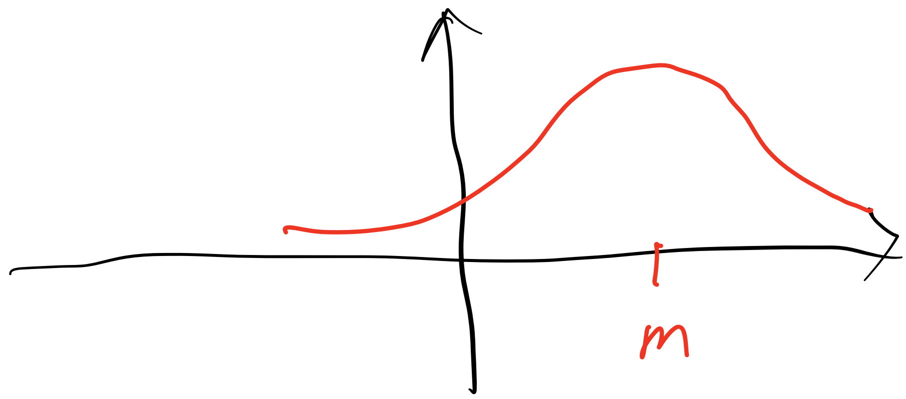
Lec 10:

Gaussian pdfs

$$f_X(u) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{u-m}{\sigma}\right)^2}$$

Notation $X \sim N(m, \sigma^2)$



Special Case: $m=0$, $\sigma^2=1$

$X \sim N(0,1)$

Is called a standard or unit Gaussian (or normal)

- Prove the constant in front is correct.

Show integrates to 1.

Let $f_X(u) = Ce^{-\frac{1}{2}\left(\frac{u-m}{\sigma}\right)^2}$

Know $1 = \int_{-\infty}^{\infty} f_X(u)\,du$

$$= C \int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(\frac{u-m}{\sigma}\right)^2} du$$

Substitute $v = \frac{u-m}{\sigma}$ $dv = \frac{du}{\sigma}$

$$= C \int_{-\infty}^{\infty} e^{-v^2/2} (\sigma dv)$$

$$= \sigma C \cdot \int_{-\infty}^{\infty} e^{-v^2/2} dv$$

$$\frac{1}{\sigma C} = \int_{-\infty}^{\infty} e^{-v^2/2} dv$$

Let $I = \int_{-\infty}^{\infty} e^{-v^2/2} dv$

Trick:

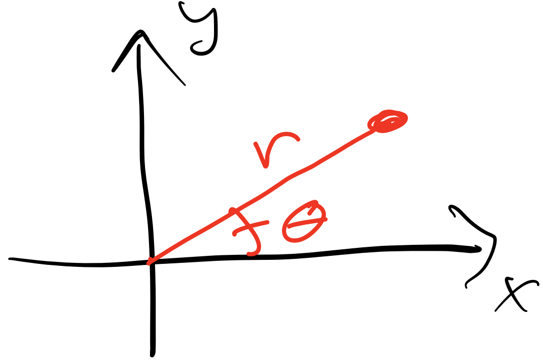
$$I^2 = \left(\int_{-\infty}^{\infty} e^{-v^2/2} dv \right)^2$$

$$= \left(\int_{-\infty}^{\infty} e^{-x^2/2} dx \right) \left(\int_{-\infty}^{\infty} e^{-y^2/2} dy \right)$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+y^2)/2} dx\, dy$$

Convert to polar coordinates

$r^2 = x^2 + y^2$



$$\int_0^{2\pi}\int_0^{\infty} e^{-r^2/2}\, r \, dr\, d\theta$$

Do θ 1st.

$$= \int_0^{\infty} r e^{-r^2/2} \underbrace{\left(\int_0^{2\pi} d\theta \right)}_{2\pi} dr$$

$$= 2\pi \int_0^{\infty} r e^{-r^2/2}\, dr$$

$$= 2\pi \left(-e^{-r^2/2} \right) \Big|_0^{\infty}$$

$$= 2\pi (1 - 0) = 2\pi$$

$\therefore \; I^2 = 2\pi \Rightarrow I = \sqrt{2\pi}$

$\therefore \; \dfrac{1}{\sigma c} = I = \sqrt{2\pi}$

$C = \dfrac{1}{\sigma\sqrt{2\pi}}$ <u>Done</u>

Expectation (expected value, average, mean)

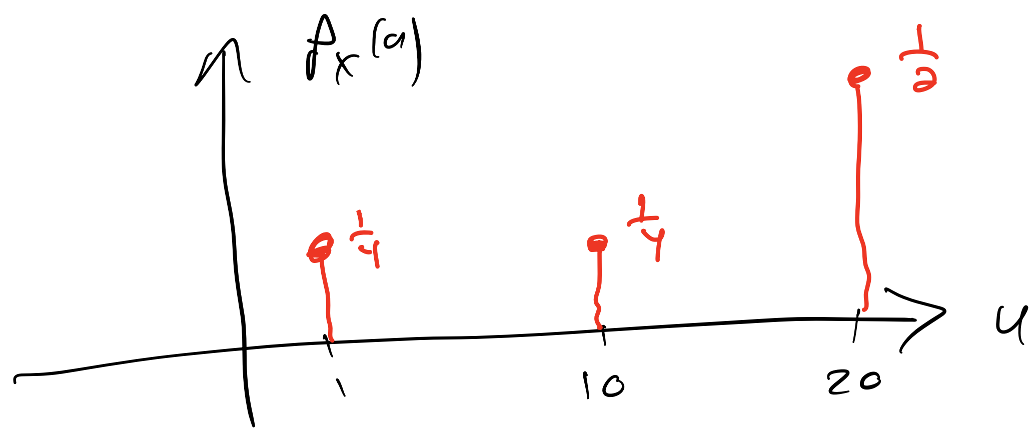
Ex: Given a set of numbers

$$a_1, a_2, a_3, \ldots, a_n$$

Their "average" is:

$$\frac{a_1 + a_2 + \cdots + a_n}{n}$$

ex: Suppose X is a discrete r.v. with pmf: $p_X(1) = \frac{1}{4}$, $p_X(10) = \frac{1}{4}$, $p_X(20) = \frac{1}{2}$



If we observe n independent values of X (for large n), you would expect roughly:

$\frac{n}{4}$ 1's, $\frac{n}{4}$ 10's, $\frac{n}{2}$ 20's

So the average would be about:

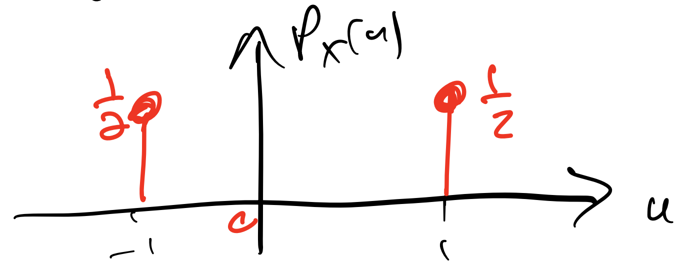
$$\frac{\left(\frac{n}{4}\cdot 1\right)+\left(\frac{n}{4}\cdot 10\right)+\left(\frac{n}{2}\cdot 20\right)}{n}$$

$$= \left(\frac{1}{4}\cdot 1\right)+\left(\frac{1}{4}\cdot 10\right)+\left(\frac{1}{2}\cdot 20\right) = 12.75$$

$\boxed{\text{Def: If } X \text{ is a discrete r.v. then its expected value (or mean) is: } E[X] = \sum_u u\cdot p_X(u)}$

Sum is really over only those u for which $p_X(u) \neq 0$.

Ex: X is uniform discrete on $\{-1, 1\}$

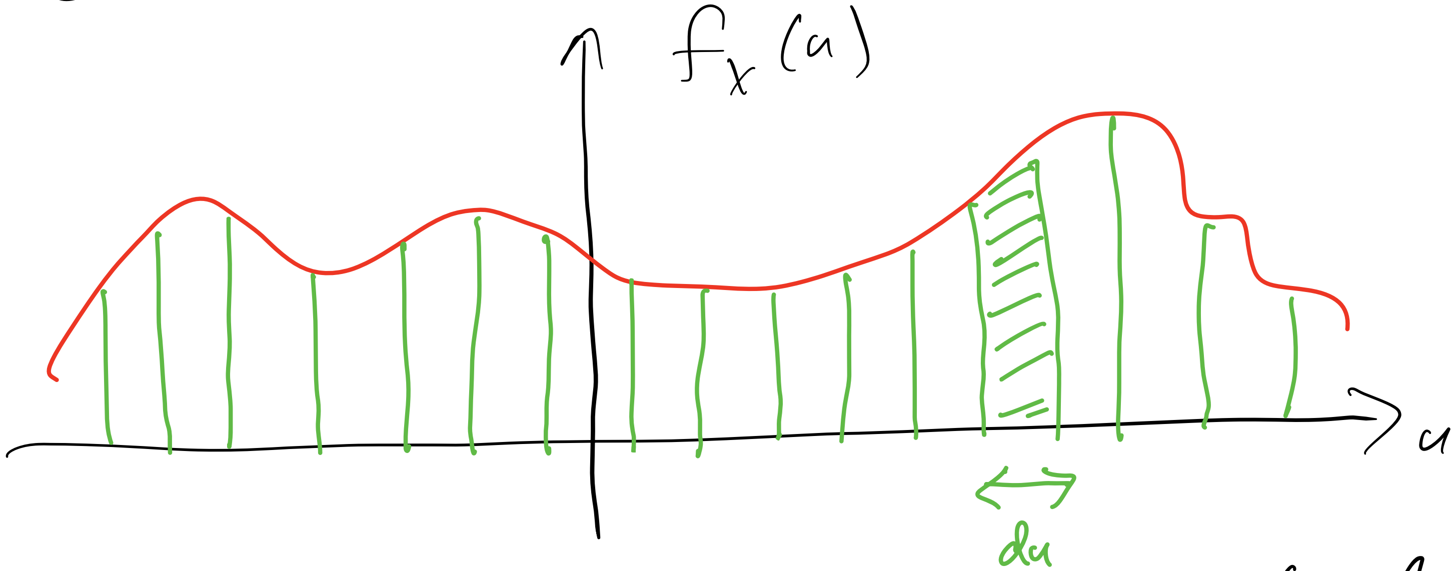


The expected value of X is:

$$E[X] = \left(1 \cdot \tfrac{1}{2}\right) + (-1)\left(\tfrac{1}{2}\right) = 0$$

The mean is 0.

Continuous r.v.'s:



As $du \to 0$, the prob of the shaded region is about $f_X(u)\,du$

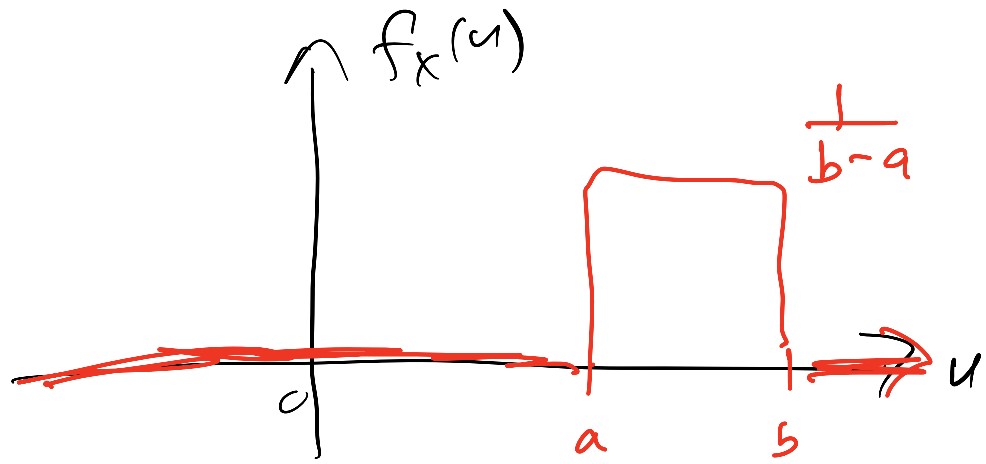
Thus, the mean is about:

$$\sum_u u f_X(u)\,du \approx \int_{-\infty}^{\infty} u f_X(u)\,du$$

Def: The expected value of a continuous r.v. is

$$E[X] = \int_{-\infty}^{\infty} u f_X(u)\,du$$

Ex: Find the mean of a uniform r.v. on $[a, b]$.



$$E[X] = \int_{-\infty}^{\infty} u f_X(u)\, du$$

$$= \int_a^b u \cdot \frac{1}{b-a}\, du = \frac{1}{b-a} \cdot \frac{u^2}{2} \Big|_a^b$$

$$= \frac{1}{b-a} \cdot \frac{b^2-a^2}{2} = \frac{\cancel{(b-a)}(b+a)}{2\cancel{(b-a)}} = \frac{a+b}{2}$$

= midpoint of interval $[a, b]$

Ex: Find mean of a binomial r.v.

Recall pmf:

$$p_X(k) = \binom{n}{k} p^k (1-p)^{n-k}$$

$$k = 0, 1, \ldots, n$$

$$E[X] = \sum_u u\, p_X(u)$$

$$= \sum_{k=0}^{n} k \cdot \binom{n}{k} p^k (1-p)^{n-k}$$

$$= \sum_{k=1}^{n} k \cdot \frac{n!}{k!\,(n-k)!} p^k (1-p)^{n-k}$$

$$= (np) \sum_{k=1}^{n} \frac{(n-1)!}{(k-1)!\,(n-1-(k-1))!} \cdot p^{k-1} (1-p)^{n-1-(k-1)}$$

Now let $j = k-1$ and $m = n-1$ and substitute

$$= (np) \underbrace{\sum_{j=0}^{m} \frac{m!}{j!\,(m-j)!} \cdot p^j (1-p)^{m-j}}_{\text{binomial pmf}}$$

$$= (np) \cdot \underline{1}$$

$$= np$$

$\therefore \quad E[X] = np$

ie. if you flip a biased coin n times, the average # of Heads is np.

<u>Ex</u>: Find the mean of a Poisson r.v. X. The pmf is:

$$p_X(k) = \frac{e^{-\lambda}\lambda^k}{k!} \quad \text{for } k = 0, 1, 2, \ldots$$

$$E[X] = \sum_{k=0}^{\infty} k\, p_X(k) = \sum_{k=0}^{\infty} k \cdot \frac{e^{-\lambda}\lambda^k}{k!}$$

$$= e^{-\lambda} \cdot \sum_{k=1}^{\infty} \frac{\lambda^k}{(k-1)!} = \lambda e^{-\lambda} \cdot \sum_{k=1}^{\infty} \frac{\lambda^{k-1}}{(k-1)!}$$

Substitute $n = k-1$

$$= \lambda e^{-\lambda} \cdot \underbrace{\sum_{n=0}^{\infty} \frac{\lambda^n}{n!}}_{\text{Taylor Series for } e^{\lambda}}$$

$$= \lambda e^{-\lambda} \cdot e^{\lambda} = \lambda$$

$$\therefore \boxed{E[X] = \lambda}$$

<u>Ex</u>: Find mean of a geometric r.v.

The pmf: $p_X(k) = p(1-p)^{k-1}$

for $k = 1, 2, 3, \ldots$

$$E[X] = \sum_{k=1}^{\infty} k\, p_X(k) = p \cdot \sum_{k=1}^{\infty} k(1-p)^{k-1}$$

Let $q = 1-p$

$$= (1-q) \cdot \underbrace{\sum_{k=1}^{\infty} k q^{k-1}}_{\text{derive from derivative of geometric sum}} = (1-q) \cdot \frac{1}{(1-q)^2}$$

$$= \frac{1}{1-q} = \frac{1}{p}$$

$$E[X] = \frac{1}{p}$$

Ex: Find mean of Gaussian $X \sim (m, \sigma^2)$

i.e. Show $E[X] = m$.

pdf is $f_X(u) = \frac{1}{\sigma\sqrt{2\pi}} \cdot e^{-\frac{1}{2}\left(\frac{u-m}{\sigma}\right)^2}$

$$E[X] = \int_{-\infty}^{\infty} u\, f_X(u)\, du$$

$$= \int_{-\infty}^{\infty} u \cdot \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{u-m}{\sigma}\right)^2} du$$

Let $y = \frac{u-m}{\sigma}$ $\quad dy = \frac{du}{\sigma}$

Substitute into integral

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} (\underbrace{\sigma y + m}_{u}) e^{-y^2/2} (\underbrace{\sigma\, dy}_{du})$$

$$= \frac{1}{\sqrt{2\pi}} \left(\sigma \underbrace{\int_{-\infty}^{\infty} y e^{-y^2/2}\, dy}_{=0} + m \int_{-\infty}^{\infty} e^{-y^2/2}\, dy \right)$$

$ye^{-y^2/2}$ is odd

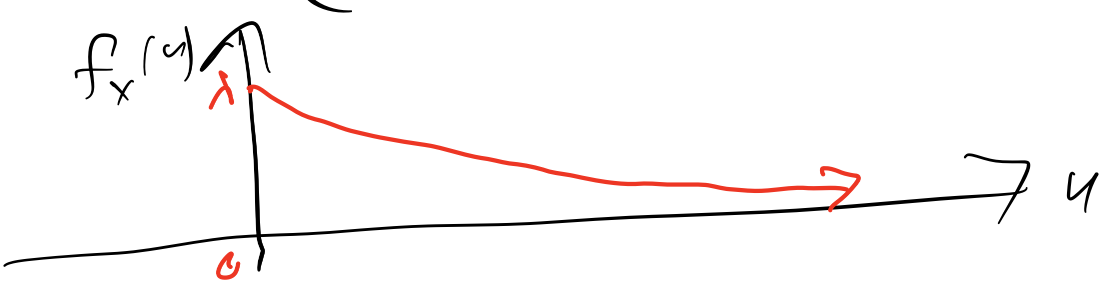
$\int_{-\infty}^{0} = -\int_{0}^{\infty}$

$$= m \int_{-\infty}^{\infty} \underbrace{\frac{1}{\sqrt{2\pi}} e^{-y^2/2}\, dy}_{\text{pdf of } N(0,1) \text{ r.v.}} = m \cdot 1 = m$$

$$E[X] = m$$

Ex: Find mean of an exponential r.v. with pdf

$$f_X(u) = \begin{cases} \lambda e^{-\lambda u} & \text{if } u > 0 \\ 0 & \text{else} \end{cases}$$



$$E[X] = \int_{-\infty}^{\infty} z f_X(z)\, dz$$

$$= \int_0^{\infty} z \cdot \lambda e^{-\lambda z}\, dz \quad = \int_0^{\infty} u\, dv$$

Integrate by parts:

$$\boxed{\begin{aligned} \text{Let } u &= z & v &= -e^{-\lambda z} \\ du &= dz & dv &= \lambda e^{-\lambda z} dz \end{aligned}}$$

$$= uv \Big|_0^{\infty} - \int_0^{\infty} v\, du$$

$$= \underbrace{-ze^{-\lambda z} \Big|_0^{\infty}}_{(0-0)} - \underbrace{\int_0^{\infty} -e^{-\lambda z} dz}_{+\, -\frac{1}{\lambda} e^{-\lambda z} \big|_0^{\infty}}$$

$$= \frac{1}{\lambda}$$

$$E[X] = \frac{1}{\lambda}$$