Final Exam

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1.

(a)

$$L(\theta; y_1, y_2, \dots, y_n) = \prod_{i=1}^{n} f(y_i;\theta) = (\theta+1)^n \prod_{i=1}^{n} y_i^{\theta}$$

$$\ln(\theta) = n\ln(\theta+1) + \theta\sum_{i=1}^{n}\ln(y_i)$$

$$\frac{d}{d\theta} = \frac{n}{\theta+1} + \sum_{i=1}^{n}\ln(y_i)$$

$$\frac{n}{\theta+1} + \sum_{i=1}^{n}\ln(y_i) = 0$$

$$\theta + 1 = -\frac{n}{\sum_{i=1}^{n}\ln(y_i)}$$

$$\hat\theta = -\frac{n}{\sum_{i=1}^{n}\ln(y_i)} - 1$$

$$\boxed{\hat\theta = -\frac{n}{\sum_{i=1}^{n}\ln(y_i)} - 1}$$

(b)

$$E(Y) = \int_0^1 y * (\theta+1)y^{\theta}dy$$

$$E(Y) = (\theta+1)\int_0^1 y^{\theta+1}dy = (\theta+1)*\frac{1}{\theta+2}$$

$$E(Y) = \frac{\theta+1}{\theta+2}$$

$$\bar Y = \frac{\theta+1}{\theta+2}$$

$$\bar Y(\theta+2) = \theta+1$$

$$\bar Y\theta + 2\bar Y = \theta + 1$$

$$\bar Y\theta - \theta = 1 - 2\bar Y$$

$$\theta(\bar Y - 1) = 1 - 2\bar Y$$

$$\boxed{\hat\theta = \frac{1-2\bar Y}{\bar Y - 1}}$$

(c)

$$\bar Y = \frac{\sum_{i=1}^{1000} Y_i}{1000}$$

```
781.7118 / 1000
```

```
## [1] 0.7817118
```

$$\hat\theta = -\frac{n}{\sum_{i=1}^{n}\ln(y_i)} - 1$$

```
1000 / [illegible]
```

```
## [1] [illegible]
```

```
[illegible]
```

```
## [1] [illegible]
```

```
[illegible] - 1
```

```
## [1] [illegible]
```

$$\boxed{\hat\theta = [illegible]}$$

$$\hat\theta = \frac{1-2\bar Y}{\bar Y - 1}$$

```
(1 - 2 * 0.7817118) / (0.7817118 - 1)
```

```
## [1] [illegible]
```

$$\boxed{\hat\theta = [illegible]}$$

2.

(a)

$$f_{XY}(x,y) = 6(1-y),\ 0 \le x \le y \le 1$$

$$P(X \le \frac{3}{4}, Y \ge \frac{1}{2})$$

First Region:

$$\int_{1/2}^{3/4}\int_0^{y} 6(1-y)dxdy$$

$$= \int_{1/2}^{3/4} 6(1-y)y\,dy$$

$$= 6\left[\frac{y^2}{2} - \frac{y^3}{3}\right]_{1/2}^{3/4}$$

$$= \frac{11}{32}$$

First region $= \frac{11}{32}$

Second Region:

$$\int_{3/4}^{1}\int_0^{3/4} 6(1-y)dxdy$$

$$= \frac{18}{4}\left[y - \frac{y^2}{2}\right]_{3/4}^{1}$$

$$= \frac{9}{64}$$

$Region2 = \frac{9}{64}$

```
11/32 + 9/64
```

```
## [1] 0.484375
```

$\boxed{0.484375}$

(b)

Find the Marginal density function of X.

$$f_X(x) = f_{XY}(x, y)dy$$

$$f_X(x) = \int_{y=x}^{1} 6(1-y)dy$$

$$= 6\int_{y=x}^{1}(1-y)dy$$

$$= 6(y - \frac{y^2}{2})]_{y=x}^{y=1}$$

$$= 6[(1-\frac{1}{2}) - (x - \frac{x^2}{2})]$$

$$= 6(\frac{1}{2}) - (x - \frac{x^2}{2})$$

$$= 6[\frac{1}{2} - x + \frac{x^2}{2}]$$

$$= 3 - 6x + 3x^2$$

$$\boxed{f_X(x) = 3 - 6x + 3x^2,\ 0 \le x \le 1}$$

(c)

$$E(X) = \int_0^1 x * f_X(x)dx$$

$$E(X) = \int_0^1 x * (3 - 6x + 3x^2)dx$$

$$= \int_0^1 (3x - 6x^2 + 3x^3)dx$$

$$= \frac{3x^2}{2} - \frac{6x^3}{3} + \frac{3x^4}{4}]_0^1$$

$$= \frac{3(1)^2}{2} - \frac{6(1)^3}{3} + \frac{3(1)^4}{4}$$

```
3/2 - 6/3 + 3/4
```

```
## [1] 0.25
```

$\boxed{E(X) = 0.25}$

To calculate the Var(X) we first compute $E(X^2)$

$$E(X^2) = \int_0^1 x^2 * f_X(x)dx$$

$$E(X) = \int_0^1 x^2 * (3 - 6x + 3x^2)dx$$

$$= \int_0^1 (3x^2 - 6x^3 + 3x^4)dx$$

$$= \frac{3x^3}{3} - \frac{6x^4}{4} + \frac{3x^5}{5}]_0^1$$

```
1 - 6/4 + 3/5
```

```
## [1] 0.1
```

$Var(X) = E(X^2) - [E(X)]^2$

```
0.1 - (0.25)^2
```

```
## [1] 0.0375
```

$\boxed{Var(X) = 0.0375}$

3.

(a)

Conditions for Independence

$$P(X = x, Y = y) = P_X(x) * P_Y(y)$$

Marginal PMF of X

$$P_X(x) = \sum_y P(X = x, Y = y)$$

$$X = 1:\ P_X(1) = 0.1 + 0.1 = 0.2$$

$$X = 2:\ P_X(2) = 0.3 + 0.3 = 0.6$$

$$X = 3:\ P_X(3) = 0.1 + 0.1 = 0.2$$

Marginal PMF of Y

$$Y = 0:\ P_Y(0) = 0.3$$

$$Y = 1:\ P_Y(1) = 0.1 + 0.1 = 0.2$$

$$Y = 3:\ P_Y(3) = 0.1 + 0.1 = 0.2$$

$$Y = 4:\ P_Y(4) = 0.3$$

Check Independence:

$$P(2,0) = P_X(2) * P_Y(0) = 0.6 * 0.3 = 0.18 \neq 0.3$$

$$\boxed{X\ and\ Y\ are\ not\ independent}$$

(b)

$$Let\ W = X^2(Y-2)$$

X	Y	P(X,Y)	$W = X^2(Y-2)$	W^2	$W * P(X,Y)$	$W^2 * P(X,Y)$
2	0	0.30	$2^2(0-2) = -8$	$(-8)^2 = 64$	$-8 * 0.30 = -2.4$	$64 * 0.30 = 19.2$
1	1	0.10	$1^2(1-2) = -1$	$(-1)^2 = 1$	$-1 * 0.10 = -0.1$	$1 * 0.10 = 0.1$
2	4	0.30	$2^2(4-2) = 8$	$(8)^2 = 64$	$8 * 0.30 = 2.4$	$64 * 0.30 = 19.2$
1	3	0.10	$1^2(3-2) = 1$	$(1)^2 = 1$	$1 * 0.10 = 0.10$	$1 * 0.10 = 0.10$
3	1	0.10	$3^2(1-2) = -9$	$(-9)^2 = 81$	$-9 * 0.10 = -0.9$	$81 * 0.10 = 8.1$
3	3	0.10	$3^2(3-2) = 9$	$(9)^2 = 81$	$9 * 0.10 = 0.9$	$81 * 0.10 = 8.1$

$$E(W) = \sum W * P(X,Y)$$

```
-2.4 + (-0.1) + 2.4 + 0.10 + (-0.9) + 0.9
```

```
## [1] -1.110223e-16
```

$$E(W^2) = \sum W^2 * P(X,Y)$$

```
19.2 + 0.10 + 19.2 + 0.10 + 8.1 + 8.1
```

```
## [1] 54.8
```

$$Var(W) = E(W^2) - [E(W)]^2$$

$$54.8 - 0 = 54.8$$

$\boxed{Var(W) = 54.8}$

(c)

P(W>0)

$$W = X^2(Y-2) > 0$$

$$X^2 > 0$$

$$Y - 2 > 0 = Y > 2$$

X	Y	P(X,Y)	$W = X^2(Y-2)$
1	3	0.10	$1^2(3-2) = 1$
2	4	0.30	$2^2(4-2) = 8$
3	3	0.10	$3^2(3-2) = 9$

$$P(W > 0) = P(1,3) + P(2,4) + P(3,3) = 0.10 + 0.30 + 0.10 = 0.50$$

$\boxed{P(W > 0 = 0.50}$

4.

(a)

$Find\ Var(\bar X)$

Given is Var(X) = 0.03 and n = 20

Formula for $Var(\bar X) = \frac{\sigma^2}{n}$

```
0.03 / 20
```

```
## [1] 0.0015
```

$\boxed{Var(\bar X) = 0.0015}$

(b)

$$\mu_{\bar X} = E(X) = 3.5$$

$$\sigma_{\bar X} = \sqrt{Var(\bar X)} = \sqrt{\frac{Var(X)}{n}}$$

```
sqrt(0.03/20)
```

```
## [1] 0.03872983
```

Standardize

$$Z = \frac{\bar X - \mu_{\bar X}}{\sigma_{\bar X}}$$

Z_1

```
(3.39-3.5) / 0.03872983
```

```
## [1] -2.840188
```

Z_2

```
(3.6-3.5) / 0.03872983
```

```
## [1] 2.581989
```

Standard Normal Table $P(3.39 \le \bar X \le 3.6 = P(Z_1 \le \bar X \le Z_2$

$Z_1 = -2.840 = 0.0023$

$Z_2 = 2.58 = 0.9951$

$$P(Z \le 2.58) - P(Z \le -2.84) = 0.9951 - 0.0023 \approx 0.9928$$

$$\boxed{P(3.39 \le \bar X \le 3.6 \approx 0.9928}$$

(c)

Assumptions made: For the sampling distribution of X (GPA) if this is normal, the sampling distribution will also be normal regardless of the sample size. However, the CLT states that the sampling distribution will approximate a normal distribution as the sample size n becomes sufficiently large.

5.

(a)

$$M(t) = \sum_{x=1}^{10} e^{tx}cx$$

$$M(t) = c\sum_{x=1}^{10} xe^{tx} = 1$$

$$M(0) = c\sum_{x=1}^{10} xe^{0} = 1$$

$$M(0) = c(1+2+3+4+5+6+7+8+9+10) = 1$$

$$c(55) = 1$$

$$c = \frac{1}{55}$$

E(X) is equal to the first derviative of the MGF

$$M'(t) = c\sum_{x=1}^{10} x^2e^{tx}dt$$

$$M'(0) = \sum_{x=1}^{10} x^2e^{tx}(\frac{1}{55})$$

$$= \sum_{x=1}^{10} x^2 * \frac{1}{55}$$

```
(1/55) * (1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 + 7^2 + 8^2 + 9^2 + 10^2)
```

```
## [1] 7
```

$$\boxed{E(X) = 7}$$

$$M''(t) = c\sum_{x=1}^{10} x^3e^{tx}$$

Evaluate at t = 0

$$M''(0) = \sum_{x=1}^{10} cx^3$$

$$= \sum_{x=1}^{10}\frac{1}{55} * x^3$$

```
(1/55) * (1^3 + 2^3 + 3^3 + 4^3 + 5^3 + 6^3 + 7^3 + 8^3 + 9^3 + 10^3)
```

```
## [1] 55
```

$$Var(X) = E(X^2) - [E(X)]^2$$

```
55 - (7^2)
```

```
## [1] 6
```

$$\boxed{Var(X) = 6}$$

(b)

$$\sum_{x=1}^{10} P(X = x) = 1$$

$$\sum_{x=1}^{10} cx = 1$$

We found $c = \frac{1}{55}$

$$\sum_{x=1}^{10} P(X = x) = \sum_{x=1}^{10}\frac{x}{55} = \frac{1}{55}\sum_{x=1}^{10} x = \frac{1}{55} * 55 = 1$$

$$\boxed{pmf\ of\ X:\ P(X = x) = \frac{x}{55}, x = 1, 2, 3, \dots, 10}$$

(c)

Using the pmf of X P(X=x) = $\frac{x}{55},\ x = 1, 2, 3, \dots, 10$

$$P(X > 8) = P(X = 9) + P(X = 10)$$

$$P(X = 9) = \frac{9}{55}$$

$$P(X = 10) = \frac{10}{55}$$

```
9/55 + 10/55
```

```
## [1] 0.3454545
```

$$\boxed{P(X > 8) = \frac{19}{55}}$$