Astronomy 400B Lecture 6: Collisionless Boltzmann Equation

Brant Robertson

February, 2015

1 Distribution Function

The *distribution function* $f(\mathbf{x},\mathbf{v},t)$ gives the probability density in six-dimensional *phase space* $(\mathbf{x},\mathbf{v})$ of having an object in the volume $d\mathbf{x}d\mathbf{v}$. The number density $n(\mathbf{x},t)$ of objects is the volume integral of the distribution function

$$n(\mathbf{x},t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\mathbf{x},\mathbf{v},t) dv_x dv_y, dv_z. \tag{1}$$

We can use this expression to define moments of the velocity distribution, such as the average velocity

$$\langle \mathbf{v}(\mathbf{x},t)\rangle = \frac{1}{n(\mathbf{x},t)}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathbf{v} f(\mathbf{x},\mathbf{v},t) dv_x dv_y, dv_z. \tag{2}$$

For a collisionless system where objects cannot be created or destroyed, the number density of objects in a given volume will follow the continuity equation

$$\frac{\partial n}{\partial t} + \frac{\partial (nv)}{\partial x} = 0. \tag{3}$$

This equation simply describes the mass conservation of objects, such that the time rate of change of the number density is balanced by the advection of spatial gradients in the number density.

The *collisionless Boltzmann equation* that describes the probability density of objects in phase space is more complicated because it must describe a time variation in the velocity as well as the spatial coordinates. In one dimension, we have that

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} + \frac{dv}{dt}(x,v,t)\cdot\frac{\partial f}{\partial v} = 0 \tag{4}$$

The acceleration of an object will only depend on position in a background potential, so we have that $dv/dt = -\partial\Phi/\partial x$, and

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} - \frac{\partial \Phi}{\partial x}(x,t)\cdot\frac{\partial f}{\partial v} = 0. \tag{5}$$

In three dimensions, we can write

$$\frac{\partial f(\mathbf{x},\mathbf{v},t)}{\partial t} + \mathbf{v}\cdot\nabla f - \nabla\Phi\cdot\frac{\partial f}{\partial \mathbf{v}} = 0. \tag{6}$$

We usually don't deal with this equation in this form, but often will take moments with respect to e.g., velocity

$$\frac{\partial n(x,t)}{\partial t} + \frac{\partial}{\partial x}[n(x,t)\langle v(x,t)\rangle] - \frac{\partial \Phi}{\partial x}(x,t)[f]_{-\infty}^{\infty} = 0 \tag{7}$$

The last term will be zero if f is well behaved, and we get back Equation 3.

If we integrate Equation 5 multiplied by v, we instead find

$$\frac{\partial}{\partial t}[n(x,t)\langle v(x,t)\rangle] + \frac{\partial}{\partial x}[n(x,t)\langle v^2(x,t)\rangle] = -n(x,t)\frac{\partial \Phi}{\partial x} \tag{8}$$

If we define the velocity dispersion as

$$\langle v^2(x,t)\rangle = \langle v(x,t)\rangle^2 + \sigma^2, \tag{9}$$

apply this definition, and then divide by $n(x,t)$, we have

$$\frac{d\langle v\rangle}{dt} + \langle v\rangle\frac{\partial\langle v\rangle}{\partial x} = -\frac{\partial\Phi}{\partial x} - \frac{1}{n}\frac{\partial}{\partial x}[n\sigma^2(x,t)]. \tag{10}$$

2 Disk Mass and Jeans Equation

Consider Equation 10 applied to the Galactic disk. Well above the disk plane, $\langle v_z\rangle = 0$ and, if we assume the disk is stable, $\partial\langle v_z\rangle/\partial t = 0$. We can then write

$$\frac{\partial}{\partial z}[n(z)\sigma_z^2] = -n(z)\frac{\partial\Phi}{\partial z} \tag{11}$$

This equation is sometimes called the *Jeans Equation*. To proceed, we need to find some way of expressing the disk potential. We can use Poisson's Equation in cylindrical polar coordinates assuming polar symmetry, and we find that

$$4\pi G\rho(R,z) = \nabla^2\Phi(R,z) = \frac{\partial^2\Phi}{\partial z^2} + \frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial\Phi}{\partial R}\right). \tag{12}$$

We can substitute $\partial\Phi/\partial R = V^2(R)/R$, which gives

$$4\pi G\rho(R,z) = \frac{d}{dz}\left\{-\frac{1}{n(z)}\frac{d}{dz}[n(z)\sigma_z^2]\right\} + \frac{1}{R}\frac{d}{dR}[V^2(R)]. \tag{13}$$

The last term is small since the Galactic rotation curve is nearly flat. Integrating over z, we then have

$$2\pi G\Sigma(<z) \equiv 2\pi G\int_{-z}^{z}\rho(z')dz' \approx -\frac{1}{n(z)}\frac{d}{dz}[n(z)\sigma_z^2] \tag{14}$$

By measuring the vertical distribution of stars in the disk and the variation of σ_z with height above the disk we find that $\Sigma(<z) \approx 50-60\ M_\odot\ \mathrm{pc}^2$ near the solar circle. The mass in gas and stars is about $\Sigma_{gs}(<z) \approx 40-55\ M_\odot\ \mathrm{pc}^2$.

3 Integrals of Motion

If the potential does not change with time, then the distribution function is also constant. In this case, we can define *integrals of motion* that remain constant along an orbit. Examples are the total energy $E(\mathbf{x},v) = \mathbf{v}^2/2 + \Phi(\mathbf{x})$, or the total angular momentum $\mathbf{L}$ in a spherical potential, or the angular momentum along the symmetry axis l_z in an axisymmetric potential.

For any quantity $\mathcal{I}$ that is constant along the orbit, we have

$$\frac{d}{dt}\mathcal{I}(\mathbf{x},\mathbf{v}) \equiv \frac{\partial\mathbf{x}}{\partial t}\cdot\nabla\mathcal{I} + \frac{\partial\mathbf{v}}{\partial t}\cdot\frac{\partial\mathcal{I}}{\partial\mathbf{v}} = 0 \tag{15}$$

or

$$\mathbf{v}\cdot\nabla\mathcal{I} - \nabla\Phi\cdot\frac{\partial\mathcal{I}}{\partial\mathbf{v}} = 0. \tag{16}$$

Note that Equation 15 looks like Equation 6, with the substitution $f \to \mathcal{I}$. So the phase space density around a moving object is also a constant. So what can we do now?

Well, remember from the discussion of epicyclical frequency that for nearly circular orbits

$$\ddot{z} \approx -\nu^2(R_g)z, \tag{17}$$

such that motion in z is basically independent of motion in (R,ϕ). This result means that the energy in the vertical motion

$$E_z = \Phi(R_0,z) + v_z^2/2 \tag{18}$$

is an integral of motion. We therefore have that

$$f(z, v_z) = f(E_z) = f\left(\Phi(R_0, z) + \frac{1}{2}v_z^2\right) \tag{19}$$

If we guess at the form of $f(E_z)$, we can integrate to find $n(z)$ and σ_z. For instance, assume

$$f(E_z) = \frac{n_0}{\sqrt{2\pi\sigma^2}} \exp(-E_z/\sigma^2) \text{ for } E_z < 0 \tag{20}$$

(if $E_z > 0$, the stars would be unbound). If we integrate over v_z, we have

$$n(z) = n_0 \exp\left[-\Phi(R_0, z)/\sigma^2\right]; \; \sigma_z = \sigma. \tag{21}$$

For a spherical potential $\Phi(r)$, any $f(E, \mathbf{L})$ that doesn't have unbound stars will describe some density distribution that generates the potential. If we adopt $f(E)$ without an $\mathbf{L}$ dependence, the velocity distribution will be isotropic.

3.1 Isothermal Distribution

Consider the isothermal distribution

$$f_I(E) = \frac{n_0}{(2\pi\sigma^2)^{3/2}} \exp\left\{-\left[\Phi(r) + \frac{v^2}{2}\right]\Big/\sigma^2\right\} \tag{22}$$

The spherical and vertical disk isothermal solutions for the number density are the same, e.g.,

$$n(r) = n_0 \exp\left[-\Phi(r)/\sigma^2\right] \tag{23}$$

and we have

$$4\pi G\rho(r) = \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\Phi}{dr}\right) = 4\pi G m n_0 \exp\left[-\frac{\Phi(r)}{\sigma^2}\right] \tag{24}$$

The calculation of $\Phi(r)$ necessarily requires a radial integral from $r = 0$. At the center, if the potential is smooth then the radial force has to be zero and $d\Phi(r = 0)/dr = 0$. But no matter how $\Phi(r = 0)$ is chosen, the mass will be infinite. If the mass had been finite, then the escape speed would drop below σ at some radius and not all the stars would be bound.

3.2 King Model

To avoid these problems near the escape speed for finite mass, we can take the King model, also known as the lowered isothermal model, which is given by

$$f_K(E) = \frac{n_0}{(2\pi\sigma)^{3/2}} \exp\left[-\left(\Phi(r) + \frac{v^2}{2}\right)\Big/\sigma^2 - 1\right]. \tag{25}$$

This model has a truncation radius beyond which the number density declines dramatically, but the interior is nearly isothermal. This model provides a good representation of globular clusters.

3.3 Angular Momentum

We can modify Equation 25 to give

$$f_A(E, L) = f_K(E) \exp\left[-\mathbf{L}^2/(2\sigma^2 r_a^2)\right] \tag{26}$$

where r_a is the *anisotropy radius.* Outside of r_a the stars have radial orbits, as the large angular momentum orbits (e.g., circular) are exponentially suppressed.

Flattened systems can be described by distribution functions that depend on L_z, e.g., $f(E, L_z)$. For example, if

$$f(E, L_z) = \tilde{f}(E) L_z^2 \text{ for } E > 0 \tag{27}$$

for some function $\tilde{f}$, then few stars will have orbits that approach the z axis (which requires $L_z \approx 0$) while many will have nearly circular orbits with large L_z. Note that this requires

$$n(\mathbf{x})\langle v_z^2 \rangle \equiv \int f\left[\Phi(\mathbf{x}) + \frac{\mathbf{v}^2}{2}, Rv_\phi\right] dv_R dv_\phi dv_z = n(\mathbf{x})\langle v_R^2 \rangle. \tag{28}$$

But for the Galactic disk, we saw that $\langle v_R^2 \rangle > \langle v_z^2 \rangle$. So for the Galactic disk the distribution function cannot just be $f(E, L_z)$ but must instead rely on a *third integral of motion*. One can show that for an axisymmetric potential $\Phi(R, z)$ there is no other function of position and velocity other than E and L_z that is conserved along an orbit, which is perplexing!