Keck_Brandon_STAT620_HW5

Brandon Keck

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3.2-12 Let X equal the number of alpha particle emissions of carbon-14 that are counted by a Geiger counter each second. Assume that the distribution of X is Poisson with mean 16. Let W equal the time in seconds before the seventh count is made.

(a) Give the distribution of W.

X is the number of alpha particle emissions counted each second and follows a Poisson distribution with mean 16. X ~ Poisson(λ = 16)

W follows a Gamma distribution with shape parameter $\alpha = 7$ and rate parameter $\lambda = 16$

$\boxed{W \sim Gamma(7, 16)}$

(b) Find P(W ≤ 0.5). HINT: Use Equation 3.2-1 with λw = 8.

We are give that $\lambda w = 8$ and to use the Equation 3.2-1 which is

$$F(w) = 1 - \sum_{k=0}^{\alpha-1} \frac{\lambda w^k e^{-\lambda w}}{k!}$$

$$1 - \sum_{k=0}^{7-1} \frac{8^k e^{-k}}{k!}$$

```
k <- c(0,1,2,3,4,5,6)

1 - sum(8^k * exp(-8) / factorial(k))
```

```
## [1] 0.6866257
```

$\boxed{0.6866257}$

3.2-13. If X is χ2(23), find the following:

(a) P(14.85 < X < 32.01).

```
pchisq(32.01, 23) - pchisq(14.85, 23)
```

```
## [1] 0.7999933
```

$$P(14.85 < X < 32.01) = \boxed{0.7999933}$$

(b) Constants a and b such that P(a < X < b) = 0.95 and P(X < a) = 0.025.

```
a <- qchisq(0.025, 23)
a
```

```
## [1] 11.68855
```

```
b <- qchisq(0.975, 23)
b
```

```
## [1] 38.07563
```

$$a = \boxed{11.68855}$$

$$b = \boxed{38.07563}$$

(c) The mean and variance of X.

The mean of the chi-square distribution is equal to the degrees of freedom which is r. The value of r is 23.

$$\boxed{\mu = 23}$$

The formula for the variance of a chi-square distribution is

$$\sigma^2 = 2r$$

$$2 * 23 = 46$$

$$\boxed{2r = 46}$$

(d) χ2 0.05(23) and χ2 0.95(23).

```
qchisq(0.05, 23)
```

```
## [1] 13.09051
```

```
qchisq(0.95, 23)
```

```
## [1] 35.17246
```

$$\boxed{\chi^2_{0.05} = 13.09051}$$

$$\boxed{\chi^2_{0.95} = 35.17246}$$

3.3-5. If X is normally distributed with a mean of 6 and a variance of 25, find

(a) P(6 ≤ X ≤ 14).

```
pnorm(14, mean = 6, sd = 5) - pnorm(6, mean = 6, sd = 5)
```

```
## [1] 0.4452007
```

$\boxed{0.4452007}$

(b) P(4 ≤ X ≤ 14).

```
pnorm(14, mean = 6, sd = 5) - pnorm(4, mean = 6, sd = 5)
```

```
## [1] 0.6006224
```

$\boxed{0.6006224}$

(c) P(−4 < X ≤ 0).

```
pnorm(0, mean = 6, sd = 5) - pnorm(-4, mean = 6, sd = 5)
```

```
## [1] 0.09231954
```

$\boxed{0.09231954}$

(d) P(X > 15).

```
1 - pnorm(15, mean = 6, sd = 5)
```

```
## [1] 0.03593032
```

$\boxed{0.03593032}$

(e) P(|X − 6| < 5)

P(-5 < X -6 < 5) = P(1 < X < 11)

```
pnorm(11, 6, sd = 5) - pnorm(1, 6, sd = 5)
```

```
## [1] 0.6826895
```

$\boxed{0.6826895}$

(f) P(|X − 6| < 10).

p(-10 < X-6 < 10) = P(-4 < X < 16)

```
pnorm(16, 6, sd = 5) - pnorm(-4, 6, sd = 5)
```

```
## [1] 0.9544997
```

$\boxed{0.9544997}$

(g) P(|X − 6| < 15).

P(-15 < X - 6 < 15) = P(-9 < X < 21)

```
pnorm(21, mean = 6, sd = 5) - pnorm(-9, mean = 6, sd = 5)
```

```
## [1] 0.9973002
```

$\boxed{0.9973002}$

(h) P(|X − 6| < 12.4).

```
pnorm(18.4, 6, sqrt(25)) - pnorm(-6.4, 6, sqrt(25))
```

```
## [1] 0.9868618
```

$\boxed{0.9868618}$

3.3-8. Let the distribution of X be N(μ, σ2). Show that the points of inflection of the graph of the pdf of X occur at x = μ ± σ.

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}$$

$$we\ must\ utilize\ the\ chain\ rule\ f'(g(x)) * g'(x)$$

$$setting\ u = -\frac{(x-\mu)^2}{2\sigma^2}$$

$$= -\frac{2(x-\mu)}{2\sigma^2}$$

$$= \frac{x-\mu}{\sigma^2}$$

$$\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2} * u}$$

$$= \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2} * -\frac{x-\mu}{\sigma^2}}$$

$$= -\frac{x-\mu}{\sigma^3\sqrt{2\pi}} e^{-\frac{x-\mu}{2\sigma^2}}$$

Now we must take the derivative of f'(x) to get f''(x).

$$u(x) = -\frac{(x-\mu)}{\sigma^3\sqrt{2\pi}}$$

$$v(x) = e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$$we\ can\ apply\ the\ product\ rule\ (fg)' = f'g + fg'$$

$$f''(x) = u'(x) * v(x) + u(x) * v'(x)$$

$$u'(x) = -\frac{1}{\sigma^3\sqrt{2\pi}}$$

$$v(x) = e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$$u(x) = -\frac{x-\mu}{\sigma^3\sqrt{2\pi}}$$

$$v'(x) = e^{-\frac{(x-\mu)^2}{2\sigma^2} * -\frac{x-\mu}{\sigma^2}}$$

$$f''(x) = -\frac{1}{\sigma^3\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2} + -\frac{(x-\mu)^2}{2\sigma^2}} * e^{-\frac{(x-\mu)^2}{2\sigma^2} * -\frac{x-\mu}{\sigma^2}}$$

$$f''(x) = \frac{e^{-\frac{(x-\mu)^2}{2\sigma^2}}}{\sigma^5\sqrt{2\pi}}(\sigma^2 - (x-\mu)^2)$$

$$\frac{e^{-\frac{(x-\mu)^2}{2\sigma^2}}}{\sigma^5\sqrt{2\pi}}(\sigma^2 - (x-\mu)^2) = 0$$

$$\sigma^2 - (x-\mu)^2 = 0$$

$$(x-\mu)^2 = \pm\sigma^2$$

$$x - \mu = \pm\sigma$$

$$Therefore, \boxed{x = \mu \pm \sigma}$$

3.3-9. Find the distribution of W = X^2 when

(a) X is N(0, 4),

$$\sigma = \sqrt{4} = 2$$

So X ~ N(0,4) X has a mean of 0 and a sd of 2

$$f(x) = \frac{1}{2\sqrt{2\pi}} e^{-\frac{x^2}{8}}$$

$$F_W(w) = P(W \leq w) = P(X^2 \leq w)$$

$$P(X \leq \pm\sqrt{w})$$

$$F_X(\pm\sqrt{w})$$

$$f_W(w) = \frac{dF_W(w)}{dw}$$

$$f_W(w) = \frac{1}{\sqrt{w}} * f_X(\sqrt{w})$$

$$f_X(x) = \frac{1}{2\sqrt{2\pi}} e^{-\frac{x^2}{8}}$$

$$\frac{1}{\sqrt{w}} * \frac{1}{2\sqrt{2\pi}} e^{-\frac{w}{8}}$$

$$f_W(w) = \frac{e^{-\frac{w}{8}} w^{\frac{1}{2}-1}}{\Gamma\frac{1}{2} * 8^{\frac{1}{2}}}$$

$$Thus,\ W\ follws\ a\ gamma\ distribution\ with\ parameters\ \alpha = \frac{1}{2}\ and\ \theta = 8$$

(b) X is N(0, σ2).

$$F_W(w) = P(W \leq w) = P(X^2 \leq w)$$

$$P(X \leq \pm\sqrt{w})$$

$$F_X(\pm\sqrt{w})$$

$$f_W(w) = \frac{dF_W(w)}{dw}$$

$$f_W(w) = \frac{1}{\sqrt{w}} * f_X(\sqrt{w})$$

$$f_X(x) = \frac{1}{2\sqrt{2\pi}} e^{-\frac{x^2}{8}}$$

$$\frac{1}{\sqrt{w}} * \frac{1}{2\sqrt{2\pi}} e^{-\frac{w}{8}}$$

$$f_W(w) = \frac{e^{-\frac{w}{8}} w^{\frac{1}{2}-1}}{\Gamma\frac{1}{2} * \sqrt{2\sigma^2}}$$

$\boxed{W \sim \Gamma(\frac{1}{2}, 2\sigma^2)}$

3.3-11. A candy maker produces mints that have a label weight of 20.4 grams. Assume that the distribution of the weights of these mints is N(21.37, 0.16).

(a) Let X denote the weight of a single mint selected at random from the production line. Find P(X > 22.07).

```
1 - pnorm(22.07, mean = 21.37, sd = 0.4)
```

```
## [1] 0.04005916
```

$\boxed{0.04005916}$

(b) Suppose that 15 mints are selected independently and weighed. Let Y equal the number of these mints that weigh less than 20.857 grams. Find P(Y ≤ 2).

Find the probability that a single mints weighs less than 20.857 grams

```
pnorm(20.857, 21.37, 0.4)
```

```
## [1] 0.09983365
```

```
pbinom(2, size = 15, prob = 0.09983365)
```

```
## [1] 0.8165799
```

$\boxed{0.8165799}$