ECE 203

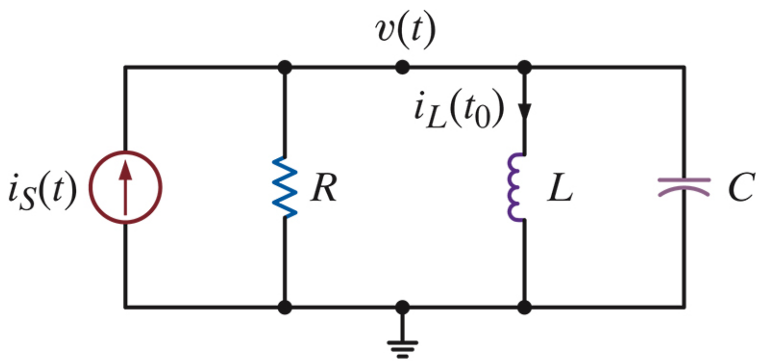
Circuits I

2^{nd} Order Transient Circuits

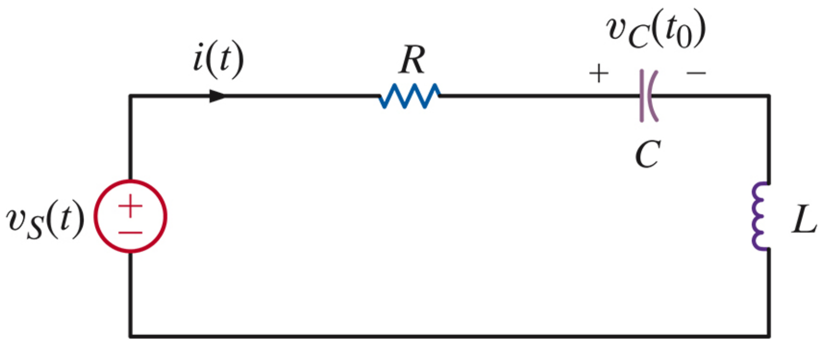
Lecture 11-1

Second Order Circuits

Second order circuits are circuits with two energy storage elements in any combination

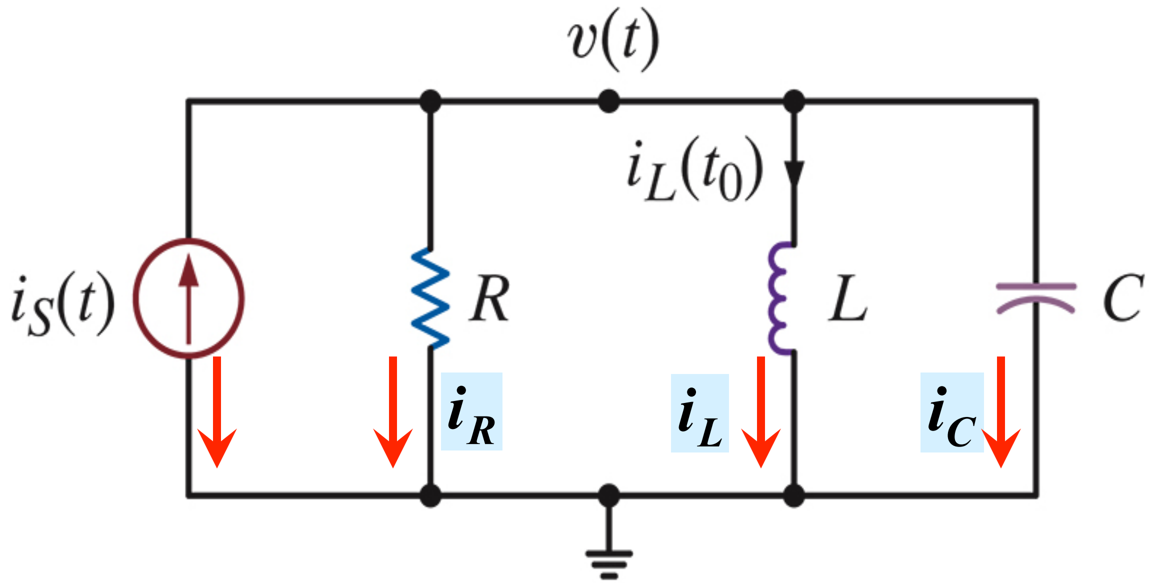


Example 1



Example 2

Parallel RLC Circuit



Single Node-pair: Use KCL

$$-i_S + i_R + i_L + i_C = 0$$

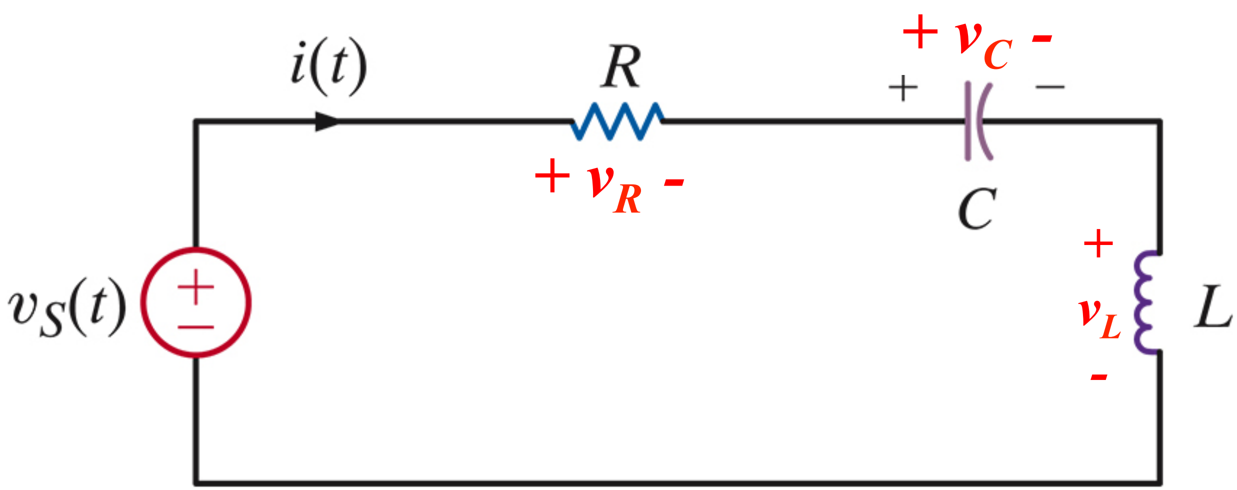
$$i_R = \frac{v(t)}{R}; \quad i_L = \frac{1}{L}\int_{t_0}^{t} v(x)dx + i_L(t_0); \quad i_C = C\frac{dv}{dt}(t)$$

$$\frac{v}{R} + \frac{1}{L}\int_{t_0}^{t} v(x)dx + i_L(t_0) + C\frac{dv}{dt}(t) = i_S$$

Differentiating

$$C\frac{d^2v}{dt^2} + \frac{1}{R}\frac{dv}{dt} + \frac{v}{L} = \frac{di_S}{dt}$$

Series RLC Circuit



Single Loop: Use KVL

$$-v_S + v_R + v_C + v_L = 0$$

$$v_R = Ri;\ v_C = \frac{1}{C}\int_{t_0}^{t} i(x)dx + v_C(t_0);\ \ v_L = L\frac{di}{dt}(t)$$

$$Ri + \frac{1}{C}\int_{t_0}^{t} i(x)dx + v_C(t_0) + L\frac{di}{dt}(t) = v_S$$

Differentiating

$$L\frac{d^2 i}{dt^2} + R\frac{di}{dt} + \frac{i}{C} = \frac{dv_S}{dt}$$

General Response in Second Order Circuits

Including the initial conditions the model for the capacitor voltage or the inductor current will be shown to be of the form

$$\frac{d^2x(t)}{dt^2} + a_1\frac{dx(t)}{dt} + a_2x(t) = f(t) \quad ; \; x(t=0) = x_0 \; ; \; dx/dt(t=0) = x'_0$$

Differential Equation Review:

A fundamental theorem of differential equations states that the general solution to this equation can be written as:

$$x(t) = x_p(t) + x_c(t)$$

Where $x_p(t)$ is the *particular solution*, or *forced response*, and $x_c(t)$ is the *complementary solution* or *natural response*.

$$\frac{d^2x_p(t)}{dt^2} + a_1\frac{dx_p(t)}{dt} + a_2x_p(t) = f(t)$$

$$\frac{d^2x_c(t)}{dt^2} + a_1\frac{dx_c(t)}{dt} + a_2x_c(t) = 0$$

General Response in Second Order Circuits

Let's again assume that $f(t) = A$, i.e. a constant:

$$\frac{d^2x_p(t)}{dt^2} + a_1\frac{dx_p(t)}{dt} + a_2x_p(t) = A$$

Homogenous Equation →

$$\frac{d^2x_c(t)}{dt^2} + a_1\frac{dx_c(t)}{dt} + a_2x_c(t) = 0$$

The solution to this differential equation is:

$$\begin{cases} x_p(t) = K_1 \quad \textbf{where} \quad K_1 = \dfrac{A}{a_2} \\ x_c(t) = Ke^{st} \end{cases}$$

But how to compute *S* and *K*?

Let's take a closer look at the homogeneous equation.

Damping Factor & Natural Frequency

The homogenous equation can be rewritten as:

Homogenous Equation →

$$\frac{d^2 x_c(t)}{dt^2} + a_1 \frac{dx_c(t)}{dt} + a_2 x_c(t) = 0$$

$$\frac{d^2 x_c(t)}{dt^2} + 2\zeta\omega_0 \frac{dx_c(t)}{dt} + \omega_0^2 x_c(t) = 0$$

We write the homogenous equation in this form because ζ and ω_0 are two important parameters in the second order circuits. We define:

$$a_2 = \omega_0^2 \quad \Rightarrow \quad \omega_0 = \sqrt{a_2} \qquad \omega_0 = Natural \;\; frequency$$

$$a_1 = 2\zeta\omega_0 \quad \Rightarrow \quad \zeta = \frac{a_1}{2\sqrt{a_2}} \qquad \zeta = Damping \;\; factor$$

Characteristic Equation

Now, let's find the value of s such that it satisfy the homogenous equation:

Homogenous Equation

$$\frac{d^2 x_c(t)}{dt^2} + 2\zeta\omega_0 \frac{dx_c(t)}{dt} + \omega_0^2 x_c(t) = 0$$

$$x_c(t) = Ke^{st}$$

There are two values of s (or a repeated value)

$$\Rightarrow s^2Ke^{st} + 2\zeta\omega_0 sKe^{st} + \omega_0^2 Ke^{st} = 0$$

$$\Rightarrow s^2 + 2\zeta\omega_0 s + \omega_0^2 = 0$$

This is called the <u>Characteristic Equation</u>. The solution to this equation, gives the values of s that will satisfy the homogenous equation.

Example

Determine the characteristic equation, damping factor and natural frequency of:

$$4\frac{d^2x}{dt^2}(t) + 8\frac{dx}{dt}(t) + 16x(t) = 0$$

$$\frac{d^2x}{dt^2}(t) + 2\frac{dx}{dt}(t) + 4x(t) = 0$$

Characteristic Equation

$$s^2 + 2s + 4 = 0$$

$$s^2 + 2\zeta\omega_0 s + \omega_0^2 = 0$$

$$\begin{cases} \omega_0 = 2 & \textbf{Natural Frequency} \\ \zeta = 0.5 & \textbf{Damping Factor} \end{cases}$$

Analysis of Homogeneous Equation

Normalized form of homogenous equation:

$$\frac{d^2 x_c(t)}{dt^2} + 2\zeta\omega_0 \frac{dx_c(t)}{dt} + \omega_0^2 x_c(t) = 0$$

$x_c(t) = Ke^{st}$ **is the solution to this equation only and only if s satisfies the characteristic equation (for your exercise prove it):**

$$\Rightarrow \quad \boxed{s^2 + 2\zeta\omega_0 s + \omega_0^2 = 0}$$

The solutions are:

$$s_{1,2} = -\omega_0\zeta \pm \omega_0\sqrt{\zeta^2 - 1}$$

There are two values of s (or a repeated value)

Case 1: $\zeta > 1 \quad \Rightarrow \quad s_{1,2} = -\omega_0\zeta \pm \omega_0\sqrt{\zeta^2 - 1}$ **real & distinct roots**

Case 2: $\zeta < 1 \quad \Rightarrow \quad s_{1,2} = -\omega_0\zeta \pm j\omega_0\sqrt{1 - \zeta^2}$ **complex conjugate roots**

Case 3: $\zeta = 1 \quad \Rightarrow \quad s_1 = s_2 = -\omega_0\zeta$ **real & equal roots**

Case I: Real and Distinct Roots (Overdamped)

Now consider the general equation:

$$\frac{d^2x(t)}{dt^2} + 2\zeta\omega_0 \frac{dx(t)}{dt} + \omega_0^2 x(t) = A \quad \Rightarrow \quad s^2 + 2\zeta\omega_0 s + \omega_0^2 = 0$$

Case 1: $\zeta > 1 \quad \Rightarrow \quad s_{1,2} = -\omega_0\zeta \pm \omega_0\sqrt{\zeta^2 - 1}$ **real & distinct roots**

$$\boxed{x(t) = x_p(t) + x_c(t) = K_0 + K_1 e^{s_1 t} + K_2 e^{s_2 t}}$$

$$x_p(t) = K_0 = \frac{A}{\omega_0^2}$$

K_0 is A/ω_0^2 and K_1 and K_2 are determined by the initial condition:

$$x(t=0) = x_0 \ ; \ dx/dt(t=0) = x'_0$$

Case II: Complex Conjugate Roots (Underdamped)

Again, consider the general equation:

$$\frac{d^2x(t)}{dt^2} + 2\zeta\omega_0\frac{dx(t)}{dt} + \omega_0^2 x(t) = A \quad \Rightarrow \quad s^2 + 2\zeta\omega_0 s + \omega_0^2 = 0$$

Case 2: $\zeta < 1 \quad \Rightarrow \quad s_{1,2} = -\omega_0\zeta \pm j\omega_0\sqrt{1-\zeta^2}$ **complex conjugate roots**

$$\boxed{x(t) = x_p(t) + x_c(t) = K_0 + e^{-\sigma t}\left(K_1\cos(\omega_d t) + K_2\sin(\omega_d t)\right)}$$

$$x_p(t) = K_0 = \frac{A}{\omega_0^2} \qquad \sigma = \zeta\omega_0 \qquad \omega_d = \omega_0\sqrt{1-\zeta^2}$$

K_0 is A/ω_0^2 and K_1 and K_2 are determined by the initial condition.

$$x(t=0) = x_0 \;;\; dx/dt(t=0) = x'_0$$

Case III: Real and Equal Roots (Critically Damped)

Again, consider the general equation:

$$\frac{d^2x(t)}{dt^2} + 2\zeta\omega_0 \frac{dx(t)}{dt} + \omega_0^2 x(t) = A \quad \Rightarrow \quad s^2 + 2\zeta\omega_0 s + \omega_0^2 = 0$$

Case 3: $\zeta = 1 \quad \Rightarrow \quad s_1 = s_2 = -\omega_0\zeta$ **real & equal roots**

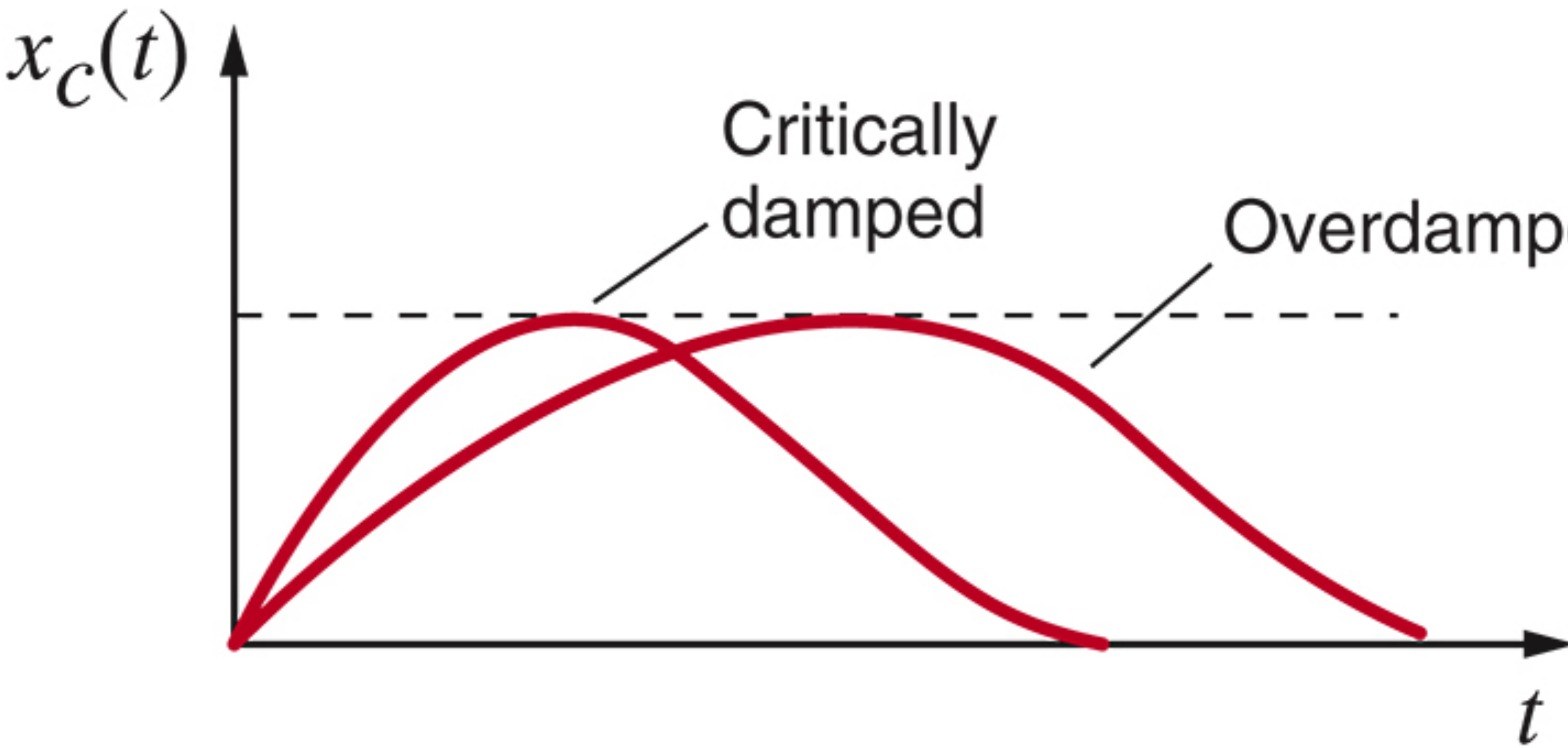
$$\boxed{x(t) = x_p(t) + x_c(t) = K_0 + \left(K_1 + K_2 t\right)e^{-\omega_0\zeta t}}$$

$$x_p(t) = K_0 = \frac{A}{\omega_0^2}$$

K_0 is A/ω_0^2 and K_1 and K_2 are determined by the initial condition.

$$x(t{=}0) = x_0 \;\; ; \;\; dx/dt(t{=}0) = x'_0$$

Overdamped & Critically Damped Waveforms



Underdamped Waveforms