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MATH 438: Introduction to Complex Variables
Proofs

3. Chapter 3

3.3. **Proof**
Assume f is differentiable at z. Thus $\lim_{h\to 0}\frac{f(z+h)-f(z)}{h}$ does not approach ∞, and therefore $f(z+h)-f(z)$ must approach 0. Which implies f is continous at z. ■

3.10. (a)

$$\begin{aligned}\frac{dz^2}{dz} &= \frac{(z+dz)^2-z^2}{dz}\\ &= \frac{z^2+2zdz+dz^2-z^2}{dz}=\frac{2zdz+dz^2}{dz}\\ &= 2z+dz\\ &= 2z\end{aligned}$$

(b)

$$\begin{aligned}\frac{dz^3}{dz} &= \frac{(z+dz)^3-z^3}{dz}\\ &= \frac{z^3+3z^2dz+3dz^2z+dz^3-z^3}{dz}=\frac{3z^2dz+3dz^2z+dz^3}{dz}\\ &= 3z^2+3zdz+dz^2\\ &= 3z^2\end{aligned}$$

3.17. Assume 3.15

$$\begin{aligned}f &= u+\mathbf{i}v\\ f_x &= u_x+\mathbf{i}v_x\\ f_y &= u_y+\mathbf{i}v_y\\ f_x &= f_y/\mathbf{i} \text{ by 3.15}\\ u_x+\mathbf{i}v_x &= (u_y+\mathbf{i}v_y)/\mathbf{i}=v_y-\mathbf{i}u_y\\ &\implies u_x=v_y \text{ and } u_y=-v_x\end{aligned}$$

3.20.

$$\begin{aligned}f &= =u+0\mathbf{i}\\ f_x &= u_x\\ f_y &= u_y\\ u_x=f_x &= -\mathbf{i}f_y=-\mathbf{i}u_y \text{ by Complex Cauchy-Riemann}\\ &\implies u_x=u_y=0 \text{ since } u_x,u_y\in\mathbb{R}\end{aligned}$$

3.21. (a)

$$\begin{aligned}u_r &= \cos(\theta)\\ v_r &= \sin(\theta)\\ u_\theta &= -r\sin(\theta)\\ v_\theta &= r\cos(\theta)\\ ru_r &= r\cos(\theta)=v_\theta\\ rv_r &= r\sin(\theta)=-u_\theta\end{aligned}$$

3.25. Let f be a C^1 analytic function.

$$\begin{aligned}\frac{\partial f}{\partial \bar{z}} &= 0.5(f_x - f_y/i) \\ &= 0.5(f_x - f_x) = 0 \text{ by Cauchy-Riemann}\end{aligned}$$

3.31.

$$\begin{aligned}e^z &= e^{x+y\mathbf{i}} \\ (e^z)_x &= e^{x+y\mathbf{i}} \\ (e^z)_y &= \mathbf{i}e^{x+y\mathbf{i}} \\ (e^z)_x &= (e^z)_y/\mathbf{i} \\ &\Longrightarrow e^z \text{ is analytic}\end{aligned}$$

3.34.

$$\begin{aligned}|e^z| &= |e^x(\cos(y) + \mathbf{i}\sin(y))| \\ &= |e^x|\,|\cos(y) + \mathbf{i}\sin(y)| \\ &= |e^x|\end{aligned}$$

3.35. (a) All of $\mathbb{C}$ except for 0

(b)

$$\begin{aligned}e^{z+2\pi\mathbf{i}} &= e^{x+(y+2\pi)\mathbf{i}} \\ &= e^x(\cos(y+2\pi) + \mathbf{i}\sin(y+2\pi)) \\ &= e^x(\cos(y) + \mathbf{i}\sin(y)) \\ &= e^z\end{aligned}$$

(c)

(d) Since e^z is periodic of period $2\pi\mathbf{i}$ there exists $n \in \mathbb{Z}$ such that $0 \leq y + 2n\pi\mathbf{i} < 2\pi$ and $e^{z+2n\pi\mathbf{i}} = e^z$

3.36. Let $z = t(x + y\mathbf{i}) = xt + yt\mathbf{i}$

$$\begin{aligned}e^z &= e^{xt+yt\mathbf{i}} \\ &= e^{xt}(\cos(yt) + \mathbf{i}\sin(yt))\end{aligned}$$

e^z is a spiral in the complex plane, with radius e^{xt} and angle yt

3.37. **Proof** $\overline{e^z} = e^{\bar{z}}$

$$\begin{aligned}\overline{e^z} &= \overline{e^x(\cos(y) + \mathbf{i}\sin(y))} \\ &= e^x(\cos(y) - \mathbf{i}\sin(y)) \\ &= e^x(\cos(-y) + \mathbf{i}\sin(-y)) \\ &= e^{\bar{z}}\end{aligned}$$

■

3.38.

$$\begin{aligned}e^z &= e^x(\cos(y) + \mathbf{i}\sin(y)) \\ e^w &= e^u(\cos(v) + \mathbf{i}\sin(v))\end{aligned}$$

3.39. **Proof** $\sin(z)$ **is entire**

$$\begin{aligned}
\sin(z) &= (e^{\mathbf{i}z} - e^{-\mathbf{i}z})/(2\mathbf{i}) \\
\sin_x(z) &= (\mathbf{i}e^{\mathbf{i}z} + \mathbf{i}e^{-\mathbf{i}z})/(2\mathbf{i}) = (e^{\mathbf{i}z} + e^{-\mathbf{i}z})/(2) \\
\sin_y(z) &= (-e^{\mathbf{i}z} - e^{-\mathbf{i}z})/(2\mathbf{i}) \\
1/\mathbf{i} &= -\mathbf{i} \\
\sin_x(z) &= \sin_y(z)/\mathbf{i}
\end{aligned}$$

■

Proof $\cos(z)$ **is entire**

$$\begin{aligned}
\cos(z) &= (e^{\mathbf{i}z} + e^{-\mathbf{i}z})/2 \\
\cos_x(z) &= (\mathbf{i}e^{\mathbf{i}z} - \mathbf{i}e^{-\mathbf{i}z})/2 \\
\cos_y(z) &= (-e^{\mathbf{i}z} + e^{-\mathbf{i}z})/2 \\
1/\mathbf{i} &= -\mathbf{i} \\
\cos_x(z) &= \cos_y(z)/\mathbf{i}
\end{aligned}$$

■

3.40.

$$\begin{aligned}
\sin(z) &= (e^{\mathbf{i}z} - e^{-\mathbf{i}z})/(2\mathbf{i}) \\
\overline{\sin z} &= \overline{(e^{\mathbf{i}z} - e^{-\mathbf{i}z})/(2\mathbf{i})} \\
&= \overline{(e^{\mathbf{i}(x+y\mathbf{i})} - e^{-\mathbf{i}(x+y\mathbf{i})})/(2\mathbf{i})} = \overline{(e^{-y+x\mathbf{i}} - e^{y-x\mathbf{i}})/(2\mathbf{i})} \\
&= \overline{(e^{-y}(\cos(x) + \mathbf{i}\sin(x)) - e^{y}(\cos(-x) + \mathbf{i}\sin(-x)))/(2\mathbf{i})} \\
&= \overline{(e^{-y}(-\mathbf{i}\cos(x) + \sin(x)) - e^{y}(-\mathbf{i}\cos(-x) + \sin(-x)))/(2)} \\
&= (e^{-y}(\mathbf{i}\cos(x) + \sin(x)) - e^{y}(\mathbf{i}\cos(x) - \sin(x)))/2 \\
\sin(\bar{z}) &= (e^{\mathbf{i}\bar{z}} - e^{-\mathbf{i}\bar{z}})/(2\mathbf{i}) = (e^{\mathbf{i}(x-y\mathbf{i})} - e^{-\mathbf{i}(x-y\mathbf{i})})/(2\mathbf{i}) \\
&= (e^{y+x\mathbf{i}} - e^{-y-x\mathbf{i}})/(2\mathbf{i}) \\
&= (e^{y}(\cos(x) + \mathbf{i}\sin(x)) - e^{-y}(\cos(-x) + \mathbf{i}\sin(-x)))/(2\mathbf{i}) \\
&= (e^{y}(-\mathbf{i}\cos(x) + \sin(x)) - e^{-y}(-\mathbf{i}\cos(-x) + \sin(-x)))/2 \\
&= (-e^{y}(\mathbf{i}\cos(x) - \sin(x)) + e^{-y}(\mathbf{i}\cos(x) + \sin(x)))/2 \\
\overline{\sin z} &= \sin(\bar{z})
\end{aligned}$$

3.51.

$$\begin{aligned}
1 &= \frac{dz}{dz} \\
&= \frac{de^{\log z}}{dz} \\
&= e^{\log z}\frac{d\log z}{dz} \\
&= z\frac{d\log z}{dz} \\
\frac{1}{z} &= \frac{d\log z}{dz}
\end{aligned}$$

3.55.

$$\begin{aligned}
\frac{dz^w}{dz} &= \frac{de^{w\log z}}{dz} \\
&= e^{w\log z}\frac{dw\log z}{dz} \\
&= z^w w/z \\
&= wz^{w-1}
\end{aligned}$$

3.57. (a) 0

(b) $-\ln(2) - \mathbf{i}\pi/2$

(c) $-\ln(2) + \mathbf{i}\pi$

(d) $-\pi/2$

(e) $\ln(2)/2 + 3\mathbf{i}\pi/4$

(f) $1.60944 + 0.927295\mathbf{i}$

3.59. (a) $\cos(\pi\sqrt{2}) + \mathbf{i}\sin(\pi\sqrt{2})$

(b) $2\cos(\ln(2)) - 2\mathbf{i}\sin(\ln(2))$

(c) $2^{1/\sqrt{2}}\cos(\frac{\pi\sqrt{2}}{4}) + 2^{1/\sqrt{2}}\mathbf{i}\sin(\frac{\pi\sqrt{2}}{4})$

(d) $e^{-\pi/2}$

(e) $\pi\mathbf{i}$

(f) $e^{\pi/2}$