Lec 08

Last time : CDFs

Various properties

Some of these :

$$P(a < X \le b) = F(b) - F(a)$$

$\vdots$

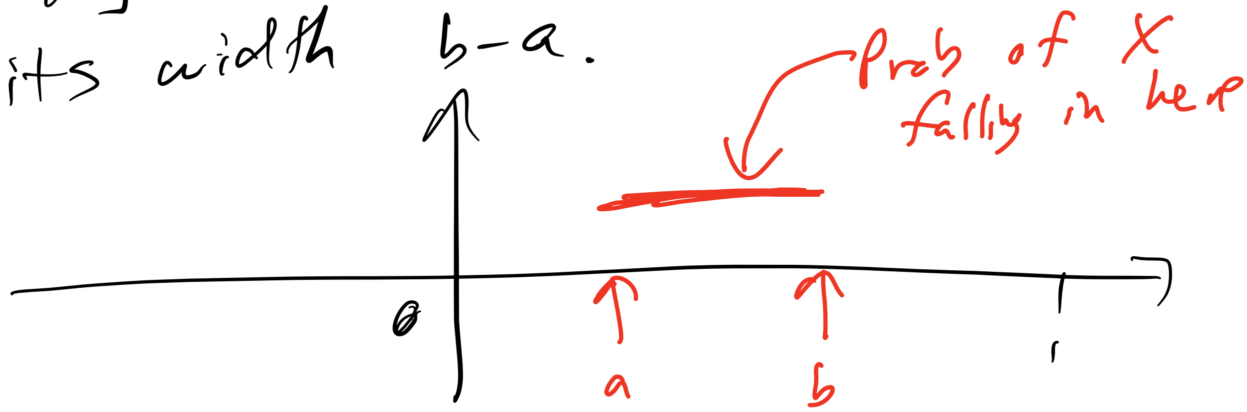
$$P(X = u) = F(u) - F(u^-)$$

$$= \text{"jump at } u\text{"}$$

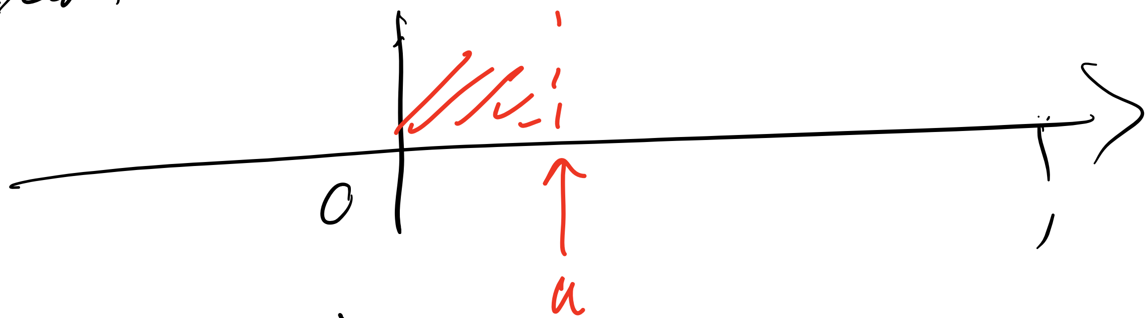
Ex : In computer languages like C or C++, etc there's a random number generator, such as drand48(); which is type double. It's approximately any real number in [0, 1]. Let's model it as a r.v. and find its CDF.

Assumption : No preference for any particular value in [0, 1].

we'll assume that the prob that rv X lies in some subinterval $[a,b]$ contained in $[0,1]$ equals its width $b-a$.



In particular



$$P(X \in [0,u]) = u - 0 = u$$

if $0 \le u \le 1$.

$$= P(X \le u) = F(u) \quad \text{CDF}$$

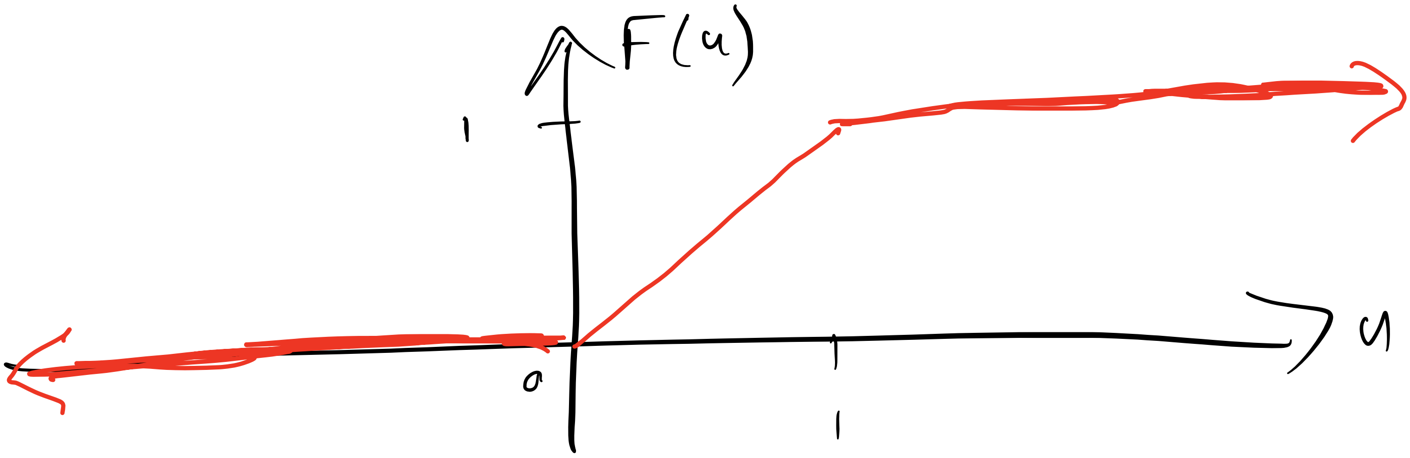
when $u \in [0,1]$

Other cases:

If $u < 0$ then $P(X \le u) = 0$

If $u > 1$, then $P(X \le u) = 1$

Plot CDF for all u



Notice there's no jumps in $F(u)$.
So F is continuous everywhere.

Thus, for <u>every</u> u, we have

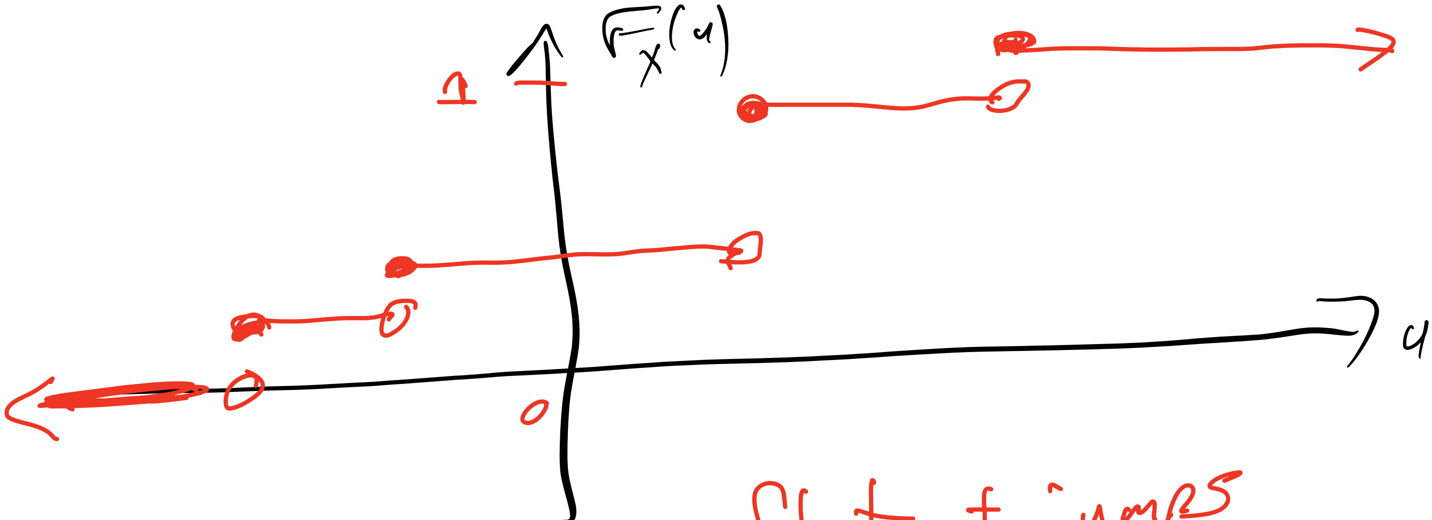
$$P(X=u) = 0$$ since no jumps.

ie. X has zero probability of equalling any particular real number u, but X always equals some real number u.

We will focus primarily on 2 types of rvs in ECE109: discrete rv
continuous rv

A **discrete rv** X is a rv which takes on a finite (or countably discrete infinite) set of values.

The CDF of any discrete r.v. is a "staircase function".



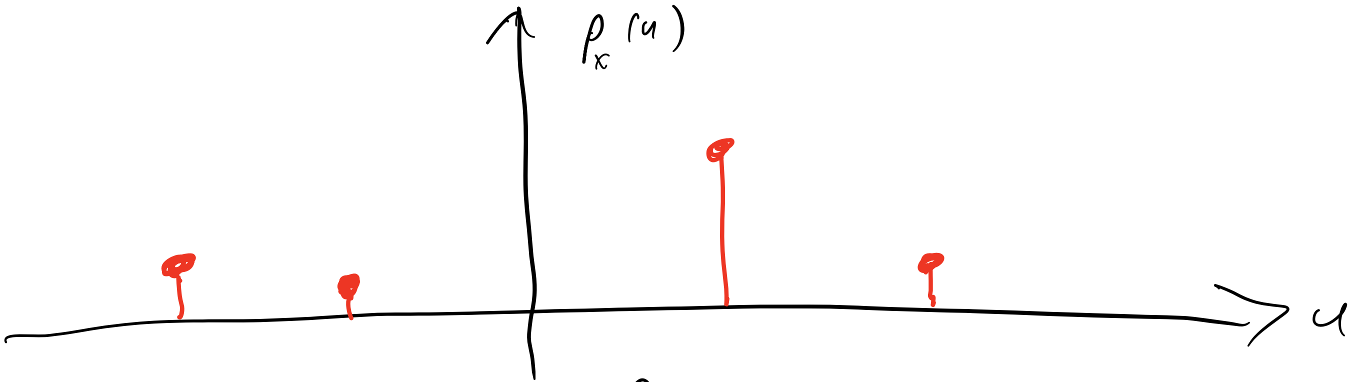
flat + jumps

The **probability mass function (pmf)** of a discrete rv X is:

$$p_X(u) = P(X=u)$$

Small "p" is pmf name

Large "P" is "probability"



$p_X(u)$ is the jump of the CDF at u.
Heights are the probs.
Use vertical bars for ease of viewing.

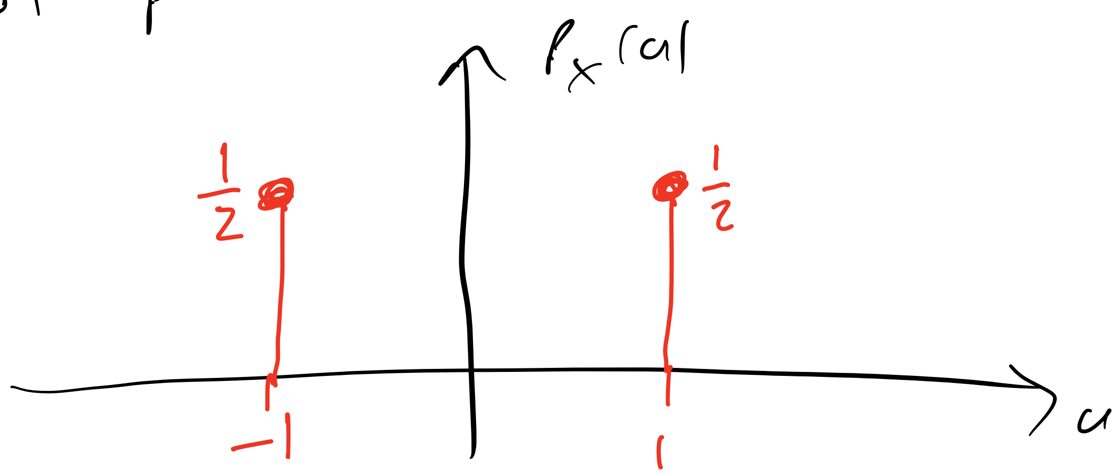
Facts: $p_X(u) \geq 0$ for all u.

$$\sum_u p_X(u) = 1$$

Ex: Flip one fair coin

Let $X = \begin{cases} 1 & \text{if Heads} \\ -1 & \text{if Tails} \end{cases}$

Plot pmf of X.



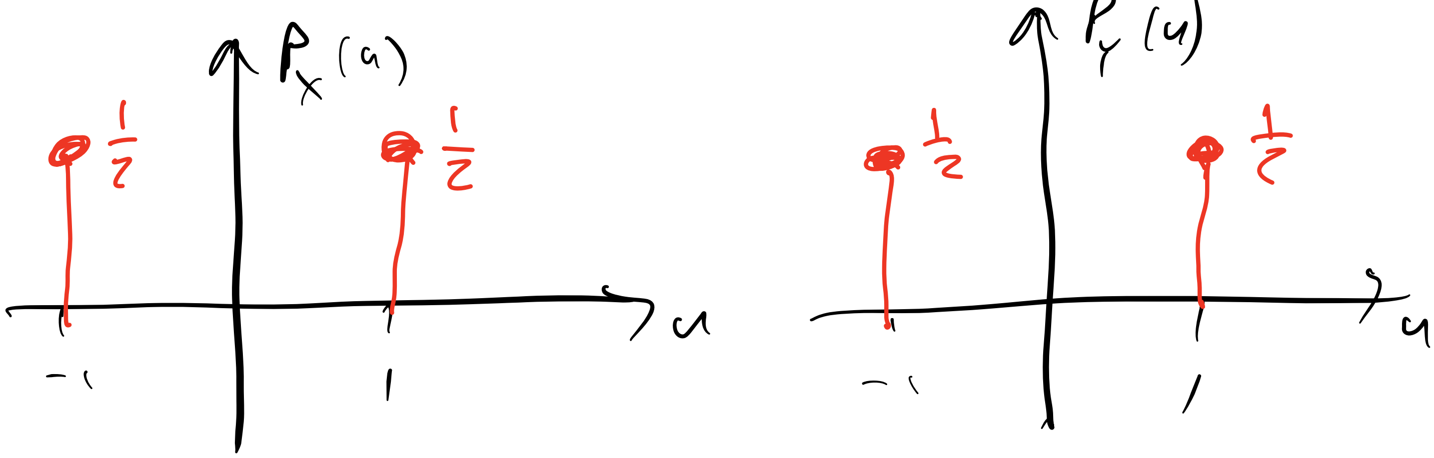
Ex Roll a fair die.

Define 2 rvs:

$$X = \begin{cases} -1 & \text{if die is even} \\ 1 & \text{if die is odd} \end{cases}$$

$$Y = \begin{cases} -1 & \text{if die is } \leq 3 \\ 1 & \text{if die is } \geq 4 \end{cases}$$

Plot the pmfs:



The pmfs of X and Y are the same.
But the rvs X and Y are <u>different</u>.

e.g. $X(2) = X(4) = X(6) = -1$
$X(1) = X(3) = X(5) = 1$

but

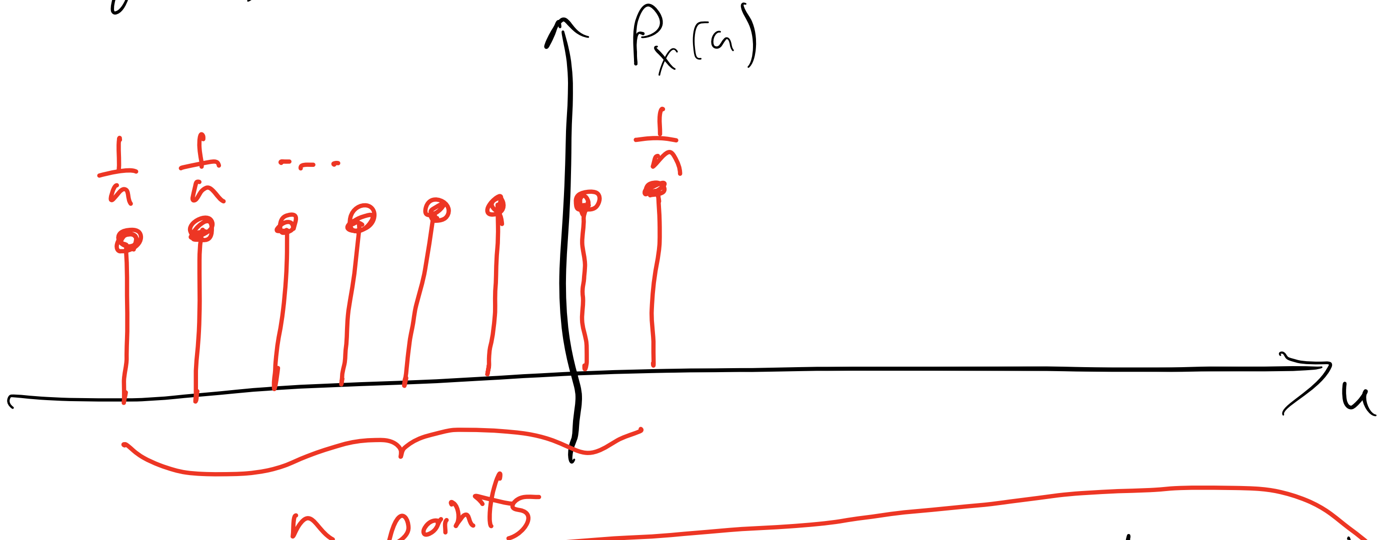
$Y(1) = Y(2) = Y(3) = -1$
$Y(4) = Y(5) = Y(6) = 1$

For example $Y(1) \neq X(1)$

Let's look at some special "named" rvs:

① Uniform Discrete r.v.

The pmf has a finite number of equally spaced and equal height values.



② Binomial r.v. (with parameters n, p)

The pmf is:

$$P_X(K) = \binom{n}{K} p^K (1-p)^{n-K}$$

for $K = 0, 1, 2, \ldots, n$

X represents the # of Heads you get if a biased coin with P(Heads) = p is flipped n times.

Recall from algebra the "binomial theorem"

$$(x+y)^n = \sum_{k=0}^{n} x^k y^{n-k} \binom{n}{k}$$

eg: $(x+y)^1 = x+y$

$(x+y)^2 = x^2 + 2xy + y^2$

$(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$

$\vdots$ use pascal's triangle

$$\therefore \left(\underbrace{p}_{x} + \underbrace{(1-p)}_{y}\right)^n = \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k}$$

$$\| \qquad\qquad\qquad \|$$

$$1 \qquad\qquad \sum_{k=0}^{n} p_X(k)$$

③ Poisson r.v. (with parameter λ)

The pmf is:

$$p_X(k) = \frac{\lambda^k e^{-\lambda}}{k!} \quad \text{for } k = 0, 1, 2, 3 \ldots$$

used to model arrival times, failure rates, etc.

$e^{-\lambda}$ assures us that prob adds to 1.

$$\sum_{k=0}^{\infty} P_X(k) = \sum_{k=0}^{\infty} \frac{\lambda^k e^{-\lambda}}{k!} = e^{-\lambda} \cdot \underbrace{\sum_{k=0}^{\infty} \frac{\lambda^k}{k!}}_{\text{Taylor Series for } e^{\lambda}}$$

$$= e^{-\lambda} \cdot e^{\lambda} = 1$$

④ Geometric r.v. (with parameter p)

The pmf is

$$P_X(k) = p(1-p)^{k-1}$$

for $k = 1, 2, 3, \ldots$

X is the # of flips of a biased coin until 1st Head appears. $p = P(\text{Heads})$

i.e. $\underbrace{T, T, T, \ldots, T}_{k-1 \text{ flips}}, H$