Introduction to Finance

by George Blazenko

Chapter 6 *The Math of Finance*

"Time is money." Benjamin Franklin

"The greatest invention of mankind is that of compound interest. — *Albert Einstein*

"Great moments in science: Einstein discovers that time is actually money." Read more at: https://www.brainyquote.com/quotes/gary_larson_102131

In Chapter Six You Will Learn:

1. What is a present value (PV)?
2. What is a future value (FV)?
3. Calculate PV with different cash-flow patterns.
4. Calculate FV with different deposit patterns.
5. Internal Rate of Return (IRR).
6. In five cases, calculate IRR with a formula!
7. If you cannot calculate IRR with a formula, can you calculate it with a financial calculator? Often, yes!
8. What is an effective rate of return?
9. What is a nominal rate of return?
10. When I buy a financial asset that I sell before maturity and reinvest payments it makes between when I buy and sell, can I calculate my annualized holding period rate of return compounded "m" times per annum? Yes, I can!
11. Can I calculate outstanding balance, interest, principal reduction for a mortgage? Yes, you can!
12. Can I incorporate tax into PV problems? Yes, you can!

Chapter Six Contents

(6.1) Introduction..

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In chapters to this point in this E-book, we have investigated short-term financial problems: what is a firm's tax liability for the current year or how much short term borrowing will be required by the end of the year. Investigation of longer-term financial problems, like decisions on capital expenditure, requires a new set of financial-analysis tools. Long-term financial problems invariably involve some element of valuation. For example, what is the value of new common shares sold to new shareholders in a private corporation or what lump sum deposit is required to satisfy a life style in retirement? These problems are applications of discounted cash flow analysis. We introduce in this chapter the fundamental tools required to solve these and other problems in long-term financial planning.

Two important investment problems are the foundation for most long-term financial analysis: *present-value* and *future-value.*

6.1.1 Present Value

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An investor makes a single deposit – typically today – into an investment plan. Over time, the investor makes *withdrawals* from his/her investment account according to a predefined schedule in such a way as to drive the account balance to zero at the time of the last withdrawal. The planned schedule of withdrawals recognizes that periodic interest is also paid on the outstanding account balance over time. The amount the investor must deposit to make the planned set of withdrawals is called the *present value* of the investment plan.

Present value calculations have applications far beyond the hypothetical investment account described in the above paragraph. The term "value" is significant because the investor has effectively purchased a financial asset. Promised future payments on the financial asset are principal and interest, which are calculated from the account balance. Because the single deposit required to sustain future withdrawals is the purchase price of the financial asset, the term "value" is appropriately applied. Generally, "value" can be thought of as the amount required to purchase an asset or the amount received upon sale. The importance of present value calculations arises out of the trading of assets. For many markets and many types of transactions, present value calculations are essential tools for any financial analyst to establish the amount that he or she should bid to buy or offer to sell an asset.

Definition of PV: 4 Minutes

6.1.2 Future Value

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A *future value* is an amount in an investment plan at a specific time in the future from a planned scheduled set of *deposits*. The balance in the plan arises not only from the deposits but also from interest that is paid periodically on the outstanding balance. The critical element of any future value calculation is deciding the date at which the balance should be measured. Future value calculations are common in investment analysis and personal financial planning.

FV Definition: 6 Minutes

(6.2) PV and FV in Simple Environments

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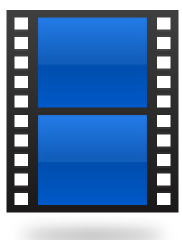
If you have just purchased your TI BA II Plus financial calculator, then it is factory-preset to display only two decimal places (presumably to round dollar amounts to the penny). For interest-rate and rate of return calculations, two decimal places is insufficient and, thus, the instructional-video that follows shows you how to set your TI BA II Plus to display nine decimal places.



TI BA II PLUS Display

6.2.1 The Holding Period Rate of Return (HPRR)

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One of the most basic, but also one of the most important, calculations in investment analysis is the *holding period return* (HPRR). A HPRR is an effective rate of return because it measures the rate of growth of your wealth as the result of owning a financial asset over a period that we refer to as the holding period. The holding period is generally between when you buy a financial asset and when you sell it (possibly sell it hypothetically because your wealth increases even if you do not realize capital gains on financial assets). Suppose you buy a financial asset for **$PV** (Present Value) today. The length of time over which you own the financial asset is the holding period. The holding period can be any interval of time and is not necessarily a year. Let the sum of the sale price of the asset and reinvested disbursements with interest at the end of the holding period be **$FV** (i.e., Future Value). If we denote the holding period rate of return as **r**, then,

$$r = \frac{FV - PV}{PV} \tag{6.1}$$

As an application of this equation, suppose that you buy a common share for $10.00 and sell it for $12.00 in one week. Your holding period rate of return for the week is (12–10)/10 = 20%. This calculation seems easy enough but in more complicated situations, you should interpret **$PV** as the amount put at risk in an investment (i.e., the "out of pocket" expenditure required to undertake the investment possibly after transactions costs and after tax effects). What you put at risk for a financial asset will often be its purchase price (PP) at the beginning of the holding period. Likewise, you should interpret **$FV** as the net amount receivable from the asset upon hypothetical or actual disposal at the end of the holding period (after transactions costs and taxes and with reinvested distributions received over the holding period).

6.2.2 FV of $C after a Holding Period

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Suppose an investment account offers a rate of return of ***r*** for a stated investment term. If you make a single deposit of **$C** dollars today into the account (i.e., the present value of your investment is **$C**), how much do you have at the end of the investment term? We obtain the answer to this question by rearranging the holding period rate of return in equation (6.1) with the present value replaced by **$C**:

$$FV = C\,(1 + r) \tag{6.2}$$

The term 1+**r** represents principal plus interest per dollar invested. More formally, principal plus interest (either per dollar invested or in total) can be described as the *return* on the investment. The offered interest rate is often referred to as the *effective rate of interest* because it represents the rate at which your account grows over the holding period. The amount you have in your account at the end of a holding period per dollar invested less $1 for principal is the effective rate of interest on the account for the holding period. It is always the effective rate of interest (over possibly differ holding periods) that we use in FV and PV calculations. Because the interest rate attainable on an amount invested today depends upon current market conditions, the interest rate used in any future value calculation is a current market rate of interest (i.e., determined by market forces in the financial markets at the current time).

As a specific example, suppose that a bank offers a 25% rate of interest per dollar invested for a two-year term. If you invest $1,200,000 today, how much do you have in your account at the end of the holding period? The answer to this question is the future value of $1,200,000 invested at 25%. Substituting these values into equation (6.2), we find that your account balance after two years is $1,200,000*(1.25) = $1,500,000. In this example, you might be surprised that we use an interest rate over a two-year term. In financial markets, interest is generally quoted on a yearly basis.

FV of $C for One Period: 3 Minutes

6.2.3 PV of $C After One Period

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You want to have **$C** in an investment account after a specific holding period (i.e., the future value of your investment plan is **$C**). If your investment account offers a rate of return of r for this term, what amount must you deposit today? The answer to this question can be obtained by using a rearrangement of the holding period rate of return in equation (6.1) with the future value replaced by **$C**:

$$PV = \frac{C}{1+r} \tag{6.3}$$

In equation (6.3), because the interest rate r is non-negative, the present value PV is lesser than the future withdrawal **$C,** and therefore, the interest rate used to relate the present value and the future withdrawal is referred to as the *discount factor* or the *discount rate*. Notice that because the present value must be deposited at current market rates of interest to generate **$C** at the end of the holding period, the discount factor is a current market rate of interest.

As an example, suppose that a bank offers a 25% rate of interest per dollar invested for a two-year term. If you want $1,500,000 in your account in two years, how much must you deposit today? The answer to this question is the present value of $1,500,000 discounted at 25%. Substituting these values into equation (6.3), we find that the amount you should deposit today is $1,500,000/(1.25) = $1,200,000.

It should be clear from our discussion in the current and the previous subsection that present value and future value calculations are *inverse* mathematical operations. This fact arises from their relationship in the holding period rate of return calculation in equation (6.1).

PV of $C in One Period: 3 Minutes

6.2.4 FV of $C after *n* Sub-Periods

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Suppose that you are interested in the future value of $C over a specific holding period but the interest rate quoted by the investment is in terms of a sub-period. For example, you may want to invest for three years but the bank quotes interest yearly. To answer this FV question, you need to know the return per dollar invested over your holding period. If the sub-period effective rate of interest is r (i.e., this is the rate at which your account grows over a sub-period), and if n is the number of sub-periods in your holding period, then return per dollar invested over your holding period is $\mathbf{(1+r)^n}$. This result is obtained by repeated application of equation (6.2). The future value of **$C** invested is, thus,

$$FV = C\left(1+r\right)^{n} \tag{6.4}$$

As an example, suppose you deposit $1,200,000 for a three-year term at an annual rate of interest of 10%. Then your account balance in 3 years is $1{,}200{,}000(1.1)^3$ = $1,597,200.

This calculation multiplies an effective rate for a sub-period (that is, 10% per annum) by the number of sub-periods in the holding period (that is, three years) to produce the effective rate of return for the holding period (three years). However, in going from an effective rate over a shorter period (a year) to an effective rate over a longer period (three years), we multiply by three geometrically rather than arithmetically. These are both effective rates but over different time intervals. The rate of growth of your wealth per annum is 10% and the rate of growth of your wealth per three years is $(1.1)^3$-1=33.1%. So, your wealth in three years is 1,200,000*1.331 = $1,597,200.

FV of $C in "n" Sub-Periods: 4 Minutes

6.2.5 PV of $C After *n* Sub-Periods

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Suppose you want to withdraw **$C** from an investment plan after a specific holding period but the interest is quoted by sub-period. For example, you want to withdraw funds in three years but the bank quotes interest per-annum. We know from our analysis in the previous section that your holding period return per dollar invested is $\mathbf{(1+r)^n}$ where **n** is the number of sub-periods in your holding period. The present value of your investment of **$C** is, therefore,

$$PV = \frac{C}{(1+r)^n} \tag{6.5}$$

As an example, suppose you want to withdraw $1,500,000 from an investment plan after three years when the annual rate of interest of 10%. The amount you must deposit today is $1,500,000/$(1.10)^3$ = $1,126,972.20.

PV of $C in "n" Sub-Periods: 3 Minutes

6.2.6 FV of an Annuity

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An annuity is a set of finite and constant payments made every period over a specific term. For example, you might make 10 annual deposits into an investment plan and you want to know the amount in your plan at termination. The future value of the ten deposits is the sum of the future values of each individual deposit. Suppose that the effective interest rate for a period is r and you plan to make **n** deposits of **$C** each period into your investment plan.

FV of an Annuity: 26 Minutes
(Ordinary, Due, and k periods after last deposit)

If you measure your investment account *at the time of the last deposit* your balance is:

$$FV = \frac{C}{r}\left[(1+r)^n - 1\right] \tag{6.6}$$

As an example, suppose you make 25 per annum deposits of $5000 when the rate of interest on your investment plan is expected to be 5% per year. How much do you have in your account at the time of the last deposit? Substituting the appropriate numbers, your account balance at the time of the last deposit is $\$5,000 \times [(1.05)^{25} - 1]/0.05 = \$238,635.49$.

 **FV of an Annuity Measured at Last Deposit with EXCEL**	 **FV of an Annuity Measured at Last Deposit with TI BA II PLUS**

Notice that in our above investment planning, we are undoubtedly thinking about making our first deposit of $5,000 today. In this case, our last deposit is 24 years from today. If on the other hand, you plan for some reason to make the first deposit one year from today, the last deposit is in 25 years. In both of these cases, because we measure our account balance at the time of the last deposit, our account balance is the same. This fact illustrates that in future value calculations, it is not important when you begin your investment plan but rather how long each deposit is in your plan relative to the date at which you measure your account. In the above example, we might plan, for example, to make our first deposit 10.75 years from today. Nonetheless, the amount in our investment plan at the time of the last deposit 24 years after the first remains $238,635.49.

For an annuity of payments into an investment plan, if you measure your account balance *one period after* the last deposit rather than at the time of the last deposit, your future value is (1+r) times as large as the future value in equation (6.6). The balance earns interest in your account for one additional period. The future value of an annuity where the account balance is measured one period after the last deposit, is therefore (using a little algebra),

$$FV = \frac{C}{r}\left[(1+r)^{n+1} - (1+r)\right] \qquad (6.7)$$

In our above example, if we measure the account balance one year after the last deposit, our future value is $\$238,635.49 \times 1.05 = \$250,567.26$.

FV of an Annuity Measured One Period After Last Deposit with TI BA II PLUS

6.2.7 PV of an Annuity

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You want to make **n** withdrawals from your investment plan (starting in one period) for each of the next **n** periods. If the rate of interest on your investment plan for a period is r, how much do you need in your account today? Equivalently, what is the value of an **n**-period annuity with the first payment in one period if the effective rate of interest for one period is r? The answer to this PV question is contained in the following calculation:

$$PV = \frac{C}{r}\left[1 - \frac{1}{(1+r)^n}\right] \tag{6.8}$$

For example, suppose you want to make withdrawals of $5,000 per annum for 10 years, starting in one year, when the rate of interest paid on your account is 10% per annum. How much must you deposit into your account today? Substituting these numbers into the formula for the present value of an annuity, the amount you need is

$$PV = \frac{5000}{0.1}\left[1 - \frac{1}{(1+0.1)^{10}}\right] = \ \$30,722.84.$$

PV of an Annuity Discussion: 14 Minutes

The videos below illustrate three different ways that you can calculate the PV of an ordinary annuity in practice.

MIRCOSOFT EXCEL and PV of an Ordinary Annuity	**Scientific Calculator and PV of an Ordinary Annuity**	**TI BA II Plus and PV of an Ordinary Annuity**

In the above investment plan, it is natural to think of making a lump-sum deposit today and then begin making withdrawals starting in one period. However, suppose that for some reason, you want to make the first withdrawal at the time of your lump-sum deposit. The net required deposit is then PV – 5000 = $5000*[1–1/(1.1)^9]/0.1$ = \$28,795.12. The gross deposit is \$28,795.12 plus \$5000, which equals \$33,795.12. The gross amount you need to deposit is greater than in the original example because you make *all* withdrawals from your account earlier and therefore you earn less interest on funds in the account over the course of the investment plan.

When the first payment on an annuity is one period from the present, the annuity is called *ordinary*. When the first payment is immediate, the annuity is called an annuity *due*. An annuity due is, as we just demonstrated, more valuable than an ordinary annuity.

TI BA II Plus and PV of an Annuity Due

6.2.8 Deferred and Accelerated PVs

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Suppose that you have been promised an annuity of **n** equal periodic payments but the payments do not begin until **k** periods from today (k is a non-negative number) and the period effective rate of interest is **r**. This set of payments is an ordinary annuity (as described above) but the first payment is deferred (or accelerated relative to ordinary if 0<k<1). We can apply equation (6.8) to this set of payments, but then the present value is calculated as of one period prior to the first payment. Notice that in the ordinary annuity described above where the first payment is one period from today, the present value is also calculated as of one period prior to the first payment – i.e., today. The present value of a deferred annuity is therefore equal to the result of equation (6.8) discounted by another k–**1** periods. The present value of a deferred annuity is therefore

$$PV = \frac{\frac{C}{r}\left[1 - 1\big/(1+r)^{n}\right]}{(1+r)^{k-1}} \tag{6.9}$$

Notice that if k is equal to one, so that in fact the annuity is not deferred but an ordinary annuity with the first payment exactly one period from today, then, equation (6.9) reduces to the present value of an ordinary annuity in equation (6.8). On the other hand, if k equals zero, then the first payment is immediate so that equation (6.9) is the value of an ordinary annuity in equation (6.8) times (1+*r*). Any ordinary PV calculation can be converted to a PV "due" calculation (payments at the beginning of the period) by multiplying by (1+*r*).

Suppose you are promised an annuity of 10 payments of $1000 per annum where the first payment is in 7 years and the per annum interest rate is 10%. How much should you be willing to pay for this financial asset? In this case, n=10, k=7, C=1000, and r = 0.10. Substituting these numbers into equation (6.9), the present value is $1,000 × [1–1/(1.10)^{10}] ÷ [0.1 × (1.10)^{6}] = $3,468.45.

The PV calculation in equation (6.9) also works if k is not an integer. In particular, equation (6.9) works if k is a number between 0 and 1. The interpretation of k remains the same: k is the number of periods prior to the first payment on the annuity. If k is between 0 and 1, then there is less than a full period to the first payment. We call this situation an annuity accelerated relative to ordinary because the first payment is in less than a period.

Suppose you are promised an annuity of 10 payments of $1000 per annum where the first payment is in 7 months and the per annum interest rate is 10%. How much should you be willing to pay for this financial asset? In this case, n=10, k=7/12, C=1000, and r = 0.10. Substituting these numbers into equation (6.9), the present value is

$$\frac{\dfrac{1000}{0.1}\left[1-\dfrac{1}{(1.1)^{10}}\right]}{(1+r)^{7/12-1}} = \$6{,}393.49.$$

Deferral calculations like Equation (6.9) are possible for other present values beyond an annuity. For example, a similar calculation is possible for the present value of a deferred perpetuity or the present value of a deferred, but growing perpetuity. In chapter 7, we will use deferral calculations like Equation (6.9) to value a bond when the first coupon is less than a full coupon payment interval from today.

PV of a Deferred/Accelerated Annuity: 23 Minutes

6.2.9 PV of an Perpetuity

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Case 1: Ordinary Perpetuity

Suppose a financial asset promises a constant periodic payment indefinitely when the first payment is in one period and the period effective rate of interest is **r**. The present value of this set of payments is

$$PV = \frac{C}{r} \tag{6.10}$$

Because a constant perpetuity is the same as an annuity of payments where the number of payments approaches infinity, equation (6.10) can be obtained from equation (6.8) by taking the limit as **n** approaches infinity ($n \to \infty$).

Canadian Pacific Railway issued a bond in 1872 that promises a fixed annual payment indefinitely, that is, without maturity. These bonds still exist but they are extremely hard to find (they exist in pension plans around the world), so, don't expect to buy one of these bonds. Suppose the annual payment on each bond is $30 (per-period payments made to bondholders are *coupons*), that the next coupon is due in one year, and that the market rate of interest for equally risky bonds is 8% The market rate of interest for a bond is called its *yield to maturity*, although, the CPR bond has no maturity). What is the value of the CP bond? To answer this question, begin by noting that the bond is perpetuity. The value of the bond is therefore PV = 30/0.08 = \$375. The longevity of this bond is particularly attractive for estate planning (i.e., leaving something of value to your ungrateful grandchildren).

Rather than the CPR bond, a more common financial asset that is a perpetuity is a preferred share. Preferred shares typically have no maturity. You can buy many preferred shares that trade like common shares on stock exchanges around the world. However, in Canada because of favorable tax treatment of dividends received by individuals, preferred shares are particularly popular in Canada. We study the valuation of both preferred and common shares in some detail in chapter 8 on equities and equity markets.

PV of an Ordinary Perpetuity: 7 Minutes

Case 2: Perpetuity Due

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Suppose an investment returns \$C per period indefinitely starting immediately (rather than in one period). This is a perpetuity "due." We know from section 6.2.8 we can convert the PV for a stream of ordinary payments (first payment in one period) to the PV of the equivalent "due" case (first payment immediate) by multiplying by (1+r) where r is the opportunity cost rate of return. So, the PV of a perpetuity due is,

$$PV = \frac{C}{r}(1+r) \qquad (6.11)$$

A perpetuity due is always more valuable than an ordinary perpetuity because investors would always like to get the first payment sooner (immediately) rather than later (in one period).

PV of a Perpetuity Due: 3 Minutes

Case 3: Perpetuity Deferred/Accelerated

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Suppose that you have been promised a perpetuity of equal periodic payments but the payments do not begin until **k** periods from today (k is a non-negative number) and the period effective rate of interest is **r**. The first payment is deferred relative to ordinary if k>1 and accelerated relative to ordinary if 0<k<1. The present value of a deferred/accelerated perpetuity is equation (6.10) discounted by another k–**1** periods.

$$PV = \frac{C/r}{(1+r)^{k-1}} \qquad (6.12)$$

Notice that if k is equal to one, so that in fact the perpetuity is not deferred but an ordinary annuity with the first payment exactly one period from today, then, equation (6.12) reduces to the present value of an ordinary perpetuity in equation (6.10). On the other hand, if k equals zero, then the first payment is immediate so that equation (6.12) is the value of an ordinary perpetuity in equation (6.10) times (1+*r*). Any ordinary PV calculation can be converted to a PV "due" calculation (payments at the beginning of the period) by multiplying by (1+*r*).

PV of a Deferred/Accelerated Perpetuity: 3 Minutes

6.2.10 PV of a Growing Perpetuity

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Case 1: Ordinary Growing Perpetuity

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Suppose you are interested in a financial asset which makes promised payments that you expect to grow indefinitely into the future. A common share is a good example of such a financial asset. Suppose that the first payment, $C, is expected one period from now (of course, these payments are dividends if the financial asset is a common share). Assume that each payment is expected to be **1+g** times greater than the previous. Finally, let the periodic interest rate be r. As long as r > **g**, the present value of this set of payments is

$$PV = \frac{C}{r-g} \qquad r > g \qquad (6.13)$$

Suppose you buy a common share of a firm which promises to pay dividends yearly (more commonly, common shares pay dividends quarterly). The upcoming dividend is expected to be \$1.2 per share and is due in one year. The growth factor in dividends is expected to be 2% per annum. The average rate of return on shares of equivalent risk to this particular common share is about 12% per annum. What is the tradable value of the common share? The answer to this question is obtained by substituting C = 1.2, g = 0.02, and r = 0.12 into equation (6.13). The tradable value is PV= 1.2/(0.12–0.02) = \$12.

Suppose, alternatively, that the common share is expected to pay a dividend of \$1.2 almost immediately. What is the tradable value of the common share? In this case, the value of the share is the \$1.2 to be received immediately plus the present value of remaining dividends using equation (6.13). However, you should carefully note that the dividend in one year is expected to be 2% higher than the dividend immediately received. The value of the share is PV = 1.2 + 1.2*1.02/(0.12–0.02) = \$13.44.

PV of a Growing Perpetuity: 11 Minutes

Case 2: Growing Perpetuity Due

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Suppose an investment returns \$C per period indefinitely starting immediately (rather than in one period) and then each payment thereafter is 1+g greater than the previous. This is a growing perpetuity "due." We know from section 6.2.8 we can convert the PV for a stream of ordinary payments (first payment in one period) to the PV of the equivalent "due" case (first payment immediate) by multiplying by (1+r) where r is the opportunity cost rate of return. So, the PV of a growing perpetuity due is,

$$PV = \frac{C}{r-g}(1+r) \qquad r > g \qquad (6.14)$$

PV of a Growing Perpetuity Due: 4 Minutes

Case 3: Growing Perpetuity Deferred/Accelerated

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Suppose that you have been promised a perpetuity of equal periodic payments but the payments do not begin until **k** periods from today (k is a non-negative number). The first payment is $C and each subsequent payment is 1+g greater. The period effective rate of interest is **r**. This growing perpetuity is deferred relative to ordinary if k>1 and accelerated relative to ordinary if 0<k<1. The present value of a deferred/accelerated growing perpetuity is equation (6.13) discounted by another k–**1** periods.

$$PV = \frac{\dfrac{C}{r-g}}{(1+r)^{k-1}} \qquad r > g \qquad (6.15)$$

Notice that if k is equal to one, so the perpetuity is not deferred but an ordinary with the first payment exactly one period from today, then, equation (6.15) reduces to the present value of an ordinary perpetuity in equation (6.13). On the other hand, if k equals zero, then the first payment is immediate so that equation (6.15) is the value of an ordinary perpetuity in equation (6.13) times (1+*r*). Any ordinary PV calculation can be converted to a PV "due" calculation (payments at the beginning of the period) by multiplying by (1+*r*).

PV of a Deferred/Accelerated Growing Perpetuity: 6 Minutes

6.2.11 PV of a Growing Annuity

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Consider a financial asset with a specified term where the payments grow over time. Suppose that the first payment of $C is expected one period from now, the period interest rate is r, each payment is expected to be 1+**g** times greater than the previous, and there are a total of **n** payments (the value of the n'th payment is $C*(1+g)^{n-1}$). If $r \neq g$, the present value of this set of payments is,

$$PV = \frac{C}{r-g}\left[1-\left(\frac{1+g}{1+r}\right)^{n}\right] \qquad (6.16)$$

As an example, suppose that there is a financial asset that offers a per annum rate of return of 6% compounded quarterly with 27 quarterly payments and the first in exactly one quarter. The first payment is \$100 and each subsequent payment is 2% greater than the previous. In this case, the effective return on the financial asset per quarter is 0.06/4 = 1.5%. Then, r = 0.015, g = 0.02, C = 100, and n =27. Substitute these values into equation (6.14). The value of the financial asset is

$$PV = \left[\frac{100}{0.015 - 0.02}\right] * \left[1 - \left(\frac{1.02}{1.015}\right)^{27}\right] = \$2{,}837.65$$

Note that because $g > r$ in this example, each term in the product on the right hand side of the above calculation is negative. The product of two negative numbers is, however, positive. Of course the present value of a set of positive payments should be positive.

PV of a Growing Annunity, r not Equal g: 10 Minutes

On the other hand, rather than $r \neq g$, if $r = g$, the present value of a growing annuity of n payments that start one period from today with a payment of \$C and each subsequent payment is 1+g times greater than the previous is,

$$PV = \frac{n * C}{1 + r} \qquad (6.17)$$

Consider a financial asset that offers a per annum rate of return of 6% compounded quarterly with 27 payments and the first in exactly one quarter. The first payment is \$100 and each subsequent payment is 1.5% greater than the previous. In this case, the effective return on the financial asset per quarter is 0.06/4 = 1.5% and, at the same time, g = 0.015. Thus, r = g. In addition, C = 100, and n =27. Substitute these values into equation (6.15). The value of the financial asset is $27*100/1.015 = \$2,660.10$.

PV of a Growing Annunity, r Equal g: 10 Minutes

6.2.12 FV of a Growing Annuity

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Suppose you plan to make n periodic deposits into an investment account. The account offers an effective return of “r” per period. Your first deposit is $C and each subsequent deposit is 1+g times greater than the previous. Your last deposit is n-1 periods from the first and equals $C*(1+g)^{n-1}$. If $r \neq g$, the future value of your deposits with interest immediately after the n’th deposit is,

$$FV = \left[\frac{C}{r-g}\right]\left[(1+r)^n - (1+g)^n\right] \qquad (6.18)$$

For example, your investment account offers a per annum return of 6% compounded quarterly. Starting today you plan to make 27 quarterly deposits. The first deposit is $100 and each subsequent deposit is 2% greater than the previous. In this case, the effective return on the financial asset per quarter is 0.06/4 = 1.5%. Then, r = 0.015, g = 0.02, C = 100, and n =27. Substitute these values into equation (6.16) to find that your account balance immediately after then 27’th deposit is,

$$FV = \left[\frac{100}{0.015 - 0.02}\right] * \left[(1.015)^{27} - (1.02)^{27}\right] = \$4,241.73$$

Note that because $g > r$ in this example, each term in the product on the right hand side of the above calculation is negative. The product of two negative numbers is, however, positive. Of course the future value of a set of positive payments should be positive.

FV of a Growing Annuity, r Not Equal g: 6 Minutes

On the other hand, rather than $r \neq g$, if $r = g$, the future value of a growing annuity of n payments at the time of the last payment with a first payment of \$C and each subsequent payment 1+g times greater than the previous is,

$$FV = C * n * (1 + r)^{n-1} \tag{6.19}$$

Consider an investment account that offers a per annum rate of return of 6% compounded quarterly. You plan 27 quarterly deposits. The first deposit is \$100 and each subsequent deposit is 1.5% greater than the previous. The effective return on the investment account per quarter is 0.06/4 = 1.5% and, at the same time, g = 0.015. Thus, r = g. In addition, C = 100, and n =27. Substitute these values into equation (6.17). Your account balance at the time of the 27'th deposit is $FV = 100 * 27 * (1.015)^{26} = \$3,976.32$.

FV of a Growing Annuity, r Equal g: 4 Minutes

6.2.13 “The” Return on an Investment

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A common finance question is "what is the return on this business investment?" There are a number of ways to answer this question but the most common answer is the *internal rate of return*. The internal rate of return (IRR) is that *hypothetical* opportunity cost rate of return that makes NPV equal zero. IRR is a hypothetical opportunity cost rate of return because discounted cash flow analysis (that is, PV) requires the one and only opportunity cost rate of return that makes NPV greater or lesser than zero but almost surely not exactly equal zero.

Why does IRR make NPV equal zero? The answer is that if NPV>0, the business investment is better (in terms of shareholder wealth creation) than an equivalent financial investment with equal risk. If NPV<0 the business investment is worse than an equivalent financial investment with equal risk. So, only if NPV=0 is the business investment equivalent to an equal-risk financial investment. If equivalent, the business investment should have the same return as the financial investment (otherwise, it would not be equivalent). So IRR is a business investment return because it is equivalent to an equal-risk financial investment with respect to shareholder wealth creation.

As an example, suppose you predict that an investment generates, $50, $60, $70, and $80 in one, two, three, and four years from today respectively. The required expenditure to initiate the investment is \$100. What is the rate of return on this investment? We answer this question by finding the internal rate of return:

$$-100 + \frac{50}{(1+IRR)} + \frac{60}{(1+IRR)^2} + \frac{70}{(1+IRR)^3} + \frac{80}{(1+IRR)^4} = 0$$

We can solve this equation by: (1) trial and error, (2) a financial calculator, or (3) an IRR financial function in a spreadsheet computer software package. No one in this day and age uses method (1). It is just too much work to get an accurate solution. So, if you use method (2) or (3), you find the IRR is about 48% per annum. Not a bad return!

IRR with a Financial Calculator: 12 Minutes

An attractive feature of stating the benefits of an investment as a rate of return is that all of us have at least a rudimentary/ballpark understanding of what constitutes a good or bad return in financial markets. Even without a precise evaluation of equally risky financial assets, we know that 48% is much better than we could typically earn on average in financial markets, and therefore, 48% seems like an exceptionally good rate of return. For this reason, often internal financial analysis which is carried out within a corporation never goes as far as the valuation question that is answered in the NPV calculation. While there is information in the NPV calculation, because NPV is denominated in dollars (and it is easy to lose perspective when millions of dollars flash before your eyes) it is harder to ascertain from this value whether the investment is tremendously good or only marginally good. For example, suppose that the opportunity cost of investment is 10% per annum. The NPV of the above investment is then (according to my calculations) \$102.27. It is difficult to recognize from this value before we do the IRR calculation that the investment appears to be tremendously good.

In real asset evaluation, projects are acceptable if their IRR exceeds the financial market opportunity cost. In our example, because 48% > 10%, the investment appears to be acceptable.

Let's do a former final-exam question that requires a financial calculator for solution. We begin by finding the IRR per the payment interval, which is per month. That is we find the IRR/month. In the first instance an IRR is an effective rate per the payment interval because we always use effective rates in present and future value calculations and we calculate the IRR from a NPV calculation. After the first instance we can transform the IRR and state it in any way we want, including an effective rate over a period of time other than the payment interval or as a nominal rate. So, after we find the IRR/month in the solution below, you will have to read section 6.2.15 to learn about effective and nominal rates and making conversions between them to understand the complete solution to this question.

You buy a financial asset today for $100 that pays $100 in one month and $100 in twenty nine months.

(i) Find the per annum IRR on this investment.

(ii) Find the per annum IRR compounded quarterly on this investment.

Solution	**IRR with EXCEL and Cash-Flow Frequencies**	**IRR with TI BA II Plus and Cash-Flow Frequencies: 14 Minutes**

6.2.14 Simple IRR Calculations

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There are a number of special cases for which the IRR calculation is relatively simple in the sense that we can develop a formula. In this subsection, we discuss four of these cases (and a number of minor variants). In end-of-chapter problem #78 you study a fifth special case, the IRR for a two-period investment that requires the quadratic equation, which you have probably forgotten from your high-school algebra classes.

First, recall that the IRR is that hypothetical opportunity cost rate of return that makes NPV equal zero. There is only one opportunity cost rate of return for a business investment that depends upon interest rates in the economy and risk of the business investment. Investors, influenced by these factors, determine opportunity cost rates of return by trading in financial markets. The IRR is the return on a business investment that is equivalent to a financial investment with respect to return, risk, and wealth creation (thus, the NPV=0 condition). Recall, also, that the notation we usually use for an opportunity cost rate of return is "r." Thus for the IRR calculation in the special cases of this sub-section, we use a PV from above but rather than the "r" in the formula, we substitute the IRR and then set NPV to zero. So, with this methodology in mind, let us proceed to the special cases.

SPECIAL IRR CASE 1:

A ONE-PERIOD INVESTMENT

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In this subsection, we calculate the IRR for a one period investment that requires an expenditure of \$I today and returns \$C (once) in one period. Equation (6.3) is the PV for this investment and, thus, we find the IRR with,

$$NPV = PV - I = \frac{C}{1+IRR} - I = 0$$

Solve this equation to find that the IRR for a one-period investment is:

$$IRR = \frac{C}{I} - 1$$

As an example, suppose that a financial asset cost $1,200,000 and pays $1,500,000 in two years. Notice that we are working backwards from an example we did in section 6.2.3. The rate of return on this investment over the two-year holding period is,

$$IRR = \frac{C}{I} - 1 = \frac{1,500,000}{1,200,000} - 1 = 25\%$$

IRR for a One Period Investment: 11 Minutes

SPECIAL IRR CASE 2:

PER SUB-PERIOD IRR for ONE PERIOD INVESTMENT

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By convention, financial markets always state rates of return per annum. So, the return for the example above for SPECIAL IRR CASE 1 has the peculiarity that it is a return for a two-year holding period. We can calculate the per-period IRR for an investment that requires an expenditure of $I and pays $C (once) in *n* periods.

Equation (6.5) is the PV for this investment and, thus, we can find the IRR with,

$$NPV = PV - I = \frac{C}{(1+IRR)^n} - I = 0$$

Solve this equation to find that the per-period IRR is:

$$IRR = \left(\frac{C}{I}\right)^{\frac{1}{n}} - 1$$

As an example, suppose that a financial asset cost $1,200,000 and pays $1,500,000 in two years. The per-annum rate of return for this investment is

$$IRR = \left(\frac{1,500,000}{1,200,000}\right)^{\frac{1}{2}} - 1 = 1.25^{1/2}\text{-}1 = 11.80\%$$

This calculation divides the holding period rate of return (that is, 25%) by two because there are two sub-periods in the holding period. However, in going from an effective rate over a longer period (two years) to an effective rate over a shorter period (per annum), we divide by two geometrically rather than arithmetically.

IRR per Period for $C in "n" Periods: 7 Minutes

SPECIAL IRR CASE 3:

The IRR for a TYPICAL BUSINESS INVESTMENT

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A typical business investment requires an expenditure of \$I and returns \$C per period indefinitely starting in one period. The amount \$C depends upon profitability of the business venture (in a way we discuss below).

Equation (6.10) is the PV for this investment and, thus, we find the IRR with,

$$NPV = PV - I = \frac{C}{IRR} - I = 0$$

Solve this equation to find the per-period IRR is:

$$IRR = \frac{C}{I}$$

In section 2.6.4 of chapter two, we noted that both ROIC and ROE are special IRRs. For ROIC, C=(1-t)*(EBITDA-deprec) and I=IC (invested capital). In the case of ROE, C=NI (net income) and I=BVE (book value of equity). So, both ROIC and ROE are special IRRs, which presume the pattern of payments in an ordinary perpetuity.

IRR for an Ordinary Perpetuity: 4 Minutes

We can also calculate the IRR for a slight variant of this business investment. We can calculate the IRR for a perpetuity due.

From the PV in equation (6.11) we can calculate the IRR of a perpetuity due,

$$NPV = PV - I = \frac{C}{IRR}(1 + IRR) - I = 0$$

Solve this equation to find the per-period IRR is:

$$IRR = \frac{C}{I - C}$$

This formula is similar to the IRR for an ordinary perpetuity but you divide by net expenditure ($I-$C) rather than the expenditure ($I).

Suppose an investment costs $1,000 and pays $100 per period indefinitely with the first payment immediate. Then, the rate of return on this investment is,

$$IRR = \frac{100}{1,000 - 100} = 11.11\%$$

IRR for a Perpetuity Due: 3 Minutes

SPECIAL IRR CASE **4:**

The IRR for a GROWING PERPETUITY

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We can find the per-period IRR of an investment that costs \$I that pays \$C in one period with subsequent payments 1+g greater than the previous per-period indefinitely.

This situation describes neither a typical business investment nor an entire business. Typical business investments do not grow organically (simply with the passage of time). In addition, businesses in their entirety grow only with incremental business investments beyond the initial \$I we describe above. Rather than a business investment, some financial investments grow organically (or at least we expect), like, for example, a common share and, thus, the analysis here is most suited to a common share. This discussion tells us that we have to be careful. The common-share of a business differs from the business itself. More on these issues in chapters 8, 9, and 10 to follow.

Of course, there are exceptions to the general rule that business investments do not grow organically but, in the entire universe of businesses, these business investments are relatively rare. A stand of timber or an apple orchard, for example, are business investments that grow organically (pun intended ─ ignoring fertilizers and other minor incremental investments). A publicly traded company that has business investments that grow organically is Acadian Timber

Corp with stock-ticker ADN on the Toronto Stock Exchange (TSX). Acadian Timber is a leading supplier of primary forest products in Eastern Canada and the Northeastern US.

Equation (6.13) gives the PV of a growing ordinary perpetuity, so, we can find the IRR from,

$$NPV = PV - I = \frac{C}{IRR - g} - I = 0$$

Solve this equation to find the per-period IRR is:

$$IRR = \frac{C}{I} + \text{g}$$

Suppose an investment costs \$1,000 and pays \$100 in one period with subsequent payments that are 2% greater than previous indefinitely. The IRR of this investment is,

$$IRR = \frac{100}{1{,}000} + 0.02 = 12\%$$

IRR for an Ordinary Growing Perpetuity: 8 Minutes

We can also calculate the IRR for a slight variant of this investment. Suppose an investment requires an expenditure of \$I and returns \$C immediately (rather than in one period) and each subsequent per-period payment is 1+g greater than the previous indefinitely. This is a growing perpetuity "due." We know from section 6.2.8 we can convert the PV for a stream of ordinary payments (first payment in one period) to the PV of the equivalent "due" case (first payment immediate) by multiplying by (1+r) where r is the opportunity cost rate of return. So, the PV of a growing perpetuity due is (presuming r>g),

$$PV = \frac{\mathrm{C}}{\mathrm{r\text{-}g}}\ (1+\mathrm{r})$$

A perpetuity due is always more valuable than an ordinary perpetuity because investors always like to get the first payment sooner (immediate) rather than later (in one period). From this PV we can calculate the IRR of a growing perpetuity due,

$$NPV = PV - I = \frac{C}{IRR - g}(1 + IRR) - I = 0$$

After a little algebra, you can solve this equation to find the per-period IRR of a growing perpetuity due is:

$$IRR = \frac{C + g * I}{I - C}$$

Suppose an investment costs $1,000 and pays $100 immediately with subsequent payments that are 2% greater than previous indefinitely. The IRR of this investment is,

$$IRR = \frac{C + g * I}{I - C} = \frac{100 + 0.02 * 1{,}000}{1{,}000 - 100} = 13.33\%$$

IRR for Growing Perpetuity Due: 5 Minutes

6.2.15 Effective and Nominal Rates

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Nominal Versus Effective Rates: 24 Minutes

In all of the present and future value calculations to this point in this chapter, we have used the effective rate of interest over a variety of different holding periods. The effective rate of interest, or equivalently, the effective rate of return, is the rate of growth of your wealth as the result of an investment over a particular period of time. So, an effective rate is the rate of growth of your wealth for a day, a week, a month, a quarter, a year, or 10 years, for example. The adjective "effective" describes the fact that this rate might arise in part from interest paid on interest over the holding period. The effective rate of interest gives "effect" over the holding period to the way in which interest is calculated and added to the investment account. There is always a period of time associated with an effective rate but not necessarily a *compound* period.

Suppose that an investment account calculates and adds interest **m** times per annum. We refer to **m** as the number of compound periods per annum. The interval between interest payments is called the compound period. At the end of a compounding period, the amount added to the account is the opening balance times the effective rate for the compounding period. By convention in financial markets, nominal rates of interest or, equivalently, nominal rates of return are always stated per annum. A nominal rate of interest always comes to us as two pieces of information but this information can be stated in two ways. A nominal rate of return is a particular per annum rate (10% for example) and either the number of compound periods during a year (for example, **m**=365, **m**=12, **m**=4, **m**=2, or **m**=1) or the length of a compound period (daily, monthly, quarterly, semi-annually, or annually). The nominal rate of interest is sometimes called the *annual percentage rate* (APR), which summarizes the way in which interest is calculated and added to an investment account.

The APR equals the effective periodic rate charged (in percent) for one compound period multiplied by the number of compound periods per year. The APR is related to the effective rate of interest over one compound period by this equation:

$$APR = \mathrm{m} \times \text{ (Effective Rate for a Compound Period)} \qquad (6.20)$$

This equation represents the interest added to your investment account per dollar of opening balance. For example, suppose you have an opening balance of $1,000,000, the annual percentage rate is 10 per cent and interest is compounded monthly. The amount added to your account in interest at the end of the month is $1,000,000× 0.10/12 = $8,333.33.

The nominal rate of interest is the rate that is "named" in a financial contract, and therefore, it is also called the *quoted* rate or *the contract rate of interest*. In financial markets, interest rates are generally quoted yearly, and therefore, the nominal rate of interest is a per annum rate.

The nominal rate of interest is *always* accompanied by a *compounding period*. In some markets, because it becomes tedious to continuously repeat the compounding period, it may not be explicitly stated. In these markets, participants need to be aware of the compounding period. As an example, the federal Interest Act requires that Canadian mortgages be quoted using semi-annual compounding. However, mortgage lenders seldom offer this information when quoting rates.

The nominal rate of interest describes the mechanical way in which interest is calculated and added to an investment account. Because bankers are rather mechanical and methodical, they love nominal rates of interest. However, investors (or borrowers) are more interested in the rate at which their wealth grows (or is depleted) over a year, and therefore, they are more interested in the *effective annual rate* (EAR). Investors (borrowers) are concerned with nominal rates of interest only to the extent they may be needed to determine the EAR.

The inverse of equation (6.16) gives the effective rate per one compound period from a nominal rate:

$$\text{Effective Rate for One Compound Period} = \frac{APR}{m}$$

The Effective Rate for One Compound Period: 6 Minutes

The effective interest rate for **n** compounding periods is:

$$\left(1+\frac{APR}{m}\right)^{n}-1 \tag{6.21}$$

The Effective Rate for N Compound Periods: 7 Minutes

This equation is an application of the future value of $1 over a holding period composed of **n** sub-periods. The sub-period in the current context is a compounding period. Notice that as a special case of equation (6.17), if the holding period is for one year, then n = m, and the effective *annual* rate (EAR) is:

$$EAR = \left(1 + \frac{APR}{m}\right)^m - 1 \tag{6.22}$$

Because the EAR includes interest paid on interest whereas the APR does not, effective rates per annum are always greater than or equal to the nominal rate.

As an application of equations (6.21) and (6.22), if the nominal rate of interest is 10% per annum compounded monthly, find the effective rate of interest for holding periods of 3, 12, and 24 months. By substituting, APR=0.1, m=12, and n=3, 12, and 24 into equation (6.17) you can verify, that the effective rates are 2.52%, 10.47%, and 22.03% respectively.

The EAR from A Nominal Rate: 5 Minutes

To understand compound interest, you must be able to work backwards and forwards between effective rates over any holding period and nominal rates per annum compounded for any desired compounding period. For example, if the EAR is 10%, what is the nominal rate per annum compounded monthly, quarterly, and semi-annually? Using equation (6.22), substitute EAR = 0.1, and then respectively m =12, 4, and 2. You should be able to verify that the nominal rates per annum compounded monthly, quarterly, and semi-annually are 9.569%, 9.645%, and 9.76% per annum respectively. Notice in this example, to get the same EAR, the nominal rate per annum compounded monthly is the lowest rate because there is more interest on interest when the number of compounding periods per year is greater. Because there is more interest on interest with more compounding periods, you can calculate interest with a lesser rate and still have the same amount in your account balance at the end of the year.

You should think of the effective rate of interest as the “market rate of interest” in the economy (at least for the class of financial assets under consideration). Because of competition in financial asset markets, all assets with equivalent risk earn the same market rate of interest. For example, institutions that offer lower effective lending rates for equivalent deposit products will soon lose market share. Further, there exists only a single market rate of interest (the EAR) regardless of how interest is calculated and added to investment accounts because investors’ primary concern is with effective rates (i.e., the rate at which their account grows). As an example of this phenomenon, suppose that, contrary to our prediction, three different banks pay interest on deposits at these quoted interest rates, 10% compounded semi-annually, 9.8% compounded quarterly, and 9.75% compounded monthly. If you intend to lend funds for one year, which account do you prefer? Other things equal, you should find that the last two banks lose market share to the first. Clearly, in this case, nominal interest rates will change until each of the three accounts pays the same EAR, even though the compounding periods differ.

A Nominal Rate From the EAR: 5 Minutes

Now that you can calculate effective rates from nominal rates and vice versa you should try some of our previous calculations in this new environment. For example, if the market rate of interest in the economy is 10% per annum compounded quarterly, what is the present value of an ordinary annuity of $100 per month for 36 months? First, we have to find the effective rate of interest for a one-month period. The effective rate for a quarter is 0.10/4 = 2.5%. If r represents the effective rate for a month, then $(1+r)^{3} = 1.025$. Solving this equation, we find that the effective rate is 0.8265% per month. The present value of the annuity is, therefore, PV = $100*[1–1/(1.008265)^{36}]/0.008265 = \$3102.77$. On the other hand, suppose that the interest rate is 10% compounded monthly, what is the present value of the same annuity? In this case, PV = $100*[1–1/(1+0.1/12)^{36}]/(0.1/12) = \$3099.12$.

For a future value question, suppose that the rate of interest in the economy is 10% per annum. What is the future value at the time of the last deposit of $100 deposited every month for 36 months? First, note that because we are given neither a context (i.e., a market) nor a compounding period, the per annum rate must be an effective rate of interest. If r is the monthly effective rate, then $(1+r)^{12} = 1.1$. Solving this equation, the monthly effective rate is 0.7974%. The future value at the time of the last deposit is therefore $100*[(1.007974)^{36}–1]/0.007974 =$ $4150.99.

6.2.16 Identify Effective Versus Nominal Rates?

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How can you tell if a rate is effective or nominal? The answer is that if the rate is correctly stated, then you can tell with one word: "compounded." If the rate is stated as "compounded" then it is a nominal rate. If you do not see the word "compounded" then it is an effective rate. Notice, however, even if you do not see the word "compounded" because an effective rate is the rate of growth of your wealth over a period of time, there is always a period associated with an effective rate.

Let us do some examples:

1. "10% per annum" is an effective rate because the word "compounded" does not describe this amount,

2. "10% per quarter" is an effective rate because the word "compounded" does not describe this rate,

3. "10% per annum compounded quarterly" is a nominal because the word "compounded" describes this rate.

Identify a Nominal Versus an Effective Rate: 4 Minutes

Unfortunately, financial markets are not quite as easy as we describe above. Sometimes rates are described poorly or wrongly. Sometimes, by habit, nominal rates in particular markets rates are informally described by active participants without detailing the compound information. The Canadian mortgage market is an example that we investigate in some detail in section 6.3.3 below. A “3% mortgage rate,” is really “3% per annum compounded semi-annually” because all Canadian mortgage rates must be stated by law as a per annum rate compounded semi-annually regardless of the payment period (which is often monthly). In formal mortgage contracts, rates are stated with great precision and include the compound information. In informal settings, the compound information is often omitted for simplicity of discussion. Thus, a market context will often tell you that a rate is nominal. For example, if a discussion is about a Canadian mortgage, even if it is not formally stated, a rate must be per annum compounded semi-annually because of legal requirements.

6.2.17 Effective RoR Over a Holding Period

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Oftentimes in financial problems you must restate an effective rate of return on an investment for some time period other than the holding period. For example, if you know the rate of return for a holding period, you might want to know the rate of return per sub-period within the holding period. Alternatively, if you know the effective rate of growth of your investment over a sub-period, you might want to know the rate of growth over a longer period. These problems require a conversion of one effective rate of return to another over an alternative period.

The basic relation that is required for this conversion is the future value of one dollar invested for n sub-periods where **n** is the number of sub-periods in the holding period. When converting an effective rate for a sub-period to an effective rate for a longer holding period, notice that a power greater than one is used in this relation.

$$\textbf{Effective Rate for a Holding Period} = (1 + \textit{effective rate for a subperiod})^n - 1 \qquad (6.23)$$

As an example, suppose that you know that the effective rate of return on an investment is 2% for a quarter. The EAR is then, $(1+0.02)^4 - 1 = 8.24\%$.

Effective to Effective, Short to Long: 5 Minutes

Alternatively, suppose that you want to know the rate of return per sub-period on an investment that is made over a longer holding period. Rearranging equation (6.19),

$$\textbf{Effective Rate for a Sub-period} = (\mathbf{1} + \textbf{Effective rate for a Holding Period})^{\mathbf{1/n}} - \mathbf{1} \qquad (6.24)$$

In this equation, there are **n** sub-periods in the holding period. When converting an effective rate over a longer holding period to an effective rate for a sub-period, notice that a power lesser than one is used in this relation. This equation is often used to annualize a rate of return, for comparison purposes, on an investment that was made over a longer holding period.

As an example of how one might use the above relation, consider how the magic of compounding can be used (consciously or unconsciously) to exaggerate the performance of an investment. The former Dean of the Faculty of Business at Simon Fraser University has an annoying habit of praising the performance of the McGill University academic pension plan. When Stanley left the employment of McGill University in 1981 (as a Professor of Marketing), he left behind his defined-contribution pension plan to be invested on his behalf by the McGill University academic pension plan. Stanley often marvels at the performance of this investment plan and he boasts that the value of his money in this plan has increased fivefold between 1981 and 1996.

On the face of it, a fivefold increase in wealth does seem like an incredibly great investment achievement. However, in order to benchmark this performance, we need to annualize this rate of return, because returns in financial markets that we can use for benchmarking quote on an annual basis. Let us suppose that Stanley left behind $100,000 at McGill at the beginning of 1981. The holding period rate of return between 1981 and 1996 is (500,000–100,000)/100,000 =4.0 (i.e., 400%). Using equation (6.20), the annualized rate of return on this 16-year investment is $(1+4)^{1/16}-1 = 10.58\%$.

Not a bad rate of return, Stanley, but not an astonishingly rate of return, either! Between 1981 (a high interest rate period) and 1996, most balanced funds in Canada would have matched or bettered this performance. Perhaps you should have brought along your funds to Simon Fraser University, Stanley!

Effective to Effective, Long to Short: 6 Minutes

6.2.18 Is IRR a Nominal or Effective Rate?

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Good question. The answer is that in the first instance the IRR is an effective rate (over the payment interval of the investment). However, after the first instance you can convert an effective rate and state it any way you want. Thus, you can state an IRR as a nominal rate. So, you have to be careful. We can state an IRR either as an effective rate or a nominal rate.

An IRR is an effective rate in the first instance because of the way it is defined. Recall that an IRR is that hypothetical opportunity cost rate of return that make NPV=0. In any NPV calculation, including the one that calculates an IRR, the opportunity cost rate of return is an effective rate over the payment interval of the investment. Why? We always use effective rates in any PV or NPV calculation. An opportunity cost rate of return is the rate of growth of your wealth for an alternative financial asset investment with the same risk. Rate of growth of your wealth defines an effective rate and, thus, an IRR is an effective rate in the first instance because it comes out of a NPV calculation.

However, remember you have to be careful! After the first instance, we can state an IRR any way we want including as a nominal rate. Perhaps an example might clarify. Below is a former final exam question.

Example: You buy a financial asset today for \$100 that pays \$100 in one month and \$100 in twenty nine months.

Required:

(iii) Find the IRR/month,
(iv) Find the per annum IRR,
(v) Find the per annum IRR compounded quarterly.

Solution

The equation that defines the IRR is,

$$-100+\frac{100}{(1+IRR)}+\frac{100}{(1+IRR)^{29}}=0,$$

which determines the IRR as an effective rate/month because the payment interval is per/month (although payments between the second and the twenty-eighth are zero).

Use a financial calculator or a spread-sheet function to determine that the IRR/month is 8.98632% (the above embedded solutions gives the detailed keystrokes for a TI BA II Plus calculator to solve this question). Notice that this is an effective rate because we do not use the word "compounded" either explicitly or implicitly. In the first instance, we determine the IRR as an effective rate. In part (ii) and (iii) of this question, we convert the IRR/month to an IRR/annum and to an IRR/annum compounded quarterly, respectively. In the first instance the IRR is an effective rate (that is, part i) but we convert it and state it as a nominal rate in part (iii).

6.2.19 Continuous Compounding

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If interest is paid into your investment account in a continuous stream, then the rate of growth of your account over an instant is called the continuously compounded rate. Notice that the continuously compounded rate is a nominal rate of interest. Because, interest is calculated and added to your account instant by instant, the number of compounding periods in the year is infinite (**$m \to \infty$**). No investment accounts or financial assets pay interest continuously but there are some brokerage accounts that calculate and pay interest daily which is close to continuous compounding. On the other hand, daily interest savings accounts, which are available at most banks and trust companies in Canada, calculate interest on the outstanding balance from the close of business on the previous day, but they typically add interest to your account only at the end of the month. The effect of this process is closer to monthly compounding rather than continuous compounding. Despite the fact that there are no investment accounts or financial assets that offer interest in a continuous stream, continuous compounding is nonetheless important. For example, for real asset investments that we study in chapter 9, firms earn operating cash flows sequentially in a stream that is often best represented as continuous rather than at the end of an arbitrarily period. We can discount these cash flow streams with continuously compounded discount rates.

The relationship between the EAR and the continuously compounded rate *i* is:

$$EAR = e^{i} - 1 \tag{6.25}$$

where "e" is the transcendental number approximately equal to 2.7182818…

As an example of this relation, suppose that the continuously compounded rate on a financial asset is 10% per annum. What is the EAR? Substitute $i = 10\%$ into equation (6.21). Then, the EAR is 10.51709% per annum.

EAR from the Continuously Compounded Rate: 11 Minutes

We can rearrange equation (6.21) and write the per annum continuously compounded rate i in terms of the EAR,

$$i = \ln\left(1 + EAR\right) \tag{6.26}$$

As an illustration of this equation, what is the continuously compounded rate if the EAR is 10%? Substituting, EAR=0.1 into equation (6.21), you should be able to verify that the continuously compounded rate is 9.53102%.

The Continuously Compounded Rate from the EAR: 5 Minutes

An extension of equation (6.21) is the future value of \$C invested for n years,

$$\text{FV} = C * e^{i*n} \tag{6.27}$$

This equation is one plus equation (6.26) multiplied by itself n times. Suppose that the continuously compounded per annum rate is 10%. What is the future value of \$100 invested for 2.5 years. Your account balance after 2.5 years is $100 * e^{0.1*2.5} = 100 * e^{0.25} = \$128.40$.

The inverse of equation (6.27) is the PV of \$C after n years. That is, what is the lump sum deposit today into an account that pays a continuously compounded per annum rate i so that \$C can be withdrawn n years from today. The answer to this question is

$$\text{PV} = C * e^{-i*n} \tag{6.24}$$

For example, if C=\$100 and the continuously compounded rate is 10% per annum, then a deposit of $100 * e^{-0.1*2.5} = 100 * e^{-0.25} = \$77.88$ grows to \$100 after 2.5 years. The amount \$77.88 is said to "fund" \$100 after 2.5 years.

(6.3) Applications

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In the following subsections a number of applications of the principles of discounted cash flow analysis in a variety of different economic environments are investigated.

6.3.1 Amortizing a Term Loan

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A term loan has the defining characteristic that payments are periodic and constant for a finite term. In other words, a term loan is an annuity of payments. Because payments are constant, part of each payment is interest and part is a reduction of principal. There are a number of reasons that one might wish to decompose each payment into interest and principal reduction. First, the original amount borrowed less principal reduction in past payments is the *outstanding balance* which is the amount the lender requires (plus possible penalties) as relieve from further payments (i.e., the amount you need to "pay off" the loan). Second, for financial accounting purposes, only the interest portion of payments is expensible. Third, if the original borrowing was undertaken for the purpose of generating income (as recognized by Canada Revenue Agency), then the interest portion but not the principal portion of each payment is tax deductible.

There are a number of ways to decompose each payment in a term loan into interest and principal repayment. The first way is to design and complete a loan amortization table. In the loan amortization table, determination of the outstanding balance is called retrospective, because it is calculated as the original amount borrowed less principal reduction from all past payments. To illustrate a loan amortization table, consider the following example.

Suppose you borrow $1000, which is to be repaid in three equal payments at the end of each of the next three years. The nominal rate of interest on this contract is 6% compounded once per annum.

The first thing we do in this example is to find the payments. Because the bank has purchased a financial asset, from their perspective, $1000 is equal to the present value of three equal payments with the first payment in one year, $\$1000 = C^*[1–1/(1.06)^3]/0.06$. Solving this equation, C = $374.11. The loan amortization table is given below:

Loan Amortization Table

Year	Payment	Interest	Principal Repayment	Balance
0	–	–	–	$1000.00
1	$ 374.11	$ 60.00	$ 314.11	685.89
2	374.11	41.15	332.95	352.93
3	374.11	21.76	352.93	0.00
			$1000.00	

Interest in each payment is calculated as the outstanding balance at the beginning of the period times the contract rate of interest – 6%. Each principal repayment is the payment less the interest portion. The outstanding balance is the balance at the beginning of the period minus principal reduction associated with the payment.

Loan Amortization Table: 13 Minutes

Loan Amortization with BA II Plus

An alternative way to find the outstanding balance at any time during the life of the loan is to use a prospective approach. In the prospective calculation, the outstanding balance is the present value of the *remaining* payments at the *contract rate of interest* (regardless of what has happened to market rates of interest since the loan was first negotiated). For example, immediately after the first payment, when there remain two further payments (the first in one year), the outstanding balance is $374.11*[1–1/1.06^2]/0.06 = \$685.89$.

Prospective Approach to the Outstanding Balance: 7 Minutes

Rather than use the loan amortization table, principal reduction at the **k**'th payment in a term loan which has **n** payments at its inception is equal to the payment on the term loan discounted at the contract rate of interest for **n–k+1** periods.

$$\text{Principal Reduction at the } k\text{'th of } n \text{ payments} = \frac{Per\ Period\ Payment}{(1+i)^{n-k+1}}$$

where i is the mortgage rate per payment interval.

For example, in the above term loan, principal reduction at the first payment is:
$374.11/(1.06)^{3-1+1} = \$314.11$.

Principal Repayment at a Payment: 8 Minutes

6.3.2 Term Loan Prepayment

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Suppose that, in the term loan described in the previous subsection, immediately after the first payment you receive some unexpected cash and are considering whether you should pay off the loan. Paying off a loan prior to maturity is called prepaying. You can view prepayment as an investment: in exchange for immediate cash, you are relieved of the obligation to make future loan payments. Whether you should prepay the loan depends upon the "return to prepayment" and your opportunity cost which is the rate of return for the funds that you unexpectedly have available. Suppose the market rate of interest has increased from 6 percent per annum to 10% per annum since you originally negotiated your loan. Your opportunity cost to prepaying your mortgage is, therefore, 10% per annum. On the other hand, what is the return to prepaying your mortgage? Your prepayment is a current expenditure of $685.89 (the outstanding balance) in exchange for relieve from payments of $374.11 and $374.11 in the upcoming two years. The return to your investment is the IRR on this set of cash flows: $-685.89 + 374.11\,[1-1/(1+IRR)^2]/IRR = 0$. You can solve this simple equation or notice that the outstanding balance is the discounted value of remaining payments on your mortgage at the contract rate of interest. In our example, therefore, the return to prepaying your mortgage is 6%. Because our opportunity cost of 10% exceeds our return to prepaying, our best use of funds is to invest elsewhere rather than prepay.

Rather than use returns to solve the above prepayment problem, we can also use the techniques of discounted cash flow analysis. The present value cost of making continued payments on our loan (i.e., invest our excess cash elsewhere) is $374.11\,[1-1/1.1^2]/0.1 = \$649.28$. Notice that in this calculation, the market rate of interest of 10% is used because this is the opportunity cost of the excess cash. An interpretation of this dollar amount is that one way we can effectively pay our liability is to deposit $649.28 into an investment account at 10% per annum. With interest, we can make withdrawals of $374.11 and $374.11 one and two years from today and use these funds to make our final payments on the loan. The cost today, therefore, of paying off the loan by investing elsewhere is $649.28. Because this is lesser than the outstanding balance of $685.89, it is cheaper to kill off the liability by investing elsewhere.

Suppose that, in the example above, the market rate of interest has decreased rather than increased since you originally negotiated your loan from 6 percent to 3% per annum. The return to prepaying your mortgage remains 6% (because the outstanding balance and the remaining payments on the loan are unaffected by whether the market rate of interest increases or decreases). Because the opportunity cost of 3% is lesser than the return to prepaying, the best use of funds is to prepay the loan rather than invest elsewhere. The present value cost of making continued payment on our loan (i.e., invest our excess cash elsewhere) is $374.11\ [1-1/1.03^2]/0.03 = \$715.85$. Notice that in this calculation, the market rate of interest of 3% is used because this is the opportunity cost of the excess cash. An interpretation of this dollar amount is that one way we can effectively pay-off our liability is to deposit $715.89 into an investment account at 3% per annum. With interest, we can make withdrawals of $374.11 and $374.11 one and two years from today and use these funds to make our final payments on the loan. The cost today, therefore, of paying off the loan by investing elsewhere is $715.89. Because this is greater than the outstanding balance of $685.89, it is cheaper to kill off the liability by prepaying.

Often, lenders include prepayment penalties or restrictions in term loan contracts. From the above example, can you develop an economic argument that justifies these impediments to prepayment?

Consider the poor bank. What will they do with the outstanding balance when you prepay? Of course, they will reinvest their funds by lending to others at the new market rate of interest. However, from the above example, when are you most likely to prepay? You prepay when market rates have decreased rather than increased because the opportunity cost of using your funds for prepayment is lesser. In the above example, you prepay when market rates fall to 3% but you make continued payments and invest elsewhere when market rates increase to 10%.

When interest rates are low, other things equal, lenders face greater prepayment rates from borrowers. When the lenders reinvest the proceeds of the early principal payments, they reinvest at lower rates of return. On the other hand, when market rates are high and they would prefer to have their funds back to reinvest at higher market rates, they find that the rate of prepayment from borrowers is lesser. The effect of prepaying when it is to the advantage of borrowers is to reduce the revenues of lenders. As compensation or protection against this loss of revenue, lenders impose prepayment penalties and/or prepayment restrictions.

6.3.3 The Canadian Mortgage Market

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There are a number of distinctive features of the Canadian mortgage market. First, unlike the United States, interest on mortgage payments where the mortgage is for your home (rather than for the purpose of investment income) is not tax deductible. Second, as a requirement of the federal Interest Act, rates in the Canadian mortgage market are stated as APR compounded semi-annually. Rates are quoted in this manner even though payments are unlikely to be made semi-annually. Presumably, this method of stating mortgage rates is mandated to allow consumers an unhindered ability to comparison shop for mortgages between financial institutions. Finally, the rate of interest on a mortgage is periodically reset to the current market rate. The length of time over which the rate on your mortgage is fixed is called the *term* of the mortgage. Common terms for Canadian mortgages are 6 months, 1 year, 2 years and five years. The longer period over which payments are made to pay-off an mortgage in its entirety is the *amortization* period.

As an example of the Canadian mortgage market, consider the following problem. Suppose you borrow $60,000 from the National Bank of Canada at a mortgage rate of 10% per annum with a 5-year term and a 25-year amortization period. You will make monthly payments. At the end of the term of the mortgage, rates in the mortgage market decrease to 9% per annum. By how much will your monthly payments decrease?

To answer this question, first note that because the question is about the Canadian mortgage market, the quoted rates must be compounded semi-annually even though payments are made monthly. At the time of the original borrowing, the effective rate for a one-month period is therefore $(1+0.10/2)^{1/6}-1$ = 0.81649% per month. The present value of payments at this rate must equal $\$60{,}000 = C*[1-1/(1.0081649)^{300}]/0.0081649$. Solving this equation, you can verify that the original payments are \$536.69 per month. After five years the outstanding balance is $\$536.69*[1-1/(1.0081649)^{240}]/0.0081649 = \$56{,}394.86$. At the new market mortgage rate of 9% per annum, the effective rate for a month is $(1+0.09/2)^{1/6}-1$ = 0.73631% per month. The discounted value of new monthly payments at this new monthly rate must equal the outstanding balance: $\$56{,}394.86 = \$C\times[1-1/(1.0073631)^{240}]/0.0073631$. Solving this equation, the new monthly payments are \$501.46 per month.

Canadian Mortgage Market: 30 Minutes

6.3.4 Retirement Planning with Inflation

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Harry is doing some retirement planning. Harry is 41 years old and he wants to retire when he is 65. Harry's current salary is \$100,000 per annum. In order to maintain his current lifestyle in his retirement, Harry wants an income that has *purchasing power* equal to \$100,000 per year at the current price level. In other words, in his retirement, Harry wants his real dollar income to be exactly what it is today. Inflation is expected to be 2.2% per annum into the indefinite future.

Harry's saving toward retirement are held in an investment account with a current balance of \$25,000. In addition, starting in exactly one year, Harry will make equal annual deposits to this account until his retirement. The 24'th and last deposit is exactly 24 years from today. Harry expects a rate of return of 10% per annum on this investment account (nominal compounded once per year).

In exactly 24 years, Harry plans to buy a 20-year annuity that makes annual payments. These payments will be Harry's income in his retirement. The first payment on this annuity will be in exactly 24 years. There will be a total of 20 annual payments on this annuity. The payments on the annuity will not be constant but will increase at a rate of 2.2% per annum. In addition, each of these payments will have a today's dollar purchasing power of \$100,000. Recall that Harry wants to consume \$100,000 per year in today's dollars in his retirement. Harry expects a rate of return of 8% per annum on the annuity (nominal compounded once per year).

What equal annual deposits must Harry make to his investment account in order to finance his retirement plan? (Ignore mortality and tax considerations in this problem).

To solve this problem, first of all recognize that Harry's annuity payments must be greater than \$100,000 to account for the effect of expected inflation. In 24 years, Harry's first payment must be $(1.022)^{24}*100{,}000 = \$168{,}586.00$. Annuity payments must, thereafter, increase at 2.2% per annum to offset the effect of expected inflation. The amount that Harry needs in his account in exactly 24 years can be determined by using the formula for the present value of a growing annuity.

$$\text{required balance in 24 years} = 168{,}586 + \frac{168{,}586 \times 1.022}{0.08-0.022}\left[1-\left(\frac{1.022}{1.08}\right)^{19}\right]$$

$$= \$2{,}098{,}404.86$$

Let the annual deposits to Harry's investment account be described as "$***C***." The future value of the current account balance plus the future value of the deposits, both at an investment rate of 10% per annum, must equal the above requirement:

$$\$25{,}000 \times (1.10)^{24} + \frac{C}{0.10}\left[(1.10)^{24} - 1\right] = \$2{,}098{,}404.86$$

Solving this equation for ***C***, you can verify that Harry must make annual deposits of $20,929.00.

As an extension of this retirement problem, we could also have recognized that over the next 24 years, Harry's wage (or self-employment) income is likely to grow, and therefore, his annual contributions to his investment account are also likely to grow.

6.3.5 Tax Deferred Savings Versus Pay-down Mortgage

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One of the most important personal financial problems in Canada is whether homeowners should invest in their Registered Retirement Savings Plans (RRSP's) or pay down their mortgages. Like many financial problems, the optimal decision depends on the rates of return to alternative uses of funds. We already know half of the solution to this problem. In the above sub-section of this electronic book on prepayment of term loans, we learned that the rate of return to funds used for prepayment is the contract rate of interest on the loan (in the case at hand, the mortgage). We now investigate the rate of return to a dollar invested in a RRSP by considering the following example.

Suppose that your marginal personal tax rate is 40% and that you can invest in interest-bearing financial assets in your RRSP at the current market rate of 9% per annum. When you invest a dollar into your RRSP you get a deduction from taxable income in the amount of $1. The dollar invested is deferred income and it is taxed when you close your RRSP account in, say, 30 years at the time of your retirement. Also interest that you earn on the \$1 invested while it is in your RRSP account is not taxed. However, this interest is taxed when you close your RRSP. If you expect to receive 9% per annum on the interest bearing financial asset, what is the annualized holding period return?

Let us begin by finding the holding period return for 30 years. The general form of this calculation is given in equation (6.1). The expenditure required to undertake the RRSP investment is only 60¢ on the dollar because the RRSP contribution is tax deductible. You put \$1 into your RRSP but as the result of this investment you get $0.4 \times \$1 = \$0.4$ back from the government in the form of a tax refund (or a reduction in your tax payable). Notice that you do not get this refund if you use your \$1 to pay down your mortgage. The effective cost of making a \$1 RRSP investment is, therefore, \$0.6. The \$1 you invest in your RRSP grows to $(1.09)^{30} =$ \$13.27. This amount represents principal plus interest on the \$1 investment. Notice that your account grows at 9% per annum because interest is not taxed while it is in your RRSP. However, when you close your RRSP (in 30 years) the \$1 invested is taxed because it is deferred taxable income and, also, all of the interest earned is taxed. In other words, all of the \$13.27 (principal plus interest) is taxed. There is some uncertainty as to what your marginal tax rate will be in 30 years, but let us presume that it remains 40%. In this case, the future value of your \$1 RRSP investment after tax is $0.6 \times 13.27 = \$7.96$. Your 30 year holding period rate of return is, therefore, $(7.96–0.6)/0.6 = 1226.67\%$. Adding one for principal, and taking the 1/30'Th power, your annualized 30-year holding period return is $(1+12.2667)^{1/30}–1 = 9\%$ per annum. Notice that RRSP tax rules have the effect of completely eliminating tax on your investment!! You effectively invest at the before-tax market rate of interest.

We are now in a position to determine whether you should invest in your RRSP or pay down your mortgage. The answer to this question depends upon a comparison between the return to investing in interest-bearing financial assets for your RRSP and the rate you pay on your mortgage. The current market rate on interest-bearing financial assets is the rate of return for investments into your RRSP and your mortgage rate is the return to paying down your mortgage.

You should compare these two rates and invest where you get the higher return. As an example, your mortgage rate might be 11% per annum and the current market rate of interest for interest bearing financial assets might be 9% (you borrowed for your home a number of years ago when market interest rates were higher). In this case, because your mortgage rate is higher, your better investment is to pay down your mortgage. Notice that paying down your mortgage does not preclude you from making an RRSP investment in the future. Tax rules beginning in 1993 allow unused RRSP contributions to be carried forward and used in the future.

(6.4) Personal Taxes and DCF[1]

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As a general rule, personal taxes can be integrated into discounted cash flow analysis if they are included in both the cash flows to be discounted in the numerator and the opportunity cost in the denominator. Because these tax effects often cancel, discounted cash flow analysis with and without personal taxes is often equivalent. There are important exceptions, however, which arise for example in the Canadian home mortgage market where interest is not tax deductible but the opportunity cost of funds is after-tax. To illustrate the application of personal taxes in discounted cash flow analysis, consider the following problem.

You wish to mortgage your home for $50,000. A bank offers you the following payment options: (a) $526.61 monthly for 25 years or (b) $136.37 weekly for 15.5 years. For either set of payments, the effective annual rate of interest is 12.685%. You can verify this assertion by discounting either set of payments with the appropriate effective rate over the payment interval

[1] This section investigates the use of the discounted cash flow methodology for personal financial planning problems that oftentimes require an assessment of the tax implications of the decision at hand. This section can be omitted on the first reading of this chapter of this electronic book or for more introductory finance courses. Problems at the end of chapter that are associated with this chapter are identified as such and can be ignored if this section has not been studied.

and you will find that the result is $50,000. This discussion implies that borrowing rates in the economy are 12.685%. Oftentimes in financial markets, borrowing and lending rates diverge. Suppose that the current lending rate for individuals is 8.5% per annum. The *spread* between borrowing and lending is the difference between these two rates. Suppose in addition, that your marginal personal tax rate (federal and provincial) is 48%. If you have sufficient employment income to make payments under either plan, which do you prefer?

Notice that because the mortgage is for your home, the interest you pay is not tax deductible, and therefore, the cost of making payments is the full amount of the payment (i.e., the payments must be made out of after-tax dollars). On the other hand, your opportunity cost for funds is an after-tax rate because if you invest elsewhere in interest bearing financial assets you must pay tax on interest earned. Your opportunity cost for a year is 0.52*8.5% = 4.42%. The effective after-tax opportunity cost for a month is $(1.0442)^{1/12}-1 = 0.36108\%$. The effective after-tax opportunity cost for a week is $(1.0442)^{1/52}-1 = 0.08321\%$. The discounted values of future payments at these rates are:

$$\text{monthly:} \qquad \frac{526.61}{0.0036108}\left[1-\frac{1}{(1.0036108)^{300}}\right] = \$96{,}379.19$$

$$\text{weekly:} \qquad \frac{136.37}{0.0008321}\left[1-\frac{1}{(1.0008321)^{806}}\right] = \$80{,}057.39$$

A number of things are noteworthy about these calculations. First, both present values are greater than the amount that we borrow. The reason for this result is that our opportunity cost is counted after taxes but the interest on our mortgage is not tax deductible. We could effectively pay off our mortgage by investing $96,379.19 at 8.5% in an investment account, pay tax on the interest and make monthly withdrawals of $526.61 to make mortgage payments so that at the time of our last payment the amount in our investment account would equal zero.

Alternatively we could pay off our mortgage by investing \$80,057.39 at 8.5%, in an investment account, pay tax on the interest, make weekly withdrawals of \$136.37 to make mortgage payments so that at the time of our last payment the amount in our investment account would equal zero. The present value cost of paying off our mortgage in these ways is high because we must pay tax on interest earned when we invest elsewhere at 8.5%. Present values greater than the amount we borrow is simply an indication that if we really want to pay down our mortgage with excess funds, rather than invest elsewhere at 8.5%, we are better off paying the mortgage directly with the lender (taking advantage of prepayment opportunities). The rate of return to prepaying is the contract rate on the mortgage. In the case of both sets of payments, this rate is 12.6825% per annum (you can verify this calculation). Because the rate to prepaying directly with the lender is substantially greater than the rate to investing elsewhere (after-tax), the best use of excess funds is to prepay the mortgage.

Second, the present value for weekly payments is lesser than the present value for monthly payments. The implication of this comparison is that the weekly payment stream is preferable. However, you should note that the major reason for this result is that the amortization period is lesser. We could increase the attractiveness of the monthly payment stream by shorting the amortization period. While financial institutions make a big deal of weekly versus monthly payments, in fact, the present value cost of making weekly versus monthly payments is virtually identical for equal amortization periods. Weekly payments are only very slightly preferred because you pay off your mortgage just slightly sooner with weekly payments compared to monthly payments. In Canada, there is generally a strong incentive to pay off one's mortgage as soon as possible. In the example, paying off the mortgage in 15.5 years rather than 25 years saves you about \$96,379.19 – \$80,057.39 = \$16,321.80 in discounted future payments. Of course, your after-tax opportunity cost is the appropriate rate at which to do this discounting. Even though financial institutions also like to quote, in advertising, undiscounted interest savings, this practice reflects neither the time value of money nor an appropriate after-tax opportunity cost for you. You would never do such a thing, would you?

6.4.1 A Second Example

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As a second example of personal taxes in discounted cash flow analysis, consider the following problem. You borrow $10,000 at 13% per annum and promise to repay with three equal payments beginning in exactly one year. Because the loan is for your investments (residential apartment buildings), the interest portion of the payments is tax deductible. Immediately after you negotiate this loan, two unexpected things happen. First, you receive some unexpected cash (you discover in a pile of junk a tea set owned by Queen Victoria herself which you sell for $50,000) which is sufficiently large to pay the loan at the outstanding balance (without penalty) if you choose to do so. Second, borrowing and lending rates in the economy increase to 14% and 12% per annum respectively. Your marginal personal tax-rate is 35%. Should you pay off the outstanding balance of loan or should you invest all of your newfound wealth at 12% per annum?

To answer this question, begin by decomposing each payment into the amount of interest and principal repayment. This task is represented in the following extension of a loan amortization table:

Year	Total Payment	Interest Payment	Principal Repayment	Remaining Balance	After –Tax Cost of Payments
0	–	–	–	10,000.00	–
1	4,235.22	1,300.00	2,935.22	7,064.78	$2,935.22+(1–.35)×$1,300 = $3,780.22
2	4,235.22	918.42	3,316.80	3,747.98	$3,316.80+(1–.35)×$918.42 = $3913.77
3	4,235.22	487.24	3,747.98	0.00	$3,747.98+(1–.35)×$487.24 = $4064.69

After-tax Cost of Payments

Notice that in the last column of the above table, the after-tax cost of making payments is principal reduction plus one minus the tax rate times interest. Interest is less costly than principal reduction because interest is deductible in the calculation of taxable income. The implication of this deduction is that the effective cost of making interest payments is reduced by the tax rate times interest (the deduction reduces taxes payable by the tax rate times interest).

The rate of return to prepaying the loan at the outstanding balance of $10,000 is not the contract rate of interest. The reason is that if you prepay the mortgage the benefit is not the full amount of the payment. If you prepay, you forgo the benefit of the tax deduction on interest. The benefit of prepayment is, therefore, the after-tax cost of making future payments. The rate of return to prepayment is the IRR on the required investment relative to the future after-tax benefits:

$$-10{,}000 = \frac{3{,}780.22}{1+IRR} + \frac{3{,}913.77}{(1+IRR)^2} + \frac{4{,}064.69}{(1+IRR)^3} = 0$$

When you solve this equation, you can verify that the return to prepayment is the after-tax contract rate of interest. In other words, the return is the contract rate of interest times one minus the tax rate. In the example at hand, the rate of return to prepayment is 0.65*13 = 8.45% per annum. The opportunity cost rate of return is the after-tax return to investing elsewhere: 0.65*0.12 = 7.2% per annum. Notice that because the notion of opportunity cost requires investing elsewhere, the appropriate rate to use in this calculation is your lending rate and not the borrowing rate. The rate of return to prepayment is greater than the rate of return to alternative investments, and therefore, the optimal decision is to prepay. Because we are comparing after-tax rates to after-tax rates, the before tax rate on our loan – 13% per annum – exceeds the before tax rate to investing elsewhere – 12% – as long as lending rates in the economy do not increase and exceed the contract rate on the loan. In this case, the best use of funds is to prepay.

6.4.2 Example Continued

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Often we can make decisions by comparing rates of return to alternative uses of funds. We used this methodology in the example above. When dollar amounts become involved, however, you may be forced to do some discounted cash flow analysis. To illustrate this possibility, let us determine the maximum prepayment penalty that you would pay and still prefer to prepay your loan. To determine this amount we must compare the present value cost of making continued payments on the loan to the sum of the outstanding balance plus the prepayment penalty. First, let us find the present value cost of making continued payments. The amount we need is the deposit we could make in an investment plan that allows us to make yearly withdrawals of \$3,780.22, \$3,913.77, and \$4,064.69 in the next three years, to pay taxes over time on interest received, and then to have precisely zero left in the investment account. This amount is the present value of these after-tax cash flows at our after-tax opportunity cost. The present value cost of continued payments is:

$$\frac{3780.22}{1.072} + \frac{3913.77}{(1.072)^2} + \frac{4064.49}{(1.072)^3} = \$10231.32$$

The maximum prepayment penalty is the difference between the present value of continued payments and the outstanding loan balance: \$10231.32 – \$10,000 = \$231.32. If the prepayment penalty exceeds this amount, it is cheaper to repay the loan, according to our present value calculations, by making continued payments into the future.

(6.5) Summary

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This chapter investigates discounted cash flow techniques. To introduce the application of these tools, we examine a number of important personal financial planning problems.

Two important investment problems are at the root of a large portion of all financial analysis. In a present value problem, an investor makes a single deposit – typically today – into an investment plan. Over time, the investor makes withdrawals according to a predefined schedule in such a way as to drive the investment account balance to zero at the time of the last withdrawal. The planned schedule of withdrawals recognizes that periodic interest is paid on the outstanding account balance over time. The amount that the investor must deposit to make the planned set of withdrawals is called the *present value* of the investment plan. A *future value* is an amount in an investment plan at a specific time in the future that results from making a scheduled set of deposits. The balance in the plan arises not only from the set of deposits but also from interest that is paid periodically on the outstanding balance. The critical element of any future value calculation is deciding the date at which the balance should be measured.

Present value calculations arise in many financial asset transactions. In many markets and for many types of transactions, financial analyst depend on present value calculations to establish the appropriate bid at which to buy an asset or the appropriate offer at which to sell an asset. Future value calculations are commonly used in investment analysis and personal financial planning.

In this chapter, discounted cash flow analysis was applied to a number of important personal financial planning problems. Why do banks charge penalties or restrict borrowers' opportunities to prepay loans and mortgages? If you have extra and excess cash for investment purposes, should you invest in interest bearing financial assets or should you pay down your mortgage? Should you invest first into your RRSP or your mortgage? How does tax deductibility of interest affect the return on prepayment?

Many of these decisions are affected or influenced by your personal tax rate. This chapter shows that as a general rule, personal taxes can be integrated into discounted cash flow analysis if they are included in both the cash flows to be discounted (in the numerator of DCF) and the opportunity cost (in the denominator of DCF).

(6.6) Problems

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1. ***Effective and Nominal Rates***

You have an investment that has an annual rate of return of 15%. Find the following:

a) the semi-annual rate
b) the quarterly rate
c) the monthly rate
d) the per-annum nominal rate compounded semi-annually
e) the per annum nominal rate compounded quarterly
f) the per annum nominal rate compounded monthly
g) Repeat this question using different annual rates: 14%, 13% and 12%.

Note: In the embedded solution we use the spread-sheet function NOMINAL to calculate a nominal rate from an EAR. Of course, you can also calculate the nominal rates from first principles. For example, multiply the semi-annual rate by 2 to get the per-annum rate compounded semi-annually.

Solution

2. ***Internal Rate of Return.***

You purchase a financial asset today for $200,000 that makes the payments listed below. What is the per annum return on your investment?

i) $100,000 in one year
ii) $90,000 in two years
iii) $70,000 in four years
iv) $60,000 in five years
v) $50,000 in six years, *and*
vi) $40,000 in seven years.

Solution

<u>NOTE:</u> You can solve this question on a financial calculator. Alternatively, in the embedded solution above we solve the question with the IRR spread-sheet function.

3. ***Personal Financial Planning and Retirement*.**
Starting today, you plan to deposit $6,000 per year for the next 30 years into your bank account (30 deposits). Thirty years from today, you plan to retire. At that time you plan to make withdrawals for twenty years (20 withdrawals, with the first withdrawal 30 years from today). If you expect interest rates to be 8% per annum into the future, what will be your yearly withdrawal in your retirement?

Solution

4. ***Retirement planning***
For the past five years, you have made semi-annual deposits of $2000 in a savings plan. The first deposit was exactly five years ago and the last deposit was six months ago. There were ten deposits in all. The plan has paid 8 percent per annum compounded quarterly. Today, you made the first of 12 semi-annual withdrawals. Each of the first six withdrawals will be equal to one another and each of the seventh through twelfth withdrawals will be equal to one another. Each withdrawal from the first set of six will be twice as large as a withdrawal from the second set of six. Starting today, you expect the interest on your plan to be 7% compounded semi-annually. How much do you expect your withdrawals to be?

Solution

5. ***Deferred/Accelerated Annuity***
a) you expect to receive $650 every three years into the future (as described below). The yearly interest rate is 8%. What is the present value if:
i) the first payment is three years from today and there are twenty payments.
ii) the first payment is received today and payments are expected forever.

b) Now suppose you wish to pay $650 into a savings account every three years (as described below). The yearly interest rate is 8%. What is the future value at the time of the last payment if:

i) the first deposit is today and there are 30 payments.

ii) the first payment is 2 years from today and there are 30 payments.

Solution

6. ***Effective versus nominal rates.***

Suppose the EAR for a savings account is 10.25%. What is the per annum rate of interest compounded monthly? What is the continuously compounded rate of interest?

Solution

7. ***Mortgages***

In July 1982, interest rates were extremely high. Mortgage rates were between 16 and 19 percent. At the same time, banks were paying very high interest rates on deposits. In order to free up funds from older low interest rate mortgages, a number of banks offered various plans to encourage people to pay off their existing low rate mortgages more rapidly. John Stubbs borrowed from the Bank of Burnaby a number of years ago at 10.875% per annum compounded monthly. The bank has now offered John the opportunity to increase his monthly mortgage payments (combined principal and interest) from $852.39. In return, the bank will reduce the contract rate of interest on the loan from the original 10.875% per annum compounded monthly to 8.0% per annum, compounded monthly. Under the new payment plan, the loan would be paid off in ten years. Under the existing terms, the mortgage would be paid off in 27 years and 10 months.

NOTES: Ignore semi-annual compounding of Canadian mortgage rates, and assume instead that all rates are compounded monthly. Ignore secondary issues; base your answers only on the principles of discounted cash flow analysis.

a) What is the current loan balance?
b) If under the new plan, the bank uses the current balance from part (a) to calculate principal and interest payments, what will be John's new total monthly payment?
c) If current market interest rates are 15% per annum compounded monthly, should John accept the bank's offer?

Solution

8. ***Present value of a declining annuity.***
a) The value of services provided by your house is expected to be $7,500 at the end of this year (measure services at ends of years). The value of services is expected to decline at a rate of 3% thereafter (i.e., depreciation, which is a negative growth factor) until your house is demolished exactly twenty years from now (there are no further services rendered by the house after this time).
If current interest rates are 10% and the value of your house is calculated as the present value of future services, what is the fair market value of your house today? Assume that all market participants use the same theory of value.
b) Use the information from part (a) to answer this question. If you enter into a contract to sell your house exactly 12 years from today, but you ask for payment immediately, how much should you demand?
NOTE: Assume the purchaser uses the same theory of value and think carefully about the future services to be received by the purchaser.

Solution

9. ***Present value of a growing perpetuity.***
a) The per annum value of services provided by your house is expected to be $8,000 at the end of this year (measure services at ends of years). The value of these services is expected to grow at 8% per year in perpetuity. If the annual interest rate is 12%, what is the market value of your home if this value is calculated as the discounted value of all future services provided?
b) Use information from part A to answer this question. Suppose that you enter into the following financial arrangement with an insurance company. You will live in your house for exactly eleven years, at which time you will relinquish ownership to the insurance company.

Assume annual interest rates are 12% and that they are expected to remain constant over the life of the financial contract. How much should you demand as a lump-sum payment from the insurance company to be paid when ownership is transferred?
NOTES: You establish the price today to be paid in 11 years. The financial asset described is called a *reverse mortgage*, and it is an example of a forward *contract*.

Solution

10. ***Account Balance.***
As the result of an aggressive investment plan, you have $1,000,000 in a fully insured bank account. You will make *no additional deposits* to your account. Instead, starting today, you plan to make constant annual withdrawals for thirty years (thirty-one withdrawals) to finance your retirement. At the time of the final withdrawal, you intend to have a zero balance in your account. Your bank guarantees interest at 10 percent per annum, compounded monthly. How much money will you have in the account exactly 20 years from today, immediately after the twenty-first withdrawal?

Solution

11. ***Taxes in PV (Section 6.4 question)***[2]
Five years ago, you negotiated a ten-year, $10,000 home improvement loan (interest on home-improvement loans is not tax-deductible). The loan calls for level (constant) annual payments that started exactly one year after the loan was negotiated. You have just made the fifth payment and the sixth payment is due exactly one year from today. The loan contract allows you to make extra principal payments without penalty at any time, up to the outstanding balance. The stated interest rate on your loan is 13% per annum. The market rate for similar loans is currently 11% per annum. On the other hand, the best rate that individuals can earn before taxes on similar investments by *lending* is 10% per annum.
To take advantage of lower interest rates, you wish to borrow enough to pay off your home-improvement loan. Your brother-in-law has excess cash that he is willing to lend to you (and he

[2] Do not do this question unless you have studied section 6.4. At a first reading, or for more elementary finance courses, this section may not be required.

likes you to be fully aware of this fact). Your brother-in-law wants you to accept a five-year term loan that requires level annual payments to start in exactly one year.

By lending to you, your brother-in-law plans to break the law (the scoundrel!). He will not report on his tax return the interest income he receives from you. If your brother-in-law does not lend to you, he will invest in financial assets that pay the market interest rate, on which he must pay taxes. Assume that the personal tax rate for all individuals in the economy is 40%.

a) How much money must you borrow from your brother-in-law to fully retire your home-improvement loan?

b) What is the *minimum* yearly payment that your brother-in-law will accept to repay the amount you borrow from him? (Recall that the term of the new loan is five years).

c) If you accept the loan from your brother-in-law and agree to make the annual payments that answer part (b) above, what interest rate is implied? Is this loan more attractive or less attractive than loans that are available elsewhere? Explain.

d) On funds borrowed from your brother-in-law, what is the *maximum* yearly payment that is acceptable to you?

Solution

12. ***After-tax return on an RRSP investment.***

Suppose that your marginal combined personal tax rate is 40% (federal and provincial with surtaxes). The rate of interest in the economy is 9% per annum (for any term you choose). If you invest into an RRSP (registered retirement savings plan), you get a tax deduction *equal to the amount of your investment* (within prescribed limits). Taxable income is thereby reduced by the amount of the investment, and therefore, the effective cost of any RRSP contribution is lessened. Your contribution is eventually taxed as income, but only after you collapse your RRSP in the future (for the purpose of living off the after-tax proceeds in your retirement). Also, interest earned in RRSP accounts is taxed not when it is received, but only when the plan is collapsed at your retirement (when you are 65 years old). You are now 25 years old and your retirement will commence in exactly 40 years. Incorporating these tax rules, what is the *per annum* after-tax rate of return you will earn on a 40-year RRSP investment?

Solution

13. ***Taxes in PV (Section 6.4 question).***[3]
The remaining amortization period on your *home* mortgage is 15 years (180 monthly payments). You live in Surrey (and you are proud of it) and you pay Canadian income taxes. You just made a mortgage payment and the next payment is due in exactly one month. Your monthly payment is $590.84. The interest rate on your mortgage is 8.5% per annum, compounded monthly. Individuals may lend at 12% per annum, compounded monthly by holding interest-bearing securities. Your marginal combined personal tax rate is 30%. Your mortgage allows you two options (each without penalty), either to increase your monthly payment by any amount (the incremental payment reduces loan principal dollar for dollar), or to pay off your outstanding loan balance at any time. If you have extra cash to invest, should you pay down your mortgage or should you invest elsewhere in taxable, interest bearing financial assets?

Solution

14. ***Mortgages.***
A fixed rate mortgage of $100,000 has just been arranged for 25 years at an annual interest rate of 12% per annum, compounded monthly. The mortgage calls for monthly payments to be made at month end.

a) What is the amount of each payment?

b) Immediately after the 99'th payment is made, what will be the outstanding loan balance?

c) What is the amount of principal repayment made with the 75'th, the 89'th and the 151'st payments?

Solution

15. ***Present Values.***
Financial markets have recently developed new types of securities called IO's (interest only) and PO's (principal only). A bank or a trust company that has lent funds in a term loan (or a mortgage) sells the interest portion of all future payments that it expects to receive (as they arise) on the term loan to another investor (the IO). In addition, the bank or trust company also sells the principal portion of all future payments that it expects to receive (as they arise) on the term loan to a second investor (a PO).

[3] See footnote to Problem 11.

Suppose a bank has just lent against a three-year term loan for $10,000 at 12% per annum, compounded annually, with the first payment due in exactly one year. Suppose investors are willing to accept a rate of return of 11.5% compounded annually on such investments. Ignore tax effects, prepayment options, and default risk.

a) At what prices should the bank sell the IO and the PO?

b) Why is the sum of the IO and PO greater than the value of the underlying term loan?

c) Suppose that immediately after the term loan is negotiated and the IO and PO are sold, lending rates in the economy increase from 11.5% to 12.5% per annum. Which security loses a larger proportion of its value, the IO or the PO?

d) Why is the PO more sensitive to changes in market interest rates? Make a clear explanation without any numbers.

Solution

16. ***Taxes in PV (Section 6.4 question).***[4]

Current lending rates in the economy are 10% per annum. Borrowing rates are 13% per annum. You plan to borrow $10,000 for the purpose of investing in financial assets, and therefore, the interest payments are tax-deductible. You will pay off the loan with three level payments – a portion of each payment is principal and a portion of each is interest – starting in one year. Your personal tax rate is 40%. Suppose that immediately after negotiating the loan, two unexpected things happen: first, you receive an amount of cash sufficient to pay off the loan; and second, lending and borrowing rates both increase by 2% per annum (to 12% and 15% per annum, respectively).

These rates are appropriate for any investment period you choose. Should you prepay the loan (pay the outstanding balance) if you can do so without penalty? What is the maximum penalty that you would incur and still prepay?

Solution

17. ***Daily interest savings accounts.***

The stated interest rate on a daily interest savings account is 6% per annum. A daily interest

[4] See footnote to Problem 11.

savings account does not have daily compounding because while interest is calculated daily – based on the current account balance – interest is not added to your account until the end of the month. Suppose that you put $10,000 into your daily interest savings account. If you do not make any withdrawals, how much do you have in your account after one year? You may assume that each month has the same number of days (i.e., 30 for example).

Solution

18. ***Taxes in PV (Section 6.4 question).***[5]
Exactly five years ago you mortgaged your home for $80,000. The fixed-rate mortgage calls for monthly payments at 8% per annum, compounded monthly. At origination, the mortgage required monthly payments for 10 years. The first payment was made exactly 4 years and 11 months ago and you have just made the 60'th payment. Current lending rates in the economy on interest bearing financial assets is 13.75% per annum. Your rate of tax on interest income is 40% (combined federal and provincial taxes). Recently, you have come into some unexpected cash. You now have sufficient cash to prepay (i.e., payoff) your entire mortgage if you choose to do so. However, if you prepay, the bank requires not only the outstanding balance, but also a penalty equal to the sum of the interest portions of your next three payments that otherwise would have been paid.
Should you prepay your mortgage or should you invest your excess funds in interest bearing financial assets? Recall that in Canada, interest on home mortgages is not tax-deductible.

Solution

[5] See footnote to Problem 11.

19. ***Prepayment penalties.***
Explain why many loan contracts restrict or impose penalties for prepayment of the outstanding balance prior to maturity of the contract.

Solution

20. ***Canadian Mortgage Market.***
Explain why in the Canadian mortgage market, there is generally a strong incentive for borrowers to repay their mortgages as quickly as their mortgage contracts will allow. Under what financial conditions does this strong incentive tend to dissipate?

Solution

21. ***Real estate investment.***
You plan to buy an apartment building with forty rental units. The average monthly rent of a unit is $500. You expect the building to be fully rented. Expenses (i.e., amortized repair and replacement, maintenance, landscaping, etc.) are 35% of gross revenues. The apartment building is offered for sale at $1,250,000. Additional non-deductible expenditures required in order to purchase the building are 10% of the purchase price. The marginal combined tax rate on this business venture is 50%. You expect no significant growth in rental income or in expenses for the indefinite future. Ignore capital cost allowance that might, in fact, be available on the building.

a) If you pay cash for the building, what is your after-tax effective annual return on your investment? Express your answer as an annual rate based on monthly compounding.
Suppose that you plan to finance 70% of the expenditure required to begin your business venture at 8.5% per annum, compounded monthly. In other words, you expect to pay 0.085/12 on each borrowed dollar in interest, every month indefinitely into the future. Note that the entire payment is interest. There is no principal reduction on a loan that lasts indefinitely into the future. What is your after-tax effective annual return on your investment? Express your answer as an annual rate based on monthly compounding.

Solution

22. ***Mortgages.***
You borrowed from the Bank of Montreal in order to purchase your Burnaby home. Your 10-year mortgage calls for equal monthly payments. The first payment was due one month after you receive the borrowed funds. The contractual rate of interest on the mortgage is 12% per annum, compounded monthly. The principal portion of the 39'th payment is $583.72. How much did you originally borrow?

Solution

23. ***Mortgages.***
Today, you borrowed $B from a bank and promised to make equal monthly payments starting in one month for 10 years (120 payments in total). The principal reduction portion of the 59'th payment is $294.18626. The principal reduction portion of the 83'rd payment is $373.53849. How much did you originally borrow?

Solution

24. ***Taxes in PV (Section 6.4 question).***[6]
You plan to buy a Chrysler minivan for $30,000. The minivan is for your personal use and is unrelated to your business as a rock star. You have $10,000 to use as a down payment and you plan to finance the difference with equal monthly payments for 36 months (starting in one month). The effective annual rate of interest on your loan is 12.25% per annum. You, as an individual, can lend funds in the economy at 11.5% per annum by buying interest-bearing financial assets. Your marginal combined personal tax rate is 22.5%. What is the present value cost to you of financing the minivan purchase rather than using cash?

Solution

25. ***Mortgage.***
Bill (Slick Willy) Clinton is on a budget. The maximum monthly house payment his wife can afford is $800 per month. They have $30,000 in personal savings to use as a down payment. A bank has agreed to provide a mortgage at 9.75% per annum, compounded semi-annually but paid monthly. The amortization period would be 25 years.

a) What is the maximum price that the Clintons can afford to pay for a home?
b) What will be the outstanding balance on the loan immediately after the 60'th payment?
c) What is the interest portion of the 12'th payment?

Solution

26. ***Future values with differing investment terms.***
Tricky Dick Financial Corporation, Ltd. offers you the following investment opportunities.
a) If you deposit $92.59 today, they will pay 8% per annum on this deposit for a one-year term (i.e., principal plus interest on this deposit is paid in one year). You expect to be able to reinvest at 8.75 % per annum compounded quarterly for a three-year term (i.e., your sole payoff on the reinvestment is three years after the reinvestment).
b) If you deposit an additional $84.66 today, they will pay 8.5% per annum compounded semi-annually on this extra deposit for a two-year term (i.e., principal plus interest on this deposit is paid in two years). You expect to be able to reinvest at 8.5% per annum compounded

[6] See footnote to Problem 11.

semi-annually for a two-year term (i.e., your sole payoff on the reinvestment is two years after reinvestment).

c) If in addition to the first two deposits, you deposit an additional $77.13 today, they will pay 8.75% per annum compounded quarterly on this extra deposit for a three-year term (i.e., principal plus interest on this deposit is paid in three years). You expect to be able to reinvest at 8.0% pre annum for a one-year term (i.e., your sole payoff on the reinvestment is one year after reinvestment).

d) Finally, if in addition to the first three deposits, you deposit an additional $69.86 today, they will pay 9.0% per annum compounded monthly on this deposit for a four-year term (i.e., principal plus interest on this deposit is paid in three years).

You contract today to make all four of the above deposits with the associated interest factors. You boss asks you to calculate the overall per annum return on this four year investment plan.

Solution

27. ***The Effective Rate of Interest and Periodic Payments***.

Section 347 of the Criminal Code of Canada makes it illegal to enter into an agreement to receive interest payments that exceed 60% per annum of the funds borrowed. Courts have interpreted this 60% rate as an *effective* annual rate. In other words, it is illegal to enter into a contract where the effective annual rate is greater than 60%.

Your brother wants to borrow $10,000. You agree, but you propose a set of constant monthly payments for a one-year period to pay off the loan. The first payment is in one month and there will be a total of 12 payments.

a) What is the maximum payment you can require of your brother without violating section 347 of the Criminal Code?

b) You start to feel a little guilty at the thought of charging your brother a 60 per cent effective annual rate (but not too guilty). To mask the severity of this charge, you plan to *quote* to your brother (without changing the effective annual rate or the amount of the monthly payment) the lesser of the nominal per annum rate compounded daily and the nominal per annum rate compounded monthly (use 365 days in a year). Describe the rate of interest you plan to quote to your brother (including a numerical value for this rate). Explain.

Solution

28. ***Taxes in PV (Section 6.4 question).***[7]

Seven years ago, you borrowed $150,000 from the Bank of Spanish Turbot. The original loan term was ten years. Repayment requires ten equal annual payments. The first payment was one year after you received the borrowed funds. The interest rate on the loan is 12.625% per annum. You have just made the seventh payment. The purpose of the loan was to purchase an apartment building that you now operate as a sole proprietorship. Your marginal personal tax rate is 45%. Current borrowing rates in the economy are 10.25% per annum. Lending rates are 8.25% per annum. Because interest rates in the economy have fallen recently, you are looking for ways to reduce your annual payments. A contract term in your loan allows you to renegotiate your loan (for the remaining term) at the new borrowing rate in the economy if you pay a penalty equal to the interest portion of the payment you have just made. Should you take advantage of the option under your loan contract or should you continue to make payments under the original contract terms?

Solution

[7] See footnote to Problem 11.

29. ***Annualized HPRR***
You plan to purchase a piece of land for $100,000, hold it for 10 years, and then sell it for $260,000. Sales commissions for purchase and sale are 2% of the transaction prices. Your marginal tax rate is 45%. Recognizing the appropriate inclusion rate for capital gains, what is the annualized after-tax rate of return on your investment?

Solution

30. ***Effective and Nominal Rates*** **.**
If the effective rate of interest over a three-month holding period is 2%, what is the nominal per annum rate of interest compounded daily (assume 365 days in a year).

Solution

31. ***Mortgage.***
You make monthly payments on your mortgage of $1000. The rate of interest is 10% per annum compounded monthly. There are exactly 120 remaining payments on your mortgage. You have just made a payment and the next and upcoming payment is in exactly one month. What is the interest portion of the payment that you will make exactly two years from today? What is the outstanding balance on your mortgage at that time (immediately after the payment)?

Solution

32. ***Perpetuity.***
You are considering the purchase of a financial asset that has an effective annual rate of interest of 12%. Payments are semi-annual indefinitely with the first payment is in exactly one month. The price of the financial asset is $1000. What are the semi-annual payments?

Solution

33. ***An annuity with reinvestment.***
You purchased a financial asset 10 years ago for $10,000. The interest payments were $1,000 per annum. The first interest payment was received one year after the purchase of the financial asset and, today, you received the 10'th interest payment. Over the course of the past 10 years, you reinvested all of the interest payments at 10% per annum. The current market value of the financial asset is $10,000. Including the reinvested interest payments, what is the *annualized* holding period rate of return on your investment?

Solution

34. ***Present values and continuously compounded returns.***
The continuously compounded per annum rate of interest in the economy is 8%. You have been offered a financial asset which pays $100 in seven months, $100 in twelve months, and $100 in nineteen months. What is the value of this financial asset?

Solution

35. ***Nominal and effective interest rates.***
Suppose that the nominal rate of interest compounded monthly is 10 percent per annum. What is the nominal rate of interest compounded semi-annually?

Solution

36. ***Mortgage.***
You have 15 years of monthly payments remaining on your mortgage. The effective rate of interest on your mortgage is 10% per annum. The next and upcoming payment is in exactly one month (i.e., you have just made a mortgage payment). The interest portion of this payment will be $100. What is the outstanding balance on your mortgage in exactly one year, immediately after you have made your payment at that time?

Solution

37. ***Perpetuity.***
You are considering the purchase of a financial asset that offers $63.75 every six months into the indefinite future. The first payment is exactly three months from today. The effective annual rate of return on this investment is 6%. What is the purchase price of the financial asset?

Solution

38. ***FV of a Growing Annuity.***
Your per annum salary is now $100,000 and is expected to grow at 4% into the foreseeable future. Beginning today, you plan to invest 5% of your salary per year into an investment plan that pays 5.5% per annum after tax. How much will you have in this investment plan exactly 20 years from today at the time of the 21'st deposit? (Hint: present values and future values are inverse operations).

Solution

39. ***Nominal and Effective Rates.***
Both the current market rate of interest and the rate of interest on a term loan are 8.5 percent per annum compounded monthly.

a) If the loan has just been negotiated, the amount borrowed is $10,000, and constant payments are to be made monthly for eight full year (first payment in exactly one month for a total of 96 payments), what should the monthly payment be?

b) Suppose that the borrower and the lender renegotiate the loan (today) so that constant payments are to be made quarterly rather than monthly. The loan would still be paid off in eight full years (first payment in exactly one quarter for a total of 32 quarters). What should the amount of the quarterly payments be?

c) Other things equal, which of the above payment schemes, (a) or (b), would be preferred by (i) the borrower, (ii) the lender.

Solution

40. ***Deferred Perpetuity and Nominal Rates***
Consider a deferred perpetuity of $100 *per quarter*. The first payment is in exactly *five months*, and then payments are every quarter indefinitely.

a) What is the current value of this perpetuity if the opportunity cost is 10% per annum, compounded semi-annually?

b) … if the opportunity cost is 10% per annum compounded monthly?

Solution

41. ***Perpetuities and Compounding Periods.***
Consider an ordinary perpetuity of \$100 *per quarter*. The *first* payment to be received on this perpetuity comes in one quarter. Suppose that the value of the perpetuity is \$4,125.

a) What is the per annum rate of return on your investment compounded monthly?

b) … compounded semi-annually?

Solution

42. ***Canadian Mortgages and Prepayment.***
George borrows \$100,000 for a home mortgage. The bank earns an effective rate of return of 8% per annum on its investment. Payments on the mortgage are monthly with the first payment due one month after George receives the borrowed funds. The mortgage matures ten years after the loan origination day.

a) What is the outstanding balance immediately after the 24'th payment?

b) Immediately after the 24'th payment, George has sufficient funds to repay the outstanding mortgage balance and the bank imposes no penalty for prepayment. Discuss the *financial market* conditions in Canada that make early retirement of mortgage an attractive choice.

c) Suppose instead that the bank charges a penalty for prepayment. Immediately after the 24’th payment, suppose that mortgage rates have fallen to 7.25% per annum, compounded semi-annually. In addition, suppose that if George chooses to do so, he can invest in financial assets to earn this rate. Without approximating, and ignoring tax considerations, calculate the maximum prepayment penalty that George should accept.

d) Canadian banks are required by law to *quote* their mortgage rates with semi-annual compounding. What is the per annum rate compounded semi-annually on your mortgage?

Solution

43. ***FV of an Annuity.***
Exactly 14 years ago, you began an investment plan. You made equal deposits every quarter for eleven years. Your first deposit was exactly 14 years ago and your last deposit was exactly 3 years ago. Over this fourteen year period, your investment account has earned 8.2% per annum compounded monthly. Your current account balance is $250,000. What were your quarterly deposits?

Solution

44. ***The Canadian Mortgage Market and Refinancing***.
On January 1, 1997 you borrowed $100,000 from a Burnaby branch of the Bank of Montreal for a new home mortgage. You used the proceeds to help you purchase your new home in the Kensington neighborhood of Burnaby. The quoted interest rate is 6.95 percent per annum compounded semi-annually for a five-year term with a twenty five-year amortization period. Payments are monthly with the first payment one month after you receive the $100,000. At the end of the mortgage's initial term, you plan to "refinance" with a second five-year term and a 20-year amortization. At that time, in the Canadian mortgage market, if rates happen to increase to 10 percent per annum compounded semi-annually (also for a five year term) by how much will your monthly payments increase?

Solution

45. ***Reinvestment***
You buy a financial asset for $1250 that offers a per annum rate of return of 10% compounded quarterly. This financial asset makes a per annum payment to you indefinitely into the future (the first in exactly one year). You buy this financial asset today and reinvest the payments you receive from it (over time) at 9.5% per annum compounded semi-annually. You sell the financial asset for $1000 in exactly 10 years immediately after having received the payment on the financial asset at that time. In total, you received 10 payments on this financial asset over your holding period of 10 years.
<u>Required:</u> Including the reinvestments, and the interest on the reinvestments, what is the annualized rate of return on your investment compounded monthly for the 10 year holding period?

Solution

46. ***Mortgages.***
On January 1, 1998 you borrow a certain sum of money from a Burnaby branch of the Bank of Montreal for a new home mortgage. You use the proceeds to help you purchase your new home in the Kensington neighborhood of Burnaby. The quoted interest rate is 6.95 percent per annum compounded semi-annually for a five-year term. In other words, the contract rate on the mortgage is fixed over this five-year period. Monthly payments are expected to be made for an amortization period that extends beyond the five-year term. Payments are monthly with the first payment one month after you receive the borrowed funds. Monthly payments are \$900.86. At the end of the five-year term, the outstanding balance will be \$111,651.64. Ignore tax considerations in this problem. How much did you originally borrow?

Solution

47. ***Mortgages.***
Humungous Bank wants an 8 percent effective annual rate on its mortgages. You borrow \$100,000 and promise to repay with equal monthly payments. The first payment is exactly one month after you receive the \$100,000. You plan to make payments for 10 years (exactly 120 payments). The payments are fixed and do not vary over the life of the mortgage.

a) What is the outstanding balance immediately after the 60'th payment?

b) By Canadian law, the Humungous bank must *quote* the rate that it charges as an annual rate compounded semi-annually. What is the per annum rate compounded semi-annually on your mortgage?

Solution

48. ***HPRR***

Today, you purchase a financial asset, which offers a rate of return of 6.8 percent per annum compounded monthly. This financial asset makes 12 equal quarterly payments of $100 each with the first in exactly one quarter. There are no other payments or cash flows on this financial asset. You buy the financial asset today and reinvest the payments at 6.2 percent per annum compounded monthly. The reinvestment period for any payment is between receipt of the payment and the last payment on the financial asset. Your overall holding-period for this investment is from today to the last payment on the financial asset. What is your annualized holding period rate of return, compounded quarterly, on your investment (including reinvested payments)?

Solution

49. ***HPRR***

Today, you purchase a financial asset that offers an effective annual rate of return of 8.8 percent. This financial asset makes 10 semi-annual payments (every six months) with the first in exactly six months. There are no other payments or cash flows on this financial asset. You buy the financial asset today and reinvest the payments at 6.2 percent per annum compounded monthly. The reinvestment period for any payment is between receipt of the payment and the last payment on the financial asset. Your overall holding-period for this investment is from today to the last payment on the financial asset. The increase in your wealth over this holding period as the result of your investment is $100,000. What is your annualized holding period rate of return, compounded weekly (52 weeks in a year), on your investment (including reinvested payments)? Ignore taxes in this problem.

Solution

50. ***Retirement Planning***

Fourteen years and nine months ago, you began an investment-plan. You made semi-annual deposits of $100. Your first deposit was fourteen years and nine months ago and your last deposit was three months ago. Your deposits earned a rate of return of 9.2 percent per annum compounded monthly. You plan no further $100 deposits to your account. Instead, today, you

plan to take the proceeds of your investment account and buy a financial asset that makes a set of equal quarterly payments. The first of these payments is in two months and the last is in seven years and eight months. There are no other payments on this financial asset. The rate of return on this financial asset is 9 percent per annum compounded semi-annually. As you receive the payments on this financial asset, you reinvest them at a rate of 10.1 per cent per annum compounded quarterly.
Required: What is your wealth, seven years and eight months from today, immediately after the final payment on the financial asset at that time?

Solution

51. ***Retirement Planning***
Today, you purchase a financial asset for $10,000 that offers a rate of return of 5.75 percent per annum compounded quarterly. This financial asset makes 14 equal quarterly payments with the first in exactly one month. There are no other payments or cash flows on this financial asset. You buy the financial asset today and reinvest the payments when received at 5.3 percent per annum compounded monthly. The reinvestment period for any payment is between receipt of the payment and four months after the last payment on the financial asset. Your overall holding-period for this investment is from today to four months after the last payment on the financial asset. What is your annualized holding period rate of return, compounded semi-annually, on your investment (including reinvested payments)?

Solution

52. ***Effective Versus Nominal Rates***
Explain how a nominal rate of interest differs from an effective rate of interest.

Solution

53. ***Annualized Rate of Return***

For each of the following investments, find *independently* the annualized rate of return compounded semi-annually:

(i) Invest $1,000 today and receive $1,500 in exactly 31 months.

(ii) Invest $5,000 today and receive $100 per month indefinitely with the first such amount exactly one month from today.

(iii) Invest $5,000 today and receive payments every six months indefinitely into the future. The payments grow at the rate of 1% for each six-month period. In other words, each payment is 1% greater than the previous one. The first payment is $400 and will be received exactly six months from today.

Solution

54. ***HPRR***

Today, you purchase a financial asset. This financial asset offers a rate of return of 8.2 percent per annum compounded quarterly. This rate is not expected to change over the life of this particular financial asset. Twenty equal semi-annual payments of $200 each are expected on this financial asset with the first in two months (thereafter, the payments are received every six months). There are no other cash flows or payments on this financial asset. You reinvest the payments when received at 7.9% per annum compounded monthly.

Required: What is the annualized holding period rate of return compounded monthly on your investment between today and two months after the 10'th payment?

Solution

55. ***HPRR***

Today, you purchase a financial asset. This financial asset offers a rate of return of ***i*** percent per annum compounded monthly. This rate is not expected to change over the life of this particular financial asset. Equal quarterly payments are expected on this financial asset indefinitely into the future (i.e., in perpetuity) with the first payment exactly one quarter from today. There are no other cash flows or payments on this financial asset. You reinvest the payments when received at 8% per annum compounded monthly. The annualized holding period rate of return compounded monthly on your investment between today and immediately after the receipt of the 20'th payment is 8.05 percent.

Required: Determine ***i*** the per annum rate of return offered on the financial asset (compounded monthly).

Solution

56. ***HPRR***

Today, you purchase a financial asset that promises quarterly payments of $150 each indefinitely with the first payment exactly one quarter from today. There are no other cash flows or payments on this financial asset. When you purchase the financial asset it offers a rate of return of 5.6% per annum compounded semi-annually. Immediately after you receive the fifteenth payment, you sell the financial asset. At the sale, the financial asset offers a rate of return of 5.9% per annum compounded quarterly. Between your purchase and your sale of the financial asset you reinvest the payments that you receive in interest bearing financial assets that offer a rate of return of 4.2 percent per annum compounded monthly.

a). What is your purchase price for the financial asset?

b). What is your sale price for the financial asset?

c). At the time of your sale of the financial asset, how much do you have (with interest) from reinvested payments?

d). What is the holding period rate of return for your overall investment?

e). What is the annualized holding period rate of return for your investment compounded monthly?

Solution

57. ***HPRR***

The three following investments are *unrelated* (thus, each part of this question below will be marked entirely independently).

Find the per annum rate of return compounded semi-annually for each of the following investments.

(i) Invest $125 today and receive one payment of $200 in exactly 17 months.

(ii) Invest $10,000 today and receive $90 per month indefinitely with the first such amount exactly one month from today.

(iii) Invest $5,000 today and receive payments every quarter indefinitely. The payments grow at the rate of 2% per quarter. In other words, each payment is 2% greater than the previous. The first payment is $100 and will be received immediately.

Solution

58. ***HPRR***

Today, you purchase a financial asset that promises indefinite monthly payments of $250 each with the first payment to be received immediately. There are no other cash flows or payments on this financial asset. When you purchase the financial asset it offers a rate of return of 6.2% per annum compounded semi-annually. Immediately after you receive the twenty-second payment, you sell the financial asset. At the sale, the financial asset offers a rate of return of 6.4% per annum compounded semi-annually. Between your purchase and your sale of the financial asset you reinvest the payments you receive in other interest bearing financial assets that offer a rate of return of 5.0 percent per annum compounded monthly.

a). What is your purchase price for the financial asset?

b). What is your sale price for the financial asset?

c). At the time of your sale of the financial asset, how much do you have (with interest) from reinvested payments?

d). What is the holding period rate of return for your overall investment?

e). What is the annualized holding period rate of return for your investment compounded quarterly.

Solution

59. ***HPRR***

The three following investments are *unrelated* (thus, each part of the question below will be marked entirely independently).

Each of the following investments offers a per annum rate of return, compounded semi-annually, of 6.3 percent.

(i) Invest $10,000 today and receive one payment of $A in exactly 17 months.

(ii) Invest $10,000 today and receive $A per month indefinitely with the first such amount to be received immediately.

(iii) Invest $10,000 today and receive payments every quarter indefinitely into the future. The payments grow at the rate of 1% each quarter. In other words, each payment is 1% greater than the previous. The first payment is $A and will be received immediately.

Determine the amount A in each of the above 3 cases.

Solution

60. ***HPRR***

Today, you purchase a financial asset that promises 35 semi-annual payments of $150 each with the first payment exactly four months from today. There are no other cash flows or payments on this financial asset. When you purchase the financial asset it offers a rate of return of 5.6% per annum compounded quarterly. Four months after you receive the eighteenth payment, you sell the financial asset. At the sale, the financial asset offers a rate of return of 5.9% per annum compounded quarterly. Between your purchase and your sale of the financial asset you reinvest payments in interest bearing financial assets that offer a rate of return of 5.2 percent per annum compounded monthly.

a). What is your purchase price for the financial asset?

b). What is your sale price for the financial asset?

c). At the time you sell the financial asset, how much do you have (with interest) from reinvested payments?

d). What is the holding period rate of return for your overall investment between purchase and sale of the financial asset (including payments reinvested)?
e). What is the annualized holding period rate of return compounded quarterly for your investment?

Solution

61. ***Savings Plan***
For the past ten years, you have made semi-annual deposits of $Y into savings plan "A". The first deposit was exactly ten years ago and the last deposit was six months ago. The plan has paid 6 percent per annum compounded quarterly. Today, you purchase a financial asset with your account balance. The financial asset offers a rate of return of 5.9 percent per annum compounded quarterly. There are thirteen quarterly payments on this financial asset with the first in exactly two months. As you receive these payments, you deposit them into savings plan "B" that pays 5.8% per annum compounded monthly. Immediately after you reinvest the last (that is, thirteenth) payment from the financial asset into savings plan B, your account balance is $50,000.

Required:

a. What is the amount of the quarterly payment on the financial asset?
b. What is the purchase price of the financial asset?
c. What was your semi-annual deposit Y into savings account A?

Solution

62. ***Savings Plan***
For the past ten years, you have made semi-annual deposits of $10,000 into a savings plan. The first deposit was exactly ten years ago and the last deposit was today. The plan has paid 5% per annum compounded continuously. However, beginning today, the savings plan will pay interest at the rate of 6 percent per annum compounded monthly. You plan no further deposits to your savings plan. Instead, you plan to make 30 equal quarterly withdrawals of $X each from your

saving plan with the first withdrawal in exactly one month. At the time of the last withdrawal, you expect your account balance to be zero.

Required: Immediately after your tenth withdrawal, what is your account balance?

Solution

63. ***HPRR***

Today, you purchase a financial asset that promises 23 equal quarterly payments of $150 each with the first payment exactly five months from today. There are no other cash flows or payments on this financial asset. When you purchase the financial asset it offers a rate of return of 7% per annum compounded semi-annually. Two months after you receive the seventeenth payment, you sell the financial asset. At the sale, the financial asset offers a rate of return of 5% per annum compounded continuously. Between your purchase and your sale of the financial asset you reinvest payments in interest bearing financial assets that offer a rate of return of 6% per annum compounded monthly.

Required: What is the annualized holding period rate of return compounded monthly for your investment between your purchase and sale?

Solution

64. ***Savings Plan***

For the past ten years, you have made semi-annual deposits of $X into a savings plan. The first deposit was exactly ten years ago and the last deposit was today. The plan has paid 5% per annum compounded continuously. However, beginning today, the savings plan will pay interest at the rate of 6 percent per annum compounded monthly. You plan no further deposits to your savings plan. Instead, you plan to make 30 equal quarterly withdrawals of $10,000 each with the first withdrawal in exactly one month. At the time of the last withdrawal, you expect your account balance to be zero.

Required:

(a) Immediately after your tenth withdrawal from your savings plan, what is your account balance?
(b) What were your semi-annual deposits, $X, into your savings plan?

Solution

65. ***HPRR***
Today, you purchase a financial asset that promises 28 equal semi-annual payments of $250 each with the first payment exactly 7 months from today. There are no other cash flows or payments on this financial asset. When you purchase the financial asset it offers a rate of return of 7% per annum compounded semi-annually. One month after you receive the 13'th payment, you sell the financial asset. At the sale, the financial asset offers a rate of return of 5% per annum compounded quarterly. Between your purchase and your sale of the financial asset you reinvest payments in interest bearing financial assets that offer a rate of return of 6% per annum compounded continuously.
Required: What is the annualized holding period rate of return compounded quarterly on your investment between purchase and sale?

Solution

66. ***HPRR and PV/FV of a Growing Annuity***
You purchase a financial asset today for $A. The financial asset offers semi-annual payments indefinitely where the first payment is six months from today (thereafter, each payment is received in six month intervals). The first payment is $B and each subsequent payment is 1% greater than the previous. You reinvest the payments at an interest rate of 5.25% per annum compounded monthly. You sell the financial asset for $10,000 four months after you receive the sixteenth payment. At this time, the financial asset offers a per annum rate of return of 6.3% compounded monthly. The annualized holding period rate of return on your investment between your purchase and your sale of the financial asset is 7.0 percent per annum compounded quarterly.

Required: When you purchase the financial asset, it offers what per annum rate of return compounded monthly?

Solution

67. ***HPRR and PV/FV of a Growing Annuity***
Today, you purchase a financial asset. The financial asset offers a rate of return of 5.9 percent per annum compounded continuously. There are fifty semi-annual payments with the first equal to \$Z in exactly two months. Each subsequent payment is 4.2% greater than the previous. As you receive these payments, you deposit them into a savings plan that pays 5.5% per annum compounded monthly. Two months after you deposit the fourteenth payment from the financial asset into the savings plan you sell the financial asset. At this time, the financial asset offers a per annum rate of return of 5.0 per cent per annum compounded semi-annually. Also at this time, the sum of your savings account balance (fourteen deposits with interest) plus the sale price of the financial asset is \$225,000.

Required: What was your purchase price for the financial asset?

Solution

68. ***HPRR and Growing Annuity***
Today, you purchase a financial asset. The financial asset offers a rate of return of 6 percent per annum compounded continuously. The financial asset offers fifty-five semi-annual payments with the first equal to \$120 in exactly four months. Each subsequent payment is 3.9% greater than the previous. As you receive these payments, you deposit them into a savings plan that pays 5.4% per annum compounded monthly. Four months after you deposit the twenty-sixth payment, you sell the financial asset. At this time, the financial asset offers a per annum rate of return of 5 per cent per annum compounded semi-annually.

Required: What is the annualized continuously compounded holding period rate of return on your investment between purchase and sale of the financial asset including reinvested payments with interest?

Solution

69. ***HPRR and PV/FV of a Growing Annuity***

Today, you purchase a financial asset. This financial asset offers a rate of return of ***i*** percent per annum compounded continuously. This rate does not change over your investment holding period described below. The financial asset offers indefinite semi-annual payments with the first equal to $C in exactly six months. Each subsequent payment is 1.2% greater than the previous. As you receive these payments, you deposit them into a savings plan that pays 5.2% per annum compounded continuously. Immediately after you receive the twenty-fifth payment on the financial asset, you sell it. Your annualized holding period rate of return on your investment between purchase and sale of the financial asset, including reinvested payments with interest is 6.5% per annum compounded continuously.

Required: Find ***i*** the per annum rate of return offered on the financial asset (compounded continuously).

Solution

70. ***HPRR and PV/FV of a Growing Annuity***

Today, you purchase a financial asset. This financial asset offers a rate of return of ***i*** percent per annum compounded continuously. This rate does not change over your investment holding period described below. The financial asset offers indefinite semi-annual payments with the first equal to $100 in exactly six months. Each subsequent payment is 1.2% greater than the previous. As you receive these payments, you deposit them into a savings plan that pays 5.2% per annum compounded continuously. Immediately after you receive the twenty-fifth payment on the financial asset, you sell the financial asset for $10,000.

Required:

(a) Find ***i*** the per annum rate of return offered on the financial asset (compounded continuously).
(b) Find your annualized continuously compounded holding period rate of return between purchase and sale of the financial asset, including reinvested payments with interest.

Solution

71. ***Mortgage***
On Dec 31, 2001 you borrowed a certain sum from the Royal Bank of Canada in a mortgage agreement. The mortgage calls for monthly payments starting at the end of January 2002. The mortgage rate is 5.6 percent per annum compounded semi-annually for a 5-year term. The amortization is 25 years (300 payments). Today is Dec 31, 2006, and you have just made your 60'th mortgage payment.

You have renegotiated your mortgage with the Royal Bank for a second 5 year term with an amortization period of 20 years (240 payments). Your first payment in this second term will be one month from today. The mortgage rate over the second term is 5.0 percent per annum compounded semi-annually and your monthly payment is $700.00.

Required: How much did you originally borrow on December 31, 2001?

Solution

72. ***Mortgage***
On Dec 31, 2002 you borrowed a certain sum in a mortgage agreement. The mortgage calls for monthly payments of $1,000 starting at the end of January 2003. The amortization is 25 years (300 payments). The interest portion of the 21'st payment is $752.54. Today is Dec 31, 2007 and you have just made the 60'th mortgage payment.

Required: What is the outstanding balance immediately after your Dec 31, 2007 payment?

Solution

73. ***IRR***

The three following investments are *unrelated* (thus, each part of this question will be marked entirely independently).

Find the per annum rate of return continuously compounded for each of the following investments.

(i) Invest $105 today and receive one payment of $220 in exactly 37 months.

(ii) Invest $10,000 today and receive $90 per month indefinitely, with the first payment immediately received.

(iii) Invest $5,000 today and receive payments every quarter indefinitely. The payments grow at the rate of 2% per quarter. In other words, each payment is 2% greater than the previous. The first payment is $100 and will be received immediately.

Solution

74. ***HPRR and a Growing Annuity***

You purchase a financial asset today that offers a per annum rate of return of 6 percent compounded quarterly. The financial asset makes quarterly payments, the first one quarter from today, is $100. The last payment is exactly 27 quarters from today. Each payment, after the first, is 1.5% greater than the previous. As you receive payments on the financial asset, you reinvestment them at 6 percent per annum compounded quarterly. You sell the financial asset immediately after the 13’th payment. At that time, the financial asset offers a rate of return of 6 per cent per annum compounded quarterly.

Required: Find the holding period rate of return on your investment between your purchase and your sale of the financial asset including reinvested payments with interest.

Solution

75. ***Mortgage***

On Dec 31, 2002 you borrowed a certain sum in a mortgage agreement. The mortgage calls for monthly payments starting at the end of January 2003. The mortgage rate is 5.5 percent per annum compounded monthly for a 5-year term. The amortization is 25 years (300 payments). Today is Dec 31, 2007 and you have just made the 60'th mortgage payment.

You have renegotiated your mortgage with the Royal Bank for a second 5 year term with an amortization period of 20 years (240 payments). Your first monthly payment in this second term is one month from today. The mortgage rate over the second term is 5.9 percent per annum compounded monthly. The interest portion of the 28'th payment in the second term (that is, the 88'th payment from Dec 31, 2002) is $1,000.

Required: How much did you originally borrow on December 31, 2002?

Solution

76. ***IRR***

The three following investments are *unrelated* (thus, each part of this question will be marked entirely independently).

Each of the following three investments has a continuously compounded rate of return of 8% per annum.

(i) Invest $105 today and receive one payment of $A in exactly 37 months.

(ii) Invest $10,000 today and receive $A per month indefinitely with the first such amount exactly one month from today.

(iii) Invest $5,000 today and receive quarterly payments indefinitely. The payments grow at the rate of 1% per quarter. In other words, each payment is 1% greater than the previous. The first payment is $A and will be received exactly one quarter from today.

In each of the three above cases, determine the amount $A.

Solution

77. ***HPRR and a Growing Annuity***

Today, you purchase a financial asset. The financial asset offers a rate of return of 6 percent per annum compounded semi-annually. There are fifty semi-annual payments with the first equal to \$100 in exactly six months. Each subsequent payment is 3% greater than the previous. As you receive these payments, you deposit them into a savings plan that pays 6% per annum compounded semi-annually. Immediately after you deposit the fourteenth payment from the financial asset into the savings plan, you sell the financial asset. At this time, the financial asset offers a rate of return of 6 per cent per annum compounded semi-annually.

Required: What is your holding period rate of return between your purchase and sale of the financial asset including received payments with interest?

Solution

78. IRR for a "Two-Period" Investment.

An investment costs I=\$100 today that pays A=\$60 in one year and B=\$70 in two years (a two period investment).

a. Write the equation that defines the IRR for this particular investment (that is, the NPV=0 equation)

b. Use your financial calculator to find the per-annum IRR.

c. Use the formula $IRR = \frac{A + \sqrt{A^2 + 4 * I * B}}{2 * I} - 1$ to verify the IRR from part "a". This equation is the quadratic solution of the equation that defines the IRR for a two-period investment.

Solution

(6.7) Chapter Index