

# Directed tilings of the Euclidean and hyperbolic plane

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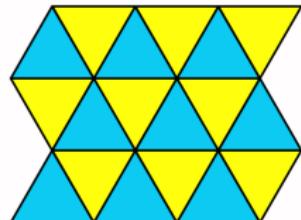
MAA Spring 2025 MD-DC-VA Section

# Regular tilings of the plane

Euclidean tilings:

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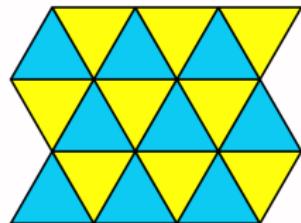
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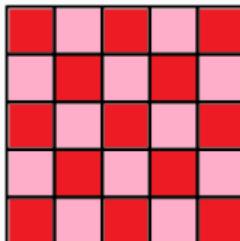
{3,6}

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Euclidean tilings:



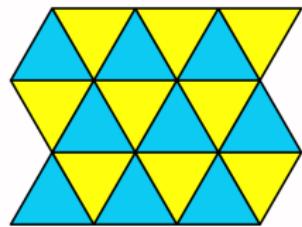
$\{3,6\}$



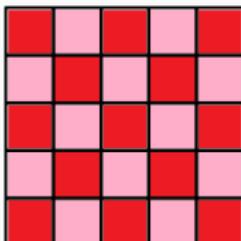
$\{4,4\}$

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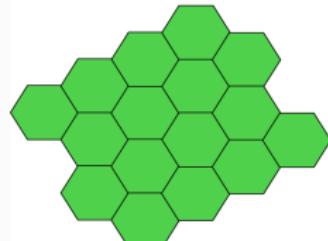
Euclidean tilings:



{3,6}



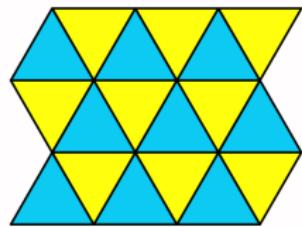
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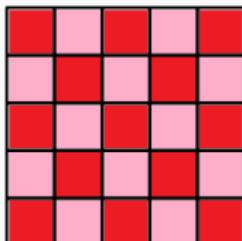
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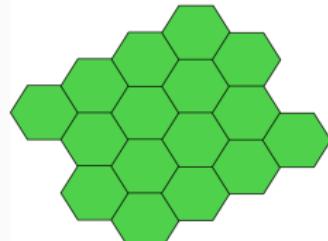
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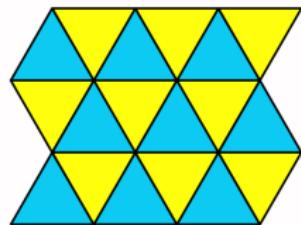


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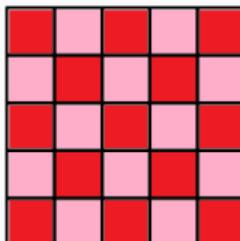
Hyperbolic Tilings:

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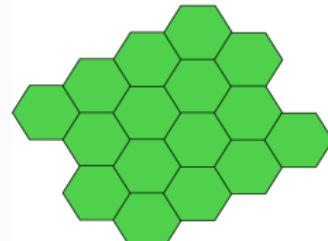
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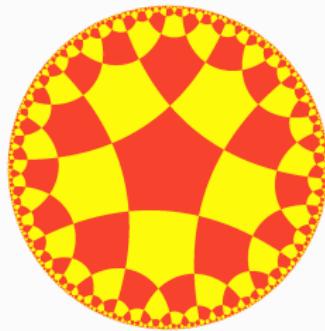


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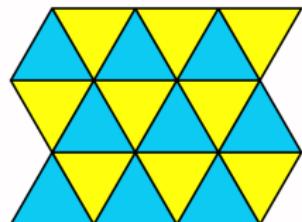
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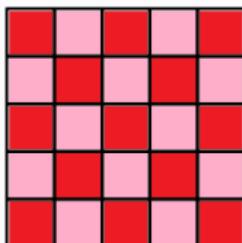
$\{5,4\}$

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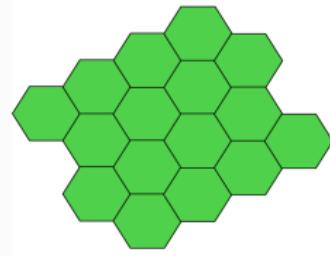
Euclidean tilings:



$\{3,6\}$

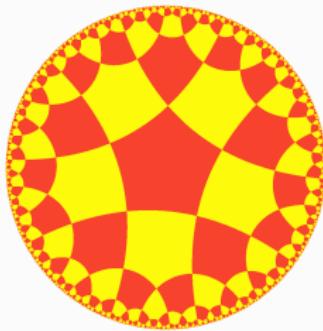


$\{4,4\}$

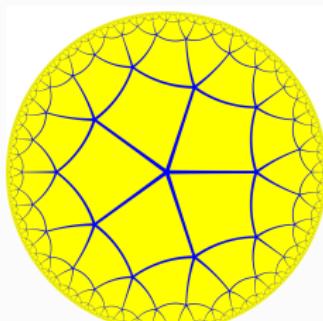


$\{6,3\}$

Hyperbolic Tilings:



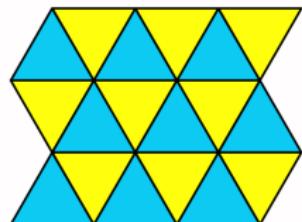
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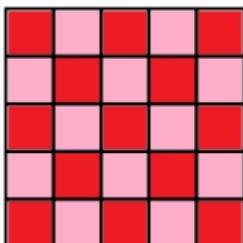
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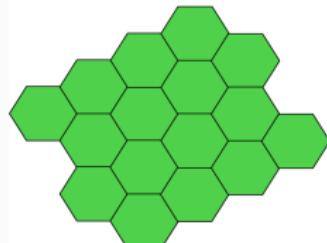
Euclidean tilings:



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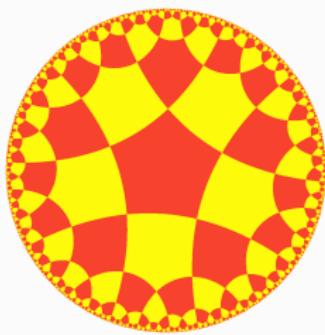


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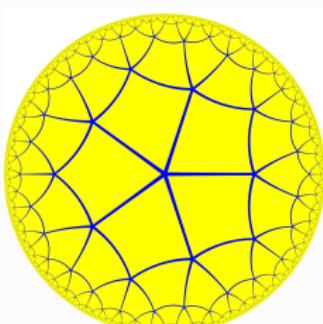


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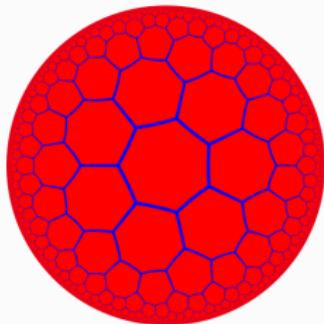
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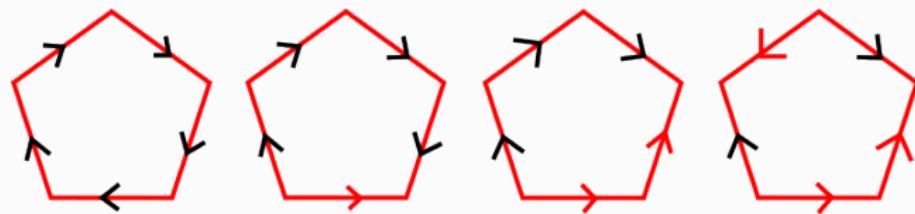
$\{7,3\}$

# Directed tiles

Edge direction patterns, up to rotation and reflection:

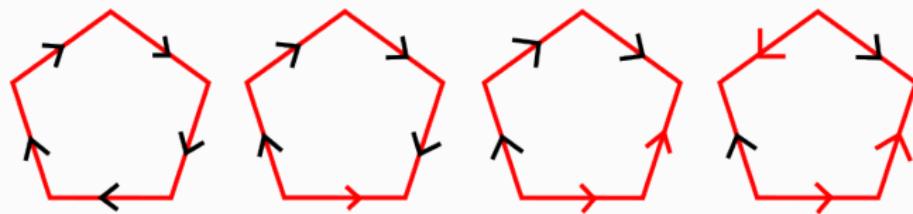
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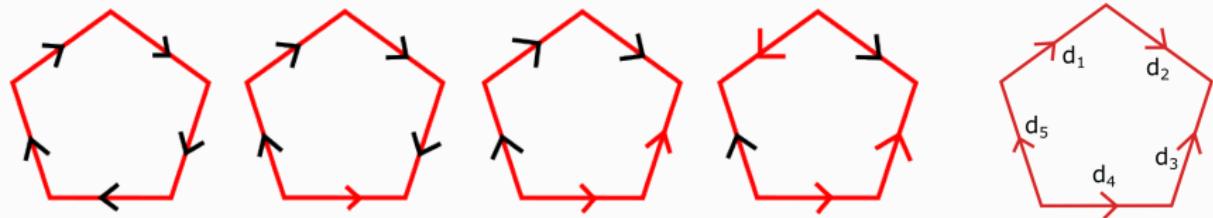
Edge direction patterns, up to rotation and reflection:



Labeled endpoints & edges facilitate computational models:

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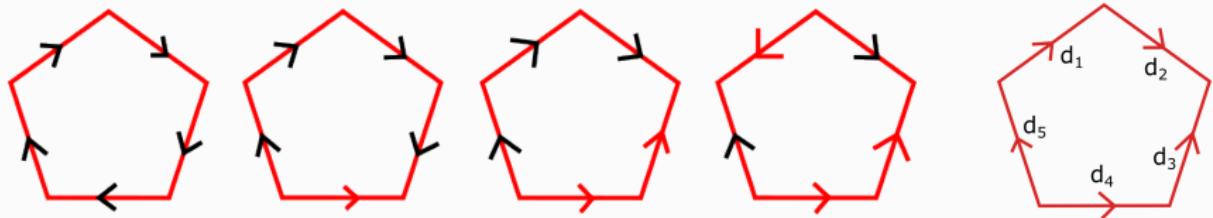
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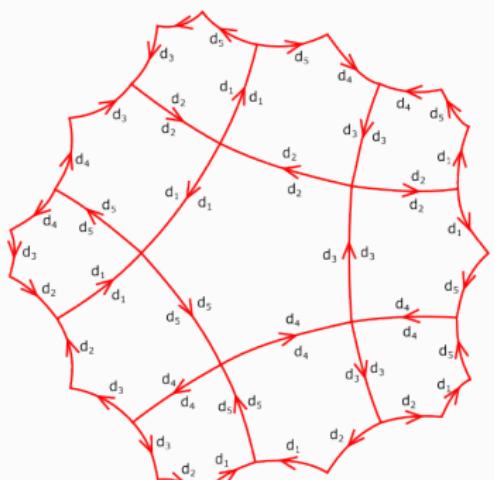
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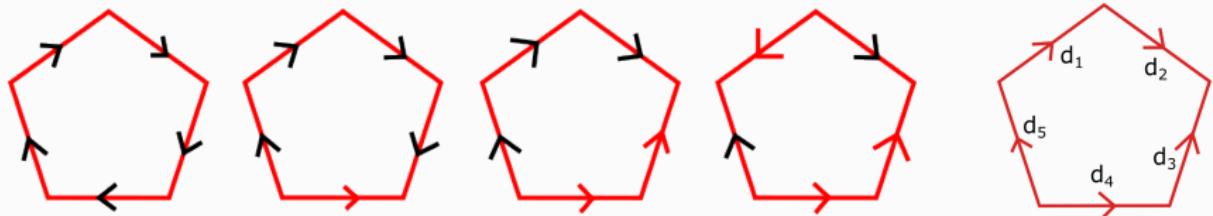


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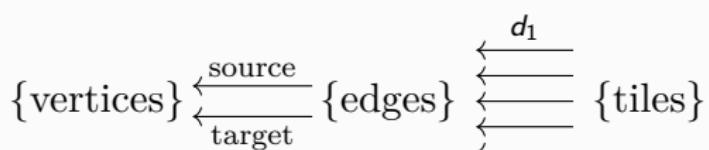
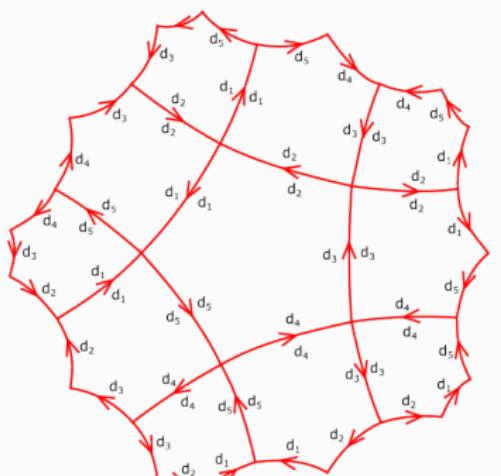


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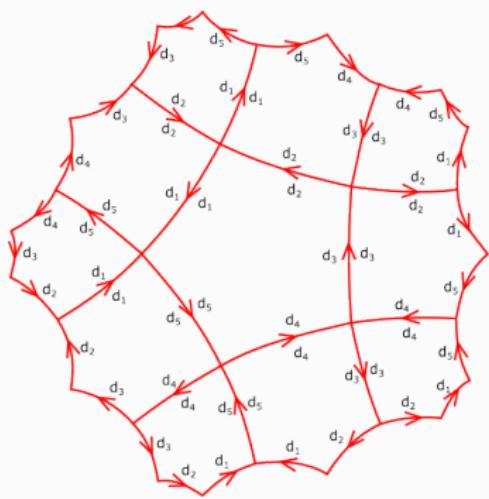
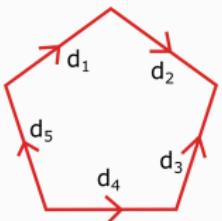


# Directed tilings

There are many different directed  $\{m, n\}$  tilings:

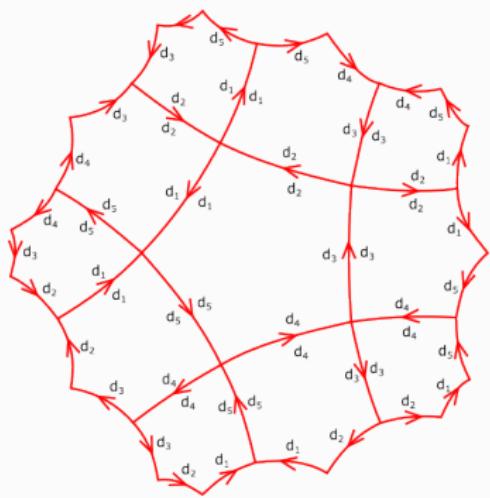
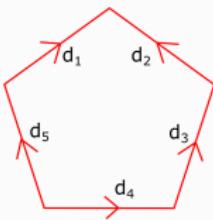
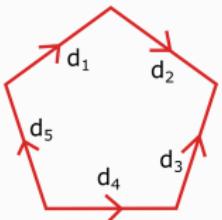
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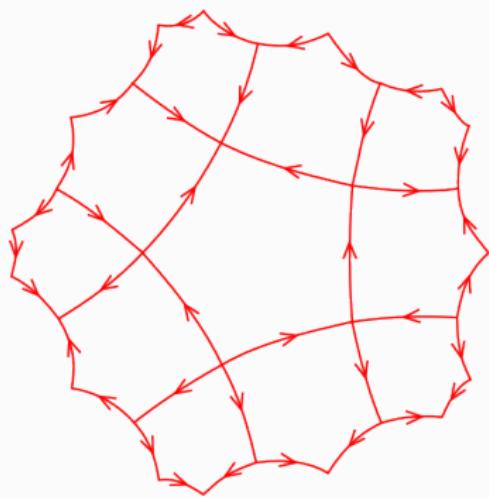
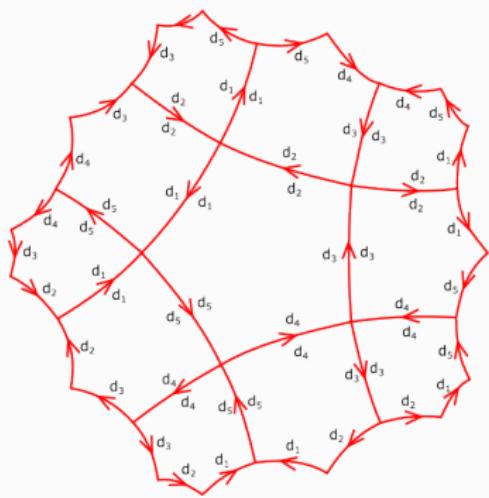
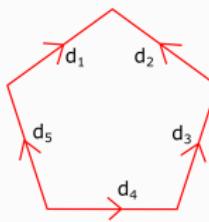
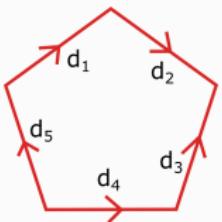
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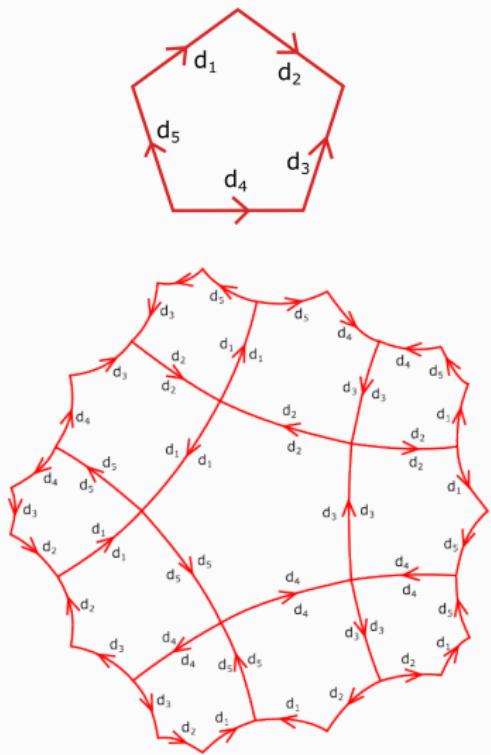
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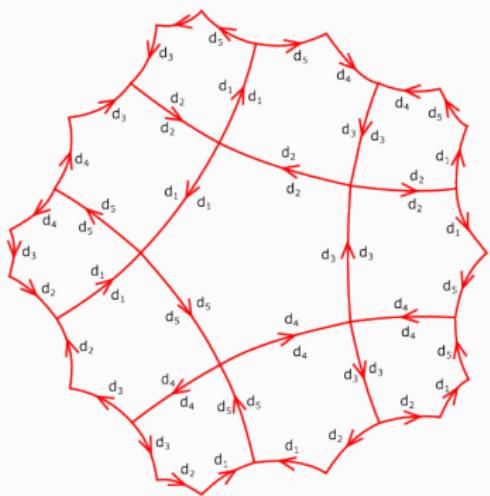
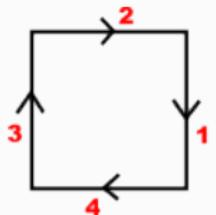
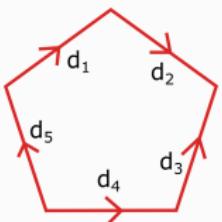
# Reflective tilings

Whenever  $n$  is even, there is a reflective directed  $\{m, n\}$  tiling:



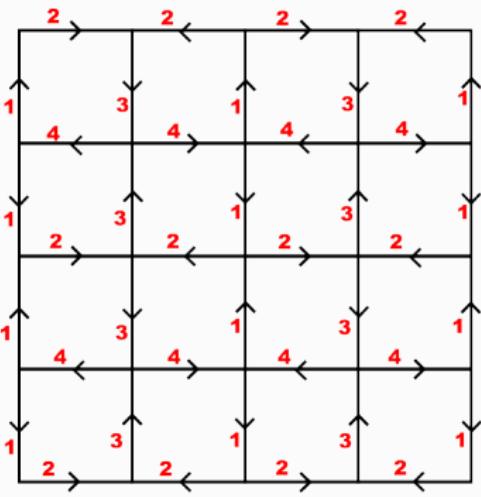
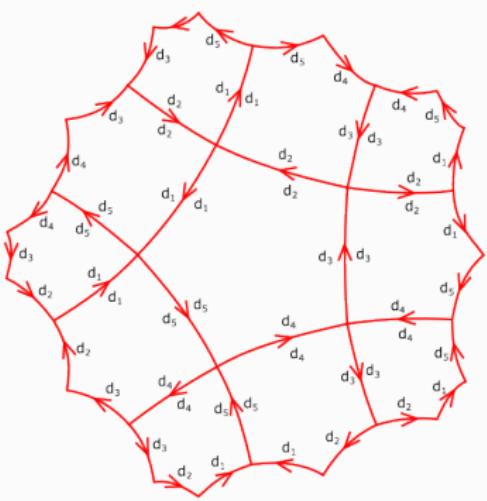
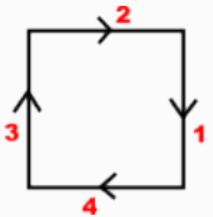
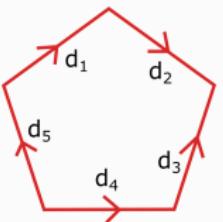
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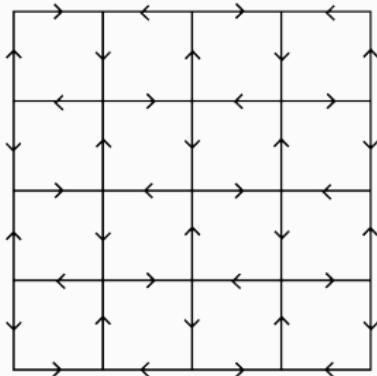
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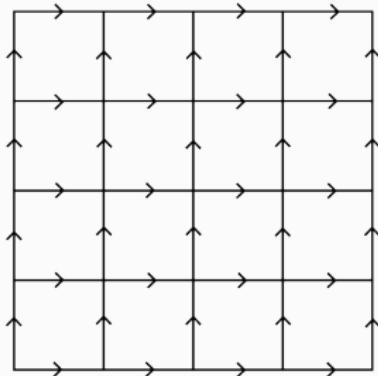
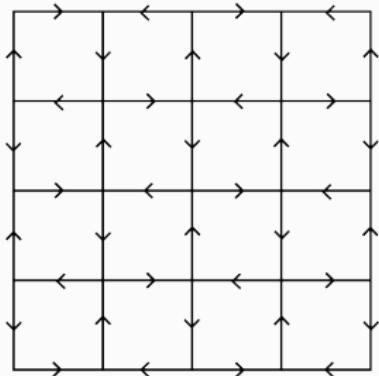
# Reflection generated tilings

How to change the directions of a reflective tiling:



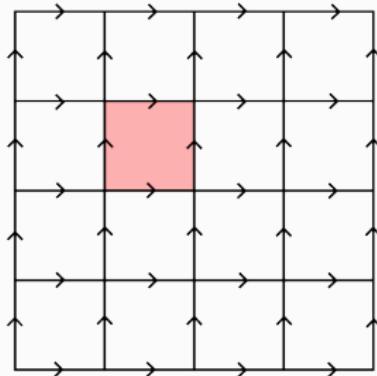
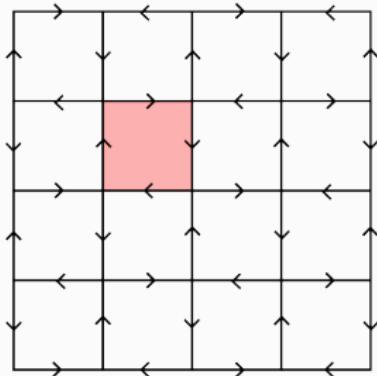
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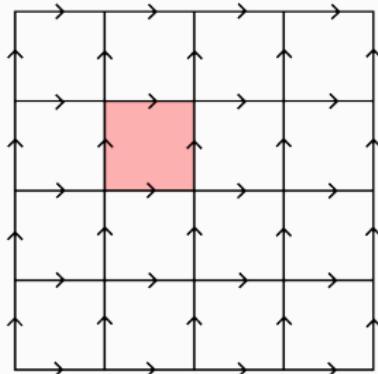
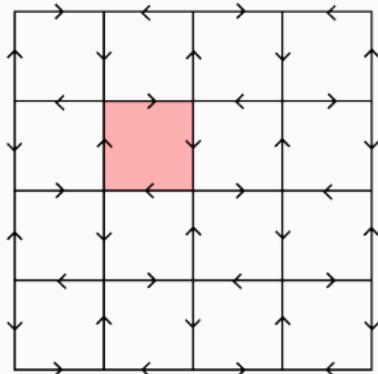
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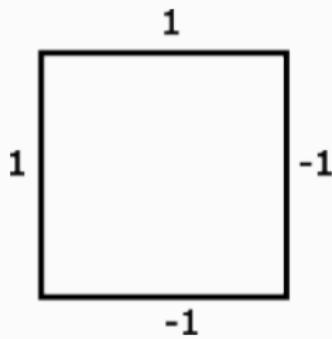


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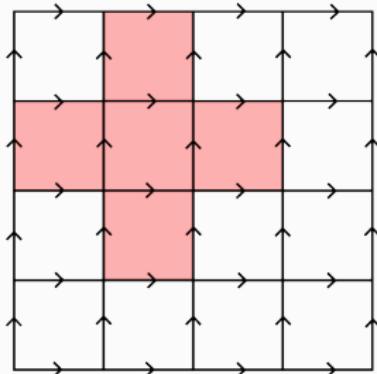
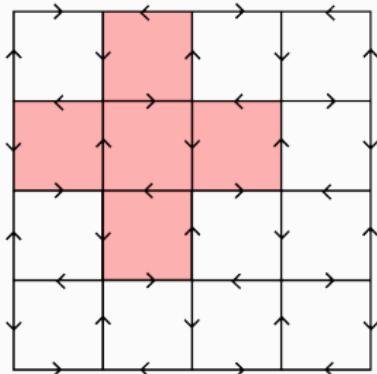


Reverse some edges of one tile:

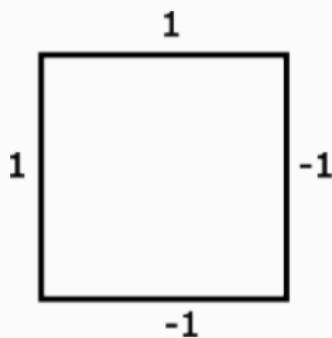


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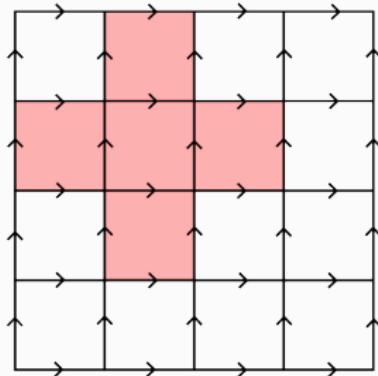
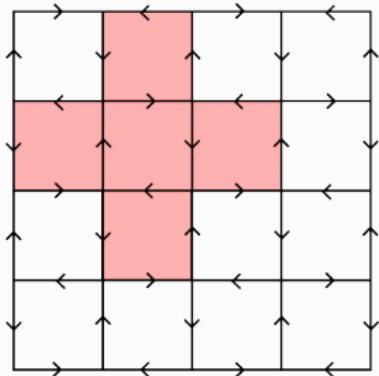


Reverse some edges of one tile:

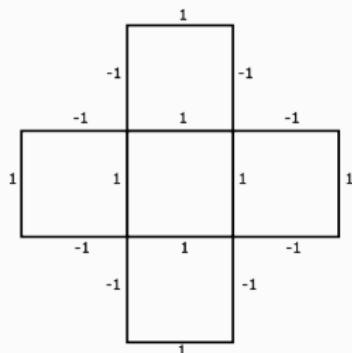


# Reflection generated tilings

How to change the directions of a reflective tiling:



Reflect that tile outward and reverse more edges:



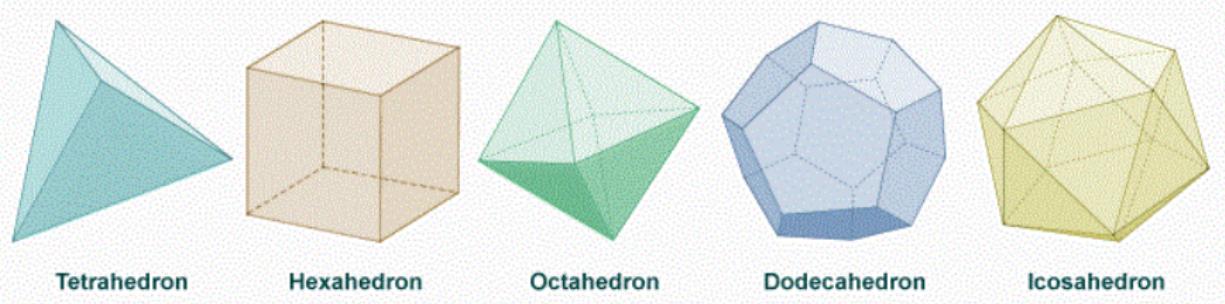
# Where next?

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- Tilings of the sphere (polyhedra) and other surfaces

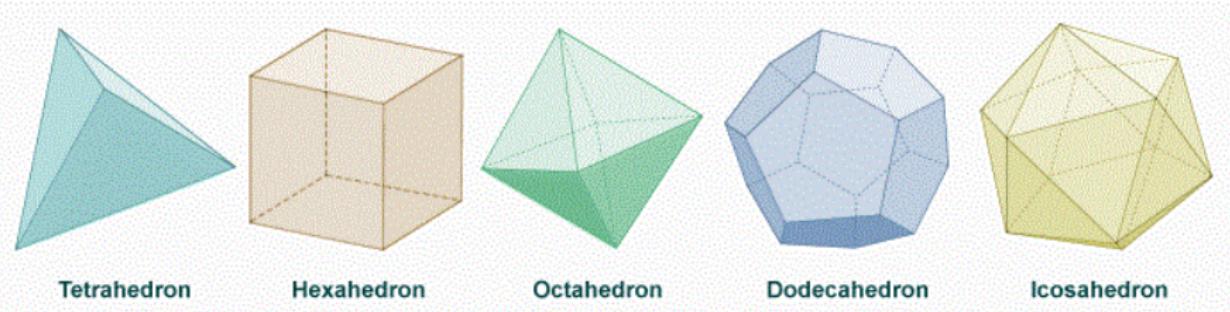
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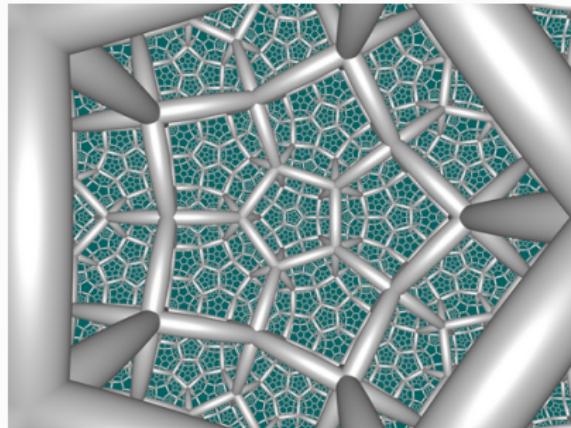
# Where next?

- Tilings of the sphere (polyhedra) and other surfaces
- 3-Dimensional tilings



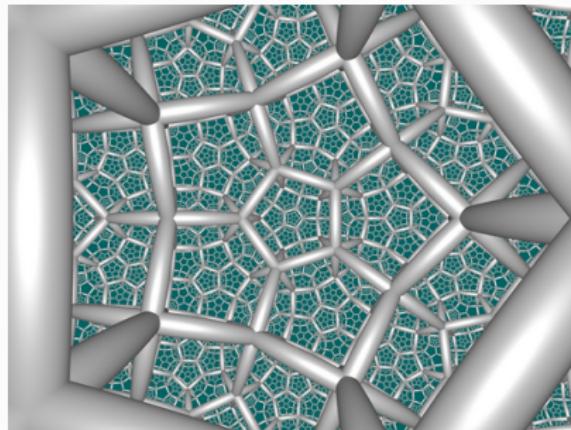
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- Tilings of the sphere (polyhedra) and other surfaces
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# Where next?

- Tilings of the sphere (polyhedra) and other surfaces
- 3-Dimensional tilings
- Irregular tilings and subdivisions



# Where next?

- Tilings of the sphere (polyhedra) and other surfaces
- 3-Dimensional tilings
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# References

"Categorical Tiling Theory" - DiLeo, Sessoms, S., work in progress

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Thanks for coming!

[https://en.wikipedia.org/wiki/Order-4\\_pentagonal\\_tiling](https://en.wikipedia.org/wiki/Order-4_pentagonal_tiling)

[https://en.wikipedia.org/wiki/Order-5\\_square\\_tiling](https://en.wikipedia.org/wiki/Order-5_square_tiling)

[https://en.wikipedia.org/wiki/Heptagonal\\_tiling](https://en.wikipedia.org/wiki/Heptagonal_tiling)

<https://www.technologyuk.net/mathematics/geometry/platonic-solids.shtml>

[https://en.wikipedia.org/wiki/Order-4\\_dodecahedral\\_honeycomb](https://en.wikipedia.org/wiki/Order-4_dodecahedral_honeycomb)