Lec 05:

# Independence of events

Ex: Flip a coin twice

$$S = \{HH, HT, TH, TT\}$$

Define 2 events:

$E$ = "1st flip is Heads" $= \{HH, HT\}$

$F$ = "flips are different" $= \{HT, TH\}$

$$P(E) = \frac{2}{4} = \frac{1}{2}$$

$$P(E|F) = \frac{P(EF)}{P(F)} = \frac{P(\{HT\})}{P(\{HT, TH\})}$$

$$= \frac{1/4}{2/4} = \frac{1}{2}$$

So, in this case,

$$P(E) = P(E|F) = \frac{1}{2}$$

$\Rightarrow$ $E, F$ are independent

Let's multiply both sides by $P(F)$.

$$P(E)P(F) = P(E/F)P(F) = P(EF)$$

i.e. $P(EF)$ factors into $P(E)P(F)$.

**Def**: Events $E, F$ are independent if
$$\boxed{P(EF) = P(E)P(F)}$$

If $P(F) \neq 0$, then this implies
$$P(E|F) = P(E).$$

**Consequences**:

If $E, F$ are indep, then

1. $E, F^c$ are indep
2. $E^c, F$ are indep
3. $E^c, F^c$ are indep

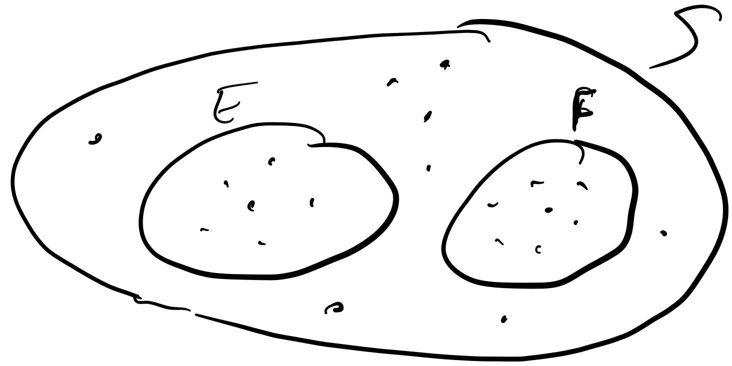
**Proof of 1**: assume $P(E) \neq 0$.

$$\begin{aligned} P(EF^c) &= P(E)P(F^c|E) \\ &= P(E)(1 - P(F|E)) && \text{by cond prob facts} \\ &= P(E)(1 - P(F)) && \text{by indep} \\ &= P(E)P(F^c) \end{aligned}$$

$\Rightarrow E, F^c$ are indep

# Disjoint vs independence

Suppose $E, F$ are disjoint and $P(E) \neq 0$, $P(F) \neq 0$




$\therefore \quad EF = \phi$

$P(EF) = P(\phi) = 0$

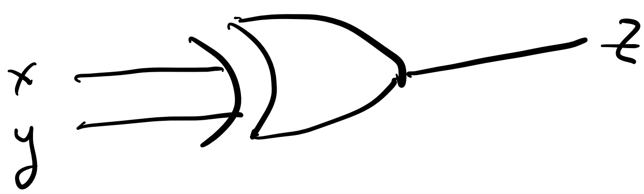
$P(E)P(F) \neq 0$

$\therefore \quad \underbrace{P(EF)}_{0} \neq \underbrace{P(E)P(F)}_{\neq 0}$

$\Rightarrow$ E F are not independent.

I.e. $P(E) \neq 0$ but $P(E|F) = 0$.

Ex: XOR gate




$x, y, z \in \{0, 1\}$

XOR does this:

$z = 1$ if and only if $x \neq y$.

Suppose $x, y$ are chosen with equal probs $\frac{1}{2}$ and independently.

Define events:

$A = $ "$x = 1$"
$B = $ "$y = 1$"
$C = $ "$z = 1$"

By assumption $A, B$ are independent events

$$P(A) = P(B) = \frac{1}{2}.$$

$$C = \underbrace{AB^c}_{\substack{x=1\\y=0}} \cup \underbrace{A^cB}_{\substack{x=0\\y=1}}$$

disjoint union

$$\begin{aligned} P(C) &= P(AB^c \cup A^cB) \\ &= P(AB^c) + P(A^cB) \quad \text{by disjoint} \\ &= P(A)P(B^c) + P(A^c)P(B) \quad \text{by indep of } A, B \\ &= \frac{1}{2}\cdot\frac{1}{2} + \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{2} \end{aligned}$$

$$\begin{aligned} AC &= \text{"}x=1 \text{ and } z=1\text{"} \\ &= \text{"}x=1,\ y=0,\ z=1\text{"} \\ &= \text{"}x=1,\ y=0\text{"} \\ &= AB^c \end{aligned}$$

$$P(AC) = P(AB^c) = P(A)P(B^c) = \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{4}$$

$$\therefore \underbrace{P(AC)}_{\frac{1}{4}} = \underbrace{P(A)}_{\frac{1}{2}}\,\underbrace{P(C)}_{\frac{1}{2}}$$

$\Rightarrow A, C$ are independent.

Also, by symmetry $B, C$ independent.

But $A, C$ (also $B, C$) are <u>not</u> physically independent.

**Def**: Events $A, B, C$ are <u>independent</u> if all of the following are true:

1. $A, B$ indp
2. $B, C$ indp
3. $A, C$ indp

} pairwise independent

4. $P(ABC) = P(A)P(B)P(C)$

<u>Note</u>: If $A, B, C$ pairwise indp then

$$P(ABC) = P(A)P(B)P(C)$$

is equivalent to $\boxed{P(A|BC) = P(A)}$

Since: multiply by $P(BC)$

$$\underbrace{P(A|BC)P(BC)}_{= P(ABC)} = \underbrace{P(A)P(BC)}_{= P(A)P(B)P(C)}$$

It's possible to satisfy pairwise independence (#1, 2, 3) but not condition #4.

Consider the XOR example.

$$ABC = \text{"}x=1, y=1, z=1\text{"} = \emptyset$$

$$P(ABC) = P(\emptyset) = 0$$

But $P(A)P(B)P(C) = \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2} = \frac{1}{8} \neq 0$

$\therefore$ Condition # 4 <u>not</u> satisfied.

$\therefore$ A, B, C are pairwise indep

but A, B, C are <u>not</u> indep.

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<u>Def</u>: Events A, B are <u>conditionally independent given event C</u>, if:

$$P(AB|C) = P(A|C)\,P(B|C)$$

assuming $P(C) \neq 0$.

<u>Note</u>: If A, B are indep given C, then:

$$P(A|BC) = \frac{P(ABC)}{P(BC)} = \frac{P(AB|C)\cancel{P(C)}}{P(B|C)\cdot\cancel{P(C)}}$$

$$= P(A|C)$$

This is similar to $P(A|B) = P(A)$
but with the extra "given C".

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<u>Ex</u> Flip 2 coins.

Define events:

A = "1st coin is Heads"

B = "2nd coin is Heads"

C = "1st and 2nd coins both Heads"

Note: $C = AB$

① $A, B$ are indep (by assumption)

② $P(A|C) = \frac{P(AC)}{P(C)} = \frac{P(AAB)}{P(C)} = \frac{P(AB)}{P(C)} = \frac{P(C)}{P(C)} = 1$

$P(A|BC) = \frac{P(ABC)}{P(BC)} = \frac{P(C)}{P(C)} = 1$

$\therefore P(A|C) = P(A|BC) = 1$

$\Rightarrow A, B$ are indep given $C$.

③ $P(A|C^c) = \frac{P(AC^c)}{P(C^c)} = \frac{P(HT)}{1-P(HH)} = \frac{1/4}{1-1/4}$

$= \frac{1/4}{3/4} = \frac{1}{3}$

$P(A|BC^c) = \frac{P(ABC^c)}{P(BC^c)} = \frac{P(\phi)}{P(TH)} = 0$

$\therefore \underbrace{P(A|C^c)}_{1/3} \neq \underbrace{P(A|BC^c)}_{0}$

$\Rightarrow A, B$ are <u>not</u> indep given $C^c$.

<u>Ex</u>: Let a sample space be $S = \{1, 2, 3, 4, 5\}$ equiprobable outcomes.

Define events: $A = \{1, 2, 5\}$

$B = \{1, 3, 5\}$

$C = \{1, 2, 3, 4\}$

① $P(AB) = P(\{1,5\}) = \frac{2}{5}$

$P(A) = P(B) = \frac{3}{5}$

$\underbrace{P(AB)}_{2/5} \neq \underbrace{P(A)P(B)}_{\frac{3}{5}\cdot\frac{3}{5}} \Rightarrow A, B$ not independent

② $P(AB|C) = P(\{1,5\} | \{1,2,3,4\}) = \frac{1}{4}$

$P(A|C) = P(\{1,2,5\} | \{1,2,3,4\}) = \frac{2}{4} = \frac{1}{2}$

$P(B|C) = P(\{1,3,5\} | \{1,2,3,4\}) = \frac{2}{4} = \frac{1}{2}$

$\therefore \underbrace{P(AB|C)}_{1/4} = \underbrace{P(A|C)}_{1/2} \cdot \underbrace{P(B|C)}_{1/2}$

$\Rightarrow A, B$ are indep given $C$.

Summary:

A, B not indep, but they become indep when given C.

Next time: independent trials

i.e. repeat experiment multiple times "independently".