Lec 17:

2 functions of 2 rvs.

$$Z = g(X,Y)$$
$$W = h(X,Y)$$

Find joint pdf $f_{Z,W}(u,v)$ in terms of the joint pdf $f_{X,Y}(u,v)$.

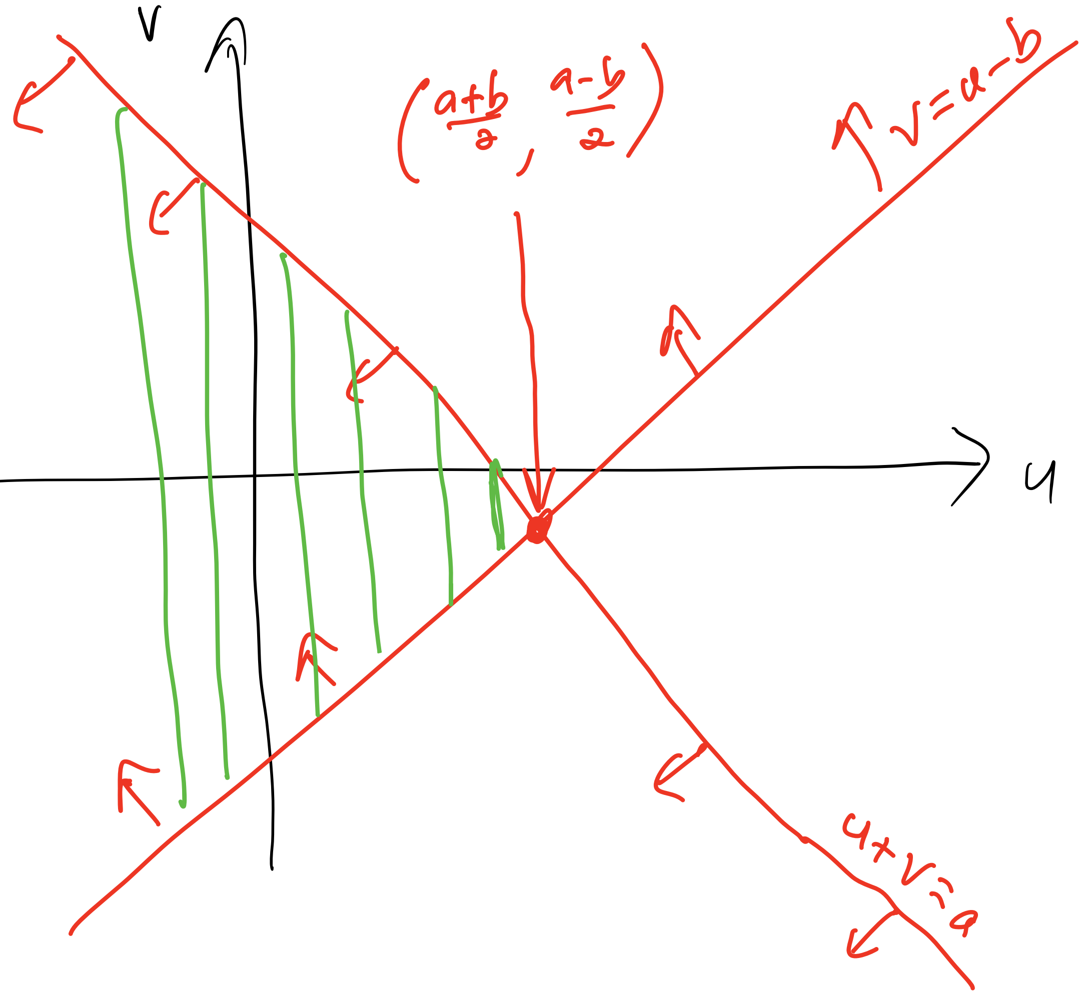
Ex: $Z = X+Y$
$W = X-Y$

Set up joint CDF of $Z, W$.
Then differentiate twice.

$$F_{Z,W}(a,b) = P(Z \le a, W \le b)$$
$$= P(X+Y \le a, X-Y \le b)$$
$$= P(Y \le -X+a, Y \ge X-b)$$
$$= P((X,Y) \in T)$$

where $T = \{(u,v) : v \le -u+a, v \ge u-b\}$

T $\Longrightarrow$ integrate in here




$$= \iint f_{X,Y}(u,v)\,dv\,du$$

$$= \int_{-\infty}^{\frac{a+b}{2}} \int_{u-b}^{a-u} f_{X,Y}(u,v)\,dv\,du$$

want $\dfrac{\partial^2 F_{Z,W}(a,b)}{\partial a \partial b} = f_{Z,W}(a,b)$

$$\frac{\partial}{\partial a} F_{Z,W}(a,b) = \frac{d}{da}\Big( \qquad \Big)$$

$$= \frac{d}{da} \int_{-\infty}^{\frac{a+b}{2}} \left( \int_{u-b}^{a-u} f_{X,Y}(u,v)\,dv \right) du$$

Intersection of the 2 lines:

$u-b = a-u$

$2u = a+b$

$u = \frac{a+b}{2}$

$v = \frac{a-b}{2}$

(use Leibniz – 2nd term is zero)

$$= \underbrace{\left( \int_{\left(\frac{a+b}{2}\right)-b}^{a-\left(\frac{a+b}{2}\right)} f_{X,Y}\left(\frac{a+b}{2}, v\right) dv \right) \cdot \underbrace{\frac{\partial}{\partial a}\left(\frac{a+b}{2}\right)}_{1/2}}_{\text{1st term of Leibniz}}$$

$$+ \underbrace{\int_{-\infty}^{\frac{a+b}{2}} \left( \frac{d}{da} \int_{a-b}^{a-u} f_{X,Y}(u,v)\, dv \right) du}_{\text{3rd term of Leibniz}}$$

Note: In 1st term, upper limit is

$$a - \left(\frac{a+b}{2}\right) = \frac{a-b}{2}$$

Lower limit is $\left(\frac{a+b}{2}\right) - b = \frac{a-b}{2}$

$\therefore$ This integral is zero!

$$= \int_{-\infty}^{\frac{a+b}{2}} \left( \frac{d}{da} \int_{u-b}^{a-u} f_{X,Y}(u,v)\, dv \right) du$$

(use Leibniz, 2nd + 3rd terms are zero)

$$= \int_{-\infty}^{\frac{a+b}{2}} f_{X,Y}(u, a-u)\, du$$

we just computed

$\therefore$ joint density is

$$f_{Z,W}(a,b) = \frac{\partial^2 F_{Z,W}(a,b)}{\partial a\, \partial b} = \frac{\partial}{\partial b}\left( \frac{\partial F_{Z,W}(a,b)}{\partial a} \right)$$

$$= \frac{d}{db}\int_{-\infty}^{\frac{a+b}{2}} f_{X,Y}(u, a-u)\,du$$

(use Leibniz, 2nd + 3rd terms zero)

$$= f_{X,Y}\left(\frac{a+b}{2}, a-\frac{a+b}{2}\right)\cdot\frac{\partial}{\partial b}\left(\frac{a+b}{2}\right)$$

$$= \boxed{\frac{1}{2} f_{X,Y}\left(\frac{a+b}{2}, \frac{a-b}{2}\right)}$$

<u>Special Case</u>:

$X, Y$ are iid uniform on $[0,1]$

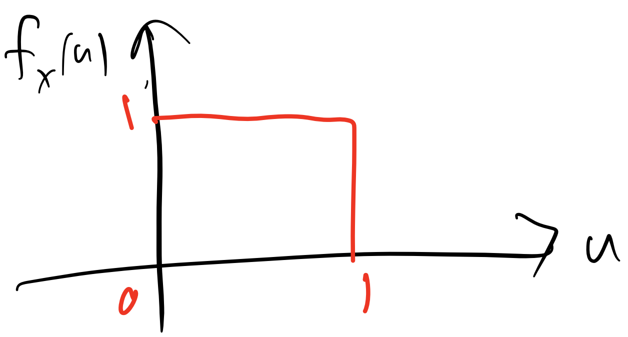
$Z = X+Y$

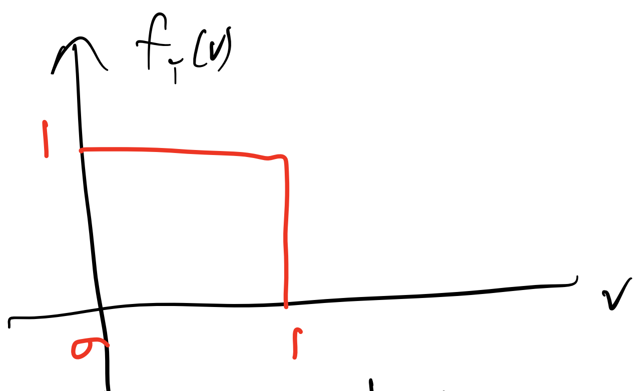
$W = X-Y$

From above we get:

$$f_{Z,W}(a,b) = \frac{1}{2} f_{X,Y}\left(\frac{a+b}{2}, \frac{a-b}{2}\right)$$

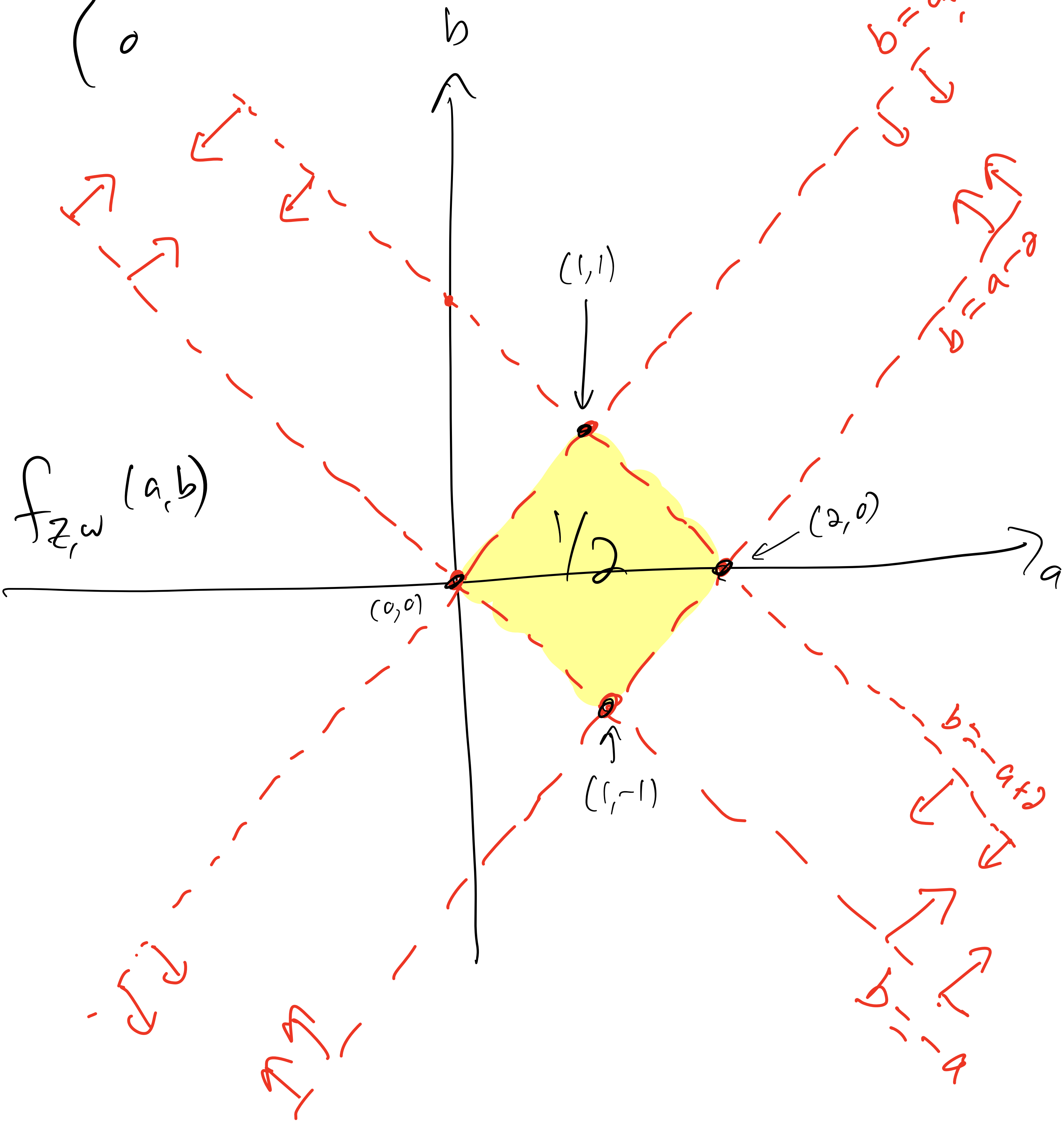
$$= \frac{1}{2} f_X\left(\frac{a+b}{2}\right)\cdot f_Y\left(\frac{a-b}{2}\right)$$




$$= \begin{cases} \frac{1}{2} & \text{if } 0 \le \frac{a+b}{2} \le 1 \ \underline{\text{and}}\ 0 \le \frac{a-b}{2} \le 1 \\ 0 & \text{else} \end{cases}$$

$$= \begin{cases} \frac{1}{2} & \text{if } -a \le b \le -a+2 \ \underline{\text{and}} \ a-2 \le b \le a \\ 0 \end{cases}$$



We've seen before:

If $Y = aX + b$ ($a, b$ constants)

then $E[Y] = aE[X] + b$

$Var(Y) = a^2 \cdot Var(X)$

Expectation of functions of multiple r.v.s.

Suppose $Z = g(X_1, X_2, \ldots, X_n)$

$$E[Z] = E[g(X_1, \ldots, X_n)]$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} g(u_1, \ldots, u_n) f_{X_1, \ldots, X_n}(u_1, \ldots u_n)\, du_1 \ldots du_n$$

n-fold integral (continuous)

For discrete case, get $n$ $\Sigma$'s.

For $n = 2$:

$$E[g(X,Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(u,v) f_{X,Y}(u,v)\, du\, dv$$

(continuous)

or

$$\boxed{\sum_v \sum_u g(u,v)\, p_{X,Y}(u,v) \quad \text{(discrete.)}}$$

Ex: Expected value of a sum of r.v.s.

Take $X, Y$ continuous.

$$\underbrace{E[X+Y]}_{g(X,Y)} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (u+v)\, f_{X,Y}(u,v)\, du\, dv$$

$$= \iint u\, f_{X,Y}(u,v)\, du\, dv + \iint v\, f_{X,Y}(u,v)\, du\, dv$$

$$= \int u \underbrace{\left( \int f_{X,Y}(u,v)\, dv \right)}_{f_X(u)} du + \int v \underbrace{\left( \int f_{X,Y}(u,v)\, du \right)}_{f_Y(v)} dv$$

$$= \int_{-\infty}^{\infty} u\, f_X(u)\, du + \int_{-\infty}^{\infty} v\, f_Y(v)\, dv$$

$$= E[X] + E[Y]$$

$$\therefore \boxed{E[X+Y] = E[X] + E[Y]}$$

Always true, don't need independence.

Same result is true for discrete rvs.

**Def**: The <u>covariance</u> of rvs $X, Y$ is:

$$\mathrm{Cov}(X,Y) = E\left[(X-E[X])(Y-E[Y])\right]$$

**Note**: $\mathrm{Cov}(X,X) = E\left[(X-E[X])^2\right] = \mathrm{Var}(X)$

Easier way to calculate covariance in some/many cases:

Let $m_X = E[X]$ $m_Y = E[Y]$

$$\mathrm{Cov}(X,Y) = E\left[(X-m_X)(Y-m_Y)\right]$$
$$= E[XY] - E[m_X Y] - E[m_Y X] + E[m_X m_Y]$$
$$= E[XY] - m_X \cdot m_Y - m_Y \cdot m_X + m_X m_Y$$
$$= E[XY] - m_X m_Y$$

$$\therefore \boxed{\mathrm{Cov}(X,Y) = E[XY] - E[X]E[Y]}$$

**Facts**: $\mathrm{Cov}(X, Y+Z) = \mathrm{Cov}(X,Y) + \mathrm{Cov}(X,Z)$

$\mathrm{Cov}(aX, bY) = ab \cdot \mathrm{Cov}(X,Y)$

$$E[XY] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u \cdot v \cdot f_{X,Y}(u,v)\,du\,dv$$

(continuous)

$$E[XY] = \sum_u \sum_v uv \cdot p_{X,Y}(u,v)$$
(discrete)

$$\boxed{\begin{array}{l}\text{Def: The } \underline{\text{correlation coefficient}} \text{ of } X, Y \text{ is:} \\[1em] \rho_{X,Y} = \dfrac{\mathrm{Cov}(X,Y)}{\sigma_X \sigma_Y}\end{array}}$$

Note:

$$\rho_{X,Y} = E\left[ \underbrace{\left( \frac{X - E[X]}{\sigma_X} \right)}_{\substack{\text{mean}=0 \\ \text{var}=1}} \underbrace{\left( \frac{Y - E[Y]}{\sigma_Y} \right)}_{\substack{\text{mean}=0 \\ \text{var}=1}} \right]$$

If $\rho_{X,Y} = 0$, then we say rvs $X, Y$ are uncorrelated.

i.e. $\mathrm{Cov}(X,Y) = 0$

Equivalently, $E[XY] = E[X]E[Y]$.

It turns out that $-1 \le \rho_{X,Y} \le 1$

ie. $|\rho_{X,Y}| \leq 1$

always true.