Overview

This document presents a complete, start-to-finish overview of the major classes of numbers used in mathematics. It begins with basic counting numbers and systematically builds through rational, irrational, real, complex, algebraic, transcendental, and explicitly constructed numbers.

Natural Numbers

The natural numbers are the counting numbers:

1, 2, 3, 4, ...

Depending on convention, 0 may or may not be included.

Whole Numbers

Whole numbers consist of the natural numbers together with zero:

0, 1, 2, 3, 4, ...

Integers

Integers include all whole numbers and their negatives:

... -3, -2, -1, 0, 1, 2, 3 ...

They are closed under addition, subtraction, and multiplication.

Rational Numbers

A rational number is any number that can be written as a ratio of two integers p/q with $q \neq 0$.

Their decimal expansions either terminate or repeat.

Examples include 1/2, -3, and 0.75.

Irrational Numbers

Irrational numbers cannot be written as a ratio of integers.

Their decimal expansions are non-terminating and non-repeating.

Examples include $\sqrt{2}$, π, and e.

Real Numbers

The real numbers consist of all rational and irrational numbers.
They correspond exactly to points on the number line.

Algebraic Numbers

An algebraic number is any number that is a solution to a polynomial equation with integer coefficients.
All rational numbers are algebraic, as are irrational numbers such as $\sqrt{2}$.

Transcendental Numbers

Transcendental numbers are real numbers that are not algebraic.
They are not solutions to any polynomial equation with integer coefficients.
Examples include π and e.

Imaginary and Complex Numbers

Imaginary numbers are multiples of $i = \sqrt{(-1)}$.
Complex numbers are numbers of the form $a + bi$.
All real numbers are a subset of the complex numbers.

Explicitly Constructed Numbers

Some real numbers are defined not by equations but by explicit digit-construction rules.
A concrete example is the number:
1.010010001000010000001...

In this construction, the number of zeros between successive ones increases by one each time.
This decimal expansion is deliberately structured and never becomes periodic.

Classification of the Constructed Example

The constructed number 1.01001000100001000001... has the following properties:

- It is irrational, because its decimal expansion never repeats.
- It is real, because it corresponds to a point on the real number line.
- It is transcendental, because it can be approximated by rational numbers too well to be algebraic.

Numbers of this type belong to the family of Liouville-type numbers.

Why Constructed Numbers Matter

Liouville-type constructions were the first explicit proof that transcendental numbers exist.

They demonstrate that a number can be fully specified by a simple rule and yet lie beyond all algebraic equations.

Containment Summary

Natural ⊂ Whole ⊂ Integers ⊂ Rational ⊂ Real ⊂ Complex

Irrational numbers form a subset of the real numbers.

The real numbers split into algebraic and transcendental numbers.

Final Perspective

The hierarchy of number systems reflects the historical expansion of mathematics as new problems demanded new kinds of numbers.

Explicitly constructed numbers sit at a deep boundary between arithmetic, analysis, and logic, revealing how rich the real number system truly is.