Partial evaluations and the compositional structure of the bar construction

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Abstract. An algebraic expression like $3+2+6$ can be evaluated to 11, but it can also be *partially evaluated* to $5+6$. In categorical algebra, such partial evaluations can be defined in terms of the 1-skeleton of the bar construction for algebras of a monad. We show that this partial evaluation relation can be seen as the relation internal to the category of algebras generated by relating a formal expression to its result. The relation is transitive for many monads which describe commonly encountered algebraic structures, and more generally for BC monads on Set, defined by the underlying functor and multiplication being weakly cartesian. We find that this is not true for all monads: we describe a finitary monad on Set for which the partial evaluation relation on the terminal algebra is not transitive.

With the perspective of higher algebraic rewriting in mind, we then investigate the compositional structure of the bar construction in all dimensions. We show that for algebras of BC monads, the bar construction has fillers for all *directed acyclic configurations* in Δ^n, but generally not all inner horns. We introduce several additional *completeness* and *exactness* conditions on simplicial sets which correspond via the bar construction to composition and invertibility properties of partial evaluations, including those arising from *weakly cartesian* monads. We characterize and produce factorizations of pushouts and certain commutative squares in the simplex category in order to provide simplified presentations of these conditions and relate them to more familiar properties of simplicial sets.

Contents

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1. Introduction

In this paper we study compositional and combinatorial aspects of the bar construction for algebras over several types of monads, motivated by the idea that edges in the bar construction can be interpreted as *partial evaluations* of formal algebraic expressions [**FP20**]. This leads us to introduce various new classes of simplicial sets with a compositional flavour similar to but distinct from quasicategories. A careful analysis of pushouts in the simplex category Δ lets us characterize these classes of simplicial sets in terms of filler conditions.

Partial evaluations and the bar construction. In more detail, the *bar construction* associates to every Eilenberg-Moore algebra a simplicial object in the category of algebras, playing the role of a universal resolution of the algebra [**Tri07**]. For the case of monads on Set, the resulting simplicial set can be interpreted operationally in terms of *partial evaluations*. Just as one can say that the formal sum $2+3+4$ can be evaluated to 9, one can also say that it can be *partially evaluated* to $5+4$. A way to make this precise is by using the 1-dimensional structure of the bar construction of $\mathbb{N}$ as an algebra of a suitable monad, namely the monad of commutative monoids (see Section 4.2), in which such a partial evaluation is represented by an edge (1-simplex). More generally, the bar construction can be seen as a simplicial set where the 0-simplices are formal expressions specified by the monad, the 1-simplices are partial evaluations between two such formal expressions, and the higher-dimensional simplices have to do with higher substitutions; for example, we will interpret the 2-simplices as composition rules for partial evaluations. The relation induced by the existence of partial evaluations can be seen as the relation internal to the category of algebras, generated by linking a formal expression to its result. Partial evaluations themselves can be seen them as a categorification (or a *proof-relevant version*) of this idea.

We will present these ideas in more detail in Sections 3 and 4.2. We also refer to our earlier work [**FP20**], and note that an example of a partial evaluation in the context of the bar construction had already appeared earlier in notes by Baez on cohomology and computation [**Bae**].

Compositional structure of partial evaluations. Whenever the monad under consideration is cartesian, which happens for example whenever it is presented by a plain operad, the bar construction of every algebra is known to be the nerve of a category (Remark 4.1.1). This means, in particular, that partial evaluations can be composed uniquely, and that their composition operation is strictly unital and associative. A large class of monads appearing in algebra, as well as in probability and other fields, are however only *weakly cartesian*, or have even less rigid properties such as the property BC (see Section 2). Examples of weakly cartesian monads on

Set are all those monads which are presented by a *symmetric* operad, such as the monad of commutative monoids.

For the algebras of BC monads, the bar construction generally satisfies weaker filling conditions than the nerve of a category, making the resulting composition "operation" no longer well-defined. As we will see, the bar construction of these monads is generally not even a quasicategory, as not all inner horns admit a filler above dimension 2. It nevertheless satisfies filler conditions reminiscent of a compositional structure; we call simplicial sets satisfying the relevant filler conditions *inner span complete*. For example, the convex combinations monad—also known as the distribution monad in probability terms—is BC, and therefore its algebras have inner span complete bar constructions. Building on the relation to second-order stochastic dominance developed in [**FP20**], it is natural to wonder whether this inner span completeness also has significance for probability theory.

The monad of commutative monoids, and more generally all weakly cartesian monads, have bar constructions which produce simplicial sets satisfying filler conditions stronger than inner span completeness, intuitively by requiring that any suitably degenerate simplex arises from units of the compositional structure. We call simplicial sets of this kind *inner complete*. This stronger property, as we shall see, is in a sense *orthogonal* to being a quasicategory: an inner complete simplicial set is a quasicategory if and only if it is the nerve of a category.

The existence and (non-)uniqueness of composites are not the only interesting properties that partial evaluations may have. Other properties are, for example, invertibility or the lack thereof. We define and study those monads for which partial evaluations can always (respectively never) be inverted. These again give rise to classes of simplicial sets satisfying suitable filler conditions, corresponding to new types of compositional structures. We call these simplicial sets *span complete* (respectively *pushout complete*).

New compositional structures. Our motivation for defining the higher compositional structures given by *inner span complete*, *inner complete*, *pushout complete* and *span complete* simplicial sets has been our study of partial evaluations and the bar construction, based on concrete examples for monads such as the commutative monoid monad and the distribution monad from probability. So similar to the situation with higher rewriting theory, here we have compositional structures motivated purely by computational properties of certain algebraic structures. To the best of our knowledge, none of these structures has been considered before. We hope that they will be of independent interest also in contexts other than the bar construction.

As our choice of terminology indicates, the definitions of these compositional structures fit a pattern, which consists of the following. In each case, a collection of commuting squares in the simplex category Δ is selected; a simplicial set $X : \Delta^{\mathrm{op}} \to$

Set is then defined to be *complete* with respect to these squares if it sends all of them to weak pullbacks in Set (Definition 2.1.1), or in geometrical terms if fillers of a certain type exist. We also consider the stronger *exact* version of this condition, by which we mean that these squares are sent to pullbacks in Set, or equivalently that the required fillers must exist uniquely. In this form, the exact simplicial sets relative to the distinguished squares in Δ are then precisely the models of a *limit sketch* specified by these squares.[1] While the exact simplicial sets with respect to those squares are then precisely the models of the limit sketch, the merely complete ones can then be thought of as "weak models" for the same limit sketch, as now the functor $\Delta^{\mathrm{op}} \to \mathsf{Set}$ is only required to send the distinguished cones to weak limit cones.

Summarizing our various completeness conditions specified by concrete classes of squares in Δ, we have the following diagram of full subcategories of simplicial sets:

$$\begin{tikzcd}
\mathsf{Set} \arrow[r, hook] \arrow[d, hook] & \mathsf{Gpd} \arrow[r, hook] \arrow[d, hook] & \mathsf{Kan} \arrow[r, hook] \arrow[d, hook] & \mathsf{SpC} \arrow[ddl, hook] \\
\mathsf{PE} \arrow[r, hook] \arrow[d, hook] & \mathsf{Cat} \arrow[r, hook] \arrow[d, hook] & \mathsf{qCat} \arrow[d, hook] & \\
\mathsf{PC} \arrow[r, hook] & \mathsf{IC} \arrow[r, hook] & \mathsf{ISpC} &
\end{tikzcd} \tag{1.1}$$

Here, the prefixes $\mathsf{ISp}, \mathsf{I}, \mathsf{P}, \mathsf{Sp}$ each refer to a collection of commuting squares in Δ, respectively abbreviating the words

$$\mathsf{I}\textit{nner}\ \mathsf{Sp}\textit{an}, \quad \mathsf{I}\textit{nner}, \quad \mathsf{P}\textit{ushout}, \quad \mathsf{Sp}\textit{an}.$$

The designated squares in Δ are respectively the following:

- ▷ Pushout squares of "inner" spans of face maps (Section 6);
- ▷ Pushout squares of "inner" spans of more general maps (Section 7);
- ▷ All pushout squares (Section 8);
- ▷ Commuting squares of face maps which are pushouts of finite sets, but not necessarily pushouts in Δ (Section 9).

The suffixes C, E refer to completeness or exactness relative to those squares, respectively. Overall, each combination of prefix and suffix therefore specifies a full

[1]In general, a limit sketch is a collection of diagrams in a category $\mathbb{C}$ each with a choice of cone, and a model of the sketch is a functor $\mathbb{C} \to \mathsf{Set}$ sending that cone to a limit cone (see [**BW90**] for more detail). In our setting, $\mathbb{C} = \Delta^{op}$ and the diagrams are given by spans in Δ where a cocone amounts to a commuting square containing that span.

subcategory of simplicial sets. However, the diagram does not list the categories lSpE, IE and SpE explicitly, since we will prove that

$$\mathsf{lSpE} = \mathsf{IE} = \mathsf{Cat}, \qquad \mathsf{Sp} = \mathsf{Gpd},$$

meaning that these coincide with the category of (nerves of) categories or groupoids, respectively.

All other classes of simplicial sets in the diagram are standard, with Set denoting the discrete simplicial sets, Kan the Kan complexes, and qCat the quasicategories. It is also worth noting that all of the rectangular subdiagrams of (1.1) are pullbacks: the lower right square by Proposition 7.1.3, the lower left and upper left ones as a consequence of Theorem 8.3.2, and the upper right one is obvious. Also the outer trapezoid is a pullback by Proposition 9.4.3.

Relation to $(\infty, 1)$-categories. It is worth noting that the compositional structures we study, while still weaker analogues of categories, are meaningfully different from quasicategories and are not expected to relate to $(\infty, 1)$-category theory. In fact, they are not even "homotopically meaningful": even though the classes of simplicial sets we introduce are characterized by filler conditions, there is no model structure on simplicial sets with monomorphisms as cofibrations and any of our classes as the fibrant objects. To see this, it is enough to remark that all of the classes of simplicial sets we define have lifts against the spine inclusions

$$\Delta^1 \sqcup_{\Delta^0} \overset{n}{\cdots} \sqcup_{\Delta^0} \Delta^1 \hookrightarrow \Delta^n,$$

and the minimal Cisinski model structure with the spine inclusions as trivial cofibrations is the Joyal model structure [**Ara14**, Theorem 5.20]. Thus if e.g. span complete simplicial sets were the fibrant objects in a Cisinski model structure, then they would all in particular have to be quasicategories. But this is not the case, as we will see in Example 9.1.4.

Outline.

- ▷ In Section 2 we provide some background on monads and some standard definitions, such as that of a weak pullback. We also outline the relevant weak exactness conditions on monads (Section 2.1) and their algebras (Section 2.2) that we use in the rest of the paper. Some of these conditions are standard, while some appear for the first time here (to our knowledge).
- ▷ In Section 3 we recall the concept of partial evaluations from [**FP20**], which can be seen as an operational interpretation of the bar construction in low dimension, and study its compositional properties further. In particular, we define the partial evaluation relation and show that it is the smallest relation internal to the category of algebras which relates a formal expression to its

result (Proposition 3.2.2), and then we proceed to give some criteria for when and how partial evaluations can be composed (Section 3.3) and for when they can be reversed (Sections 3.4 and 3.5).

- ▷ In Section 4 we formally introduce the bar construction (Section 4.1) and begin our study of its compositional properties. We use the commutative monoid monad to give counterexamples to several natural hypotheses, including the general non-uniqueness of composites (Theorem 4.3.1) and the nonexistence of fillers for inner horns (Theorem 4.4.1). This prepares the ground for the second half of the paper, in which we develop a number of compositional properties which *do* hold for the bar constructions of various classes of monads which include the commutative monoid monad.
- ▷ Before introducing our new compositional structures in detail, Section 5 develops a number of technical results on pushouts in the simplex category Δ which will be of relevance to us in the later sections. In particular, we prove decomposition and classification theorems for pushouts in Δ. This subsequently facilitates convenient characterizations for when a simplicial set admits certain fillers.
- ▷ In Section 6 we define *inner span complete* simplicial sets (Definition 6.1.1) and show how these generalize quasicategories in a way which still facilitates a (non-unique) composition of 1-simplices (Proposition 6.1.4). We introduce a combinatorial notion of directed acyclicity for subcomplexes of the n-simplex (Definition 6.2.6), and we show that every directed acyclic configuration in an inner span complete simplicial set has a filler (Theorem 6.2.10). We then show that the bar construction of an algebra of a BC monad is inner span complete (Theorem 6.3.1), and discuss examples such as the commutative monoid monad and the distribution monad.

  Finally, we introduce *inner span exact* simplicial sets, which are defined as inner span complete simplicial sets in which the relevant fillers are unique (Definition 6.4.1), and show that these are exactly the nerves of categories (Theorem 6.4.4).
- ▷ In Section 7 we define the stricter class of *inner complete* simplicial sets (Definition 7.1.1), and show that a quasicategory is inner complete if and only if it is the nerve of a category (Proposition 7.1.3). We also show that the bar construction for weakly cartesian monads is always inner complete (Theorem 7.2.1), and demonstrate this structure in the example of the trivial commutative monoid. The corresponding *inner exact* simplicial sets are shown to also be precisely the nerves of categories (Theorem 7.3.3).

- ▷ In Section 8 we define *pushout complete* simplicial sets as those simplicial sets $\Delta^{\mathrm{op}} \to \mathsf{Set}$ which take all (weak) pushouts to weak pullbacks (Definition 8.1.1). We use our simplified characterization of pushout complete simplicial sets (Theorem 8.1.2) to show that every weakly cartesian and positive monad has pushout complete bar constructions for its algebras (Proposition 8.2.1). We also show that the similarly defined *pushout exact* simplicial sets, which take pushouts in Δ to pullbacks in Set, are precisely the nerves of categories in which identity morphisms are irreducible (Theorem 8.3.2).
- ▷ In Section 9 we define *span complete* simplicial sets (Definition 9.1.1), which generalize Kan complexes by requiring weaker filler conditions that still ensure a non-unique composition of 1-simplices with non-unique inverses, and describe the conditions on monads and algebras necessary for a bar construction to be span complete (Theorem 9.2.1). We then show that the corresponding *span exact* simplicial sets are precisely the nerves of groupoids (Corollary 9.3.3), and further that any simplicial set which is both span complete and pushout complete must be discrete (Proposition 9.4.3).

Here is how the sections depend on one another.

$$\begin{tikzcd}
 & \text{Section } 2 \arrow[d] & & \\
 & \text{Section } 3 \arrow[d] & & \\
 & \text{Section } 4 \arrow[dl] \arrow[d] \arrow[dr] \arrow[drr] & \text{Section } 5 \arrow[dll] \arrow[dl] \arrow[d] \arrow[dr] & \\
\text{Section } 6 \arrow[r, dashed] & \text{Section } 7 \arrow[r, dashed] & \text{Section } 8 \arrow[r, dashed] & \text{Section } 9
\end{tikzcd}$$

Here, a normal arrow denotes an actual mathematical dependency, while a dashed arrows denotes dependency on a conceptual and intuitive level which will merely help with understanding but is not technically necessary.

Relevant background. We assume familiarity with the theory of monads, their algebras, and the basic idea of how to do categorical algebra in terms of finitary monads on Set. We also assume familiarity with simplicial sets, but provide a brief recap next in the context of setting up notation. Some parts also assume familiarity with the basic definitions of quasicategory theory [**Joy02**].

Notation and terminology. Throughout the paper, Δ denotes the simplex category, i.e. the category of nonempty finite ordinals

$$\overline{n} \coloneqq \{0, \ldots, n\}$$

for $n \in \mathbb{N}$ as objects and monotone maps as morphisms. Similarly, Δ_+ denotes the augmented simplex category, i.e. the category of finite ordinals and monotone maps, where we also include the empty ordinal $\overline{-1} \coloneqq \emptyset$. In either case, its *generating face maps* are the images of the morphisms

$$d^{n,i} \; : \; \overline{n-1} \longrightarrow \overline{n}$$

for $i = 0, \ldots, n$, given by the inclusion of $\overline{n-1}$ into $\overline{n}$ omitting the element i. The *generating degeneracy maps* are likewise the images of the morphisms

$$s^{n,i} \; : \; \overline{n+1} \longrightarrow \overline{n}$$

for $i = 0, \ldots, n$, given by the map which hits i twice but otherwise acts like the identity. A face map or degeneracy map in general is a composite of generating ones.

A simplicial set is then a functor $\Delta^{\mathrm{op}} \to \mathsf{Set}$, and an augmented simplicial set is a functor $\Delta_+^{\mathrm{op}} \to \mathsf{Set}$. As usual, when the simplicial set under consideration is clear from the context, then we also denote the application of face and degeneracy maps using subscripts, $d_{n,i}$ and $s_{n,i}$, or merely d_i and s_i.

We generally specify a finitary monad on Set in terms of the algebraic theory that it presents. For example, the *commutative monoid monad* will be used throughout the paper for illustration.

2. Preliminaries on monads

In this section we list some conditions on monads and on their algebras, which in the rest of this work will be both applied and given an operational motivation. Some of these conditions are known in the literature (see the given references), and some are introduced here for the first time.

Motivation for these conditions can be given in terms of *partial evaluations*, see Section 3.

2.1. Lifting conditions for monads. We start by recalling the notions of cartesian and weakly cartesian monads, which we will use in the rest of the paper. The interested reader can find more details in [**Web04**] and [**CHJ14**].

Definition 2.1.1. *A diagram*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "m"] \\ C \arrow[r, "n"'] & D \end{tikzcd} \tag{2.1}$$

in a category C *is called a* weak pullback*, or* weakly cartesian square*, if for every object* S *and every commutative diagram*

$$\begin{tikzcd} S \arrow[drr, "p"] \arrow[ddr, "q"'] & & \\ & & B \arrow[d, "m"] \\ & C \arrow[r, "n"] & D \end{tikzcd}$$

in C *there exists an arrow* $S \to A$ *making the following diagram commute.*

$$\begin{tikzcd} S \arrow[drr, "p"] \arrow[ddr, "q"'] \arrow[dr] & & \\ & A \arrow[r, "f"'] \arrow[d, "g"] & B \arrow[d, "m"] \\ & C \arrow[r, "n"] & D \end{tikzcd}$$

Note that if we moreover require the map $S \to A$ to be unique, then we get the ordinary notion of pullback (or cartesian square). We sometimes also say *strong* pullback to emphasize the contrast to weak pullbacks.

If we are in the category Set, the diagram (2.1) is a weak pullback if and only if for every $b \in B$ and $c \in C$ with $m(b) = n(c)$ there exists $a \in A$ such that $f(a) = b$ and $g(a) = c$.

Definition 2.1.2. *Let* $F : \mathsf{C} \to \mathsf{D}$ *be a functor. We call* F cartesian *if it preserves pullbacks, and* weakly cartesian *if it preserves weak pullbacks.*

If C has pullbacks, then $F : \mathsf{C} \to \mathsf{D}$ is weakly cartesian equivalently if it sends pullbacks to weak pullbacks.

Definition 2.1.3. *Let* $F, G : \mathsf{C} \to \mathsf{D}$ *be functors. A natural transformation* $\alpha : F \Rightarrow G$ *is called* cartesian *(resp.* weakly cartesian*) if for every morphism* $f : X \to Y$ *of* C*, the naturality square*

$$\begin{tikzcd} FX \arrow[r, "Ff"] \arrow[d, "\alpha_X"] & FY \arrow[d, "\alpha_Y"] \\ GX \arrow[r, "Gf"] & GY \end{tikzcd}$$

is cartesian (resp. weakly cartesian).

Definition 2.1.4. *A monad* (T, η, μ) *is called* BC *if* T *preserves weak pullbacks and* μ *is weakly cartesian.*

"BC" stands for "Beck-Chevalley", and follows the terminology of [**CHJ14**]. As we will see, the BC property, and in particular weak cartesianness of μ, is closely related to the problem of composing partial evaluations (see Section 3.3 for the details).

Example 2.1.5. The convex combinations monad or distribution monad[2] is BC. It is known that the multiplication transformation is weakly cartesian ([**FP20**, Proposition 6.4]). To show moreover that the functor D preserves weak pullbacks, we will use a construction sometimes known as *conditional product*[3]. Let

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"] & B \arrow[d, "m"] \\ C \arrow[r, "n"] & E \end{tikzcd}$$

be a (strong) pullback in Set. In particular, we have

$$A \cong \coprod_{e \in E} m^{-1}(e) \times n^{-1}(e). \tag{2.2}$$

Now consider its D-image.

$$\begin{tikzcd} DA \arrow[r, "Df"] \arrow[d, "Dg"] & DB \arrow[d, "Dm"] \\ DC \arrow[r, "Dn"] & DE \end{tikzcd}$$

Let now $p \in DB$ and $q \in DC$ be finitely supported distributions, and suppose that $Dm(p) = Dn(q)$, i.e. that for all $e \in E$,

$$\sum_{b \in m^{-1}(e)} p(b) = \sum_{c \in n^{-1}(e)} q(c).$$

Denote by $r \in DE$ the resulting distribution on E. Now define the distribution $s \in DA$ as follows. Using (2.2), we can write every element of A as a pair (b, c), with $b \in B$ and $c \in C$ such that $m(b) = n(c)$. Now, for each such (b, c), let $e \coloneqq m(b) = n(c)$, and set

$$s(b, c) \coloneqq \begin{cases} \dfrac{p(b) \cdot q(c)}{r(e)} & \text{if } r(e) > 0, \\ 0 & \text{otherwise.} \end{cases}$$

[2] See e.g. [**FP20**, Section 6.2] for the detailed definition.

[3] We refer to Simpson's [**Sim18**, Section 6] for a categorical treatment which is especially close to what we use here.

It is straightforward to verify that this satisfies the relevant normalization condition $\sum_{(b,c)\in A} s(b,c) = 1$. We then have that for each $b \in B$ and $c \in C$,

$$\begin{aligned} Df(s)(b) &= \sum_{(b',c)\in f^{-1}(b)} s(b,c) = \sum_{c\in C \text{ s.t. } m(b)=n(c)} \frac{p(b)\cdot q(c)}{r(e)} \\ &= p(b)\,\frac{\sum_{c\in n^{-1}(e)} q(c)}{\sum_{c\in n^{-1}(e)} q(c)} = p(b), \end{aligned}$$

and analogously $Dg(s)(c) = q(c)$. Hence D is indeed a weakly cartesian functor.

Definition 2.1.6. *A monad (T, η, μ) is called* (weakly) cartesian *if T preserves weak pullbacks and both η and μ are (weakly) cartesian.*

Example 2.1.7. The monad of monoids is cartesian [**CHJ14**, Observation 2.1(d)]. More generally, every monad arising from a (non-symmetric) operad is cartesian [**Lei04**, Section C.1].

The monad of commutative monoids is weakly cartesian [**CHJ14**, Example 8.2]. More generally, every monad arising from a symmetric operad is weakly cartesian [**SZ14**]. Further examples and nonexamples of weakly cartesian monads can be found again in [**CHJ14**].

While the definitions above have previously appeared in the literature, the following are new (as far as we know).

Definition 2.1.8. *A monad T is called* strictly positive *if the following square is a pullback for all X.*

$$\begin{tikzcd} X \arrow[r, equal] \arrow[d, "\eta\eta"] & X \arrow[d, "\eta"'] \\ TTX \arrow[r, "\mu"] & TX. \end{tikzcd} \tag{2.3}$$

Since the upper horizontal map is an identity, the diagram above is a pullback if and only if it is a weak pullback.

For a cartesian monad, we next show that it is enough to check this condition on the terminal set $X = 1 = \{*\}$, so that it is strictly positive if and only if $\eta\eta(*)$ is the only element of $TT1$ which multiplies to $\eta(1)$.

Proposition 2.1.9. *Let (T, η, μ) be a monad on* Set *such that η or μ is cartesian. Then the square* (2.3) *is a pullback for all sets X if and only if it is for $X = 1$.*

Proof. The "only if" direction is trivial. For the "if" direction, suppose that the square (2.3) is a pullback for $X = 1$. Let X be any set, and denote by $u : X \to 1$ the unique map. We can enlarge (2.3) to the following diagram.

$$
\begin{tikzcd}
X \arrow[rrr, "\eta\eta"] \arrow[dddd, equal] \arrow[dr, "u"] & & & TTX \arrow[dl, "TTu"'] \arrow[dddd, "\mu"] \\
& 1 \arrow[r, "\eta\eta"] \arrow[dd, equal] & TT1 \arrow[dd, "\mu"] & \\
& & & \\
& 1 \arrow[r, "\eta"] & T1 & \\
X \arrow[rrr, "\eta"] \arrow[ur, "u"] & & & TX \arrow[ul, "Tu"']
\end{tikzcd}
\tag{2.4}
$$

Then if $x \in X$ and $k \in TTX$ are such that $\eta(x) = \mu(k)$, we need to show that $k = \eta\eta(x)$.

Suppose first that μ is cartesian. Then since the right square is a pullback, the claim $\eta\eta(x) = k$ follows if we can show that both these elements coincide when mapped to $TT1$ and to TX. While the latter holds by the assumption $\eta(x) = \mu(k)$, the former follows from commutativity of the upper square and the assumption that the inner square is a pullback.

Suppose now that η is cartesian. Then since $1 \to T1$ is trivially a monomorphism and monomorphisms are stable under pullback, the following naturality diagram being a pullback implies that all components of η are monomorphisms.

$$
\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, "\eta"] & 1 \arrow[d, "\eta"] \\
TX \arrow[r, "Tu"] & T1
\end{tikzcd}
$$

In terms of our $x \in X$ and $k \in TTX$, since the bottom and right squares of (2.4) commute, we have $\eta(*) = \mu(TTu(k))$. Since the inner square is a pullback, this implies that $\eta\eta(\bullet) = TTu(k)$. Since also the top square is a pullback by composition of pullbacks, it follows that there is a unique $y \in X$ such that $\eta\eta(y) = k$. Now

$$\eta(x) = \mu(k) = \mu(\eta\eta(y)) = \eta(y),$$

and since η is injective, we conclude $x = y$, and therefore the desired $\eta\eta(x) = k$. $\square$

As we will see (Section 3.5), strictly positive monads can again be interpreted in terms of partial evaluations, and give conditions for when these are "irreversible".

Example 2.1.10. For M a monoid, consider the M-set monad $M \times -$ on Set. Per the above, this monad is strictly positive if and only if the diagram

$$\begin{tikzcd} 1 \arrow[r, equal] \arrow[d] & 1 \arrow[d] \\ M \times M \arrow[r] & M \end{tikzcd}$$

where the arrows denote the obvious structure maps, is a pullback. In other words, the monad $M \times -$ is strictly positive if and only if the unit element of M cannot be factored nontrivially.

Example 2.1.11. The monads of monoids and commutative monoids are not strictly positive. Indeed with X a set and any $x \in X$, consider $x \in X$ itself together with the "doubly formal expression"

$$\boxed{\boxed{x}} + \boxed{\bullet} \in TTX,$$

where $\bullet \in TX$ denotes the neutral element, and each box denotes a level formality corresponding to an application of T. (We will develop this notation for elements of $T^n X$ more formally in Section 4.2.) Then these two elements show that the square (2.3) is not a pullback, since both elements map to $\boxed{x} \in TX$, but the doubly formal element under consideration differs from $\eta\eta(x) = \boxed{\boxed{x}}$.

Example 2.1.12. On the other hand, the semigroup monad T is strictly positive, as a consequence of the previous Proposition 2.1.9: it is the monad associated to a (non-symmetric) operad and therefore cartesian; furthermore, we have $T1 = \mathbb{N}_{>0}$, and $TT1$ can therefore be identified with the set of nonempty lists of positive integers, in such a way that $\mu : TT1 \to T1$ is the map which takes a list of positive integers and forms their sum. Based on this, it is straightforward to see that the strict positivity condition holds for the terminal algebra 1.

Similarly, the commutative semigroup is both weakly cartesian and strictly positive, but not strongly cartesian. Indeed it is weakly cartesian by virtue of being the monad associated to a symmetric operad (Example 2.1.7), namely the one with exactly one operation in each positive arity and no operation in arity zero. Moreover, the unit of the monad is monic, which implies that η is a cartesian transformation. Hence by Proposition 2.1.9, we only need to check that the square (2.3) is cartesian for $X = 1$, which holds in an analogous manner to the semigroup monad case with multisets instead of lists.

Example 2.1.13. Let be T the monoid monad and 1 the trivial one-element monoid. Then $\mathrm{Bar}_T(1)$ is the nerve of the augmented simplex category Δ_+, for example by [**Bat08**, Corollary 7.2.1].

Definition 2.1.14. *A monad T is called* left reversible *if the associativity square is a weak pullback for all X.*

$$\begin{tikzcd} T^3X \arrow[r, "T\mu"] \arrow[d, "\mu"'] & T^2X \arrow[d, "\mu"] \\ T^2X \arrow[r, "\mu"] & TX. \end{tikzcd} \tag{2.5}$$

Example 2.1.15. Following up on Example 2.1.10, consider again the M-action monad $M \times -$ for a monoid M. We now show that this monad is left reversible if and only if M is a group.

First, the above square then becomes

$$\begin{tikzcd} M \times M \times M \times X \arrow[r, "\mathrm{id}_M \times \mu \times \mathrm{id}_X"] \arrow[d, "\mu \times \mathrm{id}_X"'] & M \times M \times X \arrow[d, "\mu \times \mathrm{id}_X"] \\ M \times M \times X \arrow[r, "\mu \times \mathrm{id}_X"] & M \times X \end{tikzcd}$$

This square is a weak pullback for all X if and only if it is so for $X = 1$, so we focus on this case. In this case, the condition is that for all $(a, b) \in M \times M$ in the upper right and $(c, d) \in M \times M$ in the upper left which satisfy the compatibility condition

$$\mu(a, b) = \mu(c, d),$$

there must be $(a', e, d') \in M \times M \times M$ in the upper left such that

$$a = a', \qquad b = \mu(e, d'), \qquad c = \mu(a, e), \qquad d = d'.$$

Denoting the monoid multiplication $\mu : M \times M \to M$ as usual by juxtaposition instead, the condition simplifies to requiring that for every equation $ac = bd$ in M, there must exist $e \in M$ with $ae = c$ and $ed = b$.

Considering the special case $a = d$ and $b = c = 1$ in this condition makes e satisfy $ae = 1$ and $ea = 1$. This shows that a monoid with left reversible action monad must be a group. Conversely, if M is a group then we can take $e = a^{-1}c = bd^{-1}$. Note that this e is unique, and therefore that (2.5) is automatically a strong pullback if M is a group.

Example 2.1.16. As a consequence of Example 2.2.3, it will follow that the group monad on Set is left reversible, as is the abelian group monad.

2.2. Lifting conditions for algebras. So far we have considered lifting conditions applicable for a monad, which can in particular be instantiated on all algebras. We now move on to lifting conditions at the level of individual algebras.

Definition 2.2.1. *Let T be a monad on* Set. *We call a T-algebra (A, e)* indiscrete *if the algebra square*

$$\begin{tikzcd} T^2A \arrow[r, "Te"] \arrow[d, "\mu"'] & TA \arrow[d, "e"] \\ TA \arrow[r, "e"] & A \end{tikzcd}$$

is weakly cartesian.

Note that T is left reversible if and only if every free T-algebra is indiscrete.

In terms of partial evaluations, we will see that indiscrete algebras give partial evaluations that can always be reversed and induce the equivalence relation of having equal total evaluation (Proposition 3.4.1). This motivates our terminology *indiscrete*.

Example 2.2.2. Continuing on from Example 2.1.15, let G be a group. Then the G-sets, which are the algebras of the monad $G \times -$, are all indiscrete algebras.

Indeed we show that the algebra square

$$\begin{tikzcd} G \times G \times A \arrow[r, "\mathrm{id}_G \times e"] \arrow[d, "\mu \times \mathrm{id}_A"'] & G \times A \arrow[d, "e"] \\ G \times A \arrow[r, "e"] & A \end{tikzcd}$$

is a (strong) pullback. So let (g, a) and (h, b) in $G \times A$ be such that $e(g, a) = e(h, b)$, that is $ga = hb$. Take the element $(g, g^{-1}h, b) \in G \times G \times A$. We have

$$(\mu \times \mathrm{id}_A)(g, g^{-1}h, b) \;=\; (gg^{-1}h, b) \;=\; (h, b)$$

and

$$(\mathrm{id}_G \times e)(g, g^{-1}h, b) \;=\; (g, g^{-1}hb) \;=\; (g, g^{-1}hb) \;=\; (g, g^{-1}ga) \;=\; (g, a).$$

No other element of $G \times G \times A$ would give us the desired result: the first component must be g in order to map by $G \times e$ to (g, a), while the third component must be b to map by $\mu \times A$ to (h, b); then again because $\mu \times \mathrm{id}_A$ sends our triple to (h, b), the second component must be $g^{-1}h$.

Example 2.2.3. As we will see in Corollary 3.4.5, every model of a *Mal'cev theory* is an indiscrete algebra of the corresponding monad. For example, since the theory of groups is a Mal'cev theory, every group is an indiscrete algebra of the group monad. Likewise, every abelian group is an indiscrete algebra of the abelian group monad.

We will also be interested in the following condition for algebras, which for partial evaluations provides the dual properties to left reversibility.

Definition 2.2.4. *Let T be a monad on* Set*. We call a T-algebra (A, e)* right reversible *if the square*

$$\begin{tikzcd} T^3A \arrow[r, "T^2e"] \arrow[d, "T\mu"'] & T^2A \arrow[d, "Te"] \\ T^2A \arrow[r, "Te"] & TA \end{tikzcd} \tag{2.6}$$

is weakly cartesian.

If T is a weakly cartesian functor, then an indiscrete algebra is obviously also right reversible, since the relevant square is precisely the T-image. Conversely, right reversibility turns out to imply indiscreteness with no further assumptions.

Proposition 2.2.5. *Any right reversible algebra is indiscrete.*

This statement becomes obvious from the perspective of partial evaluations, where any two expressions with the same total evaluation have partial evaluations to the same degenerate expression and right reversibility provides a triangle filler to that pair of partial evaluations. We now unwind this argument in the following elementary proof.

Proof. For A a right reversible T-algebra, we have the following diagram in Set:

$$\begin{tikzcd} & T^2A \arrow[rr, "Te"] \arrow[d, "\mu"] \arrow[ddl, "\eta"] & & TA \arrow[d, "e"] \arrow[ddl, "\eta"] \\ & TA \arrow[rr, "e"] \arrow[ddl, "\eta"] & & A \arrow[ddl, "\eta"] \\ T^3A \arrow[rr, "T^2e"] \arrow[d, "T\mu"'] \arrow[uur, blue, bend left, "\mu"] & & T^2A \arrow[d, "Te"] \arrow[uur, blue, bend left, "\mu"] & \\ T^2A \arrow[rr, "Te"] \arrow[uur, blue, bend left, "\mu"] & & TA & \end{tikzcd} \tag{2.7}$$

where the black diagram commutes, as do the squares with parallel blue maps, and all three of the blue maps μ satisfy $\mu\eta = \mathrm{id}$. The front square is a weak pullback by assumption, and the goal is to show the same for the back square using a diagram chase.

For any X, consider any pair of maps $f, g : X \to TA$ with $ef = eg$. We then have maps $\eta f, \eta g : X \to T^2A$ with

$$(Te)\eta f = \eta e f = \eta e g = (Te)\eta g.$$

Since the front square is a weak pullback, we get an induced map $\gamma : X \to T^3A$ with $(T^2e)\gamma = \eta f$ and $(T\mu)\gamma = \eta g$. It then only remains to show that $\mu\gamma : X \to T^2A$ satisfies the two relevant conditions

$$(Te)\mu\gamma = f, \qquad \mu\mu\gamma = g.$$

Indeed, we have

$$(Te)\mu\gamma = \mu(T^2e)\gamma = \mu\eta f = f,$$

and similarly

$$\mu\mu\gamma = \mu(T\mu)\gamma = \mu\eta g = g,$$

as was to be shown. □

3. Partial evaluations and their compositional properties

We here recall the definition of partial evaluations together with some of their basic properties from [**FP20**], and we also prove a number of new results, in particular that the partial evaluation relation is the smallest relation internal to Eilenberg-Moore algebras which relates every formal expression to its result.

3.1. Partial evaluations. Following [**FP20**], our starting point is the simple observation that a formal expression like $3 + 4 + 5$ can not only be totally evaluated to 12, but it can also be "partially evaluated" to $7 + 5$, and that the theory of monads provides a convenient framework for giving a general definition of partial evaluations. If T is a monad on Set and $e : TA \to A$ is a T-algebra, then elements of TA are formal expressions; and a formal expression $t_0 \in TA$ can be *partially evaluated* to a formal expression $t_1 \in TA$ if there is $\tau \in TTA$ such that

$$t_0 = \mu(\tau), \qquad t_1 = (Te)(\tau). \tag{3.1}$$

This intuitively means that μ is a doubly formal expression which results in t_0 upon removing the outer level of formality, and results in t_1 upon evaluating the inner level of formality. For the above example, we may take T to be the commutative monoid monad and

$$\begin{aligned} t_0 &= \boxed{3} + \boxed{4} + \boxed{5}, \\ t_1 &= \boxed{7} + \boxed{5}, \\ \tau &= \boxed{\boxed{3} + \boxed{4}} + \boxed{\boxed{5}}, \end{aligned} \tag{3.2}$$

where the boxings represent the levels of formality; we will explain this notation in more detail in Section 4.2.

The equations (3.1) can also be understood in terms of the T-algebra diagram

$$\begin{tikzcd} TTA \arrow[r, "Te"] \arrow[d, "\mu"] & TA \arrow[d, "e"] \\ TA \arrow[r, "e"] & A \end{tikzcd}$$

which has the given elements t_0 and t_1 in the lower left and upper right corners, and the element τ lifts both of these to the upper left (whenever it exists). This makes it obvious that $e(t_0) = e(t_1)$ is a necessary condition for t_1 to be a partial evaluation of t_0.

Whenever we are only interested in the existence of a partial evaluation from t_0 to t_1, then we speak of the *partial evaluation relation*. However, in this paper we will go further and in particular study properties of the *partial evaluation witness* τ.

Remark 3.1.1. For probability monads, the partial evaluation relation has long been studied in probability theory and economics, where it is known as *second-order stochastic dominance* [**FP20**, Section 6].

3.2. Compatibility with T-algebraic structure. If A is an algebra of a monad T and $R \subseteq A \times A$ is a relation, then R is *internal* if it is a T-subalgebra of $A \times A$ [**Bor94**], where $A \times A$ carries the usual componentwise T-structure corresponding to the product of T-algebras.

Definition 3.2.1. *Let $f, g : X \to Y$ be functions. The* relation generated by f and g *is the relation on Y given by the set-theoretical image of the pairing map $(f, g) : X \to Y \times Y$.*

This terminology is convenient in that it allows us to say that the partial evaluation relation for a T-algebra (A, e) is the relation generated by μ and $Te : TTA \to TA$.

Proposition 3.2.2. *Let T be a monad on Set and (A, e) any T-algebra. The partial evaluation relation on TA is an internal relation, and moreover it is the smallest internal relation which relates a formal expression to its (total) result.*

We present the proof below based on the following technical lemma.

Lemma 3.2.3. *Let T be a monad on Set. Then the relation generated by morphisms of T-algebras is internal. Moreover, if (A, e) is a T-algebra, then the smallest* internal *relation larger or equal than the (set-theoretical) relation generated by a pair of maps $f, g : X \to A$ for any set X is the relation generated by the following parallel pair of composites,*

$$\begin{tikzcd} TX \arrow[r, shift left, "Tf"] \arrow[r, shift right, "Tg"'] & TA \arrow[r, "e"] & A \end{tikzcd} \tag{3.3}$$

Note that these composites are the mates of f and g—sometimes denoted by $f^\sharp$ and $g^\sharp$—under the usual monadic adjunction,

$$\mathsf{Set}^T(TX, A) \cong \mathsf{Set}(X, A).$$

Proof of Lemma 3.2.3. First of all, the pairing $(p, q) : A \to B \times B$ of two morphisms of algebras $p, q : A \to B$ is again a morphism of algebras. The relation generated by p and q is the set-theoretic image of this map, and since the forgetful functor $U : \mathsf{Set}^T \to \mathsf{Set}$ preserves image factorizations [**Bor94**, Theorem 4.3.5], it follows that this image is a T-subalgebra.

Now let $f, g : X \to A$. The relation generated by $e \circ Tf$ and $e \circ Tg$ is internal, as we have just shown, and a straightforward argument involving $\eta : X \to TX$ shows that it contains the relation generated by f and g. Suppose now that an internal relation $R \subseteq A \times A$ contains the one generated by f and g, i.e. that the map $(f, g) : X \to A \times A$ factors through R. We have the following commutative diagram,

$$\begin{tikzcd} TX \arrow[r, "Tp"'] \arrow[rr, bend left, "T(f,g)"] & TR \arrow[r, "Ti"'] \arrow[d, "e"] & T(A \times A) \arrow[d, "e"] \\ X \arrow[r, "p"] \arrow[rr, bend right, "(f,g)"'] & R \arrow[r, hook, "i"] & A \times A \end{tikzcd} \tag{3.4}$$

where $i : R \to A \times A$ is the inclusion (which is a morphism of algebras), and p is the unique map such that $(f, g) = i \circ p$. By commutativity of (3.4), the map $e \circ T(f, g)$ factors through R, and so the relation R contains the image of $e \circ T(f, g)$.

Now, the image of $e \circ T(f, g)$ is the relation generated by the pair (3.3), since $e \circ T(f, g) = (e \times e) \circ (Tf, Tg)$. To see this, recall that the structure map of the product algebra $e : T(A \times A) \to A \times A$ is given by the composite

$$T(A \times A) \xrightarrow{\nabla} TA \times TA \xrightarrow{e \times e} A \times A,$$

where the map ∇ is the unique map which makes the following diagram commute,

$$\begin{tikzcd} & TA \\ T(A \times A) \arrow[ur, "T\pi_1"] \arrow[r, dashed, "\nabla"] \arrow[dr, "T\pi_2"'] & TA \times TA \arrow[u, "\pi_1"'] \arrow[d, "\pi_2"] \\ & TA \end{tikzcd}$$

where $\pi_1, \pi_2 : A \times A \to A$ are the product projections. Now by the commutativity of the following diagram,

$$
\begin{tikzcd}
 & & TA \arrow[r, "e"] & A \\
TX \arrow[r, "T(f,g)"] \arrow[urr, bend left, "Tf"] \arrow[drr, bend right, "Tg"'] & T(A \times A) \arrow[r, "\nabla"] \arrow[ur, "T\pi_1"] \arrow[dr, "T\pi_2"'] & TA \times TA \arrow[r, "e\times e"] \arrow[u, "\pi_1"] \arrow[d, "\pi_2"'] & A \times A \arrow[u, "\pi_1"'] \arrow[d, "\pi_2"] \\
 & & TA \arrow[r, "e"] & A
\end{tikzcd}
$$

and by the universal property of the product $A \times A$, we conclude that $e \circ T(f, g) = (e \times e) \circ \nabla \circ T(f, g) = (e \circ Tf, e \circ Tg)$.

Overall, we have therefore shown that R contains the internal relation generated by the pair (3.3) consisting of $e \circ Tf$ and $e \circ Tg$. This relation in turn contains the relation generated by f and g. Since R was an arbitrary internal relation containing the one generated by f and g, it follows that the relation generated by (3.3) is the smallest internal relation generated by f and g. □

Proof of Proposition 3.2.2. The maps $\mu, Te : TTA \to TA$ are morphisms of algebras, and so by Lemma 3.2.3, the relation they generate is internal.

Consider now the following parallel pair.

$$
\begin{tikzcd}
TA \arrow[r, "e"'] \arrow[rr, bend left, "\mathrm{id}"] & A \arrow[r, "\eta"'] & TA
\end{tikzcd}
$$

The relation generated by these maps is the one that links a formal expression to its total result. Note that the lower map is *not* a morphism of algebras in general, because η is not. The *internal* relation generated by these maps, by Lemma 3.2.3 and instantiating (3.3) for the free algebra (TA, μ), is given by the following pair of composites,

$$
\begin{tikzcd}
TTA \arrow[r, "Te"'] \arrow[rr, bend left, "\mathrm{id}"] & TA \arrow[r, "T\eta"'] & TTA \arrow[r, "\mu"] & TA
\end{tikzcd}
$$

which since $\mu \circ T\eta = \mathrm{id}$ (right unitality triangle of the monad) is equal to the following,

$$
\begin{tikzcd}
TTA \arrow[r, shift left, "\mu"] \arrow[r, shift right, "Te"'] & TA,
\end{tikzcd}
$$

which generates the partial evaluation relation by definition. □

Corollary 3.2.4. *The map $e : TA \to A$, being the coequalizer of μ and $Te : TTA \to TA$, is the quotient algebra of TA obtained by identifying formal expressions with their results.*

3.3. Composition of partial evaluations. Suppose now that we have three formal expressions $t_0, t_1, t_2 \in TA$, and that t_1 is a partial evaluation of t_0 with witness τ_{01}, and likewise that t_2 is a partial evaluation of t_1 with witness τ_{12}. Then does it follow that t_2 is also a partial evaluation of t_1? In other words, is the partial evaluation relation transitive? And if so, is there a canonical choice of witness constructed in terms of τ_{01} and τ_{12}?

In [**FP20**], we had shown that if T is a weakly cartesian monad, then the partial evaluation relation is indeed transitive. In fact, the proof goes through in general for BC monads, and can be illustrated in terms of the following diagram.

$$\begin{tikzcd}
\{*\} \arrow[dr, dotted, "\Theta"] \arrow[rrrd, dashed, bend left, "\tau_{12}"] \arrow[ddd, dashed, bend right, "\tau_{01}"'] & & & & \\
& T^3A \arrow[rr, "TTe"] \arrow[dr, "T\mu"] \arrow[dd, "\mu"'] & & T^2A \arrow[dr, "Te"] \arrow[dd, "\mu"] & \\
& & T^2A \arrow[rr, "Te"] \arrow[dd, "\mu"] & & TA\ t_2 \arrow[dd, "e"] \\
& T^2A \arrow[rr, "Te"] \arrow[dr, "\mu"'] & & \underset{t_1}{TA} \arrow[ur] \arrow[dr, "e"] & \\
& & \underset{t_0}{TA} \arrow[rr, "e"] \arrow[ur] \arrow[uurr] & & A
\end{tikzcd} \tag{3.5}$$

This diagram of ordinary black arrows commutes by general properties of monads. The blue arrows indicate the partial evaluations, keeping in mind that these are not morphisms in the same way as the other arrows are. Since the back square is a naturality square for μ, it is possible to lift $\tau_{01} \in T^2A$ and $\tau_{12} \in T^2A$ to an element $\Theta \in T^3A$ as soon as μ is a weakly cartesian transformation, which in particular holds in a BC monad. Then using commutativity of the diagram, it is easy to see that $\tau_{02} \coloneqq (T\mu)(\Theta)$ is a partial evaluation witness from t_0 to t_2. We intuitively think of the new witness τ_{02} as a *composite* of the witnesses τ_{01} and τ_{12}, and we therefore also call Θ a *composition strategy*.

We have hence shown the following:

Lemma 3.3.1. *If T is a BC monad and A any T-algebra, then the partial evaluation relation on TA is transitive.*

It is natural to ask whether this transitivity holds in general. This turns out not to be the case, but finding a counterexample has been surprisingly tricky.

Theorem 3.3.2. *There is a finitary monad T on* Set *together with a T-algebra A such that the partial evaluation relation on TA is not transitive.*

The following proof presents an explicit example. The way in which we found this example owes a lot to work of Clementino, Hofmann and Janelidze: we first constructed a semiring satisfying conditions (a)–(c) but not (d)–(e) of their [**CHJ14**, Theorem 8.10]. However, the following presentation is largely self-contained.

Proof. Let S be the commutative semiring[4] $S := \mathbb{N}[X]/\langle X^2 = 2\rangle$. This means that the elements of S are of the form[5]

$$a + bX$$

for $a, b \in \mathbb{N}$, with componentwise addition, and multiplication such that

$$(a_1 + b_1X)(a_2 + b_2X) = (a_1a_2 + 2b_1b_2) + (a_1b_1 + a_2b_2)X.$$

This semiring can be realized concretely as the smallest subsemiring of $(\mathbb{R}_+, +, \cdot)$ containing the number $\sqrt{2}$, so that $a + bX$ corresponds to $a + b\sqrt{2}$.[6]

The equation

$$X \cdot X = 1 + 1 \tag{3.6}$$

in S will be what makes the counterexample work, together with the following two facts:

- $\triangleright$ $X \in S$ is additively indecomposable: if $X = r + s$ with $r, s \in S$, then $r = 0$ or $s = 0$.
- $\triangleright$ There is no $r \in S$ with $Xr = 1$: writing $r = a + bX$ for a putative such r, we find that Xr must have constant coefficient ≥ 2 if $b \neq 0$, which we do not want; but if $b = 0$, then Xr is merely a multiple of X, which is not what we want either.

[4]Recall that a *semiring* (sometimes called a *rig*) is defined as a set with addition and multiplication operations like on a ring, but without the requirement that additive inverses must exist. $\mathbb{N}[X]$ is the semiring of polynomials in one variable X with natural number coefficients and the usual addition and multiplication of polynomials.

[5]If so desired, we could also make S finite by imposing $2 + 1 = 2$ in addition, so that every semiring element could be represented as above with $a, b \in \{0, 1, 2\}$. This would result in $|S| = 9$, and the same argument would go through and produce a more minimal counterexample. But we will not do this in order to keep the example as simple as possible.

[6]We thank Martti Karvonen for having pointed this out to us.

Now let T be the S-semimodule monad. This means that TX for $X \in \mathsf{Set}$ is the set of finitely supported functions $X \to S$, and we interpret and denote these as formal S-linear combinations. The monad structure of T is the obvious one which makes T-algebras into S-semimodules; we refer to [**CHJ14**, Section 6] for more details.

Let $A \coloneqq \{*\}$ the one element T-algebra, i.e. the zero S-semimodule. We will use box notation as in (3.2). Then we have:

- ▷ $\boxed{*} \in TA$ partially evaluates to $2\boxed{*} \in TA$, as witnessed by
$$\boxed{\bullet} + \boxed{\boxed{*}} \in TTA.$$
Indeed this is the formal S-linear combination given by the formal sum of the "empty expression" $0 = \bullet \in TA$ and $\eta * = \boxed{*} \in TA$. Now applying μ to this doubly formal expression removes the outer brackets, giving $0 + \boxed{*} = \boxed{*}$. While applying Te amounts to removing the inner brackets, including the evaluation of $\bullet$ to $*$, giving the desired $\boxed{*} + \boxed{*}$.
- ▷ $2\boxed{*} \in TA$ partially evaluates to $X\boxed{*} \in TA$, as witnessed by
$$X\boxed{X\boxed{*}} \in TTA.$$
Indeed, removing the outer brackets gives $X^2\boxed{*} = 2\boxed{*}$, while removing the inner brackets results in $X\boxed{*}$ since $X \cdot * = *$.
- ▷ Thus if the partial evaluation relation was transitive, also $\boxed{*} \in TA$ would have to partially evaluate to $X\boxed{*} \in TA$.
- ▷ To see that this is not the case, note that elements $\tau \in TTA$ are finitely supported functions $\tau : TA \to S$. Using the definition of the functor T on the algebra map e gives a description of $(Te)(\tau) \in TA$ as a finitely supported function $A \to S$, namely
$$(Te)(\tau) = \left(* \mapsto \sum_{t \in TA} \tau(t) \right).$$
In other words, since A is the zero module, $Te : TTA \to TA$ simply sums up all values of the function τ.

  Now suppose that such a $\tau : TA \to S$ witnesses the putative partial evaluation from $\boxed{*}$ to $X\boxed{*}$. Since the sum of values of τ must be X, by using the fact that X is additively indecomposable in S, we conclude that we must have $\tau(t) = X\delta_{t,r}$ for some $r \in TA$.

  On the other hand, applying μ to this τ then results in
$$\sum_t \tau(t)\, t \cdot \boxed{*} = Xr \cdot \boxed{*}.$$

In order for this to be equal to just $\boxed{*}$, we need to have $Xr = 1$. But this is impossible in S as also noted above. $\square$

3.4. Reversing partial evaluations. For the commutative monoid monad, there is a partial evaluation from $\boxed{3}+\boxed{4}+\boxed{5}$ to $\boxed{7}+\boxed{5}$, but there is none the other way around. Thus, as its name already indicates, the partial evaluation relation is typically not symmetric. However, there also are monads for which it is symmetric on *any* of its algebras. In the following, we give some criteria for when this and related phenomena occur. Recall the notion of indiscrete algebra from Definition 2.2.1.

Proposition 3.4.1. *Let T be a monad on* Set*, and let (A, e) be a T-algebra. Then A is indiscrete if and only if the partial evaluation relation is an equivalence relation. In this case, the equivalence relation obtained is the kernel pair of $e : TA \to A$.*

The final statement means that for $t_0, t_1 \in TA$, there is a partial evaluation from t_0 to t_1 if and only if these two expressions have the same result, $e(t_0) = e(t_1)$. Equivalently, the quotient of the equivalence relation is exactly A.

Proof. Suppose that (A, e) is indiscrete, meaning that the algebra square

$$\begin{tikzcd} TTA \arrow[r, "Te"] \arrow[d, "\mu"] & TA \arrow[d, "e"] \\ TA \arrow[r, "e"] & A \end{tikzcd}$$

is a weak pullback. This means exactly that two formal expressions in TA have the same result in A if and only if there exists a partial evaluation between them.

Conversely, suppose that the partial evaluation relation for A is an equivalence. Since two expressions that have different results cannot admit a partial evaluation between them, the partial evaluation relation must be finer or equal than the kernel pair of e. Just as well, since there always exists a partial evaluation from any formal expression to its result (total evaluation), the equivalence relation is necessarily coarser or equal than the kernel pair of e. Hence the two equivalence relations coincide. $\square$

Example 3.4.2. Let G be a group and $G \times -$ the G-action monad. Then the algebras of this monad are indiscrete (Example 2.2.2), and hence the partial evaluation relation for G-sets is an equivalence relation. More concretely, for a G-set A it is the relation on $G \times A$ given by $(g, a) \sim (h, b)$ if and only if $ga = hb$.

The same turns out to be the case for the group monad and the abelian group monad, where one can also intuitively "invert" things. These are instances of a more general statement which we now turn to, based on the following classical notion of universal and categorical algebra [BB04].

Definition 3.4.3. *A* Mal'cev operation *on a set A is a ternary operation $m : A \times A \times A \to A$ such that for each $a, b \in A$,*

$$m(a, b, b) \ = \ a \qquad \text{and} \qquad m(a, a, b) \ = \ b.$$

A Mal'cev theory *is an algebraic theory which contains a Mal'cev operation.*

In the theory of groups, there is a Mal'cev operation given by

$$m(a, b, c) \ := \ a\, b^{-1}\, c.$$

Therefore any theory whose algebras are groups, with possibly extra structures or properties, is a Mal'cev theory. These includes the theories of groups, abelian groups, rings, commutative rings, and modules over a fixed ring, but not, for example, the theory of monoids, commutative monoids, semirings, and semimodules over a semiring which is not a ring. An example of a Mal'cev theory which is not a theory of particular groups is the theory of *heaps* (closely related to *torsors*).

The following well-known statement is why we are interested in Mal'cev theories, together with the usual correspondence between models of an algebraic theory and the algebras of the associated monad.

Proposition 3.4.4 (e.g. [**BB04**, Chapter 2])**.** *An algebraic theory is Mal'cev if and only if every internal reflexive relation in the category of T-algebras is an equivalence relation.*

Recall that an *internal* relation is one which is compatible with the algebraic operations, or equivalently one in which the relation itself is a model of the theory (or equivalently a T-algebra).

Since the partial evaluation relation is an internal reflexive relation (Proposition 3.2.2), we therefore obtain the following by Proposition 3.4.1.

Corollary 3.4.5. *Let be the monad T on* Set *associated to a Mal'cev theory. Then every T-algebra A is indiscrete, and the partial evaluation relation on TA is an equivalence relation.*

Note that the converse is not true: for G a group, the theory of G-actions is not Mal'cev, since there is no operation of arity two or higher, but the partial evaluation relation is still an equivalence relation (Example 3.4.2).

3.5. Irreversibility of partial evaluations.

Finally, we consider some conditions on the monad and the algebra which amount to a certain kind of irreversibility of partial evaluations. Recall from Definition 2.1.8 that an algebra of a monad T is strictly positive if the square

$$\begin{tikzcd} A \arrow[r, "\eta\eta"] \arrow[d, equal] & TTA \arrow[d, "\mu"] \\ A \arrow[r, "\eta"'] & TA. \end{tikzcd}$$

is a pullback. This also has some significance for partial evaluations.

Proposition 3.5.1. *Let T be any monad on* Set *and (A, e) a T-algebra. Then the following are equivalent:*

(a) A is strictly positive.

(b) For $a \in A$, the only partial witness away from the trivial formal expression $\eta(a) \in TA$ is $\eta\eta(a) \in TTA$.

PROOF. Straightforward unfolding of definitions. □

Note that the element $\eta\eta(a) \in TTA$ is the canonical reflexivity witness of the partial evaluation from $\eta(a)$ to itself. In particular, (b) implies that if $\eta(a)$ partially evaluates to any $t \in TA$, then $t = \eta(a)$.

If (A, e) is strictly positive and the functor T preserves weak pullbacks, then also the following diagram is a pullback,

$$\begin{tikzcd} TA \arrow[r, "T(\eta\eta)"] \arrow[d, equal] & T^3A \arrow[d, "T\mu"] \\ TA \arrow[r, "T\eta"'] & TTA. \end{tikzcd}$$

In terms of partial evaluations, this square being a pullback means exactly that the identity partial evaluation cannot be expressed as a nontrivial composite (i.e. it can only be written as the composition of twice itself). To see this, recall that given a composition strategy $\Theta \in T^3A$, the resulting composite partial evaluation witness is given by $(T\mu)(\Theta)$. Now suppose that for some $t \in TA$ we have $(T\mu)(\Theta) = (T\eta)(t)$, i.e. the identity partial evaluation at t arises in this way from the composition strategy Θ. Then the pullback condition says that necessarily $\Theta = (T(\eta\eta))(\alpha)$.

Finally, the following result shows that strict positivity and reversibility of an algebra rarely come together.

Proposition 3.5.2. *Let T be any monad on* Set *and let (A, e) be a strictly positive and indiscrete T-algebra. Then the partial evaluation relation of A is the identity relation and $e : TA \to A$ is an isomorphism.*

PROOF. Let $t \in TA$ be a formal expression, and let $a := e(t)$ be its total result. By reversibility there exists not only a partial evaluation from t to $\eta(a)$, but also one from $\eta(a)$ to t. By strict positivity, the partial evaluation from $\eta(a)$ to t must be an identity, which means that $t = \eta(a)$. This is true for all $t \in TA$, which means in particular that e is injective, and so (since it is split epi) an isomorphism. $\square$

Proposition 9.4.3 strengthens this result to all right reversible algebras and to more general simplicial sets than those arising as bar constructions.

4. The bar construction and the quest for its compositional structure

Consider again the diagram (3.5) involving the composition strategy Θ. The three blue arrows which illustrate the partial evaluations indicate that it may be beneficial to think of Θ as a *triangle* or *2-simplex* in a structure where the elements of TA are vertices, the elements of T^2A are edges between these vertices representing partial evaluation witnesses, the 2-simplices are composition strategies, etc. We do not need to look far in order to obtain a general definition for what this structure is, since it is well-known: the bar construction. We refer to Trimble's exposition [**Tri07**] for a more extensive treatment of the bar construction and its categorical properties.

4.1. The bar construction. Given a monad (T, μ, η) on Set and a T-algebra (A, e), the bar construction gives a free resolution of that algebra in the form of an augmented simplicial set, i.e. a functor

$$\mathrm{Bar}_T(A) \; : \; \Delta_+^{\mathrm{op}} \longrightarrow \mathsf{Set},$$

defined as follows. On objects,

$$\mathrm{Bar}_T(A)(n) := T^{n+1}A$$

for all $\overline{n} \in \Delta_+$, including $n = -1$. Thus an n-dimensional simplex in the bar construction is an element of $T^{n+1}A$, i.e. a formal expression with elements from A and $n+1$ levels of formality. The generating face maps

$$d_{n,i} \; : \; T^{n+1}A \longrightarrow T^nA,$$

are given by T^ne for $i = 0$ and by $T^{n-i}\mu$ for $1 \le i \le n$, resulting in the diagram

$$\cdots \quad T^4A \;\begin{matrix}\xrightarrow{T^3e}\\ \xrightarrow{T^2\mu}\\ \xrightarrow{T\mu}\\ \xrightarrow{\mu}\end{matrix}\; T^3A \;\begin{matrix}\xrightarrow{T^2e}\\ \xrightarrow{T\mu}\\ \xrightarrow{\mu}\end{matrix}\; T^2A \;\begin{matrix}\xrightarrow{Te}\\ \xrightarrow{\mu}\end{matrix}\; TA \xrightarrow{\;e\;} A.$$

The degeneracy maps

$$s_{n,i} \; : \; T^{n+1}A \longrightarrow T^{n+2}A$$

are given by $T^{n-i+1}\eta$ for $0 \leq i \leq n$, resulting in the diagram

$$\begin{tikzcd}
\cdots & T^4A & T^3A \arrow[l, shift right=4, "T\eta"'] \arrow[l, "T^2\eta"] \arrow[l, shift left=4, "T^3\eta"] & T^2A \arrow[l, shift right=2, "T\eta"'] \arrow[l, shift left=2, "T^2\eta"] & TA \arrow[l, "T\eta"'] & A.
\end{tikzcd}$$

Taken together, the face and degeneracy maps define the augmented simplicial set $\mathrm{Bar}_T(A) : \Delta_+^{\mathrm{op}} \to \mathsf{Set}$. Its restriction to Δ is a simplicial set which we also denote $\mathrm{Bar}_T(A)$ by abuse of notation; throughout the paper $\mathrm{Bar}_T(A)$ will refer to the latter since the augmentation plays no role for us.

Remark 4.1.1. It is well-known that if T is a cartesian monad, then $\mathrm{Bar}_T(A)$ is the nerve of a category.[7]

Indeed if we assume merely that μ is strongly cartesian, the following diagram shows that $X_n \cong X_1 \times_{X_0} \overset{n}{\cdots} \times_{X_0} X_1$:

$$\begin{tikzcd}
&&&&& T^{n+1}A \arrow[dl, "\mu"'] \arrow[dr, "T^ne"] \\
&&&& T^nA \arrow[dl, "\mu"'] \arrow[dr, "T^{n-1}e"] && T^nA \arrow[dl, "\mu"'] \arrow[dr, "T^{n-1}e"] \\
&&& T^{n-1}A \arrow[dl] \arrow[dr] && T^{n-1}A \arrow[dl] \arrow[dr] && T^{n-1}A \arrow[dl] \arrow[dr] \\
&& T^3A \arrow[dl, "\mu"'] \arrow[dr, "T^2e"] && \cdots \arrow[dl, "\mu"'] \arrow[dr, "T^2e"] && \cdots \arrow[dl, "\mu"'] \arrow[dr, "T^2e"] && T^3A \arrow[dl, "\mu"'] \arrow[dr, "T^2e"] \\
& T^2A \arrow[dl, "\mu"'] \arrow[dr, "Te"] && T^2A \arrow[dl, "\mu"'] \arrow[dr, "Te"] && \cdots \arrow[dl, "\mu"'] \arrow[dr, "Te"] && T^2A \arrow[dl, "\mu"'] \arrow[dr, "Te"] && T^2A \arrow[dl, "\mu"'] \arrow[dr, "Te"] \\
TA && TA && TA & \cdots & TA && TA && TA
\end{tikzcd}$$

Each square in the diagram is a naturality square for μ, hence a pullback, and any cone over the bottom two rows of the diagram induces unique maps to each subsequent row moving upwards. $\mathrm{Bar}_T(A)$ is thus the nerve of a category for any monad T with μ strongly cartesian.

Example 4.1.2. More concretely, for a monoid M let $M \times -$ be the M-set monad. For an M-set A, the bar construction $\mathrm{Bar}_{M\times -}(A)$ is the nerve of a category. A straightforward unfolding of the definition of the first few levels of the bar construction (similar to Example 2.1.15) shows that this category has pairs $(x, a) \in M \times A$ as

[7]This seems to be a folklore observation for which it is difficult to find a reference. The only explicit mention that we are aware of is a comment by Trimble on the n-Category Café blog at https://golem.ph.utexas.edu/category/2007/05/on_the_bar_construction.html#c009955.

objects, with morphisms $(x, a) \to (y, b)$ corresponding to the monoid elements $z \in M$ satisfying $x = yz$ and $b = za$, and composing by multiplication.

Remark 4.1.3. If T is a cartesian monad and A a T-algebra with cartesian algebra square, then $\mathrm{Bar}_T(A)$ is even the nerve of an equivalence relation, namely of the kernel pair of $e : TA \to A$ as in Proposition 3.4.1. This is because by definition, a T-algebra (A, e) is cartesian if and only if the parallel pair

$$TTA \underset{Te}{\overset{\mu}{\rightrightarrows}} TA$$

is a kernel pair of e.

On the other hand, it is easy to give examples in which the category of which $\mathrm{Bar}_T(A)$ is the nerve does have distinct parallel arrows. This happens for example for $\mathrm{Bar}_{M\times-}(1)$, where M is a suitable monoid and the singleton set 1 carries the unique M-action, since then the relevant category is given by the elements $a, b \in M$ as objects and such that the morphisms $a \to b$ are in bijection with the elements $c \in M$ satisfying $a = cb$.

Based on the definition of the bar construction, it is immediate that a 1-simplex in $\mathrm{Bar}_T(A)$ from vertex $t_0 \in TA$ to vertex $t_1 \in TA$ is the same thing as a partial evaluation witness from t_0 to t_1. Moreover for $t_0, t_1, t_2 \in TA$ and partial evaluation witnesses $\tau_{01}, \tau_{12} \in TTA$, the composition strategies Θ (Section 3.3) are exactly those 2-simplices in $\mathrm{Bar}_T(A)$ whose face obtained by deleting vertex 2 is τ_{01}, and whose face obtained by deleting vertex 0 is τ_{12}. In the parlance of quasicategory theory, finding such a Θ for given τ_{01} and τ_{12} hence amounts to *filling an inner 2-horn* [**Joy02**].

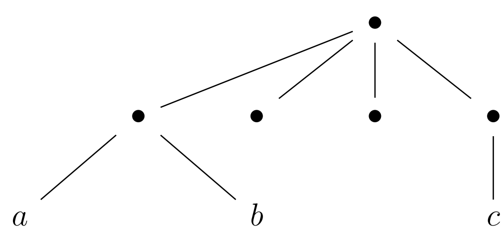
This motivates our quest of trying to understand to what extent the bar construction, considered as a simplicial set, can be thought of as a higher compositional structure. As we have seen in the previous section, all inner 2-horns can be filled if T is a BC monad; on the other hand for a general monad T and two composable 1-simplices in the bar construction of a T-algebra, there may not even be a third 1-simplex pointing directly from the source of the first to the target of the second (Theorem 3.3.2). This theme will continue throughout the paper: we will find good compositionality properties for the bar construction as long as T satisfies suitable lifting conditions, but not in general.

4.2. The commutative monoid monad and its bar construction. With the commutative monoid monad on Set serving as a recurring example in what follows, we now give a more explicit description of its bar construction in some detail, in particular making precise the idea that higher levels of formality correspond to iterated "bracketing" or "boxing" of expressions. Although this description applies in very much the same way to all monads coming from symmetric operads (and similarly from non-symmetric operads), we focus on the commutative monoid monad for simplicity.

Variants of the following considerations are very well-known in operad theory. Nevertheless, we surprisingly have not found any reference containing the relevant statements in the precise form that we need, and we therefore offer our own detailed exposition in what follows. Similar considerations in somewhat different contexts can be found e.g. in works of Ching [**Chi05**, Section 4] or Kock [**Koc19**, Section 2.3.5].

So let $T : \mathsf{Set} \to \mathsf{Set}$ be the commutative monoid monad. For any $X \in \mathsf{Set}$, the set TX is the set of finite multisubsets of X, or equivalently of finitely supported maps $X \to \mathbb{N}$. We can thus identify the elements of TX with non-planar rooted trees of height at most 1, where the leaves are labelled by elements of X. The tree consisting only of the root then corresponds to the neutral element $0 \in TX$ representing the empty multiset.

Now upon applying T multiple times, it follows that $T^n X$ for $n \in \mathbb{N}$ can be identified with the set of non-planar rooted trees of height at most n and with leaves at depth n labelled by elements of X; see also the literature on *operadic trees* for more explanation [**Koc19**, Section 1.5]. For example for $n = 2$, a typical element of $T^2 X$ is represented by a tree that looks like



(4.1)

for some $a, b, c \in X$. In terms of multiset notation, we could also denote this element of $T^2 X$ as

$$\{\{a, b\}, \{\}, \{\}, \{c\}\}.$$

But since this type of expression can get cumbersome to write, we equivalently use boxed expressions such as

$$\boxed{\boxed{a} + \boxed{b}} + \boxed{\bullet} + \boxed{\bullet} + \boxed{\boxed{c}},$$

were the tree structure is now encoded in the boxing, so that each level of boxing represents a level of formality. The fact that the trees are non-planar now means that it is understood that the individual summands within a box can be arranged arbitrarily but not moved across box boundaries. We denote unlabelled leaves by $\bullet$, as in the tree diagrams. These correspond to the neutral element $0 \in T^k X$ when there are $n - k$ levels of boxing around $\bullet$. The elements of $T^n X$ hence are identified with equivalence classes—with respect to the commutative monoid laws—of boxed expressions with up to n levels of boxing and elements of X at exactly level n. In particular, elements of TX already have one level of boxing.

In terms of the trees or boxed expressions picture, we then have the following:

- ▷ For a function $f : X \to Y$, the induced map $T^n f : T^n X \to T^n Y$ takes such a tree with bottom leaves labelled in X and replaces these labels by applying f to each of them, and similarly in the boxed expressions notation.
- ▷ The multiplication $\mu : T^2 X \to TX$, which takes a multiset of multisets and maps it to their multiset union, takes a tree of height at most 2 and removes all nodes at level 1, leaving their child nodes in place and connecting them up directly with the root node. More generally, $T^k \mu : T^{n+1} X \to T^n X$ for $k = 0, \ldots, n-1$, takes a tree of height at most $n+1$, removes all nodes at depth $k+1$, and similarly connects their children to their parent nodes.
- ▷ The unit $\nu : X \to TX$ takes an element of X to the corresponding singleton multiset $\{x\}$. Hence in the tree picture, $T^k \eta : T^n X \to T^{n+1} X$ for $k = 0, \ldots, n$ replaces every node at depth k by a pair of nodes, one at depth k and one at depth $k+1$, such that the latter is the only child node of the former.

We thus have all the tools in hand to do concrete computations in the bar construction of the commutative monoid monad: we perform them on the corresponding trees, while usually using boxed expression notation for these trees.

Remark 4.2.1. The commutative monoid monad has the following convenient property, which is obvious from the trees picture: if $t_0 \in TA$ partially evaluates to $t_1 \in TA$, then the number of terms in t_1 is at least as large as the number of terms in t_0, with equality if and only if $t_0 = t_1$. (Here, the number of terms can be conveniently defined by applying the functor T to the map $A \to 1$ and composing with the obvious isomorphism $T1 \cong \mathbb{N}$.)

4.3. Nonuniqueness of composite partial evaluations. A related question concerns the uniqueness of composition strategies. Upon composing two partial evaluation witnesses using composition strategies, is the resulting composite partial evaluation witness well-defined, i.e. independent of the choice of composition strategy? Equivalently, is $\mathrm{Bar}_T(A)$ such that fillers for inner 2-horns have unique third faces? This is not the case:

Theorem 4.3.1. *There is a finitary monad T on* Set *together with a T-algebra A for which $\mathrm{Bar}_T(A)$ contains an inner 2-horn with two different fillers such that also their outer 1-faces are different.*

Note that this is a phenomenon which cannot occur in the nerve of a category.

Proof. Again we construct a concrete example, this time with the commutative monoid monad T and T-algebra $A := (\mathbb{N}, +)$. Consider the elements of TTA given by

$$\begin{aligned}
\alpha &:= \boxed{\boxed{2}+\boxed{2}} + \boxed{\boxed{3}+\boxed{3}} + \boxed{\boxed{3}+\boxed{1}}, \\
\beta &:= \boxed{\boxed{4}+\boxed{6}} + \boxed{\boxed{4}}.
\end{aligned}$$

These form an inner 2-horn, because

$$(Te)(\alpha) = \mu(\beta) = \boxed{4} + \boxed{6} + \boxed{4}.$$

This horn admits two distinct fillers given by

$$\begin{aligned}
\delta &:= \boxed{\boxed{\boxed{2}+\boxed{2}} + \boxed{\boxed{3}+\boxed{3}}} + \boxed{\boxed{\boxed{3}+\boxed{1}}}, \\
\delta' &:= \boxed{\boxed{\boxed{2}+\boxed{2}}} + \boxed{\boxed{\boxed{3}+\boxed{3}} + \boxed{\boxed{3}+\boxed{1}}}.
\end{aligned}$$

Indeed, removing the outer boxes easily gives $\mu(\delta) = \mu(\delta') = \alpha$, while removing the inner boxes shows that $(T^2e)(\delta) = (T^2e)(\delta') = \beta$, proving that both δ and δ' fill the horn. The resulting outer 1-faces arise by removing the intermediate level of boxing,

$$\begin{aligned}
(T\mu)(\delta) &= \boxed{\boxed{2}+\boxed{2}+\boxed{3}+\boxed{3}} + \boxed{\boxed{3}+\boxed{1}}, \\
(T\mu)(\delta') &= \boxed{\boxed{2}+\boxed{2}} + \boxed{\boxed{3}+\boxed{3}+\boxed{3}+\boxed{1}},
\end{aligned}$$

which are indeed distinct parallel 1-cells. $\square$

4.4. Non-fillable inner horns. Since inner 2-horns in the bar construction have fillers for BC monads, it is natural to ask whether inner horns have fillers in general under suitable assumptions on the monad. By Remark 4.1.1, this is clearly the case for $\mathrm{Bar}_T(A)$ whenever T is a cartesian monad since the nerve of a category trivially has all inner horn fillers. In the following, we will consider inner 3-horns; these come in the following two kinds.

An inner 3-horn of the first kind in a bar construction consists of three 2-simplices $\alpha, \gamma, \delta \in T^3A$ satisfying the equations

$$(T\mu)(\alpha) = \mu(\gamma), \qquad (TTe)(\alpha) = \mu(\delta), \qquad (TTe)(\gamma) = (TTe)(\delta). \tag{4.2}$$

A filler is then an element $\varepsilon \in T^4A$ which recovers the given 2-simplices via

$$\alpha = \mu(\varepsilon), \qquad \gamma = (T^2\mu)(\varepsilon), \qquad \delta = (T^3e)(\varepsilon).$$

In other words, given compatible elements α, γ, δ of the red part of the diagram

$$
\begin{tikzcd}
(\varepsilon \in)\, T^4A \arrow[rr] \arrow[dr] \arrow[dd] & & (\delta \in)\, T^3A \arrow[dr] \arrow[dd] & \\
& (\gamma \in)\, T^3A \arrow[rr] \arrow[dd] & & T^2A \arrow[dd] \\
(\alpha \in)\, T^3A \arrow[rr] \arrow[dr] & & T^2A \arrow[dr] & \\
& T^2A \arrow[rr] & & TA
\end{tikzcd}
\tag{4.3}
$$

there is a common lift ε along the blue arrows.

An inner 3-horn of the second kind consists of three 2-simplices $\alpha, \beta, \delta \in T^3A$ satisfying the equations

$$\mu(\alpha) = \mu(\beta), \qquad (T^2e)(\alpha) = \mu(\delta), \qquad (T^2e)(\beta) = (T\mu)(\delta).$$

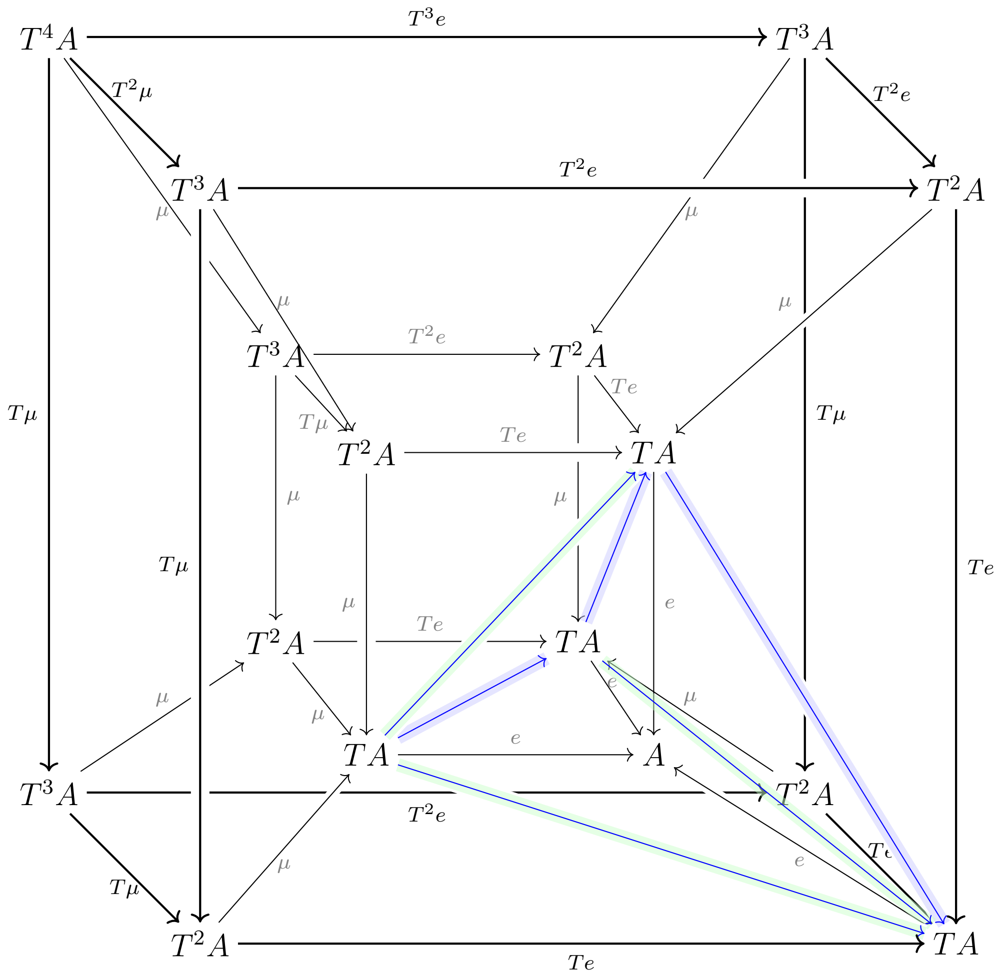
A filler is then an element $\varepsilon \in T^4A$ which recovers the given 2-simplices via

$$\mu(\varepsilon) = \alpha, \qquad (T\mu)(\varepsilon) = \beta, \qquad (T^3e)(\varepsilon) = \delta.$$

In other words, given compatible elements α, β, δ of the red part of the diagram

$$
\begin{tikzcd}
(\varepsilon \in)\, T^4A \arrow[rrr] \arrow[dr] \arrow[ddd] & & & (\delta \in)\, T^3A \arrow[dl] \arrow[ddd] \\
& (\alpha \in)\, T^3A \arrow[r] \arrow[d] & T^2A \arrow[d] & \\
& T^2A \arrow[r] & TA & \\
(\beta \in)\, T^3A \arrow[ur] \arrow[rrr] & & & T^2A \arrow[ul]
\end{tikzcd}
\tag{4.4}
$$

there is a common lift ε along the blue arrows. Here, we have drawn the above diagrams in this form since they both appear in roughly their respective shape as subdiagrams of the hypercube



where the simplices of the bar construction can be conveniently visualized in terms of the schematic blue tetrahedron, extending (3.5) by one dimension.

Theorem 4.4.1. *There is a weakly cartesian finitary monad T on* Set *together with a T-algebra A for which* $\mathrm{Bar}_T(A)$ *contains one of each kind of inner 3-horn without a filler.*

In fact, the examples that we construct in the proof will show that not only is it impossible to fill the interior of the 3-simplex, but even the remaining 2-simplex face cannot be filled.

Proof. We again use the commutative monoid monad T and the bar construction of the T-algebra $A := (\mathbb{N}, +)$. For the first kind of inner 3-horn as described above, consider

$$\begin{aligned}
\alpha &\coloneqq \boxed{\boxed{\boxed{2}+\boxed{2}}} + \boxed{\boxed{\boxed{2}+\boxed{2}}} + \boxed{\boxed{\boxed{3}+\boxed{1}}}, \\
\gamma &\coloneqq \boxed{\boxed{\boxed{2}+\boxed{2}} + \boxed{\boxed{2}+\boxed{2}}} + \boxed{\boxed{\boxed{3}+\boxed{1}}}, \\
\delta &\coloneqq \boxed{\boxed{\boxed{4}}} + \boxed{\boxed{\boxed{2}+\boxed{2}} + \boxed{\boxed{3}+\boxed{1}}}.
\end{aligned}$$

We verify that these 2-simplices indeed assemble to an inner 3-horn. We have that μ removes the outer boxes, $T\mu$ removes the mid-level boxes, and T^2e removes the inner boxes (and possibly evaluates the sums). Hence indeed,

$$(T\mu)(\alpha) \;=\; \boxed{\boxed{2}+\boxed{2}} + \boxed{\boxed{2}+\boxed{2}} + \boxed{\boxed{3}+\boxed{1}} \;=\; \mu(\gamma),$$

$$(T^2e)(\alpha) \;=\; \boxed{\boxed{4}} + \boxed{\boxed{2}+\boxed{2}} + \boxed{\boxed{3}+\boxed{1}} \;=\; \mu(\delta),$$

and moreover,

$$\begin{aligned}
(T^2e)(\gamma) &= \boxed{\boxed{4}+\boxed{4}} + \boxed{\boxed{4}} \\
(T^2e)(\delta) &= \boxed{\boxed{4}} + \boxed{\boxed{4}+\boxed{4}}
\end{aligned} \tag{4.5}$$

resulting in $(T^2e)(\gamma) = (T^2e)(\delta)$, since these two expressions differ only by rearranging the summands.[8]

Hence α, γ and δ indeed define an inner 3-horn. Now if there existed a filler $\varepsilon \in T^4A$, then in particular there would have to be a $\beta \in T^3A$ playing the role of the remaining 2-simplex, i.e. satisfying the equations

$$(T\mu)(\beta) = (T\mu)(\gamma), \qquad (T^2e)(\beta) = (T\mu)(\delta).$$

Let's see why such a β cannot exist. First of all,

$$\begin{aligned}
(T\mu)(\gamma) &= \boxed{\boxed{2}+\boxed{2}+\boxed{2}+\boxed{2}} + \boxed{\boxed{3}+\boxed{1}}, \\
(T\mu)(\delta) &= \boxed{\boxed{4}} + \boxed{\boxed{2}+\boxed{2}+\boxed{3}+\boxed{1}}.
\end{aligned}$$

[8]This is where we are using that T is not cartesian, but only weakly cartesian; the same example would not work with T being the monoid monad.

Both terms consist of two outer boxes, and therefore β must consist of two outer boxes as well. Thus, up to permutation,

$$\begin{tikzcd}[column sep=small]
& \beta = \boxed{\cdots}+\boxed{\cdots} \arrow[dl, mapsto, "T\mu"'] \arrow[dr, mapsto, "T^2 e"] & \\
\boxed{\boxed{3}+\boxed{1}} + \boxed{\boxed{2}+\boxed{2}+\boxed{2}+\boxed{2}} & & \boxed{\boxed{4}} + \boxed{\boxed{2}+\boxed{2}+\boxed{3}+\boxed{1}}
\end{tikzcd}$$

Applying Te to either desired result gives $\boxed{4}+\boxed{8}$. This shows that the two outer boxes of β must match up with the other outer boxes as follows: in the first box of β, there must be an element of TTA witnessing a partial evaluation from $\boxed{3}+\boxed{1}$ to $\boxed{4}$. In the second slot of β, we need to have a witness of a partial evaluation from $\boxed{2}+\boxed{2}+\boxed{2}+\boxed{2}$ to $\boxed{2}+\boxed{2}+\boxed{3}+\boxed{1}$. The proof is now complete upon noting that there is not such partial evaluation, for example because any nontrivial partial evaluation must strictly decrease the number of terms (Remark 4.2.1).

Concerning the second kind of inner 3-horn, consider similarly the terms

$$\alpha \;:=\; \boxed{\boxed{\boxed{2}+\boxed{2}}+\boxed{\boxed{2}+\boxed{2}}} + \boxed{\boxed{\boxed{3}+\boxed{1}}},$$

$$\beta \;:=\; \boxed{\boxed{\boxed{2}+\boxed{2}}} + \boxed{\boxed{\boxed{2}+\boxed{2}}+\boxed{\boxed{3}+\boxed{1}}},$$

$$\delta \;:=\; \boxed{\boxed{\boxed{4}+\boxed{4}}} + \boxed{\boxed{\boxed{4}}}.$$

We have that

$$\mu(\alpha) \;=\; \boxed{\boxed{2}+\boxed{2}} + \boxed{\boxed{2}+\boxed{2}} + \boxed{\boxed{3}+\boxed{1}} \;=\; \mu(\beta)$$

$$(T^2e)(\alpha) \;=\; \boxed{\boxed{4}+\boxed{4}} + \boxed{\boxed{4}} \;=\; \mu(\delta)$$

and

$$(T^2e)(\beta) \;=\; \boxed{\boxed{4}} + \boxed{\boxed{4}+\boxed{4}}$$

$$(T\mu)(\delta) \;=\; \boxed{\boxed{4}+\boxed{4}} + \boxed{\boxed{4}}$$

which, as before, differ only by a permutation, and so are equal as elements of TTA. Therefore α, β and δ form an inner 3-horn. As before, we can show that we cannot even find a 2-simplex $\gamma \in T^3A$ such that $\mu(\gamma) = (T\mu)(\alpha)$ and $(T\mu)(\gamma) = (T\mu)(\beta)$. Since

$$
\begin{aligned}
(T\mu)(\alpha) &= \boxed{\boxed{2}+\boxed{2}+\boxed{2}+\boxed{2}} + \boxed{\boxed{3}+\boxed{1}} \\
(T\mu)(\beta) &= \boxed{\boxed{2}+\boxed{2}} + \boxed{\boxed{2}+\boxed{2}+\boxed{3}+\boxed{1}}
\end{aligned}
$$

this would mean that we would have:

$$
\begin{tikzcd}
& \gamma \arrow[dl, mapsto, "\mu"'] \arrow[dr, mapsto, "T\mu"] & \\
\boxed{\boxed{2}+\boxed{2}+\boxed{2}+\boxed{2}} + \boxed{\boxed{3}+\boxed{1}} & & \boxed{\boxed{2}+\boxed{2}} + \boxed{\boxed{2}+\boxed{2}+\boxed{3}+\boxed{1}}
\end{tikzcd}
$$

Hence upon considering TA as a free T-algebra, the term on the right would have to be a partial evaluation of the term on the left. By the number of terms counting of Remark 4.2.1, this is again not the case. $\square$

5. Pushouts in Δ and their decomposition

The results of the previous section have all been negative: we have only found counterexamples to the compositional structure of the bar construction in dimensions > 1. In the remaining sections we will focus on what can be said in the positive for suitably well-behaved monads. The resulting compositional properties will be formulated in terms of filler conditions, the analysis of which requires a good understanding of (weak) pushouts in Δ. We thus dedicate this section to a classification of these pushouts, and to proving results on how these pushouts decompose into more basic ones.

Recall that we denote the objects of Δ as $\overline{n} = \{0, \dots, n\}$. Throughout this section, we also use the term "joint image" to refer to the union of the images of several maps, and we often say "strong pushout" instead of "pushout" in order to emphasize the difference with "weak pushout", as we have already done for pullbacks.

To illustrate the subtlety of pushouts in Δ, consider the following commutative square in Δ.

$$
\begin{tikzcd}
\overline{1} \arrow[r, "02"] \arrow[d] & \overline{2} \arrow[d] \\
\overline{0} \arrow[r, equal] & \overline{0}
\end{tikzcd}
$$

This square is a pushout in Δ, although its image in Set, under the usual forgetful functor $\Delta(\overline{0}, -) : \Delta \to \mathsf{Set}$, is not. It is useful to have a name for all those pushouts which do not display this (initially) counterintuitive behavior.

Definition 5.0.1. *A pushout in Δ is* concrete *if it is preserved by the forgetful functor $\Delta \to \mathsf{Set}$.*

5.1. Weak pushouts in Δ. We start with the reduction of weak pushouts in Δ to strong pushouts. This reduction is a well-known property of weak pushouts in categories with pushouts. Since Δ however does not have all pushouts, we present the proof in detail.

Lemma 5.1.1. *Every weak pushout in Δ has a split monomorphism from a strong pushout of the same span.*

$$
\begin{tikzcd}
\overline{m} \arrow[r, "f"] \arrow[d, "g"'] & \overline{p} \arrow[d, "h"'] \arrow[ddr, "\phi"] & \\
\overline{q} \arrow[r, "k"] \arrow[drr, "\psi"'] & \overline{n} \arrow[dr, dashed] & \\
& & \overline{t}
\end{tikzcd}
$$

Proof. Suppose that the square involving $\overline{n}$ is a weak pushout. Let $i : \overline{s} \hookrightarrow \overline{n}$ be the inclusion of the joint image of h, k in $\overline{n}$, with the obvious (now jointly surjective) maps $h' : \overline{p} \to \overline{s}$ and $k' : \overline{q} \to \overline{s}$. We argue that this gives a strong pushout.

First, $\overline{s}$ is clearly also a weak pushout of the span consisting of f and g, as for any ϕ, ψ as above, composing the induced map from $\overline{n}$ with i gives a commuting diagram out of $\overline{s}$. To see that this weak pushout is strong, it remains to be shown that any two induced maps α, β out of $\overline{n}$ become equal when composed with this inclusion. But this follows immediately, as h' and k' are jointly epimorphic by construction.

Applying the weak pushout property of the original square produces a map $j : \overline{n} \to \overline{s}$ which makes the following diagram commute.

$$
\begin{tikzcd}
\overline{m} \arrow[r, "f"] \arrow[d, "g"'] & \overline{p} \arrow[d, "h"'] \arrow[ddr, "h'"] \arrow[dddrr, bend left, "h"] & & \\
\overline{q} \arrow[r, "k"] \arrow[drr, "k'"'] \arrow[ddrrr, bend right, "k"'] & \overline{s} \arrow[dr, dashed, "i"] & & \\
& & \overline{n} \arrow[dr, dashed, "j"] & \\
& & & \overline{s}
\end{tikzcd}
$$

Thus the uniqueness part of the strong pushout property implies that $ji = \mathrm{id}_{\overline{s}}$. $\square$

This implies that a functor out of Δ^{op} is weakly cartesian so long as it sends strong pushouts in Δ to weak pullbacks. The proof of this lemma also tells us that a weak pushout in Δ is strong if and only if h, k are jointly surjective.

We now proceed to characterize strong pushouts in Δ by decomposing them into simpler parts. The key observation is that a morphism $f : \overline{r} \to \overline{n}$ in Δ *decomposes* $\overline{n}$ into $r + 2$ pieces, namely the spinal subsimplices

$$\begin{aligned}
\overline{n}_0 &:= \{0, \dots, f(0)\}, \\
\overline{n}_i &:= \{f(i-1), \dots, f(i)\} \quad \text{for } 1 \leq i \leq r, \\
\overline{n}_{r+1} &:= \{f(r), \dots, n\},
\end{aligned}$$

and that simplices resembling these pieces can be assembled into $\overline{n}$ through a sort of product which we now describe.

5.2. $\star$-Decompositions and $\star$-products. Before getting to the characterization of pushouts in Δ, we introduce our main technical tool for reducing the problem to simpler cases by letting us decompose diagrams in Δ into families of simpler diagrams of the same shape: the $\star$-decomposition. There is a partially defined inverse operation which we call $\star$-product, and which behaves similarly to a monoidal structure. We will subsequently exploit the fact that both the $\star$-decomposition and the $\star$-product are defined as colimits in order to argue that they are well-behaved with respect to pushouts.

Definition 5.2.1. *Let $\overline{r}/\Delta$ denote the* undercategory *of $\overline{r}$ in Δ, whose objects are maps $\overline{r} \to \overline{n}$ in Δ and whose morphisms $(\overline{r} \to \overline{n}) \to (\overline{r} \to \overline{m})$ are maps $\overline{n} \to \overline{m}$ commuting with the maps from $\overline{r}$. We denote by $\star : \overline{r}/\Delta \to \Delta$ the forgetful functor sending $\overline{r} \to \overline{n}$ to $\overline{n}$ and forgetting the commuting property of the morphisms.*

As alluded to above, we will consider a map $f : \overline{r} \to \overline{n}$ as a decomposition of $\overline{n}$ into $r + 2$ pieces, which we call *$\star$-components*. We recall now a general categorical property of undercategories, which we make extensive use of in this section.

Lemma 5.2.2. *$\star$ is a discrete opfibration. That is, for any map $f : \overline{n} \to \overline{m}$ in Δ and a lift of $\overline{n}$ to $g : \overline{r} \to \overline{n}$ in $\overline{r}/\Delta$, there is a unique lift of $\overline{m}$ to $h : \overline{r} \to \overline{m}$ in $\overline{r}/\Delta$ such that f lifts to a map from g to h in $\overline{r}/\Delta$.*

Proof. Define h to be the composite fg and this follows immediately. Concretely, this decomposes $\overline{m}$ into the $\star$-components

$$\begin{aligned}
\overline{m}_0 &:= \{0, \dots, f(g(0))\}, \\
\overline{m}_i &:= \{f(g(i-1)), \dots, f(g(i))\} \quad \text{for } 1 \leq i \leq r, \\
\overline{m}_{r+1} &:= \{f(g(r)), \dots, m\}.
\end{aligned}$$

$\square$

We will call a lift of $\overline{n}$ to $\overline{r}/\Delta$ a $\star$-*decomposition* of $\overline{n}$, so that this lemma shows that $\star$-decompositions push forward along maps in Δ. This lets us further extend a $\star$-decomposition on $\overline{n}$ to more general diagrams in Δ.

Corollary 5.2.3. *Let J be a category with an initial object I, and $D : \mathsf{J} \to \Delta$ a diagram. Then every $\star$-decomposition of $D(I)$ extends to a $\star$-decomposition of the whole diagram D; that is, D lifts along $\star$ to a diagram $D' : \mathsf{J} \to \overline{r}/\Delta$.*

Proof. This is a standard property of discrete opfibrations. □

This allows us to $\star$-decompose spans and squares in Δ according to a $\star$-decomposition of their initial object. In order to more easily show that $\star$ preserves pushouts, we provide an equivalent definition as a colimit.

Lemma 5.2.4. *Consider diagrams in Δ as below, where we specify a morphism out of the singleton set $\overline{0} = \{0\}$ by its image in the target:*

$$\begin{tikzcd}
 & \overline{n}_0 & & \cdots & & \overline{n}_{r+1} & \\
\overline{0} \arrow[ur, "0"] & & \overline{0} \arrow[ul, "n_0"'] \arrow[ur, "0"] & \cdots & \overline{0} \arrow[ul, "n_r"'] \arrow[ur, "0"] & & \overline{0} \arrow[ul, "n_{r+1}"']
\end{tikzcd}$$

We denote the shape of these diagrams by $\wedge^{r+2}$. Then:

(a) $\overline{r}/\Delta$ is isomorphic to the category $\mathrm{Fun}_\star(\wedge^{r+2}, \Delta)$ of diagrams of this form and natural transformations.

(b) The functor $\mathrm{Fun}_\star(\wedge^{r+2}, \Delta) \cong \overline{r}/\Delta \xrightarrow{\star} \Delta$ is naturally isomorphic to the colimit functor sending such a diagram to its colimit $\overline{n_0 + \cdots + n_{r+1}}$.

Proof. (a) Given such a diagram, construct a map $g : \overline{r} \to \overline{n_0 + \cdots + n_{r+1}}$ sending i to $n_0 + \cdots + n_i$ for $0 \leq i \leq r$. Conversely given a map $g : \overline{r} \to \overline{n}$, construct such a diagram by setting

$$\begin{aligned}
n_0 &:= g(0) \\
n_i &:= g(i) - g(i-1) \quad \text{for } 1 \leq i \leq r, \\
n_{r+1} &:= n - g(r).
\end{aligned}$$

These constructions are easily checked to be inverse to one another, so it only remains to extend this bijection to morphisms in $\overline{r}/\Delta$ and $\mathrm{Fun}_\star(\wedge^{r+2}, \Delta)$.

Natural transformations in $\mathrm{Fun}_\star(\wedge^{r+2}, \Delta)$ from the diagram given by $(\overline{n}_0, \ldots, \overline{n}_{r+1})$ to another one given by $(\overline{m}_0, \ldots, \overline{m}_{r+1})$ correspond to tuples of maps $f_i : \overline{n}_i \to \overline{m}_i$ such that $f_0, \ldots, f_r$ preserve the maximum element and $f_1, \ldots, f_{r+1}$ preserve the minimum element. Morphisms in $\overline{r}/\Delta$ from

$g : \overline{r} \to \overline{n}$ to $h : \overline{r} \to \overline{m}$ amount to a family of monotone maps like this:

$$\begin{aligned} f_0 &: \{0, \dots, g(0)\} &&\longrightarrow \{0, \dots, h(0)\} \\ f_i &: \{g(i-1), \dots, g(i)\} &&\longrightarrow \{h(i-1), \dots, h(i)\} \quad \text{for } 1 \leq i \leq r, \\ f_{r+1} &: \{g(r), \dots, n\} &&\longrightarrow \{h(r), \dots, m\}, \end{aligned}$$

such that $f_0, \dots, f_r$ preserve the maximum element and $f_1, \dots, f_{r+1}$ preserve the minimum element. These two types of morphisms are in an obvious bijection, matching the bijection on objects defined above.

(b) The composite functor $\mathrm{Fun}_\star(\wedge^{r+2}, \Delta) \cong \overline{r}/\Delta \xrightarrow{\star} \Delta$ indeed sends the pictured diagram to $\overline{n_0 + \cdots + n_{r+1}}$, so it remains to show that this is a colimit.

A cocone from this diagram to some $\overline{m}$ consists of $r+1$ elements $x_0, \dots, x_r$ in $\overline{m}$ and monotone maps $f_i : \overline{n}_i \to \overline{m}$ for $i = 0, \dots, r+1$ satisfying

$$f_{i+1}(0) = x_i, \qquad f_i(n_i) = x_i$$

whenever $i = 0, \dots, r$. This data uniquely determines a map $f : \overline{n_0 + \cdots + n_{r+1}} \to \overline{m}$ by defining, for any $i = 0, \dots, r+1$ and $0 \leq j \leq n_i$,

$$f(n_0 + \cdots + n_{i-1} + j) \coloneqq f_i(j),$$

where the above compatibility conditions between the f_i guarantee that this is well-defined. f is by definition monotone on every subset from $n_0 + \cdots + n_{i-1}$ to $n_0 + \cdots + n_i$, which implies monotonicity overall. f restricts to f_i along the inclusions $\overline{n}_i \to \overline{n_0 + \cdots + n_{r+1}}$ sending 0 to $n_0 + \cdots + n_{i-1}$ and n_i to $n_0 + \cdots + n_i$. Since these inclusions are moreover jointly surjective, this property uniquely determines f. □

This equivalent perspective motivates the following alternative notation for $\star$.

Definition 5.2.5. *For any finite sequence $n_0, \dots, n_{r+1} \in \mathbb{N}$, define the* $\star$-product *$\overline{n}_0 \star \cdots \star \overline{n}_{r+1}$ as $\overline{n_0 + \cdots + n_{r+1}}$. Furthermore, for maps $f_i : \overline{n}_i \to \overline{m}_i$ in Δ with $i = 0, \dots, r+1$, their $\star$-product*

$$f_0 \star \cdots \star f_{r+1} \; : \; \overline{n}_0 \star \cdots \star \overline{n}_{r+1} \longrightarrow \overline{m}_0 \star \cdots \star \overline{m}_{r+1} \tag{5.1}$$

is defined precisely when the $f_0, \dots, f_r$ preserve maximum elements and the $f_1, \dots, f_{r+1}$ preserve minimum elements.

We can also express the extraction of each $\star$-component as a colimit. While perhaps the more intuitive relationship is the inclusion $\overline{n}_i \to \overline{n}_0 \star \cdots \star \overline{n}_{r+1}$, more helpful for proving that $\star$-products reflect pushouts is the surjective map $\overline{n}_0 \star \cdots \star \overline{n}_{r+1} \to \overline{n}_i$ acting as the identity on the component $\overline{n}_i$ and as the constant map to 0 or n_i on the components $\overline{n}_j$ for $j < i$ or $i < j$ respectively.

Lemma 5.2.6. *Given $g : \overline{r} \to \overline{n}$ exhibiting $\overline{n}$ as $\overline{n}_0 \star \cdots \star \overline{n}_{r+1}$ and $0 \leq i \leq r+1$, let*

$$g_i^-, g_i^+ \; : \; \overline{1} \longrightarrow \overline{n}$$

be the maps with

$$g_i^-(0) = 0, \qquad g_i^+(0) = n_0 + \cdots + n_i,$$
$$g_i^-(1) = n_0 + \cdots + n_{i-1}, \qquad g_i^+(1) = n.$$

Then the $\star$-component $\overline{n}_i$ is the colimit of the following diagram.

$$\begin{tikzcd}
& \overline{1} \arrow[dl] \arrow[dr, "g_i^-"] & & \overline{1} \arrow[dl, "g_i^+"'] \arrow[dr] & \\
\overline{0} & & \overline{n}_0 \star \cdots \star \overline{n}_{r+1} & & \overline{0}
\end{tikzcd}$$

Proof. A cocone out of this diagram is precisely a map $\overline{n}_0 \star \cdots \star \overline{n}_{r+1} \to \overline{m}$ constant on each of the subobjects $\overline{n}_0 \star \cdots \star \overline{n}_{i-1}$ and $\overline{n}_{i+1} \star \cdots \star \overline{n}_{r+1}$. These maps are in obvious bijection with maps $\overline{n}_i \to \overline{m}$, so $\overline{n}_i$ is the colimit of the diagram. $\square$

5.3. Characterization of pushouts in Δ. With the machinery of $\star$-decompositions and $\star$-products in place, we can now apply it to pushouts.

Proposition 5.3.1. *Let*

$$\begin{tikzcd}
\overline{m}_i \arrow[r, "f_i"] \arrow[d, "g_i"'] & \overline{p}_i \\
\overline{q}_i &
\end{tikzcd}$$

for $i = 0, \ldots, r+1$ be a sequence of spans in Δ whose $\star$-product exists. Then:

(a) If the above spans all have pushouts, then the $\star$-product of these pushout squares exists and is a pushout square for the $\star$-product span.

(b) Conversely, if the $\star$-product span has a pushout, then so do the above spans, and the $\star$-product of their pushouts is again the pushout of the $\star$-product span.

(c) Both (a) and (b) also hold with "pushout" replaced by "concrete pushout".

Proof. We first show that a commuting square

$$\begin{tikzcd}
\overline{m} \arrow[r, "f"] \arrow[d, "g"'] & \overline{p} \arrow[d, "h"] \\
\overline{q} \arrow[r, "k"] & \overline{n}
\end{tikzcd}$$

with jointly surjective h and k is such that if f and g preserve the maximum element, then so do h and k. Indeed since the square commutes, the assumption on f and g implies that it is enough that one of h or k preserves the maximum element and the other one follows. But clearly at least one does since h and k must be jointly surjective. A similar argument shows that if f and g preserve the minimum element, then so do h and k.

(a) The $\star$-product of the pushout squares exists since the relevant preservation conditions are implied by the statement from the previous paragraph. Using the description of $\star$-products as colimits then shows that the resulting $\star$-product square is a pushout as well since colimits commute with colimits.

(b) For $0 \leq i \leq r+1$, consider the following diagram $D : \mathsf{J} \to \mathsf{Span}(\Delta)$, where Span denotes the category of spans and J is the shape of the diagram in Lemma 5.2.6. The objects in J sent to $\overline{0}$ and $\overline{1}$ in Lemma 5.2.6 are sent by D to the constant spans at $\overline{0}$ and $\overline{1}$ respectively, and the object sent to $\overline{n}_0 \star \cdots \star \overline{n}_{r+1}$ in Lemma 5.2.6 is sent to the $\star$-product span of the postulated sequence, with the analogous maps as in Lemma 5.2.6 for each of q, m, p.

Each of these spans has a pushout, with the constant spans pushing out to $\overline{0}$ and $\overline{1}$ respectively and the $\star$-product span having a pushout by assumption. The functor $\mathsf{J} \to \Delta$ picking out the pushout objects has a colimit since it is of the form in Lemma 5.2.6, selecting the ith component of the pushout. The diagram D therefore has an overall colimit. As colimits commute with colimits, this means that the ith component of the pushout of the $\star$-product span is the pushout of the ith span above.

(c) The concrete version of (a) follows from the fact that the characterizing colimits of $\star$-products are also colimits in Set. For the concrete version of (b), suppose that we are given a set X together with $\phi : \overline{p}_i \to X$ and $\psi : \overline{q}_i \to X$ such that $\phi f_i = \psi g_i$, but ϕ and ψ do not factor across $\overline{n}$. Then consider the set functions

$$\phi' : \overline{p}_1 \star \cdots \star \overline{p}_r \longrightarrow X,$$

which takes every vertex below $p_1 + \ldots + p_{i-1}$ to $\phi(p_1 + \ldots + p_{i-1})$, every vertex above $p_1 + \ldots + p_i$ to $\phi(p_1 + \ldots + p_i)$, and let ϕ' act like ϕ in between. Also let

$$\psi' : \overline{q}_1 \star \cdots \star \overline{q}_r \longrightarrow X,$$

be defined analogously. Then the assumed universal property of the $\star$-product square implies the unique factorization of ϕ and ψ across $\overline{n}_i$. $\square$

Definition 5.3.2. *The* canonical $\star$-decomposition *of $\overline{n}$ is* $\mathrm{id} : \overline{n} \to \overline{n}$ *in $\overline{n}/\Delta$, or equivalently the expression of $\overline{n}$ as $\overline{0} \star \overline{1} \star \overline{1} \star \cdots \star \overline{1} \star \overline{0}$ (n copies of $\overline{1}$).*

Applying the previous proposition to the canonical $\star$-decomposition of $\overline{m}$ gives the following helpful tool for arguments involving pushouts in Δ.

Corollary 5.3.3. *A span as below has a pushout in Δ if and only if each of its $r+2$ many $\star$-components with respect to the canonical $\star$-decomposition of $\overline{r}$ has a pushout, in which case its pushout is the $\star$-product of those pushouts.*

$$\begin{tikzcd} \overline{r} \arrow[r, "f"] \arrow[d, "g"'] & \overline{p} \\ \overline{q} \end{tikzcd}$$

This lets us reduce the characterization of pushouts in Δ to those as in the span above where r is 0 or 1, summarized in the following theorem.

Theorem 5.3.4. *A span*

$$\begin{tikzcd} \overline{r} \arrow[r, "f"] \arrow[d, "g"'] & \overline{p} \\ \overline{q} \end{tikzcd}$$

has a pushout in Δ if and only if the following three conditions hold:

(a) for every i with $1 \leq i \leq r$, we have $f(i) \leq f(i-1)+1$ or $g(i) \leq g(i-1)+1$;

(b) $f(0) = 0$ or $g(0) = 0$;

(c) $f(r) = p$ or $g(r) = q$.

Property (a) fails if $f(i+1) > f(i)+1$ and $g(i+1) > g(i)$. Thus intuitively, the condition says that f and g should not both "add an extra element" in between two consecutive elements of $\overline{r}$; the reason that the pushout then does not exist is that these two elements could not be totally ordered in a canonical way. The same issue arises when neither f nor g hit the minimum (or maximum) element of their codomains, which is how we think of (b) and (c).

Proof. It is easy to see that this condition holds for the given span if and only if it holds for each span in $\star$-decomposition obtained from the canonical $\star$-decomposition of $\overline{r}$. Hence by Corollary 5.3.3, it suffices to prove this statement only for the components of the $\star$-decomposition. Each one of these $\star$-components has one of the following special properties, which results in three cases:

(i) $r = 0$ and $f(0) = g(0) = 0$.

(ii) $r = 0$ and $f(0) = p$ and $g(0) = q$.

(iii) $r = 1$ and $f(0) = g(0) = 0$ and $f(1) = p$ and $g(1) = q$.

We start with case (i), which then also covers case (ii) by symmetry. In case (i), conditions (a) and (b) hold automatically. Hence the claim is that the pushout exists if and only if (c) holds, which now is equivalent to $p = 0$ or $q = 0$. If this holds, then we have $f = \mathrm{id}_{\overline{0}}$ or $g = \mathrm{id}_{\overline{0}}$, respectively, so that the pushout trivially exists. On the other hand if $p > 0$ and $q > 0$, then we need to show that no commutative square

$$\begin{tikzcd} \overline{0} \arrow[r, "0"] \arrow[d, "0"'] & \overline{p} \arrow[d, "h"] \\ \overline{q} \arrow[r, "k"] & \overline{n} \end{tikzcd}$$

is a pushout. The reason is that $1 \in \overline{p}$ and $1 \in \overline{q}$ must be ordered in $\overline{n}$ in an arbitrary way which will end up opposite in some other commutative square: assuming $h(1) \le k(1)$ without loss of generality, choose $\phi : \overline{p} \to \overline{1}$ and $\psi : \overline{q} \to \overline{1}$ satisfying

$$\phi(0) = 0 = \psi(0), \qquad \phi(1) = 1, \qquad \psi(1) = 0.$$

Then we have $\phi f = \psi g$, but this square does not factor through the putative pushout, since this would require $\phi(1) \le \psi(1)$ by $h(1) \le k(1)$, using monotonicity of the induced map.

It remains to treat case (iii). Now conditions (b) and (c) hold automatically, and (a) only applies for $i = 1$, and is now equivalent to $p \le 1$ or $q \le 1$. Suppose first that the condition holds, which amounts to $q \le 1$ without loss of generality. $q = 1$ means that $g = \mathrm{id}_{\overline{1}}$, so that the pushout again exists trivially. For $q = 0$, we show that the following square is a pushout:

$$\begin{tikzcd} \overline{1} \arrow[r, "f"] \arrow[d, "g"'] & \overline{p} \arrow[d, "h"] \arrow[ddr, dashed, "\phi"] & \\ \overline{0} \arrow[r, equal] \arrow[drr, dashed, "\psi"'] & \overline{0} & \\ & & \overline{m} \end{tikzcd} \tag{5.2}$$

If ϕ and ψ as above satisfy $\phi f = \psi g$, then necessarily $\phi(0) = \phi(p) = \psi(0) \in \overline{m}$. But as ϕ is monotonic, this means that it must be constant. Therefore $\psi : \overline{0} \to \overline{m}$ must be the unique induced map with $\psi h = \phi$, and therefore serves as the unique factorization in the universal property of a pushout.

Finally if (a) fails, then $p > 1$ and $q > 1$. Similar to the above, we then show that every commutative square

$$\begin{tikzcd} \overline{0} \arrow[r, "f"] \arrow[d, "g"'] & \overline{p} \arrow[d, "h"] \\ \overline{q} \arrow[r, "k"] & \overline{n} \end{tikzcd}$$

fails the relevant universal property. Assuming again $h(1) \leq k(1)$ without loss of generality, choose $\phi : \overline{p} \to \overline{1}$ and $\psi : \overline{q} \to \overline{1}$ with

$$\phi(0) = 0 = \psi(0), \qquad \phi(p) = 1 = \psi(q), \qquad \phi(1) = 1, \qquad \psi(1) = 0.$$

We have $\phi f = \psi g$. But if the above square was a pushout, then we would need to have $\phi(1) \leq \psi(1)$ by $h(1) \leq k(1)$. □

That the four nearly trivial cases considered in the proof are enough to show that any span satisfying the given properties has a pushout illustrates the utility of the $\star$-decomposition.

Note that the proof not only characterizes those spans which have a pushout, but moreover can be used to explicitly construct the pushout as a $\star$-product for every span which has one. In particular, this construction shows that pushouts of face maps are again face maps. Moreover, using the fact that a $\star$-product of concrete pushouts is again a concrete pushout by Proposition 5.3.1(c), we obtain the following.

Corollary 5.3.5. *A commutative square of face maps as below is a pushout if and only if it is a pushout in* Set*, and if for every* $i = 0, \ldots, n-1$*, the edge* $\{i, i+1\} \subseteq \bar{n}$ *is in the image of* h *or* k*.*

$$\begin{tikzcd} \overline{m} \arrow[r, "f"] \arrow[d, "g"'] & \overline{p} \arrow[d, "h"] \\ \overline{q} \arrow[r, "k"] & \overline{n} \end{tikzcd}$$

Thinking of $\bar{n}$ as the formal n-simplex, the extra condition states that h and k must cover the spine of $\overline{n}$.

We now proceed to describe another decomposition of pushout squares in Δ into the simplest possible building blocks.

Definition 5.3.6. *The* defect δ_f *of a map* $f : \overline{n} \to \overline{m}$ *in* Δ *is*

$$\delta_f \coloneqq (|\overline{n}| - |\operatorname{im}(f)|) + (|\overline{m}| - |\operatorname{im}(f)|) = n + m + 2 - 2|\operatorname{im}(f)|.$$

The defect counts the number of elements in the domain identified by f with another element plus the number of elements in the codomain outside the image. The idea is to measure how far a map in Δ is from an identity, which has defect 0, in a fairly symmetric way that behaves uniformly across different values of n and m. More precisely, the defect is additive with respect to $\star$:

Lemma 5.3.7. *For maps* $f : \overline{n}_0 \to \overline{m}_0$ *with* $f(n_0) = m_0$ *and* $g : \overline{n}_1 \to \overline{m}_1$ *with* $g(0) = 0$*, we have* $\delta_{f\star g} = \delta_f + \delta_g$*.*

Note that the equations $f(n_0) = m_0$ and $g(0) = 0$ are relevant only for ensuring that the $\star$-product $f \star g$ exists.

PROOF. First observe that $f \star g : \overline{n_0 + n_1} \to \overline{m_0 + m_1}$. Since both the left and right parts of $f \star g$ have $m_0 \in \overline{m_0 + m_1}$ in their image, and the images are otherwise disjoint, we have $|\operatorname{im}(f \star g)| = |\operatorname{im}(f)| + |\operatorname{im}(g)| - 1$. We then calculate

$$\begin{aligned}\delta_{f\star g} &= n_0 + n_1 + m_0 + m_1 + 2 - 2|\operatorname{im}(f)| - 2|\operatorname{im}(g)| + 2 \\ &= (n_0 + m_0 + 2 - 2|\operatorname{im}(f)|) + (n_1 + m_1 + 2 - 2|\operatorname{im}(g)|) \\ &= \delta_f + \delta_g.\end{aligned}$$ □

The maps with defect 1 are exactly the generating face and degeneracy maps, since they can either identify one pair of elements in the domain or map injectively into a codomain with one additional element. From this perspective, the defect of f can be seen as counting the minimal number of generating maps in Δ that f factors into, since each identification in the domain requires a generating degeneracy and each element in the codomain outside the image requires a generating face map. A factorization of f into such a minimal number of generators is what we call *efficient*, and in this case the defects of the factors (all 1) add up to the total defect of f. More generally, we declare the following.

Definition 5.3.8. *A factorization* $f = h \circ g$ *of a map in* Δ *is* efficient *if* $\delta_f = \delta_h + \delta_g$*.*

As an example of an inefficient factorization, consider the composition $\overline{0} \xrightarrow{0} \overline{2} \xrightarrow{011} \overline{1}$, which composes to $\overline{0} \xrightarrow{0} \overline{1}$. The composite has defect 1, but the factors have respective defects 2 and 1 adding up to 3, hence this factorization is not efficient. Any map f has an efficient factorization into δ_f generating maps as described above; this can be chosen such that the generating face maps follow the degeneracies, as the Reedy factorization[9] of a map is efficient. In fact, let $f = ds$ be the Reedy factorization, so that s is a degeneracy map and d a face map. Then δ_f is the sum of the degree changes of d and s, which determine the number of face and degeneracy maps in such an efficient factorization of f into generators.

For our purposes, efficiency of a factorization guarantees that the factors of a map do not take unnecessarily large steps that could prevent the factorization from extending to pushout squares of the composite.

Proposition 5.3.9. *For a pushout square as below left and an efficient factorization* $f = f_1 \circ f_0$*, the square factors into a horizontal composite of pushout squares as below right.*

[9] See for example [**Rie13**, Section 14.2].

$$\begin{tikzcd} \overline{r} \arrow[r, "f"] \arrow[d, "g"'] & \overline{p} \arrow[d, "h"'] \\ \overline{q} \arrow[r, "k"] & \overline{n} \end{tikzcd} \qquad \begin{tikzcd} \overline{r} \arrow[r, "f_0"] \arrow[d, "g"'] & \overline{\ell} \arrow[r, "f_1"] \arrow[d, "g'"'] & \overline{p} \arrow[d, "h"'] \\ \overline{q} \arrow[r, "k_0"] & \overline{m} \arrow[r, "k_1"] & \overline{n} \end{tikzcd}$$

From now on, let us call a pushout square *trivial* if its underlying span contains at least one identity map.

PROOF. By Lemma 5.3.7 and Corollary 5.3.3, we can again use the $\star$-decomposition to reduce to the cases in the proof of Theorem 5.3.4.

(i) Suppose $r = 0$ and $f(0) = g(0) = 0$. Then in order for the left square to be a pushout, we must have $p = 0$ or $q = 0$.

If $p = 0$, then $f = \mathrm{id}_{\overline{0}}$, and hence there is no nontrivial efficient factorization of f. If $q = 0$, then $g = \mathrm{id}_{\overline{0}}$, and the square factors as the composite of trivial pushout squares for f_0 and f_1.

(ii) Suppose $r = 0$ and $f(0) = p$ and $g(0) = q$. This follows from the previous case by symmetry.

(iii) Suppose $r = 1$ and both f and g preserve minimum and maximum. Then in order for the left square to be a pushout, we must have $p \leq 1$ or $q \leq 1$.

If $p = 1$, then $f = \mathrm{id}_{\overline{1}}$, and $f_0 = f_1 = \mathrm{id}_{\overline{1}}$ follows. Similarly if $p = 0$, then f is the unique map $\overline{1} \to \overline{0}$. This map has no nontrivial efficient factorization since one of the factors must have defect 0, so that one of f_0 and f_1 coincides with f and the other is an identity.

If $q = 1$, then $g = \mathrm{id}_{\overline{1}}$ and the square factors into trivial pushout squares. Finally, $q = 0$ is a more interesting case, making the square be of the form

$$\begin{tikzcd} \overline{1} \arrow[r, "0p"] \arrow[d] & \overline{p} \arrow[d] \\ \overline{0} \arrow[r, equal] & \overline{0} \end{tikzcd}$$

The given efficient factorization of f is then any factorization of the face map $0p$ into two min and max preserving face maps $\overline{1} \xrightarrow{f_0} \overline{p}' \xrightarrow{f_1} \overline{p}$. The vertical map $g' : \overline{p}' \to \overline{0}$ makes the left square a pushout by the arguments given for Theorem 5.3.4, and the remaining map k_1 is then induced by the universal property of that new pushout square. The fact that the right square is a pushout then follows by the pushout lemma. □

The defect also plays nicely with pushouts as follows.

Lemma 5.3.10. *For a pushout square in Δ as below, assume that f is a face map or g is a degeneracy map. Then $\delta_k \leq \delta_f$, and $\delta_k = \delta_f$ if and only if the pushout is concrete.*

$$\begin{tikzcd} \overline{r} \arrow[r, "f"] \arrow[d, "g"'] & \overline{p} \arrow[d, "h"] \\ \overline{q} \arrow[r, "k"] & \overline{n} \end{tikzcd}$$

Proof. By the $\star$-additivity of the defect from Lemma 5.3.7 and Corollary 5.3.3, it again suffices to check this property on each $\star$-component of the square. This reduces the problem again to the cases described in the proof of Theorem 5.3.4: either the pushout square is trivial, in which case parallel maps have the same defect $\delta_k = \delta_f$ on that component, or it is of the following form:

$$\begin{tikzcd} \overline{1} \arrow[r, "0p"] \arrow[d] & \overline{p} \arrow[d] \\ \overline{0} \arrow[r, equal] & \overline{0} \end{tikzcd}$$

In this square, the top map has defect $|p-1|$, and the bottom map has defect 0. Thus with f on top and k at the bottom, we have $\delta_k \leq \delta_f$ in this case as well. If f is on the left and k on the right, then g is the top map $0p$, which satisfies the degeneracy assumption only if $p \leq 1$. But then $\delta_k \leq \delta_f$ holds also.[10]

Concerning the additional statement on $\delta_k = \delta_f$, the above square is a pushout in Set if and only if $p = 1$. Thus this claim follows also by considering the same cases. □

These results can be combined to prove that any pushout in Δ can be factored into a grid of pushout squares with both maps in the spans involved being generating faces or degeneracies. We call a pushout square in Δ a *basic pushout* if it is of this form, i.e. if its span consists of generating faces or degeneracies.

Proposition 5.3.11. *Every pushout in Δ can be obtained from the basic pushouts by horizontal and vertical composition of pushouts.*

Furthermore:

(a) *If the original pushout is one of two face maps, then this can be achieved with only basic pushouts of face maps.*

[10]The only case from the proof of Theorem 5.3.4 in which $\delta_k \leq \delta_f$ does not hold is this final case with $p > 1$, but then f is not a face map and g is not a degeneracy map, in contrast to the current assumption.

(b) If the original pushout is concrete, then this can be achieved with only concrete basic pushouts.

PROOF. Consider a pushout in Δ as in the diagram

$$\begin{tikzcd} \overline{r} \arrow[r, "f"] \arrow[d, "g"'] & \overline{p} \arrow[d, "h"] \\ \overline{q} \arrow[r, "k"] & \overline{n} \end{tikzcd}$$

First, assume that f is a generating face map. We factor g efficiently into generating face maps. This factorization extends to vertically factor the pushout square by Proposition 5.3.9 as pictured below. By repeated application of Lemma 5.3.10, since $\delta_f = 1$ all of the horizontal maps f_i and k have defect 1 or 0. Therefore, each of the factor squares is either a basic pushout square or a trivial pushout square.

$$\begin{tikzcd} \overline{r} \arrow[r, "f"] \arrow[d, "g_1"'] & \overline{p} \arrow[d, "h_1"] \\ \overline{r_1} \arrow[r, "f_1"] \arrow[d, "g_2"'] & \overline{p} \arrow[d, "h_2"] \\ \vdots \arrow[d, "g_{\delta_g - 1}"'] & \vdots \arrow[d, "h_{\delta_g - 1}"] \\ \overline{r_{\delta_g - 1}} \arrow[r, "f_{\delta_g - 1}"] \arrow[d, "g_{\delta_g}"'] & \overline{p} \arrow[d, "h_{\delta_g}"] \\ \overline{q} \arrow[r, "k"] & \overline{n} \end{tikzcd}$$

Next, assume that f is a generating degeneracy map and factor g efficiently into generating degeneracies followed by generating faces. By Proposition 5.3.9, this factorization extends to vertically factor the pushout square, as pictured above, where the maps g_i are some number of generating degeneracy maps followed by generating face maps. By the previous case, each square containing a face map factor of g factors into basic pushouts as desired. By repeated application of Lemma 5.3.10, since f has defect 1, each horizontal map f_i atop a square with g_{i+1} a degeneracy has δ_{f_1} as 1 or 0, so all such squares are basic or trivial pushouts.

Finally, for an arbitrary pushout square in Δ as above, factoring f (or g) efficiently into generators extends to a factorization of the entire square into pushout squares with one map a generating face or degeneracy, again by Proposition 5.3.9. In conclusion, any pushout square in Δ can be factored into basic and trivial pushout squares, as was to be shown for the first statement.

We prove the two additional statements. If the original pushout is that of a span consisting of two face maps, then going through the steps of the previous arguments shows that the resulting factorization is one of pushouts of face maps only, based also on the fact that face maps can be factored efficiently into generating face maps. A similar argument, using the equality criterion of Lemma 5.3.10, proves the statement about concrete pushouts. $\square$

We now spell out the description of the basic pushouts more directly. Theorem 5.3.4 implies that the basic pushouts come in the following types, where we list the corresponding range of the indices below each diagram.

(i) Pushouts of two generating face maps: these are given by the diagrams

$$\begin{tikzcd} \overline{n-2} \arrow[r, "d^i"] \arrow[d, "d^{j-1}"] & \overline{n-1} \arrow[d, "d^j"] \\ \overline{n-1} \arrow[r, "d^i"] & \overline{n} \end{tikzcd} \tag{5.3}$$

$$(0 \leq i < j-1 \leq n-1)$$

(Although the diagram still commutes as part of the simplicial identities for $i = j-1$, it is then no longer a pushout, with the one nontrivial $\star$-component in its span being $\overline{2} \xleftarrow{02} \overline{1} \xrightarrow{02} \overline{2}$.)

(ii) Mixed pushouts of one generating face and one generating degeneracy map: these are given by the diagrams

$$\begin{tikzcd} \overline{n} \arrow[r, "d^i"] \arrow[d, "s^{j-1}"'] & \overline{n+1} \arrow[d, "s^j"'] \\ \overline{n-1} \arrow[r, "d^i"] & \overline{n} \end{tikzcd} \qquad \begin{tikzcd} \overline{n+1} \arrow[r, "d^{i+1}"] \arrow[d, "s^i"'] & \overline{n+2} \arrow[d, "s^i s^{i+1}"', "s^i s^i"] \\ \overline{n} \arrow[r, equal] & \overline{n} \end{tikzcd} \qquad \begin{tikzcd} \overline{n} \arrow[r, "d^{i+1}"] \arrow[d, "s^j"'] & \overline{n+1} \arrow[d, "s^j"'] \\ \overline{n-1} \arrow[r, "d^i"] & \overline{n} \end{tikzcd} \tag{5.4}$$

$$(0 \leq i < j \leq n) \qquad (0 \leq i \leq n) \qquad (0 \leq j < i \leq n)$$

Again the commutativity of the first and the third squares are among the simplicial identities, while for the second the simplicial identities are relevant in noting that the right vertical arrow is $s^i s^{i+1} = s^i s^i$.

Note that since every generating degeneracy map s^j is a split epimorphism, the image of such a square under any contravariant functor (such as a simplicial set $X : \Delta^{\mathrm{op}} \to \mathsf{Set}$) is a pullback already if it is merely a weak pullback, since then the images of the s^j are all monomorphisms.

(iii) Pushouts of two generating degeneracy maps: these are given by the diagrams

$$\begin{array}{cc} \begin{tikzcd} \overline{n+2} \arrow[r, "s^i"] \arrow[d, "s^{j+1}"] & \overline{n+1} \arrow[d, "s^j"] \\ \overline{n+1} \arrow[r, "s^i"] & \overline{n} \end{tikzcd} & \begin{tikzcd} \overline{n+1} \arrow[r, "s^i"] \arrow[d, "s^i"] & \overline{n} \arrow[d, equal] \\ \overline{n} \arrow[r, equal] & \overline{n} \end{tikzcd} \\ (0 \leq i \leq j \leq n) & (0 \leq i \leq n) \end{array} \tag{5.5}$$

By Proposition 3.8 in [**BR13**] and the fact that Δ is an elegant Reedy category with degeneracy maps as its degree-lowering subcategory, every pushout of this type is even a split pushout[11], and therefore sent to a pullback by *any* contravariant functor out of Δ.

It is easily checked by Lemma 5.3.10 that all of these basic pushout squares are concrete except for the middle squares in (5.4), which are all $\star$-products of identities on either side of the motivating non-concrete pushout square at the beginning of this section.

5.4. $\star$-Decomposing basic pushouts. In the spirit of reducing pushouts in Δ to a smaller generating set, we can further decompose the basic pushout squares using $\star$, though this will only be relevant for characterizing simplicial sets with strong pullback properties.

Lemma 5.4.1. *Every basic pushout square in Δ is a $\star$-product of trivial pushout squares and the following two pushout squares.*

$$\begin{tikzcd} \overline{1} \arrow[r, "02"] \arrow[d] & \overline{2} \arrow[d] \\ \overline{0} \arrow[r, equal] & \overline{0} \end{tikzcd} \qquad \begin{tikzcd} \overline{1} \arrow[r] \arrow[d] & \overline{0} \arrow[d, equal] \\ \overline{0} \arrow[r, equal] & \overline{0} \end{tikzcd}$$

Furthermore:

(a) Every basic pushout square of two face maps is a $\star$-product of only trivial pushout squares.

(b) Every concrete basic pushout is a $\star$-product of only trivial pushout squares and the pushout square above right.

[11] Which Bergner and Rezk call a "strong pushout" [**BR13**], a term we prefer to use for an ordinary pushout to contrast with weak pushouts.

Proof. It is easy to see from Lemma 5.3.7 that if a $\star$-product of pushout squares is basic, then so are the $\star$-factors. Thus it is enough to prove this for basic pushout squares which in addition arise as canonical $\star$-components as in the proof of Theorem 5.3.4. Following the arguments there and assuming that f and g are both either a generating face or generating degeneracy map results only in the following possibilities: all pushout squares are trivial in cases (i) and (ii); while in case (iii), we obtain the above two pushout squares in addition.

The first additional statement follows upon noting that only cases (i) and (ii) occur for pushouts of two face maps. The second additional statement follows similarly as all basic pushouts are concrete, except for the middle ones in (5.4). $\square$

In order for this lemma to be useful for providing simpler descriptions of the various classes of simplicial sets we consider in the following sections, we describe how $\star$-products of pushouts are preserved by simplicial sets.

Proposition 5.4.2. *Assume a simplicial set $X : \Delta^{\mathrm{op}} \to \mathsf{Set}$ sends squares of the form below to pullbacks.*

$$\begin{tikzcd} \overline{0} \arrow[r, "a"] \arrow[d, "0"'] & \overline{a} \arrow[d] \\ \overline{b} \arrow[r] & \overline{a+b} \end{tikzcd}$$

If X sends in addition a sequence of pushout squares in Δ admitting a $\star$-product each to pullbacks, then X also sends their $\star$-product pushout square to a pullback.

Proof. Consider a sequence of pushout squares

$$\begin{tikzcd} \overline{m}_i \arrow[r] \arrow[d] & \overline{p}_i \arrow[d] \\ \overline{q}_i \arrow[r] & \overline{n}_i \end{tikzcd}$$

in Δ for $i = 0, \dots, r+1$, and assume they are all sent to pullbacks by X and that they satisfy the relevant preservation condition for their $\star$-product to exist. Let $\overline{m}, \overline{p}, \overline{q}, \overline{n}$ make up the $\star$-product pushout square.

The pushout pictured in the proposition is exactly the colimit defining $\overline{a} \star \overline{b}$. The assumption that X preserves it means that the resulting diagram

$$\begin{tikzcd} X_{a+b} \arrow[r] \arrow[d] & X_a \arrow[d] \\ X_b \arrow[r] & X_0 \end{tikzcd}$$

is a pullback. Iterating these observations shows that X_n is the limit of the diagram

$$
\begin{tikzcd}[column sep=small]
& X_{n_0} \arrow[dl, "0"'] \arrow[dr, "n_0"] & & \cdots \arrow[dl, "0"'] \arrow[dr, "n_r"] & & X_{n_{r+1}} \arrow[dl, "0"'] \arrow[dr, "n_{r+1}"] & \\
X_0 & & X_0 & \cdots & X_0 & & X_0
\end{tikzcd}
$$

defined to be the image under X of the colimit diagram which defines the $\star$-product $\overline{n} = \overline{n}_0 \star \cdots \star \overline{n}_{r+1}$. Therefore as limits commute with limits in Set, if the following squares are pullbacks

$$
\begin{tikzcd}
X_{n_i} \arrow[r] \arrow[d] & X_{p_i} \arrow[d] \\
X_{q_i} \arrow[r] & X_{r_i}
\end{tikzcd}
$$

then X_n is the pullback of the corresponding cospan $X_p \to X_r \leftarrow X_q$. $\square$

5.5. Balanced squares of face maps. We end this section with some considerations involving non-pushout squares, which will turn out to be relevant for us only in Section 9. Our theme here is to factor general commutative squares of face maps in Δ into basic squares, similarly to how we have factored pushout squares into basic pushouts squares. Throughout the following, all maps will be face maps in Δ, with or without further explicit mention.

To begin, recall that every face map in Δ is a composite of generating face maps $d^i : \overline{n-1} \to \overline{n}$ for $0 \leq i \leq n$. The relations between these generating face maps are given by the relevant simplicial identities,

$$
\begin{tikzcd}
\overline{n-2} \arrow[r, "d^i"] \arrow[d, "d^{j-1}"'] & \overline{n-1} \arrow[d, "d^j"] \\
\overline{n-1} \arrow[r, "d^i"] & \overline{n}
\end{tikzcd}
\qquad (0 \leq i < j \leq n) \tag{5.6}
$$

We are now interested in characterizing those squares which can be factored into grids of these simplicial identities squares together with trivial squares, by which we mean those having one of the following two forms.

$$
\begin{tikzcd}
\overline{m} \arrow[r, equal] \arrow[d, "f"] & \overline{m} \arrow[d, "f"'] \\
\overline{n} \arrow[r, equal] & \overline{n}
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\overline{m} \arrow[r, "f"] \arrow[d, equal] & \overline{n} \arrow[d, equal] \\
\overline{m} \arrow[r, "f"] & \overline{n}
\end{tikzcd}
$$

Let us say that a square as below is *balanced* if $r+n=p+q$ and h,k are jointly surjective.

$$\begin{tikzcd} \overline{r} \arrow[r, "f"] \arrow[d, "g"'] & \overline{p} \arrow[d, "h"] \\ \overline{q} \arrow[r, "k"] & \overline{n} \end{tikzcd}$$

In terms of the notion of defect from Section 5.3, the equation $r+n=p+q$ is equivalent to $\delta_f=\delta_k$ and also to $\delta_g=\delta_h$. Relating back pushouts as the main theme of this section, we also have the following simple characterization.

Lemma 5.5.1. *A square of face maps in Δ as above is a pushout in* Set *if and only if it is jointly surjective and $p+q=r+n$.*

$$\begin{tikzcd} \overline{r} \arrow[r, "f"] \arrow[d, "g"'] & \overline{p} \arrow[d, "h"] \\ \overline{q} \arrow[r, "k"] & \overline{n} \end{tikzcd}$$

PROOF. If the square is a pushout of finite sets, then $\overline{n}\cong\overline{p}\cup_{\overline{r}}\overline{q}$, so $n=p+q-r$ and the square is jointly surjective. If the square is jointly surjective and $p+q=r+n$, then by joint surjectivity the induced map $\overline{p}\cup_{\overline{r}}\overline{q}\to\overline{n}$ is a surjection, but as $n=p+q-r$, this is a surjection between finite sets of the same cardinality, so it must be an isomorphism and the square a pushout of sets. $\square$

Note that a pushout of monomorphisms of sets is also a pullback, and a more general result for bicartesian squares in Δ without restricting to face maps could be proven similarly, though it will not be relevant to this paper.

The simplest examples of non-balanced squares are the following.

Definition 5.5.2. *A lower connection square is one of the form below left, and an upper connection square is one of the form below right.*

$$\begin{tikzcd} \overline{m} \arrow[r, equal] \arrow[d, equal] & \overline{m} \arrow[d, "f"] \\ \overline{m} \arrow[r, "f"] & \overline{n} \end{tikzcd} \qquad \begin{tikzcd} \overline{m} \arrow[r, "f"] \arrow[d, "f"'] & \overline{n} \arrow[d, equal] \\ \overline{n} \arrow[r, equal] & \overline{n} \end{tikzcd} \tag{5.7}$$

We call these squares *connection squares* in reference to the connection maps in some varieties of cubical sets [**BS76**]. In fact, there is a cubical nerve functor [**Jar06**, Example 19] from categories to cubical sets with connections, where the 1-cube corresponding to a morphism f is sent by the connection maps to precisely the 2-cubes corresponding to the squares above.

Proposition 5.5.3. *A commuting square of face maps in Δ can be factored into a grid of squares of the two forms below, together with trivial squares, if and only if it is balanced.*

$$
\begin{tikzcd}
\overline{n-2} \arrow[r, "d^i"] \arrow[d, "d^{j-1}"'] & \overline{n-1} \arrow[d, "d^j"] \\
\overline{n-1} \arrow[r, "d^i"] & \overline{n}
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\overline{n-2} \arrow[r, "d^{j-1}"] \arrow[d, "d^i"'] & \overline{n-1} \arrow[d, "d^i"] \\
\overline{n-1} \arrow[r, "d^j"] & \overline{n}
\end{tikzcd}
$$

$$(0 \leq i < j \leq n) \qquad\qquad (0 \leq i < j \leq n)$$

Any trivial square factors further into trivial squares involving only generating face maps, so one can also clearly restrict to these if desired.

Proof. Using Lemma 5.5.1, the "only if" direction follows by composition of pushouts.

For the "if" direction, we consider a balanced square as below and prove the existence of a factorization by induction on the total defect $\delta_f + \delta_g = \delta_h + \delta_k$.

$$
\begin{tikzcd}
\overline{r} \arrow[r, "f"] \arrow[d, "g"'] & \overline{p} \arrow[d, "h"] \\
\overline{q} \arrow[r, "k"] & \overline{n}
\end{tikzcd}
$$

The balanced squares which we try to factor into are precisely those in which all four maps have defect ≤ 1. Therefore using induction and reflection symmetry along the diagonal, it is enough to show that if $\delta_f = \delta_k > 1$, then the square can be factored nontrivially into two other balanced squares.

Thus with $i \in \overline{p} \setminus \mathrm{im}(f)$ any vertex which gets "added by f", we can factor f into $d^i : \overline{p-1} \to \overline{p}$ composed with a unique $f' : \overline{r} \to \overline{p-1}$, as in the diagram below. The same is true for k with respect to $h(i) \in \overline{n}$, which is not in $\mathrm{im}(k)$ as the original square is a pullback and $h(i)$ is by assumption not in $\mathrm{im}(hf)$. We then have the following factorization into squares, where h' is defined as the restriction of h along d^i, ensuring that both squares commute.

$$
\begin{tikzcd}
\overline{r} \arrow[r, "f'"] \arrow[d, "g"'] & \overline{p-1} \arrow[r, "d^i"] \arrow[d, "h'"] & \overline{p} \arrow[d, "h"] \\
\overline{q} \arrow[r, "k'"] & \overline{n-1} \arrow[r, "d^{h(i)}"] & \overline{n}
\end{tikzcd}
$$

The objects in both squares clearly satisfy the size condition, so it only remains to show that they are jointly surjective in order to prove that they are balanced. In the right square, the only element not in the image of $d^{h(i)}$ is $h(i)$, which is definitionally in the image of h. In the left square, restricting h along d^i excludes only $h(i)$ from the joint image, but $h(i)$ is also excluded from $\overline{n-1}$ by factoring through $d^{h(i)}$. $\square$

Remark 5.5.4. Although we will not need this, it is worth noting that this factorization implies the factorization of *any* square of face maps into the simplicial identities squares along with connection and trivial squares.

Indeed given any commutative square as in the proof above, we construct a diagram as follows.

$$\begin{tikzcd}
\overline{r} \arrow[rrr, "f"] \arrow[dr] \arrow[ddd, "g"] & & & \overline{p} \arrow[ddl, "h'"'] \arrow[ddd, "h"] \\
& \overline{r}' \arrow[urr, "f'"'] \arrow[ddl, "g'"'] & & \\
& & \overline{n}' \arrow[dr] & \\
\overline{q} \arrow[urr, "k'"] \arrow[rrr, "k"] & & & \overline{n}
\end{tikzcd}$$

Here, $\overline{n}' \to \overline{n}$ is the inclusion of the joint image of h and k, and $\overline{r}'$ is the intersection of $\overline{p}$ and $\overline{q}$ as subobjects of $\overline{n}$, and all new arrows are the obviously induced ones. Hence the inner square is balanced by construction, and has a factorization of the desired type by Proposition 5.5.3.

Finally, rewriting the above diagram in the following form then establishes the desired factorization overall.

$$\begin{tikzcd}
\overline{r} \arrow[r] \arrow[d] & \overline{r}' \arrow[r, "f'"] \arrow[d, equal] & \overline{p} \arrow[r, equal] \arrow[d, equal] & \overline{p} \arrow[d, equal] \\
\overline{r}' \arrow[r, equal] \arrow[d, "g'"] & \overline{r}' \arrow[r, "f'"] \arrow[d, "g'"] & \overline{p} \arrow[r, equal] \arrow[d, "h'"] & \overline{p} \arrow[d, "h'"] \\
\overline{q} \arrow[r, equal] \arrow[d, equal] & \overline{q} \arrow[r, "k'"] \arrow[d, equal] & \overline{n}' \arrow[r, equal] \arrow[d, equal] & \overline{n}' \arrow[d] \\
\overline{q} \arrow[r, equal] & \overline{q} \arrow[r, "k'"] & \overline{n}' \arrow[r] & \overline{n}
\end{tikzcd}$$

The upper and lower connection squares described above are examples of *symmetric* commutative squares, meaning that the two morphisms from the source are the same, as are the two morphisms into the sink. By the following and Lemma 5.5.1, not just the connection squares, but *all* symmetric squares of face maps fail to be pushouts of sets, except for the "doubly trivial" ones which consist only of identities.

Lemma 5.5.5. *A symmetric commutative square of face maps in Δ is jointly surjective if and only if it is an upper connection.*

Proof. In the square below, suppose that h, h jointly cover all of $\overline{n}$.

$$\begin{tikzcd} \overline{r} \arrow[r, "f"] \arrow[d, "f"'] & \overline{p} \arrow[d, "h"] \\ \overline{p} \arrow[r, "h"] & \overline{n} \end{tikzcd}$$

Then this means equivalently that h is surjective, and therefore $h = \mathrm{id}_n$ since h is at the same time a face map. □

We now turn to the conditions under which a simplicial set sends connection squares and squares which are not necessarily jointly surjective to (weak) pullbacks in Set.

Proposition 5.5.6. *Given a symmetric square of face maps in Δ as below left and a simplicial set $X : \Delta^{\mathrm{op}} \to \mathsf{Set}$:*

(a) *If the square is a lower connection ($f = id$), then the square below right is a weak pullback, and a strong pullback if and only if X_g is an isomorphism.*
(b) *If the square is an upper connection ($g = id$), then the square below right is a weak pullback if and only if it is a strong pullback if and only if X_f is an isomorphism.*

$$\begin{tikzcd} \overline{r} \arrow[r, "f"] \arrow[d, "f"'] & \overline{p} \arrow[d, "g"] \\ \overline{p} \arrow[r, "g"] & \overline{n} \end{tikzcd} \qquad \begin{tikzcd} X_n \arrow[r, "X_g"] \arrow[d, "X_g"'] & X_p \arrow[d, "X_f"] \\ X_p \arrow[r, "X_f"] & X_r \end{tikzcd}$$

Proof. Recall from the dual proof of Lemma 5.1.1 that since Set has strong pullbacks, a square as above right is a weak pullback if and only if the induced map $X_n \to X_p \times_{X_r} X_p$ is a (necessarily split) epimorphism; and that all face maps in Δ are split monomorphisms, which are therefore sent to split epimorphisms by X.

(a) In this case, the strong pullback evaluates to $X_p \times_{X_r} X_p \cong X_p$, and the induced map $X_g : X_n \to X_p$ is always an epimorphism since g is a face map. Therefore the square above right is automatically a weak pullback, and a strong pullback precisely when X_g is an isomorphism.

(b) In this case, the induced map $X_n = X_p \to X_p \times_{X_r} X_p$ is a monomorphism, so for the square to be a weak pullback this map must be an isomorphism, making the square a strong pullback. $\square$

For every commutative square of face maps such as the outer square below, the joint image of h, k in $\overline{n}$ forms an ordered set isomorphic to some $\overline{m}$, giving the factorization below where the inner square is jointly surjective, and the outer square is jointly surjective precisely when ℓ is the identity:

$$\begin{tikzcd}
\overline{r} \arrow[r, "f"] \arrow[d, "g"'] & \overline{p} \arrow[d, "h'"'] \arrow[ddr, "h"] & \\
\overline{q} \arrow[r, "k'"] \arrow[drr, "k"'] & \overline{m} \arrow[dr, dashed, "\ell"] & \\
& & \overline{n}
\end{tikzcd}$$

Proposition 5.5.7. *A simplicial set X sends the outer square above to a weak pullback if and only if it sends the inner square to a weak pullback. If X sends the outer square to a strong pullback, then X_ℓ is an isomorphism.*

Proof. Applying X results in the commuting diagram below.

$$\begin{tikzcd}
X_n \arrow[dr, dashed, "X_\ell"] \arrow[drr, bend left, "X_h"] \arrow[ddr, bend right, "X_k"'] & & \\
& X_m \arrow[r, "X_{h'}"'] \arrow[d, "X_{k'}"] & X_p \arrow[d, "X_f"] \\
& X_q \arrow[r, "X_g"'] & X_r
\end{tikzcd}$$

Since X_ℓ is a split epimorphism, if the square with X_m is a weak pullback then so is the square with X_n. Thus if the square with X_m is a weak pullback, then so is the square with X_n. Conversely, if the square with X_n is a weak pullback, then clearly so is the other one, since lifts can be constructed by composition with X_ℓ.

If the square with X_n is a strong pullback, then in particular X_h and X_k are jointly monic, and therefore X_ℓ must already be a monomorphism by commutativity. But X_ℓ is already split epi, so in this case it is an isomorphism. $\square$

This proof generalizes to any pair of squares in Δ with fixed span related by a face map. The jointly surjective squares are those with no such inner squares. In fact, the first claim in Proposition 5.5.6 follows from this as the inner jointly surjective square is trivial, hence always sent to a pullback.

In conclusion, a simplicial set X cannot send upper connection squares to weak pullbacks without sending them to strong pullbacks, and weak cartesianness of its action on squares which are not jointly surjective is entirely dependent on its action on their inner jointly surjective squares. Furthermore, X sending any of these types of squares to strong pullbacks forces certain structure maps to be isomorphisms.

Corollary 5.5.8. *If a simplicial set X sends either all upper connection squares or all lower connection squares to strong pullbacks, then X is discrete.*

PROOF. By Proposition 5.5.6, for any face map f in Δ, if either of the upper or lower connection squares containing f are sent to strong pullbacks by X, then X_f is an isomorphism. Therefore all face maps of X are isomorphisms, hence so are all degeneracy maps of X as they have isomorphisms as retracts. Therefore X is discrete. $\square$

We get a similarly destructive result if any square which is not jointly surjective is sent to a strong pullback

Corollary 5.5.9. *If X sends any square as below which is not jointly surjective to a strong pullback, then all face and degeneracy maps between $X_m, X_{m+1}, \ldots, X_n$ are isomorphisms.*

$$\begin{tikzcd} \overline{r} \arrow[r, "f"] \arrow[d, "g"'] & \overline{p} \arrow[d] \arrow[ddr, "h"] & \\ \overline{q} \arrow[r] \arrow[drr, "k"'] & \overline{m} \arrow[dr, dashed, "l"] & \\ & & \overline{n} \end{tikzcd} \tag{5.8}$$

PROOF. If X sends the above square to a strong pullback, by Proposition 5.5.7, X_l is an isomorphism. X_l factors into generating face maps $d_{i_1} \cdots d_{i_{n-m}}$, which are all isomorphisms as well by induction on the argument that $d_{i_{n-m}}$ is surjective as a face map and injective as the first factor of an isomorphism. Finally, if any $d_i : X_n \to X_{n-1}$ is an isomorphism then its two sections s_i, s_{i-1} are also isomorphisms, in which case d_{i-1}, d_{i+1} are isomorphisms, so proceeding by induction all face and degeneracy maps between X_{n-1} and X_n are isomorphisms. This argument then applies to all structure maps between $X_m, \ldots, X_n$. $\square$

6. Inner span completeness and the bar construction for BC monads

In this section, we consider simplicial sets which satisfy relatively weak filler conditions, namely those corresponding to the (basic) pushouts of two face maps in Δ. We study which other filler conditions are implied by these, and for which monads the bar construction produces this type of simplicial set.

6.1. Inner span complete simplicial sets. We start by considering those simplicial sets which preserve pushouts of two face maps, and then reducing this to the basic face map pushouts from (8.1).

Definition 6.1.1. *A simplicial set $X : \Delta^{\mathrm{op}} \to \mathsf{Set}$ is* inner span complete *if it maps pushouts of face maps in Δ to weak pullbacks in Set. We write ISpC for the full subcategory of inner span complete simplicial sets in sSet.*

The image under X of a pushout square of face maps in Δ, as shown below left, is equivalent to the existence of the dotted extension of any diagram of the form below right (in simplicial sets). This shows that inner span completeness is defined in terms of *filler conditions*.

$$\begin{tikzcd} X_n \arrow[r, "X_h"] \arrow[d, "X_k"'] & X_p \arrow[d, "X_f"] \\ X_q \arrow[r, "X_g"] & X_r \end{tikzcd} \qquad\qquad \begin{tikzcd} \Delta^p \;{}_f\sqcup_g \Delta^q \arrow[r] \arrow[d, hook, "{h,k}"'] & X \\ \Delta^n \arrow[ur, dashed] & \end{tikzcd}$$

Inclusions of this form—from (the pushout in simplicial sets of) a span of simplices into a higher dimensional simplex—are called *inner span inclusions*. By Corollary 5.3.5, these inclusions are characterized by the property that their image contains the entire spine of the larger simplex.

Following Section 5, this definition has several equivalent formulations. For instance, Lemma 5.1.1 shows that a simplicial set is inner span complete if and only if it sends all *weak* pushouts of face maps in Δ to weak pullbacks of sets. The following more minimal criterion for checking inner span completeness follows directly from Proposition 5.3.11 and the particular form of the basic pushouts of face maps given in (5.3).

Proposition 6.1.2. *A simplicial set $X : \Delta^{\mathrm{op}} \to \mathsf{Set}$ is inner span complete if and only if each square of the form*

$$\begin{tikzcd} X_n \arrow[r, "d_i"] \arrow[d, "d_j"'] & X_{n-1} \arrow[d, "d_{j-1}"] \\ X_{n-1} \arrow[r, "d_i"] & X_{n-2} \end{tikzcd}$$

is a weak pullback whenever $j - i > 1$.

In terms of filler conditions, this condition means that given two $(n-1)$-simplices which agree on their respective ith and $(j-1)$th faces, then there exists an n-simplex which has those two as its jth and ith faces, respectively. So X is inner span complete if for any diagram of simplicial sets as below, there exists the dotted extension:

$$\begin{tikzcd} \Delta^{n-1} {}_{d^i}\sqcup_{d^{j-1}} \Delta^{n-1} \arrow[r] \arrow[d, hook, "{d^j,d^i}"'] & X \\ \Delta^n \arrow[ur, dashed] & \end{tikzcd}$$

This phrasing suggests that inner span complete simplicial sets can be interpreted as a higher compositional structure; as we will see, they are strictly more general than quasicategories [**Joy02**].

We call the relevant inclusion

$$\Delta^{n-1} {}_{d^i}\sqcup_{d^{j-1}} \Delta^{n-1} \xrightarrow{d^j,d^i} \Delta^n$$

for $j - i > 1$ that of a *basic inner span*. In summary, we have shown the following.

Corollary 6.1.3. *A simplicial set is inner span complete if and only if it has fillers of all basic inner spans.*

We now turn to the relation to quasicategories.

Proposition 6.1.4. *Quasicategories are inner span complete.*

Proof. Consider a basic inner span in a quasicategory X of the faces of the n-simplex, omitting respectively the ith and kth vertices where $j - i > 1$. Choose k to lie between i and j. Observe that the basic inner span contains precisely the faces of Δ^n not containing the edge from i to j. By [**Lur09**, Lemma 4.4.5.5] applied with $J = \{i, j\}$, the basic inner span inclusion is inner anodyne, and therefore has a filler. □

The converse to this statement, that an inner span complete simplicial set is a quasicategory, is not true: below we show that the bar construction of an algebra of a BC monad is inner span complete, so that the unfilled inner horns of Section 4.4 provide counterexamples in bar constructions of BC monads.

6.2. Fillers for directed acyclic configurations. In this subsection we prove the existence of additional fillers in an inner span complete simplicial set X, obtained by iterating the filling condition for basic inner spans.

For the moment we will work in the undirected context, considering *simplicial complexes* in the sense of collection of subsets of a finite ground set which are downward closed. In the directed context below, we will then consider simplicial subsets of the n-simplex $\Delta_n = \Delta(-, \overline{n}) : \Delta^{\mathrm{op}} \to \mathsf{Set}$.

A vertex in a simplicial complex is *extremal* if it is contained in only one maximal simplex. A *combinatorial sphere* is an abstract simplicial complex with at least 3 vertices, containing precisely the proper subsets of the ground set. One can visualise it geometrically as a hollow triangle, or a hollow tetrahedron, or in general the boundary of a simplex.

The following definitions and the characterization of Theorem 6.2.3 are well-known, but they do not seem to be easy to find in the literature in this exact form. Much of the related literature is in the area of relational database theory, where often more general hypergraphs rather than simplicial complexes are considered[12], resulting in greater generality and complexity than what we need here.

Definition 6.2.1 ([**Gra79**]). *An abstract simplicial complex S is* Graham acyclic *if it satisfies the following recursive acyclicity definition: S is empty, or S contains an extremal vertex v and $S \setminus \{v\}$ is Graham acyclic.*

Here, $S \setminus \{v\} \coloneqq \{A \setminus \{v\} \mid A \in S\}$ denotes the new abstract simplicial complex obtained by removing v from all simplices. Applying this recursive elimination of vertices to a simplicial complex is called *Graham reduction*. For a Graham acyclic simplicial complex, the reduction results in the empty complex[13], while otherwise the process terminates at a non-empty complex.

Definition 6.2.2. *A* chordal graph *is an undirected graph in which all cycles with at least 4 edges have a chord, i.e. an edge which connects two vertices non-adjacent in the cycle.*

Applying this definition repeatedly shows that in a chordal graph, a cycle of any length can be triangulated, which is why chordal graphs are also sometimes called *triangulated graphs*.

The following characterization theorem is well-known in its hypergraph version in the literature on acyclic database schemes, see e.g. [**BFMY83**] or [**Mai83**, Theorem 13.2], while the proof is somewhat simpler in our setting involving simplicial complexes. The condition (a) is easy to check algorithmically, while condition (b) is useful for mathematical proofs. Condition (c) is the one which will facilitate our reduction to inner span fillers in the directed case below, and is generally useful when working with algebraic or combinatorial structures on simplicial complexes, such as the tables in a database.

Theorem 6.2.3. *The following are equivalent for a simplicial complex S:*

(a) S is Graham acyclic.

(b) Every combinatorial sphere in S has a filler, and the 1-skeleton of S is a chordal graph.

[12] See e.g. [**BFMY83**], or [**Mai83**, Chapter 13] for a textbook account.

[13] It is known that Graham reduction can be performed in any order, i.e. it is impossible to get stuck.

(c) *S has the* running intersection property*: the maximal simplices of S can be ordered as $T_1, \ldots, T_m$ such that for every $k = 1, \ldots, m$ there is $j < k$ with*

$$T_k \cap \left(\bigcup_{i=1}^{k-1} T_i \right) \subseteq T_j .$$

Moreover if S is connected, then the $T_1, \ldots, T_m$ in (c) can be chosen such that $T_k \cap \bigcup_{i=1}^{k-1} T_i$ is nonempty for all $k = 2, \ldots, m$.

We provide a proof for convenience.

Proof. (a) $\Rightarrow$ (b): We use induction on the number of vertices of S, with the statement being trivial if S is empty. For the induction step, suppose that S is Graham acyclic with extremal vertex v.

Now consider a combinatorial sphere in S. If this sphere does not contain v, then it has a filler by the induction assumption applied to $S \setminus \{v\}$. If this sphere contains v, then it must also have a filler, since otherwise v would be contained in more than one maximal simplex.

Similarly, consider a cycle of length ≥ 4 in the 1-skeleton of S. If v is not part of this cycle, then it again has a chord by the induction assumption, so suppose that v is a vertex in the cycle. Then both neighboring vertices of v in the cycle are also members of the unique maximal simplex containing v, and therefore so is the edge between these vertices, resulting in a chord.

(b) $\Rightarrow$ (a): A vertex v in a graph G is called *simplicial* if every two neighboring vertices of v are themselves adjacent. A standard graph-theoretic result is that a graph is chordal if and only if there is an ordering $\{v_1, \ldots, v_n\}$ of its vertices such that each v_i is simplicial in the subgraph induced by the vertices $\{v_1, \ldots, v_i\}$ [**Wes96**, Theorem 5.3.17]. In the case of a simplicial complex whose 1-skeleton is a chordal graph, as long as there are no unfilled combinatorial spheres, such an ordering can be reversed to provide an ordering for the Graham reduction process. This is because every complete subgraph of the 1-skeleton has to be a simplex in S by assumption, in particular making v_n extremal in S.

(c) $\Rightarrow$ (a): If the running intersection property holds, then putting $k = m$ shows that there is a $j < m$ such that $T_m \cap \left(\bigcup_{i=1}^{m-1} T_i \right) \subseteq T_j$. This implies that there is some vertex $v \in T_m$ which does not belong to any of the other maximal simplices from 1 to $m - 1$, making v extremal. Considering the reduced complex $S \setminus \{v\}$, there are now two possibilities: it may be that $S \setminus \{v\}$ has maximal simplices $T_1, \ldots T_{m-1}$ as maximal simplices, in which case the running intersection property still holds trivially; or $S \setminus \{v\}$ may in addition have the maximal simplex $T_m \setminus \{v\}$, in which case the running intersection property still holds with T_m replaced by $T_m \setminus \{v\}$ in the new ordering. In either case, the induction assumption finishes the argument, again with the empty simplicial complex as the base case.

(a) $\Rightarrow$ (c): We once more use induction on the number of vertices, where the empty base case is obvious. For the induction step, suppose that v is an extremal vertex in S, belonging to a unique maximal simplex T. Since $S \setminus \{v\}$ is still Graham acyclic, the induction hypothesis shows that there exists an ordering $T_1, \ldots T_{m-1}$ of the maximal simplices of $S \setminus \{v\}$ which satisfies the running intersection property.

Then we again have two cases. First, if $T \setminus \{v\}$ is still maximal in $S \setminus \{v\}$, then it must coincide with some T_k. Then $T_1, \ldots, T_k \cup \{v\}, \ldots, T_m$ is an ordering of the maximal simplices of S which witnesses the running intersection property. Second, if $T \setminus \{v\}$ is no longer maximal in $S \setminus \{v\}$, then it must be properly contained in some T_j. Then the sequence of maximal simplices

$$T_1, \ldots, T_m, T$$

witnesses the running intersection property for S, because of $T \cap \bigcup_{i=1}^m T_i = T \setminus \{v\} \subseteq T_j$.

Moreover, the final claim on connectedness follows by an inspection of the previous argument: if S is connected, then so is $S \setminus \{v\}$, and it is straightforward to check that every $T_k \cap \bigcup_{i=1}^{k-1} T_i$ is nonempty provided that this holds likewise on $S \setminus \{v\}$, which it does by the induction assumption. $\square$

Remark 6.2.4. It should be noted that the acyclicity property characterized by Theorem 6.2.3 is not homotopy invariant, and in particular distinct from notions of acyclicity familiar from algebraic topology. This applies similarly to our directed version coming up next.

Definition 6.2.5. *We call a connected simplicial complex $S \subseteq 2^{\overline{n}}$ with $\bigcup S = \overline{n}$ satisfying the conditions of Theorem 6.2.3 an* acyclic configuration *inside the n-simplex.*

We now propose versions of the above acyclicity notion in our directed setting. A *directed simplicial complex* S is a downward closed collection of subsets of a finite nonempty totally ordered set, which without loss of generality we take to be given by

$$\overline{n} = \{0, \ldots, n\} = \bigcup S,$$

thereby identifying a directed simplicial complex on n vertices with a simplicial subcomplex of the n-simplex. As before we write $S \setminus \{v\} = \{A \setminus \{v\} \mid A \in S\}$, where now this reduced directed simplicial set lives on $\overline{n-1}$, so that all vertices beyond v must be relabeled by -1.

All notions for which we have not introduced directed versions, such as extremality of a vertex, are used in the undirected sense defined above.

Definition 6.2.6. *A directed simplicial complex $S \subseteq 2^{\overline{n}}$ is* directed Graham acyclic *if $n = 0$, or if S has an extremal vertex $v \in \overline{n}$ such that*

(a) *If $v > 0$, then $\{v-1, v\} \in S$.*

(b) *If $v < n$, then $\{v, v+1\} \in S$.*

(c) *$S \setminus \{v\}$ is again directed Graham acyclic.*

We then have a characterization analogous to that of Theorem 6.2.3.

Theorem 6.2.7. *The following are equivalent for a directed simplicial complex $S \subseteq 2^{\{0,\ldots,n\}}$ with $\bigcup S = \{0, \ldots, n\}$:*

(a) *S is directed Graham acyclic.*

(b) *Every combinatorial sphere in S has a filler, and the 1-skeleton of S is a chordal graph which contains the entire spine.*

(c) *S has the* directed running intersection property*: the maximal simplices of S can be ordered as $T_1, \ldots, T_m$ such that for every $k = 1, \ldots, m$ there is $j < k$ with*

$$\left(\bigcup_{i=1}^{k-1} T_i\right) \cap T_k \;\subseteq\; T_j,$$

and for every two vertices $v < w$ which are consecutive in $\bigcup_{i=1}^{k} T_i$, we have $\{v, w\} \subseteq T_k$ or $\{v, w\} \subseteq \bigcup_{i=1}^{k-1} T_i$.

Note that each one of these conditions implies its undirected counterpart given in Theorem 6.2.3.

Proof. It is enough to show that the additional conditions relative to Theorem 6.2.3 imply each other, assuming that the underlying undirected simplicial complex of S is acyclic.

Assuming (a), a simple induction argument indeed shows that S contains the whole spine. For if $v \in \overline{n}$ is as in Definition 6.2.6, then $S \setminus \{v\}$ can be assumed to contains its entire spine by the induction assumption, and the extra condition on v then implies that S also contains the additional spinal edges not implied by those of $S \setminus \{v\}$. Conversely if (b) holds, then the conditions $\{v-1, v\} \in S$ for $v > 0$ and $\{v, v+1\} \in S$ for $v < n$ are part of the assumption that S contains the entire spine.

For the equivalence between (b) and (c), it is now enough to prove that the extra condition in (c) is equivalent to S containing the entire spine, provided that the undirected acyclicity of Theorem 6.2.3 holds. Thus if (c) holds, we now argue that $\{v, v+1\} \in S$ for every $v < n$. To this end, consider the smallest k with

$\{v, v+1\} \subseteq \bigcup_{i=1}^{k} T_i$. Then the assumption implies the desired $\{v, v+1\} \subseteq T_k$, since $\{v, v+1\} \subseteq \bigcup_{i=1}^{k-1} T_i$ would contradict the minimality of k.

In the other direction, suppose that S satisfies the undirected running intersection property and contains the entire spine. Let $v < w$ be two vertices consecutive in $\bigcup_{i=1}^{k} T_i$. We will use backwards induction on k to prove the desired property $\{v, w\} \subseteq T_k$ or $\{v, w\} \subseteq \bigcup_{i=1}^{k-1} T_i$, or equivalently that the induced subcomplex on $\bigcup_{i=1}^{k} T_i$ contains its entire spine. This is clear in the base case $k = m$: for then we must have $w = v + 1$, so that the containing the entire spine assumption applies.

For $k < m$, suppose first that v and w are still consecutive in $\bigcup_{i=1}^{k+1} T_i$. Since the induced subcomplex on $\bigcup_{i=1}^{k+1} T_i$ contains the entire spine by the induction assumption, we must have $\{v, w\} \subseteq T_h$ for some $h \leq k + 1$. For $h \leq k$ we are done, so assume $h = k + 1$. Then the running intersection property implies that there is $j \leq k$ with $T_{k+1} \cap \bigcup_{i=1}^{k} T_i \subseteq T_j$. We therefore also conclude that $\{v, w\} \subseteq T_j$, which is enough.

Finally if v and w are no longer consecutive in $\bigcup_{i=1}^{k+1} T_i$, then there are nonzero many elements $u_1, \ldots, u_\ell \in T_{k+1} \setminus \bigcup_{i=1}^{k} T_i$ such that the sequence

$$v, u_1, \ldots, u_\ell, w$$

consists of consecutive vertices in $\bigcup_{i=1}^{k+1} T_i$. The induction assumption together with $u_\star \notin \bigcup_{i=1}^{k} T_i$ then gives us that the edge formed by any two consecutive vertices in this list is in T_{k+1}. But then also $\{v, u_1, \ldots, u_\ell, w\} \subseteq T_{k+1}$, and in particular $\{v, w\} \subseteq T_{k+1} \cap \bigcup_{i=1}^{k} T_i$. But then again the running intersection property implies $\{v, w\} \subseteq T_j$ for some $j \leq k$, as was to be shown. □

Definition 6.2.8. *We call a directed simplicial complex $S \subseteq 2^{\overline{n}}$ with $\bigcup S = \overline{n}$ satisfying the conditions of Theorem 6.2.3 a* directed acyclic configuration *inside the n-simplex.*

Note that the connectivity requirement which we had made in the undirected case (Definition 6.2.5) is now automatic.

Example 6.2.9. All inner spans define directed acyclic configurations, as follows. Suppose that the diagram below is a pushout of face maps in Δ.

$$\begin{tikzcd} \overline{r} \arrow[r, "f"] \arrow[d, "g"'] & \overline{p} \arrow[d, "h"] \\ \overline{q} \arrow[r, "k"] & \overline{n} \end{tikzcd}$$

Then consider the directed simplicial complex on $\overline{n}$ given by

$$S := \{A \subseteq \overline{n} \mid A \subseteq \mathrm{im}(h) \ \vee \ A \subseteq \mathrm{im}(k)\}.$$

This S is a directed acyclic configuration: the underlying undirected complex of S is a union of two simplices glued along a common face, and therefore acyclic in the undirected sense of Theorem 6.2.3. Since it moreover contains the entire spine by Corollary 5.3.5, directed acyclicity follows.

In particular, all basic inner span inclusions define directed acyclic configurations: for $0 \leq i < j-1 \leq n-1$, the directed simplicial complex

$$S := \{A \subseteq \overline{n} \mid i \notin A \ \vee\ j \notin A\}.$$

is directed acyclic. Using the same S with $i = j-1$ would not work, since then the spine condition would be violated due to the spinal edge $\{j-1, j\}$ not being a member of S.

The relevance of directed acyclicity in our context is the following general result.

Theorem 6.2.10. *If X is an inner span complete simplicial set, then X has fillers for all directed acyclic configurations.*

Proof. We use condition (c) of Theorem 6.2.7 as the relevant characterization of directed acyclic configurations. We will show by induction on k that every induced subcomplex on vertices $\bigcup_{i=1}^{k} T_i$ has a filler. There is nothing to prove in the base case $k = 1$, so assume $k > 1$. Then the induced subcomplex on vertices $\bigcup_{i=1}^{k-1} T_i$ has a filler by the induction assumption. But now with the inclusion maps as morphisms, the diagram[14]

$$\begin{tikzcd} T_k \cap \bigcup_{i=1}^{k-1} T_i \arrow[r] \arrow[d] & T_k \arrow[d] \\ \bigcup_{i=1}^{k-1} T_i \arrow[r] & \bigcup_{i=1}^{k} T_i \end{tikzcd}$$

is a diagram of face maps in Δ, where the additional condition of the directed running intersection property in (c) guarantees that the square is a pushout, using Corollary 5.3.5. Since $T_k \cap \bigcup_{i=1}^{k-1} T_i \subseteq T_j$ for some j, we know that the given two simplices have the same face on $T_k \cap \bigcup_{i=1}^{k-1} T_i$, namely the corresponding face of the given simplex on T_i. We can therefore apply the definition of inner span completeness to obtain a simplex which has the already given simplices on $\bigcup_{i=1}^{k-1} T_i$ and T_k as faces, which finishes the induction step. $\square$

Example 6.2.11. Consider the spine inclusions of the edges $0 \to 1 \to \cdots \to n$ into Δ^n for $n \geq 1$. The spines contain no cycles or combinatorial spheres and certainly

[14]Note that the consecutive vertices condition in the directed running intersection property guarantees that the set-theoretic intersection $T_k \cap \bigcup_{i=1}^{k-1}$ is nonempty.

contain the entire spine. Thus this defines a directed acyclic configuration, more explicitly given by the maximal simplices

$$T_1 = \{0, 1\}, \quad \ldots, \quad T_n = \{n-1, n\}.$$

Theorem 6.2.10 thus implies that in an inner span complete simplicial set, every string of n edges is the spine of an n-simplex. This is a weak version of the "(1-)Segal condition" which characterizes nerves of categories.

Example 6.2.12. Consider any triangulation of the $(n+1)$-gon for $n \geq 2$, with vertices labeled in order from 0 to n. The edges and triangles of the triangulation form a directed acyclic configuration in the n-simplex.

Indeed the configuration contains the spine of the n-simplex, as the spinal edges are among the outer edges of the n-gon. As a triangulation, the 1-skeleton of this configuration is a chordal graph, and the only combinatorial spheres are the filled triangles. By Theorem 6.2.7(b), the directed acyclicity follows.

By Theorem 6.2.10, we hence conclude that an inner span complete simplicial set has fillers for all of these inclusions. When these fillers are unique, they correspond to the "2-Segal condition" of [**DK12**].

Example 6.2.13. Indeed any triangulation of a polytope on vertices $0, \ldots, n$ has fillers of combinatorial spheres, and forms a directed acyclic configuration of the n-simplex if its 1-skeleton is a chordal graph containing the spinal edges.

For example, the cyclic polytope on vertices $0, 1, \ldots, n$ in d-dimensional space (with $n \geq d$) is the convex hull of those points on the moment curve $(t, t^2, \ldots, t^d)$ [**Gal63**].A version of the "d-Segal condition" alluded to in [**DK12**] is having unique lifts against the inclusion into the n-simplex of any d-simplex triangulation of the d-dimensional cyclic polytope on $0, \ldots, n$. When $d > 3$, by [**Gal63**, Theorem 1] the 1-skeleton of each cyclic polytope is a complete graph, so these triangulations form directed acyclic configurations. When $d = 3$, by [**Pog17**, Proposition 1.4] the 1-skeleton of the cyclic polytope on $0, \ldots, n$ contains the spinal edges along with edges from 0 to any vertex and from any vertex to n. As the only edges between vertices other than 0 and n are the spinal edges from i to $i+1$, any cycle must then contain 0 or n, which has an edge to every vertex in the cycle, so the 1-skeleton is chordal and any triangulation by 3-simplices forms an acyclic configuration.

Since Theorem 6.2.10 guarantees fillers for all configurations of this type, we conclude that inner span complete simplicial sets satisfy a weak version of the d-Segal condition for all d.

6.3. The bar construction for BC monads. Our original motivation has been to investigate the compositional structure of the bar construction. Correspondingly, we next prove a criterion for when the bar construction of a monad results in an inner span complete simplicial set.

Let (T, η, μ) be a BC monad, meaning that the functor T preserves weak pullbacks and the transformation μ is weakly cartesian. Conditions such as these translate directly to weak pullback properties of the standard commuting squares of generating structure maps in the bar constructions for algebras of T. For instance, the relations for face maps between the set of triangles $X_2 = T^3A$, edges $X_1 = T^2A$ and vertices $X_0 = TA$ are given by the following squares:

$$\begin{tikzcd} & & TA & & \\ & T^2A \arrow[ur, "Te"] & T^3A \arrow[l, "\mu"'] \arrow[r, "T^2e"] \arrow[d, "T\mu"] & T^2A \arrow[ul, "\mu"'] \arrow[dr, "Te"] & \\ TA & & T^2A \arrow[ll, "\mu"'] \arrow[rr, "Te"] & & TA \arrow[from=2-2, to=3-1, "\mu"'] \end{tikzcd} \tag{6.1}$$

In order from bottom left to top to bottom right, these squares are instances for $\mathrm{Bar}_T(A)$ of the simplicial identities for face maps in dimension two, namely the following.

$$\begin{tikzcd} X_2 \arrow[r, "d_1"] \arrow[d, "d_2"'] & X_1 \arrow[d, "d_1"] \\ X_1 \arrow[r, "d_1"] & X_0 \end{tikzcd} \qquad \begin{tikzcd} X_2 \arrow[r, "d_0"] \arrow[d, "d_2"'] & X_1 \arrow[d, "d_1"] \\ X_1 \arrow[r, "d_0"] & X_0 \end{tikzcd} \qquad \begin{tikzcd} X_2 \arrow[r, "d_0"] \arrow[d, "d_1"'] & X_1 \arrow[d, "d_0"] \\ X_1 \arrow[r, "d_0"] & X_0 \end{tikzcd}$$

If μ is weakly cartesian, then the top naturality square in (6.1) is a weak pullback, hence so is the middle square above. There is no reason to expect that the left or right squares would be weak pullbacks.

More generally, the square

$$\begin{tikzcd} T^{n+1}A \arrow[r, "T^{n-i}\mu"] \arrow[d, "T^{n-j}\mu"'] & T^nA \arrow[d, "T^{n-j}\mu"] \\ T^nA \arrow[r, "T^{n-i-1}\mu"] & T^{n-1}A \end{tikzcd}$$

is T^{n-j} applied to a naturality square of μ for any $0 \leq i < j-1 \leq n-1$, where for $i = 0$, it is understood that the horizontal maps are T^ne and $T^{n-1}e$, respectively. The BC assumption implies that this square is weakly cartesian. Hence for $X = \mathrm{Bar}_T(A)$, the following square is a weak pullback when $i < j-1$:

$$\begin{tikzcd} X_n \arrow[r, "d_i"] \arrow[d, "d_j"'] & X_{n-1} \arrow[d, "d_{j-1}"] \\ X_{n-1} \arrow[r, "d_i"] & X_{n-2} \end{tikzcd}$$

The analogous squares for $i = j-1$ are those given by associativity of μ, multiplicativity of e, or some functor power T^k applied to such a square, and do not need to be weak pullbacks in general.

In summary, the weak pullbacks among generating face maps in $\mathrm{Bar}_T(A)$ are then precisely those arising from basic pushout squares of face maps in Δ, which by Corollary 6.1.3 implies the following result.

Theorem 6.3.1. *For any algebra A of a BC monad T, the bar construction* $\mathrm{Bar}_T(A)$ *is inner span complete.*

Since inner span complete simplicial sets in particular have fillers for inner 2-horns, this reproduces and generalizes the transitivity of the partial evaluation relation for BC monads from Lemma 3.3.1.

Example 6.3.2. We saw in Example 2.1.5 that the distribution monad is BC. By Theorem 6.3.1, this implies that for all its algebras, the bar construction is inner span complete. At the lowest level, this in particular implies that the partial evaluation relation (known also as *second-order stochastic dominance*, see [**FP20**]) is transitive. Inner span completeness is a stronger property than just inducing a transitive relation, and this may reflect a more profound structure at the level of random variables, generalizing the compositional nature of conditional expectation; see again [**FP20**] for the relationship between partial evaluations and conditional expectation.

However, a detailed analysis of the probabilistic meaning of inner span completeness is beyond the scope of this paper, and ideally would have to be carried out in categories other than Set, facilitating the treatment of measure-theoretic probability.

6.4. Inner span exact simplicial sets. For any class of simplicial sets defined by filler conditions, there is a corresponding class for which those fillers are unique, where diagrams previously required to be weak limits are now required to be strong limits. In terms of our definition of inner span completeness, it is thus natural to consider the following.

Definition 6.4.1. *A simplicial set $X : \Delta^{\mathrm{op}} \to \mathsf{Set}$ is* inner span exact *if it maps pushouts of face maps in Δ to strong pullbacks in Set.*

We will encounter this pattern of *complete* and *exact* simplicial sets with respect to some class of squares in Δ also in the remaining three sections of this paper.

By the same reasoning as in the case of weak pullbacks, a simplicial set X is inner span exact if and only if every square of the form below left is a strong pullback, or equivalently that in any diagram as below right the dotted extension exists uniquely.

$$\begin{tikzcd} X_n \arrow[r, "d_i"] \arrow[d, "d_j"'] & X_{n-1} \arrow[d, "d_{j-1}"] \\ X_{n-1} \arrow[r, "d_i"] & X_{n-2} \end{tikzcd} \qquad \begin{tikzcd} \Delta^{n-1} {}_{d^i}\sqcup_{d^{j-1}} \Delta^{n-1} \arrow[r] \arrow[d, hook, "{d^j,d^i}"'] & X \\ \Delta^n \arrow[ur, dashed] \end{tikzcd}$$

Our goal is now to characterize the inner span exact simplicial sets as precisely the nerves of categories.

Lemma 6.4.2. *If X is an inner span complete simplicial set, then X has unique fillers for all directed acyclic configurations.*

Proof. This follows by the same argument as the proof of Theorem 6.2.10, where we showed the (non-unique) existence of fillers in inner span complete simplicial sets, while additionally keeping track of the uniqueness in each step. $\square$

Lemma 6.4.3. *A simplicial set $X : \Delta^{\mathrm{op}} \to \mathsf{Set}$ is inner span exact if and only if it sends pushouts of the form*

$$\begin{tikzcd} \overline{0} \arrow[r, "p"] \arrow[d, "0"'] & \overline{p} \arrow[d] \\ \overline{q} \arrow[r] & \overline{p+q} \end{tikzcd}$$

to pullbacks.

Proof. The "only if" part is obvious since the squares are pushouts of face maps, so we focus on the "if". Every pushout of face maps $\star$-decomposes into trivial pushout squares by Lemma 5.4.1. Therefore by Proposition 5.4.2, X sends all pushout squares of face maps to pullbacks, and this makes it inner span exact. $\square$

Theorem 6.4.4. *A simplicial set is inner span exact if and only if it is the nerve of a category.*

Proof. By Lemma 6.4.2 and Example 6.2.11, in an inner span exact simplicial set we have $X_n \cong X_1 \times_{X_0} \overset{n}{\cdots} \times_{X_0} X_1$, which is precisely the Segal condition for being a nerve.

The converse follows from the previous Lemma 6.4.3, as the nerve of a category satisfies the Segal condition $X_n \cong X_1 \times_{X_0} \overset{n}{\cdots} \times_{X_0} X_1$ for all n, which implies that $X_n \cong X_p \times_{X_0} X_q$ for every $p + q = n$. $\square$

This result suggests that the inner span complete simplicial sets of Definition 6.1.1 can be thought of as compositional structures which generalize categories in a meaningful way that is less restrictive than quasicategories.

Remark 6.4.5. The above result is similar in flavor to that of [**FGK+19**], which shows (in a more general homotopical setting) that 2-Segal simplicial sets satisfy certain unitality conditions. These 2-Segal sets are defined as being exact with respect to pushout squares of face maps in Δ in which the vertical maps are both outer face maps, meaning d_0 or $d_n : \overline{n-1} \to \overline{n}$, and the unitality conditions amount to exactness with respect to a similar subset of the basic mixed pushout squares in Δ given in (5.4). Our techniques for describing exactness properties are less applicable to the 2-Segal setting however, as there the defining class of squares in Δ does not contain the squares of Lemma 6.4.3 and is not closed under $\star$-products or $\star$-decompositions.

It is interesting to note that the basic squares used in the proof of Theorem 6.4.4 are disjoint from those that define 2-Segal sets, the former being those sent to pullbacks by 1-Segal sets. In the setting of exactness properties, the 1-Segal condition implies the 2-Segal condition, but when those conditions are weakened to completeness properties, requiring only that the relevant squares are sent to weak pullbacks, they become independent of one another. It would perhaps not be surprising if the remaining basic pushout squares of face maps in Δ can be partitioned by the number $d > 2$ for which they are among the exactness conditions for d-Segal sets, but this is beyond the scope of this paper.

One may also wonder under which conditions on the monad the bar construction produces an inner span exact simplicial set. Since the inner span exact simplicial sets are exactly the nerves of categories (Theorem 6.4.4), cartesianness of the monad provides a simple sufficient condition (Remark 4.1.1).

7. Inner completeness and the bar construction for weakly cartesian monads

We can further impose the condition that also suitable pushout squares involving degeneracy maps in Δ are sent to weak pullbacks, which as we will see more faithfully models the structure of bar constructions of algebras of weakly cartesian monads.

7.1. Inner complete simplicial sets. The inner span complete simplicial sets from the previous section only preserve (weakly) the pushouts of face maps in Δ. We now consider a larger class.

Definition 7.1.1. *A simplicial set $X : \Delta^{\mathrm{op}} \to \mathsf{Set}$ is called* inner complete *if it maps concrete pushouts in Δ to weak pullbacks in Set.*

In particular, an inner complete simplicial set is inner span complete, as by Proposition 5.3.11 and the subsequent description of concrete basic pushouts all pushouts of face maps in Δ are concrete. The following analogue of Corollary 6.1.3 gives a more concrete characterization.

Proposition 7.1.2. *A simplicial set X is inner complete if it is inner span complete (Corollary 6.1.3) and the following squares are (necessarily strong) pullbacks,*

$$\begin{tikzcd} X_n \arrow[r, "d_i"] \arrow[d, "s_j"'] & X_{n-1} \arrow[d, "s_{j-1}"'] \\ X_{n+1} \arrow[r, "d_i"] & X_n \end{tikzcd} \qquad \begin{tikzcd} X_n \arrow[r, "d_i"] \arrow[d, "s_j"'] & X_{n-1} \arrow[d, "s_j"'] \\ X_{n+1} \arrow[r, "d_{i+1}"] & X_n \end{tikzcd}$$

$$(0 \leq i < j \leq n) \qquad (0 \leq j < i \leq n)$$

PROOF. This again follows from Proposition 5.3.11 together with the observation that all basic pushouts are concrete except for the middle ones in (5.4), and recalling that the pushouts of two degeneracy maps from (5.5) are preserved by every functor. □

The right square above being a weak pullback means that whenever the $(i+1)$th n-simplex face of an $(n+1)$-simplex x is the jth degeneracy of an $(n-1)$-simplex, then x is itself the jth degeneracy of a (necessarily unique) n-simplex. More geometrically, this means that when $d_{i+1}x$ is degenerate in such a way that its $[j, j+1]$ edge is degenerate, which is also the $[j, j+1]$ edge of x, then this degeneracy extends to all of x. The interpretation of the left square is similar.

More generally, if x is a simplex in an inner complete simplicial set and some face of x is degenerate along a *spinal* edge of x, then x itself is degenerate. This is because every pushout diagram in Δ as below with $r \geq 1$ and f a face map such that $f(j+1) = f(j) + 1$ is a concrete pushout.

$$\begin{tikzcd} \overline{r} \arrow[r, "f"] \arrow[d, "s_j"] & \overline{p} \arrow[d, "s_k"] \\ \overline{r-1} \arrow[r, "h"] & \overline{p-1} \end{tikzcd}$$

If we did not assume the degenerate edge in the face of x to be spinal in x, and still were to postulate that the degeneracy extends to x, then we would obtain a much stronger condition, with a single degenerate edge in an n-simplex (namely the $[0, n]$ edge) enough to imply that the entire simplex is degenerate from a 0-simplex. We do not assume this stronger property of inner complete simplicial sets here, since in particular it does typically not hold for nerves of categories. However, we will consider it for the *pushout complete* simplicial sets of the next section.

Returning to the comparison with quasicategories, in general an inner horn (say 2-horn) has many different fillers. For instance, any two "composites" of two edges in a quasicategory are related by a 2-simplex with one spinal edge degenerate. In an inner complete simplicial set, such a 2-simplex must itself be degenerate, implying that all composites are unique. The following result strengthens this observation.

Proposition 7.1.3. *An inner complete simplicial set which is also a quasicategory is the nerve of a category.*

Thus, the two ways of weakening the compositionality present in categories to quasicategories and to inner complete simplicial sets are intuitively orthogonal.

Proof. Let X be such a simplicial set. Using either structure, we see that X_n is a weak limit of the wide span with strong limit $X_1 \times_{X_0} \cdots_n \times_{X_0} X_1$, meaning that every string of n composable 1-simplices is the spine of at least one n-simplex. It is enough to prove that such an n-simplex is unique.

Assume by induction that $X_n \cong X_1 \times_{X_0} \cdots_n \times_{X_0} X_1$, where the isomorphism is implemented by the obvious map assigning to every n-simplex its spine. This indeed holds trivially for $n = 1$ as a base case. For the induction step, it is then enough to prove that $X_{n+1} \cong X_n \times_{X_0} X_1$, with the isomorphism implemented by the formation of the initial face and final edge on $[n, n+1]$. Thus suppose that $x, y \in X_{n+1}$ are such that $d_{n+1}x = d_{n+1}y$, and also such that their final edges coincide. What we need to prove is that $x = y$.

By the induction assumption, we also have $d_i x = d_i y$ for every $i = 0, \ldots, n$, since $d_i x$ and $d_i y$ have the same spine. We will argue that since X is a quasicategory, there is an nth $(n+2)$-horn filler $H \in X_{n+2}$ satisfying

$$d_i H = s_n d_i x = s_n d_i y \qquad \forall i < n,$$

as well as $d_{n+1}H = x$ and $d_{n+2}H = y$. To check that we are indeed dealing with an nth $(n+2)$-horn, we verify the relevant overlap conditions in detail.

(i) Concerning the final faces,

$$\begin{aligned} d_{n+1}d_{n+2}H &= d_{n+1}y \\ &= d_{n+1}x \\ &= d_{n+1}d_{n+1}H. \end{aligned}$$

(ii) For $i < j < n$, we have

$$\begin{aligned} d_i d_j H &= d_i s_n d_j x \\ &= s_{n-1} d_i d_j x \\ &= s_{n-1} d_{j-1} d_i x \\ &= d_{j-1} s_n d_i x \\ &= d_{j-1} d_i H. \end{aligned}$$

(iii) Finally for $i < n$, we get

$$\begin{aligned} d_i d_{n+1} H &= d_i x \\ &= d_n s_n d_i x \end{aligned}$$

$$= d_n d_i H,$$

and similarly

$$\begin{aligned} d_i d_{n+2} H &= d_i y \\ &= d_{n+1} s_n d_i y \\ &= d_{n+1} d_i H. \end{aligned}$$

These identities show that H indeed exists as a filler of an $(n+2)$-horn missing the nth face. But then $d_0 H = s_n d_0 x = s_n d_0 y$, and since the diagram

$$\begin{tikzcd} X_{n+1} \arrow[r, "d_0"] \arrow[d, "s_{n+1}"] & X_n \arrow[d, "s_n"] \\ X_{n+2} \arrow[r, "d_0"] & X_{n+1} \end{tikzcd}$$

is a pullback, we obtain $H = s_{n+1} z$ for some $z \in X_{n+1}$ and $d_0 x = d_0 y = d_0 z$. But then

$$\begin{aligned} x &= d_{n+1} H \\ &= d_{n+1} s_{n+1} z \\ &= z \\ &= d_{n+2} s_{n+1} z \\ &= d_{n+2} H \\ &= y. \end{aligned}$$

This proves the desired $x = y$, completing the induction step. □

7.2. The bar construction for weakly cartesian monads. We now consider which monads give rise to inner complete simplicial sets via their bar construction. Consider again a monad (T, η, μ), and assume now that it is weakly cartesian. In addition to the weak pullback squares which follow from the BC property for T, we therefore also have that η is a weakly cartesian transformation. This yields in particular the following weak pullback squares,

$$\begin{tikzcd} T^{n+1} A \arrow[r, "T^{n-i}\mu"] \arrow[d, "T^{n-j+1}\eta"'] & T^n A \arrow[d, "T^{n-j+1}\eta"] \\ T^{n+2} A \arrow[r, "T^{n-i+1}\mu"] & T^{n+1} A \end{tikzcd}$$

$$(0 \le i < j \le n)$$

where again $T^n \mu$ needs to be replaced by $T^n e$ for $i = 0$. Moreover, the squares

$$\begin{tikzcd} T^{n+1}A \arrow[r, "T^{n-i}\mu"] \arrow[d, "T^{n-j+1}\eta"'] & T^nA \arrow[d, "T^{n-j}\eta"] \\ T^{n+2}A \arrow[r, "T^{n-i}\mu"] & T^{n+1}A \end{tikzcd}$$

$$(0 \leq j < i \leq n)$$

which correspond to T^{n-i} applied to a naturality square of μ, are weak pullbacks already for any BC monad. Therefore, we have the following:

Theorem 7.2.1. *For any algebra A of a weakly cartesian monad T, the bar construction $\mathrm{Bar}_T(A)$ is an inner complete simplicial set.*

Since weakly cartesian monads include those arising from symmetric operads (Example 2.1.7), this result applies to many monads describing commonly occurring algebraic structures. It applies in particular to the commutative monoid monad, for which we have studied in particular the bar construction $\mathrm{Bar}_T(\mathbb{N})$ in some detail in Section 4 and found counterexamples to various hypotheses about its compositional structure, such as the non-uniqueness of composites. We now have a positive result about its compositional structure, and many other bar constructions of a similar flavor, namely that of being an inner complete simplicial set. Since this implies in particular inner span completeness, we can also apply Theorem 6.2.10 here, and conclude that fillers for acyclic configurations exist in $\mathrm{Bar}_T(\mathbb{N})$, and more generally in all bar constructions of weakly cartesian monads.

7.3. Inner exact simplicial sets. Just as with inner span complete simplicial sets, we consider the variant of inner complete simplicial sets in which all of the relevant pullback squares are strong.

Definition 7.3.1. *A simplicial set X is* inner exact *if it sends all concrete pushouts in Δ to strong pullbacks in Set.*

As an inner exact simplicial set is inner span exact, these are also nerves of categories by Theorem 6.4.4. The converse follows from the following characterization.

Lemma 7.3.2. *A simplicial set $X : \Delta^{\mathrm{op}} \to \mathsf{Set}$ is inner exact if and only if it sends pushouts of the form*

$$\begin{tikzcd} \overline{0} \arrow[r, "p"] \arrow[d, "0"'] & \overline{p} \arrow[d] \\ \overline{q} \arrow[r] & \overline{p+q} \end{tikzcd}$$

to pullbacks.

Proof. This statement, like Lemma 6.4.3, follows from Proposition 5.3.11, Proposition 5.4.2, and Lemma 5.4.1, which shows that honest basic pushout squares are $\star$-products of trivial pushout squares and pushout squares preserved by every functor out of Δ. □

Theorem 7.3.3. *A simplicial set is inner exact if and only if it is the nerve of a category.*

In particular, inner exact simplicial sets also coincide with inner span exact simplicial sets. Similar to the inner span case, we can therefore also argue that inner complete simplicial sets are compositional structures which meaningfully generalize categories.

8. Pushout completeness and the bar construction for strictly positive monads

In this section, we introduce and characterize *pushout complete simplicial sets*, along with the exact variant, and investigate which monads give rise to these via their bar construction.

8.1. Pushout complete simplicial sets.

While the definitions of inner (span) complete simplicial sets from the previous two sections have considered the weak preservation of certain pushouts, we now consider *all* pushouts in Δ, as a further natural strengthening of the conditions encountered in the previous two sections.

Definition 8.1.1. *A simplicial set $X : \Delta^{\mathrm{op}} \to \mathsf{Set}$ is* pushout complete *if it maps all pushouts in Δ to weak pullbacks in Set.*

In other words, a simplicial set is pushout complete if and only if it is *weakly cartesian* as a functor $\Delta^{\mathrm{op}} \to \mathsf{Set}$. Together with Proposition 5.3.11, the description of basic pushouts results in the following more explicit necessary and sufficient conditions for a simplicial set to be weakly cartesian.

Theorem 8.1.2. *Let $X : \Delta^{op} \to \mathsf{Set}$ be a simplicial set. Then X is pushout complete if and only if all squares of the form*

$$\begin{tikzcd} X_n \arrow[r, "d_i"] \arrow[d, "d_j"'] & X_{n-1} \arrow[d, "d_{j-1}"] \\ X_{n-1} \arrow[r, "d_i"] & X_{n-2} \end{tikzcd} \tag{8.1}$$

$$(0 \leq i < j - 1 \leq n - 1)$$

are weak pullbacks and all squares of the form

$$
\begin{tikzcd}
X_n \arrow[r, "d_i"] \arrow[d, "s_j"'] & X_{n-1} \arrow[d, "s_{j-1}"] \\
X_{n+1} \arrow[r, "d_i"] & X_n
\end{tikzcd}
\qquad
\begin{tikzcd}
X_n \arrow[r, equal] \arrow[d, "s_{i+1}s_i"'] & X_n \arrow[d, "s_i"] \\
X_{n+2} \arrow[r, "d_{i+1}"] & X_{n+1}
\end{tikzcd}
\qquad
\begin{tikzcd}
X_n \arrow[r, "d_i"] \arrow[d, "s_j"'] & X_{n-1} \arrow[d, "s_j"] \\
X_{n+1} \arrow[r, "d_{i+1}"] & X_n
\end{tikzcd}
\tag{8.2}
$$

$$(0 \leq i < j \leq n) \qquad (0 \leq i \leq n) \qquad (0 \leq j < i \leq n)$$

are (automatically strong) pullbacks.

Note that this coincides with the characterization of inner completeness given in Definition 7.1.1, apart from the (weak) pullback square in the middle. In particular, every pushout complete simplicial set is trivially inner complete.

Example 8.1.3. The nerve of a category $\mathbb{C}$ is an inner complete simplicial set. As we will show more carefully in Section 8.3, it is pushout complete if and only if the middle squares in (8.2) are pullbacks as well. The 1-Segal condition implies that it is enough to check this for $n = 0$. Thus the nerve of a category is pushout complete if and only if every identity morphism is irreducible, i.e. cannot be factored nontrivially.

This is reminiscent of Example 2.1.10, where we had characterized the strict positivity of a monoid action monad in terms of the irreducibility of the unit element of the monoid.

8.2. The bar construction for strictly positive monads. Recall from Definition 2.1.8 that a monad (T, μ, η) is *strictly positive* if for all X the square

$$
\begin{tikzcd}
X \arrow[r, equal] \arrow[d, "\eta\eta"'] & X \arrow[d, "\eta"] \\
T^2X \arrow[r, "\mu"] & TX
\end{tikzcd}
\tag{8.3}
$$

is a (necessarily strong) pullback.

If X is the bar construction of a T-algebra A, the square below left corresponds to the square below right.

$$
\begin{tikzcd}
X_n \arrow[r, equal] \arrow[d, "s_{i+1}s_i"'] & X_n \arrow[d, "s_i"] \\
X_{n+2} \arrow[r, "d_{i+1}"] & X_{n+1}
\end{tikzcd}
\qquad
\begin{tikzcd}
T^{n+1}A \arrow[r, equal] \arrow[d, "T^{n-i+1}(\eta\eta)"'] & T^{n+1}A \arrow[d, "T^{n-i+1}\eta"] \\
T^{n+3}A \arrow[r, "T^{n-i+1}\mu"] & T^{n+2}A
\end{tikzcd}
$$

$$(0 \leq i \leq n) \qquad (0 \leq i \leq n)$$

These squares are all obtained by repeated application of T to the strict positivity square above, and are therefore all pullbacks when T is weakly cartesian and strictly positive. This along with Proposition 5.3.11 and Theorem 8.1.2 suffices to prove the following.

Proposition 8.2.1. *The bar constructions for algebras of a strictly positive weakly cartesian monad are pushout complete simplicial sets.*

Example 8.2.2. For M a monoid, the M-set monad $M \times -$ is cartesian. The bar construction of an M-set A is therefore the nerve of a category, which we had described in Example 4.1.2. Equivalently, $\mathrm{Bar}_{M\times -}(A)$ is an inner exact simplicial set. If the unit of M is irreducible, then the monad $M \times -$ is strictly positive (Example 2.1.10), and therefore $\mathrm{Bar}_{M\times -}(A)$ is pushout complete in this case.

By Examples 2.1.11 and 2.1.12, also all semigroups and commutative semigroups have pushout complete bar constructions, when considered as algebras of the semigroup or commutative semigroup monad.

8.3. Pushout exact simplicial sets. While pushout complete simplicial sets send pushouts in Δ to weak pullbacks, *pushout exact* simplicial sets send pushouts in Δ to strong pullbacks in Set. As all of the pullbacks involving degeneracy maps are automatically strong, this is equivalent to a pushout complete simplicial set being inner (span) exact.

Lemma 8.3.1. *A simplicial set $X : \Delta^{\mathrm{op}} \to \mathsf{Set}$ is pushout exact if and only if it sends the pushouts*

$$\begin{tikzcd} \overline{0} \arrow[r, "p"] \arrow[d, "0"'] & \overline{p} \arrow[d] \\ \overline{q} \arrow[r] & \overline{p+q} \end{tikzcd} \qquad\qquad \begin{tikzcd} \overline{1} \arrow[r, "02"] \arrow[d] & \overline{2} \arrow[d] \\ \overline{0} \arrow[r, equal] & \overline{0} \end{tikzcd} \tag{8.4}$$

to pullbacks.

We call the right square the *positivity square*.

Proof. The "only if" part is obvious since both squares are pushouts in Δ, so we focus on the "if". By Proposition 5.4.2, Lemma 5.4.1, and Proposition 5.3.11, it suffices to show that X sends the above squares and the first square in Lemma 5.4.1 to pullbacks. However, that square is a split pushout and therefore automatically preserved, so that just these squares are sufficient. $\square$

Theorem 8.3.2. *A simplicial set is pushout exact if and only if it is the nerve of a category in which no two nontrivial morphisms compose to an identity.*

PROOF. Let $X : \Delta^{\mathrm{op}} \to \mathsf{Set}$ be pushout exact. Then X is the nerve of a category by Theorem 6.4.4. The second pullback square in Lemma 8.3.1 states that any 2-simplex in X with composite edge degenerate is itself doubly degenerate. As X is the nerve of a category, such a 2-simplex describes a pair of composable morphisms whose composite is an identity arrow, and such a simplex being doubly degenerate means exactly that both of those morphisms must then also be identities. $\square$

As any nerve of a category is inner span exact, a nerve is pushout complete precisely when it is pushout exact.

9. Span completeness and invertibility properties

Lastly, we discuss one remaining way in which simplicial sets may preserve weak pullbacks. Intuitively speaking, these add weak left and right inverses to the compositional properties of inner span complete simplicial sets, namely by requiring some additional squares which are not pushouts in Δ to be mapped to (weak) pullbacks in Set.

9.1. Span complete simplicial sets. The purpose of this section is to strengthen the definition of inner span complete simplicial sets to also include simplex fillers for non-inner spans, by demanding that a simplicial set X send to weak pullbacks a broader class of commutative squares of face maps in Δ than merely the pushout squares.

The results of Propositions 5.5.6 and 5.5.7 exhibit the perils and wastes of asking for this class to include all commutative squares of face maps in Δ, which by Corollaries 5.5.8 and 5.5.9 would have even more destructive implications later on when these span fillers will be assumed unique. A safe choice then is to exclude these redundant or destructive squares and restrict our consideration to balanced squares of face maps in Δ, which will account for all of the spans which, neglecting the order on the faces of a simplex, resemble geometrically the inner spans considered in the previous sections.

Definition 9.1.1. *A simplicial set X is span complete if it sends all balanced squares of face maps in Δ to weak pullbacks.*

We then have the following characterization.

Proposition 9.1.2. *A simplicial set $X : \Delta^{\mathrm{op}} \to \mathsf{Set}$ is span complete if and only if all of the squares describing the simplicial relations between face maps are weak pullbacks. These comprise the "inner" squares below left and the "outer" squares below right.*

$$\begin{tikzcd} X_n \arrow[r, "d_i"] \arrow[d, "d_j"'] & X_{n-1} \arrow[d, "d_{j-1}"] \\ X_{n-1} \arrow[r, "d_i"] & X_{n-2} \end{tikzcd} \qquad \begin{tikzcd} X_n \arrow[r, "d_i"] \arrow[d, "d_{i+1}"'] & X_{n-1} \arrow[d, "d_i"] \\ X_{n-1} \arrow[r, "d_i"] & X_{n-2} \end{tikzcd}$$

$$(0 \leq i < j-1 \leq n-1) \qquad\qquad (0 \leq i < n)$$

Proof. The "only if" direction follows as these squares are the images under X of balanced squares in Δ. The "if" direction follows from Proposition 5.5.3 and composition of pullbacks. ☐

Here, the distinction of inner and outer squares is useful since the inner ones are weak pullbacks already for inner span complete simplicial sets (Section 6). In the present context, the description of span completeness in terms of lifting properties amounts to the corresponding inner span and "outer span" fillers

$$\begin{tikzcd} \Delta^{n-1} \,{}_{d^i}\sqcup_{d^{j-1}} \Delta^{n-1} \arrow[r] \arrow[d, hook, "{d^j, d^i}"'] & X \\ \Delta^n \arrow[ur, dashed] \end{tikzcd} \qquad \begin{tikzcd} \Delta^{n-1} \,{}_{d^i}\sqcup_{d^i} \Delta^{n-1} \arrow[r] \arrow[d, hook, "{d^{i+1}, d^i}"'] & X \\ \Delta^n \arrow[ur, dashed] \end{tikzcd}$$

The distinction between inner and outer span fillers is similar to that between inner and outer horn fillers. In line with this analogy, just as quasicategories are inner span complete (Proposition 6.1.4), Kan complexes are span complete.

Proposition 9.1.3. *All Kan complexes are span complete.*

Proof. It suffices to show that the span inclusions $\Delta^{n-1} \sqcup_{\Delta^{n-2}} \Delta^{n-1} \hookrightarrow \Delta^n$ are trivial cofibrations in the Quillen model structure on simplicial sets, which follows immediately as both the domain and codomain geometrically realize to contractible spaces, so the realization of the inclusion map is a weak equivalence. ☐

The converse does not hold, however, by the following counterexample.

Example 9.1.4. Span complete simplicial sets are closely related to the examples of *compositories* given in [**FF16**], and they can be used to construct examples of span complete simplicial sets which have inner and outer horns without fillers. Recall that a compository is defined to be a simplicial set X for which inner spans of the type

$$\begin{tikzcd} \Delta^p \,{}_{r'}\sqcup_{\ell'} \Delta^q \arrow[r] \arrow[d, hook, "{\ell, r}"'] & X \\ \Delta^n \arrow[ur, dashed] \end{tikzcd} \tag{9.1}$$

have specified fillers, and these fillers satisfy certain coherences in the form of associativity and partial naturality properties. Here, $\ell : \Delta^p \to \Delta^n$ is the left inclusion $d^n \cdots d^{p+1}$ and $r : \Delta^q \to \Delta^n$ is the right inclusion $d^0 \cdots d^0$, and similarly ℓ' and r' are the left and right inclusions of the intersection simplex Δ^{p+q-n}.

However, most of the known and interesting examples of compositories are actually (augmented) *symmetric* simplicial sets in a natural way, or equivalently *gleaves* on FinSet [**FF16**, Section 5]. And every gleaf on FinSet is in particular a span complete simplicial set: the filler condition above clearly holds for $i = 0$ and $j = n$ as part of the algebraic structure carried by a gleaf, and this is enough to prove the filler condition in general by symmetry.

To have a more concrete example, we now sketch the main example of [**FF16**] coming from probability theory. Fix a finite set S, representing the set of possible values of some random variables; the finiteness assumption on S is merely for simplicity. Then define an n-simplex to be a joint distribution of $(n+1)$ many S-valued random variables, i.e. a probability measure on the cartesian product $S^{\times(n+1)}$. Taking the pushforward of this measure along a projection to a subproduct is then what defines the faces of such a simplex, and similarly taking the pushforward along the diagonal $S \to S \times S$ is involved in the definition of the degeneracy maps, which intuitively amounts to duplicating the value of a variable. We thus obtain the desired simplicial set in which the n-simplices are the joint distributions of $(n+1)$ random variables. The fillers of (9.1) then acquire the following significance. Suppose that we are given a set of $n+1$ random variables and write it as the union of subsets containing $p+1$ and $q+1$ random variables, respectively. Then if we are given a joint distribution of the $p+1$ variables and a joint distribution of the $q+1$ variables, and if these two agree when marginalized to the subsubset of variables contained in both subsets, then there is joint distribution for all $n+1$ variables which marginalizes to the given ones. And indeed there is a distinguished choice for this overall joint distribution: we can make all variables contained in the first subset but not the second to be conditionally independent of those in the second set but not the first, conditionally with respect to the variables contained in both, which is exactly the conditional product that we used in Example 2.1.5.

This simplicial set does not have fillers for (inner or outer) 3-horns as soon as $|S| \geq 2$. For then, we can consider without loss of generality $S = \{0, 1\}$, and four random variables A, B, C, D where $D = 0$ with probability 1, and the other three such that they take either value with probability 1/2, but are correlated such that they take *opposite* values with probability 1. This defines joint distributions of ABD, ACD and BCD, thereby forming a 3-horn in the corresponding symmetric simplicial set. Similar to the examples in the proof of Theorem 4.4.1, this 3-horn is such that already its missing 2-face cannot be filled: there is no joint distribution of ABC which would make all three variables take opposite values with probability 1, since there is not even a single assignment of values in S which would assign opposite values to every pair.

We also refer to [**Fri20**, Section 12], where a more general construction of gleaves of this type has been proposed. This in particular applies in the context of measure-theoretic probability.

Analogous to inner span complete simplicial sets, span completeness provides fillers for any acyclic subcomplex of a simplex, now in the undirected sense of Theorem 6.2.3, and in particular without the spine condition.

Theorem 9.1.5. *A simplicial set is span complete if and only if it has fillers for all acyclic configurations in the n-simplex (for every n).*

Proof. The "if" direction follows from the acyclicity of spans, which is the undirected analogue of Example 6.2.9. The "only if" direction follows by a completely analogous argument to Theorem 6.2.10 in the undirected setting, based on the observation that every pullback of face maps in Δ is a balanced square and therefore sent to a weak pullback by the span complete simplicial set under consideration. The connectedness is relevant for ensuring that $T_k \cap \bigcup_{i=1}^{k-1} T_i$ is nonempty for every k, which indeed holds by the final statement in Theorem 6.2.3. □

9.2. Bar constructions of left reversible monads and right reversible algebras. Just as with inner span complete, inner complete, and pushout complete simplicial sets, we can describe conditions on monads T and algebras A which ensure that $\mathrm{Bar}_T(A)$ is span complete, involving in particular the reversibility properties of Definitions 2.1.14 and 2.2.4.

Theorem 9.2.1. *For any right reversible algebra A of a left reversible BC monad T, the bar construction $\mathrm{Bar}_T(A)$ is span complete.*

Note that for BC monads, right reversibility is equivalent to indiscreteness by Proposition 2.2.5 and weak pullback preservation, so we could just as well assume that A is indiscrete.

Proof. Let $X = \mathrm{Bar}_T(A)$. By Proposition 9.1.2, it suffices to show that the following squares are weak pullbacks:

$$\begin{array}{ccc} X_n & \xrightarrow{d_i} & X_{n-1} \\ {\scriptstyle d_j}\downarrow & & \downarrow{\scriptstyle d_{j-1}} \\ X_{n-1} & \xrightarrow{d_i} & X_{n-2} \end{array} \qquad\qquad \begin{array}{ccc} X_n & \xrightarrow{d_i} & X_{n-1} \\ {\scriptstyle d_{i+1}}\downarrow & & \downarrow{\scriptstyle d_i} \\ X_{n-1} & \xrightarrow{d_i} & X_{n-2} \end{array}$$

$$(0 \le i < j-1 \le n-1) \qquad\qquad (0 \le i < n)$$

Since T is BC, by Theorem 6.3.1 the left squares above are indeed weak pullbacks, so only the right squares remain. These squares are either of the form below left for $n \geq 2$ and $0 < i < n$, or of the form below right for $n \geq 2$ and $i = 0$.

$$\begin{tikzcd} T^{n+1}A \arrow[r, "T^{n-i}\mu"] \arrow[d, "T^{n-i-1}\mu"'] & T^n A \arrow[d, "T^{n-i-1}\mu"] \\ T^n A \arrow[r, "T^{n-i-1}\mu"] & T^{n-1}A \end{tikzcd} \qquad \begin{tikzcd} T^{n+1}A \arrow[r, "T^n e"] \arrow[d, "T^{n-1}\mu"'] & T^n A \arrow[d, "T^{n-1}e"] \\ T^n A \arrow[r, "T^{n-1}e"] & T^{n-1}A \end{tikzcd} \tag{9.2}$$

By right reversibility and weak pullback preservation, the squares above right are weak pullbacks, and by left reversibility and weak pullback preservation, the squares above left are weak pullbacks. □

9.3. Span exact simplicial sets. Just as inner span exact simplicial sets are precisely the nerves of categories, the analogous span exact simplicial sets coincide with the nerves of groupoids.

Definition 9.3.1. *A simplicial set X is span exact if it sends all balanced squares of face maps in Δ to strong pullbacks.*

As with all of our exactness conditions, Proposition 5.4.2 reduces the criteria for checking span exactness.

Lemma 9.3.2. *A simplicial set $X : \Delta^{\mathrm{op}} \to \mathsf{Set}$ is span exact if and only if it sends the squares*

$$\begin{tikzcd} \overline{0} \arrow[r, "p"] \arrow[d, "0"'] & \overline{p} \arrow[d] \\ \overline{q} \arrow[r] & \overline{p+q} \end{tikzcd} \qquad \begin{tikzcd} \overline{0} \arrow[r, "d^0"] \arrow[d, "d^0"'] & \overline{1} \arrow[d, "d^0"] \\ \overline{1} \arrow[r, "d^1"] & \overline{2} \end{tikzcd} \qquad \begin{tikzcd} \overline{0} \arrow[r, "d^1"] \arrow[d, "d^1"'] & \overline{1} \arrow[d, "d^1"] \\ \overline{1} \arrow[r, "d^2"] & \overline{2} \end{tikzcd} \tag{9.3}$$

to pullbacks.

Proof. By Proposition 5.5.3, any balanced square of face maps can be factored into simplicial identity squares and trivial squares. Therefore it suffices to show that if X sends the squares above to pullbacks, then it sends these simplicial identity squares to pullbacks. By Proposition 5.4.2, since X sends the squares above left to pullbacks, it further suffices to check this only for the $\star$-components of simplicial identity squares. These are all either trivial, one of the squares above center or right, or the following:

$$\begin{tikzcd} \overline{1} \arrow[r, "d^1"] \arrow[d, "d^1"'] & \overline{2} \arrow[d, "d^2"] \\ \overline{2} \arrow[r, "d^1"] & \overline{3} \end{tikzcd} \tag{9.4}$$

To see that under the given assumptions X sends this square to a pullback, first observe that the following two composite squares are the same:

$$\begin{tikzcd} \overline{0} \arrow[r, "d^0"] \arrow[d, "d^0"'] & \overline{1} \arrow[r, "d^1"] \arrow[d, "d^1"'] & \overline{2} \arrow[d, "d^2"'] \\ \overline{1} \arrow[r, "d^0"] & \overline{2} \arrow[r, "d^1"] & \overline{3} \end{tikzcd} \qquad \begin{tikzcd} \overline{0} \arrow[r, "d^0"] \arrow[d, "d^0"'] & \overline{1} \arrow[r, "d^0"] \arrow[d, "d^1"'] & \overline{2} \arrow[d, "d^2"'] \\ \overline{1} \arrow[r, "d^0"] & \overline{2} \arrow[r, "d^0"] & \overline{3} \end{tikzcd} \tag{9.5}$$

The right square in the right composite is the $\star$-product of the following squares:

$$\begin{tikzcd} \overline{1} \arrow[r, equal] \arrow[d, "d^1"'] & \overline{1} \arrow[d, "d^1"'] \\ \overline{2} \arrow[r, equal] & \overline{2} \end{tikzcd} \qquad \begin{tikzcd} \overline{0} \arrow[r, "d^0"] \arrow[d, equal] & \overline{1} \arrow[d, equal] \\ \overline{0} \arrow[r, "d^0"] & \overline{1} \end{tikzcd} \tag{9.6}$$

X sends both of these trivial squares to pullbacks, so by Proposition 5.4.2 it also sends their $\star$-product to a pullback. As the left square in both composites is the center square in the proposition, X sends the composite square to a pullback. Therefore, in the following composite of squares in Set, both the outer rectangle and right square are pullbacks, hence by the pullback lemma so is the left square.

$$\begin{tikzcd} X_3 \arrow[r, "d_1"] \arrow[d, "d_2"'] & X_2 \arrow[r, "d_0"] \arrow[d, "d_1"'] & X_1 \arrow[d, "d_0"'] \\ X_2 \arrow[r, "d_1"] & X_1 \arrow[r, "d_0"] & X_0 \end{tikzcd} \tag{9.7}$$

□

Next, we show that in the nerve a category, the three squares above being sent to pullbacks corresponds to, respectively, composition, left inverses, and right inverses, making the category into a groupoid. The additional square (9.4) then corresponds to the fact that in a groupoid, any commutative square has a unique diagonal filler. In fact, this is a complete description of span exact simplicial sets:

Corollary 9.3.3. *A simplicial set is span exact if and only if it is the nerve of a groupoid.*

Proof. For the "only if" direction, by Lemma 9.3.2 and Theorem 6.4.4, a span exact simplicial set X is the nerve of a category, which we call $\mathbb{C}$. By Lemma 9.3.2, the following squares are pullbacks:

$$\begin{tikzcd} X_2 \arrow[r, "d_1"] \arrow[d, "d_0"'] & X_1 \arrow[d, "d_0"'] \\ X_1 \arrow[r, "d_0"] & X_0 \end{tikzcd} \qquad \begin{tikzcd} X_2 \arrow[r, "d_2"] \arrow[d, "d_1"'] & X_1 \arrow[d, "d_1"'] \\ X_1 \arrow[r, "d_1"] & X_0 \end{tikzcd} \tag{9.8}$$

For any morphism $f : x \to y$ in $\mathbb{C}$, the left square above fills in the span (f, id_y) to a 2-simplex given by a composable pair $y \xrightarrow{g} x \xrightarrow{f} y$ with composite id_y, so f has a right inverse. Likewise the right square above fills in the span (id_x, f) to a 2-simplex given by a composable pair $x \xrightarrow{f} y \xrightarrow{h} x$ with composite id_x, so f also has a left inverse. Left and right inverses in a category must agree, so f is an isomorphism. Since f was arbitrary, we conclude that $\mathbb{C}$ is a groupoid.

For the "if" direction, let $\mathbb{C}$ be a groupoid. Then by Theorem 6.4.4, its nerve X is inner span exact. By Lemma 9.3.2, to show X is span exact it suffices to check that the above two squares are pullbacks. Indeed, for any pair of maps $y \xrightarrow{f} z \xleftarrow{g} x$ in $\mathbb{C}$ the sequence $x \xrightarrow{f^{-1}g} y \xrightarrow{f} z$ has composite g, and is unique among such sequences with second map f and composite g, so the left square above is a pullback. Similarly any pair of maps $y \xleftarrow{f} x \xrightarrow{g} z$ can be uniquely filled into a composition triangle $x \xrightarrow{f} y \xrightarrow{gf^{-1}} z$ with composite g, so also the right square above is a pullback. □

9.4. Fully complete simplicial sets. To complete the classification of pullback properties in simplicial sets, we discuss the case of a simplicial set with all of the completeness properties discussed so far.

Definition 9.4.1. *A simplicial set is* fully complete *if it is both pushout complete and span complete.*

As we will see in Proposition 9.4.3, every fully complete simplicial set is discrete; thus the point of this subsection is mainly to illustrate that the properties of being pushout complete and being span complete are orthogonal in this sense. This generalizes that a category which is both a groupoid and such that every identity morphism is irreducible has only identity morphisms.

Lemma 9.4.2. *The following squares are (automatically strong) pullbacks in a fully complete simplicial set X.*

$$\begin{tikzcd} X_n \arrow[r, equal] \arrow[d, "s_i"'] & X_n \arrow[d, equal] \\ X_{n+1} \arrow[r, "d_i"] & X_n \end{tikzcd} \qquad\qquad \begin{tikzcd} X_n \arrow[r, equal] \arrow[d, "s_i"'] & X_n \arrow[d, equal] \\ X_{n+1} \arrow[r, "d_{i+1}"] & X_n \end{tikzcd}$$

$$(0 \leq i \leq n) \qquad\qquad\qquad (0 \leq i < n)$$

Proof. The squares can be factored as below, where the top squares are pullbacks since X is pushout complete and the bottom squares are pullbacks since X is span complete.

$$\begin{tikzcd} X_n \arrow[r, equal] \arrow[d, "s_i s_i"'] & X_n \arrow[d, "s_i"] \\ X_{n+2} \arrow[r, "d_{i+1}"] \arrow[d, "d_i"'] & X_{n+1} \arrow[d, "d_i"] \\ X_{n+1} \arrow[r, "d_i"] & X_n \end{tikzcd} \qquad \begin{tikzcd} X_n \arrow[r, equal] \arrow[d, "s_{i+1}s_i = s_i s_i"'] & X_n \arrow[d, "s_i"] \\ X_{n+2} \arrow[r, "d_{i+1}"] \arrow[d, "d_{i+2}"'] & X_{n+1} \arrow[d, "d_{i+1}"] \\ X_{n+1} \arrow[r, "d_{i+1}"] & X_n \end{tikzcd}$$

$$(0 \leq i \leq n) \qquad\qquad (0 \leq i < n)$$

□

Proposition 9.4.3. *A simplicial set X is fully complete if and only if it is discrete.*

PROOF. This follows immediately from the above lemma, since the squares in there can be pullbacks only if every generating degeneracy map s_i is an isomorphism. □

For example when x is a 1-simplex, then outer span filling equips x with an "inverse" x' together with a 2-simplex from x and x' to the identity in the form of the degenerate 02-edge. By pushout completeness this 2-simplex must be degenerate from a point. Therefore x too is degenerate.

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