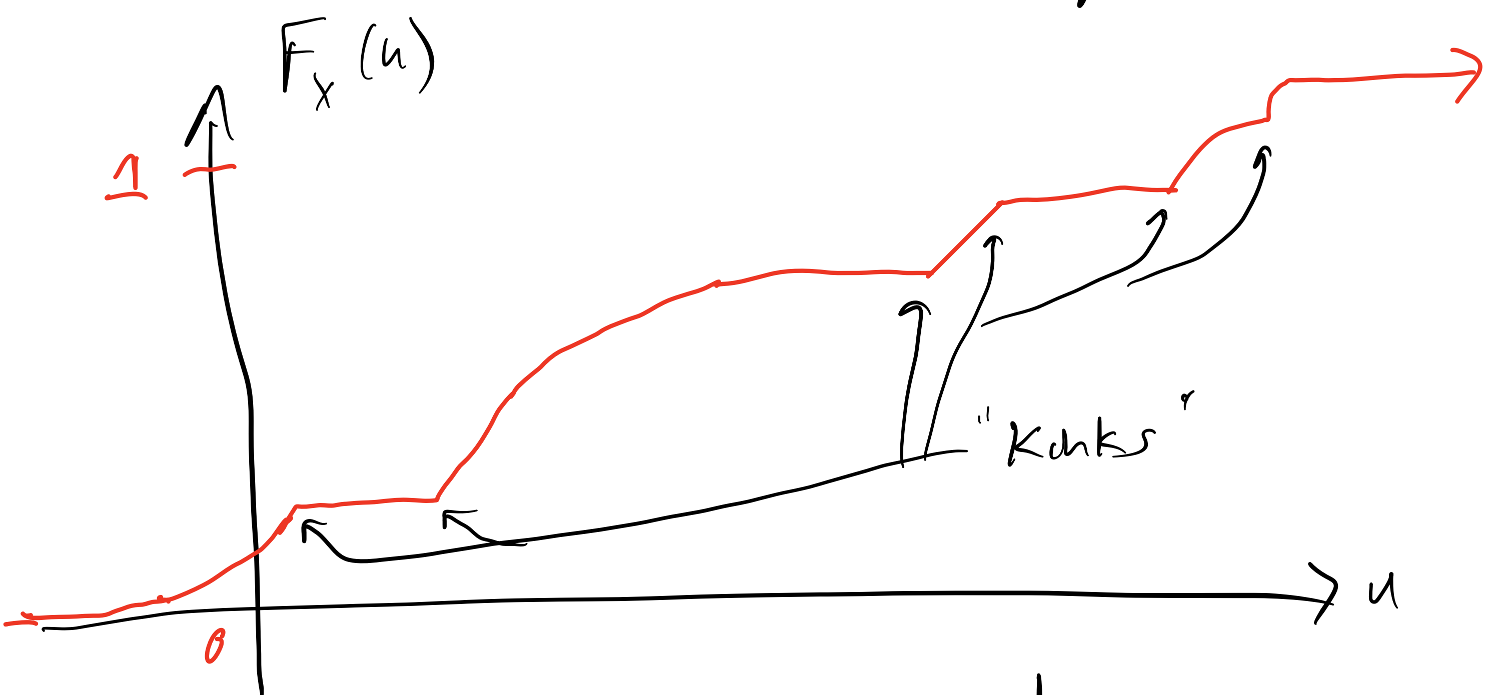
Lec 09:

Last time - discrete r.v.'s

Def: A continuous r.v. is one whose CDF satisfies:

1. Continuous everywhere (no "jumps")
2. Differentiable, except possibly at a finite or countably infinite set of points.



Since all CDFs are monotone non-decreasing, their derivatives can never be negative.

Def: The probability density function (pdf) of a continuous r.v. X is:

$$f_X(u) = \frac{dF_X(u)}{du}.$$

At kinks in CDF, we define pdf $f_X(u)$ to be any nonnegative value.

Always use lower case "f" for pdfs
" " upper " "F" " CDFs

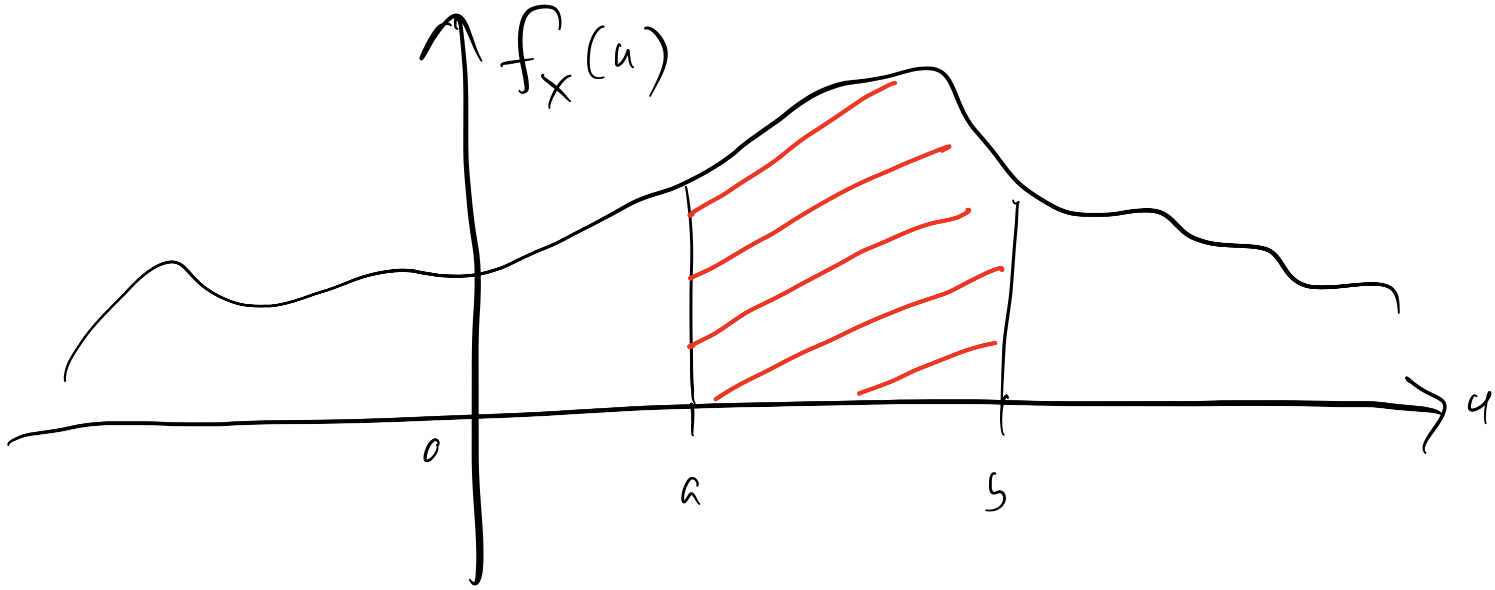
Properties of pdfs

(1) $\int_a^b f_X(u)\,du = \int_a^b \left(\frac{dF_X(u)}{du}\right) du$

$= F_X(u)\Big|_a^b = F_X(b) - F_X(a)$

$= P(a < X \leq b)$

Doesn't matter if these are $<$ or $\leq$ since CDF is continous.



② $f_X(u) \geq 0$

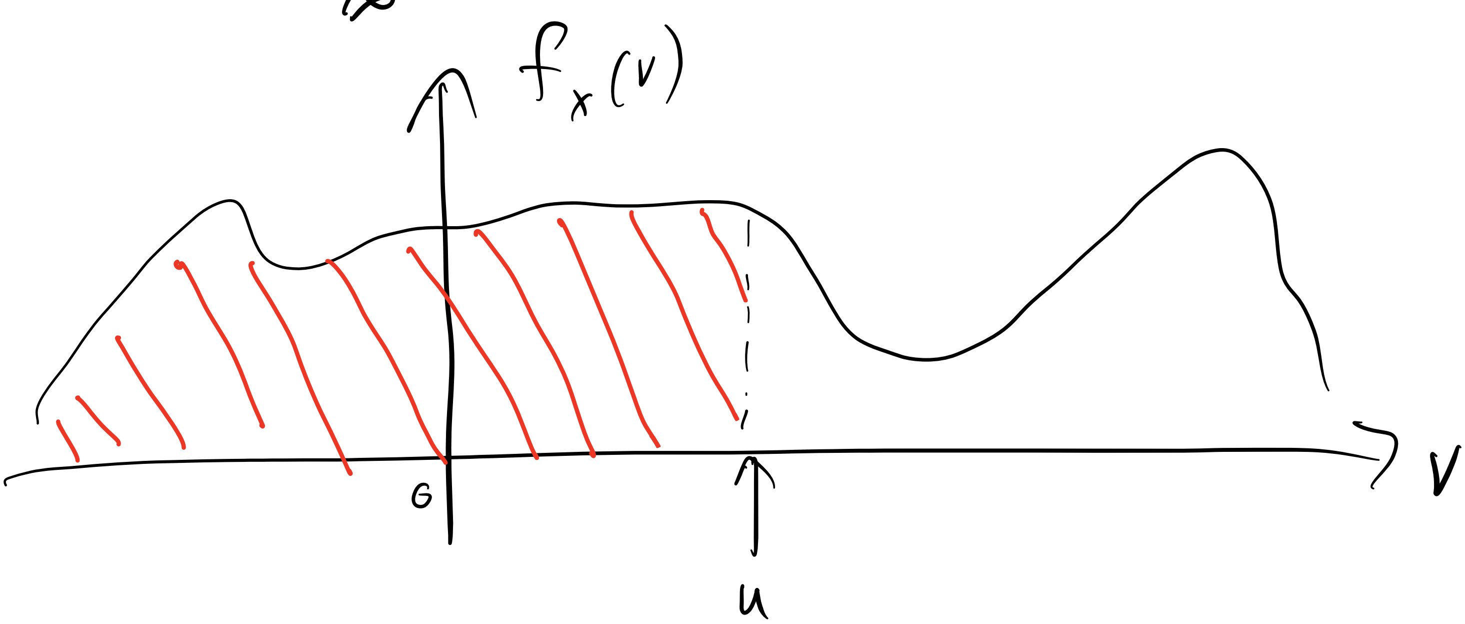
③ $\int_{-\infty}^{\infty} f_X(u)\,du = P(-\infty < X < \infty) = 1$

④ CDF:

$$F_X(u) = P(X \leq u) = P(-\infty < X < u)$$

$$= \int_{-\infty}^{u} f_X(v)\,dv$$

Gets the CDF from the pdf



(5) Suppose $A \subseteq \mathbb{R}$ (set of real numbers)

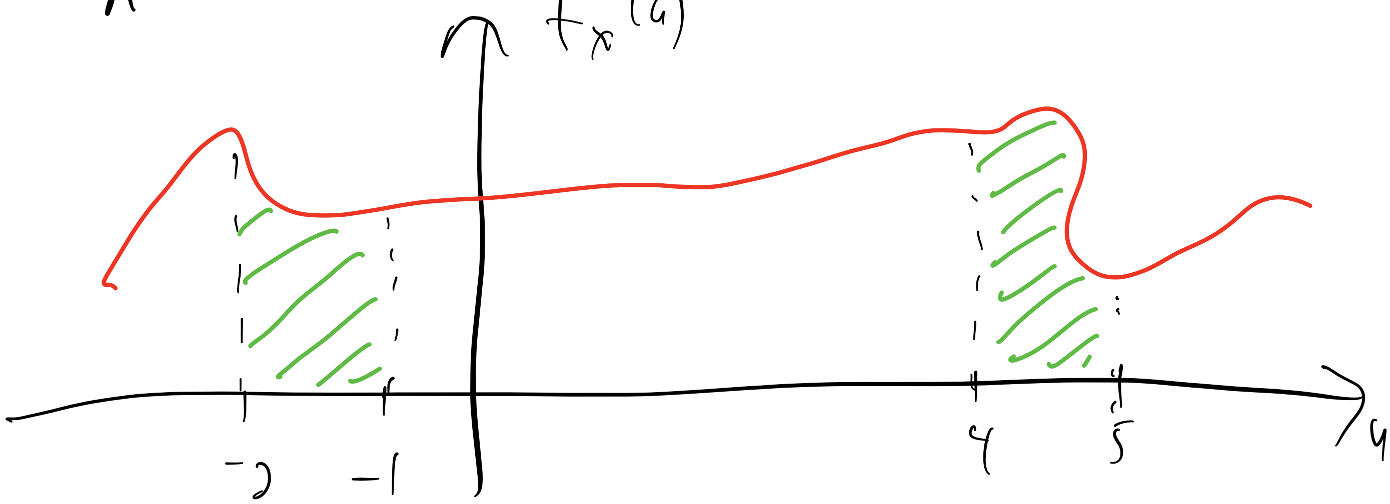
$$P(X \in A) = \int_A f_X(u)\, du$$

The "A" means integral is over all points in A.

For example, if $A = [-2,-1] \cup [4,5]$

then

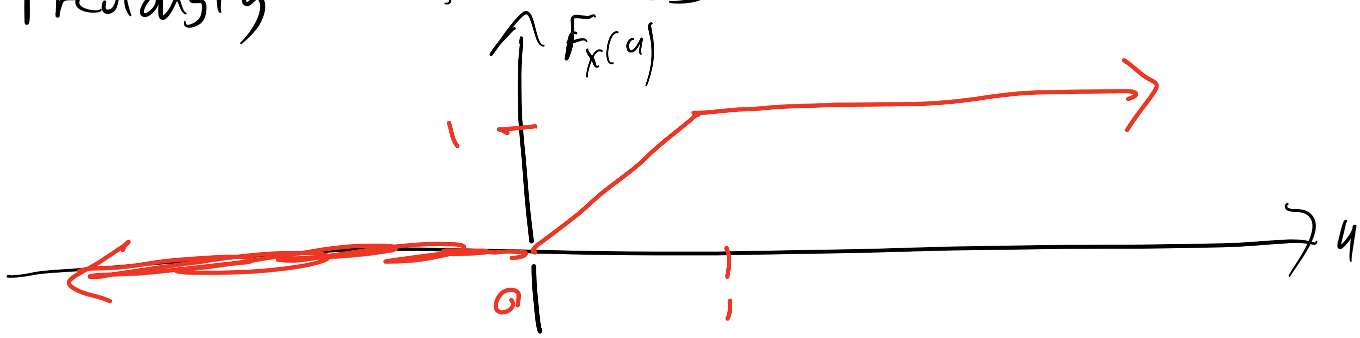
$$\int_A f_X(u)\, du = \int_{-2}^{-1} f_X(u)\, du + \int_4^5 f_X(u)\, du$$



Ex: Uniform rv on interval $[0,1]$.

eg. random function on computer.

Previously we found CDF to be



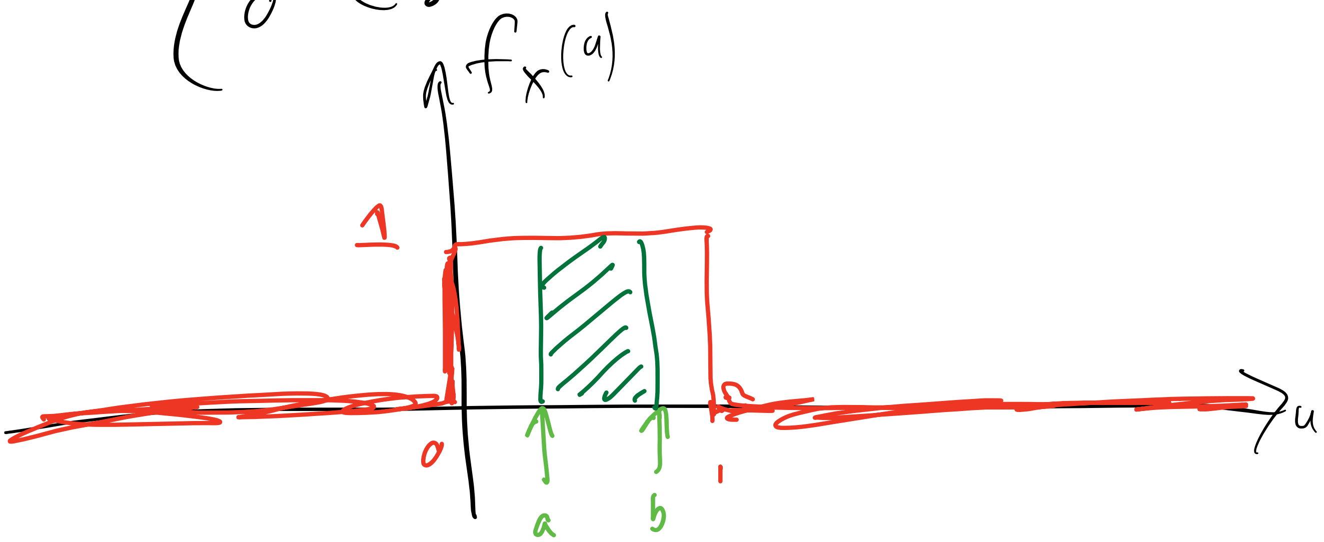
Find pdf.

$$f_X(u) = \frac{d}{du} F_X(u)$$

$$= \frac{d}{du} \begin{cases} 0 & \text{if } u < 0 \\ u & \text{if } 0 \le u \le 1 \\ 1 & \text{if } u > 1 \end{cases}$$

$$= \begin{cases} 0 & \text{if } u < 0 \\ 1 & \text{if } 0 \le u \le 1 \\ 0 & \text{if } u > 1 \end{cases}$$

$$= \begin{cases} 1 & \text{if } 0 \le u \le 1 \\ 0 & \text{else} \end{cases}$$



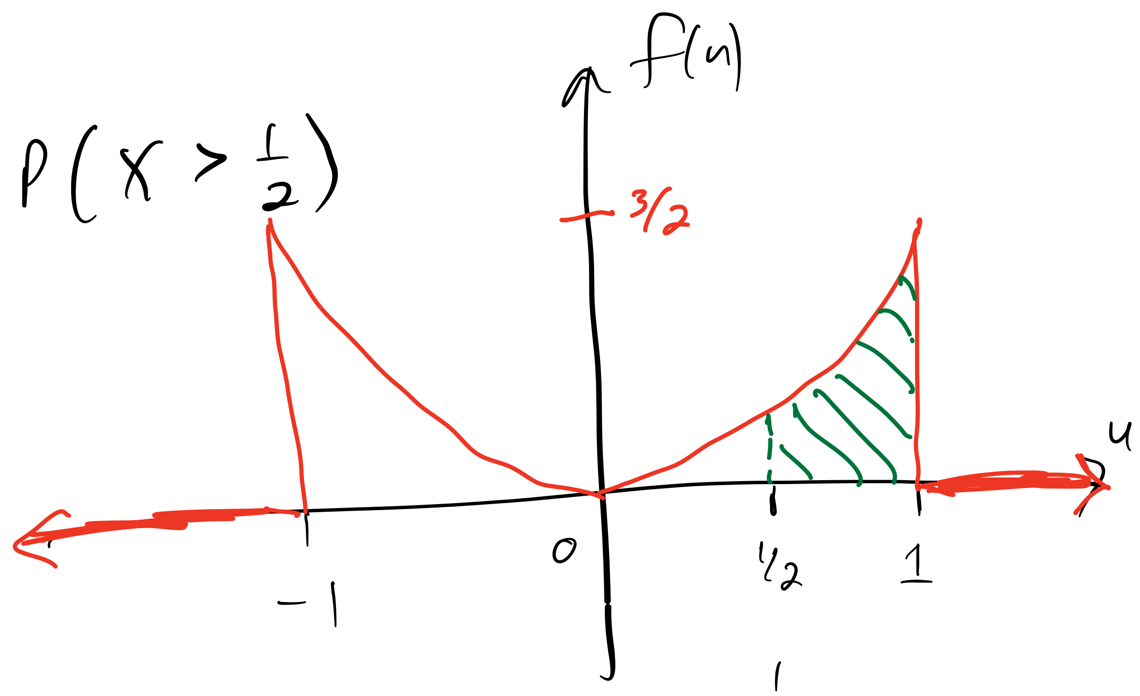
Suppose $0 \le a \le b \le 1$

$$P(a < X < b) = \int_a^b f_X(u)\,du = \int_a^b 1 \cdot du$$

$$= b - a$$

Ex The pdf of a rv X is:

$$f(u) = \begin{cases} \frac{3}{2}u^2 & \text{if } |u|<1 \\ 0 & \text{else} \end{cases}$$



① Find $P(X > \frac{1}{2})$

$$P(X > \tfrac{1}{2}) = \int_{1/2}^{\infty} f(u)\,du = \int_{1/2}^{1} \tfrac{3}{2}u^2\,du$$

$$= \tfrac{1}{2}u^3 \Big|_{1/2}^{1} = \tfrac{1}{2}\left(1^3 - \tfrac{1}{2^3}\right) = \tfrac{1}{2}\left(1 - \tfrac{1}{8}\right) = \frac{7}{16}$$

② what is $P(X > \frac{1}{2} \mid X > \frac{1}{4})$?

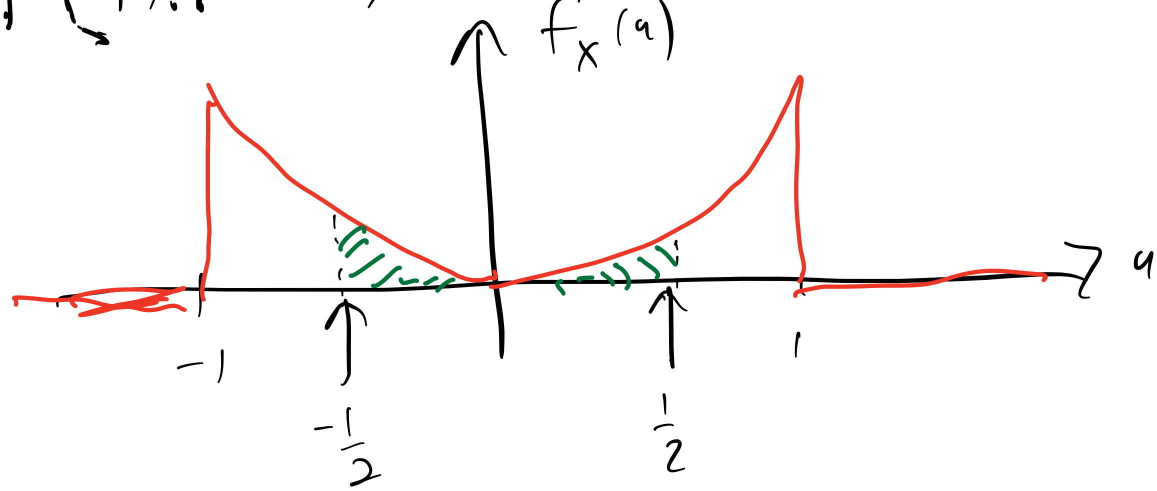
$$P(X > \tfrac{1}{2} \mid X > \tfrac{1}{4}) = \frac{P(X > \frac{1}{2} \text{ and } X > \frac{1}{4})}{P(X > \frac{1}{4})}$$

$$= \frac{P(X > \frac{1}{2})}{P(X > \frac{1}{4})} = \frac{7/16}{\int_{1/4}^{1} \frac{3}{2}u^2\,du} = \frac{7/16}{\frac{1}{2}u^3 \Big|_{1/4}^{1}}$$

$= \frac{8}{9}$ (work it out)

(3) What is $P(|X| < \frac{1}{2})$?

$P(|X| < \frac{1}{2}) = P(-\frac{1}{2} < X < \frac{1}{2})$



$= \int_{-1/2}^{1/2} \frac{3}{2} u^2 \, du = \ldots$

or an easier way...

$= 1 - 2 \cdot P(X > \frac{1}{2})$

$= 1 - 2 \cdot \frac{7}{16} = 1 - \frac{7}{8} = \frac{1}{8}$

(4) What is $P(8X^2 - 6X + 1 > 0)$?

$P(8X^2 - 6X + 1 > 0)$

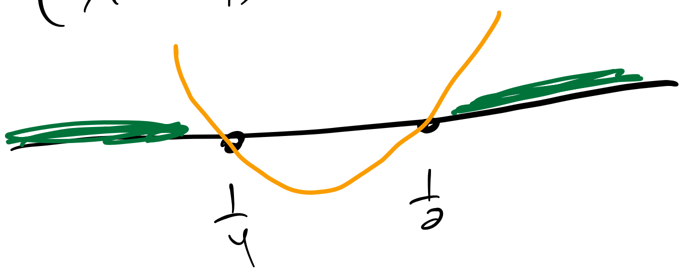
$= P((4X-1)(2X-1) > 0)$

$$= P\begin{pmatrix} 4x-1>0 \ \underline{\text{and}} \ 2x-1>0 \\ \text{OR} \\ 4x-1<0 \ \underline{\text{and}} \ 2x-1<0 \end{pmatrix}$$

$$= P\begin{pmatrix} x>\frac{1}{4} \ \underline{\text{and}} \ x>\frac{1}{2} \\ \text{OR} \\ x<\frac{1}{4} \ \underline{\text{and}} \ x<\frac{1}{2} \end{pmatrix}$$

$$= P\left(x>\frac{1}{2} \ \underline{\text{OR}} \ x<\frac{1}{4}\right)$$ ← disjoint events

$$= P\left(X>\frac{1}{2}\right) + P\left(X<\frac{1}{4}\right)$$



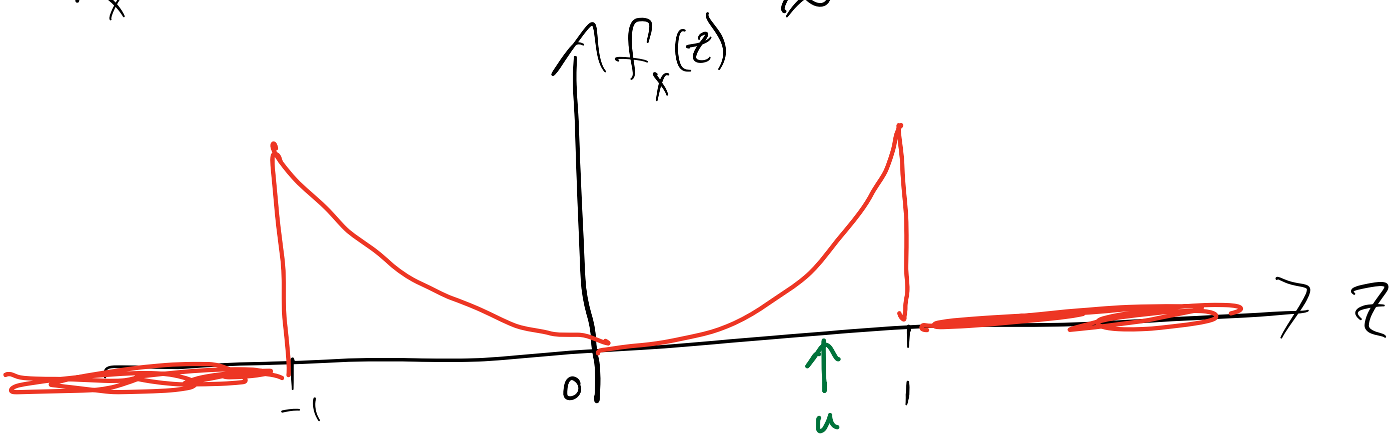
$$= \frac{7}{16} + \int_{-1}^{1/4} \frac{3}{2}u^2\,du$$

(from before)

$$= \frac{7}{16} + \frac{1}{2}u^3\Big|_{-1}^{1/4} = \frac{7}{16} + \frac{1}{2} + \frac{1}{128} = \frac{125}{128}$$

⑤ Find CDF of X.

$$F_X(u) = P(X \le u) = \int_{-\infty}^{u} f(z)\,dz$$



3 cases:

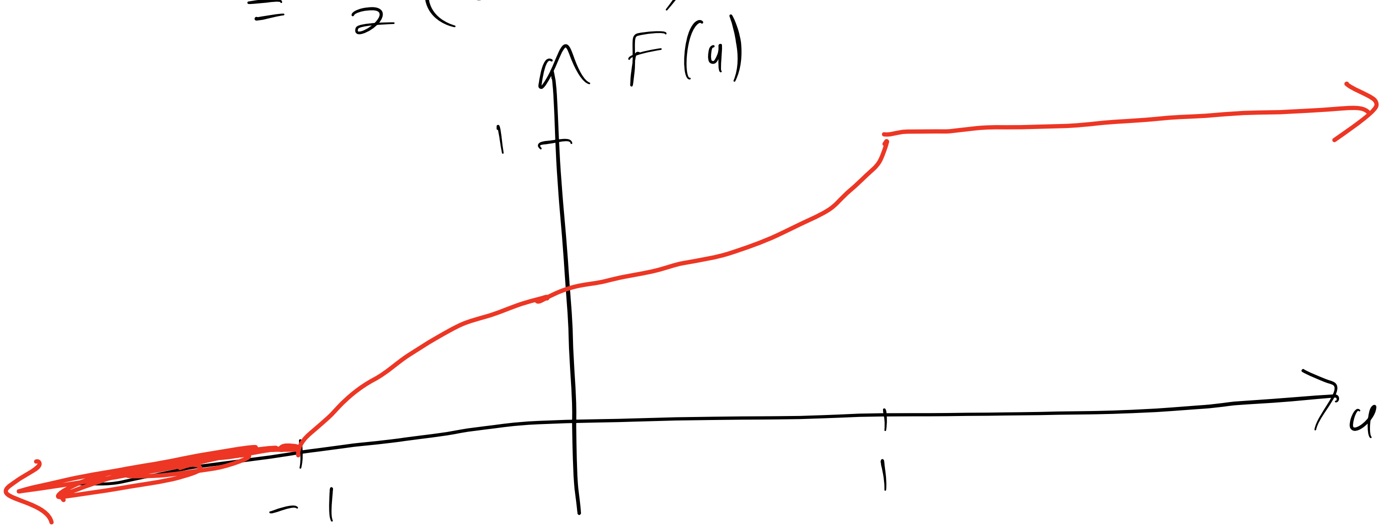
If $u \le -1$, then $F_X(u) = 0$.

If $u \geq 1$, then $F_X(u) = 1$

If $-1 < u < 1$, then

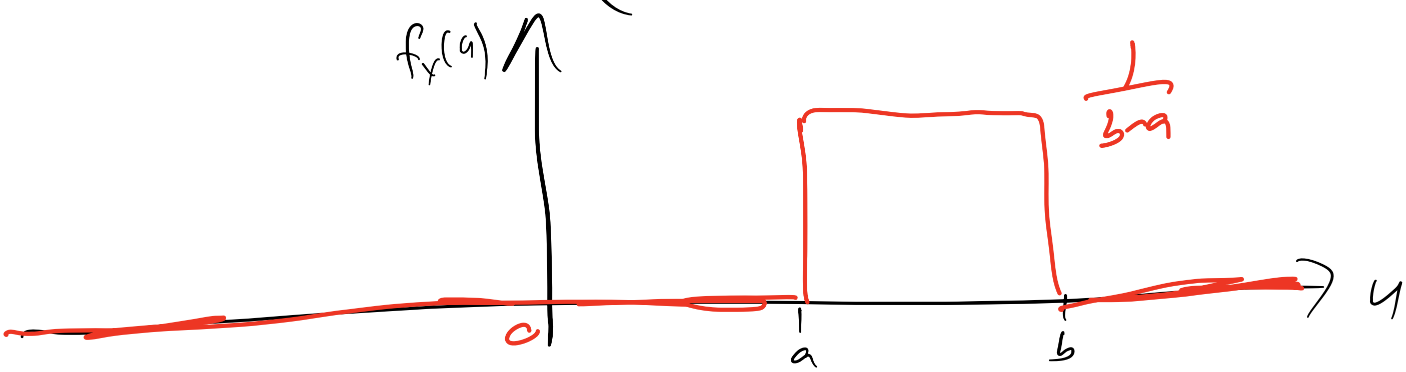
$$F_X(u) = \int_{-1}^{u} \frac{3}{2} z^2 \, dz = \frac{1}{2} z^3 \Big|_{-1}^{u}$$

$$= \frac{1}{2}(u^3 + 1)$$



<u>Def</u> A <u>uniformly distributed r.v.</u> on an interval (a,b) has pdf:

$$f_X(u) = \begin{cases} \frac{1}{b-a} & \text{if } a < u < b \\ 0 & \text{else} \end{cases}$$



$$\text{Area} = \frac{1}{b-a}\cdot(b-a) = 1$$

<u>Def</u>: A <u>Gaussian</u> (or "normal") r.v. X has pdf:

$$f_X(u) = \frac{1}{\sigma\sqrt{2\pi}} \cdot e^{-\frac{1}{2}\left(\frac{u-m}{\sigma}\right)^2}$$

for $-\infty < u < \infty$

The parameters are named as follows:

m = "mean"

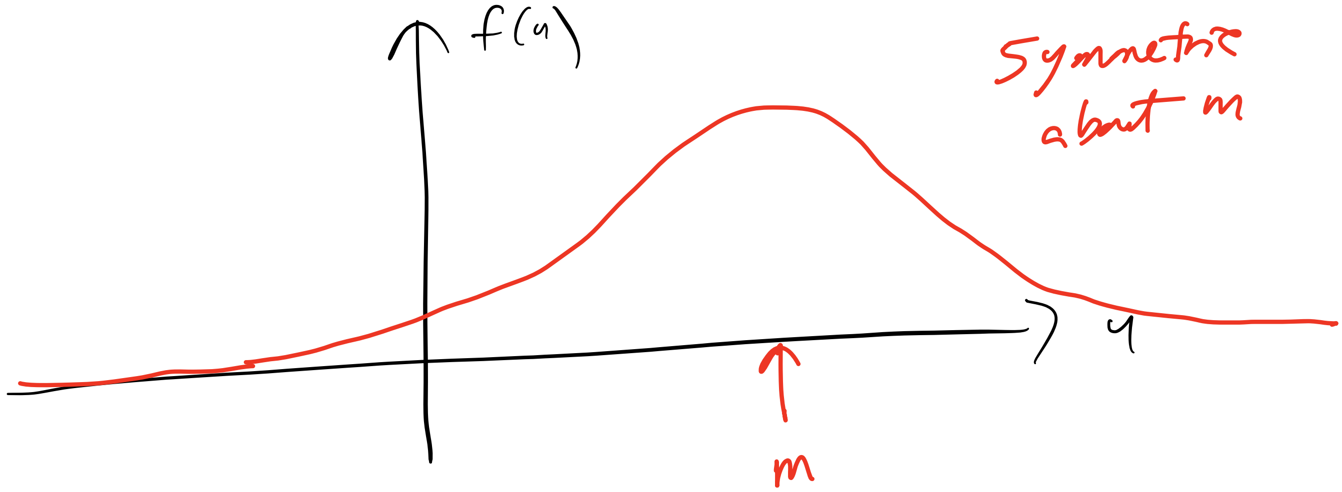
σ^2 = "variance"

σ = "standard deviation"

<u>Notation</u>: we write

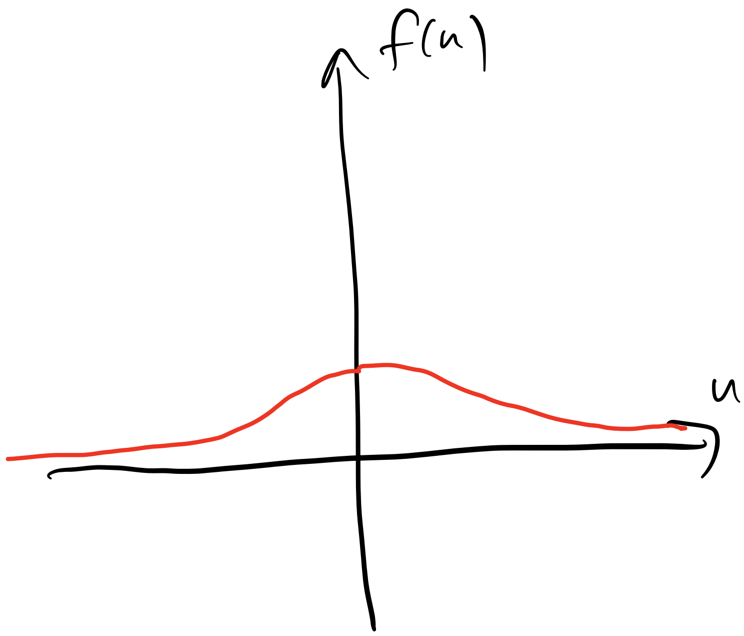
$$\boxed{X \sim N(m, \sigma^2)}$$

to indicate X has the pdf above



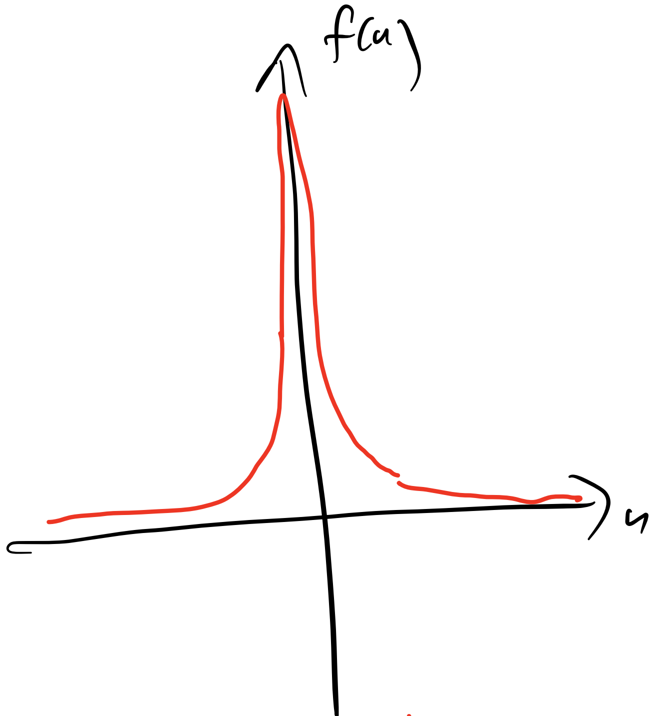
Take $m = 0$:

Large σ



Short + fat

Small σ



tall + thin