Astronomy 400B Lecture 5: Stellar Orbits

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1 Motion Under Gravity

Newton's Law of Gravity point mass M attracts another mass m separated by distance $\mathbf{r}$, causing a change in momentum $m\mathbf{v}$ of:

$$\frac{d}{dt}(m\mathbf{v}) = -\frac{GmM}{r^3}\mathbf{r} \tag{1}$$

where G is Newton's gravitational constant. For an N-body system, we have

$$\frac{d}{dt}(m_i\mathbf{v}_i) = -\sum_{j\neq i}\frac{Gm_im_j}{|\mathbf{x}_i - \mathbf{x}_j|^3}(\mathbf{x}_i - \mathbf{x}_j) \tag{2}$$

This equation can be re-written as

$$\frac{d}{dt}(m\mathbf{v}_i) = -m\nabla\Phi(\mathbf{x}_i) \tag{3}$$

where

$$\Phi(\mathbf{x}_i) = -\sum_{i\neq j}\frac{Gm_j}{|\mathbf{x}_i - \mathbf{x}_j|} \tag{4}$$

is the gravitational potential supplied by the point mass distribution at positions $\mathbf{x}_i$. Note we have chosen to define the potential such that $\Phi(x) \to 0$ as $x \to \infty$ but this is arbitrary. Note that

$$\nabla = \left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial x}\right] \tag{5}$$

1.1 Continuous Matter Distributions

Now consider a continuous distribution of matter density $\rho(\mathbf{x})$. The potential generated by $\rho(\mathbf{x})$ is given by

$$\Phi(\mathbf{x}) = -\int \frac{G\rho(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|}d^3\mathbf{x}' \tag{6}$$

Note that the integral is performed over $\mathbf{x}'$. The force $\mathbf{F}$ per unit mass is

$$\mathbf{F}(\mathbf{x}) = -\nabla\Phi(\mathbf{x}) = -\int \frac{G\rho(\mathbf{x}')(\mathbf{x} - \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^3}d^3\mathbf{x}' \tag{7}$$

1.2 Poisson's Equation

Take Equation 6 and apply the Laplacian operator

$$\nabla^2 \equiv \nabla \cdot \nabla = \left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right] \tag{8}$$

to both sides. Remembering that the operator acts on $\mathbf{x}$ and not $\mathbf{x}'$, we have

$$\nabla^2\Phi(\mathbf{x}) = -\int G\rho(\mathbf{x}')\nabla^2\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right)d^3\mathbf{x}'. \tag{9}$$

We can evaluate this by noting that

$$\nabla\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right) = -\frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3}, \nabla^2\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right) = 0. \tag{10}$$

So we conclude that outside of a very small region around $\mathbf{x}$, $\nabla^2\Phi(\mathbf{x}) = 0$. Let's take a spherical region $S(\epsilon)$ of radius ϵ centered on $\mathbf{x}$. In proceeding, let's note that

$$\nabla^2 f(|\mathbf{x}-\mathbf{x}'|) = \nabla^2_{\mathbf{x}'} f(|\mathbf{x}-\mathbf{x}'|) \tag{11}$$

for any function $f(|\mathbf{x}-\mathbf{x}'|)$. If we take ϵ to be small enough such that $\rho(\mathbf{x}) \approx$ a constant, then we can write

$$\begin{aligned}\nabla^2\Phi(\mathbf{x}) &\approx -G\rho(\mathbf{x})\int_{S(\epsilon)}\nabla^2\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right)d^3\mathbf{x}' \\ &= -G\rho(\mathbf{x})\int_{S(\epsilon)}\nabla^2_{\mathbf{x}'}\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right)dV'. \end{aligned}\tag{12}$$

Now we get to use the *divergence* theorem

$$\int \nabla^2 f dV = \oint \nabla f \cdot dS, \tag{13}$$

which allows us to write Equation 12 as

$$-G\rho(\mathbf{x})\int_{S(\epsilon)}\nabla^2_{\mathbf{x}'}\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right)dV' = -G\rho(\mathbf{x})\oint_{S(\epsilon)}\nabla_{\mathbf{x}'}\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right)\cdot d\mathbf{S}' \tag{14}$$

By applying Equation 10 and the identity $\nabla_{\mathbf{x}'} f = -\nabla f$, we have

$$\begin{aligned}-G\rho(\mathbf{x})\oint_{S(\epsilon)}\nabla_{\mathbf{x}'}\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right)\cdot d\mathbf{S}' &= -G\rho(\mathbf{x})\oint_{S(\epsilon)}\left(\frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3}\right)\cdot d\mathbf{S}' \\ &= 4\pi G\rho(\mathbf{x}) \end{aligned}\tag{15}$$

1.3 Inside a Uniform Shell

The gravitational force inside a spherical shell of uniform density is zero. The potential is a constant.

See Figure 3.1 of Sparke and Gallagher.

The opening angle OA is the same as OB, so the ratio of the enclosed mass is $(\mathrm{SA}/\mathrm{SB})^2$. Since the ratio of the forces scale like the inverse of this ratio (from the inverse square law), the force contributions of the A and B patches are equal and opposite.

1.4 Gravitational Potential Outside a Uniform Spherical Shell

See Figure 3.2 of Sparke and Gallagher

We are calculating the potential a uniform spherical shell of mass M and radius a. Consider a point P a distance r. The contribution of a narrow cone of opening solid angle $\Delta\Omega$ around another point Q' is

$$\Delta\Phi[\mathbf{x}(P)] = -\frac{GM}{|\mathbf{x}(P)-\mathbf{x}(Q')|}\frac{\Delta\Omega}{4\pi} \tag{16}$$

Now consider the potential Φ' at point P' at a radius a away from the center of a shell of the same mass M but with a radius r. The contribution $\Delta\Phi'$ from the material in the same cone of solid angle $\Delta\Omega$ but at point Q a distance r away is

$$\Delta\Phi'[\mathbf{x}(P')] = -\frac{GM}{|\mathbf{x}(P')-\mathbf{x}(Q)|}\frac{\Delta\Omega}{4\pi} \tag{17}$$

but since $PQ' = P'Q$, $\Delta\Phi[\mathbf{x}(P)] = \Delta\Phi'[\mathbf{x}(P')]$. When we integrate over 4π, we have

$$\Phi[\mathbf{x}(P)] = \Phi'[\mathbf{x}(P')] = \Phi'[\mathbf{x} = 0] = -\frac{GM}{r} \tag{18}$$

The force associated with this spherical shell is just $F(r) = \nabla\Phi[\mathbf{x}(P)] = -\frac{GMm}{r}$. So the force outside the shell is the same as for a point mass at distance r.

Inside a spherical mass distribution $\rho(r)$, the centripetal acceleration that allows for a circular orbit must be the radial gravitational force inwards. On a circular orbit, in terms of the circular velocity V this acceleration is just

$$a = \frac{V^2(r)}{r} = -F(r) = \frac{GM(< r)}{r^2}. \tag{19}$$

For a point mass, $V(r) \propto r^{-1/2}$. No extended distribution can have a circular velocity curve that declines more rapidly than $\propto r^{-1/2}$.

Note that the potential of a distributed mass density $\rho(\mathbf{x})$ is not the same as for a point mass. Instead, we have

$$\Phi(r) = -\left[\frac{GM(< r)}{r} + 4\pi G\int_r^\infty \rho(r')r'dr'\right]. \tag{20}$$

But as long as the spherical mass distribution has a finite size, eventually we will have

$$\Phi(\mathbf{x}) \to -\frac{GM_{\rm tot}}{|\mathbf{x}|} \tag{21}$$

at large enough radius.

1.5 Moving Through a Potential

If we are moving through a background potential $\Phi(\mathbf{x})$ with velocity $\mathbf{x}$, the potential we experience changes with time according to $d\Phi/dt = \mathbf{v}\cdot\nabla\Phi(\mathbf{x})$. We can re-write Newton's equation as

$$\mathbf{v}\cdot\frac{d}{dt}(m\mathbf{v}) + m\mathbf{v}\cdot\nabla\Phi(\mathbf{x}) = 0 = \frac{d}{dt}\left[\frac{1}{2}m\mathbf{v}^2 + m\Phi(\mathbf{x})\right] \tag{22}$$

Therefore, the total energy

$$E \equiv \frac{1}{2}m\mathbf{v}^2 + m\Phi(\mathbf{x}) = \text{ const} \tag{23}$$

. We can write of course that $E = KE + PE$, where $KE = \frac{1}{2}m\mathbf{v}^2$ and $PE = m\Phi(\mathbf{x})$. The kinetic energy cannot be negative, and we adopt $\Phi(\mathbf{x}) \to 0$ as $\mathbf{x} \to \infty$. At position $\mathbf{x}$, an orbit is unbound only if the total energy $E > 0$. The speed at this place in the orbit must exceed the escape speed, which is found by setting Equation 23 to zero. We then have

$$v_e^2 = -2\Phi(\mathbf{x}). \tag{24}$$

1.6 Angular Momentum

The angular momentum of an orbit is $L = \mathbf{x} \times m\mathbf{v}$. The time rate of change is

$$\frac{dL}{dt} = \mathbf{x} \times \frac{d}{dt}(m\mathbf{v}) = -m\mathbf{x} \times \nabla\Phi. \tag{25}$$

For a spherically symmetric distribution, the force is central and $dL/dt = 0$ (angular momentum is conserved). In an axisymmetric distribution, on the component of L parallel to the symmetry axis is conserved.

1.7 Individual Energy is Not Conserved in a Time-Dependent Potential

In a many-body system, the total energy of each star is not individually conserved. The time derivative of the kinetic energy of star i is

$$\sum_i \mathbf{v}_i \cdot \frac{d}{dt}(m_i \mathbf{v}_i) = \frac{d}{dt} KE = -\sum_{i,j;i\neq j} \frac{G m_i m_j}{|\mathbf{x}_i - \mathbf{x}_j|^3}(\mathbf{x}_i - \mathbf{x}_j)\cdot \mathbf{v}_i \tag{26}$$

Doing the same calculation on star j and taking the dot product with $\mathbf{v}_j$ gives

$$\frac{1}{2}\sum_j \frac{d}{dt}(m_j \mathbf{v}_j \cdot \mathbf{v}_j) = -\sum_{i,j;i\neq j} \frac{G m_i m_j}{|\mathbf{x}_i - \mathbf{x}_j|^3}(\mathbf{x}_j - \mathbf{x}_i)\cdot \mathbf{v}_j \tag{27}$$

Adding the RHS of these two equations gives

$$-\sum_{i,j;i\neq j} \frac{G m_i m_j}{|\mathbf{x}_i - \mathbf{x}_j|^3}(\mathbf{x}_i - \mathbf{x}_j)\cdot(\mathbf{v}_i - \mathbf{v}_j) = \sum_{i,j;i\neq j} \frac{d}{dt}\left(\frac{G m_i m_j}{|\mathbf{x}_i - \mathbf{x}_j|}\right). \tag{28}$$

The potential energy PE is a sum of pairs of potentials from individual objects

$$PE = -\frac{1}{2}\sum_{i,j;i\neq j} \frac{G m_i m_j}{|\mathbf{x}_i - \mathbf{x}_j|} = \frac{1}{2}\sum_i m_i \Phi(\mathbf{x}_i) = \frac{1}{2}\int \rho(\mathbf{x})\Phi(\mathbf{x}) dV. \tag{29}$$

We divided by two so every object contributes only once.

We can now see that, for the whole collection of objects

$$2\frac{d}{dt}\left[KE - \frac{1}{2}\sum_{i,j;i\neq j} \frac{G m_i m_j}{|\mathbf{x}_i - \mathbf{x}_j|}\right] = 0. \tag{30}$$

This means the total energy of the system is conserved.

1.8 External Forces

Consider the total force on an object i in a many body system under the influence of an external force $\mathbf{F}_{\rm ext}$.

$$\sum_i \frac{d}{dt}(m_i \mathbf{v}_i)\cdot \mathbf{x}_i = -\sum_{i,j;i\neq j} \frac{G m_i m_j}{|\mathbf{x}_i - \mathbf{x}_j|^3}(\mathbf{x}_i - \mathbf{x}_j)\cdot \mathbf{x}_i + \sum_i \mathbf{F}^i_{\rm ext}\cdot \mathbf{x}_i. \tag{31}$$

The force on the jth object is

$$\sum_j \frac{d}{dt}(m_j \mathbf{v}_j)\cdot \mathbf{x}_j = -\sum_{i,j;i\neq j} \frac{G m_i m_j}{|\mathbf{x}_i - \mathbf{x}_j|^3}(\mathbf{x}_j - \mathbf{x}_i)\cdot \mathbf{x}_j + \sum_j \mathbf{F}^j_{\rm ext}\cdot \mathbf{x}_j \tag{32}$$

The left hand sides of these equations are equal, and are equal to

$$\frac{1}{2}\sum_i \frac{d^2}{dt^2}(m_i \mathbf{x}_i \cdot \mathbf{x}_i) - \sum_i m_i \mathbf{v}_i \cdot \mathbf{v}_i = \frac{1}{2}\frac{d^2 I}{dt^2} - 2KE \tag{33}$$

where the moment of inertia I is

$$I \equiv \sum_i m_i \mathbf{x}_i \cdot \mathbf{x}_i. \tag{34}$$

By averaging the force on i and j, we find

$$\frac{1}{2}\frac{d^2 I}{dt^2} - 2KE = PE + \sum_i \mathbf{F}^i_{\rm ext}\cdot \mathbf{x}_i \tag{35}$$

and averaging this over a short time interval $0 < t < \tau$ gives

$$\frac{1}{2\tau}\left[\frac{dI}{dt}(\tau) - \frac{dI}{dt}(0)\right] = 2\langle KE\rangle + \langle PE\rangle + \sum_i \langle \mathbf{F}^i_{\rm ext}\cdot \mathbf{x}_i\rangle \tag{36}$$

If all objects in the system are bound, then $|\mathbf{x}_i \cdot \mathbf{v}_i|$ and dI/dt will be finite. As $\tau \to \infty$, the LHS goes to zero. Then we have

$$2\langle KE\rangle + \langle PE\rangle + \sum_i \langle \mathbf{F}^i_{\rm ext}\cdot \mathbf{x}_i\rangle = 0 \tag{37}$$

2 Two-Body Relaxation

A potential can be thought of as a combination of a smooth and steep potential wells. We can calculate the average time between strong encounters with the steep potential wells supplied near individual stars. Suppose that the stars have mass m and typical velocities V. If two stars come within r of one another, their kinetic energies increase to balance the change in potential energy. We say there is a *strong encounter* if the change in potential energy is comparable to their starting kinetic energy. In other words

$$\frac{Gm^2}{r} \gtrsim \frac{mV^2}{2} \tag{38}$$

and the radius must be smaller than

$$r \lesssim r_s \equiv \frac{2Gm}{V^2}. \tag{39}$$

In the solar neighborhood, $V \approx 30$ km s^{-1}, and taking $m \sim 0.5M_\odot$ we have $r_s \approx 1$ AU. For the Sun, there has been no strong encounter for ~ 4.5 Gyr. Over a time t, the Sun has an encounter with all stars in a cylinder with volume $\pi r_s^2 V t$. If there is a number density of n, then we are interested in the time when $n\pi r_s^2 V t = 1$. This time is

$$t_s = \frac{V^3}{4\pi G^2 m^3 n} \approx 4\times 10^{12}\ \text{yr}\ \left(\frac{V}{10\ \text{km}\ s^{-1}}\right)^3 \left(\frac{m}{M_\odot}\right)^{-2}\left(\frac{n}{1\ \text{pc}^{-3}}\right)^{-1}. \tag{40}$$

For $n \approx 0.1$ pc^{-3}, then $t_s \sim 10^{15}$ yr.

2.1 Weak Encounters

Instead of strong encounters, we need to calculate the effects of weaker encounters on indivudal stars. We do this via the impulse approximation, that tells us how to approximate the change in velocity from a weak encounter. As one object passes another, the perpendicular force between them is

$$\mathbf{F}_\perp = \frac{GmMb}{(b^2+V^2t^2)^{3/2}} = M\frac{dV_\perp}{dt}. \tag{41}$$

Integrating over time, we find that

$$\Delta V_\perp = \frac{1}{M}\int_{-\infty}^{\infty} \mathbf{F}_\perp(t)dt = \frac{2Gm}{bV}. \tag{42}$$

Slower approaches result in larger perpendicular velocity changes. The path of M is bent via

$$\alpha = \frac{\Delta V_\perp}{V} = \frac{2Gm}{bV^2}. \tag{43}$$

A weak encounter requires b to be larger than r_s.

The number of stars with mass m passing M with separations between b and $b+\Delta b$ is the product of the number density n and the volume $Vt\cdot 2\pi b\Delta b$. We multiply by $\Delta V_\perp^2$ and integrate over b to find

$$\langle \Delta V_\perp^2\rangle = \int_{b_{min}}^{b_{max}} nVt\left(\frac{2Gm}{bV}\right)^2 2\pi b db = \frac{8\pi G^2 m^2 nt}{V}\ln\left(\frac{b_{max}}{b_{min}}\right). \tag{44}$$

After a time t_{relax} such that $\langle \Delta V_\perp^2 \rangle = V^2$, the expected perpendicular velocity becomes about equal to its original forward speed. The initial path is forgotten! Defining $\Lambda \equiv (b_{max}/b_{min})$, we find the relaxation time is much shorter than the strong interaction time as

$$t_{\text{relax}} = \frac{V^3}{8\pi G^2 m^2 n \ln \Lambda} = \frac{t_s}{2 \ln \Lambda} \tag{45}$$

$$t_{\text{relax}} \approx \frac{2 \times 10^9 \text{ yr}}{\ln \Lambda} \left(\frac{V}{10 \text{ km } s^{-1}} \right)^3 \left(\frac{m}{M_\odot} \right)^{-2} \left(\frac{n}{10 \text{ pc}^{-3}} \right)^{-1}. \tag{46}$$

So what is Λ? Well, if $b < r_s$ the method can't be correct and we usually take $b_{min} = r_s$. We can then take b_{max} to be the size of the whole system. For the Sun, $r_s = 1$ AU. If we take 300 pc $\leq b_{max} \leq$ 30 kpc, then $\Lambda \approx 18 - 22$. So the exact value of Λ doesn't matter, as $t_{\text{relax}} \approx 10^{13}$ yr.

In a cluster consisting of N stars with mass m and typical V, the average separation is about half the size of the system R. We have

$$\frac{1}{2} N m V^2 \sim \frac{G(Nm)^2}{2R}, \ \Lambda = \frac{R}{r_s} \sim \frac{GmN}{V^2} \cdot \frac{V^2}{2Gm} \sim \frac{N}{2}. \tag{47}$$

The crossing time of the system is $t_{cross} \sim R/V$. Since $N = 4n\pi R^3/3$, we have

$$\frac{t_{relax}}{t_{cross}} \sim \frac{V^4 R^2}{6NG^2 m^2 \ln \Lambda} \sim \frac{N}{6 \ln(N/2)}. \tag{48}$$

For $N \sim 10^{11}$ stars, the relaxation will take 10^9 crossing times. For globulars with 10^6 stars, the relaxation time for the whole system is $t_{\text{relax}} \sim 10^4 t_{cross}$.

2.2 Effects of Two-Body Relaxation

Consider a cluster of stars, each with mass m. The *Maxwellian* distribution gives the fraction f of stars with velocities between v and $v + \Delta v$ as

$$4\pi f_M(E) v^2 \Delta v \tag{49}$$

where

$$f_M(E) \propto \exp\left(-\frac{E}{kT} \right) = \exp\left\{ - \left[m\Phi(\mathbf{x}) + \frac{mv^2}{2} \right] \Big/ kT \right\}. \tag{50}$$

The temperature is related to the average kinetic energy as

$$\frac{1}{2} m \langle \mathbf{v}^2(\mathbf{x}) \rangle = \frac{3}{2} kT. \tag{51}$$

But there are stars at large energies, and these will evaporate. The kinetic energy of stars that escape will typically be

$$\left\langle \frac{1}{2} m v_2^2(\mathbf{x}) \right\rangle = -\frac{1}{N} \sum_i m_i \Phi(\mathbf{x}_i) = -\frac{2}{N} PE = \frac{4}{N} KE = 6kT \tag{52}$$

The fraction of stars that escape will be

$$\int_{\sqrt{12kT/m}}^{\infty} f(E) v^2 dv \Big/ \int_0^{\infty} f(E) v^2 dv = 0.0074 \approx \frac{1}{136} \tag{53}$$

The evaporation time will be $t_{evap} \sim 136 t_{relax}$. For globulars this is longer than the age of the universe, but for open clusters it is a few gigayears.

3 Orbits of Disk Stars and Epicycles

Now we will treat orbits in axisymmetric potentials, and we will use cylindrical polar coordinates (R, ϕ, z). In an axisymmetric potential, $\partial\Phi/\partial\phi = 0$, and angular momentum about the z-axis is conserved. We adopt l_z at the angular momentum per unit mass, we can write

$$\frac{d}{dt}(R^2\dot{\phi}) = 0; \; l_z \equiv R^2\dot{\phi} = \text{ constant.} \tag{54}$$

The radial equation of motion is

$$\ddot{R} = R\dot{\phi}^2 - \frac{\partial\Phi}{\partial R} = -\frac{\partial\Phi_{\rm eff}}{\partial R} \tag{55}$$

$$\Phi_{\rm eff} \equiv \Phi(R,z) + \frac{l_z^2}{2R^2}. \tag{56}$$

Multiplying 55 by $\dot{R}$ and integrating informs us that for a star in the midplane $z = 0$

$$\frac{1}{2}\dot{R}^2 + \Phi_{\rm eff}(R, z=0; l_z) = \text{ constant.} \tag{57}$$

If $l_z \neq 0$, then there is an angular momentum barrier at the center of the galaxy that limits R to some minimum value.

The vertical equation of motion is

$$\ddot{z} = -\frac{\partial\Phi}{\partial z}(R,z) = -\frac{\partial\Phi_{\rm eff}}{\partial z}(R,z) \tag{58}$$

If the potential is an even function about $z = 0$ there is no vertical force in the midplane.

Let's expand the potential about the average radius R_g of a star in a Taylor series and truncate at the first term

$$\ddot{z} = -z\left[\frac{\partial^2\Phi}{\partial z^2}(R_g, z)\right]_{z=0} \equiv -\nu^2(R_g)z \tag{59}$$

where ν is an angular frequency. The solution to this equation is a harmonic oscillator

$$z(t) = Z\cos(\nu t + \theta) \tag{60}$$

where Z and θ are constants.

Circular orbits with $\dot{R} = 0$ are only possible at a radius R_g where $\Phi_{\rm eff} =$ constant. At this radius

$$\frac{\partial\Phi}{\partial R}(R_g, z=0) = \frac{l_z^2}{R_g^3} = R_g\Omega^2(R_g) \tag{61}$$

where $\Omega(R)$ is the angular speed of a circular orbit in the disk plane. If $\Phi_{\rm eff}$ has a minimum at R_g, then the circular orbit will have the lowest l_z. Any larger l_z will oscillate about the circular orbit, in an *epicycle* about a point that moves with $\Omega(R_g)$ in a circular orbit of radius R_g.

Let's set $R = R_g + x$ in Equation 55, assume that $x \ll R$, and expand to first order in x/R and z/R. We find that

$$\ddot{x} \approx -x\left[\frac{\partial^2\Phi_{\rm eff}}{\partial R^2}\right]_{R_g} \equiv -\kappa^2(R_g)x \tag{62}$$

and

$$x(t) \approx X\cos(\kappa t + \psi). \tag{63}$$

The constant κ is called the epicyclic frequency when $\kappa^2 > 0$. If $\kappa^2 < 0$, the orbit is unstable.

Recalling that in a circular orbit

$$R\Omega^2(R) = \frac{\partial\Phi(R, z=0)}{\partial R} \tag{64}$$

we can write

$$\kappa^2(R) = \frac{d}{dR}[R\Omega^2(R)] + \frac{3l_z^2}{R^4} = \frac{1}{R^3}\frac{d}{dR}[(R^2\Omega)^2] = -4B\Omega \tag{65}$$

where B is Oort's constant. In the solar neighborhood $B < 0$ and κ^2 is positive and the Solar orbit is stable. If $R^2\Omega(R)$, the angular momentum of a circular orbit, increases with R then all larger circular orbits are stable. Near a black hole this isn't the case, and the inner most circular orbit is at $R = 6GM/c^2$.

3.1 Azimuthal Motion in Epicycles

Since R changes in an epicyclical orbit, the azimuthal velocity must change to compensate to keep l_z constant. This means that

$$\dot{(\phi)} = \frac{l_z}{R^2} = \frac{\Omega(R_g)R_g^2}{(R_g+x)^2} \approx \Omega(R_g)\left(1 - \frac{2x}{R_g}\right). \tag{66}$$

We can integrate this equation to find that the azimuthal position with time varies as

$$\phi(t) = \phi_0 + \Omega(R_g)t - \frac{1}{R_g}\frac{2\Omega}{\kappa} X \sin(\kappa t + \psi) \tag{67}$$

The first two terms are circular motion starting from ϕ_0 with constant angular frequency $\Omega(R_g)$. The third term is a harmonic oscillation with the same frequency κ as the x motion but $\pi/2$ out of phase, and larger by a factor $2\Omega/\kappa$. The motion is also retrograde!

3.2 κ vs. Ω

For a point mass $\kappa = \Omega$. For a uniform sphere $\kappa = 2\Omega$. The galaxy has $\Omega < \kappa < 2\Omega$, and near the sun $\kappa \approx 1.4\Omega$.

3.3 Relative Speed of Nearby Stars

The relative speeds of stars with $R_g > R_0$ are

$$v_y = R_0[\dot{\phi} - \Omega(R_0)] \approx R_0\left[\Omega(R_g) - 2x\frac{\Omega(R_g)}{R_g} - \Omega(R_0)\right] \tag{68}$$

but since $R_0 = R_g + x$ and dropping terms in x^2 we have

$$v_y \approx -x\left[2\Omega(R_0) + R_0\left(\frac{d\Omega}{dR}\right)_{R_0}\right] = -\frac{\kappa^2 x}{2\Omega} = 2Bx \tag{69}$$

Taking the average of stars we can measure, we find

$$\langle v_y^2\rangle = \left(\frac{\kappa^2}{2\Omega}\right)^2\langle x^2\rangle = \frac{\kappa^2}{4\Omega^2}\langle v_x^2\rangle \tag{70}$$

and since $\kappa < 2\Omega$, $\langle v_y^2\rangle < \langle v_x^2\rangle$. For the thin disk, we find that $2 \lesssim \langle v_x^2\rangle/\langle v_y^2\rangle \lesssim 3$.