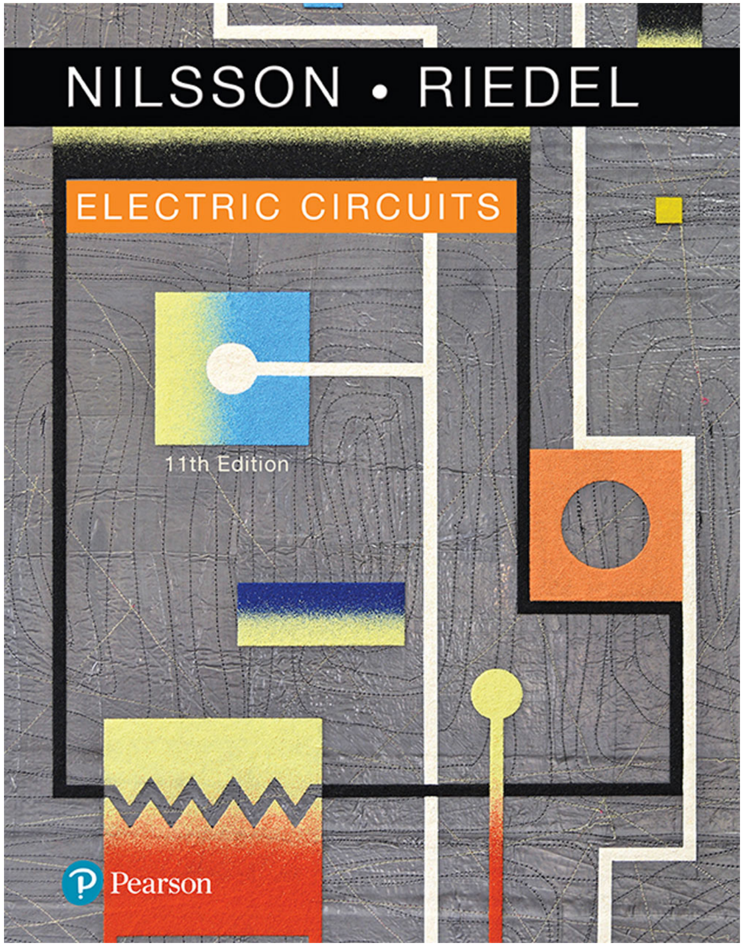
Electric Circuits

Eleventh Edition



Chapter 7

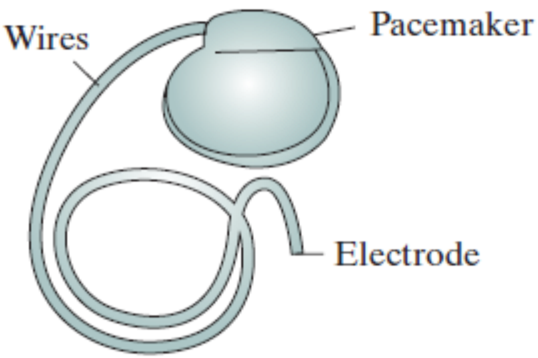
Response of First-Order *RL* and *RC* Circuits

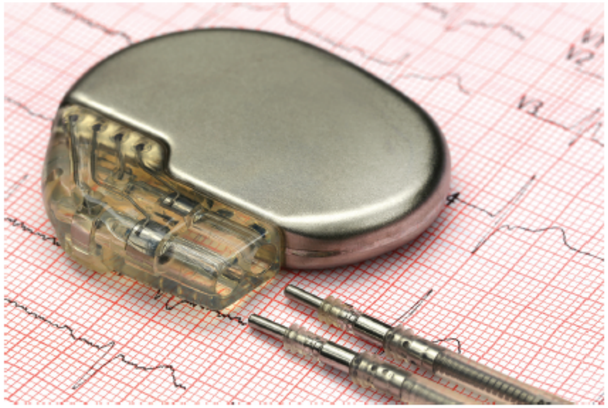
Pearson

Learning Objectives

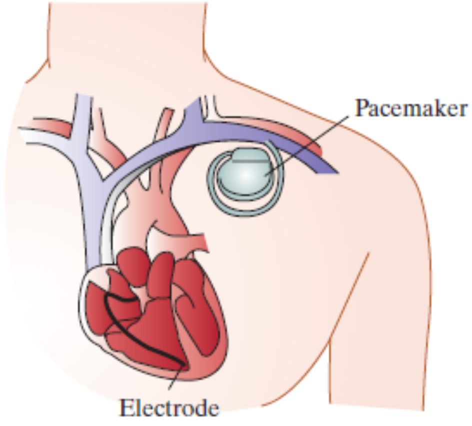
- The Natural Response of an *RL* Circuit
- The Natural Response of an *RC* Circuit
- The Step Response of *RL* and *RC* Circuits
- A General Solution for Step and Natural Responses
- Sequential Switching
- Unbounded Response
- The Integrating Amplifier*

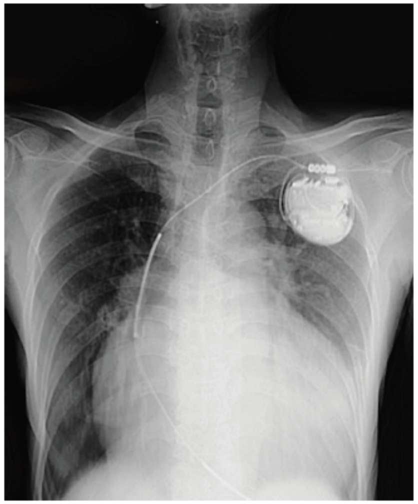
Practical Perspective - Artificial Pacemaker





Swapan Photography/Shutterstock





Tewan Banditrukkanka/Shutterstock

RL & RC Circuits

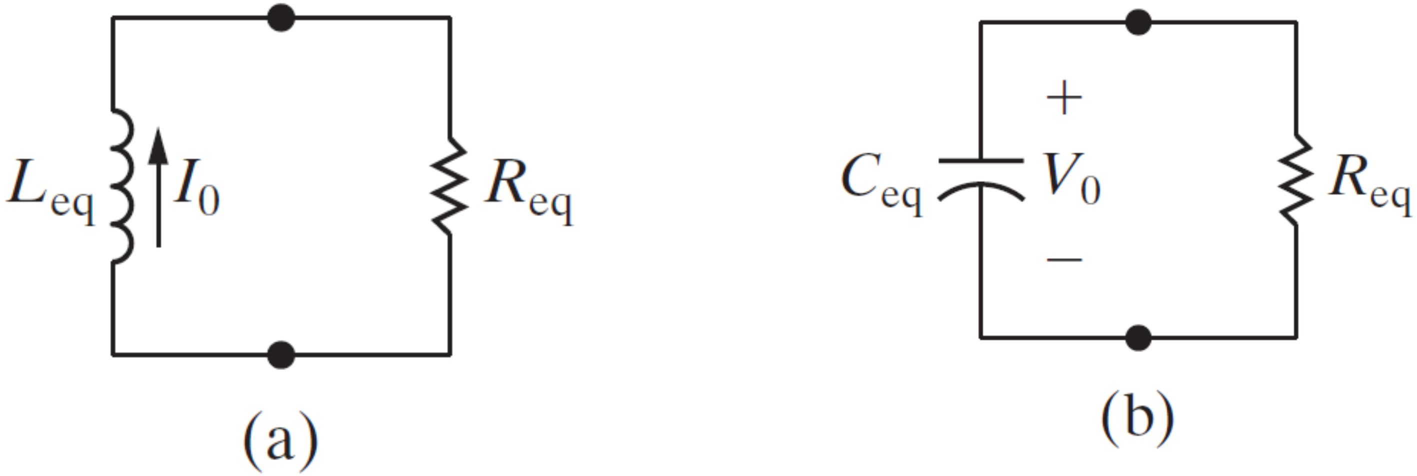


Figure 7.1: The two forms of the circuits for natural response. (a) RL circuit. (b) RC circuit.

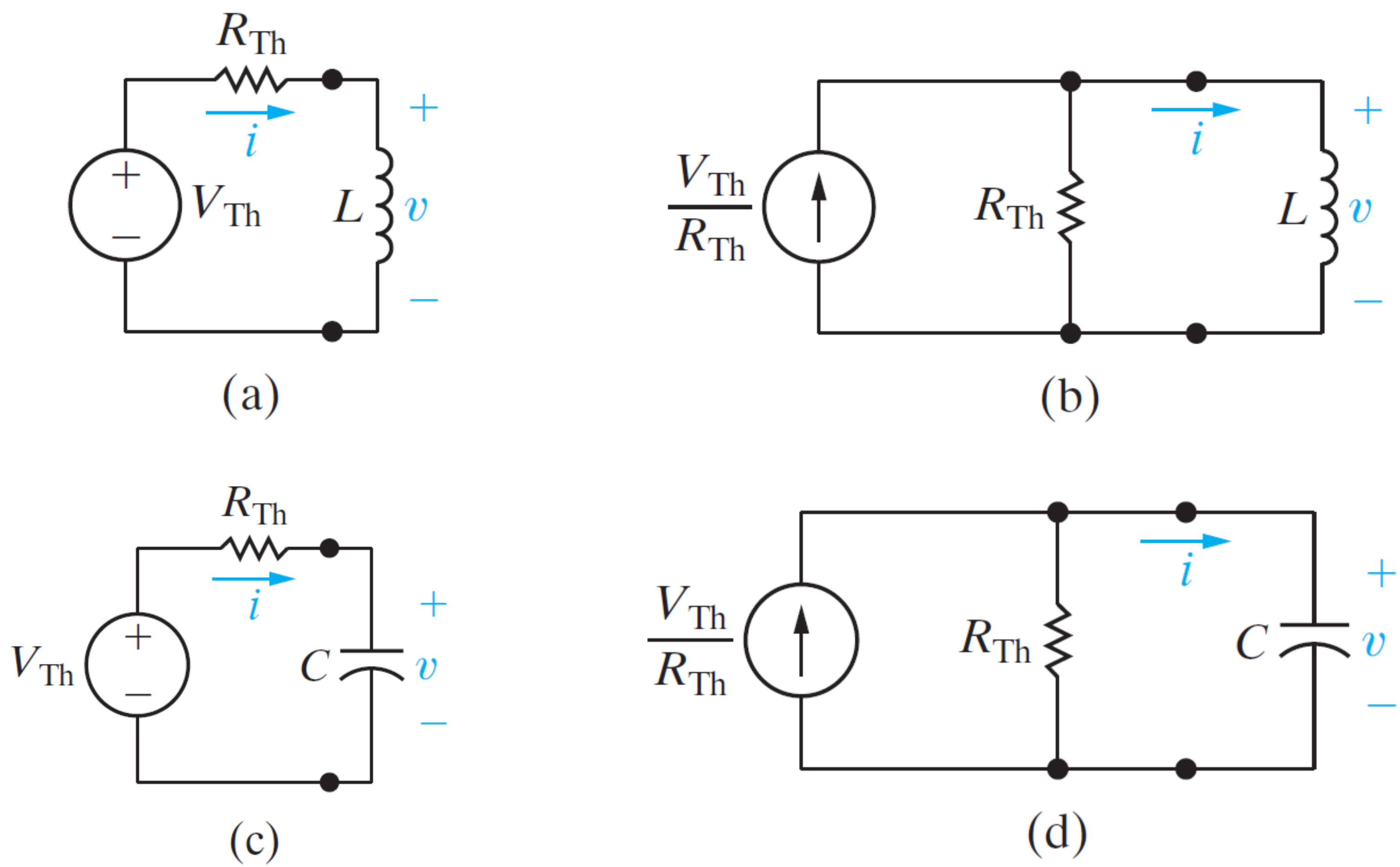


Figure 7.2: Four possible first-order circuits. (a) An inductor connected to a Thévenin equivalent. (b) An inductor connected to a Norton equivalent. (c) A capacitor connected to a Thévenin equivalent. (d) A capacitor connected to a Norton equivalent.

7.1 Natural Response of RL Circuit

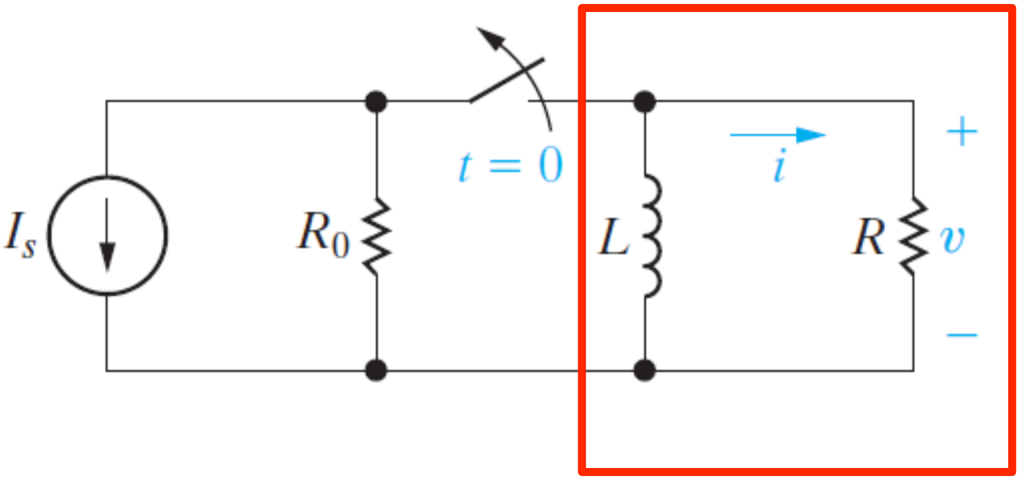


Figure 7.3: An RL circuit.

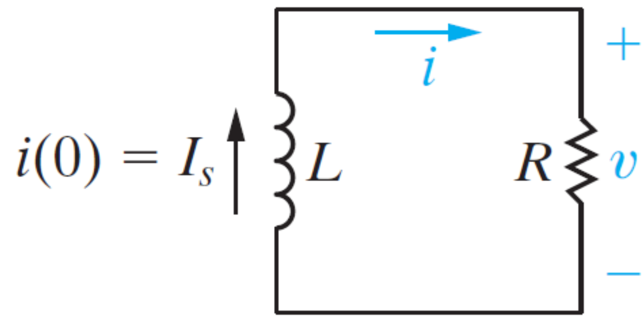


Figure 7.4: The circuit shown in Fig. 7.3, for $t \geq 0$.

a) The switch has been in a closed position for a long time: *all currents and voltages have reached a constant value*.
b) The inductor appears as a short circuit ($L di/dt = 0$) prior to the release of the stored energy.

Expression for the Current

$$L\frac{di}{dt} + Ri = 0$$

$$\frac{di}{i} = -\frac{R}{L}dt$$

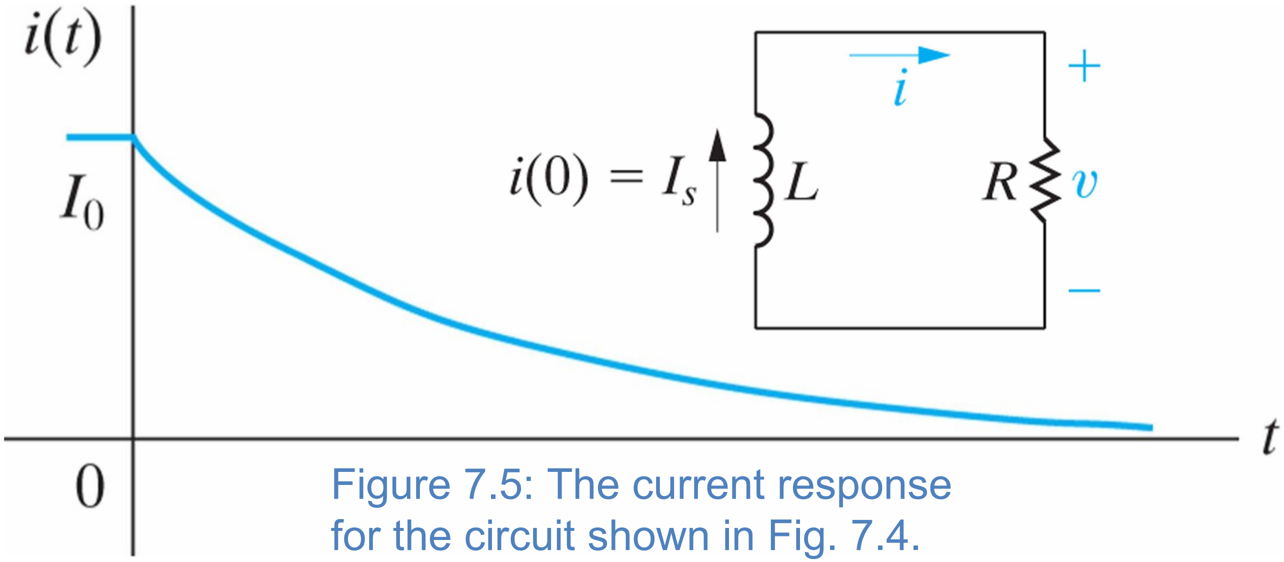


Figure 7.5: The current response for the circuit shown in Fig. 7.4.

$$\int_{i(t_0)}^{i(t)} \frac{dx}{x} = -\frac{R}{L}\int_{t_0}^{t} dy \quad \Rightarrow \quad \ln\frac{i(t)}{i(0)} = -\frac{R}{L}t$$

$$\Rightarrow \quad i(t) = i(0)e^{-(R/L)t}$$

Expression for the Power and Energy

$$v = iR = I_0 R e^{-(R/L)t}, \qquad t \geq 0^+$$

$$\boxed{p = vi}$$

$$p = I_0^2 R e^{-2(R/L)t}$$

$$w = \int_0^t p\,dx = \int_0^t I_0^2 R e^{-2(R/L)x}\,dx$$

$$= \frac{1}{2(R/L)} I_0^2 R(1 - e^{-2(R/L)t})$$

$$= \frac{1}{2} L I_0^2 (1 - e^{-2(R/L)t}), \qquad t \geq 0.$$

The Significance of the Time Constant

$$i(t) = i(0)e^{-\boxed{(R/L)}t} \Rightarrow \tau = \text{time constant} = \frac{L}{R}$$

$$i(t) = I_0 e^{-t/\tau}, \quad t \geq 0,$$

$$v(t) = I_0 R e^{-t/\tau}, \quad t \geq 0^+,$$

$$p = I_0^2 R e^{-2t/\tau}, \quad t \geq 0^+,$$

$$w = \frac{1}{2} L I_0^2 (1 - e^{-2t/\tau}), \quad t \geq 0.$$

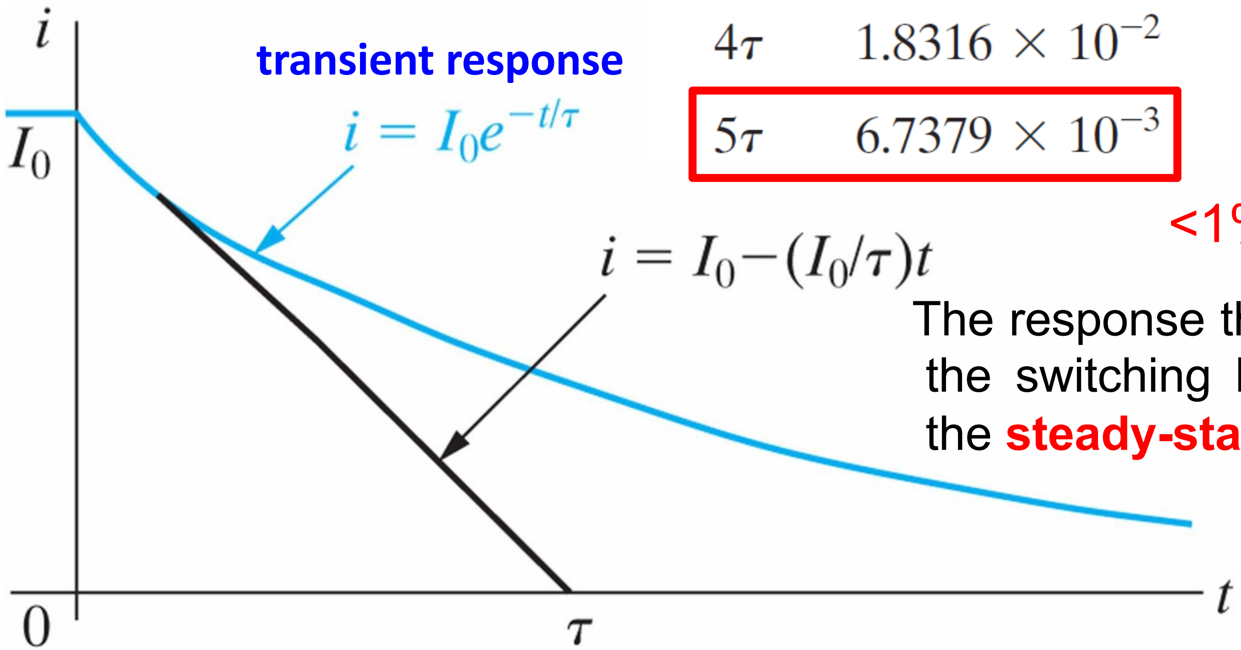
The time constant is an important parameter in first-order circuits, *it is convenient to think of the time elapsed after switching in terms of integral multiples of τ*: one time constant after the inductor starts to release its stored energy to the resistor, the current has been reduced to $e^{-1} \approx 0.37$ of its initial value.

Steady-State Response

TABLE 7.1 Value of $e^{-t/\tau}$
for t integral multiples of τ

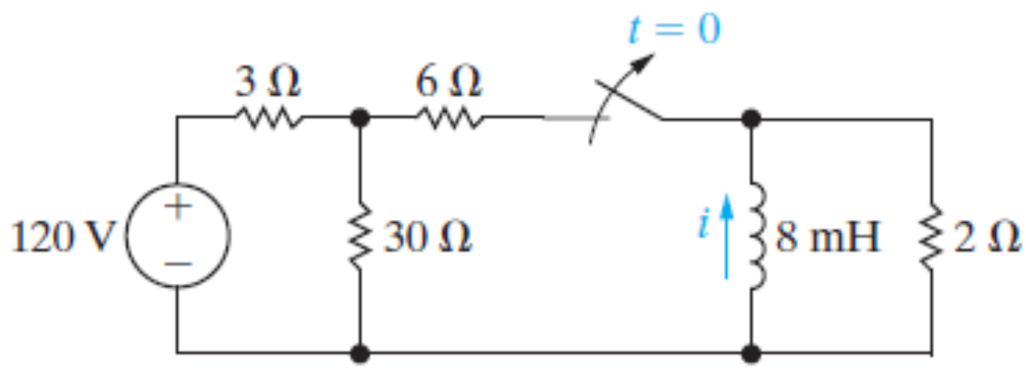
t	$e^{-t/\tau}$	t	$e^{-t/\tau}$
τ	3.6788×10^{-1}	6τ	2.4788×10^{-3}
2τ	1.3534×10^{-1}	7τ	9.1188×10^{-4}
3τ	4.9787×10^{-2}	8τ	3.3546×10^{-4}
4τ	1.8316×10^{-2}	9τ	1.2341×10^{-4}
$\boxed{5\tau}$	$\boxed{6.7379 \times 10^{-3}}$	10τ	4.5400×10^{-5}

<1%



The response that exists *a long time* after the switching has taken place is called the **steady-state response**

Assessment Problems 7.1



The switch in the circuit has been closed for a long time and is opened at t = 0.

a) Calculate initial value of i.

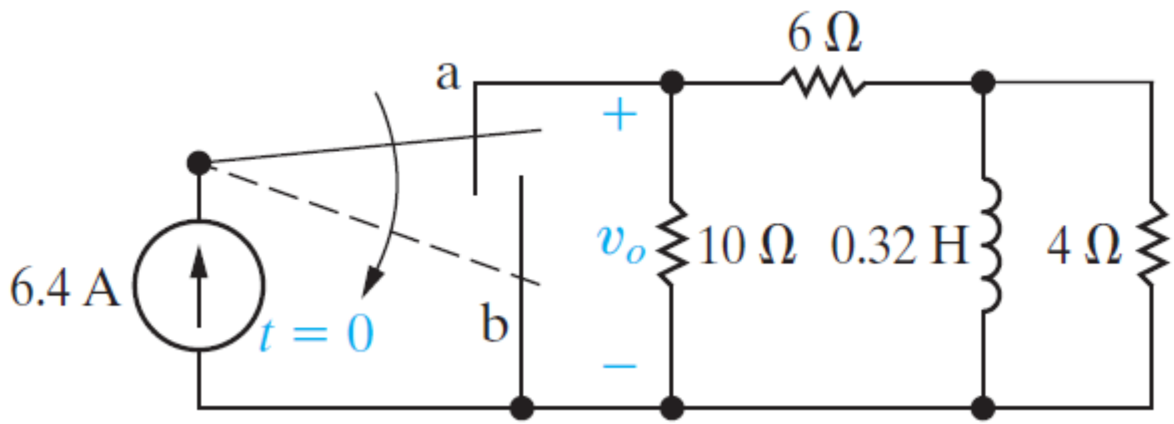
b) Calculate initial energy stored in inductor.

c) What is τ?

d) What is the numerical expression for $i(t)$?

e) What percentage of the initial energy stored has been dissipated in the 2 Ω resistor 5 ms after the switch is opened?

Assessment Problems 7.2



At t = 0, the switch is moved from a to b.

a) Calculate v_o for $t \geq 0^+$.

b) What percentage of the initial energy stored in the inductor is eventually dissipated in the 4 Ω resistor?

7.2 Natural Response of RC Circuit

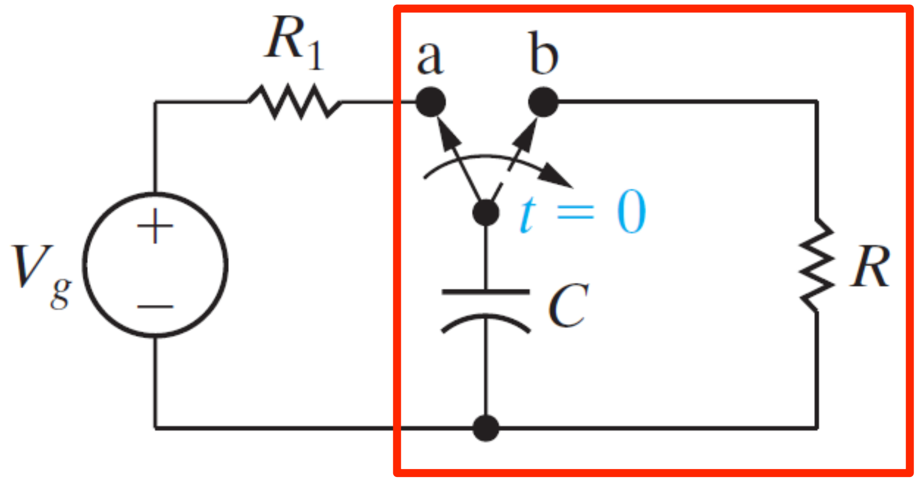


Figure 7.12: An RC circuit.

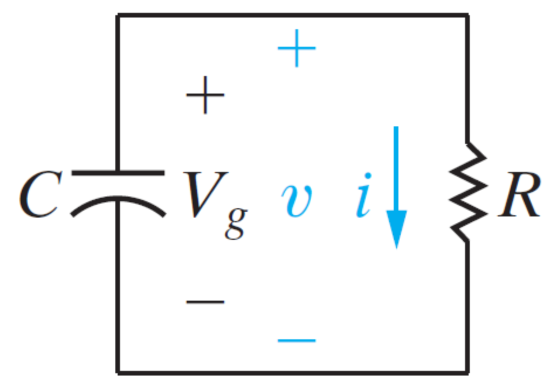


Figure 7.13: The circuit shown in Fig. 7.12, after switching.

a) The switch has been in a closed position for a long time: *all currents and voltages have reached a constant value*.
b) The capacitor appears as an open circuit ($Cdv/dt = 0$) prior to the release of the stored energy.

$$C\frac{dv}{dt} + \frac{v}{R} = 0$$

$$v(t) = v(0)e^{-t/RC}, \quad t \geq 0$$

$$\boxed{\tau = RC}$$

$$v(t) = V_0 e^{-t/\tau}, \quad t \geq 0$$

$$i(t) = \frac{v(t)}{R} = \frac{V_0}{R} e^{-\tau/t}, \quad t \geq 0^+$$

$$p(t) = vi = \frac{V_0^2}{R} e^{-2\tau/t}, \quad t \geq 0^+$$

$$w = \frac{1}{2} C V_0^2 \left(1 - e^{-2\tau/t}\right), \quad t \geq 0$$

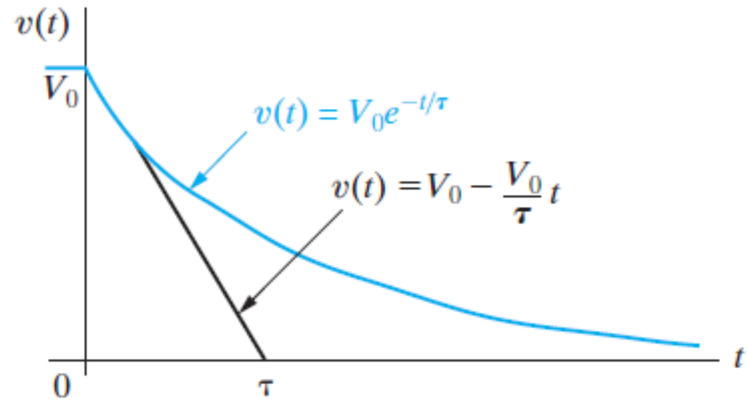
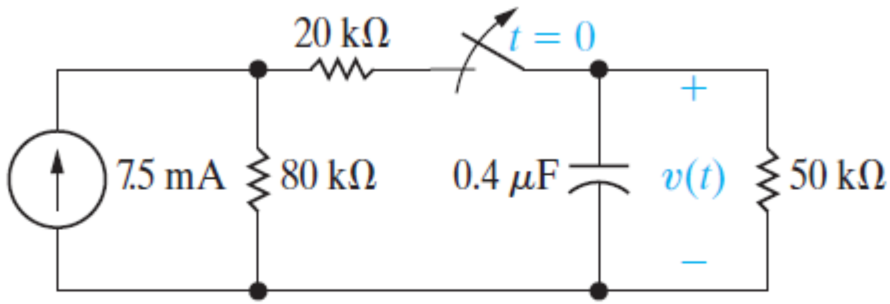


Figure 7.14: The natural response of an *RC* circuit.

Assessment Problems 7.3



The switch in the circuit has been closed for a long time and is opened at t = 0. Find:

a) the initial value of $v(t)$,

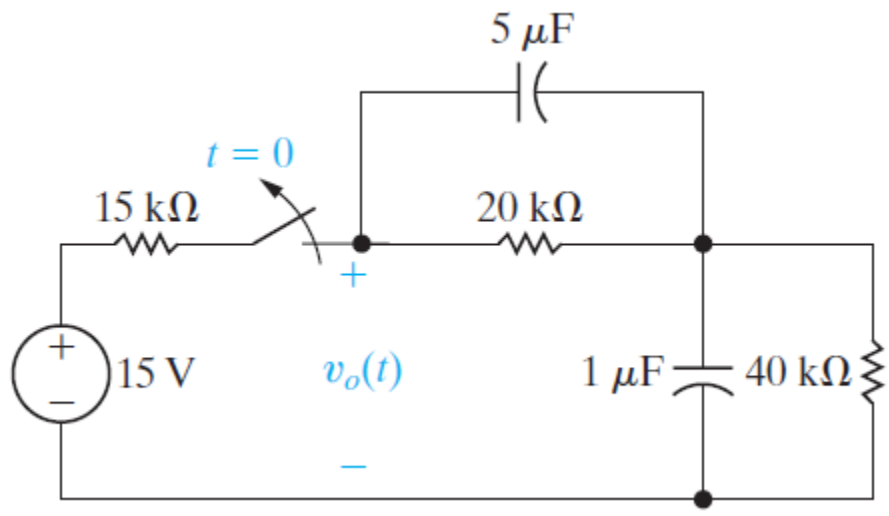
b) the time constant for t > 0,

c) $v(t)$ after the switch is opened

d) the initial energy stored in the capacitor, and

e) the time required to dissipate 75% of the initial energy.

Assessment Problems 7.4



The switch in the circuit has been closed for a long time and is opened at t = 0.

a) Find v_o for t ≥ 0.

b) What percentage of the initial energy stored in the circuit has been dissipated after 60 ms?

7.3 Step Response of RL & RC Circuits

Step Response of RL Circuit

KVL:

$$V_s = Ri + L\frac{di}{dt}$$

$$\frac{di}{\boxed{dt}} = \frac{-Ri + V_s}{L} = \frac{-R}{L}\boxed{\left(i - \frac{V_s}{R}\right)}$$

$$\frac{di}{i - (V_s/R)} = \frac{-R}{L}dt$$

$$\int_{I_0}^{i(t)} \frac{dx}{x - (V_s/R)} = \frac{-R}{L}\int_0^t dy$$

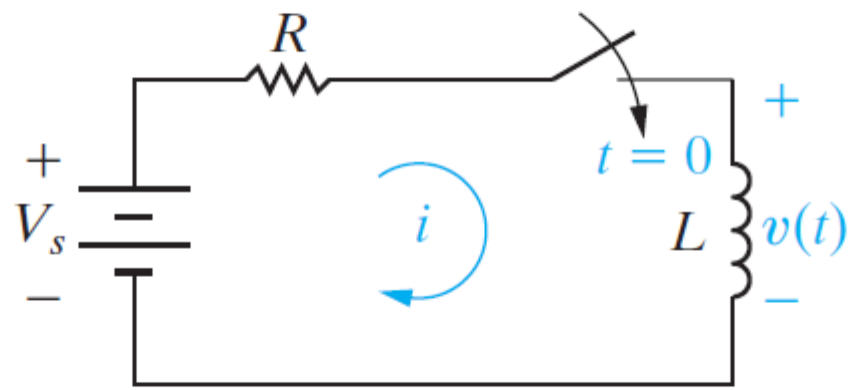


Figure 7.20: A circuit used to illustrate the step response of a first-order *RL* circuit.

Pearson

$$\ln \frac{i(t) - (V_s/R)}{I_0 - (V_s/R)} = \frac{-R}{L}t$$

$$\frac{i(t) - (V_s/R)}{I_0 - (V_s/R)} = e^{-(R/L)t}$$

$$i(t) = \frac{V_s}{R} + \left(I_0 - \frac{V_s}{R}\right)e^{-(R/L)t}$$

$$\boxed{i(t) = I_{\mathrm{f}} + (I_0 - I_{\mathrm{f}})e^{-t/\tau}}$$

Step Response Curve

$I_0 = 0$

$$i(t) = \frac{V_s}{R} - \frac{V_s}{R}e^{-(R/L)t}$$

$$i(\tau) = \frac{V_s}{R} - \frac{V_s}{R}e^{-1} \approx 0.6321\frac{V_s}{R}$$

$$\frac{di}{dt} = \frac{-V_s}{R}\left(\frac{-1}{\tau}\right)e^{-t/\tau} = \frac{V_s}{L}e^{-t/\tau}$$

$$\frac{di}{dt}(0) = \frac{V_s}{L}$$

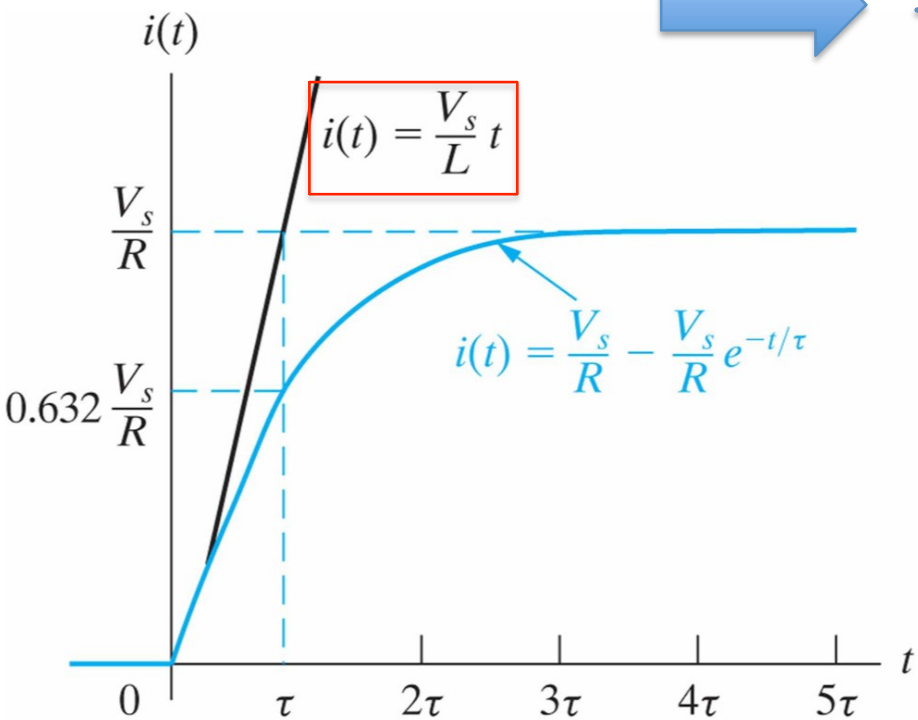


Figure 7.26: The step response of the *RL* circuit shown in Fig. 7.20 when $I_0 = 0$.

$$i(t=\tau) = \frac{V_s}{L}\underbrace{\boxed{\frac{L}{R}}}_{\tau} = \frac{V_s}{R}$$

The voltage across an inductor is $L\,di/dt$, so for $t > 0^+$, we have:

$$v = L\left(\frac{-R}{L}\right)\left(I_0 - \frac{V_s}{R}\right)e^{-(R/L)t} = (V_s - I_0R)e^{-(R/L)t}$$

$I_0 = 0$

$$v = V_s e^{-(R/L)t}$$

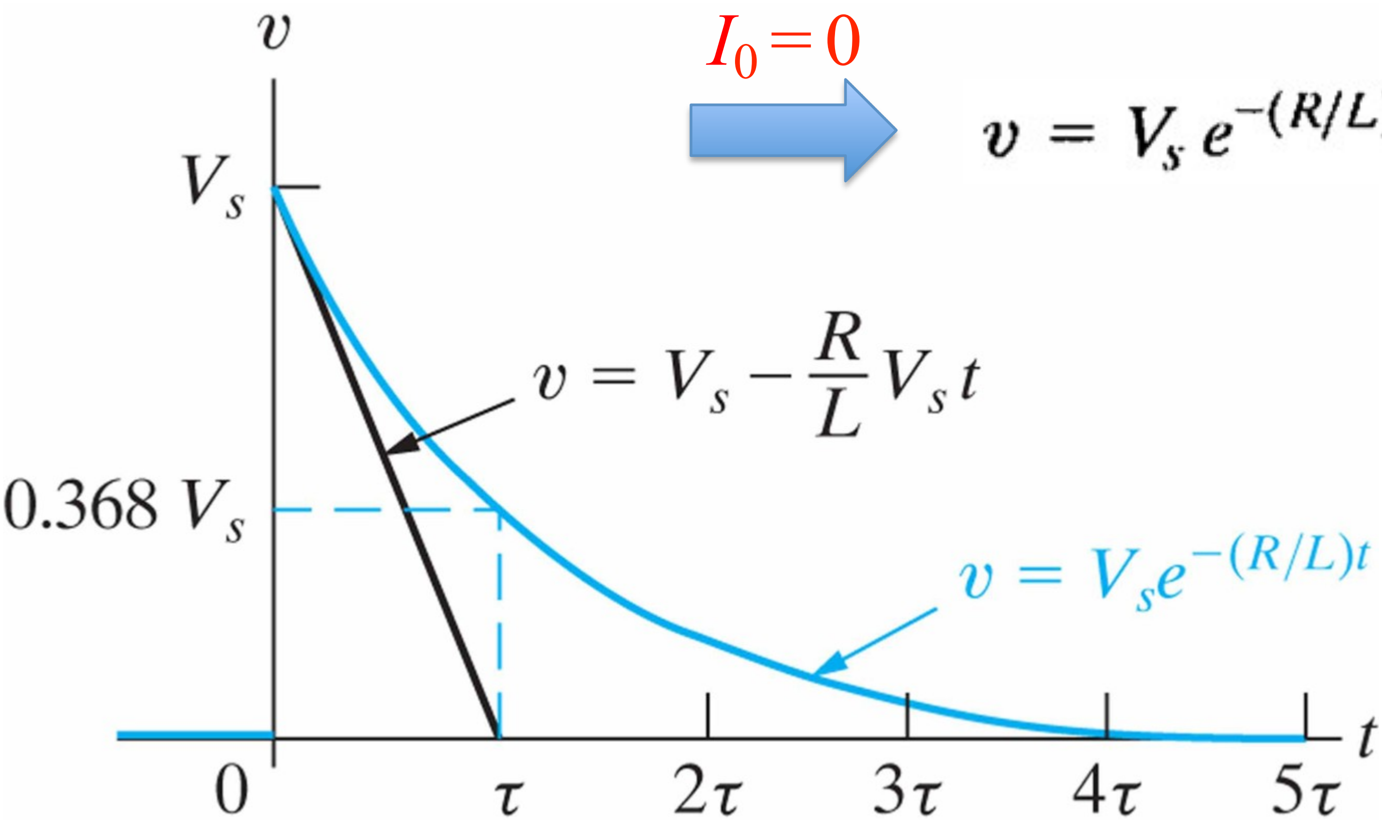
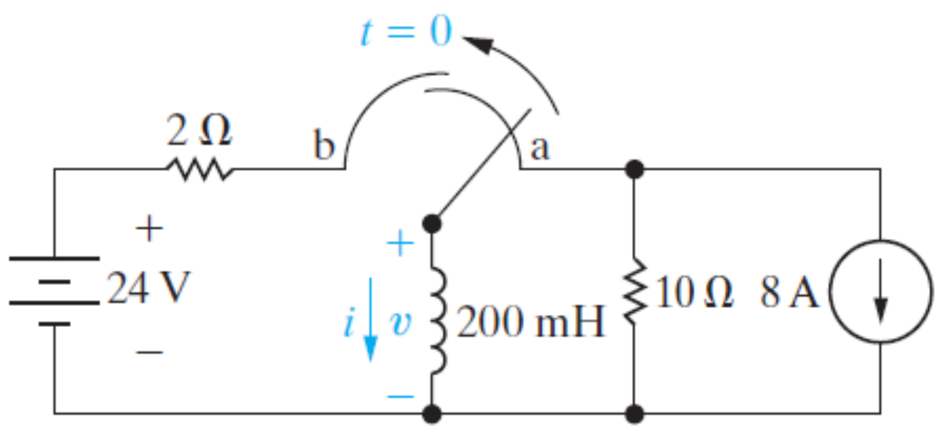


Figure 7.27: Inductor voltage versus time.

Assessment Problems 7.5



a) Find $i(0+)$.

b) Find $v(0+)$.

c) What is τ?

d) Find $i(t)$.

e) Find $v(t)$.

Step Response of RC Circuit

$$C\frac{dv_C}{dt} + \frac{v_C}{R} = I_s$$

$$\frac{dv_C}{dt} + \frac{v_C}{RC} = \frac{I_s}{C}$$

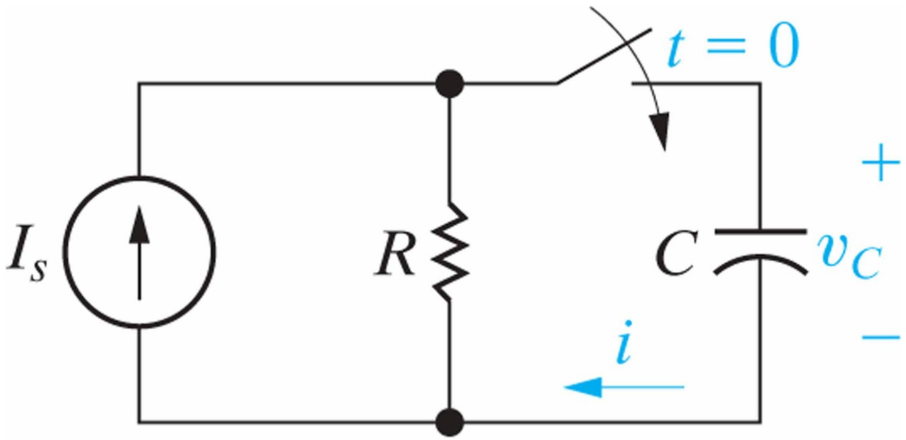


Figure 7.28: A circuit used to illustrate the step response of a first-order *RC* circuit.

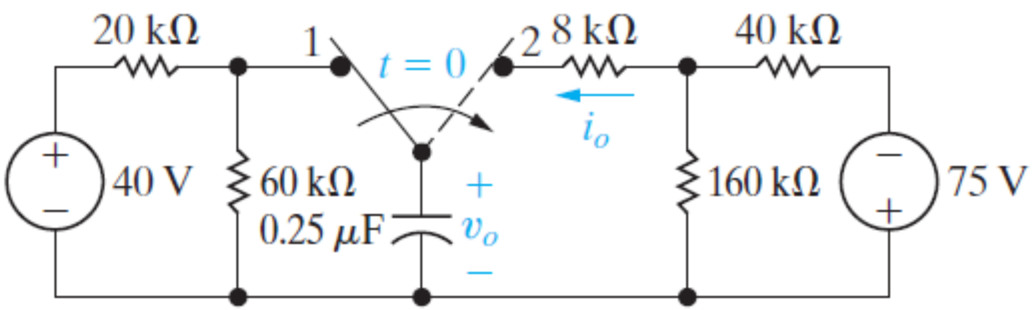
By comparing with Eq. 7.17

$$v_C = I_sR + (V_0 - I_sR)e^{-t/RC}, \quad t \geq 0$$

$$\boxed{v(t) = V_f + (V_0 - V_f)e^{-t/\tau}}$$

$$\frac{di}{dt} + \frac{1}{RC}i = 0 \quad \Rightarrow \quad i = \left(I_s - \frac{V_0}{R}\right)e^{-t/RC}, \quad t \geq 0^+$$

Assessment Problems 7.6



Find the expression for the voltage across the 160 kΩ resistor. Call this voltage v_A, and assume that the reference polarity for the voltage is positive at the upper terminal of the 160 kΩ resistor.

7.4 General Solution for Step and Natural Response

$$\frac{dx}{dt} + \frac{x}{\tau} = K \quad \Rightarrow \quad x_f = K\tau$$

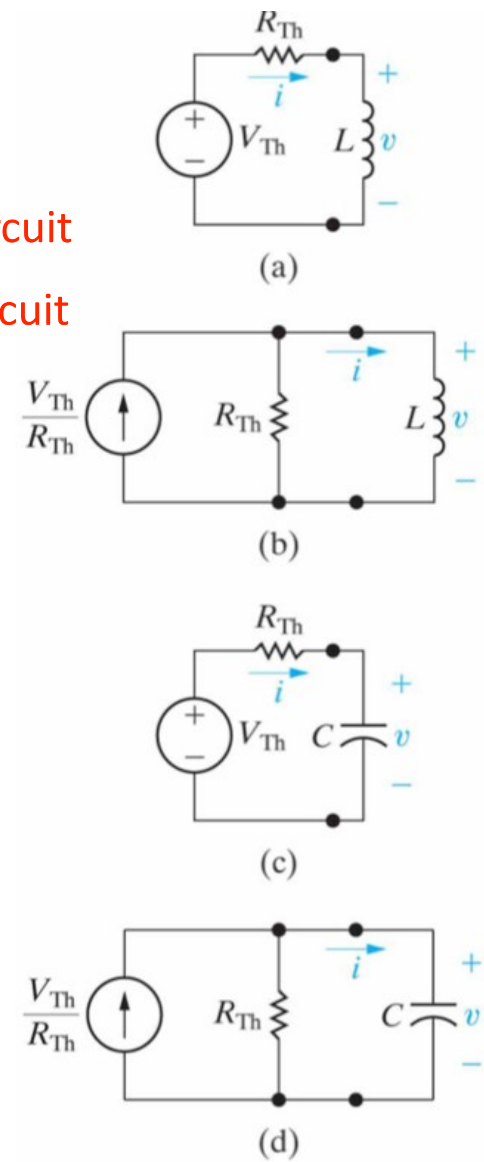
$\tau = L/R$ for RL circuit

$\tau = RC$ for RC circuit

$$\frac{dx}{dt} = \frac{-x}{\tau} + K = \frac{-(x - K\tau)}{\tau} = \frac{-(x - x_f)}{\tau}$$

$$\Rightarrow \quad \frac{dx}{x - x_f} = \frac{-1}{\tau} dt$$

$$\int_{x(t_0)}^{x(t)} \frac{du}{u - x_f} = -\frac{1}{\tau}\int_{t_0}^{t} dv$$



$$\boxed{x(t) = x_f + [x(t_0) - x_f]e^{-(t-t_0)/\tau}}$$

$$\begin{array}{r} \text{the unknown} \\ \text{variable as a} \\ \text{function of time} \end{array} = \begin{array}{r} \text{the final} \\ \text{value of the} \\ \text{variable} \end{array}$$

$$+ \left[\begin{array}{r} \text{the initial} \\ \text{value of the} \\ \text{variable} \end{array} - \begin{array}{r} \text{the final} \\ \text{value of the} \\ \text{variable} \end{array} \right] \times e^{\frac{-[t - (\text{time of switching})]}{(\text{time constant})}}$$

Helpful Notes

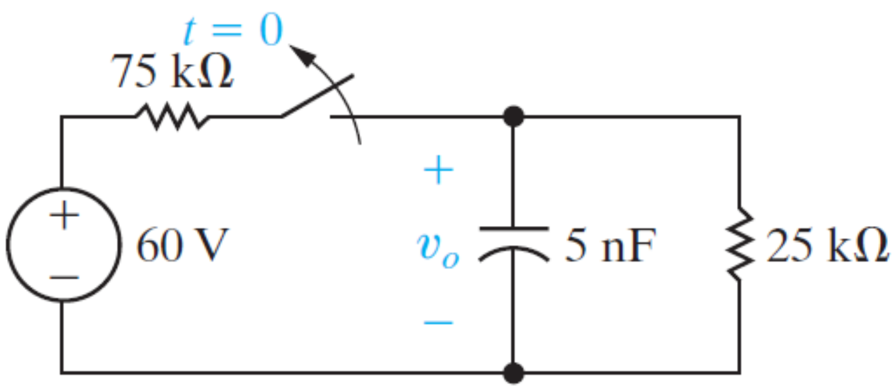
1. Identify the variable of interest for the circuit. For *RC* circuits, it is most convenient to choose the capacitive voltage; for *RL* circuits, it is best to choose the inductive current.

2. Determine the initial value of the variable, which is its value at t_0. Note that if you choose capacitive voltage or inductive current as your variable of interest, it is not necessary to distinguish between $t = t_0^-$ and $t = t_0^+$. This is because they both are continuous variables. If you choose another variable, you need to remember that its initial value is defined at $t = t_0^+$.

3. Calculate the final value of the variable as $t \to \infty$.

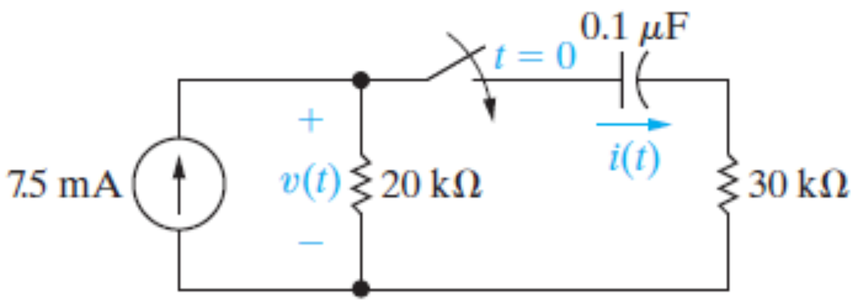
4. Calculate the time constant for the circuit.

Assessment Problems 7.7



The switch in the circuit has been closed for a long time. At t = 0 the switch opens and stays open. Find the expression for $v_o(t)$ for $t \geq 0$.

Assessment Problems 7.8



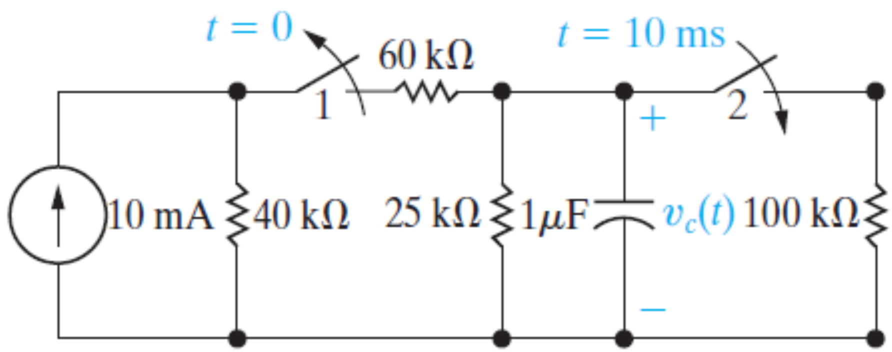
The switch in the circuit has been open for a long time. The initial charge on the capacitor is zero. At t = 0 the switch is closed. Find the expression for:

a) $i(t)$ for $t \geq 0^{+}$ and
b) $v(t)$ for $t \geq 0^{+}$.

7.5 Sequential Switching

- **Definition**: switching occurs more than once in a circuit.
- **Method**: derive the expressions for $v(t)$ and $i(t)$ for a given position of the switch or switches and then use these solutions to determine the initial conditions for the next position of the switch or switches.
- Note: anything but inductive currents and capacitive voltages can change instantaneously at the time of switching.

Assessment Problems 7.9



Switch 1 has been closed and switch 2 has been open for a long time. At t = 0, switch 1 is opened. Then, 10 ms later, switch 2 is closed. Find:

a) $v_C(t)$ for $0 \leq t \leq 10$ ms,

b) $v_C(t)$ for $t \geq 10$ ms,

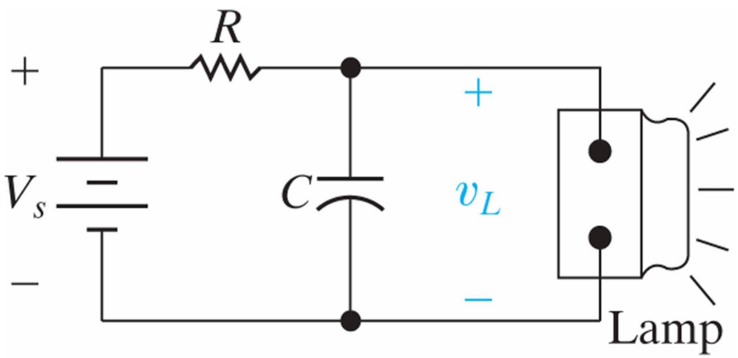
c) the total energy dissipated in the 25 kΩ resistor,

d) the total energy dissipated in the 100 kΩ resistor.

7.6 Unbounded Response

- Under some conditions the circuit might grow, rather than decay, exponentially.
- This is known as unbounded response.
- In such circuits, the Thévenin equivalent resistance is negative.
- Generates a negative time constant.
- Hence, voltages and currents increase without limit.
- This ultimately leads to a breakdown of the circuit.

Practical Perspective - A Flashing Light Circuit



The lamp in this circuit starts to conduct whenever the lamp voltage reaches a value $V_{\max}$. During the time the lamp conducts, it can be modeled as a resistor whose resistance is R_L. The lamp will continue to conduct until the lamp voltage drops to the value $V_{\min}$. When the lamp is not conducting, it behaves as an open circuit.