Eilenberg–MacLane Spectra as Thom Spectra

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1 Introduction

The goal of these two lectures is to prove a theorem, originally due to Bökstedt:

Theorem 1 (Bökstedt)**.**

$$\mathrm{THH}_*(\mathbb{F}_p) = \mathbb{F}_p[x], \quad |x| = 2$$

This was originally proved by a tedious spectral sequences argument, but that's not the approach we'll take. Instead, we'll take advantage of two different theorems.

Theorem 2 (Hopkins–Mahowald)**.** *There is an equivalence of $\mathbb{E}_2$-ring spectra*

$$H\mathbb{F}_p \simeq Mf_p,$$

where $f_p \colon \Omega^2 S^3 \to B\,GL_1(\widehat{\mathbb{S}_p})$ is the map determined by $1 - p \in \pi_1 B\,GL_1(\widehat{\mathbb{S}_p}) \simeq \mathbb{Z}_p^\times$.

Theorem 3 (Blumberg–Cohen–Schlichtkrull)**.** *Let $f \colon X \to B\,GL_1(R)$ be an $\mathbb{E}_2$-map of spaces. Assume that the $\mathbb{E}_2$-structure on Mf extends to an $\mathbb{E}_3$-structure. Then there is an equivalence of $\mathbb{E}_1$-R-module spectra*

$$\mathrm{THH}(Mf/R) \simeq Mf \otimes BX_+$$

Proof of Theorem 1. We compute only $\mathrm{THH}(\mathbb{F}_p)$. The computation of $\mathrm{THH}(\mathbb{Z})$ is similar. By Theorem 1, it suffices to compute $\mathrm{THH}(Mf_p)$. By Theorem 3,

$$\mathrm{THH}(H\mathbb{F}_p) \simeq \mathrm{THH}(Mf_p) \simeq Mf_p \otimes \Sigma^\infty_+ B(\Omega^2 S^3) \simeq Mf_p \otimes \Sigma^\infty_+ \Omega S^3 \simeq H\mathbb{F}_p \otimes \Sigma^\infty_+ \Omega S^3.$$

So as a spectrum, $\mathrm{THH}(H\mathbb{F}_p) \simeq H\mathbb{F}_p \otimes \Sigma^\infty_+ \Omega S^3_+$. The homotopy of this spectrum is (by definition!) the $\mathbb{F}_p$-homology of ΩS^3, which can be calculated by the Serre spectral sequence associated to the fibration

$$\Omega S^3 \to \mathrm{Map}([0,1], S^3) \to S^3.$$

$$E^2_{s,t} = H_s(S^3; H_t(\Omega S^3; \mathbb{F}_p)) \implies H_{s+t}(\mathrm{Map}([0,1], S^3); \mathbb{F}_p) = 0.$$

I leave this as an exercise (see Hatcher's spectral sequences book). $\square$

2 Thom Spectra

Let R be an $\mathbb{E}_{n+1}$-ring spectrum. There is a category of modules R-**Mod**, which is an $\mathbb{E}_n$-monoidal ∞-category if R is an $\mathbb{E}_{n+1}$-algebra. So R-**Mod** has a tensor product.

Definition 1. The **Picard groupoid** of R is the subcategory of R-**Mod** consisting of all invertible R-modules for objects and equivalences between them for morphisms.

More precisely, $\mathbf{Pic}(R)$ consists of R-modules M such that there exists another R-module N with $M \otimes_R N \simeq R$, but the module N and the equivalence $M \otimes_R N \simeq R$ is not part of the data of the object $M \in \mathbf{Pic}(R)$.

Definition 2. A **local system of R-modules** over a space X is a functor $X \to \mathbf{Pic}(R)$.

Remark 3. This is a lot like the definition of a presheaf, which is a functor from the opposite of the topology on X to a category $\mathcal{C}$. Except now we consider the whole space X with all it's higher homotopy data, where $X \simeq X^{op}$ as quasicategories (since it's an ∞-groupoid).

Definition 4. Given a local system of R-modules, $f\colon X \to \mathbf{Pic}(R)$, the associated Thom spectrum is

$$Mf = \operatorname{colim}(i \circ f) = \operatorname{colim}(X \xrightarrow{f} \mathbf{Pic}(R) \xrightarrow{i} R\text{-}\mathbf{Mod}).$$

To construct Thom spectra, we need a better way to get our hands on these maps $X \to \mathbf{Pic}(R)$. To that end, we introduce the "group of units" of a ring spectrum. Most Thom spectra we consider will factor through the classifying space of this "group of units".

Definition 5. Let R be an $\mathbb{E}_1$-ring spectrum. The **space of units** of R is the homotopy pullback of spaces

$$\begin{tikzcd} GL_1(R) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \Omega^\infty(R) \arrow[d] \\ (\pi_0 R)^\times \arrow[r] & \pi_0 R \end{tikzcd}$$

This is an $\mathbb{E}_1$-algebra in spaces, and if R is an E_∞-ring spectrum, then $GL_1(R)$ is an $\mathbb{E}_\infty$-algebra in spaces.

The homotopy groups of $GL_1(R)$ aren't too bad to compute either, since homotopy commutes with pullback diagrams.

$$\pi_n \, GL_1(R) = \begin{cases} (\pi_0 R)^\times & n = 0 \\ \pi_n R & n > 0. \end{cases}$$

Proposition 6. *There is an equivalence of spaces between* $B\,GL_1(R)$ *and the component of* ***Pic***(R) *containing* R*.*

We can now give some examples of Thom spectra.

Example 7. If $X = BG$ is the classifying space of some group, then we can recognize Mf as the homotopy coinvariants R_{hG}. This is because the homotopy coinvariants are defined by the colimit

$$\operatorname{colim}(BG \to \mathbf{Sp}),$$

where the unique point of BG is sent to R.

Example 8. If $X = \Omega^2\Sigma^2 Y$ for some space Y (i.e. X is the free $\mathbb{E}_2$-algebra on Y), then $\mathbb{E}_2$-maps $\Omega^2\Sigma^2 Y \to B\,GL_1(R)$ are in bijection with maps $Y \to B\,GL_1(R)$, by the universal property of free $\mathbb{E}_2$-algebras.

In particular, if $Y = S^1$, to define a map $\Omega^2 S^3 \to B\,GL_1(R)$, it suffices to pick an element in $\pi_1 B\,GL_1(R) \cong \pi_0\,GL_1(R) \cong \pi_0(R)^\times$.

This is how we will construct $H\mathbb{F}_p$ as a Thom spectrum.

Example 9. The classical Thom spectrum MO is constructed from the J homomorphism $J\colon BO \to B\,GL_1(\mathbb{S})$.

Proposition 10. *For a fixed ring spectrum* R*, the Thom spectrum construction defines a functor* $M\colon \mathcal{S}_{/B\,GL_1(R)} \to R$-**Mod** *that is both colimit-preserving and symmetric monoidal with respect to the Cartesian monoidal structure on its domain.*

Lemma 11 (Thom Isomorphism Theorem)**.** *Let* $f\colon X \to B\,GL_1(R)$ *be an* n*-fold loop map. Any* $\mathbb{E}_n$ R*-algebra map* $Mf \to A$ *gives rise to an isomorphism of* $\mathbb{E}_n$ A*-algebras* $A \otimes_R Mf \simeq A \otimes \Sigma^\infty_+ X$*.*

3 Main Theorem

Theorem 1 (Hopkins–Mahowald)**.** *There are an equivalences of* $\mathbb{E}_2$*-ring spectra*

$$H\mathbb{F}_p \simeq Mf_p,$$

where $f_p\colon \Omega^2 S^3 \to B\,GL_1(\widehat{\mathbb{S}_p})$ *is the map determined by* $1-p \in \pi_1 B\,GL_1(\widehat{\mathbb{S}_p}) \simeq \mathbb{Z}_p^\times$*.*

Proof. We have that $Mf_p \simeq (\widehat{\mathbb{S}_p})_{h\Omega^3 S^3}$. Hence, $\pi_0 Mf_p$ is the coinvariants of the $\pi_0\Omega^3 S^3 \cong \mathbb{Z}$ action on $\pi_0\widehat{\mathbb{S}_p} \cong \mathbb{Z}_p$, where 1 acts by multiplication by $1-p$.

$$\pi_0 Mf_p \cong {}^{\mathbb{Z}_p}/_{\langle 1-(1-p)\rangle} = \mathbb{Z}_p/p \cong \mathbb{F}_p.$$

Therefore, the zeroth Postnikov section of Mf_p defines an $\mathbb{E}_2$-map

$$\phi\colon Mf_p \to H\mathbb{F}_p.$$

We want to prove that ϕ is an equivalence. Since both $Mf_p \simeq (\widehat{\mathbb{S}_p})_{h\Omega^3 S^3}$ and $H\mathbb{F}_p$ are p-complete and connective, it suffices to check that ϕ is an isomorphism on $H\mathbb{F}_p$-homology, i.e. we must check that

$$\phi_*\colon H_*(Mf_p;\mathbb{F}_p) \to H_*(H\mathbb{F}_p, \mathbb{F}_p)$$

is an isomorphism.

Note that ϕ is also a map of $\mathbb{E}_2$ $\widehat{\mathbb{S}_p}$-algebras, again because both Mf_p and $H\mathbb{F}_p$ are p-complete and connective. Therefore, by Lemma 11, we have $Mf_p \otimes H\mathbb{F}_p \cong \Sigma^\infty_+ \Omega^2 S^3 \otimes H\mathbb{F}_p$. By the Dyer–Lashof operations, it suffices to show that ϕ_* is a homology isomorphism in degrees 0 and 1. This is easy to verify. □

Corollary 2. *There is an equivalence of $\mathbb{E}_2$-ring spectra*

$$H\mathbb{Z}_p \simeq M(f_p \circ \pi),$$

where f_p is as before and $\pi\colon \tau_{\geq 2}\Omega^2 S^3 \to \Omega^2 S^3$ is the universal covering map.

Corollary 3. *There is an equivalence of $\mathbb{E}_2$-ring spectra between $H\mathbb{Z}$ and the Thom spectrum of the map*

$$\tau_{\geq 2}\Omega^2 S^3 \to \prod_{p \text{ prime}} \tau_{\geq 2} B\,GL_1(\widehat{\mathbb{S}_p}) \simeq \tau_{\geq 2} B\,GL_1(\mathbb{S}) \to B\,GL_1(\mathbb{S}).$$

Proof. Notice that because $\tau_{\geq 2}\Omega^2 S^3$ is simply connected, $\pi_0 Mf \simeq \mathbb{Z}$ and there is a map $Mf \to H\mathbb{Z}$. It suffices to check that this induces isomorphisms on homology both rationally and for each prime. Rationally, this is a simple check, and p-adically, this was done in the previous corollary. □

Then we may use Theorem 3 to compute $THH(\mathbb{Z})$ and $THH(\mathbb{Z}_p)$:

$$THH(\mathbb{Z}) \simeq H\mathbb{Z} \otimes \Sigma^\infty_+ \tau_{\geq 3}\Omega S^3$$
$$THH(\mathbb{Z}_p) \simeq H\mathbb{Z}_p \otimes \Sigma^\infty_+ \tau_{\geq 3}\Omega S^3$$