Introduction to the Finite Element Method

Introductory Course on Multiphysics Modelling

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Contents

1 Introduction **1**
1.1 Motivation and general concepts 1
1.2 Major steps of finite element analysis 3

2 Strong and weak forms **3**
2.1 Model problem 3
2.2 Boundary-value problem and the strong form 4
2.3 The weak form 5
2.4 Associated variational problem 7

3 Galerkin method **8**
3.1 Discrete (approximated) problem 8
3.2 System of algebraic equations 8

4 Finite element model **9**
4.1 Discretization and (linear) shape functions 9
4.2 Lagrange interpolation functions 10
4.3 Finite element system of algebraic equations 10
4.3.1 Matrix of the system 10
4.3.2 Right-hand-side vector 13
4.4 Imposition of the essential boundary conditions 13
4.5 Results of analytical and FE solutions 14

1 Introduction

1.1 Motivation and general concepts

The **Finite Element Method (FEM)** is:

[generally speaking] a powerful computational technique for the solution of differential and integral equations that arise in various fields of engineering and applied sciences;

[mathematically] a generalization of the classical variational (Ritz) and weighted-residual (Galerkin, least-squares, etc.) methods.

Motivation
Most of the real problems:

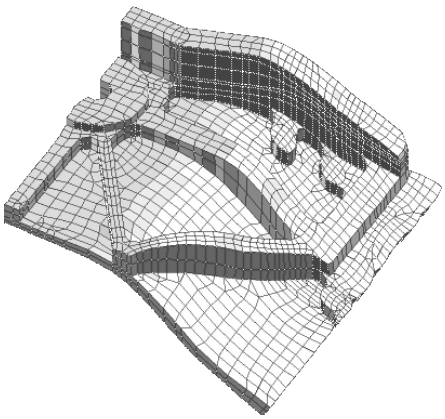
- are defined on domains that are geometrically complex,
- may have different boundary conditions on different portions of the boundary.

Therefore, it is usually impossible (or difficult):

1. to find a solution analytically (so one must resort to approximate methods),
2. to generate approximation functions required in the traditional variational methods.

An answer to these problems is a **finite-element approach** which consists in representing domains with irregular shapes by a collection of finite elements.

Main concept of FEM
A given domain can be viewed as an assemblage of simple geometric shapes, called **finite elements**, for which it is possible to systematically generate the approximation functions needed in the solution of differential equations by any of the variational and weighted-residual methods.



Remarks:

- The approximation functions are also called **shape functions** or **interpolation functions** since they are often constructed using ideas from interpolation theory.
- The finite element method is a piecewise (or element-wise) application of the variational and weighted-residual methods.
- For a given BVP, it is possible to develop different finite element approximations (or **finite element models**), depending on the choice of a particular variational and weighted-residual formulation.

1.2 Major steps of finite element analysis

The major steps in the finite element analysis of a typical problem are presented below.

1. **Discretization of the domain** into a set of finite elements (mesh generation).
2. **Weighted-integral or weak formulation** of the differential equation over a typical finite element (subdomain).
3. **Development of the finite element model** of the problem using its weighted-integral or weak form. The finite element model consists of a set of algebraic equations among the unknown parameters (*degrees of freedom*) of the element.
4. **Assembly of finite elements** to obtain the global system (i.e., for the total problem) of algebraic equations – for the unknown global degrees of freedom.
5. **Imposition of essential boundary conditions**.
6. **Solution of the system of algebraic equations** to find (approximate) values in the global degrees of freedom.
7. **Post-computation** of solution and quantities of interest.

2 Strong and weak forms

2.1 Model problem

Consider the following **(ordinary) differential equation**

$$\textbf{(O)DE:} \quad -\frac{\mathrm{d}}{\mathrm{d}x}\left(\alpha(x)\,\frac{\mathrm{d}u(x)}{\mathrm{d}x}\right)+\gamma(x)\,u(x)=f(x) \quad \text{for } x\in(a,b) \tag{1}$$

where

- $\alpha(x)$, $\gamma(x)$, $f(x)$ are the known data of the problem: the first two quantities result from the *material properties* and *geometry* of the problem whereas the third one depends on *source* or *loads*,
- $u(x)$ is the solution to be determined; it is also called **dependent variable** of the problem (with x being the **independent variable**).

The domain of this 1D problem is an interval (a,b), and the points $x=a$ and $x=b$ are the boundary points where **boundary conditions** are imposed, e.g.,

$$\textbf{BCs:} \quad \begin{cases} \left(q(a)\,n_x(a)=\right)-\alpha(a)\,\dfrac{\mathrm{d}u}{\mathrm{d}x}(a)=\hat{q}\,, & \text{(Neumann b.c.)} \\ u(b)=\hat{u}\,. & \text{(Dirichlet b.c.)} \end{cases} \tag{2}$$

Here,

- $\hat{q}$ and $\hat{u}$ are the given boundary values,

- n_x is the component of the outward unit vector normal to the boundary. In the 1D case there is only one component and: $n_x(a) = -1$, $n_x(b) = +1$.

Moreover,

- $q(x) \equiv \alpha(x) \dfrac{\mathrm{d}u(x)}{\mathrm{d}x}$ is the so-called **secondary variable** specified on the boundary by the **Neumann boundary condition** $(2)_1$ also known as the **second kind** or **natural** boundary condition,

- $u(x)$ is the **primary variable** specified on the boundary by the **Dirichlet boundary condition** $(2)_2$ also known as the **first kind** or **essential** boundary condition.

Examples of different physical problems

The model problem can describe different physical problems (formulated as 1-dimensional). The table below contains a list of fields of study in which the model equation arises, with meaning of various parameters and variables.

u (primary var.)	α (material data)	f (source, load)	q (secondary var.)
Heat transfer			
temperature	thermal conductance	heat generation	heat
Flow through porous medium			
fluid-head	permeability	infiltration	source
Flow through pipes			
pressure	pipe resistance	0	source
Flow of viscous fluids			
velocity	viscosity	pressure gradient	shear stress
Elastic cables			
displacement	tension	transversal force	point force
Elastic bars			
displacement	axial stiffness	axial force	point force
Torsion of bars			
angle of twist	shear stiffness	0	torque
Electrostatics			
electric potential	dielectric constant	charge density	electric flux

2.2 Boundary-value problem and the strong form

Let:

- $\Omega = (a, b)$ be an open set (an open interval in case of 1D problems);

- Γ be the boundary of Ω, that is, $\Gamma = \{a, b\}$;

- $\Gamma = \Gamma_q \cup \Gamma_u$ where, e.g., $\Gamma_q = \{a\}$ and $\Gamma_u = \{b\}$ are disjoint parts of the boundary (i.e., $\Gamma_q \cap \Gamma_u = \emptyset$) relating to the Neumann and Dirichlet boundary conditions, respectively;

- (the data of the problem): $f:\Omega\to\Re,\ \alpha:\Omega\to\Re,\ \gamma:\Omega\to\Re$;
- (the values prescribed on the boundary): $\hat{q}:\Gamma_q\to\Re,\ \hat{u}:\Gamma_u\to\Re$.

Boundary-value problem (BVP):
Find u =? satisfying

$$\textbf{differential eq.:}\qquad -\left(\alpha u'\right)'+\gamma u=f \quad \text{in } \Omega=(a,b), \tag{3}$$

$$\textbf{Neumann b.c.:}\qquad \alpha u' n_x=\hat{q} \quad \text{on } \Gamma_q=\{a\}, \tag{4}$$

$$\textbf{Dirichlet b.c.:}\qquad u=\hat{u} \quad \text{on } \Gamma_u=\{b\}. \tag{5}$$

Definition 1 (Strong form)**.** The classical **strong form** of a boundary-value problem described by a second-order (partial) differential equation with boundary conditions consists of:

- the **differential equation** of the problem,
- the **Neumann boundary conditions**, i.e., the natural conditions imposed on the secondary dependent variable (which involves the first derivative of the dependent variable).

The Dirichlet (essential) boundary conditions must be satisfied *a priori*, that is, the solution is to be found in the space of all twice-differentiable functions satisfying the Dirichlet conditions.

2.3 The weak form

Derivation of the equivalent weak form consists of the three steps presented below.

1. Write the ***weighted-residual*** **statement** for the domain equation

$$\int_a^b \Big[-\left(\alpha u'\right)'+\gamma u-f\Big]\delta u\,\mathrm{d}x=0. \tag{6}$$

   Here:

   - δu (the weighting function) belongs to (the space of) **test functions**,
   - u (the solution) belongs to (the space of) **trial functions**.

2. Trade differentiation from u to δu using ***integration by parts***

$$\Big[-\alpha u'\delta u\Big]_a^b+\int_a^b\Big[\alpha u'\delta u'+\gamma u\,\delta u-f\,\delta u\Big]\,\mathrm{d}x=0 \tag{7}$$

where the boundary term may be written as

$$\begin{aligned}\Big[-\alpha\,u'\,\delta u\Big]_a^b &= \Big[-\alpha\,u'\,\delta u\Big]_{x=b} - \Big[-\alpha\,u'\,\delta u\Big]_{x=a} \\ &= \Big[-\alpha\,u'\,n_x\,\delta u\Big]_{x=b} + \Big[-\alpha\,u'\,n_x\,\delta u\Big]_{x=a} = \Big[-\alpha\,u'\,n_x\,\delta u\Big]_{x=\{a,b\}}.\end{aligned} \tag{8}$$

The integration by parts *weakens the differentiability requirement* for the trial functions u (i.e., for the solution).

3. **Use the Neumann boundary condition** ($\alpha\,u'\,n_x = \hat{q}$ on Γ_q) and the property of test function ($\delta u = 0$ on Γ_u) for the boundary term

$$\Big[-\alpha\,u'\,n_x\,\delta u\Big]_{x=\{a,b\}} = \Big[-\underbrace{\alpha\,u'\,n_x}_{\hat{q}}\,\delta u\Big]_{x=a} + \Big[-\alpha\,u'\,n_x\,\underbrace{\delta u}_{0}\Big]_{x=b} = \Big[-\hat{q}\,\delta u\Big]_{x=a}. \tag{9}$$

In this way, the **weak (variational) form** is obtained.

Weak form

$$\Big[-\hat{q}\,\delta u\Big]_{x=a} + \int_a^b \Big[\alpha\,u'\,\delta u' + \gamma\,u\,\delta u - f\,\delta u\Big]\,\mathrm{d}x = 0. \tag{10}$$

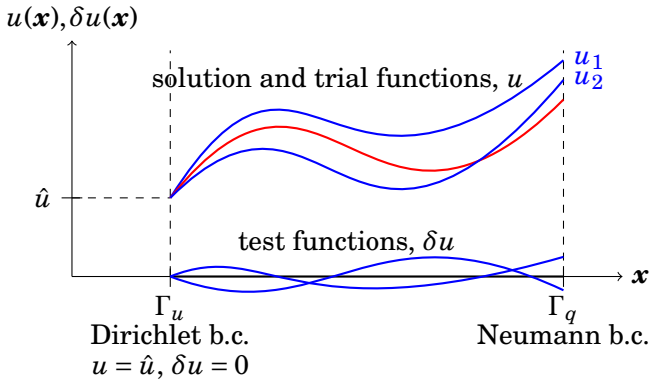
The weak form is ***mathematically equivalent*** to the strong one, that is, if u is a solution to the strong (local, differential) formulation of a BVP, it also satisfies the corresponding weak (global, integral) formulation for any δu (admissible, i.e., sufficiently smooth and $\delta u = 0$ on Γ_u).

The weak integral form requires that:

- The essential boundary conditions must be explicitly satisfied by the trial functions: $u = \hat{u}$ on Γ_u. (In case of displacement formulations of many mechanical and structural engineering problems this is called **kinematic admissibility requirement**.)
- Consequently, the test functions must satisfy the adequate *homogeneous* essential boundary conditions: $\delta u = 0$ on Γ_u.
- The trial functions u (and test functions, δu) need only to be continuous. (Remember that in the case of strong form the continuity of the first derivative of solution u was required.)

Remarks:

- The strong form can be derived from the corresponding weak formulation if we take more demanding assumptions for the smoothness of trial functions (i.e., one-order higher differentiability).
- In variational methods, any test function is a variation defined as the difference between any two trial functions. Since any trial function satisfy the essential boundary conditions, the requirement that $\delta u = 0$ on Γ_u follows immediately (see Fig. 1).



u_1, u_2 – arbitrary trial functions

$$\delta u = u_1 - u_2 \quad \text{and} \quad \left.\begin{array}{ll} u_1 = \hat{u} & \text{on } \Gamma_u \\ u_2 = \hat{u} & \text{on } \Gamma_u \end{array}\right\} \quad \rightarrow \quad \delta u = 0 \quad \text{on } \Gamma_u$$

Figure 1: Test and trial functions.

2.4 Associated variational problem

Let:

- $\mathcal{U}$, $\mathcal{W}$ are functional spaces. The first one is called the **space of solution** (or trial functions), the other one is the **space of test functions** (or weighting functions),
- $\mathcal{A}$ is a **bilinear form** defined on $\mathcal{U} \times \mathcal{W}$,
- $\mathcal{F}$ is a **linear form** defined on $\mathcal{W}$,
- $\mathcal{P}$ is a certain **functional** defined on $\mathcal{U}$.

The weak form is equivalent to a variational problem.

Weak form vs. variational problem

$$\text{Weak formulation:} \quad \text{Find } u \in \mathcal{U} \text{ so that} \quad \mathcal{A}(u, \delta u) = \mathcal{F}(\delta u) \quad \forall\ \delta u \in \mathcal{W}. \tag{11}$$

$$\text{Variational problem:} \quad \text{Find } u \in \mathcal{U} \text{ which minimizes } \mathcal{P}(u). \tag{12}$$

Example 2. In the case of the model problem:

$$\mathcal{A}(u, \delta u) = \int_a^b \left[\alpha\, u' \delta u' + \gamma\, u\, \delta u \right] \mathrm{d}x, \qquad \mathcal{F}(\delta u) = \int_a^b f\, \delta u \, \mathrm{d}x + \left[\hat{q}\, \delta u \right]_{x=a}.$$

The weak form is the statement of the **principle of the minimum total potential energy**

$$\boxed{\delta \mathcal{P}(u) = 0}, \qquad \delta \mathcal{P}(u) = \mathcal{A}(u, \delta u) - \mathcal{F}(\delta u) \tag{13}$$

where:

- δ is now the **variational symbol**,

- $\mathcal{P}(u)$ is the **potential energy** defined by the **quadratic functional**

$$\mathcal{P}(u) = \frac{1}{2}\mathcal{A}(u,u) - \mathcal{F}(u). \tag{14}$$

This definition holds only when the **bilinear form is *symmetric*** in u and δu since:

$$\frac{1}{2}\delta\mathcal{A}(u,u) = \frac{1}{2}\Big(\underbrace{\mathcal{A}(\delta u,u)}_{\mathcal{A}(u,\delta u)} + \mathcal{A}(u,\delta u)\Big) = \mathcal{A}(u,\delta u), \qquad \delta\mathcal{F}(u) = \mathcal{F}(\delta u). \tag{15}$$

Example 3. In the case of the model problem:

$$\mathcal{P}(u) = \frac{1}{2}\mathcal{A}(u,u) - \mathcal{F}(u) = \int_a^b \left[\frac{\alpha}{2}\left(u'\right)^2 + \frac{\gamma}{2}u^2 - f\,u\right]\mathrm{d}x - \Big[\hat{q}\,u\Big]_{x=a},$$

$$\delta\mathcal{P}(u) = \mathcal{A}(u,\delta u) - \mathcal{F}(\delta u) = \int_a^b \Big[\alpha\,u'\,\delta u' + \gamma\,u\,\delta u - f\,\delta u\Big]\mathrm{d}x - \Big[\hat{q}\,\delta u\Big]_{x=a}.$$

3 Galerkin method

3.1 Discrete (approximated) problem

If the problem is *well-posed* one can try to find an **approximated solution** u_h by solving the so-called **discrete problem** which is an approximation of the corresponding variational problem.

Discrete (approximated) problem

$$\begin{aligned} &\text{Find } u_h \in \mathcal{U}_h \text{ so that} \\ &\mathcal{A}_h(u_h,\delta u_h) = \mathcal{F}_h(\delta u_h) \quad \forall\,\delta u_h \in \mathcal{W}_h\,. \end{aligned} \tag{16}$$

Here:

- $\mathcal{U}_h$ is a finite-dimension space of functions called **approximation space** whereas u_h is the **approximate solution** (i.e., *approximate* to the *original* problem).
- δu_h are **discrete test functions** from the **discrete test space** $\mathcal{W}_h$. In the Galerkin method $\boxed{\mathcal{W}_h = \mathcal{U}_h}$. (In general, $\mathcal{W}_h \neq \mathcal{U}_h$.)
- $\mathcal{A}_h$ is an approximation of the bilinear form $\mathcal{A}$.
- $\mathcal{F}_h$ is an approximation of the linear form $\mathcal{F}$.

3.2 System of algebraic equations

In the Galerkin method ($\mathcal{W} = \mathcal{U}$) *the same* **shape functions**, $\phi_i(x)$, are used to ***interpolate*** the approximate solution as well as the (discrete) test functions:

$$u_h(x) = \sum_{j=1}^{N} \theta_j\,\phi_j(x), \qquad \delta u_h(x) = \sum_{i=1}^{N} \delta\theta_i\,\phi_i(x). \tag{17}$$

Here, θ_i are called **degrees of freedom**.

Using this interpolation for the approximated problem leads to a system of algebraic equations (as described below).

- The left-hand and right-hand sides of the problem equation yield:

$$\mathcal{A}_h(u_h,\delta u_h)=\sum_{i=1}^{N}\sum_{j=1}^{N}\mathcal{A}_h(\phi_j,\phi_i)\,\theta_j\,\delta\theta_i=\sum_{i=1}^{N}\sum_{j=1}^{N}A_{ij}\,\theta_j\,\delta\theta_i\,, \tag{18}$$

$$\mathcal{F}_h(\delta u_h)=\sum_{i=1}^{N}\mathcal{F}_h(\phi_i)\,\delta\theta_i=\sum_{i=1}^{N}F_i\,\delta\theta_i\,, \tag{19}$$

  where the (bi)linearity property is used, and the **coefficient matrix** (stiffness matrix) and the **right-hand-side vector** are defined as follow, respectively:

$$A_{ij}=\mathcal{A}_h(\phi_j,\phi_i),\qquad F_i=\mathcal{F}_h(\phi_i). \tag{20}$$

- Now, the approximated problem may be written as

$$\sum_{i=1}^{N}\sum_{j=1}^{N}\Big[A_{ij}\,\theta_j-F_i\Big]\delta\theta_i=0\quad\forall\,\delta\theta_i. \tag{21}$$

- It is (always) true if the expression in brackets equals zero which gives the **system of algebraic equations** (for θ_j =?):

$$\sum_{i=1}^{N}A_{ij}\,\theta_j=F_i\,. \tag{22}$$

Example 4. In the case of our model problem we have

$$A_{ij}=\mathcal{A}_h(\phi_j,\phi_i)=\int_a^b\Big[\alpha\,\phi_i'\,\phi_j'+\gamma\,\phi_i\,\phi_j\Big]\,\mathrm{d}x\,,$$

$$F_i=\mathcal{F}_h(\phi_i)=\int_a^b f\,\phi_i\,\mathrm{d}x+\Big[\hat{q}\,\phi_i\Big]_{x=a}.$$

4 Finite element model

4.1 Discretization and (linear) shape functions

Figure 2 presents linear approximation functions (the shape functions) for the domain interval. The procedure of constructing such linear interpolants is described below.

- The domain interval is divided into $N-1$ **finite elements** (subdomains).
- There are N **nodes**, each with only 1 **degree of freedom (DoF)**.

- **Local (or element) shape function** is (most often) defined on an element in this way that it is equal 1 in a particular node and 0 in the other(s). So, there are only two linear interpolation functions in 1D finite element. Higher-order interpolation functions involve additional nodes inside element.
- **Global shape function** ϕ_i is defined on the whole domain as:
  - local shape functions on (neighboring) elements sharing the node (or DoF) i,
  - identically equal zero on all other elements.

Shape functions for internal nodes ($i = 2, \ldots, (N-1)$):

$$\phi_i = \begin{cases} \dfrac{x - x_{i-1}}{h_{i-1}} & \text{for } x \in \Omega_{i-1}\,, \\ \dfrac{x_{i+1} - x}{h_i} & \text{for } x \in \Omega_i\,, \\ 0 & \text{otherwise.} \end{cases} \tag{23}$$

Shape functions for boundary nodes ($i = 1$ or N):

$$\phi_1 = \begin{cases} \dfrac{x_2 - x}{h_1} & \text{for } x \in \Omega_1\,, \\ 0 & \text{otherwise,} \end{cases} \qquad \phi_N = \begin{cases} \dfrac{x - x_{N-1}}{h_{N-1}} & \text{for } x \in \Omega_{N-1}\,, \\ 0 & \text{otherwise.} \end{cases} \tag{24}$$

First derivatives of shape functions (see Fig. 3) are discontinuous at interfaces (points) between elements (in the case of linear interpolation they are element-wise constant):

$$\phi_1' = \begin{cases} -\dfrac{1}{h_1} & \text{for } x \in \Omega_1\,, \\ 0 & \text{otherwise,} \end{cases} \quad \phi_i' = \begin{cases} \dfrac{1}{h_{i-1}} & \text{for } x \in \Omega_{i-1}\,, \\ -\dfrac{1}{h_i} & \text{for } x \in \Omega_i\,, \\ 0 & \text{otherwise.} \end{cases} \quad \phi_N' = \begin{cases} \dfrac{1}{h_{N-1}} & \text{for } x \in \Omega_{N-1}\,, \\ 0 & \text{otherwise.} \end{cases} \tag{25}$$

4.2 Lagrange interpolation functions

Figure 4 presents the linear and quadratic Lagrange interpolation polynomials.

4.3 Finite element system of algebraic equations

4.3.1 Matrix of the system

- The symmetry of the bilinear form $\mathcal{A}$ involves the symmetry of the matrix of the FE system of algebraic equations, i.e., $A_{ij} = A_{ji}$.
- A component A_{ij} (corresponding to the degrees of freedom i and j) is defined as an integral (over the problem domain) of a sum of a product of shape functions, ϕ_i and ϕ_j, and a product of their derivatives, ϕ_i' and ϕ_j'.
- The product of two shape functions (or their derivatives) is nonzero only on the elements that contain the both corresponding degrees of freedom (since a shape function corresponding to a particular degree of freedom is nonzero only on the elements sharing it).

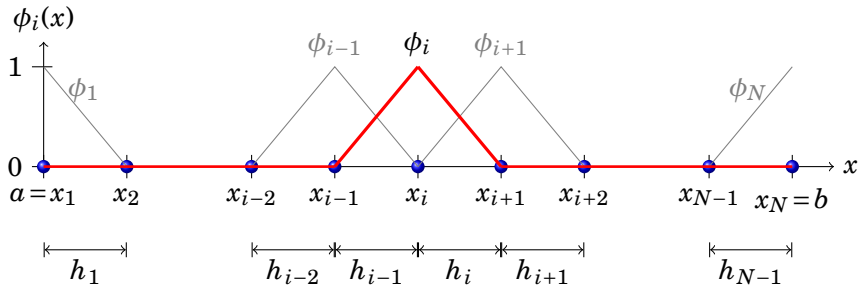


Figure 2: Finite element discretization and interpolation by linear shape functions of a 1-dimensional domain (i.e., the interval $[a,b]$).

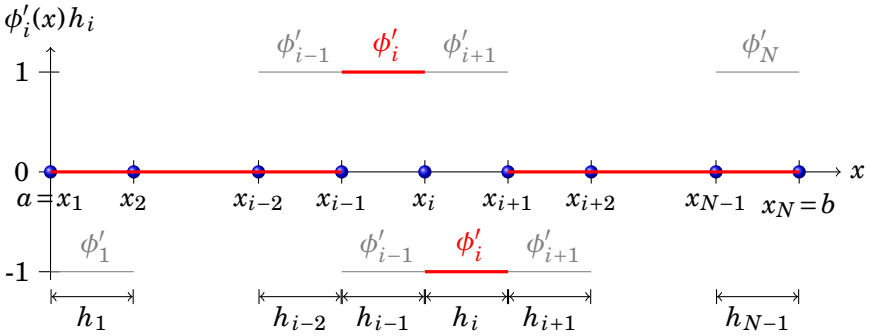
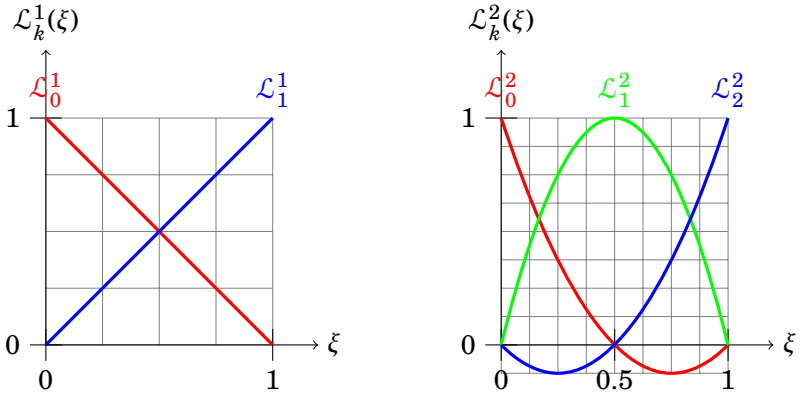


Figure 3: First derivatives of linear shape functions.



1st order (linear)

$$\mathcal{L}_0^1(\xi) = 1 - \xi,$$
$$\mathcal{L}_1^1(\xi) = \xi,$$

2nd order (quadratic)

$$\mathcal{L}_0^2(\xi) = (2\xi - 1)(\xi - 1),$$
$$\mathcal{L}_1^2(\xi) = 4\xi(1 - \xi),$$
$$\mathcal{L}_2^2(\xi) = \xi(2\xi - 1).$$

Figure 4: Lagrange interpolation polynomials of the first (linear) and second-order (quadratic). In the latter case an additional node is needed inside the element at $\xi = 0.5$.

- Therefore, the integral can be computed as a sum of the integrals defined only over these finite elements that share the both degrees of freedom (since the contribution from all other elements is null):

$$A_{ij} = \sum_{e \in \mathcal{E}} A_{ij}^{(e)} = \sum_{e \in \mathcal{E}(i,j)} A_{ij}^{(e)}. \tag{26}$$

Here: $\mathcal{E}$ – the set of all finite elements, $\mathcal{E}(i,j)$ – the set of finite elements that contain the (both) degrees of freedom i and j.

For a 1D problem approximated by finite elements with linear shape functions the matrix of the system will be *tridiagonal*:

$$A_{ij} = \begin{cases} A_{11}^{(1)} & \text{for } i = j = 1, \\ A_{ii}^{(i-1)} + A_{ii}^{(i)} & \text{for } i = j = 2, \ldots, (N-1), \\ A_{NN}^{(N-1)} & \text{for } i = j = N, \\ A_{i,i+1}^{(i)} & \text{for } |i-j| = 1, \\ 0 & \text{for } |i-j| > 1. \end{cases} \tag{27}$$

For our model problem the nonzero elements of the matrix are:

$$A_{11} = \int_{x_1}^{x_2} \left[\alpha \left(\phi_1'\right)^2 + \gamma \phi_1^2 \right] \mathrm{d}x = \int_{x_1}^{x_1+h_1} \frac{\alpha + \gamma \left(x_1 + h_1 - x\right)^2}{h_1^2} \mathrm{d}x, \tag{28a}$$

$$A_{ii} = \int_{x_{i-1}}^{x_{i+1}} \left[\alpha \left(\phi_i'\right)^2 + \gamma \phi_i^2 \right] \mathrm{d}x = \int_{x_i - h_{i-1}}^{x_i} \frac{\alpha + \gamma \left(x - x_i + h_{i-1}\right)^2}{h_{i-1}^2} \mathrm{d}x + \int_{x_i}^{x_i+h_i} \frac{\alpha + \gamma \left(x_i + h_i - x\right)^2}{h_i^2} \mathrm{d}x, \tag{28b}$$
$$i = 2, \ldots, (N-1),$$

$$A_{NN} = \int_{x_{N-1}}^{x_N} \left[\alpha \left(\phi_N'\right)^2 + \gamma \phi_N^2 \right] \mathrm{d}x = \int_{x_N - h_{N-1}}^{x_N} \frac{\alpha + \gamma \left(x - x_N + h_{N-1}\right)^2}{h_{N-1}^2} \mathrm{d}x, \tag{28c}$$

$$A_{i,(i+1)} = \int_{x_i}^{x_{i+1}} \left[\alpha \phi_i' \phi_{i+1}' + \gamma \phi_i \phi_{i+1} \right] \mathrm{d}x = \int_{x_i}^{x_i+h_i} \frac{-\alpha + \gamma \left(x_i + h_i - x\right)\left(x - x_i\right)}{h_i^2} \mathrm{d}x, \tag{28d}$$
$$i = 1, \ldots, (N-1).$$

For a homogeneous material when $\alpha(x) = \text{const} = \alpha$ and $\gamma(x) = \text{const} = \gamma$, our (tridiagonal) matrix is defined as:

$$A_{ij} = \begin{cases} \frac{\alpha}{h_1} + \frac{\gamma h_1}{3} & \text{for } i = j = 1, \\ \frac{\alpha}{h_{i-1}} + \frac{\gamma h_{i-1}}{3} + \frac{\alpha}{h_i} + \frac{\gamma h_i}{3} & \text{for } i = j = 2, \ldots, (N-1), \\ \frac{\alpha}{h_{N-1}} + \frac{\gamma h_{N-1}}{3} & \text{for } i = j = N, \\ -\frac{\alpha}{h_i} + \frac{\gamma h_i}{6} & \text{for } |i-j| = 1, \\ 0 & \text{for } |i-j| > 1. \end{cases} \tag{29}$$

4.3.2 Right-hand-side vector

Right-hand-side vector:

$$F_i = \sum_{e\in\mathcal{E}} F_i^{(e)} = \sum_{e\in\mathcal{E}(i)} F_i^{(e)}. \tag{30}$$

Here: $\mathcal{E}$ – the set of all finite elements, $\mathcal{E}(i)$ – the set of finite elements that contain the degree of freedom i.

For our model problem we have:

$$F_1 = \int_{x_1}^{x_2} f\,\phi_1\,\mathrm{d}x + \Big[\hat{q}\,\phi_1\Big]_{x=x_1} = \int_{x_1}^{x_1+h_1} \frac{f\left(x_1+h_1-x\right)}{h_1}\,\mathrm{d}x + \hat{q}\,, \tag{31a}$$

$$F_i = \int_{x_{i-1}}^{x_{i+1}} f\,\phi_i\,\mathrm{d}x = \int_{x_i-h_{i-1}}^{x_i} \frac{f\left(x-x_i+h_{i-1}\right)}{h_{i-1}}\,\mathrm{d}x + \int_{x_i}^{x_i+h_i} \frac{f\left(x_i+h_i-x\right)}{h_i}\,\mathrm{d}x\,, \quad i=2,\ldots,(N-1), \tag{31b}$$

$$F_N = ?\ \text{(to be computed as a reaction to the essential b.c. imposed in this node).} \tag{31c}$$

Finally, for a uniform source (load), i.e., when $f(x) = \text{const} = f$, the r.h.s. vector is:

$$F_i = \begin{cases} \frac{f\,h_1}{2} + \hat{q} & \text{for } i=1\,, \\ \frac{f\left(h_{i-1}+h_i\right)}{2} & \text{for } i=2,\ldots,(N-1)\,, \\ F_N = ? & \text{for } i=N \text{ (a reaction to the essential b.c.).} \end{cases} \tag{32}$$

4.4 Imposition of the essential boundary conditions

In general, the assembled matrix A_{ij} is *singular* and the system of algebraic equations is undetermined. To make it solvable **the essential boundary conditions need to be imposed**.

Let $\mathcal{B}$ be the set of all degrees of freedom where the essential boundary conditions are applied, that is, for $n \in \mathcal{B}$: $\theta_n = \hat{\theta}_n$ where $\hat{\theta}_n$ is a given value. In practice, the essential BCs are imposed as described below.

- Compute a new r.h.s. vector

$$\tilde{F}_i = F_i - \sum_{n\in\mathcal{B}} A_{in}\hat{\theta}_n \quad \text{for } i=1,\ldots,N. \tag{33}$$

- Set $\tilde{F}_n = \hat{\theta}_n$.
- Set $\tilde{A}_{nn} = 1$ and all other components in the n-th row and n-th column to zero, i.e.,

$$\tilde{A}_{ni} = \tilde{A}_{in} = \delta_{in} \quad \text{for } i=1,\ldots,N. \tag{34}$$

Now, the new (sightly modified) system of equations, $\boxed{\tilde{A}_{ij}\theta_i = \tilde{F}_j}$, is solved for θ_i. Eventually, a reaction reaction (force, source) is computed

$$F_n = \sum_{i=1}^{N} A_{ni}\,\theta_i\,. \tag{35}$$

For our model problem the essential b.c. are imposed only in the last node (i.e., the N-th DoF) where a known value $\hat{\theta}_N$ is given, so the modified matrix and r.h.s. vector can be formally written as follows:

$$\tilde{A}_{ij} = \begin{cases} A_{ij} & \text{for } i,j = 1,\ldots,(N-1), \\ \delta_{Nj} & \text{for } i = N,\ j = 1,\ldots,N, \\ \delta_{iN} & \text{for } i = 1,\ldots,N,\ j = N, \end{cases} \quad \tilde{F}_i = \begin{cases} F_i - A_{iN}\hat{\theta}_N & \text{for } i = 1,\ldots,(N-1), \\ \hat{\theta}_N & \text{for } i = N. \end{cases} \tag{36}$$

After the solution of the modified system, the reaction may be computed:

$$F_N = \sum_{i=1}^{N} A_{Ni}\,\theta_i = A_{N,(N-1)}\,\theta_{N-1} + A_{NN}\,\hat{\theta}_N\,. \tag{37}$$

4.5 Results of analytical and FE solutions

Data for the problem:

$$\alpha(x) = 1, \qquad \gamma = 3, \qquad f(x) = 1.$$
$$a = 0, \quad q(0) = \hat{q} = 1, \qquad b = 2, \quad u(2) = \hat{u} = 0,$$