Solutions: Mathematical Physics Final Exam

Question 1: Algebra of Vectors

Consider the following two vectors in $\mathbb{R}^3$:

$$\mathbf{A} = 3\hat{\imath} - 2\hat{\jmath} + \hat{k}, \quad \mathbf{B} = \hat{\imath} + 4\hat{\jmath} - 2\hat{k}.$$

a) Compute the dot product $\mathbf{A} \cdot \mathbf{B}$.

$$\mathbf{A} \cdot \mathbf{B} = 3 \times 1 + (-2) \times 4 + 1 \times (-2) = 3 - 8 - 2 = -7.$$

b) Compute the cross product $\mathbf{A} \times \mathbf{B}$.

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 3 & -2 & 1 \\ 1 & 4 & -2 \end{vmatrix} = \hat{\imath} \begin{vmatrix} -2 & 1 \\ 4 & -2 \end{vmatrix} - \hat{\jmath} \begin{vmatrix} 3 & 1 \\ 1 & -2 \end{vmatrix} + \hat{k} \begin{vmatrix} 3 & -2 \\ 1 & 4 \end{vmatrix}.$$

$$\begin{aligned} &= \hat{\imath}\,[(-2)(-2) - (1)(4)] - \hat{\jmath}\,[(3)(-2) - (1)(1)] + \hat{k}\,[(3)(4) - (-2)(1)] \\ &= \hat{\imath}\,[4 - 4] - \hat{\jmath}\,[-6 - 1] + \hat{k}\,[12 + 2] \\ &= 0\hat{\imath} + 7\hat{\jmath} + 14\hat{k}. \end{aligned}$$

Therefore,

$$\mathbf{A} \times \mathbf{B} = 7\hat{\jmath} + 14\hat{k}.$$

c) Find the angle between the two vectors $\mathbf{A}$ and $\mathbf{B}$.

The angle θ between the vectors is given by:

$$\cos\theta = \frac{\mathbf{A} \cdot \mathbf{B}}{|\mathbf{A}||\mathbf{B}|}$$

First, compute the magnitudes of $\mathbf{A}$ and $\mathbf{B}$:

$|\mathbf{A}| = \sqrt{3^2 + (-2)^2 + 1^2} = \sqrt{9 + 4 + 1} = \sqrt{14}, \quad |\mathbf{B}| = \sqrt{1^2 + 4^2 + (-2)^2} = \sqrt{1 + 16 + 4} = \sqrt{21}.$

Now, compute the cosine of the angle:

$$\cos\theta = \frac{-7}{\sqrt{14} \times \sqrt{21}} = \frac{-7}{\sqrt{294}}.$$

Hence,

$$\theta = \cos^{-1}\left(\frac{-7}{\sqrt{294}}\right).$$

Question 2: Coupled Linear First-Order Differential Equations

Consider the coupled system of differential equations:

$$\frac{dx}{dt} = 4x + y, \quad \frac{dy}{dt} = -2x + y.$$

a) Write the system in matrix form: $\frac{d\vec{X}}{dt} = A\vec{X}$.

$$\vec{X} = \begin{bmatrix} x \\ y \end{bmatrix}, \quad A = \begin{bmatrix} 4 & 1 \\ -2 & 1 \end{bmatrix}, \quad \frac{d\vec{X}}{dt} = A\vec{X}$$

b) Find the eigenvalues and eigenvectors of matrix A.

The eigenvalues are the solutions to $\det(A - \lambda I) = 0$:

$$\det \begin{bmatrix} 4-\lambda & 1 \\ -2 & 1-\lambda \end{bmatrix} = (4-\lambda)(1-\lambda) + 2 = \lambda^2 - 5\lambda + 6 = 0.$$

Solving this quadratic equation gives the eigenvalues:

$$\lambda = 2, 3.$$

Now, find the eigenvectors: - For $\lambda = 2$, solve $(A - 2I)\vec{v} = 0$:

$$\begin{bmatrix} 2 & 1 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

This gives the eigenvector $\vec{v_1} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$.

- For $\lambda = 3$, solve $(A - 3I)\vec{v} = 0$:

$$\begin{bmatrix} 1 & 1 \\ -2 & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

This gives the eigenvector $\vec{v_2} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

c) Solve the system for $x(t)$ and $y(t)$, assuming initial conditions $x(0) = 2$, $y(0) = 1$.

The general solution is:

$$\vec{X}(t) = c_1 e^{2t} \begin{bmatrix} 1 \\ -2 \end{bmatrix} + c_2 e^{3t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Applying initial conditions:

$$c_1 \begin{bmatrix} 1 \\ -2 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

This gives the system:

$$\begin{cases} c_1 + c_2 = 2 \\ -2c_1 - c_2 = 1 \end{cases}$$

Solving this system yields $c_1 = -3$, $c_2 = 5$. Thus, the solution is:

$$\vec{X}(t) = -3e^{2t} \begin{bmatrix} 1 \\ -2 \end{bmatrix} + 5e^{3t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

d) Describe the behavior of the solution in the phase plane.

Since both eigenvalues are real and positive ($\lambda = 2, 3$), and the eigenvectors are linearly independent, the system represents an **unstable node**. The solutions will exponentially diverge from the origin along the directions defined by the eigenvectors.

Question 3: AC RL Circuit and Complex Impedance

An alternating voltage source $V(t) = V_0 e^{i\omega t}$ is applied across a resistor R and an inductor L connected in series.

a) Using the concept of complex impedance, derive the total impedance Z of the RL circuit in terms of R, L, and the angular frequency ω.

The impedance of the resistor is $Z_R = R$, and the impedance of the inductor is $Z_L = i\omega L$. The total impedance is the sum of these two:

$$Z = Z_R + Z_L = R + i\omega L.$$

b) Using Ohm's law in the complex form $V(t) = I(t)Z$, find the expression for the current $I(t)$ in the circuit.

From Ohm's law, we have:

$$I(t) = \frac{V(t)}{Z} = \frac{V_0 e^{i\omega t}}{R + i\omega L}.$$

Multiplying numerator and denominator by the complex conjugate of Z, we get:

$$I(t) = \frac{V_0 e^{i\omega t}(R - i\omega L)}{R^2 + (\omega L)^2}.$$

Thus, the current is:

$$I(t) = \frac{V_0}{\sqrt{R^2 + (\omega L)^2}} e^{i(\omega t - \theta)},$$

where $\theta = \tan^{-1}\left(\frac{\omega L}{R}\right)$ is the phase shift.

c) Find the phase shift between the voltage and the current in the circuit.

The phase shift is given by $\theta = \tan^{-1}\left(\frac{\omega L}{R}\right)$. This phase shift represents the time delay between the voltage and current waveforms, which depends on the relative contributions of the resistor and inductor to the total impedance.

Question 4: Fourier Analysis

Let $f(x)$ be a periodic function with period T, defined as:

$$f(x) = \begin{cases} 1, & 0 \leq x < \frac{T}{2}, \\ -1, & \frac{T}{2} \leq x < T. \end{cases}$$

a) Compute the Fourier coefficients a_0, a_n, and b_n for this function.

The Fourier coefficients are computed using the formulas:

$$a_0 = \frac{2}{T} \int_0^T f(x)\,dx = \frac{2}{T} \left(\int_0^{T/2} 1\,dx + \int_{T/2}^T (-1)\,dx \right) = 0.$$

For $n \geq 1$,

$$a_n = \frac{2}{T} \int_0^T f(x) \cos\left(\frac{2\pi n x}{T}\right) dx = 0,$$

and

$$b_n = \frac{2}{T} \int_0^T f(x) \sin\left(\frac{2\pi n x}{T}\right) dx = \frac{4}{n\pi} \left(1 - (-1)^n\right).$$

b) Find the Fourier series representation of $f(x)$.

The Fourier series is:

$$f(x) = \sum_{n=1}^{\infty} \frac{4}{n\pi} \sin\left(\frac{2\pi n x}{T}\right).$$

c) Discuss the convergence of the Fourier series for this function.

The Fourier series for this piecewise function converges to $f(x)$ at all points except where $f(x)$ is discontinuous (at $x = T/2$). At these points, the series converges to the average of the left-hand and right-hand limits of the function, according to the Dirichlet conditions.

Question 5: Vector Calculus

Let $\mathbf{F} = x^2\hat{\imath} + 2xy\hat{\jmath} + yz\hat{k}$ be a vector field.

a) Compute the divergence of $\mathbf{F}$, $\nabla \cdot \mathbf{F}$.

$$\nabla \cdot \mathbf{F} = \frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial y}(2xy) + \frac{\partial}{\partial z}(yz) = 2x + 2x + y = 2x + y.$$

b) Compute the curl of $\mathbf{F}$, $\nabla \times \mathbf{F}$.

$$\nabla\times\mathbf{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & 2xy & yz \end{vmatrix} = \hat{i}\left(\frac{\partial yz}{\partial y} - \frac{\partial(2xy)}{\partial z}\right) - \hat{j}\left(\frac{\partial yz}{\partial x} - \frac{\partial(x^2)}{\partial z}\right) + \hat{k}\left(\frac{\partial(2xy)}{\partial x} - \frac{\partial(x^2)}{\partial y}\right).$$

Simplifying, we get:

$$\nabla \times \mathbf{F} = \hat{i}(z-0) - \hat{j}(0-0) + \hat{k}(2y-0) = z\hat{i} + 2y\hat{k}.$$

c) Evaluate the line integral of $\mathbf{F}$ along the curve C parameterized by $\mathbf{r}(t) = \langle t, t^2, t^3\rangle$ for t from 0 to 1.

The line integral is given by:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 \mathbf{F}(\mathbf{r}(t)) \cdot \frac{d\mathbf{r}(t)}{dt} dt.$$

First, compute $\frac{d\mathbf{r}(t)}{dt} = \langle 1, 2t, 3t^2\rangle$. Next, substitute $\mathbf{r}(t) = \langle t, t^2, t^3\rangle$ into $\mathbf{F}$:

$$\mathbf{F}(t, t^2, t^3) = \langle t^2, 2t^3, t^5\rangle.$$

Then, the dot product is:

$$\mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = t^2(1) + 2t^3(2t) + t^5(3t^2) = t^2 + 4t^4 + 3t^7.$$

Finally, integrate with respect to t:

$$\int_0^1 (t^2 + 4t^4 + 3t^7)\, dt = \left[\frac{t^3}{3} + \frac{4t^5}{5} + \frac{3t^8}{8}\right]_0^1 = \frac{1}{3} + \frac{4}{5} + \frac{3}{8}.$$

The final answer is:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \frac{1}{3} + \frac{4}{5} + \frac{3}{8} = \frac{40+24+15}{120} = \frac{79}{120}.$$