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MATH 438: Introduction to Complex Variables
Assignment 4

2. **Proof** $1 + 2z + 3z^2 + ... + nz^{n-1} = \frac{1-z^n}{(1-z)^2} - \frac{nz^n}{1-z}$

$$\begin{aligned} 1 + 2z + 3z^2 + ... + nz^{n-1} &= \sum_{k=1}^{n} \left(kz^{k-1}\right) \\ &= \sum_{k=1}^{n} \left(\frac{dz^k}{dz}\right) = \frac{d}{dz}\left(\sum_{k=1}^{n} z^k\right) \\ &= \frac{d}{dz}\left(\frac{z^{n+1} - z}{z - 1}\right) \\ &= \frac{1 - z^n}{(1-z)^2} - \frac{nz^n}{1-z} \end{aligned}$$

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6. **Proof**

$$\begin{aligned} H(z) &= \int_0^1 \frac{h(t)}{t-z}\ dt \\ H'(z) &= \lim_{w\to 0} \frac{H(z+w) - H(z)}{w} \\ &= \lim_{w\to 0} \frac{\int_0^1 \frac{h(t)}{t-z-w}\ dt - \int_0^1 \frac{h(t)}{t-z}\ dt}{w} = \lim_{w\to 0} \frac{\int_0^1 \left(\frac{h(t)}{t-z-w} - \frac{h(t)}{t-z}\right)\ dt}{w} \\ &= \lim_{w\to 0} \int_0^1 \left(\frac{h(t)}{w(t-z-w)} - \frac{h(t)}{w(t-z)}\right)\ dt \\ &= \lim_{w\to 0} \int_0^1 \frac{h(t)}{(t-z)(t-w-z)}\ dt \\ &= \int_0^1 \frac{h(t)}{(t-z)^2}\ dt \end{aligned}$$

$H(z)$ is analytic because it's derivative exists for all z.
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5. **Proof**

$$\begin{aligned} &\cos(z) = 0.5(e^{\mathbf{i}z} + e^{-\mathbf{i}z}) \\ &g_n(z) = -\mathbf{i}\log(z \pm \sqrt{z^2-1}) = -\mathbf{i}\ln\left|z \pm \sqrt{z^2-1}\right| + \mathrm{Arg}(z) + 2\pi n \end{aligned}$$

The derivative of the nth branchs of $g(z)$, $g_n(z)$ should all be equal, since the n term's deriviatve is zero.
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9. **Proof**

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\begin{aligned}
\int\int_D |f'(z)|^2 \; dx \; dy &= \int\int_D \left|\frac{d(x+yi)^2}{dx}\right|^2 \; dx \; dy = \int\int_D \left|\frac{d(x^2-y^2+2xyi)}{dx}\right|^2 \; dx \; dy \\
&= \int\int_D |2x+2yi|^2 \; dx \; dy = \int\int_D 4x^2+4y^2 \; dx \; dy \\
&= 4\int_0^1\int_0^{2\pi} r^2 r \; dr \; d\theta \\
&= 2\pi \\
&= 2(\text{Area of a circle with radius 1})
\end{aligned}
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