Lec 04:

Last time - conditional prob

$$P(E|F) = \frac{P(EF)}{P(F)}$$

with $P(F) \neq 0$

Recall axioms of probability

1. $0 \leq P(E) \leq 1$
2. $P(S) = 1$
3. If $E_1, E_2, \ldots$ are pairwise disjoint events, then
   $$P(E_1 \cup E_2 \cup \cdots) = \sum_n P(E_n)$$

For conditional probability we get similar axioms:

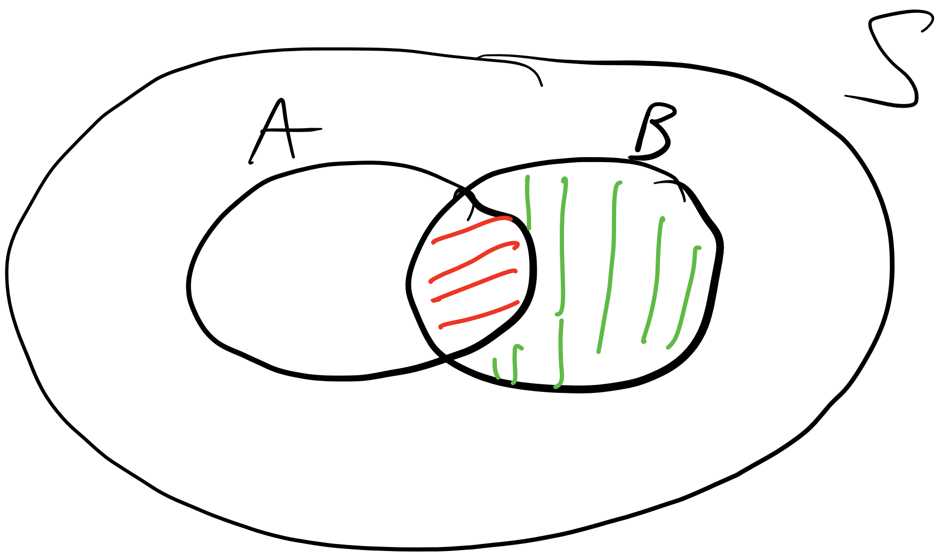
1) $0 \le P(E|F) \le 1$

2) $P(S|F) = 1$

3) If $E_1, E_2, \ldots$ are pairwise disjoint, then

$$P((E_1 \cup E_2 \cup \ldots) | F) = \sum_n P(E_n | F)$$

Partition Rule of Probability



$$B = \underbrace{BA}_{\text{red}} \cup \underbrace{BA^c}_{\text{green}}$$

$\uparrow$ disjoint union

$$P(B) = P(BA \cup BA^c)$$
$$= P(BA) + P(BA^c)$$

$$= \boxed{P(B|A)P(A) + P(B|A^c)P(A^c)}$$

View A as "case 1"
A^c as "case 2"

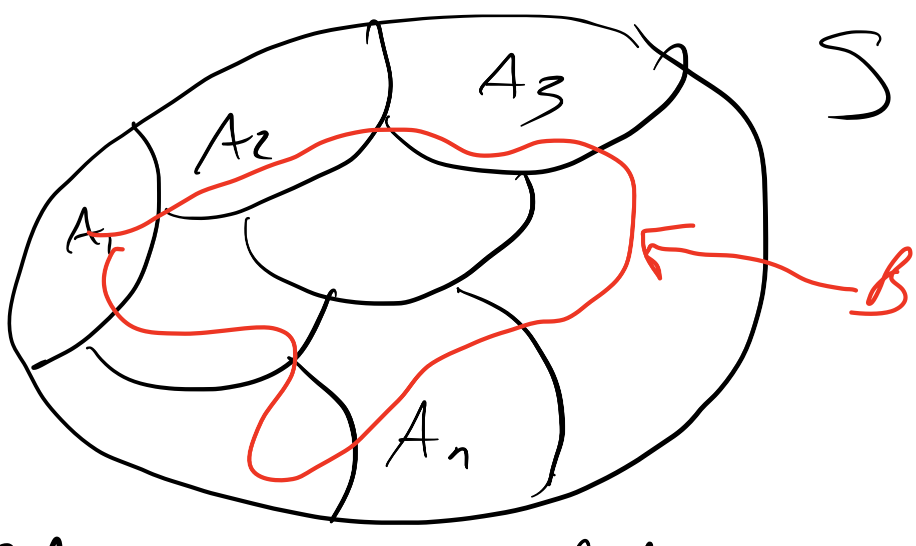
Can generalize to n cases:

Suppose $A_1, A_2, \dots, A_n$ are events that partition S.

ie. A_i's are a disjoint cover of S.

disjoint: $i \neq j \Rightarrow A_i A_j = \phi$

cover: $S = A_1 \cup A_2 \cup \dots \cup A_n$



$B = BA_1 \cup BA_2 \cup \dots \cup BA_n$

disjoint unions

$P(B) = P(BA_1 \cup \dots \cup BA_n)$

$= P(BA_1) + \cdots + P(BA_n)$

$$= \boxed{P(B|A_1)P(A_1) + \cdots + P(B|A_n)P(A_n)}$$

n cases

Case for $n=2$ we did earlier.

There $A = A_1$ and $A^c = A_2$

Can extend this to conditional probs:

$$\boxed{P(B|E) = P(B|A_1E)P(A_1|E) + \cdots + P(B|A_nE)P(A_n|E)}$$

Ex: Roll 1 fair 6-sided die.

If the die comes up ≥ 3, then we win. If not, then we flip a fair coin and we win if Heads, lose if Tails.

What is the prob of winning.

Soln: Define events

$E =$ "die is ≥ 3"

$F =$ "we win"

We want $P(F)$.

$$P(F) = P(F|E)P(E) + P(F|E^c)P(E^c)$$

Case 1 Case 2

$$P(E) = P(\text{die is } 3,4,5,6) = \frac{4}{6} = \frac{2}{3}$$

$$P(F|E) = 1$$

$$P(F|E^c) = \frac{1}{2}$$

$$P(E^c) = 1 - P(E) = 1 - \frac{2}{3} = \frac{1}{3}$$

$$\therefore P(F) = \left(1 \cdot \frac{2}{3}\right) + \left(\frac{1}{2} \cdot \frac{1}{3}\right) = \frac{5}{6}$$

2nd Question: What is the prob the die is ≥ 3, given we won?

Soln: want $P(E|F)$

Recall: $P(EF) = P(F)\,P(E|F)$
$= P(E)\,P(F|E)$

Know: $P(E) = \frac{2}{3}$
$P(F|E) = 1$
$P(F) = \frac{5}{6}$

$$\therefore \frac{5}{6} \cdot P(E|F) = \frac{2}{3} \cdot 1 \implies P(E|F) = \frac{2/3}{5/6} = \frac{4}{5}$$

Ex: A box contains 2 red and 3 green marbles. Pick 2 marbles at random (without replacement).

If both are green, we win.
If both are red, we lose.
Otherwise, we pick a 3rd marble and we win if red, lose if green.
What is the prob of winning?

Soln: Break into 3 cases.

Define events:

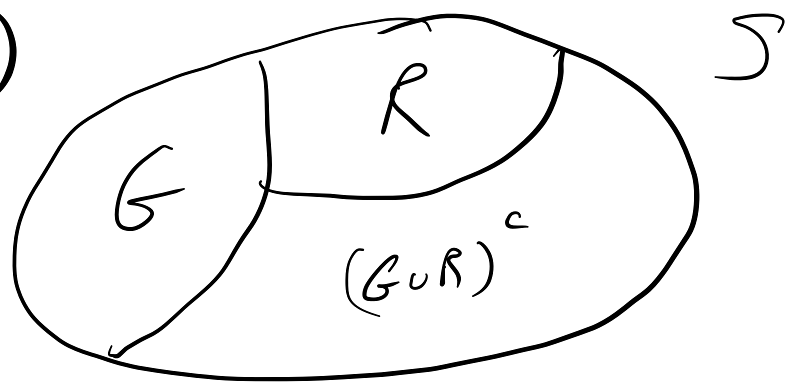
W = "we win"

G = "both marbles are green"

R = "both marbles are red"

Want $P(W)$

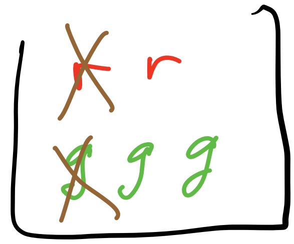
3 events partition S



By DeMorgan
$(G \cup R)^c = G^c R^c$

3 cases for winning:

$$P(w) = \underbrace{P(w \mid G)P(G)}_{\text{case 1}} + \underbrace{P(w \mid R)P(R)}_{\text{case 2}} + \underbrace{P(w \mid G^c R^c)P(G^c R^c)}_{\text{case 3}}$$



Know: $P(w \mid G) = 1$

$P(w \mid R) = 0$

$P(w \mid G^c R^c) = \frac{1}{3}$

$$P(G^c R^c) = P((G \cup R)^c) = 1 - P(G \cup R)$$

disjoint

$$= 1 - P(G) - P(R)$$

$$= 1 - \frac{3}{5}\cdot\frac{2}{4} - \frac{2}{5}\cdot\frac{1}{4} = \frac{3}{5}$$

Plug these all in.

$$P(w) = \left(1 \cdot \frac{3}{10}\right) + (0 \cdot ?) + \left(\frac{1}{3}\cdot\frac{3}{5}\right) = \frac{1}{2}$$

Chain Rule of Probability

Using the def of cond. prob, we know:

$$P(A_1 A_2) = P(A_1)P(A_2 \mid A_1)$$

For 3 events:

$$P(A_1 A_2 A_3) = P(A_1 (A_2 A_3))$$
$$= P(A_1) P(A_2 A_3 | A_1)$$

<u>Note</u>: $P(A_3 | A_1 A_2) = \dfrac{P(A_1 A_2 A_3)}{P(A_1 A_2)}$

$$= \frac{P(A_2 A_3 | A_1) \cancel{P(A_1)}}{P(A_2 | A_1) \cancel{P(A_1)}}$$

Substitute in for $P(A_2 A_3 | A_1) \longrightarrow$

$$P(A_1 A_2 A_3) = P(A_1) P(A_2 | A_1) P(A_3 | A_1 A_2)$$

Can keep going and in general:

$$\boxed{P(A_1 A_2 \cdots A_n) = P(A_1) P(A_2 | A_1) P(A_3 | A_1 A_2) \cdots P(A_n | A_1 A_2 \cdots A_{n-1})}$$

<u>Ex</u>: Given a standard deck of 52 cards. What is the prob of picking an Ace, and then a red card (no replacement)?

Sol'n: Define events

$A =$ "1st card is an Ace"
$R =$ "2nd card is red"
$B =$ "1st card is red"

Want $P(AR)$

Know: $P(AR) = P(A)\,P(R|A) = \frac{1}{13} P(R|A)$

$$P(R|A) = \underbrace{P(R|BA)\,P(B|A)}_{\text{case 1}} + \underbrace{P(R|B^cA)\,P(B^c|A)}_{\text{case 2}}$$

$$= \left(\frac{25}{51} \cdot \frac{1}{2}\right) + \left(\frac{26}{51} \cdot \frac{1}{2}\right) = \frac{1}{2}$$

Final answer

$$P(AR) = \frac{1}{13} \cdot \frac{1}{2} = \frac{1}{26}$$

Baye's formula

Suppose $A_1, A_2, \ldots, A_n$ is a partition of sample space S.
Suppose we know $P(B|A_i)$ for all i
Want $P(A_i|B)$

Can write: $P(A_i|B) = \dfrac{P(B|A_i)\,P(A_i)}{P(B)}$

Note: $P(B) = \sum_{j=1}^{n} P(B|A_j)P(A_j)$

$$\boxed{P(A_i|B) = \frac{P(B|A_i)\,P(A_i)}{\sum_{j=1}^{n} P(B|A_j)P(A_j)}}$$

Independence of events

physical independence
statistical independence

Main idea: Events E, F are independent if the prob of E stays the same even after learning that F occurred.