Physics 340 Final Exam: Mathematical Physics

Instructions

Answer all five questions. Show all work for full credit. Each question is worth 20 points.

Question 1: Algebra of Vectors (20 points)

Consider the following two vectors in $\mathbb{R}^3$:

$$\mathbf{A} = 3\hat{i} - 2\hat{j} + \hat{k}, \quad \mathbf{B} = \hat{i} + 4\hat{j} - 2\hat{k}.$$

a) Compute the dot product $\mathbf{A} \cdot \mathbf{B}$.

b) Compute the cross product $\mathbf{A} \times \mathbf{B}$.

c) Find the angle between the two vectors $\mathbf{A}$ and $\mathbf{B}$.

Continue your solution here

Question 2: Coupled Linear First-Order Differential Equations (20 points)

Consider the coupled system of differential equations:

$$\frac{dx}{dt} = 4x + y, \quad \frac{dy}{dt} = -2x + y.$$

a) Write the system in matrix form: $\frac{d\vec{X}}{dt} = A\vec{X}$.

b) Find the eigenvalues and eigenvectors of matrix A.

c) Solve the system for $x(t)$ and $y(t)$, assuming initial conditions $x(0) = 2$, $y(0) = 1$.

d) Describe, in words, the behavior of this system.

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Question 3: AC RL Circuit and Complex Impedance (15 points)

An alternating voltage source $V(t) = V_0 e^{i\omega t}$ is applied across a resistor R and an inductor L connected in series.

a) (5 points) Using the concept of complex impedance, derive the total impedance Z of the RL circuit in terms of R, L, and the angular frequency ω.

b) (5 points) Using Ohm's law in the complex form $V(t) = I(t)Z$, find the expression for the current $I(t)$ in the circuit. Express the current in terms of V_0, ω, R, and L.

c) (5 points) Find the phase shift between the voltage and the current in the circuit. What does this phase shift physically represent in terms of the behavior of the resistor and the inductor?

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Question 4: Fourier Analysis (20 points)

Let $f(x)$ be a periodic function with period T, defined as:

$$f(x) = \begin{cases} 1, & 0 \leq x < \frac{T}{2}, \\ -1, & \frac{T}{2} \leq x < T. \end{cases}$$

a) Compute the Fourier coefficients a_0, a_n, and b_n for this function.

b) Find the Fourier series representation of $f(x)$.

c) Discuss the convergence of the Fourier series for this function.

Continue your solution here

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Question 5: Vector Calculus (20 points)

Let $\mathbf{F} = x^2\hat{i} + 2xy\hat{j} + yz\hat{k}$ be a vector field.

a) Compute the divergence of $\mathbf{F}$, $\nabla \cdot \mathbf{F}$.

b) Compute the curl of $\mathbf{F}$, $\nabla \times \mathbf{F}$.

c) Evaluate the line integral of $\mathbf{F}$ along the curve C parameterized by $\mathbf{r}(t) = \langle t, t^2, t^3 \rangle$ for t from 0 to 1.

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