# Math 100B Homework 5

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**Problem 1: a)** The likelihood function:

$$L(\lambda|x_1, x_2, \ldots, x_n) = \prod Pr\{X = xi\} = \prod \frac{2^{xi}\lambda^{xi}e^{-2\lambda}}{xi!}$$

The log-likelihood function:

$$\begin{aligned}\ln L(\lambda|x_1, x_2, \ldots, x_n) &= \ln\left[\prod \frac{2^{xi}\lambda^{xi}e^{-2\lambda}}{xi!}\right]\\ &= \sum \ln\left[\frac{2^{xi}\lambda^{xi}e^{-2\lambda}}{xi!}\right]\\ &= \sum \left[xi\ln(2) + xi\ln(\lambda) - 2\lambda - \ln(xi!)\right]\end{aligned}$$

Differentiated and set to 0:

$$\begin{aligned}\frac{d}{d\lambda}\ln L(\lambda|x_1, x_2, \ldots, x_n) &= \sum\left[\frac{xi}{\lambda} - 2\right] = 0\\ \Rightarrow \sum \frac{xi}{\lambda} &= 2n\\ \Rightarrow \lambda &= \frac{\sum xi}{2n}\end{aligned}$$

**b)** The likelihood function:

$$L(p|x_1, x_2, \ldots, x_m) = \prod \binom{n}{xi} p^{xi}(1-p)^{n-xi}$$

The log-likelihood function:

$$\begin{aligned}\ln L(p|x_1, x_2, \ldots, x_m) &= \ln\left[\prod \binom{n}{xi} p^{xi}(1-p)^{n-xi}\right]\\ &= \sum \ln\left[\binom{n}{xi} p^{xi}(1-p)^{n-xi}\right]\\ &= \sum\left[\ln\binom{n}{xi} + xi\ln(p) + (n-xi)\ln(1-p)\right]\end{aligned}$$

Differentiated and set to 0:

$$\frac{d}{dp}\ln L(p|x_1,x_2,\ldots,x_m)=\sum\left[\frac{xi}{p}-\frac{n-xi}{1-p}\right]=0$$

$$\Rightarrow \sum xi - p\sum n = 0$$

$$\Rightarrow p=\frac{\sum xi}{m\cdot n}$$

**c)** The likelihood function:

$$\begin{aligned}L(\mu,\sigma^2|x_1,x_2,\ldots,x_n)&=\prod f(x_i;\mu,\sigma^2)\\&=\prod\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{(x_i-\mu)^2}{2\sigma^2}}\\&=(2\pi\sigma^2)^{-\frac{n}{2}}e^{-\frac{1}{2\sigma^2}\sum(x_i-\mu)^2}\end{aligned}$$

The log-likelihood function:

$$\begin{aligned}\ln L(\mu,\sigma^2|x_1,x_2,\ldots,x_n)&=\ln\left[(2\pi\sigma^2)^{-\frac{n}{2}}e^{-\frac{1}{2\sigma^2}\sum(x_i-\mu)^2}\right]\\&=-\frac{n}{2}\ln(2\pi\sigma^2)-\frac{1}{2\sigma^2}\sum(x_i-\mu)^2\\&=-\frac{n}{2}\ln(2\pi)-\frac{n}{2}\ln(\sigma^2)-\frac{1}{2\sigma^2}\sum(x_i-\mu)^2\end{aligned}$$

Partial differentiated with respect to each parameter, and set to 0:

$$\frac{d}{d\mu}\ln L(\mu,\sigma^2|x_1,x_2,\ldots,x_n)=\frac{1}{\sigma^2}\sum(x_i-\mu)=0$$

$$\Rightarrow \mu=\frac{1}{n}\sum x_i$$

$$\frac{d}{d\sigma^2}\ln L(\mu,\sigma^2|x_1,x_2,\ldots,x_n)=-\frac{n}{2\sigma^2}+\frac{1}{2(\sigma^2)^2}\sum(x_i-\mu)^2=0$$

$$\Rightarrow \sigma^2=\frac{1}{n}\sum(x_i-\mu)^2$$

**d)** The likelihood function:

$$L(\theta|x_1,x_2,\ldots,x_n)=\prod f(x_i;\theta)=\prod\left(\frac{\theta}{3}e^{-\frac{\theta}{3}x_i}\right)$$

The log-likelihood function:

$$\begin{aligned}\ln L(\theta|x_1,x_2,\ldots,x_n)&=\ln\left[\prod\left(\frac{\theta}{3}e^{-\frac{\theta}{3}x_i}\right)\right]\\&=\sum\ln\left[\frac{\theta}{3}e^{-\frac{\theta}{3}x_i}\right]\\&=\sum\left[\ln(\frac{\theta}{3})-\frac{\theta}{3}x_i\right]\end{aligned}$$

Differentiated and set to 0:

$$\frac{d}{d\theta} \ln L(\theta|x_1, x_2, \ldots, x_n) = \sum \left[\frac{1}{\theta} - \frac{x_i}{3}\right] = 0$$
$$\Rightarrow \frac{n}{\theta} = \frac{1}{3}\sum x_i$$
$$\Rightarrow \theta = \frac{3n}{\sum x_i}$$

**Problem 2:** Unbiasedness: An estimator is unbiased when its expected value is the same as the actual value of a parameter. This is a property of the estimator's sampling distribution, and it holds regardless of the sample size. This means that on average, over many trials, the estimator provides correct estimates of the parameter. For instance, the sample mean is an unbiased estimator of the population mean because the expected value of the sample mean equals the population mean.

Consistency: A consistent estimator provides increasingly accurate estimates of the parameter with increasing sample size. For example, the sample mean is a consistent estimator for the population mean, which is a result of the Law of Large Numbers.

Efficiency: Among a class of unbiased estimators, an estimator is said to be efficient if it has the smallest variance. This means that it provides estimates that are closer to the true parameter value than those of other unbiased estimators.

Sufficiency: An estimator is sufficient for a parameter if it contains all the information relevant to the parameter being estimated. In other words, once we know our sufficient statistic $f(x)$, there's no additional information about $\theta$ that can be extracted from the data. This means we can simplify our task of estimating $\theta$ by focusing on the statistic f(x) rather than the entire dataset.

**Problem 3:** The Cramer-Rao Inequality provides a lower bound on the variance of an unbiased estimator. When an unbiased estimator achieves this lower bound, it is said to be efficient.