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MATH 438: Introduction to Complex Variables
Assignment 1

5. a **Proof** $1+z+z^2+...+z^n = \frac{1-z^{n+1}}{1-z}$

$$\begin{aligned}
1+z+z^2+...+z^n &= \sum_{j=0}^{n}(z^j) \\
&= \frac{1-z}{1-z}\sum_{j=0}^{n}(z^j) \\
&= \frac{1}{1-z}\left(\sum_{j=0}^{n}(z^j) - z\sum_{j=0}^{n}(z^j)\right) = \frac{1}{1-z}\left(\sum_{j=0}^{n}(z^j) - \sum_{j=0}^{n+1}(z^j)\right) \\
&= \frac{1}{1-z}\left(z^0 - z^{n+1}\right) \\
&= \frac{1-z^{n+1}}{1-z}
\end{aligned}$$

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6. a **Proof** $\prod_{j=0}^{n-1}(z-\omega_j) = z^n - 1$

The roots of the function $z^n - 1$ are $\omega_0, \omega_1, ..., \omega_{n-1}$. Therefore, the functions $\prod_{j=0}^{n-1}(z-\omega_j)$ and $z^n - 1$ must be proportional. Since the coefficient of the z^n term of $\prod_{j=0}^{n-1}(z-\omega_j)$ must be 1, the functions must be equal.

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b **Proof** $\omega_0 + ... + \omega_{n-1} = 0$

$$\begin{aligned}
\omega_j &= \exp\left(\frac{2\pi j i}{n}\right) \\
\omega_0 + ... + \omega_{n-1} &= \exp(0) + \exp\left(\frac{2\pi i}{n}\right) + \exp\left(\frac{4\pi i}{n}\right) + ... + \exp\left(\frac{2\pi(n-1)i}{n}\right) \\
&= \exp\left(\frac{2\pi i}{n}\right)^0 + \exp\left(\frac{2\pi i}{n}\right)^1 + \exp\left(\frac{2\pi i}{n}\right)^2 + ... + \exp\left(\frac{2\pi i}{n}\right)^{n-1} \\
&= \sum_{j=0}^{n-1}\exp\left(\frac{2\pi i}{n}\right)^j \\
&= \frac{1-\exp\left(\frac{2\pi i}{n}\right)^n}{1-\exp\left(\frac{2\pi i}{n}\right)} = \frac{1-\exp(2\pi i)}{1-\exp\left(\frac{2\pi i}{n}\right)} \\
&= 0
\end{aligned}$$

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c **Proof** $\prod_{j=0}^{n-1} \omega_j = (-1)^{n-1}$

$$\begin{aligned}\prod_{j=0}^{n-1} \omega_j &= \prod_{j=0}^{n-1} \exp\left(\frac{2\pi j i}{n}\right) \\ &= \exp\left(\sum_{j=0}^{n-1} \frac{2\pi j i}{n}\right) = \exp\left(\frac{2\pi i}{n} \sum_{j=0}^{n-1} j\right) \\ &= \exp\left(\frac{2\pi i}{n} \frac{n(n-1)}{2}\right) \\ &= \exp\left(\pi(n-1)i\right) \\ &= (-1)^{n-1}\end{aligned}$$

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d **Proof** $\sum_{j=0}^{n-1} \omega_j^k = \begin{cases} 0, & 1 \le k \le n-1 \\ n, & k = n \end{cases}$

Case 1: $k = n$

$$\begin{aligned}\sum_{j=0}^{n-1} \omega_j^n &= \sum_{j=0}^{n-1} 1 \\ &= n\end{aligned}$$

Case 2: $1 \le k \le n-1$

$$\begin{aligned}\sum_{j=0}^{n-1} \omega_j^k &= \sum_{j=0}^{n-1} \exp\left(\frac{2\pi j i}{n}\right)^k \\ &= \sum_{j=0}^{n-1} \exp\left(\frac{2\pi j k i}{n}\right) \\ &= \sum_{j=0}^{n-1} \exp\left(\frac{2\pi k i}{n}\right)^j \\ &= \frac{1 - \exp\left(\frac{2\pi k i}{n}\right)^n}{1 - \exp\left(\frac{2\pi k i}{n}\right)} = \frac{1 - \exp\left(2\pi k i\right)}{1 - \exp\left(\frac{2\pi k i}{n}\right)} \\ &= 0\end{aligned}$$

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