Notes on the Seifert-Van Kampen Theorem

Branton Demoss
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I. Prerequisites

A normal subgroup N of a group G, is a subgroup $N \subset G$ such that $n \in N$ if $gng^{-1} \in N, \forall g \in G$. That is, the elements of N are invariant under conjugation by elements of G. For example, note that every subgroup of an abelian group is a normal group.

First isomorphism theorem: Let G and H be groups, and let $\phi : G \to H$ be a homomorphism. It follows that the kernel of ϕ is a normal subgroup of G, the image of ϕ is a subgroup of H, and the image of ϕ is isomorphic to the quotient group G / ker(ϕ). In particular, if ϕ is surjective then H is isomorphic to G / ker(ϕ).

In group theory, a word is any written product of group elements and their inverses. For example, if x, y and z are elements of a group G, then xy, $z^{-1}xzz$ and $y^{-1}zxx^{-1}yz^{-1}$ are words in the set {x, y, z}. The free product of a set of groups G, indexed by α is denoted as: $*_\alpha G_\alpha$, and is the group of all combinations of words of constituent elements of both groups. Note that this operation is not, in general, commutative (it is commutative only when one of the groups is trivial).

Let X be a topological space, and let x_0 be a point in X. We define a set of continuous functions called loops with base point x_0:

$$\{f : [0,1] \to X | f(0) = x_0 = f(1)\}$$

and call the set of equivalence classes of these loops the fundamental group at the point x_0. For path connected spaces the fundamental groups at different points are the same up to isomorphism, and we denote this group as

$$\pi(X)$$

II. Statement of Theorem

Let $\mathcal{X}$ be the union of sets, $\bigcup_\alpha \mathcal{A}_\alpha = \mathcal{X}$ such that $\forall \alpha, \beta$: $\mathcal{A}_\alpha \cap \mathcal{A}_\beta \neq \emptyset$ and $\mathcal{A}_\alpha \cap \mathcal{A}_\beta$ is open and path connected. We claim that $\exists \phi : *_\alpha \pi(A_\alpha) \twoheadrightarrow \pi(X)$

Furthermore, ker(ϕ) = N, such that $N = \langle i_{\alpha\beta}(\omega) i^{-1}_{\beta\alpha}(\omega) \rangle$ with $\omega \in \pi(A_\alpha \cap A_\beta)$ such that

$$i_{\alpha\beta} : \omega \rightarrowtail \pi(A_\alpha) \text{ and } i^{-1}_{\beta\alpha} : \pi(A_\beta) \to \omega$$

And $\phi \rightsquigarrow \pi(X) \simeq *_\alpha \pi(A_\alpha)/N$

Proof that ϕ is surjective:

Given a loop $f : I \to X$ based at x_0 there is a partition $0 = s_1 < s_2 < ... < s_n = 1$ of $[0,1]$ such that each interval $[s_i, s_{i+1}]$ is mapped to a single space A_α. This is because each $s \in [0,1]$ has a neighborhood that is mapped completely to a unique A_α in X. Compactness of I implies that there exists a finite covering by these neighborhoods, such that their boundaries determine our partition.
Denote the A_α containing $f([s_{i-1}, s_i])$ by A_i, and let $f_i = f|_{[s_{i-1}, s_i]}$. Then, $f = f_1 * f_2 * ... * f_n$, where each path given by f_i exists solely in the corresponding A_i.
Since $A_i \cap A_{i+1}$ is path connected by hypothesis, $\exists$ a path g_i from x_0 to $f(s_i) \in A_i \cap A_{i+1}$.
Now it is clear we can write a loop which is a composition (product) of loops which each individually lie entirely in one of the constituent spaces, thus $[f]$ is in the image of ϕ, as it is the product of equivalence classes of loops, which is the image under ϕ of a word in $*_\alpha \pi(A_\alpha)$. Thus ϕ is surjective.

$\square$

It is more difficult to show that ϕ is injective. For further details see, for example, Hatcher's textbook $Algebraic\,Topology$, pp. 45-46

III. Intuition

The essential notion of the Seifert - van Kampen theorem is the fact that the fundamental group of a space can be obtained from the free product of constituent spaces. The free product is obviously always larger, but sometimes equal to the fundamental group of the union of the spaces. In particular, the key idea is to construct a loop in the total space the is a product of loops which indiviually exist solely in one of the spaces. The non-commutativity (except in the trivial case) of the free product tells us that, metaphorically, going north then east isn't always the same as going east then north.

IV. Knots?

We may be able to model a knot as an embedding $K : S^1 \subset S^3$, then look for homeomorphisms between knots to classify "equivalent" knots. We might look for equivalence classes of knots, and in joining knots at a point could use Van Kampen to learn the newly created knot's fundamental group as the free product of the constituent knots.