Rudin Real and Complex Analysis Chapter 3

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Problem 11

To begin take the lebesgue integral of $fg \geq 1$, $\int_\Omega fg d\mu \geq \int_\Omega d\mu$.

$$\int_\Omega fg d\mu \geq \int_\Omega d\mu$$

Looking at the right side of the inequality,

$$\begin{aligned}
\int_\Omega d\mu &= \sup \int_\Omega s d\mu \\
&= \sup \sum_{i=1}^{n} \alpha_i \mu(A_i \cap \Omega) \\
&= \mu(\Omega) = 1
\end{aligned}$$

In the above $\alpha_i = 1$ and A_i is all values from Ω that give the same α_i. From this since $\int_\Omega f(x) d\mu = \int_\Omega d\mu \ f(x) = 1$ which is just a horizontal line thus $\mu(A_i \cap \Omega) = \mu(\Omega)$. Also note that since $f(x)$ only produces one value there is only one α_i. From above we have shown that clearly $\int_\Omega fg d\mu \geq 1$. Next since it is given that f and g are positive measurable functions we can utilize hoelders inequality. First we begin with $p = 1$ and $q = \infty$ and then with $p = \infty$ and $q = 1$.

$$\begin{aligned}
1 &\leq \int_\Omega fg d\mu \\
&\leq \Big\{ \int_\Omega f^p d\mu \Big\}^{\frac{1}{p}} \Big\{ \int_\Omega g^q d\mu \Big\}^{\frac{1}{q}} \\
&\leq \int_\Omega f d\mu
\end{aligned}$$

$$
\begin{aligned}
1 &\leq \int_{\Omega} fg d\mu \\
&\leq \Big\{ \int_{\Omega} f^p d\mu \Big\}^{\frac{1}{p}} \Big\{ \int_{\Omega} g^q d\mu \Big\}^{\frac{1}{q}} \\
&\leq \int_{\Omega} g d\mu
\end{aligned}
$$

Since $\int_{\Omega} g d\mu \geq 1$ and $\int_{\Omega} f d\mu \geq 1$ it is clear that $\int_{\Omega} f d\mu \cdot \int_{\Omega} g d\mu \geq 1$.

Problem 24

(a)

Break this up into 2 cases. The first case is $f < g$

$$\int ||f|^p - |g|^p| d\mu \leq$$

Problem 25

$\int_E (\log(f)) d\mu \leq \mu(E) \log(\frac{1}{\mu(E)})$

To begin note $\int_E f d\mu \leq \int_X f d\mu = 1$ in the case $E \subset X$, this comes directly from definition and more so from the fact that μ and f are both positive functions. To see this I am looking at definition 1.23,

$$\int_X f d\mu = \sup \sum_{i=1}^{n} \alpha_i \mu(A_i \cap X)$$

Where $A_i = \{x : f(x) = \alpha_i\}$ so clearly $\alpha_i = f(x)$. Now if we consider any subset $E \subset X$ coupled with the fact that f and μ are positive as stated above and in the origninal question it is obvious that the sum produced from $\int_X f d\mu$ would be bigger. Next consider the right side of the inequality, we can rewrite this,

$$
\begin{aligned}
\mu(E) \log\left(\frac{1}{\mu(E)}\right) &= \mu(E)(\log(1) - \log(\mu(E))) \\
&= -\mu(E) \log(\mu(E))
\end{aligned}
$$

But what can we say about $\mu(E)$?

$$\begin{aligned}\mu(E) &\leq \mu(X)\\ &= \mu(\bigcup_{i=1}^{n} E_i)\end{aligned}$$

So arbitrarily I am just breaking X into disjoint sets of size E so one of the E_i's should actually be E and by using the definition of a measure this holds. But,

$$\begin{aligned}\mu(X) &= \mu(E_1 \cap X) + \mu(E_2 \cap X) + \cdots + \mu(E_n \cap X)\\ &< \alpha_1\mu(E_1 \cap X) + \alpha_2\mu(E_2 \cap X) + \cdots + \alpha_n\mu(E_n \cap X)\\ &\leq \int_X f d\mu = 1\end{aligned}$$

This implies $0 < \mu(E) < 1$, since $\alpha_i > 0$. Utilizing what we've just shown it is clear that $\log(\mu(E)) < 0$ since $\log(x) < 0$ for $0 < x < 1$ and further more we can infer that $-\mu(E)\log(\mu(E)) > 0$. Keeping all this in mind we can take our inequality and apply Jensen's inequality (theorem 3.3) and show,

$$\begin{aligned}\int_E (\log(f))d\mu &\leq \mu(E)\log(\frac{1}{\mu(E)})\\ e^{\int_E (\log(f))d\mu} &\leq e^{\mu(E)\log(\frac{1}{\mu(E)})}\\ \int_E e^{(\log(f))d\mu} &\leq e^{\mu(E)\log(\frac{1}{\mu(E)})}\\ \int_E f d\mu &\leq e^{\mu(E)\log(\frac{1}{\mu(E)})}\end{aligned}$$

To finish this by definition $e^x > 1$ for $x > 0$ and what I mentioned in the first sentence all the above inequalities hold.

Problem 26

To begin let's look at $f(x)$. Since it is given that $f(x)$ is a positive measurable function its image is a subset or equal to $[0, \infty]$. Next we need to consider

$\log(f(x))$ in particular this function is negative when $0 < f(x) < 1$. To approach this problem I am considering two scenarios, first the case when $0 < f(x) < 1$ and second the case when $f(x) > 1$. Note that when $f(x) = 0$ or $f(x) = 1$ then both integrals are equal (i.e either $-\infty$ or 0). So in the first case when $0 < f(x) < 1$ I will use hoelder's inequality first with $p = 1$ and $q = \infty$, then vice-versa.

$$\begin{aligned}
\int_0^1 f(x)\log(f(x))dx &\leq \int_0^1 f(x)(-\log(f(x)))dx \\
&\leq \left\{\int_0^1 f(x)^p dx\right\}^{\frac{1}{p}} \left\{\int_0^1 (-\log(f(x)))^q dx\right\}^{\frac{1}{q}} \\
&\leq \left\{\int_0^1 f(x)^1 dx\right\}^{\frac{1}{1}} \lim_{q\to\infty} \left\{\int_0^1 (-\log(f(x)))^q dx\right\}^{\frac{1}{q}} \\
&\leq \int_0^1 f(x)dx
\end{aligned}$$

In the first inequality I write $-\log(f(x))$ to obtain a function that has a positive image so I can use Hoelder's inequality. I obtain my final inequality by noting that $\lim_{q\to\infty} \frac{1}{q} = 0$. Now with $p = \infty$ and $q = 1$,

$$\begin{aligned}
\int_0^1 f(x)\log(f(x))dx &\leq \int_0^1 f(x)(-\log(f(x)))dx \\
&\leq \left\{\int_0^1 f(x)^p dx\right\}^{\frac{1}{p}} \left\{\int_0^1 (-\log(f(x)))^q dx\right\}^{\frac{1}{q}} \\
&\leq \lim_{p\to\infty} \left\{\int_0^1 f(x)^p dx\right\}^{\frac{1}{p}} \left\{\int_0^1 (-\log(f(x)))^1 dx\right\}^{\frac{1}{1}} \\
&\leq \int_0^1 -\log(f(x))dx
\end{aligned}$$

From this we get the inequalitiess,

$$\begin{aligned}
\int_0^1 f(x)(-\log(f(x)))dx &\le \int_0^1 f(x)dx \int_0^1 -\log(f(x))dx \\
-\int_0^1 f(x)\log(f(x))dx &\le -\int_0^1 f(x)dx \int_0^1 \log(f(x))dx \\
\int_0^1 f(x)\log(f(x))dx &\ge \int_0^1 f(x)dx \int_0^1 \log(f(x))dx
\end{aligned}$$

In the case when $0 < f(x) < 1$. When dealing with $f(x) > 0$ we can use hoelder's inequaltiy without making adjustments, again starting with $p = 1$ and q approaching infinity,

$$\begin{aligned}
\int_0^1 f(x)\log(f(x))dx &\le \Big\{\int_0^1 f(x)^p dx\Big\}^{\frac{1}{p}} \Big\{\int_0^1 \log(f(x))^q dx\Big\}^{\frac{1}{q}} \\
&\le \int_0^1 f(x)dx
\end{aligned}$$

Now with $p = \infty$, $q = 1$,

$$\begin{aligned}
\int_0^1 f(x)\log(f(x))dx &\le \Big\{\int_0^1 f(x)^p dx\Big\}^{\frac{1}{p}} \Big\{\int_0^1 \log(f(x))^q dx\Big\}^{\frac{1}{q}} \\
&\le \int_0^1 \log(f(x))dx
\end{aligned}$$

Of course we can change our term of integration from x to the respective t and s since they are only place holders which gives us our result, $\int_0^1 f(x)\log(f(x))dx \le \int_0^1 f(s)ds \int_0^1 \log(f(t))dt$.

Thoughts

- It doesn't account for a positive measurable function that outputs values both less than and greater than 1.