# Astronomy 400B Lecture 5: Stellar Orbits

Brant Robertson

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## 1 Motion Under Gravity

Newton's Law of Gravity point mass $M$ attracts another mass $m$ separated by distance $\mathbf{r}$, causing a change in momentum $m\mathbf{v}$ of:

$$\frac{d}{dt}(m\mathbf{v}) = -\frac{GmM}{r^3}\mathbf{r} \tag{1}$$

where $G$ is Newton's gravitational constant. For an $N$-body system, we have

$$\frac{d}{dt}(mi\mathbf{v}_i) = -\sum_{j\neq i}\frac{Gm_im_j}{|\mathbf{x}_i - \mathbf{x}_j|^3}(\mathbf{x}_i - \mathbf{x}_j) \tag{2}$$

This equation can be re-written as

$$\frac{d}{dt}(m\mathbf{v}_i) = -m\nabla\Phi(\mathbf{x}_i) \tag{3}$$

where

$$\Phi(\mathbf{x}_i) = -\sum_{i\neq j}\frac{Gm_j}{|\mathbf{x}_i - \mathbf{x}_j|} \tag{4}$$

is the gravitational potential supplied by the point mass distribution at positions $\mathbf{x}_i$. Note we have chosen to define the potential such that $\Phi(x) \to 0$ as $x \to \infty$ but this is arbitrary. Note that

$$\nabla = \left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial x}\right] \tag{5}$$

### 1.1 Continuous Matter Distributions

Now consider a continuous distribution of matter density $\rho(\mathbf{x})$. The potential generated by $\rho(\mathbf{x})$ is given by

$$\Phi(\mathbf{x}) = -\int \frac{G\rho(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|}d^3\mathbf{x}' \tag{6}$$

Note that the integral is performed over $\mathbf{x}'$. The force $\mathbf{F}$ per unit mass is

$$\mathbf{F}(\mathbf{x}) = -\nabla\Phi(\mathbf{x}) = -\int \frac{G\rho(\mathbf{x}')(\mathbf{x} - \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^3}d^3\mathbf{x}' \tag{7}$$

### 1.2 Poisson's Equation

Take Equation 6 and apply the Laplacian operator

$$\nabla^2 \equiv \nabla \cdot \nabla = \left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right] \tag{8}$$

to both sides. Remembering that the operator acts on $\mathbf{x}$ and not $\mathbf{x}'$, we have

$$\nabla^2\Phi(\mathbf{x}) = -\int G\rho(\mathbf{x}')\nabla^2\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right)d^3\mathbf{x}'. \tag{9}$$

We can evaluate this by noting that

$$\nabla\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right), \nabla^2\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right) = 0. \tag{10}$$

So we conclude that outside of a very small region around $\mathbf{x}$, $\nabla^2\Phi(\mathbf{x}) = 0$. Let's take a spherical region $S(\epsilon)$ of radius $\epsilon$ centered on $\mathbf{x}$. In proceeding, let's note that

$$\nabla^2 f(|\mathbf{x}-\mathbf{x}'|) = \nabla^2_{\mathbf{x}'} f(|\mathbf{x}-\mathbf{x}'|) \tag{11}$$

for any function $f(|\mathbf{x}-\mathbf{x}'|)$. If we take $\epsilon$ to be small enough such that $\rho(\mathbf{x}) \approx$ a constant, then we can write

$$\begin{aligned}\nabla^2\Phi(\mathbf{x}) &\approx -G\rho(\mathbf{x})\int_{S(\epsilon)}\nabla^2\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right)d^3\mathbf{x}' \\ &= -G\rho(\mathbf{x})\int_{S(\epsilon)}\nabla^2_{\mathbf{x}'}\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right)dV'. \end{aligned}\tag{12}$$

Now we get to use the *divergence* theorem

$$\int \nabla^2 f dV = \oint \nabla f \cdot dS, \tag{13}$$

which allows us to write Equation 12 as

$$-G\rho(\mathbf{x})\int_{S(\epsilon)}\nabla^2_{\mathbf{x}'}\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right)dV' = -G\rho(\mathbf{x})\oint_{S(\epsilon)}\nabla_{\mathbf{x}'}\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right)\cdot d\mathbf{S}' \tag{14}$$

By applying Equation 10 and the identity $\nabla_{\mathbf{x}'} f = -\nabla f$, we have

$$\begin{aligned}-G\rho(\mathbf{x})\oint_{S(\epsilon)}\nabla_{\mathbf{x}'}\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right)\cdot d\mathbf{S}' &= -G\rho(\mathbf{x})\oint_{S(\epsilon)}\left(\frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3}\right)\cdot d\mathbf{S}' \\ &= 4\pi G\rho(\mathbf{x}) \end{aligned}\tag{15}$$

## 1.3 Inside a Uniform Shell

The gravitational force inside a spherical shell of uniform density is zero. The potential is a constant.

**See Figure 3.1 of Sparke and Gallagher.**

The opening angle OA is the same as OB, so the ratio of the enclosed mass is $(\mathrm{SA}/\mathrm{SB})^2$. Since the ratio of the forces scale like the inverse of this ratio (from the inverse square law), the force contributions of the A and B patches are equal and opposite.

## 1.4 Gravitational Potential Outside a Uniform Spherical Shell

**See Figure 3.2 of Sparke and Gallagher**

We are calculating the potential a uniform spherical shell of mass $M$ and radius $a$. Consider a point $P$ a distance $r$. The contribution of a narrow cone of opening solid angle $\Delta\Omega$ around another point $Q'$ is

$$\Delta\Phi[\mathbf{x}(P)] = -\frac{GM}{|\mathbf{x}(P)-\mathbf{x}(Q')|}\frac{\Delta\Omega}{4\pi} \tag{16}$$

Now consider the potential $\Phi'$ at point $P'$ at a radius $a$ away from the center of a shell of the same mass $M$ but with a radius $r$. The contribution $\Delta\Phi'$ from the material in the same cone of solid angle $\Delta\Omega$ but at point $Q$ a distance $r$ away is

$$\Delta\Phi'[\mathbf{x}(P')] = -\frac{GM}{|\mathbf{x}(P')-\mathbf{x}(Q)|}\frac{\Delta\Omega}{4\pi} \tag{17}$$

but since $PQ' = P'Q$, $\Delta\Phi[\mathbf{x}(P)] = \Delta\Phi'[\mathbf{x}(P')]$. When we integrate over $4\pi$, we have

$$\Phi[\mathbf{x}(P)] = \Phi'[\mathbf{x}(P')] = \Phi'[\mathbf{x} = 0] = -\frac{GM}{r} \tag{18}$$

The force associated with this spherical shell is just $F(r) = \nabla\Phi[\mathbf{x}(P)] = -\frac{GMm}{r}$. So the force outside the shell is the same as for a point mass at distance $r$.

Inside a spherical mass distribution $\rho(r)$, the centripetal acceleration that allows for a circular orbit must be the radial gravitational force inwards. On a circular orbit, in terms of the circular velocity $V$ this acceleration is just

$$a = \frac{V^2(r)}{r} = -F(r) = \frac{GM(< r)}{r^2}. \tag{19}$$

For a point mass, $V(r) \propto r^{-1/2}$. No extended distribution can have a circular velocity curve that declines more rapidly than $\propto r^{-1/2}$.

Note that the potential of a distributed mass density $\rho(\mathbf{x})$ is not the same as for a point mass. Instead, we have

$$\Phi(r) = -\left[\frac{GM(< r)}{r} 4\pi G \int_r^\infty \rho(r')r'dr'\right]. \tag{20}$$

But as long as the spherical mass distribution has a finite size, eventually we will have

$$\Phi(\mathbf{x}) \to -\frac{GM_{\rm tot}}{|\mathbf{x}|} \tag{21}$$

at large enough radius.

### 1.5 Moving Through a Potential

If we are moving through a background potential $\Phi(\mathbf{x})$ with velocity $\mathbf{x}$, the potential we experience changes with time according to $d\Phi/dt = \mathbf{x} \cdot \nabla\Phi(\mathbf{x})$. We can re-write Newton's equation as

$$\mathbf{x} \cdot \frac{d}{dt}(m\mathbf{v}) + m\mathbf{v} \cdot \nabla\Phi(\mathbf{x}) = 0 = \frac{d}{dt}\left[\frac{1}{2}m\mathbf{v}^2 + m\Phi(\mathbf{x})\right] \tag{22}$$

Therefore, the total energy

$$E \equiv \frac{1}{2}m\mathbf{v}^2 + m\Phi(\mathbf{x}) = \text{ const} \tag{23}$$

. We can write of course that $E = KE + PE$, where $KE = \frac{1}{2}m\mathbf{v}^2$ and $PE = m\Phi(\mathbf{x})$. The kinetic energy cannot be negative, and we adopt $\Phi(\mathbf{x}) \to 0$ as $\mathbf{x} \to \infty$. At position $\mathbf{x}$, an orbit is unboutnd only if the total energy $E > 0$. The speed at this place in the orbit must exceed the escape speed, which is found by setting Equation 23 to zero. We then have

$$v_e^2 = -2\Phi(\mathbf{x}). \tag{24}$$

### 1.6 Angular Momentum

The angular momentum of an orbit is $L = \mathbf{x} \times m\mathbf{v}$. The time rate of change is

$$\frac{dL}{dt} = \mathbf{x} \times \frac{d}{dt}(m\mathbf{v}) = -m\mathbf{x} \times \nabla\Phi. \tag{25}$$

For a spherically symmetric distribution, the force is central and $dL/dt = 0$ (angular momentum is conserved). In an axisymmetric distribution, on the component of $L$ parallel to the symmetry axis is conserved.

## 1.7 Total Energy is Not Conserved in a Time-Dependent Potential

In a many-body system, the total energy of each star is not individually conserved. The time derivative of the kinetic energy of star $i$ is

$$\sum_i \mathbf{v}_i \cdot \frac{d}{dt}(m_i \mathbf{v}_i) = \frac{d}{dt} KE = -\sum_{i,j;i\neq j} \frac{G m_i m_j}{|\mathbf{x}_i - \mathbf{x}_j|^3}(\mathbf{x}_i - \mathbf{x}_j) \cdot \mathbf{v}_i \tag{26}$$

Doing the same calculation on star $j$ and taking the dot product with $\mathbf{x}_j$ gives

$$\frac{1}{2}\sum_j (m_j \mathbf{v}_j \cdot \mathbf{v}_j) = -\sum_{i,j;i\neq j} \frac{G m_i m_j}{|\mathbf{x}_i - \mathbf{x}_j|^3}(\mathbf{x}_j - \mathbf{x}_i) \cdot \mathbf{x}_j \tag{27}$$

Adding the RHS of these two equations gives

$$-\sum_{i,j;i\neq j} \frac{G m_i m_j}{|\mathbf{x}_i - \mathbf{x}_j|^3}(\mathbf{x}_j - \mathbf{x}_i) \cdot (\mathbf{v}_i - \mathbf{v}_j) = \sum_{i,j;i\neq j} \left( \frac{G m_i m_j}{|\mathbf{x}_i - \mathbf{x}_j|} \right). \tag{28}$$

The potential energy $PE$ is a sum of pairs of potentials from individual objects

$$PE = -\frac{1}{2}\sum_{i,j;i\neq j} \frac{G m_i m_j}{|\mathbf{x}_i - \mathbf{x}_j|} = \frac{1}{2}\sum_i m_i \Phi(\mathbf{x}_i) = \frac{1}{2}\int \rho(\mathbf{x})\Phi(\mathbf{x}) dV. \tag{29}$$

We divided by two so every object contributes only once.

We can now see that, for the whole collection of objects

$$2\frac{d}{dt}\left[ KE - \frac{1}{2}\sum_{i,j;i\neq j} \frac{G m_i m_j}{|\mathbf{x}_i - \mathbf{x}_j|} \right] = 0. \tag{30}$$

This means the total energy of the system is conserved.

## 1.8 External Forces

Consider the total force on an object $i$ in a many body system under the influence of an external force $\mathbf{F}_{\text{ext}}$.

$$\sum_i \frac{d}{dt}(m_i \mathbf{v}_i) \cdot \mathbf{x}_i = -\sum_{i,j;i\neq j} \frac{G m_i m_j}{|\mathbf{x}_i - \mathbf{x}_j|^3}(\mathbf{x}_i - \mathbf{x}_j) \cdot \mathbf{x}_i + \sum_i \mathbf{F}^i_{\text{ext}} \cdot \mathbf{x}_i. \tag{31}$$

The force on the $j$th object is

$$\sum_j \frac{d}{dt}(m_j \mathbf{v}_j) \cdot \mathbf{x}_j = -\sum_{i,j;i\neq j} \frac{G m_i m_j}{|\mathbf{x}_i - \mathbf{x}_j|^3}(\mathbf{x}_j - \mathbf{x}_i) \cdot \mathbf{x}_j + \sum_j \mathbf{F}^j_{\text{ext}} \cdot \mathbf{x}_j \tag{32}$$

The left hand sides of these equations are equal, and are equal to

$$\frac{1}{2}\sum_i \frac{d^2}{dt^2}(m_i \mathbf{x}_i \cdot \mathbf{x}_i) - \sum_i m_i \mathbf{v}_i \cdot \mathbf{v}_i = \frac{1}{2}\frac{d^2 I}{dt^2} - 2KE \tag{33}$$

where the moment of inertia $I$ is

$$I \equiv \sum_i m_i \mathbf{x}_i \cdot \mathbf{x}_i. \tag{34}$$

By averaging the force on $i$ and $j$, we find

$$\frac{1}{2}\frac{d^2 I}{dt^2} - 2KE = PE + \sum_i \mathbf{F}^i_{\text{ext}} \cdot \mathbf{x}_i \tag{35}$$

and averaging this over a short time interval $0 < t < \tau$ gives

$$\frac{1}{2\tau}\left[\frac{dI}{dt(\tau)} - \frac{dI}{dt}(0)\right] = 2\langle KE\rangle + \langle PE\rangle + \sum_i \langle \mathbf{F}^i_{\text{ext}} \cdot \mathbf{x}_i\rangle \tag{36}$$

If all objects in the system are bound, then $|\mathbf{x}_i \cdot \mathbf{v}_i|$ and $dI/dt$ will be finite. As $\tau \to \infty$, the LHS goes to zero. Then we have

$$2\langle KE\rangle + \langle PE\rangle + \sum_i \langle \mathbf{F}^i_{\text{ext}} \cdot \mathbf{x}_i\rangle = 0 \tag{37}$$