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MATH 438: Introduction to Complex Variables
Proofs

1. Chapter 1

1.1. a. **Proof** $(a+bi)(a-bi) = a^2 + b^2$

$$\begin{aligned} (a+bi)(a-bi) &= a^2 - abi + abi - b^2i^2 \\ &= a^2 - b^2i^2 \\ &= a^2 + b^2 \end{aligned}$$

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b. **Proof** $(a+bi)^2 = a^2 - b^2 + 2abi$

$$\begin{aligned} (a+bi)^2 &= (a+bi)(a+bi) \\ &= a^2 + abi + abi + b^2i^2 \\ &= a^2 - b^2 + 2abi \end{aligned}$$

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1.2. a **Proof** $zw = wz$
Let $z = x + yi$ and $w = u + vi$

$$\begin{aligned} zw &= (x+yi)(u+vi) \\ &= ux + xvi + yui + yvi^2 \\ &= ux - vy + (vx + uy)i \\ wz &= (u+vi)(x+yi) \\ &= ux + uyi + vxi + vyi^2 \\ &= ux - vy + (vx + uy)i \\ zw &= wz \end{aligned}$$

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b **Proof** $z(r+w) = zr + zw$
Let $z = x + yi$, $w = u + vi$, and $r = p + qi$

$$\begin{aligned} z(r+w) &= (x+yi)(p+qi+u+vi) \\ &= px + qxi + ux + vxi + pyi + i^2yq + uyi + yvi^2 \\ &= px - yq - vy + (qx + vx + py + uy)i \\ zr + zw &= (x+yi)(p+qi) + (x+yi)(u+vi) \\ &= px + qxi + pyi + i^2yq + ux + vxi + uyi + vyi^2 \\ &= px - yq - vy + (qx + py + vx + uy)i \\ z(r+w) &= zr + zw \end{aligned}$$

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c **Proof** $r(wz) = (rw)z$
Let $z = x + yi$, $w = u + vi$, and $r = p + qi$

$$\begin{aligned}
r(wz) &= (p+qi)((u+vi)(x+yi)) \\
&= (p+qi)(ux - vy + (vx+uy)i) \\
&= pux - pvy + p(vx+uy)i + quxi - qvyi - q(vx+uy) \\
&= pux - pvy - qvx - quy + (pvx + puy + qux - qvy)i \\
(rw)z &= ((p+qi)(u+vi))(x+yi) \\
&= (pu - qv + (pv+qu)i)(x+yi) \\
&= pux + puyi - qvx - qvyi + x(pv+qu)i - y(pv+qu) \\
&= pux - pvy - qvx - quy + (puy - qvy + pvx + qux)i \\
r(wz) &= (rw)z
\end{aligned}$$

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1.3. a **Proof** $\overline{z+w} = \overline{z} + \overline{w}$
Let $z = x + yi$ and $w = u + vi$

$$\begin{aligned}
\overline{z+w} &= \overline{x + u + (y+v)i} \\
&= x + u - (y+v)i \\
\overline{z} + \overline{w} &= \overline{x+yi} + \overline{u+vi} \\
&= x - yi + u - vi \\
&= x + u - (y+v)i \\
\overline{z+w} &= \overline{z} + \overline{w}
\end{aligned}$$

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b **Proof** $\overline{zw} = \bar{z}\bar{w}$
Let $z = x + yi$ and $w = u + vi$

$$\begin{aligned}
\overline{zw} &= \overline{(x+yi)(u+vi)} \\
&= \overline{xu - vy + (xv+uy)i} \\
&= xu - vy - (xv+uy)i \\
\bar{z}\bar{w} &= (\overline{x+yi})(\overline{u+vi}) \\
&= (x-yi)(u-vi) \\
&= xu - vy - (vx+uy)i \\
\overline{zw} &= \bar{z}\bar{w}
\end{aligned}$$

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1.4. a **Proof** $|z|^2 = z\overline{z}$
Let $z = x + yi$

$$\begin{aligned}
|z|^2 &= |x+yi|^2 \\
&= x^2 + y^2 \\
z\overline{z} &= (x+yi)\overline{(x+yi)} \\
&= (x+yi)(x-yi) \\
&= x^2 - xyi + xyi + y^2 \\
&= x^2 + y^2 \\
|z|^2 &= z\overline{z}
\end{aligned}$$

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b **Proof** $|zw| = |z|\,|w|$
Let $z = x + yi$ and $w = u + vi$

$$\begin{aligned}
|zw| &= |(x+yi)(u+vi)| \\
&= |xu - vy + (xv + uy)i| \\
&= \sqrt{(xu-vy)^2 + (xv+uy)^2} \\
&= \sqrt{u^2x^2 - 2uvxy + v^2y^2 + u^2y^2 + 2uvxy + v^2x^2} \\
&= \sqrt{u^2x^2 + v^2y^2 + u^2y^2 + v^2x^2} \\
|z|\,|w| &= |x+yi|\,|u+vi| \\
&= \sqrt{x^2+y^2}\sqrt{u^2+v^2} \\
&= \sqrt{(x^2+y^2)(u^2+v^2)} \\
&= \sqrt{x^2u^2 + x^2v^2 + y^2u^2 + y^2v^2} \\
|zw| &= |z|\,|w|
\end{aligned}$$

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c **Proof** $|z| = |\overline{z}|$
Let $z = x + yi$

$$\begin{aligned}
|z| &= |x+yi| \\
&= \sqrt{x^2+y^2} \\
|\overline{z}| &= |\overline{x+yi}| \\
&= |x-yi| \\
&= \sqrt{x^2+y^2} \\
|z| &= |\overline{z}|
\end{aligned}$$

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1.5. **Proof** $z^{-1} = \frac{\overline{z}}{|z|^2} \Leftrightarrow zz^{-1} = 1$
Let $z \neq 0$

$$\begin{aligned}
z^{-1} &= \frac{\overline{z}}{|z|^2} \\
zz^{-1} &= \frac{z\overline{z}}{|z|^2} \\
&= \frac{|z|^2}{|z|^2} \text{ by 1.4} \\
&= 1
\end{aligned}$$

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1.6. a **Proof** $\overline{\frac{z}{w}} = \frac{\bar{z}}{\bar{w}}$
Let $z = x + yi$ and $w = u + vi$

$$\begin{aligned}
\overline{(\frac{z}{w})} &= \overline{(\frac{x+yi}{u+vi})} \\
&= \overline{(\frac{x+yi}{u+vi}\frac{u-vi}{u-vi})} \\
&= \overline{(\frac{xu+yv+(uy-xv)i}{u^2+v^2})} \\
&= \frac{xu+yv-(uy-xv)i}{u^2+v^2} \\
\frac{\bar{z}}{\bar{w}} &= \frac{\overline{x+yi}}{\overline{u+vi}} \\
&= \frac{x-yi}{u-vi} \\
&= \frac{(x-yi)(u+vi)}{(u-vi)(u+vi)} \\
&= \frac{xu+yv-(uy-xv)i}{u^2+v^2} \\
\overline{(\frac{z}{w})} &= \frac{\bar{z}}{\bar{w}}
\end{aligned}$$

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b **Proof** $\left|\frac{z}{w}\right| = \frac{|z|}{|w|}$
Let $z = x + yi$ and $w = u + vi$

$$\begin{aligned}
\left|\frac{z}{w}\right| &= \left|\frac{x+yi}{u+vi}\right| \\
&= \left|\frac{(x+yi)(u-vi)}{(u+vi)(u-vi)}\right| \\
&= \left|\frac{xu+vy+(uy-xv)i}{u^2+v^2}\right| \\
&= \left|\frac{xu+vy}{u^2+v^2}+\frac{(uy-xv)i}{u^2+v^2}\right| \\
&= \sqrt{(\frac{xu+vy}{u^2+v^2})^2+(\frac{uy-xv}{u^2+v^2})^2} \\
&= \sqrt{\frac{(xu+vy)^2+(uy-vx)^2}{(u^2+v^2)^2}} \\
&= \sqrt{\frac{(x^2+y^2)(u^2+v^2)}{(u^2+v^2)^2}} \\
&= \sqrt{\frac{x^2+y^2}{u^2+v^2}} \\
&= \frac{\sqrt{x^2+y^2}}{\sqrt{u^2+v^2}} \\
&= \frac{|z|}{|w|}
\end{aligned}$$

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1.7. a **Proof** $\left|\frac{z-1}{z+1}\right|^2 = \frac{(x-1)^2+y^2}{(x+1)^2+y^2}$
Let $z = x + yi$

$$\begin{aligned}
\left|\frac{z-1}{z+1}\right|^2 &= \left|\frac{x-1+yi}{x+1+yi}\right|^2 \\
&= \frac{|x-1+yi|}{|x+1+yi|} \text{ by 1.6.b} \\
&= \frac{(x-1)^2+y^2}{(x+1)^2+y^2}
\end{aligned}$$

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b **Proof** $\left|\frac{1+4i}{4+i}\right| = 1$

$$\begin{aligned}
\left|\frac{1+4i}{4+i}\right| &= \frac{|1+4i|}{|4+i|} \text{ by 1.6.b} \\
&= \frac{\sqrt{1^2+4^2}}{\sqrt{4^2+1^2}} \\
&= 1
\end{aligned}$$

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c **Proof** $|\cos(\theta) + i\sin(\theta)| = 1$

$$\begin{aligned}
|\cos(\theta) + i\sin(\theta)| &= \sqrt{\cos^2(\theta) + \sin^2(\theta)} \\
&= 1
\end{aligned}$$

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d **Proof** $\left|\frac{3+4i}{4+3i}\right| = 1$

$$\begin{aligned}
\left|\frac{3+4i}{4+3i}\right| &= \frac{|3+4i|}{|4+3i|} \text{ by 1.6.b} \\
&= \frac{\sqrt{3^2+4^2}}{\sqrt{4^2+3^2}} \\
&= 1
\end{aligned}$$

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1.8. a **Proof** $\frac{1}{2+i} = \frac{2}{5} - \frac{1}{5}i$

$$\begin{aligned}
\frac{1}{2+i} &= \frac{1}{2+i}\frac{2-i}{2-i} \\
&= \frac{2-i}{2^2+1^2} \\
&= \frac{2}{5} - \frac{1}{5}i
\end{aligned}$$

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b **Proof** $\frac{(1+i)^2}{3+2i} = \frac{4}{13} + \frac{6}{13}i$

$$\begin{aligned}
\frac{(1+i)^2}{3+2i} &= \frac{2i}{3+2i} \\
&= \frac{2i}{3+2i}\frac{3-2i}{3-2i} \\
&= \frac{4+6i}{3^2+2^2} \\
&= \frac{4}{13} + \frac{6}{13}i
\end{aligned}$$

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c **Proof** $\frac{2+i}{3+4i} = \frac{2}{5} - \frac{1}{5}i$

$$\begin{aligned} \frac{2+i}{3+4i} &= \frac{2+i}{3+4i}\frac{3-4i}{3-4i} \\ &= \frac{10-5i}{3^2+4^2} \\ &= \frac{2}{5} - \frac{1}{5}i \end{aligned}$$

$\blacksquare$

d **Proof** $\frac{1}{z^2} = \frac{\overline{z^2}}{(z\overline{z})^2}$

Let $z = x + yi$

$$\begin{aligned} \frac{1}{z^2} &= \frac{1}{(x+yi)^2} \\ &= \frac{1}{x^2-y^2+2xyi} \\ &= \frac{1}{x^2-y^2+2xyi}\frac{x^2-y^2-2xyi}{x^2-y^2-2xyi} \\ &= \frac{x^2-y^2-2xyi}{(x^2-y^2)^2} \\ &= \frac{\overline{z^2}}{(z\overline{z})^2} \end{aligned}$$

$\blacksquare$

1.9. **Proof** $2c = z\xi + \overline{z\xi} \Leftrightarrow c = ax - by$

Let $z = x + yi$ and $\xi = a + bi$

$$\begin{aligned} 2c &= z\xi + \overline{z\xi} \\ &= (x+yi)(a+bi) + \overline{(x+yi)(a+bi)} \\ &= ax - by + (ay+bx)i + \overline{ax - by + (ay+bx)i} \\ &= ax - by + (ay+bx)i + ax - by - (ay+bx)i \\ &= 2(ax - by) \\ c &= ax - by \end{aligned}$$

c must be real, since a, b, x, and y are real. The slope is $\frac{Re(\xi)}{Im(\xi)}$ $\blacksquare$

1.10. **Proof** $z\bar{z} - \xi\bar{z} - \bar{\xi}z = c$ **is the equation of a circle centered at** ξ **with radius** $\sqrt{c + |\xi|^2}$

$$\begin{aligned} z\bar{z} - \xi\bar{z} - \bar{\xi}z &= c \\ z\bar{z} - \xi\bar{z} - \bar{\xi}z + \xi\bar{\xi} &= c + \xi\bar{\xi} \\ (z-\xi)\overline{(}x-\xi) &= c + |\xi|^2 \\ |z-\xi|^2 &= c + |\xi|^2 \\ |z-\xi| &= \sqrt{c + |\xi|^2} \end{aligned}$$

$\blacksquare$

1.11. a $|z - 1 - i| = 2$

b The statements $x = 5$ and $ax - by = c$ are equivalent when $a = 1$, $b = 0$, and $c = 5$. Therefore, by 1.9, the complex equation is $z + \overline{z} = 10$.

c $y = -2 \Leftrightarrow ax - by = c$ when $a = 0, b = -1$, and $c = -2$. Therefore, by 1.9, the complex equation is $-iz + \overline{-iz} = -4$

d $y-2x=0 \Leftrightarrow \overline{ax-by}=c$ when $a=-2, b=1$, and $c=0$. Therefore, by 1.9, the complex equation is $(-2+i)z+\overline{(-2+i)z}=0$.

1.12. a $z+\overline{z}=2 \Leftrightarrow 2x=2 \Leftrightarrow x=1$

1.13. a $Arg(1)=0$. $arg(1)=2\pi k$

b $Arg(-3)=\pi$. $arg(-3)=\pi+2\pi k$

c $Arg(-1+i)=\frac{3\pi}{4}$. $arg(-1+i)=\frac{3\pi}{4}+2\pi k$

d $Arg(3+3i)=\frac{\pi}{4}$. $arg(3+3i)=\frac{\pi}{4}+2\pi k$

e $Aarg(1-\sqrt{3}i)=-\frac{\pi}{3}$. $arg(1-\sqrt{3}i)=-\frac{\pi}{3}+2\pi k$

f $Arg(-4i)=-\frac{\pi}{2}$. $arg(-4i)=-\frac{\pi}{2}+2\pi k$

1.14. a $1=1$

b $-3=3e^{i\pi}$

c $-1+i=\sqrt{2}e^{i\frac{3\pi}{4}}$

d $3+3i=3\sqrt{2}e^{i\frac{\pi}{4}}$

e $1-\sqrt{3}i=2e^{-i\frac{\pi}{3}}$

f $-4i=4e^{-i\frac{\pi}{2}}$

1.15. **Proof** $z \neq 0, \theta \in arg(z) \implies Rez=|z|\cos(\theta) and Imz=|z|\sin(\theta)$ Let $\theta \in arg(z)$. The polar form for z can then be written as.

$$\begin{aligned} |z|\,e^{i\theta} &= |z|\,(\cos(\theta)+i\sin(\theta)) \\ &= |z|\cos(\theta)+i\,|z|\sin(\theta) \\ Rez &= |z|\cos(\theta) \\ Imz &= |z|\sin(\theta) \end{aligned}$$

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1.16. $\overline{z}$ is the reflection of z over the real axis. Therefore, $Arg(\overline{z})=-Arg(z)$ if z isn't on the negative real axis, including zero. Same with $arg(\overline{z})=-arg(z)$.

1.23. a $(1+i)^5=\sqrt{2}^5(\cos(\frac{5\pi}{4})+i\sin(\frac{5\pi}{4}))$

b $(1+\sqrt{3}i)^5=32(\cos(\frac{5\pi}{3})+i\sin(\frac{5\pi}{3}))$

c $(1+i)^{24}=4096(\cos(\frac{24\pi}{4})+i\sin(\frac{24\pi}{4}))$

1.24.

1.25.

1.26. a **Proof** $\left|e^{i\theta}\right|=1$

$$\begin{aligned} \left|e^{i\theta}\right| &= |\cos(\theta)+i\sin(\theta)| \\ &= \sqrt{\cos^2(\theta)+\sin^2(\theta)} \\ &= 1 \end{aligned}$$

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b **Proof** $\overline{e^{i\theta}}=e^{-i\theta}$

$$\begin{aligned} \overline{e^{i\theta}} &= \overline{\cos(\theta)+i\sin(\theta)} \\ &= \cos(\theta)-i\sin(\theta) \\ &= \cos(\theta)+i\sin(-\theta) \\ &= \cos(-\theta)+i\sin(-\theta) \\ &= e^{-i\theta} \end{aligned}$$

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1.27. a **Proof** $e^{i\theta}e^{i\phi} = e^{i(\theta+\phi)}$

$$\begin{aligned} e^{i\theta}e^{i\phi} &= (\cos(\theta) + i\sin(\theta))(\cos(\phi) + i\sin(\phi)) \\ &= \cos(\theta+\phi) + i\sin(\theta+\phi) \\ &= e^{i(\theta+\phi)} \end{aligned}$$

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1.28. a **Proof** $e^z e^w = e^{z+w}$

$$\begin{aligned} e^z e^w &= e^{x+yi}e^{u+vi} \\ &= e^x e^{yi} e^u e^{vi} \\ &= e^{x+u}e^{vi+yi} \\ &= e^{x+u+vi+yi} \\ &= e^{z+w} \end{aligned}$$

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1.33. **Proof**

$$\begin{aligned} 1 &= z^n \\ &= (|z|(\cos(arg(z)) + i\sin(arg(z))))^n \\ &= |z|^n(\cos(narg(z)) + i\sin(narg(z))) \\ &= \cos(narg(z)) + i\sin(narg(z)) && |z| \text{ must be } 1 \\ &= \cos(narg(z)) \\ 2\pi k &= narg(z) \\ (2\pi k)/n &= arg(z) \\ z &= e^{iarg(z)} = e^{\frac{2\pi ik}{n}} \end{aligned}$$

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1.34. $1, i, -1, -i$

1.35. $1, (1+i\sqrt{3})/2, (-1+i\sqrt{3})/2, -1, (-1-i\sqrt{3})/2, (1-i\sqrt{3})/2$