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MATH 438: Introduction to Complex Variables
Proofs

2. Chapter 2

2.1. a. Real dot: 8
Complex dot: $8 + \mathbf{i}$
b. Real dot: 8
Complex dot: $8 - \mathbf{i}$
c. Real dot: 1
Complex dot: $i + 2\mathbf{i}$
d. Real dot: 2
Complex dot: $2 - \mathbf{i}$
e. Real dot: x
Complex dot: $x + y\mathbf{i}$
f. Real dot: y
Complex dot: $y - x\mathbf{i}$

2.2.

$$\begin{aligned} Re(z\bar{w}) &= Re(xu + yv + \mathbf{i}(yu - vx)) = xu + yv \\ Re(\bar{z}w) &= Re(xu + yv + \mathbf{i}(vx - uy)) = xu + yv \end{aligned}$$

2.3. **Proof** $|z \cdot w| \leq |z|\,|w|$

$$\begin{aligned} |Re(z)| &\leq |z| \\ |Im(z)| &\leq |z| \\ |z \cdot w| &= |Re(z\bar{w})| \\ &\leq |z\bar{w}| \\ &= |z|\,|w| \end{aligned}$$

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2.4. a **Proof** $|z + w|^2 = |z|^2 + |w|^2 + 2Re(z\bar{w})$

$$\begin{aligned} |z + w|^2 &= (z + w)\overline{(z + w)} \\ &= z\bar{z} + w\bar{w} + z\bar{w} + \bar{z}w \\ &= |z|^2 + |w|^2 + 2Re(z\bar{w}) \end{aligned}$$

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b **Proof** $|z - w|^2 = |z|^2 + |w|^2 - 2Re(z\bar{w})$

$$\begin{aligned} |z - w|^2 &= (z - w)\overline{(z - w)} \\ &= z\bar{z} + w\bar{w} - z\bar{w} - \bar{z}w \\ &= |z|^2 + |w|^2 - 2Re(z\bar{w}) \end{aligned}$$

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2.5. **Proof** $|z + w| \leq |z| + |w|$

$$\begin{aligned} |z + w|^2 &= |z|^2 + |w|^2 + 2Re(z\bar{w}) \\ &\leq |z|^2 + |w|^2 + |z|\,|w| \\ &= (|z| + |w|)^2 \\ &\Longrightarrow \\ |z + w| &\leq |z| + |w| \end{aligned}$$

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2.6. **Proof** $||z| - |w|| \leq |z - w|$

$$\begin{aligned} |z| &= |z - w + w| \\ &\leq |z - w| + |w| \\ |z| - |w| &\leq |z - w| \end{aligned}$$

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2.7. Proof by induction using triangle inequality.

2.8.

2.9.

$$\begin{aligned} p(z) &= a_n z^n + a_{n-1} z^{n-1} + ... + a_1 z + a_0 = 0 \\ p(\bar{z}) &= a_n \bar{z}^n + a_{n-1} \bar{z}^{n-1} + ... + a_1 \bar{z} + a_0 \\ &= \overline{P(z)} \\ &= 0 \end{aligned}$$

2.10. **Proof**
Pick $\epsilon > 0$. Since $|w_n| \to 0$, $\exists N$ such that $n \geq N \implies |w_n| < \epsilon$. Therefore, $|z_n - z| \leq |w_n| < \epsilon$ $\implies z_n \to z$.
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2.11. Proof showing it z_n can't get close enought to z otherwise.

2.12. For each $\epsilon, \exists N$ such that $|z_n - z| < \epsilon$ for $n > N$. Thus $|z_n| < \epsilon + |z| \implies z_n$ is bounded for $n > N$. Since $z_n n \in 1..N$ is finite, it must be bounded. Therefore z_n is bounded.

2.13.

2.14. $\frac{1-1}{1+1} = 0$

2.15. $\frac{z_n^2 - 1}{z_n - 1} = z_n + 1 = 2$

2.16. 1

2.17. 0

2.18. $z_n \to \infty$ if $\forall K > 0\ \exists N$ such that $n > N \implies |z_n| \geq K$

2.25. The sum of two continuous functions must be continuous, so $f(z)$ is continuous.
Since u and $\mathbf{i}v$ are on seperate axis, their addition can't be continuous if either one isn't continuous.

2.28.

$$\begin{aligned} z &= \frac{z - w}{h} \\ &= \frac{z}{h} - \frac{w}{h} \\ z\frac{h-1}{h} &= -\frac{w}{h} \\ z &= -\frac{w}{h-1} \\ &= \frac{w}{1-h} \end{aligned}$$