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**MATH 438: Introduction to Complex Variables**
**Assignment 3**

3. **Proof $n^n z^n$ converges only for $z = 0$**
By De Moirve's formula, we find

$$\begin{aligned} n^n z^n &= n^n |z|^n (\cos(n\theta) + i\sin(n\theta)) \\ &= (n|z|)^n (\cos(n\theta) + i\sin(n\theta)) \end{aligned}$$

Which is periodic with respect to $n$, which will not converge unless $n^n |z|^n$ approaches 0. Since the sequence is always zero when $z = 0$, the sequece converges. When $z \neq 0$, since $n$ is approaching infinity, there will exist $n_0$ such that for all $n \geq n_0$, $(n|z|) > 2$. Since $2^n$ doesn't converge to 0, the function will not converge, because the function will be greater than $2^n$ for all $n > n_0$.
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5. **Proof $b_n = 1 + 1/2 + 1/3 + ... + 1/n - \ln n, n \geq 1$ is decreasing**
Let $n \geq 1$

$$\begin{aligned} b_n &= 1 + 1/2 + 1/3 + ...1/n - \ln n \\ &= \sum_{i=1}^{n} \left(\frac{1}{i}\right) - \ln n \\ b_{n+1} - b_n &= \sum_{i=1}^{n+1} \left(\frac{1}{i}\right) - \ln(n+1) - \sum_{i=1}^{n} \left(\frac{1}{i}\right) + \ln(n) \\ &= \frac{1}{n+1} - \ln(n+1) + \ln(n) = \frac{1}{n+1} + \ln\frac{n}{n+1} \\ &< 0 \end{aligned}$$

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**Proof $a_n = 1 + 1/2 + 1/3 + ... + 1/(n-1) - \ln n, n \geq 1$ is increasing**
Let $n \geq 1$

$$\begin{aligned} a_n &= 1 + 1/2 + 1/3 + ...1/(n-1) - \ln n \\ &= \sum_{i=1}^{n-1} \left(\frac{1}{i}\right) - \ln n \\ a_{n+1} - a_n &= \sum_{i=1}^{n} \left(\frac{1}{i}\right) - \ln(n+1) - \sum_{i=1}^{n-1} \left(\frac{1}{i}\right) + \ln(n) \\ &= \frac{1}{n} - \ln(n+1) + \ln(n) = \frac{1}{n} + \ln\frac{n}{n+1} \\ &> 0 \end{aligned}$$

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**Proof $a_n$ $b_n$ converge to the same limit**
The defintion, $a_n = b_n - 1/n \implies b_n - a_n = 1/n$. Since $1/n$ approaches 0 as $n$ approaces infinity, $a_n$ and $b_n$ must approach the same value. Since $a_n$ is increasing while $b_n$ is decreasing we get that the two sequences converge to a finite number via the squeeze therom.
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**Proof** $0.5 < \gamma < 0.6$
Since $a_n$ is increasing, and $a_7 \approx 0.5041 > 0.5$, $\gamma > 0.5$, and since $b_n$ is decreasing, and $b_{50} \approx 0.587182 < 0.6$, $\gamma < 0.6$

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19. **Proof**
Since $|p(z)|$ is bounded below by 0, we know that the set of points $kz \in \mathbb{C} : p(z)$ must have an infimum. Since $|p(z)|$ is continous, $k$ must contain it's infimum. Therefore we know that $|p(z)|$ must attain it's minimum value at some point $z_0 \in \mathbb{C}$.

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**Proof**
Let $p(z) = 1 + az^m + ...$, where $m \geq 1$ and $a \neq 0$. Assume $p(z)$ is at a minimum at $z = 0$, where $p(z) = 1$. We know that $z^m = |z| (\cos(m\theta) + i \sin(m\theta))$. If we let $z_1 = (1/a)e^{i\pi}$ we find $|p(z_1)| = 0$, which contradicts $p(0)$ being a minimum.

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