(b)

$$C^n(w) = -\sum_{k=1}^{K} y_k^n \ln(\hat{y}_k^n) \qquad \hat{y}_k^n = \frac{e^{z_k}}{\sum_{k'}^{K} e^{z_{k'}}} \qquad z_k = \sum_{i}^{I} w_{k,i} \cdot x_i$$

$$\frac{\partial C^n(w)}{\partial w_{kj}} = \frac{\partial C^n(w)}{\partial \hat{y}_k^n} \times \frac{\partial \hat{y}_k^n}{\partial z_k} \times \frac{\partial z_k}{\partial w_{kj}}$$

$\frac{\partial z_{k'}}{\partial w_{k'j}} = x_j$ as all other w in matrix treated as const.

$$\frac{\partial \hat{y}_k^n}{\partial z_{k'}} = \frac{\partial}{\partial z_{k'}} \frac{e^{z_k}}{\sum_{k'}^{K} e^{z_{k'}}}$$

$$\frac{\partial \hat{y}_k^n}{\partial z_{k'}} = \frac{\left(\sum e^{z_{k'}}\right)\frac{\partial}{\partial z_{k'}} e^{z_k} - e^{z_k} \frac{\partial}{\partial z_{k'}} \sum_{k'}^{K} e^{z_{k'}}}{\left(\sum e^{z_{k'}}\right)^2}$$

when $k = k'$,

$$\frac{\partial \hat{y}_k^n}{\partial z_{k'}} = \frac{\left(\sum e^{z_{k'}}\right) e^{z_{k'}} - e^{z_{k'}} \cdot e^{z_{k'}}}{\left(\sum e^{z_{k'}}\right)^2}$$

$$= \frac{e^{z_{k'}}}{\sum e^{z_{k'}}} - \left(\frac{e^{z_{k'}}}{\sum e^{z_{k'}}}\right)^2$$

$$= \hat{y}^n_{k'} - \left(\hat{y}^n_{k'}\right)^2 = \hat{y}^n_{k'}\left(1 - \hat{y}^n_{k'}\right)$$

when $k \neq k'$

$$\frac{\partial \hat{y}^n_k}{\partial z_{k'}} = \frac{\sum e^{z_{k'}} \times 0 - e^{z_k} \cdot e^{z_{k'}}}{\left[\sum_{k'}^{K} e^{z_{k'}}\right]^2}$$

$$= - \frac{e^{z_k}}{\sum_{k'}^{K} e^{z_{k'}}} \times \frac{e^{z_{k'}}}{\sum_{k'}^{K} e^{z_{k'}}}$$

$$= - \hat{y}^n_k \times \hat{y}^n_{k'}$$

$$\frac{\partial C^n(w)}{\partial \hat{y}^n_k} = - \sum_{k'=1}^{K} \frac{\partial}{\partial \hat{y}^n_{k'}} \left(y^n_{k'}\right) \ln\left(\hat{y}^n_{k'}\right)$$

$$= - \sum_{k'=1}^{K} \frac{y^n_{k'}}{\hat{y}^n_{k'}}$$

$$\frac{\partial C^n(w)}{\partial w_{kj}} = - \sum_{k'=1}^{K} \frac{y^n_{k'}}{\hat{y}^n_{k'}} \times \frac{\partial \hat{y}^n_k}{\partial z_{k'}} \times x_j$$

$$= - x_j \sum_{k'=1}^{K} \frac{y^n_{k'}}{\hat{y}^n_{k'}} \times \frac{\partial \hat{y}^n_k}{\partial z_{k'}}$$ Only 1 instance when $k = k'$, rest $k \neq k'$

$$= - x_j \left[\frac{y^n_{k'}}{\hat{y}^n_{k'}} \times \hat{y}^n_{k'}\left(1 - \hat{y}^n_{k'}\right) + \sum_{k'=1,\, k' \neq k}^{K} \frac{y^n_{k'}}{\hat{y}^n_{k'}} \left(- \hat{y}^n_k \times \hat{y}^n_{k'}\right) \right]$$

$$= -x_j\left[y_{k'}^n(1-\hat{y}_{k'}^n) + \sum_{k'=1, k'\neq k}^{K} (-y_{k'}^n \hat{y}_k^n) \right]$$

$$= x_j\left[y_{k'}^n \hat{y}_{k'}^n - y_{k'}^n + \sum_{k'=1, k'\neq k}^{K} \hat{y}_k^n \times y_{k'}^n \right]$$

$$= x_j\left[-y_{k'}^n + y_{k'}^n \hat{y}_k^n + \sum_{k'=1, k'\neq k}^{K} \hat{y}_k^n \times y_{k'}^n \right],$$ as in $y_{k'}^n \hat{y}_{k'}^n$, $k=k'$, $\therefore \hat{y}_{k'}^n = \hat{y}_k^n$,

$$= x_j\left[-y_{k'}^n + \hat{y}_k^n \left[y_{k'}^n + \sum_{k'=1, k'\neq k}^{K} y_{k'}^n \right]\right]$$

$$= x_j\left[-y_{k'}^n + \hat{y}_k^n \left(\sum_{k'=1}^{K} y_{k'}^n \right)\right]$$

$$= -x_j\left[y_k^n - \hat{y}_k^n (1) \right]$$ as $k=k'$ in that instance and $\sum_{k'=1}^{K} y_{k'} = 1$

$$= -x_j\left[y_k^n - \hat{y}_k^n \right] \quad \text{(Shown)}$$