij} \qquad d\boldsymbol{\phi}(\boldsymbol{v}) = \frac{\partial\boldsymbol{\phi}}{\partial v_i}dv_i \tag{8-18}$$

or

$$d\phi(\boldsymbol{v}) = \frac{\partial\phi}{\partial v_i}\boldsymbol{e}_i\cdot(dv_j\boldsymbol{e}_j)$$
$$d\phi(\boldsymbol{T}) = \frac{\partial\phi}{\partial T_{ij}}\boldsymbol{e}_i\otimes\boldsymbol{e}_j\cdot\cdot(dT_{kl}\boldsymbol{e}_k\otimes\boldsymbol{e}_l) \tag{8-19}$$
$$d\boldsymbol{\phi}(\boldsymbol{v}) = \frac{\partial\boldsymbol{\phi}}{\partial v_i}\otimes\boldsymbol{e}_i\cdot(dv_j\boldsymbol{e}_j)$$

from which we obtain the following explicit forms for gradients when the bases are orthonormal constants:

$$\left(\phi(\boldsymbol{v})\right)\bar{\nabla}_{\boldsymbol{v}} = \frac{\partial \phi}{\partial v_i}\boldsymbol{e}_i \qquad \text{Gradient of a scalar wrt a vector is a vector}$$

$$\left(\phi(\boldsymbol{T})\right)\bar{\nabla}_{\boldsymbol{T}} = \frac{\partial \phi}{\partial T_{ij}}\boldsymbol{e}_i \otimes \boldsymbol{e}_j \qquad \text{Gradient of a scalar wrt a tensor is a tensor} \tag{8-20}$$

$$\left(\boldsymbol{\phi}(\boldsymbol{v})\right)\bar{\nabla}_{\boldsymbol{v}} = \frac{\partial \boldsymbol{\phi}}{\partial v_i} \otimes \boldsymbol{e}_i \qquad \text{Gradient of a vector wrt a vector is a tensor}$$

**Example (4):**

Suppose $\boldsymbol{u}$ is a constant vector and

$$\phi(\boldsymbol{v}) = \boldsymbol{u}\cdot\boldsymbol{v} + sin(\boldsymbol{v}\cdot\boldsymbol{v}) = u_j v_j + sin(v_j v_j) \tag{8-21}$$

Then

$$\begin{aligned} \phi\bar{\nabla}_{\boldsymbol{v}} &= \frac{\partial \phi}{\partial v_i}\boldsymbol{e}_i \\ \frac{\partial \phi}{\partial v_i} &= \frac{\partial u_j v_j}{\partial v_i} + cos(v_j v_j)\frac{\partial (v_k v_k)}{\partial v_i} \end{aligned} \tag{8-22}$$

We use the relations

$$\frac{\partial u_j}{\partial v_i} = 0_{ij} \qquad \frac{\partial v_k}{\partial v_i} = \delta_{ik} \tag{8-23}$$

where $0_{ij}$ denotes the component of the null tensor. Then (8-22) yields

$$\phi\bar{\nabla}_{\boldsymbol{v}} = u_j\boldsymbol{e}_j + cos(v_j v_j)2v_k\boldsymbol{e}_k = \boldsymbol{u} + \{2\,cos(\boldsymbol{v}\cdot\boldsymbol{v})\}\boldsymbol{v} \tag{8-24}$$

**Example (5):**

Suppose $\boldsymbol{R}$ is a constant tensor and

$$\phi(\boldsymbol{T}) = \boldsymbol{R}\cdot\cdot\boldsymbol{T} + tr(\boldsymbol{T}) + exp(\boldsymbol{T}\cdot\cdot\boldsymbol{T}) = R_{kl}T_{kl} + T_{kk} + exp(T_{kl}T_{kl}) \tag{8-25}$$

Then

$$\begin{aligned} \phi\bar{\nabla}_{T} &= \frac{\partial \phi}{\partial T_{ij}}\boldsymbol{e}_i \otimes \boldsymbol{e}_j \\ \frac{\partial \phi}{\partial T_{ij}} &= \frac{\partial R_{kl}}{\partial T_{ij}}T_{kl} + \frac{\partial T_{kl}}{\partial T_{ij}}R_{kl} + \frac{\partial T_{kk}}{\partial T_{ij}} + \frac{(T_{kl}T_{kl})}{\partial T_{ij}}exp(T_{kl}T_{kl}) \end{aligned} \tag{8-26}$$

But

$$\frac{\partial R_{kl}}{\partial T_{ij}} = 0_{klij} \qquad \frac{\partial T_{kl}}{\partial T_{ij}} = \delta_{ki}\delta_{lj} \qquad \frac{\partial T_{kk}}{\partial T_{ij}} = \delta_{ki}\delta_{kj} = \delta_{ij} \qquad \frac{(T_{kl}T_{kl})}{\partial T_{ij}} = 2T_{ij} \quad (8\text{-}27)$$

Then (8-26) becomes

$$\phi\bar{\nabla}_T = \left[R_{ij} + \delta_{ij} + 2T_{ij}\, exp(T_{kl}T_{kl})\right]\boldsymbol{e}_i \otimes \boldsymbol{e}_j = \boldsymbol{R} + \boldsymbol{I} + 2\boldsymbol{T}\, exp(\boldsymbol{T} \cdot\cdot \boldsymbol{T}) \qquad (8\text{-}28)$$

**Example (6):**

Suppose $\boldsymbol{u}$ is a constant vector, $\boldsymbol{R}$ is a constant vector and

$$\boldsymbol{\phi}(\boldsymbol{v}) = \boldsymbol{R}\cdot\boldsymbol{v} + \boldsymbol{v}(\boldsymbol{v}\cdot\boldsymbol{R}\cdot\boldsymbol{u}) = \left[R_{ik}v_k + v_i(R_{kl}v_k u_l)\right]\boldsymbol{e}_i \qquad (8\text{-}29)$$

Then

$$\begin{aligned}
&\left(\boldsymbol{\phi}(\boldsymbol{v})\right)\bar{\nabla}_{\boldsymbol{v}} = \frac{\partial \boldsymbol{\phi}}{\partial v_j} \otimes \boldsymbol{e}_j = \frac{\partial \phi_i}{\partial v_j}\boldsymbol{e}_i \otimes \boldsymbol{e}_j \\
&\frac{\partial \phi_i}{\partial v_j} = \frac{\partial}{\partial v_j}\left[R_{ik}v_k + v_i(R_{kl}v_k u_l)\right] = \left[R_{ik}\delta_{jk} + \delta_{ij}(R_{kl}v_k u_l) + v_i(R_{kl}\delta_{jk}u_l)\right] \qquad (8\text{-}30) \\
&\qquad = \left[R_{ij} + \delta_{ij}(R_{kl}v_k u_l) + v_i(R_{jl}u_l)\right]
\end{aligned}$$

and

$$\left(\boldsymbol{\phi}(\boldsymbol{v})\right)\bar{\nabla}_{\boldsymbol{v}} = \boldsymbol{R} + \boldsymbol{I}(\boldsymbol{v}\cdot\boldsymbol{R}\cdot\boldsymbol{u}) + \boldsymbol{v}\otimes(\boldsymbol{R}\cdot\boldsymbol{u}) \qquad (8\text{-}31)$$

This subsection has been an exposure to gradients with respect to scalars, vectors and tensors, all of which are used in the formulation of constitutive equations. The key point is that gradients map the total differential of the primary variable to a total differential of the secondary, or dependent, variable.

## 8.3 Gradients with Respect to the Position Vector

Now, instead of a gradient with respect to any vector, we focus on the very special case of a gradient with respect to the position vector. If gradient is used with no adjective it is typically implicitly assumed that the gradient is with respect to the position vector and no subscript is attached to the gradient operator. If $\boldsymbol{r}$ denotes the position vector, and $\Psi$ is a generic function of $\boldsymbol{r}$ then the gradient maps the total differential of the primary variable, $d\boldsymbol{r}$, to the total differential of the dependent variable, $d\Psi$, as follows:

$$d\Psi = (\Psi)\bar{\nabla}_{\boldsymbol{r}}\cdot d\boldsymbol{r} = (\Psi)\bar{\nabla}\cdot d\boldsymbol{r} \qquad (8\text{-}32)$$

For a constant basis, we choose to represent the position vector as

$$\boldsymbol{r} = x_i\boldsymbol{e}_i \qquad (8\text{-}33)$$

so that the gradient becomes

$$(\Psi)\bar{\nabla} = \frac{\partial \Psi}{\partial x_i} \otimes \boldsymbol{e}_i = \frac{\partial \Psi}{\partial x_j} \otimes \boldsymbol{e}_j \tag{8-34}$$

where the tensor product is discarded if $\Psi$ is a scalar. We introduce the "comma" notation to denote the partial derivative with respect to $x_i$, i.e.,

$$\frac{\partial \Psi}{\partial x_i} = \Psi_{,i} \tag{8-35}$$

Then gradients of scalar and vector functions assume the forms

$$\begin{aligned} &(\Psi)\bar{\nabla} = \Psi_{,i} \otimes \boldsymbol{e}_i && \Psi \text{ - scalar} \\ &(\boldsymbol{v})\bar{\nabla} = v_{i,j}\, \boldsymbol{e}_i \otimes \boldsymbol{e}_j && \boldsymbol{v} = v_i \boldsymbol{e}_i \text{ - vector} \end{aligned} \tag{8-36}$$

Note that the gradient of a vector is a second-order tensor and that the index j indicating the partial derivative is associated with the second, or last, base vector of tensor basis.

In a similar fashion, the gradient of a second-order tensor is a third-order tensor:

$$(\boldsymbol{T})\bar{\nabla} = \boldsymbol{T}_{ij,k}\, \boldsymbol{e}_i \otimes \boldsymbol{e}_j \otimes \boldsymbol{e}_k \tag{8-37}$$

Again, the index for partial differentiation is associated with the last base vector with the definition of gradient adopted here.

## 8.4 Divergence, Curl and Laplacian Operators

The divergence is obtained from the gradient by dotting, or contracting on the last two base vectors in the expression for the gradient. With the use of the contraction operator, the divergence of a vector and divergence of a scalar are

$$\begin{aligned} div(\boldsymbol{v}) &= C_{12}\left\{(\boldsymbol{v})\bar{\nabla}\right\} = v_{i,j}\boldsymbol{e}_i \cdot \boldsymbol{e}_j = v_{i,j}\delta_{ij} = v_{i,i} \\ div(\boldsymbol{T}) &= C_{23}\left\{(\boldsymbol{T})\bar{\nabla}\right\} = T_{ij,k}\boldsymbol{e}_i \otimes (\boldsymbol{e}_j \cdot \boldsymbol{e}_k) = T_{ij,k}\boldsymbol{e}_i\delta_{jk} = T_{ij,j}\boldsymbol{e}_i \end{aligned} \tag{8-38}$$

The following notation is typically used for the divergence:

$$\begin{aligned} \boldsymbol{v} \cdot \bar{\nabla} &= tr\left(\boldsymbol{v}\bar{\nabla}\right) = v_{i,i} \\ \boldsymbol{T} \cdot \bar{\nabla} &= C_{23}\left(\boldsymbol{T}\bar{\nabla}\right) = T_{ij,j}\boldsymbol{e}_i \end{aligned} \tag{8-39}$$

The curl operator is similar to the divergence operator but with a cross product used instead of the dot product. The result is expressed two ways for a vector and a tensor as follows:

$$\text{curl}(\boldsymbol{v}) = \boldsymbol{\varepsilon} \cdot\cdot (\boldsymbol{v}\bar{\nabla}) = (\varepsilon_{ijk}\boldsymbol{e}_i \otimes \boldsymbol{e}_j \otimes \boldsymbol{e}_k) \cdot\cdot (v_{m,n}\boldsymbol{e}_m \otimes \boldsymbol{e}_n) = \varepsilon_{imn}v_{m,n}\boldsymbol{e}_i$$

$$\text{curl}(\boldsymbol{v}) = \boldsymbol{v} \times \bar{\nabla} = v_{j,k}\boldsymbol{e}_j \times \boldsymbol{e}_k = \varepsilon_{ijk}v_{j,k}\boldsymbol{e}_i$$

$$\text{curl}(\boldsymbol{T}) = C_{25}C_{36}[\boldsymbol{\varepsilon} \otimes (\boldsymbol{T}\bar{\nabla})] = C_{25}C_{36}[((\varepsilon_{ijk}\boldsymbol{e}_i \otimes \boldsymbol{e}_j \otimes \boldsymbol{e}_k) \otimes (\boldsymbol{T}_{mn,p}\boldsymbol{e}_m \otimes \boldsymbol{e}_n \otimes \boldsymbol{e}_p)] \quad (8\text{-}40)$$

$$= [((\varepsilon_{ijk}\boldsymbol{e}_i) \otimes (\boldsymbol{T}_{mn,p}\boldsymbol{e}_m)]\delta_{jn}\delta_{kp} = \varepsilon_{inp}\boldsymbol{T}_{mn,p}\boldsymbol{e}_i \otimes \boldsymbol{e}_m$$

$$\text{curl}(\boldsymbol{T}) = \boldsymbol{T} \times \bar{\nabla} = T_{ij,k}\boldsymbol{e}_i \otimes (\boldsymbol{e}_j \times \boldsymbol{e}_k) = \varepsilon_{jkl}T_{ij,k}\boldsymbol{e}_i \otimes \boldsymbol{e}_l$$

Note that the curl of a vector is a vector and the curl of a second-order tensor is a second-order tensor. Because of the definition of the gradient adopted here (the base vector associated with the derivative on the right), this definition of curl will have a sign opposite to that used in most texts.

Various operators are defined by applying the gradient operator a second time. If $\phi$ is a scalar, then $\phi\bar{\nabla}$ is a vector. If we apply the divergence operator to this vector, the result is called the Laplacian operator acting on a scalar.

$$(\phi)\bar{\nabla}^2 \equiv \left(\phi\bar{\nabla}\right) \cdot \bar{\nabla} = (\phi)\bar{\nabla} \cdot \bar{\nabla} = \phi,_{ii} \qquad (8\text{-}41)$$

In a similar fashion, the Laplacian acting on a vector yields

$$(\boldsymbol{v})\bar{\nabla}^2 = (\boldsymbol{v})\bar{\nabla} \cdot \bar{\nabla} = v_{i,jk}\boldsymbol{e}_i(\boldsymbol{e}_j \cdot \boldsymbol{e}_k) = v_{i,jj}\boldsymbol{e}_i \qquad (8\text{-}42)$$

One example of another operator that involves two gradients is the divergence of the curl of a vector or

$$div\left\{curl(\boldsymbol{v})\right\} = \left(\boldsymbol{v} \times \bar{\nabla}\right) \cdot \bar{\nabla} = \varepsilon_{ijk}v_{i,jl}(\boldsymbol{e}_k \cdot \boldsymbol{e}_l) = \varepsilon_{ijk}v_{i,jk} = 0 \qquad (8\text{-}43)$$

The result is zero because $\varepsilon_{ijk}$ is skew symmetric with respect to any two indices, and in particular with respect to the indices $j$ and $k$ where as $v_{i,jk}$ is symmetric with respect to the indices $j$ and $k$. This is an application of the theorem that $tr(\boldsymbol{R} \cdot \boldsymbol{S}) = 0$ if $\boldsymbol{R}$ is symmetric and $\boldsymbol{S}$ is skew-symmetric.

## 8.5 An Application to Line Integrals

Often a situation arises where physical data indicate that a line integral for all closed paths is zero, i.e.,

$$\oint \boldsymbol{F} \cdot d\boldsymbol{r} = 0 \qquad \text{For all paths} \qquad (8\text{-}44)$$

This integral might be a measure of energy with for a force field where $\boldsymbol{F}$ depends of the position $\boldsymbol{r}$. Another way of saying the same thing is that

$$\int_{P_1}^{P_2} \boldsymbol{F} \cdot d\boldsymbol{r} = 0 \qquad \text{For all paths and points } P_1 \text{ and } P_2 \text{ coincide} \qquad (8\text{-}45)$$

The only way for either (8-44) or (8-45) to be satisfied is that the argument of the integral must be a total differential of a scalar function, $\phi(\boldsymbol{r})$, for this example, and then

$$\int_{P_1}^{P_2} d\phi = \phi\Big|_{P_1}^{P_2} = 0 \qquad \text{For all paths and points } P_1 \text{ and } P_2 \text{ coincide} \tag{8-46}$$

But with the use of a gradient, the total differential becomes

$$d\phi = \phi\bar{\bar{\nabla}} \cdot d\boldsymbol{r} \tag{8-47}$$

It follows that if (8-44) holds, then the force term, $\boldsymbol{F}$, must be the gradient of a scalar function, or

$$\boldsymbol{F} = \phi\bar{\bar{\nabla}} \tag{8-48}$$

The same type of argument holds in a more general context. Let the two second-order tensors, $\boldsymbol{\sigma}$ and $\boldsymbol{e}$, denote the stress and strain tensors, respectively. Elastic bodies exhibit the property that if the body is loaded and unloaded, no matter in what fashion, the energy stored in the body, $W$, is zero. Suppose the stress and strain tensors do not depend on position (homogeneous deformation). Then to within a scalar multiple the work applied to the body is

$$W = \int_{P_1}^{P_2} \boldsymbol{\sigma} \cdot\cdot d\boldsymbol{e} \tag{8-49}$$

for any path in strain space. For any closed loop, points $P_1$ and $P_2$ coincide and $W = 0$. The same argument as the one used above. The argument of the integral must be the total differential of a scalar function of strain, $\psi(\boldsymbol{e})$, and the stress is the gradient with respect to strain:

$$\boldsymbol{\sigma} = \Psi\bar{\bar{\nabla}}_{\boldsymbol{e}} \tag{8-50}$$

and the function $\Psi$ is called the elastic internal energy.

### 8.6 Concluding Remarks

The conventional gradient, curl, divergence and Laplacian operators are based on the assumption that the independent variable is the position vector. These expressions have been summarized on the assumption that the vector and tensor bases are constant and orthonormal.

However gradients with respect to vectors other than the position vector, and with respect to tensors, can arise in the development of constitutive equations. It is shown that the concept of a gradient can be generalized. Then the total differential of a dependent variable is obtained when the generalized gradient operates on the total differential of the primary, or independent, variable.