Lec 14:

Last time: joint pdf

$$P((X,Y) \in T) = \iint_T f_{X,Y}(u,v)\,du\,dv$$

If $f(u,v)$ is constant on T, we say f is "uniform on T".

If $f(u,v) = c$ on T.

Then

$$P((X,Y) \in T) = \iint_T c \cdot du\,dv$$
$$= c \cdot \iint_T du\,dv$$
$$= c \cdot \text{Area}(T)$$

Recall, with discrete r.v.s,

$$\sum_{u,v} P_{X,Y}(u,v) = 1$$

If X, Y are continuous r.v.s, and we take $T = \mathbb{R}^2$ (entire plane), then we know

$P((X,Y) \in T) = 1$

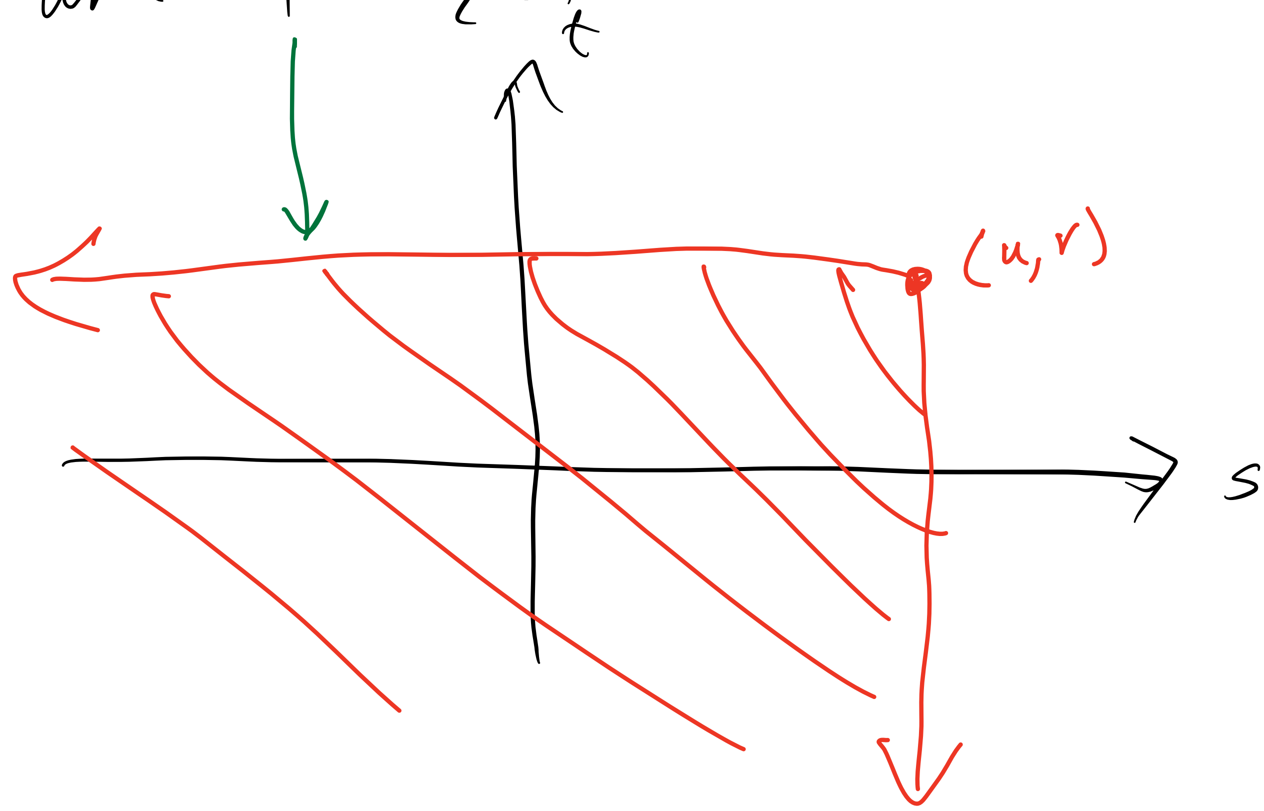
$$\therefore \boxed{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(u,v)\,du\,dv = 1}$$

How can we get the joint pdf from the joint CDF?

Joint CDF:

$$F_{X,Y}(u,v) = P(X \le u, Y \le v)$$
$$= P((X,Y) \in T)$$

where $T = \{(s,t) \in \mathbb{R}^2 : s \le u, t \le v\}$



$$= \iint_{T} f_{X,Y}(s,t)\,ds\,dt$$

$$= \int_{-\infty}^{v}\int_{-\infty}^{u} f_{X,Y}(s,t)\,ds\,dt$$

This gives joint CDF from joint pdf.

Now let's find other direction.

Take derivatives w.r.t v and then u.

$$\frac{\partial}{\partial v} F_{X,Y}(u,v) = \frac{\partial}{\partial v} \int_{-\infty}^{v} \left(\int_{-\infty}^{u} f_{X,Y}(s,t)\,ds \right) dt$$

(use Leibniz rule: 2nd + 3rd terms are zero)

$$= \int_{-\infty}^{u} f_{X,Y}(s,v)\,ds$$

Now differentiate w.r.t u ...

$$\frac{\partial}{\partial u}\left(\frac{\partial}{\partial v} F_{X,Y}(u,v) \right) = \frac{\partial}{\partial u} \int_{-\infty}^{u} f_{X,Y}(s,v)\,ds$$

(use Leibniz:

$$= f_{X,Y}(u,v)$$

$$\therefore \boxed{f_{X,Y}(u,v) = \frac{\partial^2 F_{X,Y}(u,v)}{\partial u \, \partial v}}$$

Can take derivatives $\frac{\partial}{\partial u}$ and $\frac{\partial}{\partial v}$ in either order

How can we get the marginal pdfs from the joint pdf?

Suppose X, Y have joint pdf $f_{X,Y}(u,v)$

We know joint CDF is:

$$F_{X,Y}(u,v) = \int_{-\infty}^{v} \int_{-\infty}^{u} f_{X,Y}(s,t) \, ds \, dt$$

We know marginal CDF of X is:

$$F_X(u) = F_{X,Y}(u,\infty)$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{u} f_{X,Y}(s,t) \, ds \, dt$$

The marginal pdf of X is the derivative of the marginal CDF of X.

$$f_X(u) = \frac{d}{du} F_X(u)$$

$$= \frac{d}{du} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{u} f_{X,Y}(s,t)\, ds \right) dt$$

(use Leibniz : 1st, 2nd terms zero)

$$= \int_{-\infty}^{\infty} \left(\frac{d}{du} \int_{-\infty}^{u} f_{X,Y}(s,t)\, ds \right) dt$$

(use Leibniz : 2nd, 3rd terms zero)

$$= \int_{-\infty}^{\infty} f_{X,Y}(u,t)\, dt$$

Let's rename t to be v to look nicer:

$$\boxed{f_X(u) = \int_{-\infty}^{\infty} f_{X,Y}(u,v)\, dv}$$

"Integrate out the other variables"

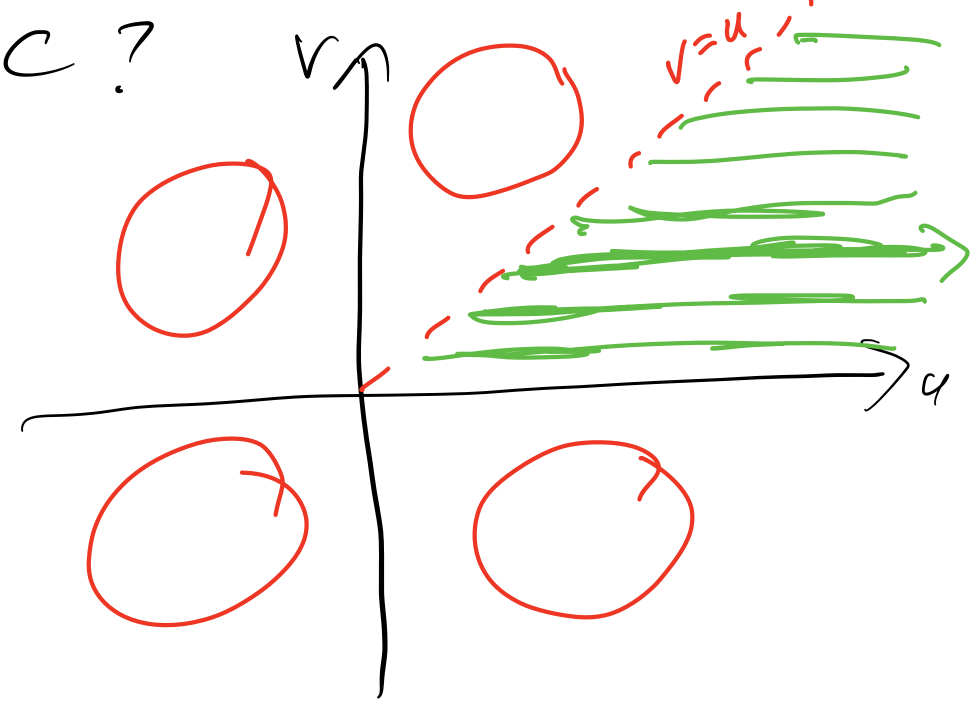
Also,

$$\boxed{f_Y(v) = \int_{-\infty}^{\infty} f_{X,Y}(u,v)\, du}$$

<u>Example</u> : Given rvs X, Y with joint pdf

$$f_{X,Y}(u,v) = \begin{cases} ce^{-u} & \text{if } 0 < v < u \\ 0 & \text{else} \end{cases}$$

① What is c ?



Solve for c:

Set $1 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(u,v)\,du\,dv$

$= \int_{0}^{\infty}\int_{v}^{\infty} ce^{-u}\,du\,dv$

$= c\int_{0}^{\infty}\left(-e^{-u}\right)\Big|_{v}^{\infty} dv$

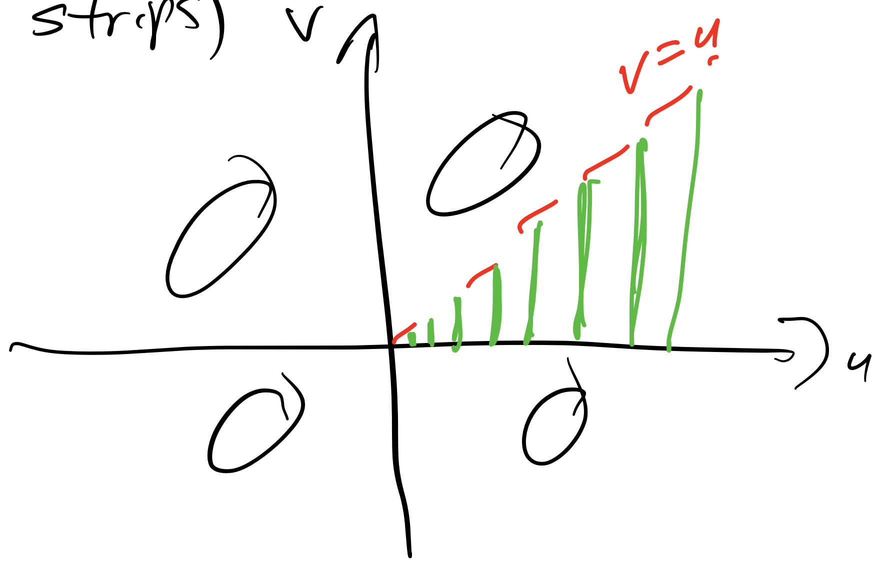
$= c\int_{0}^{\infty} e^{-v}\,dv$

$= c\left(-e^{-v}\right)\Big|_{0}^{\infty}$

$= c\,(0 - -1) = c$

$\therefore \boxed{c = 1}$

Alternatively, integrating $dv\,du$ gives: (vertical strips)



Set

$$1 = \int_0^\infty \int_0^u Ce^{-u}\,dv\,du$$

$= \ldots$ have to integrate by parts.
verify same answer.

(2) Find marginal pdfs of X, Y:

We saw earlier that

$$f_X(u) = \int_{-\infty}^{\infty} f_{X,Y}(u,v)\,dv$$

Note $f_X(u) = 0$ if $u \leq 0$.

So suppose $u > 0$:

$$f_X(u) = \int_0^u e^{-u}\,dv = ve^{-u}\Big|_0^u = ue^{-u}$$

$$\therefore f_X(u) = \begin{cases} u e^{-u} & \text{if } u>0 \\ 0 & \text{else} \end{cases}$$

marginal pdf of X.

For Y:

Note $f_Y(v) = 0$ if $v \le 0$.

Suppose $v > 0$.

$$f_Y(v) = \int_v^\infty e^{-u}\,du = -e^{-u}\Big|_v^\infty = e^{-v}$$

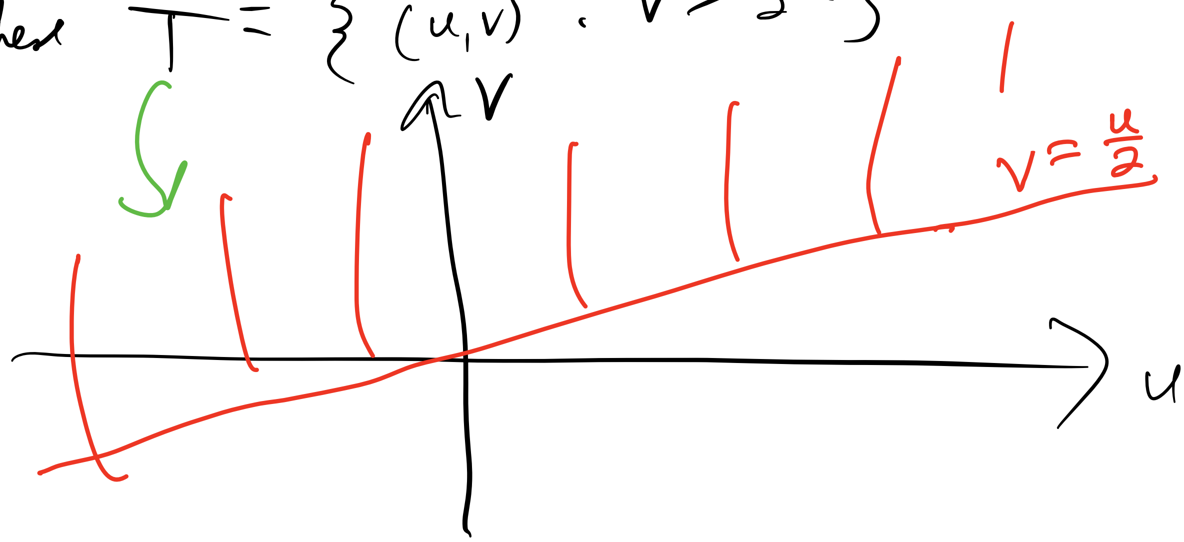
$$\therefore f_Y(v) = \begin{cases} e^{-v} & \text{if } v>0 \\ 0 & \text{else} \end{cases}$$

(3) Find $P(X < 2Y)$.

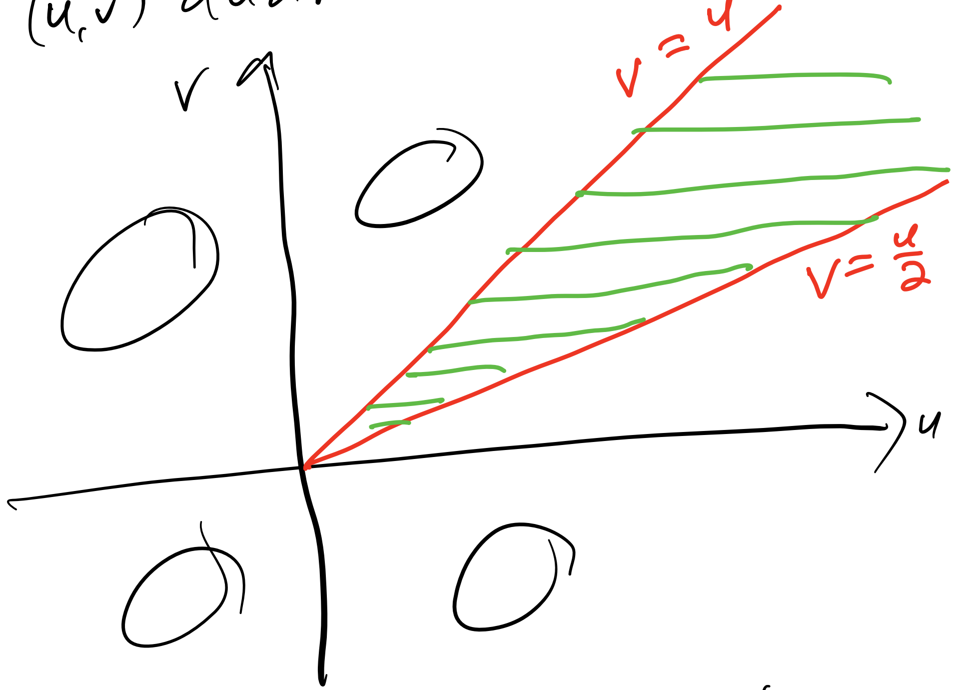
$$P(X<2Y) = P(Y > \tfrac{1}{2}X)$$

$$= P((X,Y) \in T)$$

where $T = \{(u,v) : v > \tfrac{1}{2}u\}$



$$= \iint_T f_{X,Y}(u,v)\,du\,dv$$



Integrate in wedge (green)

Note in this example: $P(Y > \frac{1}{2}X) = P(Y < X < 2Y)$

$$= \int_0^\infty \int_v^{2v} e^{-u}\,du\,dv$$

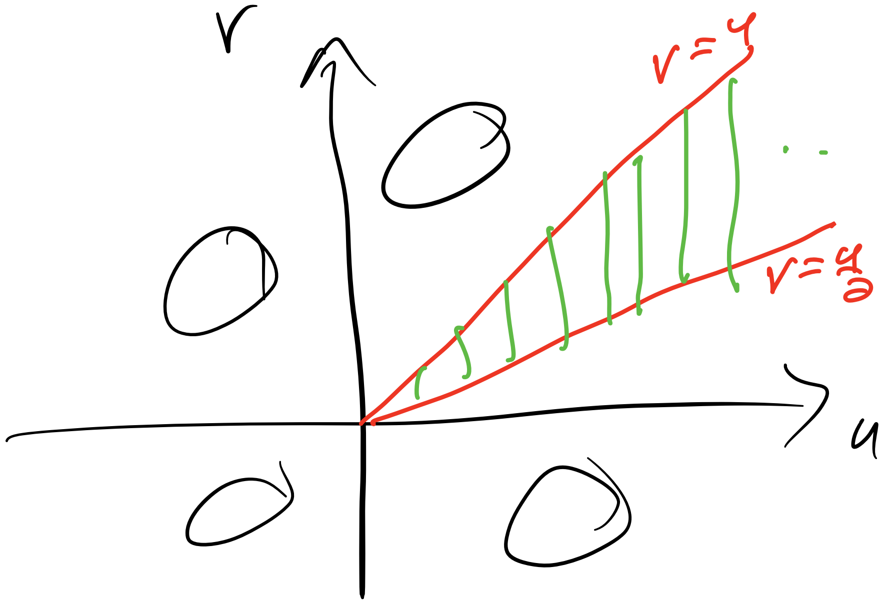
$$= \int_0^\infty \left(-e^{-u}\right)\Big|_v^{2v} dv$$

$$= \int_0^\infty \left(e^{-v} - e^{-2v}\right) dv$$

$$= \left(-e^{-v} + \frac{1}{2}e^{-2v}\right)\Big|_0^\infty$$

$= 1 - \frac{1}{2} = \frac{1}{2}$

Alternatively, with vertical strips $(dv\,du)$



$P(X < 2Y)$

$$= \int_0^\infty \int_{u/2}^{u} e^{-u}\, dv\, du$$

integrate by parts

$= \dots$