C

Continuum Mechanics Summary

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§C.1. Introduction

This Appendix summarizes the basic relations of three-dimensional continuum mechanics for linear elastic solids. These include strain-displacement, constitutive and equilibrium equations. Both indicial and full notations are used.

§C.2. The Strain-Displacement Equations

Let $u_i(x_j)$ denote the components of the displacement vector field $\mathbf{u}(x_j)$. Then the infinitesimal strains are given by

$$e_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}) \tag{C.1}$$

These are the components of the strain tensor $[e_{ij}] = \underline{\mathbf{e}}$, which written in full is

$$\begin{aligned}
e_{11} &= u_{1,1} = \frac{\partial u_1}{\partial x_1} \\
e_{22} &= u_{2,2} = \frac{\partial u_2}{\partial x_2} \\
e_{33} &= u_{3,3} = \frac{\partial u_3}{\partial x_3} \\
e_{12} &= \tfrac{1}{2}(u_{1,2} + u_{2,1}) = \tfrac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) \\
e_{23} &= \tfrac{1}{2}(u_{2,3} + u_{3,2}) = \tfrac{1}{2}\left(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\right) \\
e_{31} &= \tfrac{1}{2}(u_{3,1} + u_{1,3}) = \tfrac{1}{2}\left(\frac{\partial u_3}{\partial x_1} + \frac{\partial u_1}{\partial x_3}\right)
\end{aligned} \tag{C.2}$$

§C.2.1. The Linear Strain Tensor

The infinitesimal or linear strain tensor in the x_i coordinate system is

$$\underline{\mathbf{e}} = [e_{ij}] = \begin{bmatrix} e_{11} & e_{12} & e_{13} \\ & e_{22} & e_{23} \\ symm & & e_{33} \end{bmatrix} \tag{C.3}$$

In another coordinate system x'_j related to x_i by the transformation

$$x_i = a_{ij} x'_j \tag{C.4}$$

the strain components become

$$e'_{ij} = a_{im} a_{jn} e_{mn} \tag{C.5}$$

§C.2.2. The Engineering Notation

The standard engineering notation uses x, y, z for x_1, x_2, x_3 and u_x, u_y, u_z for u_1, u_2, u_3, respectively. Then the engineering strains are related to the displacements by

$$\begin{aligned}
e_{xx} &= e_{11} = \frac{\partial u_x}{\partial x} \\
e_{yy} &= e_{22} = \frac{\partial u_y}{\partial y} \\
e_{zz} &= e_{33} = \frac{\partial u_z}{\partial z} \\
\gamma_{xy} &= 2e_{12} = \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} \\
\gamma_{yz} &= 2e_{23} = \frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y} \\
\gamma_{zx} &= 2e_{31} = \frac{\partial u_z}{\partial x} + \frac{\partial u_x}{\partial z}
\end{aligned} \tag{C.6}$$

The linear strain tensor in terms of engineering strains is

$$[e] = \begin{bmatrix} e_{xx} & \frac{1}{2}\gamma_{xy} & \frac{1}{2}\gamma_{xz} \\ & e_{yy} & \frac{1}{2}\gamma_{yz} \\ symm & & e_{zz} \end{bmatrix} \tag{C.7}$$

§C.3. Compatibility Equations

The strain tensor $\underline{\mathbf{e}}$ has 6 independent components. The displacement field has 3 independent components. It follows that there must be 3 independent conditions between the e_{ij}. These expressions arise from the condition of compatibility of deformation. In the three-dimensional case these compatibility equations are

$$e_{ij,k\ell} + e_{k\ell,ij} = e_{ik,j\ell} + e_{j\ell,ik} \tag{C.8}$$

For the two dimensional case only one equation survives

$$e_{11,22} + e_{22,11} = 2e_{12,12} \tag{C.9}$$

which in standard notation is

$$\frac{\partial^2 e_{xx}}{\partial y^2} + \frac{\partial^2 e_{yy}}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial_x \partial_y} \tag{C.10}$$

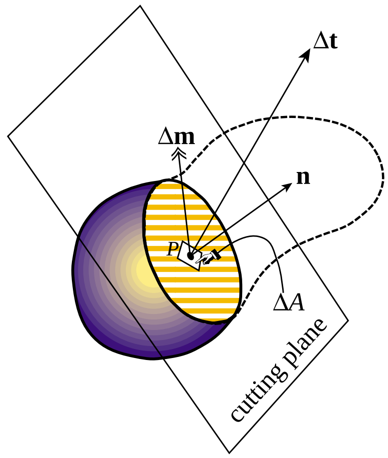

FIGURE C.1. Plane cut through a body for defining the interior force resultants at point P.

§C.4. The Stress Vector

Consider a continuum body and an interior point $P(x_i)$. Make a cut through P with a plane with exterior normal $\mathbf{n}$, as illustrated in Figure C.1.

The *stress vector* at P for direction $\mathbf{n}$ is defined as

$$\mathbf{t}_n = \lim_{\Delta A \to 0} \frac{\Delta \mathbf{t}}{\Delta A}, \tag{C.11}$$

where ΔA is a differential area surrounding P on the cutting plane (see Figure C.1).

The *couple stress vector* for direction $\mathbf{n}$ is

$$\mathbf{m}_n = \lim_{\Delta A \to 0} \frac{\Delta \mathbf{m}}{\Delta A}. \tag{C.12}$$

It is optional to include $\mathbf{m}_n$ in the theory of stress. Doing so leads to the so-called *polar material* models. In classical continuum mechanics it is generally assumed that $\mathbf{m}_n = 0$, which corresponds to non-polar materials. Polar material models are generally considered only when continua are subjected to strong electromagnetic fields.

§C.5. The Stress Tensor

Consideration of the equilibrium of an elemental tetrahedron at P whose faces are normal to x_1, x_2, x_3 and $\mathbf{n}$ leads to the expression

$$t_i = \sigma_{ij} n_j, \tag{C.13}$$

where t_i is the component of $\mathbf{t}$ in the x_i direction, and n_j are the components of $\mathbf{n}$. The nine values σ_{ij} are the components of the Cauchy stress tensor

$$\underline{\boldsymbol{\sigma}} = [\sigma_{ij}] = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}. \tag{C.14}$$

For non-polar materials this tensor is *symmetric*. That is, $\sigma_{ij} = \sigma_{ji}$.

§C.6. Equilibrium Equations

Consider the equilibrium of a body of volume V and surface S subject to the following actions

(a) Body force field $\mathbf{f}$ of components b_i in V

(b) Acceleration field $\mathbf{a} = d^2\mathbf{u}/dt^2 = \ddot{\mathbf{u}}$ (t = time) of components a_i in V

(c) Stress vectors $\mathbf{t}$ of components t_i on S

Dynamic equilibrium along any direction x_i requires

$$\int_S t_i \, dS + \int_V b_i \, dV = \int_V \rho a_i \, dV, \tag{C.15}$$

where ρ is the body density. Substitute $t_i = \sigma_{ij} n_j$ in the surface integral:

$$\int_S \sigma_{ij} n_j \, dS + \int_V b_i \, dV = \int_V \rho a_i \, dV. \tag{C.16}$$

To transform the surface integral to a volume integral we use Gauss' divergence theorem. For any vector field $\mathbf{a}$:

$$\int_S \mathbf{a}.\mathbf{n} \, dS = \int_V \mathbf{div}.\mathbf{a} \, dV. \tag{C.17}$$

or in component form

$$\int_S a_j n_j \, dS = \int_V \frac{\partial a_j}{\partial x_j} \, dV. \tag{C.18}$$

Consequently the equilibrium integral (C.15) may be reduced to

$$\int_V [\sigma_{ij,j} + b_i - \rho a_i] \, dV = 0, \tag{C.19}$$

for an arbitrary volume. Because the volume is arbitrary we must have

$$\sigma_{ij,j} + b_i - \rho a_i = 0. \tag{C.20}$$

These are the three differential equations of dynamic equilibrium, which are obtained by setting the free index i to 1, 2 and 3. These are also called the internal equilibrium equations, or balance equations. If the medium is at rest or moving uniformly with respect to an inertial frame, the accelerations vanish and we obtain the equations of *static equilibrium*

$$\sigma_{ij,j} + b_i = 0. \tag{C.21}$$

Example C.1. If $i = 1$ the first static equilibrium equation along axis x_1 is

$$\sigma_{1j,j} + b_1 = 0 \tag{C.22}$$

or, written in full

$$\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} + \frac{\partial \sigma_{13}}{\partial x_3} + b_1 = 0. \tag{C.23}$$

In conventional (engineering) notation:

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + b_x = 0. \tag{C.24}$$

§C.7. Constitutive Equations

Constitutive equations characterize the behavior of the material of mechanical bodies. These equations are relations that constrain the space of deformations of a body (as defined by the strain tensor) and the state of internal forces (as defined by the stress tensor). The relations hold at each point of the body. They are generally partial differential equations, or even integrodifferential equations, in space and time.

The simplest type of constitutive equations are homogeneous linear algebraic relations that connect the stress and stress tensors at each point. This type of constitutive equation characterize the *linear elastic solids*, which are the only ones we shall consider in this course.

§C.7.1. Elastic Solids

An elastic solid is a body characterized by two configurations:

(1) The *natural state* or *undeformed state*, which is taken by the solid in the absence of applied forces

(2) The *deformed state*, attainable by any *reversible* process.

The concepts just outlined are closely associated with the idea of *stored energy*, *strain energy* or *stress energy*. This expresses mathematically that the behavior of an elastic solid is independent of the preceding history of the material. In other words, the state of stress depends only on the state of strain, and not on the path followed to get to that strain.

§C.7.2. The Strain Energy

Let us define $\mathcal{U}$ as the strain energy per unit of deformed volume, also called the *strain energy density*. This is generally a function of position (x_i) and of the deformation of the body at that position. Since in linear elasticity the deformation may be characterized by the strain tensor ϵ_{ij}

$$\mathcal{U} = \mathcal{U}(x_i, e_{ij}). \tag{C.25}$$

Additional properties of this function are:

(a) $\mathcal{U}$ is a scalar invariant: it is unaffected by rigid body displacements and by the orientation of the global RCC system.

(b) If $\mathcal{U}$ is independent of x_i over a volume V, the material is said to be *homogeneous* in V.

(c) The material is said to be *isotropic* if the stress-strain law is independent of directions in the material. If the material is isotropic, the strain energy density must be a function only of the invariants of the strain tensor.

Remark C.1. Isotropy must not be confused with invariance. All materials in classical mechanics satisfy the invariance principle (material properties do not depend on the observer), but not all materials are isotropic.

§C.7.3. Principle of Virtual Work

Consider a body in static equilibrium. The principle of virtual work (PVW) states that the virtual work done by all forces acting on that body during a virtual displacement* $\delta\mathbf{u}$ must be zero. Mathematically,

$$\delta W_i + \delta W = 0, \tag{C.26}$$

where W_i and W are the virtual work done by the internal and external forces, respectively. The principle can be derived mathematically by taking the equilibrium equations and stress boundary conditions

$$\sigma_{ij,j} + f_i = 0, \qquad \sigma_{ij} n_j = \hat{t}_i, \tag{C.27}$$

multiply the first by δu_i, integrate over V and apply the divergence theorem which for symmetric σ_{ij} yields

$$\int_V (\sigma_{ij,j} + f_i)\,\delta\bar{u} = \int_V [-\sigma_{ij}\delta\tfrac{1}{2}(u_{i,j} + u_{j,i}) + f_i\,\delta u_i]\,dV + \int_S \sigma_{ij} n_j\,\delta u_i\,dS = 0. \tag{C.28}$$

Split the surface integral over $S_t \cup S_u$. The integral over the latter vanishes. The former can be transformed using the second of (F.3) weighted by δu_i and integrated over S_t to produce

$$\int_V \sigma_{ij}\delta e_{ij}\,dV = \int_V f_i\,\delta u_i + \int_{S_t} \hat{t}_i\,\delta u_i\,dS. \tag{C.29}$$

But $\delta W_i = -\delta U$, where U is the total stored strain energy:

$$U = \int_V \mathcal{U}\,dV. \tag{C.30}$$

Comparing with (C.5) we get

$$\delta U = \int_V \sigma_{ij}\,\delta e_{ij}\,dV, \qquad \delta\mathcal{U} = \sigma_{ij}\delta e_{ij}. \tag{C.31}$$

§C.7.4. Hyperelastic Constitutive Equations

For a homogeneous linear elastic body possessing a strain energy density (such material is called "hyperelastic" in the literature)

$$\mathcal{U} = \mathcal{U}(e_{ij}). \tag{C.32}$$

$$\delta\mathcal{U} = \frac{\partial\mathcal{U}}{\partial e_{ij}}\delta e_{ij} = \frac{\partial U}{\partial e_{11}}\delta e_{11} + \frac{\partial\mathcal{U}}{\partial e_{12}}\delta e_{12} + \cdots = \sigma_{ij}\,\delta e_{ij}. \tag{C.33}$$

Since $\delta\mathcal{U} = \sigma_{ij}\,\delta e_{ij}$ we must have

$$\sigma_{ij} = \frac{\partial\mathcal{U}}{\partial e_{ij}}. \tag{C.34}$$

* A virtual displacement is a change in the geometric configuration of the body compatible with all kinematic constraints, made while keeping all forces and stress acting on the body frozen.

These are the constitutive equations of a linear *hyperelastic* material.

For an isotropic linear elastic material, $\mathcal{U}$ must be a quadratic form in e_{ij} that preserves the invariants of the strain tensor. It is shown in books on elasticity that this condition leads to the *generalized Hooke's law*

$$\sigma_{ij} = \lambda e_{ii}\delta_{ij} + 2\mu e_{ij}. \tag{C.35}$$

Here λ and μ are called the Lamé coefficients; both of which have dimensions of stress. In engineering applications the material coefficients E, G, and ν (modulus of elasticity, shear modulus, and Poissons's ratio, respectively) are more commonly used. These are related to λ and μ by

$$E = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu}, \qquad \nu = \frac{\lambda}{2(\lambda + \mu)}, \qquad G = \frac{E}{2(1+\nu)}. \tag{C.36}$$

Conversely,

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}, \qquad \mu = G = \frac{E}{2(1+\nu)}. \tag{C.37}$$