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Example: Given the sample space:

$$S = \{H, TH, TTH, TTTH, ...\}$$

Flip a fair coin until 1st head appears, then stop. Let us define the random variable X = number of tosses until 1st head:

$$X(H) = 1$$
$$X(TH) = 2$$
$$X(TTH) = 3$$
$$\vdots$$
$$P(X = 1) = P(H) = \frac{1}{2}$$
$$P(X = 2) = P(TH) = \frac{1}{2^2}$$
$$\vdots$$
$$P(X = n) = P(T...TH) = \frac{1}{2^n}$$

Define a second random variable Y to indicate oddness:

$$Y = \begin{cases} 1 \text{ if X is odd} \\ 0 \text{ if X is even} \end{cases} \quad Y(H) \quad = 1 \text{ odd}$$
$$Y(TH) = 0 \text{ even}$$
$$Y(TTH) = 1 \text{ odd}$$
$$Y(TTTH) = 0 \text{ even}$$
$$\vdots$$

What is $P(Y = 0)$?

$$\begin{aligned} P(Y = 0) &= P(Xiseven) \\ &= P(X = 2) + P(X = 4) + P(X = 6) + ... \\ &= \frac{1}{2^2} + \frac{1}{2^4}\frac{1}{2^6} + ... \text{ geometric series} \\ &= \frac{\frac{1}{4}}{1 - \frac{1}{4}} * \frac{4}{4} = \frac{1}{4 - 1} = \frac{1}{3} \end{aligned}$$

Consequently,

$$P(Y = 1) = 1 - P(Y = 0) = 1 - \frac{1}{3} = \frac{2}{3}$$

Note:

$$\begin{aligned} \{Y = 0\} &= \{u \in S : Y(u) = 0\} \\ &= \{TH, TTTH, TTTTTH, ...\} \end{aligned}$$

Notation:

Let $A \subseteq R$ (i.e. a set of real numbers). We will use the following notation:

$$\boxed{\{X \in A\} = \{u \in S : X(u) \in A\}}$$

In the previous example we can write

$$\begin{aligned}\{Y = 0\} &= \{Y \in \{0\}\} \\ \{X \leq 4\} &= \{X \in (-\infty, 4]\} \\ \{-1 \leq X < 7\} &= \{X \in [-1, 7)\} \\ &\vdots\end{aligned}$$

Example: **Flip biased coin 3 times. $P(H) = q$. Define random variable X = number of heads (i.e. 0, 1, 2, 3). What is $P(X \leq 1)$?**

$$\{X \leq 1\} \text{ is the event } \{TTT, HTT, THT, TTH\}$$

$$\begin{aligned}P(X \leq 1) &= P(\{TTT, HTT, THT, TTH\}) \\ &= (1-q)^3 + 3q(1-q)^2\end{aligned}$$

Fundamental question about a random variable X is this:

What is $P(X \leq \text{something})$?

Recall the CDF (Cummulative Distribution Function) of random variable X :

$$F_x(u) = P(X \leq u) \text{ for } -\infty < u < \infty$$

- Always use uppercase F for CDF.
- The X indicates which random variable
- Try not to use X as the argument. Any other variable is ok. e.g. u, v, w, a, b, c

Example: **Flip a fair coin 2 times.**

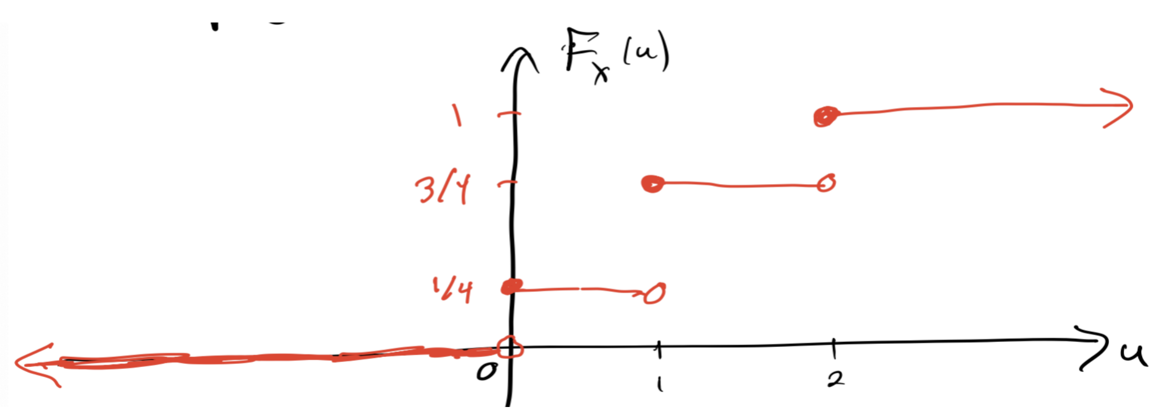
$$S = \{HH, HT, TH, TT\}$$

Define random variable X = number of heads $\in \{0, 1, 2\}$. Find CDF of X.

$$P(X = 0) = P(TT) = \frac{1}{4}$$

$$P(X = 1) = P(HT \text{ or } TH) = \frac{1}{2}$$

$$P(X = 2) = P(HH) = \frac{1}{4}$$



Cases:

$$u < 0 => P(X \le u) = 0$$

$$u = 0 => P(X \le u) = P(X \le 0) = P(X = 0) = \frac{1}{4}$$

$$0 \le u < 1 => P(X \le u) = P(X = 0) = \frac{1}{4}$$

$$1 \le u < 2 => P(X \le u) = P(X = 0 \text{ or } X = 1) = \frac{1}{4} + \frac{1}{2} = \frac{3}{4}$$

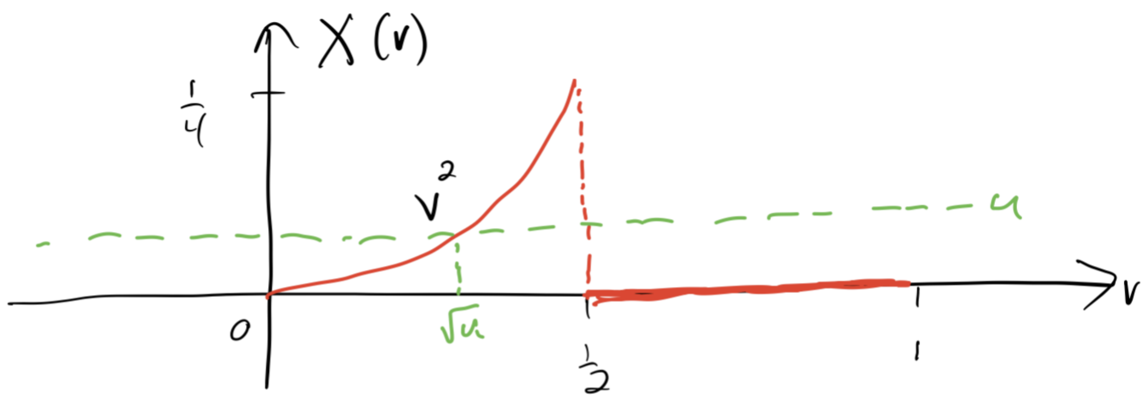
$$u \ge 2 => P(X \le u) = 1$$

Example: Suppose an experiment has sample space

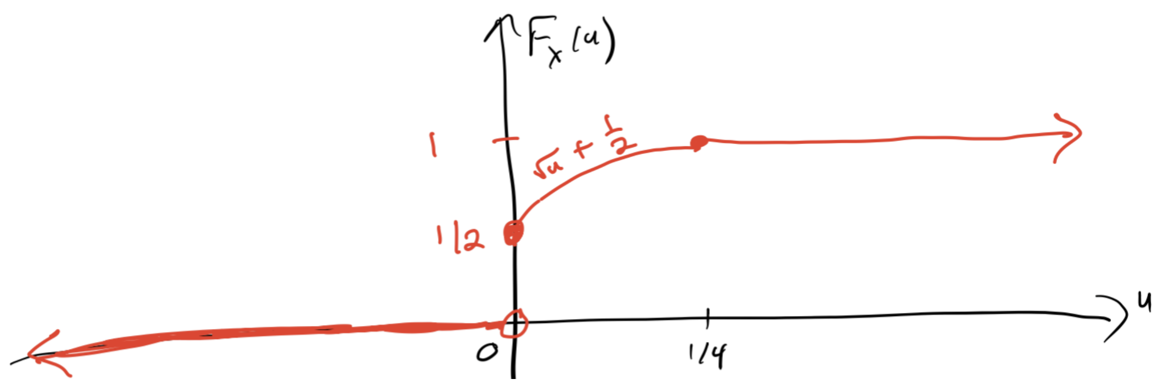
$$S = [0, 1]$$

Suppose $P([a, b]) = b - a$ whenever $0 \le a \le b \le 1$

Define a random variable X as in the following diagram:



What is the CDF of X? Need to compute $P(X \le u)$ for all u.



<u>Cases:</u>

$$
\begin{aligned}
u < 0 &: \text{It's never true that } X \le u. \text{ So } F_x(u) = P(X \le u) = 0 \\
u = 0 &: P(X \le 0) = P(X = 0) = P(\{0\} \cup [\frac{1}{2}, 1]) \\
&= P(\{0\}) + P([\frac{1}{2}, 1]) \\
&= P(\{[0,0]\}) + P([\frac{1}{2}, 1]) \\
&= 0 + \frac{1}{2} = \frac{1}{2} \\
u \ge \frac{1}{4} &: P(X \le u) = P([0,1]) = 1 \\
0 \le u < \frac{1}{4} &: P(X \le u) = P([0, \sqrt{u}] \cup [\frac{1}{2}, 1]) \\
&= \sqrt{u} + \frac{1}{2}
\end{aligned}
$$

Some properties of CDFs:

1. $0 \le F_x(u) \le 1$
2. If $a < b$, then $F(a) \le F(b)$. Since $P(X \le b) = P(X \le a) + P(a < X \le b)$
3. $F(\infty) = \lim_{u \to \infty} F(u) = 1$
4. $F(-\infty) = \lim_{u \to -\infty} F(u) = 0$
5. F(u) is right-continuous. If you approach u from the right, it's continuous, but not necessarily from the left.

Some computation facts:

$$
\begin{aligned}
P(X > u) &= P(\{X \le u\}^c) \\
&= 1 - P(X \le u) \\
&= 1 - F(u) \\
P(X < u) &= P(X \le u) - P(X = u) \\
&= F(u^-) \\
P(X = u) &= 1 - P(X < u) = 1 - F(u^-) \\
P(X = u) &= P(X \le u) - P(X < u) \\
&= F(u) - F(u^-) \\
P(a < X \le b) &= P(X \le b) - P(X \le a) \\
&= F(b) - F(a) \\
P(a < X < b) &= F(b^-) - F(a) \\
P(a \le X < b) &= F(b^-) - F(a^-) \\
P(a < X \le b) &= F(b^-) - F(a^-)
\end{aligned}
$$

If $F(u)$ is continuous at $u = a$, then there is no jump at a. Thus, $P(a < X \le b) = P(a \le X \le b)$ since $F(a) = F(a^+) = F(a^-)$.

$$\therefore P(X = a) = 0$$