Math 151A: Problem Set 6

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Problem 1: (T) Quadrature

Let $h = \frac{b-a}{3}$, $x_0 = a$, $x_1 = a + h$, and $x_2 = b$. Find the degree of precision of the quadrature formula:

$$\int_a^b f(x)\,\mathrm{dx} \approx \frac{9h}{4} f(x_1) + \frac{3h}{4} f(x_2)$$

You can check your work with symbolic software, but to receive full credit you must show your work.

Solution:

For the degree-0 polynomial $p_0(x) = 1$, the true value is:

$$\int_a^b p_0(x)\,\mathrm{dx} = \int_a^b \mathrm{dx} = b - a = 3h$$

And the approximation is:

$$\frac{9h}{4} p_0(x_1) + \frac{3h}{4} p_0(x_2) = \frac{9h}{4} \cdot 1 + \frac{3h}{4} \cdot 1 = 3h$$

So, the approximation is exact for $n = 0$.

For the degree-1 polynomial $p_1(x) = x$:

$$\begin{aligned}
\int_a^b p_1(x)\,\mathrm{dx} &= \int_a^b x\,\mathrm{dx} \\
&= \frac{1}{2}b^2 - \frac{1}{2}a^2 \\
&= \frac{1}{2}(a+3h)^2 - \frac{1}{2}a^2 \\
&= 3ah + \frac{9h^2}{2}
\end{aligned}$$

And the approximation is:

$$\begin{aligned}\frac{9h}{4}p_1(x_1) + \frac{3h}{4}p_1(x_2) &= \frac{9h}{4}(a+h) + \frac{3h}{4}b \\ &= \frac{9h}{4}a + \frac{9h^2}{4} + \frac{3h}{4}(a+3h) \\ &= \frac{9h}{4}a + \frac{9h^2}{4} + \frac{3h}{4}a + \frac{9h^2}{4} \\ &= 3ah + \frac{9h^2}{2}\end{aligned}$$

So, the approximation is also exact for $n = 1$.

For the degree-2 polynomial $p_2(x) = x^2$:

The exact integral is:

$$\begin{aligned}\int_a^b p_2(x)\,\mathrm{dx} &= \int_a^b x^2\,\mathrm{dx} \\ &= \frac{1}{3}b^3 - \frac{1}{3}a^3 \\ &= \frac{1}{3}(a+3h)^3 - \frac{1}{3}a^3 \\ &= 3a^2h + 3ah^2 + 9h^3\end{aligned}$$

The approximation for $p_2(x)$ is:

$$\begin{aligned}\frac{9h}{4}p_2(x_1) + \frac{3h}{4}p_2(x_2) &= \frac{9h}{4}(a+h)^2 + \frac{3h}{4}(a+3h)^2 \\ &= \frac{9h}{4}(a^2+2ah+h^2) + \frac{3h}{4}(a^2+6ah+9h^2) \\ &= 3a^2h + 3ah^2 + 9h^3\end{aligned}$$

So, the approximation is also exact for $n = 2$.

For the degree-3 polynomial $p_3(x) = x^3$ (and I really hope this is the last one):

The exact integral is:

$$\begin{aligned}\int_a^b p_3(x)\,\mathrm{dx} &= \int_a^b x^3\,\mathrm{dx} \\ &= \frac{1}{4}b^4 - \frac{1}{4}a^4 \\ &= \frac{1}{4}(a+3h)^4 - \frac{1}{4}a^4 \\ &= \frac{3}{4}h(4a^3 + 18a^2h + 36ah^2 + 27h^3)\end{aligned}$$

The approximation for $p_3(x)$ is:

$$\begin{aligned}
\frac{9h}{4}p_3(x_1) + \frac{3h}{4}p_3(x_2) &= \frac{9h}{4}(a+h)^3 + \frac{3h}{4}(a+3h)^3 \\
&= \frac{9h}{4}(a^3 + 3a^2h + 3ah^2 + h^3) + \frac{3h}{4}(a^3 + 9a^2h + 27ah^2 + 27h^3) \\
&= \frac{3}{4}h(4a^3 + 18a^2h + 36ah^2 + 30h^3)
\end{aligned}$$

Then the approximation is not exact for $n = 3$.

Therefore, the given quadrature formula has degree-2 precision.

Problem 2: (T) More Quadrature

Find the constants a, x_0, and x_1 so that the quadrature formula:

$$\int_0^1 f(x)\,\mathrm{dx} \approx \frac{1}{2}f(x_0) + af(x_1)$$

has the highest possible degree of precision.

Solution:

For the degree-0 polynomial $p_0(x) = 1$, the true value is:

$$\int_0^1 p_0(x)\,\mathrm{dx} = \int_0^1 \mathrm{dx} = 1$$

And the approximation is:

$$\frac{1}{2}p_0(x_0) + ap_0(x_1) = \frac{1}{2}\cdot 1 + a \cdot 1 = \frac{1}{2} + a$$

For the approximation to be exact, we have $\frac{1}{2} + a = 1$, which implies $a = \frac{1}{2}$.

For the degree-1 polynomial $p_1(x) = x$:

$$\int_0^1 p_1(x)\,\mathrm{dx} = \int_0^1 x\,\mathrm{dx} = \frac{1}{2}$$

And the approximation is:

$$\frac{1}{2}p_1(x_0) + ap_1(x_1) = \frac{1}{2}x_0 + ax_1$$

For the approximation to be exact, we have $\frac{1}{2}x_0 + ax_1 = \frac{1}{2}$. With $a = \frac{1}{2}$, this gives:

$$x_0 + x_1 = 1$$

Then the exact integral for $p_2(x) = x^2$ is:

$$\int_0^1 p_2(x)\,\mathrm{dx} = \int_0^1 x^2\,\mathrm{dx} = \frac{1}{3}$$

The approximation is:

$$\frac{1}{2}p_2(x_0) + ap_2(x_1) = \frac{1}{2}x_0^2 + ax_1^2 = \frac{1}{2}x_0^2 + \frac{1}{2}x_1^2$$

Which gives:

$$x_0^2 + x_1^2 = \frac{2}{3}$$

Then we have the system of linear equations:

$$x_0 + x_1 = 1$$
$$x_0^2 + x_1^2 = \frac{2}{3}$$

Solving this system yields:

$$x_0 = \frac{1}{6}(3 - \sqrt{3}), x_1 = \frac{1}{6}(3 + \sqrt{3})$$
$$x_0 = \frac{1}{6}(3 + \sqrt{3}), x_1 = \frac{1}{6}(3 - \sqrt{3})$$

Noting that both x_0 and x_1 should be numbers between 0 and 1, neither of these solutions satisfy this requirement. Therefore, we fail to achieve degree-2 precision for our given quadrature rule. Hence, the maximum degree of precision we can achieve is 1, so long as $a = \frac{1}{2}$ and $x_0 + x_1 = 1$ for $x_0, x_1 \in [0, 1]$.

Problem 3: (T) Even More Quadrature

Determine the constants a, b, c, and d that will produce a quadrature formula:

$$\int_{-1}^{1} f(x)\,\mathrm{dx} \approx af(-1) + bf(1) + cf'(-1) + df'(1)$$

that is exact to degree three.

Solution:

For $f(x) = 1$, we have:

$$\int_{-1}^{1} \mathrm{dx} = 2 = af(-1) + bf(1) + cf'(-1) + df'(1) \Rightarrow a + b = 2$$

For $f(x) = x$, we have:

$$\int_{-1}^{1} x\,\mathrm{dx} = 0 = af(-1) + bf(1) + cf'(-1) + df'(1) \Rightarrow -a + b + c + d = 0$$

For $f(x) = x^2$, we have:

$$\int_{-1}^{1} x^2\,\mathrm{dx} = \frac{2}{3} = af(-1) + bf(1) + cf'(-1) + df'(1) \Rightarrow a + b - 2c + 2d = \frac{2}{3}$$

For $f(x) = x^3$, we have:

$$\int_{-1}^{1} x^3\,\mathrm{dx} = 0 = af(-1) + bf(1) + cf'(-1) + df'(1) \Rightarrow -a + b + 3c + 3d = 0$$

Then the system of equations to solve is:

$$\begin{aligned} a + b &= 2 \\ -a + b + c + d &= 0 \\ a + b - 2c + 2d &= \frac{2}{3} \\ -a + b + 3c + 3d &= 0 \end{aligned}$$

Solving this system we find:

$$
\begin{aligned}
a &= 1 \\
b &= 1 \\
c &= \frac{1}{3} \\
d &= -\frac{1}{3}
\end{aligned}
$$

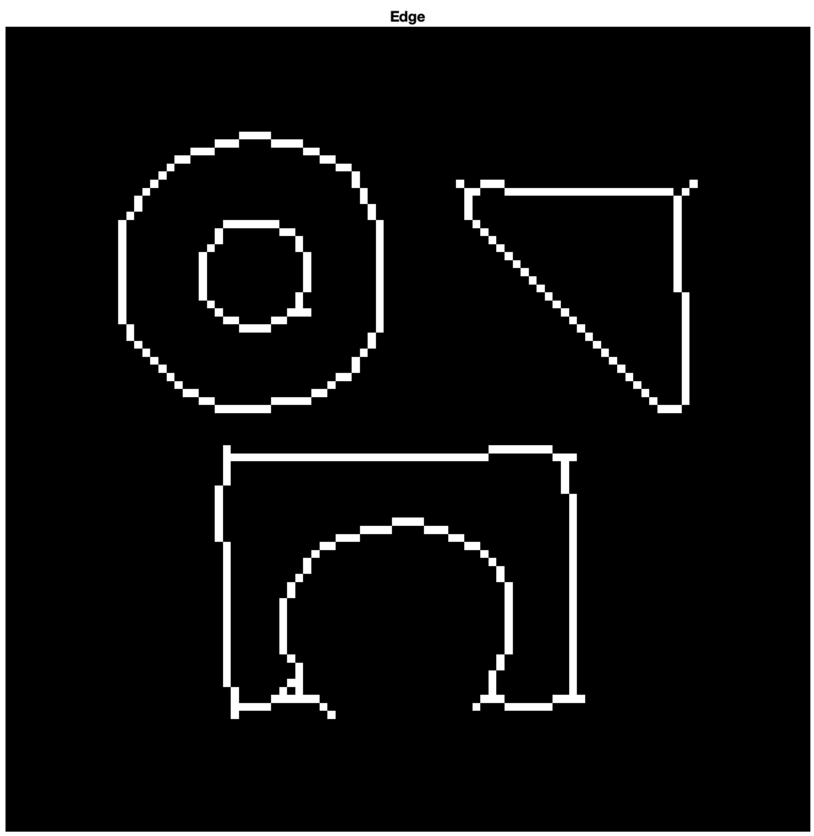
Then the quadrature formula that is exact to degree-3 is:

$$\int_{-1}^{1} f(x)\,\mathrm{dx} \approx f(-1) + f(1) + \frac{1}{3}f'(-1) + -\frac{1}{3}f'(1)$$

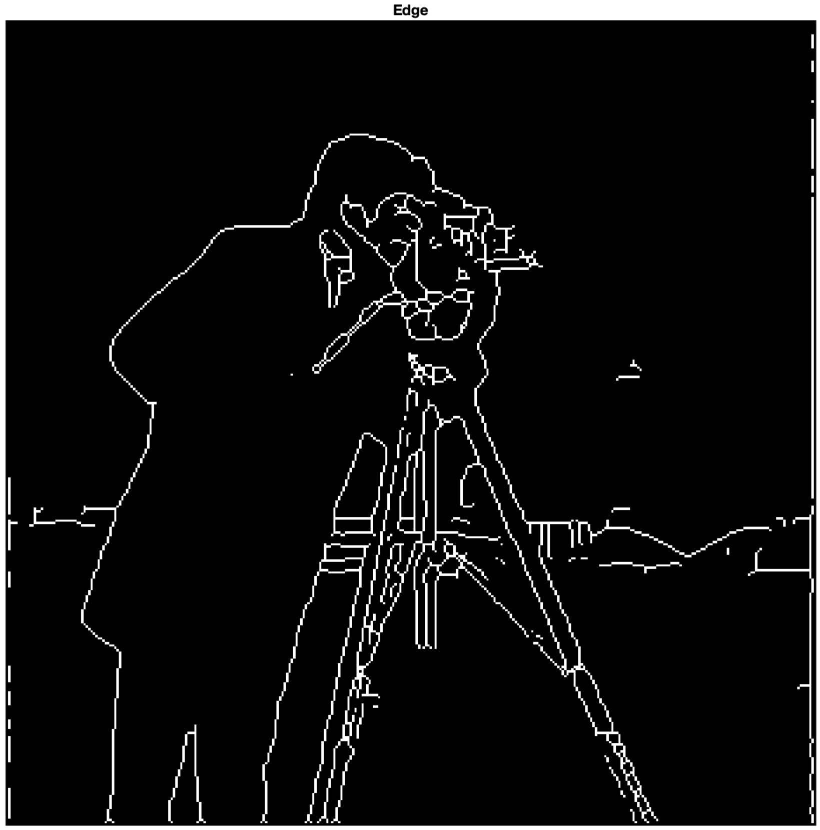
Problem 4: (C) Edge Detector

Complete the template code "MyEdgeDetector.m" by filling in the missing lines. You will need to choose a finite difference method (either forward, backward, or central), and tune the parameters ('thres' and the parameters in 'G' on line 25). The two example images provided are '3.gif' and 'cameraman.png' from class. You will need to include a third example of your choosing (see line 15). You will need to submit **four images**: the edge maps for the two examples provided, the original grayscale image for your third example, and the edge map for the third example. Attach your code for full credit.

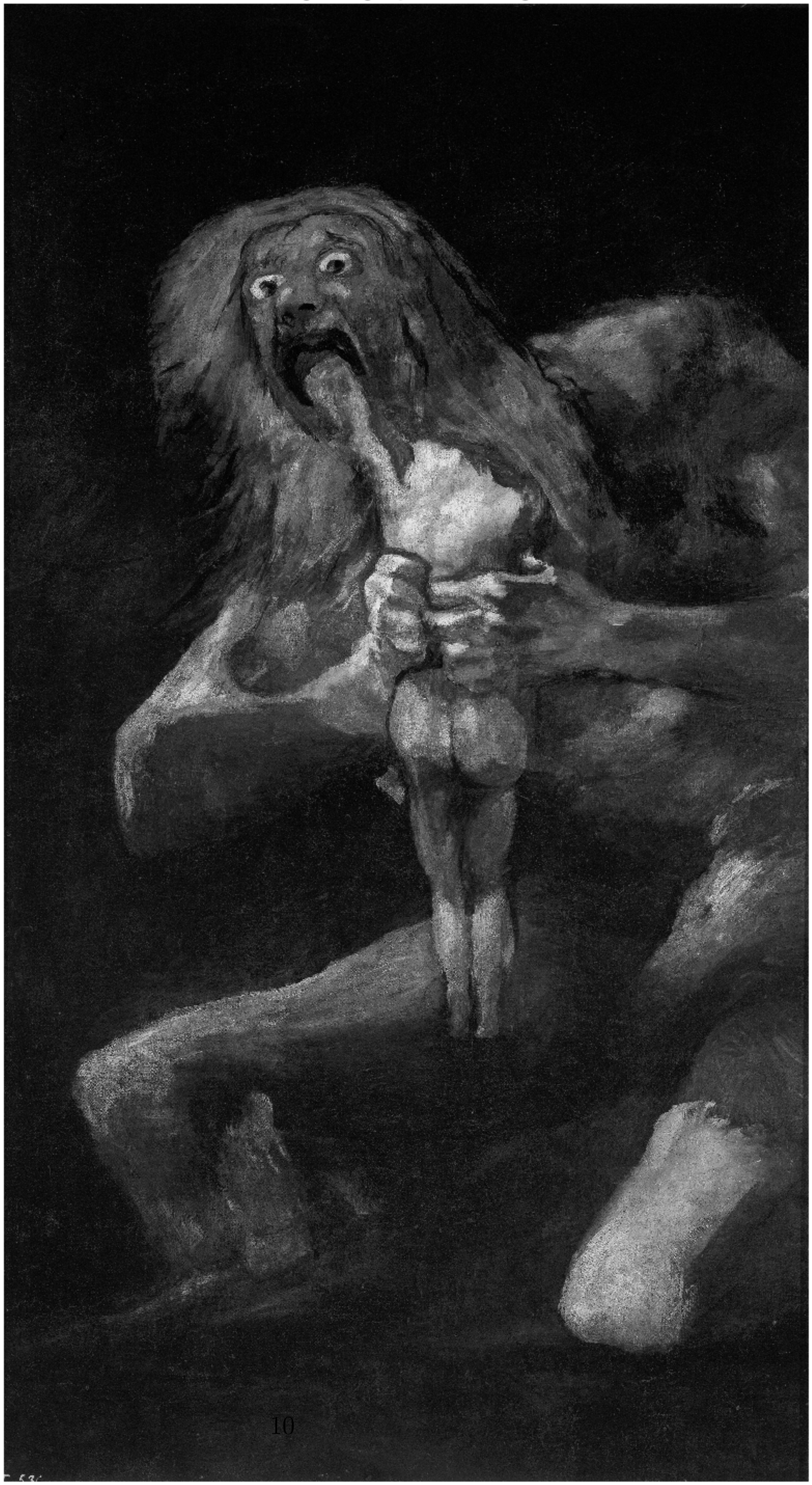
Solution:

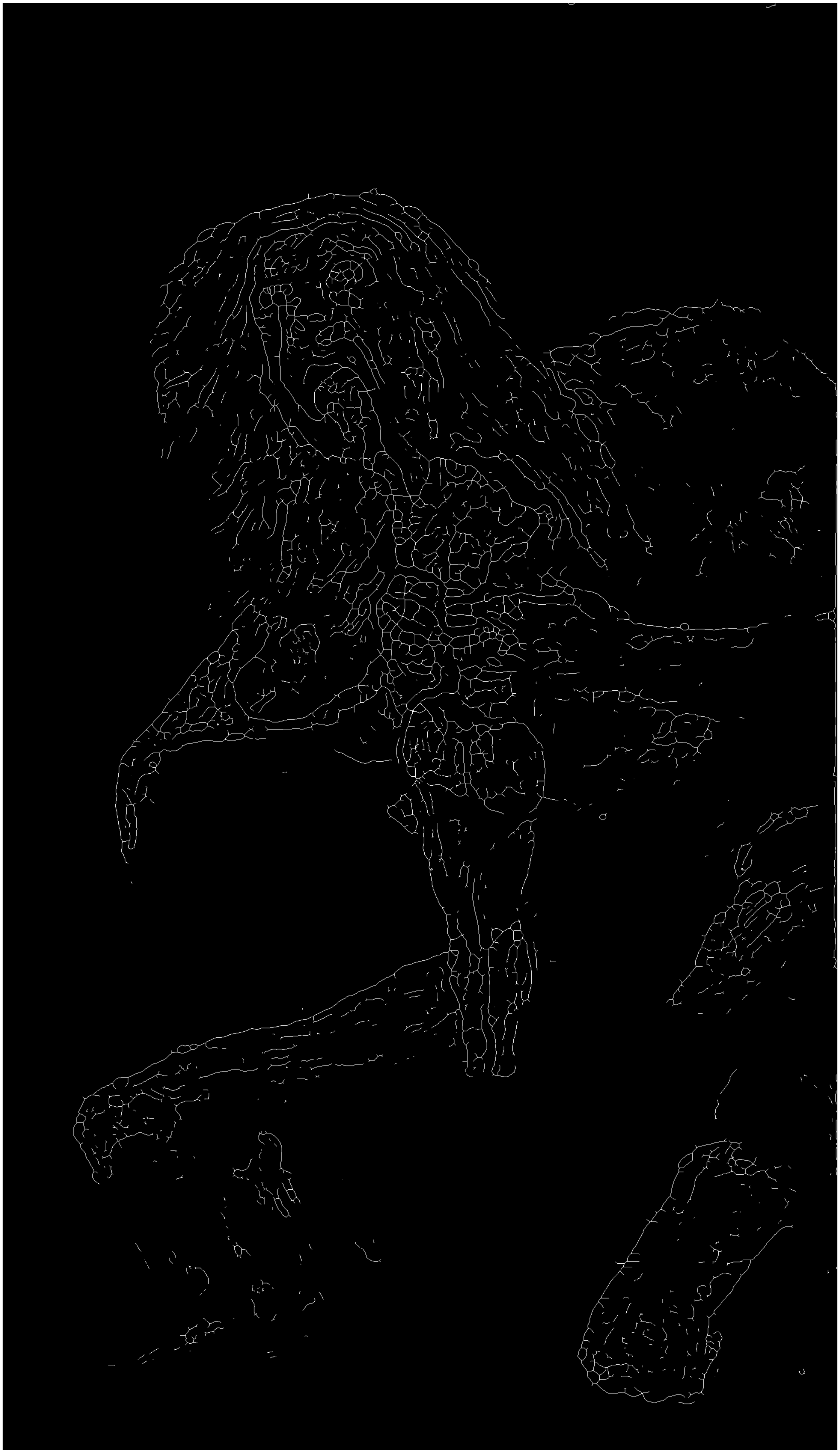


Edge



Original grayscale image

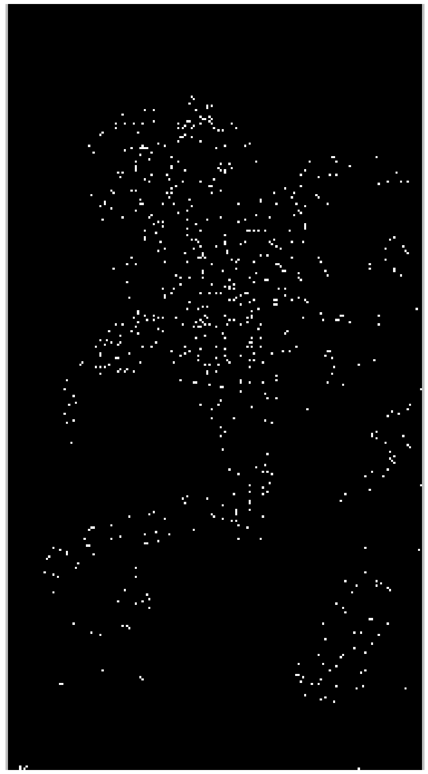




```
function MyEdgeDetector
%Pick an example and case
Example = 3;

if Example == 1
    fc = imread('3.gif');
    f0 = double(fc);
    thres = 0.35;
elseif Example == 2
    fc = imread('cameraman.png');
    f0 = double(rgb2gray(fc));
    thres = 0.25;
elseif Example == 3
    fc = imread('saturn.png');
    f0 = double(rgb2gray(fc));
    thres = 0.13;
end

[n, m] = size(f0);

%Filter Noise from Image
G = fspecial('gaussian',20,4.2); %tuned to fit saturn.png
f = imfilter(f0,G,'replicate');

%Compute Derivatives
Dx = zeros(n,m); % df/dx
Dy = zeros(n,m); % df/dy
D = zeros(n,m); % |Df|

%Loop to compute derivatives and length of gradient
for j = 2:n-1
    for k = 2:m-1
        Dx(j,k) = (f(j,k+1) - f(j,k-1))/2; % Central difference in x
        Dy(j,k) = (f(j+1,k) - f(j-1,k))/2; % Central difference in y
        D(j,k) = sqrt((Dx(j,k)).^2+(Dy(j,k)).^2);
    end
end

%Rescale the norm of the gradient to be between 0 and 1
D = (D - min(D(:)))/(max(D(:)) - min(D(:)));

%Threshold the Norm of Gradient to find large jumps in image
TDf = double(D > thres);

%Thin the binary image, see bwmorph for help.
Edge = bwmorph(TDf, 'Skel', inf);

%Plot Edges
imagesc(Edge); colormap(gray);
axis off; axis image; title 'Edge';
saveas(gcf,'ex3.png');
```

```
end
```

Edge



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