Measuring things

When we do experiments, we measure things. Each thing that we measure has an <u>uncertainty</u> associated with it. Always!

$$
\begin{array}{ccc}
x & \pm & \delta x \\
t & \pm & \delta t \\
m & \pm & \delta m
\end{array}
$$

The question is: if we use these measurements to calculate something else, what is the uncertainty in that new quantity? Math can help us:

Suppose we measure two things:

$$x \pm \delta x$$
$$y \pm \delta y$$

Then, suppose there is some new quantity, $z = f(x,y)$

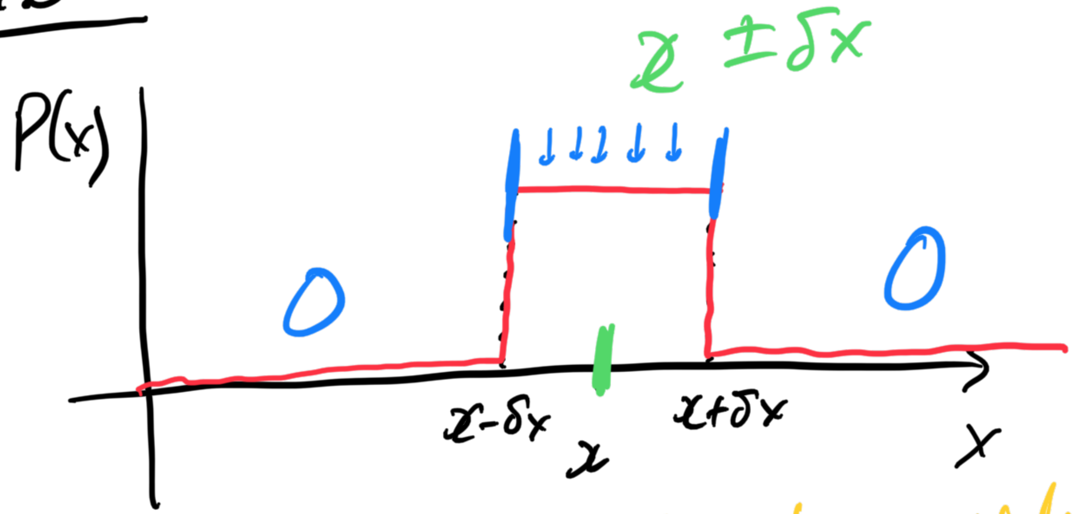
e.g.

$$z = x + y$$
$$z = \frac{x^2}{y^3}$$
$$z = \sin^{-1}\left(\frac{x}{y}\right)$$
$$\vdots$$

What is δz?

The answer depends on how we measured $\delta x, \delta y$ in the first place.

<u>Case 1</u> : Uniform Uncertainties.



This says that the <u>true</u> value of x is somewhere between $x - \delta x$ and $x + \delta x$, and it is <u>equally probable</u> that it be anywhere in this range.

This applies when we make a <u>single</u> measurement with some instrument that has a <u>digital</u> scale, e.g. stop watch, meter-stick, etc.

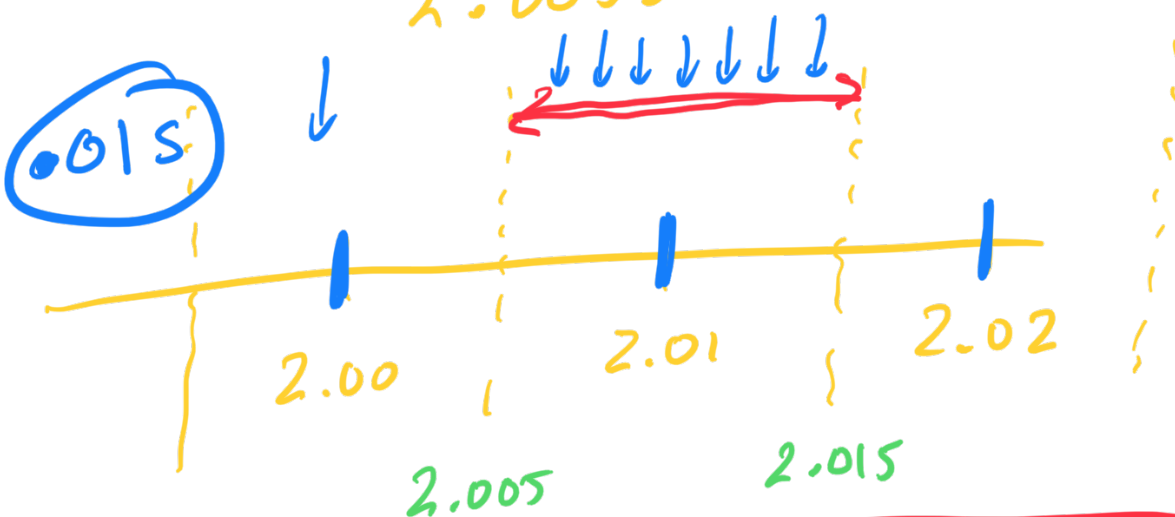
Example: we measure some

$$t = 2.01\,s$$

with a stop watch. This says, implicitly, that

$t \neq 2.00\,s$, $t \neq 2.02\,s$

i.e it is somewhere between 2.005 s and 2.015 s



$$\boxed{\pm \frac{1}{2}\,(\text{smallest division of the instrument})}$$

For uniform uncertainties, we

For ...

Calculate δz as follows:

Calculus.

works every time.

$$\boxed{\delta z = \left|\frac{\partial z}{\partial x}\right| \delta x + \left|\frac{\partial z}{\partial y}\right| \delta y}$$

(Chain Rule of Differentiation)

Examples:

$$\boxed{z = x + y}$$

100 ± 5 5%

$+$

50 ± 10 20%

150 ± 15

10%

$$\frac{\partial z}{\partial x} = 1$$

$$\frac{\partial z}{\partial y} = 1$$

$$\boxed{\delta z = \delta x + \delta y}$$

Truths

$98 - 98 = 0$

100 ± 2

95 ± 3

5 ± 5

100%

$z = x - y$

$$\frac{\partial z}{\partial x} = 1 \quad \left|\frac{\partial z}{\partial x}\right| = 1$$

$$\frac{\partial z}{\partial y} = -1 \quad \left|\frac{\partial z}{\partial y}\right| = 1$$

$$\boxed{\delta x = \delta x + \delta y}$$

100 ± 5 5%

∂y $\quad\left| \partial y \right|$

$$- \quad 50 \pm 10 \quad 20\%$$

$$\overline{\quad 50 \pm 15 \quad 30\% \quad}$$

$$z = xy$$

$$\frac{\partial z}{\partial x} = y$$

$$\frac{\partial z}{\partial y} = x$$

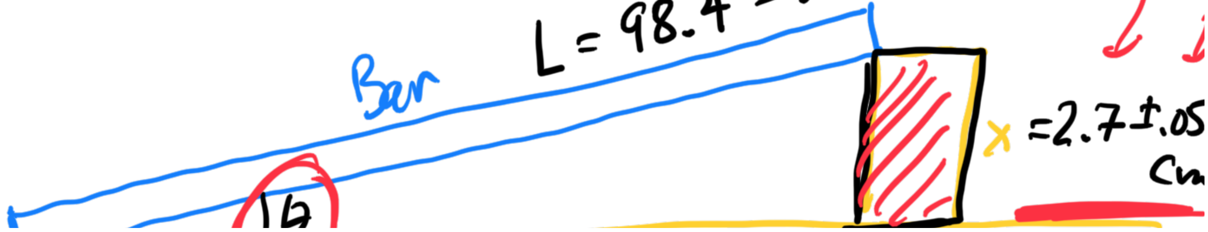
$$\frac{\delta z}{z} = \frac{y\,\delta x}{xy} + \frac{x\,\delta y}{xy}$$

$$\times 100 \left(\frac{\delta z}{z}\right) = \times 100 \left(\frac{\delta x}{x}\right) + \times 100 \left(\frac{\delta y}{y}\right)$$

% error in z — % error in x — % error in y

Complicated Example !!

meter stick $(1\,\text{mm}) = .1\,\text{cm}$ $\quad \pm \frac{1}{2}$ (subdiv) $\quad .05\,\text{cm}$



Brick

SOHCAHTOA

From trigonometry, we have that

$$\boxed{\sin\theta = \frac{x}{L}} \qquad \theta = \sin^{-1}\left(\frac{x}{L}\right) = 1.57^\circ$$

What is $\delta\theta$?

Use implicit differentiation:

$$\cos\theta\ \delta\theta = \left|\frac{1}{L}\right|\delta x + \left|-\frac{1}{L^2}\right|\delta L$$

$$\delta\theta = \frac{1}{\cos\theta}\left(\left|\frac{1}{L}\right|\delta x + \frac{1}{L^2}\delta L\right.$$

$x = 2.7$

$\delta x = .05$

$L = 98.4$

$$\cos\theta = \frac{\sqrt{L^2 - x^2}}{}$$

$\delta L = .005$

$$\boxed{\delta\theta = \frac{L}{\sqrt{L^2 - x^2}}\left(\frac{\delta x}{L} + \frac{\delta L}{L^2}\right)}$$

$$\theta = \sin^{-1}\left(\frac{x}{L}\right) = \sin^{-1}\left(\frac{2.7}{98.4}\right)$$

$$z = \cosh\left(\log\left(\frac{\sin x}{L^3}\right)\right) \; ?$$

$$= 1.572^\circ$$

$$\sin\theta = \frac{x_{\text{min}}}{L_{\text{max}}}$$

$$\delta\theta = \underline{0.029^\circ}$$

$$\therefore \boxed{\theta = (1.572 \pm 0.029)^\circ}$$

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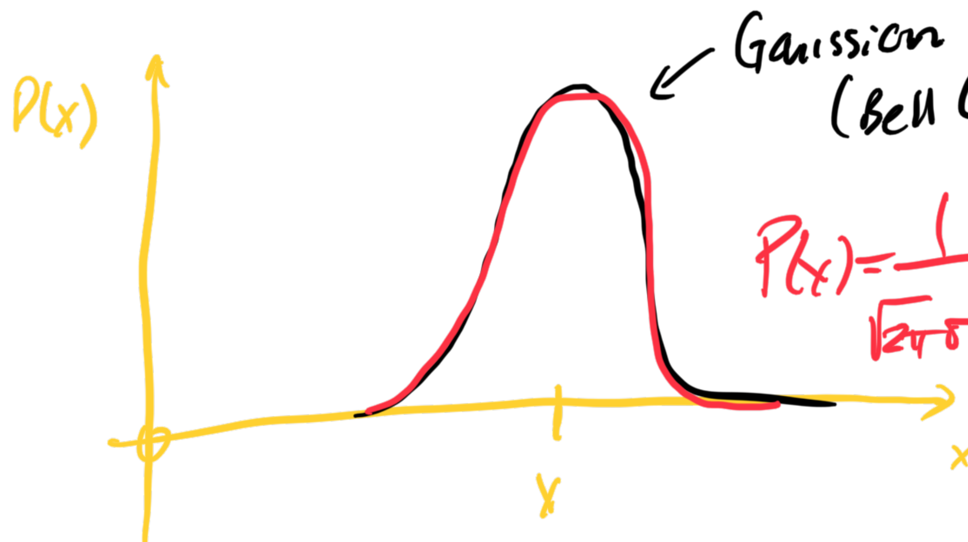
(2) Gaussian / Statistical Uncertainties.

In other experiments, we measure some quantity by making many measurements and looking at the distribution of values.



$$P(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-\bar{x})^2}{2\sigma^2}}$$

The "δx" that we quote is actually the standard deviation of the Gaussian, σ.

In this case, we calculate δz as follows:

calculus happens

$$\boxed{\delta z = \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 (\delta x)^2 + \left(\frac{\partial z}{\partial y}\right)^2 (\delta y)^2}}$$

Examples :

$$z = x + y$$

$$\delta z = \sqrt{(\delta x)^2 + (\delta y)^2}$$

$$z = xy$$

$$\delta z = \sqrt{y^2 \delta x^2 + x^2 \delta y^2}$$

etc.

⋮