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Week 2, Lecture 01-17-23

Example: **Given a box with 6 pennies and 8 quarters, pick 5 of the coins at random (without replacement). What is the probability that we choose 2 pennies and 3 quarters?**
There are a total of $\binom{14}{5}$ 5-tuples of coins. How many of these choices are "good"? i.e. 2 pennies, 3 quarters. There are $\binom{6}{2}$ ways of picking 2 pennies and $\binom{8}{3}$ ways of picking 3 quarters.

$$\therefore \text{the total number of good 5-tuples is the product} \binom{6}{2}\binom{8}{3}$$

$$\therefore \text{using equiprobability (i.e. all 5-tuples have the same probability)}$$

$$\therefore \text{Probability} = \frac{\binom{6}{2}\binom{8}{3}}{\binom{14}{5}}$$

Example: **Toss a coin 3 times. What is the probability we get exactly 2 heads?**
The sample space of all possible outcomes can be defined as:

$$S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$$

$$P(\{HHT, HTH, THH\}) = \frac{3}{8}$$

What is the probability we get exactly 2 heads, given that the first two flips are not both heads?

$$S = \{\cancel{HHH}, \cancel{HHT}, HTH, HTT, THH, THT, TTH, TTT\}$$

Now, there is only two possible outcomes (i.e. HTH and THH) and only 6 to choose from.
Let us define the events:

$$E = \text{"Exactly two heads occur"}$$
$$F = \{HHT, HHH\}^c$$

Intuitively, the probability is $\frac{2}{6} = \frac{1}{3}$.

$$\boxed{\text{We write } P(E|F) \text{ to mean } P(E) \text{ given } P(F)}$$

$$\boxed{\textbf{Definition: } \text{If } P(F) > 0, \text{ then define } P(E|F) = \frac{P(EF)}{P(F)}}.$$

This is also called the conditional probability of E given F and is intuitive given a venn diagram.

Example: **Roll 2 dice. Find the probability both dice are even given their sum is greater than or equal to 10.**
Let us define the events:

$$\begin{aligned} E &= \text{Both dice are even} \\ F &= \text{Sum is } \geq 10 \\ &= \{(6,6), (6,5), (5,6), (6,4), (4,6), (5,5)\} \end{aligned}$$

We want $P(E|F)$.

$$EF = \{(6,6), (6,4), (4,6)\}$$

$$P(EF) = \frac{|EF|}{|S|} = \frac{3}{36} = \frac{1}{12}$$

$$P(F) = \frac{|F|}{|S|} = \frac{6}{36} = \frac{1}{6}$$

$$P(E|F) = \frac{P(EF)}{P(F)} = \frac{\frac{3}{36}}{\frac{6}{36}} = \frac{3}{6} = \frac{1}{2}$$

Now find the probability the sum = 7, given the sum ≠ 6. Let us define the events:

$$\begin{aligned} E &= \text{Sum} = 7 \\ F &= \text{Sum} \neq 6 \end{aligned}$$

We want $P(E|F)$.

$$\begin{aligned} E &= \{(1,6),(6,1),(2,5),(5,2),(4,3),(3,4)\} \\ F &= \{(1,5),(5,1),(4,2),(2,4),(3,3)\}^c \\ P(F^c) &= \frac{5}{36} \\ P(F) &= 1 - \frac{5}{36} = \frac{31}{36} \end{aligned}$$

Note: $E \subseteq F$ implies $EF = E$. Therefore,

$$\begin{aligned} P(EF) &= P(E) = \frac{6}{36} \\ P(EF) &= \frac{P(EF)}{P(F)} = \frac{\frac{6}{36}}{\frac{31}{36}} = \frac{6}{31} \end{aligned}$$

Special Cases:

1. If E, F are disjoint, then $EF = 0$, so $P(EF) = 0$.
   $\therefore P(E|F) = \frac{P(EF)}{P(F)} = 0$
2. If $E \subseteq F$, then $EF = E$. So $P(F|E) = \frac{P(EF)}{P(E)} = \frac{P(E)}{P(E)} = 1$

Potential useful property:

$$\boxed{P(EF) = P(E|F)P(F) = P(F|E)P(E)}$$

Example: **A box contains 3 blue, 4 red, and 7 green marbles. One marble is chosen at random and it is not red. What is the probability that it is blue?**

Let us define the events:

$$\begin{aligned} E &= \text{Marble is blue} \\ F &= \text{Marble is not red} \end{aligned}$$

We want $P(E|F)$. We know that

$$\begin{aligned} P(F^c) &= P(\text{Marble is red}) \\ &= \frac{4}{3+4+7} = \frac{4}{14} \\ P(F) &= 1 - \frac{4}{14} = \frac{10}{14} \end{aligned}$$

We claim that $E \subseteq F$, as the even that a blue marble is chosen implies that the chosen marble is not red, whereas if the chosen marble is not red, this does not imply that the marble is blue. Resultantly,

$$EF = E$$

$$P(EF) = P(E) = \frac{3}{14}$$

$$\therefore P(E|F) = \frac{P(EF)}{P(F)} = \frac{\frac{3}{14}}{\frac{10}{14}} = \frac{3}{10}$$

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