Algebra of Vectors

① Addition

Consider $\vec{a} = a_x \hat{i} + a_y \hat{j} + a_z \hat{k}$

$\vec{b} = b_x \hat{i} + b_y \hat{j} + b_z \hat{k}$

(which we can always do, if we follow steps ① and ② in every problem)

What is $\vec{c} = \vec{a} + \vec{b}$

Answer:

$\vec{c} = (a_x + b_x)\hat{i}$

$$+ (a_y + b_y)\hat{j}$$

$$+ (a_z + b_z)\hat{k}$$

Why? Because 1.2 apples + 3.8 apples

= 5.0 apples !!

So, $1.2\,\hat{i} + 3.8\,\hat{i} = 5.0\,\hat{i}$

etc.

⋮

② Subtraction: What is $\vec{a} - \vec{b}$?

Well, $\vec{c} = \vec{a} - \vec{b}$

$$= (a_x - b_x)\hat{i} +$$

$$(a_y - b_y)\hat{j} +$$

$$(a_z - b_z)\hat{k}$$

Simple !! But, note this:

$\vec{a} + \vec{b} = \vec{b} + \vec{a}$ Symmetric!

$$\vec{a} - \vec{b} \neq \vec{b} - \vec{a}$$ Not symmetric!!

Vectors are symmetric under addition,
but anti-symmetric under subtraction.
(since $\vec{a} - \vec{b} = -(\vec{b} - \vec{a})$)

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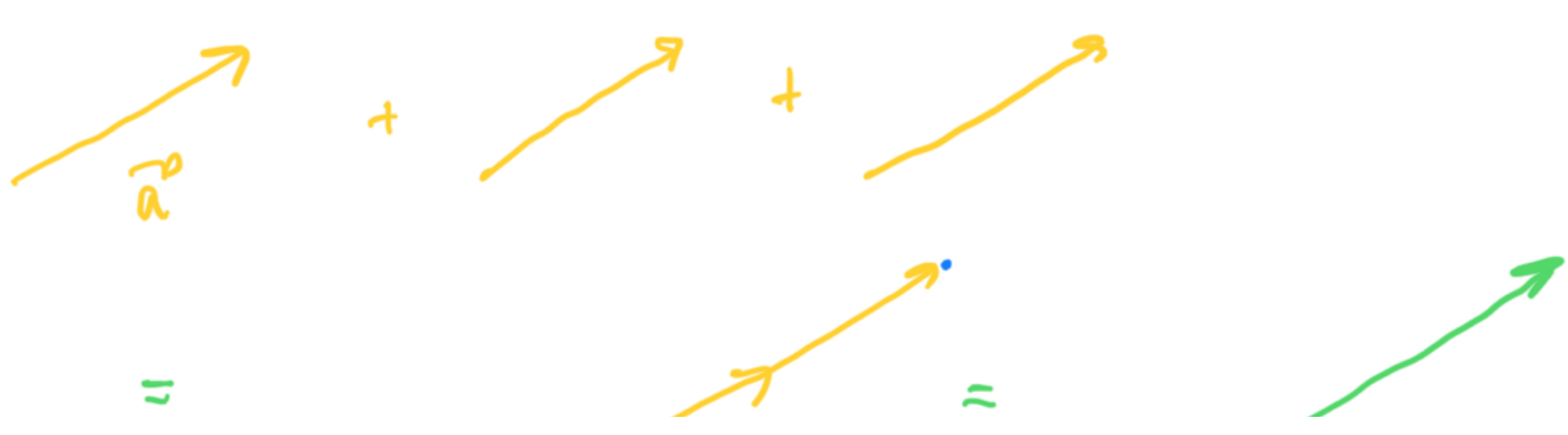
③ Multiplication

- Hmm... what does this even mean?

a) Multiplication by a Scalar

- ~~just like you would think...~~
repeated addition.

$3\vec{a} =$



b) The Scalar Product

Remember when we calculated the length of a vector ....

$$|\vec{a}| = \sqrt{a_x^2 + a_y^2 + a_z^2}$$

$$|\vec{a}|^2 = a_x^2 + a_y^2 + a_z^2$$

We can think of generating this by considering multiply a vector <u>by itself</u>, component by component.

$$\begin{aligned}
\vec{a} &= a_x\hat{i} + a_y\hat{j} + a_z\hat{k} \\
& \quad\ \times \qquad\ \ \times \qquad\ \ \times \\
\vec{a} &= a_x\hat{i} + a_y\hat{j} + a_z\hat{k} \\
\uparrow & \quad\ = \qquad\ \ = \qquad\ \ = \\
\vec{a}\cdot\vec{a} &= a_x^2 + a_y^2 + a_z^2
\end{aligned}$$

We can extend this to any two vectors:

$$\boxed{\vec{a} \cdot \vec{b} \equiv a_x b_x + a_y b_y + a_z b_z}$$

Scalar product of two Vectors.

It is called the scalar product because the result is a scalar (just a number)

def scalar_product($\vec{a}$, $\vec{b}$)

vector vector

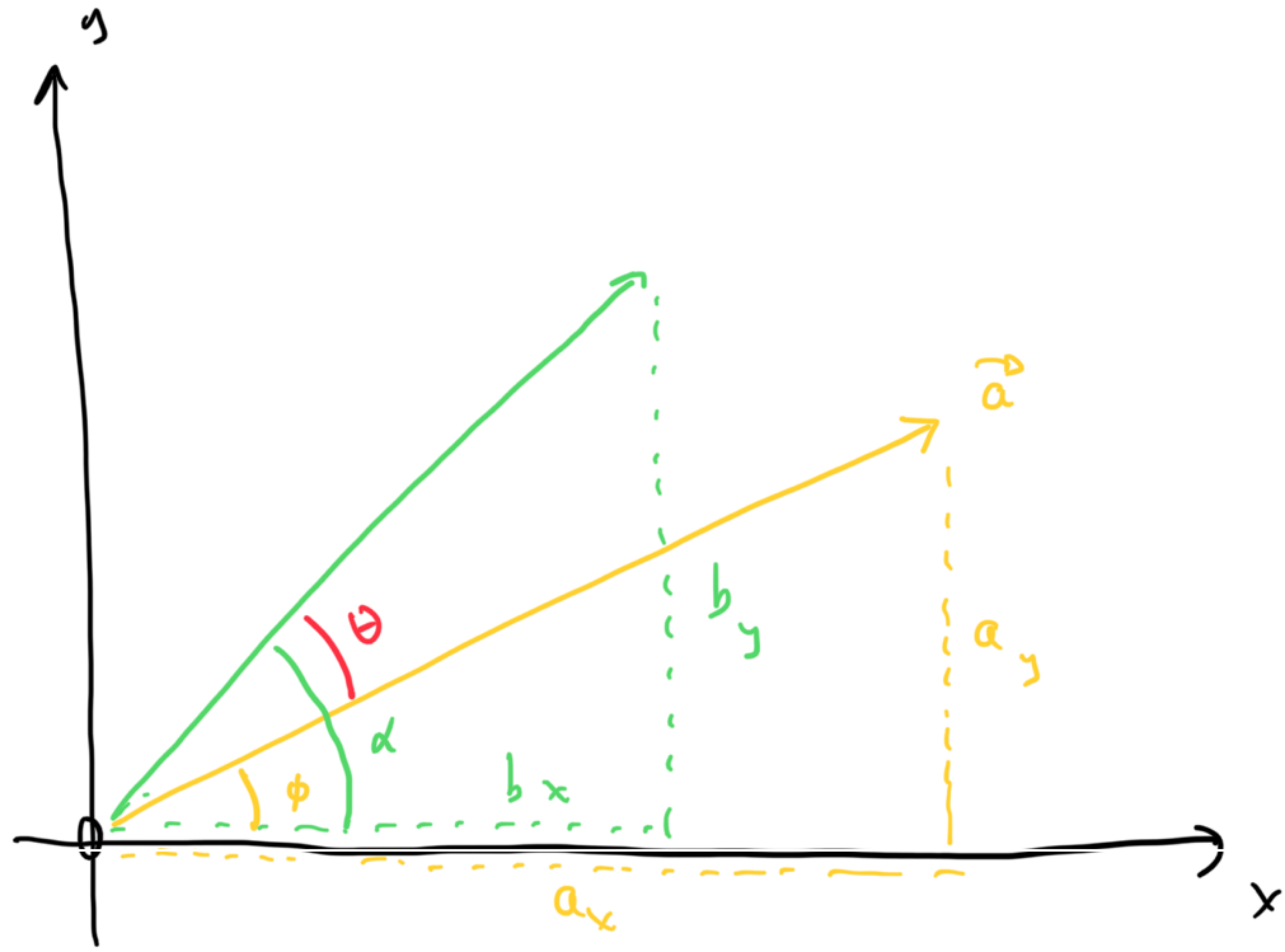
⋮

return c

scalar

And now, some cool geometry ...

Two general vectors.



$$\boxed{\theta = \alpha - \phi}$$

$a_y = |\vec{a}| \sin \phi$

$a_x = |\vec{a}| \cos \phi$

$b_y = |\vec{b}| \sin \alpha$

$b_x = |\vec{b}| \cos \alpha$

$$\begin{aligned}
\vec{a} \cdot \vec{b} &\equiv a_x b_x + a_y b_y \\
&= (|\vec{a}| \cos \phi)(|\vec{b}| \cos \alpha) \\
&\quad + (|\vec{a}| \sin \phi)(|\vec{b}| \sin \alpha) \\
&= |\vec{a}||\vec{b}| \left[\cos \phi \cos \alpha + \sin \phi \sin \alpha \right]
\end{aligned}$$

$$= \cos(\alpha - \phi) \quad \text{this identity}$$

$$= \cos\theta$$

$$\therefore \boxed{\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos\theta}$$

Why is this important? Well, $\cos(90°) = 0$!! So, if $\vec{a} \cdot \vec{b} = 0$, then the two vectors are orthogonal!!

$$\vec{a} = 3\hat{i} - 2\hat{j} + 5\hat{k}$$

$$\vec{b} = 2\hat{i} - 2\hat{j} - 2\hat{k}$$

$$\vec{a} \cdot \vec{b} = (3)(2) + (-2)(-2) + (5)(-2)$$

$$= 6 + 4 - 10$$

$$= 0 \,!$$

$$\therefore \boxed{\vec{a} \perp \vec{b}}$$

$\therefore \quad \boxed{a \perp b}$

Reason 2 that the Scalar product is important (in physics)....
there are tons of physical laws that use this concept.

Ex. $W = \vec{F} \cdot \vec{\Delta x}$

work — scalar!!

force — (vector)

displacement — (vector)

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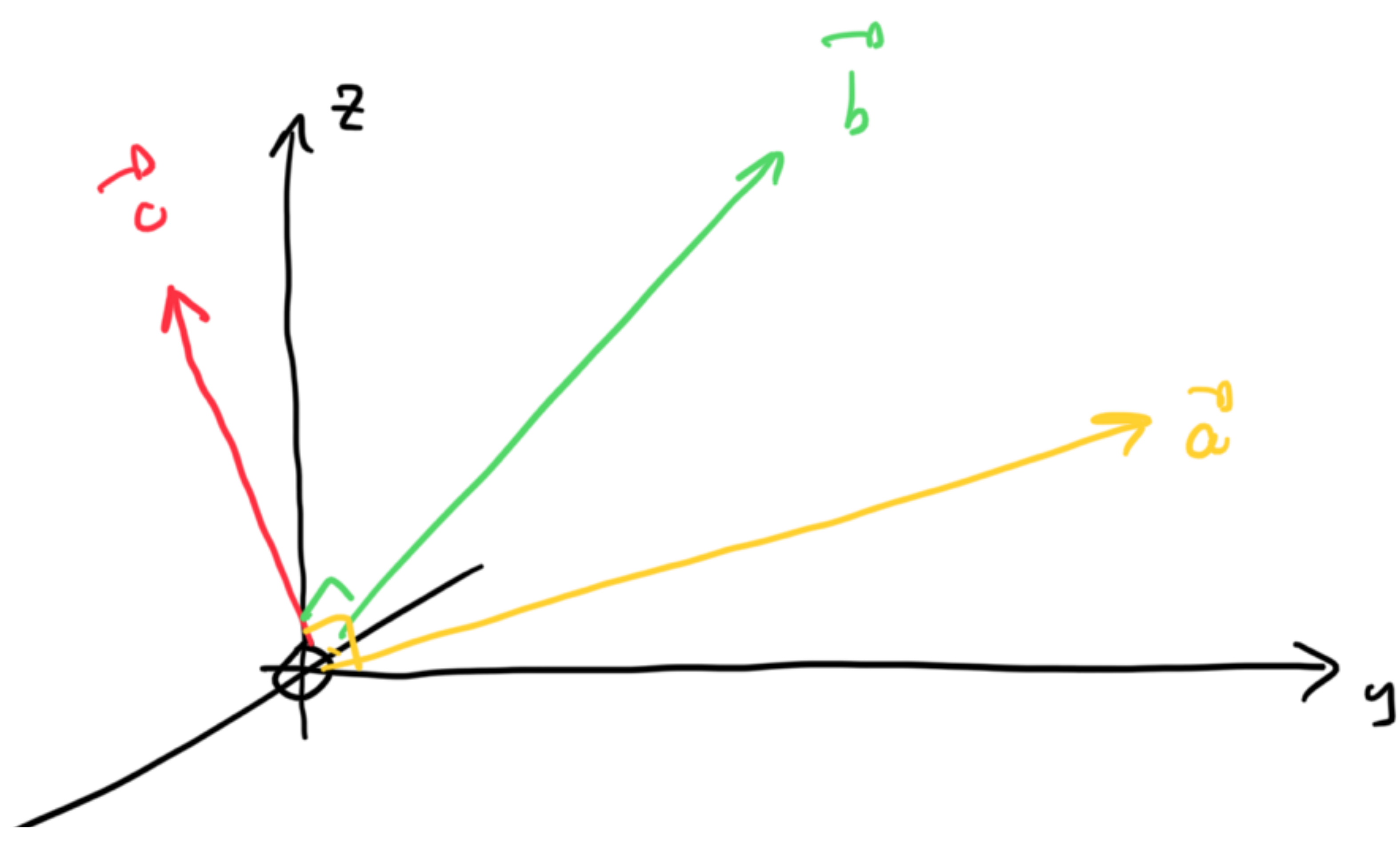
c) The Vector Product



x

Problem: Given $\vec{a}$ and $\vec{b}$, find a vector, $\vec{c}$, which is perpendicular to both $\vec{a}$ and $\vec{b}$, and has length = 1.

$$\vec{a} = a_x \hat{i} + a_y \hat{j} + a_z \hat{k}$$

$$\vec{b} = b_x \hat{i} + b_y \hat{j} + b_z \hat{k}$$

$$\vec{c} = c_x \hat{i} + c_y \hat{j} + c_z \hat{k}$$

$$\vec{a} \cdot \vec{c} = 0 = a_x c_x + a_y c_y + a_z c_z \quad (1)$$

$$\vec{b} \cdot \vec{c} = 0 = b_x c_x + b_y c_y + b_z c_z \quad (2)$$

$$|\vec{c}| = \sqrt{c_x^2 + c_y^2 + c_z^2} = 1 \quad (3)$$

3 equations in 3 unknowns! :)

The algebra is kinda hard :(

?

$\Big\downarrow$ a miracle occurs

$$\vec{c} = \pm \frac{\vec{d}}{|\vec{d}|} \quad \text{where} \quad \vec{d} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$$

Ex:

$$\vec{a} = 3\hat{i} + 2\hat{j} + \hat{k}$$

$$\vec{b} = 2\hat{i} + \hat{j} + 4\hat{k}$$

$$\vec{d} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 2 & 1 \\ 2 & 1 & 4 \end{vmatrix} = \hat{i}\,(2\cdot 4 - 1\cdot 1) - \hat{j}\,(3\cdot 4 - 2\cdot 1) + \hat{k}\,(3\cdot 1 - 2\cdot 2)$$

$$\vec{d} = 7\hat{i} - 10\hat{j} - \hat{k}$$

$$|\vec{d}| = \sqrt{7^2 + 10^2 + 1^2} = \sqrt{150}$$

$$= 5\sqrt{6}$$

$\vec{d}$ $\quad | \quad 7 \quad -10 \quad -1 \quad |$

$$\frac{\vec{d}}{|\vec{d}|} = \left(\frac{\;}{5\sqrt{6}}, \frac{\;}{5\sqrt{6}}, \frac{\;}{5\sqrt{6}} \right)$$

$$\vec{C} = \pm \left(\frac{7}{5\sqrt{6}}, \frac{-10}{5\sqrt{6}}, -\frac{1}{5\sqrt{6}} \right)$$

The vector $\vec{d}$ is known as the <u>Vector product</u> of two vectors.

$$\vec{d} = \underbrace{\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}}_{\text{vectors in}}$$

vector out!

$$= \hat{i}\,(a_y b_z - b_y a_z)$$
$$- \hat{j}\,(a_x b_z - b_x a_z)$$
$$+ \hat{k}\,(a_x b_y - b_x a_y)$$

$+ \hat{k} (a_x b_y - b_x a_y)$

all of these terms involve <u>products</u> of the vector components of $\vec{a}$ and $\vec{b}$.

<u>Fun fact</u>:

$$\boxed{|\vec{d}| = |\vec{a}||\vec{b}| \sin\theta}$$

So, if $\vec{a}$ and $\vec{b}$ are <u>parallel</u> to one another, then $|\vec{d}| = 0$, hence $\vec{d} = 0$.

We write the vector product as:

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$$

Again, there are physics quantities that depend on this vector product.

Ex:

$$\vec{\tau} = \vec{r} \times \vec{F}$$

torque

moment arm

force.

$$\vec{F}_m = q\vec{v} \times \vec{B}$$

mag. force

charge

velocity

mag. field

$$\vec{S} = \vec{E} \times \vec{B}$$

Poynting Vector (Energy Flow)

electric field

magnetic field

Conclusion:

Some Geometry Question → do math → Discover new and interesting Quanity

What is the angle between two vectors? $\dashrightarrow$ Scalar Product

What is the vector $\perp$ to two vectors $\dashrightarrow$ Vector Product

We will see this over and over again!! ..

Summary:

$$\vec{a} = a_x \hat{i} + a_y \hat{j} + a_z \hat{k}$$

$$\vec{b} = b_x \hat{i} + b_y \hat{j} + b_z \hat{k}$$

Norm $\quad |\vec{a}| = \sqrt{a_x^2 + a_y^2 + a_z^2}$

Addition $\quad \vec{a} + \vec{b} = (a_x + b_x)\hat{i} + (a_y + b_y)\hat{j} + (a_z + b_z)\hat{k}$

Scalar Product $\quad \vec{a} \cdot \vec{b} = a_x b_x + a_y b_y + a_z b_z = |\vec{a}||\vec{b}| \cos \theta_{AB}$

Vector Product

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$$

$$|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}| \sin\theta$$

One more thing..... we see that calculating scalar and vector products can be useful and interesting in physics, and we have techniques (above) for doing this. There is another approach, using unit vectors, that is also interesting and is sometimes useful.

Based on what we have above, it is easy to see that:

$$\hat{i} \cdot \hat{i} = 1 \qquad \hat{i} \cdot \hat{j} = 0$$

$$\hat{j} \cdot \hat{j} = 1 \qquad \hat{i} \cdot \hat{k} = 0$$

$\hat{k}\cdot\hat{k} = 1 \qquad \hat{j}\cdot\hat{k} = 0$

We can calculate $\vec{a}\cdot\vec{b}$ as:

$$\vec{a}\cdot\vec{b} = \left(a_x\hat{i} + a_y\hat{j} + a_z\hat{k}\right)\cdot\left(b_x\hat{i} + b_y\hat{j} + b_z\hat{k}\right)$$

$$\begin{aligned} &= a_x b_x \overset{1}{\hat{i}\cdot\hat{i}} + a_x b_y \overset{0}{\cancel{\hat{i}\cdot\hat{j}}} + a_x b_z \overset{0}{\cancel{\hat{i}\cdot\hat{k}}} \\ &\quad + a_y b_x \overset{0}{\cancel{\hat{j}\cdot\hat{i}}} + a_y b_y \overset{1}{\hat{j}\cdot\hat{j}} + a_y b_z \overset{0}{\cancel{\hat{j}\cdot\hat{k}}} \\ &\quad + a_z b_x \overset{0}{\cancel{\hat{k}\cdot\hat{i}}} + a_z b_y \overset{0}{\cancel{\hat{k}\cdot\hat{j}}} + a_z b_z \underset{1}{\hat{k}\cdot\hat{k}} \end{aligned}$$

$$= a_x b_x + a_y b_y + a_z b_z$$

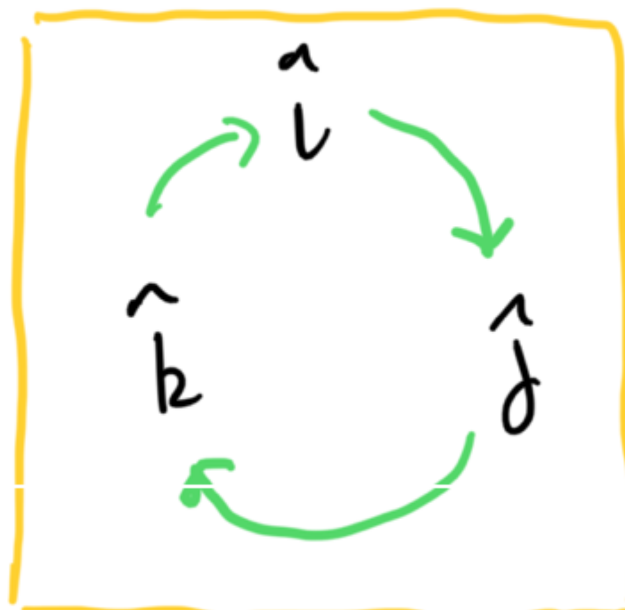
as expected!!

We can consider cross products similarly:

$\hat{i}\times\hat{i} = 0 \qquad \hat{i}\times\hat{j} = \hat{k} \qquad \hat{j}\times\hat{i} = -\hat{k}$

$\hat{j}\times\hat{j} = 0 \qquad \hat{i}\times\hat{k} = -\hat{j} \qquad \hat{k}\times\hat{i} = \hat{j}$

$\hat{k} \times \hat{k} = 0 \qquad \hat{j} \times \hat{k} = \hat{i} \qquad k \times j = -i$



$$\vec{a} \times \vec{b} = \left(a_x \hat{i} + a_y \hat{j} + a_z \hat{k}\right) \times \left(b_x \hat{i} + b_y \hat{j} + b_z \hat{k}\right)$$

$$\begin{aligned}
= \; & a_x b_x \overset{0}{\cancel{\hat{i} \times \hat{i}}} + a_x b_y \overset{\hat{k}}{\hat{i} \times \hat{j}} + a_x b_z \overset{-\hat{j}}{\hat{i} \times \hat{k}} \\
+ \; & a_y b_x \underset{-\hat{k}}{\hat{j} \times \hat{i}} + a_y b_y \overset{0}{\cancel{\hat{j} \times \hat{j}}} + a_y b_z \underset{\hat{i}}{\hat{j} \times \hat{k}} \\
+ \; & a_z b_x \underset{\hat{j}}{\hat{k} \times \hat{i}} + a_z b_y \underset{-\hat{i}}{\hat{k} \times \hat{j}} + a_z b_z \underset{0}{\cancel{\hat{k} \times \hat{k}}}
\end{aligned}$$

$$\begin{aligned}
= \; & \hat{i} \left(a_y b_z - a_z b_y \right) \\
& + \hat{j} \left(a_z b_x - a_x b_z \right) \\
& + \hat{k} \left(a_x b_y - a_y b_x \right)
\end{aligned}$$

exactly as expected.