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ECE 109

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- Gaussian pdfs

$$f_x(u) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}(\frac{\sigma - m}{\sigma})^2}$$

- Special case: m = 0, $\sigma^2 = 1$

$$X - N(0, 1)$$

  It is called a standard or unit Gaussian (or normal)

- Prove the constant in front is correct. Show that it integrates to 1.
- Let

$$f_x(u) = Ce^{\frac{1}{2}(\frac{u-c}{\sigma})^2}$$

- Know

$$1 = \int_{-\infty}^{\infty} f_x(u)du$$

- Trick to get the indefinite integral (use u substitution and look at the square of the integral) results in:

$$\begin{aligned} I^2 &= (\int_{-\infty}^{\infty} e^{-v^2/2} dv)^2 \\ &= (\int_{-\infty}^{\infty} e^{-x^2/2} dx)(\int_{-\infty}^{\infty} e^{-y^2/2} dy) \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{(x^2+y^2)/2} dxdy \end{aligned}$$

- Convert to polar coordinates (remember that $r^2 = x^2 + y^2$)

$$\begin{aligned} &= \int_0^{2\pi} \int_0^{\infty} e^{-r^2/2} dr d\theta \\ &= 2\pi \int_0^{\infty} \\ &\therefore I^2 = 2\pi => I = \sqrt{2\pi} \\ &\therefore \frac{1}{\sigma c} = I = \sqrt{2\pi} \\ C &= \frac{1}{\sigma\sqrt{2\pi}} \end{aligned}$$

- Expectation (expected value, average, mean)
- Example: Given a set of numbers $a_1, a_2, a_3, ..., a_n$, their average is

$$\frac{a_1 + a_2 + a_3 + ... + a_n}{n}$$

- Suppose X is a discrete random variable with pmf:

$$P_x(1) = \frac{1}{4}, P_x = \frac{1}{4}, P_x(20) = \frac{1}{2}$$

If we observe a independent value of X (for large n), you would expect roughly $\frac{n}{4}$ 1's, $\frac{n}{4}$ 10's, and $\frac{n}{2}$ 20's. So the average would be about:

$$\frac{\frac{n}{4} + \frac{n}{4} * 10 + \frac{n}{2} * 20}{n} = 12.75$$

Definition:

If X is a discrete random variable, then its expected value is: $E(X) = \sum_u u * P_x(u)$

The sum is really over only those u for which $P_x(u)$ is nonzero.

- Example: X is uniform discrete on $\{-1, 1\}$
  The expected value of X is:
  $$E(X) = (1 * \frac{1}{2}) + (-1 * \frac{1}{2}) = 0$$
  This mean is 0
- Continuous random variables:
  As $du \to 0$, the probability of the shaded region is approximately $f_x(u)du$. Thus, the mean is about:
  $$\sum_u u f_x(u) du = \int_{-\infty} \infty u f_x(u) du$$

Definition:

If X is a discrete random variable, then its expected value is: $E(X) = u f_x(u) du$

- Example: Find the mean of a uniform random variable on the interval from a to b.
  $$\begin{aligned} E(X) &= \int_{-\infty}^{\infty} u f_x d(u) du \\ &= \int_a^b du \\ &= \frac{1}{b-a} * \frac{b^2 - a^2}{2} = \frac{a+b}{2} \\ &= \text{ The midpoint of the interval from a to b.} \end{aligned}$$
- Example: Find the mean of a binomial random variable. Recall pmf.
  $$P_x(K) = \binom{n}{k} p^K (1-p)^{n-K}$$
  $$\begin{aligned} E(X) &= \sum_u u P_x(u) \\ &= \sum_{k=0}^{n} K * \binom{n}{k} p^k (1-p)^{n-k} \\ &= \sum_{k=1}^{n} K * \frac{n!}{k!(n-k)!} p^k (1-p)^{n-k} \\ &= (np) \sum_{k=1}^{n} \frac{(n-1)!}{(k-1)!(n-1-(k+1)!} * p^{k-1} (1-p)^{n-1-(k+1)} \\ &= (np) \sum_{j=0}^{m} \frac{m!}{j!(m-j)!} * p^j (1-p)^{m-j} \\ &= (np) * 1 = np \\ &\therefore E(X) = np \end{aligned}$$

i.e. If you flip a biased coin n times, the average number of heads is np.

- Example: Find the mean of a poisson random variable X. The pmf is:

$$P_x(K) = \frac{e^{-\lambda}\lambda^K}{k!}$$

$$\begin{aligned}
E(X) &= \sum_{k=0}^{\infty} K P_x(K) = \sum_{k=0}^{\infty} * \frac{e^{-\lambda}\lambda^K}{k!} \\
&= \lambda e^{-\lambda} * \sum_{k=1}^{\infty} \frac{\lambda^{k-1}}{(k-1)!} \\
&= \lambda e^{-}\lambda * e^{\lambda} = \lambda \\
&\therefore E(X) = \lambda
\end{aligned}$$

- Example: Find the mean of a geometric random variable. The pmf:

$$P_x(K) = p(1-p)^{k-1}$$

- Example: Find the mean of a Gaussian random variable. The pmf:
- Example: Find the mean of a exponential random variable. The pmf: