Topology and Knot Theory

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Abstract

We present an introduction to knot theory suitable for undergraduate students: In section 1, we review basic definitions of general topology and the notion of a fundamental group is introduced. In section 2 we partially prove the essential van Kampen theorem and show how it relates to the study of algebraic topology. In section 3, we redefine the fundamental group using the notion of the groupoid, and the relation between loop space and the groupoid is revealed. In section 4, basic definitions of knot theory including Reidemeister moves and ambient isotopy are introduced, we discuss knot invariants, specifically knot polynomials. In section 5, we reinterpret knots as tensor contractions, and show how this leads naturally to quantum field theory through the Yang-Baxter equation. In section 6, the knot groups of right and left-handed trefoils are calculated explicitly as an example.

Contents

1 Point Set Topology

1.1 Topological Space

We begin with the definition of a topology:

Definition 1.1.1. Given a set X, we call a collection $\mathcal{T}$ of subsets of X a topology, if it satisfies:

1. $\emptyset$, $X \in \mathcal{T}$.
2. For any $U \in \mathcal{T}$, $\mathcal{T}$ is closed under arbitrary unions of U's.
3. For any $U \in \mathcal{T}$, $\mathcal{T}$ is closed under finite intersections of U's.

Remark. A space X with a topology $\mathcal{T}_x$ is called a topological space.

Definition 1.1.2. A space X with topology $\mathcal{T}_x$ is called Hausdorff if for any x, $y \in X$ with $x \neq y$, there exist two open sets U, $V \in \mathcal{T}_x$ and $x \in U$, $y \in V$ such that $U \cap V = \emptyset$.

Remark. A Hausdorff space is a space where we can separate points by open sets.

1.2 Continuity And Continuous Function

Definition 1.2.1. Given two topological spaces X, Y with topologies $\mathcal{T}_x$ and $\mathcal{T}_y$ respectively, let $f : X \longrightarrow Y$ be a map. We say f is continuous if for any $V \in \mathcal{T}_y$, $f^{-1}(V) \in \mathcal{T}_x$.

Definition 1.2.2. Let $f : X \longrightarrow Y$. We say f is continuous at $x \in X$ if for every open set V in Y with $f(x) \in V$, there exists an open set U in X such that $f(U) \subset V$.

Proposition 1.2.3. *A function $f : X \longrightarrow Y$ is continuous if and only if it is continuous at every point in X.*

Proof. If the function f is continuous in X, then it is obvious that such function is continuous at every point in X.

Assume f is continuous at every point $x \in X$, so for every point x with $f(x) \in V$ where V is an open set in Y we have $f(U_x) \subset V$ with $x \in U_x$. Thus $U_x \subset f^{-1}(V)$, we can write

$$f^{-1}(V) = \bigcup_{x \in f^{-1}(V)} U_x$$

since the chosen open set V is arbitrary, we see that the function f is continuous in X. $\square$

Definition 1.2.4. A continuous bijective map $f : X \longrightarrow Y$ is called a homeomorphism if $f^{-1} : Y \longrightarrow X$ is continuous as well.

Remark. If there exists a homeomorphism between X and Y, we say X and Y are homeomorphic.

1.3 Connected and Path-Connected

Definition 1.3.1. Given a topological space X. We say X is connected if whenever $X = U \cup V$ where U and V are disjoint open sets, we have $\{U, V\}$=$\{\emptyset$, $X\}$. If there exist other disjoint open sets $\{U$, $V\}$such that $X = U \cup V$, then we call U, V the separation of X.

Remark. X is connected if and only if the only separation of X is $\{\emptyset$, $X\}$.

Proposition 1.3.2. *A space X is connected if and only if the only sets that are both open and closed are $\emptyset$ and X.*

Proof. Suppose X is connected and U is a subset of X which is both closed and open but not $\emptyset$ and X, so $V = X - U$ is both open and closed but not $\emptyset$ and X. Thus $\{U, V\}$ provides a separation of X. Here is a contradiction, so there must be no such U and V.

The other side of the proof is obvious. $\square$

Definition 1.3.3. Given a continuous map $f : [a, b] \longrightarrow X$ and $x \in X$, $y \in X$ where X is a topological space. Then f is called a path from x to y if $f(a) = x$, $f(b) = y$. A space X is called path-connected if given any two points $x, y \in X$, there exists a path between x and y.

1.4 Compactness

Definition 1.4.1. Let X be a space and $\{U_i\}$ a collection of open sets. $\{U_i\}$ is called an open cover of X if $X = \bigcup_{i \in I} U_i$. The space X is compact if every open cover contains a finite subcover. (i.e. $U_1, U_2, ..., U_n \in \{U_i\}$ such that $X = U_1 \cup U_2 \cup ... \cup U_n$)

Proposition 1.4.2. *Every closed subset of a compact space is compact.*

Proof. Given a compact set X, suppose Y is a closed subset of X. Let $\{U_i\}$ be any open cover of Y. Let $\mathcal{U}' = \{U_i\} \cup \{X - Y\}$, we can tell that $\mathcal{U}'$ is an open cover of X so it has finite subcover $U'_1, U'_2, ...U'_n$. If $\{X - Y\}$ is among any U'_i, throw them out. Hence Y is compact. $\square$

Proposition 1.4.3. *Every compact subset of a Hausdorff space is closed.*

Proof. Given a Hausdorff space X and a compact subset Y of X. Let $x_0 \in X - Y$, the fact that X is Hausdorff allows us to find open sets U_{x_0} and V_y with $x_0 \in U_{x_0}$ and $y \in V_y$ such that $U_{x_0} \cap V_y = \emptyset$. Let $\{V_{y_i}\}_{i \in I}$ be an arbitrary open cover of Y, since Y is compact so there exist subcover such that $\bigcup_{i=1}^{n} V_{y_i}$ which is disjoint from the open set $U = U_{x_{0_1}} \cap U_{x_{0_2}} \cap ... \cap U_{x_{0_n}}$ with $x_0 \in U$. Since x_0 is chosen arbitrarily, so U is an open set in $X - Y$. Thus $X - Y$ is open, so Y is closed as the proposition claimed. $\square$

Proposition 1.4.4. *Let $f : X \longrightarrow Y$ be a continuous map. If X is compact, then $f(X)$ is compact as well.*

Proof. Suppose the map $f : X \longrightarrow Y$ is continuous and X is compact. Let U be an open subset of X such that $f(U) = V$ where V is an open subset of Y. Let $\{U_i\}_{i \in I}$ be an arbitrary open cover of U, there exists a finite subcover such that $U = \bigcup_{i=1}^{n} U_i$ since X is compact. Let $\{V_i\}_{i \in I}$ be an arbitrary open cover of V. Since the map f is continuous, so we have $f^{-1}(V_i) \in \{U_1, U_2, ..., U_n\}$. If this is not the case, we throw such V_i out. Thus U is chosen arbitrarily, so $f(X)$ is compact. $\square$

1.5 Quotient Map and Quotient Space

Definition 1.5.1. Given two topological spaces X and Y. Let $\pi : X \longrightarrow Y$ be a surjective map. We call π a quotient map provided that $U \in \mathcal{T}_y$ if and only if $\pi^{-1}(U) \in \mathcal{T}_x$.

Definition 1.5.2. A partition A of a set X is a collection of subsets $\{U_i\}_{i \in I}$ such that $X = \bigcup_{i \in I} U_i$ and $U_i \cap U_j = \emptyset$ if $i \neq j$.

Definition 1.5.3. Let X be a topological space and A a partition of X. Let $\pi : X \longrightarrow Y$ be a surjective map provided that $\pi(x_i) = U_i$ where U_i is an element of A which uniquely contains x_i. If we let $\mathcal{T}_A$ denote the quotient topology on A arising from the quotient map π, then we call A the quotient space of X with respect to A.

1.6 Homotopy and The Fundamental Group

We've now developed the language needed to introduce the notion of homotopy. A more detailed discussion will be given in Section 3, but we briefly define it here:

Definition 1.6.1. Given two continuous maps f_0 and f_1 from X to Y. A homotopy between them is a continuous map F such that

$$F : X \times [0, 1] \longrightarrow Y$$

provided that for all $x \in X$, $F(x, 0) = f_0(x)$ and $F(x, 1) = f_1(x)$. If there exists a homotopy F between f_0 and f_1, then we say f_0 and f_1 are homotopic and write $f_0 \simeq f_1$.

Remark. By defining a homotopy we are defining an equivalence relation. So in the terminology of abstract algebra, we can define a homotopy on a group to form a quotient group in terms of cosets. As we will see, this is indeed the way to define the fundamental group.

Definition 1.6.2. Let (X, x_0) and (Y, y_0) be spaces with fixed base points. Assume that the maps $f_0, f_1 : X \longrightarrow Y$ are base-point-preserving. A homotopy F between f_0 and f_1 is based if $F(x_0, t) = y_0$ for all $t \in [0, 1]$.

Definition 1.6.3. Let (X, x_0) be a space with a base point x_0, and let $*$ be a base point on the circle S^1 . The set of all continuous, base-point-preserving maps for $(S^1, *)$ to (X, x_0) is called the loop space of (X, x_0) and is denoted by $\Omega(X, x_0)$. The based homotopy defines an equivalence relation on $\Omega(X, x_0)$ and the quotient space formed is denoted by $\pi_1(X, x_0)$.

Theorem 1.6.4. *Concatenation of loops defines a group operation on* $\pi_1(X, x_0)$.

Proof. Proof is intentionally omitted here. □

Remark. In other words, concatenation of loops has inverse, identity and follows the rule of associativity. So $\pi_1(X, x_0)$ along with concatenation is called the fundamental group.

2 Seifert-van Kampen Theorem

The Seifert-van Kampen theorem links the study of topological spaces with the study of groups by giving a way to define the fundamental group of a space as the product of the fundamental groups of the constituent spaces.

2.1 Pre-Requisites

We briefly present some background algebra before tackling the theorem:

A normal subgroup N of a group G, is a subgroup $N \subset G$ such that $n \in N$ if

$gng^{-1} \in N, \forall g \in G$. That is, the elements of N are invariant under conjugation by elements of G. For example, note that every subgroup of an abelian group is a normal subgroup.

First isomorphism theorem: Let G and H be groups, and let $\phi : G \to H$ be a homomorphism. It follows that the kernel of ϕ is a normal subgroup of G, the image of ϕ is a subgroup of H, and the image of ϕ is isomorphic to the quotient group G / $\ker(\phi)$. In particular, if ϕ is surjective then H is isomorphic to G / $\ker(\phi)$.

In group theory, a word is any written product of group elements and their inverses. For example, if x, y and z are elements of a group G, then xy, $z^{-1}xzz$ and $y^{-1}zxx^{-1}yz^{-1}$ are words in the set {x, y, z}. The free product of a set of groups $\{G_\alpha\}$ is denoted as: $*_\alpha\{G_\alpha\}$, and is the group of all finite combinations of words of constituent elements of both groups. Note that this operation is not, in general, commutative (it is commutative only when one of the groups is trivial).

We now wish to generalize the notion of the based fundamental group: Let X be a topological space, and let x_0 be a point in X. As before, we define a set of continuous functions called loops with base point x_0:

$$\{f : [0,1] \to X | f(0) = x_0 = f(1)\}$$

and call the set of equivalence classes of these loops the fundamental group at the point x_0. For path connected spaces the fundamental groups at different points are equivalent up to isomorphism, and we denote this group as

$$\pi(X)$$

2.2 Statement of Theorem

Theorem 2.2.1. *Let X be the union of sets, $\bigcup_\alpha A_\alpha = X$ such that $\forall \alpha, \beta$: $A_\alpha \cap A_\beta \neq \emptyset$ and $A_\alpha \cap A_\beta$ is open and path connected. We claim that $\exists \phi : *_\alpha \pi(A_\alpha) \twoheadrightarrow \pi(X)$*

Furthermore, ker(ϕ) = N, such that $N = \langle i_{\alpha\beta}(\omega) i_{\beta\alpha}^{-1}(\omega) \rangle$ with $\omega \in \pi(A_\alpha \cap A_\beta)$ such that

$$i_{\alpha\beta} : \omega \rightarrowtail \pi(A_\alpha) \text{ and } i_{\beta\alpha}^{-1} : \pi(A_\beta) \to \omega$$

And $\phi \rightsquigarrow \pi(X) \simeq *_\alpha \pi(A_\alpha)/N$

2.3 Proof that ϕ is surjective:

Proof. Given a loop $f : I \to X$ based at x_0 there is a partition $0 = s_1 < s_2 < ... < s_n = 1$ of $[0,1]$ such that each interval $[s_i, s_{i+1}]$ is mapped to a single space A_α. This is because each $s \in [0,1]$ has a neighborhood that is mapped completely to a unique A_α in X. Compactness of I implies that there exists a finite covering by these neighborhoods, such that their boundaries determine our partition.

Denote the A_α containing $f([\,s_{i-1}, s_i]\,)$ by A_i, and let $f_i = f|_{[\,s_{i-1},s_i]}$. Then, $f = f_1 * f_2 * ... * f_n$, where each path given by f_i exists solely in the corresponding A_i. Since $A_i \cap A_{i+1}$ is path connected by hypothesis, $\exists$ a path g_i from x_0 to $f(s_i) \in A_i \cap A_{i+1}$.

Now it is clear we can write a loop which is a composition (product) of loops which each individually lie entirely in one of the constituent spaces, thus $[\,f]$ is in the image of ϕ, as it is the product of equivalence classes of loops, which is the image under ϕ of a word in $*_\alpha \pi(A_\alpha)$. Thus ϕ is surjective.

$\square$

It is more difficult to show that ϕ is injective. For further details see, for example, [3] (pp. 45-46).

2.4 Intuition

The essential notion of the Seifert-van Kampen theorem is the fact that the fundamental group of a space can be obtained from the free product of constituent spaces. The free product is obviously always larger, but sometimes equal to the fundamental group of the union of the spaces. In particular, the key idea is to construct a loop in the total space the is a product of loops which indiviually exist solely in one of the spaces. The non-commutativity (except in the trivial case) of the free product tells us that, metaphorically, going north then east isn't always the same as going east then north.

3 Groupoids And The Fundamental Group

First recall the definition of homotopy.

Definition 3.0.1. Let $f_0, f_1 : X \longrightarrow Y$ be continuous maps. A continuous map $F : X \times [0,1] \longrightarrow Y$ is called a homotopy if

$$F(x,0) = f_0(x), F(x,1) = f_1(x)$$

for all $x \in X$. And we write $f_0 \simeq f_1$.

In section 1.1.6, we mentioned without proof that concatenation defines a group operation in $\pi_1(X, x_0)$. Now given two paths $f, g : [0,1] \longrightarrow X$, we define $(f * g)(t)$ in the form such that

$$(f * g)(t) = \begin{cases} f(2t), 0 \le t \le 1/2 \\ g(2t-1), 1/2 \le t \le 1 \end{cases}$$

Suppose X now is path-connected, then all paths $f : [0,1] \longrightarrow X$ are homotopic. Say we have a constant path $i_p : [0,1] \longrightarrow X$ such that $i_p(t) = p \in X$ for all $t \in [0,1]$. As we will later see that i_p can be considered as an "identity" which indicates that $i_p * f = f$ provided that $\alpha(f) = p$ (the origin of f is p). Since all paths are homotopic, so we have $f \simeq i_p$ for all paths f in X. There is only one homotopy class in such case and it is trivial. To avoid such situation, it becomes necessary to modify the original definition of homotopy. But before that, I want to first introduce the definition of deformation retraction which states how a function can "evolve" with "time".

Definition 3.0.2. A deformation retraction of a space X onto a subspace A is a homotopy $F : X \times [0,1] \longrightarrow X$ such that $F(x,0) = x$ for all $x \in X$, $F(X,1) = A$ and $F(A,t) = A$ for all $t \in [0,1]$.

Informally speaking, a deformation retraction of X onto a subspace A is a homotopy from the identity map of X to a retraction of X onto A, a retraction is a map $r : X \longrightarrow A$ such that $r(X) = A, r|A = \mathbb{1}$.

In general, we call a homotopy $F : X \times [0,1] \longrightarrow Y$ relative to a subspace $A \subset X$ if the restriction to A is independent of $t \in [0,1]$. And we denote it by homotopy rel A.

Definition 3.0.3. Let $A \subset X$ and $f_0, f_1 : X \longrightarrow Y$ be two continuous maps with $f_0|A = f_1|A$. We write

$$f_0 \simeq f_1 \text{ rel } A$$

if there exists a continous map $F : X \times [0,1] \longrightarrow Y$ with $f_0 \simeq f_1$ and

$$F(a,t) = f_0(a) = f_1(a)$$

for all $a \in A, t \in [0,1]$.

Here we introduced the definition of relative homotopy (homotopy rel some subsets). And the original definition of homotopy can be modified into the way that homotopy rel $\emptyset$. Such homotopy is called free homotopy.

Definition 3.0.4. The equivalence class of paths $f : [0,1] \longrightarrow X$ rel {0,1} is called path class and is denoted by $[f]$.

Note. *{0,1} is considered as the boundary of [0,1].*

In section 1.1.6, we mentioned that the fundamental group $\pi_1(X, x_0)$ is a quotient space of the loop space $\Omega(S^1, *)$. As we will see later that the elements in $\pi_1(X, x_0)$ are just path classes defined above. But what restrictions should be posed onto these path classes? In order to answer such question, we need first introduce the definition of the origin and the end of a path. Also the definition of groupoid.

Definition 3.0.5. Let $f : [0,1] \longrightarrow X$ be a path from x_0 to x_1. The origin of f is x_0 and is denoted by $\alpha(f)$, the end of f is x_1 and is denoted by $\omega(f)$. A path is closed if $\alpha(f) = \omega(f)$.

Note. *We can also define origin and end on path classes* $[f]$ *that are denoted by* $\alpha[f]$ *and* $\omega[f]$.

What then provides the "resources" of a fundamental group $\pi_1(X, x_0)$?

Definition 3.0.6. Let X be a space. The set of all path classes in X under the binary operation $[f][g] = [f * g]$(though not always defined) is called a groupoid, if it satisfies the followings:

(i). For every path class $[f]$, it has an origin $\alpha[f] = p$ and an end $\omega[f] = q$. We have
$$[i_p][f] = [f] = [f][i_q];$$

(ii). Associativity holds whenever possible;

(iii). If $\alpha[f] = p$ and $\omega[f] = q$, then we have
$$[f][f^{-1}] = [i_p], [f^{-1}][f] = [i_q].$$

Note. *A groupoid is not a group since the operation may not be defined.*

Definition 3.0.7. The fundamental group $\pi_1(X, x_0)$ with a basepoint x_0 is defined as

$$\pi_1(X, x_0) = \{[f] : [f] \text{ is path class with } \alpha[f] = x_0 = \omega[f]\}$$

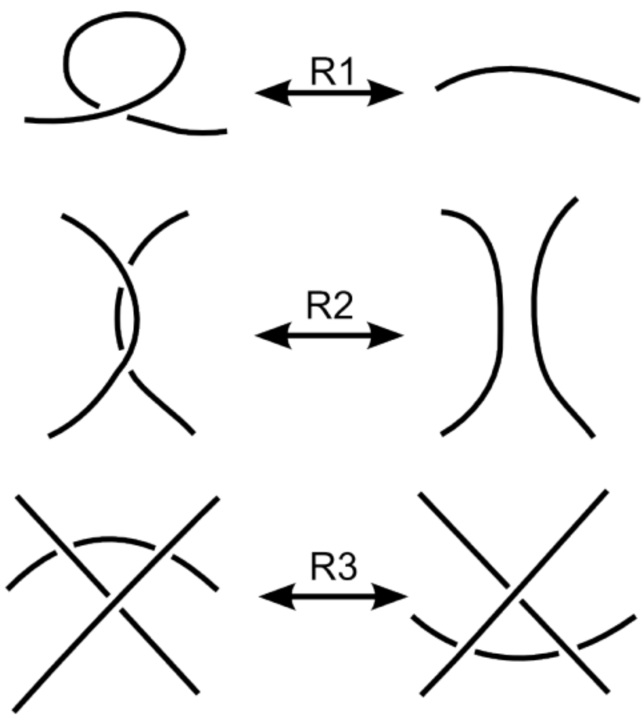
with the binary operation
$$[f][g] = [f * g].$$

4 Knot Theory

4.1 Reidemeister Moves And Ambient Isotopy

There are certain continuous deformations which leave the topological properties of knots invariant. These are called the Reidemeister moves.

Definition 4.1.1. The Reidemeister moves are local continuous deformations of knots that appear as such:



The equivalence relation on diagrams generated by all Reidemeister moves is called an ambient isotopy.

In section 3, we mentioned the definition of deformation retraction. To those who don't care too much graphically like me, I want those readers to notice here that ambient isotopy is just a special case of deformation retraction or at least we can make an analogy when we come to precisely define ambient isotopy.

Definition 4.1.2. Let g, h be two embeddings of a manifold M in manifold N. An ambient isotopy is a homotopy $F : N \times [0,1] \longrightarrow N$ considered to take g to h such that $F_0 = \mathbb{1}$, F_t is a homeomorphism from N to itself for all $t \in [0,1]$ and $F_1 \circ g = h$.

Notice here that how similar these two definitions are.

Definition 4.1.3. Given two knots K_0 and K_1, we say K_0 and K_1 are ambient isotopic if there exists an isotopy $h : \mathbb{R}^3 \times [0,1] \longrightarrow \mathbb{R}^3$ such that $h(K_0, 0) = K_0$ and $h(K_1, 1) = K_1$

When we refer to a knot, we are really referring to the whole equivalence class of knots which are ambient isotopic to one another. When we say two knots are different, we really mean they are elements of different knot equivalence classes.

In principle, because of Reidemeister's Theorem (which states that two knots are equivalent if their diagrams can be connected by a sequence of Reidemeister moves), we can classify all knots by the diagrams generated by any sequence of Redemeister moves on a given knot. In practice, however, it is exceedingly difficult to find an explicit sequence of Reidemeister moves between two diagrams, and it is even harder to show that **no** sequence of Reidemeister moves exists between two diagrams! Therefore, we must be more clever in showing that two knots exist within different equivalence classes. Finally, we are ready to tackle the problem of classifying and differentiating between

4.2 Knot Invariants

Because of the difficulty of finding an explicit sequence of Reidemeister moves between knot diagrams, and the difficulty in proving no such sequence exists, we need more powerful tools to classify knots. The most prominent tool is called a "knot invariant", and is defined as such:

Definition 4.2.1. A knot invariant is any function i of knots which only depends on their equivalence class. If K_1 and K_2 are two equivalent knots, $K_1 \cong K_2$, then $i(K_1) = i(K_2)$, which implies that if $i(K_1) \neq i(K_2)$ then $K_1 \ncong K_2$

Unfortunately, no complete knot invariant has been found. That is, for all invariants i, while $i(K_1) \neq i(K_2) \implies K_1 \ncong K_2$, the converse does not necessarily hold.

4.3 The Bracket Polynomial and Its Generalizations

Our first explicit knot invariant is known as the Bracket Polynomial and is defined as such:

Definition 4.3.1. For a given knot K, its bracket polynomial $\langle K \rangle$ is given by

$$\langle K \rangle = \langle K \rangle(A, B, d) = \sum_{\sigma} \langle K|\sigma \rangle d^{||\sigma||}$$

where A, B and d are commuting algebraic variables and $||\sigma||$ represents the number which is one less than the number of loops in σ. Here σ runs over all the states of K.

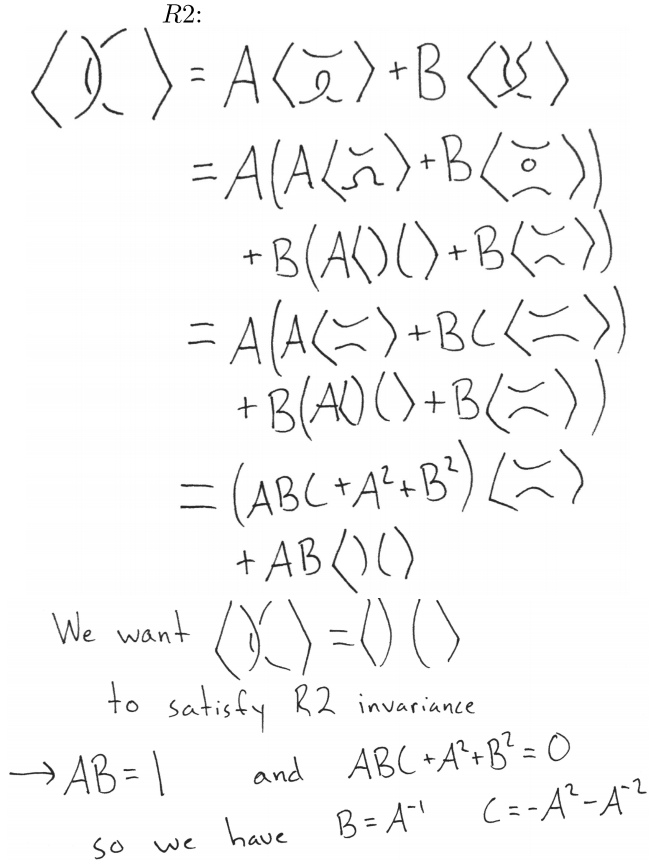
We next introduce the pictorial representation of the bracket polynomial, which is defined as follows:

1. Satisfies the Skein Relation:

2. $\langle \bigcirc K \rangle = C\langle K \rangle$, where $\langle \bigcirc K \rangle$ is the disjoint union of a knot K and the crossingless diagram of the unknot.

3. Normalization: $\langle \bigcirc \rangle = 1$

To show the bracket polynomial is a knot invariant, it is sufficient to show that it is invariant under the Reidemeister moves. We show invariance under $R2$:



It is straightforward to show the the bracket polynomial is not invariant under $R1$ moves, hence we fix this issue by generalizing to the normalized bracket polynomial.

Definition 4.3.2. Let K be an oriented link diagram. The writhe of K, $\omega(K)$, is given by the equation $\omega(K) = \sum_p \epsilon(p)$ where p runs over all crossings in K, and $\epsilon(p)$ is the sign of the crossing:

Figure 1: The left crossing has $\epsilon = +1$, while the right has $\epsilon = -1$.

Definition 4.3.3. The normalized bracket polynomial $\mathcal{L}$ of a knot K is given by

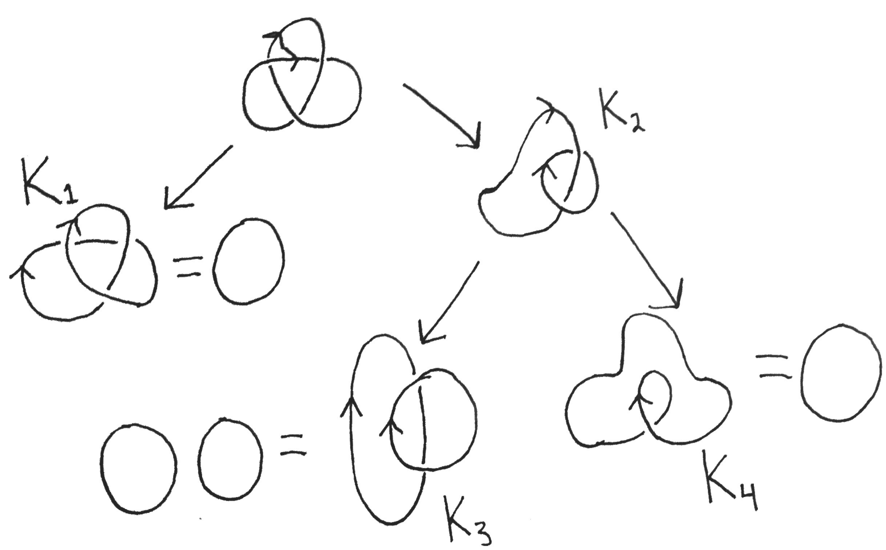
$$\mathcal{L}_K = (-A^3)^{-\omega(K)}\langle K\rangle$$

4.4 The Alexander Polynomial

The first knot polynomial was discovered by James Alexander in 1928. It is invariant under all three Reidemeister moves, applies to oriented knots, and is defined as such:

1. $\Delta(\bigcirc) = 1$ for every diagram of the unknot
2. $\Delta(L_+) - \Delta(L_-) + (t^{\frac{1}{2}} - t^{-\frac{1}{2}})\Delta(L_{\bigcirc}) = 0$

At a crossing of whichever knot you wish to compute the Alexander polynomial of, determine whether the knot is an L_+ or L_- knot, and rearrange the above equation to compute the leftover terms according to the Skein relations. We present an example of the diagrammatic breakdown of the right-handed trefoil knot:



Hence the algebraic representation of the above is:

$$
\begin{aligned}
\Delta_{\text{Trefoil}} &= \Delta_{K_1} - (t^{\frac{1}{2}} - t^{-\frac{1}{2}})\Delta_{K_2} && (1)\\
&= 1 - (t^{\frac{1}{2}} - t^{-\frac{1}{2}})(\Delta_{K_3} + (t^{\frac{1}{2}} - t^{-\frac{1}{2}})\Delta_{K_4}) && (2)\\
&= 1 - (t^{\frac{1}{2}} - t^{-\frac{1}{2}})(0 + (t^{\frac{1}{2}} - t^{-\frac{1}{2}}))1 && (3)\\
&= t^{-1} - 1 + t && (4)
\end{aligned}
$$

We claim without proof that the Alexander polynomial is the same for the left-handed trefoil. Indeed, one can show that the Alexander polynomial cannot distinguish at all between chiral knots (those which are not ambient isotopic to their mirror image).

4.5 The Jones Polynomial and Its Generalizations—Kauffman Polynomial

Definition 4.5.1. The 1-variable Jones polynomial, $V_K(t)$, is a Laurent polynomial in the variable $\sqrt{t}$ assigned to an oriented link K which fulfills the following properties:

1. If K is ambient isotopic to K', then $V_K(t) = V_{K'}(t)$.
2. $V_{\bigcirc} = 1$.
3. $t^{-1}V_{K_+} - tV_{K_-} = (\sqrt{t} - \frac{1}{\sqrt{t}})V_{K_0}$.

Theorem 4.5.2. *Let $\mathcal{L}_K(A) = (-A^3)^{-\omega(K)}\langle K\rangle$. Then we have*

$$V_K(t) = \mathcal{L}_K(t^{-1/4})$$

which indicates that the normalized bracket yields the 1-variable Jones polynomial.

Proof. Give two bracket polynomials $\langle K_+\rangle$ and $\langle K_-\rangle$. According to the basic properties of bracket polynomials, we can write these two polynomials in the form

$$
\begin{aligned}
\langle K_+\rangle &= A\langle K_0\rangle + B\langle K_0'\rangle\\
\langle K_-\rangle &= B\langle K_0\rangle + A\langle K_0'\rangle
\end{aligned}
$$

By setting $B = A^{-1}$, we have

$$A\langle K_+\rangle - A^{-1}\langle K_-\rangle = (A^2 - A^{-2})\langle K_0\rangle$$

Let $\omega = \omega(K_0)$ and $\alpha = -A^{-3}$, so we have $\omega(K_+) = \omega + 1$ and $\omega(K_-) = \omega - 1$. Hence, we have

$$A\langle K_+\rangle\alpha^{-\omega} - A^{-1}\langle K_-\rangle\alpha^{-\omega} = (A^2 - A^{-2})\langle K_0\rangle\alpha^{-\omega}$$

Thus,

$$A\alpha\langle K_+\rangle\alpha^{-(\omega+1)} - A^{-1}\alpha^{-1}\langle K_-\rangle\alpha^{-(\omega-1)} = (A^2 - A^{-2})\langle K_0\rangle\alpha^{-\omega}$$

which can be rewritten in the form

$$-A^4\mathcal{L}_{K_+} + A^{-4}\mathcal{L}_{K_-} = (A^2 - A^{-2})\mathcal{L}_{K_0}$$

By setting $A = t^{-1/4}$, we have

$$t^{-1}\mathcal{L}_{K_+} - t\mathcal{L}_{K_-} = (\sqrt{t} - \frac{1}{\sqrt{t}})\mathcal{L}_{K_0}$$

And property 1 and 2 follow immediately from the property of the reduced bracket polynomial. $\square$

There is another generalization of Jones polynomial, namely the Kauffman polynomial.

Definition 4.5.3. The Kauffman polynomial $F_K(\alpha, z)$ is a normalization of a polynomial, L_K, defined for unoriented links and fulfills the following properties:

1. If K is regularly isotopic to K', then $L_K(\alpha, z) = L'_K(\alpha, z)$.
2. $L_{\bigcirc} = 1$.
3. $L + L = z(L + L)$.
4. $L = \alpha L$ and $L = \alpha^{-1} L$

Therefore the Kauffman polynomial $F_K(\alpha, z)$ is defined by the formula

$$F_K(\alpha, z) = \alpha^{-\omega(K)} L_K(\alpha, z)$$

The following proposition is stated without proof.

Proposition 4.5.4. *We have*

$$\langle K\rangle(A) = L_K(-A^3, A + A^{-1})$$

and

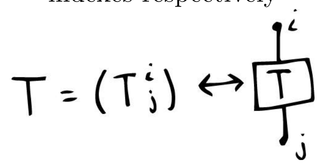
$$V_K(t) = F_K(-t^{-3/4}, t^{-1/4} + A^{1/4})$$

5 The Yang Baxter Equation

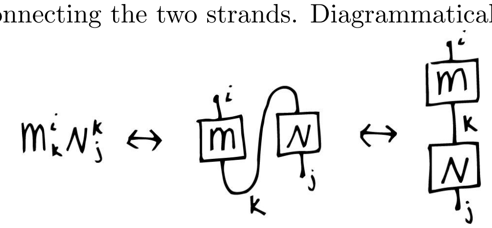
Now we have built up our knot-theoretic toolset sufficiently that we can ask, "What is the point?" Though these structures are interesting in their own right, it turns out that our study of knots leads very naturally to the famous Yang-Baxter equation from physics. To get there we must first introduce a diagrammatic notation to deal with tensors. We will not motivate such a discussion, but it will quickly become apparent how we can use knot theory and tensor diagrams to derive Yang-Baxter.

5.1 Tensor Diagrams

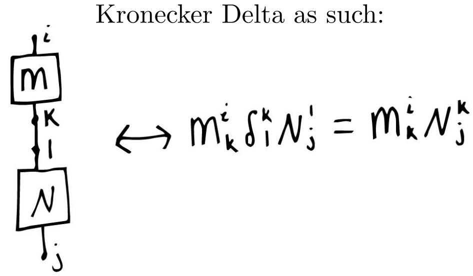
A tensor $T = (T^i_j)$ with indices i and j can be represented as a box with strands coming out of the top and bottom, representing the upper and lower indexes respectively



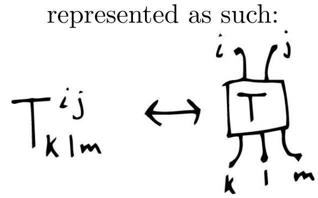
Which permits a very convenient representation of matrix multiplication. We represent the multiplication of two matrices M and N as $(MN^i_j) = \sum_k M^i_k N^k_j = M^i_k N^k_j$(assuming Einstein summation notation). We can then represent a sum over a common index of two matrices as a line connecting the two strands. Diagrammatically:



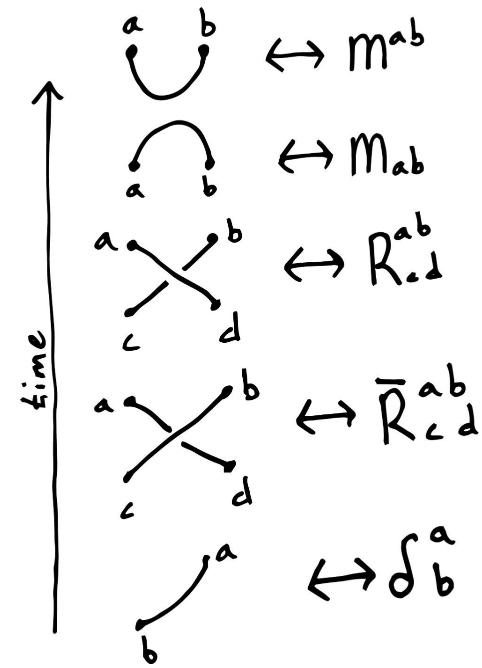
Equivalently, any line connecting two strands may be interpreted as the Kronecker Delta as such:



Most generally, any tensor like object with multiple upper and lower indices is represented as such:



Now that we have our tensor diagram language, we can see that a knot is a contracted tensor, which is represented according to the following rules:

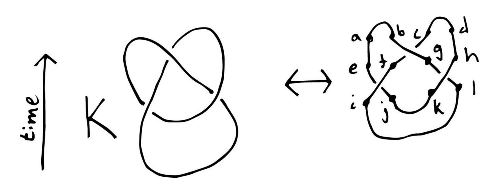


Where we have oriented our diagrams with respect to an upward time direction, which hints at our eventual connection with quantum field theory...

5.2 Knots Reinterpreted

With our new rules to interpret local sections of knots algebraically as tensors according to the rules laid out above, we can clearly see that a knot shall represent a contracted tensor. We shall soon see how this leads naturally to quantum mechanics.

For example, here is the trefoil knot represented accordingly:



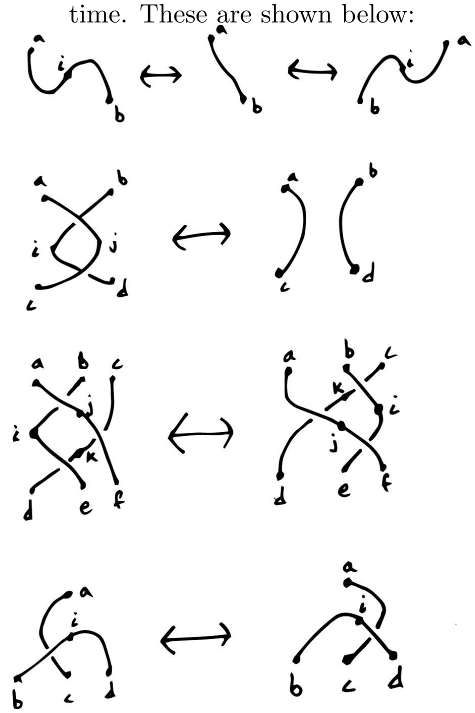
The tensor contraction $t(K)$ for the trefoil is
$t(K) = M_{ab}M_{cd}\delta^a_e\delta^d_h R^{bc}_{fg}\bar{R}^{ef}_{ij}\bar{R}^{gh}_{kl}M^{jk}M^{il}$

If we think of a knot sitting in a space-time plane, then each section is a quantum mechanical event. We know see that this interpretation is very much the same as Feynman diagrams, where minima are the creation of particles, maxima are annihilations, and crossings are particle interactions. It's now clear that a knot is some vacuum to vacuum process. In quantum mechanics to get the probability amplitude of some process we usually sum over the product of all possible intermediate states. So a knot contraction as above is the expectation of the processes given in the specific knot diagram!

5.3 Topological Invariance and Yang-Baxter

A natural question to ask at this point is how our tensor contractions function under the Reidemeister moves. Or rather, what happens if we make an algebraic demand on our tensors to remain consistent with the topological invariance of the Reidemeister moves?

Looking at the regular isotopies generated by the Reidemeister moves we need special diagrams of the moves to account for our orientation with respect to time. These are shown below:



Now we force the tensor contraction $t(K)$ to be invariant under these moves, which gets us the following equations:

$$M^{ai}M_{ib} = \delta^a_b = M_{bi}M^{ia} \tag{5}$$

$$R^{ab}_{ij}\bar{R}^{ij}_{cd} = \delta^a_c\delta^b_d \tag{6}$$

$$R^{ab}_{ij}R^{jc}_{kf}R^{ik}_{de} = R^{bc}_{ki}R^{ak}_{dj}R^{ji}_{ef} \tag{7}$$

$$\bar{R}^{ai}_{bc}M_{id} = R^{ia}_{cd}M_{bi} \tag{8}$$

Which implies that if R and $\bar{R}$ are inverse, and if M^{ab} and M_{ab} are also inverse, then the regular isotopy is satisfied.

The third Reidemeister move gives us the most interesting equation:

$$R^{ab}_{ij}R^{jc}_{kf}R^{ik}_{de} = R^{bc}_{ki}R^{ak}_{dj}R^{ji}_{ef}$$

Which is, in fact, the famous Yang-Baxter equation.
It is quite astonishing that our study of knots and topology has resulted in an equation which is of great use in statistical and theoretical physics. Our topological argument of invariance under the third Reidemeister move generated just the algebraic constraints needed to derive Yang-Baxter. This is an amazing fact, and the intuition comes from interpreting knots as vacuum to vacuum Feynman diagrams.

6 Knot Groups

6.1 The Wirtinger Representation

We start by introducing the so-called Wirtinger Representation, which will describe the relation to be defined at each crossing of a knot when we try to translate a knot into the language of groups. For instance, given a simple crossing as follows, we at least know two points:

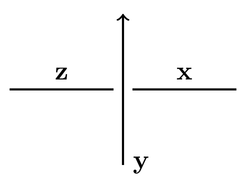


Figure 2: it is unnecessary to include the orientation of the horizontal line

1. **x**, **y** and **z** are elements;
2. **xy=yz**

So we have the crossing given in the language of groups as

$$\langle x, y, z | xy = yz \rangle$$

Remark. We find the relation in this case **xy=yz** through the right hand law. We can imagine an arrow through the crossing into the page. Let your thumb point in the direction of the arrow, then curl your fingers, and do the multiplication following the direction your fingers curl. Every crossing can be analyzed in this fashion.

6.2 Left And Right-handed Trefoils

Let's first take a look at the right-handed trefoil. For such a trefoil, we know at least two points:

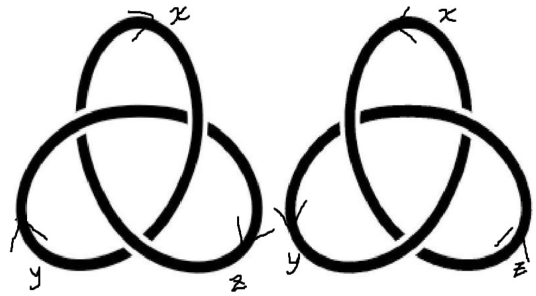


Figure 3: The left trefoil is right-handed, while the right trefoil is left-handed

1. we have elements **x**, **y** and **z** in the group;
2. the relations are **zy=yz**, **yx=xz** and **xz=zy**.

So in the language of groups, we have the right-handed trefoil in the form

$$\langle x, y, z | zy = yx, yx = xz, xz = zy \rangle$$

Furthermore, we notice that $yx = xz$, indicating that $z = x^{-1}yx$. By plugging this z into the other two relations, we have:

$$zy = yx \Longrightarrow x^{-1}yxy = yx \Longrightarrow yxy = xyx$$

and

$$xz = zy \Longrightarrow xx^{-1}yx = x^{-1}yxy \Longrightarrow yx = x^{-1}yxy \Longrightarrow xyx = yxy$$

we find out that they are the same which indicates that we can rewrite the group of right-handed trefoil in the form:

$$\langle x, y | xyx = yxy \rangle$$

Likewise, for the left-handed trefoil listed in figure 2, we have:

1. we have elements **x**, **y** and **z**;
2. we have relations **yx=xz**, **xz=zy** and **zy=yx**.

Set $z = x^{-1}yx$ as in last case, we have

$$xz = zy \Longrightarrow xx^{-1}yx = x^{-1}yxy \Longrightarrow xyx = yxy$$

and

$$zy = yx \Longrightarrow x^{-1}yxy = yx \Longrightarrow yxy = xyx$$

which indicates that we have left-handed trefoil in the form of group such that

$$\langle x, y | xyx = yxy \rangle$$

we notice here that the group of left and right-handed trefoil are the same.

7 Acknowledgments

We would like to thank Professor Markus Pflaum for his helpfulness in directing our studies, and for his interesting lectures on applications of topology. We would also like to thank the University of Colorado Boulder Mathematics Department for their generous funding of the REU program we've participated in, and to Elijah Jarocki for facilitating the creation of key diagrams.

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