Lec 12:

Quiz next week on Tues 6:30pm
(Mon - holiday)

Functions of a r.v.

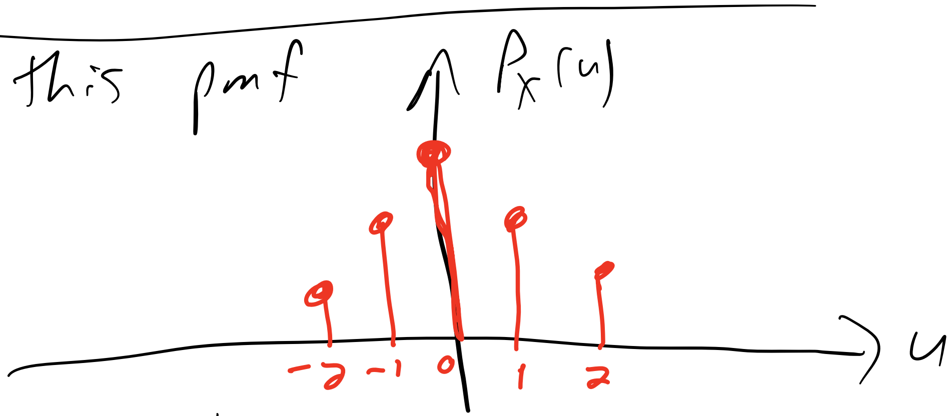
Ex: Let X be a discrete r.v. with pmf $P_X(u)$. Let $Y = aX + b$

a, b constants
$a \neq 0$

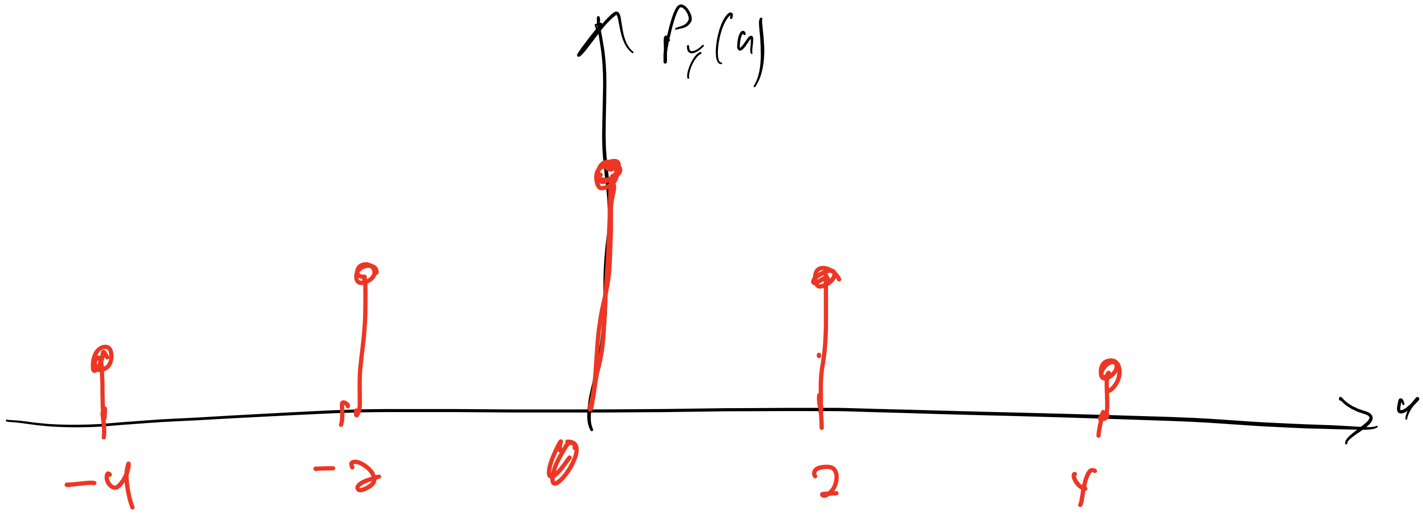
Find pmf of Y.

$$P_Y(u) = P(Y = u) = P(aX + b = u)$$
$$= P\left(X = \frac{u-b}{a}\right) = P_X\left(\frac{u-b}{a}\right)$$

Suppose X has this pmf

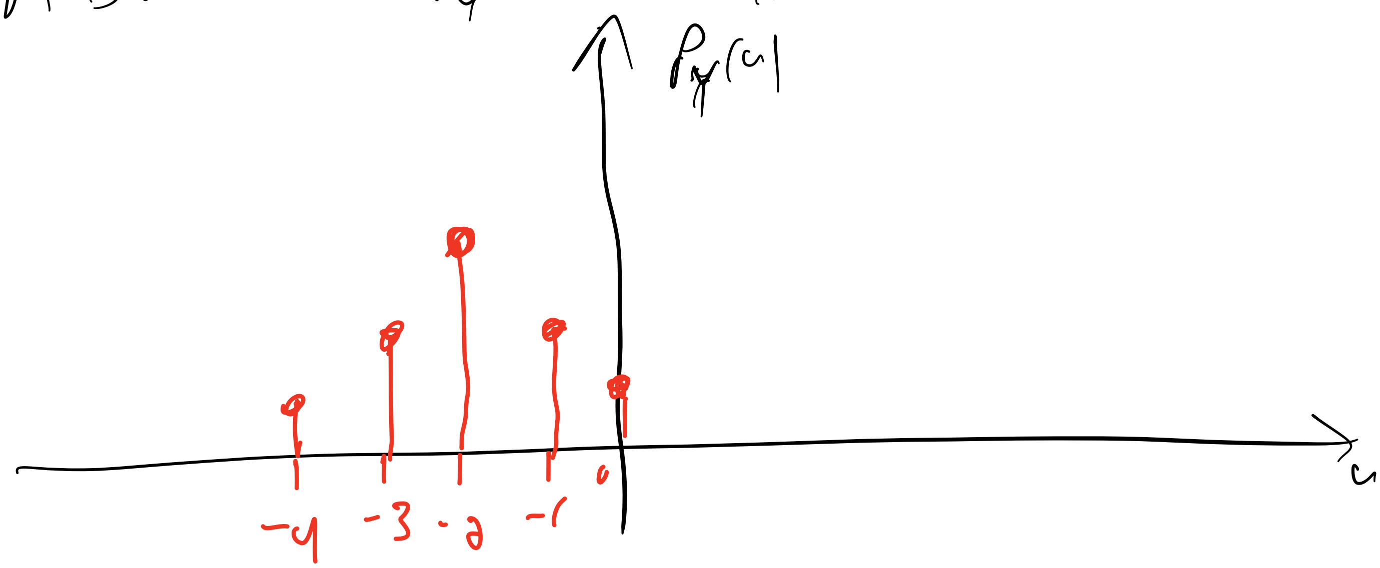


Suppose $Y = 2X$
(i.e. $a = 2$, $b = 0$)
The answer is $P_Y(u) = P_X\left(\frac{u}{2}\right)$



Now suppose $Y = X - 2$ (i.e. $a = 1$, $b = -2$)

Answer is $P_Y(u) = P_X(u+2)$



Continuous version of the last example...

X has pdf $f_X(u)$

$Y = aX + b$ $(a > 0)$

Find pdf of Y in terms of pdf of X.

General idea:

1. Set up the CDF of Y
2. Differentiate

The CDF of Y is:

$$F_Y(u) = P(Y \leq u)$$
$$= P(aX + b \leq u)$$
$$= P\left(X \leq \frac{u-b}{a}\right)$$
$$= F_X\left(\frac{u-b}{a}\right)$$

The density of Y is:

$$f_Y(u) = \frac{d}{du} F_Y(u) = \frac{d}{du} F_X\left(\frac{u-b}{a}\right)$$

(use chain rule)

$$= f_X\left(\frac{u-b}{a}\right) \cdot \frac{d}{du}\left(\frac{u-b}{a}\right)$$

$$= \frac{1}{a} \cdot f_X\left(\frac{u-b}{a}\right) \quad \checkmark$$

Alternatively, we could write

$$F_Y(u) = \int_{-\infty}^{\frac{u-b}{a}} f_X(z)\, dz$$

The pdf of Y is:

$$f_Y(u) = \frac{d}{du} \int_{-\infty}^{\frac{u-b}{a}} f_X(z)\, dz$$

Leibniz Rule

$$\frac{d}{du} \int_{h(u)}^{g(u)} f(z,u)\, dz$$

$$= f(g(u), u) \cdot g'(u)$$ 1st term

$$- f(h(u), u) \cdot h'(u)$$ 2nd term

$$+ \int_{h(u)}^{g(u)} \left(\frac{\partial}{\partial u} f(z,u) \right) dz$$ 3rd term

For the pdf of Y that we set up we have

$$f_Y(u) = \frac{d}{du} \int_{-\infty}^{\frac{u-b}{a}} f_X(z)\, dz$$

(now use Leibniz)

$$\underbrace{f_X\left(\frac{u-b}{a}\right) \cdot \frac{d}{du}\left(\frac{u-b}{a}\right)}_{\text{1st term}} - \underbrace{0}_{\text{2nd term}} + \underbrace{0}_{\text{3rd term}}$$

$$= \frac{1}{a} f_X\left(\frac{u-b}{a}\right) \quad = \quad$$ same answer as before using Chain Rule.

Special Case $\quad X \sim N(m, \sigma^2)$ Gaussian

$$Y = aX + b \qquad a > 0$$

We know density of X is:

$$f_X(u) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{u-m}{\sigma}\right)^2}$$

what is pdf of Y?

$$f_Y(u) = \frac{1}{a} f_X\left(\frac{u-b}{a}\right)$$

$$= \frac{1}{a\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{\left(\frac{u-b}{a}\right)-m}{\sigma}\right)^2}$$

$$= \frac{1}{(a\sigma)\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{u-b-am}{a\sigma}\right)^2}$$

$$= \frac{1}{(a\sigma)\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{u-(b+am)}{(a\sigma)}\right)^2}$$

$$\therefore \quad Y \sim N(b+am,\ a^2\sigma^2)$$

i.e. Y is also Gaussian.

But $E[Y] = b + am$

$$\sigma_Y^2 = a^2\sigma^2$$

- General fact about variance

Let $Y = aX + b$

Find variance of Y in terms of the variance of X.

$$Var(Y) = E[Y^2] - (E[Y])^2$$

Know: $E[Y] = E[aX+b] = aE[X] + b$

$$E[Y^2] = E[(aX+b)^2] = E[a^2X^2 + 2abX + b^2]$$

$$= a^2E[X^2] + 2abE[X] + b^2$$

$$\therefore Var(Y) = a^2E[X^2] + 2abE[X] + b^2 - (aE[X] + b)^2$$

$$= a^2(E[X^2] - (E[X])^2)$$

$$= a^2 \cdot Var(X)$$

In other notation,

$$\boxed{\sigma_Y^2 = a^2 \sigma_X^2}$$

Ex: Let X has pdf $f_X(u)$.
Let $Y = X^4$. Find pdf of Y.

Soln: Set up the CDF of Y, differentiate.

$$F_Y(u) = P(Y \le u) = P(X^4 \le u)$$
$$= P(-u^{\frac{1}{4}} \le X \le u^{\frac{1}{4}})$$
$$= \int_{-u^{1/4}}^{u^{1/4}} f_X(z)\,dz$$

If $u<0$, then clearly $F_Y(u)=0$ so take $u>0$.

The pdf of Y is:

$$f_Y(u) = \frac{d}{du}F_Y(u) = \frac{d}{du}\int_{-u^{1/4}}^{u^{1/4}} f_X(z)\,dz$$

(use Leibniz: not 3rd term is zero)

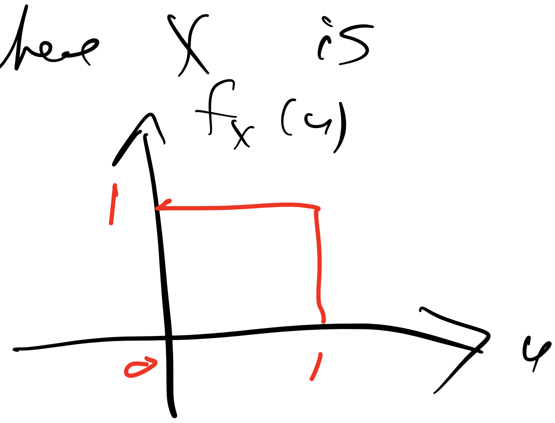
$$= \underbrace{f_X(u^{\frac{1}{4}})\frac{d}{du}(u^{\frac{1}{4}})}_{\text{1st term}} - \underbrace{f_X(-u^{\frac{1}{4}})\frac{d}{du}(-u^{\frac{1}{4}})}_{\text{2nd term}}$$

$$+ \underbrace{0}_{\text{3rd term}}$$

$$f_X(u^{\frac{1}{4}})\left(\tfrac{1}{4}u^{-3/4}\right) - f_X(-u^{\frac{1}{4}})\left(-\tfrac{1}{4}u^{-3/4}\right)$$

$$= \frac{1}{4} u^{-3/4} \left(f_X(u^{\frac{1}{4}}) + f_X(-u^{1/4}) \right) \quad \text{for } u > 0$$

Look at special case where X is uniform on $[0,1]$.



Take $u \geq 0$.

Then $f_X(-u^{\frac{1}{4}}) = 0$ since $-u^{\frac{1}{4}} \leq 0$.

Then $f_Y(u) = \frac{1}{4} u^{-3/4} f_X(u^{1/4})$

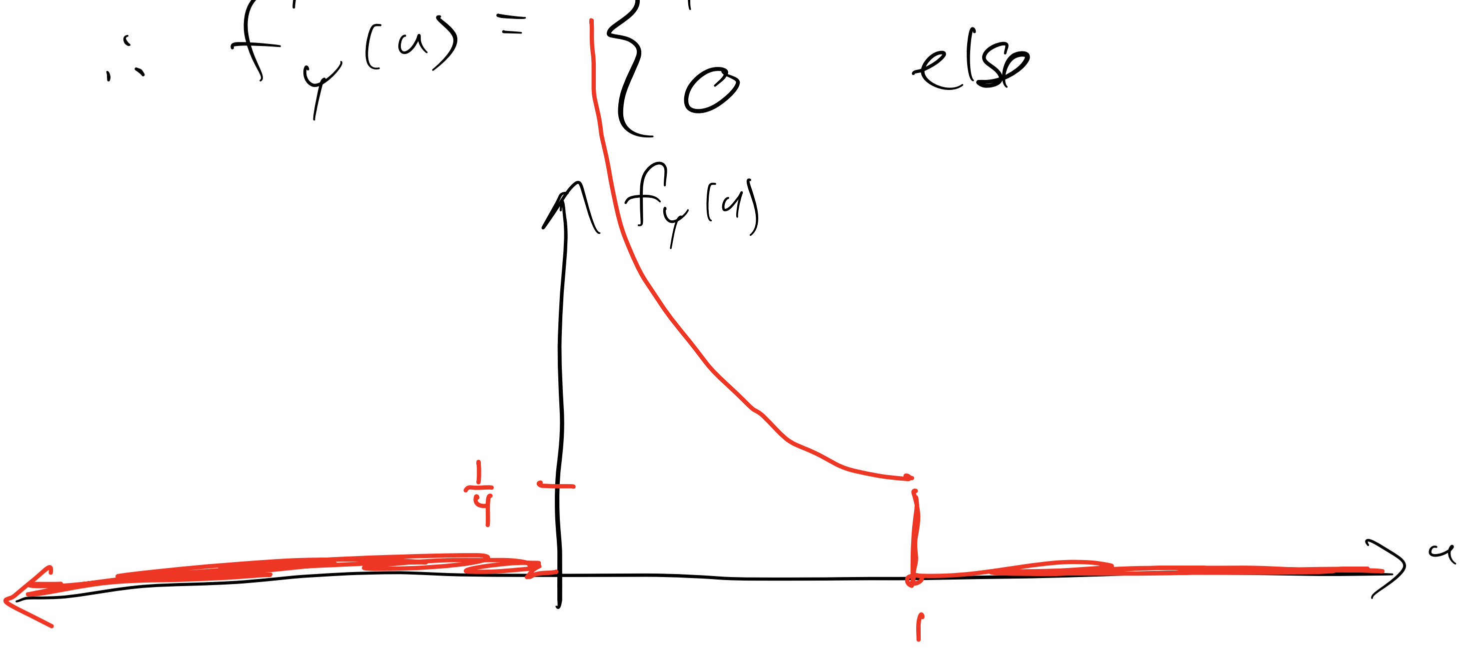
If $u > 1$, then $u^{1/4} > 1$, $\Rightarrow f_X(u^{1/4}) = 0$

$\therefore f_Y(u) = 0$ whenever $u \notin [0,1]$

Assume now $u \in [0,1]$

In this case, $f_X(u^{\frac{1}{4}}) = 1$

$$\therefore f_Y(u) = \begin{cases} \frac{1}{4} u^{-3/4} & \text{if } u \in [0,1] \\ 0 & \text{else} \end{cases}$$



New topic.
Joint statistics of 2 or more r.v.s.

<u>Def</u>: The <u>joint CDF</u> of r.v.s X, Y is:

$$F_{X,Y}(u,v) = P(X \leq u, Y \leq v)$$

Can generalize to joint CDF of n rvs: $X_1, \ldots, X_n$ as follows:

$$F_{X_1, X_2, \ldots, X_n}(u_1, u_2, \ldots, u_n) =$$
$$P(X_1 \leq u_1, X_2 \leq u_2, \ldots, X_n \leq u_n)$$