Astronomy 400B Lecture 5: Stellar Orbits

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1 Motion Under Gravity

Newton's Law of Gravity point mass M attracts another mass m separated by distance $\mathbf{r}$, causing a change in momentum $m\mathbf{v}$ of:

$$\frac{d}{dt}(m\mathbf{v}) = -\frac{GmM}{r^3}\mathbf{r} \tag{1}$$

where G is Newton's gravitational constant. For an N-body system, we have

$$\frac{d}{dt}(m_i\mathbf{v}_i) = -\sum_{j\neq i}\frac{Gm_i m_j}{|\mathbf{x}_i - \mathbf{x}_j|^3}(\mathbf{x}_i - \mathbf{x}_j) \tag{2}$$

This equation can be re-written as

$$\frac{d}{dt}(m\mathbf{v}_i) = -m\nabla\Phi(\mathbf{x}_i) \tag{3}$$

where

$$\Phi(\mathbf{x}_i) = -\sum_{i\neq j}\frac{Gm_j}{|\mathbf{x}_i - \mathbf{x}_j|} \tag{4}$$

is the gravitational potential supplied by the point mass distribution at positions $\mathbf{x}_i$. Note we have chosen to define the potential such that $\Phi(x) \to 0$ as $x \to \infty$ but this is arbitrary. Note that

$$\nabla = \left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial x}\right] \tag{5}$$

1.1 Continuous Matter Distributions

Now consider a continuous distribution of matter density $\rho(\mathbf{x})$. The potential generated by $\rho(\mathbf{x})$ is given by

$$\Phi(\mathbf{x}) = -\int \frac{G\rho(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|}d^3\mathbf{x}' \tag{6}$$

Note that the integral is performed over $\mathbf{x}'$. The force $\mathbf{F}$ per unit mass is

$$\mathbf{F}(\mathbf{x}) = -\nabla\Phi(\mathbf{x}) = -\int \frac{G\rho(\mathbf{x}')(\mathbf{x} - \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^3}d^3\mathbf{x}' \tag{7}$$

1.2 Poisson's Equation

Take Equation 6 and apply the Laplacian operator

$$\nabla^2 \equiv \nabla \cdot \nabla = \left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right] \tag{8}$$

to both sides. Remembering that the operator acts on $\mathbf{x}$ and not $\mathbf{x}'$, we have

$$\nabla^2\Phi(\mathbf{x}) = -\int G\rho(\mathbf{x}')\nabla^2\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right)d^3\mathbf{x}'. \tag{9}$$

We can evaluate this by noting that

$$\nabla\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right) = -\frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3}, \nabla^2\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right) = 0. \tag{10}$$

So we conclude that outside of a very small region around $\mathbf{x}$, $\nabla^2\Phi(\mathbf{x}) = 0$. Let's take a spherical region $S(\epsilon)$ of radius ϵ centered on $\mathbf{x}$. In proceeding, let's note that

$$\nabla^2 f(|\mathbf{x}-\mathbf{x}'|) = \nabla^2_{\mathbf{x}'} f(|\mathbf{x}-\mathbf{x}'|) \tag{11}$$

for any function $f(|\mathbf{x}-\mathbf{x}'|)$. If we take ϵ to be small enough such that $\rho(\mathbf{x}) \approx$ a constant, then we can write

$$\begin{aligned}\nabla^2\Phi(\mathbf{x}) &\approx -G\rho(\mathbf{x})\int_{S(\epsilon)}\nabla^2\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right)d^3\mathbf{x}' \\ &= -G\rho(\mathbf{x})\int_{S(\epsilon)}\nabla^2_{\mathbf{x}'}\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right)dV'. \end{aligned} \tag{12}$$

Now we get to use the *divergence* theorem

$$\int\nabla^2 f dV = \oint\nabla f\cdot dS, \tag{13}$$

which allows us to write Equation 12 as

$$-G\rho(\mathbf{x})\int_{S(\epsilon)}\nabla^2_{\mathbf{x}'}\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right)dV' = -G\rho(\mathbf{x})\oint_{S(\epsilon)}\nabla_{\mathbf{x}'}\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right)\cdot d\mathbf{S}' \tag{14}$$

By applying Equation 10 and the identity $\nabla_{\mathbf{x}'} f = -\nabla f$, we have

$$\begin{aligned}-G\rho(\mathbf{x})\oint_{S(\epsilon)}\nabla_{\mathbf{x}'}\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right)\cdot d\mathbf{S}' &= -G\rho(\mathbf{x})\oint_{S(\epsilon)}\left(\frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3}\right)\cdot d\mathbf{S}' \\ &= 4\pi G\rho(\mathbf{x}) \end{aligned} \tag{15}$$

1.3 Inside a Uniform Shell

The gravitational force inside a spherical shell of uniform density is zero. The potential is a constant.

See Figure 3.1 of Sparke and Gallagher.

The opening angle OA is the same as OB, so the ratio of the enclosed mass is $(\mathrm{SA}/\mathrm{SB})^2$. Since the ratio of the forces scale like the inverse of this ratio (from the inverse square law), the force contributions of the A and B patches are equal and opposite.

1.4 Gravitational Potential Outside a Uniform Spherical Shell

See Figure 3.2 of Sparke and Gallagher

We are calculating the potential a uniform spherical shell of mass M and radius a. Consider a point P a distance r. The contribution of a narrow cone of opening solid angle $\Delta\Omega$ around another point Q' is

$$\Delta\Phi[\mathbf{x}(P)] = -\frac{GM}{|\mathbf{x}(P)-\mathbf{x}(Q')|}\frac{\Delta\Omega}{4\pi} \tag{16}$$

Now consider the potential Φ' at point P' at a radius a away from the center of a shell of the same mass M but with a radius r. The contribution $\Delta\Phi'$ from the material in the same cone of solid angle $\Delta\Omega$ but at point Q a distance r away is

$$\Delta\Phi'[\mathbf{x}(P')] = -\frac{GM}{|\mathbf{x}(P')-\mathbf{x}(Q)|}\frac{\Delta\Omega}{4\pi} \tag{17}$$

but since $PQ' = P'Q$, $\Delta\Phi[\mathbf{x}(P)] = \Delta\Phi'[\mathbf{x}(P')]$. When we integrate over 4π, we have

$$\Phi[\mathbf{x}(P)] = \Phi'[\mathbf{x}(P')] = \Phi'[\mathbf{x} = 0] = -\frac{GM}{r} \tag{18}$$

The force associated with this spherical shell is just $F(r) = \nabla\Phi[\mathbf{x}(P)] = -\frac{GMm}{r}$. So the force outside the shell is the same as for a point mass at distance r.

Inside a spherical mass distribution $\rho(r)$, the centripetal acceleration that allows for a circular orbit must be the radial gravitational force inwards. On a circular orbit, in terms of the circular velocity V this acceleration is just

$$a = \frac{V^2(r)}{r} = -F(r) = \frac{GM(< r)}{r^2}. \tag{19}$$

For a point mass, $V(r) \propto r^{-1/2}$. No extended distribution can have a circular velocity curve that declines more rapidly than $\propto r^{-1/2}$.

Note that the potential of a distributed mass density $\rho(\mathbf{x})$ is not the same as for a point mass. Instead, we have

$$\Phi(r) = -\left[\frac{GM(< r)}{r} + 4\pi G\int_r^\infty \rho(r')r'dr'\right]. \tag{20}$$

But as long as the spherical mass distribution has a finite size, eventually we will have

$$\Phi(\mathbf{x}) \to -\frac{GM_{\rm tot}}{|\mathbf{x}|} \tag{21}$$

at large enough radius.

1.5 Moving Through a Potential

If we are moving through a background potential $\Phi(\mathbf{x})$ with velocity $\mathbf{x}$, the potential we experience changes with time according to $d\Phi/dt = \mathbf{v}\cdot\nabla\Phi(\mathbf{x})$. We can re-write Newton's equation as

$$\mathbf{v}\cdot\frac{d}{dt}(m\mathbf{v}) + m\mathbf{v}\cdot\nabla\Phi(\mathbf{x}) = 0 = \frac{d}{dt}\left[\frac{1}{2}m\mathbf{v}^2 + m\Phi(\mathbf{x})\right] \tag{22}$$

Therefore, the total energy

$$E \equiv \frac{1}{2}m\mathbf{v}^2 + m\Phi(\mathbf{x}) = \text{ const} \tag{23}$$

. We can write of course that $E = KE + PE$, where $KE = \frac{1}{2}m\mathbf{v}^2$ and $PE = m\Phi(\mathbf{x})$. The kinetic energy cannot be negative, and we adopt $\Phi(\mathbf{x}) \to 0$ as $\mathbf{x} \to \infty$. At position $\mathbf{x}$, an orbit is unboutnd only if the total energy $E > 0$. The speed at this place in the orbit must exceed the escape speed, which is found by setting Equation 23 to zero. We then have

$$v_e^2 = -2\Phi(\mathbf{x}). \tag{24}$$

1.6 Angular Momentum

The angular momentum of an orbit is $L = \mathbf{x} \times m\mathbf{v}$. The time rate of change is

$$\frac{dL}{dt} = \mathbf{x} \times \frac{d}{dt}(m\mathbf{v}) = -m\mathbf{x} \times \nabla\Phi. \tag{25}$$

For a spherically symmetric distribution, the force is central and $dL/dt = 0$ (angular momentum is conserved). In an axisymmetric distribution, on the component of L parallel to the symmetry axis is conserved.

1.7 Total Energy is Not Conserved in a Time-Dependent Potential

In a many-body system, the total energy of each star is not individually conserved. The time derivative of the kinetic energy of star i is

$$\sum_i \mathbf{v}_i \cdot \frac{d}{dt}(m_i \mathbf{v}_i) = \frac{d}{dt} KE = -\sum_{i,j;i\neq j} \frac{G m_i m_j}{|\mathbf{x}_i - \mathbf{x}_j|^3}(\mathbf{x}_i - \mathbf{x}_j) \cdot \mathbf{v}_i \tag{26}$$

Doing the same calculation on star j and taking the dot product with $\mathbf{x}_j$ gives

$$\frac{1}{2}\sum_j (m_j \mathbf{v}_j \cdot \mathbf{v}_j) = -\sum_{i,j;i\neq j} \frac{G m_i m_j}{|\mathbf{x}_i - \mathbf{x}_j|^3}(\mathbf{x}_j - \mathbf{x}_i) \cdot \mathbf{x}_j \tag{27}$$

Adding the RHS of these two equations gives

$$-\sum_{i,j;i\neq j} \frac{G m_i m_j}{|\mathbf{x}_i - \mathbf{x}_j|^3}(\mathbf{x}_j - \mathbf{x}_i) \cdot (\mathbf{v}_i - \mathbf{v}_j) = \sum_{i,j;i\neq j} \left(\frac{G m_i m_j}{|\mathbf{x}_i - \mathbf{x}_j|} \right). \tag{28}$$

The potential energy PE is a sum of pairs of potentials from individual objects

$$PE = -\frac{1}{2}\sum_{i,j;i\neq j} \frac{G m_i m_j}{|\mathbf{x}_i - \mathbf{x}_j|} = \frac{1}{2}\sum_i m_i \Phi(\mathbf{x}_i) = \frac{1}{2}\int \rho(\mathbf{x})\Phi(\mathbf{x}) dV. \tag{29}$$

We divided by two so every object contributes only once.

We can now see that, for the whole collection of objects

$$2\frac{d}{dt}\left[KE - \frac{1}{2}\sum_{i,j;i\neq j} \frac{G m_i m_j}{|\mathbf{x}_i - \mathbf{x}_j|} \right] = 0. \tag{30}$$

This means the total energy of the system is conserved.

1.8 External Forces

Consider the total force on an object i in a many body system under the influence of an external force $\mathbf{F}_{\text{ext}}$.

$$\sum_i \frac{d}{dt}(m_i \mathbf{v}_i) \cdot \mathbf{x}_i = -\sum_{i,j;i\neq j} \frac{G m_i m_j}{|\mathbf{x}_i - \mathbf{x}_j|^3}(\mathbf{x}_i - \mathbf{x}_j) \cdot \mathbf{x}_i + \sum_i \mathbf{F}^i_{\text{ext}} \cdot \mathbf{x}_i. \tag{31}$$

The force on the jth object is

$$\sum_j \frac{d}{dt}(m_j \mathbf{v}_j) \cdot \mathbf{x}_j = -\sum_{i,j;i\neq j} \frac{G m_i m_j}{|\mathbf{x}_i - \mathbf{x}_j|^3}(\mathbf{x}_j - \mathbf{x}_i) \cdot \mathbf{x}_j + \sum_j \mathbf{F}^j_{\text{ext}} \cdot \mathbf{x}_j \tag{32}$$

The left hand sides of these equations are equal, and are equal to

$$\frac{1}{2}\sum_i \frac{d^2}{dt^2}(m_i \mathbf{x}_i \cdot \mathbf{x}_i) - \sum_i m_i \mathbf{v}_i \cdot \mathbf{v}_i = \frac{1}{2}\frac{d^2 I}{dt^2} - 2KE \tag{33}$$

where the moment of inertia I is

$$I \equiv \sum_i m_i \mathbf{x}_i \cdot \mathbf{x}_i. \tag{34}$$

By averaging the force on i and j, we find

$$\frac{1}{2}\frac{d^2 I}{dt^2} - 2KE = PE + \sum_i \mathbf{F}^i_{\text{ext}} \cdot \mathbf{x}_i \tag{35}$$

and averaging this over a short time interval $0 < t < \tau$ gives

$$\frac{1}{2\tau}\left[\frac{dI}{dt(\tau)} - \frac{dI}{dt}(0)\right] = 2\langle KE\rangle + \langle PE\rangle + \sum_i \langle \mathbf{F}^i_{\rm ext}\cdot\mathbf{x}_i\rangle \tag{36}$$

If all objects in the system are bound, then $|\mathbf{x}_i\cdot\mathbf{v}_i|$ and dI/dt will be finite. As $\tau\to\infty$, the LHS goes to zero. Then we have

$$2\langle KE\rangle + \langle PE\rangle + \sum_i \langle \mathbf{F}^i_{\rm ext}\cdot\mathbf{x}_i\rangle = 0 \tag{37}$$

2 Two-Body Relaxation

A potential can be thought of as a combination of a smooth and steep potential wells. We can calculate the average time between strong encounters with the steep potential wells supplied near individual stars. Suppose taht the stars have mass m and typical velocities V. If two stars come within r of one another, their kinetic energies increase to balance the change in potential energy. We say there is a *strong encounter* if the change in potential energy is comparable to their starting kinetic energy. In other words

$$\frac{Gm^2}{r} \gtrsim \frac{mV^2}{2} \tag{38}$$

and the radius must be smaller than

$$r \lesssim r_s \equiv \frac{2Gm}{V^2}. \tag{39}$$

In the solar neighborhood, $V\approx 30$ km s^{-1}, and taking $m\sim 0.5M_\odot$ we have $r_s\approx 1$ AU. For the Sun, there has been no strong encounter for ~ 4.5 Gyr. Over a time t, the Sun has an encounter with all stars in a cylinder with volume $\pi r_s^2 Vt$. If there is a number density of n, then we are interested in the time when $n\pi r_s^2 Vt = 1$. This time is

$$t_s = \frac{V^3}{4\pi G^2 m^3 n} \approx 4\times 10^{12}\ \text{yr}\ \left(\frac{V}{10\ \text{km}\ s^{-1}}\right)^3 \left(\frac{m}{M_\odot}\right)^{-2} \left(\frac{n}{1\ \text{pc}^{-3}}\right)^{-1}. \tag{40}$$

For $n\approx 0.1$ pc^{-3}, then $t_s\sim 10^{15}$ yr.

2.1 Weak Encounters

Instead of strong encounters, we need to calculate the effects of weaker encounters on indivudal stars. We do this via the impulse approximation, that tells us how to approximate the change in velocity from a weak encounter. As one object passes another, the perpendicular force between them is

$$\mathbf{F}_\perp = \frac{GmMb}{(b^2+V^2t^2)^{3/2}} = M\frac{dV_\perp}{dt}. \tag{41}$$

Integrating over time, we find that

$$\Delta V_\perp = \frac{1}{M}\int_{-\infty}^{\infty} \mathbf{F}_\perp(t)dt\frac{2Gm}{bV}. \tag{42}$$

Slower approaches result in larger perpendicular velocity changes. The path of M is bent via

$$\alpha = \frac{\Delta V_\perp}{V} = \frac{2Gm}{bV^2}. \tag{43}$$

A weak encounter requires b to be larger than r_s.

The number of stars with mass m passing M with separations between b and $b+\Delta b$ is the product of the number density n and the volume $Vt\cdot 2\pi b\Delta b$. We multiply by $\Delta V_\perp^2$ and integrate over b to find

$$\langle \Delta V_\perp^2\rangle = \int_{b_{min}}^{b_{max}} nVt\left(\frac{2Gm}{bV}\right)^2 2\pi b db = \frac{8\pi G^2 m^2 nt}{V}\ln\left(\frac{b_{max}}{b_{min}}\right). \tag{44}$$

After a time t_{relax} such that $\langle \Delta V_\perp^2 \rangle = V^2$, the expected perpendicular velocity becomes about equal to its original forward speed. The initial path is forgotten! Defining $\Lambda \equiv (b_{max}/b_{min})$, we find the relaxation time is much shorter than the strong interaction time as

$$t_{\text{relax}} = \frac{V^3}{8\pi G^2 m^2 n \ln \Lambda} = \frac{t_s}{2 \ln \Lambda} \tag{45}$$

$$t_{\text{relax}} \approx \frac{2 \times 10^9 \text{ yr}}{\ln \Lambda} \left(\frac{V}{10 \text{ km } s^{-1}} \right)^3 \left(\frac{m}{M_\odot} \right)^{-2} \left(\frac{n}{10 \text{ pc}^{-3}} \right)^{-1}. \tag{46}$$

So what is Λ? Well, if $b < r_s$ the method can't be correct and we usually take $b_{min} = r_s$. We can then take b_{max} to be the size of the whole system. For the Sun, $r_s = 1$ AU. If we take 300 pc $\leq b_{max} \leq$ 30 kpc, then $\Lambda \approx 18 - 22$. So the exact value of Λ doesn't matter, as $t_{\text{relax}} \approx 10^{13}$ yr.

In a cluster consisting of N stars with mass m and typical V, the average separateiion is about half the size of the system R. We have

$$\frac{1}{2} N m V^2 \sim \frac{G(Nm)^2}{2R}, \ \Lambda = \frac{R}{r_s} \sim \frac{GmN}{V^2} \cdot \frac{V^2}{2Gm} \sim \frac{N}{2}. \tag{47}$$

The crossing time of the system is $t_{cross} \sim R/V$. Since $N = 4n\pi R^3/3$, we have

$$\frac{t_{relax}}{t_{cross}} \sim \frac{V^4 R^2}{6 N G^2 m^2 \ln \Lambda} \sim \frac{N}{6 \ln(N/2)}. \tag{48}$$

For $N \sim 10^{11}$ stars, the relaxation will take 10^9 crossing times. For globulars with 10^6 stars, the relaxation time for the whole system is $t_{\text{relax}} \sim 10^4 t_{cross}$.