ENGR2910 - Circuit Analysis I

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Chapter 1

Circuit Variables

1.1 Electrical Engineering: An Overview

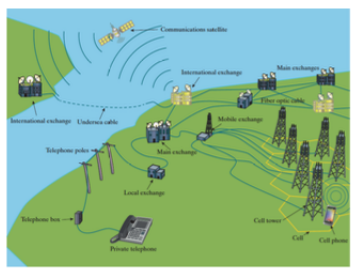


Figure 1.1: Telephone System

Five classifications of electrical systems:

1. Communication Systems
2. Computer Systems
3. Control Systems
4. Power Systems
5. Signal-Processing Systems

Circuit Theory

Three assumptions:

- Electrical effects happen instantaneously throughout a system; tis assumption is called the lumped-parameter system)

- The net charge on every component in the system is always zero.
- There is not magnetic coupling between the components of a system. However, we will allow magnetic coupling within a component.

1.2 International System of Units

Quantity	Basic unit	Symbol
Length	meter	m
Mass	kilogram	Kg
Time	second	s
Electric current	ampere	A
Thermodynamic temperature	kelvin	K
Amount of substance	mole	mol
Luminous intensity	candela	cd

Figure 1.2: Scientific Units

Quantity	Unit Name (Symbol)	Formula
Frequency	hertz (Hz)	s^{-1}
Force	newton (N)	$kg \cdot m/s^2$
Energy or work	joule (J)	$N \cdot m$
Power	watt (W)	J/s
Electric charge	coulomb (C)	$A \cdot s$
Electric potential	volt (V)	J/C
Electric resistance	ohm (Ω)	V/A
Electric conductance	siemens (S)	A/V
Electric capacitance	farad (F)	C/V
Magnetic flux	weber (Wb)	$V \cdot s$
Inductance	henry (H)	Wb/A

Figure 1.3: Derived Units

Circuit Analysis: An Overview

All engineering designs begin with a need that may include a Circuit Model before a physical prototype:

Voltage and Current

Definition of Voltage (v)

$$v = \frac{dw}{dq} \tag{1.1}$$

Prefix	Symbol	Power
atto	a	10^{-18}
femto	f	10^{-15}
pico	p	10^{-12}
nano	n	10^{-9}
micro	μ	10^{-6}
milli	m	10^{-3}
centi	c	10^{-2}
deci	d	10^{-1}
deka	da	10
hecto	h	10^{2}
kilo	k	10^{3}
mega	M	10^{6}
giga	G	10^{9}
tera	T	10^{12}

Figure 1.4: Powers of 10

where w is energy in joules and q is the charge in coulombs. Note that the charge of one electron (e is

$$e = 1.60022 \times 10^{-19} C \tag{1.2}$$

Definition of Current (i)

$$i = \frac{dq}{dt}, \tag{1.3}$$

where q is charge in coulombs and t is time in seconds.

Note, the direction of current is defined by the direction of flow of positive charge.

DC vs AC

Direct current is constant with time. Alternating current varies (sinusoidally) with time and reverses direction

Ideal Basic Circuit Element

The ideal circuit element has three attributes:

1. It has only two terminals
2. It is described mathematically in terms of current and/or voltage
3. It can not be subdivided to make other elements

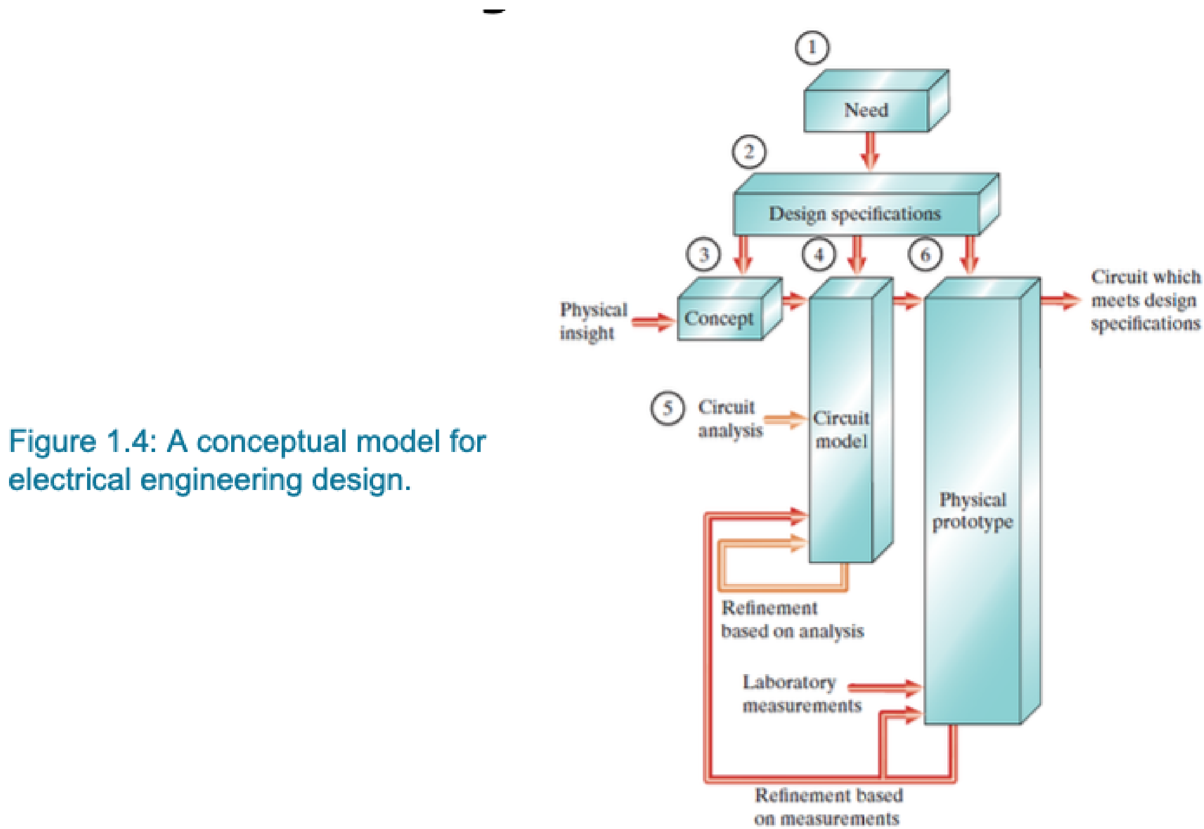


Figure 1.5: Conceptual Model for Electrical Engineering Design

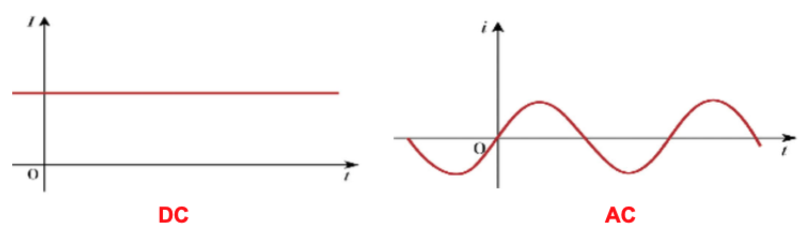


Figure 1.6: DC vs AC

WARNING: Positive Sign Convention

Whenever the reference direction for the current in an element is in the direction of the reference voltage drop across the element, use a positive sign in any expression that relates he voltage to current.

1.3 Power and Energy

Power is the energy per unit time

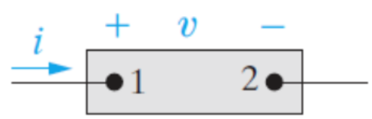


Figure 1.7: Ideal basic circuit element

$$p = \frac{dw}{dt}, \tag{1.4}$$

where p s power in watts, w is energy in joules, and t is time in seconds. And, where $1W = 1\frac{J}{s}$.

$$p = \frac{dw}{dt} = (\frac{dw}{dq})(\frac{dq}{dt}) \tag{1.5}$$

therefore,

$$p = vi \tag{1.6}$$

Note, by convention, power is positive ($p > 0$) if power is being delivered, and power is negative if power is being extracted from the circuit.

Law of Conservation of Energy

$$\sum p = 0 \tag{1.7}$$

Energy is the capacity to do work (measured in J)

$$w = \int_{t_0}^{t} pdt = \int_{t_0}^{t} vidt \tag{1.8}$$

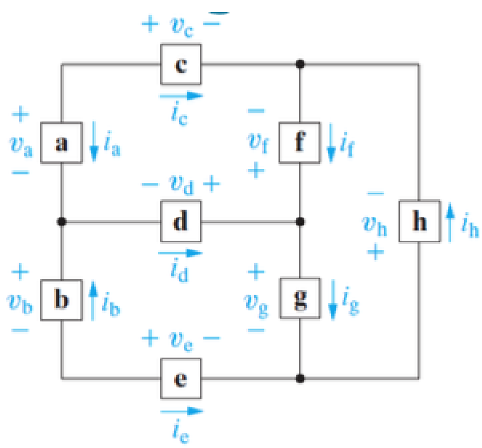


Figure 1.7: Circuit model for power distribution in a home, with voltages and currents defined.

Component	v(V)	i(A)
a	120	−10
b	120	9
c	10	10
d	10	1
e	−10	−9
f	−100	5
g	120	4
h	−220	−5

$$p_a = v_a i_a = (120)(-10) = -1200 \text{ W} \qquad p_b = -v_b i_b = -(120)(9) = -1080 \text{ W}$$

$$p_c = v_c i_c = (10)(10) = 100 \text{ W} \qquad p_d = -v_d i_d = -(10)(1) = -10 \text{ W}$$

$$p_e = v_e i_e = (-10)(-9) = 90 \text{ W} \qquad p_f = -v_f i_f = -(-100)(5) = 500 \text{ W}$$

$$p_g = v_g i_g = (120)(4) = 480 \text{ W} \qquad p_h = v_h i_h = (-220)(-5) = 1100 \text{ W}$$

Figure 1.8: Balancing Power Example

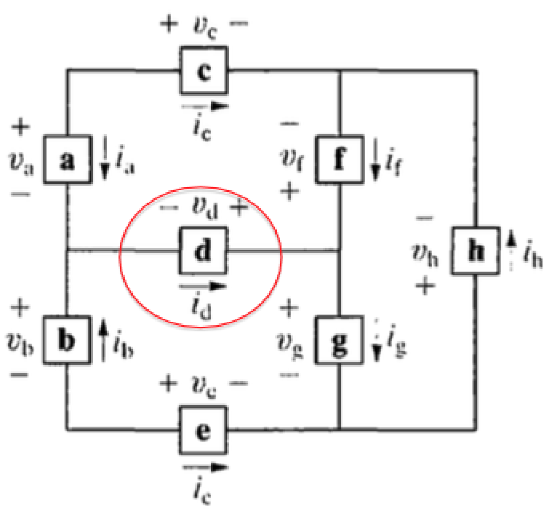
$$p_{\text{supplied}} = p_a + p_b + p_d = -1200 - 1080 - 10 = -2290\ \text{W}$$

$$p_{\text{absorbed}} = p_c + p_e + p_f + p_g + p_h$$

$$= 100 + 90 + 500 + 480 + 1100 = 2270\ \text{W}$$

$$p_{\text{supplied}} + p_{\text{absorbed}} = -2290 + 2270 = -20\ \text{W}$$

Something is wrong—if the values for voltage and current in this circuit are correct, the total power should be zero!



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e	−10	−9
f	−100	5
g	120	4
h	−220	−5

Figure 1.9: Balancing Power Correction

Chapter 2

Circuit Elements

2.1 Voltage and Current Sources

When we speak of circuit elements, it is important to differentiate between the physical device itself and the mathematical model which we will use to analyze its behavior in a circuit. We will use the expression circuit element to refer to the mathematical model. All the simple circuit elements that we will consider can be classified according to the relationship between current through the element to the voltage across the element.

Ideal Sources

- Ideal voltage source is a circuit element that maintains a prescribed voltage across its terminals regardless of the current flowing in those terminals.
- Ideal current source is a circuit element that maintains a prescribed current through its terminals regardless of the voltage across those terminals.

Independent and Dependent Sources

An independent source establishes a voltage or current in a circuit without relying on voltage or currents elsewhere in the circuit.

A dependent source establishes a voltage or current whose value depends on the value of a voltage or current elsewhere in the circuit. You cannot specify the value of a dependent source unless you know the value of the voltage or current on which it depends.

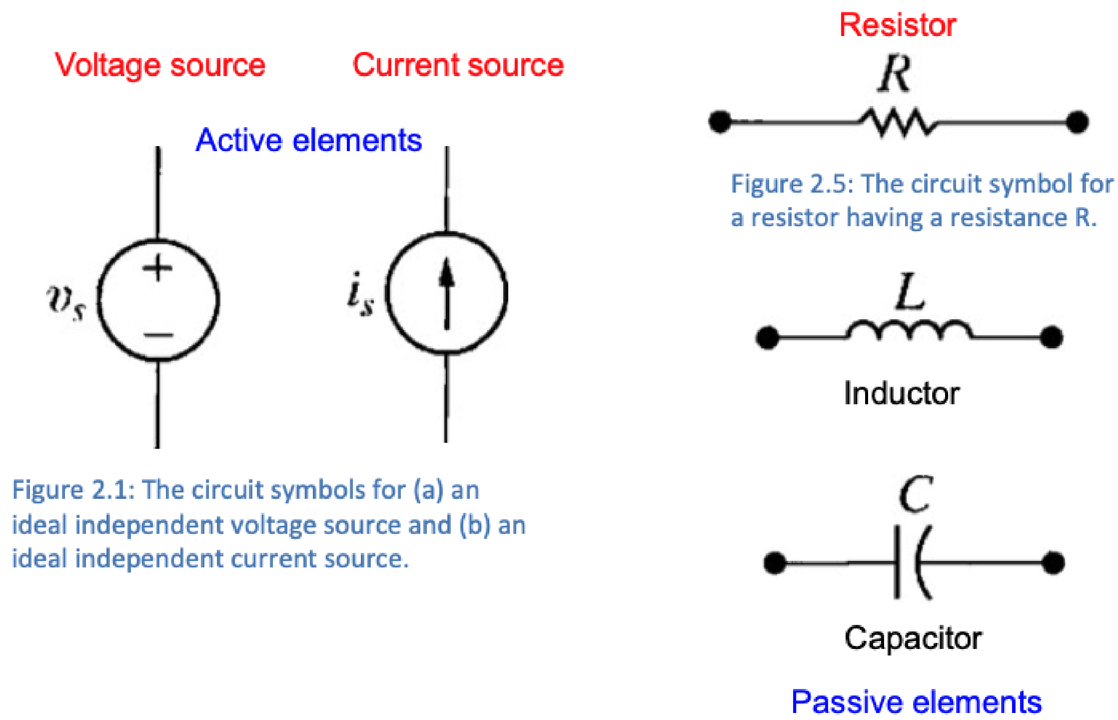


Figure 2.1: Five Basic Circuit Elements

There are four kinds of controlled sources:

- current-controlled current source (CCCS)
- voltage-controlled current source (VCCS)
- voltage-controlled voltage source, (VCVS)
- current-controlled voltage source, (CCVS)

2.2 Electrical Resistance (Ohm's Law)

Resistance is the capacity of materials to impede the flow of current or, more specifically, the flow of electric charge. The circuit element used to model this behavior is the resistor. The linear resistor is the simplest passive element.

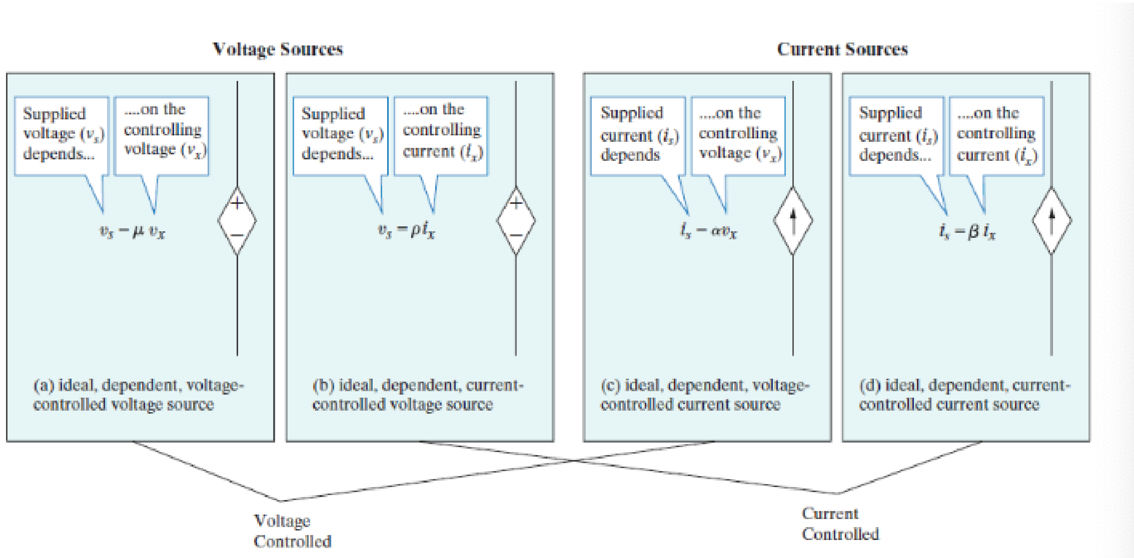


Figure 2.2: Four Controlled Sources

Ohm's Law

The relationship between Voltage and Current was empirically determined by Goerg Ohm[1]

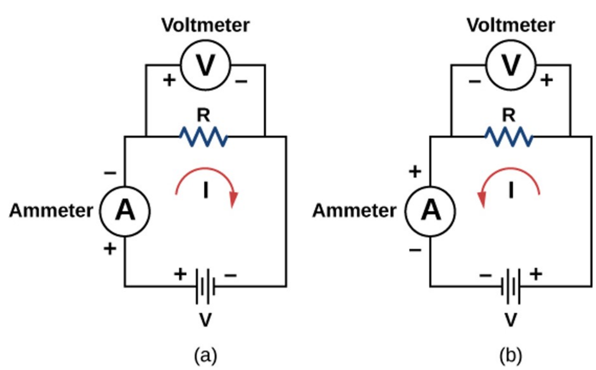


Figure 2.3: Goerg Ohm's Setup

[1]A presented in a paper published in 1827

I(A)	V(V)
−2.74	−10.00
−2.25	−9.00
−2.00	−8.00
−1.75	−7.00
−1.50	−6.00
−1.49	−5.00
−1.00	−4.00
−0.51	−3.00
−0.74	−2.00
−0.49	−1.00
+0.00	+0.00
+0.49	+1.00
+0.50	+2.00
+0.99	+3.00
+0.76	+4.00
+1.01	+5.00
+1.74	+6.00
+1.75	+7.00
+2.00	+8.00
+2.49	+9.00
+2.50	+10.00

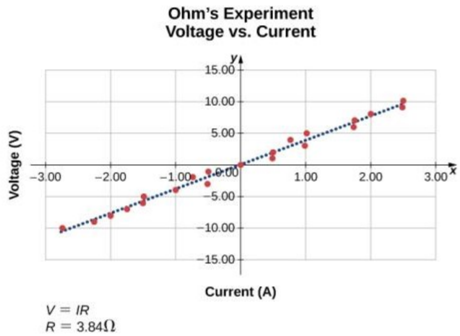


Figure 2.4: Goerg Ohm's Data

Ohm's Law

$$v = i \cdot R \tag{2.1}$$

Power

$$p = vi \tag{2.2}$$

Therefore:

$$p = (iR)i = i^2 R \tag{2.3}$$

or

$$p = v\frac{v}{R} = \frac{v^2}{R} \tag{2.4}$$

Why is this important?

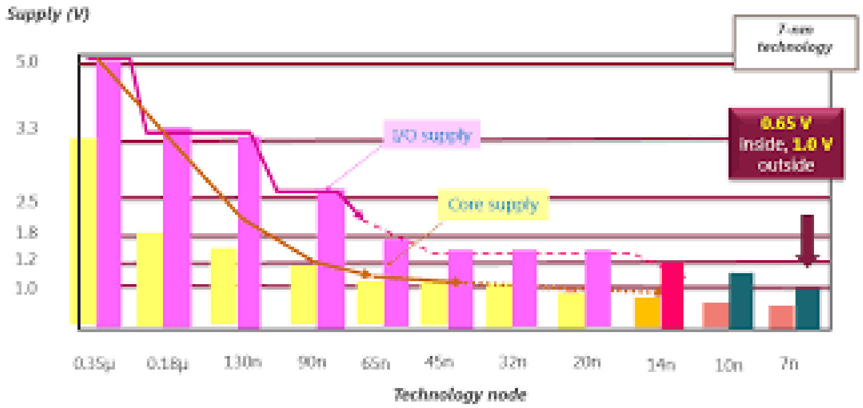


Figure 2.5: Why Care About Voltage

2.3 Constructing a circuit model

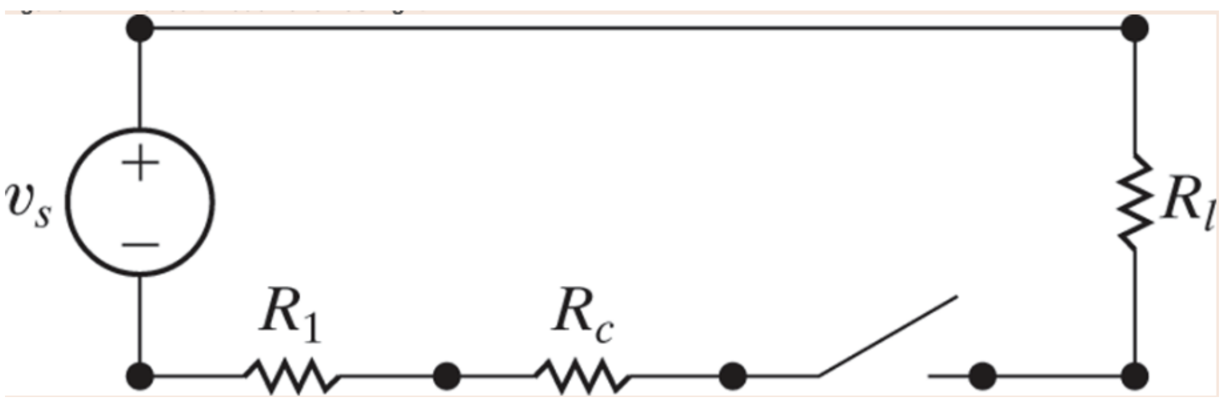


Figure 2.6: Flashlight

- Dry Cell Batteries
- Lamp (R_l)
- Switch
- Case (R_c)
- Connector Spring (R_s)

2.4 Kirchhoff's Laws

Kirchhoff's First Law (the Node Law or the Junction Rule)

The sum of all currents entering a junction must equal the sum of all currents leaving the junction.

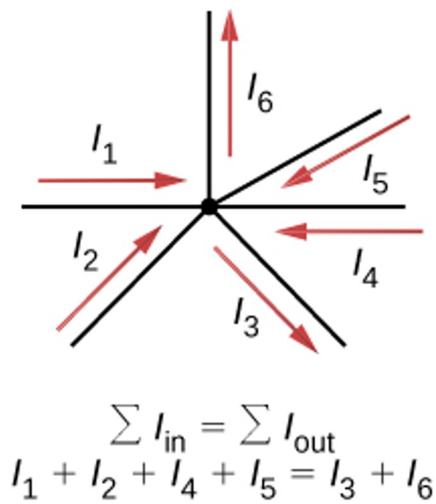


Figure 2.7: Kirchholff's Node Law

$$\sum I_{in} = \sum I_{out} \tag{2.5}$$

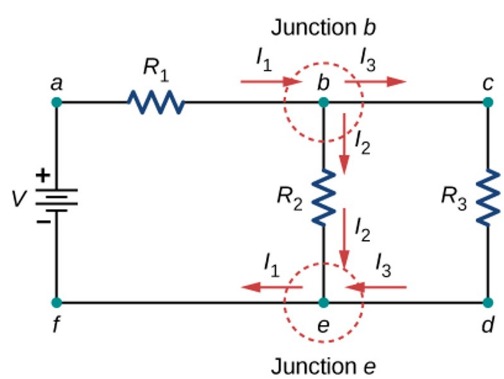


Figure 2.8: Kirchholff's Node Law Example

Kirchhoff's Second Law (the Loop Law or the Loop Rule)

The sum off all potential differences, including those supplied by voltage sources and resistive elements, around a closed loop equals zero.

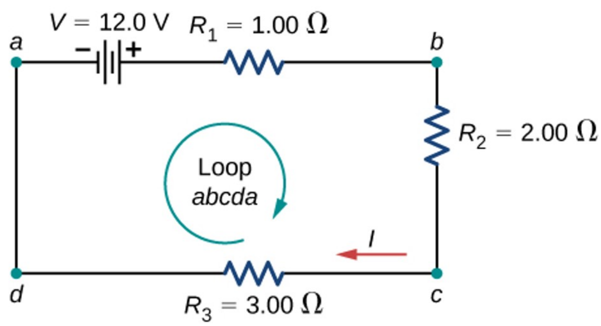


Figure 2.9: Kirchholff's Loop Law

$$\sum_{closedloop} V = 0 \tag{2.6}$$

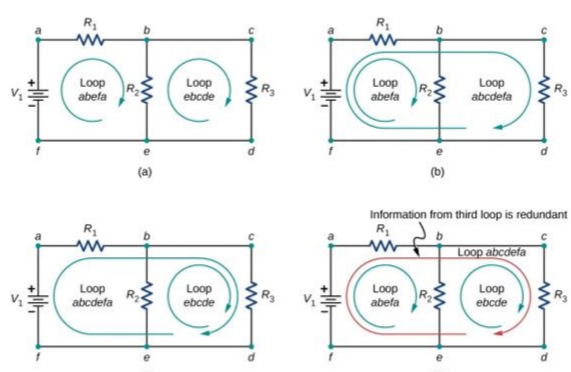


Figure 2.10: Kirchholff's Loop Law Example

2.5 Kirchhoff Examples

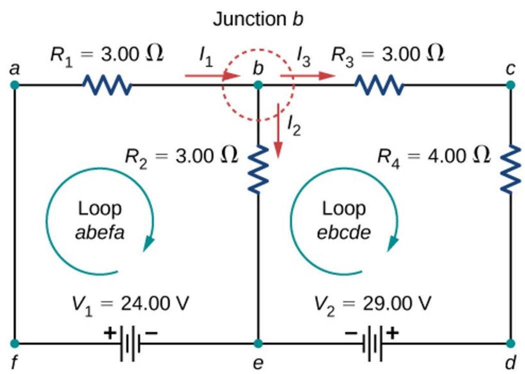


Figure 2.11: Kirchholff Example

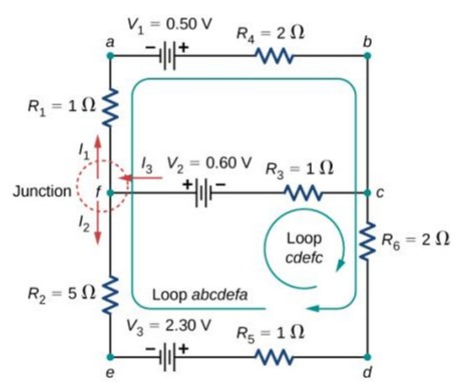


Figure 2.12: Kirchholff Examples

2.6 Circuits Containing Dependent Sources

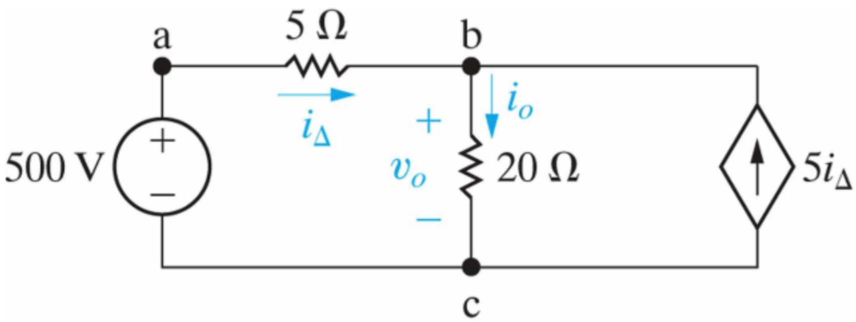


Figure 2.22: A circuit with a dependent source.

KCL $\quad i_o = i_\Delta + 5i_\Delta = 6i_\Delta$

KVL $\quad 500 = 5i_\Delta + 20i_o$

$$i_\Delta = 4 \text{ A},$$

$$i_o = 24 \text{ A}.$$

$$v_o = 20i_o = 480 \text{ V}$$

Figure 2.13: Controlled Sources

Chapter 3

Simple Resistive Circuits

3.1 Resistors in Series

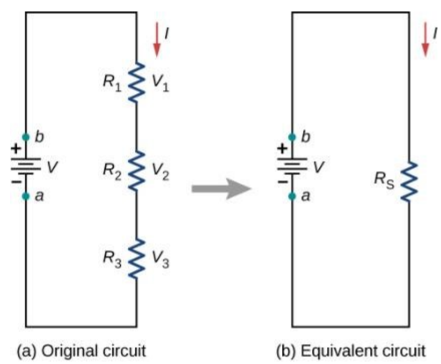


Figure 3.1: Resistors in Series

For resistors in Series

$$V = V_1 + V_2 + V_3 \tag{3.1}$$

$$V = 1R_1 + 1R_2 + 1R_3 \tag{3.2}$$

$$I = \frac{V}{R_1 + R_2 + R_3} \tag{3.3}$$

So

$$R_{eq} = \sum_{i=1}^{N} R_i \tag{3.4}$$

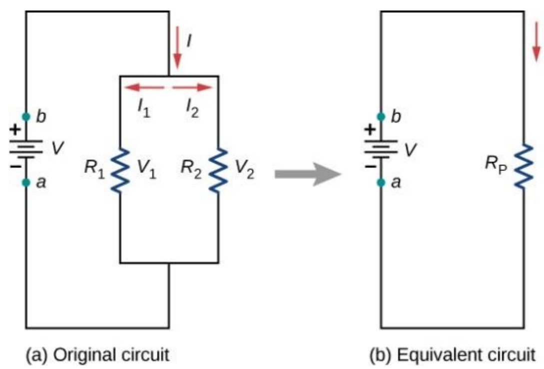


Figure 3.2: Resistors Parallel

3.2 Resistors in Parallel

$$V = V_1 = V_2 \tag{3.5}$$

$$I = I_1 + I_2 \tag{3.6}$$

$$\frac{V}{R_{eq}} = \frac{V_1}{R_1} + \frac{V_2}{R_2} \tag{3.7}$$

Because the Voltage is equal across the resistors

$$\frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2} \tag{3.8}$$

Or, more generically

$$R_{eq} = (\sum_{i=1} N \frac{1}{R_i})^{-1} \tag{3.9}$$

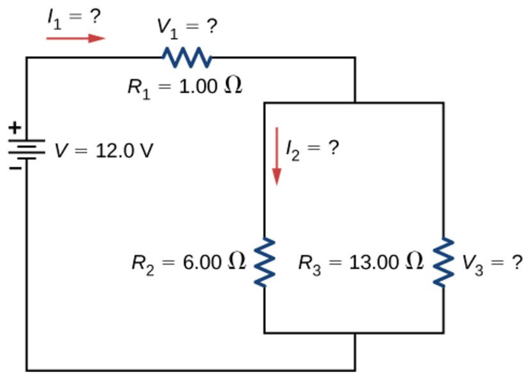
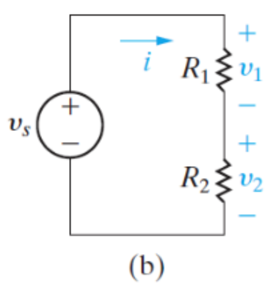


Figure 3.3: Resistors in Series and Parallel

Do Example 3.1 and Example 3.2

3.3 Divider Circuits



$$i = \frac{v_s}{R_1 + R_2}$$

$$v_1 = iR_1 = v_s \frac{R_1}{R_1 + R_2}$$

$$v_2 = iR_2 = v_s \frac{R_2}{R_1 + R_2}$$

Figure 3.4: Voltage Divider

Do Example 3.3

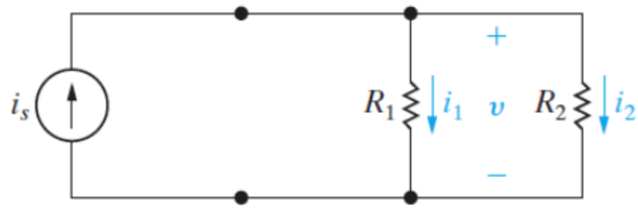


Figure 3.19: The current-divider circuit.

$$v = i_1 R_1 = i_2 R_2 = \frac{R_1 R_2}{R_1 + R_2} i_s$$

$$i_1 = \frac{R_2}{R_1 + R_2} i_s$$

$$i_2 = \frac{R_1}{R_1 + R_2} i_s$$

Figure 3.5: Current Divider

Do Example 3.6

With a Load

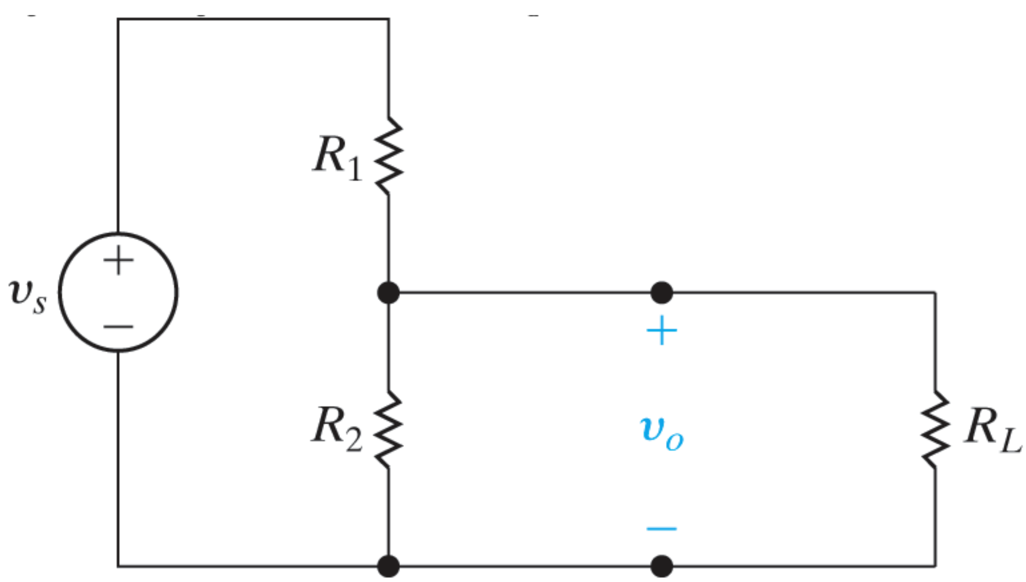


Figure 3.6: Voltage Divider with Load

$$v_0 = \frac{R_{eq}}{R_1 + R_{eq}} v_s \tag{3.10}$$

where

$$R_{eq} = \frac{R_2 R_L}{R_2 + R_L} \tag{3.11}$$

substituting

$$v_0 = \frac{R_2}{R_1[1 + \frac{R_2}{R_L}] + R_2} v_s \tag{3.12}$$

3.4 Measuring Voltage and Current

- An ammeter is an instrument designed to measure current; it is placed in series with the circuit element whose current is being measured.

- A voltmeter is an instrument designed to measure voltage; it is placed in parallel with the circuit element whose current is being measured.

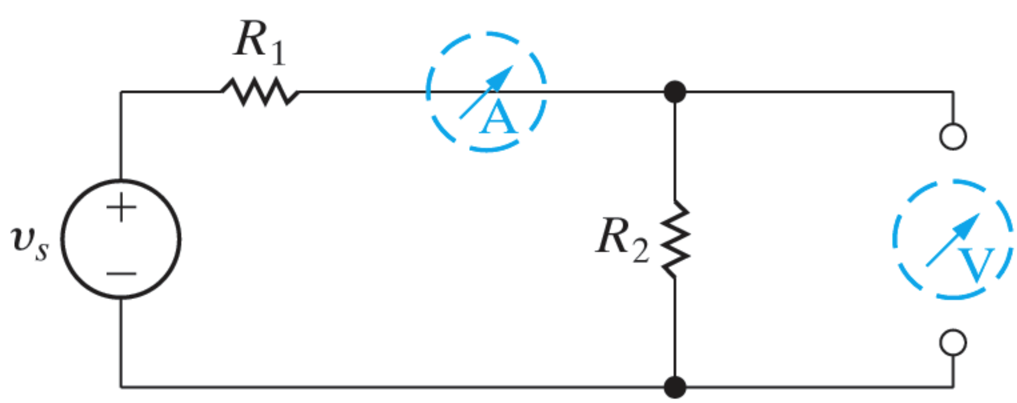


Figure 3.7: Short-circuit model for ideal ammeter, and open-circuit model for ideal volt meter

d'Arsonval meter

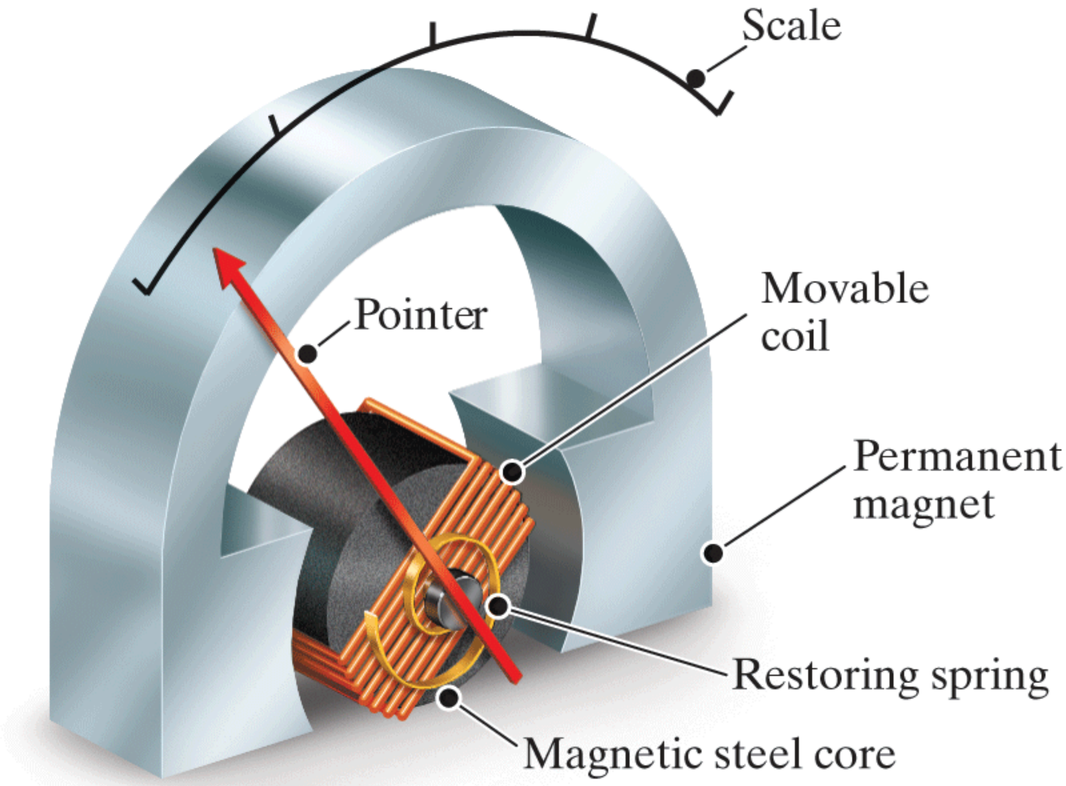


Figure 3.8: d'Arsonval meter movement

Non-ideal meters

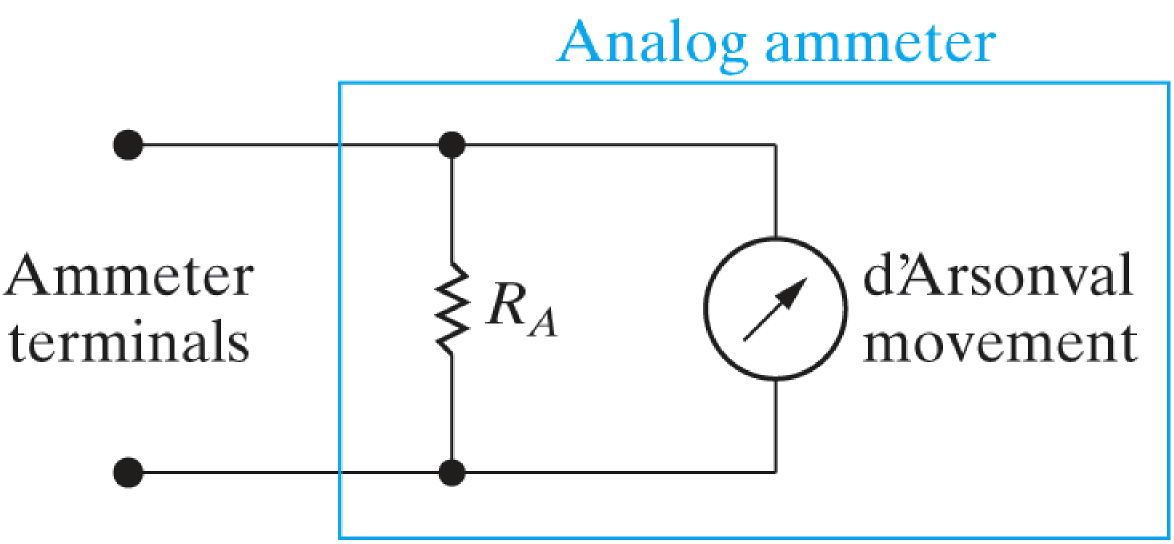


Figure 3.9: Non-Ideal Ammeter

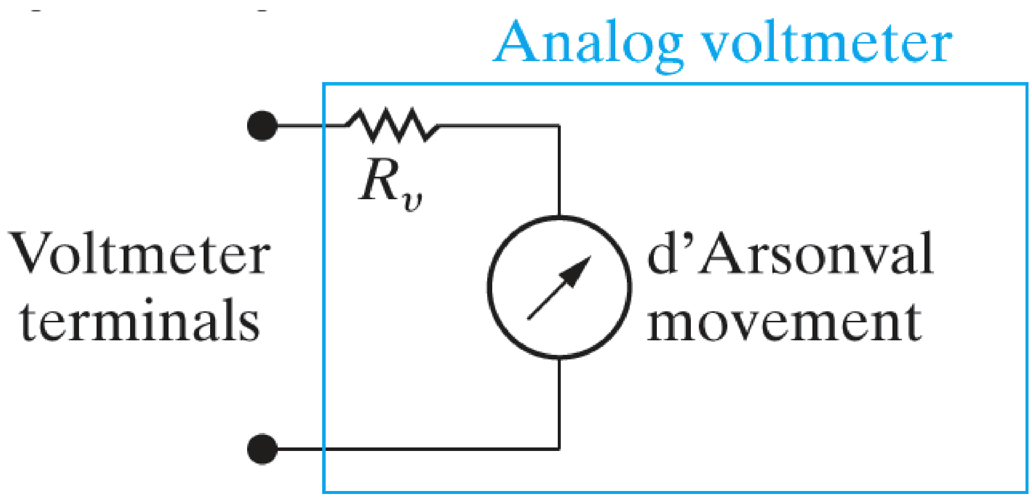


Figure 3.10: Non-ideal Voltmeter

The Wheatstone Bridge

The Wheatstone Bridge[1] is one, of many, configurations that can be used to measure resistance.

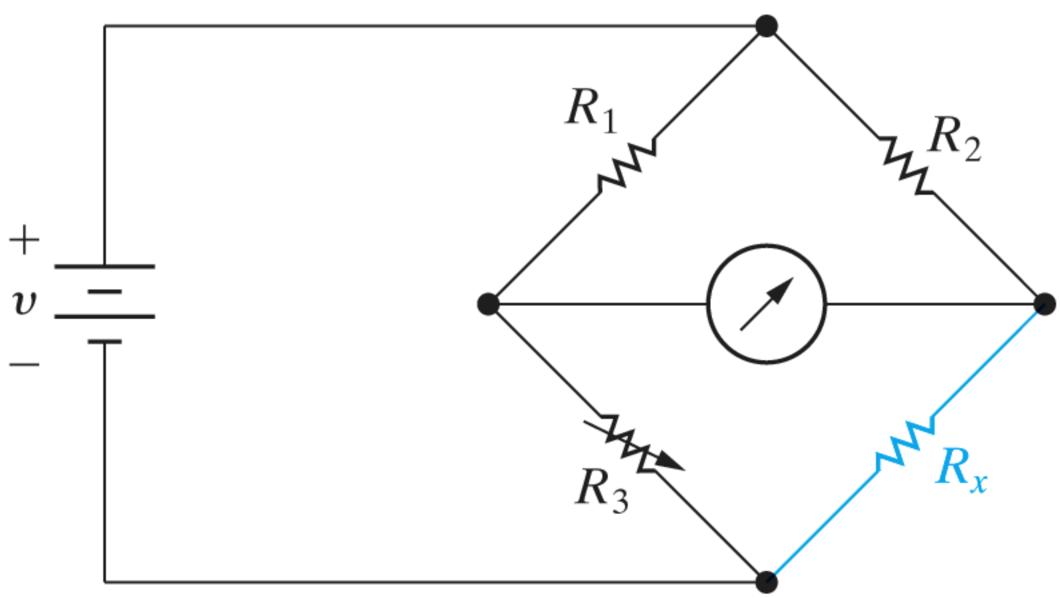


Figure 3.11: Wheatstone Bridge

[1]Sir Charles Wheatstone (6 February 1802 – 19 October 1875), was an English scientist and inventor of many scientific breakthroughs of the Victorian era. Wheatstone is best known for his contributions in the development of the Wheatstone bridge, originally invented by Samuel Hunter Christie, which is used to measure an unknown electrical resistance

To find R_x, the variable resistor R_3 is adjusted until t here is no current in the galvanometer. Then the value of unknown resistor can be found by

$$R_x = \frac{R_2}{R_1} R_3 \tag{3.13}$$

Derivation using Kirchhoff's Laws

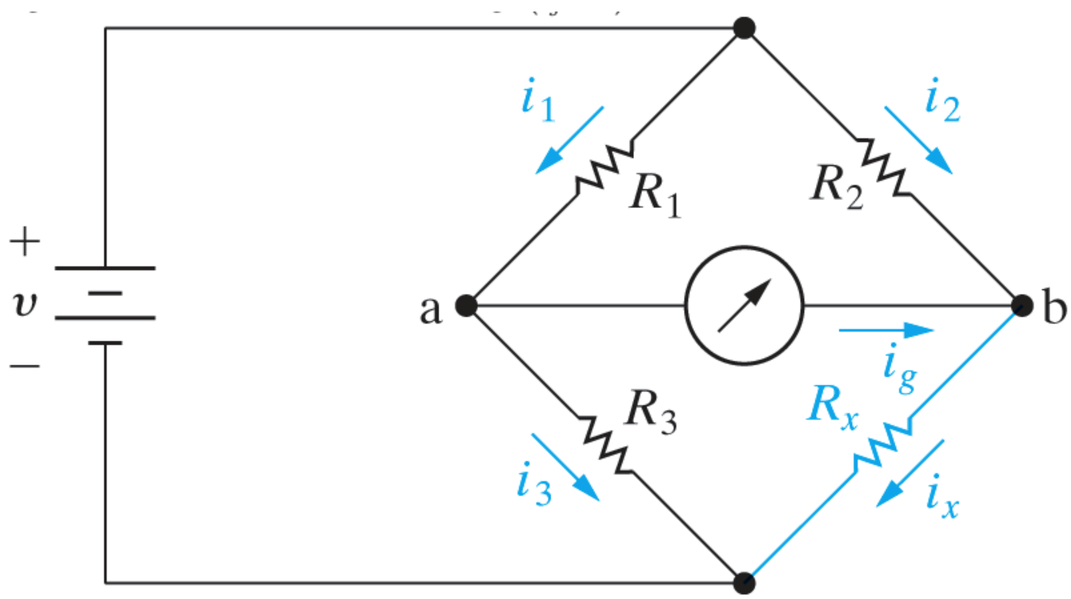


Figure 3.12: Balanced Wheatstone Bridge

Using KCL, since $i_g = 0$, at Node A:

$$i_1 = i_3 \tag{3.14}$$

And, at Node B:

$$i_2 = i_x \tag{3.15}$$

Because $i_g = 0$, this also implies the voltage drop across the detector is also zero, and thus Nodes A and B are at the same potential.

From KVL:

$$i_3R_3 - i_xR_x = 0 \tag{3.16}$$

or

$$i_3R_3 = i_xR_x \tag{3.17}$$

Likewise,

$$i_1R_1 = i_2R_2 \tag{3.18}$$

Divide the first KVL equation by the second

$$\frac{i_3R_3}{i_1R_1} = \frac{i_xR_x}{i_2R_2} \tag{3.19}$$

Since $i_1 = i_3$ and $i_2 = i_x$, solving for R_x:

$$R_x = \frac{R_2}{R_1}R_3 \tag{3.20}$$

Do Example 3.10

3.5 Delta-to-Wye (Pi-to-Tee) Equivalent Circuits

The Δ and Y structures are present in a variety of useful circuits. Hence, the $\Delta - Y$ transformation is helpful in circuit analysis.

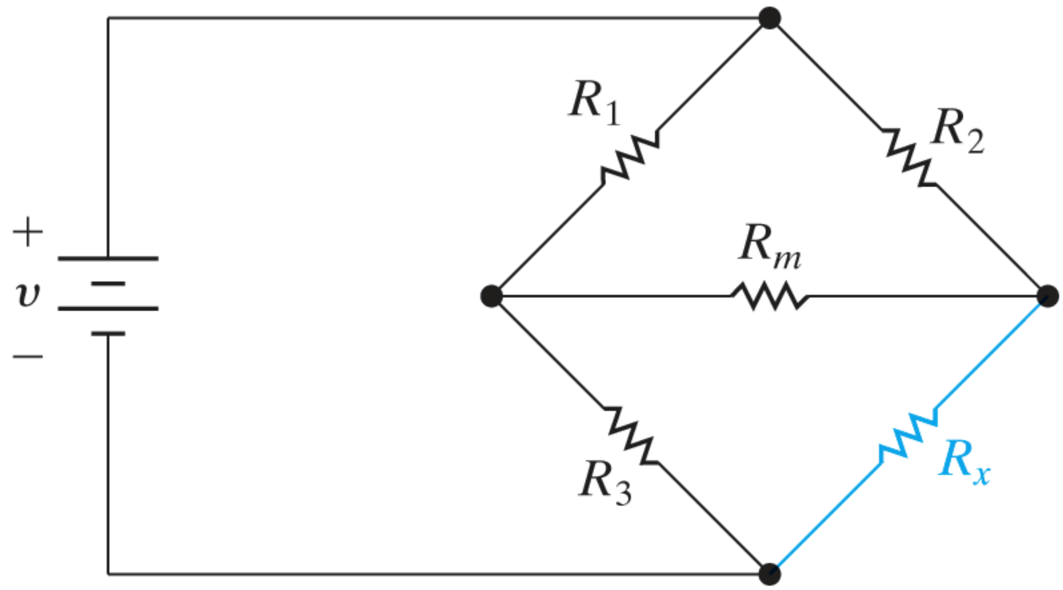


Figure 3.13: Resistive Network generated by a Wheatstone Bridge circuit

Resistors R_1, R_2, and R_m (or R_3, R_x, and R_m) form a delta (Δ) interconnect.

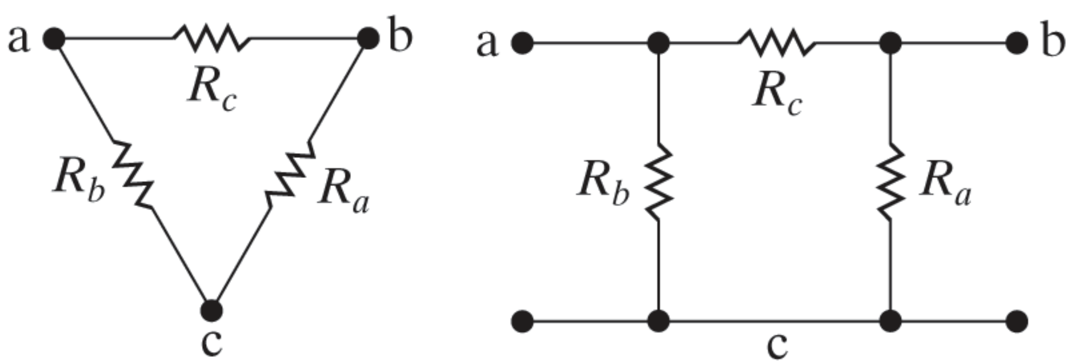


Figure 3.14: Δ configuration viewed as a π configuration

Another configuration is the wye (Y) or the electrically equivalent tee (T) interconnection

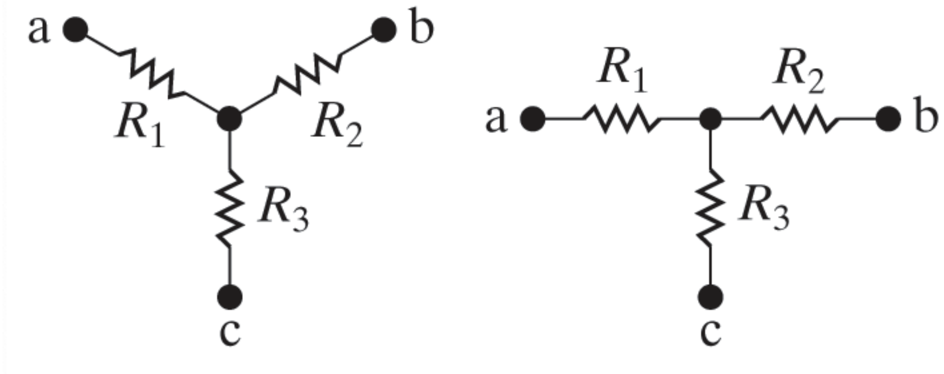


Figure 3.15: Y configuration viewed as a T configuration

The Delta to Y transformation

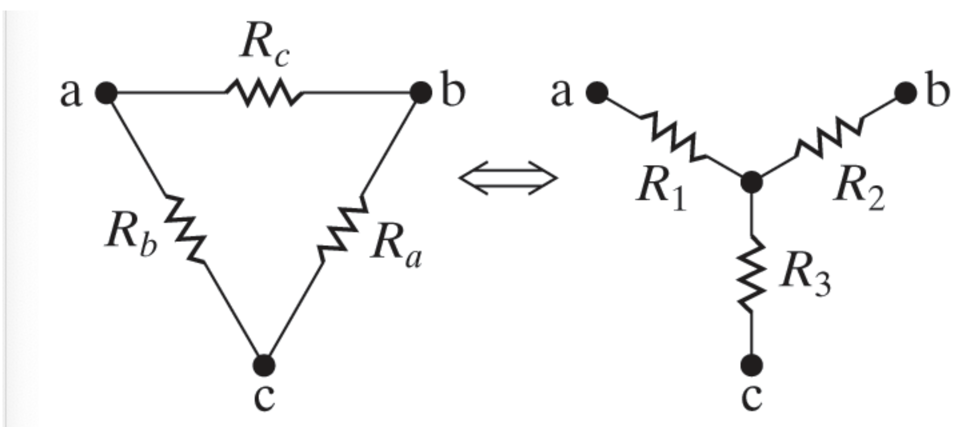


Figure 3.16: Δ to Y Transformation

The resistance between terminals needs to be the same whether they are in series or parallel. The three equivalent resistance equations are:

$$R_{ab} = \frac{R_c(R_a + R_b)}{R_a + R_b + R_c} = R_1 + R_2 \tag{3.21}$$

$$R_{bc} = \frac{R_a(R_b + R_c)}{R_a + R_b + R_c} = R_2 + R_3 \tag{3.22}$$

$$R_{ca} = \frac{R_b(R_c + R_a)}{R_a + R_b + R_c} = R_3 + R_1 \tag{3.23}$$

Through algebraic manipulation

$$R_1 = \frac{R_b R_c}{R_a + R_b + R_c} \tag{3.24}$$

$$R_2 = \frac{R_c R_a}{R_a + R_b + R_c} \tag{3.25}$$

$$R_3 = \frac{R_a R_b}{R_a + R_b + R_c} \tag{3.26}$$

Or this can be reversed

$$R_a = \frac{R_1 R_2 + R_2 R_3 + R_3 R_1}{R_1} \tag{3.27}$$

$$R_b = \frac{R_1 R_2 + R_2 R_3 + R_3 R_1}{R_2} \tag{3.28}$$

$$R_c = \frac{R_1 R_2 + R_2 R_3 + R_3 R_1}{R_3} \tag{3.29}$$

Do Example 3.11

Chapter 4

Techniques of Circuit Analysis

4.1 Terminology

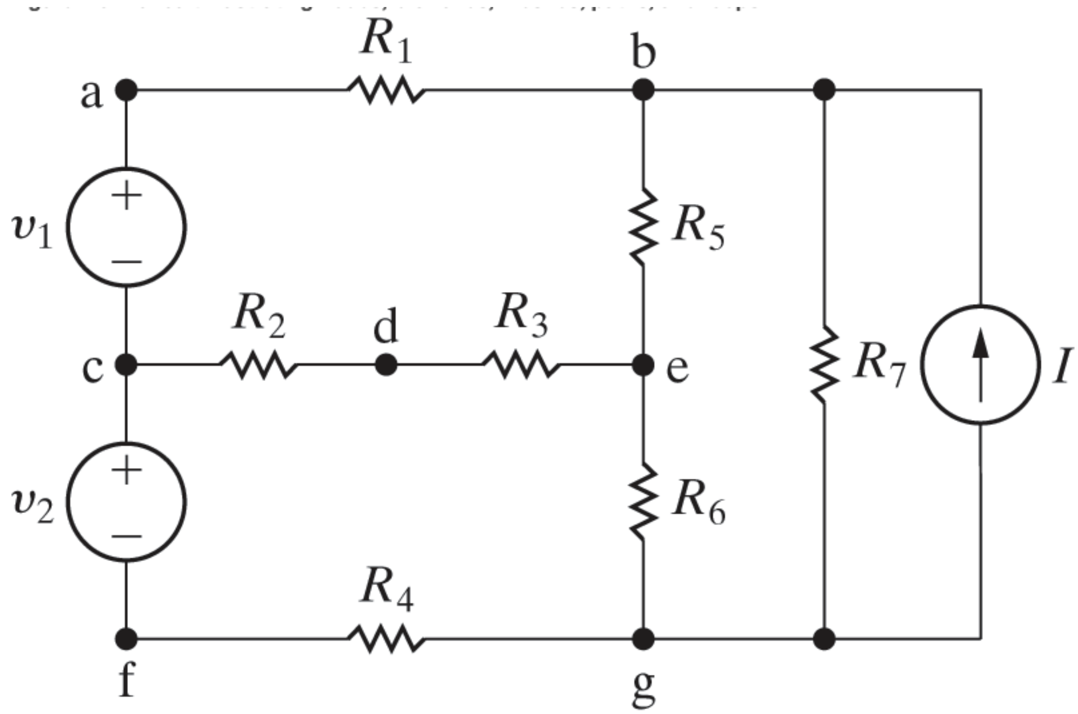


Figure 4.1: Terminology Diagram

Name	Definition	Example
node	A point where two or more elements join	a
essential node	A node where three or more elements join	b
path	A trace of adjoining basic circuit elements with no elements included more than once	$v_1 - R_1 - R_2 - R_3$
branch	A path that connects two nodes	R_1
essential branch	A path that connects two essential nodes without passing through an essential node	$v_1 - R_1$
loop	A path whose last node is the same as the starting node	$v_1 - R_1 - R_5 - R_6 - R_4 - v_2$
mesh	A loop that does not enclose any other loops	$v_1 - R_1 - R_3 - R_2$
planar circuit	A circuit that can be drawn on a plane with no crossing branches	see figures

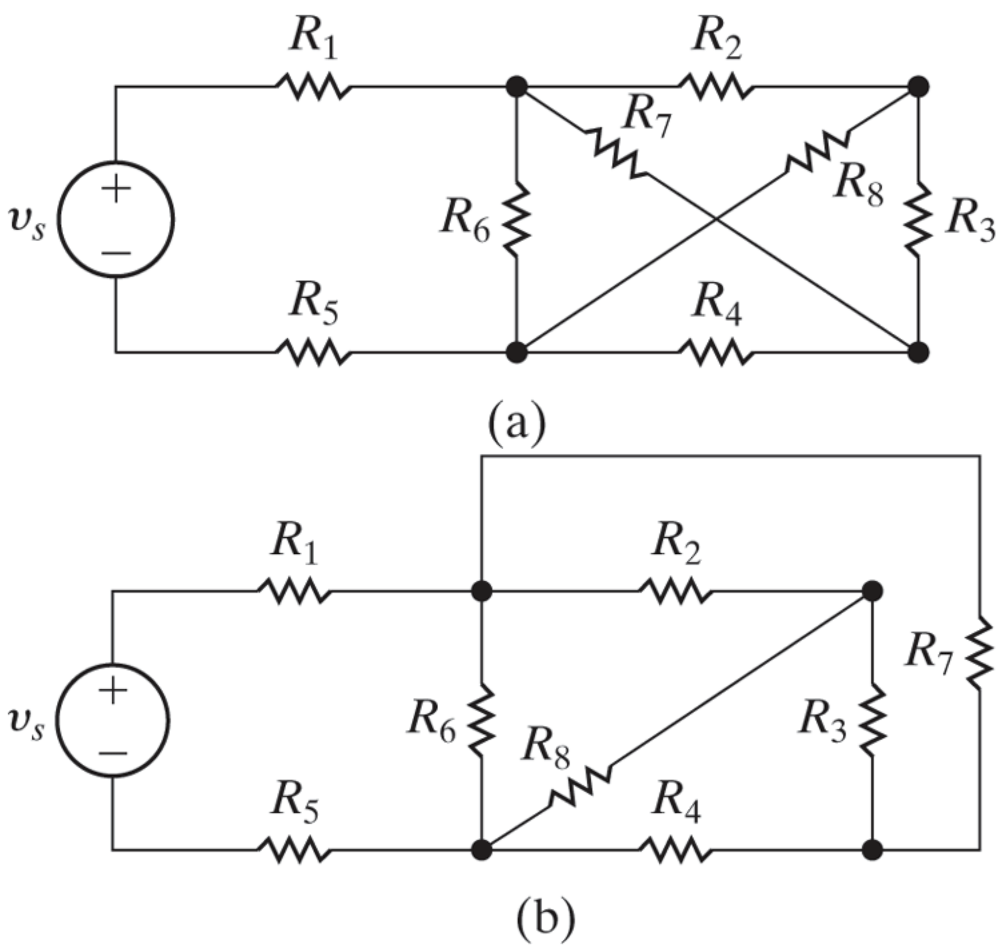


Figure 4.2: Planar Circuit

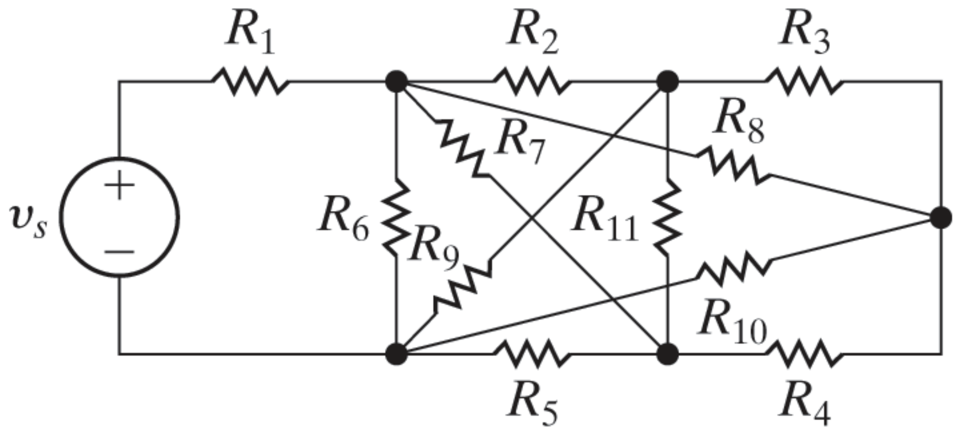


Figure 4.3: Non Planar Circuit

Do Example 4.1

Simultaneous Equations

Recall that you need x independent equations to solve a circuit with x unknown currents.

Method:

- Count the number of essential nodes, n_e
- Count the number of essential branches, b_e, where the current is unknown.
- Write $n_e - 1$ equations by applying KCL to any set of $n_e - 1$ nodes.
- Write $b_e - (n_e - 1)$ equations by applying KVL around a set of $b_e - (n_e - 1)$ loops or meshes.

Note - the voltage for each element in every loop or mesh must be known or must be described in terms of current using Ohm's law.

Do Example 4.2

4.2 Node-Voltage Method

1. Identify the essential nodes

2. Pick and label a reference node, then label the node voltages at the remaining essential nodes.

3. Write a KCL equation for every non-reference essential node

4. Solve the equations to find the node-voltage values

5. Sove the circuit using the node voltages from Step 4 to find the component currents, voltages, and power values.

Do Example 4.3

Do Assessment Problem 4.1

Left Node:

$$15 = i_{60} + i_{15} + i_1 \tag{4.1}$$

$$15 = \frac{v_1}{60} + \frac{v_1}{15} + \frac{v_1 - v_2}{5} \tag{4.2}$$

$$\frac{17}{60}v_1 - \frac{12}{60}v_2 = 15 \tag{4.3}$$

Right Node:

$$i_1 = i_2 + 5 \tag{4.4}$$

$$\frac{v_1 - v_2}{5} = \frac{v_2}{2} + 5 \tag{4.5}$$

$$\frac{1}{5}v_1 - \frac{7}{10}v_2 = 5 \tag{4.6}$$

Combining:

$$A = \begin{bmatrix} 0.2833 & -0.2000 & | & 15 \\ 0.2000 & -0.7000 & | & 5 \end{bmatrix} \tag{4.7}$$

Using in Octave rref(A):

$$rref(A) = \begin{bmatrix} 1 & 0 & | & 60 \\ 0 & 1 & | & 10 \end{bmatrix} \tag{4.8}$$

$$v_1 = 60V, v_2 = 10V, i_i = 5A \tag{4.9}$$

4.3 Node-Voltage Method and Dependent Sources

If the circuit contains dependent sources, the KCL equations must be supplemented with the constraint equations imposed by the dependent sources. The Step 3 of the Node-Voltage method is modified as seen in the example below.

Do Example 4.4

Do Assessment Problem 4.3

4.4 Node-Voltage Method - Special Cases

Known Voltages

Do Example with Figure 4.12

Supernodes

When a voltage source is between two essentail nodes, we can combine those nodes and the source to form a supernode.

Do Example with Figure 4.15

Do Example 4.5 - Node-Voltage Analysis of an Amplifier Circuit

4.5 Mesh-Current Method

Caution: this only works for Planar Circuits

The Mesh-Current Method

1. Identify the meshes with curved directed arrows that follow the perimeter of each mesh
2. Label th emesh currents for each mesh
3. Write a KVL equation for each mesh
4. Solve the equations to find the mesh current values
5. Sove the circuit using the mesh currents from Step 4 to find the component currents, voltages, and power values.

Do Example 4.6

Do Assessment Problem 4.7

4.6 The Mesh-Current Method and Dependent Sources

Do Example 4.7

4.7 The Mesh-Current Method: Special Cases

Known Current

Do Example 4.8

Supermesh

When a current source is shared between two meshes, we can combine the meshes to form a supermesh.

Do Example with Figure 4.28

4.8 Source Transformations

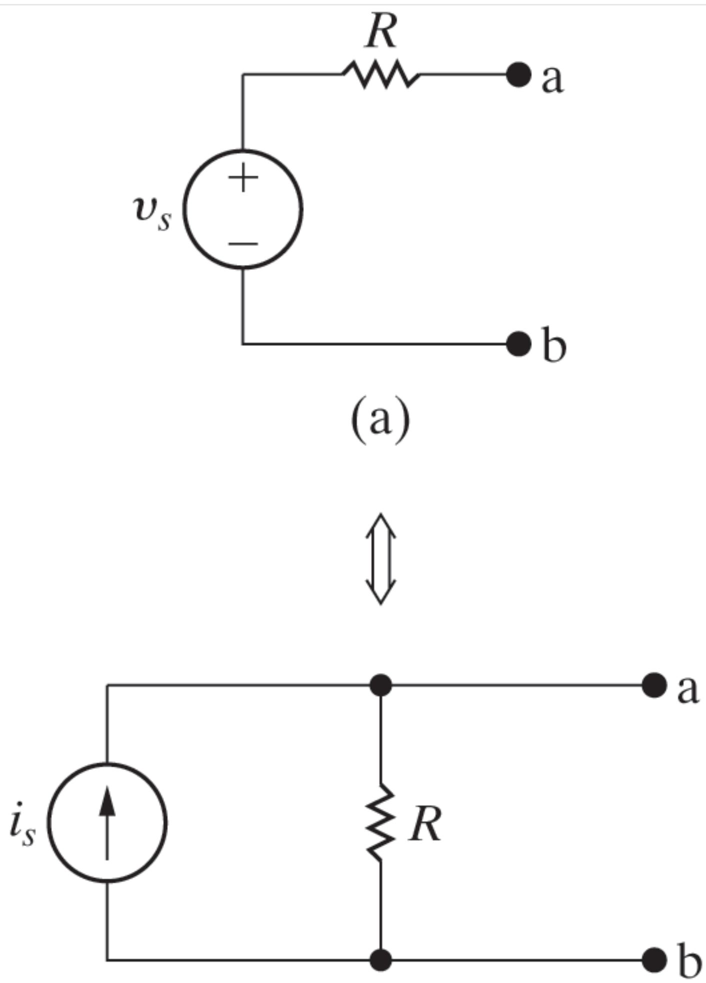


Figure 4.4: Source Transformations

Supposed we connect Resistor R_L between Nodes A and B:

From the voltage source figure,

$$i_L = \frac{v_s}{R + R_L} \tag{4.10}$$

And, from the current source figure,

$$i_L = \frac{R}{R + R_L} i_s \tag{4.11}$$

Which leads to

$$i_s = \frac{v_s}{R} \tag{4.12}$$

Do Example 4.12

4.9 Thevenin and Norton Equivalent Circuits

Thevenin Equivalent Circuit

Any circuit that contains linear elements can be represented by a Thevenin Equivalent Circuit that is the series combination of a voltage source V_{TH} and a resistor R_{TH}.

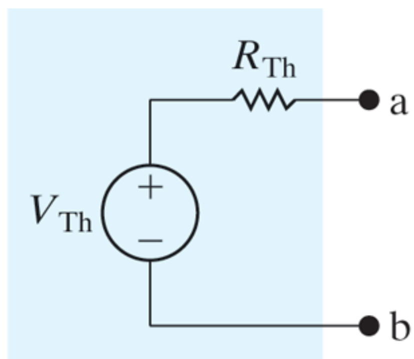


Figure 4.5: Thevenin Equivalent Circuit

To calculate V_{Th} we assume the load resistance is infinitely large (i.e., an open circuit) and calculate V_{Th}:

$$V_{Th} = V_{oc} \tag{4.13}$$

Next, we reduce the load resistance to zero (i.e., short circuit) and find i_{sc}.

Given that

$$i_{sc} = \frac{V_{Th}}{R_{Th}} \tag{4.14}$$

We find that

$$R_{TH} = \frac{V_{Th}}{i_{sc}} \tag{4.15}$$

Do Example 4.14

Norton Equivalent Circuit

Similarly, we can create a Norton Equivalent Circuit

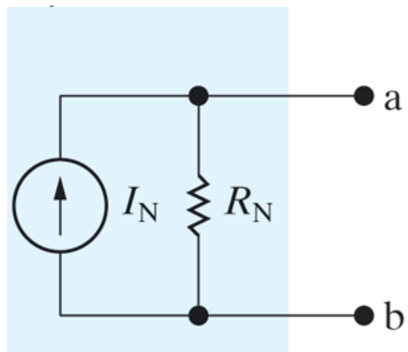


Figure 4.6: Norton Equivalent Circuit

$$I_N = i_{sc} \tag{4.16}$$

and

$$R_N = \frac{v_{oc}}{i_{sc}} = R_{Th} \tag{4.17}$$

Do Example 4.15

4.10 Maximum Power Transfer

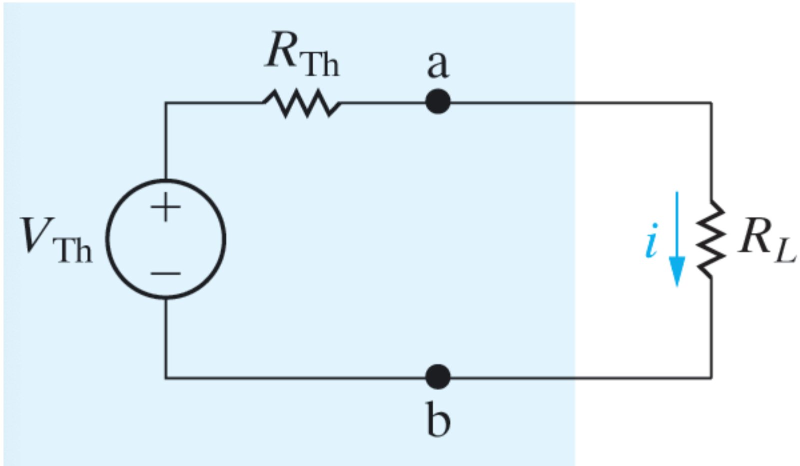


Figure 4.7: Max Power Transfer

$$p = i^2 R_L = (\frac{V_{Th}}{R_{Th} + R_L})^2 R_L \tag{4.18}$$

Take the derivative

$$\frac{dp}{dR_L} = V_{Th}^2 [\frac{(R_{Th} + R_L)^2 - 2R_L(R_{Th} + R_L)}{(R_{Th} + R_L)^4} \tag{4.19}$$

Find where the derivative is zero:

$$(R_{Th} + R_L)^2 = 2R_L(R_{Th} + R_L) \tag{4.20}$$

Solving for R_L yields

$$R_L = R_{Th} \tag{4.21}$$

And the maximum power transferred to a resistive load

$$p_{max} = \frac{V_{Th}^2}{4R_L} \tag{4.22}$$

Do Example 4.21

4.11 Superposition

A linear system obeys the principle of superposition, which states that whenever a linear system is excited, or driven, b more than one independent source of energy, the total response is the sum of the individual responses.

Do Example 4.22

Appendix A

Integration by Trig Substitution

Find the integral of

$$\int \frac{1}{(a^2+x^2)^{\frac{3}{2}}} \tag{A.1}$$

Integrate by trig substitution by setting $x = a\tan u$ which leads to

$$\frac{dx}{du} = \frac{a\tan u}{du} = \frac{a}{\cos^2 u} \tag{A.2}$$

Which leads to

$$dx = (\frac{a}{\cos^2 u})du \tag{A.3}$$

Thus

$$\int \frac{1}{(a^2+x^2)^{\frac{3}{2}}} = \int \frac{1}{(a^2+(a\tan(u))^2)^{\frac{3}{2}}}(\frac{a}{\cos^2(u)})du \tag{A.4}$$

$$= \int \frac{1}{(a^2)^{\frac{3}{2}}(1+\tan^2(u))^{\frac{3}{2}}}(\frac{a}{\cos^2(u)})du \tag{A.5}$$

$$= \int \frac{1}{(a^3)(\frac{1}{\cos^2(u)})^{\frac{3}{2}}}(\frac{a}{\cos^2(u)})du \tag{A.6}$$

$$= \frac{1}{a^2} \int \frac{1}{(\frac{1}{\cos^2(u)})^{\frac{3}{2}}} (\frac{1}{\cos^2(u)}) du \tag{A.7}$$

$$= \frac{1}{a^2} \int \frac{1}{(\frac{1}{\cos^3(u)})} (\frac{1}{\cos^2(u)}) du \tag{A.8}$$

$$= \frac{1}{a^2} \int \cos(u) du \tag{A.9}$$

$$= \frac{1}{a^2} \sin(u) + C \tag{A.10}$$

From the above $\arctan(\frac{x}{a}) = u$ so

$$= \frac{1}{a^2} \sin(\arctan(\frac{x}{a})) + C \tag{A.11}$$

$$= \frac{1}{a^2} [\frac{\frac{x}{a}}{\sqrt{1 + (\frac{x}{a})^2}}] + C \tag{A.12}$$

$$= \frac{x}{a^3} [\frac{1}{\frac{1}{a}\sqrt{a^2 + x^2}}] + C \tag{A.13}$$

Which finally leads to

$$\int \frac{1}{(a^2 + x^2)^{\frac{3}{2}}} = \frac{x}{a^2} [\frac{1}{\sqrt{a^2 + x^2}}] + C \tag{A.14}$$

Appendix B

Chain Rule

$$\frac{d}{dx}(\frac{1}{\sqrt{x^2+R^2}}) = \frac{d}{dx}(x^2+R^2)^{-\frac{1}{2}} \tag{B.1}$$

The chain rule $f(g(x))' = f'(g(x) \cdot g'(x)$. In this case $g(x) = x^2 + R^2$. From this

$$f(g(x)) = g(x)^{-\frac{1}{2}} \tag{B.2}$$

thus

$$f'(g(x)) = -\frac{1}{2}g(x)^{-\frac{3}{2}} \tag{B.3}$$

and

$$g'(x) = 2x \tag{B.4}$$

Thus

$$f(g(x))' = -\frac{1}{2}g(x)^{-\frac{3}{2}} \cdot 2x \tag{B.5}$$

$$\frac{d}{dx}(\frac{1}{\sqrt{x^2+R^2}}) = \frac{-x}{(x^2+R^2)^{\frac{3}{2}}} \tag{B.6}$$