

## Linear Dependence: The Superfluous-Vector Lemma

Grow and Shrink algorithms both test whether a vector is superfluous in spanning a vector space  $\mathcal{V}$ . Need a criterion for superfluity.

**Superfluous-Vector Lemma:** For any set  $S$  and any vector  $\mathbf{v} \in S$ , if  $\mathbf{v}$  can be written as a linear combination of the other vectors in  $S$  then  $\text{Span}(S - \{\mathbf{v}\}) = \text{Span } S$

**Proof:** Let  $S = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ . Suppose  $\mathbf{v}_n = \alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \dots + \alpha_{n-1} \mathbf{v}_{n-1}$

*To show:* every vector in  $\text{Span } S$  is also in  $\text{Span}(S - \{\mathbf{v}_n\})$ .

Every vector  $\mathbf{v}$  in  $\text{Span } S$  can be written as  $\mathbf{v} = \beta_1 \mathbf{v}_1 + \beta_2 \mathbf{v}_2 + \dots + \beta_n \mathbf{v}_n$

Substituting for  $\mathbf{v}_n$ , we obtain

$$\begin{aligned}\mathbf{v} &= \beta_1 \mathbf{v}_1 + \beta_2 \mathbf{v}_2 + \dots + \beta_n (\alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \dots + \alpha_{n-1} \mathbf{v}_{n-1}) \\ &= (\beta_1 + \beta_n \alpha_1) \mathbf{v}_1 + (\beta_2 + \beta_n \alpha_2) \mathbf{v}_2 + \dots + (\beta_{n-1} + \beta_n \alpha_{n-1}) \mathbf{v}_{n-1}\end{aligned}$$

which shows that an arbitrary vector in  $\text{Span } S$  can be written as a linear combination of vectors in  $S - \{\mathbf{v}_n\}$  and is therefore in  $\text{Span}(S - \{\mathbf{v}_n\})$ .

QED

## Defining linear dependence

**Definition:** Vectors  $\mathbf{v}_1, \dots, \mathbf{v}_n$  are *linearly dependent* if the zero vector can be written as a **nontrivial** linear combination of the vectors:

$$\mathbf{0} = \alpha_1 \mathbf{v}_1 + \dots + \alpha_n \mathbf{v}_n$$

In this case, we refer to the linear combination as a *linear dependency* in  $\mathbf{v}_1, \dots, \mathbf{v}_n$ .

On the other hand, if the *only* linear combination that equals the zero vector is the trivial linear combination, we say  $\mathbf{v}_1, \dots, \mathbf{v}_n$  are linearly *independent*.

**Example:** The vectors  $[1, 0, 0]$ ,  $[0, 2, 0]$ , and  $[2, 4, 0]$  are linearly dependent, as shown by the following equation:

$$2 [1, 0, 0] + 2 [0, 2, 0] - 1 [2, 4, 0] = [0, 0, 0]$$

*Therefore:*

$2 [1, 0, 0] + 2 [0, 2, 0] - 1 [2, 4, 0]$  is a linear dependency in  $[1, 0, 0]$ ,  $[0, 2, 0]$ ,  $[2, 4, 0]$ .

## Linear dependence

**Example:** The vectors  $[1, 0, 0]$ ,  $[0, 2, 0]$ , and  $[0, 0, 4]$  are linearly independent.

*How do we know?*

Easy since each vector has a nonzero entry where the others have zeroes.

Consider any linear combination

$$\alpha_1 [1, 0, 0] + \alpha_2 [0, 2, 0] + \alpha_3 [0, 0, 4]$$

This equals  $[\alpha_1, 2\alpha_2, 4\alpha_3]$

If this is the zero vector, it must be that  $\alpha_1 = \alpha_2 = \alpha_3 = 0$

That is, the linear combination is trivial.

We have shown the only linear combination that equals the zero vector is the trivial linear combination.

## Linear dependence in relation to other questions

How can we tell if vectors  $\mathbf{v}_1, \dots, \mathbf{v}_n$  are linearly dependent?

**Definition:** Vectors  $\mathbf{v}_1, \dots, \mathbf{v}_n$  are *linearly dependent* if the zero vector can be written as a nontrivial linear combination  $\mathbf{0} = \alpha_1 \mathbf{v}_1 + \dots + \alpha_n \mathbf{v}_n$

By linear-combinations definition,  $\mathbf{v}_1, \dots, \mathbf{v}_n$  are linearly dependent iff there is a

nonzero vector  $\begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix}$  such that  $\begin{bmatrix} \mathbf{v}_1 & \cdots & \mathbf{v}_n \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} = \mathbf{0}$

Therefore,  $\mathbf{v}_1, \dots, \mathbf{v}_n$  are linearly dependent iff the null space of the matrix is nontrivial.

This shows that the question

How can we tell if vectors  $\mathbf{v}_1, \dots, \mathbf{v}_n$  are linearly dependent?

is the same as a question we asked earlier:

How can we tell if the null space of a matrix is trivial?

# Linear dependence in relation to other questions

The question

How can we tell if vectors  $\mathbf{v}_1, \dots, \mathbf{v}_n$  are linearly dependent?

is the same as a question we asked earlier:

How can we tell if the null space of a matrix is trivial?

**Recall:** *solution set of a homogeneous linear system*

$$\begin{array}{rcl} \mathbf{a}_1 \cdot \mathbf{x} & = & 0 \\ & \vdots & \\ \mathbf{a}_m \cdot \mathbf{x} & = & 0 \end{array}$$

*is the null space of matrix*  $\left[ \begin{array}{c} \mathbf{a}_1 \\ \hline \vdots \\ \hline \mathbf{a}_m \end{array} \right]$ .

So question is same as:

How can we tell if the solution set of a homogeneous linear system is trivial?

## Linear dependence in relation to other questions

The question

How can we tell if vectors  $\mathbf{v}_1, \dots, \mathbf{v}_n$  are linearly dependent?

is the same as a question we asked earlier:

How can we tell if the null space of a matrix is trivial?

is the same as :

How can we tell if the solution set of a homogeneous linear system is trivial?

Recall:

If  $\mathbf{u}_1$  is a solution to a linear system  $\mathbf{a}_1 \cdot \mathbf{x} = \beta_1, \dots, \mathbf{a}_m \cdot \mathbf{x} = \beta_m$  then

$$\{\text{solutions to linear system}\} = \{\mathbf{u}_1 + \mathbf{v} : \mathbf{v} \in \mathcal{V}\}$$

where  $\mathcal{V} = \{\text{solutions to corresponding homogeneous linear system}$   
 $\mathbf{a}_1 \cdot \mathbf{x} = 0, \dots, \mathbf{a}_m \cdot \mathbf{x} = 0\}$

Thus the question is the same as:

How can we tell if a solution  $\mathbf{u}_1$  to a linear system is the *only* solution?

# Linear dependence and null space

The question

How can we tell if vectors  $\mathbf{v}_1, \dots, \mathbf{v}_n$  are linearly dependent?

is the same as:

How can we tell if the null space of a matrix is trivial?

is the same as:

How can we tell if the solution set of a homogeneous linear system is trivial?

is the same as:

How can we tell if a solution  $\mathbf{u}_1$  to a linear system is the *only* solution?

# Linear dependence

Answering these questions requires an algorithm.

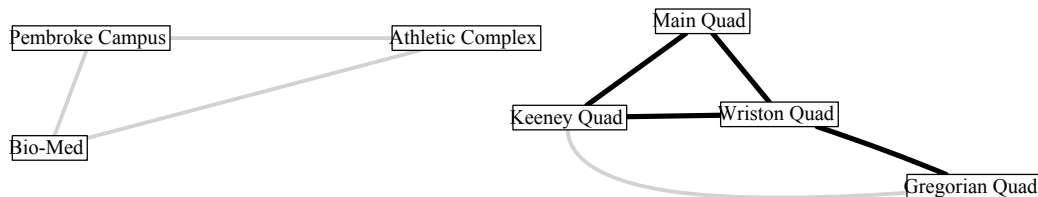
**Computational Problem:** *Testing linear dependence*

- ▶ *input:* a list  $[\mathbf{v}_1, \dots, \mathbf{v}_n]$  of vectors
- ▶ *output:* DEPENDENT if the vectors are linearly dependent, and INDEPENDENT otherwise.

We'll see two algorithms later.



## Linear dependence in *Minimum Spanning Forest*



We can get the zero vector by adding together vectors corresponding to edges that form a cycle: in such a sum, for each entry  $x$ , there are exactly two vectors having 1's in position  $x$ .

**Example:** the vectors corresponding to

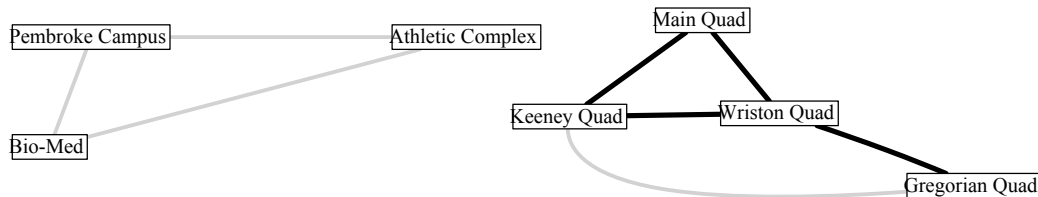
$\{\text{Main, Wriston}\}, \{\text{Main, Keeney}\}, \{\text{Keeney, Wriston}\},$

are as follows:

Pembroke	Athletic	Bio-Med	Main	Keeney	Wriston	Gregorian
			1	1		
				1	1	
			1		1	

The sum of these vectors is the zero vector.

## Linear dependence in *Minimum Spanning Forest*



Sum of vectors corresponding to edges forming a cycle can make a zero vector.  
Therefore if a subset of  $S$  form a cycle then  $S$  is linearly dependent.

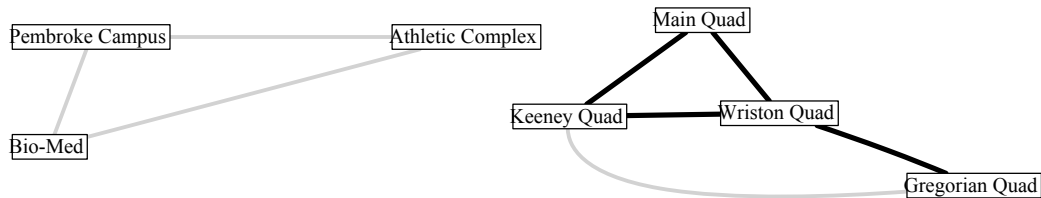
**Example:** The vectors corresponding to

$\{\text{Main, Keeney}\}$ ,  $\{\text{Main, Wriston}\}$ ,  $\{\text{Keeney, Wriston}\}$ ,  $\{\text{Wriston, Gregorian}\}$   
are linearly dependent because these edges include a cycle.

The zero vector is equal to the nontrivial linear combination

			Pembroke	Athletic	Bio-Med	Main	Keeney	Wriston	Gregorian
	1	*				1	1		
+	1	*				1		1	
+	1	*					1	1	
+	0	*						1	1

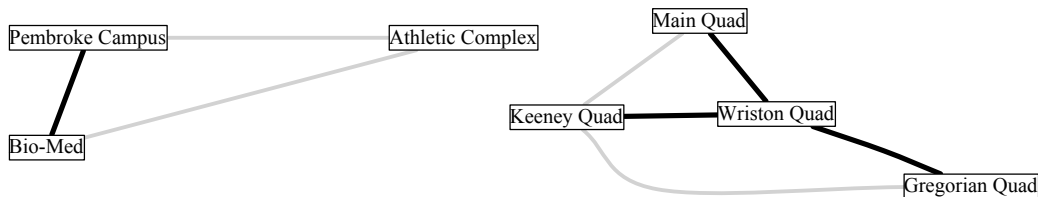
## Linear dependence in *Minimum Spanning Forest*



If a subset of  $S$  form a cycle then  $S$  is linearly dependent.

On the other hand, if a set of edges contains no cycle (i.e. is a forest) then the corresponding set of vectors is linearly independent.

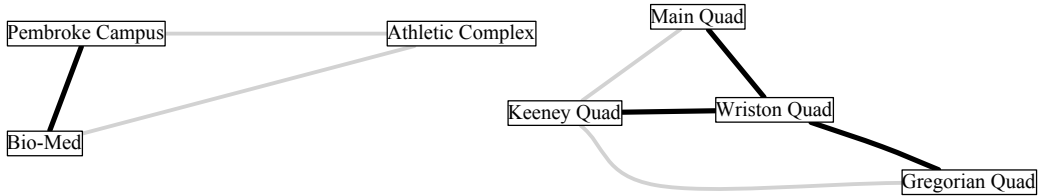
## Linear dependence in *Minimum Spanning Forest*



If a subset of  $S$  form a cycle then  $S$  is linearly dependent.

On the other hand, if a set of edges contains no cycle (i.e. is a forest) then the corresponding set of vectors is linearly independent.

## “Quiz”



Which edges are spanned?

Which sets are linearly dependent?

## Analyzing the Grow algorithm

```
def GROW( $\mathcal{V}$ )  
   $S = \emptyset$   
  repeat while possible:  
    find a vector  $\mathbf{v}$  in  $\mathcal{V}$  that is not in  $\text{Span } S$ , and put it in  $S$ .
```

**Grow-Algorithm Corollary:** The vectors obtained by the Grow algorithm are linearly independent.

In graphs, this means that the solution obtained by the Grow algorithm has no cycles (is a forest).

## Analyzing the Grow algorithm

**Grow-Algorithm Corollary:** The vectors obtained by the Grow algorithm are linearly independent.

**Proof:** For  $n = 1, 2, \dots$ , let  $\mathbf{v}_n$  be the vector added to  $S$  in the  $n^{\text{th}}$  iteration of the Grow algorithm. We show by induction that  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$  are linearly independent.

For  $n = 0$ , there are no vectors, so the claim is trivially true.

Assume the claim is true for  $n = k - 1$ . We prove it for  $n = k$ .

The vector  $\mathbf{v}_k$  added to  $S$  in the  $k^{\text{th}}$  iteration is not in the span of  $\mathbf{v}_1, \dots, \mathbf{v}_{k-1}$ .

Therefore, by the Linear-Dependence Lemma, for any coefficients  $\alpha_1, \dots, \alpha_k$  such that

$$\mathbf{0} = \alpha_1 \mathbf{v}_1 + \dots + \alpha_{k-1} \mathbf{v}_{k-1} + \alpha_k \mathbf{v}_k$$

it must be that  $\alpha_k$  equals zero. We may therefore write

$$\mathbf{0} = \alpha_1 \mathbf{v}_1 + \dots + \alpha_{k-1} \mathbf{v}_{k-1}$$

By claim for  $n = k - 1$ ,  $\mathbf{v}_1, \dots, \mathbf{v}_{k-1}$  are linearly independent, so  $\alpha_1 = \dots = \alpha_{k-1} = 0$

The linear combination of  $\mathbf{v}_1, \dots, \mathbf{v}_k$  is *trivial*. We have proved that  $\mathbf{v}_1, \dots, \mathbf{v}_k$  are linearly independent. This proves the claim for  $n = k$ . QED

## Analyzing the Shrink algorithm

def SHRINK( $\mathcal{V}$ )

$S$  = some finite set of vectors that spans  $\mathcal{V}$

    repeat while possible:

        find a vector  $\mathbf{v}$  in  $S$  such that  $\text{Span}(S - \{\mathbf{v}\}) = \mathcal{V}$ , and remove  $\mathbf{v}$  from  $S$ .

**Shrink-Algorithm Corollary:** The vectors obtained by the Shrink algorithm are linearly independent.

In graphs, this means that the Shrink algorithm outputs a solution that is a forest.

*Recall:*

**Superfluous-Vector Lemma** For any set  $S$  and any vector  $\mathbf{v} \in S$ , if  $\mathbf{v}$  can be written as a linear combination of the other vectors in  $S$  then  
 $\text{Span}(S - \{\mathbf{v}\}) = \text{Span } S$



## Analyzing the Shrink algorithm

**Shrink-Algorithm Corollary:** The vectors obtained by the Shrink algorithm are linearly independent.

**Proof:** Let  $S = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$  be the set of vectors obtained by the Shrink algorithm. Assume for a contradiction that the vectors are linearly dependent.

Then  $\mathbf{0}$  can be written as a nontrivial linear combination

$$\mathbf{0} = \alpha_1 \mathbf{v}_1 + \dots + \alpha_n \mathbf{v}_n$$

where at least one of the coefficients is nonzero.

Let  $\alpha_i$  be one of the nonzero coefficients.

By the Linear-Dependence Lemma,  $\mathbf{v}_i$  can be written as a linear combination of the other vectors.

Hence by the Superfluous-Vector Lemma,  $\text{Span}(S - \{\mathbf{v}_i\}) = \text{Span } S$ ,

so the Shrink algorithm should have removed  $\mathbf{v}_i$ .

QED