Solving a triangular system of linear equations

How to find solution to this linear system?

$$[1, 0.5, -2, 4] \cdot \mathbf{x} = -8$$

$$[0, 3, 3, 2] \cdot \mathbf{x} = 3$$

$$[0, 0, 1, 5] \cdot \mathbf{x} = -4$$

$$[0, 0, 0, 2] \cdot \mathbf{x} = 6$$

Write $\mathbf{x} = [x_1, x_2, x_3, x_4]$.
System becomes

$$\begin{array}{ccccccccc} 1x_1 & + & 0.5x_2 & - & 2x_3 & + & 4x_4 & = & -8 \\ & & 3x_2 & + & 3x_3 & + & 2x_4 & = & 3 \\ & & & & 1x_3 & + & 5x_4 & = & -4 \\ & & & & & & 2x_4 & = & 6 \end{array}$$

Solving a triangular system of linear equations: Backward substitution

$$
\begin{array}{rcrcrcrcr}
1x_1 & + & 0.5x_2 & - & 2x_3 & + & 4x_4 & = & -8 \\
 & & 3x_2 & + & 3x_3 & + & 2x_4 & = & 3 \\
 & & & & 1x_3 & + & 5x_4 & = & -4 \\
 & & & & & & 2x_4 & = & 6
\end{array}
$$

Solution strategy:

- Solve for x_4 using fourth equation.
- Plug value for x_4 into third equations and solve for x_3.
- Plug values for x_4 and x_3 into second equation and solve for x_2.
- Plug values for x_4, x_3, x_2 into first equation and solve for x_1.

Solving a triangular system of linear equations: Backward substitution

$$
\begin{array}{rcrcrcrcr}
1x_1 & + & 0.5x_2 & - & 2x_3 & + & 4x_4 & = & -8 \\
 & & 3x_2 & + & 3x_3 & + & 2x_4 & = & 3 \\
 & & & & 1x_3 & + & 5x_4 & = & -4 \\
 & & & & & & 2x_4 & = & 6
\end{array}
$$

$$
\begin{array}{lrcl}
 & 2x_4 & = & 6 \\
\text{so} & x_4 & = & 6/2 = 3 \\
 & & & \\
 & 1x_3 & = & -4 - 5x_4 = -4 - 5(3) = -19 \\
\text{so} & x_3 & = & -19/1 = -19 \\
 & & & \\
 & 3x_2 & = & 3 - 3x_3 - 2x_4 = 3 - 2(3) - 3(-19) = 54 \\
\text{so} & x_2 & = & 54/3 = 18 \\
 & & & \\
 & 1x_1 & = & -8 - 0.5x_2 + 2x_3 - 4x_4 = -8 - 4(3) + 2(-19) - 0.5(18) = -67 \\
\text{so} & x_1 & = & -67/1 = -67
\end{array}
$$

Solving a triangular system of linear equations: Backward substitution

Quiz: Solve the following system by hand:

$$\begin{array}{rcrcrcr} 2x_1 & + & 3x_2 & - & 4x_3 & = & 10 \\ & & 1x_2 & + & 2x_3 & = & 3 \\ & & & & 5x_3 & = & 15 \end{array}$$

Solving a triangular system of linear equations: Backward substitution

Quiz: Solve the following system by hand:

$$\begin{array}{ccccccc} 2x_1 & + & 3x_2 & - & 4x_3 & = & 10 \\ & & 1x_2 & + & 2x_3 & = & 3 \\ & & & & 5x_3 & = & 15 \end{array}$$

$$x_3 = 15/5 = 3$$

$$x_2 = 3 - 2x_3 = -3$$

$$x_1 = (10 + 4x_3 - 3x_2)/2 = (10 + 12 + 9)/2 = 31/2$$

Solving a triangular system of linear equations: Backward substitution

Hack to implement backsub using vectors:

- Initialize vector x to zero vector.
- Procedure will populate x entry by entry.
- When it is time to populate x_i, entries $x_{i+1}, x_{i+2}, \ldots, x_n$ will be populated, and other entries will be zero.
- Therefore can use dot-product:
  - Suppose you are computing x_2 using $[0,3,3,2] \cdot [x_1, x_2, x_3, x_4] = 3$
  - So far, vector $\mathtt{x} = [x_1, x_2, x_3, x_4] = [0, 0, -19, 3]$.
  - $x_2 := 3 - ([0,3,3,2] \cdot \mathtt{x})$

```
def triangular_solve(rowlist, b):
    x = zero_vec(rowlist[0].D)
    for i in reversed(range(len(rowlist))):
        x[i] = (b[i] - rowlist[i] * x)/rowlist[i][i]
    return x
```

Solving a triangular system of linear equations: Backward substitution

```
def triangular_solve(rowlist, b):
    x = zero_vec(rowlist[0].D)
    for i in reversed(range(len(rowlist))):
        x[i] = (b[i] - rowlist[i] * x)/rowlist[i][i]
    return x
```

Observations:

- If `rowlist[i][i]` is zero, procedure will raise `ZeroDivisionError`.
- If this never happens, solution found is the *only* solution to the system.

Solving a triangular system of linear equations: Backward substitution

```
def triangular_solve(rowlist, b):
    x = zero_vec(rowlist[0].D)
    for i in reversed(range(len(rowlist))):
        x[i] = (b[i] - rowlist[i] * x)/rowlist[i][i]
    return x
```

Our code only works when vectors in `rowlist` have domain $D = \{0, 1, 2, \ldots, n-1\}$.

For arbitrary domains, need to specify an ordering for which system is "triangular":

```
def triangular_solve(rowlist, label_list, b):
    x = zero_vec(set(label_list))
    for r in reversed(range(len(rowlist))):
        c = label_list[r]
        x[c] = (b[r] - x*rowlist[r])/rowlist[r][c]
    return x
```