## Lec 13:

Last time: joint CDF

$$F_{X,Y}(u,v) = P(X \le u, Y \le v)$$

**Ex**: Flip a fair coin twice.

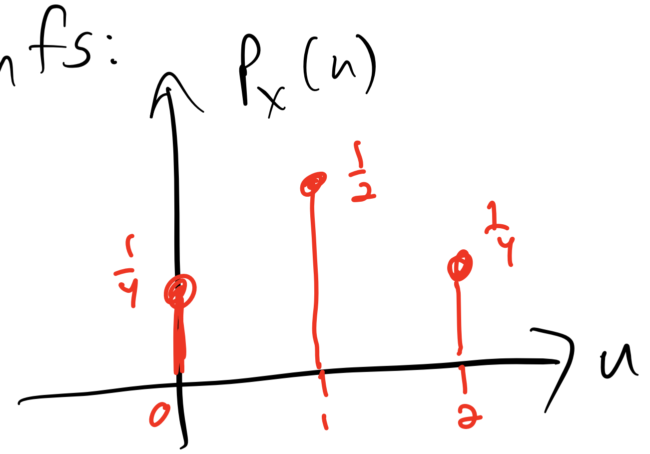
$$S = \{HH, HT, TH, TT\}$$

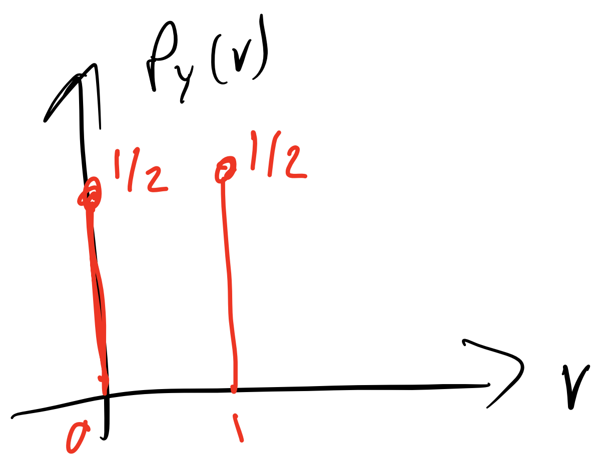
Define 2 rvs $X, Y$ as follows:

$$X = \#\text{ Heads}$$

$$Y = \begin{cases} 1 & \text{if same} \\ 0 & \text{else} \end{cases}$$

pmfs:



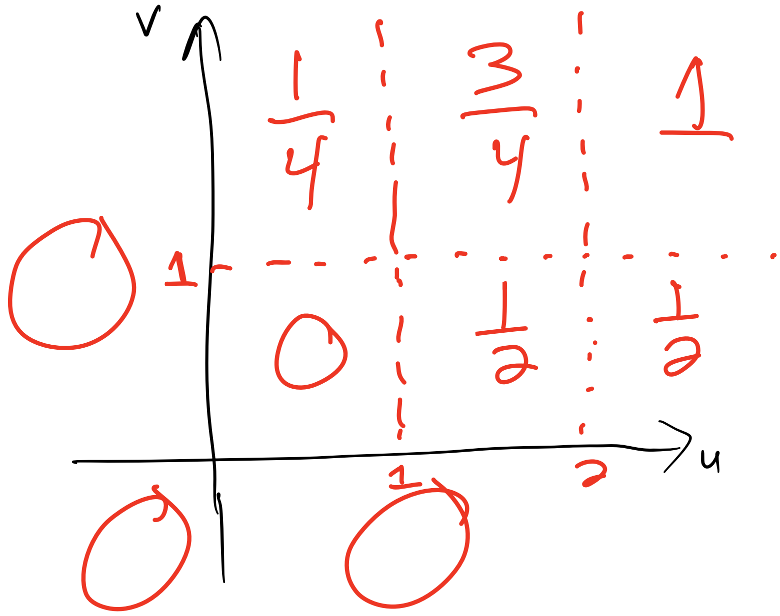

Now calculate joint CDF of $X, Y$

$$F_{X,Y}(u,v) = P(X \le u, Y \le v)$$

Cases. Top-down view

$u < 0$ or $v < 0$

$$F_{X,Y}(u,v) = P(X \le u, Y \le v) = 0$$




$0 \le u < 1$ and $v \ge 1$

$$P(X \le u, Y \le v) = P(X = 0, Y \in \{0,1\}) = P(X=0) = P(TT) = \frac{1}{4}$$

$0 \le u < 1$ and $0 \le v < 1$

$$P(X \le u, Y \le v) = P(\underbrace{X=0}_{TT}, \underbrace{Y=0}_{TH \text{ or } HT}) = P(\phi) = 0$$

$1 \le u < 2$ and $v \ge 1$

$$P(X \le u, Y \le v) = P(X \in \{0,1\}, Y \in \{0,1\})$$
$$= P(X \in \{0,1\})$$
$$= 1 - P(X = 2)$$
$$= 1 - P(HH)$$
$$= 1 - \frac{1}{4} = \frac{3}{4}$$

Work out the other cases...

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Given rvs $X, Y$ the individual CDFs $F_X(u)$ and $F_Y(v)$ are called <u>marginal</u> CDFs to distinguish them from the joint CDF $F_{X,Y}(u,v)$.

How can we get the marginal CDFs from the joint CDF?

$$F_{X,Y}(u, \infty) = P(X \leq u, Y \leq \infty)$$
$$= P(X \leq u)$$
$$= F_X(u) \leftarrow \text{marginal CDF of } X$$

Likewise,

$$F_{X,Y}(\infty, v) = F_Y(v) \leftarrow \text{marginal CDF of } Y$$

<u>Def</u>: The <u>joint pmf</u> of discrete rvs $X, Y$ is:

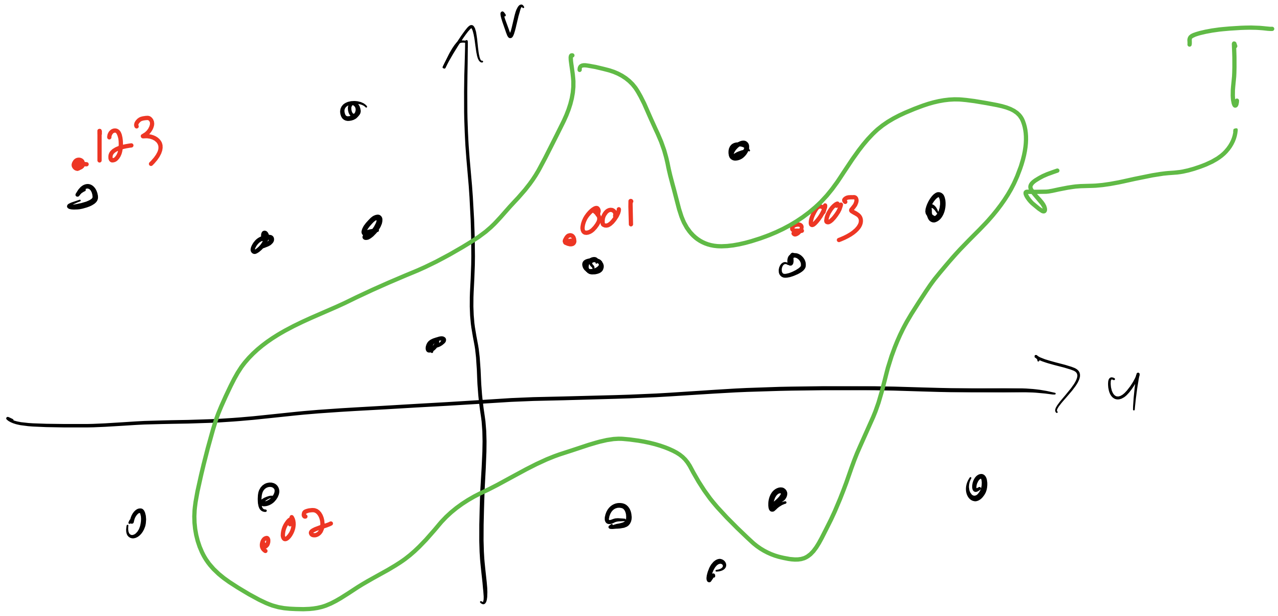
$$p_{X,Y}(u,v) = P(X=u, Y=v)$$

Key fact about joint pmfs is this:

Let $T \subseteq \mathbb{R}^2$ (ie region in the plane)

Then

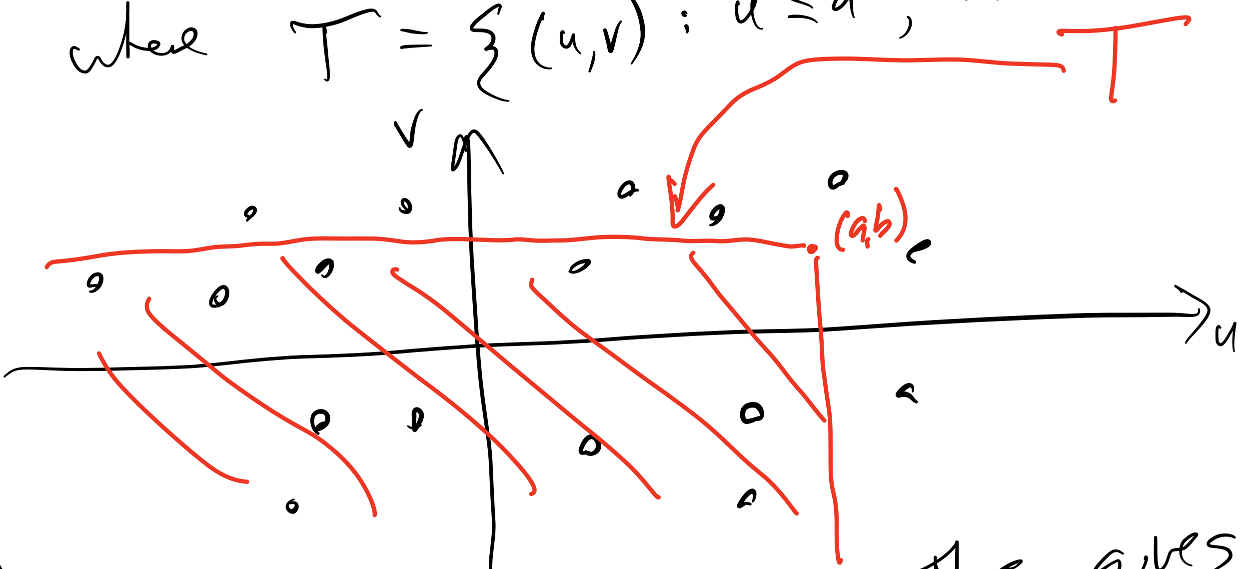
$$P((X,Y) \in T) = \sum_{(u,v) \in T} p_{X,Y}(u,v)$$




A consequence is:

$$F_{X,Y}(a,b) = P(X \le a, Y \le b)$$
$$= P((X,Y) \in T)$$

where $T = \{(u,v) : u \le a, v \le b\}$




$$= \sum_{\substack{u \le a \\ v \le b}} p_{X,Y}(u,v)$$

This gives us the joint CDF from the joint pmf.

Can we get the marginal pmfs from the joint pmf?

Yes ...

Note: $\{X=u\} = \bigcup_v \{X=u, Y=v\}$

marginal pmf:

$$\boxed{p_X(u) = P(X=u) = \sum_v P(X=u, Y=v) = \sum_v p_{X,Y}(u,v)}$$

Get marginal pmf of X from joint pmf by "summing out the other variable"

Also,

$$\boxed{p_Y(v) = \sum_u p_{X,Y}(u,v)}$$

Ex: use 2 coin flip example from before.

$X = \#$ Heads

$Y = \begin{cases} 1 & \text{if flips same} \\ 0 & \text{else} \end{cases}$

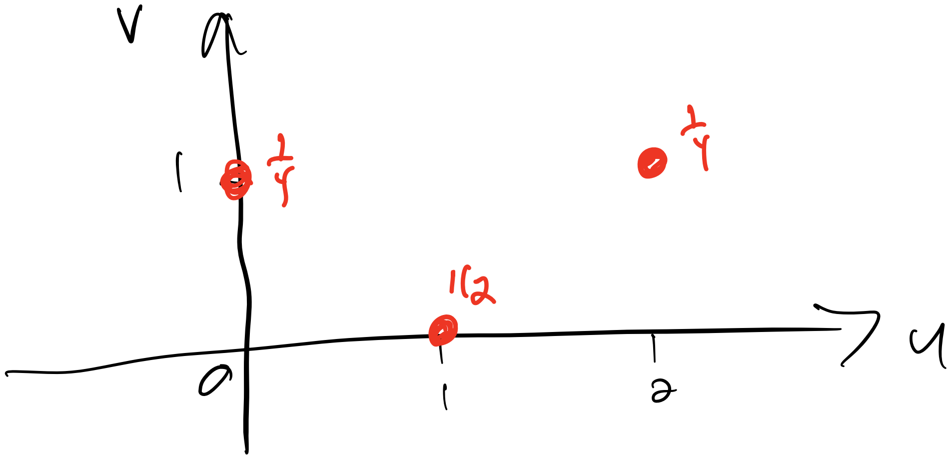
Find joint pmf:

$p_{X,Y}(0,1) = P(X=0, Y=1) = P(TT, \text{ TT or HH})$
$= P(TT) = \frac{1}{4}$

$p_{X,Y}(1,0) = P(X=1, Y=0) = P(\text{HT or TH, HT or TH})$
$= P(\text{HT or TH}) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$

$p_{X,Y}(2,1) = P(X=2, Y=1) = P(HH, \text{ HH or TT})$
$= P(HH) = \frac{1}{4}$

Total prob is $\frac{1}{4} + \frac{1}{2} + \frac{1}{4} = 1.$




Let's get marginal pmfs from joint pmf.

$$p_X(0) = \sum_v p_{X,Y}(0,v) = p_{X,Y}(0,1) = \frac{1}{4}$$

$$p_X(1) = \sum_v p_{X,Y}(1,v) = p_{X,Y}(1,0) = \frac{1}{2}$$

$$p_X(2) = \sum_v p_{X,Y}(2,v) = p_{X,Y}(2,1) = \frac{1}{4}$$

pmf of Y:

$$p_Y(0) = \sum_u p_{X,Y}(u,0) = p_{X,Y}(1,0) = \frac{1}{2}$$

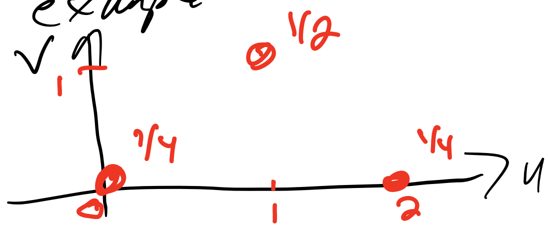
$$p_Y(1) = \sum_u p_{X,Y}(u,1) = p_{X,Y}(0,1) + p_{X,Y}(2,1) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

Can we deduce the joint pmf from marginal pmfs?

<u>No</u>, not always.

In general, there is not a <u>unique</u> joint pmf for a pair of marginal pmfs

Proof: Consider a joint pmf which is similar but different from previous example



If you work out the marginal pmfs in this case, you get exactly the same as in previous coin flipping example.

Def: A <u>joint pdf</u> of continuous rvs $X, Y$ is a function $f_{X,Y}(u,v)$ such that for any set $T \subseteq \mathbb{R}^2$ (a region in the plane), we get

$$P((X,Y) \in T) = \iint_T f_{X,Y}(u,v)\,du\,dv$$

The "T" notation below integral signs is <u>not</u> a lower limit. Rather it represents all 4 limits that are used to describe the region T in the u-v plane.