HOMEWORK #3 SOLUTION SET

① a) $-(u')' = 1, \quad 0 < x < 1, \quad u(0) = u(1) = 0$

i) THE WORK FUNCTION

$$\pi[u] = \frac{1}{2}\int_0^1 (u')^2\,dx - \int_0^1 1 \cdot u\,dx$$

$$u = \hat{u}_N = \sum_{j=1}^{N} \alpha_j \phi_j$$

$$\boxed{\pi[\hat{u}_N] = \frac{1}{2}\int_0^1 \Big(\sum_{j=1}^{N} \alpha_j \phi_j'\Big)^2 - 2\sum_{j=1}^{N} \alpha_j \phi_j \, dx}$$

ii) THE RESIDUAL FUNCTION

$$r = \mathcal{L}u - f = -(u')' - 1$$

$$\int_0^1 r v\,dx = \int_0^1 v\left(-(u')' - 1\right)dx = 0$$

$$u = \hat{u}_N = \sum_{j=1}^{N} \alpha_j \phi_j, \quad v = \sum_{k=1}^{N} \beta_k \psi_k$$

$$\boxed{\sum_{k=1}^{N} \beta_k \left[\sum_{j=1}^{N} \alpha_j \int_0^1 \psi_k (-\phi_j'')\,dx - \int_0^1 \psi_k\,dx\right] = 0}$$

iii) THE WEAK FORMULATION

FROM THE RESIDUAL FUNCTION,

$$\int_0^1 v(-(u')' - 1)\,dx = -u'v\Big|_0^1 + \int_0^1 v'u' - v\,dx = 0$$

$$\int_0^1 v'u' - v\,dx = 0$$

$$u = \hat{u}_N = \sum_{j=1}^{N} \alpha_j \phi_j, \quad v = \sum_{k=1}^{N} \beta_k \psi_k$$

$$\boxed{\sum_{k=1}^{N} \beta_k \left[\sum_{j=1}^{N} \alpha_j \int_0^1 \phi_j' \psi_k'\,dx - \int_0^1 \psi_k\,dx\right] = 0}$$

iv) $$\|e\|_{L_2} = \sqrt{\int_0^1 (\hat{u}_N - u)^2\,dx}$$

v) $$\|e\|_E = \sqrt{\frac{1}{2}\int_0^1 (\hat{u}_N' - u')^2\,dx}$$

b) $-(u')' = f, \quad 0 < x < 1, \quad u(0) = 1, \quad u'(1) = 1$

i) THE WORK FUNCTION

LET $u = u_b + \overbrace{\sum_{j=1}^{N} \alpha_j \phi_j}^{\hat{u}_N}, \quad u_b = 1 + x$. NOTE, $u_b'' = 0$.

$$\pi[\hat{u}_N] = \frac{1}{2}\int_0^1 (u')^2 dx - \int_0^1 f u \, dx$$

$$\boxed{\pi[\hat{u}_N] = \frac{1}{2}\int_0^1 \Big(\sum_{j=1}^{N} \alpha_j \phi_j'\Big)^2 - 2f \sum_{j=1}^{N} \alpha_j \phi_j \, dx}$$

ii) THE RESIDUAL FUNCTION

$r = \mathcal{L}u - f. \quad u = u_b + \overbrace{\sum_{j=1}^{N} \alpha_j \phi_j}^{\hat{u}_N}, \quad u_b = 1 + x. \quad v = \sum_{k=1}^{N} \beta_k \psi_k$

$$\int_0^1 r v \, dx = \int_0^1 v \mathcal{L}\hat{u}_N - vf + v \cancel{\mathcal{L}u_b}^{0} \, dx = 0$$

$$\boxed{\sum_{k=1}^{N} \beta_k \left[\sum_{j=1}^{N} \alpha_j \int_0^1 \psi_k \left(-(\phi_j')'\right) dx - \int_0^1 f \psi_k \, dx\right] = 0}$$

iii) THE WEAK FORM

THIS SATISFIES ONLY ESSENTIAL B.C.'S.

FROM $\int_0^1 r v \, dx = \int_0^1 v(-u'') - vf \, dx = 0$, WITH $u = 1 + \sum_{j=1}^{N} \alpha_j \phi_j$

$$-vu'\Big|_0^1 + \int_0^1 v'u' - vf \, dx = 0 \qquad v = \sum_{k=1}^{N} \beta_k \psi_k$$

$$\boxed{\sum_{k=1}^{N} \beta_k \left[\sum_{j=1}^{N} \alpha_j \int_0^1 \phi_j' \psi_k' \, dx - \int_0^1 \psi_k f \, dx - \psi_k(1) u'(1)\right] = 0}$$

iv) $\boxed{\|e\|_{L_2} = \sqrt{\int_0^1 (\hat{u}_N - u)^2 dx}}$

v) $\boxed{\|e\|_E = \sqrt{\frac{1}{2}\int_0^1 (\hat{u}_N' - u')^2 dx}}$

c) $-(Ku')' + Bu' + Cu = f, \quad 0<x<1, \quad u'(0)=1, \ u'(1)=1$

i) THE WORK FUNCTION

$$\pi[u] = \frac{1}{2}\int_0^1 K(u')^2 + Bu'u + Cu^2\,dx - \int_0^1 fu\,dx$$

$u = u_b + \hat{u}_N, \quad u_b = x, \quad \hat{u}_N = \sum_{j=1}^{N} a_j\phi_j.$ NOTE, $\mathcal{L}u_b \neq 0.$

$$-(K\hat{u}_N')' + B\hat{u}_N' + C\hat{u}_N = f \underbrace{- B - Cx}_{-\mathcal{L}u_b}$$

$$\boxed{\pi[\hat{u}_N] = \frac{1}{2}\int_0^1 K\Big(\sum_{j=1}^{N} a_j\phi_j'\Big)^2 + B\Big(\sum_{j=1}^{N} a_j\phi_j'\Big)\Big(\sum_{j=1}^{N} a_j\phi_j\Big) + C\Big(\sum_{j=1}^{N} a_j\phi_j\Big)^2 - 2(f - B - Cx)\Big(\sum_{j=1}^{N} a_j\phi_j\Big)\,dx}$$

ii) THE RESIDUAL FUNCTION

$r = \mathcal{L}u - f \qquad u = x + \hat{u}_N, \quad \hat{u}_N = \sum_{j=1}^{N} a_j\phi_j. \implies r = \mathcal{L}\hat{u}_N + \mathcal{L}x - f.$

$$\int_0^1 rv\,dx = \int_0^1 v\left(-(K\hat{u}_N')' + B\hat{u}_N' + C\hat{u}_N - f + B + Cx\right)dx = 0$$

$v = \sum_{k=1}^{N} \beta_k \psi_k$

$$\boxed{\sum_{k=1}^{N} \beta_k\left[\sum_{j=1}^{N} a_j \int_0^1 \psi_k\left(-K\phi_j'' + B\phi_j' + C\phi_j\right)dx - \int_0^1 \psi_k(f - B - Cx)\,dx\right] = 0}$$

iii) THE WEAK FORM

FROM $\int_0^1 v(-(Ku')' + Bu' + Cu - f)\,dx = 0,$

$$-vKu'\Big|_0^1 + \int_0^1 Kv'u' + Bvu' + Cvu - fv\,dx = 0$$

$u = \sum_{j=1}^{N} a_j\phi_j, \quad v = \sum_{k=1}^{N} \beta_k\psi_k$

$$\boxed{\sum_{k=1}^{N} \beta_k\left[\sum_{j=1}^{N} a_j \int_0^1 K\phi_j'\psi_k' + B\phi_j'\psi_k + C\phi_j\psi_k\,dx - \int_0^1 \psi_k f\,dx - \psi_k(1)\cdot K + \psi_k(0)K\right] = 0}$$

iv) $\boxed{\|e\|_{L_2} = \sqrt{\int_0^1 (u - \mathcal{U})^2\,dx}}$

v) $\boxed{\|e\|_E = \sqrt{\frac{1}{2}\int_0^1 K(u' - \mathcal{U}')^2 + B(u' - \mathcal{U}')(u - \mathcal{U}) + C(u - \mathcal{U})^2\,dx}}$

d) $-(3xu')' = \sin x, \quad 0<x<1, \quad u(0)=1, \quad u(1)=1$

i) THE WORK FUNCTION

$$\pi[u] = \frac{1}{2}\int_0^1 3x(u')^2\,dx - \int_0^1 u\sin x\,dx$$

$$u = 1 + \hat{u}_N, \quad \hat{u}_N = \sum_{j=1}^{N} a_j \phi_j.$$

$$\boxed{\pi[\hat{u}_N] = \frac{1}{2}\int_0^1 3x\left(\sum_{j=1}^{N} a_j\phi_j'\right)^2 - 2\left(\sum_{j=1}^{N} a_j\phi_j\right)\sin x\,dx}$$

ii) THE RESIDUAL FUNCTION

$r = \mathcal{L}u - f. \qquad u = u_b + u_N, \quad u_b = 1, \quad u_N = \sum_{j=1}^{N} a_j\phi_j.$ NOTE, $\mathcal{L}u_b = 0.$

$$\int_0^1 rv = \int_0^1 v\left(-\left(3x\sum_{j=1}^{N} a_j\phi_j'\right)' - \sin x\,dx\right) = 0$$

LET $v = \sum_{k=1}^{N} \beta_k \psi_k$

$$\boxed{\sum_{k=1}^{N}\beta_k\left[\sum_{j=1}^{N} a_j\int_0^1 \psi_k\left(-(3x\phi_j')'\right)dx - \int_0^1 \psi_k \sin x\,dx\right] = 0.}$$

iii) THE WEAK FORM

FROM $\int_0^1 v\left(-(3xu')' - \sin x\right)dx = 0$

$$\cancel{-3xu'v\Big|_0^1} + \int_0^1 3xv'u' - v\sin x\,dx = 0.$$

$$u = \sum_{j=1}^{N} a_j\phi_j, \quad v = \sum_{k=1}^{N}\beta_k\psi_k$$

$$\boxed{\sum_{k=1}^{N}\beta_k\left[\sum_{j=1}^{N} a_j\int_0^1 3x\psi_k'\phi_j'\,dx - \int_0^1 \psi_k\sin x\,dx\right] = 0}$$

iv) $\|e\|_{L_2} = \sqrt{\int_0^1 (u-\tilde{u})^2\,dx}$

v) $\|e\|_E = \sqrt{\frac{1}{2}\int_0^1 3x(u'-\tilde{u}')^2\,dx}$

(2) FOR THE WORK FUNCTION,

U MUST SATISFY BOTH ESSENTIAL & NATURAL B.C.'S

FOR THE RESIDUAL FUNCTION,

U MUST SATISFY BOTH ESSENTIAL & NATURAL B.C.'S

V MUST SATISFY HOMOGENEOUS VERSIONS OF THE GIVEN B.C.'S

FOR THE WEAK FORM,

U MUST SATISFY ESSENTIAL B.C.'S

FOR ALL, U & V MUST BE SQUARE INTEGRABLE

(3) THE DIRECT FORM (SOLVING THE WORK FUNCTION) OFTEN RESULTS IN A NONLINEAR SET OF COUPLED EQUATIONS $\left(\left(\sum_{j=1}^{N} a_j \phi_j' \right)^2 \right.$ FOR INSTANCE). THE RESIDUAL FUNCTION EXPLICITLY SATISFIES ALL B.C.'S (THOUGH SOMETIMES YIELDS $\mathcal{L}u_0$ TERMS AS EFFECTIVE FORCES. THE WEAK FORM ONLY SATISFIES ESSENTIAL B.C.'S, AND CAN INTRODUCE FLUX TERMS $(Ku'v|_0^1)$ AS EFFECTIVE FORCES.