Lec 16:

Last time independent r.v.s.

Ex: X, Y have a joint pdf.

Let $Z = X + Y$. Find pdf of Z.

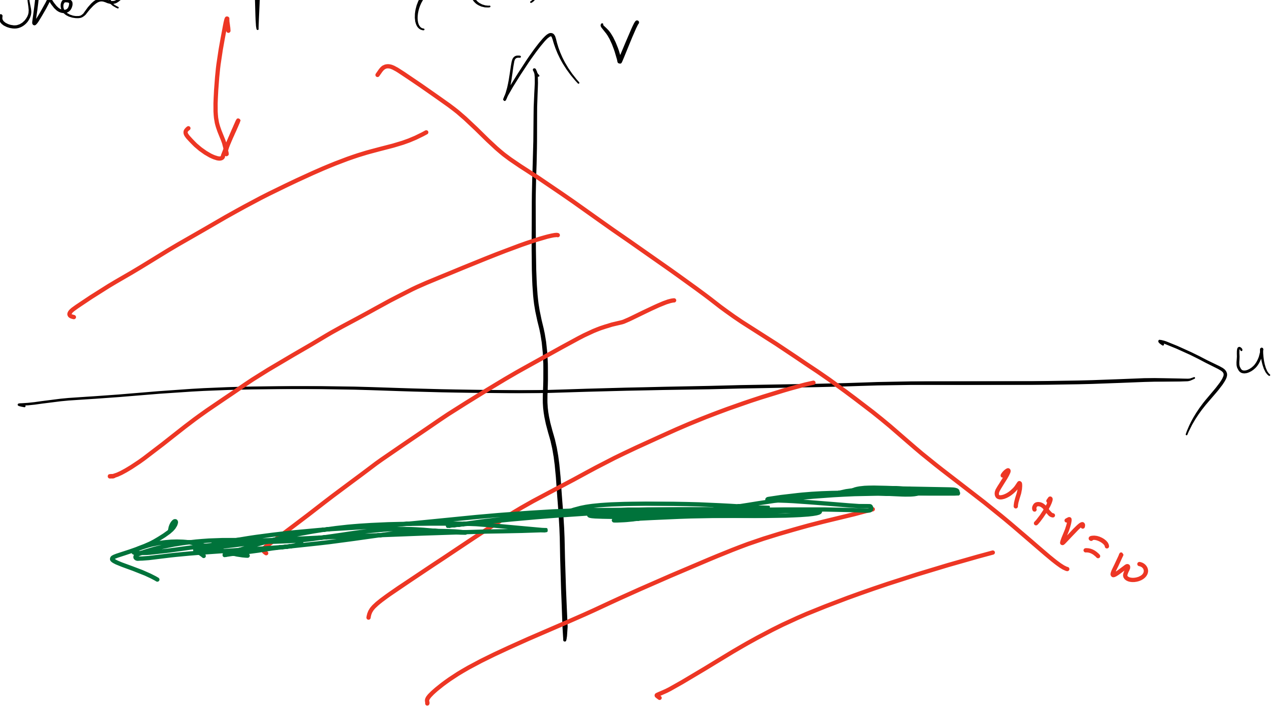
Initially, don't assume independence.

CDF:

$F_Z(w) = P(Z \le w) = P(X+Y \le w)$

$= P((X,Y) \in T)$

where $T = \{(u,v) : u+v < w\}$



$= \iint_T f_{X,Y}(u,v)\,du\,dv$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{w-v} f_{X,Y}(u,v)\, du\, dv$$

pdf of Z: differentiate CDF

$$f_Z(w) = \frac{d}{dw} F_Z(w)$$

$$= \frac{d}{dw} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{w-v} f_{X,Y}(u,v)\, du \right) dv$$

(use Leibniz: 1st + 2nd terms are zero)

$$= \int_{-\infty}^{\infty} \left(\frac{d}{dw} \int_{-\infty}^{w-v} f_{X,Y}(u,v)\, du \right) dv$$

(use Leibniz again: 2nd + 3rd terms are zero)

$$= \boxed{\int_{-\infty}^{\infty} f_{X,Y}(w-v, v)\, dv}$$

To get a 2nd equivalent form, substitute $u = w - v$, $du = -dv$

$$= \boxed{\int_{-\infty}^{\infty} f_{X,Y}(u, w-u)\, du}$$

Special case: Let's now assume X, Y are indep rvs. $\Rightarrow$ factor joint pdf

We get:

$$\boxed{\begin{aligned} f_Z(w) &= \int_{-\infty}^{\infty} f_X(w-v)\, f_Y(v)\, dv \\ &= \int_{-\infty}^{\infty} f_X(u)\, f_Y(w-u)\, du \end{aligned}}$$

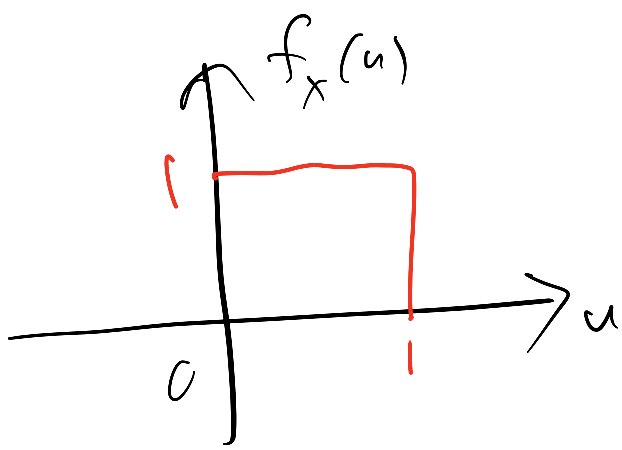
These are called "convolution integrals" (e.g. see them in ECE 45).

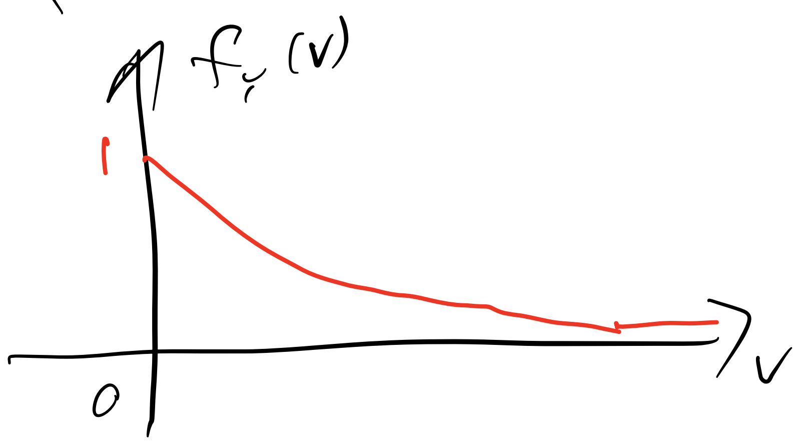
Ex: X, Y are indep rvs.

X is uniform on $[0,1]$

Y is exponential

$$f_Y(v) = \begin{cases} e^{-v} & \text{if } v \geq 0 \\ 0 & \text{else} \end{cases}$$





Let $Z = X + Y$. Find pdf of Z.

Convolution integral

$$f_Z(w) = \int_{-\infty}^{\infty} f_X(w-v) f_Y(v)\, dv$$

$$= \int_0^{\infty} f_X(w-v)\, e^{-v} dv$$

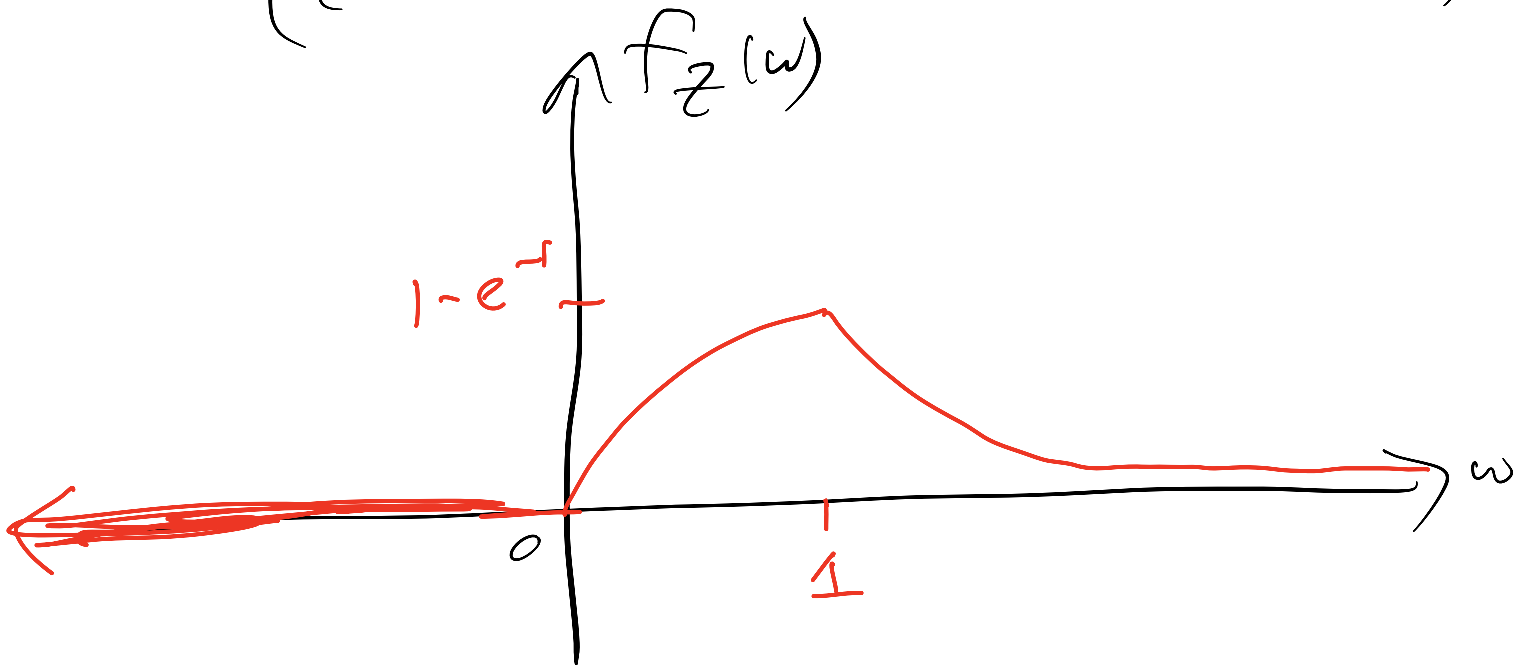
Note : $f_X(w-v) \neq 0$ when $0 < w-v < 1$

i.e. when $w-1 < v < w$

$$= \begin{cases} 0 & \text{if } w < 0 \\ \int_0^w e^{-v} dv & \text{if } 0 \le w \le 1 \\ \int_{w-1}^{w} e^{-v} dv & \text{if } w > 1 \end{cases}$$

$$= \begin{cases} 0 & \text{if } w < 0 \\ 1-e^{-w} & \text{if } 0 \le w \le 1 \\ (e-1)e^{-w} & \text{if } w > 1 \end{cases}$$

$Z = X + Y$



Ex (Maximum)

Given rvs X, Y with joint pdf.

Define $W = \max(X,Y) = \begin{cases} X & \text{if } X \geq Y \\ Y & \text{if } X < Y \end{cases}$

Find pdf of W.

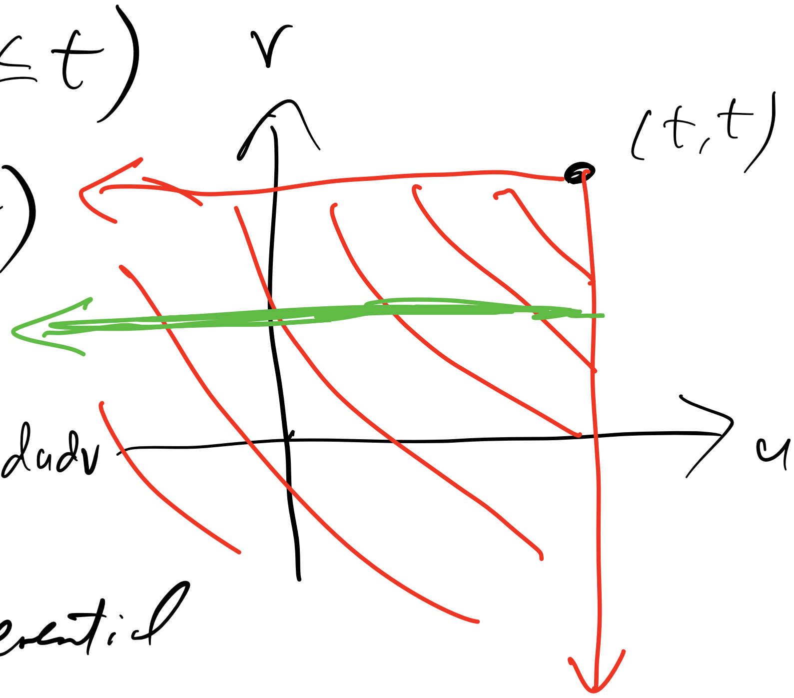
Note: $\max(X,Y) \leq t \iff X \leq t$ and $Y \leq t$

CDF of W:

$$F_W(t) = P(W \leq t) = P(\max(X,Y) \leq t)$$

$$= P(X \leq t, Y \leq t)$$

$$= F_{X,Y}(t,t)$$



$$= \int_{-\infty}^{t} \int_{-\infty}^{t} f_{X,Y}(u,v)\,du\,dv$$

pdf of W: differential

$$f_W(t) = \frac{d}{dt} F_W(t)$$

$$= \frac{d}{dt} \int_{-\infty}^{t} \left(\int_{-\infty}^{t} f_{X,Y}(u,v)\,du \right) dv$$

(use Leibniz: 2nd term is zero)

$$= \int_{-\infty}^{t} f_{X,Y}(u,t)\,du + \int_{-\infty}^{t} \left(\frac{d}{dt} \int_{-\infty}^{t} f_{X,Y}(u,v)\,du \right) dv$$

(use Leibniz 2nd + 3rd terms are zero)

$$= \boxed{\int_{-\infty}^{t} f_{X,Y}(u,t)\,du + \int_{-\infty}^{t} f_{X,Y}(t,v)\,dv}$$

Special case: X, Y are indep.

$$F_W(t) = F_{X,Y}(t,t) = F_X(t)\,F_Y(t)$$

$$f_W(t) = \frac{d}{dt} F_W(t) = \frac{d}{dt}\left(\Downarrow \right)$$

(use product rule)

$$= \boxed{F_X(t)\,f_Y(t) + f_X(t)\,F_Y(t)}$$

Now, let's do a further special case:
Assume X, Y are iid.

$$F_W(t) = F_X(t)\,F_Y(t) = \left(F_X(t)\right)^2$$

$$f_W(t) = \frac{d}{dt}\left(\swarrow \right) = 2F_X(t)\,f_X(t)$$

chain rule

Now, generalize to n iid rvs.

Let $W = \max(X_1, X_2, \dots, X_n)$

where $X_1, \dots, X_n$ are iid.

CDF W:

$$F_W(t) = P(W \le t) = P(\max(X_1, \dots, X_n) \le t)$$

$$= P(X_1 \le t, X_2 \le t, \dots, X_n \le t)$$

$$= F_{X_1, \dots, X_n}(t, t, \dots, t)$$

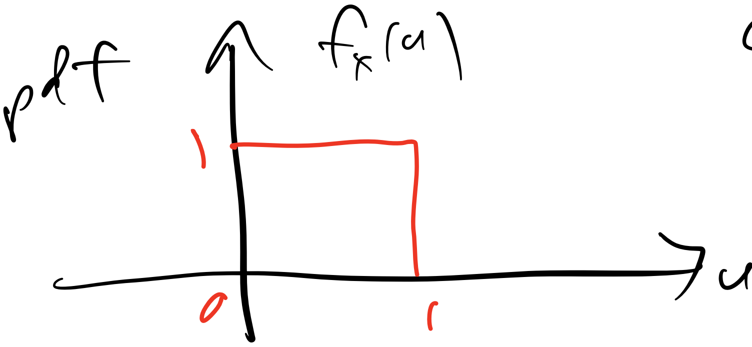
$$= F_{X_1}(t) F_{X_2}(t) \cdots F_{X_n}(t) \qquad \text{(indp)}$$

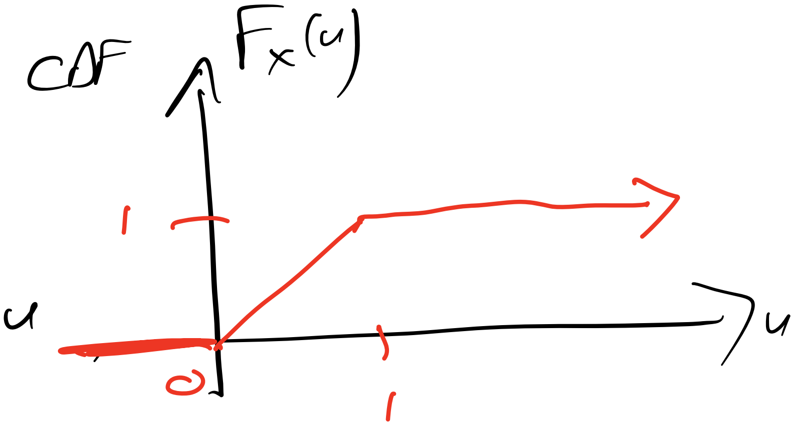
$$= \left(F_{X_1}(t)\right)^n \qquad \text{(ident dist.)}$$

pdf: $$f_W(t) = \frac{d}{dt}\left(\quad \right) = \boxed{n F_{X_1}^{n-1}(t) f_{X_1}(t)}$$

One more special case:

Suppose our n iid rvs are all uniform on $[0,1]$.



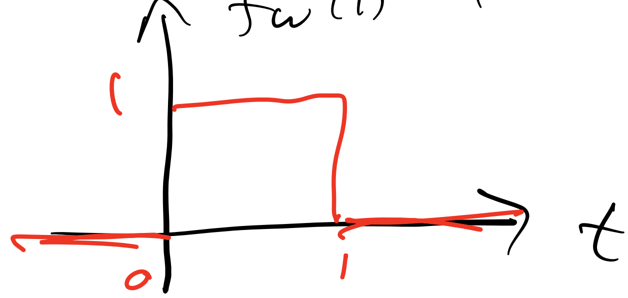


pdf of w (i.e. max) in this case is:

$$\boxed{f_w(t) = \begin{cases} nt^{n-1} & \text{if } t \in [0,1] \\ 0 & \text{else} \end{cases}}$$

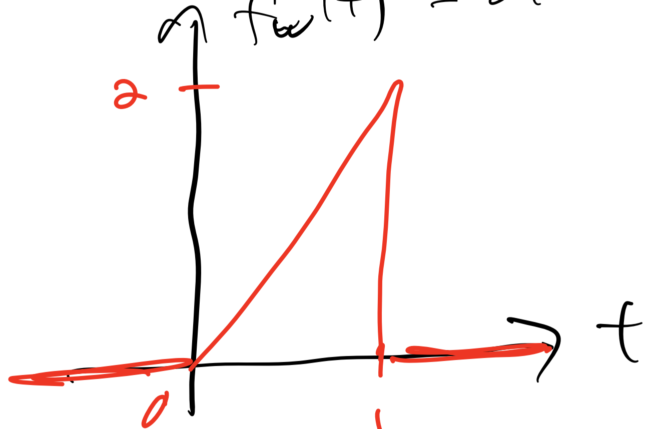
$n=1$

$f_w(t) = 1$



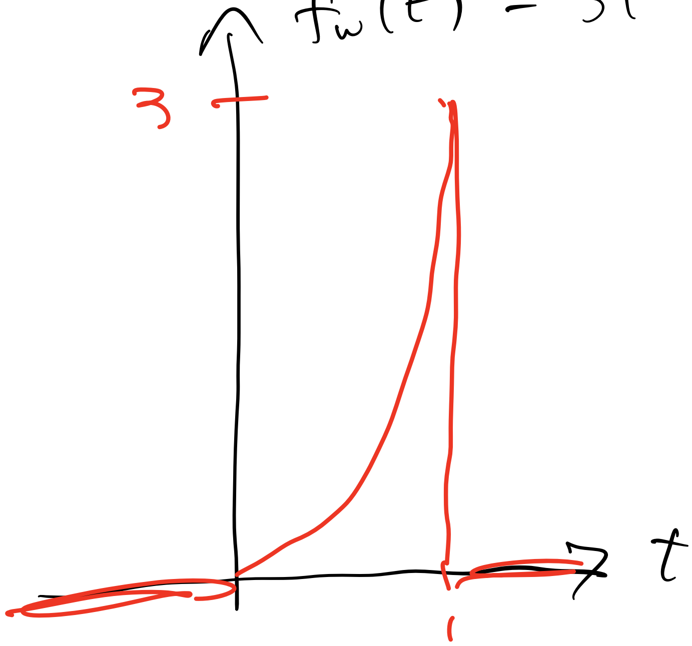
$n=2$

$f_w(t) = 2t$



$n=3$

$f_w(t) = 3t^2$



Keep going for large n, it starts to spike at 1.

Work out at home:

$Z = \min(X, Y)$ Find pdf of Z.

<u>Ex (Product)</u> X, Y have joint pdf.

Let $Z = XY$

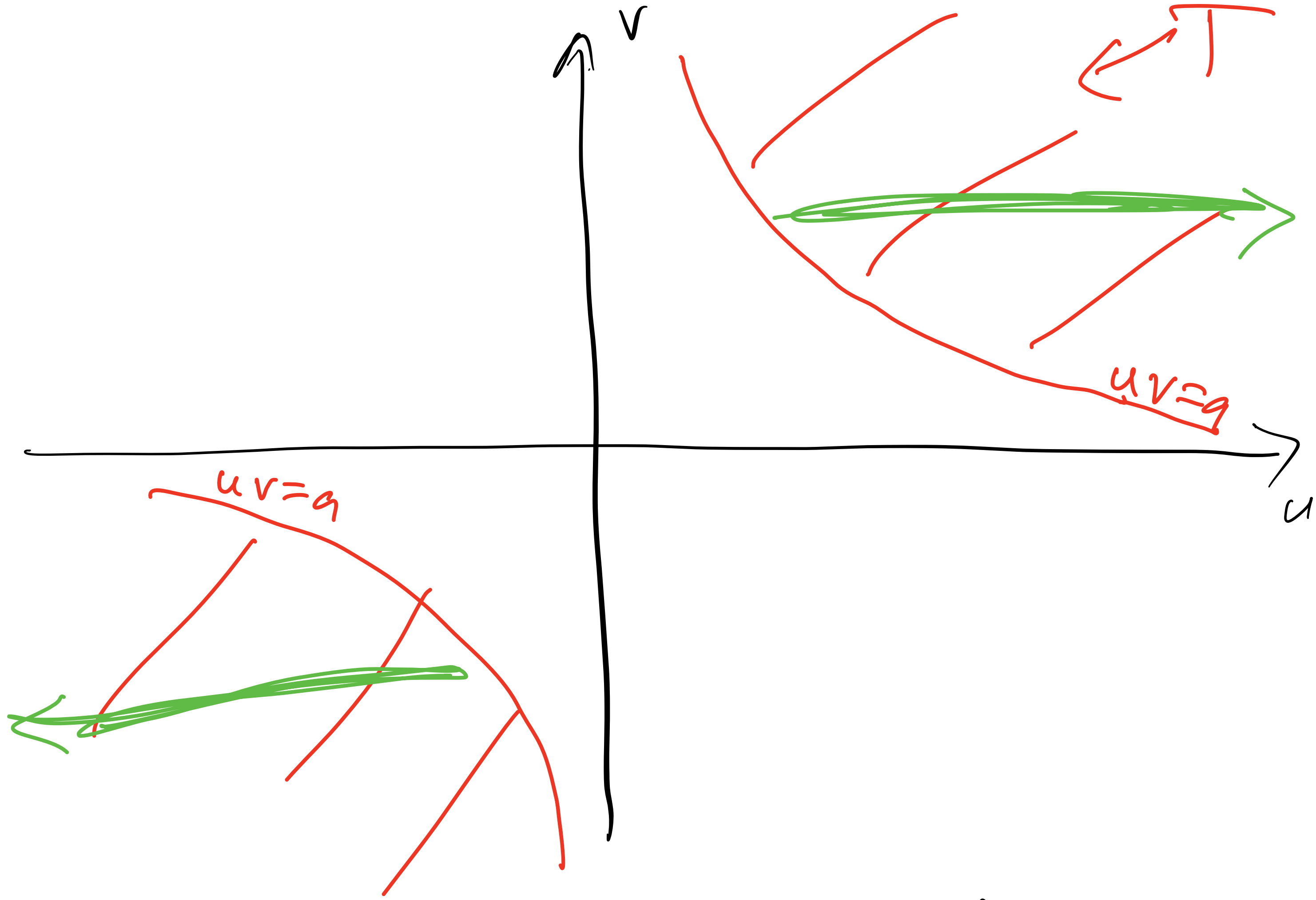
Find pdf of Z.

Look at $1 - \text{CDF}$.

assume 1st $a > 0$

$$1 - F_Z(a) = 1 - P(Z \le a)$$
$$= P(Z > a) = P(XY > a)$$
$$= P((X, Y) \in T)$$

where $T = \{(u, v) : uv > a\}$



$$1 - F_Z(a) = \iint_T f_{X,Y}(u,v)\,du\,dv$$

$$= \int_0^\infty \int_{a/v}^\infty f_{X,Y}(u,v)\,du\,dv$$ 1st quadrant

$$+ \int_{-\infty}^0 \int_{-\infty}^{a/v} f_{X,Y}(u,v)\,du\,dv$$ 3rd quadrant

Take derivative $\frac{d}{da}$

$$\frac{d}{da}\left(1 - F_Z(a)\right) = -\frac{d}{da} F_Z(a) = -f_Z(a)$$

$$= \frac{d}{da}\left(\int\int + \int\int \right)$$

use Leibniz rule twice ... work this out

$\vdots$

$$= -\int_0^{\infty} \frac{1}{v} f_{X,Y}\left(\frac{a}{v}, v\right) dv + \int_{-\infty}^{0} \frac{1}{v} f_{X,Y}\left(\frac{a}{v}, v\right) dv$$

$$= -\int_{-\infty}^{\infty} \frac{1}{|v|} \cdot f_{X,Y}\left(\frac{a}{v}, v\right) dv$$

It turns out, get same answer for $a<0$ also.

Final answer:

$$\boxed{f_Z(a) = \int_{-\infty}^{\infty} \frac{1}{|v|} \cdot f_{X,Y}\left(\frac{a}{v}, v\right) dv}$$

Next time: 2 functions of 2 r.v.s.

$$Z = g(X,Y)$$
$$W = h(X,Y)$$

Find joint pdf of Z, W.