Lec 07:

Random variables

Ex Sample space

$$S = \{H, TH, TTH, TTTH, \dots\}$$

Flip a fair coin until 1st Head appears, then stop.

Define rv X = # of tosses until 1st Head.

$$X(H) = 1$$
$$X(TH) = 2$$
$$X(TTH) = 3$$
$$\vdots$$

$$P(X=1) = P(H) = \frac{1}{2}$$
$$P(X=2) = P(TH) = \frac{1}{2^2}$$
$$\vdots$$
$$P(X=n) = P(\underbrace{T \cdots T}_{n-1} H) = \frac{1}{2^n}$$

Define a 2nd rv Y to indicate oddness.

$$Y = \begin{cases} 1 & \text{if } X \text{ is odd} \\ 0 & \text{if } X \text{ is even} \end{cases}$$

$Y(H) = 1$ odd

$Y(TH) = 0$ even

$Y(TTH) = 1$ odd

$Y(TTTH) = 0$ even

$\vdots$

What is $P(Y=0)$?

$$P(Y=0) = P(X \text{ is even})$$

$$= P(X=2) + P(X=4) + P(X=6) + \dots$$

$$= \frac{1}{2^2} + \frac{1}{2^4} + \frac{1}{2^6} + \dots$$

geometric series

$$= \frac{1/4}{1-1/4} \cdot \frac{4}{4} = \frac{1}{4-1} = \frac{1}{3}$$

$$P(Y=1) = 1 - P(Y=0) = 1 - \frac{1}{3} = \frac{2}{3}$$

Note: $\{Y=0\} = \{u \in S : Y(u) = 0\}$

$= \{TH, TTTH, TTTTTH, \dots\}$

Notation

Let $A \subseteq \mathbb{R}$ (ie a set of real numbers)

We will use the following notation:

$$\{X \in A\} = \{u \in S : X(u) \in A\}$$

events

In previous example we can write

$$\{Y = 0\} = \{Y \in \{0\}\}$$

$$\{X \leq 4\} = \{X \in (-\infty, 4]\}$$

$$\{-1 \leq X < 7\} = \{X \in [-1, 7)\}$$

$$\vdots$$

Ex: Flip biase coin 3 times.

$$P(H) = q$$

Define rv X = # Heads (ie 0, 1, 2, 3)

What is $P(X \leq 1)$?

$\{X \leq 1\}$ is the event $\{\underbrace{TTT}_{X=0}, \underbrace{HTT, THT, TTH}_{X=1}\}$

$$P(X \leq 1) = P(\qquad\qquad)$$

$$= (1-q)^3 + 3q(1-q)^2$$

TTT other 3

Fundamental question about a rv X is this:

what is $P(X \leq \text{something})$?

Recall the CDF of rv X:

$$F_X(u) = P(X \leq u)$$

for $-\infty < u < \infty$

Always use upper case "F" for CDF.

The "X" indicates which r.v.

Try not to use "x" as the argument. Any other variable is ok. eg u, v, w a, b, c

Ex Flip a fair coin 2 times.

$$S = \{HH, HT, TH, TT\}$$

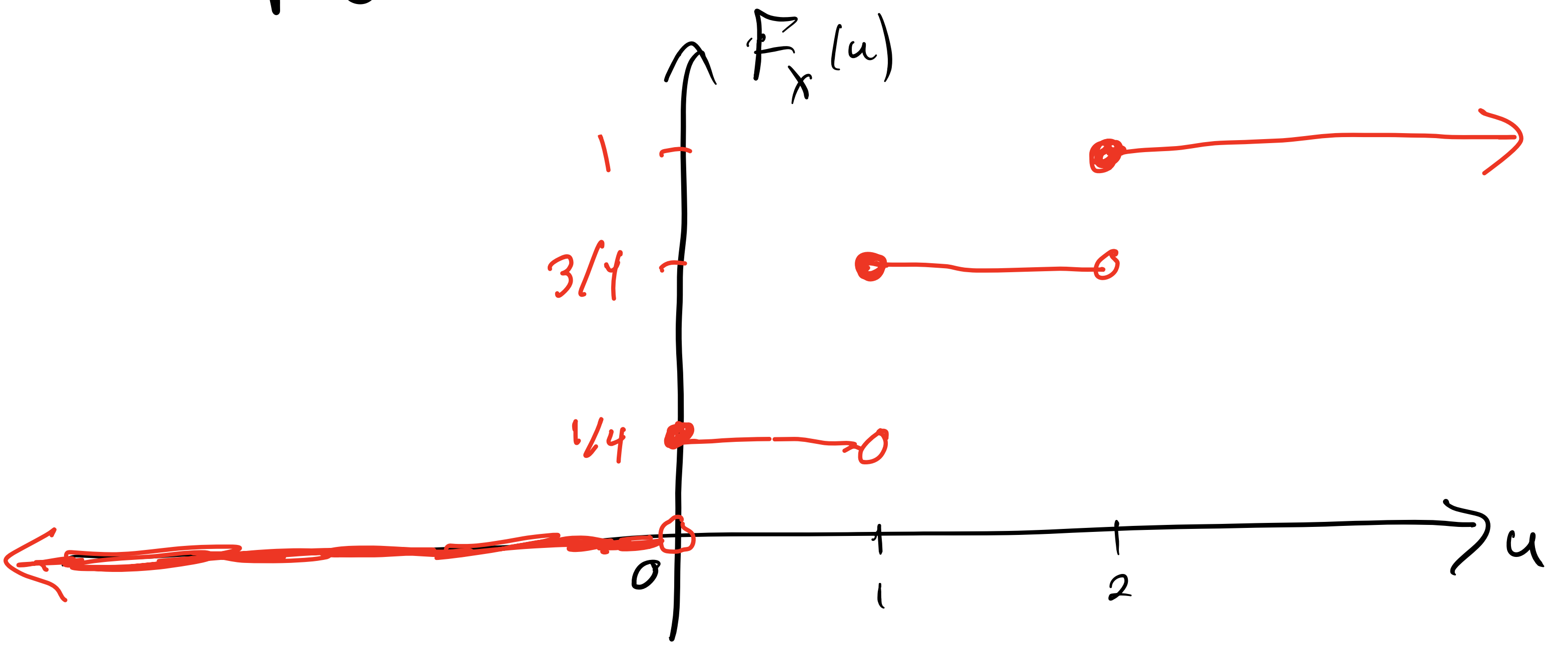
Define rv $X = \#\text{ Heads} \in \{0, 1, 2\}$

Find CDF of X

$P(X=0) = P(TT) = \frac{1}{4}$

$P(X=1) = P(HT \text{ or } TH) = \frac{1}{2}$

$P(X=2) = P(HH) = \frac{1}{4}$



<u>Cases</u> $u < 0 \Rightarrow P(X \le u) = 0$

$u = 0 \Rightarrow P(X \le u) = P(X \le 0) = P(X=0) = \frac{1}{4}$

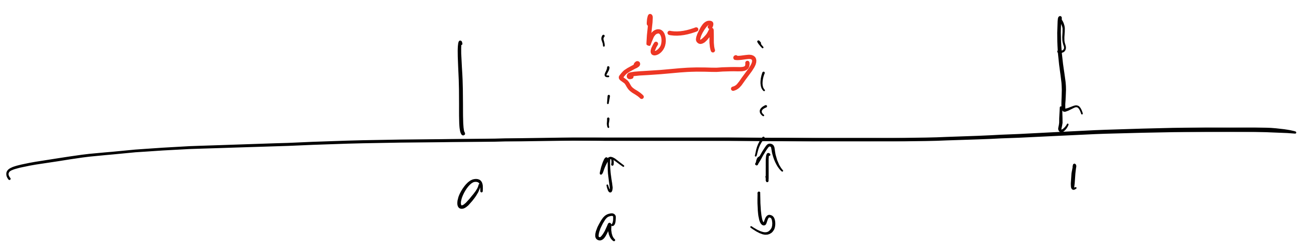
$0 \le u < 1 \Rightarrow P(X \le u) = P(X=0) = 1/4$

$1 \le u < 2 \Rightarrow P(X \le u) = P(X=0 \text{ or } X=1) = \frac{1}{4} + \frac{1}{2} = \frac{3}{4}$

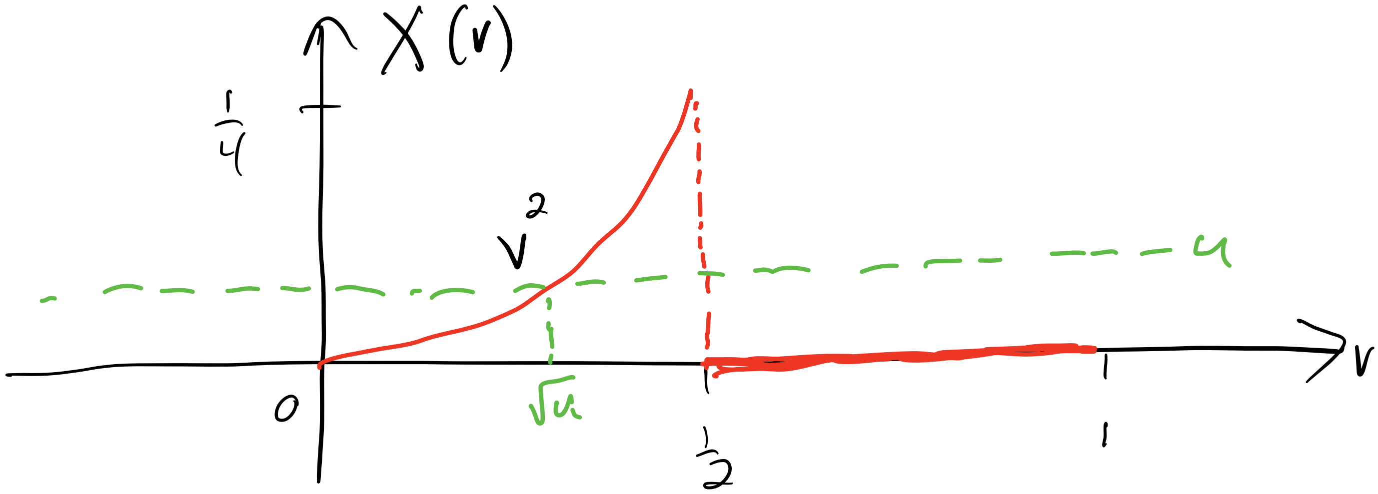
$u \geq 2 \Rightarrow P(X \le u) = 1$

<u>Ex</u>: Suppose an experiment has sample space $S = [0,1]$.

Suppose $P([a,b]) = b - a$ whenever $0 \le a \le b \le 1$.

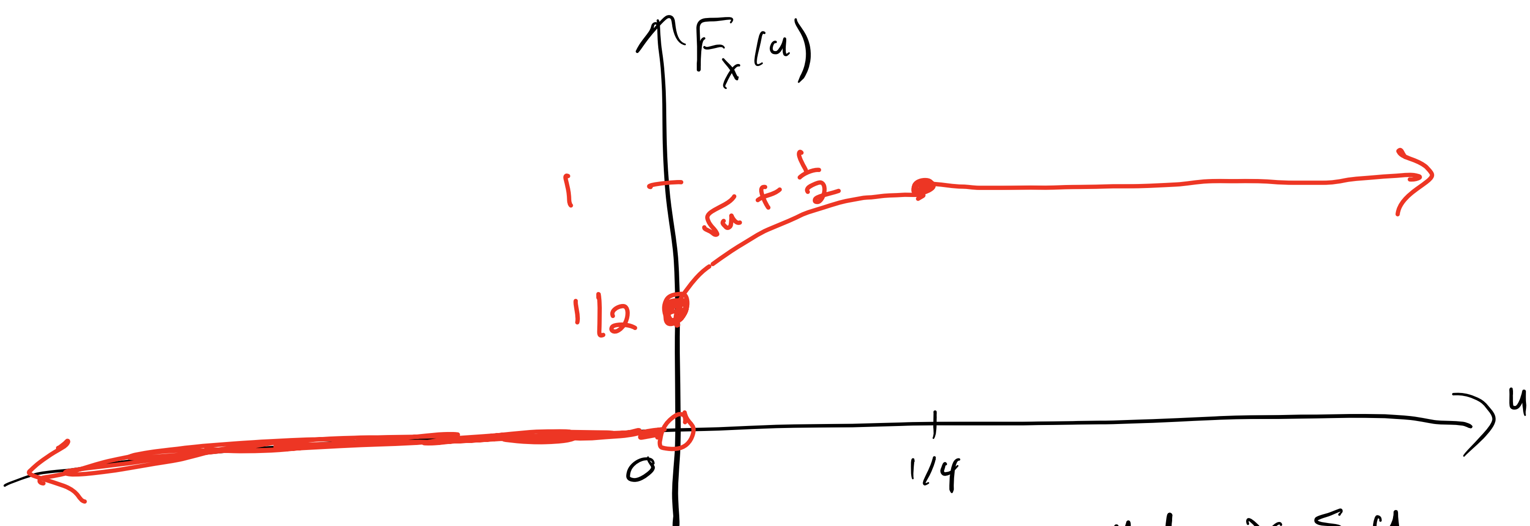


Def a r.v. X as in the following diagram:



What is the CDF of X?

Need to compute $P(X \le u)$ for all u.



<u>Cases</u>: $u < 0$: It's never true that $X \le u$.

so $F_X(u) = P(X \le u) = 0$

$u = 0$: $P(X \le 0) = P(X = 0) = P(\{0\} \cup [\frac{1}{2}, 1])$

$= P(\{0\}) + P([\frac{1}{2}, 1])$

$= P([0,0]) + P([\frac{1}{2}, 1])$

$= 0 + \frac{1}{2} = \frac{1}{2}$

$u \geq \frac{1}{4}$: $P(X \le u) = P([0,1]) = 1$

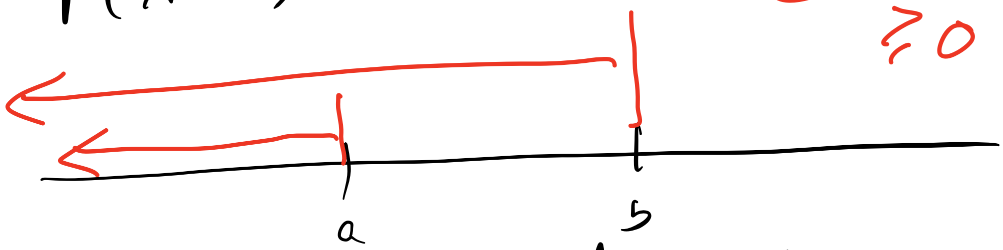
$0 \le u < \frac{1}{4}$: $P(X \le u) = P([0, \sqrt{u}] \cup [\frac{1}{2}, 1])$

$= \sqrt{u} \qquad + \frac{1}{2}$

Some Properties of CDFs:

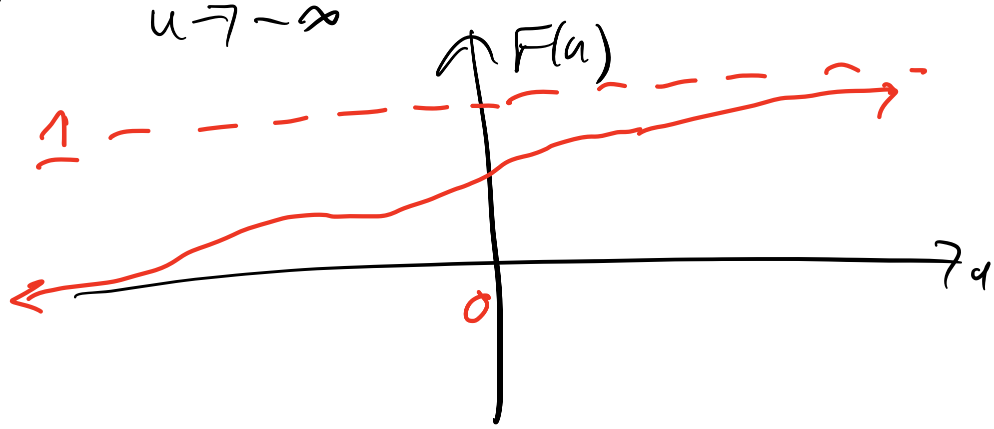
1. $0 \le F_X(u) \le 1$
2. If $a < b$, then $F(a) \le F(b)$

Since $P(X \le b) = P(X \le a) + \underbrace{P(a < X \le b)}_{\ge 0}$



i.e. F is monotone nondecreasing

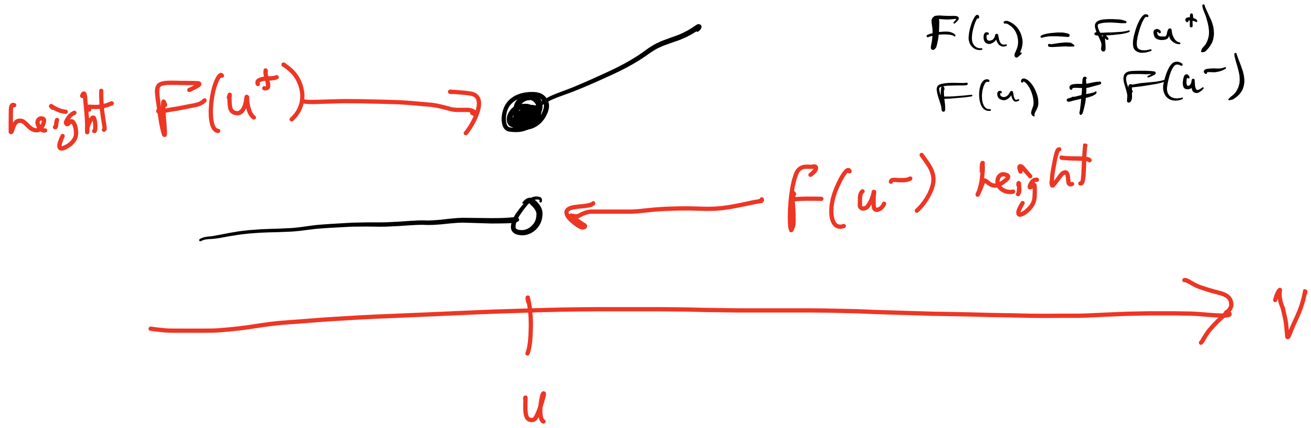
3. $F(\infty) = \lim_{u \to \infty} F(u) = 1$
4. $F(-\infty) = \lim_{u \to -\infty} F(u) = 0$



5. $F(u)$ is <u>right-continuous</u>.

If you approach u from the right, it's continuous, but not necessarily from the left.

$F(u) = F(u^+)$
$F(u) \neq F(u^-)$



Some computation facts

- $P(X > u) = P(\{X \le u\}^c)$
  $= 1 - P(X \le u)$
  $= 1 - F(u)$
- $P(X < u) = P(X \le u) - P(X = u)$
  $= F(u^-)$
- $P(X \geq u) = 1 - P(X < u) = 1 - F(u^-)$
- $P(X = u) = P(X \le u) - P(X < u)$
  $= \underbrace{F(u)}_{\text{lim on right}} - \underbrace{F(u^-)}_{\text{lim on left}}$
  $=$ "jump at u"
- $P(a < X \le b) = P(X \le b) - P(X \le a)$
  $= F(b) - F(a)$

- $P(a < X < b) = F(b^-) - F(a)$
- $P(a \le X < b) = F(b^-) - F(a^-)$
- $P(a \le X \le b) = F(b) - F(a^-)$

If $F(u)$ is continuous at $u = a$, then there is no jump at a.

Thus, $P(a < X \le b) = P(a \le X \le b)$

since $F(a) = F(a^+) = F(a^-)$

$\therefore P(X = a) = 0$ i.e no jump.