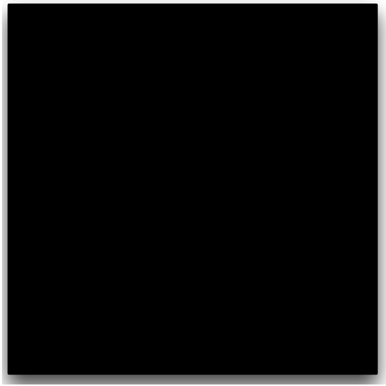
Starting to peek inside the black box

So far `solve(A, b)` is a black box.



With Gaussian elimination, we begin to find out what's inside.

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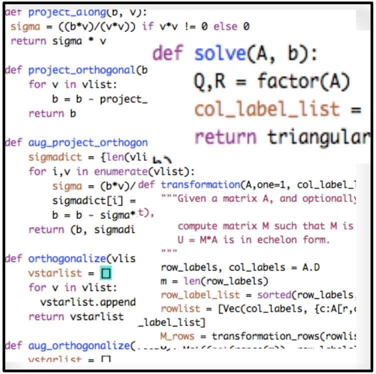

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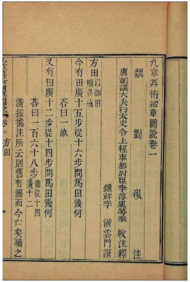
Gaussian Elimination: Origins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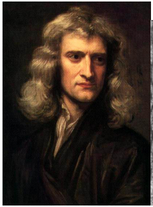
鍾祥李 潢雲門譔
方田 以御田疇界域
今有田廣十五步從十六步問爲田幾何
荅曰一畝
又有田廣十二步從十四步問爲田幾何
荅曰一百六十八步 圖從十四廣十二
潢按據注所云則舊有圖而今亡矣補之

Method illustrated in Chapter Eight of a Chinese text, *The Nine Chapters on the Mathematical Art*, that was written roughly two thousand years ago.



Rediscovered in Europe by Isaac Newton (England) and Michel Rolle (France)

Gauss called the method *eliminiationem vulgarem* ("common elimination")
Gauss adapted the method for another problem (one we study soon) and developed notation.

Gaussian elimination: Uses

- *Finding a basis for the span of given vectors*. This additionally gives us an algorithm for rank and therefore for testing linear dependence.
- *Solving a matrix equation*,which is the same as *expressing a given vector as a linear combination of other given vectors*, which is the same as *solving a system of linear equations*
- *Finding a basis for the null space of a matrix*, which is the same as *finding a basis for the solution set of a homogeneous linear system*, which is also relevant to representing the solution set of a general linear system.

Echelon form

Echelon form a generalization of triangular matrices

Example: $\begin{bmatrix} 0 & 2 & 3 & 0 & 5 & 6 \\ 0 & 0 & 1 & 0 & 3 & 4 \\ 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 9 \end{bmatrix}$

Note that

- the first nonzero entry in row 0 is in column 1,
- the first nonzero entry in row 1 is in column 2,
- the first nonzero entry in row 2 is in column 4, and
- the first nonzero entry in row 4 is in column 5.

Definition: An $m \times n$ matrix A is in *echelon form* if it satisfies the following condition: for any row, if that row's first nonzero entry is in position k then every previous row's first nonzero entry is in some position less than k.

Echelon form

Definition: An $m \times n$ matrix A is in *echelon form* if it satisfies the following condition: for any row, if that row's first nonzero entry is in position k then every previous row's first nonzero entry is in some position less than k.

This definition implies that, as you iterate through the rows of A, the first nonzero entries per row move strictly right, forming a sort of staircase that descends to the right.

2	1	0	4	1	3	9	7
0	6	0	1	3	0	4	1
0	0	0	0	2	1	3	2
0	0	0	0	0	0	0	1

$$\begin{bmatrix} 0 & 2 & 3 & 0 & 5 & 6 \\ 0 & 0 & 1 & 0 & 3 & 4 \\ 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 9 \end{bmatrix} \qquad \begin{bmatrix} 4 & 1 & 3 & 0 \\ 0 & 3 & 0 & 1 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 9 \end{bmatrix}$$

Echelon form

Definition: An $m \times n$ matrix A is in *echelon form* if it satisfies the following condition: for any row, if that row's first nonzero entry is in position k then any previous row's first nonzero entry is in some position less than k.

If a row of a matrix in echelon form is all zero then every subsequent row must also be all zero, e.g.

$$\begin{bmatrix} 0 & 2 & 3 & 0 & 5 & 6 \\ 0 & 0 & 1 & 0 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Uses of echelon form

What good is it having a matrix in echeleon form?

Lemma: If a matrix is in echelon form, the nonzero rows form a basis for the row space.

For example, a basis for the row space of

$$\begin{bmatrix} 0 & 2 & 3 & 0 & 5 & 6 \\ 0 & 0 & 1 & 0 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

is $\{[0, 2, 3, 0, 5, 6], [0, 1, 0, 3, 4]\}$.
In particular, if every row is nonzero, as in each of the matrices

$$\begin{bmatrix} 0 & 2 & 3 & 0 & 5 & 6 \\ 0 & 0 & 1 & 0 & 3 & 4 \\ 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 9 \end{bmatrix}, \begin{bmatrix} 2 & 1 & 0 & 4 & 1 & 3 & 9 & 7 \\ 0 & 6 & 0 & 1 & 3 & 0 & 4 & 1 \\ 0 & 0 & 0 & 0 & 2 & 1 & 3 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 4 & 1 & 3 & 0 \\ 0 & 3 & 0 & 1 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 9 \end{bmatrix}$$

then the rows form a basis of the row space.

Uses of echelon form

Lemma: If matrix is in echelon form, the nonzero rows form a basis for row space.

It is obvious that the nonzero rows span the row space. We need only show that these vectors are linearly independent. We prove it using the Grow algorithm:

def $\textsc{Grow}(\mathcal{V})$
  $S = \emptyset$
  repeat while possible:
    find a vector $\mathbf{v}$ in $\mathcal{V}$ that is not in Span S, and put it in S

$$\begin{bmatrix} 4 & 1 & 3 & 0 \\ 0 & 3 & 0 & 1 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 9 \end{bmatrix}$$

We run the Grow algorithm, adding rows of matrix in reverse order to S:

- Since Span $\emptyset$ does not include $[0, 0, 0, 9]$, the algorithm adds this vector to S.
- Now $S = \{[0, 0, 0, 9]\}$. Every vector in Span S has zeroes in positions 0, 1, 2, so Span S does not contain $[0, 0, 1, 7]$, so the algorithm adds this vector to S.
- Now $S = \{[0, 0, 0, 9], [0, 0, 1, 7]\}$. Every vector in Span S has zeroes in positions 0, 1, so Span S does not contain $[0, 3, 0, 1]$, so the algorithm adds it.
- Now $S = \{[0, 0, 0, 9], [0, 0, 1, 7], [0, 3, 0, 1]\}$. Every vector in Span S has a zero in position 0, so Span S does not contain $[4, 1, 3, 0]$, so the algorithm adds it, and we are done.