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ECE 109

Week 1, Lecture 01-10-23

- An **experiment** involves randomness and results in an **outcome**. Every experiment has exactly one outcome.
- Example: Experiment: Flip 2 coins
  - Outcome Possibilities: HH, HT, TH, TT
  - **Exactly one** outcome occurs.
- Example: Roll 1 die
  - Outcome possibilites are 1, 2, 3, 4, 5, 6.
- Example:
  - 1 coin flip: $S = H, T$
  - 2 coin flip: $S = HH, TT, HT, TH$
  - 3 coin flip: $S = HHH, HTT, ...HTT, TTT$
  - n coin flips: $|S| = 2^n$
- Example: Roll 2 dice
  - $S = \begin{matrix} (1,1),(1,2)... \\ (2,1),(2,2)... \end{matrix}$
  - $|S| = 36$
- An **sample space** is the set of all possible outcomes of an experiment.
- An **event** is any subset of the sample space.
- Example: Flip two coins
  - $E = HH, TT =$ both flips are the same
  - $E = HT, , TH, TT =$ At least one tail
- If $E = \phi$ (empty set), then E is called the **null event**
- If $E = \phi$ (entire sample space), then E is called the **sure event**
- We say that an event **occurred** (or happened) if the outcome of the experiment lies in the event.
- For an event $E \subseteq S$, $P(E)$ will be a probability.
- Set theory review:
  - Unions
  - Intersections
  - Complements
  - Venn Diagrams
  - DeMorgan's Law
  - Disjoint
- **Notation:** The intersection of sets A, B is usually denoted $A \cap B$. In probability, we use the abbreviated notation $AB = A \cap B$.

- Example: Flip 2 coins
  - $E = HH, HT.$
  - $F = TT, HT.$
  - $EF = HT$
  - $E \cup F = HH, TT, HT = TH^c$
  - $E^c = TT, TH$
  - Which of the events below occur?
    - E
    - E^c
    - F
    - F^c
  - Did Occur
    - E
    - E^c
    - F
    - F^c

Week 1, Lecture 01-12-23

- Experiment
- Outcome
- Sample space
- Event

Set Theory

- $(E^c)^c = E$
- DeMorgan's Law
  - Events E, F
  - $(E \cup F)^c = E^c F^c$
  - $(EF)^c = E^c \cup F^c$
  - This works for any number of sets

Example:

- ABC^c
- This is the event that:
  - A and B and C^c occur
  - A and B occur, **but** not C
- **Note:** The word "but" is almost always logically equivalent to "and."
- **Note:** In ECE109, when we use "or" we always mean inclusive, unless stated otherwise.

Example:

- Exactly one of A, B, C occurs

- $AB^cC^c \cup A^cBC^c \cup A^cB^cC$
- Order doesn't matter, unions are commutative

Definition: Probability is an assignment of a number to an event. i.e. for each event $E \subset S$, $P(E) \subset R$, where R is the set of real numbers.

Probability must satisfy 3 axioms

1. $0 \leq P(E) \leq 1$
2. $P(S) = 1$
3. If $E_1, E_2, E_3, ...$ are events that are pairwise disjoint, (i.e. $E_iE_j = 0$ whenever $i \neq j$), then $P(E_1 \cup E_2 \cup E_3 \cup ...) = P(E_1) + P(E_2) + P(E_3) + ...$

Note: May be and infinite sum

If E, F, are not disjoint, then we cannot say generally that $P(E \cup F) = P(E) + P(F)$. We would otherwise get double counting of the probability in the intersection of $E \cup F$.

Fact: $P(A^c) = 1 - P(A)$ Proof: $S = A \cup A^c$ and A and A^c are disjoint ($AA^c = 0$. Therefore, $1 = P(S) = P(A \cup A^c) = P(A) + P(A^c)$

Example:

- Given the events A, B such that:
  - $P(AB) = 0.4$
  - $P(AB^c) = 0.1$
  - $P(A \cup B) = 0.6$

Fact: $P(0) = 0$ Proof: $0 = S^c$, $P(0) = P(S^c) = 1 - P(S) = 1 - 1 = 0$

Fact: $P(E \cup F) = P(E) + P(F) - P(EF)$

Note: The last term accounts for double counting **Fact:** If $E \subseteq F$, then $P(E) \leq P(F)$. Therefore, $P(F) = P(E \cup E^cF) = P(E) + P(E^cF) \geq P(E)$. As a consequence, since $AB \subseteq A$ and $AB \subseteq B$ then $P(AB) \leq P(A)$ and $P(AB) \leq P(B)$.

Special Situation: Sometimes every outcome in a sample space has the same probability. We call this "equiprobable outcomes". If we have equally likely outcomes, then for any event E,

$$P(E) = \frac{|E|}{|S|} = \frac{\text{size of E}}{\text{size of S}}$$

This assumes S is finite.

Example:

- Pick a card randomly from a standard deck.
- $|S| = 52$.
- Let $E =$ "the card is red".
- $|E| = 26$.
- $P(E) = \frac{|E|}{|S|} = \frac{26}{52} = \frac{1}{2}$.
- Let $F =$ "the card is an ace".
- $|F| = 4$.
- $P(E) = \frac{|F|}{|S|} = \frac{4}{52} = \frac{1}{13}$.

Combinatorics studies counting set sizes. We use permutations and combinations.

Permuatations: orderings of a set.

- Given 1, 2, 3, ... , n
- There are n! different orderings.
- $n! = n(n-1)(n-2)...1$
- $n! = n(n-1)!$
- $0! = 1$ and $1! = 1$

Example:

- $n = 3$
- $3! = 3 * 2 * 1 = 6$ permutations

Combinations

$$\begin{aligned}\binom{n}{k} &= \text{"n choose k"} \\ &= \frac{n!}{k!(n-k)!} \\ &= \text{the number of subsets of size k from a set of size n}\end{aligned}$$

Example:

- How many triples of letters can we pick from A, B, C, D, E? (order does not matter).
- $\binom{5}{3} = \frac{5!}{3!(5-3)!} = 10$
- ABC, ABD, ABE, ACD, ACE, ADE, BCD, BCE, BDE, CDE
- There are 10 subsets of size 3

Recall the binomial theorem:

$$\begin{aligned}(x+y)^n &= \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} \\ (x+y)^1 &= x + y \\ (x+y)^2 &= \binom{2}{0} x^2 + \binom{2}{1} xy + \binom{2}{2} y^2\end{aligned}$$

We can use Pascal's triangle to get the coefficients