ELEC 4700

Assignment 2

Finite Difference Method

Brandon Schelhaas
101036851
February 28th, 2021
Prof. Tom Smy

Part 1: Finite Difference Method

Laplace's Equation is used to help solve electrostatic potential problems. The equation may be able to be implemented in many ways, but of these, two include hard-coding the equation by finite difference, or using a G-matrix to implement the equations used on a region. Laplace's Equation is given below:

$$\nabla^2 V = 0$$

For a 2D-region, the equation is as follows:

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$

The Finite Difference Method (FDM) implements these derivates by introducing an elemental mesh into the region, thereby discretizing the region. The elemental meshes can be implemented in any 1D, 2D, or 3D region. The derivatives are implemented with backward- and forward-difference in all directions around a certain node. This is expressed mathematically as:

$$\frac{\partial^2 V}{\partial x^2} \approx \frac{\frac{V_{n+1} - V_n}{\Delta x} - \frac{V_n - V_{n-1}}{\Delta x}}{\Delta x} = \frac{V_{n+1} - 2V_n + V_{n-1}}{\Delta x^2}$$

Then, the Finite Difference Method equation is formed as:

$$\frac{V_{nx+1} - 2V_n + V_{nx-1}}{\Delta x^2} + \frac{V_{ny+1} - 2V_n + V_{ny-1}}{\Delta y^2} = 0$$

As can be seen, this equation essentially averages all the voltages adjacent to the node of interest.

The first way to implement this in MATLAB is hard-coding this equation while using a for-loop(s) to sweep the region of interest. This was accomplished in an earlier PA session already, and will not be discussed further.

The second way, which was done in this assignment, was to implement a G-matrix to solve GV = F. The G-matrix is a special sparse matrix that implements the equation, through specific coefficients, which can then be used on a region. This special matrix has size (y^2, x^2), where y is the number of rows and x is the number of columns. The size is so big such that each row of the G-matrix can implement the necessary equation on a singular element, using MATLAB matrix multiplication.

Simple Linear Case

The first investigation of Part 1 was to solve the simple case of a 2D-region where the left side is set to 1V, the right side is set to 0V, and the top and bottom sides are floating/insulated. Given these boundary conditions, the expected solution is a linear decrease from 1V to 0V across the region.

This was solved by creating a G-matrix to implement the equations for each elemental node, a vector including the node voltages (implementing the left- and right-side voltages), and mapping a 2D-region into a 1D vector with the following equation:

$$n = row + (col - 1) * y$$

The G-matrix and node voltages are set according to the problem, and are explained, by comments, in the code. However, in summary, a generic node's voltage was calculated with the discretized FDM equation. Note, because the mesh is rectangular and unitless, the distance between nodes is identical, and thus, the equation reduces:

$$V_{nx+1} + V_{ny+1} - 4V_n + V_{nx-1} + V_{ny-1} = 0$$

Then, the floating boundary condition (BC) requires:

$$\frac{dV}{dy} = 0$$

Which implies:

$$V_n - V_{ny\pm1} = 0$$

Therefore, the coefficients of the nodal voltages were put into the G-matrix, and the voltage values into the B (F) matrix. After solving the simple case and remapping the 1D voltage vector back into a 2D matrix, the solution is shown in Figure 1.

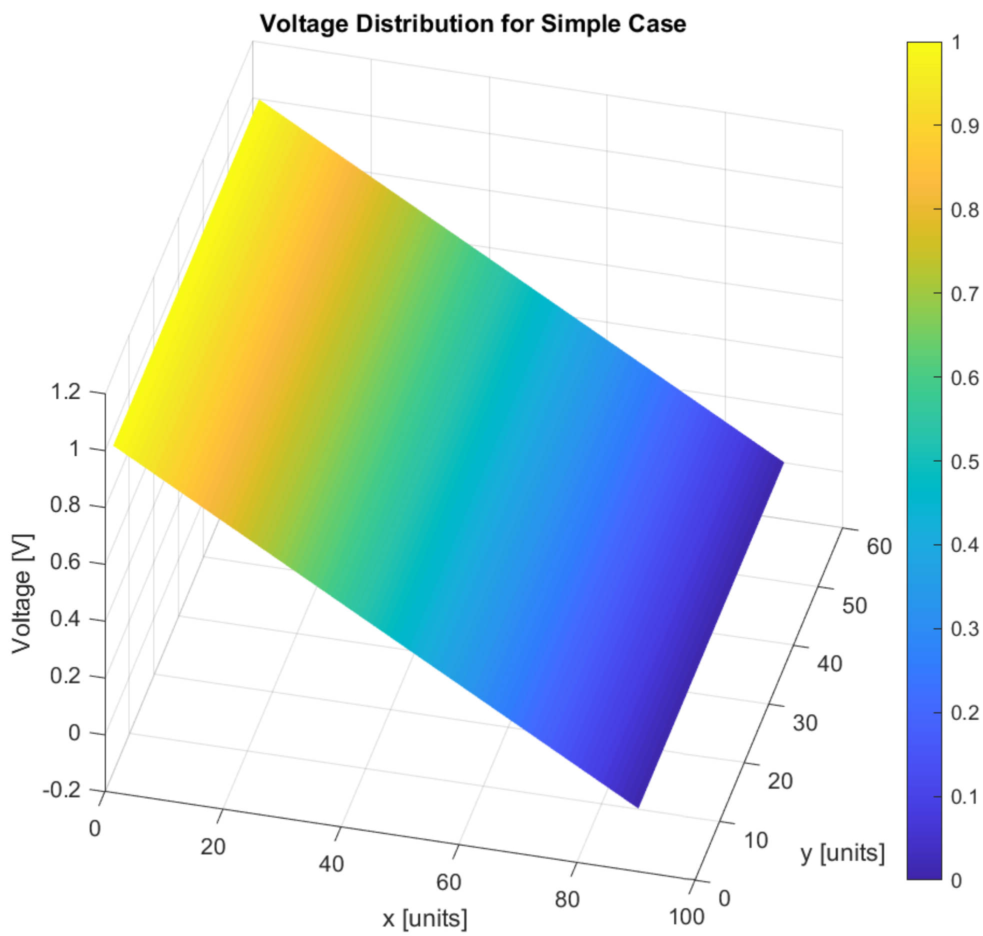

Figure 1: Voltage Distribution for Simple Case

As expected, the voltage decreases linearly across the region. Note, as required, the voltage on the left side ($x = 0$) of the region is 1V, the right side ($x = 90$) is 0V, and the top and bottom are insulating.

Complex Saddle Case

The last part of Part 1 was to solve a more complex problem where the left/right sides of the region were set to 1V, and the top/bottom sides of the region were set to 0V.

This solution can be solved and represented analytically with the following equation, given by Griffiths:

$$V(x,y) = \frac{4V_o}{\pi} \sum_{n=1,3,5\ldots}^{\infty} \frac{\cosh\left(\frac{n\pi x}{a}\right)\sin\left(\frac{n\pi y}{a}\right)}{n * cosh\left(\frac{n\pi b}{a}\right)}$$

where the axes are set by $0 \leq y \leq a$, and $-b \leq x \leq b$. Therefore, the matrix's y-dimension (ny) is equal to a, but the x-dimension has to shift from $0 \leq x \leq nx$ to $-b \leq x \leq b = -\frac{nx}{2} \leq x \leq \frac{nx}{2}$. Note that, because of the infinite sum, the nodal voltage will never be perfect; there will always be some finite error.

Regardless, this analytical solution was calculated and solved through iteration. Several sums were graphed where each solution has an increasing number of sums within the solution. These results are displayed in Figure 2.

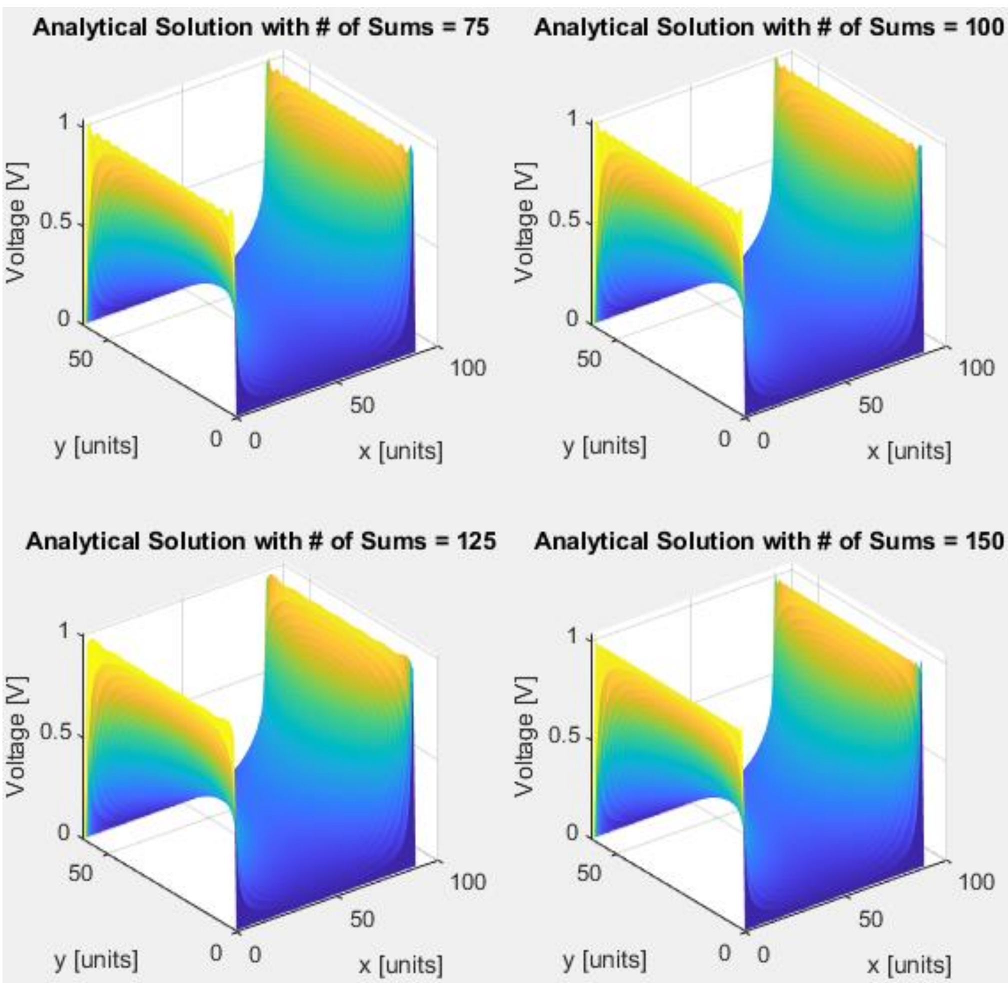

Figure 2: Analytical Solutions with Different Sums - Analytical

As can be seen, increasing the number of sums gives a more-accurate answer. Specifically, it can be noted that 75 sums cause the top 1V edges to be very wavy, whereas 150 sums are much less so and is significantly more constant – as expected. It can also be seen that the corners are inaccurate compared to the rest of the nodes, and these require significantly more sums in order to be more accurate.

The same situation can be implemented with the FDM equation, as done in Part 1A. Though, while the equations (G-matrix) are very similar, the boundary conditions must be changed. After they were changed, the result was solved and the 1D matrix was re-mapped into the 2D region. This solution is shown in Figure 3.

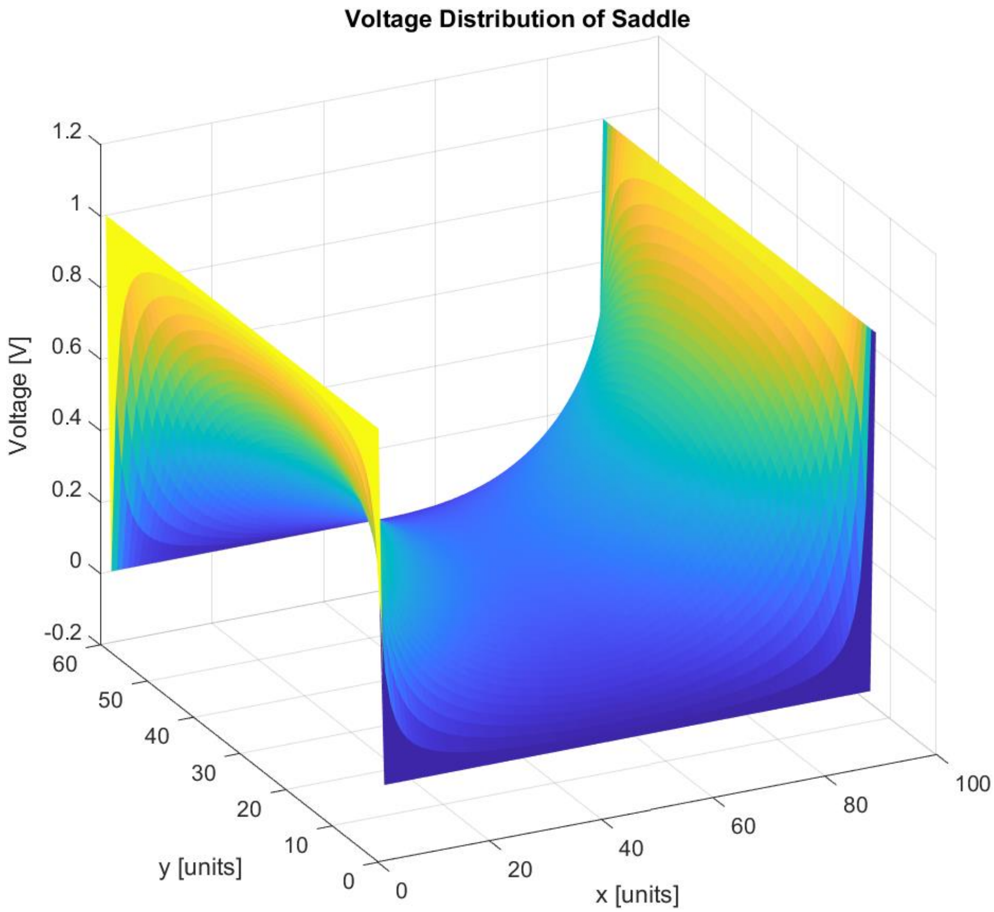

Figure 3: Voltage Distribution of Saddle - FDM

As can be seen, both the analytical and FDM methods give approximately the same solution.

Finally, the analytical and FDM methods can be compared to show the differences between the two methods. This error is shown in Figure 4.

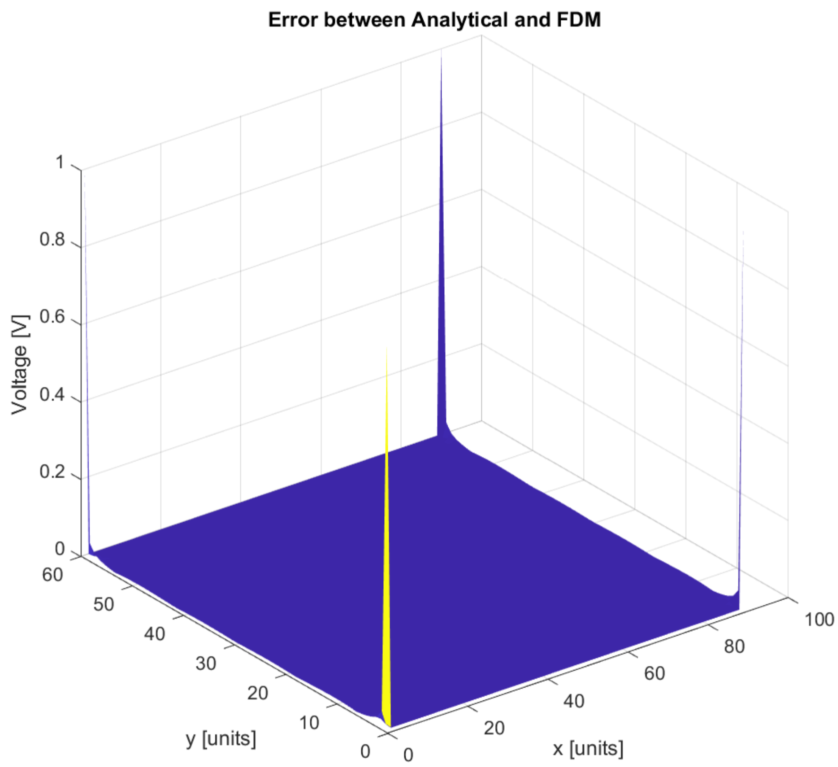

Figure 4: FDM/Analytical Error

As can be seen, most of the error is very low meaning that the FDM and Analytical solution are very consistent with each other. However, it can also be seen that the corners give the most error, as expected.

Therefore, both methods result in similar solutions, but the FDM does not show error on the boundaries. The disadvantage to the sum is that a large number of sums are required for the full solution to be accurate, as shown in the large error for 150 sums. However, the FDM method requires one to know how to setup the G-Matrix to help solve the region, whereas the analytical method is very straightforward.

Part 2: Using FDM for Current Flow

The purpose of this part was to use the FDM to solve for current flow in a region with a voltage distribution. This is done with the following equation:

$$\nabla\left(\sigma_{x,y}\nabla V\right) = 0$$

where σ is the conductivity of a region. However, this can be modelled as an orthogonal resistor network with resistors of value $\frac{1}{\sigma}$. Since no boundary conditions were specified, the simple case was used for current flow analysis. Recall, the boundary conditions were: left side = 1V, right side = 0V, top/bottom are floating, and the expected solution is a linear decrease with constant conductivity.

Current Flow

This investigation was started by defining two boxes within the 2D region, similar to Assignment 1, with different conductivities. These boxes were defined such that each box was placed in the middle third on the x-axis, and in the top and bottom thirds of the y-axis. Then, the boxes were given a conductivity of

0.01, while outside the boxes had a constant conductivity of 1. This resulted in the following conductivity map, shown in Figure 5.

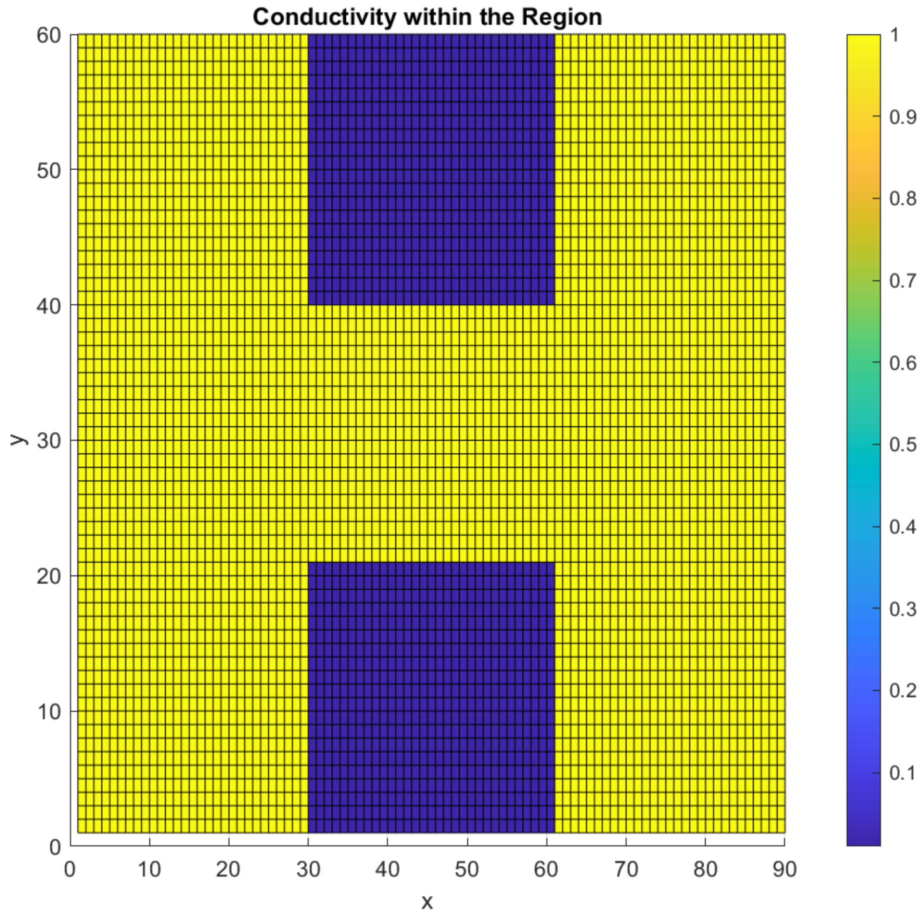

Figure 5: Plot of Sigma(x,y)

After the conductivity was defined, resistor values between the nodes were calculated, with help from Prof. Smy's Conduction code. Then, the G-matrix was re-made to include these resistor values which define the equations of the region. After this, the voltage distribution was solved once again, and its solution is shown in Figure 6.

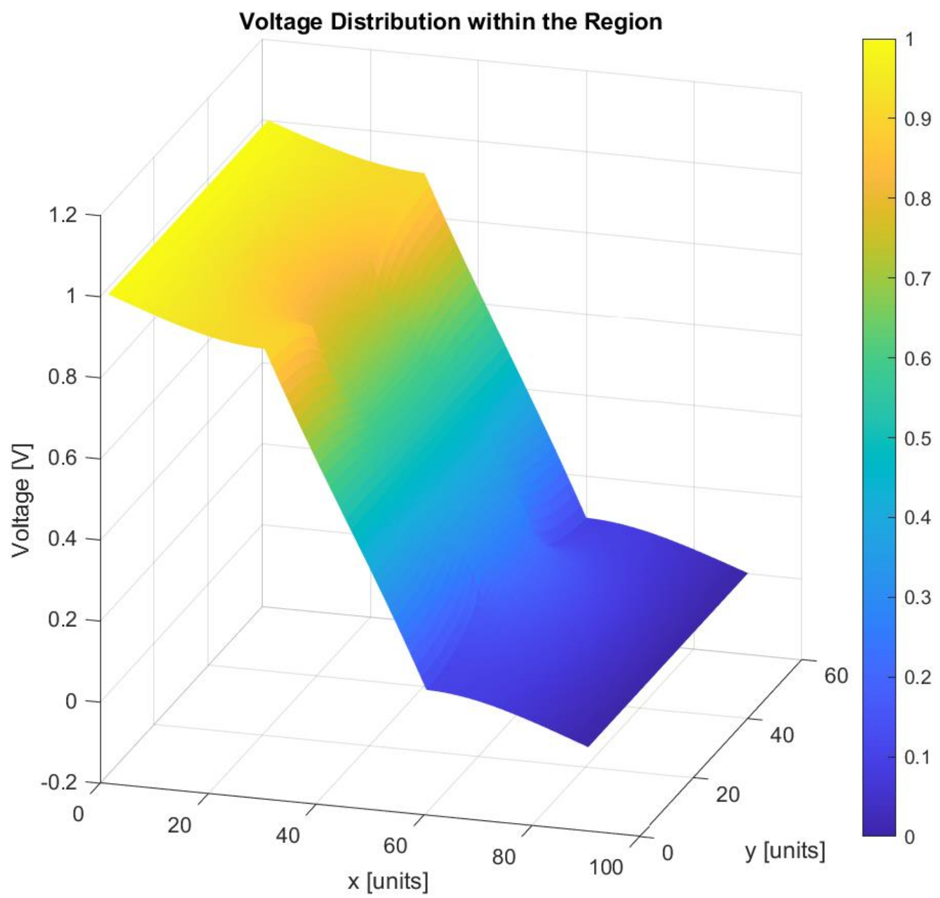

Figure 6: Plot of V(x,y)

As can be seen in the results, different conductivities affect the voltage distribution, though, this solution is as expected. This is because the boxes have a low conductivity, and therefore, high resistance. With a large resistance, there is a large voltage drop across it, which is shown in the large voltage drop across the boxes. Similarly, the regions of higher conductivity have a lower resistance, and thus, there is less of a voltage drop across the region. Therefore, this solution makes sense.

With this solution, the electric field within the region can be calculated. This is done with the following equation:

$$\vec{E} = -\nabla V$$

Using MATLAB's gradient function, the electric field components (Ex and Ey) were obtained, which gives the full Electric field of the region. This result is shown in Figure 7, plotted with the quiver function.

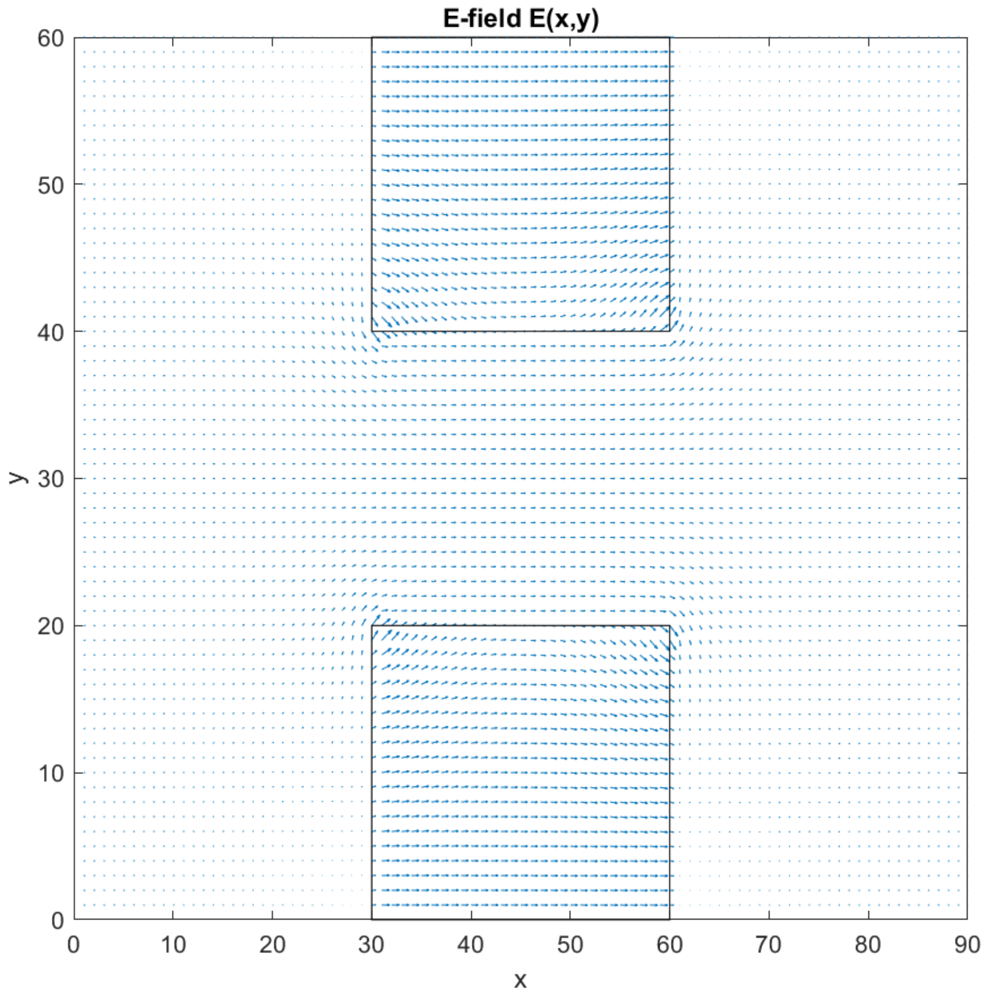

Figure 7: Plot of E(x,y)

As can be seen, this result aligns with expectations as well. That is, in regions with a large voltage drop (in this case, the boxes), the magnitude of the Electric field is larger. This is evident due to the darker coloured arrows within the boxes' regions. Similarly, outside the boxes had higher conductivity, a smaller voltage drop, and thus, a smaller Electric field. Finally, it is evident that the arrows point left-to-right, which is correct since the left side of the region is 1V, and the right side is 0V.

The Electric field can then be used to find the current with the following equation:

$$\vec{J} = \sigma \vec{E}$$

Therefore, to obtain the current, the conductivities are multiplied with the electric field, on an element-based multiplication. The current within the region is shown in Figure 8.

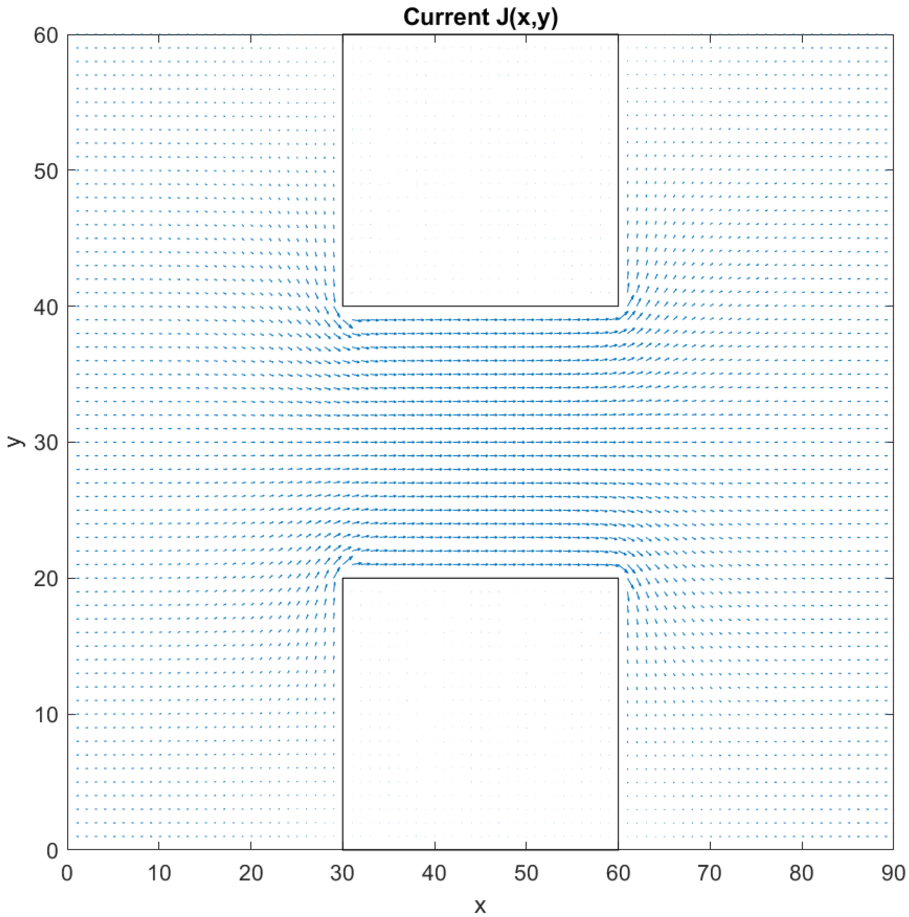

Figure 8: Plot of J(x,y)

As can be seen, this result also aligns expectations. That is, in regions with a lower conductivity (such as the boxes), there is low current. In this case, it appears it appears there is no current even though it is very small. Furthermore, the current remains nearly constant in the left and right sides. However, due to the boxes (and realistically, a lower conductivity), the current increases in the middle. This is evident in the darker-coloured arrows of the quiver plot.

Now that the currents of the region were obtained, three investigations were completed to see how factors affected current flow. Specifically, investigations on mesh density vs. current, “bottle neck” vs. current flow, and conductivity vs current flow, were completed.

Mesh Density vs. Current Flow

The purpose of this investigation is to determine how the mesh affects current flow. The expectation of this investigation is that a larger mesh provides a more accurate model of the current flow because of the discretization.

To accomplish this investigation, the mesh size needs to be changed. This was done by changing variables nx and ny which determine the size of the region. The size of the region is specifically changed because the boundary voltage and distance between nodes stays constant, due to the matrix setup. Therefore, a larger matrix ultimately implies more nodes, and this implies a more-dense region. The same script used to generate was re-used, but a loop in the main script setup the matrices before calling the specific script. The test cases are defined as shown in Table 1.

Table 1: Mesh Size Test Cases

Ny	nx (nx = 1.5ny)
20	30
40	60
80	120
100	150

Note, a test case of $ny = 60$ was not used as this was the case in the previous parts of the assignment.

The results of this investigation are shown in Figure 9.

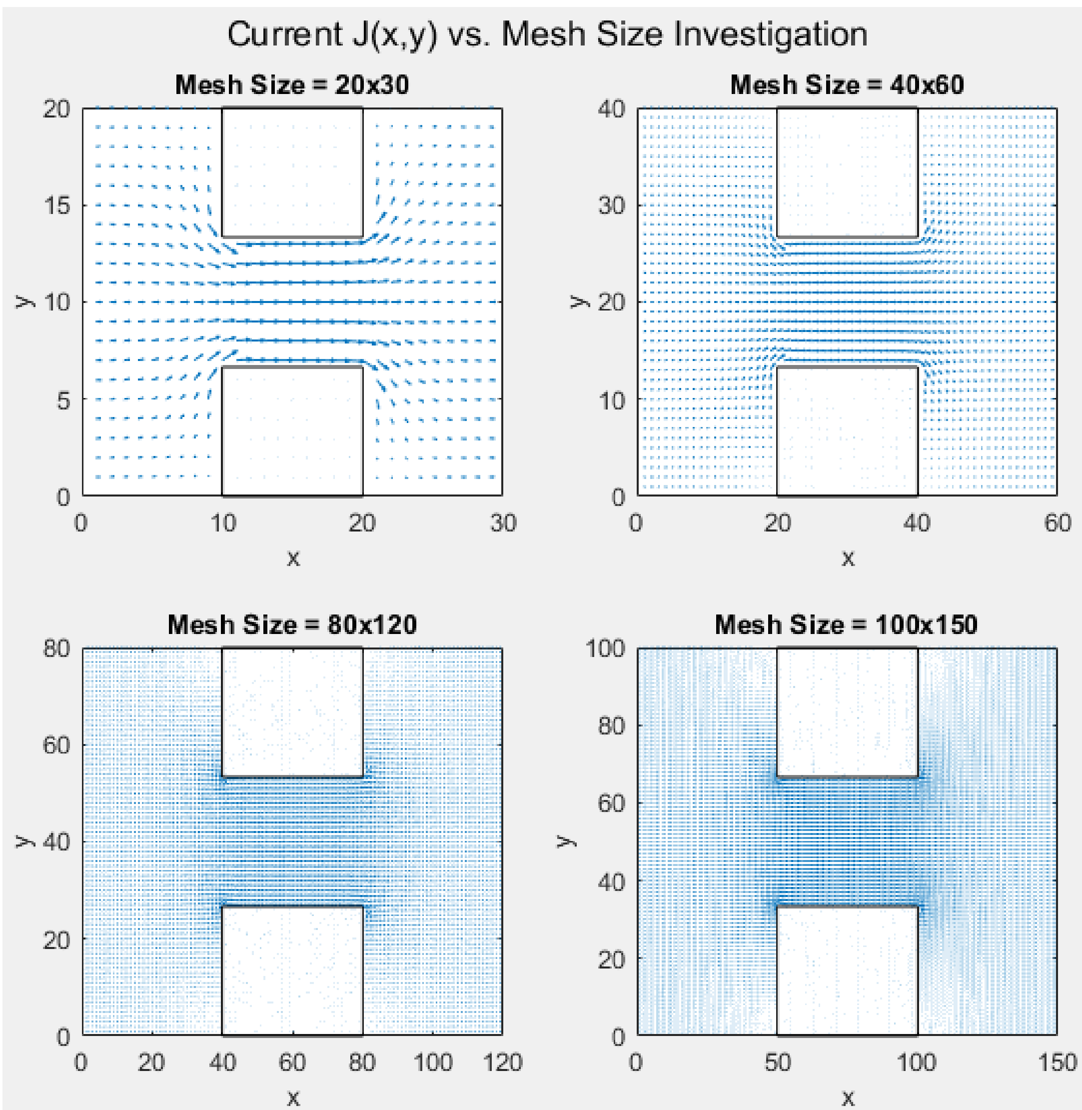

Figure 9: Mesh Investigation Results

As can be seen in the solutions, the results are fairly similar, specifically in that the magnitudes increase and decrease in the same regions/nodes. However, it is clear that a more-dense mesh (i.e., 100x150) gives a more-accurate model of the current within the region. Conversely, a less-dense mesh provides a quiver plot with larger arrows, which is ultimately due to larger nodes.

Therefore, the results align with the initial expectations.

Bottle-neck vs. Current Flow

The purpose of this investigation is to determine how the boxes/bottleneck affects current flow. The expectation of this investigation is that the magnitude of the current increases as the bottleneck narrows.

To accomplish this investigation, the height of the boxes needs to be changed. This was done by changing the y-coordinates of the boxes, ultimately accomplished by setting a “percent_open” variable which automatically set the height of the boxes based on the size of the region. The test cases were 80% open, 60% open, 40% open, and 20% open. Once again, note that this is a fraction of the total height (ny) of the region.

The results of this investigation are shown in Figure 10.

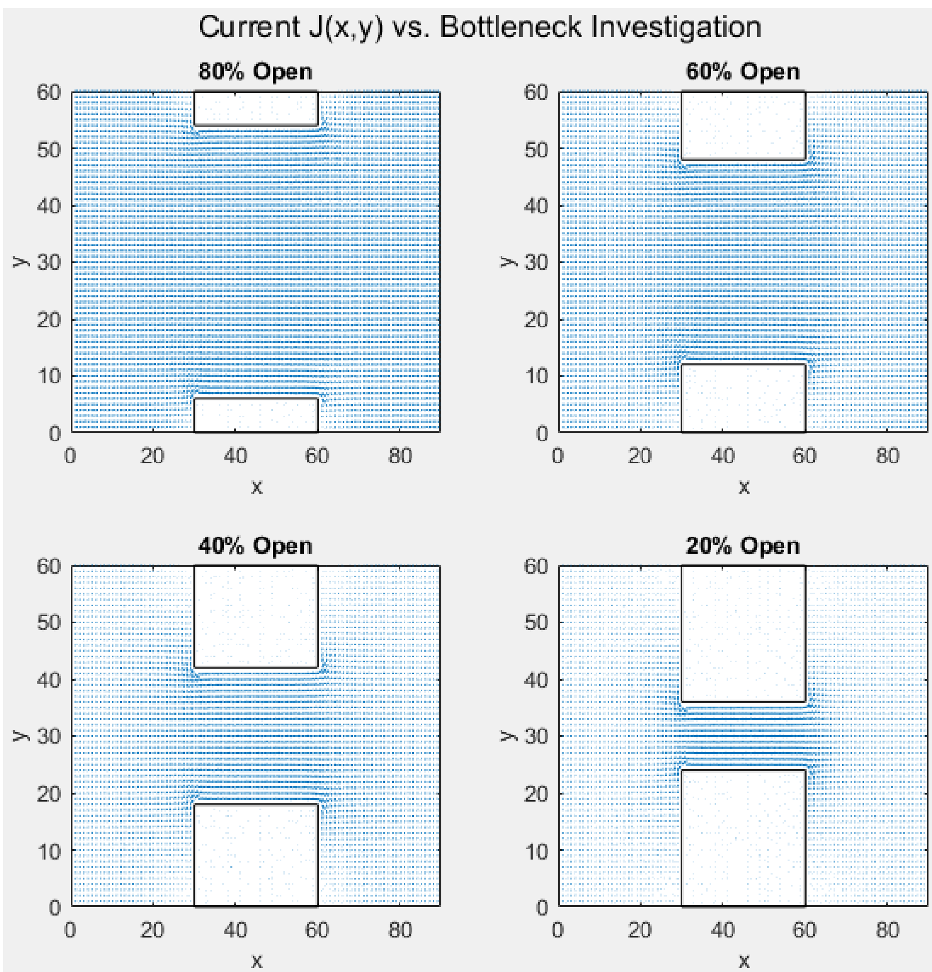

Figure 10: Bottleneck Investigation Results

As can be seen in the results, a wide bottleneck shows little change in the current ultimately due to a nearly constant conductivity. As the bottleneck narrows, the conductivity of the region changes such that the middle of the region is less conductive. This forces the current on the left side of the region to increase in magnitude as the majority of the current is forced between the bottleneck. This explains the larger magnitude in the middle, evident by the darker-coloured arrows from the quiver plot. Finally, it is also important to note that, even though the 80% plot is darker than the 20% plot, this is all relative. The 80% plot will have similar magnitude as the left and right regions of the 20% plot, but due to the larger magnitude of current in the middle of the 20% plot, the arrows are lighter on the left and right sides. This is just how the quiver function works.

Conductivity Investigation

The purpose of this investigation is to determine how changing the conductivity of the boxes affects current flow. The expectation of this investigation is that the magnitude of the current increases or decreases with the conductivity, due to electricity fundamentals.

To accomplish this investigation, the conductivity (σ) of the boxes needs to be changed. This was done by changing the conductivity element of the boxes. The test cases are outlined in Table 2.

Table 2: Conductivity Investigation Test Cases

Box 1 Conductivity	Box 2 Conductivity
0.1	0.1
10	10
0.001	100
1000	0.0001

The results of this investigation are shown in Figure 11.

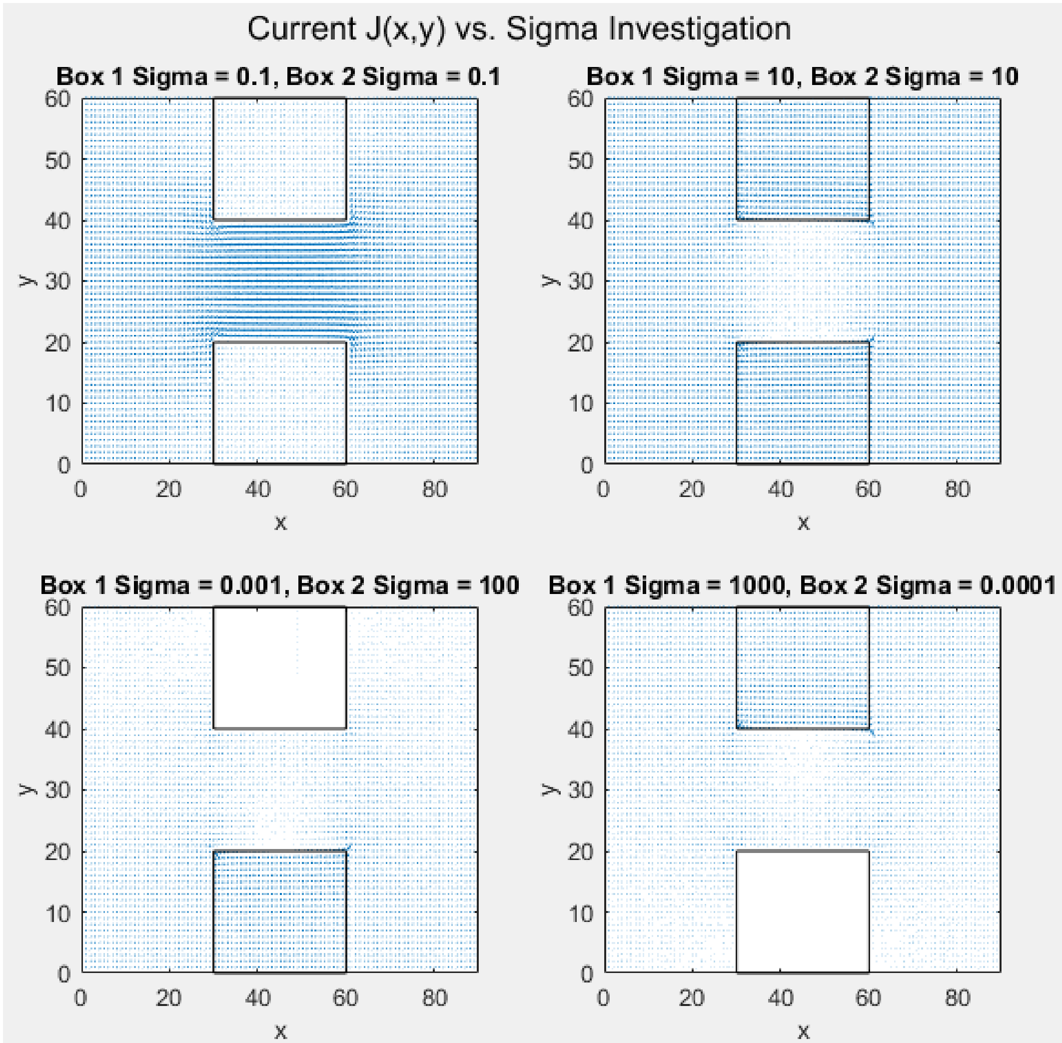

Figure 11: Conductivity Investigation Results

As can be seen, the results align with expectations. Specifically, when both conductivities are 0.1, the boxes' conductivities increased 10x the original test case. The behaviour remains consistent with the previous results, but there is more current in the box due to the increase in conductivity. In addition, the current between the boxes will have decreased due to the increase of the boxes' conductivity. Then, when both boxes have conductivity of 10, it is clear that the boxes have an increase of current, and between the boxes is drastically reduced. Next, box 1 was set to have an even smaller conductivity than the original case, and it is clear that the majority of the current flows through the bottom box with conductivity of 100. Furthermore, the current in between the boxes is reduced since box 2's conductivity is very large. The final plot shows a similar behaviour, but instead, most of the current flows through box 1 since its conductivity is 1000. The same behaviour between the boxes is also present where there is little current due to a 1000x reduction in conductivity. It can also be noted that the bottom box has nearly zero current since its conductivity is so small.

Therefore, the results align with expectations, given the various test scenarios.