Chromosome Painting

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Abstract

We consider a Moran model with recombination in a haploid population of size N. At each birth event, with probability $1-\rho_N$ the offspring copies one parent's chromosome, and with probability ρ_N she inherits a chromosome that is a mosaic of both parental chromosomes. We assume that at time 0 each individual has her chromosome painted in a different color and we study the color partition of the chromosome that is asymptotically fixed in a large population, when we look at a portion of the chromosome such that $\rho := \lim_{N\to\infty} \frac{\rho_N N}{2} \to \infty$. To do so, we follow backwards in time the ancestry of the chromosome of a randomly sampled individual. This yields a Markov process valued in the color partitions of the half-line, that was introduced by Esser et al. [2016], in which blocks can merge and split, called the partitioning process. Its stationary distribution is closely related to the fixed chromosome in our Moran model with recombination. We are able to provide an approximation of this stationary distribution when $\rho \gg 1$ and an error bound. This allows us to show that the distribution of the (renormalised) length of the leftmost block of the partition (i.e. the region of the chromosome that carries the same color as 0) converges to an exponential distribution. In addition, the geometry of this block can be described in terms of a Poisson point process with an explicit intensity measure.

1 Introduction

1.1 Motivation: a Moran model with recombination

Genetic recombination is the mechanism by which, in species that reproduce sexually, an individual can inherit a chromosome that is a mosaic of two parental chromosomes. Many classical population genetics models ignore recombination and only focus on a single locus, i.e. a location on the chromosome with a unique evolutionary history. In this setting, many analytical results are known. For example the time to fixation (i.e. the first time at which all individuals carry the same allele) or the fixation probabilities (see for example Etheridge [2011]). However, understanding the joint evolution of different loci is well known to be mathematically challenging, as one needs to take into account non-trivial correlations between loci along the chromosome. For instance, loci that are close to one another are difficult to recombine, so they often inherit their genetic material from the same parent and as a consequence, often share a similar evolutionary history. On the contrary, loci that are far from one another will tend to have different, but not independent, evolutionary histories.

To visualize the questions that will be addressed in this work, let us imagine that in the ancestral population, each individual carries a single continuous chromosome painted in a distinct color. By the blending effect of recombination, after a few generations, the chromosome of each individual looks like a mosaic of colors, each color corresponding to the genetic material inherited from a single ancestral individual. Some natural questions arise: How does the mosaic of colors that is fixed in the population look like? How many colors are there? If the leftmost locus is red (i.e. is inherited from the individual with red chromosome in the ancestral population), what is the amount of red in the mosaic and where are the red loci located? These questions are interesting from a biological point of view: for example, the number of colors in the mosaic corresponds to the number of ancestors that have contributed to an extant chromosome. Loci that are of the same color (i.e., that have been inherited from the same individual in the ancestral population) are called identical-by-descent (IBD).

It is known that changes in the population size or natural selection can alter the sizes of the IBD segments: for example genes that are under selection tend to be located within large IBD segments. This prediction can guide the detection of genes that are under selection (see for example the methods developed by Sabeti et al. [2002] or McQuillan et al. [2008]). The aim of this article is to characterize the distribution of the IBD blocks along a chromosome in the absence of selection or demography. Our results may then be used as predictions under the null hypothesis, that can serve as a standard to which compare real data, e.g., to infer selection or demography.

Also, our results may be relevant to the analysis of data obtained in experimental evolution. For example, in the experiment carried out by Teotónio et al. [2017], the authors intercrossed individuals from 16 different subpopulations of the worm *C. elegans* and let the population evolve for several generations at controlled population size. Then, each individual is genotyped, each of its variants is mapped to one of the 16 ancestor subpopulations, so as to get a represention of each DNA sequence of each individual as a partition of the sequence into 16 colors. Again, our model (or an extension to our model accomodating for the finite number of colors), might be used as a null model whose predictions can be compared to these real color mosaics.

Sampling the chromosome, seen as a continuous, single-ended strand modelled by the positive half-line, of an individual in the present population and tracing backwards in time the ancestry of every locus yields a process valued in the partitions of $\mathbb{R}^+$, called the $\mathbb{R}$-partitioning process. Here, $x \geq 0$ and $y \geq 0$ belong to the same block of the $\mathbb{R}$-partitioning process at time t if the loci at positions x and y on the sampled chromosome shared the same ancestor t units of time ago.

The $\mathbb{R}$-partitioning process is the continuum analog of the celebrated Ancestral Recombination Graph [Hudson, 1983, Griffiths, 1991, Griffiths and Marjoram, 1997]. Before giving a formal description of this object, we start by showing how it arises naturally from a multi-locus Moran model. The population size is N and each haploid individual carries a single linear chromosome of length R. At time 0, each individual has her (unique) chromosome painted in a distinct color (see Figure 1.1). Each individual reproduces at rate 1, and upon reproduction, the individual chooses a random partner in the population. Let $\rho_N \in (0, 1)$,

- With probability $1 - \rho_N R$, the offspring copies one parent's chromosome (chosen uniformly at random).
- With probability $\rho_N R$, a recombination event occurs. We assume single-crossover recombination which means that each parental chromosome is cut into two fragments. The position of the cutpoint (i.e. the crossover) is uniformly distributed along the chromosome (see Figure 1.1). The offspring copies the genetic material to the left of this point from one parent and the genetic material to the right of this point from the other parent.

The offspring then replaces a randomly chosen individual in the population. Because of recombination, at time t each chromosome is a mosaic of colors, each color corresponding to the genetic material inherited from one individual in the founding generation. (In other words, loci sharing the same color are IBD.)

Let us now consider $z = (z_0, z_1, \ldots, z_n) \in [0, R]$ corresponding to the locations of $n + 1$ loci along the chromosome (with $z_0 < z_1 < \ldots < z_n$). Forward in time, the evolution of the genetic composition of the population can be described in terms of a $(n + 1)$-locus Moran model with recombination as described in Durrett [2008], Bobrowski et al. [2010]. Backward in time, the

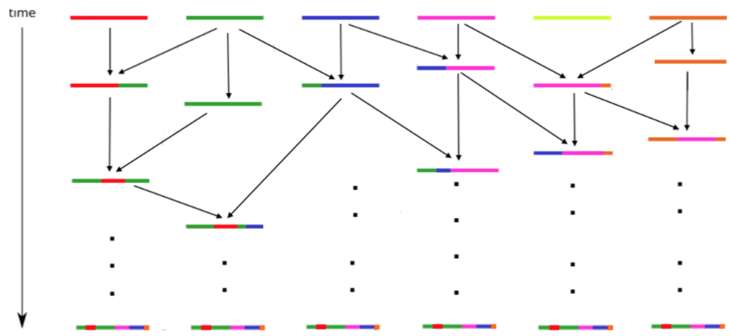


Figure 1: Moran model model with recombination.

genealogy of those loci (sampled from the same individual) is described in terms of the discrete partitioning process, introduced by Esser et al. [2016], which traces the history of the $n+1$ loci under consideration (see Figure 1.1). More precisely, the discrete partitioning process (associated to z) is a Markov process valued in the partitions of $z = \{z_0, z_1, \dots, z_n\}$ such that z_i and z_j are in the same block at time t if and only if they inherit their respective genetic material from the same individual t units of time ago. (In other words, z_i and z_j are IBD if we look t units of time in the past). In a population of size N, it can be seen that the dynamics of the discrete partitioning process are controlled by the following transitions.

- Each pair of blocks coalesces at rate $2/N + O(\rho_N/N)$.
- Each block $b = \{z_{i_1}, \dots, z_{i_k}\}$ is fragmented into $\{z_{i_1}, \dots, z_{i_j}\}$ and $\{z_{i_{j+1}}, \dots, z_{i_k}\}$ at rate $\rho_N(z_{i_{j+1}} - z_{i_j})$.
- Simultaneous splitting and coalescence events happen at rate $O(\rho_N/N)$.

The interesting scaling for this process is when time is accelerated by $N/2$ and the recombination probability scales with N in such a way that

$$\lim_{N\to\infty} \rho_N N/2 = \rho, \tag{1}$$

for some $\rho > 0$. It can readily be seen that the discrete partitioning process in a population of size N converges in distribution (in the Skorokhod topology) to a process $(\Gamma_t^{\rho,z}; t \geqslant 0)$, which is the Markov process with the following transition rates:

- Each pair of blocks coalesces at rate 1.
- Each block $b = \{z_{i_1}, \dots, z_{i_k}\}$ is fragmented into $\{z_{i_1}, \dots, z_{i_j}\}$ and $\{z_{i_{j+1}}, \dots, z_{i_k}\}$ at rate $\rho(z_{i_{j+1}} - z_{i_j})$.

In the literature, $\Gamma^{\rho,z}$ is also referred to as the Ancestral Recombination Graph (ARG) [Hudson, 1983, Griffiths, 1991, Griffiths and Marjoram, 1997] associated to z (with recombination rate ρ). The following scaling property can easily be deduced from the description of the transition rates. We assume that at time 0 all loci are sampled in the same individual, i.e. we consider the ARG started from the coarsest partition. Then

$$\forall R > 0, \quad \Gamma^{R,z} = \Gamma^{1,Rz} \text{ in distribution.} \tag{2}$$

In the following, we are going to consider a high recombination regime, i.e. that ρ is large. This relation states that this is equivalent to considering that the distances between loci of interest are large.

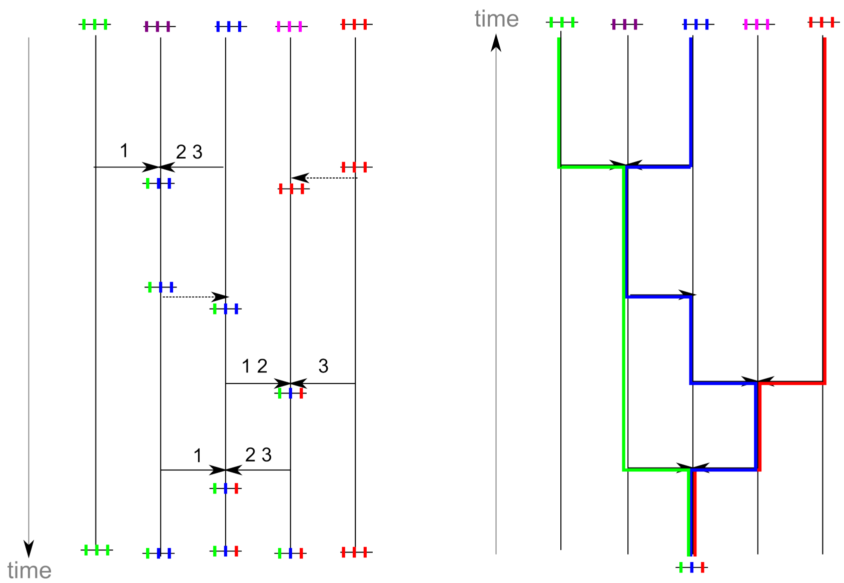


Figure 2: Duality between the Moran model and the discrete partitioning process ($N = 5,\ n = 3$). The left panel represents a realization of the Moran model with recombination. On the top we represented the chromosomes of the different individuals at generation 0, assuming that each one is painted in a different color. Arrows represent reproduction events. For reproduction events without recombination, the base of the arrow represents the parent from which the genetic material is inherited, and the arrow points at the individual that is replaced. For reproduction events with recombination (represented by two arrows), we indicate on top of each arrow the loci that are inherited from each parent. In the second panel we show how the discrete partitioning process can be obtained by reverting time. We sample the 3 loci in an individual in the present population and we follow the ancestral lineages corresponding to each of her loci. Each time an ancestral lineage finds the tip of an arrow, it jumps to the base of the arrow. If there are two arrows, the lineage corresponding to locus i jumps to the base of the arrow labelled i.

1.2 The $\mathbb{R}$-partitioning process

As the goal of this article is to characterize the distribution of the IBD blocks in a continuous chromosome in an infinite population, we extend the ARG $(\Gamma_t^{\rho,z}; t \geqslant 0)$ to the whole positive real line. To do so, we will consider partitions of $\mathbb{R}^+$. We call a segment a maximal set of connected points belonging to the same block of the partition. A partition of $\mathbb{R}^+$ is right-continuous if the segments of the partition are left-closed (right-open) intervals and the blocks correspond to disjoint unions of such intervals. Let $\mathcal{P}^{loc}$ be the set of partitions of $\mathbb{R}^+$ that are right-continuous and locally finite, i.e. such that each compact subset of $\mathbb{R}^+$ contains only a finite number of segments. For any z finite subset of $\mathbb{R}^+$, $\mathcal{P}_z$ is the set of partitions of z and t_z is the trace of z, i.e. the function $\mathcal{P}^{loc} \to \mathcal{P}_z$ such that for any $\pi \in \mathcal{P}^{loc}$, $t_z(\pi)$ is the partition of z induced by π. We define $\mathcal{F}$ as the σ-field on $\mathcal{P}^{loc}$ generated by

$$\mathcal{C} = \{\{\omega \in \mathcal{P}^{loc}, t_z(\omega) = \pi\},\ n \in \mathbb{N},\ z = (z_0, \dots, z_n) \subset \mathbb{R}^+,\ \pi \in \mathcal{P}_z\}.$$

Finally, for any measure μ on a measured space $(\Omega, \mathcal{A})$ and any $\mathcal{A}$-measurable function f, we will denote by $f \star \mu$ the pushforward of μ i.e. the measure such that $\forall B \in \mathcal{A}$, $f \star \mu(B) = \mu(f^{-1}(B))$.

Theorem 1.1. *Let μ_0 be a probability measure on $(\mathcal{P}^{loc}, \mathcal{F})$. The $\mathbb{R}$-partitioning process $(\Pi_t^\rho; t \geq 0)$ started at μ_0 is the unique càdlàg stochastic process valued in $(\mathcal{P}^{loc}, \mathcal{F})$ such that for any z, finite subset of $\mathbb{R}$, $(t_z(\Pi_t^\rho);\ t \geq 0)$ is the ARG at rate ρ for the set of loci z started at $t_z \star \mu_0$.*

The proof of this theorem can be found in Section 2. The goal of this paper is to study some properties of the invariant measure of the $\mathbb{R}$-partitioning process.

Theorem 1.2. *The $\mathbb{R}$-partitioning process $(\Pi_t^\rho; t \geq 0)$ has a unique invariant probability measure μ^ρ in $(\mathcal{P}^{loc}, \mathcal{F})$. In addition, for any finite subset z of $\mathbb{R}^+$,*

$$t_z \star \mu^\rho \ = \ \mu^{\rho,z}$$

where $\mu^{\rho,z}$ is the unique invariant measure of $\Gamma^{\rho,z}$.

We let the reader refer to Section 3 for proof of this result.

1.3 Approximation of the stationary distribution of the ARG

The ARG with more than two loci is a complex process and some authors have considered that characterizing its distribution is "computationally not tractable" (see Bobrowski et al. [2010]). Griffiths et al. [2016] and Esser et al. [2016] provided methods to compute the stationary distribution that fail when considering a large number of loci. Our goal is to provide an approximation of the stationary distribution of the ARG that is relatively easy to handle, even when we consider a large number of loci.

One of the main contributions of this paper is an explicit approximation (and an error bound for it) of the stationary distribution of the ARG $(\Gamma_t^{\rho,z}; t \geqslant 0)$ when the typical distance between the z_i's is large (or equivalently when the rate of recombination ρ is large).

We fix $z = (z_0, \dots, z_n) \subset \mathbb{R}$. We define

$$\alpha = \min_{i \neq j} |z_i - z_j|$$

and we assume that $\alpha > 0$ (or equivalently that the coordinates of z are pairwise distinct). Let $r \in \{0, \dots, n\}$, $\mathcal{P}_z^r$ is the set of partitions of z containing $n+1-r$ blocks. In particular, the only partition in $\mathcal{P}_z^0$ is π_0, the partition made of singletons. We define a "coalescence scenario of order r" as a sequence of partitions $(s_k)_{0 \leq k \leq r}$ such that s_0 is the partition made of singletons and for $1 \leq k \leq r$, s_k is a partition of order k that can be obtained from s_{k-1} by a single coagulation event. For any partition in $\pi \in \mathcal{P}_z^r$, $\mathcal{S}(\pi)$ is the set of coalescence scenarios of order r such that $s_r = \pi$.

For $\pi \in \mathcal{P}_z^r$, let $b_1, \ldots, b_{n+1-r}$ be the blocks of π. We denote by $C(\pi)$ the cover length of π defined as:

$$C(\pi) \ := \ \sum_i \max_{x,y \in b_i} |x-y|.$$

In particular, the cover length of π_0 is equal to 0.

Let $s = (s_k)_{0 \le k \le r}$ be a scenario of coalescence of order r, with $1 \le r \le n$. We define the energy of s, $E(s)$ as

$$E(s) := \prod_{i=1}^{r} C(s_i).$$

where $C(s_i)$ is the total rate of fragmentation at

In words, the energy of a scenario corresponds to the product of the successive cover lengths at each step. Finally, define

$$\forall \pi \in \mathcal{P}_z \setminus \mathcal{P}_z^0, \quad F(\pi) := \sum_{S \in \mathcal{S}(\pi)} \frac{1}{E(S)}. \tag{3}$$

Theorem 1.3. *There exists a function*

$$f^n : \mathbb{R}_*^+ \to \mathbb{R}^+, \quad \lim_{x \to \infty} f^n(x) = 0,$$

independent of the choice of $z = (z_0, \ldots, z_n)$ and ρ, such that

$$\forall \rho > 0, \ \forall k \in [n], \ \forall \pi_k \in \mathcal{P}_z^k, \quad \left| \mu^{\rho,z}(\pi_k) - \frac{1}{\rho^k} F(\pi_k) \right| \le f^n(\alpha\rho) \frac{1}{\rho^k} F(\pi_k).$$

Recall that the RHS goes to 0 either when $\rho \to \infty$ or $\alpha \to \infty$. As already mentioned (see (2)), these two scaling limits are equivalent. We let the reader refer to Section 4 for a proof of this result.

1.4 Characterization of the leftmost block of the $\mathbb{R}$-partitioning process

As an application of our approximation of $\mu^{\rho,z}$, we characterize the geometry of the leftmost block on a large scale. Motivated by the Moran model and the scaling relation (2), without loss of generality, we study the $\mathbb{R}$-partitioning process at rate 1 restricted to $[0, R]$.

For any partition π, $x \sim_\pi y$ means x and y are in the same block of π. Let Π_{eq} be the random partition with law μ^1. Let $\mathcal{L}_R(0)$ be the length of the block containing 0, rescaled by $\log(R)$. More precisely,

$$\mathcal{L}_R(0) = \frac{1}{\log(R)} \int_{[0,R]} \mathbb{1}_{\{x \sim_{\Pi_{eq}} 0\}} dx.$$

We define the random measure $\vartheta^R[a,b]$ such that

$$\forall a, b \in [0,1], \ a \le b, \quad \vartheta^R[a,b] = \frac{1}{\log(R)} \int_{R^a}^{R^b} \mathbb{1}_{\{x \sim_{\Pi_{eq}} 0\}} dx$$

so that ϑ^R encapsulates the whole information about the positions of the loci that are IBD to 0 in the *logarithmic scale* (which will be seen to be the natural scaling for the partitioning process at equilibrium). In the following ϑ^R will be considered as a random variable valued in $\mathcal{M}([0,1])$, the space of locally finite measures of $[0,1]$ equipped with the weak topology (i.e. the coarsest topology making $m \to \langle m, f \rangle$ continuous for every function f bounded and continuous). In the following, $\Longrightarrow$ denotes the convergence in distribution.

Theorem 1.4. *Consider a Poisson point process $\mathcal{P}^\infty$ on $[0,1] \times \mathbb{R}^+$ with intensity measure*

$$\lambda(x,y) = \frac{1}{x^2} \exp(-y/x) dx dy$$

and define the random measure on $\mathcal{M}([0,1])$

$$\vartheta^\infty \ := \sum_{(x_i,y_i)\in\mathcal{P}^\infty} y_i \ \delta_{x_i}.$$

Then

1. $\vartheta^R \underset{R\to\infty}{\Longrightarrow} \vartheta^\infty$ *in the weak topology.*
2. *In particular,* $\mathcal{L}_R(0) \underset{R\to\infty}{\Longrightarrow} \varepsilon(1)$ *where* $\varepsilon(1)$ *denotes the exponential distribution of parameter* 1.

This result can be interpreted as follows. As $R \to \infty$, there are distinct regions of genetic material that is IBD to 0, and at the limit, those regions are clustered into points. The locations of those regions are encapsulated by the x_i's (in the logarithmic scale) – in other words, at R^{x_i}, there is a cluster of genetic material IBD to 0 – and the coordinate y_i corresponds the amount of genetic material that is IBD to 0 present in this cluster (see Figure 1.4). Note that the positions of the segments (in the logarithmic scale) are given by the Poisson process of intensity $(1/x)dx$, which is known as "the scale invariant Poisson Process" (see for example Arratia [1998]).

We performed some numerical simulations of the partitioning process to illustrate the second part of Theorem 1.4. Figure 1.4 shows how the length of the cluster covering 0 is exponentially distributed.

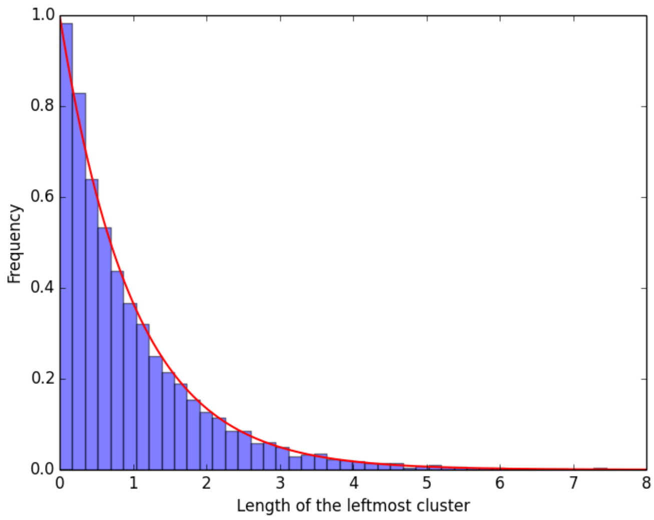


Figure 3: **Distribution of the length of the leftmost block (R = 5000).** The blue histogram represents the empirical distribution, that was obtained by simulating the partitioning process, for a chromosome of length $R = 5000$. The number of replicates is 10000. The red curve is the probability density function of an exponential distribution of parameter 1. We compared the empirical distribution to an exponential distribution using a Kolmogorov-Smirnov test, which was positive, with a p-value of 10^{-4}.

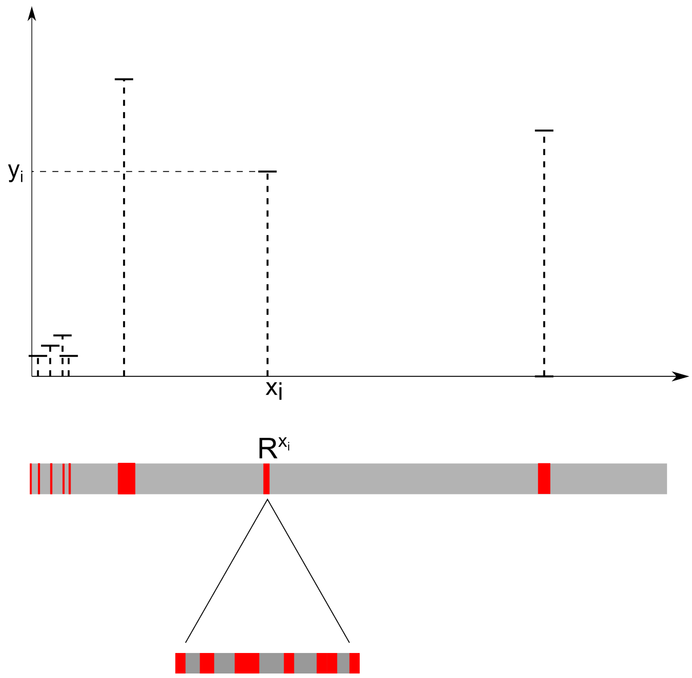


Figure 4: Example of a realization of ϑ^∞ and its interpretation. Regions of the chromosome (in the log-scale) that are IBD to 0 are represented in red. In the limit, those regions are clustered into points which can have a complex geometry on a finer scale (see lower figure). y_i is the amount of genetic material IBD to 0 in the region located at R^{x_i}.

1.5 Biological relevance

Recall that a "morgan" is a unit used to measure genetic distance. The distance between two loci is 1 morgan if the average number of crossovers is 1 per reproduction event. In other words, in a population of size N, if we consider the discrete partitioning process at rate 1, two loci z_i and z_j are at distance $\frac{2}{N}|z_i - z_j|$ morgans.

We studied the $\mathbb{R}$-partitioning process at rate 1 restricted to $[0, R]$, which should correspond to a portion (or frame) of the chromosome that is of size R/N morgans (and small enough so that the single crossing-over approximation is valid). Then, we first let the population size N tend to infinity (in order to get the partitioning process from the underlying finite population model), and then the size of the frame go to ∞ (in the $\mathbb{R}$-partitioning process). Note that since we take successive limits (first $N \to \infty$ and then $R \to \infty$), this gives no clue on how the population size and the size of the observation frame should scale with one another to ensure that the approximation is correct.

Wiuf and Hein [1997] used the same hypothesis. They explained that this approximation should be valid in human populations, where for example, the size of chromosome 1 is 2.93 morgans and the effective population size is $N = 20000$. If one looks at a frame of this chromosome of length 1 morgan (1/3 of the chromosome), then $R = 20000$.

1.6 Outline

This paper is organized as follows. In Section 2 we propose a construction of the $\mathbb{R}$-partitioning process and we prove Theorem 1.1. In Section 3 we show the existence and uniqueness of a stationary distribution for this process (Theorem 1.2). Finally, Sections 4 and 5 are devoted to the proofs of Theorems 1.3 and 1.4 respectively.

2 The $\mathbb{R}$-partitioning process

2.1 Some preliminary definitions

We start by recalling some definitions that are useful for the rest of the paper and by clarifying some notation. In the following, we consider partitions of $\mathbb{R}^+$ or of subsets of $\mathbb{R}^+$. We call a *segment* a maximal set of adjacent points belonging to the same block of the partition. For $E = \mathbb{R}^+$ or $E \subset \mathbb{R}^+$, we say that a partition ω of E is *locally finite* if for any compact subset K of E such that $K \cap E \neq \varnothing$, $t_{K\cap E}(\omega)$ contains a finite number of segments. We say that a partition is *right continuous* if the segments of the partition are left-closed (right-open) intervals (and the blocks correspond to disjoint unions of such intervals). Note that for a partition that is right continuous, infinite sequences of small intervals can only accumulate to the left of a point. For $a < b \in \mathbb{R}^+$, we denote by $\mathcal{P}^{loc}_{[a,b]}$ (resp. $\mathcal{P}^{loc}$) the set of the partitions of $[a, b]$ (resp. $\mathbb{R}^+$) that are right continuous and finite (resp. right continuous and locally finite). We define the σ-field $\mathcal{F}$ on $\mathcal{P}^{loc}$ generated by

$$\mathcal{C} = \{\{\omega \in \mathcal{P}^{loc}, t_z(\omega) = \pi\},\ n \in \mathbb{N},\ z = (z_0, \ldots, z_n) \subset \mathbb{R}^+,\ \pi \in \mathcal{P}_z\}.$$

We also need to define a distance d on $\mathcal{P}^{loc}$. To do so, we start by identifying each partition in $\mathcal{P}^{loc}$ to a function from $\mathbb{R}^+$ to itself. More precisely, we define a map $\phi : \mathcal{P}^{loc} \to D(\mathbb{R}^+, \mathbb{R}^+)$ such that, for $\pi \in \mathcal{P}^{loc}$, $\phi(\pi)$ is constructed as follows. For each block B of π and for each $x \in B$, we set $\phi(\pi)(x) := \min(B)$. Note that ϕ is injective and $\forall x \in \mathbb{R}^+,\ \phi(\pi)(x) \leq x$. Also, as $\pi \in \mathcal{P}^{loc}$, ϕ is càdlàg and has a finite number of jumps in any compact set of $\mathbb{R}^+$. Now, for any $\pi_1, \pi_2 \in \mathcal{P}^{loc}$, define

$$d(\pi_1, \pi_2) \ := \int_0^{+\infty} |\phi(\pi_1)(x) - \phi(\pi_2)(x)| \exp(-x) dx.$$

It can easily be checked that d defines a distance on $\mathcal{P}^{loc}$. For $T > 0$, we will denote by $D([0, T], \mathcal{P}^{loc})$ the Skorokhod space associated to $(\mathcal{P}^{loc}, d)$ equipped with the standard Skorokhod topology. For each partition $\pi \in \mathcal{P}^{loc}$ we define a natural ordering on its blocks. We denote by $b^0, b^1, \ldots, b^i, \ldots$ the blocks of π indexed in such a way that $\min(b^0) < \min(b^1) < \ldots$

The space $\mathcal{P}^{loc}$ is separable under d. Indeed, for $n \in \mathbb{N}^*$, let $\mathcal{S}_n$ be the set of partitions in $\pi \in \mathcal{P}^{loc}$ such that in $\pi|_{[0,n[}$ each block is a finite union of segments whose endpoints are in $[0, n[\cap\mathbb{Q}$ and $[n, +\infty[$ is included in a block of π. $S = \cup_n \mathcal{S}_n$ is countable and using standard methods, it can be shown that given $\pi \in \mathcal{P}^{loc}$ and $\epsilon > 0$, there exists a partition $\pi' \in S$ such that $d(\pi, \pi') < \epsilon$. The space $\mathcal{P}^{loc}$ is not complete but we define its completion $\bar{\mathcal{P}}^{loc}$.

In the following, we will also consider partitions of $\mathbb{Q}^+$. We define $\mathcal{P}^{loc}_{\mathbb{Q}}$ as the set of locally finite partitions of $\mathbb{Q}^+$ that are right continuous (in the sense that if $\mathbb{Q}^+ \ni x_n \downarrow x \in \mathbb{Q}^+$ then x_n is in the same segment as x for n large enough) and $\mathcal{F}_{\mathbb{Q}}$ the σ-field generated by

$$\mathcal{C}_{\mathbb{Q}} = \{\{\omega \in \mathcal{P}_{\mathbb{Q}}, t_z(\omega) = \pi\}, n \in \mathbb{N},\ z = (z_0, \ldots, z_n) \subset \mathbb{Q}^+,\ \pi \in \mathcal{P}_z\}.$$

2.2 Definition of the $\mathbb{R}$-partitioning process

We start by defining ARG for a finite set of loci Hudson [1983], Griffiths [1991], Griffiths and Marjoram [1997]. Note that here we only consider the case of single-crossover recombination. We consider a finite set of loci, whose positions in the chromosome are given by $z \subset \mathbb{R}^+$, $z = \{z_0, \ldots, z_n\}$ (with $z_0 < z_1 < \ldots < z_n$). Let $\rho > 0$. Let $(\Gamma^{\rho,z}_t; t \geq 0)$ be the Markov process on $(\mathcal{P}_z, \mathcal{F}_z)$, with the following transition rates:

- **Coagulation:** Consider $\pi_1 \in \mathcal{P}_z$ and a and b two blocks of π_1. Let $c = a \cup b$ and π_2, the partition obtained by coalescing the blocks a and b into c and letting all the other blocks unchanged. A transition from π_1 to π_2 occurs at rate:

$$q(\pi_1, \pi_2) = 1.$$

- **Fragmentation:** Now take $\pi_1 \in \mathcal{P}_z$ and a a block of π_1 containing k elements $z_{i_1}, \ldots, z_{i_k}$ such that $z_{i_1} < \ldots < z_{i_k}$. Let $j < k$. Let $b = \{z_{i_1}, \ldots, z_{i_j}\}$ and $c = \{z_{i_{j+1}}, \ldots, z_{i_k}\}$ and π_2, the partition obtained by fragmenting a into b and c. A transition from π_1 to π_2 occurs at rate:
$$q(\pi_1, \pi_2) = \rho(z_{i_{j+1}} - z_{i_j}).$$

- All these events are independent and all other events have rate 0.

This process is called the ARG at recombination rate ρ, for the set of particles (or loci) z. It is easily seen that $\Gamma_t^{\rho,z}$ has a finite state-space and is irreducible. We call $\mu^{\rho,z}$ its unique invariant probability measure, that will be characterized in Section 4.

We now want to define a process on $(\mathcal{P}^{loc}, \mathcal{F})$, called the $\mathbb{R}$-*partitioning process* so that for any z finite subset of $\mathbb{R}^+$, the trace on z is distributed as the ARG $\Gamma^{\rho,z}$.

We set $\Pi_0^{\rho,L} = \pi_0,\ \pi_0 \in \mathcal{P}_{[0,L[}^{loc}$. We assume that the blocks of this partition are indexed with the natural order defined on the previous section. The partitioning process on $\mathcal{P}_{[0,L[}^{loc}$ is generated by a sequence of independent Poisson point processes as follows:

- For all $i, j \in \mathbb{N}$, $Y^{i,j}$ is a Poisson point process of intensity 1. For $t \in Y^{i,j}$, at time t^- there is a **coagulation** event: blocks b^i and b^j are replaced by $b^i \cup b^j$. If i or j does not correspond to the index of any block, nothing happens.

- For all $i \in \mathbb{N}$, X^i is a Poisson point process on $\mathbb{R}^+ \times [0, L[$ with intensity $\rho\ dt \otimes dx$. The atoms of X^i correspond to **fragmentation** events. For $(t, x) \in X^i$, if at time t^-, $\Pi_{t^-}^{\rho,L} = \pi$, if b^i is a block of π and $x \in\ ]\min(b^i), \sup(b^i)[$, b^i is fragmented into two blocks $b^{i,-}$ and $b^{i,+}$ such that $b^{i,-} = b^i \cap [0, x[$ and $b^{i,+} = b^i \cap [x, L]$. Then $\Pi_t^{\rho,L}$ is equal to the partition obtained by replacing b^i by $b^{i,-}$ and $b^{i,+}$. If $x \notin\ ]\min(b^i), \sup(b^i)[$, nothing happens.

After each event, blocks are relabelled in such a way that they remain ordered, in the sense specified above. Recall that, with this construction, the partitions that are formed are always right continuous. Also the number of blocks of $\Pi^\rho|_{[0,L[}$ is stochastically dominated by a birth-death process which jumps from n to $n+1$ at rate $\rho L n$ and from n to $n-1$ at rate $n(n-1)/2$ with initial condition the number pf blocks in π_0, which is known to remain locally bounded (and even to have $+\infty$ as entrance boundary, see Lambert [2005]). There is the same stochastic domination between the two processes for the numbers of jump events on any fixed time interval. This shows that the number of blocks in $\Pi_t^\rho|_{[0,L[}$ is a.s. locally bounded and since the number of segments jumps at most by $+1$ at each event, the number of segments is also a.s. locally bounded. So a.s. for all t, $\Pi_t^{\rho,L} \in \mathcal{P}_{[0,L[}^{loc}$.

Finally, we define the partitioning process in $\mathbb{R}$, as the projective limit of $(\Pi_t^{\rho,L};\ t \geq 0)_{L \in \mathbb{R}^+}$ as $L \to \infty$. In fact, by construction, $\forall L' > L,\ \forall t \geq 0,\ \Pi_t^{\rho,L'}|_{[0,L[} = \Pi_t^{\rho,L}$, where $\Pi^{\rho,L'}|_{[0,L[}$ is the natural restriction of $\Pi_t^{\rho,L'}$ to $[0, L[$.

Proposition 2.1. *The $\mathbb{R}$-partitioning process, $(\Pi_t^\rho;\ t \geq 0)$ with initial measure π_0 is the unique càdlàg stochastic process valued in $(\mathcal{P}^{loc}, \mathcal{F})$ such that*
$$\forall L \geqslant 0,\ (\Pi_t^\rho \cap [0, L];\ t \geqslant 0) = (\Pi_t^{\rho,L};\ t \geqslant 0)$$
with $\Pi_0 = \pi_0$. Further for any finite subset z in $\mathbb{R}^+$, $t_z(\Pi^\rho)$ is distributed as $\Gamma^{\rho,z}$, the ARG with initial condition $t_z(\pi_0)$.

Proof. We need to check that, for any $T > 0$, $(\Pi_t^\rho;\ 0 \leq t \leq T) \in D([0, T], \mathcal{P}^{loc})$ almost surely. To do so, we need to prove that with probability 1, for every $t \in [0, T]$, for every $\epsilon > 0$, one can find $s > 0$ such that $d(\Pi_t^\rho, \Pi_{t+s}^\rho) < \epsilon$. Fix $\epsilon > 0$ and pick $L > 0$ such that $2\exp(-L)(L+1) < \epsilon$. From the Poissonian construction, for any $T > 0$, the process $\Pi^\rho|_{[0,L[}$ has a finite number of jumps in

$[0, T]$, which happen at times $t_1, \ldots, t_n$. We choose $s > 0$ such that $|t-s| < \min_i |t_{i+1} - t_i|$. Then $\Pi_t|_{[0,L[} = \Pi_{t+s}|_{[0,L[}$. As $\phi(\Pi_t^\rho|_{[0,L[}) = \phi(\Pi_t^\rho)|_{[0,L[}$, for any $x \in [0, L[$, $\phi(\Pi_t^\rho)(x) = \phi(\Pi_{t+s}^\rho)(x)$, so

$$
\begin{aligned}
d(\Pi_t^\rho, \Pi_{t+s}^\rho) &= 0 + \int_L^{+\infty} |\phi(\Pi_t^\rho)(x) - \phi(\Pi_{t+s}^\rho)(x)| e^{-x} dx \\
&\leq \int_L^{+\infty} 2x \exp(-x) dx \;=\; 2\exp(-L)(L+1) \;<\; \epsilon,
\end{aligned}
$$

and similarly for left-hand limits. So $(\Pi_t^\rho;\ 0 \leq t \leq T) \in D([0, T], \mathcal{P}^{loc})$. □

In addition, the following proposition can easily be deduced. Note that the second equality is just a trivial consequence of the first one.

Proposition 2.2 (Consistency)**.** *For all z and y, finite subsets of $\mathbb{R}^+$ such that $y \subset z$,*

$$\Gamma^{\rho,y} = \Gamma^{\rho,z}|_y,$$

where $\Gamma^{\rho,z}|_y$ denotes the restriction of $\Gamma^{\rho,z}$ to $\mathcal{P}_y$, and

$$\mu^{\rho,y} = t_y \star \mu^{\rho,z}.$$

We now turn to the proof of the main result of this section, i.e. Theorem 1.1.

Proof of Theorem 1.1. Let $(\Pi_t; t \geqslant 0)$ be a càdlàg process in $(\mathcal{P}^{loc}, \mathcal{F})$ such that for any z, finite subset of $\mathbb{R}$,

$$(t_z(\Pi_t);\ t \geq 0) \overset{d}{=} (\Gamma_t^{\rho,z};\ t \geqslant 0) \overset{d}{=} (t_z(\Pi_t^\rho);\ t \geq 0).$$

We denote by $\{z_i\}_{i \in \mathbb{N}}$ an enumeration of the rational numbers and for all $n \in \mathbb{N}$, we define $z^n := \{z_0, \ldots z_n\}$. For every $n > 1$, we have

$$t_{z^n}(\Pi^\rho) = t_{\mathbb{Q}}(\Pi^\rho)|_{z^n} \quad \text{and} \quad t_{z^n}(\Pi) = t_{\mathbb{Q}}(\Pi)|_{z^n},$$

so we have

$$(t_{\mathbb{Q}}(\Pi_t^\rho); t \geqslant 0) \overset{d}{=} (t_{\mathbb{Q}}(\Pi_t); t \geqslant 0).$$

In particular

$$\forall (t_1, \ldots, t_n) \subset \mathbb{R}^+ \quad (t_{\mathbb{Q}}(\Pi_{t_1}^\rho), \ldots, t_{\mathbb{Q}}(\Pi_{t_n}^\rho)) \overset{d}{=} (t_{\mathbb{Q}}(\Pi_{t_1}), \ldots, t_{\mathbb{Q}}(\Pi_{t_n})). \tag{4}$$

Similarly as done p.10, the number of blocks of and number of events undergone by $(t_z(\Pi_t);\ t \in [0, T])$ are stochastically dominated, uniformly in $z \subset \mathbb{Q} \cap [0, L]$, by those of a birth-death process which jumps from n to $n+1$ at rate $\rho L n$ and from n to $n-1$ at rate $n(n-1)/2$ with initial condition the number of blocks in Π_0. This shows that a.s. for all $t \geq 0$, $t_{\mathbb{Q}}(\Pi_t) \in \mathcal{P}_{\mathbb{Q}}^{loc}$ (and of course $t_{\mathbb{Q}}(\Pi_t^\rho) \in \mathcal{P}_{\mathbb{Q}}^{loc}$). Since the partitions in $\mathcal{P}^{loc}$ (resp. $\mathcal{P}_{\mathbb{Q}}^{loc}$) are right-continuous and since $\mathbb{Q}$ is dense in $\mathbb{R}$, for every $\bar{\pi} \in \mathcal{P}_{\mathbb{Q}}^{loc}$ there exists a unique $\pi \in \mathcal{P}^{loc}$ such that $t_{\mathbb{Q}}(\pi) = \bar{\pi}$. In other words, the projection map

$$t_{\mathbb{Q}} : (\mathcal{P}^{loc}, \mathcal{F}) \to (\mathcal{P}_{\mathbb{Q}}^{loc}, \mathcal{F}_{\mathbb{Q}})$$

is bijective. With a little bit of extra work, one can show that $t_{\mathbb{Q}}^{-1}$ is measurable so, from (4),

$$\forall (t_1, \ldots, t_n) \subset \mathbb{R}^+, \quad (\Pi_{t_1}^\rho, \ldots, \Pi_{t_n}^\rho) \overset{d}{=} (\Pi_{t_1}, \ldots, \Pi_{t_n}).$$

This implies that $\forall T > 0$, $(\Pi_t;\ 0 \leq t \leq T) \overset{d}{=} (\Pi_t^\rho;\ 0 \leq t \leq T)$, in the Skorokhod topology $D([0, T], \bar{\mathcal{P}}^{loc})$ (see Billingsley [1968], Theorem 16.6). So $(\Pi_t^\rho;\ t \geq 0)$ is the unique process in $D([0, T], \bar{\mathcal{P}}^{loc})$ such that for any z, finite subset of $\mathbb{R}$, $(t_z(\Pi_t^\rho); t \geq 0)$ is distributed as $(\Gamma_t^{\rho,z}; t \geq 0)$. As $\mathcal{P}^{loc} \subset \bar{\mathcal{P}}^{loc}$, the theorem is proved. □

3 Stationary measure for the $\mathbb{R}$-partitioning process

The goal of this section is to prove Theorem 1.2. The idea of the proof is to consider the stationary measure of the partitioning process on finite sets of rational numbers. Using Kolmogorov's extension theorem we define its unique projective limit in $\mathcal{P}_{\mathbb{Q}}^{loc}$. Then, using continuity arguments, we prove that there is a unique extension of this measure to the partitions of $\mathbb{R}$. Let us now go into more details. We decompose the proof into several lemmas.

Lemma 3.1. *A measure ν is invariant for $(\Pi_t^\rho;\ t \geq 0)$ iff for any finite subset z of $\mathbb{R}^+$, $\nu \circ t_z^{-1}$ is invariant for $(t_z(\Pi_t^\rho);\ t \geq 0)$.*

Proof. We obviously only prove the "if" part. We consider a probability measure ν and for each finite $z \subset \mathbb{R}^+$, we define $\nu_z := \nu \circ t_z^{-1}$. We assume that for any subset $z \in \mathbb{R}$, ν_z is invariant for $(t_z(\Pi_t^\rho))$. We assume that $\Pi_0^\rho = \pi_0$ is distributed according to ν. We want to prove that

$$\forall B \in \mathcal{F},\ \ \forall t \in \mathbb{R}^+,\ \mathbb{P}(\Pi_t^\rho \in B) = \mathbb{P}(\Pi_0^\rho \in B).$$

As $\mathcal{F}$ is the σ-field generated by $\mathcal{C}$, and $\mathcal{C}$ is closed under finite intersection, we only need to prove that for any z finite subset of $\mathbb{R}^+$,

$$\forall \pi \in \mathcal{P}_z,\ \forall t \in \mathbb{R}^+,\ \mathbb{P}(t_z(\Pi_t^\rho) = \pi) = \mathbb{P}(t_z(\Pi_0^\rho) = \pi).$$

As ν_z is invariant for $t_z(\Pi^\rho)$,

$$\forall \pi \in \mathcal{P}_z,\ \mathbb{P}(t_z(\Pi_t^\rho) = \pi) = \mathbb{P}(t_z(\Pi_0^\rho) = \pi) = \nu_z(\pi),$$

which completes the proof of Lemma 3.1. $\square$

Lemma 3.2. *There exists a unique probability measure $\bar{\mu}^\rho$ on $(\mathcal{P}_{\mathbb{Q}}, \mathcal{F}_{\mathbb{Q}})$ charging right continuous partitions such that, for every finite $z \subset \mathbb{Q}^+$,*

$$t_z \star \bar{\mu}^\rho = \mu^{\rho,z}.$$

Furthermore, $\bar{\mu}^\rho$ only charges locally finite partitions of $\mathbb{Q}^+$ and for every $x \in \mathbb{Q}^+$,

$$\bar{\mu}^\rho(x \text{ is the extremity of a segment}) = 0.$$

Proof. From Proposition 2.2, the family $(\mu^{\rho,z};\ z \subset \mathbb{Q}^+)$ is consistent in the sense that for two finite subsets $z \subset z'$ then

$$t_z \star \mu^{\rho,z'} \ = \ \mu^{\rho,z}.$$

By an application of the Kolmogorov extension theorem, there exists a unique measure $\bar{\mu}^\rho$ defined on $(\mathcal{P}_{\mathbb{Q}}, \mathcal{F}_{\mathbb{Q}})$ such that for every finite subset z in $\mathbb{Q}$ we have

$$t_z \star \bar{\mu}^\rho \ = \ \mu^{\rho,z}.$$

(To see how one can apply Kolmogorov theorem in the context of consistent random partitions, we refer the reader to Beresticky [2009], Proposition 2.1.)

We now need to prove that $\bar{\mu}^\rho$ only charges locally finite partitions of $\mathbb{Q}^+$. To do so, we follow closely Wiuf and Hein [1997]. We fix $a, b \in \mathbb{N},\ a < b$. We want to prove that, if π is a partition of $\mathbb{Q}$ distributed as $\bar{\mu}^\rho$, then $S_{[a,b]}$, the number of segments in $\pi|_{[a,b]\cap\mathbb{Q}}$ is finite almost surely. To do so, we define

$$\begin{aligned} \forall n \in \mathbb{N}^*,\ \epsilon_n &:= 2^{-n}, \\ X_{in} &:= \mathbb{1}_{((a+(i-1)\epsilon_n)\not\sim(a+i\epsilon_n))} \\ z_{in} &:= (a + (i-1)\epsilon_n, a + \epsilon_n) \in \mathbb{R}^2. \end{aligned}$$

In words, $X_{in} = 1$ if $(i-1)\epsilon_n$ and $i\epsilon_n$ belong to different segments. Let us compute the expectation of $S_{[a,b]}$. Using the monotone convergence theorem we have

$$\begin{aligned}\mathbb{E}(S_{[a,b]}) &= 1 + \mathbb{E}(\lim_{n\to\infty} \sum_{i=1}^{\lfloor 2^n(b-a)\rfloor} X_{in}) = 1 + \lim_{n\to\infty} \sum_{i=1}^{\lfloor 2^n(b-a)\rfloor} \mathbb{E}(X_{in}) \\ &= 1 + \lim_{n\to\infty} \sum_{i=1}^{\lfloor 2^n(b-a)\rfloor} \mu^{\rho, z_{in}}(\{a+(i-1)\epsilon_n\}, \{a+i\epsilon_n\}).\end{aligned}$$

The ARG at rate ρ for the set of loci z_{in} has only two types of transitions: coagulation at rate 1 and fragmentation at rate $\rho\epsilon_n$, so

$$\mu^{\rho, z_{in}}(\{a+(i-1)\epsilon_n\}, a+\{i\epsilon_n\}) = \frac{\rho\epsilon_n}{1+\rho\epsilon_n}$$

which gives

$$\mathbb{E}(S_{[a,b]}) = 1 + \lim_{n\to\infty} \sum_{i=1}^{\lfloor 2^n(b-a)\rfloor} \frac{\rho 2^{-n}}{1+\rho\epsilon_n} = 1 + \rho(b-a).$$

Then $S_{[a,b]}$ is finite almost surely, which implies that $\bar{\mu}^\rho$ only charges locally finite partitions of $\mathbb{Q}$.

For the last statement let $x \in \mathbb{Q}^+$. By the previous argument,

$$\bar{\mu}^\rho(x \text{ is the extremity of a segment}) = \lim_{\epsilon\downarrow 0} \bar{\mu}^\rho(x-\epsilon \not\sim x+\epsilon) = 0,$$

which completes the proof. $\square$

Lemma 3.3. *There exists a unique measure μ^ρ on $(\mathcal{P}^{loc}, \mathcal{F})$ such that*

$$t_{\mathbb{Q}} \star \mu^\rho \;=\; \bar{\mu}^\rho,$$

where $\bar{\mu}^\rho$ is the measure defined in Lemma 3.2.

Proof. Let $\tilde{\mathcal{P}}^{loc}_{\mathbb{Q}}$ the set of locally finite partitions of $\mathbb{Q}$ such that for all $x \in \mathbb{Q}^+$, x is not an extremity of a segment of π. Note that here we do not assume that the partitions of $\mathbb{Q}$ are right continuous. From the previous Lemma, $\bar{\mu}^\rho(\tilde{\mathcal{P}}^{loc}_{\mathbb{Q}}) = 1$. Similarly, let $\tilde{\mathcal{P}}^{loc}$ be the set of elements π of $\mathcal{P}^{loc}$ such that for all $x \in \mathbb{Q}^+$, x is not an extremity of a segment of π. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, it is easy to see that for every $\bar{\pi} \in \tilde{\mathcal{P}}^{loc}_{\mathbb{Q}}$ there exists a unique $\tilde{\pi} \in \tilde{\mathcal{P}}^{loc}$ such that $t_{\mathbb{Q}}(\pi) = \bar{\pi}$. In other words, the projection map

$$t_{\mathbb{Q}} : (\tilde{\mathcal{P}}^{loc}, \mathcal{F}) \to (\tilde{\mathcal{P}}^{loc}_{\mathbb{Q}}, \mathcal{F}_{\mathbb{Q}})$$

is bijective. (Note that the condition that there are no rational extremities for the latter statement to hold, can be understood with the following counterexample. Let $\bar{\pi}$ be the partition of $\mathbb{Q}^+$ consisting of the two blocks $[0,1] \cap \mathbb{Q}$ and $]1, +\infty) \cap \mathbb{Q}$. Then there is no right-continuous partition $\pi \in \mathcal{P}^{loc}$ such that $t_{\mathbb{Q}}(\pi) = \bar{\pi}$.) With a little bit of extra work, one can show that $t_{\mathbb{Q}}^{-1}$ is measurable. As already mentioned in the proof of Theorem 1.1, the projection map

$$t_{\mathbb{Q}} : (\tilde{\mathcal{P}}^{loc}, \mathcal{F}) \to (\tilde{\mathcal{P}}^{loc}_{\mathbb{Q}}, \mathcal{F}_{\mathbb{Q}})$$

is bijective and measurable, so the measure μ^ρ defined by

$$\mu^\rho = t_{\mathbb{Q}}^{-1} \star \left[\bar{\mu}^\rho(\cdot \cap \tilde{\mathcal{P}}^{loc}_{\mathbb{Q}})\right]$$

has mass 1 and satisfies

$$t_{\mathbb{Q}} \star \mu^\rho \;=\; \bar{\mu}^\rho.$$

To prove uniqueness, let μ on $(\mathcal{P}^{loc}, \mathcal{F})$ such that $t_{\mathbb{Q}} \star \mu \;=\; \bar{\mu}^\rho$. Because $\bar{\mu}^\rho$ only charges $\tilde{\mathcal{P}}^{loc}_{\mathbb{Q}}$,

$$t_{\mathbb{Q}} \star \mu = \bar{\mu}^\rho(\cdot \cap \tilde{\mathcal{P}}^{loc}_{\mathbb{Q}}).$$

Because μ only charges right continuous partitions, μ only charges $\tilde{\mathcal{P}}^{loc}$ (i.e., elements with no rational extrmities). Taking the pushforward of the two members of the previous equality by $t_{\mathbb{Q}}^{-1}$, we get

$$\mu(\cdot \cap \tilde{\mathcal{P}}^{loc}) = t_{\mathbb{Q}}^{-1} \star (t_{\mathbb{Q}} \star \mu) = t_{\mathbb{Q}}^{-1} \star \left[\bar{\mu}^\rho(\cdot \cap \tilde{\mathcal{P}}^{loc}_{\mathbb{Q}})\right] = \mu^\rho.$$

Since μ only charges $\tilde{\mathcal{P}}^{loc}$, $\mu = \mu^\rho$. $\square$

Proof of Theorem 1.2. We have proved that there exists a unique probability measure μ^ρ on $(\mathcal{P}^{loc}, \mathcal{F})$ such that, for any finite subset z of $\mathbb{Q}^+$, $t_z \star \mu^\rho$ is invariant for $(t_z(\Pi^\rho_t); t \geq 0)$ (by combining Lemmas 3.2 and 3.3). Using Lemma 3.1, we still need to prove that the same property holds for any finite subset $z \subset \mathbb{R}^+$. This will be shown by a continuity argument.

We fix $\rho > 0$. We denote by $\mathbb{P}^\rho$ the law of the process $(\Pi^\rho_t;\ t \geqslant 0)$, with initial condition Π^ρ_0 with law μ^ρ. We also fix $z = (z_1, \ldots, z_n) \subset \mathbb{R}^+$. For each $z^* = (z^*_1, \ldots, z^*_n) \subset \mathbb{Q}^+$, we define a function $g^* : \mathcal{P}_{z^*} \to \mathcal{P}_z$ such that, if π is a partition of z^*, $g^*(\pi)$ is the partition of z such that for every $i, j \in [n]$, $z_i \sim_{g^*(\pi)} z_j$ iff $z^*_i \sim_\pi z^*_j$. For every $t > 0$, we define the event

$$A(z^*, t) = \{\forall s \in [0, t],\ t_z(\Pi^\rho_s) \;=\; g^*(t_{z^*}(\Pi^\rho_s))\}.$$

We want to prove that for every $t > 0$ and for $\mathcal{F}_z$-measurable bounded function f on $\mathcal{P}_z$,

$$\mathbb{E}^\rho(f(t_z(\Pi^\rho_t))) \;=\; \mathbb{E}^\rho(f(t_z(\Pi^\rho_0))).$$

As μ^ρ is a measure on $\mathcal{P}^{loc}$, for every $\epsilon > 0$ one can find $z^* = (z^*_1, \ldots, z^*_n) \subset \mathbb{Q}^+$ such that

$$\mathbb{P}^\rho(A(z^*, t)^\complement)||f||_\infty < \epsilon/2 \;\text{ and }\; |\mathbb{E}^\rho(f(g^*(t_{z^*}(\Pi^\rho_0)), A(z^*, t)^\complement)| < \epsilon/2.$$

Then

$$\begin{aligned}
\mathbb{E}^\rho(f(t_z(\Pi^\rho_t))) \;&=\; \mathbb{E}^\rho(f(t_z(\Pi^\rho_t)), A(z^*, t)) + \mathbb{E}^\rho(f(t_z(\Pi^\rho_t)), A(z^*, t)^\complement) \\
&=\; \mathbb{E}^\rho(f(g^*(t_{z^*}(\Pi^\rho_t)), A(z^*, t)) + \mathbb{E}^\rho(f(t_z(\Pi^\rho_t)), A(z^*, t)^\complement).
\end{aligned}$$

As $z^* \subset \mathbb{Q}^+$, $\mu^\rho \circ t_{z^*}^{-1}$ is invariant for $t_{z^*}(\Pi^\rho_t))$,

$$\mathbb{E}^\rho(f(g^*(t_{z^*}(\Pi^\rho_t)), A(z^*, t)) = \mathbb{E}^\rho(f(g^*(t_{z^*}(\Pi^\rho_0))) - \mathbb{E}^\rho(f(g^*(t_{z^*}(\Pi^\rho_0)), A(z^*, t)^\complement).$$

Then,

$$\begin{aligned}
|\mathbb{E}^\rho(f(t_z(\Pi^\rho_t))) - \mathbb{E}^\rho(f(g^*(t_{z^*}(\Pi^\rho_0)))| \;&\leqslant \mathbb{P}^\rho(A(z^*, t)^\complement)||f||_\infty \\
&\quad + |\mathbb{E}^\rho(f(g^*(t_{z^*}(\Pi^\rho_0)), A(z^*, t)^\complement)|
\end{aligned}$$

so

$$|\mathbb{E}^\rho(f(t_z(\Pi^\rho_t))) - \mathbb{E}^\rho(f(t_z(\Pi^\rho_0))| \;<\; \epsilon,$$

and the conclusion follows by letting $\epsilon \to 0$. $\square$

To conclude this section, we state an important property of μ^ρ.

Proposition 3.4 (Scaling). *Fix $\rho > 0$. For every $\lambda \in> 0$, define $h_\lambda : \mathbb{R} \to \mathbb{R}$ such that $\forall x \in \mathbb{R},\ h_\lambda(x) = \lambda x$. Then*

$$h_\lambda \star \mu^\rho \ = \ \mu^{\lambda\rho}$$

Similarly, for any $z \subset R$,

$$h_\lambda \star \mu^{\rho,z} \ = \ \mu^{\lambda\rho,z}$$

Proof. This proposition can easily be deduced from the definition of the ARG and the scaling (2) and the construction of the $\mathbb{R}$-partitioning process given in the previous section. □

Without loss of generality, in Section 5, we will consider the partitioning process with recombination rate $\rho = 1$.

4 Proof of Theorem 1.3

Theorem 1.3, provides an approximation of the stationary measure of the discrete partitioning process when $\rho \to \infty$ or $\alpha \to \infty$, i.e. when recombination is much more frequent than coalescence. This approximation of $\mu^{\rho,z}$ is easy to handle, and that will be used in the proof of Theorem 1.4. We start by clarifying some notation that were already defined in the introduction and by introducing some new notation. In the following, we fix $z = (z_0, \ldots, z_n)$ a finite subset of $\mathbb{R}$, and we define

$$\alpha = \min_{i \neq j} |z_i - z_j|$$

and we assume that $\alpha > 0$ (or equivalently that the coordinates of z are pairwise distinct).

Definition 4.1. *We consider the ARG ρ for the set of loci z, $\Gamma^{\rho,z}$. We say that a partition $\pi \in \mathcal{P}_z$ is of order r if it can be obtained from the finest partition ($\pi_0 := \{z_0\}, \ldots, \{z_n\}$) by r successive coagulation events. We denote by $\mathcal{P}_z^k$ the subset of $\mathcal{P}_z$ containing all the partitions of order k.*

For example, for $i, j, k, l \in \{0, \ldots, n\}$:

- $\pi_0 = \{z_0\}, \ldots, \{z_n\}$ is the only partition of order 0.
- $\{z_0\}, \ldots, \{z_i, z_j\}, \ldots, \{z_n\}$ is of order 1.
- $\{z_0\}, \ldots, \{z_i, z_j, z_k\}, \ldots, \{z_n\}$ is of order 2.
- $\{z_0\}, \ldots, \{z_i, z_j\}, \{z_k, z_l\}, \ldots, \{z_n\}$ is also of order 2.
- $\{z_0, z_1, \ldots, z_n\}$ is the only partition of order n.

Note that as the number of blocks decreases by 1 at each coalescence event, in a partition of order k, there are always $n + 1 - k$ blocks, so this definition is equivalent to the one given in the Introduction.

Definition 4.2. *Let $(s_k)_{0 \leq k \leq r}$ be a sequence of r elements of $\mathcal{P}_z$. The sequence (s_k) is called a "(coalescence) scenario of order r" if:*

- *s_0 is the finest partition.*
- *For $1 \leq k \leq r$, s_k is a partition of order k that can be obtained from s_{k-1} by a single coagulation event.*

If π is a partition of order r, we denote by $\mathcal{S}(\pi)$ the set of coalescence scenarios of order r, such that $s_r = \pi$.

For example, the partition $\{z_0\},\dots,\{z_i,z_j,z_k\},\dots,\{z_n\}$ can be obtained from the finest partition with three different scenarios:

$$
\begin{array}{lll}
\{z_i\}\{z_j\}\{z_k\}\dots & \to \{z_i,z_j\}\{z_k\}\ \dots & \to \{z_i,z_j,z_k\} \\
\{z_i\}\{z_j\}\{z_k\}\dots & \to \{z_i,z_k\}\{z_j\}\ \dots & \to \{z_i,z_j,z_k\} \\
\{z_i\}\{z_j\}\{z_k\}\dots & \to \{z_k,z_j\}\{z_i\}\ \dots & \to \{z_i,z_j,z_k\}\dots
\end{array}
$$

For $\pi \in \mathcal{P}_z$, let $b_1, b_2, \dots, b_k, \dots$ be the blocks of π. We denote by $C(\pi)$ the cover length of π defined as:

$$C(\pi) \ := \ \sum_i \max_{x,y\in b_i} |x-y|.$$

In particular, the cover length of π_0 is equal to 0.

If π_1 and π_2 are two partitions in $\mathcal{P}_z$, we define $\theta(\pi_1,\pi_2)$ as the transition rate from π_1 to π_2 in the finite partitioning process $\Gamma^{1,z}$ with recombination rate $\rho = 1$ (and we set $\theta(\pi_1,\pi_2) = 0$ if the transition is not possible). By definition, in the ARG $\Gamma^{\rho,z}$ (with recombination rate ρ), the transition rate from π_1 to π_2 is $\theta(\pi_1,\pi_2)$ if the transition corresponds to a coagulation event and $\rho\theta(\pi_1,\pi_2)$ if it is a fragmentation. It can readily be seen that,

$$\forall \pi \in \mathcal{P}_z^r, \quad \sum_{\omega\in\mathcal{P}_z^{r-1}} \theta(\pi,\omega) \ = \ C(\pi).$$

In words, when $\rho = 1$, the total fragmentation rate corresponds to the cover length. For general values of ρ, the fragmentation rate is the cover length multiplied by ρ.

Also, the total coalescence rate from a partition of order k only depends on n and k (and not in the values of $z_0,\dots,z_n$ and ρ) and is given by

$$\sum_{\omega\in\mathcal{P}_z^{r+1}} \theta(\pi,\omega) = \gamma_k := \frac{(n-k)(n-k+1)}{2},$$

where γ_k corresponds to the number of pairs of blocks in a partition of order k.

Definition 4.3. *Let $s = (s_k)_{0\le k\le r}$ be a scenario of coalescence of order r, with $1 \le r \le n$. We define the energy of s, $E(s)$ as:*

$$E(s) := \prod_{i=1}^r C(s_i) = \prod_{i=1}^r \sum_{\pi\in\mathcal{P}_z^{i-1}} \theta(s_i,\pi).$$

In words, the energy of a scenario corresponds to the product of the successive cover lengths at each step.

Now we can state the main result of this section, that gives an approximation of $\mu^{\rho,z}$, when ρ or α is large. The idea behind this theorem is that, when $\rho \gg 1$ or $\alpha \gg 1$, fragmentation events occur much more often than coalescence events. This implies that the partition made of singletons is the most likely configuration and the probability of a partition decreases with its order. Define

$$\forall z \subset \mathbb{R}^+,\ \forall \pi \in \mathcal{P}_z \setminus \{\pi_0\}, \quad F(\pi) := \sum_{S\in\mathcal{S}(\pi)} \frac{1}{E(S)}. \tag{5}$$

We recall the statement of Theorem 1.3:

Theorem. *There exists a function*

$$f^n : \mathbb{R}_*^+ \to \mathbb{R}^+, \quad \lim_{x\to\infty} f^n(x) = 0,$$

independent of the choice of $z = (z_0, \ldots, z_n)$ *and* ρ, *such that*

$$\forall \rho > 0,\ \forall k \in [n],\ \forall \pi_k \in \mathcal{P}_z^k, \quad \left| \mu^{\rho,z}(\pi_k) - \frac{1}{\rho^k} F(\pi_k) \right| \leq f^n(\alpha\rho) \frac{1}{\rho^k} F(\pi_k).$$

Before proving Theorem 1.3 we need to prove some technical results. But to give the reader some intuition on this result, we will start by giving a brief sketch of the proof. Until further notice, we are going to fix $\rho > 0,\ k \in [n],\ \pi \in \mathcal{P}_z^k$ a partition of order $k \geq 1$. We will start by defining some notation:

- $t_0^+ = \inf\{t > 0,\ \Gamma_t^{\rho,z} \neq \pi_0\}$.
- $\mathcal{T}_\pi = \inf\{t > 0,\ \Gamma_t^{\rho,z} = \pi\}$, $\mathcal{T}_0 = \inf\{t > t_0^+,\ \Gamma_t^{\rho,z} = \pi_0\}$.
- $\mathbb{P}_\pi$ (resp $\mathbb{P}_0$) denotes the law of $\Gamma^{\rho,z}$ conditioned on the initial condition $\Gamma_0^{\rho,z} = \pi$ (resp $\Gamma_0^{\rho,z} = \pi_0$).

Recall that the variables defined above depend on z and ρ, but for the sake of clarity this dependency is not made explicit.

The idea behind the proof of Theorem 1.3 is to use excursion theory and a well known extension of Blackwell's renewal theorem [Blackwell, 1948] that states that

$$\mu^{\rho,z}(\pi) = \frac{\mathbb{E}_0(Y_1^\pi)}{\mathbb{E}_0(\Delta_0)}, \tag{6}$$

where Δ_0 is the time between two renewals at π_0 and Y_1^π is the time spent in π during an excursion out of π_0. (More precise definitions of these variables will be given in the proof of Theorem 1.3).

As we consider that $\alpha \gg 1$ or $\rho \gg 1$, fragmentation occurs much more often than coalescence so π_0 is the most likely configuration and $\Gamma^{\rho,z}$ spends most of the time at π_0. Then $\mathbb{E}_0(\Delta_0)$ can be approximated by the expectation of the holding time at π_0 which is $1/\gamma_0$. Also, in this regime, most excursions out of π_0 will only visit π at most one time, so $\mathbb{E}_0(Y_1^\pi)$ can be approximated by

$$\mathbb{P}_0(\mathcal{T}_\pi < \mathcal{T}_0) \frac{1}{\rho C(\pi)},$$

where $\mathbb{P}_0(\mathcal{T}_\pi < \mathcal{T}_0)$ is the probability that π is reached during the excursion out of π_0 and $\frac{1}{\rho C(\pi)}$ is approximately the expectation of the holding time at π when $\rho C(\pi) \gg \gamma_k$ (i.e. when recombination occurs much more often than coalescence).

The core of the proof is to compute $\mathbb{P}_0(\mathcal{T}_\pi < \mathcal{T}_0)$. To do so, we will consider $\bar{\Gamma}^{\rho,z}$, the embedded chain of the ARG $\Gamma^{\rho,z}$, conditioned on the initial condition $\bar{\Gamma}_0^{\rho,z} = \pi_0$. We call a "direct path" a trajectory that goes from π_0 to π in only k coalescence steps (without recombination events). Indirect paths are trajectories that are longer and that contain at least a recombination event. As we consider a high recombination regime, where coalescence occurs much more often than recombination, direct paths will be much more likely that indirect paths. (This will be formalized in Lemma 4.5.) So we can approximate $\mathbb{P}_0(\mathcal{T}_\pi < \mathcal{T}_0)$ by the sum of the probabilities of the direct paths. Then the conclusion will follow by realizing that a direct path corresponds to a scenario of coalescence and showing that $\mathbb{P}_0(\mathcal{T}_\pi < \mathcal{T}_0)$ can be approximated by $\frac{C(\pi)}{\rho^{k-1}\gamma_0} F(\pi)$. (This will be formalized in Corollary 4.6.) Finally, replacing in (6), we find that $\mu^{\rho,z}(\pi)$ can be approximated by $\frac{F(\pi)}{\rho^k}$.

Before turning to the formal proof of Theorem 1.3, we start by proving some technical results. We consider $\bar{\Gamma}^{\rho,z}$, the embedded chain of the ARG $\Gamma^{\rho,z}$. Let P_0 denote the law of $\bar{\Gamma}^{\rho,z}$ conditioned on $\bar{\Gamma}^{\rho,z}_0 = \pi_0$ and $\forall \pi' \in \mathcal{P}_z,\ P_{\pi'}$ denotes the law of $\bar{\Gamma}^{\rho,z}$ conditioned on $\bar{\Gamma}^{\rho,z}_0 = \pi'$. We will consider paths that go from π_0 to π. A path is defined as follows.

Definition 4.4. *For $j \in \mathbb{N}^*, \pi', \pi'' \in \mathcal{P}_z$, we define:*

$$\begin{aligned} G(j, \pi' \to \pi'') \ = \ &\{(\pi^{(0)} = \pi', \pi^{(1)}, \dots, \pi^{(j-1)}, \pi^{(j)} = \pi''), \\ &\pi^{(1)}, \dots, \pi^{(j-1)} \in \mathcal{P}_z \setminus \{\pi', \pi''\} \text{ such that} \\ &\theta(\pi^{(i)}, \pi^{(i+1)}) > 0 \quad \forall i \in \{0, \cdots, j-1\}\}. \end{aligned}$$

In words, $G(j, \pi' \to \pi'')$ contains every possible path (admissible for the partitioning process) that connects π' to π'' in j steps.

We are going to consider paths p that go from π_0 to π, which have at least k steps (as π is of order k).

- p is a *direct* path if $p \in G(k, \pi_0 \to \pi)$, i.e., p can only be composed of coalescence events.
- p is an *indirect* path if $p \in G(k+N, \pi_0 \to \pi),\ N \in \mathbb{N}^*$. Indirect paths contain at least one recombination event. Note that the parity of the process implies that $G(k+2N+1, \pi_0 \to \pi)$ is empty.

Lemma 4.5. *Fix $N \in \mathbb{N}^*$ and a path p in $G(k+2N, \pi_0 \to \pi)$. There exists a path $\tilde{p} \in G(k, \pi_0 \to \pi)$ such that*

$$\frac{P_0(p)}{P_0(\tilde{p})} \leq \left(\frac{(1 + \frac{\gamma_1}{\rho\alpha})^k}{\alpha\rho} \right)^N.$$

Proof of Lemma 4.5. We fix $N \in \mathbb{N}^*$ and we start with proving that

$$\begin{aligned} &\forall p \in G(k+2N, \pi_0 \to \pi), \exists\ \tilde{p} \in G(k+2(N-1), \pi_0 \to \pi), \\ &\qquad \frac{P_0(p)}{P_0(\tilde{p})} \leq \frac{(1 + \frac{\gamma_1}{\rho\alpha})^k}{\alpha\rho}. \end{aligned} \tag{7}$$

We consider a path $p \in G(k+2N, \pi_0 \to \pi)$ such that

$$p = (\pi_0, \bar{\pi}_1, \dots, \bar{\pi}_j, \tilde{\pi}_{j-1}, \pi_{i_1}, \pi_{i_2}, \dots, \pi).$$

where the indices of the $\tilde{\pi}, \bar{\pi}$'s coincide with the order of the partition (for instance, in the transition $\bar{\pi}_j \to \tilde{\pi}_{j-1}$, the order of the partition decreases by one unit, which corresponds to a fragmentation event). We do not specify the order of $\pi_{i_1}, \pi_{i_2}, \dots$ As $N \geq 1$ there is at least one recombination event ($\bar{\pi}_j \to \tilde{\pi}_{j-1}$). The path p can be decomposed into p_1 and p_2 such that:

$$\begin{aligned} p_1 \in G((j-1)+2, \pi_0 \to \tilde{\pi}_{j-1}),\ \ p_1 = (\pi_0, \bar{\pi}_1, \dots, \bar{\pi}_{j-1}, \bar{\pi}_j, \tilde{\pi}_{j-1}) \\ p_2 \in G(k+2N-(j-1)-2, \pi_{j-1} \to \pi),\ \ p_2 = (\tilde{\pi}_{j-1}, \pi_{i_1}, \pi_{i_2}, \dots, \pi). \end{aligned}$$

In words, we decompose p into two paths, p_1 that goes from π_0 until the first recombination event and p_2 that contains the rest of the path.

The idea now is to find a direct path

$$\tilde{p}_1 \in G(j-1, \pi_0 \to \tilde{\pi}_{j-1}),\ \ \tilde{p}_1 = (\pi_0, \tilde{\pi}_1, \dots, \tilde{\pi}_{j-1})$$

such that

$$\frac{P_0(p_1)}{P_0(\tilde{p}_1)} \leq \frac{1}{\alpha\rho}\left(1+\frac{\gamma_1}{\rho\alpha}\right)^{j-1} \leq \frac{1}{\alpha\rho}\left(1+\frac{\gamma_1}{\rho\alpha}\right)^{k}.$$

To do so, consider the fragmentation event that occurs between step j and step $j+1$ in p (when transitioning from $\bar{\pi}_j$ to $\tilde{\pi}_{j-1}$). $\bar{\pi}_j$ contains $n+1-j$ blocks and let $(b_1,\dots b_{n-j}, b^*)$ be the blocks of $\bar{\pi}_j$ such that b^* is the block of $\bar{\pi}_j$ that is fragmented during this fragmentation event and $z_a < z_b$ the two elements of b^* such that b^* is fragmented between z_a and z_b (i.e. such that b^* is fragmented into b^*_a and b^*_b where z_a is the rightmost element in b^*_a and z_b the leftmost element in b^*_b). We have

$$C(\bar{\pi}_j) = C(\tilde{\pi}_{j-1}) + z_b - z_a. \tag{8}$$

Let $i^* \leqslant j$ be the first step of p such that z_a and z_b are in the same block, i.e

$$i^* = \min_{i\in[j]}\{i,\ z_a \sim_{\bar{\pi}_i} z_b\}.$$

We will construct a direct path $\tilde{p}_1 = (\tilde{\pi}_0, \cdots, \tilde{\pi}_{j-1})$ in such a way that

$$\begin{aligned} \forall\ 1 \leq i < i^*,\quad C(\tilde{\pi}_i)\ &\leq\ C(\bar{\pi}_i) \\ \text{if } i^* < j-1,\ \forall\ i^* < i \leq j,\quad C(\tilde{\pi}_{i-1})\ &\leq\ C(\bar{\pi}_i), \end{aligned} \tag{9}$$

(Note that the terminal value of $\tilde{p}_1$ coincides with the terminal value of p_1 and its length is $j-1$ instead of $j+1$).) See Figure 4 for a concrete example. In words, we skip step i^*, and rearrange the path in such a way that $\tilde{p}_1$ is admissible, ends at $\tilde{\pi}_{j-1}$ and the inequalities (9) are satisfied along the way. Formally, the path $\tilde{p}_1$ is constructed as follows :

- If $i^* < j-1$, for $i \in \{i^*+1,\dots,j-1\}$, let $(b^i_1,\dots b^i_{n-i}, b^i_*)$ be the blocks of $\bar{\pi}_i$, where b^i_* is the one that contains z_a and z_b. The blocks of $\tilde{\pi}_{i-1}$ are $(b^i_1,\dots,b^i_{n-i}, b^i_{n-i+1}, b^i_{n-i+2})$ such that:

  - if $z \in b^i_*$ and $z \leq z_a$, $z \in b^i_{n-i+1}$.
  - if $z \in b^i_*$ and $z \geq z_b$, $z \in b^i_{n-i+2}$.

  If $i^* = j-1$ we skip the present step in the construction of $\tilde{p}$.

- If in $\bar{\pi}_{i^*-1}$, z_a is the rightmost element in its block and z_b the leftmost element in its block, then we define $(\tilde{\pi}_1,\dots,\tilde{\pi}_{i^*-1}) = (\bar{\pi}_1,\dots,\bar{\pi}_{i^*-1})$. With this construction $\tilde{\pi}_{i^*}$ can be obtained from $\tilde{\pi}_{i^*-1}$ by a coalescence event, so the path $\tilde{p}$ is admissible for $\Gamma^{\rho,z}$.

- Else, $(\tilde{\pi}_1,\dots,\tilde{\pi}_{i^*-1})$ are constructed from $(\bar{\pi}_1,\dots,\bar{\pi}_{i^*-1})$ in the following way. Let us denote by b_a and b_b the blocks of $\bar{\pi}_{i^*-1}$ that contain z_a and z_b respectively. For $1 \leq i \leq i^*-1$,

  - If the coalescence event between $\bar{\pi}_{i-1}$ and $\bar{\pi}_i$ involves two blocks b_c and b_d such that in π_{i^*-1}, $b_c, b_d \subset b_a$ (resp. $b_c, b_d \subset b_b$) and if b_c contains an element that is smaller than z_a and b_d contains an element is larger than z_b, then in the coalescence step between $\tilde{\pi}_{i-1}$ and $\tilde{\pi}_i$, b_c (resp. b_d) coalesces with the block containing z_a (resp. z_b). (And nothing happens to b_d - resp. b_c).
  - Otherwise the same coalescence event occurs between $\bar{\pi}_{i-1}$ and $\bar{\pi}_i$ and between $\tilde{\pi}_{i-1}$ and $\tilde{\pi}_i$.

  With this construction $\tilde{\pi}_{i^*}$ can be obtained from $\tilde{\pi}_{i^*-1}$ by a coalescence event, and as a consequence the path $\tilde{p}$ is admissible, in the sense that $\theta(\pi_i, \pi_{i+1}) > 0$ (see Figure 4 for an example).

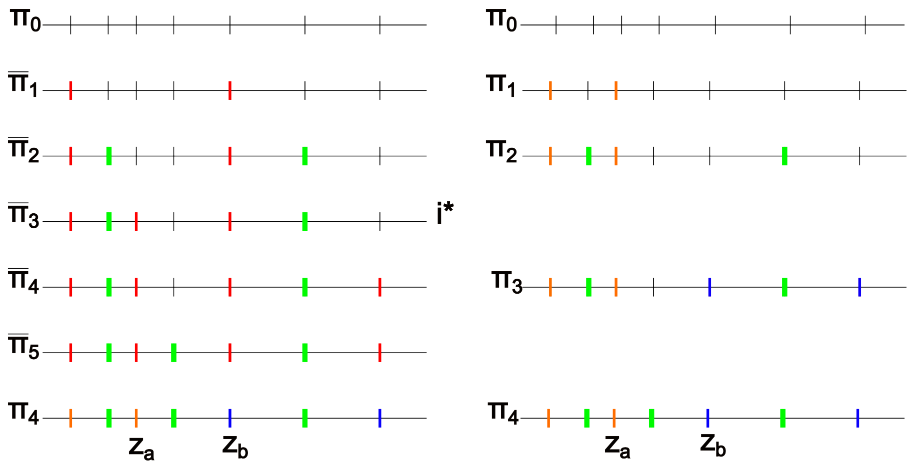


Figure 5: Example of two paths that go from π_0 to π_4, for $n = 7$. Loci in the same block are of the same color and the black loci corresponds to to loci that are in singleton blocks. The path on the right corresponds $p \in G(\pi_0 \to \pi_4, 4+2)$ and the path on the left is $\tilde{p} \in G(\pi_0 \to \pi_4, 4)$ constructed from π with the method presented above.

First,

$$P_0(\tilde{p}_1) = \frac{1}{\gamma_0 \rho^{j-2}} \prod_{i=1}^{j-1} \frac{1}{C(\tilde{\pi}_i) + \gamma_i/\rho}$$

$$P_0(p_1) = \frac{1}{\gamma_0 \rho^{j-1}} \prod_{i=1}^{j} \frac{1}{C(\bar{\pi}_i) + \gamma_i/\rho} \; \frac{\rho(z_b - z_a)}{\rho C(\bar{\pi}_j) + \gamma_j},$$

From (9), if $i^* < j-1$

$$\begin{aligned}
\frac{P_0(p_1)}{P_0(\tilde{p}_1)} &= \frac{1}{\rho} \frac{1}{C(\bar{\pi}_{i^*}) + \gamma_{i^*}} \prod_{i=1}^{i^*-1} \frac{C(\tilde{\pi}_i) + \gamma_i/\rho}{C(\bar{\pi}_i) + \gamma_i/\rho} \prod_{i=i^*+1}^{j} \frac{C(\tilde{\pi}_{i-1}) + \gamma_{i-1}/\rho}{C(\bar{\pi}_i) + \gamma_i/\rho} \; \frac{\rho(z_b - z_a)}{\rho C(\bar{\pi}_j) + \gamma_j} \\
&\leq \frac{1}{\alpha\rho} \prod_{i=i^*+1}^{j} \frac{C(\bar{\pi}_i) + \gamma_{i-1}/\rho}{C(\bar{\pi}_i) + \gamma_i/\rho} \leq \frac{1}{\alpha\rho} \prod_{i=i^*+1}^{j} \frac{1 + \frac{\gamma_{i-1}}{\rho C(\bar{\pi}_i)}}{1 + \frac{\gamma_i}{\rho C(\bar{\pi}_i)}} \\
&\leq \frac{1}{\alpha\rho} \prod_{i=i^*+1}^{j} \left(1 + \frac{\gamma_{i-1}}{\rho C(\bar{\pi}_i)}\right) \leq \frac{1}{\alpha\rho} \left(1 + \frac{\gamma_1}{\rho\alpha}\right)^k.
\end{aligned}$$

where the second inequality is a consequence of (8) and (9). The case $i^* = j-1$ follows along the same lines.

Let us define

$$\tilde{p} \in G(k+2, \pi_0 \to \pi), \;\; \tilde{p} = (\pi_0, \tilde{\pi}_1, \ldots, \tilde{\pi}_{j-1}, \pi_{i_1}, \pi_{i_2} \ldots, \pi).$$

Since

$$P_0(p) \;=\; P_0(p_1)\; P_{\tilde{\pi}_{j-1}}(p_2), \quad P_0(\tilde{p}) \;=\; P_0(\tilde{p}_1)\; P_{\tilde{\pi}_{j-1}}(p_2)$$

we have

$$\frac{P_0(p)}{P_0(\tilde{p})} = \frac{P_0(p_1)}{P_0(\tilde{p}_1)} \leq \frac{\left(1 + \frac{\gamma_1}{\rho\alpha}\right)^k}{(\alpha\rho)},$$

which completes the proof of (7). Lemma 4.5 is a trivial consequence of (7). □

Corollary 4.6. *There exists a function*

$$u^n : \mathbb{R}_*^+ \to \mathbb{R}^+, \quad \lim_{x\to\infty} u^n(x) = 0,$$

independent of the choice of z, π and ρ, such that

$$\left|\mathbb{P}_0(\mathcal{T}_\pi < \mathcal{T}_0) \ - \ \frac{C(\pi)}{\rho^{k-1}\gamma_0} F(\pi)\right| \leq u^n(\alpha R) \frac{C(\pi)}{\rho^{k-1}\gamma_0} F(\pi).$$

Proof. We have

$$\mathbb{P}_0(\mathcal{T}_\pi < \mathcal{T}_0) = \sum_{N\geq k} \sum_{p\in G(N,\pi_0\to\pi)} P_0(p).$$

As π is of order k, a path from π_0 to π has at least k steps. In addition, as the order of the partition can only increase or decrease by 1 at each step, a path from π_0 to π can only have $k + 2N$ steps, with $N \geq 0$, so

$$\mathbb{P}_0(\mathcal{T}_\pi < \mathcal{T}_0) = \sum_{p\in G(k,\pi_0\to\pi)} P_0(p) + \sum_{N\geq 1} \sum_{p\in G(k+2N,\pi_0\to\pi)} P_0(p). \tag{10}$$

We start by considering the first term in the right hand side. We consider a path p such that

$$p \in G(k, \pi_0 \to \pi), \ p = (\pi_0, \pi_1, \ldots, \pi).$$

We have:

$$P_0(p) = \frac{1}{\gamma_0} \prod_{i=1}^{k-1} \frac{1}{\rho C(\pi_i) + \gamma_i}.$$

Recall that

$$F(\pi) \ = \ \sum_{s\in\mathcal{S}(\pi)} \prod_{i=1}^{k} \frac{1}{C(s_i)}.$$

Further, paths that have k steps are only composed of coalescence events, and therefore $G(k, \pi_0 \to \pi) = \mathcal{S}(\pi)$. It follows that

$$\sum_{p\in G(k,\pi_0\to\pi)} P_0(p) - \frac{C(\pi)}{\rho^{k-1}\gamma_0} F(\pi) \ = \ \frac{1}{\rho^{k-1}\gamma_0} \sum_{s\in\mathcal{S}(\pi)} \left(\prod_{i=1}^{k-1} \frac{1}{C(s_i) + \gamma_i/\rho} - \prod_{i=1}^{k-1} \frac{1}{C(s_i)}\right)$$

and using the fact that $\gamma_i \leqslant \gamma_0$,

$$\begin{aligned}
\sum_{s\in\mathcal{S}(\pi)} \left|\prod_{i=1}^{k-1} \frac{1}{C(s_i) + \gamma_i/\rho} - \prod_{i=1}^{k-1} \frac{1}{C(s_i)}\right| &= \sum_{s\in\mathcal{S}(\pi)} \prod_{i=1}^{k-1} \frac{1}{C(s_i)} \left(1 - \prod_{i=1}^{k-1} \frac{1}{1 + \gamma_i/(\rho C(s_i))}\right) \\
&\leq \sum_{s\in\mathcal{S}(\pi)} \prod_{i=1}^{k-1} \frac{1}{C(s_i)} \left(1 - \left(\frac{1}{1 + \gamma_0/(\rho\alpha)}\right)^{k-1}\right) \\
&\leq C(\pi) F(\pi) \left(1 - \frac{1}{(1 + \gamma_0/(\rho\alpha))^{k-1}}\right)
\end{aligned}$$

so

$$\left|\sum_{p\in G(k,\pi_0\to\pi)} P_0(p) - \frac{C(\pi)}{\rho^{k-1}\gamma_0} F(\pi)\right| \ \leq \ \frac{C(\pi)}{\rho^{k-1}\gamma_0} F(\pi) \left(1 - \frac{1}{(1 + \gamma_0/(\rho\alpha))^{k-1}}\right). \tag{11}$$

To prove Proposition 4.6, we still need to consider the second term in the right hand side of (10). Using Lemma 4.5, we have

$$\sum_{N\geq 1}\sum_{p\in G(k+2N,\pi_0\to\pi)} P_0(p)$$
$$\leq \sum_{\tilde{p}\in G(k,\pi_0\to\pi)} P_0(\tilde{p}) \left(\sum_{N\geq 1}|G(k+2N,\pi_0\to\pi)|\left(\frac{(1+\frac{\gamma_1}{\rho\alpha})^k}{\alpha\rho}\right)^N\right). \tag{12}$$

To compute $|G(k+2N,\pi_0\to\pi)|$, let us recall that, at each step in a path:

- If it corresponds to a coalescence event from a partition of order j there are γ_j possibilities, and $\forall j\in\{0,\dots,n\}$ $\gamma_j\leq n(n+1)$.
- If it corresponds to a fragmentation event, there are at most $(n+1)$ blocks in the partition and each one contains at most $(n+1)$ elements, so that each block can be fragmented in n different ways.

From there, it can easily be seen that

$$|G(k+2N,\pi_0\to\pi)|\leq (n(n+1))^{k+2N}.$$

Combining this with (12), we have:

$$\sum_{N\geq 1}\sum_{p\in G(k+2N,\pi_0\to\pi)} P_0(p)$$
$$\leq \left(\sum_{p\in G(k,\pi_0\to\pi)} P_0(p)\right)(n(n+1))^k\sum_{N\geq 1}\left(\frac{n^2(n+1)^2(1+\frac{\gamma_1}{\rho\alpha})^k}{\alpha\rho}\right)^N,$$

which combined with (10) and (11), gives:

$$\left|\mathbb{P}_0(\mathcal{T}_\pi<\mathcal{T}_0)-\frac{C(\pi)}{\rho^{k-1}\gamma_0}F(\pi)\right|\leq \frac{C(\pi)}{\rho^{k-1}\gamma_0}F(\pi)u^{n,k}(\alpha\rho)$$

where $u^{n,k}$ is a function independent of z and ρ and vanishing at ∞. The conclusion follows by setting $u^n(\alpha\rho)=\max_{k\in[n]}(u^{n,k}(\alpha\rho))$. □

Before stating the last technical result that is needed in the proof of Theorem 1.3, we need to introduce some notation:

- $t_\pi^+=\inf\{t>0,\ \Gamma_t^{\rho,z}\neq\pi\}$
- $T_\pi=\inf\{t>t_\pi^+,\ \Gamma_t^{\rho,z}=\pi\}$, $T_0=\inf\{t>t_\pi^+,\ \Gamma_t^{\rho,z}=\pi_0\}$.

Lemma 4.7. *For any $n\in\mathbb{N}$, there exist two functions g^n and h^n such that*

$$\lim_{x\to\infty} g^n(x)=0,\ \lim_{x\to\infty} h^n(x)=1$$

independent on the choice of z,π and ρ such that

$$(i)\ \ \mathbb{E}_0(\mathcal{T}_0-t_0^+)\leq g^n(\alpha\rho)$$
$$(ii)\ \forall k>0,\forall\pi\in\mathcal{P}_z^k,\ \mathbb{P}_\pi(T_0<T_\pi)\geq h^n(\alpha\rho).$$

Proof. We fix $\rho > 0, n \in \mathbb{N}, z = (z_0, \ldots, z_n),\ k \in [n], \pi \in \mathcal{P}_z^k$.

The idea of the proof is to consider the stochastic process $(X_t^{\rho,z}; t \geq 0)$ valued in $\{0, \ldots, n\}$ and such that $\forall t \geq 0$, $X_t^{\rho,z}$ is the order of the partition $\Gamma_t^{\rho,z}$. This process is not Markovian, but it can easily be compared to a Markov process $(W_t^{\rho,z}, t \geq 0)$ in such a way that the excursions out of 0 of $W^{\rho,z}$ are longer than those of $X^{\rho,z}$.

More precisely, let $W^{\rho,z}$ be the birth-death process in $\{0, \cdots, n\}$ where all the death rates are equal to $\rho\alpha$ and the birth rate at state k is γ_k (note that $\gamma_n = 0$).

With these transition rates, for any $\pi_k \in \mathcal{P}_z^k$, the total coalescence rate from π_k for the process $\Gamma_t^{\rho,z}$ is the same as the birth rate from k for $W_t^{\rho,z}$. On the other hand, the total fragmentation rate for $\Gamma_t^{\rho,z}$ when $\Gamma_t^{\rho,z} = \pi_k$ is equal to $\rho C(\pi_k)$ and is always higher than the death rate at k for $W_t^{\rho,z}$. We can find a coupling between $W^{\rho,z}$ and $X^{\rho,z}$ such that the holding times at 0 of the two process are the same (as the birth rate in 0 for $W^{\rho,z}$ is the same as the coagulation rate from 0 for $\Gamma^{\rho,z}$). In addition, during an excursion out of 0, the holding time at $k > 0$ for $X^{\rho,z}$ is shorter than the holding time at k for $W^{\rho,z}$ and the embedded chain of $X^{\rho,z}$ jumps more easily to the right than the embedded chain of $W^{\rho,z}$.

Let us denote by $\bar{\mathbb{E}}_0$ the probability with respect to the distribution of $W^{\rho,z}$, conditional to $W_0^{\rho,z} = 0$, and define

$$\begin{aligned} \bar{t}_0^+ &= \inf\{t > 0, W^{\rho,z} \neq 0\} \\ \bar{\mathcal{T}}_0 &= \inf\{t > \bar{t}_0^+,\ W_t^{\rho,z} = 0\}. \end{aligned}$$

By construction, we have

$$\mathbb{E}_0(\mathcal{T}_0 - t_0^+) \ \leq\ \bar{\mathbb{E}}_0(\bar{\mathcal{T}}_0 - \bar{t}_0^+).$$

Finally, $\bar{\mathbb{E}}_0(\bar{\mathcal{T}}_0 - \bar{t}_0^+)$ only depends on $\rho\alpha$ and n and it can be checked that

$$\bar{\mathbb{E}}_0(\bar{\mathcal{T}}_0 - \bar{t}_0^+) \underset{\alpha\rho\to\infty}{\longrightarrow} 0$$

so (i) is verified.

(ii) can be handled by similar methods. Namely, let $\bar{W}^{\rho,z}$ denote the embedded chain of $W^{\rho,z}$

$$\begin{aligned} \mathbb{P}_\pi(T_0 < T_\pi) &\geq \mathbb{P}(\bar{W}_0^{\rho,z} = k, \bar{W}_1^{\rho,z} = k-1, \ldots, \bar{W}_k^{\rho,z} = 0) \\ &= \prod_{i=1}^{k} \frac{\rho\alpha}{\rho\alpha + \gamma_i} \geq \prod_{i=1}^{n} \frac{\rho\alpha}{\rho\alpha + \gamma_i} \underset{\rho\alpha\to\infty}{\longrightarrow} 1 \end{aligned}$$

where the first inequality is obtained by the same argument as in (i). This completes the proof of Lemma 4.7. □

We are now ready to prove the main results of this section.

Proof of Theorem 1.3. We will consider excursions of $(\Gamma_t^{\rho,z})$ out of π_0. Let us consider $(J_i)_{i\in\mathbb{N}}$ the renewal times at π_0 i.e. the successive jump times of $(\Gamma_t^{\rho,z})$ such that $\Gamma_{J_i}^{\rho,z} = \pi_0$ (and $\Gamma_{J_i-}^{\rho,z} \neq \pi_0$). For $i \in \mathbb{N}^*$, let us define

$$\Delta_0^i := J_i - J_{i-1}$$

the time between two renewals at π_0. The $(\Delta_0^i)_{i\in\mathbb{N}}$ are independent and identically distributed random variables. Also, for $i \in \mathbb{N}^*$, consider

$$Y_i^\pi \ := \int_{J_{i-1}}^{J_i} \mathbb{1}_{\Gamma_t^{\rho,z}=\pi} dt,$$

which corresponds to the time spent by $\Gamma^{\rho,z}$ in π during the i^{th} excursion out of π_0. The $(Y_i^\pi)_{i\in\mathbb{N}}$ are independent and identically distributed random variables.

From the ergodic theorem we have

$$\begin{aligned}\mu^{\rho,z}(\pi) &= \lim_{T\to\infty}\frac{1}{T}\int_0^T \mathbb{1}_{\{\Gamma_s^{\rho,z}=\pi\}}ds \quad \text{ a.s.}\\ &= \lim_{n\to\infty}\frac{1}{J_n}\sum_{k=1}^n\int_{J_{n-1}}^{J_n}\mathbb{1}_{\{\Gamma_s^{\rho,z}=\pi\}}ds \quad \text{a.s..}\end{aligned}$$

Since the excursions are independent from one another, using Blackwell's renewal theorem Blackwell [1948], and the law of large numbers,

$$\lim_{n\to\infty}\frac{n}{J_n}=\frac{1}{\mathbb{E}_0(\Delta_0^1)} \quad\text{and}\quad \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^n\int_{J_{n-1}}^{J_n}\mathbb{1}_{\{\Gamma_s^{\rho,z}=\pi\}}ds=\mathbb{E}_0(Y_1^\pi) \quad \text{a.s.,}$$

which easily gives

$$\mu^{\rho,z}(\pi)=\frac{\mathbb{E}_0(Y_1^\pi)}{\mathbb{E}_0(\Delta_0^1)}.$$

Let H_0 be the holding time at π_0 and H the holding time at π. H_0 follows an exponential distribution of parameter γ_0. H follows an exponential distribution of parameter $(C(\pi)+\gamma_k)$. By standard excursion theory,

$$\begin{aligned}\mathbb{E}(Y_1^\pi) &= \mathbb{P}_0(\mathcal{T}_\pi<\mathcal{T}_0)\sum_{k\geq 1}k\mathbb{E}(H)\mathbb{P}_\pi(T_\pi<T_0)^{k-1}\mathbb{P}_\pi(T_0<T_\pi)\\ &= \mathbb{P}_0(\mathcal{T}_\pi<\mathcal{T}_0)\frac{1}{(\rho C(\pi)+\gamma_k)}\frac{1}{\mathbb{P}_\pi(T_0<T_\pi)},\end{aligned}$$

and

$$\mathbb{E}(\Delta_0^1)=\mathbb{E}(H_0)+\mathbb{E}_0(\mathcal{T}_0-t_0^+) \text{ where } \mathbb{E}(H_0)=\frac{1}{\gamma_0}.$$

Combining this with Corollary 4.6, we have

$$\begin{aligned}&\left|\ \mu^{\rho,z}(\pi) - \frac{C(\pi)F(\pi)}{\gamma_0\rho^{k-1}}\frac{1}{(\rho C(\pi)+\gamma_k)}\frac{1}{\mathbb{P}_\pi(T_0<T_\pi)}\ \frac{1}{1/\gamma_0+\mathbb{E}_0(\mathcal{T}_0-t_0^+)}\right|\\ &\leq u^n(\alpha\rho)\frac{C(\pi)F(\pi)}{\gamma_0\rho^{k-1}}\frac{1}{(\rho C(\pi)+\gamma_k)}\frac{1}{\mathbb{P}_\pi(T_0<T_\pi)}\ \frac{1}{1/\gamma_0+\mathbb{E}_0(\mathcal{T}_0-t_0^+)}\\ &\leq u^n(\alpha\rho)\ \frac{F(\pi)}{\rho^k}\ \frac{1}{1+\frac{\gamma_k}{\rho C(\pi)}}\ \frac{1}{\mathbb{P}_\pi(T_0<T_\pi)}\ \frac{1}{1+\mathbb{E}_0(\mathcal{T}_0-t_0^+)\gamma_0}\end{aligned}$$

so

$$\begin{aligned}&\left|\ \mu^{\rho,z}(\pi) - \frac{F(\pi)}{\rho^k}\ \right|\\ &\leq u^n(\alpha\rho)\ \frac{F(\pi)}{\rho^k}\ \frac{1}{1+\frac{\gamma_k}{\rho C(\pi)}}\ \frac{1}{\mathbb{P}_\pi(T_0<T_\pi)}\ \frac{1}{1+\mathbb{E}_0(\mathcal{T}_0-t_0^+)\gamma_0}\\ &+\ \left|\ \frac{C(\pi)F(\pi)}{\gamma_0\rho^{k-1}}\frac{1}{(\rho C(\pi)+\gamma_k)}\frac{1}{\mathbb{P}_\pi(T_0<T_\pi)}\ \frac{1}{1/\gamma_0+\mathbb{E}_0(\mathcal{T}_0-t_0^+)}\ -\ \frac{F(\pi)}{\rho^k}\ \right|\\ &\leq\frac{F(\pi)}{\rho^k}\ \Bigg(u^n(\alpha\rho)\ \frac{1}{\mathbb{P}_\pi(T_0<T_\pi)}\ \frac{1}{1+\mathbb{E}_0(\mathcal{T}_0-t_0^+)\gamma_0}\\ &+\ \left|\ 1\ -\ \frac{1}{1+\frac{\gamma_k}{\rho C(\pi)}}\ \frac{1}{\mathbb{P}_\pi(T_0<T_\pi)}\ \frac{1}{1+\mathbb{E}_0(\mathcal{T}_0-t_0^+)\gamma_0}\ \right|\ \Bigg)\end{aligned}$$

and using Lemma 4.7 (and the fact that $\rho C(\pi) \geq \rho\alpha$), the term between parentheses can be bounded by $f^n(\alpha\rho)$, where f^n is independent on the choice of z and ρ and is such that

$$\lim_{x\to+\infty} f^n(x) = 0,$$

which completes the proof of Theorem 1.3. $\square$

5 Proof of Theorem 1.4

Thanks to Proposition 3.4 (scaling), in this section we will assume without loss of generality that $\rho = 1$ and consider the $\mathbb{R}$-partitioning process restricted to $[0, R]$. The strategy of the proof is based on the following lemma. (Note that the second point will allow us to rephrase the convergence of ϑ^R in the weak topology in terms of a moment problem).

Lemma 5.1. *(i) For every k-tuple of disjoint intervals $\{[a_i, b_i]\}_{i=1}^k$ in $[0, 1]$ and any k-tuple of integers $\{n_i\}_{i=1}^k$*

$$\mathbb{E}\left(\prod_{i=1}^{k} \vartheta^{\infty}([a_i, b_i])^{n_i}\right) = \prod_{i=1}^{k} n_i!\ b_i^{n_i-1}(b_i - a_i).$$

(ii) Let $\{\nu^R\}_{R\geq 0}$ be a sequence of random variables in $\mathcal{M}([0, 1])$ that have no atoms and such that and for every k-tuple of disjoint intervals $\{[a_i, b_i]\}_{i=1}^k$ in $[0, 1]$ and any k-tuple of integers $\{n_i\}_{i=1}^k$

$$\lim_{R\to\infty} \mathbb{E}\left(\prod_{i=1}^{k} \nu^{R}([a_i, b_i])^{n_i}\right) = \prod_{i=1}^{k} n_i!\ b_i^{n_i-1}(b_i - a_i).$$

Then $\nu^R \underset{R\to\infty}{\Longrightarrow} \vartheta^\infty$ in the weak topology.

Proof of Lemma 5.1. We start by proving (i). We fix $k = 1$. We fix $a, b \in [0, 1], a \leq b$ and we compute M, the moment generating function of $\vartheta^\infty([a, b])$.

$$\begin{aligned} M(t) &= \mathbb{E}(\exp(t\vartheta^\infty[a, b])) \\ &= \mathbb{E}(\exp(t \sum_{(x_i,y_i)\in\mathcal{P}^\infty,\ x_i\in[a,b]} y_i)). \end{aligned}$$

$M(t)$ is the Laplace functional of $\mathcal{P}^\infty$ for $f(x, y) = -ty$, so it is well known that:

$$\begin{aligned} M(t) &= \exp\left(-\int_{[a,b]\times\mathbb{R}^+} (1 - e^{ty})\lambda(x, y)dxdy\right) \\ &= \exp\left(-\int_{[a,b]} \frac{dx}{x} \int_{\mathbb{R}^+} (1 - e^{ty})\ \frac{1}{x}e^{-y/x}dy\right) \\ &= \exp\left(\int_a^b \frac{tx}{1 - tx}\frac{dx}{x}\right) = \exp\left(\log\left(\frac{1 - ta}{1 - tb}\right)\right) \\ &= \frac{1 - ta}{1 - tb}. \end{aligned}$$

Note that when $a=0$, $M(t)$ is the moment generating function of an exponential distribution of parameter $1/b$ and $M^{(n)}(0)=n!b^n$. When $a\neq 0$, we use a Taylor expansion of $M(t)$:

$$\begin{aligned}
\frac{1-ta}{1-tb} &= (1-ta)\sum_{n=0}^{\infty}(tb)^n \\
&= \sum_{n=0}^{\infty}(tb)^n - \sum_{n=1}^{\infty} ab^{n-1}t^n \\
&= 1+\sum_{n=1}^{\infty}\frac{n!\ b^{n-1}(b-a)}{n!}t^n,
\end{aligned}$$

so $M^{(n)}(0)=n!b^{n-1}(b-a)$ for $n\geqslant 1$, which implies (i). To prove this result for $k>1$, we use the fact that $\mathcal{P}^\infty$ is a Poisson point process so that, for any k-tuple of disjoint intervals $B_1,\dots B_k$, $\vartheta^\infty(B_1),\dots,\vartheta^\infty(B_k)$ are mutually independent.

We now turn to the proof of (ii). Let $\{\nu^R\}_{R\geq 0}$ be a sequence of random variables in $\mathcal{M}([0,1])$. Note that for every $x\in[0,1]$, ϑ^∞ does not charge x almost surely. From Kallenberg [2002] (Theorem 16.16 page 316), it follows that proving

$$\nu^R \underset{R\to\infty}{\Longrightarrow} \vartheta^\infty \text{ in the weak topology}$$

boils down to proving that $\forall n\in\mathbb{N}$, for any k-tuple of intervals $B_1,\dots,B_k$

$$(\nu^R(B_1),\dots,\nu^R(B_k)) \underset{R\to\infty}{\Longrightarrow} (\vartheta^\infty(B_1),\dots,\vartheta^\infty(B_k)). \tag{13}$$

To prove (13), we use a method of moments. We will apply an extension of Carleman's condition for multi-dimensional random variables Kleibler and Stoyanov [2013], Shohat and Tamarkin [1950].

Fix $n,k\in\mathbb{N}$ and for a given k-tuple of disjoint intervals $\{[a_i,b_i]\}_{i=1}^k$, define

$$M_n^k=\sum_{i=1}^k \mathbb{E}(\vartheta^\infty([a_i,b_i])^n),\quad C=\sum_{n=1}^\infty (M_n^k)^{-\frac{1}{2n}}.$$

The condition states that, if $C=\infty$ (for any choice of k and $\{[a_i,b_i]\}_{i=1}^k$ that are not necessarily disjoint), proving (13) is equivalent to proving that for $k\in\mathbb{N},\ n_1,\dots,n_k\in\mathbb{N}^k$

$$\mathbb{E}\left(\prod_{i=1}^k \nu^R([a_i,b_i])^{n_i}\right) \underset{R\to\infty}{\longrightarrow} \mathbb{E}\left(\prod_{i=1}^k \vartheta^\infty([a_i,b_i])^{n_i}\right). \tag{14}$$

From (i), we have

$$M_n^k = \sum_{i=1}^k n!\ b_i^{n-1}\ (b_i-a_i)\leq kn!$$

and since

$$\begin{aligned}
\sum_{n=1}^\infty \frac{1}{(kn!)^{\frac{1}{2n}}} &\geq \frac{1}{k}\sum_{k=1}^\infty \frac{1}{(n!)^{\frac{1}{2n}}} \\
&\geq \frac{1}{k}\sum_{k=1}^\infty \frac{1}{n^{\frac{1}{2}}}=\infty
\end{aligned}$$

we get $C = \infty$ and we can apply the extension of Carleman's condition. We use the fact that

$$\forall a \leq b \leq c, \quad \nu^R[a,c] = \nu^R[a,b] + \nu^R[b,c] \;\text{ and }\; \vartheta^\infty[a,c] = \vartheta^\infty[a,b] + \vartheta^\infty[b,c]$$

so that (14) reduces to the case where the intervals $\{[a_i, b_i]\}_{i=1}^k$ are pairwise disjoint. This completes the proof of Lemma 5.1. □

Since ϑ^R is absolutely continuous with respect to the Lebesgue measure,

$$\forall a \leq b \leq c, \quad \vartheta^R[a,c] = \vartheta^R[a,b] + \vartheta^R[b,c],$$

so, from Lemma 5.1, the proof of Theorem 1.4 boils down to proving that for every k-tuple of disjoint intervals $\{[a_i, b_i]\}_{i=1}^k$ in $[0,1]$ and any k-tuple of integers $\{n_i\}_{i=1}^k$

$$\lim_{R\to\infty} \mathbb{E}\left(\prod_{i=1}^k \vartheta^R([a_i,b_i])^{n_i}\right) \;=\; \prod_{i=1}^k n_i!\; b_i^{n_i-1}(b_i - a_i). \tag{15}$$

The rest of this section will be dedicated to the proof of this asymptotical relation. We start by fixing $k \in \mathbb{N}$, $n_1, \ldots n_k \in \mathbb{N}^k$, $n = n_1 + \ldots + n_k$ and $\{[a_i, b_i]\}_{i=1}^k$ a k-tuple of disjoint intervals. Without loss of generality we assume $a_1 < b_1 < a_2 \ldots < a_k < b_k$. For any $z = (z_0, z_1, \ldots, z_n) \subset \mathbb{R}^+$ we define $c(z)$ as the coarsest partition of z.

We start by rewriting the right hand side of the equation (14):

$$\begin{aligned}
&\mathbb{E}\left(\prod_{i=1}^k \vartheta^R([a_i,b_i])^{n_i}\right) \\
&= \frac{1}{\log(R)^n}\; \mathbb{E}_{\mu^1}\left(\int_{[R^{a_1},R^{b_1}]^{n_1}\times\ldots\times[R^{a_k},R^{b_k}]^{n_k}} \mathbb{1}_{\{0\sim z_1\sim\ldots\sim z_n\}} dz_1\ldots dz_n\right) \\
&= \frac{1}{\log(R)^n} \int_{[R^{a_1},R^{b_1}]^{n_1}\times\ldots\times[R^{a_k},R^{b_k}]^{n_k}} \mu^{z,1}(c(z))dz_1\ldots dz_n
\end{aligned} \tag{16}$$

where $\mathbb{E}_{\mu^1}$ denotes the expectation with respect to μ^1, $z_0 = 0$ and $\mu^{z,1}$ is defined as the invariant measure of the partitioning process for the set of loci $z = (z_0, \cdots, z_n)$ with a recombination rate equal to 1. Let us now give some intuition for the rest of the section. Let V_R be the volume of the integration domain above. We have

$$\mathbb{E}\left(\prod_{i=1}^k \vartheta^R([a_i,b_i])^{n_i}\right) = \frac{V_R}{\log(R)^n}\mathbb{E}_Z\left(\mu^{(z_0,Z),1}(\{z_0, Z\})\right),$$

where $\mathbb{E}_Z$ denotes the expectation with respect to $Z = (Z_1, \ldots, Z_n)$ distributed as a uniform random variable on $[R^{a_1}, R^{b_1}]^{n_1}\times\ldots\times[R^{a_k}, R^{b_k}]^{n_k}$. When $R \gg 1$, for a "typical" configuration Z, the distances between the z_i's will be of order R. As $\rho = 1$, the fragmentation rates correspond to the distances between the z_i's and are of order $R \gg 1$, whereas the coalescence rate is always 1 for each pair of blocks. In this situation, fragmentation events occur much more often than coalescence events, which is the framework of Theorem 1.3. The main idea behind (15) is to approximate the integrand using this theorem.

Let us now go into the details of the proof. We decompose the proof into four steps.

Step 1. Define

$$\begin{aligned}
C_\beta^R \;&:=\; \{z_1, \ldots, z_n \in \otimes_{i=1}^n [R^{a_i-1}, R^{b_i-1}]^{n_i} : \;\text{ s.t. if } z_0 := 0 \\
&\qquad \forall i \neq j \in \{0, \ldots, n\},\; |z_i - z_j| \geq \beta\}
\end{aligned}$$

(Note that in the rest of the proof, we will always set $z_0 = 0$.) The aim of this step is to prove the following proposition.

Proposition 5.2.

$$\forall \beta > 0, \quad \lim_{R\to\infty} \frac{1}{\log(R)^n} \int_{C_\beta^R} F(c(z))dz_1 \dots dz_n \;=\; \prod_{i=1}^{k} n_i! \; b_i^{n_i-1}(b_i - a_i), \tag{17}$$

where, as defined in Section 4,

$$F(c(z)) = \sum_{s\in\mathcal{S}(c(z))} \frac{1}{E(s)},$$

and where $E(s)$ is the product of the successive cover lengths along the coalescence scenario s.

To see why this Proposition is useful for the proof of (15), we let the reader refer to Steps 2 and 3.

In the following we fix $\beta \geq 1$, and we assume that R is large enough so that $\forall i \in [k],\ R^{b_i-1} > \beta R^{-1}$. Let Σ_n be the set of permutations of $[n]$. For $\sigma \in \Sigma_n$ define

$$C_{\beta,\sigma}^R \;:=\; \{z_1, \dots, z_n \in C_\beta^R,\ z_{\sigma(1)} < \dots . < z_{\sigma(n)}\}.$$

Recall that, as the intervals $\{[a_i, b_i]\}_{i=1}^n$ are disjoint, the z_i's belonging to $[a_j, b_j]$ are always smaller than those belonging to $[a_{j+1}, b_{j+1}]$. This means that there are only $n_1! \dots n_k!$ permutations for which $C_{\beta,\sigma}^R$ is non empty. Using the symmetry between the z_i's belonging to the same interval, we have

$$\begin{aligned}
\int_{C_\beta^R} F(c(z))\ dz_1 \dots dz_n \;&=\; \sum_{\sigma\in\Sigma_n} \int_{C_{\beta,\sigma}^R} F(c(z))\ dz_1 \dots dz_n \\
&=\; n_1! \dots n_k! \int_{C_{\beta,Id}^R} F(c(z))\ dz_1 \dots dz_n,
\end{aligned}$$

where Id is the identity permutation. To prove Proposition 5.2, it remains to show that:

$$\lim_{R\to\infty} \frac{1}{\log(R)^n} \int_{C_{\beta,Id}^R} F(c(z))dz_1 \dots dz_n \;= \prod_{i=1}^{k} b_i^{n_i-1}(b_i - a_i). \tag{18}$$

The idea now is to consider separately two different types of scenarios of coalescence.

- $\mathcal{S}_C(c(z))$ corresponds to the set of the "contiguous scenarios" i.e. the scenarios where blocks only coalesce with their neighbouring blocks (i.e where at each step the block containing z_i can only coalesce with the blocks containing z_{i-1} or z_{i+1}). This is for example the case of scenarios S_1 and S_2 in Figure 5.

- $\bar{\mathcal{S}}_C(c(z))$ contains all the other scenarios (for example S_3 and S_4 in Figure 5).

$$\begin{aligned}
\int_{C_{\beta,Id}^R} F(c(z))dz_1 \dots dz_n \;&=\; \int_{C_{\beta,Id}^R} \sum_{s\in\mathcal{S}_C(c(z))} \frac{1}{E(s)} dz_1 \dots dz_n \\
&\quad + \int_{C_{\beta,Id}^R} \sum_{s\in\bar{\mathcal{S}}_C(c(z))} \frac{1}{E(s)} dz_1 \dots dz_n.
\end{aligned} \tag{19}$$

The rest of this step is devoted to the computation of each of the terms in the RHS of this equation.

Step 1.1. The aim of Step 1.1 is to prove the following lemma

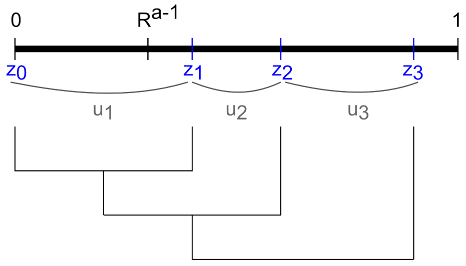
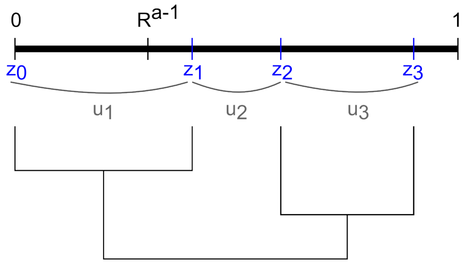
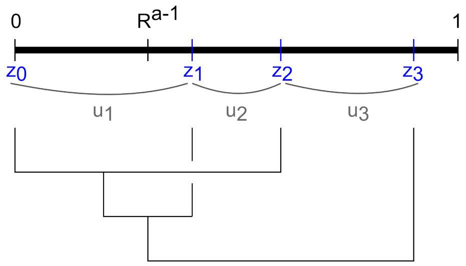
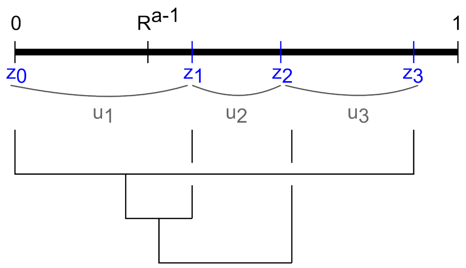
Type	Scenario of coalescence	Energy
$\mathcal{S}_1$	 S_1	$E(S_1) = (z_1 - z_0) \times (z_2 - z_0) \times (z_3 - z_0)$ $= u_1 \times (u_1 + u_2) \times (u_1 + u_2 + u_3)$ $(\tau(1) = 1,\ \tau(2) = 2,\ \tau(3) = 3)$
$\mathcal{S}_1$	 S_2	$E(S_2) = (z_1 - z_0) \times (z_1 - z_0 + z_3 - z_2)$ $\times (z_3 - z_0)$ $= u_1 \times (u_1 + u_3) \times (u_1 + u_3 + u_2)$ $(\tau(1) = 1,\ \tau(2) = 3,\ \tau(3) = 2)$
$\mathcal{S}_2$	 S_3	$E(S_3) = (z_2 - z_0) \times (z_2 - z_0) \times (z_3 - z_0)$ $= (u_1 + u_2) \times (u_1 + u_2)$ $\times (u_1 + u_2 + u_3)$
$\mathcal{S}_2$	 S_4	$E(S_4) = (z_3 - z_0) \times (z_3 - z_0) \times (z_3 - z_0)$ $= (u_1 + u_2 + u_3) \times (u_1 + u_2 + u_3)$ $\times (u_1 + u_2 + u_2)$

Figure 6: Some examples of coalescence scenarios and their energy. In these examples, $k = 1$, $b = 1$, $a_1 := a$.

Lemma 5.3.

$$\lim_{R\to\infty} \frac{1}{\log(R)^n} \int_{C^R_{\beta,Id}} \sum_{s\in\mathcal{S}_C(c(z))} \frac{1}{E(s)} dz_1 \dots dz_n \;= \prod_{i=1}^k b_i^{n_i-1}(b_i - a_i).$$

For each $i \in [n]$, we define $u_i := z_i - z_{i-1}$. It is not hard to see that each scenario $s = (s_1,\dots,s_n) \in \mathcal{S}_C(c(z))$ is characterized by a unique permutation $\tau \in \Sigma_n$ which specifies the order of coalescence of the successive contiguous blocks in such a way that

$$\frac{1}{E(s)} \;=\; \prod_{i=1}^n \frac{1}{u_{\tau(1)} + \ldots + u_{\tau(n)}}.$$

(see Figure 5 for some examples.) As a consequence, we can index each contiguous scenario by a permutation, and using the change of variables $u_i = z_i - z_{i-1}$, we get

$$\int_{C^R_{\beta,Id}} \sum_{s\in\mathcal{S}_C(c(z))} \frac{1}{E(s)}\, dz_1 \dots dz_n = \int_{U^R} \sum_{\tau\in\Sigma_n} \left(\prod_{i=1}^n \frac{du_{\tau(i)}}{u_{\tau(1)} + \ldots + u_{\tau(i)}} \right), \tag{20}$$

where U^R is defined as follows. First, let us define (see also Figure 7)

$$\begin{aligned}
w_R(1) &:= \max(\beta R^{-1}, R^{a_1-1})\\
W_R(1) &:= R^{b_1-1}\\
\forall 2 \le i \le k, w_R(i) &:= R^{a_i-1} - R^{b_{i-1}-1}\\
W_R(i) &:= R^{b_i-1} - R^{a_{i-1}-1}\\
\forall 1 \le i \le k, L_R(i) &:= R^{b_i-1} - R^{a_i-1}.
\end{aligned}$$

Finally, we set $n_0 := 0$. Under the assumption that R is large enough so that $\forall i \in [k],\; R^{b_i-1} > \beta R^{-1}$

$$\begin{aligned}
U^R :=\{\; & u_1,\dots,u_n \in \otimes_{i=1}^k \left([w_R(i), W_R(i)] \times [\beta R^{-1}, L_R(i)]^{n_i-1}\right) \;:\\
& \forall i\in[k],\; \sum_{j=n_{i-1}+2}^{n_i} u_j \le L_R(i) \;\text{ and }\; \sum_{j=1}^{n_1+\ldots+n_i} u_j \le R^{b_i-1}\}
\end{aligned}$$

In fact, by definition of $C^R_{\beta,Id}$, $\forall j \in [n]$, $\beta R^{-1} \le u_j = z_j - z_{j-1}$. In addition, the $L_R(i)$'s correspond to the lengths of the different intervals $[R^{a_i-1}, R^{b_i-1}]$, so when z_j and z_{j-1} belong to the same interval, $u_j = z_j - z_{j-1} \le L_R(i)$ (See Figure 7). The $w_R(i)$'s correspond to the distance between two contiguous intervals and the $W_R(i)$'s to the maximal distance between two points of contiguous intervals. So, when z_j and z_{j-1} belong to different intervals then $u_j = z_j - z_{j-1} \in [w_R(i), W_R(i)]$ (See Figure 7). Finally, the last inequalities come from the fact that for $j \in \{n_{i-1}+1,\dots,n_i\}$, the z_i's belong to the same interval $[a_i, b_i]$, so the sum of their distances cannot exceed the length of the interval. In addition, the distance between z_0 and z_{n_i} cannot exceed R^{b_i-1}.

To compute the RHS of (20), we start by proving the following Lemma.

Lemma 5.4. *For any $\tau \in \Sigma_n$, $\kappa > 1$, define*

$$\begin{aligned}
K_\tau^{R,\kappa} &= \{u_1,\dots,u_n \in U^R, \forall i\in[n],\; u_{\tau(i)} > \kappa \sum_{j=1}^{i-1} u_{\tau(j)}\}\\
\bar{K}_\tau^{R,\kappa} &= U^R \setminus K_\tau^{R,\kappa}
\end{aligned}$$

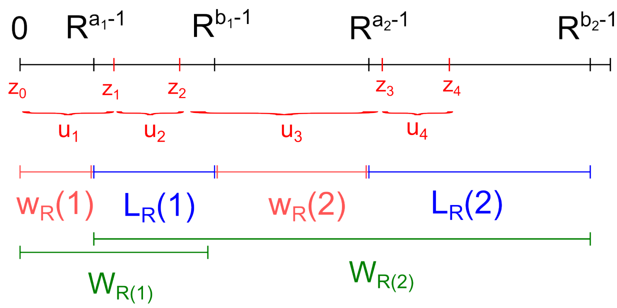


Figure 7: The set U^R.

We have

$$(i)\ \forall \kappa > 1,\ \lim_{R\to\infty} \frac{1}{\log(R)^n} \sum_{\tau\in\Sigma_n} \int_{\bar{K}_\tau^{R,\kappa}} \left(\prod_{i=1}^n \frac{du_{\tau(i)}}{u_{\tau(1)} + \ldots + u_{\tau(i)}} \right) = 0$$

$$(ii) \lim_{\kappa\to\infty} \lim_{R\to\infty} \frac{1}{\log(R)^n} \sum_{\tau\in\Sigma_n} \int_{K_\tau^{R,\kappa}} \left(\prod_{i=1}^n \frac{du_{\tau(i)}}{u_{\tau(1)} + \ldots + u_{\tau(i)}} \right) = \prod_{i=1}^k b_i^{n_i-1}(b_i - a_i).$$

Remark 5.5. *The proof of Lemma 5.4 is rather cumbersome, but the idea behind the proof is simple. In a nutshell, the idea is that, depending on the positions of the loci (the z_i's), one scenario is much more likely than the others. More precisely, for any configuration $z \in C^R_{\beta,Id}$, there exists a scenario $S_{min} \in \mathcal{S}_C(c(z))$ associated to permutation $\tau_{min} \in \Sigma_n$ such that*

$$u_{\tau_{min}(1)} \leq u_{\tau_{min}(2)} \leq \ldots \leq u_{\tau_{min}(n)}.$$

By coalescing the u_i's in the increasing order, the successive cover lengths are minimised. (i) encapsulates that the contribution of other scenarios is negligible.

Proof. We start by proving (i). We fix $\tau \in \Sigma_n$, $\kappa > 1$. We make the following change of variables. Let us define Ψ^τ such that for $1 \leq i \leq n$, $(\Psi^\tau(u_1, \ldots, u_n))_{\tau(i)} = u_{\tau(1)} + \ldots + u_{\tau(i)}$. We have

$$\int_{U^R} \left(\prod_{i=1}^n \frac{du_{\tau(i)}}{u_{\tau(1)} + \ldots + u_{\tau(i)}} \right) = \int_{v\in\Psi^\tau(U^R)} \frac{dv_1 \ldots dv_n}{v_1 \ldots v_n}. \tag{21}$$

In particular, as

$$\forall i \in [n],\quad u_{\tau(i)} \leq \kappa \sum_{j=1}^{i-1} u_{\tau(j)} \ \Leftrightarrow\ \forall i \in [n],\quad v_{\tau(i)} \leq (1+\kappa) v_{\tau(i-1)},$$

we have

$$\int_{\bar{K}_\tau^{R,\kappa}} \left(\prod_{i=1}^n \frac{du_{\tau(i)}}{u_{\tau(1)} + \ldots + u_{\tau(i)}} \right) = \int_{V_\tau^{R,\kappa}} \frac{dv_1 \ldots dv_n}{v_1 \ldots v_n}$$

where

$$V_\tau^{R,\kappa} = \{v \in \Psi^\tau(U^R), \exists i \in [n],\ v_{\tau(i)} \leq (1+\kappa) v_{\tau(i-1)}\}.$$

For every $i \in [n]$, we define

$$V_\tau^{R,\kappa}(i) = \{v \in \Psi^\tau(U^R),\ v_{\tau(i)} \leq (1+\kappa) v_{\tau(i-1)}\}.$$

We have

$$V_\tau^{R,\kappa} = \bigcup_{i=1}^n V_\tau^{R,\kappa}(i).$$

We fix $\tau \in \Sigma_n$ and $i \in [n]$. The $v_{\tau(j)}$'s are the successive cover lengths at each step of the scenario S associated to τ, so it can readily be seen that

$$\forall j \in [n], v_j \in [R^{-1}, 1] \text{ and } v_{\tau(1)} < \ldots < v_{\tau(n)},$$

which implies that

$$V_\tau^{R,\kappa}(i) \subset \{v \in [R^{-1}, 1]^n,\ v_{\tau(i-1)} < v_{\tau(i)} < (1+\kappa)v_{\tau(i-1)}\},$$

so

$$\begin{aligned}
\int_{V_\tau^{R,\kappa}(i)} \frac{dv_{\tau(1)}\ldots dv_{\tau(n)}}{v_{\tau(1)}\ldots v_{\tau(n)}} &\leq \left(\int_{R^{-1}}^1 \frac{dv}{v}\right)^{n-2} \int_{R^{-1}}^1 \frac{dv_{\tau(i-1)}}{v_{\tau(i-1)}} \int_{v_{\tau(i-1)}}^{(1+\kappa)v_{\tau(i-1)}} \frac{dv_{\tau(i)}}{v_{\tau(i)}} \\
&= \log(R)^{n-2} \int_{R^{-1}}^1 \frac{dv_{\tau(i-1)}}{v_{\tau(i-1)}} \log(1+\kappa) \\
&= \log(R)^{n-1} \log(1+\kappa),
\end{aligned}$$

which completes the proof of (i).

We now turn to the proof of (ii). We decompose the proof into four steps.

Step a. Define

$$X^R \ := \ \otimes_{i=1}^k \left([w_R(i), W_R(i)] \times [\beta R^{-1}, L_R(i)]^{n_i-1}\right).$$

Then

$$\begin{aligned}
\frac{1}{\log(R)^n} \int_{X^R} \left(\prod_{i=1}^n \frac{du_i}{u_i}\right) &= \frac{1}{\log(R)^n} \prod_{i=1}^k \int_{w_R(i)}^{W_R(i)} \frac{du}{u} \left(\int_{\beta R^{-1}}^{L_R(i)} \frac{du}{u}\right)^{n_i-1} \\
&= \frac{1}{\log(R)^n} \prod_{i=1}^k \log\left(\frac{W_R(i)}{w_R(i)}\right) \log\left(\frac{L_R(i)}{\beta R^{-1}}\right)^{n_i-1} \\
&\underset{R\to\infty}{\longrightarrow} \prod_{i=1}^k (b_i - a_i) b_i^{n_i-1}. \qquad (22)
\end{aligned}$$

Step b. Next, for every $\kappa > 1$ and for every $\tau \in \Sigma_n$, we define

$$\begin{aligned}
X_\tau^R \ &:= \{\ u_1, \ldots, u_n \in X^R \ :\ u_{\tau(1)} \leq \ldots \leq u_{\tau(n)}\}, \\
A_\tau^\kappa \ &:= \{\ u_1, \ldots, u_n, \forall i \in [n],\ u_{\tau(i)} > \kappa \sum_{j=1}^{i-1} u_{\tau(j)}\}. \\
X_\tau^{R,\kappa} \ &:= X^R \cap A_\tau^\kappa.
\end{aligned}$$

By reasoning along the same lines as in the proof of (i), one can show that

$$\lim_{R\to\infty} \frac{1}{\log(R)^n} \left| \int_{X_\tau^R} \left(\prod_{i=1}^n \frac{du_{\tau(i)}}{u_{\tau(i)}}\right) - \int_{X_\tau^{R,\kappa}} \left(\prod_{i=1}^n \frac{du_{\tau(i)}}{u_{\tau(i)}}\right) \right| = 0.$$

From Step 1, we get that for every $\kappa > 1$

$$\lim_{R\to\infty} \frac{1}{\log(R)^n} \sum_{\tau\in\Sigma_n} \int_{X_\tau^{R,\kappa}} \left(\prod_{i=1}^n \frac{du_{\tau(i)}}{u_{\tau(i)}}\right) \ = \ \prod_{i=1}^k (b_i - a_i) b_i^{n_i-1} \qquad (23)$$

Step c. We aim of this step is to prove that for every $\tau \in \Sigma_n$,

$$\lim_{R\to\infty} \frac{1}{\log(R)^n} \int_{K_\tau^{R,\kappa}} \left(\prod_{i=1}^{n} \frac{du_{\tau(i)}}{u_{\tau(i)}} \right) = \lim_{R\to\infty} \frac{1}{\log(R)^n} \int_{X_\tau^{R,\kappa}} \left(\prod_{i=1}^{n} \frac{du_{\tau(i)}}{u_{\tau(i)}} \right) \tag{24}$$

From the definition of $K_\tau^{R,\kappa}$, we have

$$K_\tau^{R,\kappa} = X_\tau^{R,\kappa} \cap (K_1^R \cap K_2^R)$$

where

$$K_1^R := \{ \ u_1, \ldots, u_n, \quad \forall i \in [k], \sum_{j=n_{i-1}+2}^{n_i} u_j \leq L_R(i) \ \},$$

$$K_2^R := \{ \ u_1, \ldots, u_n, \quad \forall i \in [k] \sum_{j=1}^{n_1+\ldots+n_i} u_j \leq R^{b_i - 1} \ \}.$$

If

$$\lim_{R\to\infty} \frac{1}{\log(R)^n} \left| \int_{X_\tau^{R,\kappa} \cap K_1^R} \left(\prod_{i=1}^{n} \frac{du_{\tau(i)}}{u_{\tau(i)}} \right) - \int_{X_\tau^{R,\kappa}} \left(\prod_{i=1}^{n} \frac{du_{\tau(i)}}{u_{\tau(i)}} \right) \right| = 0 \tag{25}$$

and

$$\lim_{R\to\infty} \frac{1}{\log(R)^n} \left| \int_{X_\tau^{R,\kappa} \cap K_2^R} \left(\prod_{i=1}^{n} \frac{du_{\tau(i)}}{u_{\tau(i)}} \right) - \int_{X_\tau^{R,\kappa}} \left(\prod_{i=1}^{n} \frac{du_{\tau(i)}}{u_{\tau(i)}} \right) \right| = 0, \tag{26}$$

then

$$\lim_{R\to\infty} \frac{1}{\log(R)^n} \left| \int_{K_\tau^{R,\kappa}} \left(\prod_{i=1}^{n} \frac{du_{\tau(i)}}{u_{\tau(i)}} \right) - \int_{X_\tau^{R,\kappa}} \left(\prod_{i=1}^{n} \frac{du_{\tau(i)}}{u_{\tau(i)}} \right) \right| = 0. \tag{27}$$

We will only prove (25), as (26) can be proved along the same lines. To do so, we define

$$Y_\tau^{R,\kappa} := \{ u_1, \ldots, u_n \in \otimes_{i=1}^k \left([w_R(i), W_R(i)] \times \left[\beta R^{-1}, \frac{L_R(i)}{1+\frac{1}{\kappa}} \right]^{n_i - 1} \right),$$
$$u_{\tau(1)} \leq \ldots \leq u_{\tau(n)} \}.$$

so that $Y_\tau^{R,\kappa} \subset X_\tau^R$. By similar computations as those used in the proof of (i), it can be shown that

$$\lim_{R\to\infty} \frac{1}{\log(R)^n} \int_{X_\tau^R \setminus Y_\tau^{R,\kappa}} \prod_{i=1}^{n} \frac{du_{\tau(i)}}{u_{\tau(i)}} = 0,$$

so

$$\lim_{R\to\infty} \frac{1}{\log(R)^n} \left| \int_{X_\tau^{R,\kappa}} \prod_{i=1}^{n} \frac{du_{\tau(i)}}{u_{\tau(i)}} - \int_{Y_\tau^{R,\kappa} \cap A_\tau^\kappa} \prod_{i=1}^{n} \frac{du_{\tau(i)}}{u_{\tau(i)}} \right| = 0, \tag{28}$$

and

$$\lim_{R\to\infty} \frac{1}{\log(R)^n} \left| \int_{X_\tau^{R,\kappa} \cap K_1^R} \prod_{i=1}^{n} \frac{du_{\tau(i)}}{u_{\tau(i)}} - \int_{Y_\tau^{R,\kappa} \cap A_\tau^\kappa \cap K_1^R} \prod_{i=1}^{n} \frac{du_{\tau(i)}}{u_{\tau(i)}} \right| = 0 \tag{29}$$

Let us show that

$$Y_\tau^{R,\kappa} \cap A_\tau^\kappa \cap K_1^R = Y_\tau^{R,\kappa} \cap A_\tau^\kappa,$$

i.e. that

$$\forall (u_1,\ldots,u_n)\in Y_\tau^{R,\kappa}\cap A_\tau^\kappa,\quad \forall i\in[k], \sum_{j=n_{i-1}+2}^{n_i} u_j \le L_R(i).$$

We fix $i\in[k]$ and we define

$$m_i := j\in\{n_{i-1}+2,\ldots,n_i\},\ \tau^{-1}(j)=\max\{\tau^{-1}(n_{i-1}+2),\ldots,\tau^{-1}(n_i)\}$$

As

$$u_{\tau(m_i)} > \kappa\sum_{j=1}^{m_i-1} u_{\tau(j)},$$

then

$$\sum_{j=n_{i-1}+2}^{n_i} u_j \le \sum_{j=1}^{m_i} u_{\tau(j)} = \left(1+\frac{1}{\kappa}\right)u_{\tau(m_i)} \le \left(1+\frac{1}{\kappa}\right)\frac{L_R(i)}{1+\frac{1}{\kappa}} = L_R(i).$$

Since $Y_\tau^{R,\kappa}\cap A_\tau^\kappa\cap K_1^R = Y_\tau^{R,\kappa}\cap A_\tau^\kappa$, combining (28) and (29), (25) is proved. Equation (26) can be proved along the same lines, so (27) is verified.

Step d. Finally, for any $\tau\in\Sigma_n$, for any $u_1,\ldots,u_n\in K_\tau^{R,\kappa}$, we have

$$\forall i\in[n],\ u_{\tau(i)}\ \le\ u_{\tau(1)}+\ldots+u_{\tau(i-1)}+\ u_{\tau(i)}\ \le\ \left(1+\frac{1}{\kappa}\right)\ u_{\tau(i)},$$

which implies that

$$\sum_{\tau\in\Sigma_n}\int_{K_\tau^{R,\kappa}}\left(\prod_{i=1}^n\frac{du_{\tau(i)}}{u_{\tau(i)}}\right) \le \sum_{\tau\in\Sigma_n}\int_{K_\tau^{R,\kappa}}\left(\prod_{i=1}^n\frac{du_{\tau(i)}}{u_{\tau(1)}+\ldots+u_{\tau(i)}}\right)$$

and

$$\sum_{\tau\in\Sigma_n}\int_{K_\tau^{R,\kappa}}\left(\prod_{i=1}^n\frac{du_{\tau(i)}}{u_{\tau(1)}+\ldots+u_{\tau(i)}}\right) \le \frac{1}{(1+\frac{1}{\kappa})^n}\sum_{\tau\in\Sigma_n}\int_{K_\tau^{R,\kappa}}\left(\prod_{i=1}^n\frac{du_{\tau(i)}}{u_{\tau(i)}}\right)$$

This, combined with Step b (see (23)) and Step c (see (24))

$$\begin{aligned}\prod_{i=1}^k(b_i-a_i)b_i^{n_i-1} &\le \lim_{R\to\infty}\frac{1}{\log(R)^n}\sum_{\tau\in\Sigma_n}\int_{K_\tau^{R,\kappa}}\left(\prod_{i=1}^n\frac{du_{\tau(i)}}{u_{\tau(1)}+\ldots+u_{\tau(i)}}\right)\\ &\le \frac{\prod_{i=1}^k(b_i-a_i)b_i^{n_i-1}}{(1+\frac{1}{\kappa})^n},\end{aligned}$$

and the conclusion follows by taking $\kappa\to\infty$. □

Proof of Lemma 5.3. From (20), we have

$$\begin{aligned}&\lim_{R\to\infty}\frac{1}{\log(R)^n}\int_{C^R_{\beta,Id}}\sum_{s\in\mathcal{S}_C(c(z))}\frac{1}{E(s)}\ dz_1\ldots dz_n\\ &=\lim_{R\to\infty}\frac{1}{\log(R)^n}\sum_{\tau\in\Sigma_k}\int_{U^R}\prod_{i=1}^n\frac{du_{\tau(i)}}{u_{\tau(1)}+\ldots+u_{\tau(i)}}\\ &=\lim_{R\to\infty}\frac{1}{\log(R)^n}\sum_{\tau\in\Sigma_k}\left(\int_{K_\tau^{R,\kappa}}\prod_{i=1}^n\frac{du_{\tau(i)}}{u_{\tau(1)}+\ldots+u_{\tau(i)}}+\int_{\bar{K}_\tau^{R,\kappa}}\prod_{i=1}^n\frac{du_{\tau(i)}}{u_{\tau(1)}+\ldots+u_{\tau(i)}}\right).\end{aligned}$$

Using Lemma 5.4 and taking $\kappa \to \infty$ in the RHS, we have

$$\lim_{R\to\infty} \frac{1}{\log(R)^n} \int_{C^R_{\beta,Id}} \sum_{s\in\mathcal{S}_C(c(z))} \frac{1}{E(s)}\, dz_1 \dots dz_n \;=\; \prod_{i=1}^{k} b_i^{n_i-1}(b_i-a_i). \tag{30}$$

$\square$

Step 1.2. The aim of this Step is to compute the second term in the RHS of (19).

Lemma 5.6.

$$\lim_{R\to\infty} \frac{1}{\log(R)^n} \int_{C^R_{\beta,Id}} \sum_{S\in\bar{\mathcal{S}}_C(c(z))} \frac{1}{E(S)} dz_1 \dots dz_n \;=\; 0.$$

Proof. We fix $z = z_0, z_1, \dots, z_n \in C^R_{\beta,Id}$. We start by considering scenarios where blocks only coalesce with neighbouring blocks, except for one step. In other words, we start by considering scenarios in $\bar{\mathcal{S}}'_C(c(z))$, the set of scenarios that contain one single coalescence event between two non-neighbouring blocks. For example, S_3 in Figure 5 is in $\bar{\mathcal{S}}'_C(c(z))$. We consider $S' = (s'_0, s'_1, \dots, s'_n) \in \bar{\mathcal{S}}'_C(c(z))$, a scenario of coalescence in which step $j > 1$, is the only coalescence event between two non neighbouring blocks. The idea is to compare S' with a scenario $S \in \mathcal{S}_C(c(z))$ and use this scenario to show that

$$\lim_{R\to\infty} \frac{1}{\log(R)^n} \int_{C^R_{\beta,Id}} \frac{1}{E(S')} dz_1 \dots dz_n = 0. \tag{31}$$

As already argued each scenario in $\mathcal{S}_C(c(z))$ is associated to a permutation τ such that, at each step i, the cover length increases by $u_{\tau(i)}$. The scenario S and the corresponding permutation τ are constructed as follows (and we let the reader refer to Figure 8 for an example, where subfigures $(i), \dots, (iv)$ correspond to each step in the following construction).

(i) For $0 \le i < j$, we set $s_i = s'_i$. Before step j, there are only coalescence events between neighbouring blocks in S'. $\tau(1), \dots, \tau(j-1)$ are constructed in such a way that

$$\forall 1 \le i < j, \quad C(s_i) = C(s'_i) = \sum_{k=1}^{i} u_{\tau(k)}.$$

(ii) At step j, in scenario S' there is a coalescence event between two non neighbouring blocks, which means that there exists $i_1 < i_2 < \dots < i_\ell$ such that $C(s'_j) = C(s'_{j-1}) + u_{i_1} + \dots + u_{i_\ell}$. s_j is the partition of order j such that $C(s_j) = C(s_{j-1}) + u_{i_1}$. We set $\tau(j) := i_1$. We have

$$C(s_j) = \sum_{k=1}^{i} u_{\tau(k)} \le C(s'_j).$$

(iii) For $j < i \le j + \ell - 1$, $\tau(i) := i_{i-j}$, i.e., we add successively $u_{i_2}, \dots, u_{i_\ell}$. We have

$$\forall j < i \le j + \ell - 1, \; C(s_i) \le C(s'_i).$$

(iv) For $j + \ell \le i \le j$, the s_i's are constructed as follows. Let $u_{r_1}, \dots, u_{r_p}$ be the u_i's that have not been added yet to the cover length of S' (i.e. the u_i's that are not in $\{u_{\tau(1)}, \dots, u_{\tau(j+\ell)}\}$), indexed in such a way that in S', u_{r_1} coalesces before u_{r_2} etc .... Then we set $u_{\tau(j+\ell+i)} = u_{r_i}$. In other words, the u_{r_i}'s are added to the cover length in S in the same order as they are added in S' (see Figure 8).

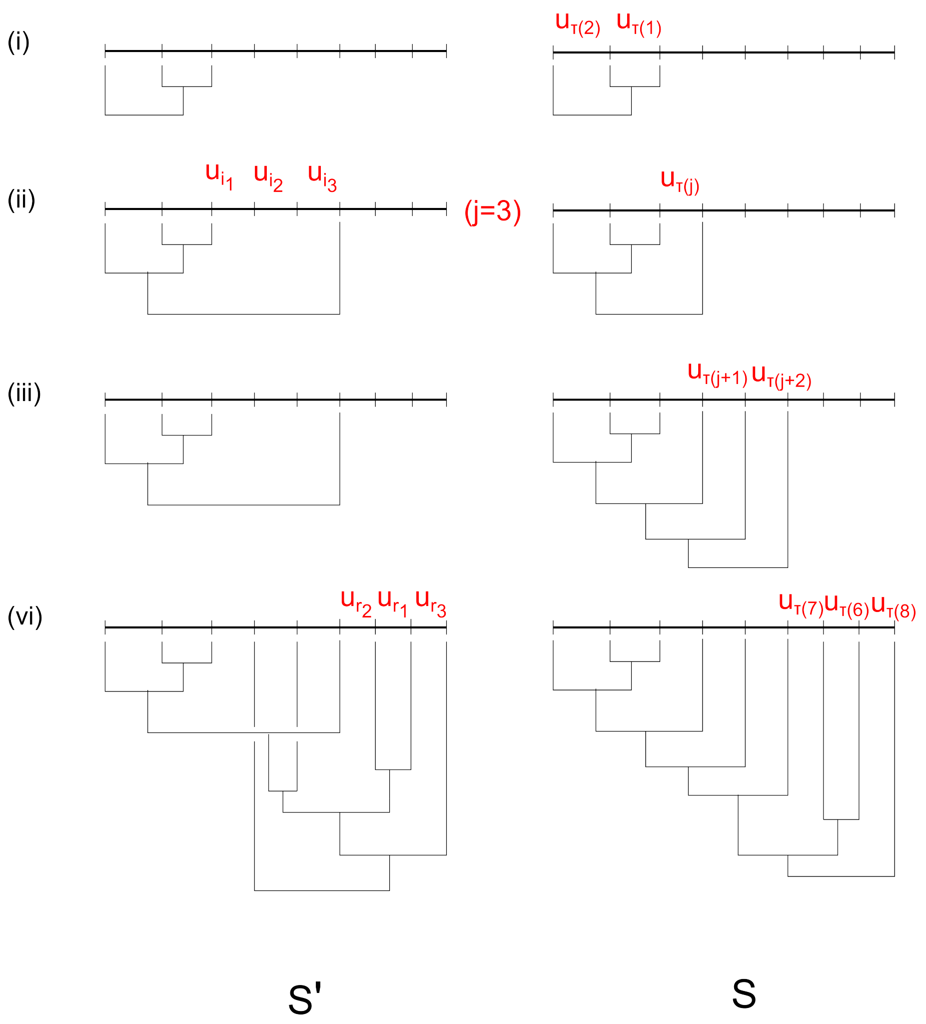


Figure 8: Example of the construction of a scenario $S \in \mathcal{S}_C(c(z))$ from a scenario $S' \in \bar{\mathcal{S}}'_C(c(z))$. The left-hand side corresponds to S' and the right-hand side to S. Steps (i) . . . (iv) correspond to the steps in construction of S from S'.

With this construction, we have

$$\begin{aligned}\frac{1}{E(S)} =& \prod_{i=1}^{n} \frac{1}{u_{\tau(1)} + \ldots + u_{\tau(i)}} \\ =& \frac{1}{v_{\tau(1)} \ldots v_{\tau(n)}}.\end{aligned}$$

where for $i \in \{1, \ldots, n\}$, $v_{\tau(i)} := \Psi^{\tau}(U^R)_{\tau(i)} = u_{\tau(1)} + \ldots + u_{\tau(i)}$. By construction, we have

$$\frac{1}{E(S')} \leq \left(\prod_{i=1}^{j-1} \frac{1}{C(s_i)} \right) \frac{1}{C(s'_j)} \left(\prod_{i=j+1}^{n} \frac{1}{C(s_i)} \right).$$

Using the fact that

$$\begin{aligned}C(s'_j) &= u_{\tau(1)} + \ldots + u_{\tau(j-1)} + u_{i_1} + \ldots + u_{i_\ell} \\ &= v_{\tau(j+\ell)},\end{aligned}$$

we have

$$\begin{aligned}\frac{1}{E(S')} \leq&= \left(\prod_{i=1}^{j-1} \frac{1}{v_{\tau(i)}} \right) \frac{1}{v_{\tau(j+\ell)}} \left(\prod_{i=j+1}^{n} \frac{1}{v_{\tau(i)}} \right) \\ \leq& \left(\prod_{i=1}^{j-1} \frac{1}{v_{\tau(i)}} \right) \frac{1}{v_{\tau(j+1)}} \left(\prod_{i=j+1}^{n} \frac{1}{v_{\tau(i)}} \right)\end{aligned}$$

where the last inequality comes from the fact that $v_{\tau(j+1)} < \ldots < v_{\tau(j)+l}$. Using the same change of variables as in (21), we have

$$\int_{C^R_{\beta,Id}} \frac{1}{E(S')} dz_1 \ldots dz_n \quad = \quad \int_{\Psi^\tau(U^R)} \frac{dv_{\tau(1)} \ldots dv_{\tau(n)}}{v_{\tau(1)} \ldots v_{\tau(j-1)} v_{\tau(j+1)} v_{\tau(j+1)} \ldots v_{\tau(n)}}$$

And, from the definition of Ψ^τ, it can easily be seen that for any $\tau \in \Sigma_n$

$$\Psi^\tau(U^R) \subset \Psi' = \{x_1, \ldots, x_n \in [R^{-1}, 1]^n, x_1 \leq \ldots \leq x_n\}$$

so

$$\begin{aligned}\int_{C^R_{\beta,Id}} \frac{dz_1 \ldots dz_n}{E(S')} &\leq \int_{\Psi'} \frac{dx_1 \ldots dx_n}{x_1 \ldots x_{j-1} x_{j+1} x_{j+1} \ldots x_n} \\ &= \int_{R^{-1}}^{1} \frac{dx_n}{x_n} \cdots \int_{R^{-1}}^{x_{j+2}} \frac{dx_{j+1}}{x_{j+1}^2} \int_{R^{-1}}^{x_{j+1}} dx_j \int_{R^{-1}}^{x_j} \frac{dx_{j-1}}{x_{j-1}} \cdots \int_{R^{-1}}^{x_2} \frac{dx_1}{x_1} \\ &= \int_{R^{-1}}^{1} \frac{dx_n}{x_n} \cdots \int_{R^{-1}}^{x_{j+2}} \frac{dx_{j+1}}{x_{j+1}^2} \int_{R^{-1}}^{x_{j+1}} \frac{\log(Rx_j)^{j-1}}{(j-1)!} dx_j \\ &\leq \frac{\log(R)^{j-1}}{(j-1)!} \int_{R^{-1}}^{1} \frac{dx_n}{x_n} \cdots \int_{R^{-1}}^{x_{j+2}} \frac{x_{j+1} - 1/R}{x_{j+1}^2} dx_{j+1} \\ &\leq \log(R)^{j-1} \int_{R^{-1}}^{1} \frac{dx_n}{x_n} \cdots \int_{R^{-1}}^{x_{j+2}} \frac{dx_{j+1}}{x_{j+1}} \;=\; \log(R)^{n-1},\end{aligned}$$

which completes the proof of (31).

To complete the proof of Lemma 5.6, we are going show that, for every scenario $S^2 \in \bar{\mathcal{S}}_C(c(z))$ with more than one step of coalescence between non-contiguous scenarios, there exist a scenario $S^3 \in \bar{\mathcal{S}}'_C(c(z))$ such that $E(S^3) \leq E(S^2)$. We fix $S^2 = (s^2_0, s^2_1, \ldots, s^2_n) \in \bar{\mathcal{S}}_C(c(z))$, and the idea is to construct $S^3 = (s^3_0, s^3_1, \ldots, s^3_n) \in \bar{\mathcal{S}}'_C(c(z))$ along the same lines as in Step 1. Let j_1 the first step of coalescence between non contiguous blocks in S^3 and j_2 the second one.

- For $0 \le i < j_2$, $s_i^3 := s_i^2$. In words, we copy all the steps, including j_1, the first step of coalescence between non neighbouring blocks.
- Steps $(s_{j_2}^3, \dots, s_n^3)$ are obtained from $(s_{j_2}^2, \dots, s_n^2)$ in the same way as S' was obtained from S in Step 1.

With this construction, $S^3 \in \bar{\mathcal{S}}'_C(c(z))$ (there is only one step of coalescence between non neighbouring blocks, which is j_1) and we have $\forall i \in [n],\ C(s_i^2) \le C(s_i^3)$, so

$$\int_{C^R_{\beta,Id}} \frac{1}{E(S^2)} dz_1 \dots dz_n \le \int_{C^R_{\beta,Id}} \frac{1}{E(S^3)} dz_1 \dots dz_n$$

As $S^3 \in \bar{\mathcal{S}}'_C(c(z))$, combining the previous equation with (31), for every scenario $S \in \bar{\mathcal{S}}_C(c(z))$,

$$\lim_{R\to\infty} \frac{1}{\log(R)^n} \int_{C^R_{\beta,Id}} \frac{1}{E(S)} dz_1 \dots dz_n = 0,$$

which completes the proof Lemma 5.6. ☐

Proof of Proposition 5.2. This is a direct consequence of Lemmas 5.3 and 5.6 and (19). ☐

Step 2. Define

$$\begin{aligned} D^R_\beta &:= \{z_1, \dots, z_n \in [R^{a_1}, R^{b_1}]^{n_1} \times \dots \times [R^{a_k}, R^{b_k}]^{n_k}, \text{ s.t. if } z_0 := 0 \\ &\qquad \forall i \neq j \in \{0, \dots, n\},\ |z_i - z_j| \ge \beta\} \\ I^R_\beta &:= \frac{1}{\log(R)^n} \int_{D^R_\beta} \mu^{z,1}(c(z)) dz_1 \dots dz_n, \end{aligned}$$

Using scaling (see Proposition 3.4) and a change of variables, we have:

$$I^R_\beta = \frac{R^n}{\log(R)^n} \int_{C^R_\beta} \mu^{z,R}(c(z)) dz_1 \dots dz_n. \tag{32}$$

Recall that $c(z)$ is a partition of order n, so from Theorem 1.3, we have:

$$\left| I^R_\beta - \frac{1}{\log(R)^n} \int_{C^R_\beta} F(c(z))\ dz_1 \dots dz_n \right| \le \frac{f^n(\beta)}{\log(R)^n} \int_{C^R_\beta} F(c(z))\ dz_1 \dots dz_n$$

and $f^n(\beta) \underset{\beta\to\infty}{\longrightarrow} 0$. By taking successive limits, first $R \to \infty$ and then $\beta \to \infty$, using Proposition 5.2

$$\lim_{\beta\to\infty} \lim_{R\to\infty} I^R_\beta = \prod_{i=1}^k n_i!\ b_i^{n_i - 1}(b_i - a_i). \tag{33}$$

Step 3. The aim of this step is to show that we can now approximate $\mathbb{E}\left(\prod_{i=1}^k \vartheta^R([a_i, b_i])^{n_i}\right)$ by I^R_β. In fact, I^R_β can be obtained from $\mathbb{E}\left(\prod_{i=1}^k \vartheta^R([a_i, b_i])^{n_i}\right)$ by removing a small fraction of the integration domain (see (16)). More precisely we will show that

Lemma 5.7.

$$\forall \beta \ge 1,\ \lim_{R\to\infty} \left(\mathbb{E}\left(\prod_{i=1}^k \vartheta^R([a_i, b_i])^{n_i} \right) - I^R_\beta \right) = 0.$$

Proof. We fix $k \in \mathbb{N},\ n_1, \ldots, n_k \in \mathbb{N},\ n = n_1 + \ldots + n_k,\ a_1, \ldots a_k \in [0,1], b_1, \ldots, b_k \in [0,1],\ a_1 < b_1 < a_2 < b_2 \ldots < a_k < b_k,\ \beta \geq 1$.

Let us define

$$\dot{\Delta}_\beta^R := \{z_1, \ldots, z_n \in [R^{a_1}, R^{b_1}]^{n_1} \times \ldots \times [R^{a_k}, R^{b_k}]^{n_k}, \text{ such that if } z_0 := 0, \\ \exists\, i, j \in \{0, \ldots, n\},\ |z_i - z_j| < \beta\}.$$

We have

$$\mathbb{E}\left(\prod_{i=1}^{k} \vartheta^R([a_i, b_i])^{n_i}\right) - I_\beta^R = \frac{1}{\log(R)^n} \int_{\dot{\Delta}_\beta^R} \mu^{z,1}(c(z))\ dz_1 \ldots dz_n.$$

Lemma 5.7 can be reformulated as follows

$$\lim_{R \to \infty} \frac{1}{\log(R)^n} \int_{\dot{\Delta}_\beta^R} \mu^{z,1}(c(z))\ dz_1 \ldots dz_n \ = \ 0.$$

By symmetry, proving this result reduces to proving that

$$\lim_{R \to \infty} \frac{1}{\log(R)^n} \int_{\Delta_\beta^R} \mu^{z,1}(c(z))\ dz_1 \ldots dz_n \ = \ 0,$$

where

$$\Delta_\beta^R := \dot{\Delta}_\beta^R \bigcap \{z_1, \ldots, z_n,\ z_0 := 0 < z_1 < \ldots < z_n\}.$$

Let $\mathbb{S}$ be the set of all subsets of $[n]$ containing at most $n-1$ elements. For $S \in \mathbb{S}$, define

$$\Delta_{\beta,S}^R = \Delta_\beta^R \bigcap \{z_1, \ldots, z_n :\ \forall i \in S,\ |z_i - z_{i-1}| \geq \beta,\ \forall i \notin S,\ |z_i - z_{i-1}| < \beta\}.$$

in such a way that

$$\Delta_\beta^R = \bigcup_{S \in \mathbb{S}} \Delta_{\beta,S}^R.$$

It follows that

$$\begin{aligned}\int_{\Delta_\beta^R} \mu^{z,1}(c(z))\ dz_1 \ldots dz_n \ &= \ \sum_{S \in \mathbb{S}} \int_{\Delta_{\beta,S}^R} \mu^{z,1}(c(z))\ dz_1 \ldots dz_n \\ &\leq \ \sum_{S \in \mathbb{S}} \int_{\Delta_{\beta,S}^R} \mu^{z,1}(\pi_S)\ dz_1 \ldots dz_n\end{aligned}$$

where $\forall S \in \mathbb{S}$, $\pi_S = \{\pi \in \mathcal{P}_z,\ \forall i, j \in S,\ \ z_i \sim_\pi z_j\}$ (and where the inequality follows from the fact that $c(z) \in \pi_S$). We define $z^S := \{z_i,\ i \in S\}$. Proposition 2.2 gives:

$$\mu^{z,1}(\pi_S) = \mu^{z^S,1}(c(z^S)).$$

Define

$$\bar{\Delta}_{\beta,S}^R := \{(z_i)_{i \in S} :\ \exists\ (z_j)_{j \in \{1, \ldots, n\} \setminus S},\ \ (z_1, \ldots, z_n) \in \Delta_{\beta,S}^R\}.$$

We let the reader convince herself that, for any $S \in \mathbb{S}$ there exists $m_1, \ldots, m_k \in \mathbb{N},\ m_1 + \ldots + m_k = |S|$, such that $\bar{\Delta}_{\beta,S}^R$ can be rewritten as

$$\bar{\Delta}_{\beta,S}^R \ = \{\bar{z}_1, \ldots \bar{z}_{|S|} \in [R^{a_1 - 1}, R^{b_1 - 1}]^{m_1} \times \ldots \times [R^{a_k - 1}, R^{b_k - 1}]^{m_k} \ : \\ \bar{z}_0 := 0 \leq \bar{z}_1 \leq \ldots \leq \bar{z}_{|S|} \text{ and } \forall\ i, j \in S,\ i \neq j,\ |\bar{z}_i - \bar{z}_j| > \beta\}.$$

This allows us to rewrite the previous inequality as

$$
\begin{aligned}
\int_{\Delta_\beta^R} \mu^{z,1}(c(z))\ dz_1 \dots dz_n \ &\leq \ \sum_{S\in\mathbb{S}} \int_{\Delta_{\beta,S}^R} \mu^{z^S,1}(c(z^S)) \underbrace{dz_1 \dots dz_i \dots}_{i\in S} \ \underbrace{\dots dz_j \dots}_{j\notin S} \\
&= \ \sum_{S\in\mathbb{S}} \int_{\bar\Delta_{\beta,S}^R} \mu^{z^S,1}(c(\bar z)) d\bar z_1 \dots d\bar z_{|S|} \left(\prod_{j\notin S} \int_{z_{j-1}}^{z_{j-1}+\beta} dz_j \right) \\
&= \ \beta^{n-|S|} \sum_{S\in\mathbb{S}} \int_{\bar\Delta_{\beta,S}^R} \mu^{z^S,1}(c(\bar z)) d\bar z_1 \dots d\bar z_{|S|} \qquad (34)
\end{aligned}
$$

where $\bar z = (\bar z_0, \dots, \bar z_{|S|})$ and $\bar z_0 = 0$. From (33), we have

$$
\lim_{R\to\infty} \frac{1}{\log(R)^{|S|}} \int_{\bar\Delta_{\beta,S}^R} \mu^{z^S,1}(c(\bar z)) d\bar z_1 \dots d\bar z_{|S|} \ = \prod_{i\in[k],\ m_i\neq 0} b_i^{m_i-1}(b_i - a_i).
$$

As $|S| < n$,

$$
\lim_{R\to\infty} \frac{1}{\log(R)^n} \sum_{S\in\mathbb{S}} \int_{\bar\Delta_{\beta,S}^R} \mu^{z^S,1}(c(\bar z)) d\bar z_1 \dots d\bar z_{|S|} \ = \ 0,
$$

which combined with (34) concludes the proof of the Lemma 5.7. □

Step 4. Conclusion. Combining (33) and with Lemma 5.7 (Step 3), we have proved that for every k-tuple of disjoint intervals $\{[a_i,b_i]\}_{i=1}^k$ and any k-tuple of integers $\{n_i\}_{i=1}^k$

$$
\lim_{R\to\infty} \mathbb{E}\left(\prod_{i=1}^k \vartheta^R([a_i,b_i])^{n_i} \right) \ = \ \prod_{i=1}^k n_i!\ b_i^{n_i-1}(b_i - a_i).
$$

So, using Lemma 5.1, ϑ^R converges to ϑ^∞ in distribution in the weak topology. In particular, we have

$$
\mathcal{L}_R(0) = \vartheta^R[0,1] + \frac{1}{\log(R)} \int_{[0,1]} \mathbb{1}_{\{x\sim_\pi 0\}} dx
$$

and

$$
\frac{1}{\log(R)} \int_{[0,1]} \mathbb{1}_{\{x\sim_\pi 0\}} dx \ \leq \ \frac{1}{\log(R)} \underset{R\to\infty}{\longrightarrow} 0
$$

so, using equation (14), we have

$$
\forall n \in \mathbb{N},\ \lim_{R\to\infty} \mathbb{E}(\mathcal{L}_R(0)^n) = \lim_{R\to\infty} \mathbb{E}(\vartheta^R[0,1]^n) = n!
$$

which are the moments of the exponential distribution of parameter 1. As in the proof of Lemma 5.1, using Carleman's condition (for $k=1$), this implies that $\mathcal{L}_R(0)$ converges in distribution to an exponential distribution of parameter 1.

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