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Week 3, Lecture 01-24-23

Example: Flip a coin twice

$$S = \{HH, HT, TH, TT\}$$

Let us define the events:

$$E = \text{"First flip is heads"} = \{HH, HT\}$$
$$F = \text{"Flips are different"} = \{HT, TH\}$$
$$P(E) = \frac{2}{4} = \frac{1}{2}$$
$$P(E|F) = \frac{P(EF)}{P(F)} = \frac{P(\{HT\})}{P(\{HT, TH\})}$$
$$= \frac{\frac{1}{4}}{\frac{2}{4}} = \frac{1}{2}$$

So, in this case,

$$P(E) = P(E|F) = \frac{1}{2}$$
$$\rightarrow E, F \text{ are independent}$$

Let's multiply both sides by $P(F)$:

$$P(E)P(F) = P(E|F)P(F) = P(EF)$$

i.e. $P(EF)$ factors into $P(E)P(F)$. If $P(F) \neq 0$, then this implies $P(E|F) = P(E)$.

Definition: Events E, F are independent if $P(EF) = P(E)P(F)$.

Consequences: If E, F are independent, then

1. E, F^c are independent
2. E^c, F are independent
3. E^c, F^c are independent

Proof of 1: Assume $P(E) \neq 0$.

$$P(EF^c) = P(E)P(F^c|E)$$
$$= P(E)(1 - P(F|E))$$
$$= P(E)(1 - P(F))$$
$$= P(E)P(F^c)$$
$$\rightarrow E, F^c \text{ are independent}$$

Disjoint versus Independence

Suppose E, F are disjoint and $P(E) \neq 0$ and $P(F) \neq 0$.

$$\therefore EF = 0$$
$$P(EF) = P(0) = 0$$
$$P(E)P(F) \neq 0$$
$$\therefore P(EF) \neq P(E)P(F)$$
$$\rightarrow P(E) \neq 0 \text{ but } P(E|F) = 0$$

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Example: Given two inputs x and y, the output of an XOR gate z is 1 if and only if $x \neq y$

$$x, y, z \in \{0, 1\}$$

Suppose x and y are chosen with equal probability $\frac{1}{2}$ and independently. Let us define the events:

$$\begin{aligned} A &== x = 1 \\ B &== y = 1 \\ C &== z = 1 \end{aligned}$$

By assumption A, B are independent events.

$$\begin{aligned} P(A) = P(B) &= \frac{1}{2} \\ C &= AB^c \cup A^cB \\ P(C) &= P(AB^c \cup A^cB) \\ &= P(AB^c) + P(A^cB) \\ &= P(A)P(B^c) + P(A^c)P(B) \\ &= \frac{1}{2} * \frac{1}{2} + \frac{1}{2} * \frac{1}{2} \\ AC &== x = 1 \text{ and } z = 1 \\ &== x = 1, y = 0, z = 1 \\ &== x = 1, y = 0 \\ &== AB^c \\ P(AC) = P(AB^c) &= P(A)P(B^c) = \frac{1}{2} * \frac{1}{2} = \frac{1}{4} \\ \therefore P(AC) &= P(A)P(C) \\ &\rightarrow A, C \text{ are independent} \end{aligned}$$

Also by symmetry, B, C are independent. But A, C (also B, C) are **not** physically indepent. It is very clear that they are part of the same XOR gate.

Definition: Events A, B, C are **independent** if all of the following are true:

1) A, B are independent
2) B, C are independent
3) A, C are independent
4) $P(ABC) = P(A)P(B)P(C)$

Note: If A, B, C are pairwise independent, then

$$P(ABC) = P(A)P(B)P(C)$$

is equivalent to $P(A|BC) = P(A)$ since:

$$\begin{aligned} P(A|BC)P(BC) &= P(A)P(BC) \\ P(ABC) &= P(A)P(B)P(C) \end{aligned}$$

It is possible to satisfy pairwise independence (conditions numbers 1, 2, and 3), but not condition number 4. Consider the XOR example.

$$\begin{aligned} ABC &== x = 1, y = 1, z = 1 = 0 \\ P(ABC) &= P(0) = 0 \end{aligned}$$

But $P(A)P(B)P(C) = \frac{1}{2} * \frac{1}{2} * \frac{1}{2} = \frac{1}{8} \neq 0$

$$\therefore \text{ condition number 4 is not satisfied.}$$

$$\therefore A, B, C \text{ are pairwise independent but } A, B, C \text{ are } \textbf{not} \text{ independent}$$

Definition: Events A, B are **conditionally independent given event C**, if:
$$P(AB|C) = P(A|C)P(B|C)$$
assuming $P(C) \neq 0$

Note: If A, B are independent given C, then:

$$\begin{aligned} P(A|BC) = \frac{P(ABC)}{P(BC)} &= \frac{P(AB|C)P(C)}{P(B|C)P(C)} \\ &= P(A|C) \end{aligned}$$

This is similar to $P(A|B) = P(A)$ but with the extra "given C".

Example: Flip 2 coins
Let us define the events:

$A =$ "First coin is heads"
$B =$ "Second coin is heads"
$C =$ "First and second coin are both heads"

Note: $C = AB$

1. A, B are independent (by assumption)
$$P(A|C) = \frac{P(AC)}{P(C)} = \frac{P(AAB)}{P(C)} = \frac{P(AB)}{P(C)} = \frac{P(C)}{P(C)} = 1$$
2. $P(A|BC) = \frac{P(ABC)}{P(BC)} = \frac{P(C)}{P(C)} = 1$
$\therefore P(A|C) = P(A|BC) = 1$
$\rightarrow A, B$ are independent given C.
$$P(A|C^c) = \frac{P(AC^c)}{P(C^c)} = \frac{P(HT)}{1 - P(HH)} = \frac{\frac{1}{4}}{1 - \frac{1}{4}} = \frac{1}{3}\frac{P(C)}{P(C)} = 1$$
3. $P(A|BC^c) = \frac{P(ABC^c)}{P(BC^c)} = \frac{P(0)}{P(TH)} = 0$
$\therefore P(A|C^c) \neq P(A|BC^c)$
$\rightarrow A, B$ are **not** independent given C^c.

Example: Let a sample space of equiprobable outcomes be:

$$S = \{1, 2, 3, 4, 5\}$$

Let us define the events:

$$A = \{1, 2, 5\}$$
$$B = \{1, 3, 5\}$$
$$C = \{1, 2, 3, 4\}$$

1. We know $P(A)$ and $P(B)$ are not independent from the following:

$$P(AB) = P(\{1,5\}) = \frac{2}{5}$$
$$P(A) = P(B) = \frac{3}{5}$$
$$P(AB) \neq P(A)P(B)$$
$$\rightarrow A, B \textbf{ not} \text{ independent}$$

2. However, given C, this changes

$$P(AB|C) = P(\{1,5\}|\{1,2,3,4\}) = \frac{1}{4}$$
$$P(A|C) = P(\{1,2,5\}|\{1,2,3,4\}) = \frac{2}{4} = \frac{1}{2}$$
$$P(B|C) = P(\{1,3,5\}|\{1,2,3,4\}) = \frac{2}{4} = \frac{1}{2}$$
$$\therefore P(AB|C) = P(A|C) * P(B|C)$$
$$\rightarrow A, B \text{ are independent given } C.$$

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Example: **Experiment: Flip 3 coins. Define event E = "exactly one head". What is the probability event E occurs exactly k times in n trials?**
This can be thought of as E occuring k times and E^c occurring $n-k$ times.

$$= (P(E))^k * (P(E^c))^{n-k}$$
$$= (P(E))^k * (1 - P(E))^{n-k}$$

There are $\binom{n}{k}$ arrangements of where the k occurrences of E could be, so the total probability is

$$\binom{n}{k}(P(E))^k * (1 - P(E))^{n-k}$$

Example: **Now we use three biased coins.**

$$E = \text{exactly one head occurs}$$
$$P(E) = P(\{HTT, THT, TTH\})$$
$$= 3q(1-q)^2$$

P(E occurs 4 times in 10 trials)
(i.e. when k = 4, n = 10)

$$\binom{10}{4}(3q(1-q)^2)^4(1 - 3q(1-q)^2)^{10-4}$$

Example: **Suppose A, B are events in an experiment and are disjoint. What is the probability A occurs before B in independent trials?**
This can happen in these ways:

$$C = (A \cup B)^c = \text{neither}$$
$$A$$
$$C, A$$
$$C, C, A$$
$$C, C, C, ..., A$$

The net probability is

$$P(A) + P(C)P(A) = (P(C))^2P(A) + (P(C))^3P(A) + ...$$
$$= P(A)(1 + P(C) + P(C)^2 + P(C)^3 + ...)$$
$$= P(A)\left(\frac{1}{1 - P(C)}\right) = \frac{P(A)}{P(A) + P(B)}$$

Example: **3 coin flips (fair coins)** Define:

$$A = 3 \text{ heads}$$
$$B = 1 \text{ head}$$

Therefore, P(exactly 3 heads before exactly 1 head):

$$\frac{P(A)}{P(A) + P(B)} = \frac{\frac{1}{8}}{\frac{1}{8} + \frac{3}{8}} = \frac{1}{4}$$

Example: **Flip biased coin (q = P(H)) repeatedly until we get at least 10 heads, then stop. What is the probability we flip the coin exactly 15 times?**
This situation occurs when we flip heads after having flipped 9 heads and 5 tails in any order for a total of 14 flips. Therefore the probability can be found by

$$\binom{14}{5}q^9(1-q)^5 * q$$

Example: **Flip the biased coin until we get at least 3 heads and 5 tails. What is the probability we flip the coin exactly 20 times?**
This situation can occur in two ways: when we flip heads after having flipped 2 heads and 17 tails, or when we flips tails after having flipped 4 tails and 15 heads. Therefore, the probability can be found by

$$\binom{19}{2}q^2(1-q)^{17} * q + \binom{19}{4}q^{15}(1-q)^4 * (1-q)$$

Definition: A random variable for a sample space S is a mapping $X : S \to \mathbb{R}$

In other words, for each $u \in S$, $X(u)$ is a real number.

Example: **S = {apple, banana, orange}**
Let

$$X(\text{apple}) = 3$$
$$X(\text{banana}) = -\pi$$
$$X(\text{orange}) = 0$$

Let

$$Y(\text{apple}) = 5$$
$$Y(\text{banana}) = 6$$
$$Y(\text{orange}) = 6$$

X, Y are random variables. This can be abbreviated as r.v.

Example: **Flip 3 coins (fair) Define r.v. X as follows:**
X counts the number of heads.

$$X(HHH) = 3$$
$$X(HTH) = X(THH) = X(HHT) = 2$$
$$X(HTT) = X(THT) = X(TTH) = 1$$
$$X(TTT) = 0$$

Define the r.v. Y as follows:

$$Y(HHH) = 1$$
$$Y(u) = 0 \text{ if } u \neq HHH$$

Convention to use upper case letters for random variables. The above information can be used to find the following probabilities:

$$P(X = 2) = \frac{3}{8}$$
$$P(X \leq 2) = \frac{7}{8}$$
$$P(Y < \frac{1}{2}) = \frac{7}{8}$$
$$P(X = Y) = \frac{1}{8}$$
$$P(X > Y) = \frac{7}{8}$$

What does $P(X = 2)$ actually mean? It means the probability of the event

$$\{u \in S : X(u) = 2\}$$
$$= \{HHT, HTH, THH\}$$

The notation "x = 2" means the above event. Similarly,

$$X \leq 2 = \{u \in S : X(u) \leq 2\}$$
$$X = Y = \{u \in S : X(u) = Y(u)\}$$

Random variables are not random. Inputs are random and thus outputs are random.

Cummulative Distribution Function (CDF) of a random variable X is $F_x(u) = P(x \leq u)$