Brandon Toner
September 17, 2021

MATH 438: Introduction to Complex Variables
Assignment 5

10.

$$
\begin{aligned}
\sum_{n=0}^{\infty}\left[\binom{\alpha}{n}z^n\right] &= \sum_{n=0}^{\infty}\left[\prod_{i=1}^{n}\left(\frac{\alpha-n+i}{i}\right)z^n\right] \\
\text{Let } s_n &= \prod_{i=1}^{n}\left(\frac{\alpha-n+i}{i}\right)z^n \\
\frac{s_{n+1}}{s_n} &= \frac{z^{n+1}\prod\limits_{i=1}^{n+1}\frac{\alpha-n-1+i}{i}}{z^n\prod\limits_{i=1}^{n}\frac{\alpha-n+i}{i}} \\
&= z\frac{\alpha}{n+1}\frac{\prod\limits_{i=1}^{n}\frac{\alpha-n-1+i}{i}}{\prod\limits_{i=1}^{n}\frac{\alpha-n+i}{i}} \\
&= z\frac{\alpha}{n+1}\prod_{i=1}^{n}\frac{\alpha-n-1+i}{\alpha-n-i} \\
&= z\frac{\alpha-n}{n+1} \\
\left|\frac{s_{n+1}}{s_n}\right| &= \left|z\frac{\alpha-n}{n+1}\right| \\
&< 1 \;\;\forall\,|z| < \left|\frac{n+1}{n-\alpha}\right| < 1 \\
&\Longrightarrow \text{Convergence for } \;|z| < 1
\end{aligned}
$$

Proof $\frac{1}{n!}\frac{d^n(1+z)^\alpha}{dz^n} = \binom{\alpha}{n}(1+z)^{\alpha-n}$

$$\begin{aligned}
\textbf{Basis: } \frac{1}{1!}\frac{d(1+z)^\alpha}{dz} &= \alpha(1+z)^{\alpha-1} \\
\binom{\alpha}{1}(1+z)^{\alpha-1} &= \alpha(1+z)^{\alpha-1} \\
\textbf{Assume } \frac{1}{n!}\frac{d^n(1+z)^\alpha}{dz^n} &= \binom{\alpha}{n}(1+z)^{\alpha-n} \\
\frac{d}{dz}\left(\frac{1}{n!}\frac{d^n(1+z)^\alpha}{dz^n}\right) &= \frac{1}{n!}\frac{d^{n+1}(1+z)^\alpha}{dz^{n+1}} \\
\frac{d}{dz}\left(\binom{\alpha}{n}(1+z)^{\alpha-n}\right) &= \binom{\alpha}{n}(\alpha-n)(1+z)^{\alpha-n-1} \\
\frac{1}{n!}\frac{d^{n+1}(1+z)^\alpha}{dz^{n+1}} &= \binom{\alpha}{n}(\alpha-n)(1+z)^{\alpha-(n+1)} \\
\frac{1}{(n+1)n!}\frac{d^{n+1}(1+z)^\alpha}{dz^{n+1}} &= \binom{\alpha}{n}\frac{\alpha-n}{n+1}(1+z)^{\alpha-(n+1)} \\
\frac{1}{(n+1)!}\frac{d^{n+1}(1+z)^\alpha}{dz^{n+1}} &= \binom{\alpha}{n+1}(1+z)^{\alpha-(n+1)}
\end{aligned}$$

■

Proof $\sum\limits_{n=0}^{\infty}\binom{\alpha}{n}z^n = (1+z)^\alpha$

The Taylor series for a function, $f(z)$ is:

$$f(z) = f(z_0) + \sum_{n=1}^{\infty}\left[\left.\frac{1}{n!}\frac{d^n f(z)}{dz^n}\right|_{z=z_0}(z-z_0)^n\right]$$

Let $z_0 = 0$. For $f(z) = (1+z)^\alpha$, we know

$$\begin{aligned}
\left.\frac{1}{n!}\frac{d^n(1+z)^\alpha}{dz^n}\right|_{z=0} &= \left.\binom{\alpha}{n}(1+z)^{\alpha-n}\right|_{z=0} \\
&= \binom{\alpha}{n}
\end{aligned}$$

Thus, the Taylor series for $f(z) = (1+z)^\alpha$ is

$$\begin{aligned}
(1+z)^\alpha &= 1 + \sum_{n=1}^{\infty}\left[\binom{\alpha}{n}z^n\right] \\
&= \sum_{n=0}^{\infty}\left[\binom{\alpha}{n}z^n\right]
\end{aligned}$$

■

11.

$$\begin{aligned}
J_n(z) &= \sum_{k=0}^{\infty} \frac{(-1)^k z^{n+2k}}{k!(n+k)!2^{n+2k}} \\
\frac{\partial J_n(z)}{\partial x} &= \sum_{k=0}^{\infty} \frac{(-1)^k (n+2k) z^{n+2k-1}}{k!(n+k)!2^{n+2k}} \\
\frac{\partial J_n(z)}{\partial y} &= \sum_{k=0}^{\infty} \frac{(-1)^k \mathbf{i}(n+2k) z^{n+2k-1}}{k!(n+k)!2^{n+2k}} \\
&= \mathbf{i}\frac{\partial J_n(z)}{\partial x} \\
&\implies J_n(z) \text{ is entire.}
\end{aligned}$$

12. For an even function, $f(z) = f(-z)$ for all z.

$$\begin{aligned}
f(z) &= \sum_{n=0}^{\infty} a_n z^n \\
f(-z) &= \sum_{n=0}^{\infty} a_n (-z)^n \\
\sum_{n=0}^{\infty} a_n z^n &= \sum_{n=0}^{\infty} a_n (-z)^n = \sum_{n=0}^{\infty} a_n (-1)^n z^n \\
&\implies (-1)^n = 1 \text{ or } a_n = 0 \\
&\implies a_n = 0 \text{ if } n \text{ is odd}
\end{aligned}$$

For an odd function, $f(z) = -f(-z)$ for all z.

$$\begin{aligned}
f(z) &= \sum_{n=0}^{\infty} a_n z^n \\
-f(-z) &= \sum_{n=0}^{\infty} -a_n (-z)^n \\
\sum_{n=0}^{\infty} a_n z^n &= \sum_{n=0}^{\infty} -a_n (-z)^n = \sum_{n=0}^{\infty} a_n (-1)^{n+1} z^n \\
&\implies (-1)^{n+1} = 1 \text{ or } a_n = 0 \\
&\implies a_n = 0 \text{ if } n \text{ is even}
\end{aligned}$$

13. Assume $f(z_0) = g(z_0) = 0$ and $g(z)$ is not identically zero.

$$\begin{aligned}
\lim_{z\to z_0} \frac{f(z)}{g(z)} &= \lim_{h\to 0} \frac{f(z_0+h)}{g(z_0+h)} = \lim_{h\to 0} \frac{f(z_0+h)-0}{g(z_0+h)-0} \\
&= \lim_{h\to 0} \frac{f(z_0+h)-f(z_0)}{g(z_0+h)-g(z_0)} \\
&= \lim_{h\to 0} \frac{(f(z_0+h)-f(z_0))h}{(g(z_0+h)-g(z_0))h} \\
&= \lim_{h\to 0} \frac{f(z_0+h)-f(z_0)}{h} \frac{h}{g(z_0+h)-g(z_0)} \\
&= \frac{f'(z_0)}{g'(z_0)}
\end{aligned}$$

1. (a)

$$\begin{aligned}
f(z) &= \frac{1}{z^2+1} \\
g(w) = f(1/w) &= \frac{w^2}{w^2+1} \\
g(w) &= w^2 - w^4 + w^6 + ... \\
&= \sum_{j=1}^{\infty} (-1)^{j+1} w^{2j}
\end{aligned}$$

(b)

$$\begin{aligned}
f(z) &= \frac{z^2}{z^3-1} \\
g(w) = f(1/w) &= \frac{w}{1-w^3} \\
&= w + w^4 + w^7 + w^1 0 + ... \\
&= \sum_{j=0}^{\infty} w^{1+3j}
\end{aligned}$$

(c)

$$\begin{aligned}
f(z) &= e^{1/z^2} \\
g(w) = f(1/w) &= e^{w^2} \\
&= 1 + w^2 + \frac{w^4}{2} + \frac{w^6}{6} + \frac{w^8}{24} + ... \\
&= \sum_{j=0}^{\infty} \frac{w^{2k}}{k!}
\end{aligned}$$

(d)

$$\begin{aligned}
f(z) &= z \sinh(1/z) \\
g(w) = f(1/w) &= \sinh(w)/w \\
&= 1 + \frac{w^2}{6} + \frac{w^4}{120} + \frac{w^6}{5040} + ... \\
&= \sum_{j=0}^{\infty} \frac{w^{2j}}{(2j+1)!}
\end{aligned}$$

2.

$$\begin{aligned}
\text{Let } g(w) &= f(1/w) \\
\lim_{z\to\infty} z[f(z) - f(\infty)] &= \lim_{w\to 0} \frac{f(1/w) - f(\infty)}{w} \\
&= \lim_{w\to 0} \frac{g(w) - g(0)}{w} \\
&= \left.\frac{dg(w)}{dw}\right|_{w=0} \\
&= \left.\frac{df(z)}{dz}\right|_{z=\infty}
\end{aligned}$$