

Finding basis for null space using orthogonal complement

To find basis for null space of an $m \times n$ matrix $A = \left[\begin{array}{c} \mathbf{a}_1 \\ \vdots \\ \mathbf{a}_m \end{array} \right]$,

find orthogonal complement of $\text{Span} \{ \mathbf{a}_1, \dots, \mathbf{a}_m \}$ in \mathbb{R}^n :

- ▶ Let $\mathbf{e}_1, \dots, \mathbf{e}_n$ be the standard basis vectors \mathbb{R}^n .
- ▶ Find the nonzero vectors among $\text{orthogonalize}([\mathbf{a}_1, \dots, \mathbf{a}_m, \mathbf{e}_1, \dots, \mathbf{e}_n])$

Algorithm for finding basis for null space

Another approach to find basis of null space of a matrix: Write matrix in terms of its columns $\mathbf{v}_0, \dots, \mathbf{v}_n$.

Here's the matrix equation expressing original vectors in terms of starred vectors:

$$\begin{bmatrix} \mathbf{v}_0 & \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \end{bmatrix} = \begin{bmatrix} \mathbf{v}_0^* & \mathbf{v}_1^* & \mathbf{v}_2^* & \cdots & \mathbf{v}_n^* \end{bmatrix} \begin{bmatrix} 1 & \alpha_{01} & \alpha_{02} & & \alpha_{0n} \\ & 1 & \alpha_{12} & & \alpha_{1n} \\ & & 1 & & \alpha_{2n} \\ & & & \ddots & \\ & & & & 1 \end{bmatrix}$$

Can transform this to express starred vectors in terms of original vectors.

$$\begin{bmatrix} \mathbf{v}_0 & \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \end{bmatrix} \begin{bmatrix} 1 & \alpha_{01} & \alpha_{02} & & \alpha_{0n} \\ & 1 & \alpha_{12} & & \alpha_{1n} \\ & & 1 & & \alpha_{2n} \\ & & & \ddots & \\ & & & & 1 \end{bmatrix}^{-1} = \begin{bmatrix} \mathbf{v}_0^* & \mathbf{v}_1^* & \mathbf{v}_2^* & \cdots & \mathbf{v}_n^* \end{bmatrix}$$

Basis for null space

$$\begin{aligned}
 & \left[\begin{array}{c|c|c|c|c|c|c} \mathbf{v}_0 & \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{v}_4 & \mathbf{v}_5 & \mathbf{v}_6 \end{array} \right] \\
 &= \left[\begin{array}{c|c|c|c|c|c|c} \mathbf{v}_0^* & \mathbf{v}_1^* & \mathbf{v}_2^* & \mathbf{v}_3^* & \mathbf{v}_4^* & \mathbf{v}_5^* & \mathbf{v}_6^* \end{array} \right] \begin{bmatrix} 1 & \alpha_{01} & \alpha_{02} & \alpha_{03} & \alpha_{04} & \alpha_{05} & \alpha_{06} \\ & 1 & \alpha_{12} & \alpha_{13} & \alpha_{14} & \alpha_{15} & \alpha_{16} \\ & & 1 & \alpha_{23} & \alpha_{24} & \alpha_{25} & \alpha_{26} \\ & & & 1 & \alpha_{34} & \alpha_{35} & \alpha_{36} \\ & & & & 1 & \alpha_{45} & \alpha_{46} \\ & & & & & 1 & \alpha_{56} \\ & & & & & & 1 \end{bmatrix}
 \end{aligned}$$

Suppose \mathbf{v}_2^* , \mathbf{v}_4^* , and \mathbf{v}_5^* are (approximately) zero vectors.

- ▶ Corresponding columns of inverse triangular matrix are nonzero vectors of the null space of the leftmost matrix.
- ▶ These columns are clearly linearly independent so they span a basis of dimension 3.
- ▶ Rank-Nullity Theorem shows that the null space has dimension 3 so these columns are a basis for null space.

Basis for null space

$$\begin{bmatrix} \mathbf{v}_0 & \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{v}_4 & \mathbf{v}_5 & \mathbf{v}_6 \end{bmatrix} \begin{bmatrix} 1 & \alpha_{01} & \alpha_{02} & \alpha_{03} & \alpha_{04} & \alpha_{05} & \alpha_{06} \\ & 1 & \alpha_{12} & \alpha_{13} & \alpha_{14} & \alpha_{15} & \alpha_{16} \\ & & 1 & \alpha_{23} & \alpha_{24} & \alpha_{25} & \alpha_{26} \\ & & & 1 & \alpha_{34} & \alpha_{35} & \alpha_{36} \\ & & & & 1 & \alpha_{45} & \alpha_{46} \\ & & & & & 1 & \alpha_{56} \\ & & & & & & 1 \end{bmatrix}^{-1} \\
 = \begin{bmatrix} \mathbf{v}_0^* & \mathbf{v}_1^* & \mathbf{v}_2^* & \mathbf{v}_3^* & \mathbf{v}_4^* & \mathbf{v}_5^* & \mathbf{v}_6^* \end{bmatrix}$$

Suppose \mathbf{v}_2^* , \mathbf{v}_4^* , and \mathbf{v}_5^* are (approximately) zero vectors.

- ▶ Corresponding columns of inverse triangular matrix are nonzero vectors of the null space of the leftmost matrix.
- ▶ These columns are clearly linearly independent so they span a basis of dimension 3.
- ▶ Rank-Nullity Theorem shows that the null space has dimension 3 so these columns are a basis for null space.

Basis for null space

$$\begin{aligned}
 & \left[\begin{array}{c|c|c|c|c|c|c} \mathbf{v}_0 & \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{v}_4 & \mathbf{v}_5 & \mathbf{v}_6 \end{array} \right] \\
 &= \left[\begin{array}{c|c|c|c|c|c|c} \mathbf{v}_0^* & \mathbf{v}_1^* & \mathbf{v}_2^* & \mathbf{v}_3^* & \mathbf{v}_4^* & \mathbf{v}_5^* & \mathbf{v}_6^* \end{array} \right] \begin{bmatrix} 1 & \alpha_{01} & \alpha_{02} & \alpha_{03} & \alpha_{04} & \alpha_{05} & \alpha_{06} \\ & 1 & \alpha_{12} & \alpha_{13} & \alpha_{14} & \alpha_{15} & \alpha_{16} \\ & & 1 & \alpha_{23} & \alpha_{24} & \alpha_{25} & \alpha_{26} \\ & & & 1 & \alpha_{34} & \alpha_{35} & \alpha_{36} \\ & & & & 1 & \alpha_{45} & \alpha_{46} \\ & & & & & 1 & \alpha_{56} \\ & & & & & & 1 \end{bmatrix}
 \end{aligned}$$

```

def find_null_space(A):
    vstarlist = orthogonalize(columns of A)
    find upper triangular matrix  $T$  such that
         $A$  equals (matrix with columns vstarlist) *  $T$ 
    return list of columns of  $T^{-1}$  corresponding to zero vectors in vstarlist
    
```

How to find matrix T ? How to find its inverse?

Augmenting orthogonalize(vlist)

We will write a procedure `aug_orthogonalize(vlist)` with the following spec:

- ▶ *input*: a list $[\mathbf{v}_1, \dots, \mathbf{v}_n]$ of vectors
- ▶ *output*: the pair $([\mathbf{v}_1^*, \dots, \mathbf{v}_n^*], [\mathbf{r}_1, \dots, \mathbf{r}_n])$ of lists of vectors such that $\mathbf{v}_1^*, \dots, \mathbf{v}_n^*$ are mutually orthogonal vectors whose span equals $\text{Span}\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$, and

$$\left[\begin{array}{c|c|c} \mathbf{v}_1 & \cdots & \mathbf{v}_n \end{array} \right] = \left[\begin{array}{c|c|c} \mathbf{v}_1^* & \cdots & \mathbf{v}_n^* \end{array} \right] \left[\begin{array}{c|c|c} \mathbf{r}_1 & \cdots & \mathbf{r}_n \end{array} \right]$$

```
def orthogonalize(vlist):  
    vstarlist = []  
    for v in vlist:  
        vstarlist.append(  
            project_orthogonal(v, vstarlist))  
    return vstarlist  
  
def aug_orthogonalize(vlist):  
    vstarlist = []  
    r_vecs = []  
    D = set(range(len(vlist)))  
    for v in vlist:  
        (vstar, alphadict) =  
            aug_project_orthogonal(v, vstarlist)  
        vstarlist.append(vstar)  
        r_vecs.append(Vec(D, alphadict))  
    return vstarlist, r_vecs
```

Using `aug_orthogonalize` to find null space

```
def find_null_space(A):  
    vstarlist = orthogonalize(columns of A)  
    find upper triangular matrix  $T$  such that  
         $A$  equals (matrix with columns vstarlist)  $\ast T$   
    return list of columns of  $T^{-1}$  corresponding to zero vectors in vstarlist
```



```
def find_null_space(A):  
    vstarlist, r_vecs = aug_orthogonalize(columns of A)  
    let  $T$  be matrix with columns given by the vectors of r_vecs  
    return list of columns of  $T^{-1}$  corresponding to zero vectors in vstarlist
```

How to find a column of T^{-1} ?

How to find a column of T^{-1} ?

$$\begin{bmatrix} \mathbf{v}_0 & \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{v}_4 & \mathbf{v}_5 & \mathbf{v}_6 \end{bmatrix} = \begin{bmatrix} \mathbf{v}_0^* & \mathbf{v}_1^* & \mathbf{v}_2^* & \mathbf{v}_3^* & \mathbf{v}_4^* & \mathbf{v}_5^* & \mathbf{v}_6^* \end{bmatrix} \begin{bmatrix} 1 & \alpha_{01} & \alpha_{02} & \alpha_{03} & \alpha_{04} & \alpha_{05} & \alpha_{06} \\ & 1 & \alpha_{12} & \alpha_{13} & \alpha_{14} & \alpha_{15} & \alpha_{16} \\ & & 1 & \alpha_{23} & \alpha_{24} & \alpha_{25} & \alpha_{26} \\ & & & 1 & \alpha_{34} & \alpha_{35} & \alpha_{36} \\ & & & & 1 & \alpha_{45} & \alpha_{46} \\ & & & & & 1 & \alpha_{56} \\ & & & & & & 1 \end{bmatrix}$$

The matrix T is square and upper triangular, with nonzero diagonal elements

$$\begin{bmatrix} T \end{bmatrix} \begin{bmatrix} T^{-1} \end{bmatrix} = \begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix}$$

To find column j of T^{-1} , solve $\begin{bmatrix} T \end{bmatrix} \begin{bmatrix} \mathbf{x} \end{bmatrix} = \mathbf{e}_j$

Use `triangular_solve`