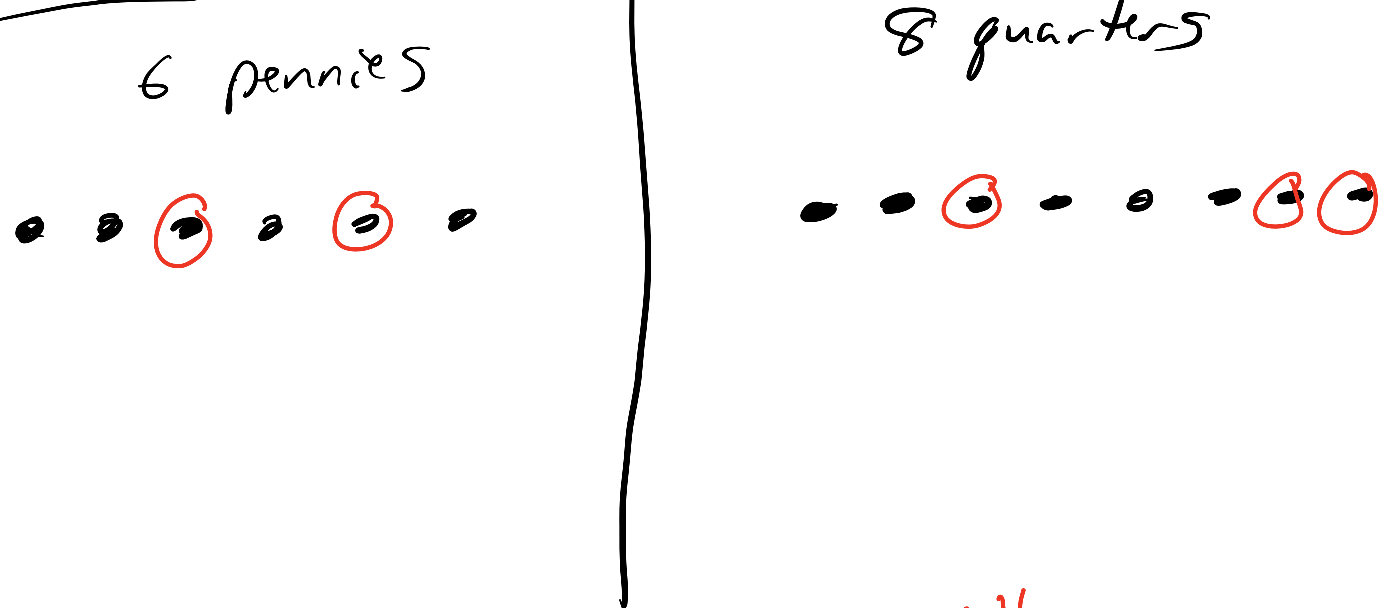
Lec 03:

Ex: Given a box with 6 pennies and 8 quarters. Pick 5 of the coins at random (without replacement). What is the prob that we choose 2 pennies and 3 quarters?

Solution:



Example of a "good" choice of 5 coins.

Total of $\binom{14}{5}$ 5-tuples of coins.

How many of these choices are "good"?

i.e. 2 pennies, 3 quarters

There are $\binom{6}{2}$ ways of picking 2 pennies and $\binom{8}{3}$ ways of picking 3 quarters

$\therefore$ total # of good 5-tuples is the product $\binom{6}{2}\binom{8}{3}$.

$\therefore$ using equiprobability (i.e. all 5-tuples have same probability)

$\therefore$ Prob $= \dfrac{\binom{6}{2}\binom{8}{3}}{\binom{14}{5}} =$ work out the numbers.

Conditional Probability

Ex: Toss a coin 3 times.

Q: what is the prob we get exactly 2 Heads?

Sample space

$S = \{HHH, \boxed{HHT}, \boxed{HTH}, HTT, \boxed{THH}, THT, TTH, TTT\}$

$P(\{HHT, HTH, THH\}) = \frac{3}{8}$.

New variation of the question.

Q: what is the prob we get exactly 2 Heads, given that the 1st two flips are not both Heads?

$$S = \{\cancel{HHH}, \boxed{\cancel{HHT}}, \boxed{HTH}, HTT, \boxed{THH}, THT, TTH, TTT\}$$

Now, there's only 2 possible outcomes (i.e. HTH and THH) and only 6 to choose from.

Define events:

E = "exactly 2 Heads occur"

$F = \{HHT, HHH\}^c$

want $P(E \text{ given } F)$.

Intuitively this prob is $\frac{2}{6} = \frac{1}{3}$.

We write $P(E|F)$ to mean $P(E \text{ given } F)$.

Notice that $P(E|F) = \dfrac{P(EF)}{P(F)}$ in this example

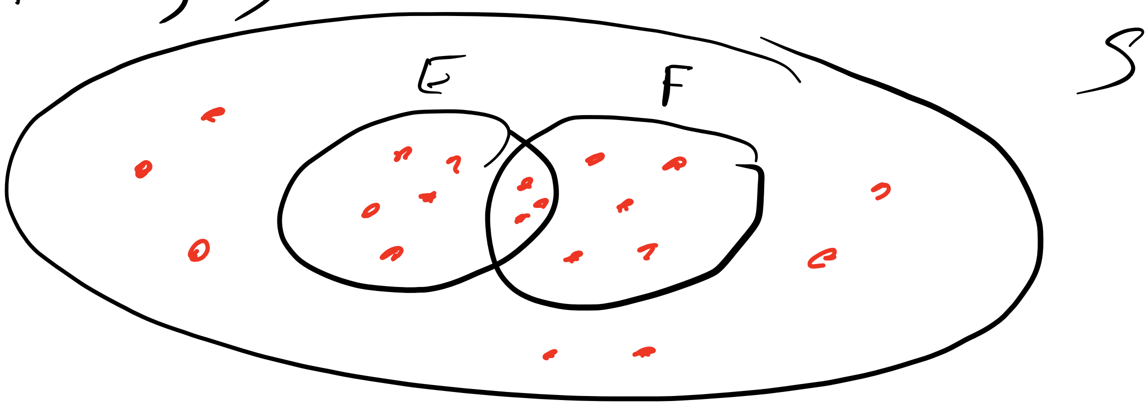
$$= \frac{2/8}{6/8} = \frac{2}{6}$$

$\boxed{\text{Def: If } P(F) > 0 \text{, then define } P(E|F) = \dfrac{P(EF)}{P(F)}}$

We also call $P(E|F)$ the "conditional probability of E given F".

In general, $P(E|F) \neq P(E)$, but it could be.

Intuitively, look at Venn diagram



Ex: Roll 2 dice.
Find the prob both dice are even, given their sum is ≥ 10.

Define events:

E = "both dice are even"

F = "sum is ≥ 10"
$= \{(6,6), (6,5), (5,6), (6,4), (4,6), (5,5)\}$

Want $P(E|F)$ equiprobable

$EF = \{(6,6), (6,4), (4,6)\}$

$P(EF) = \frac{|EF|}{|S|} = \frac{3}{36} = \frac{1}{12}$

$P(F) = \frac{|F|}{|S|} = \frac{6}{36} = \frac{1}{6}$

$$P(E|F) = \frac{P(EF)}{P(F)} = \frac{3/36}{6/36} = \frac{3}{6} = \frac{1}{2}$$

Now find the prob the sum = 7, given sum ≠ 6.

Define new events:

E = "sum = 7" want $P(E|F)$

F = "sum ≠ 6"

$E = \{(1,6), (6,1), (2,5), (5,2), (4,3), (3,4)\}$

$F = \{(1,5), (5,1), (4,2), (2,4), (3,3)\}^c$

$P(F^c) = \frac{5}{36} \Rightarrow P(F) = 1 - \frac{5}{36} = \frac{31}{36}$

Note: $E \subseteq F$

i.e. sum = 7 $\Rightarrow$ sum ≠ 6

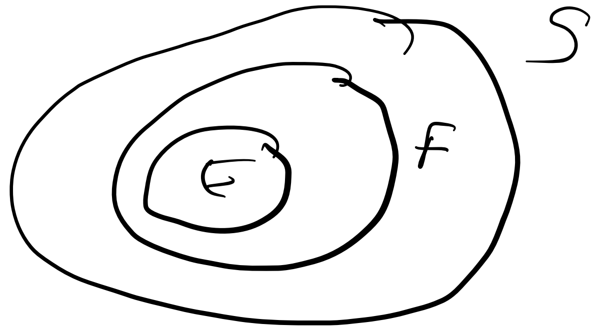
$\not\Leftarrow$

This implies that

$$EF = E$$

$$P(EF) = P(E) = \frac{6}{36}$$

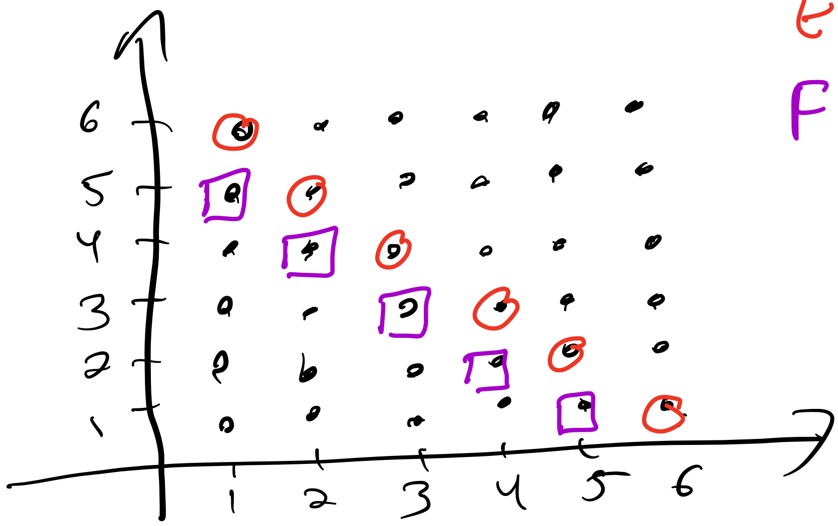
$$P(E|F) = \frac{P(EF)}{P(F)} = \frac{6/36}{31/36} = \frac{6}{31}$$



Another way to see this:

E = "sum = 7"

F^c = "sum = 6"



Start with $P(E|F) = \dfrac{P(EF)}{P(F)}$

Note $P(F|E) = \dfrac{P(EF)}{P(E)}$

$$\boxed{P(EF) = P(E|F)P(F) = P(F|E)P(E)}$$

Special Cases:

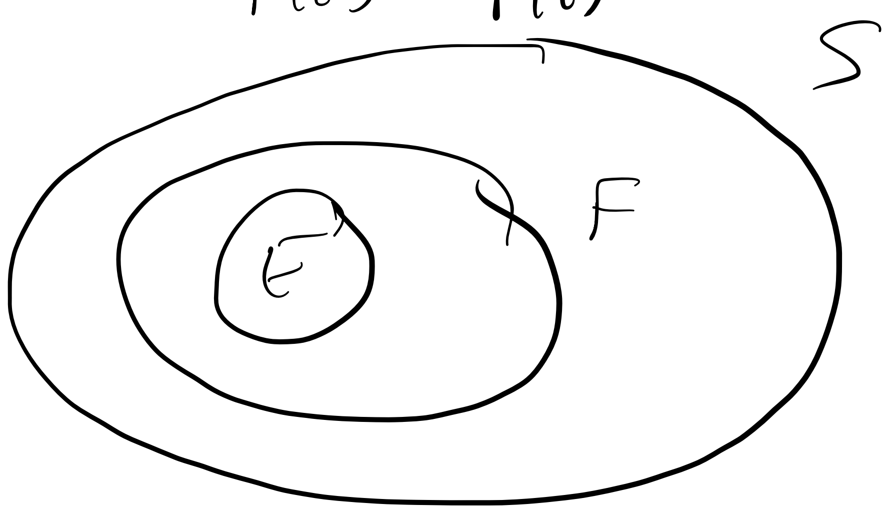
① If E, F are disjoint, then

$EF = \phi$, so $P(EF) = 0$.

$\therefore \; P(E|F) = \dfrac{P(EF)}{P(F)} = 0$

② If $E \subseteq F$, then $EF = E$.

So $P(F|E) = \dfrac{P(EF)}{P(E)} = \dfrac{P(E)}{P(E)} = 1$.



Ex: A box contains 3 blue, 4 red, 7 green marbles.

One marble is chosen at random,

and it is not red. what is the prob it is blue ?

Define events:

E = "marble is blue"

F = "marble is not red"

Want $P(E|F)$

Know $P(F^c) = P(\text{marble is red})$

$$= \frac{4}{3+4+7} = \frac{4}{14}$$

$$P(F) = 1 - \frac{4}{14} = \frac{10}{14}$$

Claim: $E \subseteq F$ blue $\Rightarrow$ not red

$\therefore EF = E$

$$P(EF) = P(E) = \frac{3}{14}$$

$$\therefore P(E|F) = \frac{P(EF)}{P(F)} = \frac{3/14}{10/14} = \frac{3}{10}$$