Lec 02:

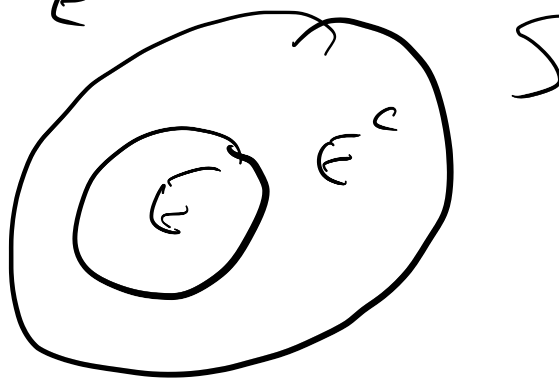
Reminder - Quiz #1 is next Tues at 6:30 pm - 6:45 pm (make sure camera works).

Last time - experiment
outcome
sample space
event

Set theory (review)

E event

Fact: $(E^c)^c = E$



DeMorgan's Law

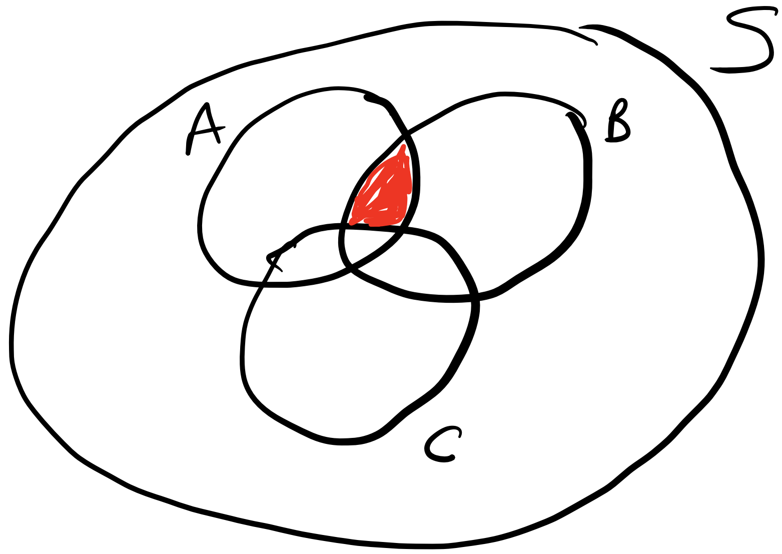
Events E, F

$$(E \cup F)^c = E^c F^c$$

$$(EF)^c = E^c \cup F^c$$

This works for any number of sets

<u>Ex</u>: ABC^c



This is the event that:

"A and B and C^c occur"

or equivalently

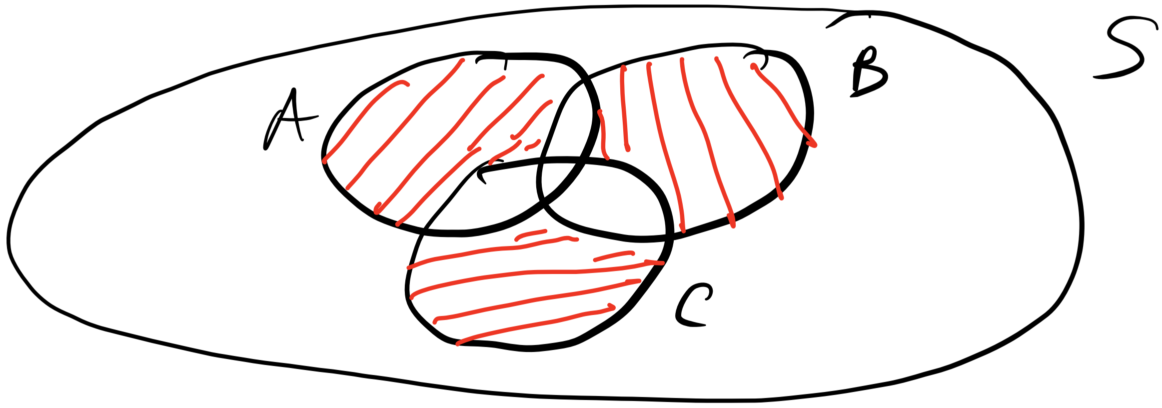
"A and B occur, <u>but</u> not C"

<u>Note</u>: The word "but" almost always means logically "and".

<u>Note</u>: In ECE109, when we use "or" we always mean inclusive, unless stated otherwise.

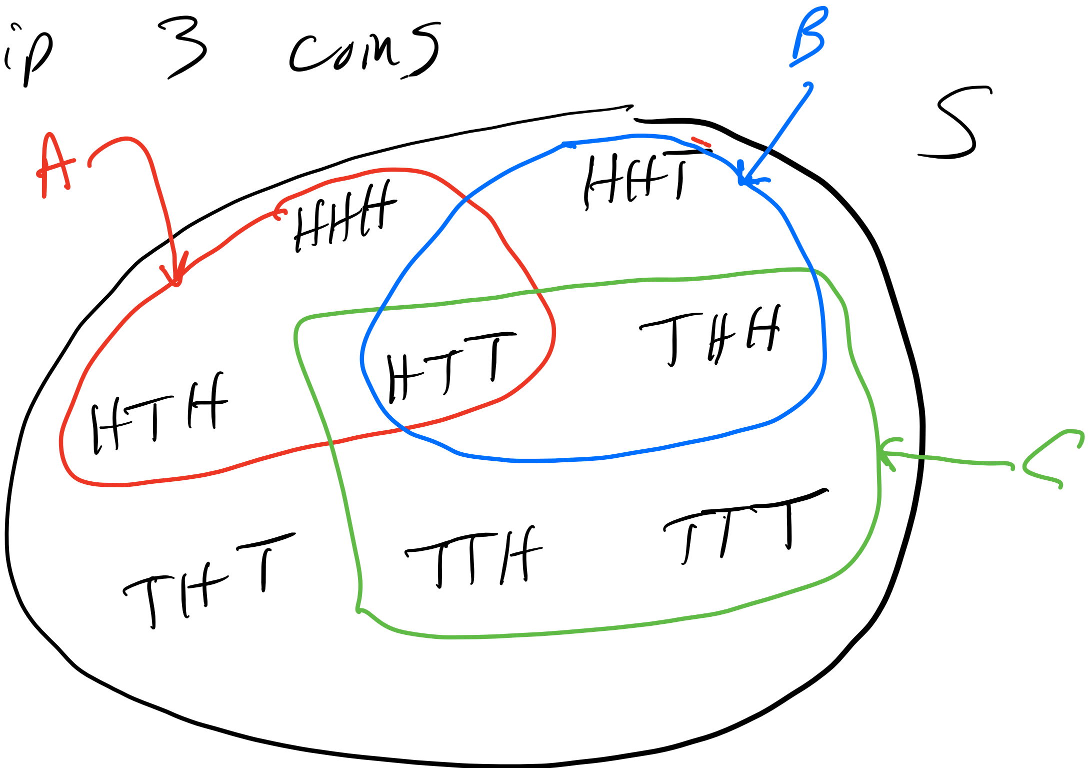
<u>Ex</u>: "Exactly one of A, B, C occurs"

$$AB^cC^c \cup A^cBC^c \cup A^cB^cC$$



Ex: Flip 3 coins

3 events A, B, C



Exactly one of A, B, C occurs is the event

$= \{HHH, HTH, HHT, TTT, TTH\}$

Def: **Probability** is an assignment of a number to an event.

ie. for each event $E \subseteq S$, $P(E) \in \mathbb{R}$

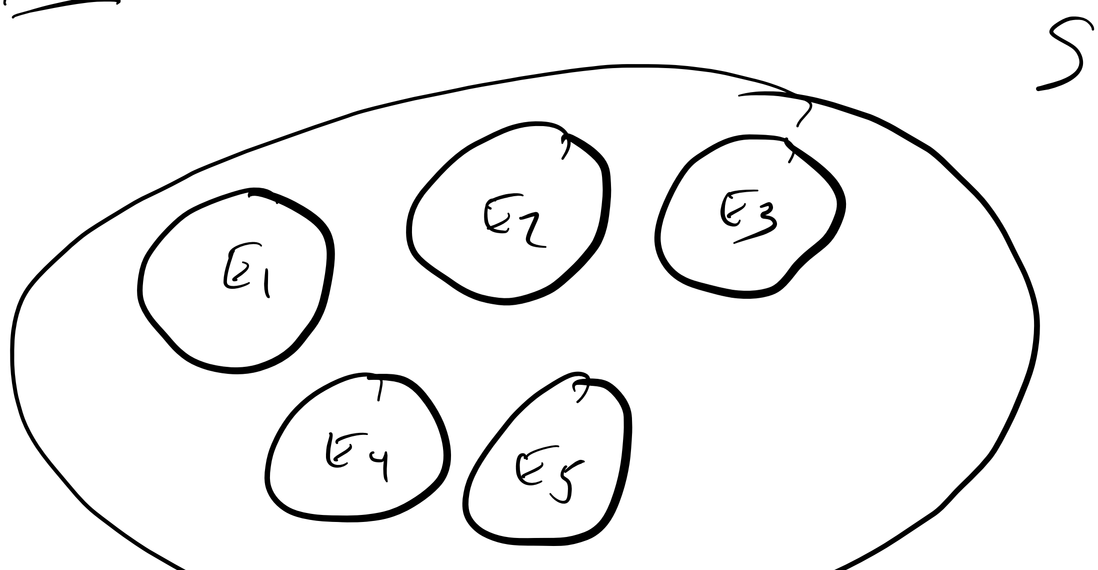
(ie set of reals ↑)

Probability must satisfy 3 axioms:

1. $0 \leq P(E) \leq 1$
2. $P(S) = 1$
3. If $E_1, E_2, E_3, \ldots$ are events that are pairwise disjoint (ie. $E_iE_j = \phi$ whenever $i \neq j$), then

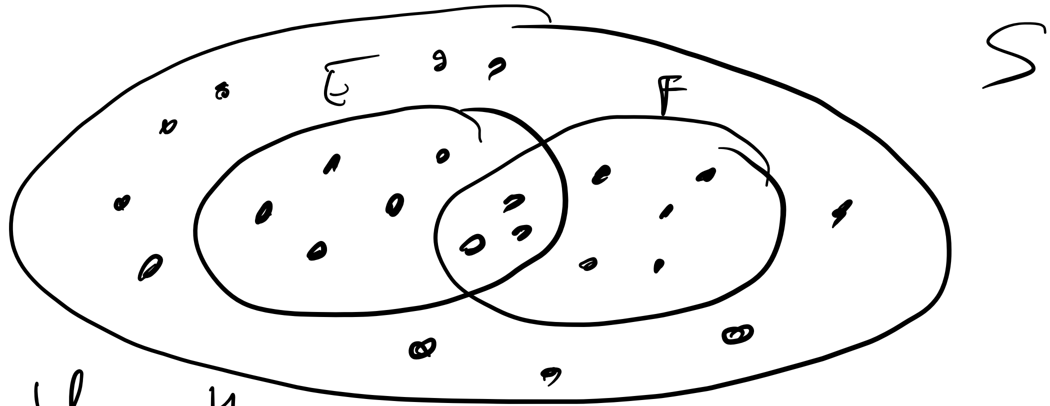
$$P(E_1 \cup E_2 \cup E_3 \cup \ldots) = P(E_1) + P(E_2) + \cdots$$

<u>Note</u>: may be an ∞ sum.



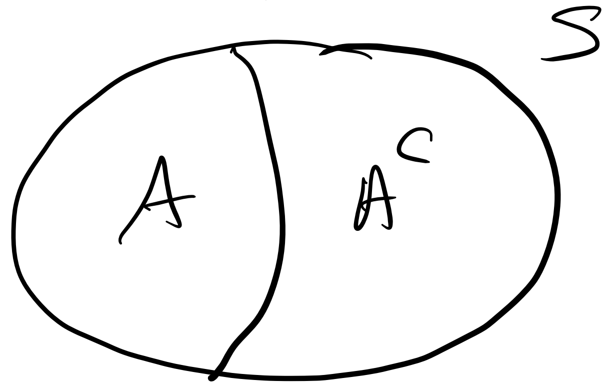
If E, F are <u>not</u> disjoint, then we <u>can</u>'t say generally that

$$P(E \cup F) = P(E) + P(F)$$



would otherwise get double counting of prob in EF.

Fact: $P(A^c) = 1 - P(A)$



Proof:

$S = A \cup A^c$

and A and A^c are disjoint.

i.e. $AA^c = \phi$.

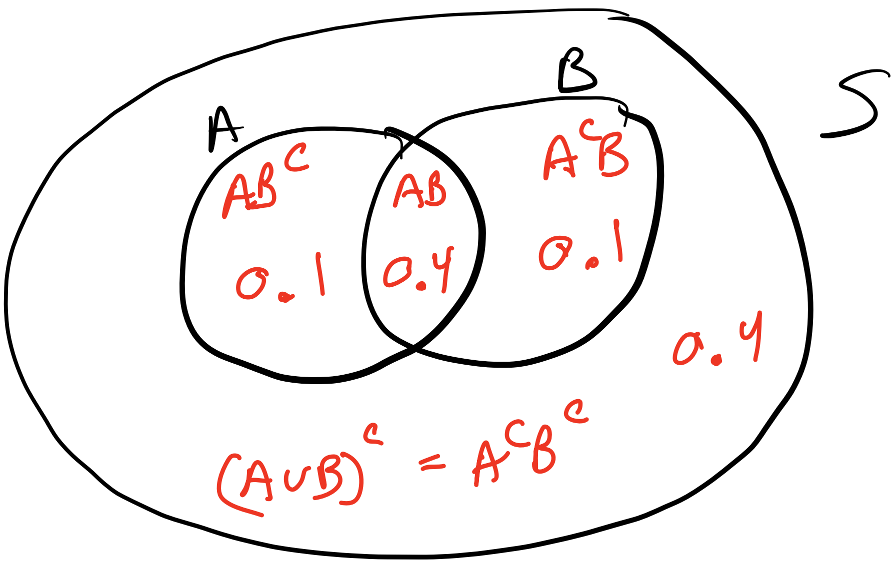
$$1 = P(S) = P(A \cup A^c)$$
$$= P(A) + P(A^c)$$

$$\therefore \quad P(A^c) = 1 - P(A)$$

Ex: Given events A, B such that:

$$P(AB) = 0.4$$
$$P(AB^c) = 0.1$$
$$P(A \cup B) = 0.6$$

What else do we know?



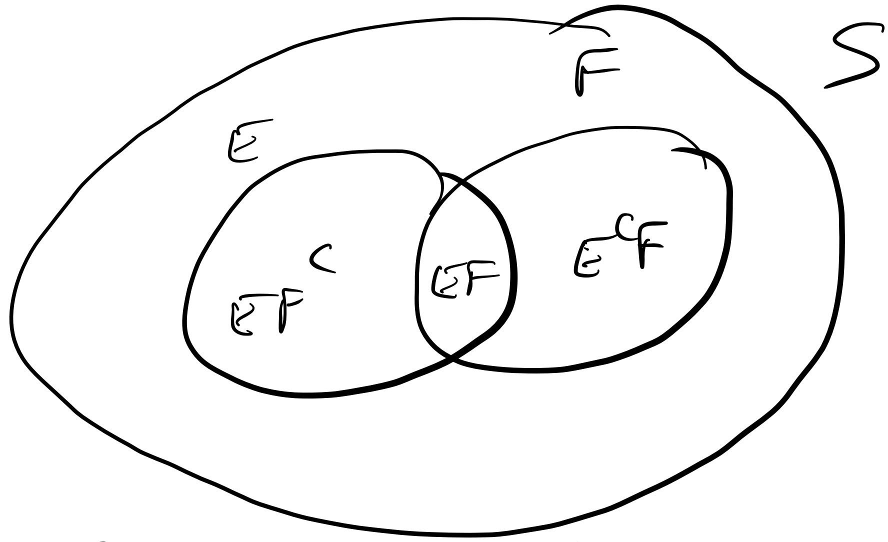
<u>Fact</u>: $P(\phi) = 0$

<u>Proof</u>: $\phi = S^c$

$$P(\phi) = P(S^c) = 1 - P(S) = 1 - 1 = 0$$

<u>Fact</u>: $P(E \cup F) = P(E) + P(F) - P(EF)$

Corrects for double counting



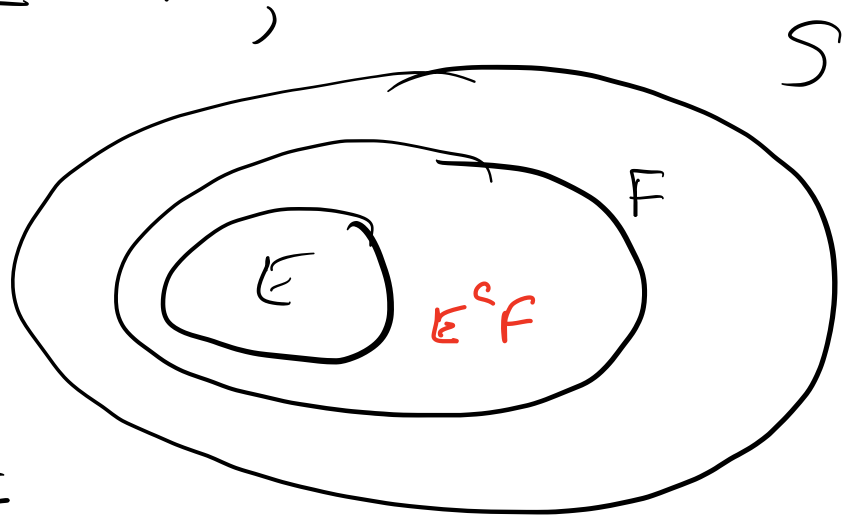
$E \cup F = EF^c \cup EF \cup E^cF$ disjoint unions

$E = EF^c \cup EF$

$F = E^cF \cup EF$

Add these probs...

Fact: If $E \subseteq F$, then $P(E) \leq P(F)$.



$F = E \cup E^cF$

disjoint union

$$P(F) = P(E \cup E^cF)$$
$$= P(E) + P(E^cF)$$
$$\geq P(E)$$

Consequence: Since $AB \subseteq A$
$AB \subseteq B$

$$\therefore \; P(AB) \leq P(A)$$
$$P(AB) \leq P(B)$$

Special Situation - Sometimes every outcome in a sample space has the same probability.

We call this "equiprobable outcomes".

If we have equally likely outcomes, then for any event E,

$$P(E) = \frac{|E|}{|S|} = \frac{\text{size of } E}{\text{size of } S}$$

(assumes S is finite)

Ex: Pick card randomly from a standard deck.

$$|S| = 52$$

Let E = "the card is red"

$|E| = 26$

$$P(E) = \frac{|E|}{|S|} = \frac{26}{52} = \frac{1}{2}$$

Let F = "card is an Ace"

$|F| = 4$

$$P(F) = \frac{|F|}{|S|} = \frac{4}{52} = \frac{1}{13}$$

Combinatorics studies counting set sizes.
We use permutations + combinations

<u>Permutations</u> are orderings of a set.

ex: $\{1, 2, 3, \ldots, n\}$

There are $n!$ diffrent orderings.

$$n! = n(n-1)(n-2) \cdots 1$$

$$n! = n(n-1)!$$

$$\boxed{0! = 1} \qquad 1! = 1$$

Ex: $n=3$

6 permutations

$3! = 3 \cdot 2 \cdot 1$
$= 6$

123
132
213
231
312
321

Combinations

$$\binom{n}{k} = \text{"}n \text{ choose } k\text{"}$$

$$= \frac{n!}{k!\,(n-k)!}$$

= the number of subsets of size k from a set of size n.

Ex: How many triples of letters can we pick from A, B, C, D, E? (order doesn't matter).

Answer: $\binom{5}{3} = \frac{5!}{3!\,(5-3)!} = \frac{5!}{3!\cdot 2!} = \frac{5\cdot 4}{2!} = 10$

ABC ACD BCD CDE

ABD ACE BCE

ABE ADE BDE

10 subsets of size 3

Recall from high school algebra:

Binomial Theorem

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

$$(x+y)^1 = x+y$$

$$(x+y)^2 = \underbrace{x^2}_{\binom{2}{0}} + \underbrace{2xy}_{\binom{2}{1}} + \underbrace{y^2}_{\binom{2}{2}}$$

$$(x+y)^3 = \underbrace{x^3}_{\binom{3}{0}} + \underbrace{3x^2y}_{\binom{3}{1}} + \underbrace{3xy^2}_{\binom{3}{2}} + \underbrace{y^3}_{\binom{3}{3}}$$

⋮

Use Pascal's triangle to get the coefficients

$$
\begin{array}{lccccccccc}
 & & & & & 1 & & & & \\
n=1 & & & & 1 & & 1 & & & \\
n=2 & & & 1 & & 2 & & 1 & & \\
n=3 & & 1 & & 3 & & 3 & & 1 & \\
n=4 & 1 & & 4 & & 6 & & 4 & & 1 \\
 & & \vdots & & & \vdots & & \vdots & &
\end{array}
$$