Lec 11:

Last time - expected values

Revisit Gaussians

CDF: $F_X(u) = P(X \le u)$

$$= \int_{-\infty}^{u} f_X(z)\, dz$$

Consider special case: $X \sim N(0,1)$

$m=0, \ \sigma^2 = 1$

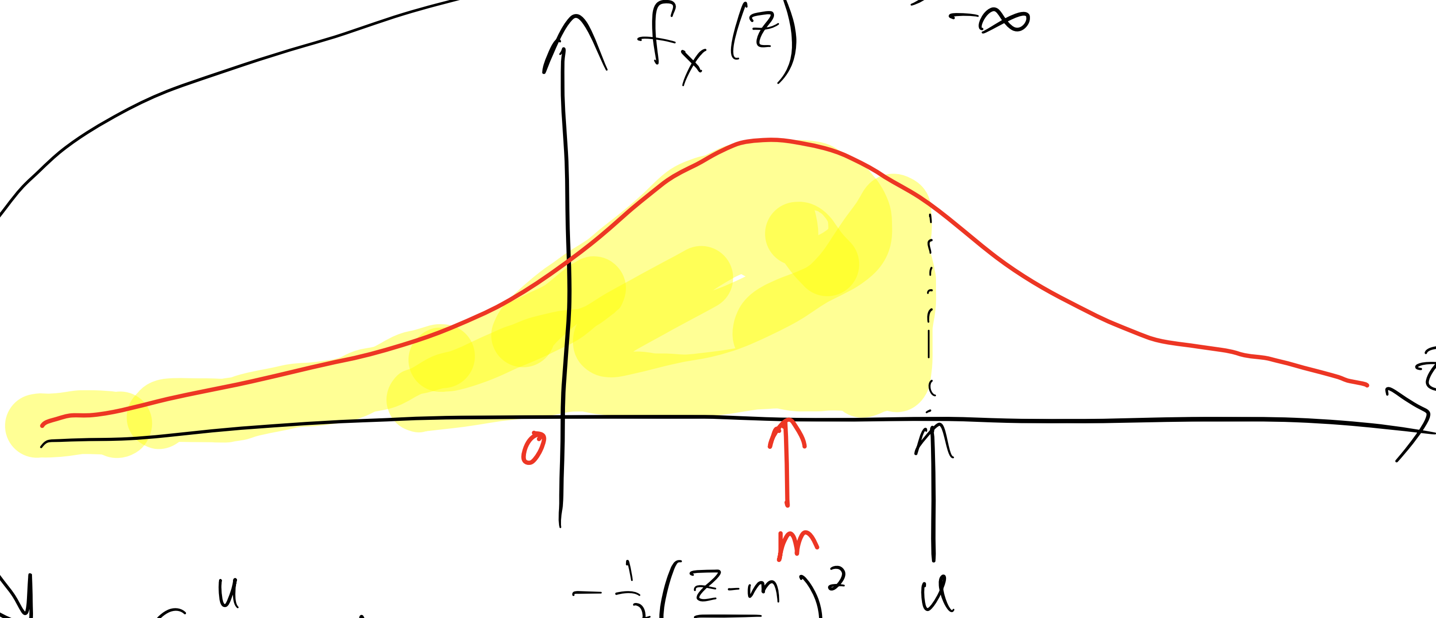
pdf: $f_X(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$

We will use special notation for the CDF in this case:

$$\boxed{\Phi(u) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{u} e^{-z^2/2}\, dz}$$

How can we get the CDF of $X \sim N(m, \sigma^2)$ in terms of Φ?

The CDF is $F_X(u) = \int_{-\infty}^{u} f_X(z)\,dz$



$$= \int_{-\infty}^{u} \frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{1}{2}\left(\frac{z-m}{\sigma}\right)^2} dz$$

$$\left(\text{Let } w = \frac{z-m}{\sigma} \qquad dw = \frac{dz}{\sigma}\right)$$

$$= \int_{-\infty}^{\frac{u-m}{\sigma}} \frac{1}{\cancel{\sigma}\sqrt{2\pi}}\, e^{-w^2/2} \left(\cancel{\sigma}\, dw\right)$$

$$= \underbrace{\int_{-\infty}^{\frac{u-m}{\sigma}} \underbrace{\frac{1}{\sqrt{2\pi}}\, e^{-w^2/2}}_{\text{pdf of } N(0,1)} dw}_{\text{CDF of } N(0,1) \text{ evaluated at } \frac{u-m}{\sigma}}$$

$$= \Phi\left(\frac{u-m}{\sigma}\right)$$

In summary,

$$\underbrace{F_X(u)}_{\text{CDF of } N(m,\sigma^2) \text{ evaluated at } u} = \underbrace{\Phi\left(\frac{u-m}{\sigma}\right)}_{\text{CDF of } N(0,1) \text{ evaluated at } \frac{u-m}{\sigma}}$$

Most computer languages have a built-in function to help calculate $\Phi(u)$. usually called "erf".

For example:

$$\boxed{\begin{aligned} \operatorname{erf}(u) &= \frac{2}{\sqrt{\pi}} \int_0^u e^{-t^2}\,dt \\ \operatorname{erfc}(u) &= 1 - \operatorname{erf}(u) \end{aligned}}$$

Here's how to convert:

$$\boxed{\Phi(u) = \frac{1}{2} + \frac{1}{2} \cdot \operatorname{erf}\left(\frac{u}{\sqrt{2}}\right)}$$

Expected values of functions of r.v.s.

Suppose $Y = g(X)$

X is a r.v.

g is a deterministic function.

Know pdf or pmf of X.

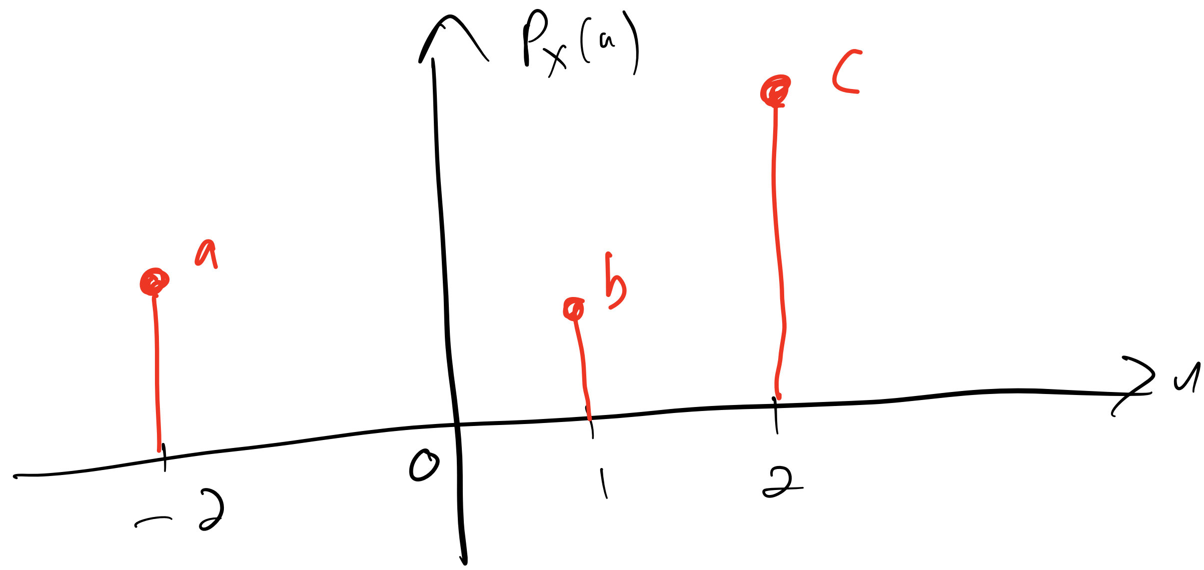
What is $E[Y]$?

i.e. what is $E[g(X)]$?

It turns out:

$$E[g(X)] = \begin{cases} \int_{-\infty}^{\infty} g(u) f_X(u)\, du & \text{(continuous)} \\ \sum_u g(u) p_X(u) & \text{(discrete)} \end{cases}$$

Ex: Suppose X is discrete with pmf



What is $E[X^2]$?

i.e. $E[g(X)]$ where $g(u) = u^2$

$$E[X^2] = (-2)^2 \cdot a + (1)^2 \cdot b + 2^2 \cdot c$$
$$= 4(a+c) + b$$

Ex Suppose $g(u) = au + b$ a, b constants

$$g(X) = aX + b$$

Continuous Case

$$E[g(X)] = E[aX + b]$$
$$= \int_{-\infty}^{\infty} g(u) f_X(u)\, du = \int_{-\infty}^{\infty} (au+b) f_X(u)\, du$$
$$= a \underbrace{\int_{-\infty}^{\infty} u f_X(u)\, du}_{E[X]} + b \underbrace{\int_{-\infty}^{\infty} f_X(u)\, du}_{1}$$

$$= aE[X] + b$$

In summary,

$$\boxed{E[aX+b] = aE[X] + b}$$

Discrete Case:

$$E[aX+b] = \sum_u (au+b)\, p_X(u)$$

$$= a \underbrace{\sum_u u\, p_X(u)}_{E[X]} + b \underbrace{\sum_u p_X(u)}_{1}$$

$$= a\,E[X] + b$$

Same form as for continuous case.

Similarly, we can show:

$$E[g_1(X) + g_2(X)] = E[g_1(X)] + E[g_2(X)]$$

<u>Def</u>: The <u>n^{th} moment</u> of a r.v. X is $E[X^n]$.

The <u>n^{th} central moment</u> of a r.v. X is $E[(X - E[X])^n]$

for $n = 1, 2, 3, \ldots$

<u>Facts</u> • 1^{st} moment is $E[X] =$ mean

- 1st Central moment is $E[X - E[X]]$

$$= E[X] - E[X] = 0$$

(from $E[aX+b]$)

- 2nd moment $(n=2)$ is $E[X^2]$

Def: The 2nd central moment is called the variance.

Notation:

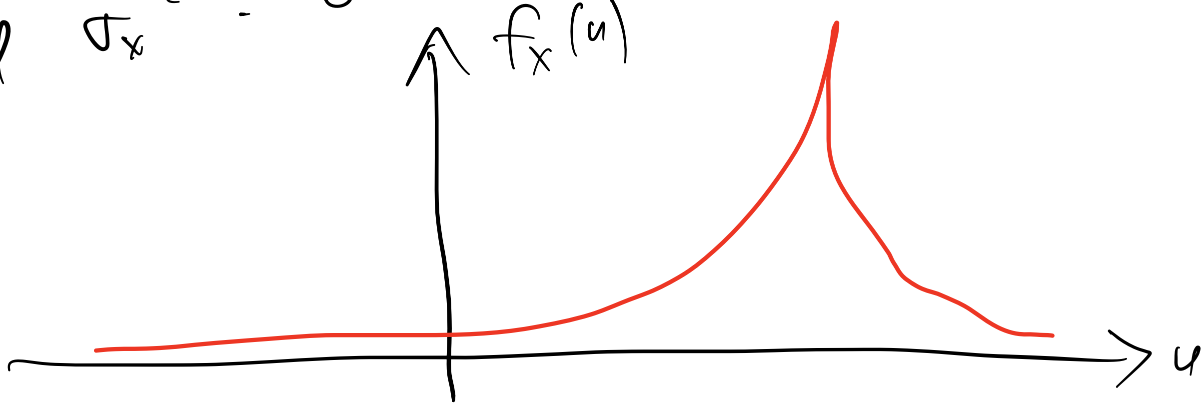
$$\sigma_X^2 = \mathrm{Var}(X) = E[(X - E[X])^2]$$

Standard deviation is

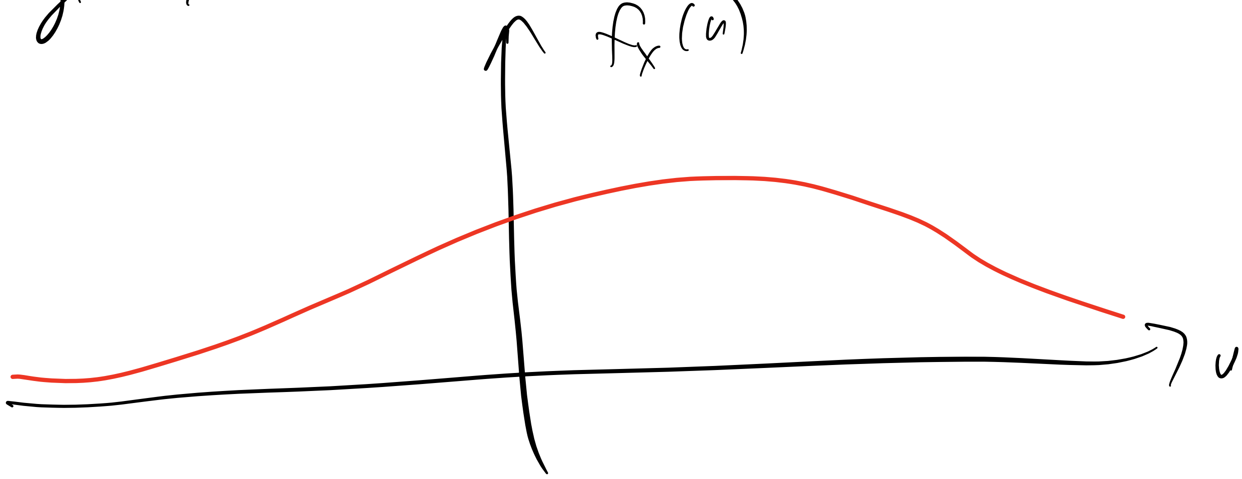
$$\sigma_X = \sqrt{\mathrm{Var}(X)}$$

Intuitively:

Small σ_X^2: Get tall, thin pdf typically



Large σ_X^2 : short, fat pdf



How can we compute the variance?

(take X continuous)

One way:

Let $m = E[X]$

$$\sigma_X^2 = E[(X-m)^2]$$

$$= E[g(X)] \quad \text{where } g(u) = (u-m)^2$$

$$= \int_{-\infty}^{\infty} g(u) f_X(u)\, du$$

$$= \int_{-\infty}^{\infty} (u-m)^2 f_X(u)\, du$$

If X is discrete:

$$\sigma_X^2 = \sum_u g(u) p_X(u) = \sum_u (u-m)^2 p_X(u)$$

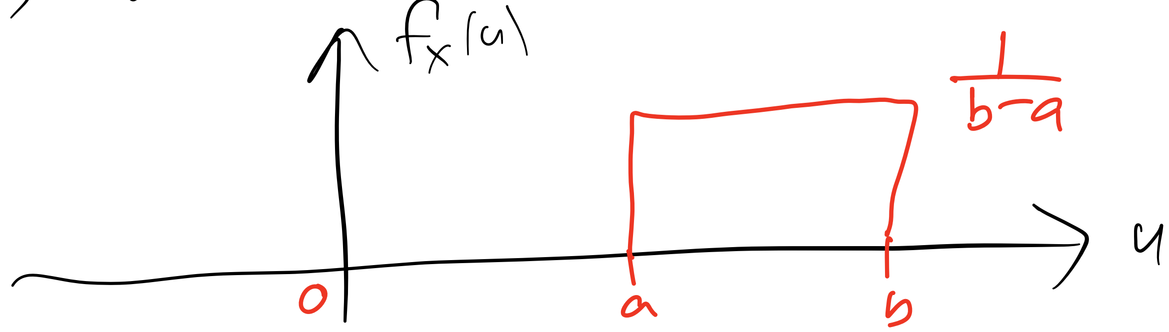
An alternate approach to calculate variance:

Let $m = E[X]$

$$\sigma_X^2 = E[(X-m)^2]$$
$$= E[X^2 - 2mX + m^2]$$
$$= E[X^2] + E[-2mX] + E[m^2]$$
$$= E[X^2] - 2m \cdot \underbrace{E[X]}_{m} + m^2$$
$$= E[X^2] - m^2$$
$$= \boxed{E[X^2] - (E[X])^2}$$

This is often easier to use. eg when $E[X] = 0$.

Ex: Compute the variance of a r.v. that is uniform on $[a, b]$.



We know $E[X] = \frac{a+b}{2}$

$$E[X^2] = \int_a^b g(u) f_X(u)\, du \quad \text{where } g(u) = u^2$$

$$= \int_a^b u^2 \cdot \frac{1}{b-a}\, du = \frac{1}{b-a} \cdot \frac{u^3}{3}\Big|_a^b$$

$$= \frac{b^3 - a^3}{3(b-a)} = \frac{\cancel{(b-a)}(b^2+ab+a^2)}{3\cancel{(b-a)}}$$

$$= \frac{1}{3}(b^2+ab+a^2)$$

$$\left(\frac{a+b}{2}\right)^2$$

$$\sigma_X^2 = E[X^2] - (E[X])^2$$

$$= \frac{1}{3}(b^2+ab+a^2) - \frac{1}{4}(a^2+2ab+b^2)$$

$$= \frac{a^2+b^2-2ab}{12} = \frac{(a-b)^2}{12}.$$

Ex: Compute variance of Gaussian $N(m, \sigma^2)$

i.e. Show $Var(X) = \sigma^2$

Solution: $Var(X) = E[(X-m)^2]$

$$= \int_{-\infty}^{\infty} (u-m)^2 f_X(u)\, du$$

$$= \int_{-\infty}^{\infty} (u-m)^2 \cdot \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{u-m}{\sigma}\right)^2} du$$

(Substitute: $y = \frac{u-m}{\sigma}$ $\quad dy = \frac{du}{\sigma}$)

$$= \int_{-\infty}^{\infty} (\sigma y)^2 \cdot \frac{1}{\sigma\sqrt{2\pi}} e^{-y^2/2} (\sigma\, dy)$$

$$= \frac{2}{\sqrt{2\pi}} \int_{0}^{\infty} (\sigma y)^2 e^{-y^2/2} dy$$

Integrate by parts ...

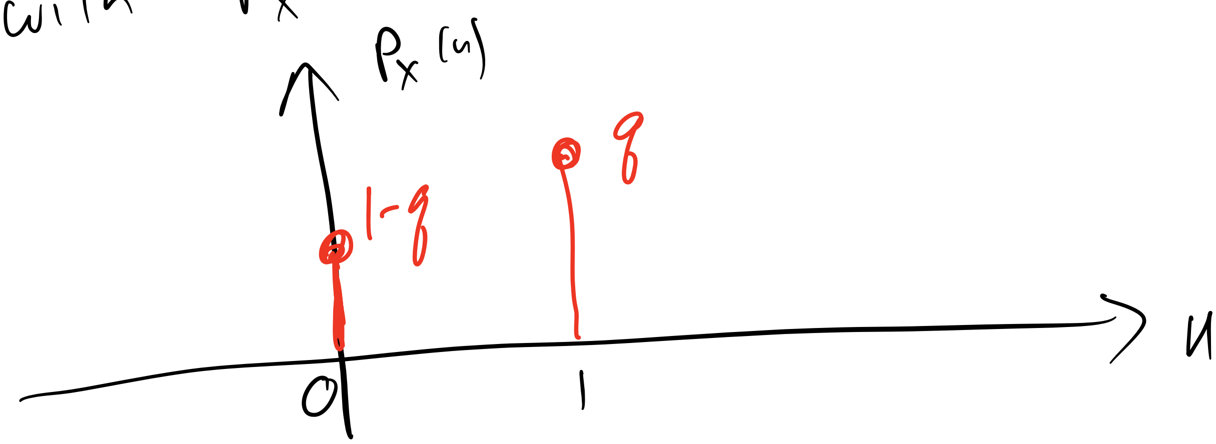
(work this out)

$\vdots$

$$= \sigma^2$$

i.e. $\mathrm{Var}(X) = \sigma^2$

<u>Ex</u>: Let X be a binary r.v. with $P_X(1) = q = 1 - P_X(0)$



Find variance of X.

$$\sigma_X^2 = E[X^2] - (E[X])^2$$

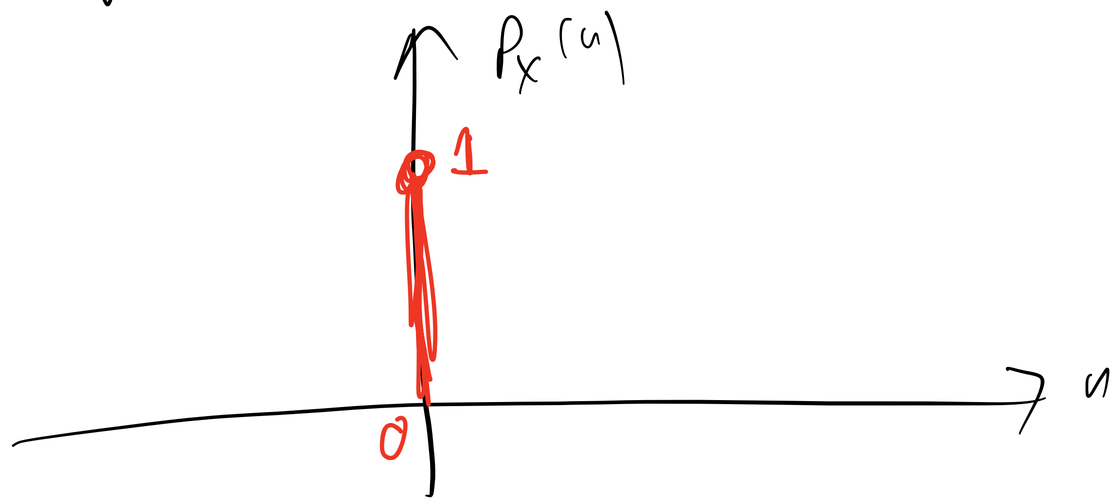
$$E[X] = \sum_u u\, p_X(u) = 0\cdot(1-q) + 1\cdot q = q$$

$$E[X^2] = \sum_u u^2 p_X(u) = 0^2\cdot(1-q) + 1^2\cdot q = q$$

$$\therefore \sigma_X^2 = q - q^2 = q(1-q)$$

In the extreme case where $q = 0$, we get a deterministic r.v. that is always 0.

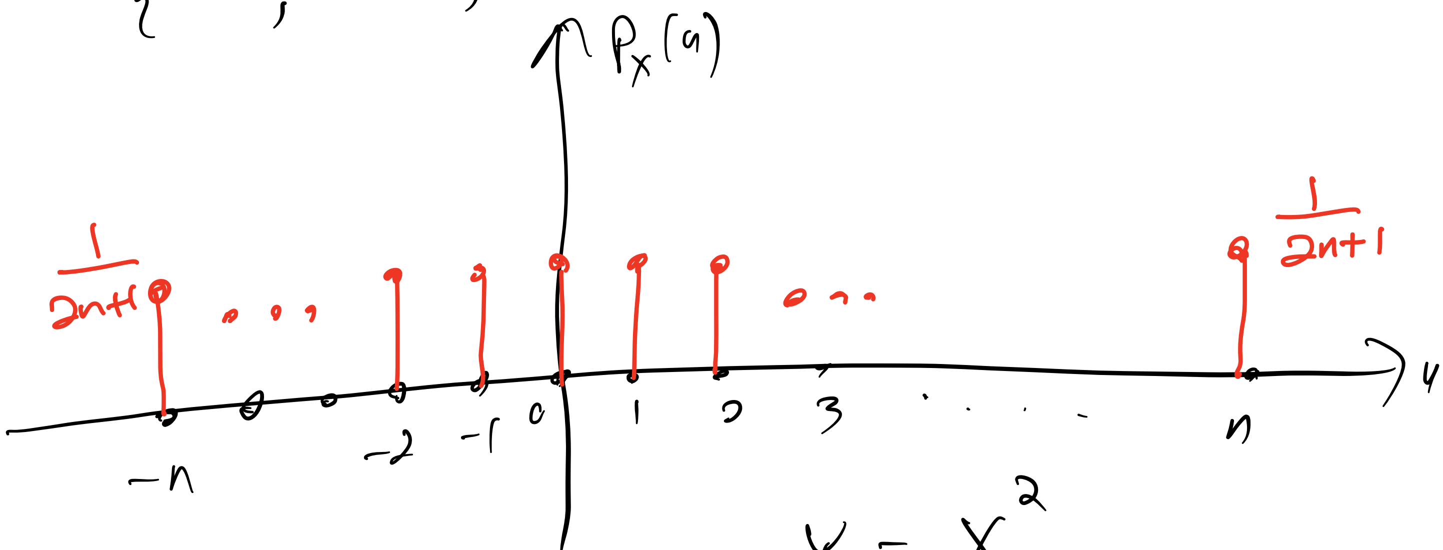
Variance is 0 in this case.



Functions of a random variable.

<u>Goal</u>: If we're given r.v. X and we form a new r.v. Y as a function of X, say $Y = g(X)$. Then find pdf or pmf of Y.

<u>Ex</u> Suppose X is discrete uniform on $\{-n, -n+1, \ldots, -1, 0, 1, 2, \ldots, n\}$



Create a new r.v. $Y = X^2$

Then Y takes on values in:

$\{0, 1, 4, 9, 16, \ldots, n^2\}$

Pmf of Y is:

$$p_Y(u) = P(Y=u) = P(X^2=u)$$
$$= P(X=\sqrt{u} \text{ or } X=-\sqrt{u})$$

$$= \begin{cases} \frac{2}{2n+1} & \text{if } u = 1, 4, 9, \ldots, n^2 \\ \frac{1}{2n+1} & \text{if } u = 0 \end{cases}$$