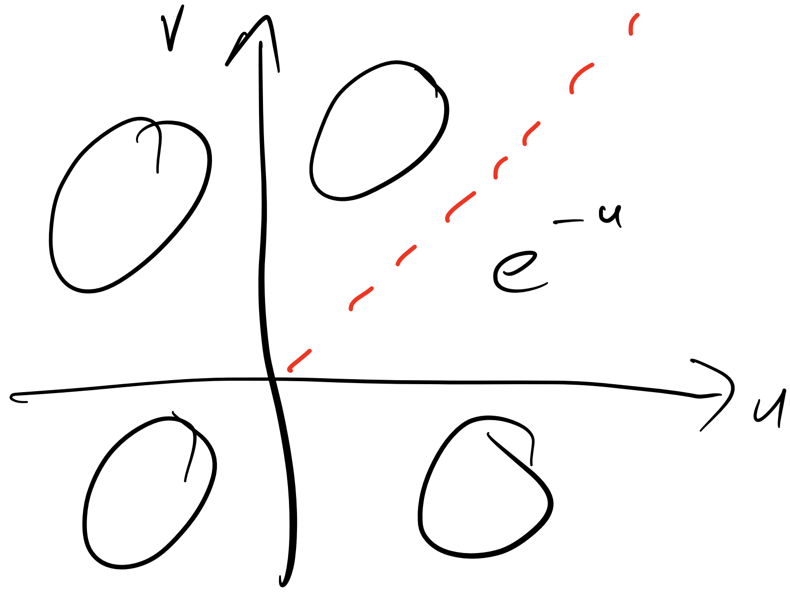
# Lec 15:

Continue example from last time.

Joint pdf

$$f_{X,Y}(u,v) = \begin{cases} e^{-u} & \text{if } 0 < v < u \\ 0 & \text{else} \end{cases}$$




One more question:

Let $a > 0$.

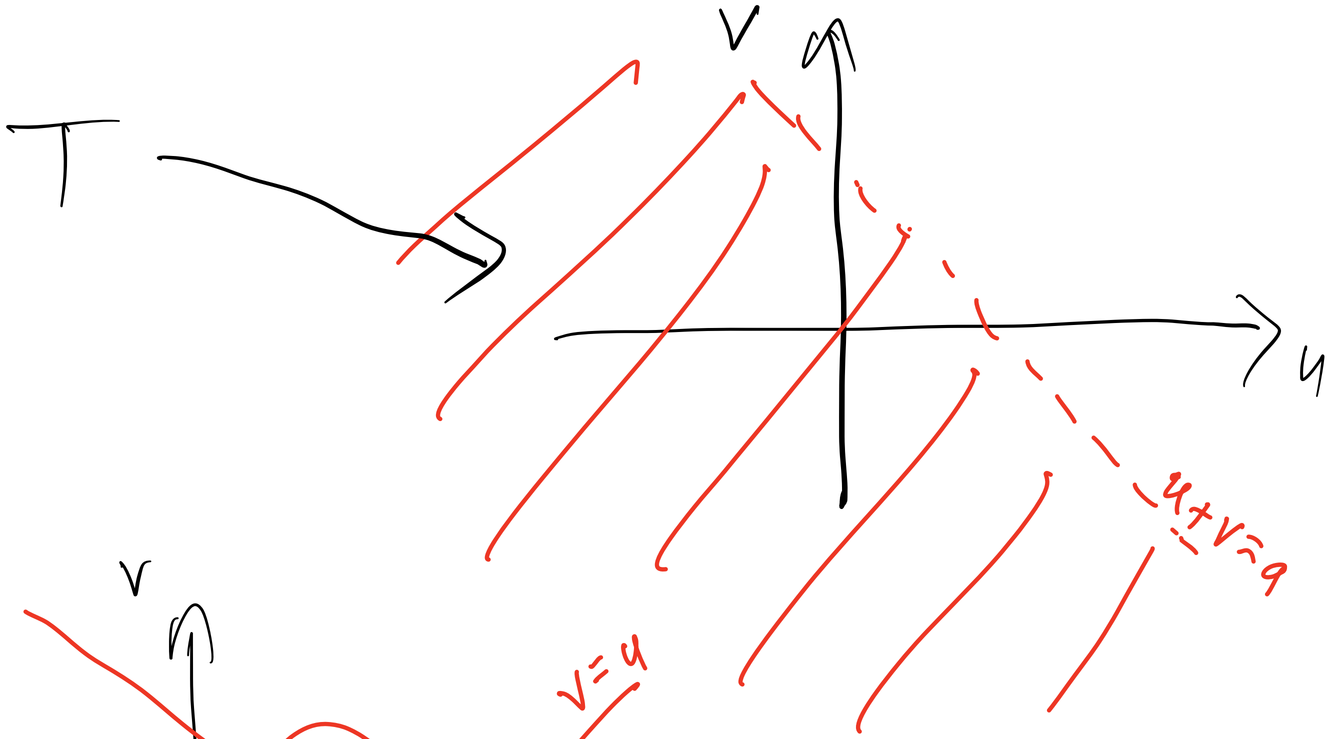
Find $P(X + Y \leq a)$

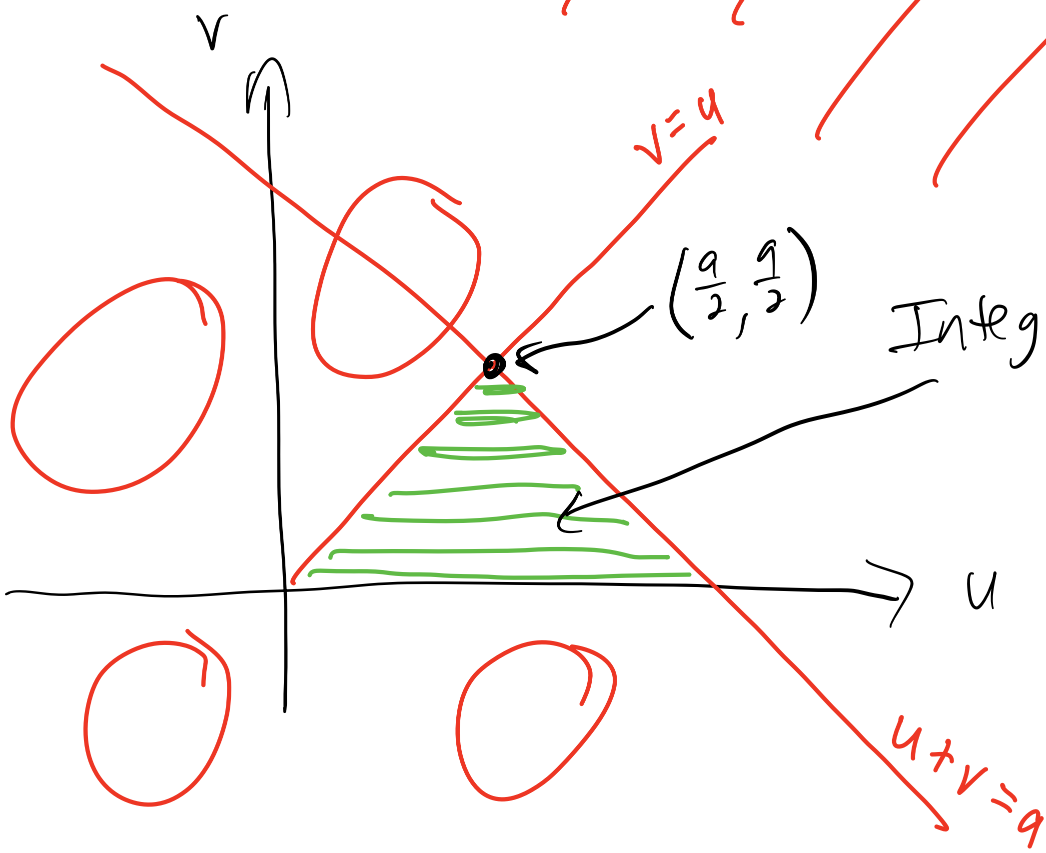
Formulate it as:

$$P((X,Y) \in T)$$

where $T = \{(u,v) : u + v \leq a\}$

$$= \iint_T f_{X,Y}(u,v)\, du\, dv$$




Integrate in green region

Lines intersect when $u = a - u$

$u = a/2$

$v = a/2$

$$P(X+Y \leq a) = P((X,Y) \in T)$$

$$= \int_0^{a/2} \int_v^{a-v} e^{-u} \, du \, dv$$

$$= \int_0^{a/2} \left(-e^{-u}\right)\Big|_v^{a-v} \, dv$$

$$= \int_0^{a/2} \left(e^{-v} - e^{v-a}\right) dv$$

$$= -e^{-v} - e^{v-a} \Big|_0^{a/2}$$

$$= \left(-e^{-a/2} - e^{\frac{a}{2}-a}\right) - \left(-1 - e^{-a}\right)$$

$$= 1 + e^{-a} - 2e^{-a/2}$$

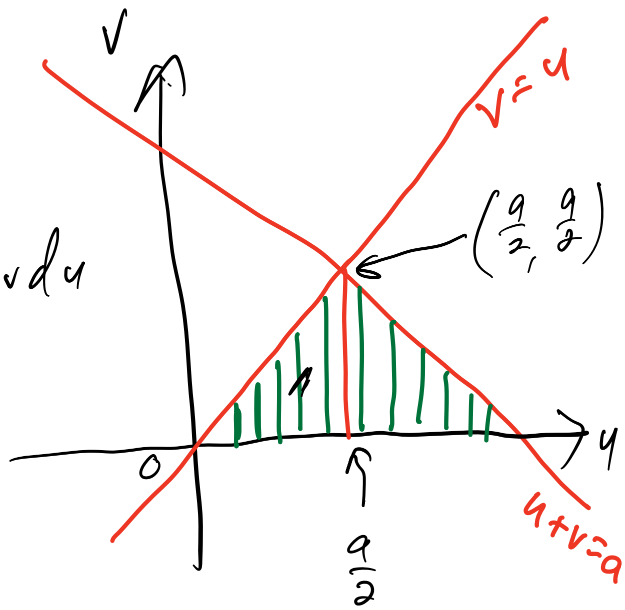
$$= \left(1 - e^{-a/2}\right)^2 \quad \text{for } a \geq 0$$

Alternatively, with vertical strips (ie $dv\,du$)

$$P(X+Y \leq a)$$

$$= \int_0^{a/2}\int_0^{u} e^{-u}\,dv\,du + \int_{a/2}^{a}\int_0^{a-u} e^{-u}\,dv\,du$$

$= \ldots$ work out




One final question for this example:

Let $Z = X+Y$

Find pdf of $Z$.

Set up CDF of $Z$, then differentiate.

$$F_Z(u) = P(Z \leq u) = P(X+Y \leq u)$$

If $u \le 0$, then $F_Z(u) = 0$

Suppose $u \geqslant 0$.

Use previous question calculation.

We computed:

$$F_Z(u) = 1 + e^{-u} - 2e^{-u/2}$$

pdf:

$$f_Z(u) = \frac{d}{du} F_Z(u) = \frac{d}{du}\left( \quad \right)$$

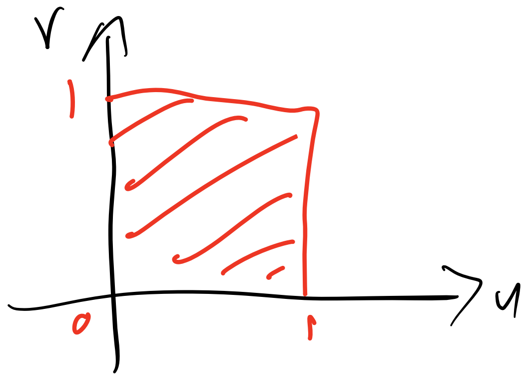
$$= \begin{cases} e^{-u/2} - e^{-u} & \text{if } u \geqslant 0 \\ 0 & \text{else} \end{cases}$$

<u>end of example</u>

<u>Ex</u>: Let $X, Y$ have joint pdf which is uniform on the unit square $[0,1]^2$.

Let $Z = X + Y$

Find pdf of $Z$.




---

On unit square

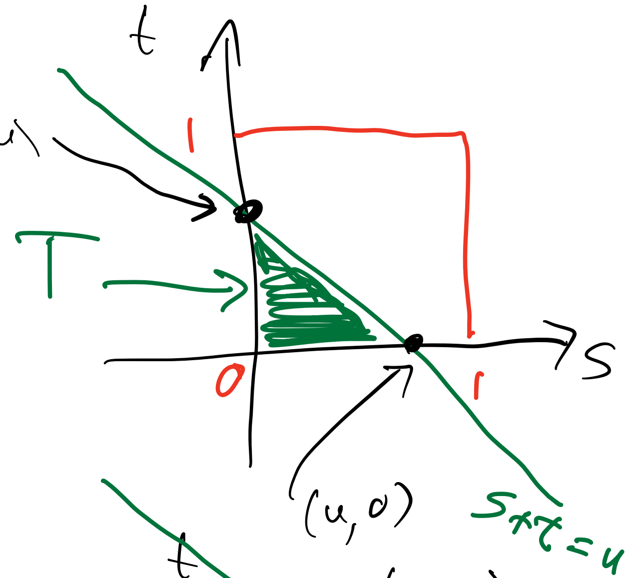
$$f_{X,Y}(u,v) = C \quad \text{and} \quad 1 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(u,v)\,du\,dv$$

$$= \int_0^1 \int_0^1 C \, du\, dv$$

$$= C \cdot \text{Area(square)} = C$$

Plan: Find CDF of $Z$, then differentiate.

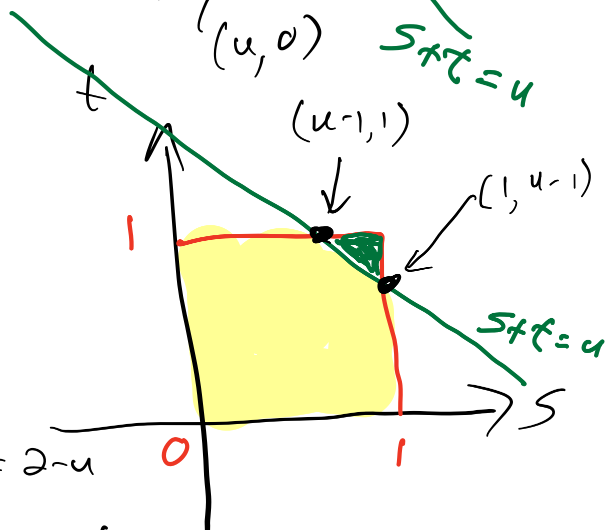
$$F_Z(u) = P(Z \le u) = P(X+Y \le u)$$
$$= P((X,Y) \in T)$$
where $T = \{(s,t) : s+t \le u\}$




Cases

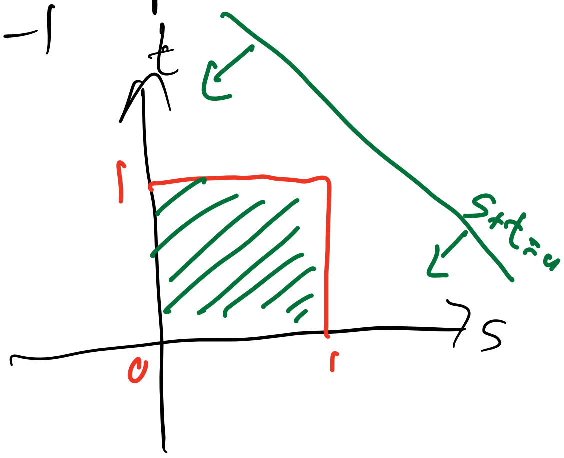
(1) $u \in [0,1]$

$$F_Z(u) = \iint_T f_{X,Y}(s,t)\, ds\, dt$$
$=$ Area(green triangle)
$= \frac{1}{2} u^2$




(2) $u \in [1,2]$

$F_Z(u) =$ Area(yellow pentagon)
$= 1 -$ Area(green triangle)
Triangle side length $= 1-(u-1) = 2-u$
$$= 1 - \frac{1}{2}(2-u)^2 = -\frac{u^2}{2} + 2u - 1$$

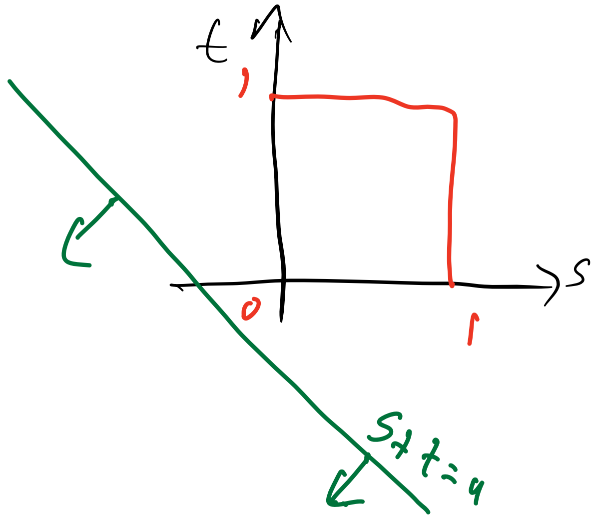



(3) $u > 2$

$F_Z(u) =$ Area(green square)
$=$

(4) $u < 0$

$F_Z(u) = \text{Area}(\emptyset) = 0$




Summarize CDF:

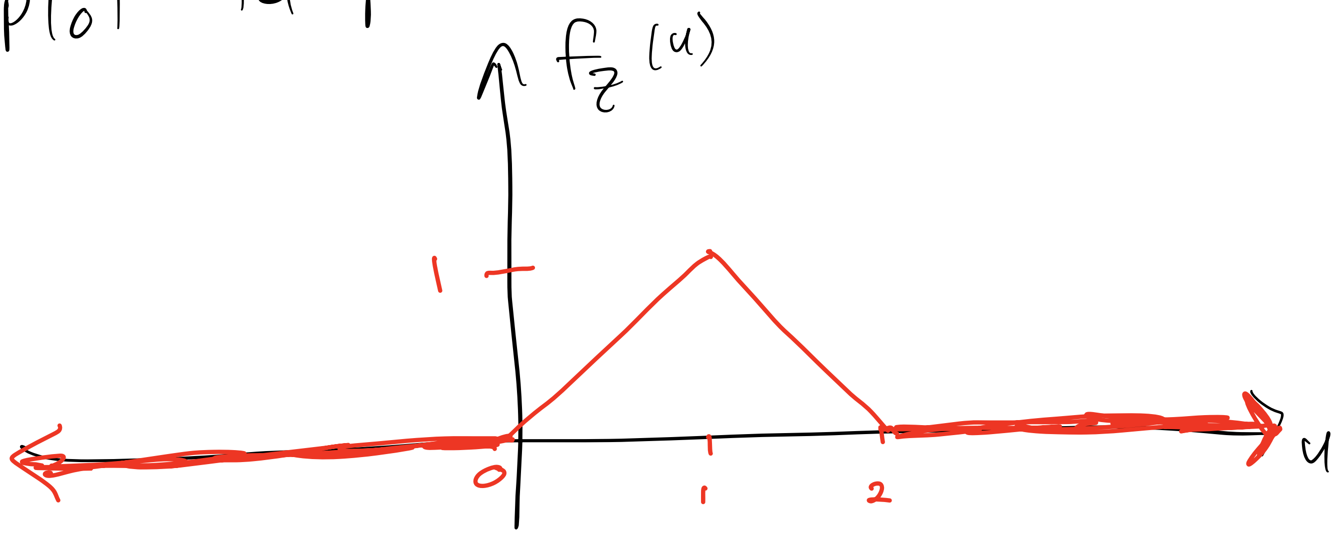
$$F_Z(u) = \begin{cases} 0 & \text{if } u < 0 \\ u^2/2 & \text{if } u \in [0,1] \\ 1 - \frac{1}{2}(2-u)^2 & \text{if } u \in [1,2) \\ 1 & \text{if } u > 2 \end{cases}$$

pdf is:

$$f_Z(u) = \frac{d}{du} F_Z(u) = \frac{d}{du}(\ )$$

$$= \begin{cases} 0 & \text{if } u < 0 \text{ or } u > 2 \\ u & \text{if } u \in [0,1] \\ 2-u & \text{if } u \in [1,2] \end{cases}$$

Plot the pdf of $Z = X+Y$



Next topic: independence of rvs.

Def: If $X, Y$ are rvs, then we say they are independent if for every pair of sets of real numbers $A, B \subseteq \mathbb{R}$, the events $\{X \in A\}$ and $\{Y \in B\}$ are independent.

i.e. $P(X \in A, Y \in B) = P(X \in A) P(Y \in B)$.

Equivalently,

1. Joint CDF factors.
   i.e. $F_{X,Y}(u,v) = F_X(u) F_Y(v)$ for all $u, v$.
2. Joint pdf factors (continuous)
   $f_{X,Y}(u,v) = f_X(u) f_Y(v)$ for all $u, v$.
3. Joint pmf factors (discrete)
   $p_{X,Y}(u,v) = p_X(u) p_Y(v)$ for all $u, v$.

Terminology: If $X, Y$ are independent and identically distributed (same pdf or pmf) then we say $X, Y$ are $\boxed{\text{i.i.d.}}$

Ex: Toss 2 fair coins

Let $X = \#$ Heads

$Y = \begin{cases} 1 & \text{if same} \\ 0 & \text{else} \end{cases}$

Are $X, Y$ independent?

Note: $P(X=1) = P(HT, TH) = \frac{1}{2}$

Suppose you learn that $Y = 0$.

$\therefore$ You know that we got $HT$ or $TH$.

This implies $X = 1$ for sure.

ie. $P(X=1 \mid Y=0) = 1 \neq \frac{1}{2}$

$\therefore P(X=1 \mid Y=0) \neq P(X=1) = \frac{1}{2}$

Calculate joint pmf:

$P_{X,Y}(1,0) = P(X=1, Y=0) = P(HT \text{ or } TH) = \frac{1}{2}$

$P_X(1) = P(HT \text{ or } TH) = \frac{1}{2}$

$P_Y(0) = P(HT \text{ or } TH) = \frac{1}{2}$

$\therefore \frac{1}{2} = P_{X,Y}(1,0) \neq P_X(1)\, P_Y(0) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$

$\therefore X, Y$ not indp r.v.s.

ie we found one particular pair $(u,v)$ s.t. the joint pmf $P_{X,Y}(u,v)$ does not factor.

Ex : Suppose $X, Y$ have joint pdf

$$f_{X,Y}(u,v) = \frac{1}{2\pi} e^{-(u^2+v^2)/2}$$

are $X, Y$ independent ?

Note :

$$f_{X,Y}(u,v) = \underbrace{\left(\frac{1}{\sqrt{2\pi}} e^{-u^2/2}\right)}_{N(0,1)} \underbrace{\left(\frac{1}{\sqrt{2\pi}} e^{-v^2/2}\right)}_{N(0,1)}$$

It's easy to show the marginal pdfs are :

by integrating joint pdf over "other" variable

$$f_X(u) = \frac{1}{\sqrt{2\pi}} e^{-u^2/2}$$

$$f_Y(v) = \frac{1}{\sqrt{2\pi}} e^{-v^2/2}$$

$\therefore f_{X,Y}(u,v) = f_X(u) f_Y(v)$ for all $u, v$.

$\therefore X, Y$ are indep rvs.