Lec 06:

Independent Trials

Repeat experiment multiple times

EX Experiment: Flip 3 coins

Define event E = "exactly 1 Head"

Q: what is the probability event E occurs exactly k times in n trials?

$$\underbrace{E, E, \ldots, E}_{k}, \underbrace{E^c, E^c, \ldots, E^c}_{n-k}$$

$$(P(E))^k \cdot (P(E^c))^{n-k}$$

$$= (P(E))^k (1 - P(E))^{n-k}$$

There are $\binom{n}{k}$ arrangements of where the k occurrences of E could be.

So the total prob is

$$\binom{n}{k} (P(E))^k (1 - P(E))^{n-k}$$

Ex: Using 3 coin flips (let's assume the coin is biased so $P(H) = q$.

E = "exactly 1 Head occurs"

$$P(E) = P(\{HTT, THT, TTH\})$$
$$= 3q(1-q)^2$$

$P(E$ occurs 4 times in 10 trials$)$

(i.e. $k=4$, $n=10$)

$$= \binom{10}{4} \left(3q(1-q)^2\right)^4 \left(1 - 3q(1-q)^2\right)^{10-4}$$

Suppose A, B are events in an experiment and are disjoint.

What is the prob A occurs before B in indep. trials?

This can happen in these ways:

Let $C = (A \cup B)^c$ = neither

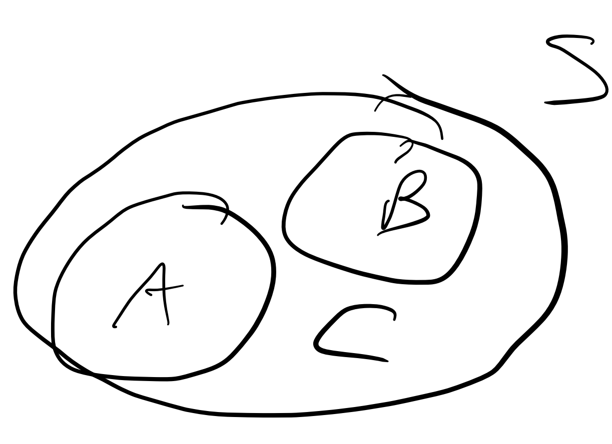
A

C, A

C, C, A

C, C, C, A

$\vdots$



Net prob is:

$$P(A) + P(C)P(A) + (P(C))^2 P(A) + (P(C))^3 P(A) + \cdots$$

$$= P(A)\left(1 + P(C) + P(C)^2 + P(C)^3 + \cdots\right)$$

$$= P(A)\left(\frac{1}{1-P(C)}\right) = \frac{P(A)}{P(A)+P(B)}$$

Ex: 3 coin flips (fair coins)

A = "3 Heads"

B = "1 Head"

P(exactly 3 Heads before exactly 1 Head)

$$= \frac{P(A)}{P(A)+P(B)} = \frac{1/8}{\frac{1}{8}+\frac{3}{8}} = \frac{\frac{1}{8}}{\frac{4}{8}} = \frac{1}{4}$$

Ex: Flip biased coin ($q = P(H)$) repeatedly until we get at least 10 Heads, then stop. What is the prob we flip the coin exactly 15 times?

$\boxed{9H \quad 5T}$ H
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$$\binom{14}{5} q^9 (1-q)^5 \cdot q$$

Ex Flip the biased coin until we get at least 3 Heads and 5 Tails. What is the prob we flip the coin exactly 20 times?

$\boxed{2H \quad 17T}$ H or $\boxed{4T \quad 15H}$ T
19 19

$$\binom{19}{2} q^2 (1-q)^{17} \cdot q + \binom{19}{4} q^{15} (1-q)^4 \cdot (1-q)$$

New topic: random variables

Convert outcomes to numbers.

Def: A random variable for a Sample space S is a mapping $X : S \to \mathbb{R}$.

I.e. for each $u \in S$, $X(u)$ is a real number.

Ex: $S = \{\text{apple}, \text{banana}, \text{orange}\}$

Let
$$X(\text{apple}) = 3$$
$$X(\text{banana}) = -\pi$$
$$X(\text{orange}) = 0$$

Let
$$Y(\text{apple}) = 5$$
$$Y(\text{banana}) = 6$$
$$Y(\text{orange}) = 6$$

X, Y are random variables.

Abbreviate as. r.v. or rv

Ex: Flip 3 coins (fair)

Define r.v. X as follows:

X counts number of Heads.

$X(HHH) = 3$
$X(HTH) = X(THH) = X(HHT) = 2$
$X(HTT) = X(THT) = X(TTH) = 1$
$X(TTT) = 0$

Define rv Y as follows:

$Y(HHH) = 1$
$Y(u) = 0$ if $u \neq HHH$

- usually we use upper case letters for r.v.'s.

Questions:

What is $P(X=2)$? $3/8$

$P(X \leq 2)$? $7/8$

$P(Y < \frac{1}{2})$? $7/8$

$P(X=Y)$? $1/8$ (ie TTT)

$P(X > Y)$? $\frac{7}{8}$ (i.e. not TTT) what does $P(X=2)$ actually mean?

It means the prob of the event

$$\{u \in S : X(u) = 2\}$$

$$= \{HHT, HTH, THH\}$$

The notation "$X=2$" means this event

Similarly,

$$\text{"}X \leq 2\text{"} = \{u \in S : X(u) \leq 2\}$$

$$\text{"}X = Y\text{"} = \{u \in S : X(u) = Y(u)\}$$

Random variables are not random.

Inputs are random and thus outputs are random.

Cummulative Distribution Function (CDF) of a random variable X is

$$F_X(u) = P(X \leq u)$$