Math 151A: Problem Set 5

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Problem 1: (T) Numerical Differentiation

Using the Lagrange polynomial approximation, we can show that:

$$f^{(4)}(x_0) = \frac{f(x_0+2h) - 4f(x_0+h) + 6f(x_0) - 4f(x_0-h) + f(x_0-2h)}{h^4} + O(h^2).$$

In this problem, you will derive this formula using the method of undetermined coefficients. Start with the following expression:

$$Af(x_0+2h) + Bf(x_0+h) + Cf(x_0) + Df(x_0-h) + Ef(x_0-2h).$$

Expand each term using Taylor polynomials of degree 5 (where the sixth order term $O(h^6)$ is the error) and choose the coefficients in order to get an approximation to the fourth derivative.

Hint: You will need to solve a 5-by-5 linear system. Once you have the coefficients, check that the sixth equation (corresponding to the terms involving "$h^5 f^{(5)}(x_0)$") is zero.

Solution: Computing degree-5 Taylor Polynomial Expansion for each term:

$$\begin{aligned}
f(x_0+2h) &= f(x_0) + 2hf'(x_0) + 2^2\frac{h^2}{2!}f''(x_0) + 2^3\frac{h^3}{3!}f'''(x_0) + 2^4\frac{h^4}{4!}f''''(x_0) + 2^5\frac{h^5}{5!}f'''''(x_0) + O(h^6)\\
f(x_0+h) &= f(x_0) + hf'(x_0) + \frac{h^2}{2!}f''(x_0) + \frac{h^3}{3!}f'''(x_0) + \frac{h^4}{4!}f''''(x_0) + \frac{h^5}{5!}f'''''(x_0) + O(h^6)\\
f(x_0) &= f(x_0)\\
f(x_0-h) &= f(x_0) - hf'(x_0) + \frac{h^2}{2!}f''(x_0) - \frac{h^3}{3!}f'''(x_0) + \frac{h^4}{4!}f''''(x_0) - \frac{h^5}{5!}f'''''(x_0) + O(h^6)\\
f(x_0-2h) &= f(x_0) - 2hf'(x_0) + 2^2\frac{h^2}{2!}f''(x_0) - 2^3\frac{h^3}{3!}f'''(x_0) + 2^4\frac{h^4}{4!}f''''(x_0) - 2^5\frac{h^5}{5!}f'''''(x_0) + O(h^6)
\end{aligned}$$

Equating:

$$
\begin{aligned}
Af(x_0+2h)+Bf(x_0+h)+Cf(x_0)+Df(x_0-h)+Ef(x_0-2h) &= \\
A[f(x_0)+2hf'(x_0)+4\frac{h^2}{2!}f''(x_0)+8\frac{h^3}{3!}f'''(x_0)+16\frac{h^4}{4!}f''''(x_0)+32\frac{h^5}{5!}f'''''(x_0)+O(h^6)] \\
+B[f(x_0)+hf'(x_0)+\frac{h^2}{2!}f''(x_0)+\frac{h^3}{3!}f'''(x_0)+\frac{h^4}{4!}f''''(x_0)+\frac{h^5}{5!}f'''''(x_0)+O(h^6)] \\
+C[f(x_0)] \\
+D[f(x_0)-hf'(x_0)+\frac{h^2}{2!}f''(x_0)-\frac{h^3}{3!}f'''(x_0)+\frac{h^4}{4!}f''''(x_0)-\frac{h^5}{5!}f'''''(x_0)+O(h^6)] \\
+E[f(x_0)-2hf'(x_0)+4\frac{h^2}{2!}f''(x_0)-8\frac{h^3}{3!}f'''(x_0)+16\frac{h^4}{4!}f''''(x_0)-32\frac{h^5}{5!}f'''''(x_0)+O(h^6)]
\end{aligned}
$$

Grouping:

$$
\begin{aligned}
&= (A+B+C+D+E)f(x_0) \\
&+(2A+B-D-2E)hf'(x_0) \\
&+(4A+B+D+4E)\frac{h^2}{2!}f''(x_0) \\
&+(8A+B-D-8E)\frac{h^3}{3!}f'''(x_0) \\
&+(16A+B+D+16E)\frac{h^4}{4!}f''''(x_0) \\
&+(32A+B-D-32E)\frac{h^5}{5!}f'''''(x_0) \\
&+O(h^6)
\end{aligned}
$$

Then the system of equation is:

$$
\begin{aligned}
A+B+C+D+E &= 0 \\
2A+B-D-2E &= 0 \\
4A+B+D+4E &= 0 \\
8A+B-D-8E &= 0 \\
16A+B+D+16E &= 24 \\
32A+B-D-32E &= 0
\end{aligned}
$$

Using Gauss-Jordan to solve the first 5 equations:

$$
\begin{aligned}
A &= 1 \\
B &= -4 \\
C &= 6 \\
D &= -4 \\
E &= 1
\end{aligned}
$$

We verify that the equation relating to $h^5\, f^{(5)}(x_0) = 0$:

$$32 * (1) + (-4) - (-4) - 32 * (1) = 0$$

Problem 2: (T) Richardson's Extrapolation

We can use lower-order formulae to generate approximations with higher accuracy. Consider the finite difference formula:

$$f'(x_0) = \frac{f(x_0+h) - f(x_0)}{h} - \frac{h}{2}f^{(2)}(x_0) - \frac{h^2}{6}f^{(3)}(x_0) + O(h^3) \tag{1}$$

which includes more truncation terms than we used in the class.

a) Replacing h with $2h$ in equation (1) yields the following formula:

$$f'(x_0) = \frac{f(x_0+2h) - f(x_0)}{2h} - hf^{(2)}(x_0) - \frac{2h^2}{3}f^{(3)}(x_0) + O(h^3). \tag{2}$$

Using equations (1) and (2), show that :

$$f'(x_0) = \frac{-f(x_0+2h) + 4f(x_0+h) - 3f(x_0)}{2h} + \frac{h^2}{3}f^{(3)}(x_0) + O(h^3)$$

b) Repeat the process from Part (a) with:

$$f'(x_0) = \frac{-f(x_0+2h) + 4f(x_0+h) - 3f(x_0)}{2h} + \frac{h^2}{3}f^{(3)}(x_0) + O(h^3)$$

to show:

$$f'(x_0) = \frac{f(x_0+4h) - 12f(x_0+2h) + 32f(x_0+h) - 21f(x_0)}{12h} + O(h^3).$$

Solution:

a)

Multiplying equation (1) by 2:

$$2f'(x_0) = \frac{2f(x_0+h) - 2f(x_0)}{h} - hf^{(2)}(x_0) - \frac{h^2}{3}f^{(3)}(x_0) + O(h^3)$$

Then subtracting equation (2) from the above equation:

$$f'(x_0) = \frac{4f(x_0+h) - 4f(x_0) - [f(x_0+2h) - f(x_0)]}{h} + \frac{h^2}{3}f^{(3)}(x_0) + O(h^3)$$

Simplifying:

$$f'(x_0) = \frac{-f(x_0+2h) + 4f(x_0+h) - 3f(x_0)}{2h} + \frac{h^2}{3}f^{(3)}(x_0) + O(h^3)$$

Which is what we wanted to show.

b) Let's call this equation (3):

$$f'(x_0) = \frac{-f(x_0+2h)+4f(x_0+h)-3f(x_0)}{2h} + \frac{h^2}{3}f^{(3)}(x_0) + O(h^3)$$

Replacing all h in equation (3) by $2h$ we get equation (4):

$$f'(x_0) = \frac{-f(x_0+4h)+4f(x_0+2h)-3f(x_0)}{4h} + \frac{4h^2}{3}f^{(3)}(x_0) + O(h^3)$$

Then multiplying equation (3) by 4:

$$4f'(x_0) = \frac{-4f(x_0+2h)+16f(x_0+h)-12f(x_0)}{2h} + \frac{4h^2}{3}f^{(3)}(x_0) + O(h^3)$$

And subtracting equation (4) we find:

$$3f'(x_0) = \frac{-12f(x_0+2h)+32f(x_0+h)-21f(x_0)+f(x_0+4h)]}{4h} + O(h^3)]$$

Which simplified is:

$$f'(x_0) = \frac{f(x_0+4h)-12f(x_0+2h)+32f(x_0+h)-21f(x_0)}{12h} + O(h^3).$$

Which is what we wanted to show.

Problem 3: (T) An Application to Parameter Estimation, Population Data

Consider the logistic growth model commonly used in biology, demography, probability, sociology, etc.:

$$\frac{d}{dt}f(t) = r\left(f(t) - f(t)^2\right)$$

where $r > 0$ is the constant growth rate parameter. Suppose we are given data on a population (in millions) over the last few years:

1lightgraylightgray

t	2011	2012	2013	2014	2015
$f(t)$	0.33000	0.33443	0.33890	0.34340	0.34792

and would like to fit the model to the data. Using the ordinary differential equation at $t = 2013$ and the forward difference (2-point right-sided approximation), the central difference (3-point centered approximation), and a 5-point approximation to the derivative (check the textbook), approximate the value of r (total of 3 approximations).

Solution: Isolating r:

$$r = \frac{df/dt}{f(t) - f(t)^2}$$

Then we can estimate r by estimating the derivative.

For the **forward difference** (2-point right-sided approximation) the approximation is:

$$r_f \approx \frac{0.34340 - 0.33890}{0.33890(1 - 0.33890)} \approx 0.02009$$

The **central difference** (3-point centered approximation) is:

$$r_c \approx \frac{0.34340 - 0.33443}{2 \times 0.33890(1 - 0.33890)} \approx 0.02002$$

And the **5-point approximation** is:

$$r_5 \approx \frac{-0.34792 + 8 \times 0.34340 - 8 \times 0.33443 + 0.33000}{12 \times 0.33890(1 - 0.33890)} \approx 0.02003$$

Problem 4: (C) Finite Differences

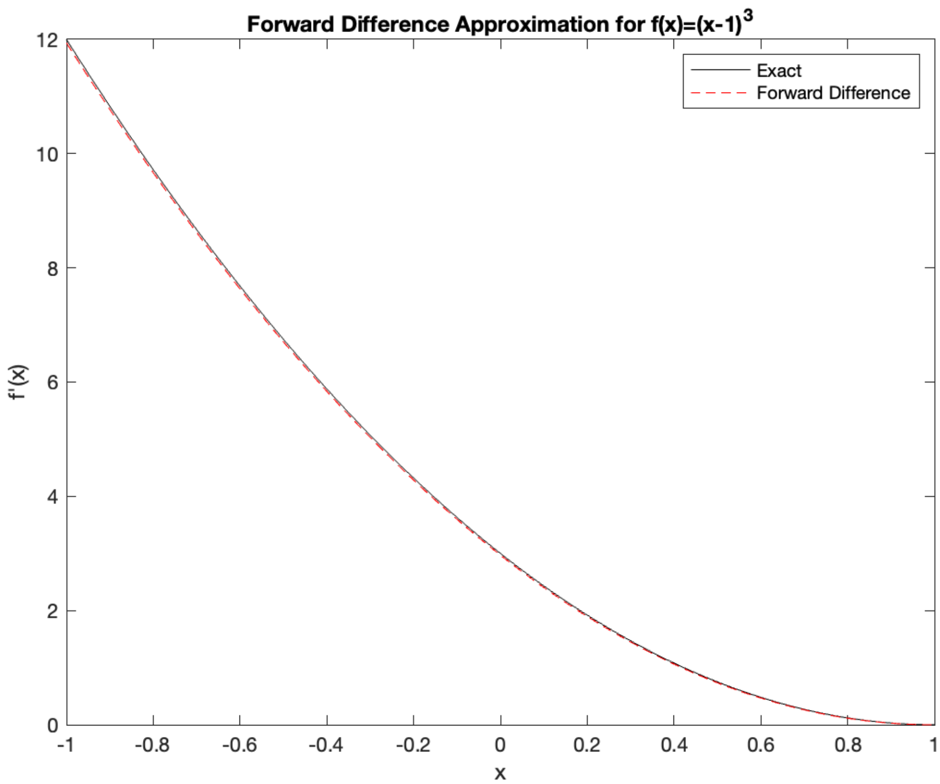
Write a program that computes an approximation to the first derivatives of:

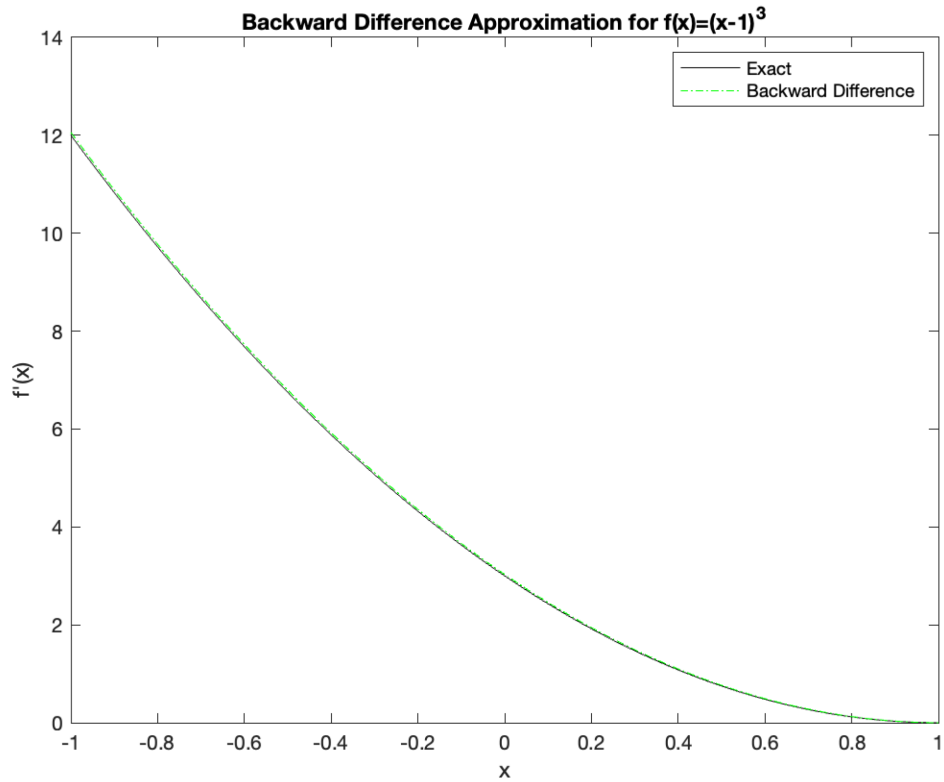
a) $f(x) = (x-1)^3$

b) $f(x) = (x-1)^2 \sin(5x)$

over the interval $x \in [-1, 1]$ using the forward, backward, and central differences with $h = 0.01$ (i.e. the grid starts at left endpoint -1 with spacing h up to the right endpoint 1). **Provide 6 plots** (i.e. your derivative approximation vs x), corresponding to each of the combinations. For credit, you must label the plots.

Solution:





Backward Difference Approximation for f(x)=(x-1)3
Exact
Backward Difference
f'(x)
x
14
12
10
8
6
4
2
0
-1
-0.8
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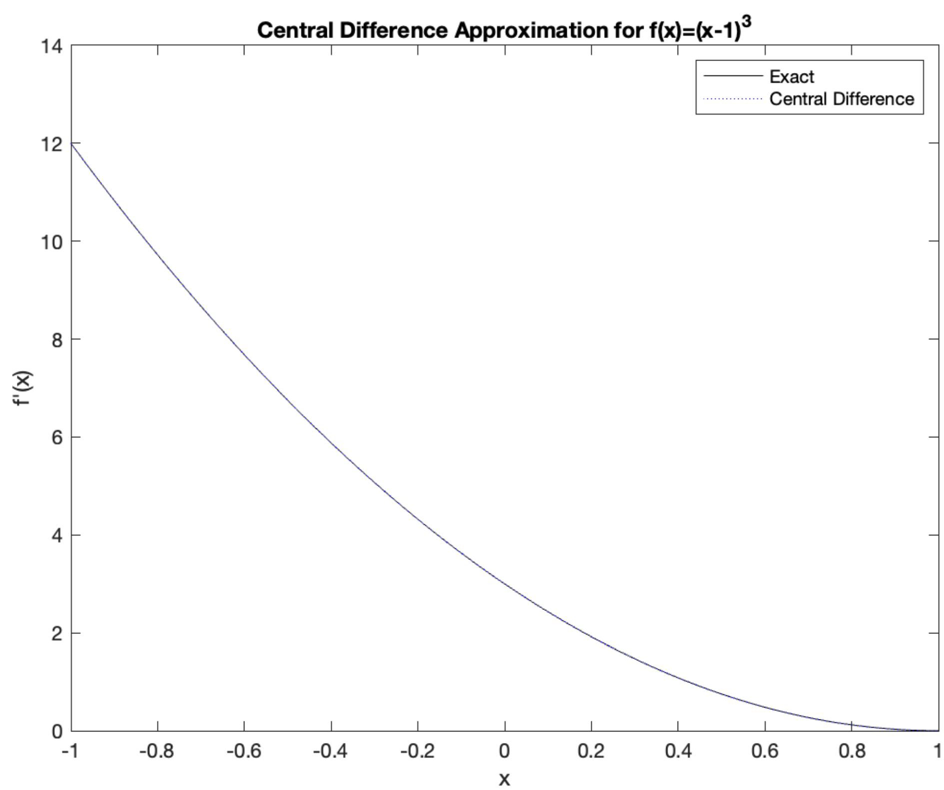

Central Difference Approximation for f(x)=(x-1)3
Exact
Central Difference
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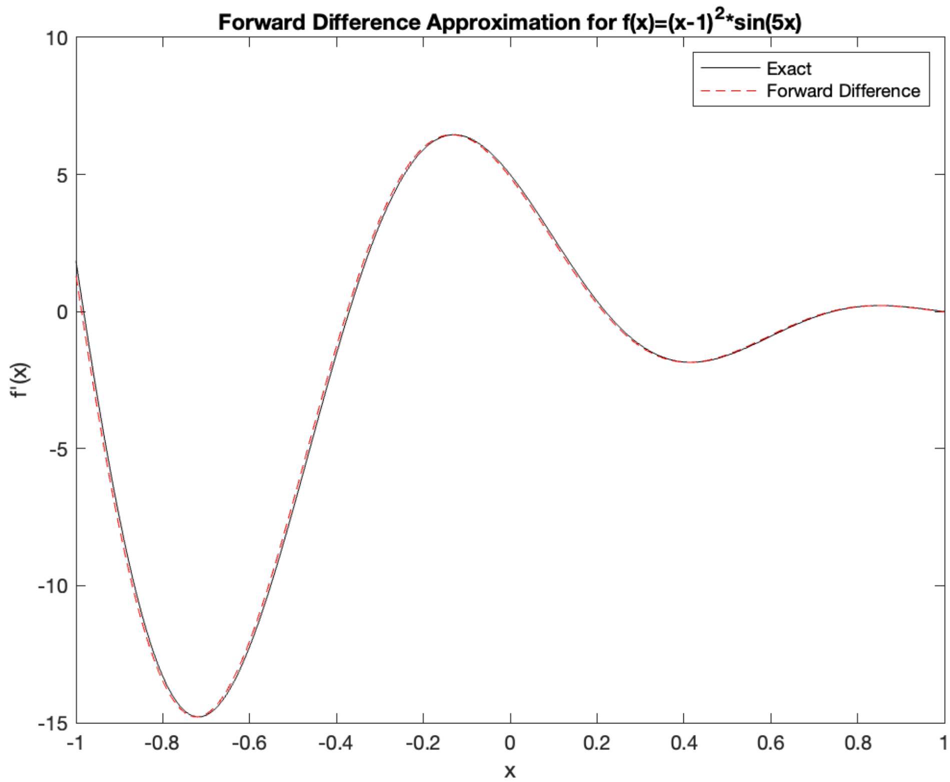
Forward Difference Approximation for f(x)=(x-1)^2*sin(5x)
Exact
Forward Difference
f'(x)
x
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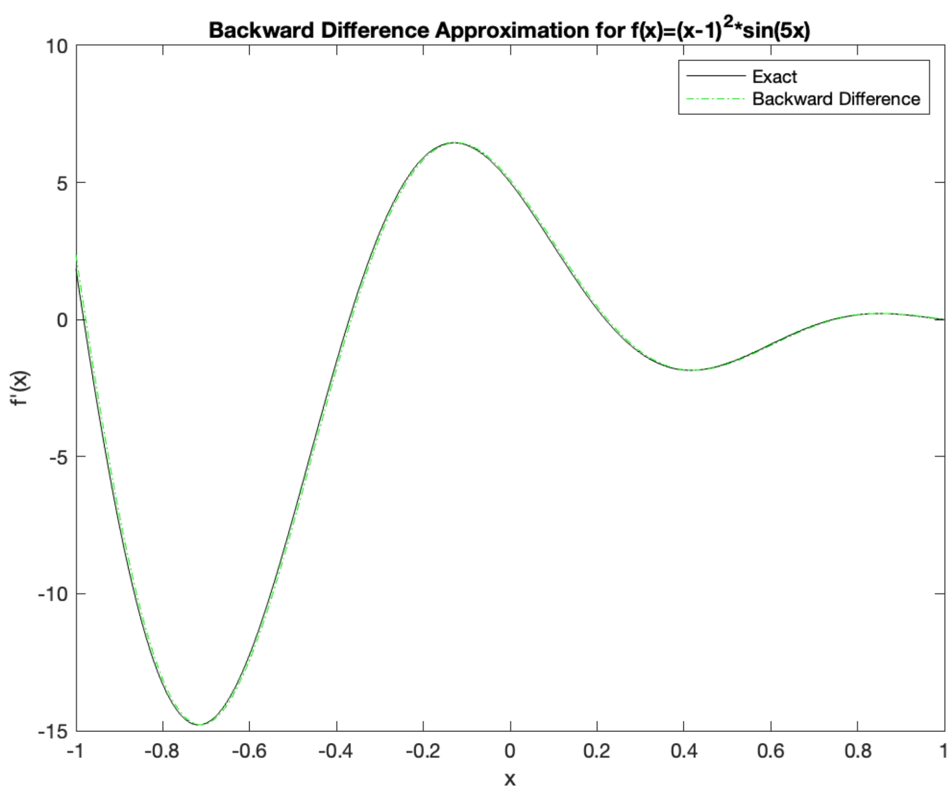
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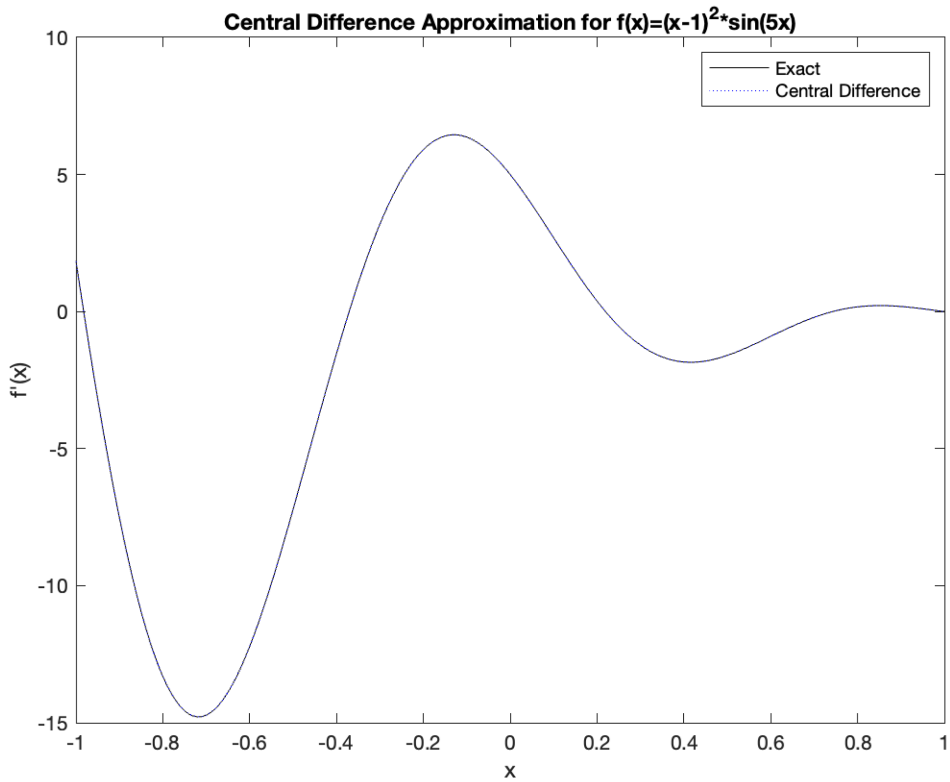
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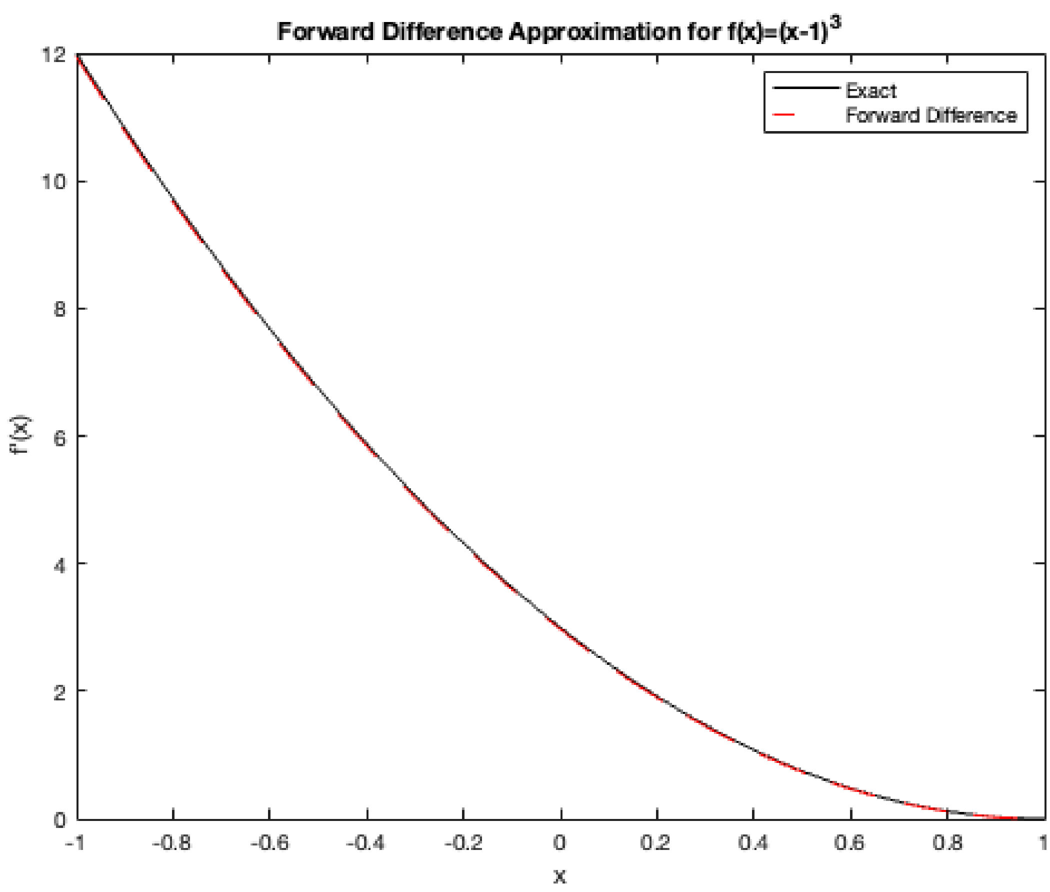
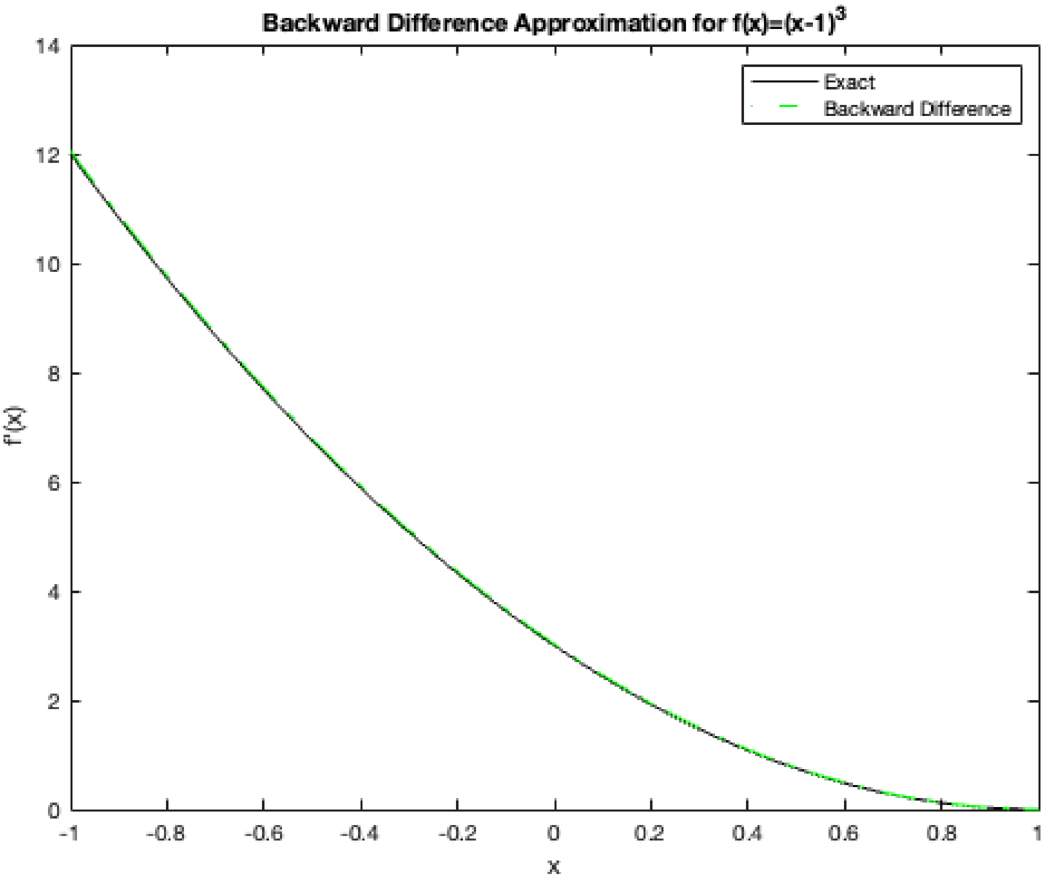
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1

```
clear;
clc

f1 = @(x) (x-1).^3;
df1 = @(x) 3.*(x-1).^2;

f2 = @(x) (x-1).^2 .* sin(5.*x);
df2 = @(x) 2.*(x-1).*sin(5.*x) + 5*(x-1).^2.*cos(5.*x);

h = 0.01;
x = -1:h:1;

% Forward Difference
df1_fd = (f1(x+h) - f1(x))/h;
df2_fd = (f2(x+h) - f2(x))/h;

% Backward Difference
df1_bd = (f1(x) - f1(x-h))/h;
df2_bd = (f2(x) - f2(x-h))/h;

% Central Difference
df1_cd = (f1(x+h) - f1(x-h))/(2*h);
df2_cd = (f2(x+h) - f2(x-h))/(2*h);

% Function 1
figure(1)
plot(x, df1(x), 'k-', x, df1_fd, 'r--')
title('Forward Difference Approximation for f(x)=(x-1)^3')
xlabel('x')
ylabel('f''(x)')
legend('Exact', 'Forward Difference')
saveas(gcf, 'figure1.png')

figure(2)
plot(x, df1(x), 'k-', x, df1_bd, 'g-.')
title('Backward Difference Approximation for f(x)=(x-1)^3')
xlabel('x')
ylabel('f''(x)')
legend('Exact', 'Backward Difference')
saveas(gcf, 'figure2.png')

figure(3)
plot(x, df1(x), 'k-', x, df1_cd, 'b:')
title('Central Difference Approximation for f(x)=(x-1)^3')
xlabel('x')
ylabel('f''(x)')
legend('Exact', 'Central Difference')
saveas(gcf, 'figure3.png')

% Function 2
figure(4)
plot(x, df2(x), 'k-', x, df2_fd, 'r--')
```

```
title('Forward Difference Approximation for f(x)=(x-1)^2*sin(5x)')
xlabel('x')
ylabel('f''(x)')
legend('Exact', 'Forward Difference')
saveas(gcf, 'figure4.png')

figure(5)
plot(x, df2(x), 'k-', x, df2_bd, 'g-.')
title('Backward Difference Approximation for f(x)=(x-1)^2*sin(5x)')
xlabel('x')
ylabel('f''(x)')
legend('Exact', 'Backward Difference')
saveas(gcf, 'figure5.png')

figure(6)
plot(x, df2(x), 'k-', x, df2_cd, 'b:')
title('Central Difference Approximation for f(x)=(x-1)^2*sin(5x)')
xlabel('x')
ylabel('f''(x)')
legend('Exact', 'Central Difference')
saveas(gcf, 'figure6.png')
```

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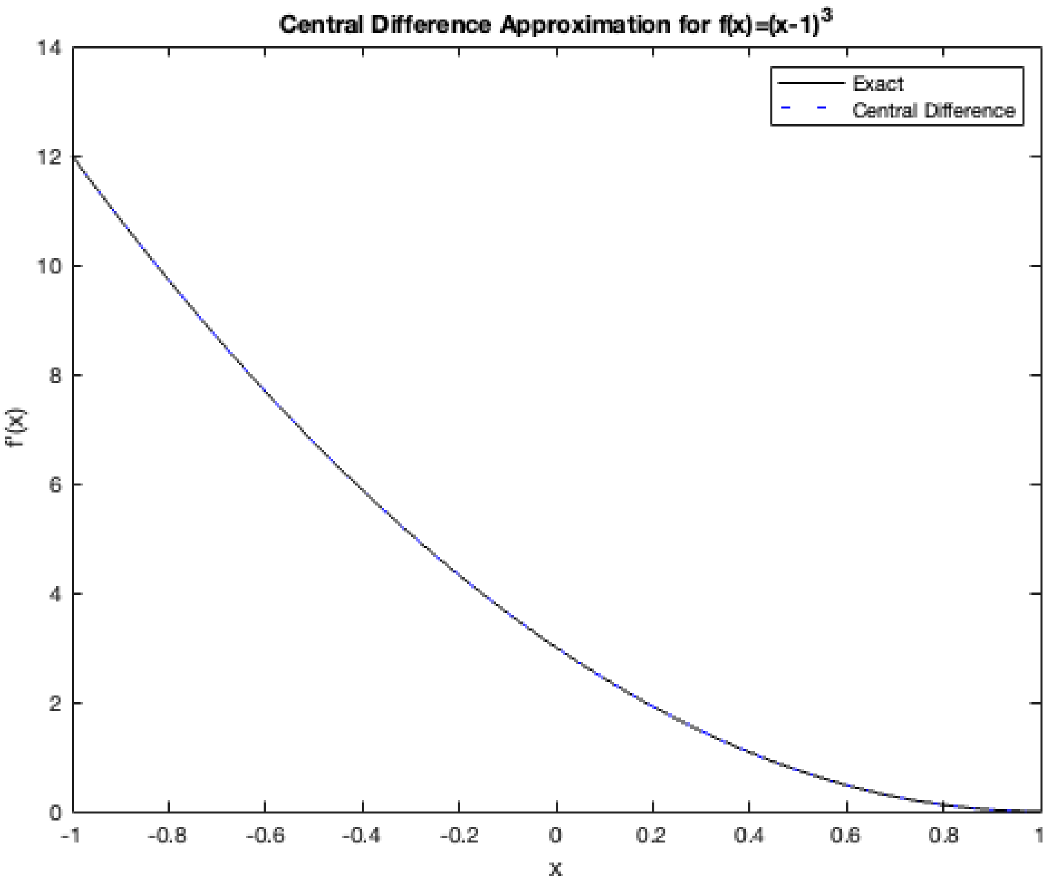
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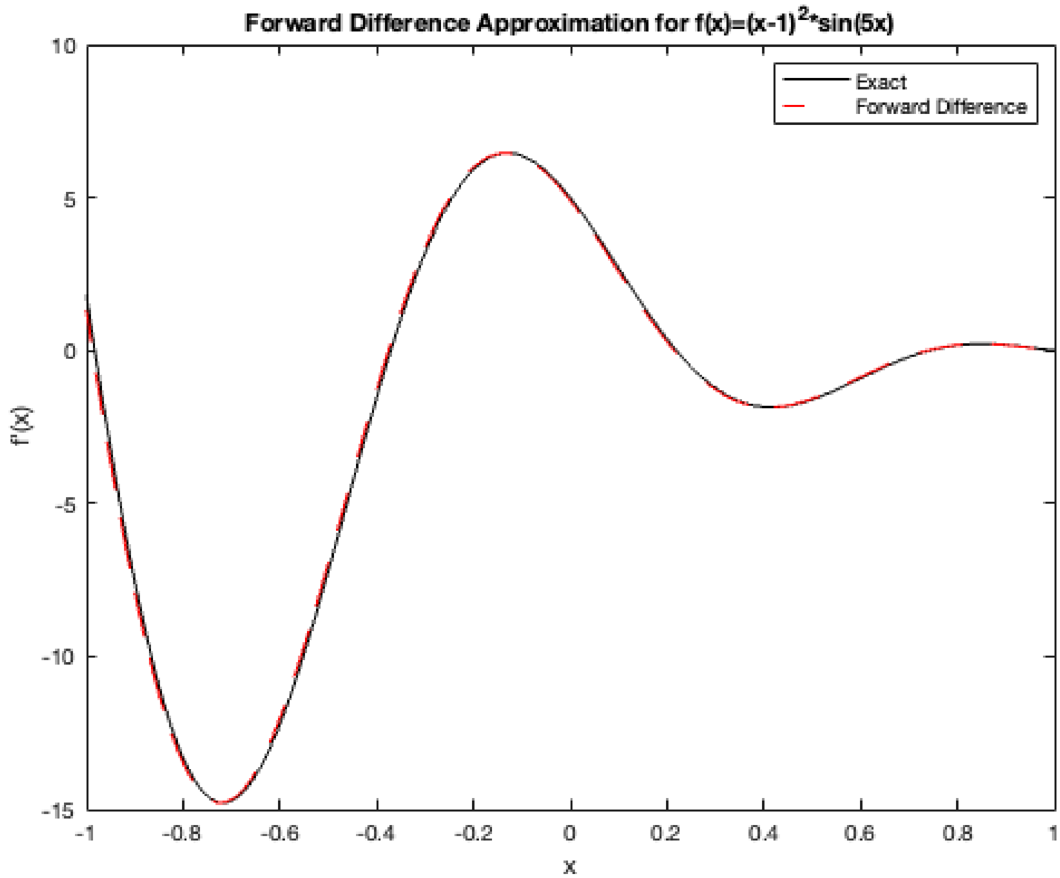
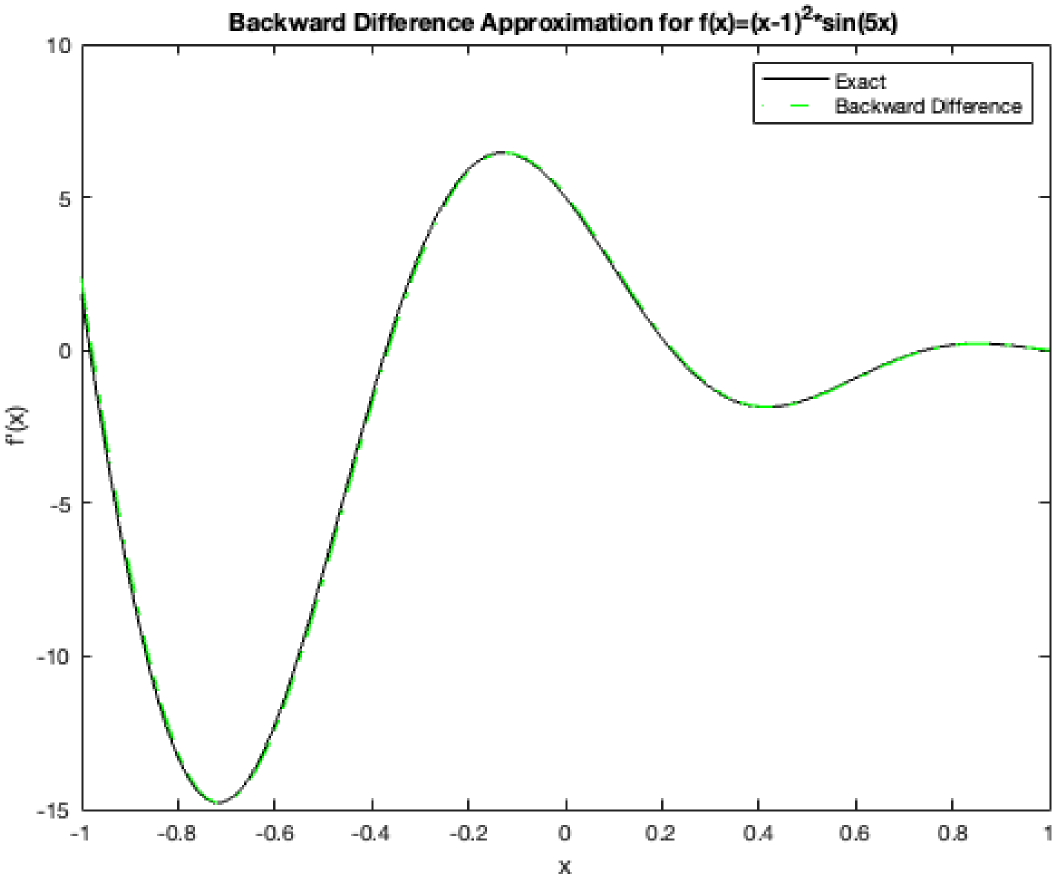


Forward Difference Approximation for f(x)=(x-1)^2*sin(5x)
Exact
Forward Difference
10
5
0
-5
-10
-15
f'(x)
-1
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1
x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Backward Difference
f'(x)
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-0.8
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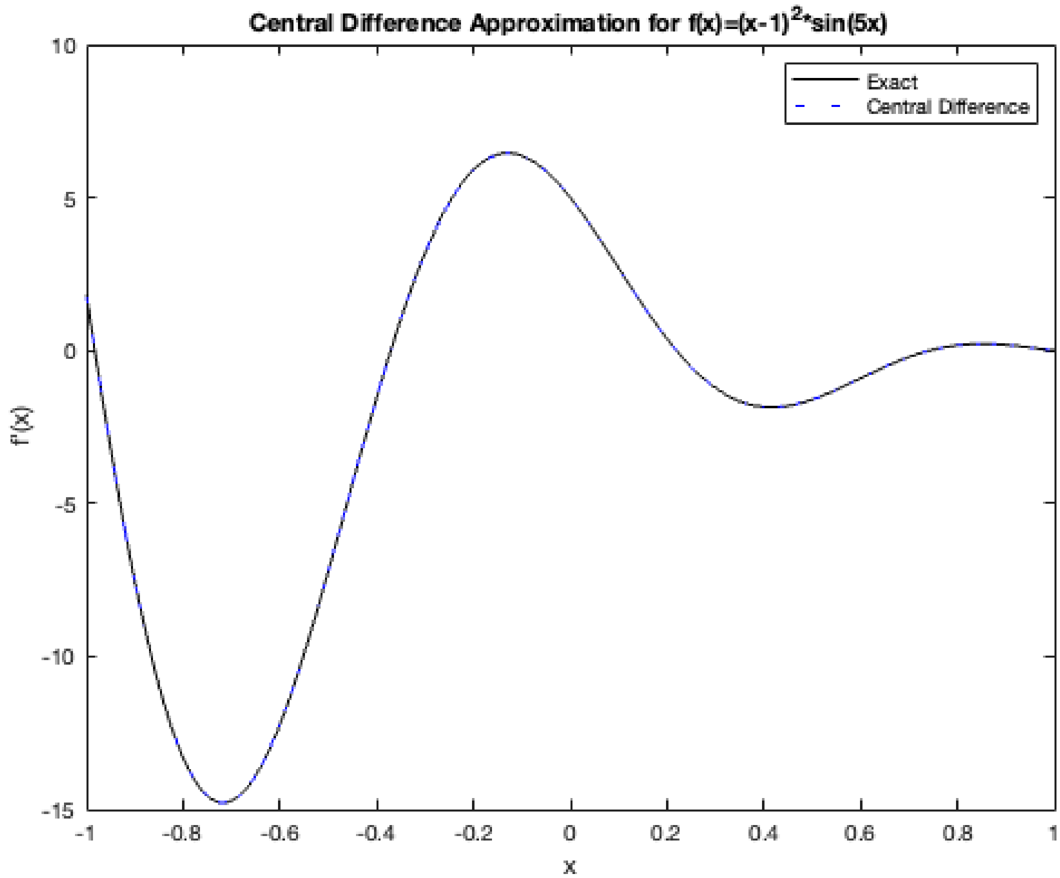

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