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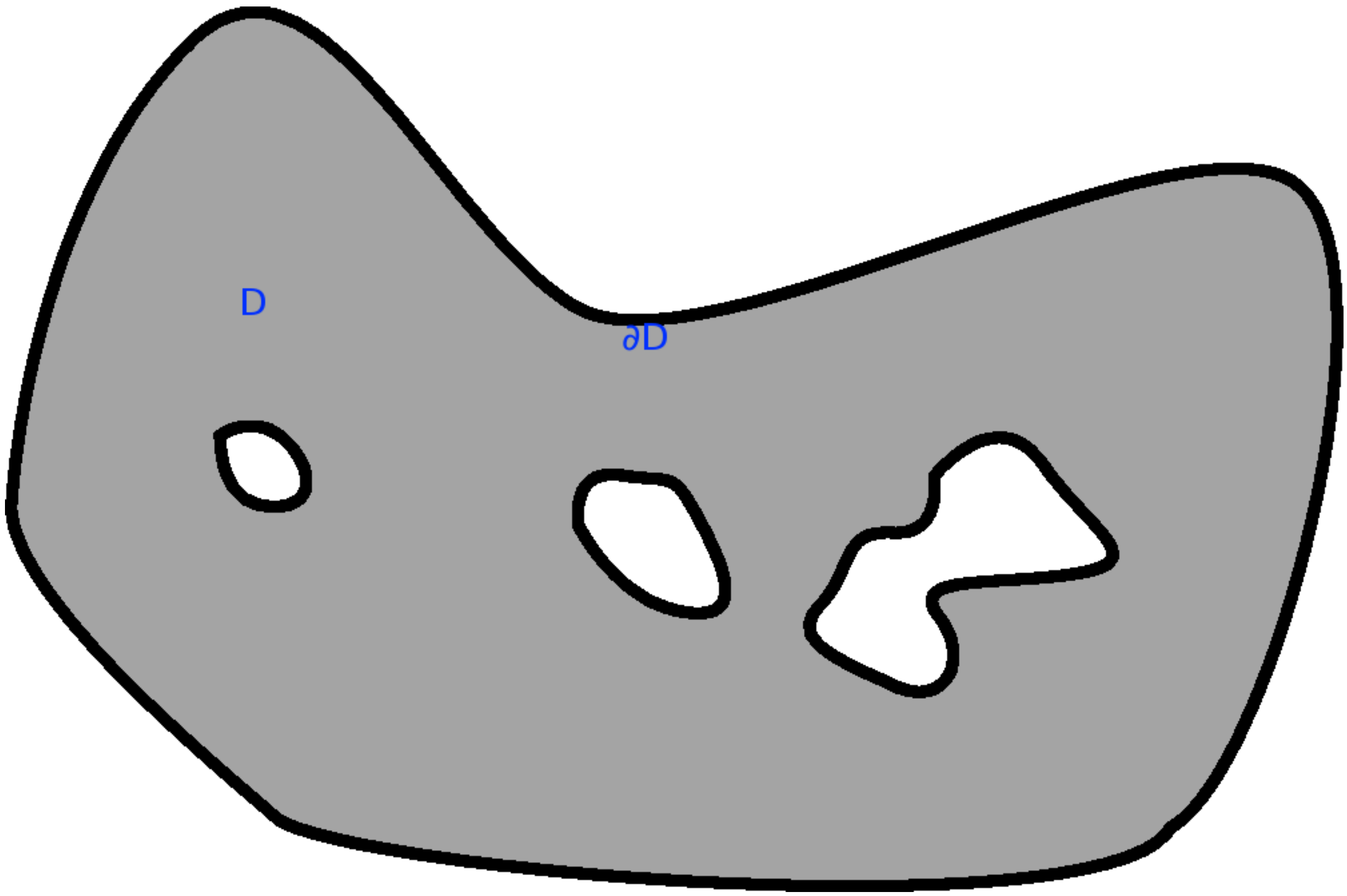
MATH 438: Introduction to Complex Variables
Final Exam

1. a) **State Green's Theorem for a multiply-connected domain D and draw a picture:**

$$\int\limits_{\partial D} u\ dx + v\ dy \quad = \quad \int\int\limits_{D} v_x - u_y\ dx\ dy$$



b) **Using Green's Theorem for a multiply-connected domain D, prove the Deformation Theorem for f analytic on $D \cup \partial D$.**
Theorem: Let $f(z)$ be analytic on domain $D \cup \partial D$, bounded by $\Gamma \cup \gamma_1 \cup ... \cup \gamma_n$. Then

$$\int\limits_{\Gamma} f(z)\ dz \quad = \quad \sum_{n=1}^{k} \int\limits_{\gamma_n} f(z)\ dz$$

Proof
Assume $\gamma_1, ..., \gamma_k$ are ordered such that γ_n and γ_{n+1} can be connected with a line segment that does not intersect any other γ_j. Draw line segments $\Phi_0, ..., \Phi_k$ such that for $1 \leq n \leq k-1$, Φ_n connects γ_n and γ_{n+1}. Φ_0 and Φ_k connect Γ to γ_1 and γ_k. D is now divided into two simply connected domains, D_1 and

D_2, by $\Gamma \cup \gamma_1 \cup ... \cup \gamma_k \cup \Phi_0 \cup ... \cup \Phi_k$. Now, by Cauchy's Theorm,

$$\begin{aligned}
\int_{\partial D_1} f(z)\ dz &= 0 \\
\int_{\partial D_2} f(z)\ dz &= 0 \\
\int_{\partial D_1 \cup \partial D_2} f(z)\ dz &= 0
\end{aligned}$$

Since $\Phi_0, ..., \Phi_k$ were traversed in opposite directions in ∂D_1 and ∂D_2

$$\begin{aligned}
\int_{\partial D} f(z)\ dz &= 0 \\
\int_{\Gamma \cup \gamma_1 \cup ... \cup \gamma_k} f(z)\ dz &= 0 \\
\int_{\Gamma} f(z)\ dz + \int_{\gamma_1 \cup ... \cup \gamma_k} f(z)\ dz &= 0 \\
\int_{\Gamma} f(z)\ dz + \sum_{n=1}^{k} \int_{\gamma_n} f(z)\ dz &= 0
\end{aligned}$$

γ_n is currently being traversed in the opposite direction of Γ, but if we let $\hat{\gamma_n}$ be γ_n traversed in the same direction as Γ,

$$\begin{aligned}
\int_{\Gamma} f(z)\ dz &- \sum_{n=1}^{k} \int_{\hat{\gamma_n}} f(z)\ dz = 0 \\
\int_{\Gamma} f(z)\ dz &= \sum_{n=1}^{k} \int_{\hat{\gamma_n}} f(z)\ dz
\end{aligned}$$

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c) **Complete: For $f(z)$ be analytic on $D \cup \partial D$, the Cauchy integral formula says**

$$f(z) = \frac{1}{2\pi \mathbf{i}} \int_{\gamma} \frac{f(\xi)}{\xi - z} d\xi \quad \forall z \text{ inside } \gamma$$

d) **The Cauchy integral formula for derivatives says**

$$f^{(k)}(z) = \frac{k!}{2\pi \mathbf{i}} \int_{\gamma} \frac{f(\xi)}{(\xi - z)^{k+1}} d\xi \quad \forall z \text{ inside } \gamma$$

e) **Prove the Cauchy integral formula for derivatives.**
Proof by induction

Base case: $k = 1$
Let z be any value inside γ. From the Cauchy integral formula, we know

$$\begin{aligned} f(z) &= \frac{1}{2\pi\mathbf{i}}\int\limits_\gamma \frac{f(\xi)}{\xi - z}d\xi \\ f^{(1)}(z) &= \frac{d}{dz}\left[\frac{1}{2\pi\mathbf{i}}\int\limits_\gamma \frac{f(\xi)}{\xi - z}d\xi\right] \\ &= \frac{1}{2\pi\mathbf{i}}\int\limits_\gamma \frac{f(\xi)}{(\xi - z)^2}d\xi \end{aligned}$$

For $k \geq 2$:

$$\begin{aligned} f^{(k)}(z) &= \frac{k!}{2\pi\mathbf{i}}\int\limits_\gamma \frac{f(\xi)}{(\xi - z)^{k+1}}d\xi \\ \frac{d}{dz}f^{(k)}(z) &= \frac{d}{dz}\left[\frac{k!}{2\pi\mathbf{i}}\int\limits_\gamma \frac{f(\xi)}{(\xi - z)^{k+1}}d\xi\right] \\ f^{(k+1)}(z) &= \frac{k!}{2\pi\mathbf{i}}\int\limits_\gamma \frac{d}{dz}\left[\frac{f(\xi)}{(\xi - z)^{k+1}}d\xi\right] \\ &= \frac{k!}{2\pi\mathbf{i}}\int\limits_\gamma (k+1)\frac{f(\xi)}{(\xi - z)^{k+2}}d\xi \\ &= \frac{(k+1)!}{2\pi\mathbf{i}}\int\limits_\gamma \frac{f(\xi)}{(\xi - z)^{(k+1)+1}}d\xi \end{aligned}$$

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2. **Find all the following integrals around the unit circle γ using the Cauchy formulas**

a)

$$\begin{aligned} \int\limits_\gamma \frac{z^3}{(2z-1)^2}dz &= \int\limits_\gamma \frac{\xi^3}{(2\xi - 1)^2}d\xi \\ &= \frac{1}{4}\int\limits_\gamma \frac{\xi^3}{(\xi - 0.5)^2}d\xi \\ &= \frac{2\pi\mathbf{i}}{4}\frac{1}{2\pi\mathbf{i}}\int\limits_\gamma \frac{\xi^3}{(\xi - 0.5)^2}d\xi \\ &= \frac{2\pi\mathbf{i}}{4}\left.\frac{d\xi^3}{d\xi}\right|_{\xi=0.5} \\ &= \frac{3\pi\mathbf{i}}{8} \end{aligned}$$

b)

$$\int_\gamma \frac{e^{2z}}{z^3} dz = \int_\gamma \frac{e^{2\xi}}{\xi^3} d\xi$$
$$= \frac{2\pi \mathbf{i}}{2!} \left.\frac{d^2 e^{2\xi}}{d\xi^2}\right|_{\xi=0}$$
$$= 4\pi \mathbf{i}$$

c)

$$\frac{1}{2\pi \mathbf{i}} \int_\gamma \frac{\cos(2z)}{z-0.5} dz = \frac{1}{2\pi \mathbf{i}} \int_\gamma \frac{\cos(2\xi)}{\xi - 0.5} d\xi$$
$$= \cos(2\xi)|_{\xi=0.5}$$
$$= \cos(1) \approx 0.540302$$

d) Since $\frac{\cos(2z)}{z-2}$ is analytic on the simply connected domain bounded by γ,

$$\frac{1}{2\pi \mathbf{i}} \int_\gamma \frac{\cos(2z)}{z-2} dz = 0$$

by the Cauchy Theorem.

3. a) **Explain what $O(z^m)$ stands for in the Taylor expansion of $f(z)$ centered at 0**

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} z^k = \sum_{k=0}^{m-1} \frac{f^{(k)}(0)}{k!} z^k + O(z^m)$$

The $O(z^m)$ in the Taylor expansion of $f(z)$ centered at 0 means the sum of the remaining terms is less than cz^m for some $c > 0$.

b) **What would you insert in the following expression to make it complete?**

$$\frac{1}{1-z} = 1 + z + z^2 + O(z^3)$$

c) **If $f(z) = a_0 + a_1 z + a_2 z^2 + O(z^3)$ and $g(z) = b_0 + b_1 z + b_2 z^2 + O(z^3)$ then complete the statement**

$$f(z)g(z) = a_0 b_0 + (a_0 b_1 + a_1 b_0) z + (a_0 b_2 + a_1 b_1 + a_2 b_0) z^2 + O(z^3)$$

d) **If $f(z) = \sum\limits_{n=0}^{\infty} a_n z^n$ how would you write $O(z^3)$ (for f) in summation notation?**

$$f(z) = \sum_{n=0}^{\infty} a_n z^n = \sum_{n=0}^{2} a_n z^n + O(z^3)$$
$$\sum_{n=3}^{\infty} a_n z^n = O(z^3)$$

4. **Calculate the terms through order seven of the power series expansion about $z = 0$ of the function $1/\cos(z)$.**

$$\begin{aligned}
f^{(0)}(0) & & & & &= 1 \\
f^{(1)}(0) &= \left.\frac{df^{(0)}(z)}{dz}\right|_{z=0} &= \frac{\sin(0)}{\cos^2(0)} & & &= 0 \\
f^{(2)}(0) &= \left.\frac{df^{(1)}(z)}{dz}\right|_{z=0} &= \frac{\sin^2(0)+1}{\cos^3(0)} & & &= 1 \\
f^{(3)}(0) &= \left.\frac{df^{(2)}(z)}{dz}\right|_{z=0} &= \frac{\sin(0)(\sin^2(0)+5)}{\cos^4(0)} & & &= 0 \\
f^{(4)}(0) &= \left.\frac{df^{(3)}(z)}{dz}\right|_{z=0} &= \frac{\sin^2(0)(\sin^2(0)+18)+5}{\cos^5(0)} & & &= 5 \\
f^{(5)}(0) &= \left.\frac{df^{(4)}(z)}{dz}\right|_{z=0} &= \frac{\sin(0)(\sin^4(0)+58\sin^2(0)+61)}{\cos^6(0)} & & &= 0 \\
f^{(6)}(0) &= \left.\frac{df^{(5)}(z)}{dz}\right|_{z=0} &= \frac{\sin^6(0)+179\sin^4(0)+478\sin^2(0)+61}{\cos^7(z)} & & &= 61 \\
f^{(7)}(0) &= \left.\frac{df^{(6)}(z)}{dz}\right|_{z=0} &= \frac{\sin(0)(\sin^6(0)+543\sin^4(0)+3111\sin^2(0)+1385)}{\cos^8(0)} & & &= 0
\end{aligned}$$

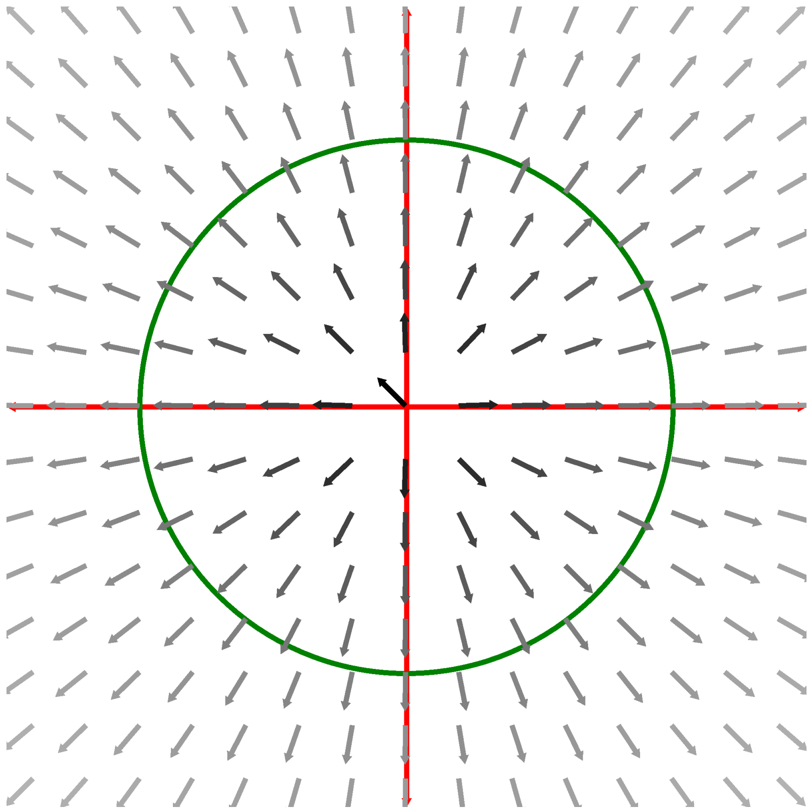
$$f(z) = 1 + \frac{z^2}{2} + \frac{5z^4}{24} + \frac{61z^6}{720} + O(z^8)$$

5. **Let $\overline{f(z)}$ be a velocity field of a fluid flowing over a domain D and let γ be a curve in D.**

   a) **State a formula for the circulation and flow rate of $\overline{f(z)}$ along and across γ in terms of a complex integral.**

$$\begin{aligned}
\text{Flow rate along } \gamma &= \int_\gamma \overline{f(z)} \cdot dz \\
\text{Flow rate across } \gamma &= \int_\gamma \overline{f(z)} \cdot -\mathbf{i}\ dz
\end{aligned}$$

   b) **Graph the velocity vector field $\overline{f(z)}$ for $f(z) = \frac{1}{z}$ and find its circulation and flux along and across the unit circle.**



Green circle is the unit circle. Arrows point in direction of $\overline{f(z)}$, and get darker as $\left|\overline{f(z)}\right| \to \infty$

$$\begin{aligned}
\text{Flow rate along } \gamma &= \int_\gamma \overline{f(z)} \cdot dz \\
&= \int_\gamma \overline{\left(\frac{1}{z}\right)} \cdot dz \\
&= \int_\gamma \overline{\left(\frac{\bar{z}}{z\bar{z}}\right)} \cdot dz = \int_\gamma \frac{z}{|z|^2} \cdot dz \\
&= \int_0^{2\pi} \frac{\cos(\theta) + \mathbf{i}\sin(\theta)}{1} \cdot (-\sin(\theta) + \mathbf{i}\cos(\theta)) d\theta \\
&= \int_0^{2\pi} -\cos(\theta)\sin(\theta) + \sin(\theta)\cos(\theta) d\theta = \int_0^{2\pi} 0 \; d\theta \\
&= 0
\end{aligned}$$

$$
\begin{aligned}
\text{Flow rate across } \gamma &= \int_{\gamma} \overline{f(z)} \cdot -\mathbf{i} dz \\
&= \int_{\gamma} \frac{z}{|z|^2} \cdot -\mathbf{i} dz \\
&= \int_0^{2\pi} \frac{\cos(\theta) + \mathbf{i}\sin(\theta)}{1} \cdot -\mathbf{i}(-\sin(\theta) + \mathbf{i}\cos(\theta)) d\theta \\
&= \int_0^{2\pi} (\cos(\theta) + \mathbf{i}\sin(\theta)) \cdot (\cos(\theta) + \mathbf{i}\sin(\theta)) d\theta \\
&= \int_0^{2\pi} \cos^2(\theta) + \sin^2(\theta) d\theta = \int_0^{2\pi} 1 \; d\theta \\
&= 2\pi
\end{aligned}
$$

6. **Find the Laurent expansions of $\frac{1}{z^2 - z}$ centered at 0**

$$
\begin{aligned}
a_n &= \frac{1}{2\pi\mathbf{i}} \int_C \frac{1}{(\xi - 1)\xi^{n+1}} - \frac{1}{\xi^{n+2}} d\xi \\
&= \frac{1}{2\pi\mathbf{i}} \int_C \frac{1}{(\xi - 1)\xi^{n+1}} d\xi - \frac{1}{2\pi\mathbf{i}} \int_C \frac{1}{\xi^{n+2}} d\xi \\
\frac{1}{2\pi\mathbf{i}} \int_C \frac{1}{(\xi - 1)\xi^{n+1}} d\xi &= \frac{1}{n!} \frac{n!}{2\pi\mathbf{i}} \int_C \frac{1}{(\xi - 1)\xi^{n+1}} d\xi \\
&= \frac{1}{n!} \frac{d^n}{d\xi^n} \left[\frac{1}{\xi - 1} \right] \Bigg|_{\xi=0} \\
&= \frac{1}{n!} \frac{(-1)^n n!}{(\xi - 1)^{n+1}} \Bigg|_{\xi=0} = \frac{(-1)^n n!}{(-1)^{n+1} n!} \\
&= -1 \\
\frac{1}{2\pi\mathbf{i}} \int_C \frac{1}{\xi^{n+2}} d\xi &= 0 \text{ because } \frac{1}{\xi^{n+2}} \text{ is analytic.} \\
a_n &= -1
\end{aligned}
$$

$$
\begin{aligned}
b_n &= \frac{1}{2\pi\mathbf{i}} \int\limits_C \frac{\xi^{n-2}}{\xi - 1} d\xi \\
&= \frac{1}{2\pi\mathbf{i}} \int\limits_0^{2\pi} \frac{(re^{\mathbf{i}\theta})^{n-2}}{re^{\mathbf{i}\theta} - 1} e^{\mathbf{i}\theta} r\mathbf{i} d\theta \\
&= \frac{r^{n-1}}{2\pi} \int\limits_0^{2\pi} \frac{e^{\mathbf{i}\theta(n-1)}}{re^{\mathbf{i}\theta} - 1} d\theta \\
b_1 &= -1 \\
b_2 &= 0 \\
b_{n+1} - b_n &= \int\limits_0^{2\pi} \mathbf{i} e^{(n-1)\theta} r^{n-1} d\theta \\
&= \frac{e^{2\pi\mathbf{i}(n-1)} - 1}{n-1} r^{n-1} \\
&= 0 \;\forall\; n \neq 1 \\
\implies b_n &= 0 \;\forall\; n \geq 2
\end{aligned}
$$

$$
\begin{aligned}
\frac{1}{z^2 - z} &= \frac{-1}{z} - 1 - z - z^2 - z^3 - z^4 \ldots \\
&= -\frac{1}{z} + \sum_{j=0}^{\infty} -z^j
\end{aligned}
$$

7. **Find the following integrals, where γ is the circle of radius 2 centered at 0**

a)

$$
\begin{aligned}
\frac{1}{2\pi\mathbf{i}} \int\limits_\gamma \frac{z^2}{z^3 - 1} dz &= \frac{1}{2\pi\mathbf{i}} \int\limits_\gamma \frac{\xi^2}{\xi^3 - 1} d\xi \\
&= \frac{1}{2\pi\mathbf{i}} \int\limits_\gamma \frac{\frac{\xi^3}{\xi^3 - 1}}{\xi} d\xi \\
&= \left. \frac{\xi^3}{\xi^3 - 1} \right|_{\xi=0} \\
&= 0
\end{aligned}
$$

b)

$$
\begin{aligned}
\frac{1}{2\pi\mathbf{i}} \int\limits_\gamma z^2 e^{\frac{1}{z}} dz &= \frac{1}{2\pi\mathbf{i}} \int\limits_\gamma \xi^2 e^{\frac{1}{\xi}} d\xi \\
&= \frac{1}{2\pi\mathbf{i}} \int\limits_\gamma \frac{\xi^2 e^{1/\xi}(\xi - 1)}{\xi - 1} d\xi \\
&= \left. \xi^2 e^{1/\xi}(\xi - 1) \right|_{\xi=1} \\
&= 0
\end{aligned}
$$

c)

$$\begin{aligned}
\frac{1}{2\pi\mathbf{i}}\int_{\gamma} z^2 e^{\frac{1}{z}}\,dz &= \frac{1}{2\pi\mathbf{i}}\int_0^{2\pi} \frac{4\mathbf{i}}{6e^{\mathbf{i}\theta}-9} - \frac{\mathbf{i}}{6e^{\mathbf{i}\theta}}d\theta \\
&= \frac{1}{2\pi\mathbf{i}}\left[\int_0^{2\pi} \frac{4\mathbf{i}}{6e^{\mathbf{i}\theta}-9}d\theta - \int_0^{2\pi}\frac{\mathbf{i}}{6e^{\mathbf{i}\theta}}d\theta\right] \\
&= \frac{1}{2\pi\mathbf{i}}\left[-\frac{8\pi\mathbf{i}}{9} - 0\right] \\
&= -\frac{4}{9}
\end{aligned}$$

8. **Calculate the residue at each isolated singularity in the complex plane of the following functions.**
$\frac{z}{(z^2+1)^2}$ has double poles at $\pm\mathbf{i}$.

$$\begin{aligned}
\mathrm{Res}\left[\frac{z}{(z^2+1)^2},\mathbf{i}\right] &= \lim_{z\to\mathbf{i}}\frac{d}{dz}\left[\frac{z(z-\mathbf{i})^2}{(z^2+1)^2}\right] \\
&= \lim_{z\to\mathbf{i}}\frac{d}{dz}\left[\frac{z}{(z+\mathbf{i})^2}\right] \\
&= \lim_{z\to\mathbf{i}}\frac{\mathbf{i}-z}{(z+\mathbf{i})^3} \\
&= \left.\frac{\mathbf{i}-z}{(z+\mathbf{i})^3}\right|_{z=\mathbf{i}} \\
&= 0
\end{aligned}$$

$$\begin{aligned}
\mathrm{Res}\left[\frac{z}{(z^2+1)^2},-\mathbf{i}\right] &= \lim_{z\to\mathbf{i}}\frac{d}{dz}\left[\frac{z(z+\mathbf{i})^2}{(z^2+1)^2}\right] \\
&= \lim_{z\to\mathbf{i}}\frac{d}{dz}\left[\frac{z}{(z-\mathbf{i})^2}\right] \\
&= \lim_{z\to\mathbf{i}}\frac{-\mathbf{i}-z}{(z-\mathbf{i})^3} \\
&= \left.\frac{-\mathbf{i}-z}{(z-\mathbf{i})^3}\right|_{z=-\mathbf{i}} \\
&= 0
\end{aligned}$$

9. **Show using the residue theory that**

$$\begin{aligned}
\int_{-\infty}^{\infty}\frac{1}{x^2+a^2}dx &= \frac{\pi}{a} \quad a>0 \\
\int_{-\infty}^{\infty}\frac{1}{x^2+a^2}dx &= 2\pi\mathbf{i}\sum_{j=0}^{m}\mathrm{Res}\left[\frac{1}{z^2+a^2},z_j\right] \quad \forall\ z_j \text{ in the upper half-plane}
\end{aligned}$$

$\frac{1}{x^2+a^2}$ has simple poles at $\pm a\mathbf{i}$, but only $a\mathbf{i}$ is in the upper half-plane.

$$\begin{aligned}
\mathrm{Res}\left[\frac{1}{z^2+a^2},a\mathbf{i}\right] &= \lim_{z\to a\mathbf{i}}\frac{z-a\mathbf{i}}{z^2+a^2} \\
&= \lim_{z\to a\mathbf{i}}\frac{1}{z+a\mathbf{i}} = \left.\frac{1}{z+a\mathbf{i}}\right|_{z=a\mathbf{i}} \\
&= \frac{1}{2a\mathbf{i}}
\end{aligned}$$

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{1}{x^2+a^2}dx &= \frac{\pi}{a} \quad a>0 \\
\int_{-\infty}^{\infty} \frac{1}{x^2+a^2}dx &= \frac{2\pi\mathbf{i}}{2a\mathbf{i}} \\
&= \frac{\pi}{2a}
\end{aligned}$$

10. **Show if $f(z)$ is analytic on the deleted neighborhood $N(a,r)/\{a\}$ and bounded then f has a removable singularity at a. Hint: Integrate across the Laurent Expansion for $f(z)$ to find a formula for the b terms. Then show the b terms are all 0.**

$$b_n = \frac{1}{2\pi\mathbf{i}} \int_C (\xi-a)^{n-1} f(\xi) d\xi$$

Now we want to show $(\xi-a)^{n-1}f(\xi)$ is analytic.

$$\begin{aligned}
\frac{\partial}{\partial x}\left[(\xi-a)^{n-1}f(\xi)\right] &= (n-1)(\xi-a)^{n-2}f(\xi) + (\xi-a)^{n-1}\frac{\partial f(\xi)}{\partial x} \\
\frac{\partial}{\partial y}\left[(\xi-a)^{n-1}f(\xi)\right] &= \mathbf{i}(n-1)(\xi-a)^{n-2}f(\xi) + (\xi-a)^{n-1}\frac{\partial f(\xi)}{\partial y} \\
&= \mathbf{i}(n-1)(\xi-a)^{n-2}f(\xi) + \mathbf{i}(\xi-a)^{n-1}\frac{\partial f(\xi)}{\partial x} \text{ since } f \text{ is analytic} \\
&= \mathbf{i}\frac{\partial}{\partial x}\left[(\xi-a)^{n-1}f(\xi)\right] \\
&\Longrightarrow (\xi-a)^{n-1}f(\xi) \text{ is analytic}
\end{aligned}$$

Now, by The Cauchy Theorem, we know $\int_C (\xi-a)^{n-1}f(\xi)d\xi = 0$. Thus $b_n = 0 \implies$ the singularity at a is removable.

11. **Show if $f(z)$ is analytic on the deleted neighborhood $N(a,r)/\{a\}$ and $\lim_{z\to a} f(z) = \infty$ then f has a pole at a. Hint: Show $\frac{1}{f(z)}$ has a removable singularity at a and thus has a zero of order m at a : $\frac{1}{f(z)} = (z-a)^m g(z) \implies f(z) = \frac{1}{(z-a)^m}\frac{1}{g(z)}$.**
Let $f(z)$ be analytic on the deleted neighborhood $N(a,r)/\{a\}$ and $\lim\limits_{z\to a} f(z) = \infty$. Let $h(z) = \frac{1}{f(z)}$.
Since $\lim\limits_{z\to a} f(z) = \infty$, $\lim\limits_{z\to a} h(z) = 0$, thus $h(z)$ has a zero of order m at a.
$\implies h(z) = (z-a)^m g(z)$
$\implies f(z) = \frac{1}{(z-a)^m}\frac{1}{g(z)}$
$\implies f(z)$ has a pole of order m at a.

12. **If $f(z)$ is analytic on the deleted neighborhood $N(a,r)/\{a\}$ an has an essential singularity at a, then the Casorati-Weierstrass Theorm says $f(z)$ approaches any value as $z \to a$, that is $\forall c \in \mathbb{C} \exists z_n \to a$ such that $f(z_n) \to c$. Illustrate this with an example. Then show that any functions with this property have an essential singularity at a.**
$f(z) = e^{(1/z)}$ has an essential singularity at 0. We want to show that $f(z)$ gets arbitrarily close to any $c \in \mathbb{C}$. Solving $f(z) = c$ for z shows $f(\frac{1}{\ln(c)}) = c \quad \forall c \neq 0$. We therefore need to show that $f(z)$ gets arbitrarily close

to 0. Let $\epsilon > 0$ be given.

$$\begin{aligned}
|f(z) - 0| &= \left| e^{(1/z)} \right| \\
&= e^{\Re(1/z)} \\
&= e^{\Re(\bar{z}/|z|^2)} \\
&= e^{\Re(z)/|z|^2} \\
&\leq e^{\Re(z)} \quad \forall 0 \leq |z|^2 \leq 1 \\
&\leq \epsilon \quad \forall \Re(z) \leq \ln(\epsilon)
\end{aligned}$$

Proof Casorati-Weierstrass Theorem

Assume that $f(z)$ has an essential singularity at a and does not get arbitrarily close to a given value $c \in \mathbb{C}$.
$\implies |f(z) - c| > \epsilon$ for some $\epsilon > 0$
$\implies g(z) = \frac{1}{f(z)-c}$ has a removable singularity (by 11.12, since $\lim_{z \to a}(z-a)g(z) = 0$)
$\implies f(z) = \frac{1}{g(z)} + c$ either has a removable singularity (if $g(a) \neq 0$), or has a pole of order k (if $g(a)$ has a zero of order k). Both situations contradict $f(z)$ having an essential singularity.

If $f(z)$ does not have an essential singularity at a, then one of 4 cases apply.

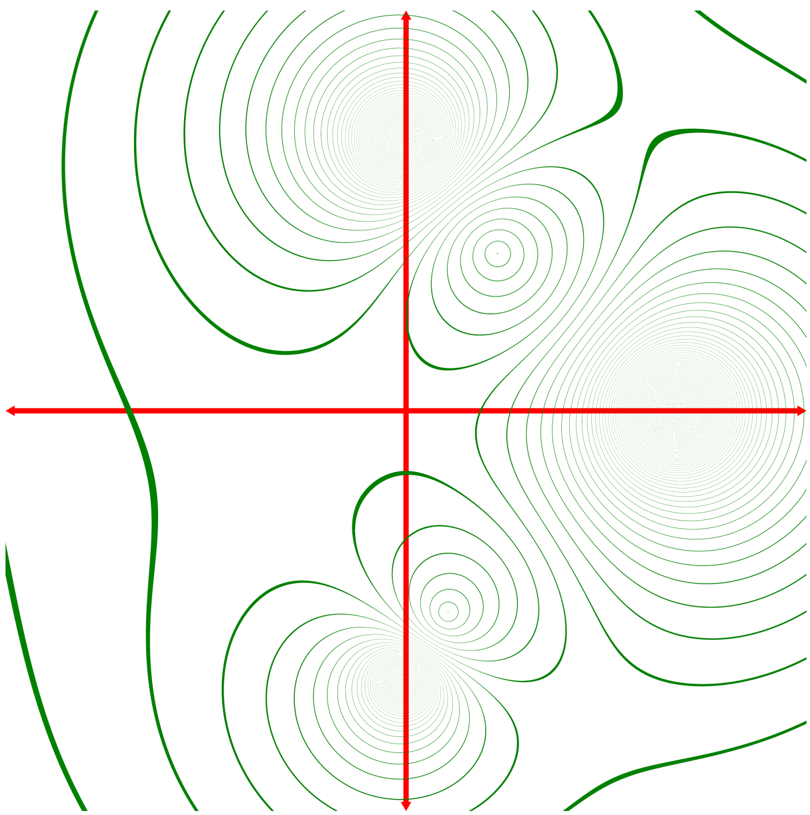
(a) It has a zero at a.
(b) It has a removable singularity at a.
(c) It has a pole of order k at a
(d) It does not have a singularity at a

For case 1, any value $c \neq 0$ will not be arbitrairly close to $f(z)$ as $z \to a$.
For case 2, $f(z)$ is bounded in a neighborhood of a, so there will exist c not in the bounds of $f(z)$ in a neighborhood of a. Thus, $f(z)$ will not get arbitrarily close to c as $z \to a$.
For case 3, $\lim_{z \to a} f(z) = \infty$. Therefore, for a neighborhood of a, $f(z)$ will be bounded away from zero, and $f(z)$ will not get arbitrarily close to 0 as $z \to a$
For case 4, $f(a)$ is defined, and has a value. Thus for all $c \neq f(a)$, $f(z)$ will not get arbitrarily close to c as $z \to a$
Therefore for all $f(z)$ that do not have an essential singularity at a, $f(z)$ does not get arbitrarily close to some value c as $z \to a$
$\blacksquare$

13. **If $f(z) = k/(z-a)$ the conjugate vector field $\overline{f(z)}$ describes the force field induced by a charge of k coulombs at point a. Then if charges $1, 2$ and 3 units are placed at the points $\mathbf{i}, -\mathbf{i}$ and 1, find the point in the plane where the force field is 0. Show that the resulting force field from the 3 charges is a gradient field and graph the equipotential lines of this force field.**

We want the sum of the forces at z to be zero, and thus we want $\sum \overline{f_j(z)} = 0$.

$$\begin{aligned}
\overline{f_1(z)} &= \overline{\frac{1}{z-\mathbf{i}}} = \frac{1}{\bar{z}+\mathbf{i}} \\
\overline{f_2(z)} &= \overline{\frac{2}{z+\mathbf{i}}} = \frac{2}{\bar{z}-\mathbf{i}} \\
\overline{f_3(z)} &= \overline{\frac{3}{z-1}} = \frac{3}{\bar{z}-1} \\
0 = f(z) &= \frac{1}{\bar{z}+\mathbf{i}} + \frac{2}{\bar{z}-\mathbf{i}} + \frac{3}{\bar{z}-1} \\
\text{Let } \xi &= \bar{z} \\
0 &= (\xi-\mathbf{i})(\xi-1) + 2(\xi+\mathbf{i})(\xi-1) + 3(\xi+\mathbf{i})(\xi-\mathbf{i}) \\
&= \xi(\xi-1) - (\xi-1)\mathbf{i} + 2\xi(\xi-1) + 2(\xi-1)\mathbf{i} + 3\xi^2 + 3 \\
&= 6\xi^2 - 3\xi + 3 + (\xi-1)\mathbf{i} \\
\xi &= \frac{\sqrt{\sqrt{1105}-32}-3}{12} + \frac{\sqrt{\sqrt{1105}+32}-1}{12}\mathbf{i} \\
z &= \frac{\sqrt{\sqrt{1105}-32}-3}{12} - \frac{\sqrt{\sqrt{1105}+32}-1}{12}\mathbf{i}
\end{aligned}$$



The green lines represent the values such that $||f(z)| \mod 0.5| <= 0.02$, or roughly the 0.5 increment level curves.