Tema 2

1\. 1. Numărul total de posibilități la o aruncare a celor 2 zaruri este $6 \cdot 6 = 36$.

Fie $F_i \to$ suma zarurilor este 5 la aruncarea i;

$G_i \to$ — " — 7 — " —

Evenimentul din enunț se poate rescrie:

$$E_n = (F_1^c \cap G_1^c) \cap (F_2^c \cap G_2^c) \cap \dots \cap (F_{n-1}^c \cap G_{n-1}^c) \cap F_n$$

$$\Rightarrow P(E_n) = P\left(\bigcap_{i=1}^{n-1} F_i^c \cap G_i^c \quad \cap F_n\right) \overset{IND}{=} P(F_1^c \cap G_1^c) \cdot P(F_2^c \cap G_2^c) \dots \cdot P(F_{n-1}^c \cdot G_{n-1}^c) \cdot P(F_n)$$

$\Omega = \{(i,j) \mid i,j \in \{1,2,\dots,6\}\} \Rightarrow |\Omega| = 6^2$

Combinații care ne dau suma 5: $\{(1,4),(4,1),(2,3),(3,2)\}$

— " — 7: $\{(1,6),(6,1),(2,5),(5,2),(3,4),(4,3)\}$

$$\Rightarrow P(F_n) = \frac{4}{36} = \frac{1}{9}$$

$$P(F_i^c \cap G_i^c) = 1 - P(F_i \cup G_i) \overset{INC}{=} 1 - P(F_i) - P(G_i) = 1 - \frac{10}{36} = \frac{26}{36} = \frac{13}{18}$$

$$\Rightarrow P(\text{suma 5 să apară înaintea sumei 7}) = P\left(\bigcup_{n=1}^{\infty} E_n\right) \overset{IND}{=} \sum_{n=1}^{\infty} P(E_n) = \sum_{n=1}^{\infty} \left(\frac{26}{36}\right)^{n-1} \cdot \frac{4}{36}$$

$$= \frac{1}{9} \sum_{i=1}^{\infty} \left(\frac{13}{18}\right)^{i-1} = \frac{1}{9} \lim_{n\to\infty} \sum_{i=0}^{n} \left(\frac{13}{18}\right)^{i-1} = \frac{1}{9} \lim_{n\to\infty} 1 \cdot \frac{\left(\frac{13}{18}\right)^n - 1}{\frac{13}{18} - 1} = \frac{1}{9} \cdot \frac{-1}{\frac{13}{18} - 1} = \frac{1}{9} \cdot \frac{1}{1 - \frac{13}{18}}$$

$$= \frac{1}{\cancel{9}} \cdot \frac{\cancel{18}^{\,2}}{5} = \frac{2}{5}$$

2\. $H_i \to$ suma zarurilor este 2 la aruncarea i

$J_n \to$ primele $n-1$ aruncări au fiecare sumă diferită de 2 și 7, la a n-a suma e 2.

Analog cazului anterior, $P(J_n) \overset{IND}{=} P(G_1^c \cap H_1^c) \cdot P(G_2^c \cap H_2^c) \dots \cdot P(G_{n-1}^c \cdot H_{n-1}^c) \cdot P(G_n)$

$P(G_n) = \frac{1}{36}$, deoarece unica pereche cu suma 2 este $\{(1,1)\}$.

$P(G_i^c \cap H_i^c) = 1 - P(G_i \cup H_i) \overset{INC}{=} 1 - P(G_i) - P(H_i) = 1 - \frac{7}{36} = \frac{29}{36}$

$$\Rightarrow P(\text{suma 2 să apară înaintea sumei 7}) = P\left(\bigcup_{n=1}^{\infty} J_n\right) \overset{IND}{=} \sum_{n=1}^{\infty} P(E_n) = \sum_{n=1}^{\infty} \left(\frac{29}{36}\right)^{n-1} \cdot \frac{1}{36}$$

$$= \frac{1}{36} \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{29}{36}\right)^{i-1} = \frac{1}{36} \lim_{n\to\infty} 1 \cdot \frac{\left(\frac{29}{36}\right)^n - 1}{\frac{29}{36} - 1} = \frac{1}{36} \cdot \frac{-1}{\frac{29}{36} - 1} = \frac{1}{36} \cdot \frac{1}{1 - \frac{29}{36}}$$

$$= \frac{1}{\cancel{36}} \cdot \frac{\cancel{36}}{7} = \frac{1}{7}$$

2\. $A_i \to$ evenimentul în care Maria este la zi cu materia după i săptămâni;

$B_i \to$ — " — în urmă — " —

$P(A_1) = 0,8$ $\quad P(A_2 | A_1) = 0,8$ $\quad P(B_2 | A_1) = 0,2$

$P(B_1) = 0,2$ $\quad P(A_2 | B_1) = 0,4$ $\quad P(B_2 | B_1) = 0,6$

$P(2)$: $\Rightarrow P(A_2) = P(A_2 | A_1) \cdot P(A_1) + P(A_2 | B_1) \cdot P(B_1) = P(A_1) \cdot 0,8 + P(B_1) \cdot 0,4 \sim 0,66$

$\Rightarrow P(B_2) =$ ~~$P(B_2|A$~~ $P(A_1) \cdot P(B_2 | A_1) + P(B_2 | B_1) \cdot P(B_1) = P(A_1) \cdot 0,2 + P(B_1) \cdot 0,6$

$P(n)$: $P(A_n) = P(A_{n-1}) \cdot 0,8 + P(B_{n-1}) \cdot 0,4$

$P(B_n) = P(A_{n-1}) \cdot P(B_n | A_{n-1}) + P(B_n | B_{n-1}) \cdot P(B_{n-1})$.

P.p. $P(n)$ adevărat, demonstrăm $P(n+1)$.

$P(A_{n+1}) = P(A_{n+1} | A_n) \cdot P(A_n) + P(A_{n+1} | B_n) \cdot P(B_n) = 0,8 \cdot \underset{\text{din ip.}}{P(A_n)} + 0,4 \cdot \underset{\text{din ip}}{P(B_n)} \xrightarrow{n\to\infty} 0,66$

$P(B_{n+1}) = P(B_{n+1} | A_n) \cdot P(A_n) + P(B_{n+1} | B_n) \cdot P(B_n) = 0,2 \cdot P(A_n) + P(B_n) \cdot 0,6$

8. $X \sim B(n,p)$ $x \in \{1,2,\dots,n\}$ $\qquad E[X] = 2Var[X]$

$P(X=k) = C_n^k p^k (1-p)^{n-k}$

$Var[X] = np(1-p)$ | -> deduse la curs $\quad E[X] \in \mathbb{N}$

$E[X] = np$

$P[X < E[X]]$

$E[X] \notin \mathbb{N} \Rightarrow p \neq 0,\ n \neq 0$ | $\Rightarrow 1 = 2 - 2p \Rightarrow$

$E[X] = 2Var[X] \Rightarrow np = 2np(1-p)$ | $\Rightarrow 2p = 1 \Rightarrow p = \frac{1}{2}$

$E[X] \notin \mathbb{N}$, $p = \frac{1}{2}$ | $\Rightarrow E[X] \in 2\mathbb{N}+1$

$P(X < E[X]) = P(X < np) = P\left(X < \frac{n}{2}\right) = \sum_{k=0}^{\frac{n}{2}-1} C_n^k p^k (1-p)^{n-k} =$ (deoarece $C_n^k = C_n^{n-k}$, $p = (1-p)$)

$= \frac{1}{2} \sum_{k=0}^{n} C_n^k p^k (1-p)^{n-k}$ $\quad \cancel{= \frac{1}{2} \cdot 2^n = 2^{n-1}}$ (

$= \frac{1}{2} \sum_{k=0}^{n} C_n^k \frac{1}{2^k} \cdot \frac{1}{2^{n-k}} = \frac{1}{2} \cdot \frac{1}{2^n} \sum_{k=0}^{n} C_n^k = \frac{1}{2} \cdot \frac{1}{2^n} \cdot 2^n = \frac{1}{2}$

$X \sim \begin{pmatrix} -1 & 0 & 1 \\ 0.3 & 0.2 & 0.5 \end{pmatrix}$ $\quad X^3 \sim \begin{pmatrix} -1 & 0 & 1 \\ 0.3 & 0.2 & 0.5 \end{pmatrix}$

$3X+7 \sim \begin{pmatrix} 4 & 7 & 10 \\ 0,3 & 0,2 & 0,5 \end{pmatrix}$ $\quad X^4 + X^2 \sim \begin{pmatrix} 0 & 2 \\ 0.5 & 0.5 \end{pmatrix}$

$X^2 \sim \begin{pmatrix} 0 & 1 \\ 0.2 & 0.8 \end{pmatrix}$

$P(X > -\frac{1}{3}) = 0,7$

$P(X < \frac{1}{4} \mid X \geq \frac{1}{2}) = \frac{P(\frac{1}{2} \leq X < \frac{1}{4}) \cdot P(X >}{P(X \geq \frac{1}{2})} = \frac{0.2}{0.5} = 0,4$

4. X v.a. cu valori în $\mathbb{N}$, $p_n = P(X=n) > 0 \ \forall n \in \mathbb{N}$

a) $\lambda > 0$:

i) $X \sim Pois(\lambda)$;

ii) $n \geq 1$: $\frac{p_n}{p_{n-1}} = \frac{\lambda}{n}$

"$\Rightarrow$" $X \sim Pois(\lambda) \Leftrightarrow p_n = e^{-\lambda} \frac{\lambda^n}{n!} \Rightarrow \frac{p_n}{p_{n-1}} = \frac{\cancel{e^{-\lambda}} \lambda^{n}}{n!} \cdot \frac{(n-1)!}{\cancel{e^{-\lambda}} \cancel{\lambda^{n-1}}} = \frac{\lambda}{n}$

"$\Leftarrow$" $\frac{p_n}{p_{n-1}} = \frac{\lambda}{n}$

$\frac{p_{n-1}}{p_{n-2}} = \frac{\lambda}{n-1}$

$\vdots$

$\frac{p_1}{p_0} = \frac{\lambda}{1}$ $\odot$

$\frac{p_n}{\cancel{p_{n-1}}} \cdots \frac{\cancel{p_1}}{p_0} = \frac{\lambda^n}{n!} \Rightarrow \frac{p_n}{p_0} = \frac{\lambda^n}{n!} \Rightarrow p_n = \frac{\lambda^n}{n!} \cdot p_0$

Știm că $\sum_{k=0}^{\infty} p_k = 1 \Leftrightarrow \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} p_0 = p_0 \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} = 1 \Rightarrow p_0 = \frac{1}{\sum_{k=0}^{\infty} \frac{\lambda^k}{k!}}$

$\lim_{n\to\infty} \frac{C_n^k}{n^k} = \lim_{n\to\infty} \frac{n!}{(n-k)!\, k!\, n^k} = \cancel{\lim_{n\to\infty} \frac{n(n-1) \ldots n(n-k)(n-k)}{(n-k)!}} =$

$= \lim_{n\to\infty} \frac{\frac{n}{n} \cdot \frac{n-1}{n} \cdot \frac{n-2}{n} \cdots \frac{n-k+1}{n}}{k!} = \cancel{\lim_{n\to\infty} \frac{n-k}{n-k}} \xrightarrow{n\to\infty} \frac{1}{k!}$

$\Rightarrow \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} = \lim_{n\to\infty} \sum_{k=0}^{n} C_n^k \frac{\lambda^k}{n^k} = \lim_{n\to\infty} (1 + \frac{\lambda}{n})^n = \lim_{n\to\infty} \left[(1+\frac{\lambda}{n})^{\frac{n}{\lambda}} \right]^{\lambda} = e^{\lim_{n\to\infty} \lambda} = e^{\lambda}$

$\Rightarrow p_0 \cdot e^{\lambda} = 1 \Rightarrow e^{-\lambda} = p_0$

$\Rightarrow p_n = e^{-\lambda} \frac{\lambda^n}{n!} \Rightarrow X \sim Pois(\lambda)$

b) i) $P(X=k) = \frac{\lambda^k}{k!} \cdot e^{-\lambda} \Rightarrow \frac{P(X=k)}{P(X=k-1)} = \frac{\lambda^{\cancel{k}} \cancel{e^{-\lambda}}}{k!} \cdot \frac{(k-1)!}{\cancel{e^{-\lambda}} \cancel{\lambda^{k-1}}} = \frac{\lambda}{k}$

$\lambda \geq k \Rightarrow P(X = k) \geq P(X = k-1)$

$\lambda \leq k \Rightarrow P(X = k) \leq P(X = k-1)$ deci $j = \lambda$ este punctul de maxim $\Big| \Rightarrow j = [\lambda]$

$j \in \mathbb{N}$

$\lambda \in \mathbb{R}$

$\Rightarrow P(X = [\lambda]) = \dfrac{\lambda^{[\lambda]}}{[\lambda]!} e^{-\lambda}$ valoarea maxima.

.) $\dfrac{P(X=k)}{P(X=k-1)} = \dfrac{\lambda}{k}$.

Analog i), maximul se atinge când $\frac{\lambda}{k} = 1 \Leftrightarrow k = [\lambda]$

X v.a. discretă $P(X=x) = \dfrac{(1-p)^k}{-k\log(p)}$ $k \geq 1$, $P(X=0)=0$, $0<p<1$

$$\mathbb{E}[X] = \sum_{k=0}^{\infty} k \cdot P(X=k) = \sum_{k=1}^{\infty} k \cdot \frac{(1-p)^k}{-k\log p} = \frac{1}{-\log p} \sum_{k=1}^{+\infty} \frac{k\cdot(1-p)^k}{k}$$

$$= -\frac{1}{\log p}\sum_{k=1}^{+\infty}(1-p)^k = -\frac{1}{\log p}\lim_{n\to\infty}\sum_{k=1}^{n}(1-p)^k = \frac{1}{-\log p}\lim_{n\to\infty}\frac{(1-p)^{n+1}-1}{1-p-1}(1-p)$$

$$= \frac{-1}{\log p}\lim_{n\to\infty}\frac{[(1-p)^n-1](1-p)}{-p} = -\frac{1}{\log p}\lim_{n\to\infty}\frac{[(1-p)^n-1](1-p)}{-p} = -\frac{1}{\log p}\left(\frac{[1-p](-1)}{-p}\right)$$

$$= \cancel{\frac{-1}{p\log p} + \frac{1}{\log p}} = \frac{\cancel{p-1}(1-p)}{p\log p}\cdot(-1)$$

$$\mathbb{E}[X^2] = \sum_{k=0}^{+\infty} k^2 P(X=k) = \sum_{k=1}^{\infty} k^2\frac{(1-p)^k}{-k\log p} = \sum_{k=1}^{\infty}\frac{k(1-p)^k}{-\log p} = -\frac{1}{\log p}\sum_{k=1} k(1-p)^k$$

$$= -\log p^{-1}(1-p)\sum_{k=1}^{+\infty} k(1-p)^{k-1} = -\log p^{-1}\cdot(1-p)\sum_{k=1}^{\infty}\left[(1-p)^k\right]' = -\log p^{-1}(1-p)\left[\sum_{k=1}(1-p)^k\right]'$$

$$= -\log p^{-1}\cdot(1-p)\cdot\left[\frac{(1-p)\cdot(-1)}{-p}\right]' = -\log p^{-1}\cdot(1-p)\cdot(-1)\left(\frac{1-p}{-p}\right)' =$$

$$= -\log p^{-1}(1-p)\left(\frac{1-p}{p}\right)' = -\log p^{-1}(1-p)\cdot\frac{(1-p)+p}{p^2} = -\log p^{-1}(1-p)\cdot\frac{1}{p^2}$$

$$= \frac{1-p}{-p\log p}$$

$$\mathrm{Var}[X] = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = -\frac{1-p}{p^2\log p} + \frac{1-p}{p\log p} = \frac{\cancel{-1}+p+\cancel{1}-p^2}{p^2\log p} = \frac{p(1-p)}{p^2\log p}$$

$$= -\frac{1-p}{p^2\log p} + \left[\frac{1-p}{p\log p}\cdot(-1)\right]^2 = -\frac{1-p}{p^2\log p} - \frac{(1-p)^2}{p^2(\log p)^2}$$

$$= \frac{-\log p + p\log p - 1 + 2p - p^2}{p^2\log p} = \frac{(1-p)(+\log p + 1 - p)}{-p^2(\log p)^2}$$

6. 10 remize succesive : $_n$='

Fie F -> câștigă Fischer 0.7 -> 1

S -> —— Spasskyh 0.3 -> 2   0 -> remiză 0.3

a) $X : \begin{pmatrix} 0 & 1 & 2 \\ 0.3 & 0.4 & 0.3 \end{pmatrix}$

* Fie n -> numărul de partide jucate până câștigă Bobby Fischer.

=> n-1, ~~nu~~ câștigă nici el, nici spassky => remiză.

$P(F) = \sum_{i=1}^{10} P(Y=i) = \sum_{i=1}^{10} 0.3^{i-1} \cdot 0.4 =$

$Y : \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 0.4 & 0.3 \cdot 0.4 & 0.3^2 \cdot 0.4 & \dots & & & & & & 0.3^9 \cdot 0.4 \end{pmatrix}$

$= 0.4 \cdot \frac{0.3^{10}-1}{0.3-1} = 0.4 \frac{1-0.3^{10}}{0.7} =$

b) Funcția de masă

Pentru o durată n, avem n-1 remize și la n-a partidă cineva câștigă.

P(remiză) = 0.3

P(cineva câștigă) = 0.7

$\Rightarrow P(Y=n) = \begin{cases} 0.3^{n-1} \cdot 0.7 & , n=\overline{1,9} \\ 0,3^9 & , n=10 \\ 0 & , \text{altfel} \end{cases}$

7. X v.a. discretă.

I Dacă $(X \geq N)$ - numărul de mașini vândute este mai mare decât N => $G = a \cdot N$

II $X < N$ => administratorul vinde X și îi rămân $N-X$ => $G = aX - b(N-X)$.

$\Rightarrow G = \begin{cases} aN, & X \geq N \\ aX - b(N-X) & , \text{altfel} \end{cases}$ G : (

$E[G] = \sum_{g_i \in G} g_i P(g_i) = \cancel{\sum_{k=0}^{N} g_i P(g_i) + \sum_{k=N}^{\infty} g_i P(g_i) = \sum_{k=0}^{N} (aX - b(N-X)) + \sum_{k=N}^{\infty} aN}$

$= \sum_{k=0}^{N-1} [ak - b(N-k)] \cdot P(X=k) + a \cdot N \cdot P(X \geq N)$

$p_1 = p_2 = \dots = p_n$ (X uniformă) $\Rightarrow P(X=x) = \frac{1}{n+1}$

$\Rightarrow E[G] = aN \cdot \sum_{x=N}^{n} \frac{1}{n+1} + \sum_{x=0}^{N} \frac{(a+b)x - b \cdot N}{n+1} = \cancel{aN(n}$

$= aN \frac{(n-N+1)}{n+1} + (a+b) \frac{N(N-1)}{2(n+1)} - \frac{bN^2}{n+1}$

$= \frac{N(a(n-N+1) + (a+b)(N-1) - bN)}{2(n+1)}$

~~E[G] maxim~~ N optim <=> E(G) maxim <=> $N[-N(a+b) + (2n+1)a - b] = 0$

Fie $f(N) = -N^2(a+b) + N[(2n+1)a - b] \Rightarrow f'(N) = -2N(a+b) + (2n+1)a - b$

$f'(N) = 0 \Rightarrow -2N(a+b) + (2n+1)a - b = 0 \Rightarrow N = \frac{b - a(2n+1)}{-2(a+b)} = \frac{a(2n+1)-b}{a(a+b)}$

$f''(N) = -2(a+b) < 0 \Rightarrow f$ convexă $\Rightarrow N = \frac{a(n+1)-b}{2(a+b)}$ punct de maxim