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ECE 109

Week 2, Lecture 01-17-23

Example: Given a box with 6 pennies and 8 quarters, pick 5 of the coins at random (without replacement). What is the probability that we choose 2 pennies and 3 quarters?
There are a total of $\binom{14}{5}$ 5-tuples of coins. How many of these choices are "good"? i.e. 2 pennies, 3 quarters. There are $\binom{6}{2}$ ways of picking 2 pennies and $\binom{8}{3}$ ways of picking 3 quarters.

$$\therefore \text{the total number of good 5-tuples is the product} \binom{6}{2}\binom{8}{3}$$

$$\therefore \text{using equiprobability (i.e. all 5-tuples have the same probability)}$$

$$\therefore \text{Probability} = \frac{\binom{6}{2}\binom{8}{3}}{\binom{14}{5}}$$

Example: Toss a coin 3 times. What is the probability we get exactly 2 heads?
The sample space of all possible outcomes can be defined as:

$$S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$$

$$P(\{HHT, HTH, THH\}) = \frac{3}{8}$$

What is the probability we get exactly 2 heads, given that the first two flips are not both heads?

$$S = \{\cancel{HHH}, \cancel{HHT}, HTH, HTT, THH, THT, TTH, TTT\}$$

Now, there is only two possible outcomes (i.e. HTH and THH) and only 6 to choose from.
Let us define the events:

$$E = \text{"Exactly two heads occur"}$$
$$F = \{HHT, HHH\}^c$$

Intuitively, the probability is $\frac{2}{6} = \frac{1}{3}$.

$$\boxed{\text{We write } P(E|F) \text{ to mean } P(E) \text{ given } P(F)}$$

$$\boxed{\textbf{Definition: } \text{If } P(F) > 0, \text{ then define } P(E|F) = \frac{P(EF)}{P(F)}}.$$

This is also called the conditional probability of E given F and is intuitive given a venn diagram.

Example: Roll 2 dice. Find the probability both dice are even given their sum is greater than or equal to 10.
Let us define the events:

$$E = \text{Both dice are even}$$
$$F = \text{Sum is } \geq 10$$
$$= \{(6,6), (6,5), (5,6), (6,4), (4,6), (5,5)\}$$

We want $P(E|F)$.

$$EF = \{(6,6), (6,4), (4,6)\}$$

$$P(EF) = \frac{|EF|}{|S|} = \frac{3}{36} = \frac{1}{12}$$

$$P(F) = \frac{|F|}{|S|} = \frac{6}{36} = \frac{1}{6}$$

$$P(E|F) = \frac{P(EF)}{P(F)} = \frac{\frac{3}{36}}{\frac{6}{36}} = \frac{3}{6} = \frac{1}{2}$$

Now find the probability the sum = 7, given the sum $\neq$ 6. Let us define the events:

$$E = \text{Sum} = 7$$
$$F = \text{Sum} \neq 6$$

We want $P(E|F)$.

$$E = \{(1,6),(6,1),(2,5),(5,2),(4,3),(3,4)\}$$
$$F = \{(1,5),(5,1),(4,2),(2,4),(3,3)\}^c$$
$$P(F^c) = \frac{5}{36}$$
$$P(F) = 1 - \frac{5}{36} = \frac{31}{36}$$

Note: $E \subseteq F$ implies $EF = E$. Therefore,

$$P(EF) = P(E) = \frac{6}{36}$$
$$P(EF) = \frac{P(EF)}{P(F)} = \frac{\frac{6}{36}}{\frac{31}{36}} = \frac{6}{31}$$

Special Cases:

1. If E, F are disjoint, then $EF = 0$, so $P(EF) = 0$.
   $\therefore P(E|F) = \frac{P(EF)}{P(F)} = 0$

2. If $E \subseteq F$, then $EF = E$. So $P(F|E) = \frac{P(EF)}{P(E)} = \frac{P(E)}{P(E)} = 1$

Potential useful property:

$$\boxed{P(EF) = P(E|F)P(F) = P(F|E)P(E)}$$

Example: **A box contains 3 blue, 4 red, and 7 green marbles. One marble is chosen at random and it is not red. What is the probability that it is blue?**

Let us define the events:

$$E = \text{Marble is blue}$$
$$F = \text{Marble is not red}$$

We want $P(E|F)$. We know that

$$P(F^c) = P(\text{Marble is red})$$
$$= \frac{4}{3+4+7} = \frac{4}{14}$$
$$P(F) = 1 - \frac{4}{14} = \frac{10}{14}$$

We claim that $E \subseteq F$, as the even that a blue marble is chosen implies that the chosen marble is not red, whereas if the chosen marble is not red, this does not imply that the marble is blue. Resultantly,

$$EF = E$$
$$P(EF) = P(E) = \frac{3}{14}$$
$$\therefore P(E|F) = \frac{P(EF)}{P(F)} = \frac{\frac{3}{14}}{\frac{10}{14}} = \frac{3}{10}$$

Week 2, Lecture 01-19-23

Recall Conditional Probability:

$$P(E|F) = \frac{P(EF)}{P(F)}$$

Recall Axioms of Probability:

1. $0 \leq P(E) \leq 1$
2. $P(S) = 1$
3. If $E_1, E_2, ...$ are pairwise disjoint events, then $P(E_1 \cup E_2 \cup ...E_n) = \sum_n P(E_n)$

Axioms of Conditional Probability:

1. $0 \leq P(E|F) \leq 1$
2. $P(S|F) = 1$
3. If $E_1, E_2, ...$ are pairwise disjoint events, then $P((E_1 \cup E_2 \cup ...E_n)|F) = \sum_n P(E_n|F)$

Partition Rule of Probability:

$$\begin{aligned} B &= BA \cup BA^c \\ P(B) &= P(BA \cup BA^c) \\ &= P(BA) + P(BA^c) \\ &= \boxed{P(B|A)P(A) + P(B|A^c)P(A^c)} \end{aligned}$$

Can be viewed as A being case 1 and A^c being case 2. This can be further generalized to n cases. Suppose $A_1, A_2, ..., A_n$ are events that partition S. (A_i's are a disjoint cover of S).

$$\begin{aligned} \text{disjoint: } & i \neq j \rightarrow A_i A_j = 0 \\ \text{cover: } & S = A_1 \cup A_2 \cup ...A_n \end{aligned}$$

$$\begin{aligned} B &= BA_1 \cup BA_2 \cup ... \cup BA_n \\ P(B) &= P(BA_1 \cup BA_2 \cup ... \cup BA_n) \\ &= P(BA_1) + P(BA_2) + ... + P(BA_n) \\ &= \boxed{P(B|A_1)P(A_1) + P(B|A_2)P(A_2) + ... + P(B|A_n)P(A_n)} \end{aligned}$$

This can be extended to conditional probability. Can be thought of as inserting the cnoditional event into the above partition rule of probability:

$$\boxed{P(B|A_1E)P(A_1|E) + P(B|A_2E)P(A_2|E) + ... + P(B|A_nE)P(A_n|E)}$$

Example: **Roll 1 fair 6-sided die. If the die comes up ≥ 3, then we win. If not, then we flip a fair coin and we win if heads and lose if tails. What is the probability of winning?**

Let us define the events:

$$\begin{aligned} E &= \text{Die is } \geq 3 \\ F &= \text{We win} \end{aligned}$$

We want $P(F)$.

$$\begin{aligned} P(F) &= P(F|E)P(E) + P(F|E^c)P(E^c) \\ P(E) &= P(\text{Die is 3,4,5 or 6}) = \frac{4}{6} = \frac{2}{3} \\ P(F|E) &= 1 \\ P(F|E^c) &= \frac{1}{2} \\ P(E^c) &= 1 - P(E) = 1 - \frac{2}{3} = \frac{1}{3} \\ \therefore P(F) &= (1 * \frac{2}{3}) + (\frac{1}{2} * \frac{1}{3}) = \frac{5}{6} \end{aligned}$$

What is the probability the die is ≥ 3, given we won?
In this case, we want $P(E|F)$. Recall,

$$\begin{aligned} P(EF) &= P(F)P(E|F) \\ &= P(E)P(F|E) \end{aligned}$$

From before, we know

$$\begin{aligned} P(E) &= \frac{2}{3} \\ P(F|E) &= 1 \\ P(F) &= \frac{5}{6} \end{aligned}$$

Plugging in,

$$\begin{aligned} P(F)P(E|F) &= P(E)P(F|E) \\ \frac{5}{6} * P(E|F) &= \frac{2}{3} * 1 \\ \therefore P(E|F) &= \frac{\frac{2}{3}}{\frac{5}{6}} = \frac{4}{5} \end{aligned}$$

<u>Example:</u> **A box contains 2 red and 3 green marbles. Pick 2 marbles at random (without replacement). If both are green, we win. If both are red, we lose. Otherwise, we pick a third marble and we win if red, lose if green. What is the probability of winning?**
Let us define the events

$$\begin{aligned} W &= \text{We win} \\ G &= \text{Both marbles are green} \\ R &= \text{Both marbles are red} \end{aligned}$$

We want $P(W)$. The three cases for winning are:

$$P(W) = P(W|G)P(G) + P(W|R)P(R) + P(W|G^cR^c)P(G^cR^c)$$

We know:

$$\begin{aligned} P(W|G) &= 1 \\ P(W|R) &= 0 \\ P(W|G^cR^c) &= \frac{1}{3} \\ P(G^cR^c) &= P((G \cup R)^c) \\ &= 1 - P(G \cup R) \quad = 1 - P(G) - P(R) = 1 - \frac{3}{5} * \frac{2}{4} - \frac{2}{5} * \frac{1}{4} = \frac{3}{5} \end{aligned}$$

Plugging in,

$$P(W) = (1 * \frac{3}{10}) + (0 * ?) + (\frac{1}{3}\frac{3}{5}) = \frac{1}{2}$$

Chain Rule of Probability:
Using the definition of conditional probability, we know:

$$P(A_1A_2) = P(A_1)P(A_2|A_1)$$

For 3 events:

$$\begin{aligned} P(A_1A_2A_3) &= P(A_1(A_2A_3)) \\ &= P(A_1)P(A_2A_3|A_1) \end{aligned}$$

Note:

$$P(A_3|A_1A_2) = \frac{P(A_1A_2A_3)}{P(A_1A_2)}$$
$$= \frac{P(A_2A_3|A_1)\cancel{P(A_1)}}{P(A_2|A_1)\cancel{P(A_1)}}$$

Substitute in for $P(A_2|A_1) \rightarrow$

$$P(A_1A_2A_3) = P(A_1)P(A_2|A_1)P(A_3|A_1A_2)$$

Can keep going:

$$\boxed{P(A_1A_2...A_n) = P(A_1)P(A_2|A_1)P(A_3|A_1A_2)...P(A_n|A_1A_2...A_{n-1})}$$

Example: **Given a standard deck of 52 cards, what is the probability of picking an ace, and then a red card (no replacement) ?**

Let us define the events

$$A = \text{First card is an ace}$$
$$R = \text{Second card is red}$$
$$B = \text{First card is red}$$

We want $P(AR)$. We know $P(AR) = P(A)P(R|A) = \frac{1}{13}P(R|A)$. Therefore all we need to find is $P(R|A)$. This can be rewritten as:

$$P(R|A) = P(R|BA)P(B|A) + P(R|B^cA)P(B^c|A)$$
$$= (\frac{25}{51} * \frac{1}{2}) + (\frac{26}{51} * \frac{1}{2}) = \frac{1}{2}$$
$$\therefore P(AR) = \frac{1}{13} * \frac{1}{2} = \frac{1}{26}$$

Baye's Formula:

Suppose $A_1, A_2, ..., A_n$ is a partition of sample space S. Suppose we know $P(B|A_i)$ for all i. We want $P(A_i|B)$. This can be written as:

$$P(A_i|B) = \frac{P(B|A_i)P(A_i)}{P(B)}$$

Note:

$$P(B) = \sum_{j=1}^{n} P(B|A_j)P(A_j)$$

$$\boxed{P(A_i|B) = \frac{P(B|A_i)P(A_i)}{\sum_{j=1}^{n} P(B|A_j)P(A_j)}}$$

Discussion 2