SIAF to SIP Conversion

Brant Robertson

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1 Overview

We want to determine the translation between the SIAF and SIP polynomial distortion models. Below, we define the expansion models and then identify like terms. We will write down the relation between the SIAF and SIP expansion coefficients succinctly at the end.

2 SIP Expansion Definition

The Simple Imaging Polynomial (SIP) expansion is defined in terms of pixel coordinates u, v relative to an origin at CRPIX1, CRPIX2 located on sky at CRVAL1, CRVAL2. The on-sky location x, y of pixel u, v is given by

$$\left(\begin{array}{c} x \\ y \end{array} \right) = \left(\begin{array}{cc} \mathtt{CD1\_1} & \mathtt{CD1\_2} \\ \mathtt{CD2\_1} & \mathtt{CD2\_2} \end{array} \right) \left(\begin{array}{c} u + f(u,v) \\ v + g(u,v) \end{array} \right). \tag{1}$$

Note that the CD matrix contains all the units (of pixel scale).

The functions $f(u,v)$ and $g(u,v)$ define the polynomial expansions that provide the distortion. These functions are given by

$$f(u,v) = \sum_{p,q} \mathtt{A}_{pq}\, u^p\, v^q, \quad p + q \leq \mathtt{A\_ORDER}, \tag{2}$$

$$g(u,v) = \sum_{p,q} \mathtt{B}_{pq}\, u^p\, v^q, \quad p + q \leq \mathtt{B\_ORDER}, \tag{3}$$

where $\mathtt{A}_{pq}$ and $\mathtt{B}_{pq}$ are the coefficients that encode the degree of distortion, and A_ORDER and B_ORDER limit the number of terms in the distortion model. Note that the sum $p+q$ is less than or equal to A_ORDER (for $f(u,v)$) or B_ORDER (for $g(u,v)$). In a record of the SIP coefficients, any unspecified coefficient is assumed to be zero.

The original SIP definition only included quadratic and higher terms, such that third-order model would be written

$$f_{old}(u,v) = \mathtt{A}_{20}u^2 + \mathtt{A}_{02}v^2 + \mathtt{A}_{11}uv + \mathtt{A}_{21}u^2v + \mathtt{A}_{12}uv^2 + \mathtt{A}_{30}u^3 + \mathtt{A}_{03}v^3, \tag{4}$$

$$g_{old}(u,v) = \mathtt{B}_{20}u^2 + \mathtt{B}_{02}v^2 + \mathtt{B}_{11}uv + \mathtt{B}_{21}u^2v + \mathtt{B}_{12}uv^2 + \mathtt{B}_{30}u^3 + \mathtt{B}_{03}v^3. \tag{5}$$

However, we will need the linear cross terms

$$f(u,v) = \mathtt{A}_{01}v + \mathtt{A}_{20}u^2 + \mathtt{A}_{02}v^2 + \mathtt{A}_{11}uv + \mathtt{A}_{21}u^2v + \mathtt{A}_{12}uv^2 + \mathtt{A}_{30}u^3 + \mathtt{A}_{03}v^3, \tag{6}$$

$$g(u,v) = \mathtt{B}_{10}u + \mathtt{B}_{20}u^2 + \mathtt{B}_{02}v^2 + \mathtt{B}_{11}uv + \mathtt{B}_{21}u^2v + \mathtt{B}_{12}uv^2 + \mathtt{B}_{30}u^3 + \mathtt{B}_{03}v^3. \tag{7}$$

Note that we expect $\mathtt{A}_{10} = 0$ and $\mathtt{B}_{01} = 0$ because they are degenerate with redefining the CD matrix elements.

To finish our SIP description, we should multiply through by the CD matrix. It will be convenient for us to use a coordinate system where the off diagonal terms are zero (aligned with the detector) and decompose the CD matrix as

$$\left(\begin{array}{cc} \mathtt{CD1\_1} & \mathtt{CD1\_2} \\ \mathtt{CD2\_1} & \mathtt{CD2\_2} \end{array} \right) = \left(\begin{array}{cc} \mathtt{CDELT1} \cdot \mathtt{PC1\_1} & 0 \\ 0 & \mathtt{CDELT2} \cdot \mathtt{PC2\_2} \end{array} \right). \tag{8}$$

Note that PC1_1 and PC2_2 account for a possible parity change between the detector and sky coordinates (they are equal to ± 1 depending on the detector), and this decomposition only helps us because of convenient definitions made by STScI in their SIAF expansion (see below).

We can then write the SIP distortion as

$$x = \mathtt{CDELT1} \cdot \mathtt{PC1\_1} \left[u + f(u,v) \right], \tag{9}$$

$$y = \mathtt{CDELT2} \cdot \mathtt{PC2\_2} \left[v + g(u,v) \right], \tag{10}$$

where $f(u,v)$ and $g(u,v)$ are given to, e.g., third order by Equations 6 and 7.

3 SIAF Expansion Definition

The SIAF expansion is given by STScI and defined in JWST Technical Report JWST-STScI-001550 (hereafter JTR1550). There are a few steps involved, and a few choice cancellations make things a bit easier for NIRCam than they could be!

First, there is a translation between the Detector coordinates (u and v from above) to the Science coordinates. There is a further translation from Science to Ideal coordinates that involves the SIAF polynomial expansion. It is important to note that the SIAF expansion coefficients are *dimensionful*.

3.1 Detector to Science

The detector to science coordinate conversion is given in §4.1 of JTR1550 as

$$\begin{aligned} XSci - XSciRef &= DetSciParity\left[(XDet - XDetRef)\cos\left(DetSciYAngle\right)\right. \\ &\quad \left. + \left(YDet - YDetRef\right)\sin\left(DetSciYAngle\right)\right], \end{aligned} \tag{11}$$

$$\begin{aligned} YSci - YSciRef &= -\left(XDet - XDetRef\right)\sin\left(DetSciYAngle\right) \\ &\quad + \left(YDet - YDetRef\right)\cos\left(DetSciYAngle\right), \end{aligned} \tag{12}$$

where the absurd notation is in the original. Note that for all NIRCam SCAs, $XDetRef = XSciRef = 1024.5$ and all $YDetRef = YSciRef = 1024.5$, so if we pick CRPIX1 $= 1024.5$, CRPIX2 $= 1024.5$, then $u \equiv (XDet - XDetRef)$ and $v \equiv (YDet - YDetRef)$. Also, for all NIRCam SCAs, $DetSciParity = -1$. Lastly, for NIRCam SCAs we have either $DetSciYAngle = 0$ or $DetSciYAngle = \pi$. We then can write

$$XSci - XSciRef = DetSciParity\cos\left(DetSciYAngle\right)u \tag{13}$$

$$YSci - YSciRef = \cos\left(DetSciYAngle\right)v \tag{14}$$

Let us define PC1_1 $\equiv DetSciParity\cos\left(DetSciYAngle\right)$ and PC2_2 $\equiv \cos\left(DetSciYAngle\right)$, so we are left with

$$XSci - XSciRef = \mathtt{PC1\_1}u \tag{15}$$

and

$$YSci - YSciRef = \mathtt{PC2\_2}v. \tag{16}$$

3.2 Science to Ideal

The SIAF expansion is written in terms of a conversion between the Science and Ideal coordinates as

$$XIdl = \sum_{i=1}^{degree}\sum_{j=0}^{i} Sci2IdlCoefX_{i,j}\left(XSci - XSciRef\right)^{i-j}\left(YSci - YSciRef\right)^{j} \tag{17}$$

$$YIdl = \sum_{i=1}^{degree}\sum_{j=0}^{i} Sci2IdlCoefY_{i,j}\left(XSci - XSciRef\right)^{i-j}\left(YSci - YSciRef\right)^{j} \tag{18}$$

where $degree = Sci2IdlDegree$ (a parameter given in the SIAF file for each detector) and again the notation is originally absurd. To give ourselves a break, we will define $X_{ij} \equiv Sci2IdlCoefX_{i,j}$ and $Y_{ij} \equiv Sci2IdlCoefY_{i,j}$ and identify $XIdl \equiv x$ and $YIdl \equiv y$.

Since the SIAF expansion is dimensionful, to match terms with the SIP expansion it will be helpful for us to write out fully the terms and then infer a recursion relationship between the coefficients in each expansion. Given Equations 17 and 18, and the equivalences defined in the previous paragraph, we can write the expansions to, e.g., $degree = 3$ as

$$\begin{aligned} x &= X_{10}\mathtt{PC1\_1}u + X_{11}\mathtt{PC2\_2}v + X_{20}\mathtt{PC1\_1}^2u^2 \\ &\quad + X_{21}\mathtt{PC1\_1}\mathtt{PC2\_2}uv + X_{22}\mathtt{PC2\_2}^2v^2 + X_{30}\mathtt{PC1\_1}^3u^3 \\ &\quad + X_{31}\mathtt{PC1\_1}^2\mathtt{PC2\_2}u^2v + X_{32}\mathtt{PC1\_1}\mathtt{PC2\_2}^2uv^2 + X_{33}\mathtt{PC2\_2}^3v^3, \end{aligned} \tag{19}$$

and

$$\begin{aligned} y &= Y_{10}\mathtt{PC1\_1}u + Y_{11}\mathtt{PC2\_2}v + Y_{20}\mathtt{PC1\_1}^2u^2 \\ &\quad + Y_{21}\mathtt{PC1\_1}\mathtt{PC2\_2}uv + Y_{22}\mathtt{PC2\_2}^2v^2 + Y_{30}\mathtt{PC1\_1}^3u^3 \\ &\quad + Y_{31}\mathtt{PC1\_1}^2\mathtt{PC2\_2}u^2v + Y_{32}\mathtt{PC1\_1}\mathtt{PC2\_2}^2uv^2 + Y_{33}\mathtt{PC2\_2}^3v^3. \end{aligned} \tag{20}$$

4 Reconciling the SIAF and SIP Expansions

We can now match terms and infer the relationship between the SIP and SIAF coefficients by setting the right hand sides of Equations 9 and 10 equal to the right hand sides of Equations 19 and 20 and grouping terms by their polynomial order in the expansion. Let us start with the x expansion,

$$\begin{aligned} &\mathtt{CDELT1}\cdot\mathtt{PC1\_1}u \\ &+\mathtt{CDELT1}\cdot\mathtt{PC1\_1}\left[\mathtt{A}_{01}v + \mathtt{A}_{20}u^2 + \mathtt{A}_{02}v^2\right. \\ &\left.+\mathtt{A}_{11}uv + \mathtt{A}_{21}u^2v + \mathtt{A}_{12}uv^2 + \mathtt{A}_{30}u^3 + \mathtt{A}_{03}v^3\right] \\ &= X_{10}\mathtt{PC1\_1}u + X_{11}\mathtt{PC2\_2}v + X_{20}\mathtt{PC1\_1}^2u^2 \\ &\quad + X_{21}\mathtt{PC1\_1}\mathtt{PC2\_2}uv + X_{22}\mathtt{PC2\_2}^2v^2 + X_{30}\mathtt{PC1\_1}^3u^3 \\ &\quad + X_{31}\mathtt{PC1\_1}^2\mathtt{PC2\_2}u^2v + X_{32}\mathtt{PC1\_1}\mathtt{PC2\_2}^2uv^2 + X_{33}\mathtt{PC2\_2}^3v^3. \end{aligned} \tag{21}$$

We want to express the SIP coefficients in terms of the SIAF coefficients. We will define for the SIAF coefficients that $\tilde{X}_{ij} = X_{ij}/\mathtt{CDELT1}$. We can identify the correspondence between coefficients as

$$\begin{aligned}
X_{10} &\equiv \mathtt{CDELT1} &(22)\\
\mathtt{A}_{01} &\equiv \tilde{X}_{11}\mathtt{PC1_1}^{-1}\mathtt{PC2_2} &(23)\\
\mathtt{A}_{20} &\equiv \tilde{X}_{20}\mathtt{PC1_1} &(24)\\
\mathtt{A}_{02} &\equiv \tilde{X}_{22}\mathtt{PC1_1}^{-1}\mathtt{PC2_2}^2 &(25)\\
\mathtt{A}_{11} &\equiv \tilde{X}_{21}\mathtt{PC2_2} &(26)\\
\mathtt{A}_{21} &\equiv \tilde{X}_{31}\mathtt{PC1_1PC2_2} &(27)\\
\mathtt{A}_{12} &\equiv \tilde{X}_{32}\mathtt{PC2_2}^2 &(28)\\
\mathtt{A}_{30} &\equiv \tilde{X}_{30}\mathtt{PC1_1}^2 &(29)\\
\mathtt{A}_{03} &\equiv \tilde{X}_{33}\mathtt{PC1_1}^{-1}\mathtt{PC2_2}^3. &(30)
\end{aligned}$$

Given that we know $\mathtt{PC1\_1}^2 = 1$, we can multiply both sides by unity and rewrite these equations conveniently as

$$\mathtt{A}_{ij} = \tilde{X}_{(i+j)j}\mathtt{PC1\_1}^{(i+1)}\mathtt{PC2\_2}^{j}. \tag{31}$$

Note that there is no $\mathtt{A}_{10}$ term in the expansion as the linear order term in u in the SIP expansion is already handled by multiplying the $\mathtt{CD}$ matrix by $(u+f, v+g)$, so $\mathtt{A}_{10} = 0$. Using that $\mathtt{PC2\_2}^2 = 1$, we can write similar equations for the y coordinate as

$$\mathtt{B}_{ij} = \tilde{Y}_{(i+j)j}\mathtt{PC1\_1}^{i}\mathtt{PC2\_2}^{(j+1)}, \tag{32}$$

where $\tilde{Y}_{ij} \equiv Y_{ij}/\mathtt{CDELT2}$. We note that $Y_{11} = \mathtt{CDELT2}$, and $B_{01} = 0$ because the linear order v expansion term is handled by $\mathtt{CD}$ matrix by $(u+f, v+g)$.

5 Summary

We have computed the correspondence between the SIAF polynomial expansion coefficients, the SIP expansion coefficients, the parity matrix $\mathtt{PC}$, and the pixel scale vector $\mathtt{CDELT}$. To compute the SIP expansion from the SIAF expansion, use Equations 31 and 32 to compute $\mathtt{A}_{ij}$ and $\mathtt{B}_{ij}$ from X_{ij}, Y_{ij}, $\mathtt{CDELT}$, and $\mathtt{PC}$. To incorporate further rotations such that the sky and detector coordinates are rotated, apply a rotation to the $\mathtt{CD}$ matrix and then compute the sky coordinates from Equation 1 with the rotated $\mathtt{CD}$ matrix (noting that $\mathtt{PC1\_1}$ and $\mathtt{PC2\_2}$ in Equations 31 and 32 are defined in the unrotated frame when the sky and detector are aligned).

6 Inverse Transforms

Unfortunately, some libraries require the pre-computed inverse transforms.

6.1 Science to Detector

The science to detector coordinate conversion is given in §4.1 of JTR1550 as

$$\begin{aligned}
XDet - XDetRef &= DetSciParity\,(XSci - XSciRef)\cos(DetSciYAngle)\\
&\quad - (YSci - YSciRef)\sin(DetSciYAngle)
\end{aligned}$$

$$\begin{aligned}
YDet - YDetRef &= DetSciParity\,(XSci - XSciRef)\sin(DetSciYAngle)\\
&\quad + (YSci - YSciRef)\cos(DetSciYAngle)
\end{aligned}$$

As before $u \equiv (XDet - XDetRef)$ and $v \equiv (YDet - YDetRef)$. Also, for all NIRCam SCAs, $DetSciParity = -1$. Lastly, for NIRCam SCAs we have either $DetSciYAngle = 0$ or $DetSciYAngle = \pi$. We then can write

$$u = DetSciParity\cos(DetSciYAngle)\,(XSci - XSciRef) \tag{33}$$

$$v = \cos(DetSciYAngle)\,(YSci - YSciRef) \tag{34}$$

Let us define $\mathtt{PC1\_1} \equiv DetSciParity\cos(DetSciYAngle)$ and $\mathtt{PC2\_2} \equiv \cos(DetSciYAngle)$. Note that $\mathtt{PC1\_1}^{-1} = \mathtt{PC1\_1}$ and $\mathtt{PC2\_2}^{-1} = \mathtt{PC2\_2}$, so we are left with

$$u = \mathtt{PC1\_1}^{-1}(XSci - XSciRef) \tag{35}$$

and

$$v = \mathtt{PC2\_2}^{-1}(YSci - YSciRef). \tag{36}$$

We can write this as

$$\begin{pmatrix} u \\ v \end{pmatrix} = \mathtt{PC}^{-1}\begin{bmatrix} (XSci - XSciRef) \\ (YSci - YSciRef) \end{bmatrix}. \tag{37}$$

6.2 Ideal to Science

The inverse SIAF transform from Ideal to Science can be written

$$XSci - XSciRef = \sum_{i=1}^{degree} \sum_{j=0}^{i} Idl2SciCoefX_{i,j} XIdl^{i-j} YIdl^{j} \tag{38}$$

$$YSci - YSciRef = \sum_{i=1}^{degree} \sum_{j=0}^{i} Idl2SciCoefY_{i,j} XIdl^{i-j} YIdl^{j} \tag{39}$$

6.3 Inverse SIP Transform

The inverse SIP transform takes the form

$$\begin{pmatrix} U \\ V \end{pmatrix} = \mathtt{CD}^{-1} \begin{pmatrix} x \\ y \end{pmatrix}. \tag{40}$$

Here, we have that

$$u = U + F(U,V) = U + \sum_{p,q} \mathtt{AP}_{pq}\, U^p\, V^q, \quad p+q \le \mathtt{A\_ORDER}, \tag{41}$$

and

$$v = V + G(U,V) = V + \sum_{p,q} \mathtt{BP}_{pq}\, U^p\, V^q, \quad p+q \le \mathtt{B\_ORDER}. \tag{42}$$

6.4 Matching Terms

$$\begin{aligned} u = \mathtt{CD1\_1}^{-1} XIdl \quad &+ \quad \sum_{p,q} \mathtt{AP}_{pq} \left(\mathtt{CD1\_1}^{-1} XIdl\right)^p \left(\mathtt{CD2\_2}^{-1} YIdl\right)^q && (43) \\ &= \quad \mathtt{PC1\_1}^{-1}(XSci - XSciRef) && (44) \\ &= \quad \mathtt{PC1\_1}^{-1} \sum_{i=1}^{degree} \sum_{j=0}^{i} Idl2SciCoefX_{i,j} XIdl^{i-j} YIdl^{j} && (45) \end{aligned}$$

Let's write out the third order expansion

$$\begin{aligned} \mathtt{CD1\_1}^{-1}x + \mathtt{AP}_{10}U + \mathtt{AP}_{01}V + \mathtt{AP}_{20}U^2 + \mathtt{AP}_{02}V^2 + \mathtt{AP}_{11}UV + \mathtt{AP}_{30}U^3 + \mathtt{AP}_{03}V^3 && (46) \\ + \mathtt{AP}_{21}U^2V + \mathtt{AP}_{12}UV^2 && (47) \\ = \mathtt{PC1\_1}^{-1}X^T_{00} + \mathtt{PC1\_1}^{-1}X^T_{10}x + \mathtt{PC1\_1}^{-1}X^T_{11}y + \mathtt{PC1\_1}^{-1}X^T_{20}x^2 && (48) \\ + \mathtt{PC1\_1}^{-1}X^T_{21}xy + \mathtt{PC1\_1}^{-1}X^T_{22}y^2 + \mathtt{PC1\_1}^{-1}X^T_{30}x^3 && (49) \\ + \mathtt{PC1\_1}^{-1}X^T_{31}x^2y + \mathtt{PC1\_1}^{-1}X^T_{32}xy^2 + \mathtt{PC1\_1}^{-1}X^T_{33}y^3 && (50) \end{aligned}$$

This is complicated by the fact that $X^T_{00} \ne 0$, but we can handle that separately. It is also the case that $X^T_{10} \ne X^{-1}_{10}$. This inequality means that $\mathtt{AP}_{10} \ne 0$. It's also the case that in this definition, the units of all the X^T_{ij} are different, and we will have to include powers of CDELT.

Let's match the terms

$$\begin{aligned} \mathtt{AP}_{00} &\equiv \mathtt{PC1\_1}^{-1}X^T_{00} && (51) \\ \mathtt{AP}_{10} &\equiv \mathtt{CDELT1}X^T_{10} - 1 && (52) \\ \mathtt{AP}_{01}\mathtt{CD2\_2}^{-1}y &\equiv \mathtt{PC1\_1}^{-1}X^T_{11}y && (53) \\ \mathtt{AP}_{20}\mathtt{CD1\_1}^{-2}x^2 &\equiv \mathtt{PC1\_1}^{-1}X^T_{20}x^2 && (54) \\ \mathtt{AP}_{02}\mathtt{CD2\_2}^{-2}y^2 &\equiv \mathtt{PC1\_1}^{-1}X^T_{22}y^2 && (55) \\ \mathtt{AP}_{11}\mathtt{CD1\_1}^{-1}\mathtt{CD2\_2}^{-1}xy &\equiv \mathtt{PC1\_1}^{-1}X^T_{21}xy && (56) \\ \mathtt{AP}_{21}\mathtt{CD1\_1}^{-2}\mathtt{CD2\_2}^{-1}x^2y &\equiv \mathtt{PC1\_1}^{-1}X^T_{31}x^2y && (57) \\ \mathtt{AP}_{12}\mathtt{CD1\_1}^{-1}\mathtt{CD2\_2}^{-2}xy^2 &\equiv \mathtt{PC1\_1}^{-1}X^T_{32}xy^2 && (58) \\ \mathtt{AP}_{30}\mathtt{CD1\_1}^{-3}x^3 &\equiv \mathtt{PC1\_1}^{-1}X^T_{30}x^3 && (59) \\ \mathtt{AP}_{03}\mathtt{CD2\_2}^{-3}y^3 &\equiv \mathtt{PC1\_1}^{-1}X^T_{33}y^3. && (60) \end{aligned}$$

This can be written succinctly as

$$\mathtt{AP}_{ij} = X^T_{(i+j)j}\mathtt{PC1\_1}^{(i+1)}\mathtt{PC2\_2}^{j}\mathtt{CDELT1}^{i}\mathtt{CDELT2}^{j}, \tag{61}$$

and correspondingly we have

$$\mathtt{BP}_{ij} = Y^T_{(i+j)j}\mathtt{PC1\_1}^{i}\mathtt{PC2\_2}^{(j+1)}\mathtt{CDELT1}^{i}\mathtt{CDELT2}^{j}. \tag{62}$$