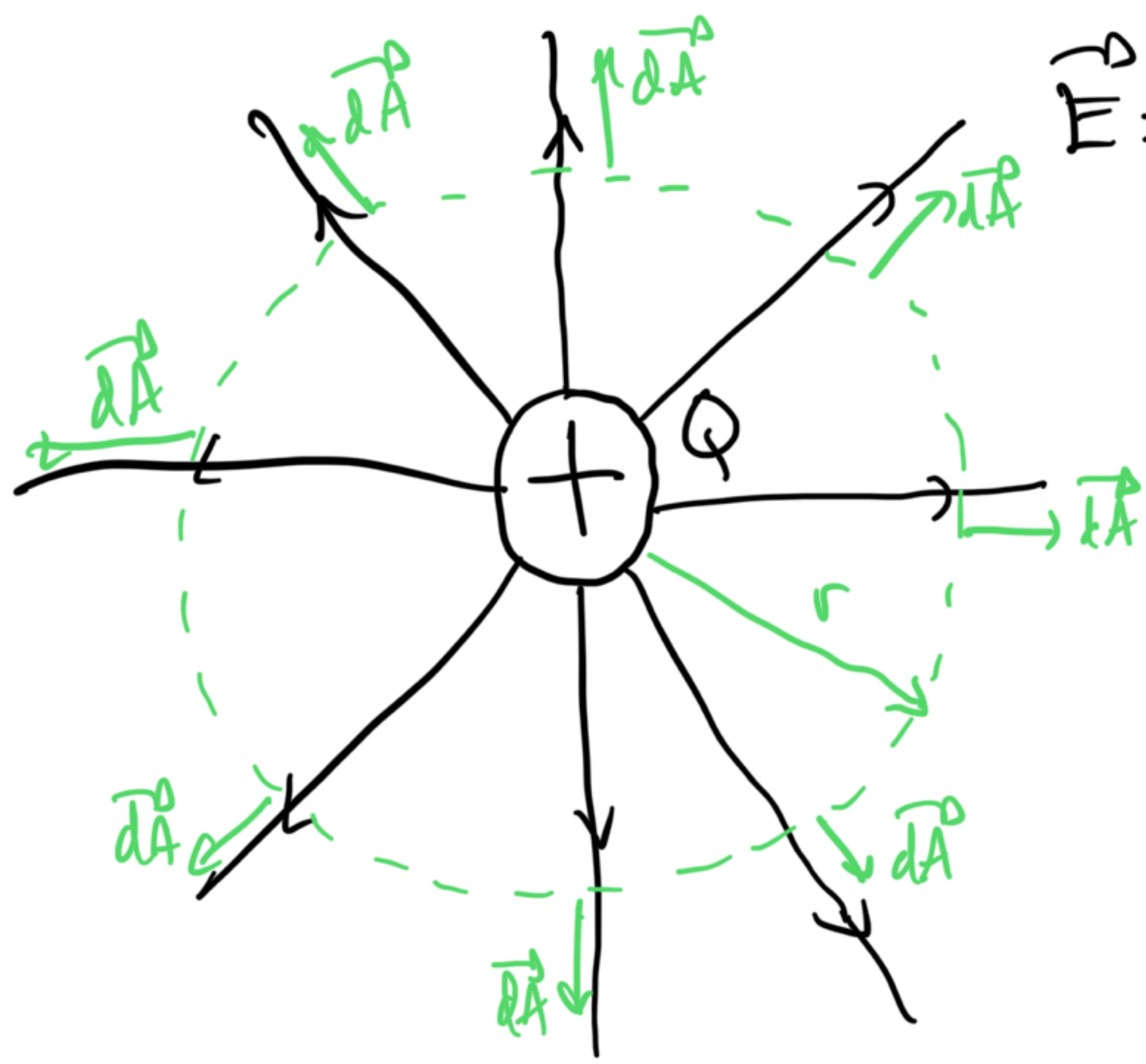
Gauss's Law

Okay, so now we sort of understand how to calculate electric flux through a surface. How can we use that to understand something more interesting?



$$\vec{E} = \frac{k_e Q}{r^2}\hat{r}$$

Imagine a fictitious sphere that surrounds the charge, completely, of radius r.

① The electric field on the surface is the same size, everywhere, and is given by $|\vec{E}| = \frac{k_e Q}{r^2}$

② The electric field points radially outward. ($\hat{r}$). The area vector $\vec{dA}$, points radially outward ($\hat{r}$)

③ The Electric flux is :

$$\Phi_E = \underbrace{\iint}_{\text{over surface}} \vec{E} \cdot \vec{dA}$$

$$= |\vec{E}| \cdot \underbrace{4\pi r^2}_{\substack{\text{Surface area} \\ \text{of} \\ \text{sphere.}}}$$

$$= \frac{k_e Q}{\cancel{r^2}} \cdot 4\pi \cancel{r^2}$$

$$\boxed{\Phi_E = 4\pi k_e Q}$$

Carl Friederich Gauss (1824) →

" The flux through any closed surface is equal to the enclosed charge within the surface, divided by ε_0

permitivity of free space
→ fundamental strength of electricity in our universe.

$$\varepsilon_0 = 8.854 \times 10^{-12}\ C^2/N \cdot m^2$$

$$\Phi_E^{\text{closed surface}} = \frac{Q_{inside}}{\varepsilon_0}$$

Gauss's Law :

$$\boxed{\oiint_S \vec{E} \cdot d\vec{A} = \frac{Q_{enclosed}}{\varepsilon_0}}$$

$$4\pi k_e \cancel{Q} = \frac{\cancel{Q}}{\varepsilon_0}$$

$$k_e = \frac{1}{4\pi\varepsilon_0} = 8.99 \times 10^9 \frac{Nm^2}{C^2}$$

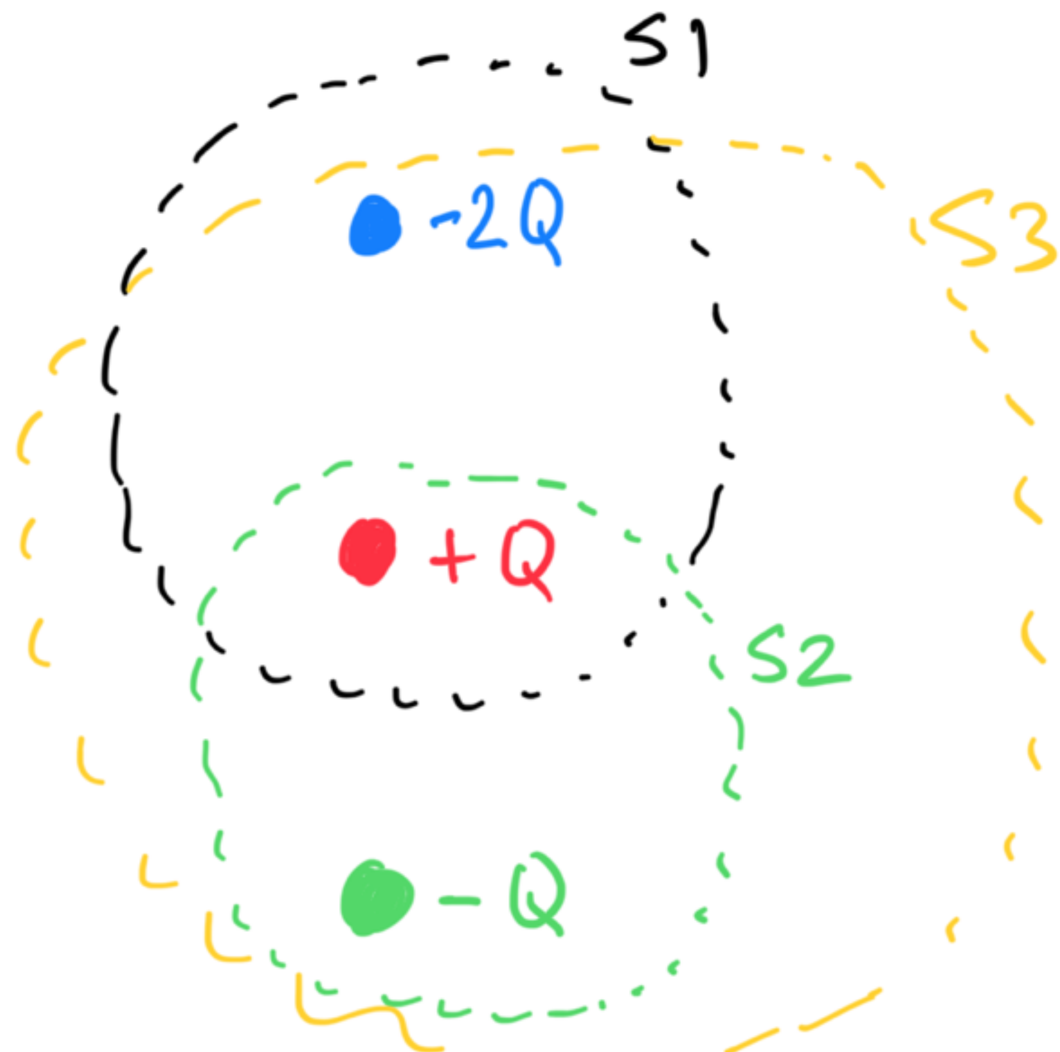
Physicists write:

$$\vec{F}_{coulomb} = \frac{1}{4\pi\varepsilon_0} \frac{q_1 q_2}{r^2} \hat{r}$$

$$E = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}\hat{r}$$

$$V = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r}$$

A3Q6

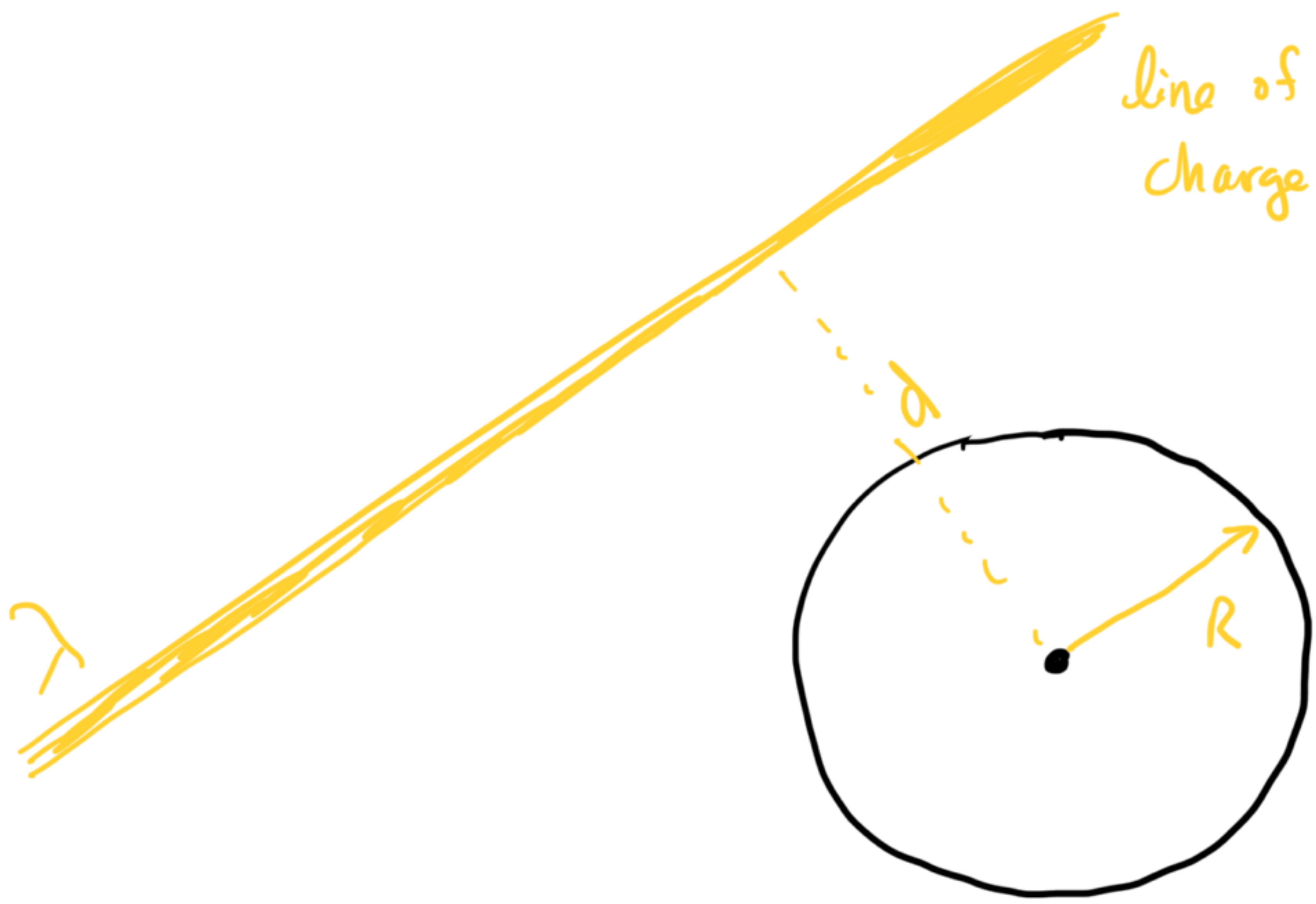


$$\Phi_E^{S1} = \frac{Q_{inside}}{\varepsilon_0} = \frac{-2Q+Q}{\varepsilon_0} = \frac{-Q}{\varepsilon_0}$$

$$\Phi_E^{S2} = \frac{Q_{inside}}{\varepsilon_0} = \frac{+Q-Q}{\varepsilon_0} = 0$$

$$\Phi_E^{S3} = \frac{Q_{inside}}{\varepsilon_0} = \frac{-2Q+Q-Q}{\varepsilon_0} = \frac{-2Q}{\varepsilon_0}$$

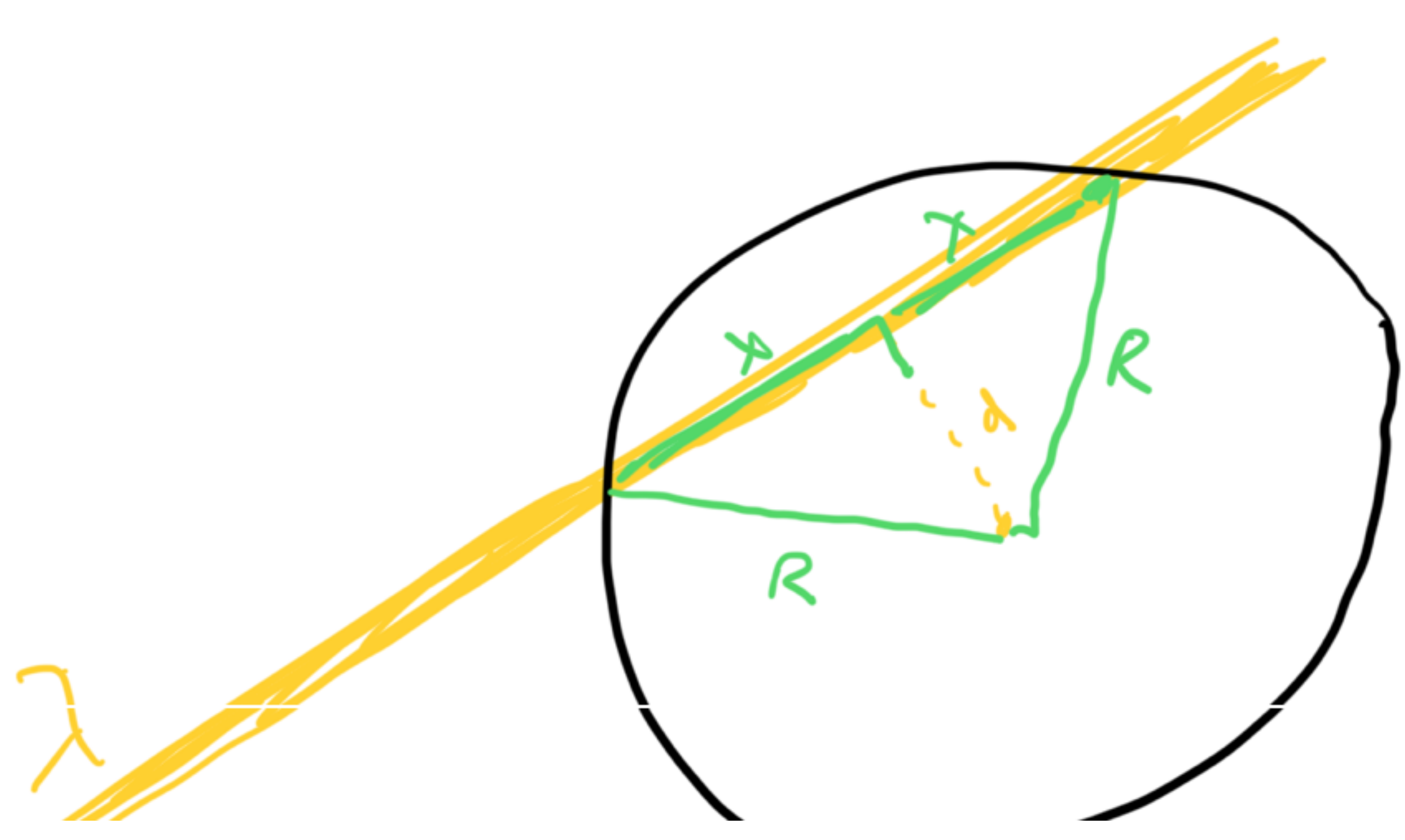
A3Q8



Case 1: $R < d$ (as shown)

$$\Phi_E = \frac{Q_{inside}}{\varepsilon_0} = 0$$

Case 2: $R > d$ ... need new diagram!



$$x^2 + d^2 = R^2$$

$$\therefore x = \sqrt{R^2 - d^2}$$

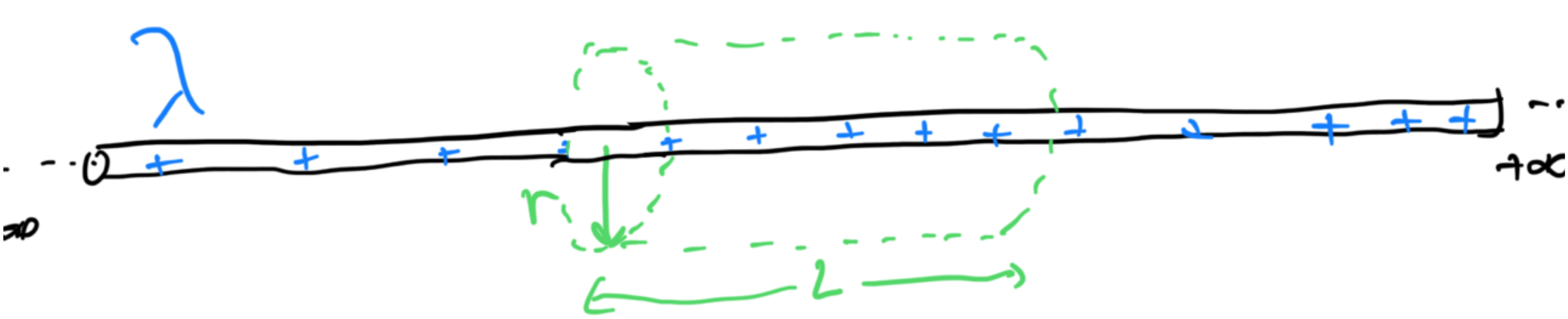
$$L = 2x = 2\sqrt{R^2 - d^2}$$

$$Q_{inside} = \lambda L = 2\lambda\sqrt{R^2 - d^2}$$

$$\therefore \Phi_E = \frac{Q_{inside}}{\varepsilon_0} = \frac{2\lambda}{\varepsilon_0}\sqrt{R^2 - d^2}$$

We can use Gauss's Law to calculate the electric field for different charge distributions!

Example 1: Infinite line of charge.



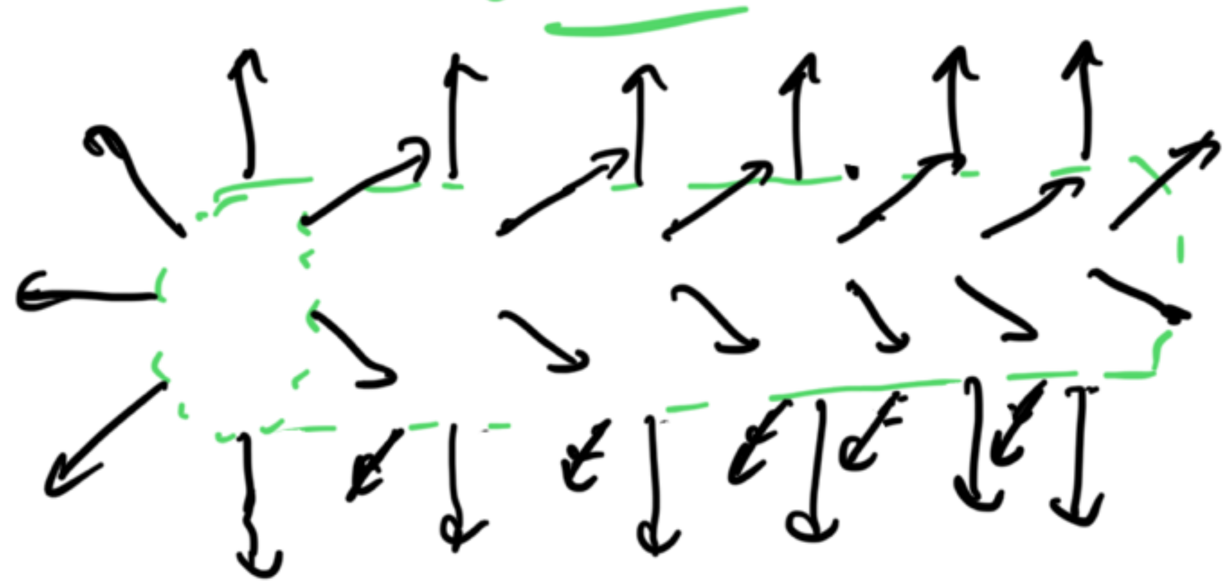
Step 1: choose a "surface" which exploits the symmetry.

Step 2:
RHS: $Q_{inside} = \lambda L$

$$\therefore \; \Phi_E^{\text{surface}} = \frac{\lambda L}{\varepsilon_0}$$

Step 3: Integrate Electric field over the surface.

→ $\vec{E}$ is constant and points outwards

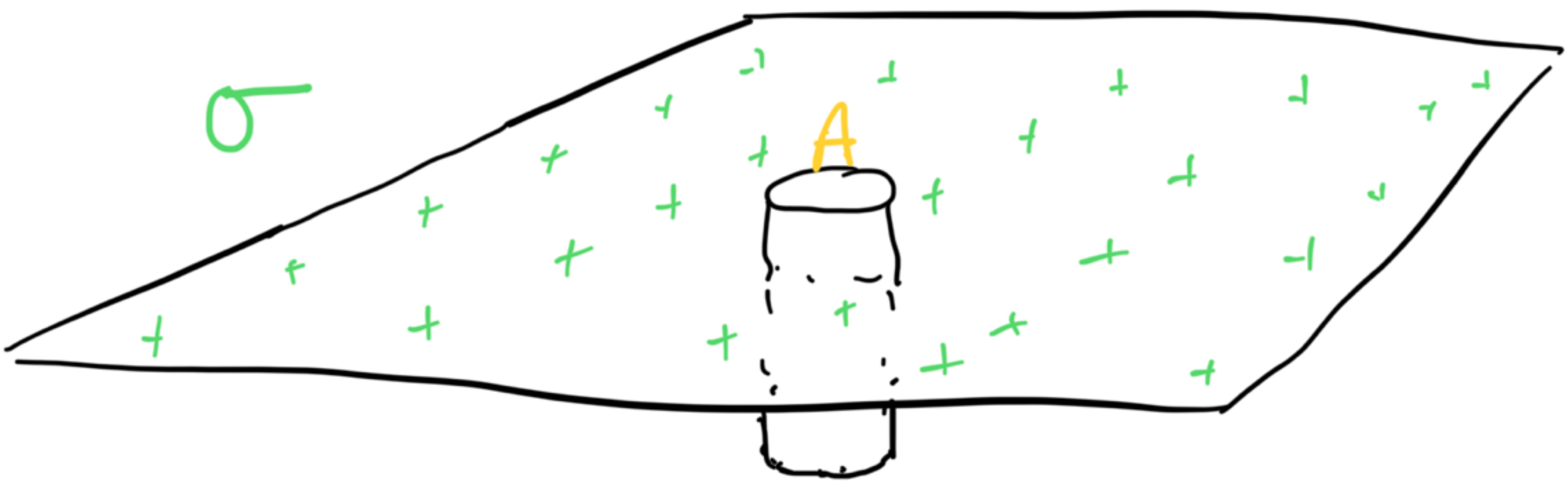


$$\oiint \vec{E} \cdot d\vec{A} = |\vec{E}| \cdot 2\pi r L$$

$$|\vec{E}| \cdot 2\pi r \cancel{L} = \frac{\lambda \cancel{L}}{\varepsilon_0}$$

$$\boxed{|\vec{E}| = \frac{\lambda}{2\pi \varepsilon_0 r}}$$

Example 2: Sheet of charge.

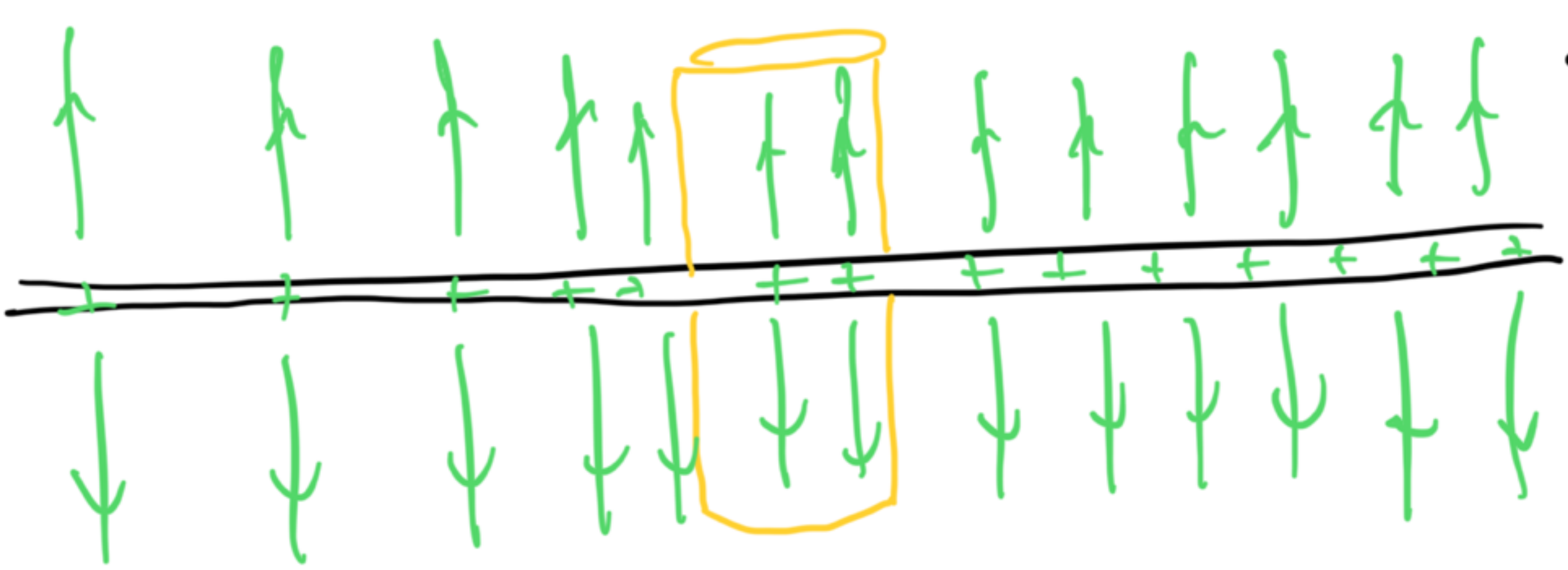


λ (1D $\rightarrow$ C/m)

σ (2D $\rightarrow$ C/m^2)

ρ (3D $\rightarrow$ C/cm^3)

Closed surface : Small cylinder of cross sectional area A, height h.



Side View

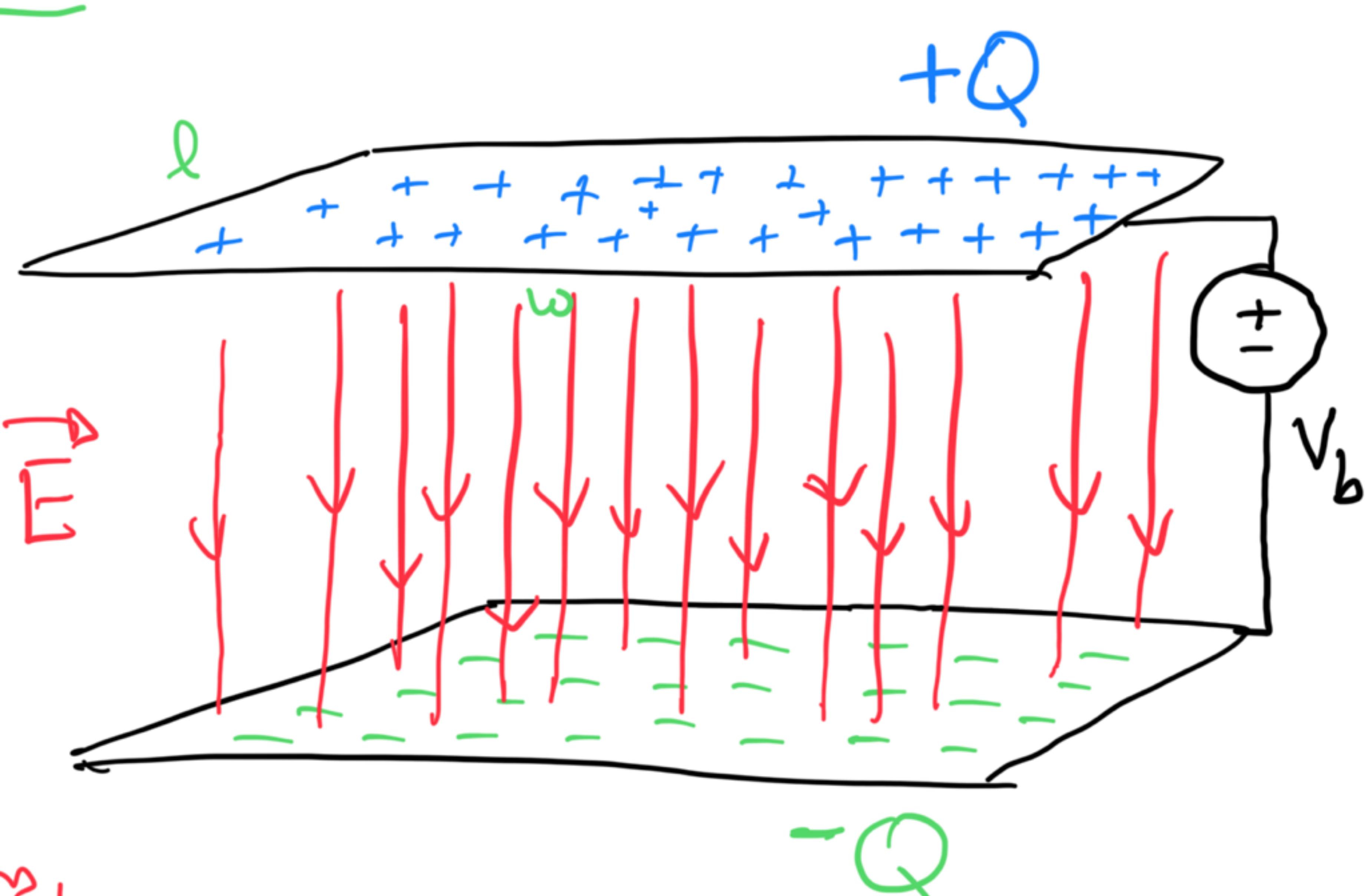
RHS: $Q_{inside} = \sigma \cdot A \rightarrow \Phi_E = \dfrac{\sigma A}{\varepsilon_0}$

LHS: $\oiint \vec{E} \cdot d\vec{A} = |\vec{E}| \underbrace{(2A)}_{\text{top and bottom faces.}}$

$$\therefore \quad |\vec{E}| \cdot 2\cancel{A} = \sigma \cancel{A} / \varepsilon_0$$

$$\boxed{|\vec{E}| = \frac{\sigma}{2\varepsilon_0}}$$

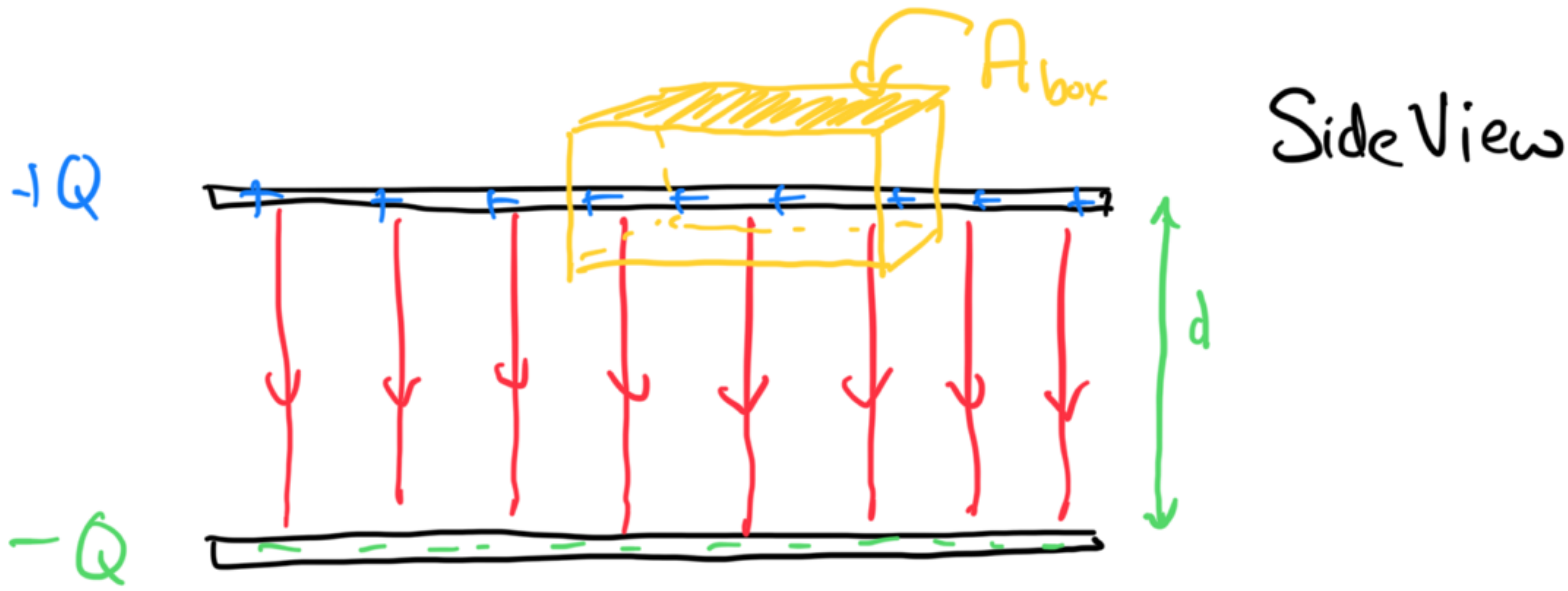
Example 3: Parallel Plate Capacitor



① $|\vec{E}|$ = constant inside

② $\vec{E} = 0$ outside

(2) $E = 0$ outside.



RHS: $\Phi_E = \dfrac{Q_{inside}}{\varepsilon_0} = \dfrac{\sigma A_{box}}{\varepsilon_0}$

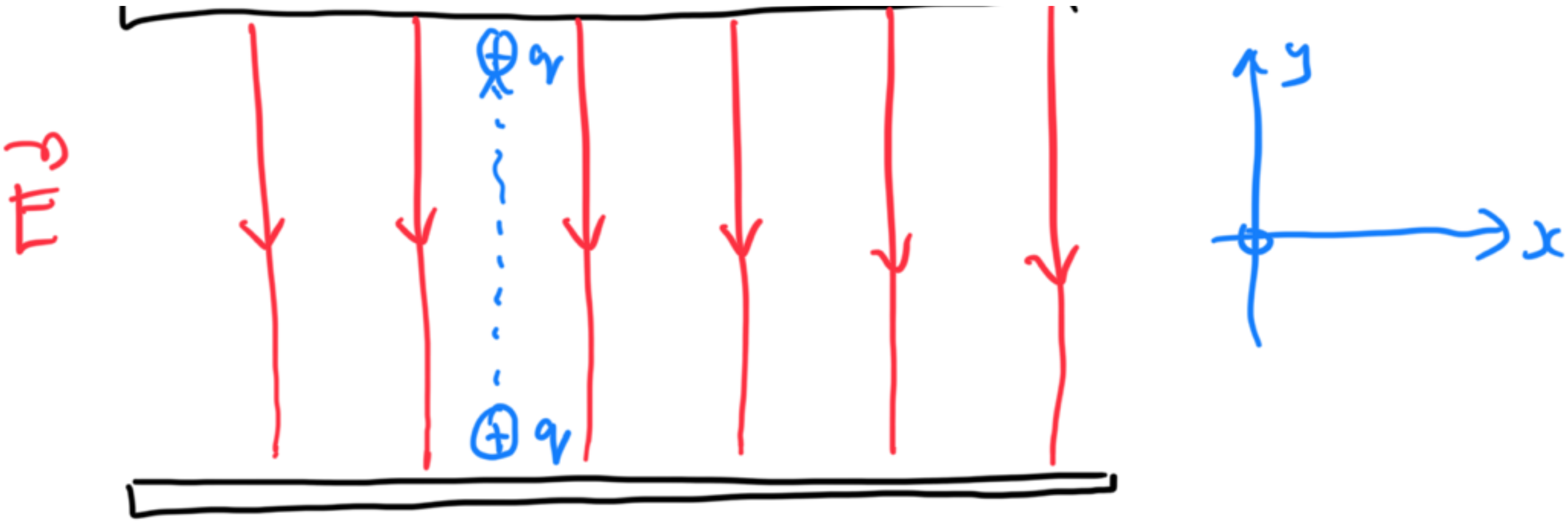
$A_{plate} = lw \quad \therefore \sigma = \dfrac{Q}{A_{plate}} = \dfrac{Q}{lw}$

LHS: $\oiint \vec{E} \cdot d\vec{A} = |\vec{E}| A_{box}$

$\therefore |\vec{E}| \cdot \cancel{A_{box}} = \dfrac{\sigma \cancel{A_{box}}}{\varepsilon_0}$

$$\boxed{|\vec{E}| = \frac{\sigma}{\varepsilon_0} = \frac{Q}{\varepsilon_0 A_{plate}}}$$

<u>Concept:</u> The electric field depends only on the battery voltage and the distance between the plates.



Imagine moving a +ve test charge from the bottom plate to the top plate.

$$W = \vec{F} \cdot \vec{\Delta x}$$
$$= (-qE\hat{j})(d\hat{j})$$
$$= -qEd$$

$$\therefore \Delta U = U_f - U_i = -W = qEd$$

$$\therefore \Delta V = \frac{\Delta U}{q} = Ed = V_{top} - V_{bottom}$$
$$= V_{battery}.$$

$$\boxed{E \cdot d = V_{battery}}$$

Question: Suppose we make a capacitor with area A and distance d between the plates. We hook up a battery of voltage V_B.

How much charge ends up on the top / bottom plates? What is Q??

$$V_{battery} = E \cdot d$$

$$E = \frac{Q}{\varepsilon_0 A}$$

$$\therefore \quad V_{battery} = \frac{Q d}{\varepsilon_0 A}$$

$$Q = \underbrace{\left(\frac{\varepsilon_0 A}{d}\right)}_{} V_{battery}.$$

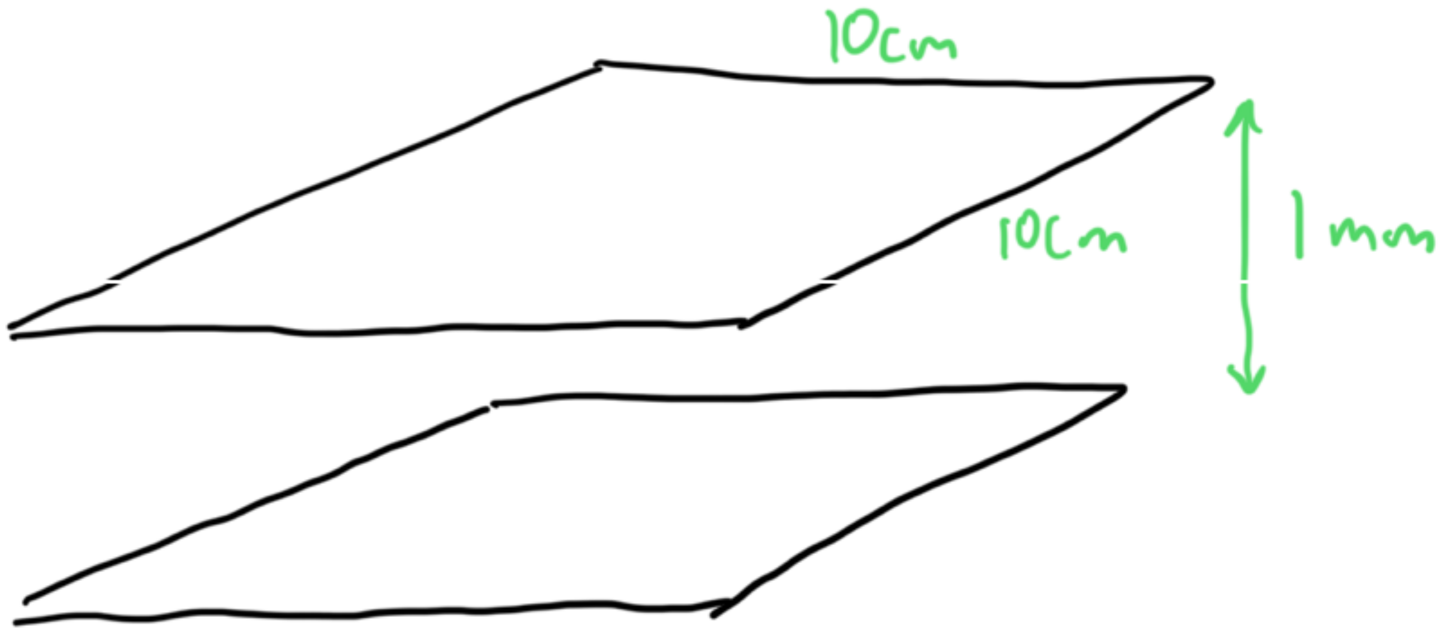
This is a property of the device.
We call this the capacitance, C

$$\boxed{C_{\substack{parallel\\plates}} \equiv \frac{\varepsilon_0 A}{d}}$$

$$Q = C \, V_{battery} \qquad \therefore \quad [C] = \frac{\text{Coulombs}}{\text{Volt}} \equiv \text{Farad}$$

Coulombs Volts

(After Michael Faraday)



$$A = (0.1\,m)(0.1\,m) = 0.01\,m^2$$

$$d = 1\,mm = 0.001\,m$$

$$C = \frac{\varepsilon_0 A}{d} = \frac{(8.854 \times 10^{-12})(.01)}{0.001}$$

$$= 8.854 \times 10^{-11}\ F$$

$$= 0.8854\ pF \quad \text{(pico Farads)}$$

Stored Energy

Kinetic Energy $\quad \frac{1}{2} m v^2$

Energy Stored in a Spring $\quad \frac{1}{2} k (\Delta x)^2$

Physics idea:

Physics Idea:

$$U_{internal} = \frac{1}{2}\begin{pmatrix}\text{fundamental} \\ \text{object} \\ \text{property}\end{pmatrix}\begin{pmatrix}\text{dynamic} \\ \text{variable}\end{pmatrix}^2$$

For a capacitor:

$$\boxed{U_{stored} = \frac{1}{2} C (\Delta V)^2}$$

Let's Explore this:

$$U_{stored} = \frac{1}{2}\left(\frac{\varepsilon_0 A}{d}\right)(Ed)^2$$

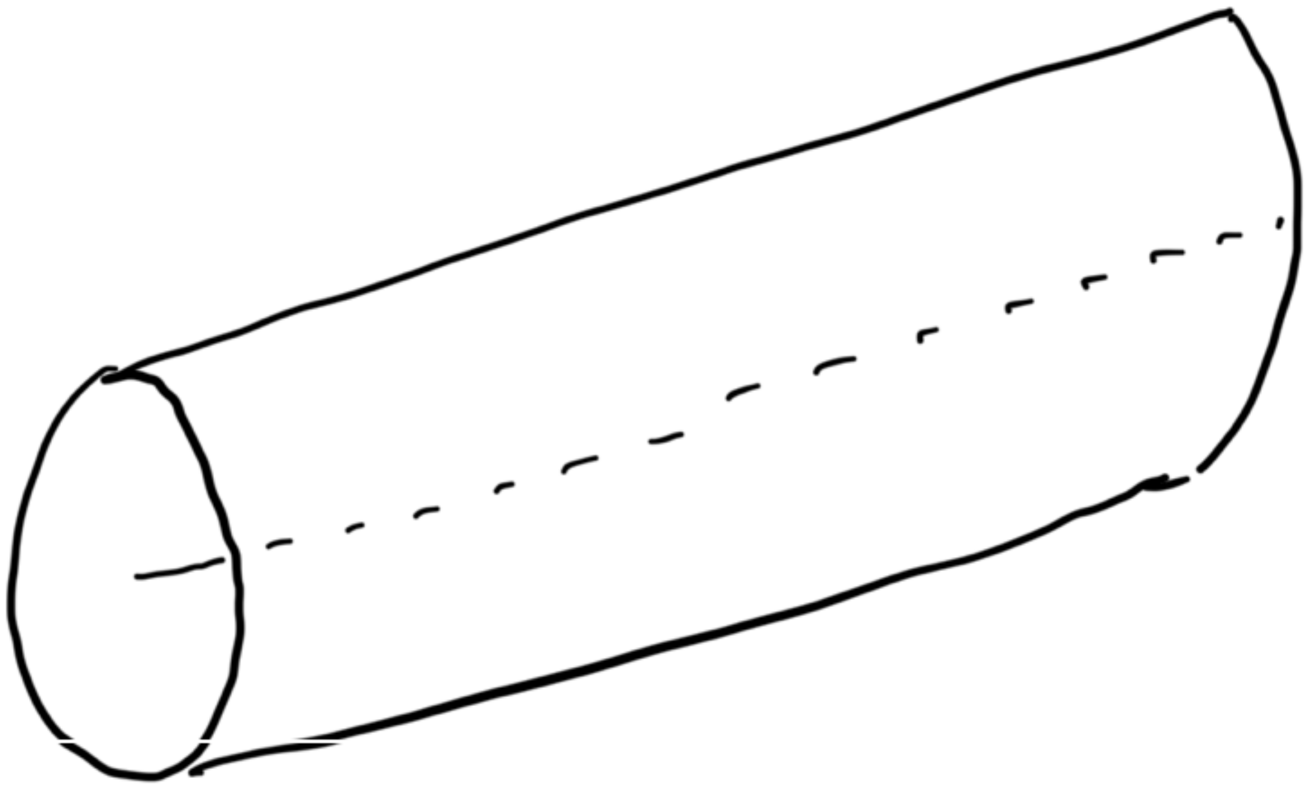
$$= \frac{1}{2}\varepsilon_0 E^2 \underbrace{(A d)}_{\text{Volume.}}$$

$$\boxed{\frac{U_{stored}}{\text{Volume}} = \frac{1}{2}\varepsilon_0 E^2}$$

This says that the electric field that exists in the air volume between the plates is a source of stored electrical energy.

This expression tells us exactly how much energy is stored there.

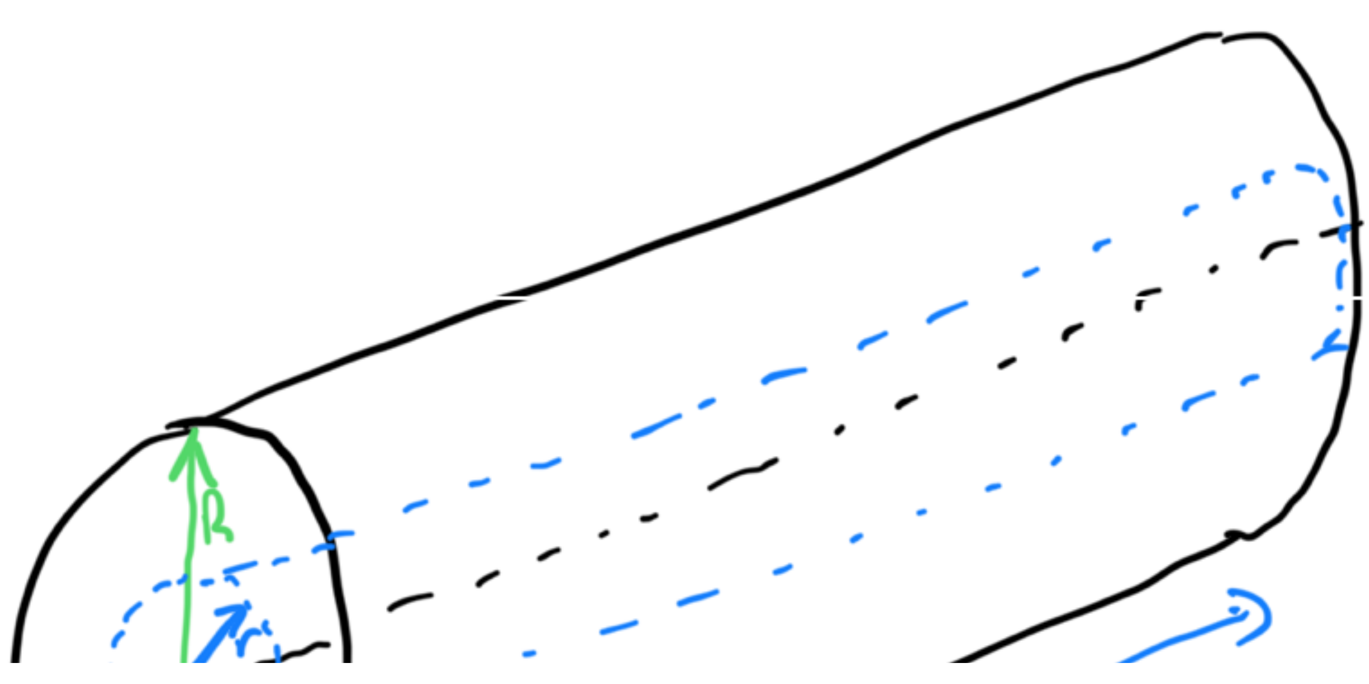
A3 Q10



Uniform charge density, $\rho \left(C/m^3 \right)$, so like a tube filled with a "gas" that is ionized.

Consider $r < R$:

① Draw a Gaussian Surface (a cylinder) of radius r $(< R)$



How much charge is inside?

$$Q_{inside} = \rho V = \rho\left(\overbrace{\pi r^2}^{\text{area}} \ell\right)$$

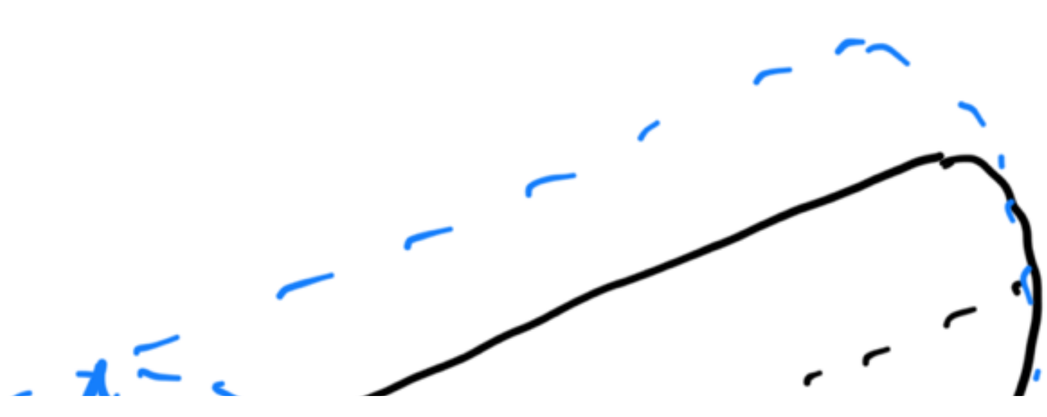
$$\therefore \Phi_E^{\text{Surface}} = \frac{Q_{inside}}{\varepsilon_0} = \frac{\rho \pi r^2 \ell}{\varepsilon_0} = RHS$$

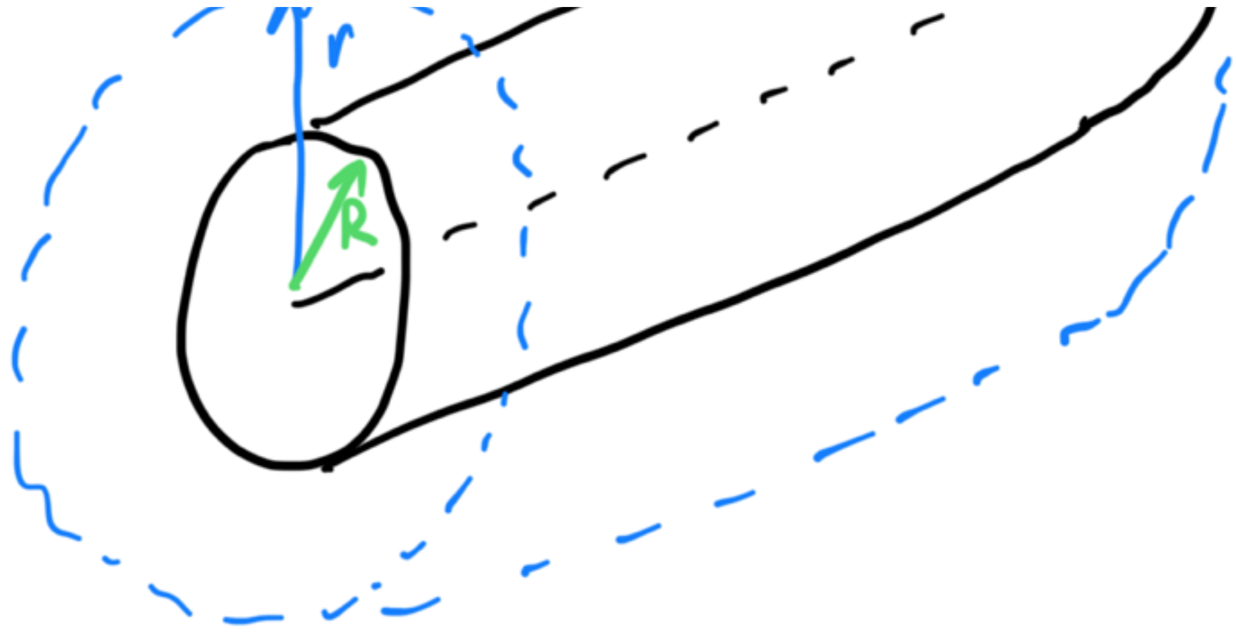
$$LHS = \oiint_{\text{Surface}} \vec{E} \cdot d\vec{A} = |\vec{E}| \cdot \underbrace{2\pi r}_{\text{circumference}} \ell$$

$$\therefore |\vec{E}| \cdot 2\cancel{\pi}\cancel{r}\cancel{\ell} = \rho \frac{(\cancel{\pi} r^{\cancel{2}} \cancel{\ell})}{\varepsilon_0}$$

$$\boxed{|\vec{E}| = \frac{\rho r}{2\varepsilon_0}}$$

Now, consider $r > R$:





How much charge is inside?

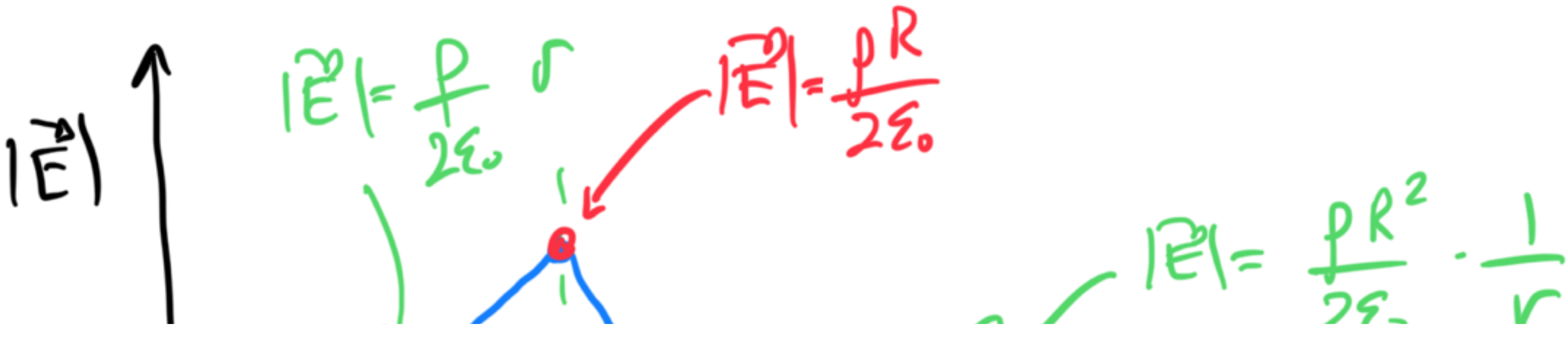
$$Q_{inside} = \rho\left(\pi R^2 l\right)$$

$$\therefore \quad \Phi_E = \frac{Q_{inside}}{\varepsilon_0} = \frac{\rho\left(\pi R^2 l\right)}{\varepsilon_0} = RHS$$

$$LHS = \underset{Surface}{\oiint} \vec{E}\cdot\vec{dA} = |\vec{E}|\cdot 2\pi r l$$

$$\therefore \quad |\vec{E}|\cdot 2\cancel{\pi} r \cancel{l} = \rho\frac{\cancel{\pi} R^2 \cancel{l}}{\varepsilon_0}$$

$$\boxed{|\vec{E}| = \frac{\rho R^2}{2\varepsilon_0 r}}$$




R
r