Math 151A: Problem Set 4

Brandon Montez

Problem 1: (T) Lagrange Polynomials and Neville's Method

Use Neville's method to obtain approximations to $f(0.1)$ using the Lagrange interpolating polynomials of degrees one, two, and three if
$f(0) = 1$, $f(0.25) = 1.65$, $f(0.5) = 2.72$, and $f(0.75) = 4.48$. You should explicitly compute the table associated with Neville's method:

x_i	$x - x_i$	$Q_{i,0}$	$Q_{i,1}$	$Q_{i,2}$	$Q_{i,3}$
0			-	-	-
0.25				-	-
0.5					-
0.75					

(The true value is $f(0.1) = 1.2214$.)

Solution:

x_i	$x - x_i$	$Q_{i,0}$	$Q_{i,1}$	$Q_{i,2}$	$Q_{i,3}$
0	0.1	1	1.26	1.0848	1.374976
0.25	-0.15	1.65	0.384	3.26112	-
0.5	-0.4	2.72	-6.1696	-	-
0.75	-0.65	4.48	-	-	-

1st degree: 1.26
2nd degree: 1.0848
3rd degree: 1.374976

Problem 2: (T) Lagrange Polynomials and Neville's Method

Suppose $x_j = j$, for $j = 0, 1, 2, 3$, and it is known that:

$$P_{0,1}(x) = 2x + 1,\ P_{0,2}(x) = x + 1,\ P_{1,2,3}(2.5) = 3$$

Determine $P_{0,1,2,3}(2.5)$.

Solution:
$P_{0,1}(2.5) = 6$
$P_{0,2}(2.5) = 3.5$
$P_{0,1,2}(2.5) = 2.875$
$P_{0,1,2,3}(2.5) = 6.0625$

Problem 3: (T) Lagrange Polynomials and Neville's Method

Suppose $x_j = 2\,j$, for $j = 0, 1, 2, 3, 4$ and it is known that:

$$P_{1,2}(1) = 2, \qquad P_{1,2,3}(1) = 1, \quad P_{1,4}(1) = 6.$$

Determine $P_{1,2,3,4}(1)$.

Solution:

$P_{1,2,3}(1) = \frac{(1-6)*2-(1-4)*P_{1,3}(1)}{(4-6)} = 1$

$\rightarrow P_{1,3}(1) = \frac{8}{3}$

$P_{1,3,4}(1) = \frac{(1-8)*\frac{8}{3}-(1-6)*6}{(6-8)}$

$\rightarrow P_{1,3,4}(1) = -\frac{17}{3}$

$P_{1,2,3,4}(1) = \frac{(1-8)*1-(1-4)*-\frac{17}{3}}{(4-8)}$

$\rightarrow P_{1,2,3,4}(1) = 6$

$$P_{1,2,3,4}(1) = 6$$

Problem 4: (T) Newton's Divided Differences

a) Find the degree-2 interpolating polynomial via Newton's divided difference for $f(x) = \frac{x}{1+x}$ using nodes $x_0 = 0$, $x_1 = 1$, and $x_2 = 2$.

b) What are the degree-2 interpolating polynomials associated with Lagrange's construction and Neville's construction? Compare them to the solution of Part (a).

Solution:

$f[x_0] = f(0) = 0$
$f[x_1] = f(1) = 0.5$
$f[x_2] = f(2) = 0.6667$

$f[x_0, x_1] = 0.5$
$f[x_1, x_2] = 0.1667$

$f[x_0, x_1, x_2] = -0.1667$

Newton's degree 2: $-0.167x^2 + 0.667x$

The Lagrange polynomial: $-0.167x^2 + 0.667x$

Neville's polynomial: $-0.167x^2 + 0.667x$

We can see that they are all the same. This is because the original function x/(1+x) is of degree $1 \leq 2$. Hence the degree-2 interpolating polynomial will be the same for each method.

Problem 5: (T) Numerical Differentiation/Finite Difference

Show that the following finite difference formula is first-order accurate:

$$f^{(2)}(x) = \frac{f(x+2\,h) - 2f(x+h) + \ f(x)}{h^2} + O(h).$$

For full credit, you must state any assumptions on $f(x)$ and justify each step in your solution. Hint: Apply Taylor's theorem to each of the terms on the right-hand side.

Solution:

$f(x)$ is assumed to be a function that is twice continuously differentiable in the interval of interest.

$$\begin{aligned}
f^{(2)}(x) &= \frac{f(x+2h) - 2f(x+h) + f(x)}{h^2} \\
&= \frac{(f(x) + 2hf'(x) + 2h^2 f''(x) + O(h^3)) - 2(f(x) + hf'(x) + \frac{h^2}{2} f''(x) + O(h^3)) + f(x)}{h^2} \\
&= \frac{2hf'(x) - 2hf'(x) + 2h^2 f''(x) - h^2 f''(x) - O(h^3)}{h^2} \\
&= \frac{h^2 f''(x) - O(h^3)}{h^2} \\
&= f''(x) - \frac{O(h^3)}{h^2}
\end{aligned}$$

Now, because the $O(h^3)$ term is of higher order than h^2 as h approaches zero, we can rewrite the expression as:

$$f''(x) - \frac{O(h^3)}{h^2} = f''(x) - O(h)$$

Finally, we remove the leading constant -1 from the big O notation, as it does not contribute any additional information, since we are talking about the magnitude of growth rate and not direction.

$$f''(x) - O(h) = f''(x) + O(h)$$

Hence, we see that the finite-difference formula is first-order accurate.