Astronomy 400B Lecture 5: Stellar Orbits

Brant Robertson

February, 2015

1 Motion Under Gravity

Newton's Law of Gravity point mass M attracts another mass m separated by distance $\mathbf{r}$, causing a change in momentum $m\mathbf{v}$ of:

$$\frac{d}{dt}(m\mathbf{v}) = -\frac{GmM}{r^3}\mathbf{r} \tag{1}$$

where G is Newton's gravitational constant. For an N-body system, we have

$$\frac{d}{dt}(mi\mathbf{v}_i) = -\sum_{j\neq i}\frac{Gm_im_j}{|\mathbf{x}_i - \mathbf{x}_j|^3}(\mathbf{x}_i - \mathbf{x}_j) \tag{2}$$

This equation can be re-written as

$$\frac{d}{dt}(m\mathbf{v}_i) = -m\nabla\Phi(\mathbf{x}_i) \tag{3}$$

where

$$\Phi(\mathbf{x}_i) = -\sum_{i\neq j}\frac{Gm_j}{|\mathbf{x}_i - \mathbf{x}_j|} \tag{4}$$

is the gravitational potential supplied by the point mass distribution at positions $\mathbf{x}_i$. Note we have chosen to define the potential such that $\Phi(x) \to 0$ as $x \to \infty$ but this is arbitrary. Note that

$$\nabla = \left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial x}\right] \tag{5}$$

1.1 Continuous Matter Distributions

Now consider a continuous distribution of matter density $\rho(\mathbf{x})$. The potential generated by $\rho(\mathbf{x})$ is given by

$$\Phi(\mathbf{x}) = -\int\frac{G\rho(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|}d^3\mathbf{x}' \tag{6}$$

Note that the integral is performed over $\mathbf{x}'$. The force $\mathbf{F}$ per unit mass is

$$\mathbf{F}(\mathbf{x}) = -\nabla\Phi(\mathbf{x}) = -\int\frac{G\rho(\mathbf{x}')(\mathbf{x} - \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^3}d^3\mathbf{x}' \tag{7}$$

1.2 Poisson's Equation

Take Equation 6 and apply the Laplacian operator

$$\nabla^2 \equiv \nabla \cdot \nabla = \left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right] \tag{8}$$

to both sides. Remembering that the operator acts on $\mathbf{x}$ and not $\mathbf{x}'$, we have

$$\nabla^2\Phi(\mathbf{x}) = -\int G\rho(\mathbf{x}')\nabla^2\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right)d^3\mathbf{x}'. \tag{9}$$

We can evaluate this by noting that

$$\nabla\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right), \nabla^2\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right) = 0. \tag{10}$$

So we conclude that outside of a very small region around $\mathbf{x}$, $\nabla^2\Phi(\mathbf{x}) = 0$. Let's take a spherical region $S(\epsilon)$ of radius ϵ centered on $\mathbf{x}$. In proceeding, let's note that

$$\nabla^2 f(|\mathbf{x}-\mathbf{x}'|) = \nabla^2_{\mathbf{x}'} f(|\mathbf{x}-\mathbf{x}'|) \tag{11}$$

for any function $f(|\mathbf{x}-\mathbf{x}'|)$. If we take ϵ to be small enough such that $\rho(\mathbf{x}) \approx$ a constant, then we can write

$$\begin{aligned}\nabla^2\Phi(\mathbf{x}) &\approx -G\rho(\mathbf{x})\int_{S(\epsilon)}\nabla^2\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right)d^3\mathbf{x}' \\ &= -G\rho(\mathbf{x})\int_{S(\epsilon)}\nabla^2_{\mathbf{x}}\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right)dV'. \end{aligned} \tag{12}$$

Now we get to use the *divergence* theorem

$$\int \nabla^2 f dV = \oint \nabla f \cdot dS, \tag{13}$$

which allows us to write Equation 12 as

$$-G\rho(\mathbf{x})\int_{S(\epsilon)}\nabla^2_{\mathbf{x}}\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right)dV' = -G\rho(\mathbf{x})\oint_S(\epsilon)\nabla\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right) \tag{14}$$