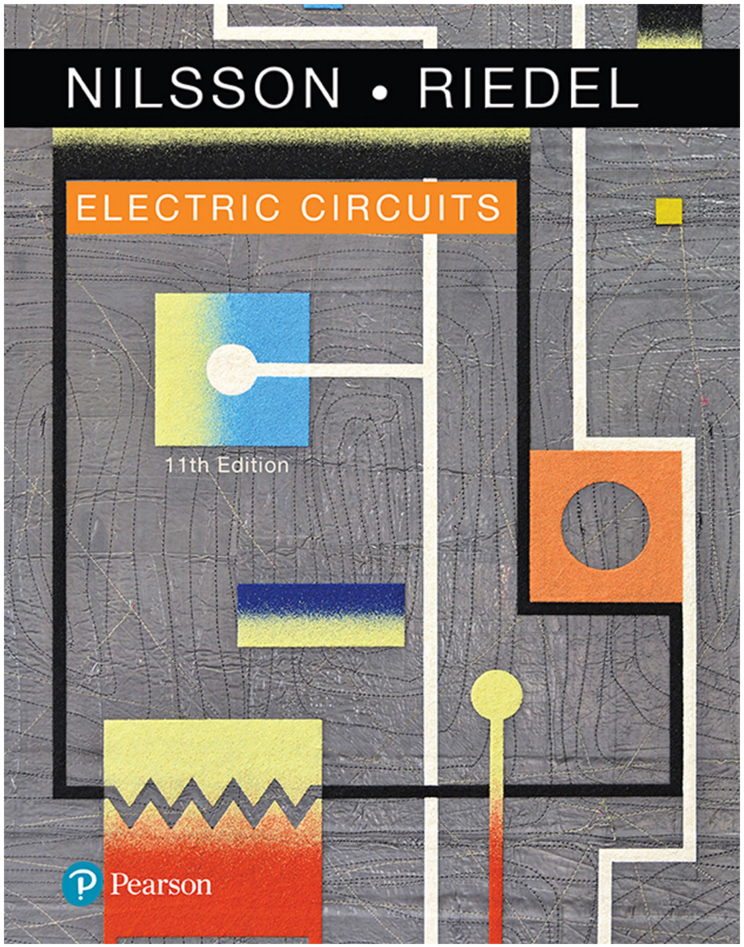
Electric Circuits

Eleventh Edition



Chapter 9

Sinusoidal Steady-State Analysis

Practical Perspective - A Household Distribution Circuit

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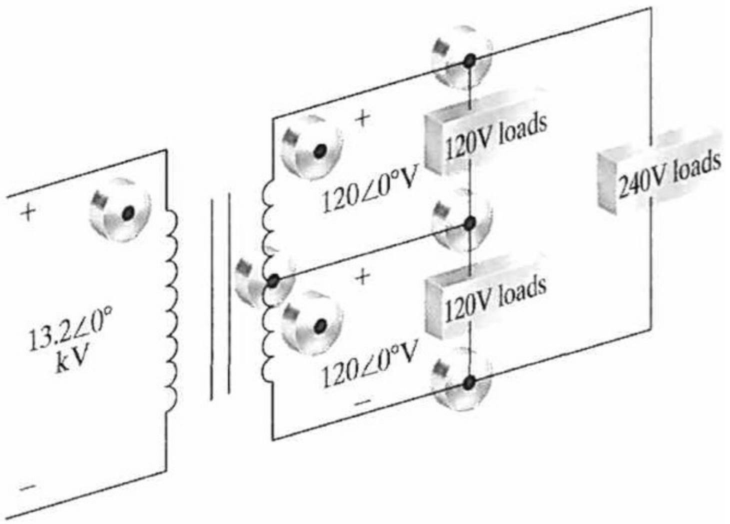
Why Sinusoidal Source?

- The generation, transmission, distribution, and consumption of electrical energy occur under essentially sinusoidal steady-state conditions.
- An understanding of sinusoidal behavior makes it possible to predict the behavior of circuits with nonsinusoidal sources.
- Steady-state sinusoidal behavior often simplifies the design of electrical systems.

A Household Distribution Circuit



Learning Objectives

- Sinusoidal Source & Response
- The Phasor
- Passive Circuits Elements in the Frequency Domain
- Kirchoff’s Laws (Frequency Domain)
- Circuit Simplifications (Frequency Domain)
- Source Transformations (Frequency Domain)
- Node-Voltage & Mesh Current Methods (Freq. Domain)
- Transformers
- Phasor Diagrams

9.1 Sinusoidal Source

- We can generate emf's using rotating coils.
- This is the basis of power generation for the transmission of electricity and communications devices.
- Devices that generate the power are typically sinusoidal, time-varying sources - Alternating Current (ac) sources.
- **A sinusoidal voltage/current source** (independent or dependent) produces a voltage/current that varies sinusoidally with time.

$$\boxed{v(t) = V_m \cos(\omega t + \phi)}$$

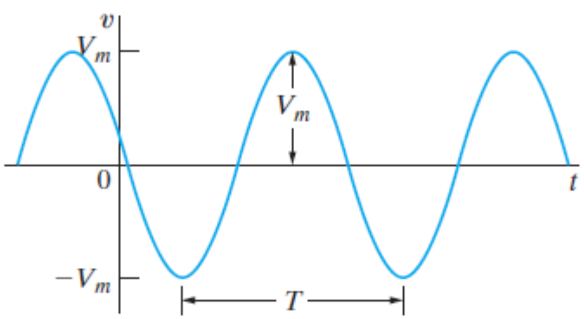


Figure 9.1: A sinusoidal voltage.

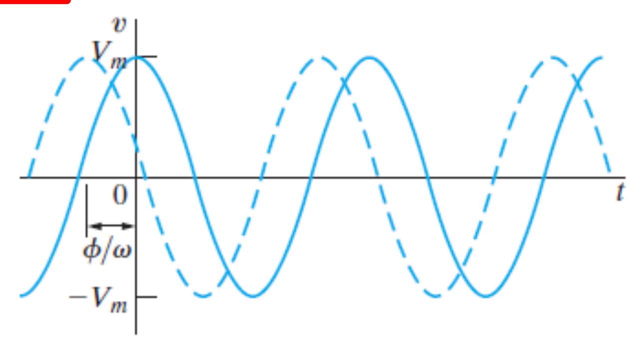


Figure 9.2: The sinusoidal voltage from Fig. 9.1 shifted to the right when $\phi = 0$.

V_m: Maximum amplitude
ω: Angular frequency
ϕ: Phase angle (radians/degrees)

$$\boxed{\omega = 2\pi f = 2\pi/T}$$

f: Frequency

T: Period

Pearson

RMS Value

- Here in the USA electrical outlets provide a voltages of 120 V or 240 V at 60 Hz (In other parts of the world: 220/240 V at 50Hz)
- Because in ac circuits the voltage and current can both be positive and negative, it is convenient to define the root mean square (rms) voltage or current.
- The rms value of a periodic function is defined as the square root of the mean value of the squared function:

$$V_{\text{rms}} = \sqrt{\frac{1}{T}\int_{t_0}^{t_0+T} V_m^2 \cos^2(\omega t + \phi)\, dt}$$

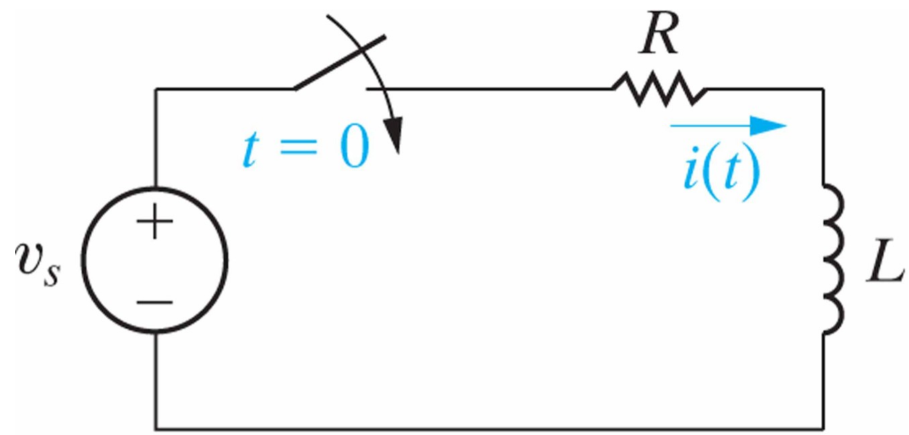
$$\Downarrow$$

$$V_{\text{rms}} = \frac{V_m}{\sqrt{2}} \longrightarrow \text{Power}$$

We can completely describe a specific sinusoidal signal if we know its *frequency*, *phase angle*, and *amplitude* (either the maximum or the rms value).

9.2 Sinusoidal Response

Figure 9.5: An *RL* circuit excited by a sinusoidal voltage source.



$$v_s = V_m \cos(\omega t + \phi)$$

$$L\frac{di}{dt} + Ri = V_m \cos(\omega t + \phi)$$

$$i = \boxed{\frac{-V_m}{\sqrt{R^2 + \omega^2 L^2}} \cos(\phi - \theta) e^{-(R/L)t}} + \boxed{\frac{V_m}{\sqrt{R^2 + \omega^2 L^2}} \cos(\omega t + \phi - \theta)}$$

Transient component

Steady-state component

As "t -> ∞"

Pearson

Steady-State Solution

- The steady-state solution is a sinusoidal function.

- The frequency of the response signal is identical to the frequency of the source signal. *This condition is always true in a linear circuit when the circuit parameters, R, L, and C, are constant.*

- The maximum amplitude of the steady-state response, in general, differs from the maximum amplitude of the source. For the circuit discussed, the maximum amplitude of the response signal is $V_m/\sqrt{(R^2+\omega^2 L^2)}$, and that of the signal source is V_m.

- The phase angle of the response signal, in general, differs from the phase angle of the source. For the circuit being discussed, the phase angle of the current is $\phi - \theta$ and that of the voltage source is ϕ.

Here we will develop an alternative method using vectors/complex numbers.

9.3 The Phasor

The **phasor** is a complex number that carries the amplitude and phase angle information of a sinusoidal function (see Appendix B).

Complex Numbers Review

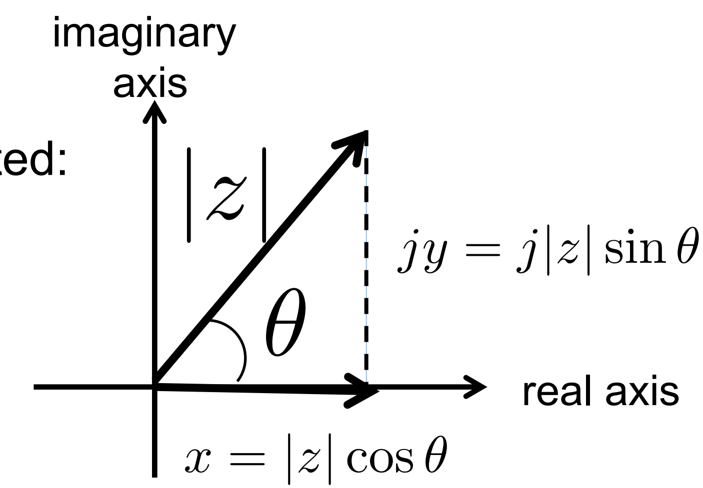
$j = \sqrt{-1}$

Rectangular and polar complex variables are related:

$$\boxed{z = x + jy = |z|e^{j\theta}}$$

real part

imaginary part

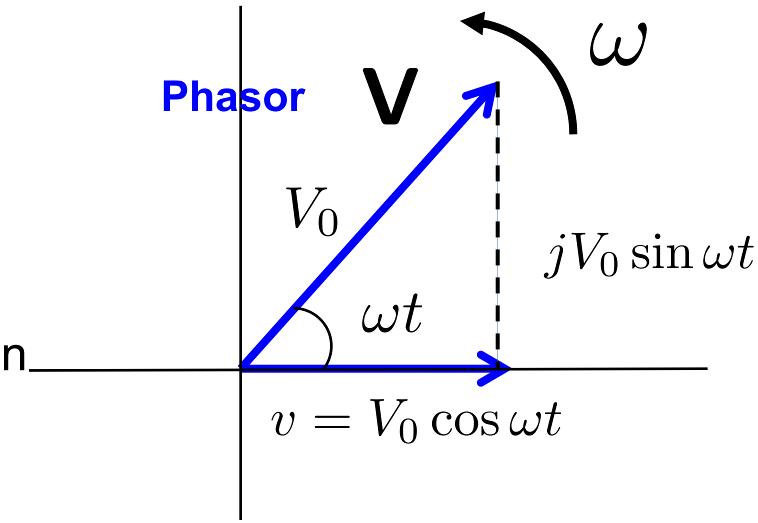


Amplitude: $$|z|^2 = zz^* = (x + jy)(x - jy) = x^2 + y^2$$

Phase: $$\tan\theta = \frac{y}{x}$$

Phasor Notation

- At any instant in time the voltage is described by its amplitude and phase angle.
- Instantaneous voltage given by projection onto real (horizontal) axis.
- We can combine this information into "one" complex number:



Phasor $\mathbf{V} = V_0 \exp(j\omega t) = \underbrace{V_0 \cos \omega t}_{\text{real part}} + j \underbrace{V_0 \sin \omega t}_{\text{imag. part}}$

(use Euler's identity: $e^{j\theta} = \cos\theta + j\sin\theta$)

$$\Longrightarrow \quad v(t) = \mathfrak{Re}(\mathbf{V}) = \mathfrak{Re}[V_0 \exp(j\omega t)] = V_0 \cos \omega t$$

Phasor & Inverse Phasor Transform

Phasor Transform & Angle Notation:

$$\mathbf{V} = V_m e^{j\phi} = \mathcal{P}\{V_m \cos(\omega t + \phi)\} \Rightarrow \mathbf{V} = V_m \cos\phi + jV_m \sin\phi$$

$$1\angle\phi^\circ \equiv 1e^{j\phi}$$

Inverse Phasor Transform:

$$\mathcal{P}^{-1}\{V_m e^{j\phi}\} = \Re\{V_m e^{j\phi} e^{j\omega t}\}$$

$$\mathbf{V} = 100\angle -26^\circ \Rightarrow v = 100\cos(\omega t - 26)$$

The **phasor transform** is useful in circuit analysis because it reduces the task of finding the maximum amplitude and phase angle of the steady-state sinusoidal response to the algebra of complex numbers.

9.4 PASSIVE CIRCUIT ELEMENTS IN THE FREQUENCY DOMAIN

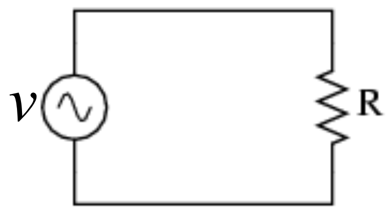
- The transient component vanishes as $t \to \infty$, so *the steady-state component of the solution must also satisfy the differential equation governing the circuit.*

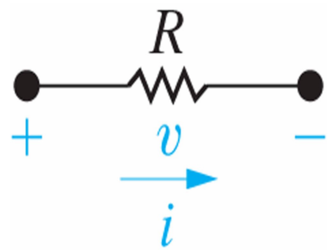
- In a linear circuit driven by sinusoidal sources, the steady-state response also is sinusoidal, and *the frequency of the sinusoidal response is the same as the frequency of the sinusoidal source.*

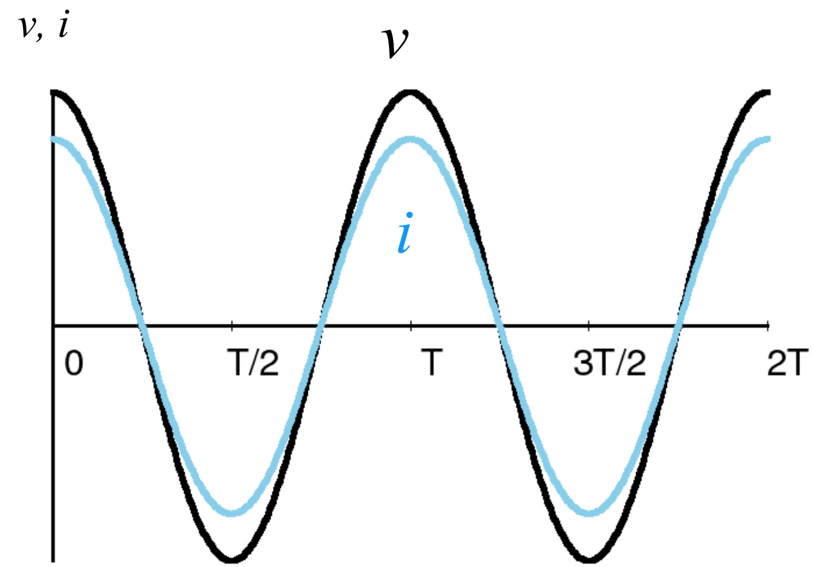
- Using the phasor notation, we can postulate that the steady-state solution is of the form $\mathrm{R}\{Ae^{j\beta}e^{j\omega t}\}$, where A is the maximum amplitude of the response and β is the phase angle of the response.

- When we substitute the postulated steady-state solution into the differential equation, the exponential terms $e^{j\omega t}$ cancel out, leaving the solution for A and β in the domain of complex numbers or frequency domain.

The V-I Relationship for a Resistor







$$v = V_0 \cos \omega t$$

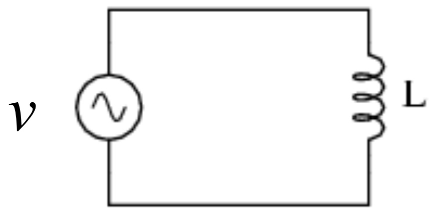
sinusoidal ac source with angular frequency w = 2πf

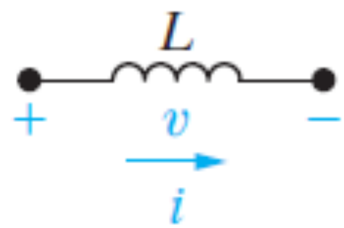
Ohm's Law $$v = iR$$

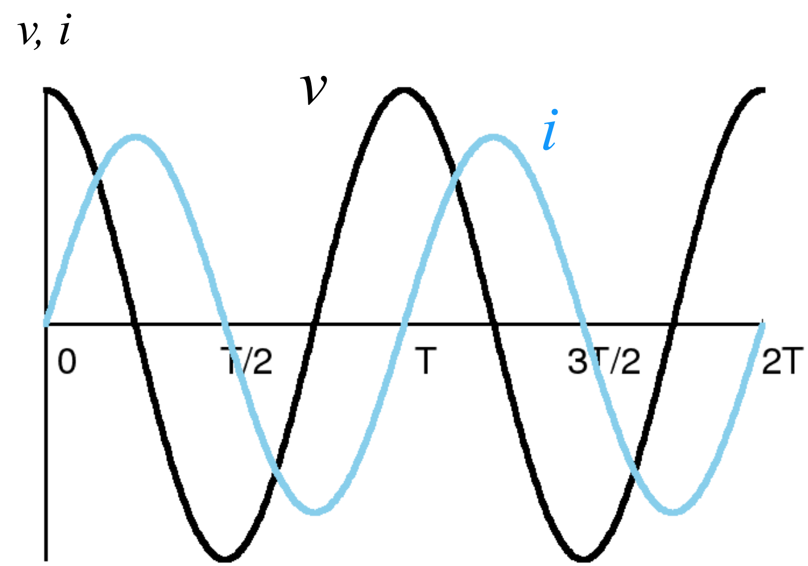
$$i = \frac{v}{R} = \frac{V_0}{R} \cos \omega t = I_0 \cos \omega t$$

current IN phase with voltage

The V-I Relationship for an Inductor







$$v = V_0 \cos \omega t$$

(integrate)

$$v = L\frac{di}{dt} \rightarrow i = \frac{V_0}{\omega L} \sin \omega t$$

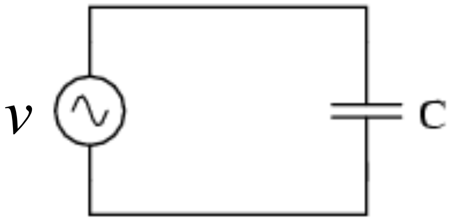
$$i = \frac{V_0}{\omega L} \cos\left(\omega t - \frac{\pi}{2}\right) = \frac{V_0}{X_L} \cos(\omega t - 90^\circ)$$

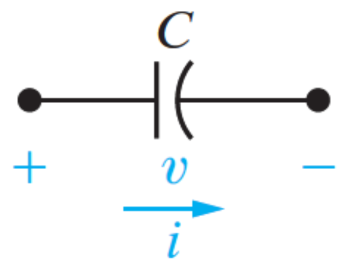
current LAGS voltage by 90°

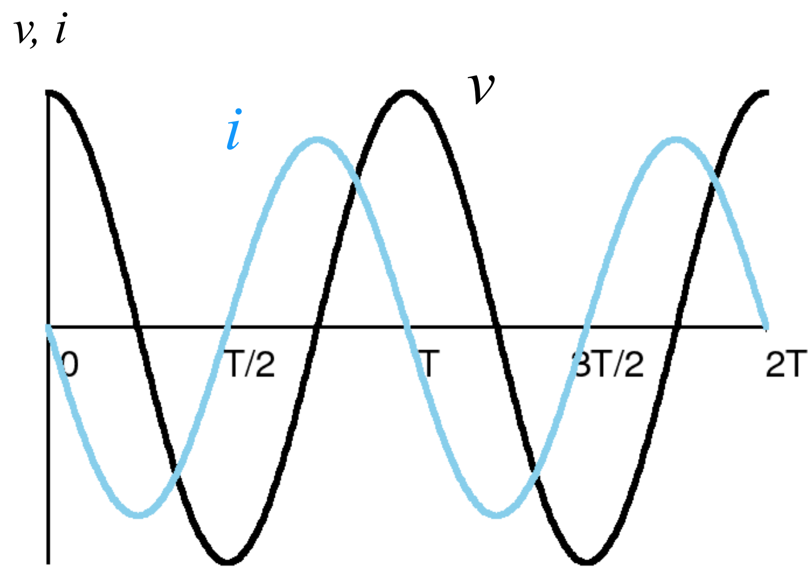
inductive reactance

$$\boxed{\frac{V_0}{I_0} = \omega L = X_L}$$

The V-I Relationship for a Capacitor







$$v = V_0 \cos \omega t$$

(differentiate)

$$v = \frac{q}{C} \rightarrow i = C\frac{dv}{dt} = -\omega C V_0 \sin \omega t$$

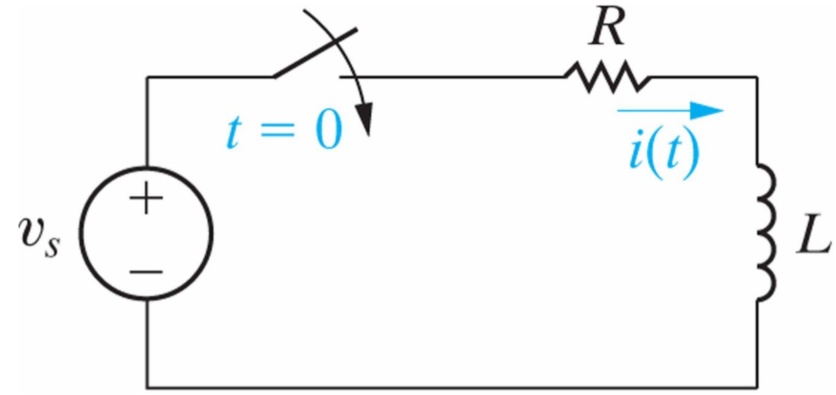
$$i = \omega C V_0 \cos\left(\omega t + \frac{\pi}{2}\right) = \frac{V_0}{X_C} \cos(\omega t + 90^\circ)$$

current LEADS voltage by 90°

capacitive reactance

$$\boxed{\frac{V_0}{I_0} = \frac{1}{\omega C} = X_C}$$

Illustration



$$i_{ss}(t) = \Re\{I_m e^{j\beta} e^{j\omega t}\}$$

$$\Re\{j\omega L I_m e^{j\beta} e^{j\omega t}\} + \Re\{R I_m e^{j\beta} e^{j\omega t}\} = \Re\{V_m e^{j\phi} e^{j\omega t}\}$$

$$\Re\{(j\omega L + R) I_m e^{j\beta} e^{j\omega t}\} = \Re\{V_m e^{j\phi} e^{j\omega t}\}$$

$$(j\omega L + R) I_m e^{j\beta} = V_m e^{j\phi} \quad \Rightarrow \quad I_m e^{j\beta} = \frac{V_m e^{j\phi}}{R + j\omega L}$$

- The phasor transform, along with the inverse phasor transform, allows you to go back and forth between the time domain and the frequency domain. Therefore, when you obtain a solution, you are either in the time domain or the frequency domain. You cannot be in both domains simultaneously. Any solution that contains a mixture of time and phasor domain nomenclature is nonsensical.

- The phasor transform is also useful in circuit analysis because it applies directly to the sum of sinusoidal function, if

$$v = v_1 + v_2 + \cdots + v_n$$

When all the voltages on the right-hand side are sinudoidal voltages of the same frequency, then

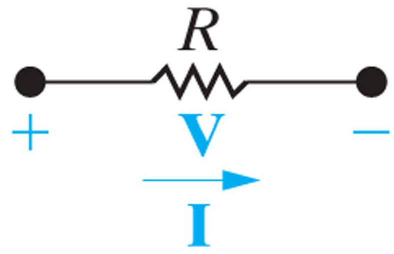
$$\mathbf{V} = \mathbf{V}_1 + \mathbf{V}_2 + \cdots + \mathbf{V}_n$$

9.12 Phasor Diagrams

- We can now apply are phasor concept to solve problems in the frequency domain.
- We do this by plotting our phasors in the complex plane.
- The benefit comes from the fact that we can simply add phasors.
- Before we do this we have to introduce the concept of impedance.
- We've already seen this in our development of inductive and capacitive reactance.
- Impedance follows from the V-I relations we just derived.

Resistance Phasor

$$\boxed{i = \frac{V_0}{R}\cos\omega t}$$

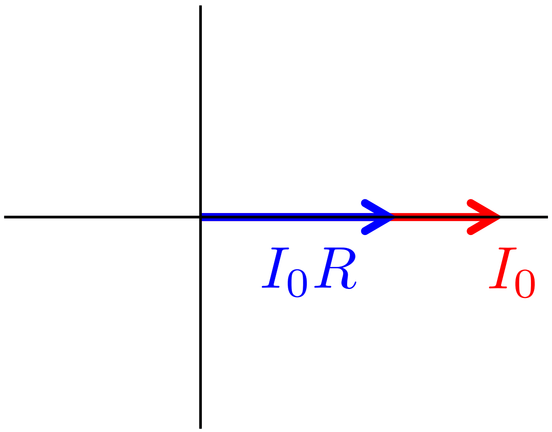


$$\mathbf{I} = \frac{V_0}{R}\exp(j\omega t)$$

$$\mathbf{V} = V_0\exp(j\omega t)$$

$$\mathbf{I} = \mathbf{V}/R$$

$$\boxed{\mathbf{V} = R\mathbf{I}}$$



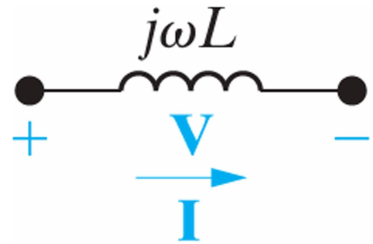
More generally

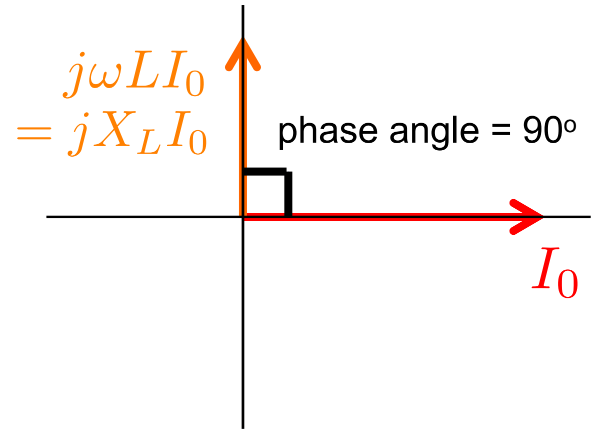
$$v = R[I_m\cos(\omega t + \theta_i)]$$

$$\mathbf{V} = RI_m e^{j\theta_i} = RI_m\underline{/\theta_i}$$

Inductive Phasor

$$i = \frac{V_0}{\omega L}\cos(\omega t - 90^\circ)$$





$$\mathbf{I} = \frac{V_0}{\omega L}\exp\left[j\left(\omega t - \frac{\pi}{2}\right)\right]$$

$$\mathbf{I} = \frac{V_0}{\omega L}\exp(j\omega t)\exp\left(-j\frac{\pi}{2}\right) = \frac{\mathbf{V}}{\omega L}\exp\left(-j\frac{\pi}{2}\right)$$

$$\mathbf{V} = \omega L \mathbf{I}\underbrace{\exp\left(j\frac{\pi}{2}\right)}_{=j}$$

$$\boxed{\mathbf{V} = jX_L \mathbf{I}}$$

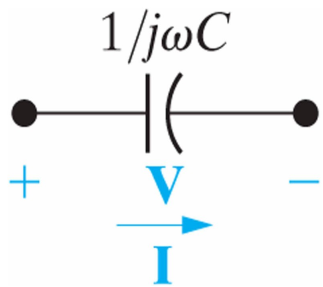
More generally

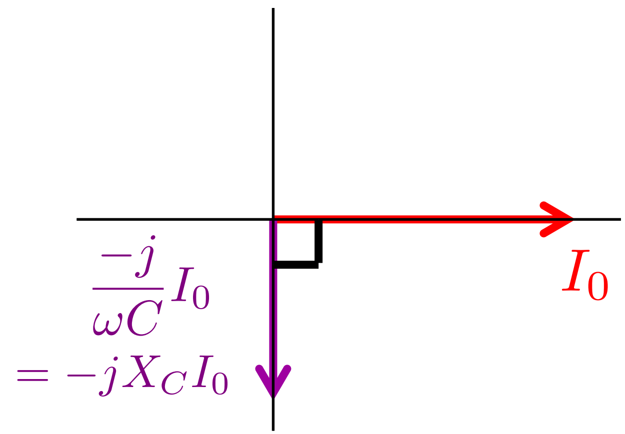
$$v = L\frac{di}{dt} = -\omega L I_m \sin(\omega t + \theta_i)$$

$$\mathbf{V} = (\omega L \underline{/90^\circ}) I_m \underline{/\theta_i} = \omega L I_m \underline{/(\theta_i + 90)^\circ}$$

Capacitive Phasor

$$i = \omega C V_0 \cos(\omega t + 90^\circ)$$





$$\mathbf{I} = \omega C V_0 \exp\left[j\left(\omega t + \frac{\pi}{2}\right)\right]$$

$$\mathbf{I} = \omega C \mathbf{V} \underbrace{\exp\left(j\frac{\pi}{2}\right)}_{=j}$$

$$\boxed{\mathbf{V} = -jX_C \mathbf{I}}$$

More generally

$$i = C\frac{dv}{dt} \qquad v = V_m \cos(\omega t + \theta_v)$$

$$\mathbf{V} = \frac{1}{\omega C} \underline{/-90^\circ} I_m \underline{/\theta_i^\circ} = \frac{I_m}{\omega C} \underline{/(\theta_i - 90)^\circ}$$

Impedance and Reactance

Resistance	Inductance	Capacitance
$\mathbf{V} = R\mathbf{I}$	$\mathbf{V} = jX_L\mathbf{I}$	$\mathbf{V} = -jX_C\mathbf{I}$

All voltage-current relations look like Ohm's Law: $\mathbf{V} = Z\mathbf{I}$

(complex) **Impedance** Z (units of ohms)

CIVIL

Circuit Element	Reactance X	Impedance Z	Phase θ	Angle Notation
Resistor	-	R	in phase	$\mathbf{V} = RI_0\,\underline{/0^\circ}$
Capacitor	1/ωC	-j/ωC	CIV Leads +90°	$\mathbf{V} = X_C I_0\underline{/-90^\circ}$
Inductor	ωL	jωL	VIL Lags -90°	$\mathbf{V} = X_L I_0\,\underline{/90^\circ}$

Combining Impedances $\mathbf{V} = Z\mathbf{I}$

- Impedance in ac circuits behaves like resistance in dc circuits.
- Use similar rules for combining impedances.
- Combining impedances in series:

$$\mathbf{V} = \mathbf{V}_R + \mathbf{V}_L + \mathbf{V}_C \qquad \text{(LRC circuit)}$$

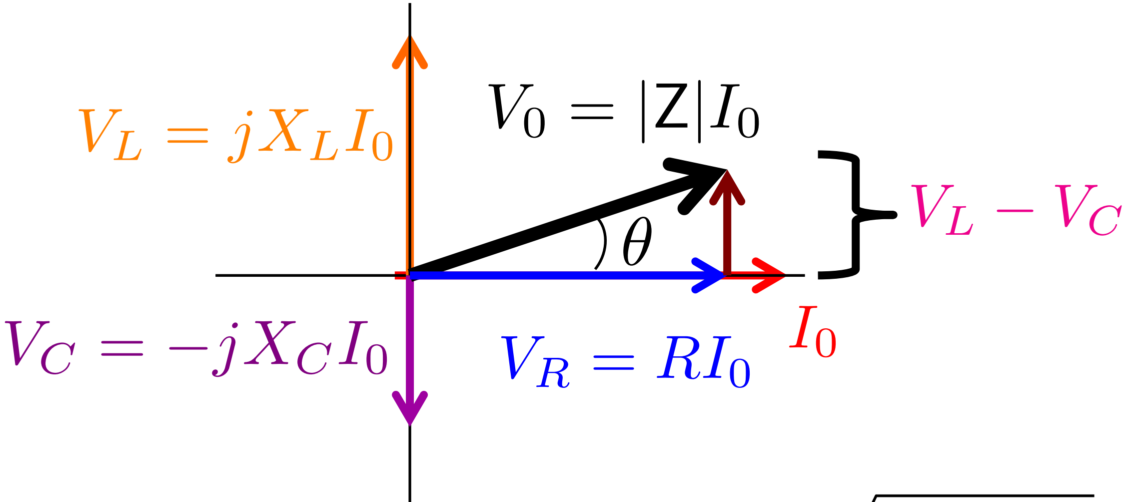
$$= Z_R\mathbf{I} + Z_L\mathbf{I} + Z_C\mathbf{I} = (Z_R + Z_L + Z_C)\mathbf{I} = Z\mathbf{I}$$

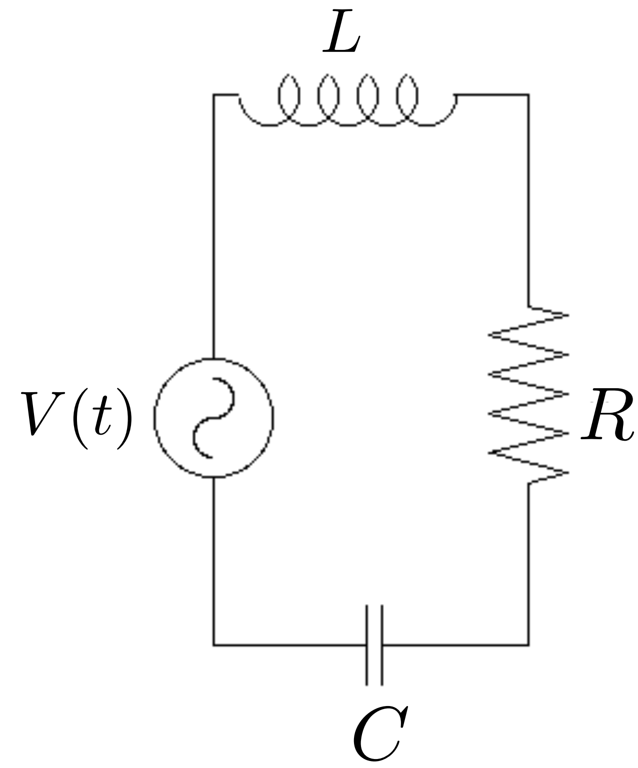
But remember impedance is a complex number...

Analysis of the LRC series circuit using phasors

$$\mathbf{V} = \mathbf{V}_R + \mathbf{V}_L + \mathbf{V}_C$$

Example: $X_L > X_C$





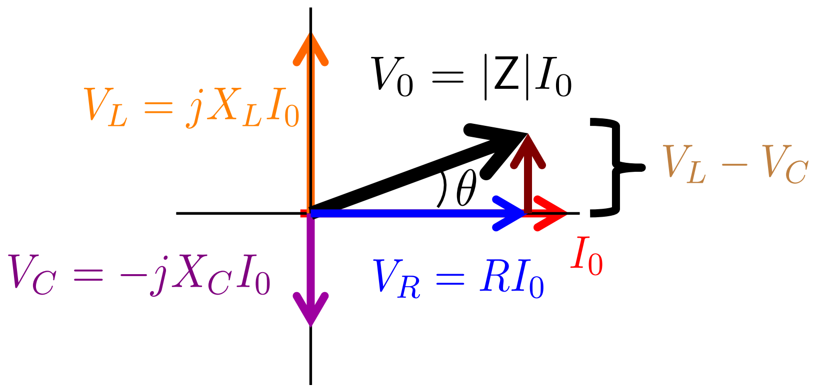
Recall: $|z| = \sqrt{x^2 + y^2}$ $\qquad \theta = \tan^{-1}\left(\frac{y}{x}\right)$

$$x = V_R \qquad y = V_L - V_C$$

Analysis of the LRC series circuit using phasors

$$V_0 = \sqrt{V_R^2 + (V_L - V_C)^2}$$
$$= I_0\sqrt{R^2 + (X_L - X_C)^2}$$

Example: $X_L > X_C$



Impedance

$$V_0 = |Z| I_0 \Rightarrow |Z| = \sqrt{R^2 + (\omega L - 1/\omega C)^2}$$

impedance of LRC circuit

Phase Angle:
$$\tan\theta = \frac{V_L - V_C}{V_R} = I_0 \frac{(X_L - X_C)}{I_0 R} = \frac{X_L - X_C}{R}$$

$$\Rightarrow \theta = \tan^{-1}\left(\frac{\omega L - 1/\omega C}{R}\right)$$

phase angle of LRC circuit

Angle Notation: $\mathbf{V} = Z I_0 \angle \theta^\circ$

EXAMPLE: RESONANCE

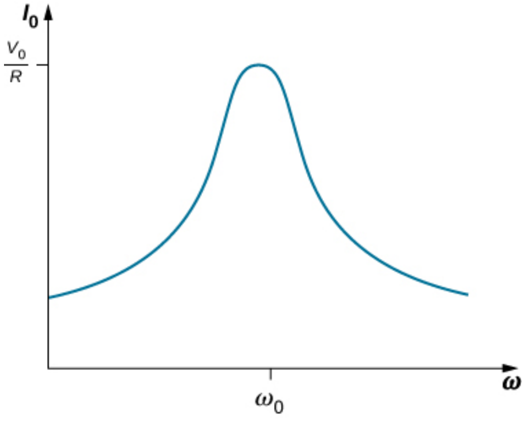
RLC CIRCUIT

- Notice from the expression for the impedance for the RLC circuit, the current can become very large for certain values of the inductance and capacitance:

$$I = \frac{V}{Z} = \frac{V}{\sqrt{R^2 + (\omega L - 1/\omega C)^2}}$$

- In particular, when $X_L = X_C$
- This defines the resonant, or center, frequency ω_0:

$$\boxed{\omega_0 = \sqrt{\frac{1}{LC}}}$$



- RLC circuits were used in devices such as radios to pick out specific radio channels.

9.5 Kirchhoff's law in the Frequency Domain

$$v_1 + v_2 + \cdots + v_n = 0$$

$$V_{m_1} \cos(\omega t + \theta_1) + V_{m_2} \cos(\omega t + \theta_2) + \cdots + V_{m_n} \cos(\omega t + \theta_n) = 0$$

$$\Re\{V_{m_1} e^{j\theta_1} e^{j\omega t}\} + \Re\{V_{m_2} e^{j\theta_2} e^{j\omega t}\} + \cdots + \Re\{V_{m_n} e^{j\theta_n} e^{j\omega t}\}$$

$$\Re\{(V_{m_1} e^{j\theta_1} + V_{m_2} e^{j\theta_2} + \cdots + V_{m_n} e^{j\theta_n}) e^{j\omega t}\} = 0$$

$$\Re\{(\mathbf{V}_1 + \mathbf{V}_2 + \cdots + \mathbf{V}_n) e^{j\omega t}\} = 0$$

KVL $\mathbf{V}_1 + \mathbf{V}_2 + \cdots + \mathbf{V}_n = 0$

KCL $\mathbf{I}_1 + \mathbf{I}_2 + \cdots + \mathbf{I}_n = 0$

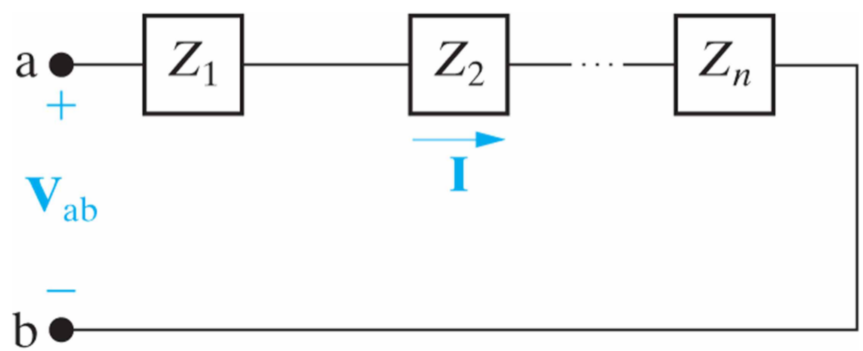
9.6 Series, Parallel, and Delta-Wye Simplification

The rules for combining impedances in series or parallel and for making delta-to-wye transformations *are the same as those for resistors*. The only difference is that *combining impedances involves the algebraic manipulation of complex numbers.*

Impedances in series:

$$\mathbf{V}_{ab} = Z_1\mathbf{I} + Z_2\mathbf{I} + \cdots + Z_n\mathbf{I}$$
$$= (Z_1 + Z_2 + \cdots + Z_n)\mathbf{I}$$

$$Z_{ab} = \frac{\mathbf{V}_{ab}}{\mathbf{I}} = Z_1 + Z_2 + \cdots + Z_n$$

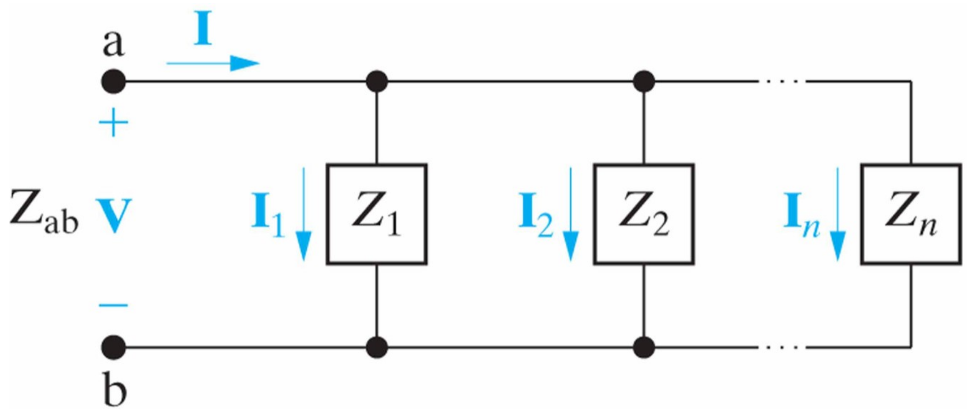


Impedances in parallel:

$$\mathbf{I} = \mathbf{I}_1 + \mathbf{I}_2 + \cdots + \mathbf{I}_n$$

$$\frac{\mathbf{V}}{Z_{ab}} = \frac{\mathbf{V}}{Z_1} + \frac{\mathbf{V}}{Z_2} + \cdots + \frac{\mathbf{V}}{Z_n}$$

$$\frac{1}{Z_{ab}} = \frac{1}{Z_1} + \frac{1}{Z_2} + \cdots + \frac{1}{Z_n} \xrightarrow{\text{Two impedance}} Z_{ab} = \frac{Z_1 Z_2}{Z_1 + Z_2}$$



susceptance

$$Y = \frac{1}{Z} = G + jB \text{ (siemens)}$$

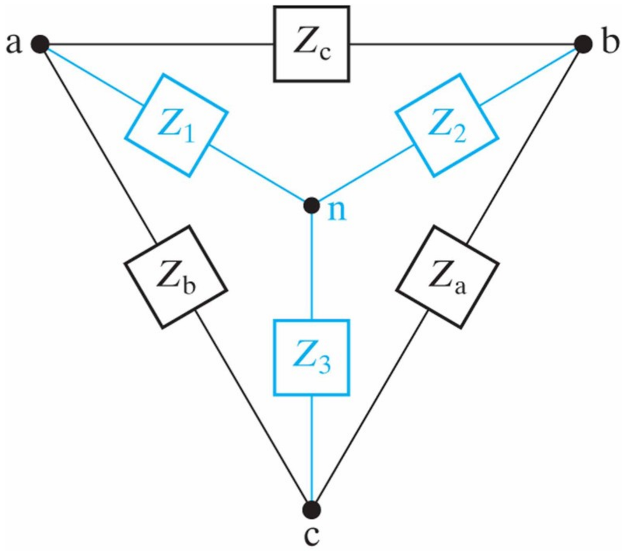
conductance

$$Y_{ab} = Y_1 + Y_2 + \cdots + Y_n$$

admittance

Circuit Element	Admittance (Y)	Susceptance
Resistor	G (conductance)	—
Inductor	$j(-1/\omega L)$	$-1/\omega L$
Capacitor	$j\omega C$	ωC

Delta-To-Wye Transformations



$$Z_1 = \frac{Z_b Z_c}{Z_a + Z_b + Z_c}$$

$$Z_2 = \frac{Z_c Z_a}{Z_a + Z_b + Z_c}$$

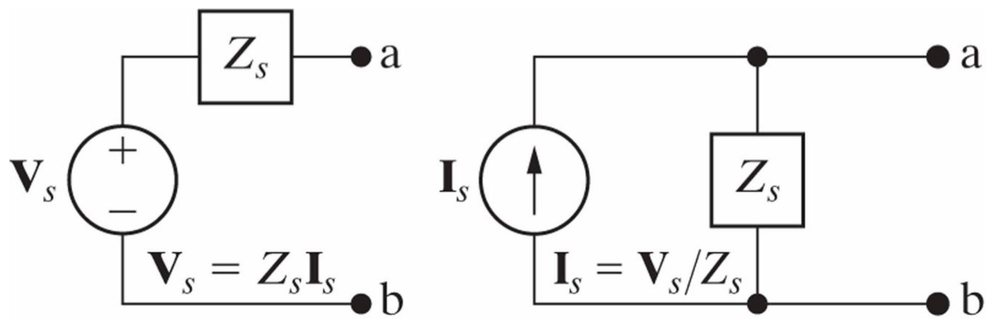
$$Z_3 = \frac{Z_a Z_b}{Z_a + Z_b + Z_c}$$

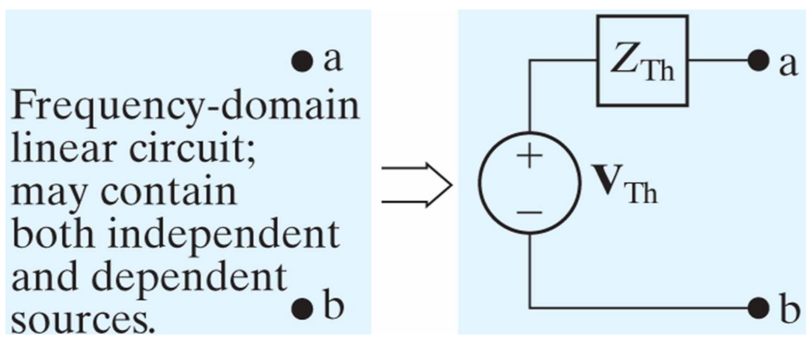
$$Z_a = \frac{Z_1 Z_2 + Z_2 Z_3 + Z_3 Z_1}{Z_1}$$

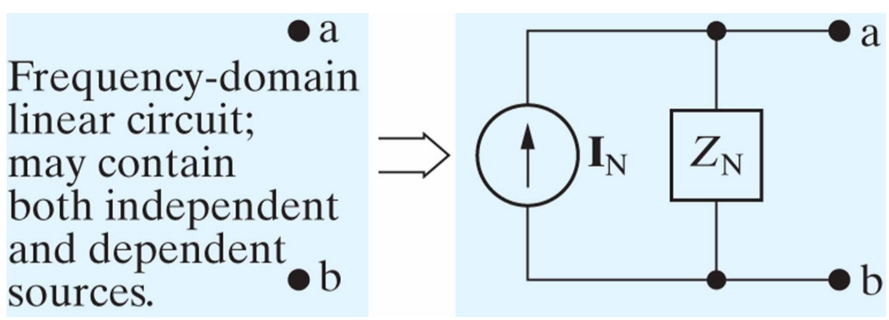
$$Z_b = \frac{Z_1 Z_2 + Z_2 Z_3 + Z_3 Z_1}{Z_2}$$

$$Z_c = \frac{Z_1 Z_2 + Z_2 Z_3 + Z_3 Z_1}{Z_3}$$

9.7 Source Transformations and Thévenin-Norton Equivalent Circuits







9.8/9 Node-Voltage & Mesh-Current Methods

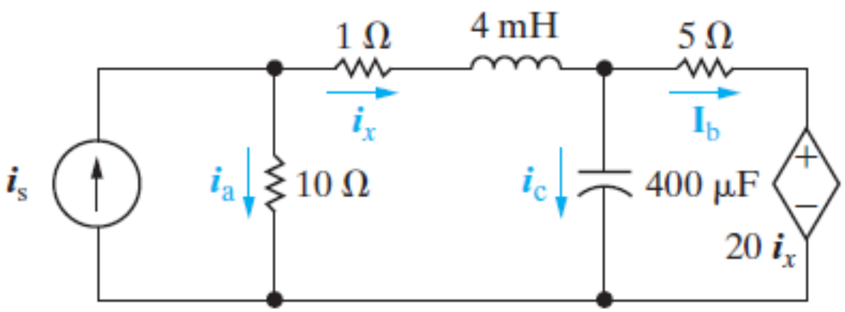


Figure 9.36: The circuit for Example 9.13.

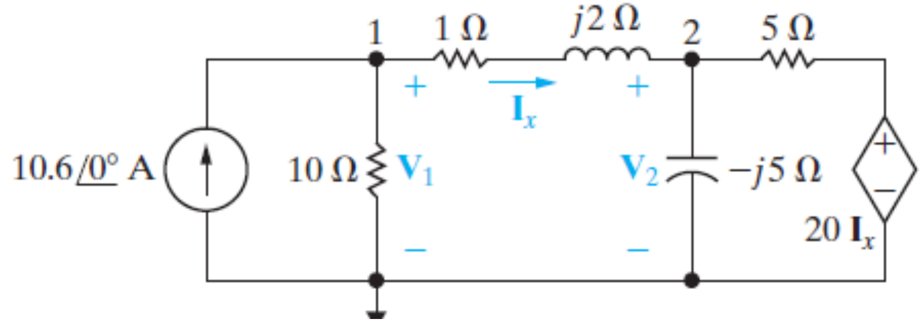


Figure 9.38: The circuit shown in Fig. 9.37, with the node voltages defined.

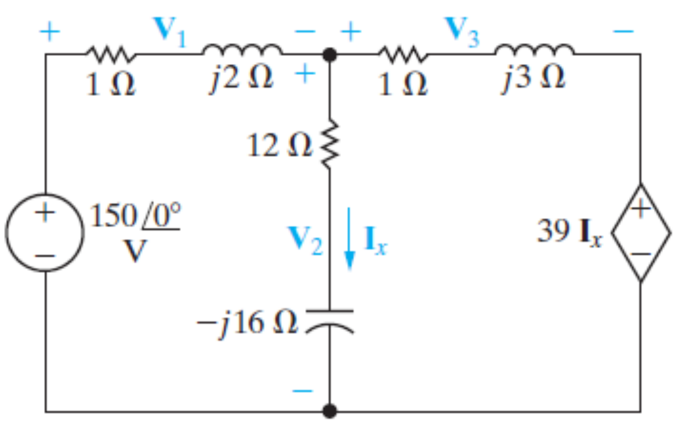


Figure 9.39: The circuit for Example 9.14.

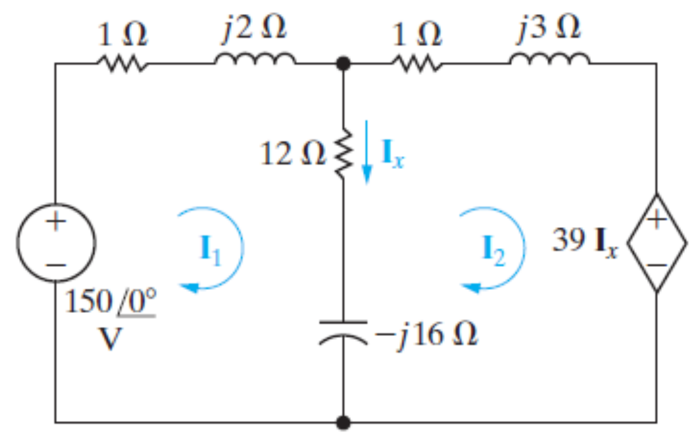
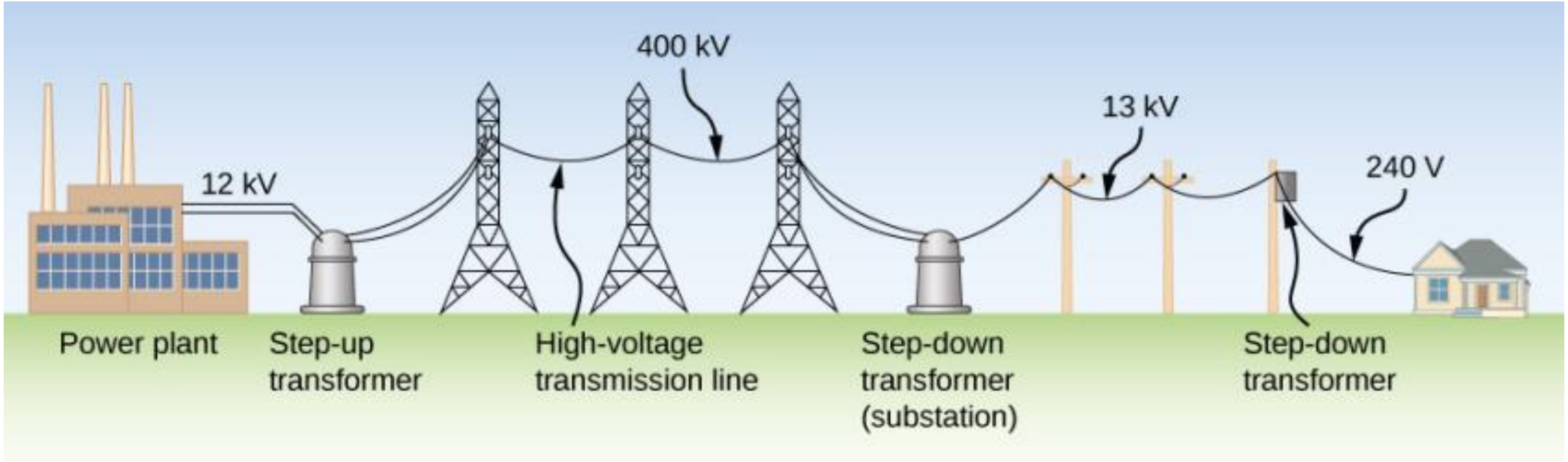


Figure 9.40: Mesh currents used to solve the circuit shown in Fig. 9.39.

9.10 The Transformer



***A transformer* is a device that is based on magnetic coupling.**

- Transformers are used in both communication and power circuits.
- **linear transformer:** is found primarily in communication circuits
- **Ideal transformer:** is used to model the ferromagnetic transformer found in power systems.

TRANSFORMERS

- A ***transformer*** is a device for increasing or decreasing an ac voltage.
- The ac current in the primary coil creates a changing magnetic field that is fed by the iron core through the primary and secondary coils.
- The resulting changing flux generates an induced emf in both the primary (self-inductance) and secondary coils (mutual-inductance).

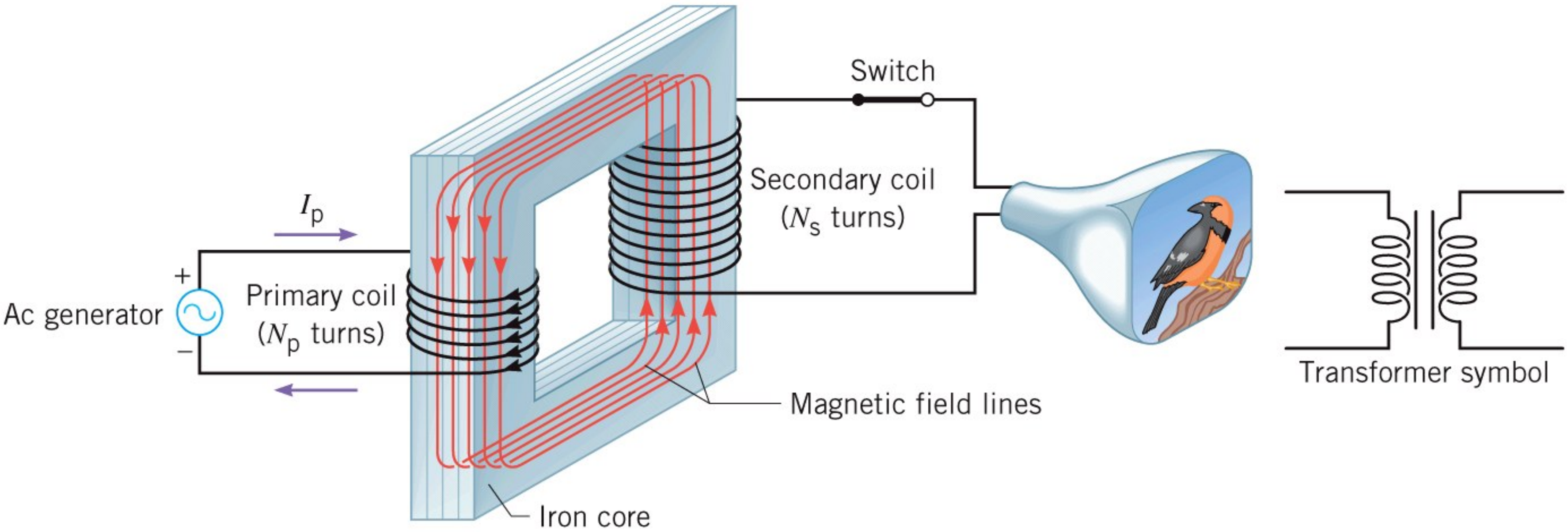


Figure 22.29

TRANSFORMER EQUATION

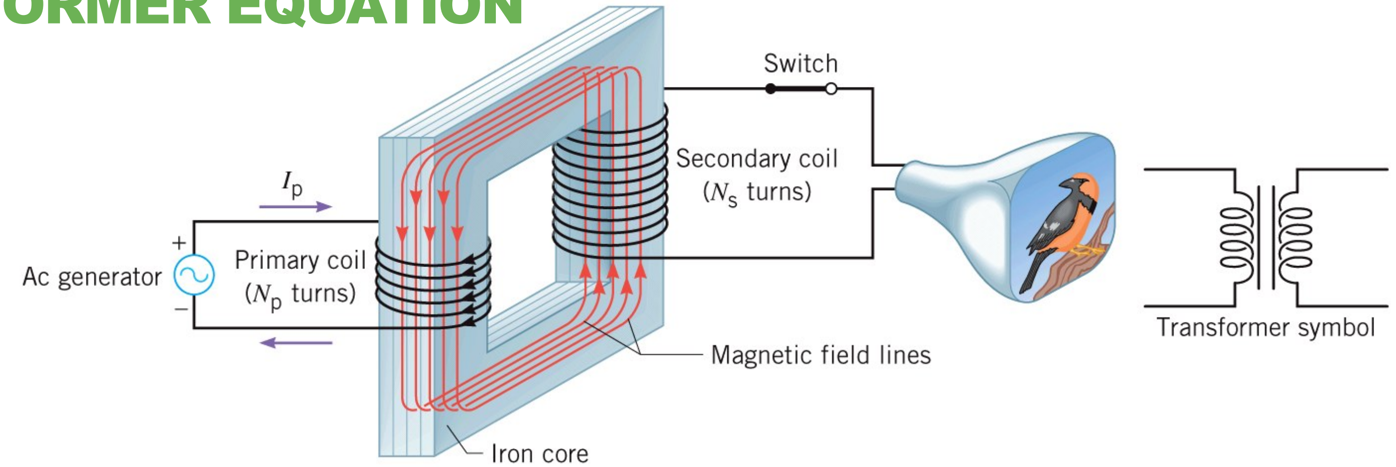


Figure 22.29

- Applying Faraday's law to both coils: $\mathcal{E}_s = -N_s \dfrac{\Delta\Phi}{\Delta t}$ $\mathcal{E}_p = -N_p \dfrac{\Delta\Phi}{\Delta t}$
- Now using the fact that the flux is (approximately) the same in both coils: $\boxed{\dfrac{\mathcal{E}_s}{\mathcal{E}_p} = \dfrac{N_s}{N_p}}$
- Finally, if the resistances in the coils are negligible, so the emfs are nearly equal to the terminal voltages, V: $\boxed{\dfrac{V_s}{V_p} = \dfrac{N_s}{N_p}}$ ***Transformer equation***

$N_s > N_p$: larger secondary voltage ⇔ step-up transformer. Step-down transformer is opposite.

POWER

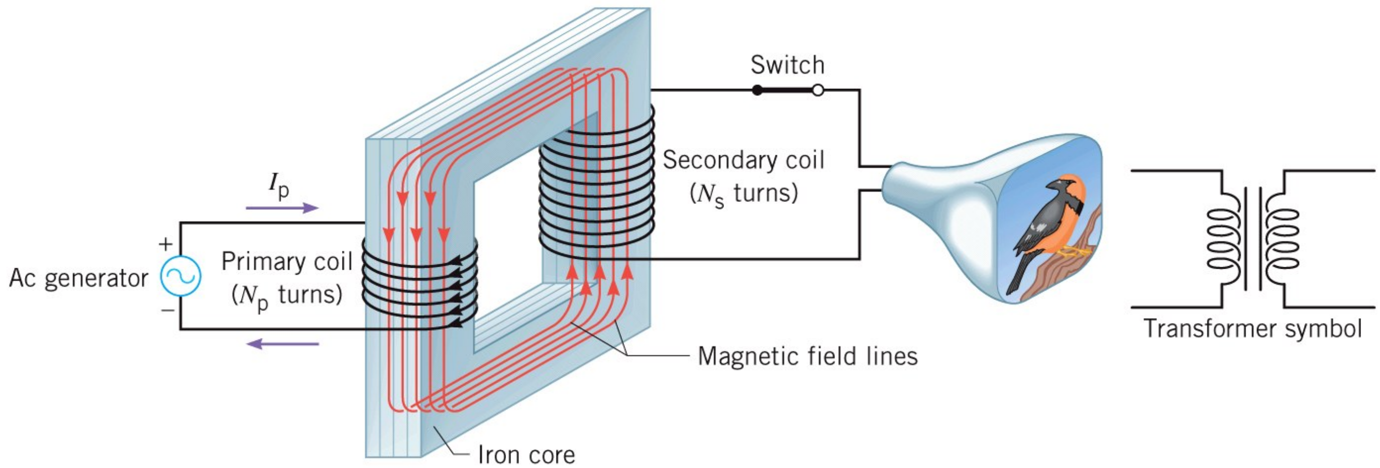


Figure 22.29

The average power delivered to each coil, P =IV, is (approximately) the same (conservation of energy):

$$\frac{I_s}{I_p} = \frac{V_p}{V_s} = \frac{N_p}{N_s}$$

Step-up transformer steps up the voltage & steps down the current.

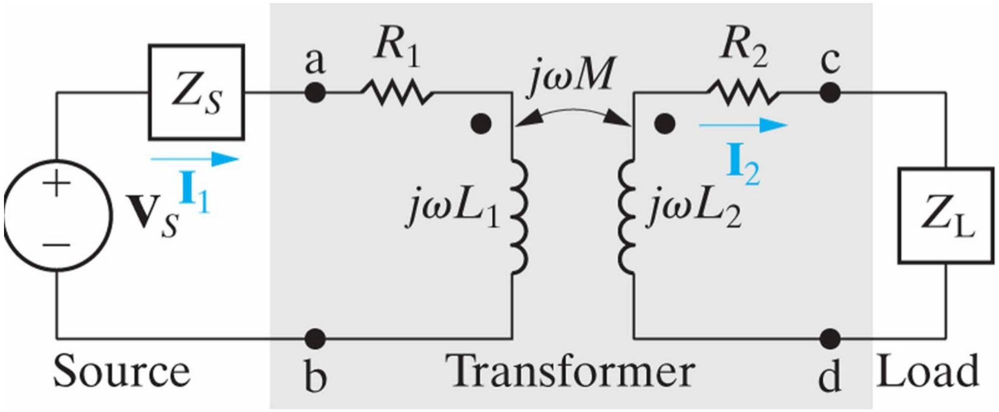
Step-down transformer steps down the voltage & steps up the current.

The Analysis of a Linear Transformer Circuit

A simple **transformer** is formed *when two coils are wound on a single core to ensure magnetic coupling*.

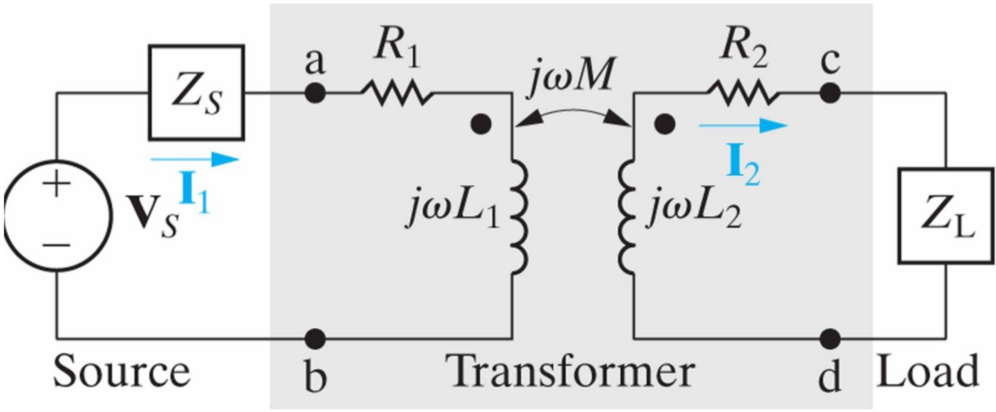
- **Primary winding:** the transformer winding connected to the source;
- **Secondary winding:** the winding connected to the load as the.

R_1 = the resistance of the primary winding
R_2 = the resistance of the secondary winding
L_1 = the self-inductance of the primary winding
L_2 = the self-inductance of the secondary winding
M = the mutual inductance



$$\mathbf{V}_s = (Z_s + R_1 + j\omega L_1)\mathbf{I}_1 - j\omega M\mathbf{I}_2$$

$$0 = -j\omega M\mathbf{I}_1 + (R_2 + j\omega L_2 + Z_L)\mathbf{I}_2$$



$$\boxed{\begin{aligned} Z_{11} &= Z_s + R_1 + j\omega L_1 \\ Z_{22} &= R_2 + j\omega L_2 + Z_L \end{aligned}}$$

$$\mathbf{I}_1 = \frac{Z_{22}}{Z_{11}Z_{22} + \omega^2 M^2}\mathbf{V}_s \qquad \mathbf{I}_2 = \frac{j\omega M}{Z_{11}Z_{22} + \omega^2 M^2}\mathbf{V}_s = \frac{j\omega M}{Z_{22}}\mathbf{I}_1$$

$$\frac{\mathbf{V}_s}{\mathbf{I}_1} = Z_{\text{int}} = \frac{Z_{11}Z_{22} + \omega^2 M^2}{Z_{22}} = Z_{11} + \frac{\omega^2 M^2}{Z_{22}}$$

$$Z_{ab} = Z_{11} + \frac{\omega^2 M^2}{Z_{22}} - Z_s = R_1 + j\omega L_1 + \frac{\omega^2 M^2}{(R_2 + j\omega L_2 + Z_L)}$$

The impedance Z_{ab} is independent of the magnetic polarity of the transformer!

Reflected Impedance

$$Z_{ab} = Z_{11} + \frac{\omega^2 M^2}{Z_{22}} - Z_s = R_1 + j\omega L_1 + \boxed{\frac{\omega^2 M^2}{(R_2 + j\omega L_2 + Z_L)}}$$

Reflected impedance Z_r

$$Z_L = R_L + jX_L$$

$$Z_r = \frac{\omega^2 M^2}{R_2 + R_L + j(\omega L_2 + X_L)} = \frac{\omega^2 M^2 [(R_2 + R_L) - j(\omega L_2 + X_L)]}{(R_2 + R_L)^2 + (\omega L_2 + X_L)^2}$$

$$= \frac{\omega^2 M^2}{|Z_{22}|^2} [(R_2 + R_L) - j(\omega L_2 + X_L)]$$

9.11 The Ideal Transformer

An **ideal transformer** consists of two magnetically coupled coils having N_1 and N_2 turns, respectively, and exhibiting these three properties:

1. The coefficient of coupling is unity ($k = 1$).
2. The self-inductance of each coil is infinite ($L_1 = L_2 = \infty$).
3. The coil losses, due to parasitic resistance, are negligible.

Exploring Limiting Values

$$Z_{ab} = Z_{11} + \frac{\omega^2 M^2}{Z_{22}} - Z_s = R_1 + j\omega L_1 + \frac{\omega^2 M^2}{(R_2 + j\omega L_2 + Z_L)}$$

$$\boxed{Z_{22} = R_2 + R_L + j(\omega L_2 + X_L) = R_{22} + jX_{22}}$$

$$Z_{ab} = R_1 + \frac{\omega^2 M^2 R_{22}}{R_{22}^2 + X_{22}^2} + j\left(\omega L_1 - \frac{\omega^2 M^2 X_{22}}{R_{22}^2 + X_{22}^2}\right) = R_{ab} + j\boxed{X_{ab}}$$

$$M^2 = L_1 L_2$$

$$X_{ab} = \omega L_1 - \frac{(\omega L_1)(\omega L_2) X_{22}}{R_{22}^2 + X_{22}^2} = \omega L_1\left(1 - \frac{\omega L_2 X_{22}}{R_{22}^2 + X_{22}^2}\right)$$

$$X_{22} = \omega L_2 + X_L$$

$$X_{ab} = \omega L_1\left(\frac{R_{22}^2 + \omega L_2 X_L + X_L^2}{R_{22}^2 + X_{22}^2}\right)$$

Factoring ωL_2 out of the numerator and denominator yields:

$$X_{ab} = \frac{L_1}{L_2} \frac{X_L + (R_{22}^2 + X_L^2)/\omega L_2}{(R_{22}/\omega L_2)^2 + [1 + (X_L/\omega L_2)]^2}$$

$L_1 \rightarrow \infty$, $L_2 \rightarrow \infty$, and $k \rightarrow 1.0$ $\quad L_1/L_2 = (N_1/N_2)^2$

$$X_{ab} = \left(\frac{N_1}{N_2}\right)^2 X_L \qquad \frac{\omega^2 M^2 R_{22}}{R_{22}^2 + X_{22}^2} = \frac{L_1}{L_2} R_{22} = \left(\frac{N_1}{N_2}\right)^2 R_{22}$$

$$Z_{ab} = R_1 + \left(\frac{N_1}{N_2}\right)^2 R_2 + \boxed{\left(\frac{N_1}{N_2}\right)^2} (R_L + jX_L)$$

Scaling factor

Determining the Voltage and Current Ratios

(a)

$$\mathbf{I}_1 = \frac{\mathbf{V}_1}{j\omega L_1}$$

$$\mathbf{V}_2 = j\omega M \mathbf{I}_1$$

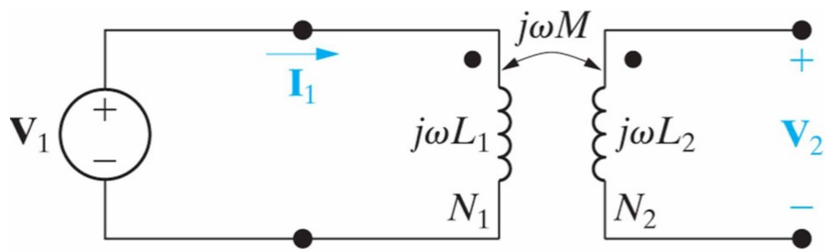
$$\mathbf{V}_2 = \frac{M}{L_1}\mathbf{V}_1$$

$M^2 = L_1 L_2$

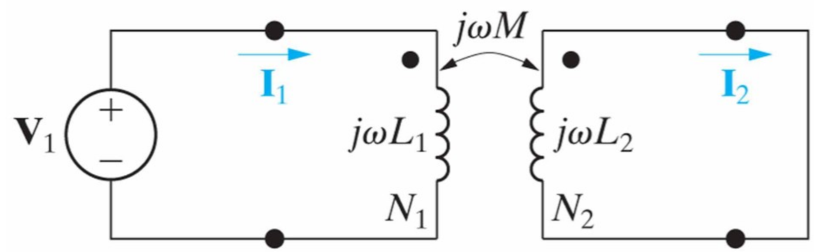
$$\mathbf{V}_2 = \sqrt{\frac{L_2}{L_1}}\mathbf{V}_1$$

$L_1/L_2 = (N_1/N_2)^2$

$$\boxed{\frac{\mathbf{V}_1}{N_1} = \frac{\mathbf{V}_2}{N_2}}$$



(a)



(b)

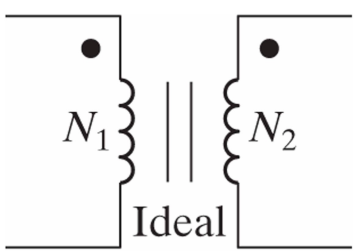
(b)

$$0 = -j\omega M \mathbf{I}_1 + j\omega L_2 \mathbf{I}_2$$

$$\frac{\mathbf{I}_1}{\mathbf{I}_2} = \frac{L_2}{M} = \frac{L_2}{\sqrt{L_1 L_2}} = \sqrt{\frac{L_2}{L_1}} = \frac{N_2}{N_1}$$

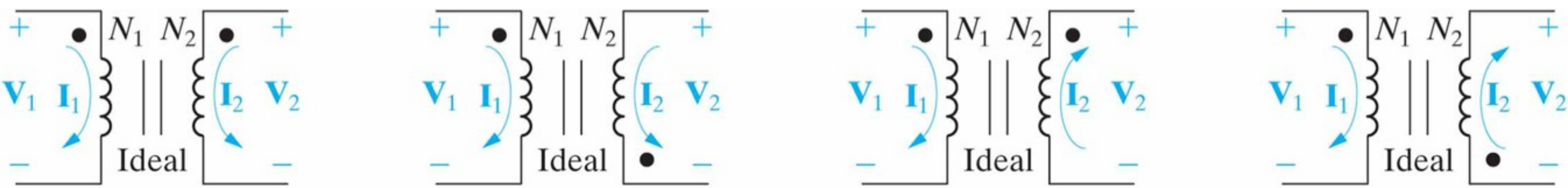
$$\mathbf{I}_1 N_1 = \mathbf{I}_2 N_2$$

$R_1 = R_2 = 0$



Determining the Polarity of the Voltage and Current Ratios

a) If the coil voltages $\mathbf{V}_1$ and $\mathbf{V}_2$ *are both positive or negative* at the dot-marked terminal, *use a plus sign*, Otherwise, use a negative sign.
b) If the coil currents $\mathbf{I}_1$ and $\mathbf{I}_2$ are *both directed into or out* of the dot; marked terminal, *use a minus sign*. Otherwise, use a plus sign.



$\frac{\mathbf{V}_1}{N_1} = \frac{\mathbf{V}_2}{N_2}$, $N_1\mathbf{I}_1 = -N_2\mathbf{I}_2$

(a)

$\frac{\mathbf{V}_1}{N_1} = -\frac{\mathbf{V}_2}{N_2}$, $N_1\mathbf{I}_1 = N_2\mathbf{I}_2$

(b)

$\frac{\mathbf{V}_1}{N_1} = \frac{\mathbf{V}_2}{N_2}$, $N_1\mathbf{I}_1 = N_2\mathbf{I}_2$

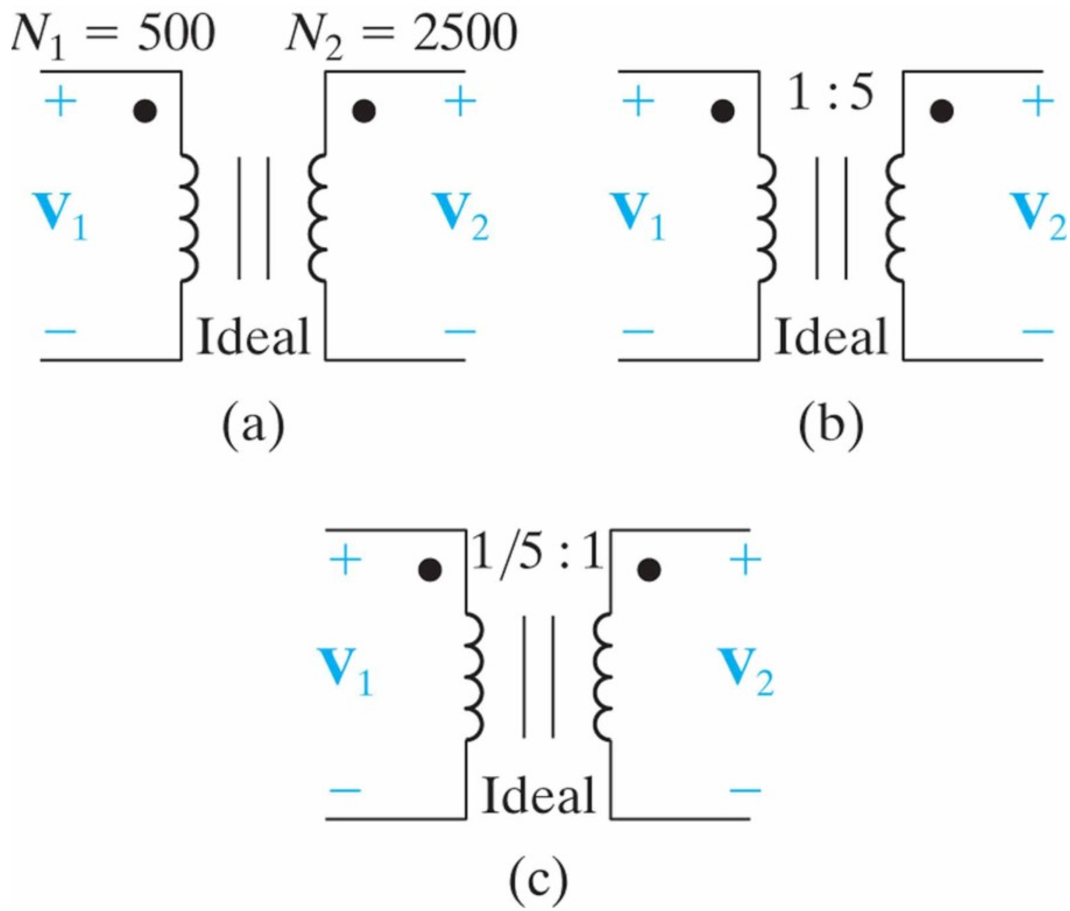
(c)

$\frac{\mathbf{V}_1}{N_1} = -\frac{\mathbf{V}_2}{N_2}$, $N_1\mathbf{I}_1 = -N_2\mathbf{I}_2$

(d)

$$a = \frac{N_2}{N_1}$$

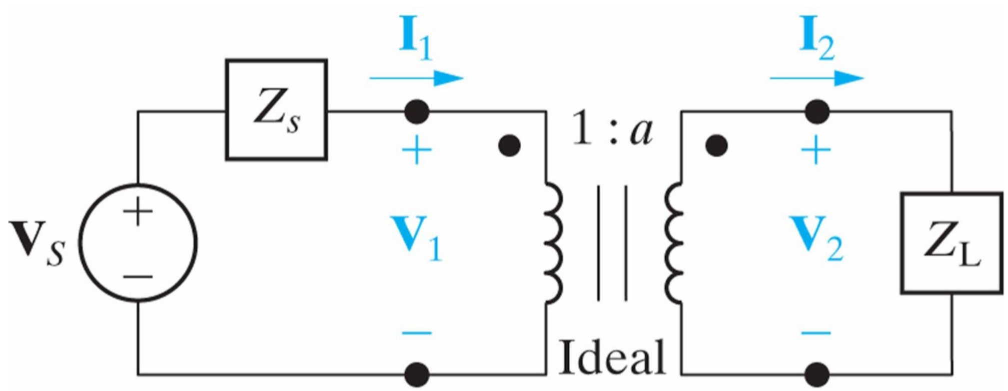
Important parameter for ideal transformer!



Ideal Transformer Used for Impedance Matching

$$\mathbf{V}_1 = \frac{\mathbf{V}_2}{a} \qquad \mathbf{I}_1 = a\mathbf{I}_2$$

$$Z_{IN} = \frac{\mathbf{V}_1}{\mathbf{I}_1} = \frac{1}{a^2}\frac{\mathbf{V}_2}{\mathbf{I}_2} = \frac{1}{a^2}Z_L$$



Thus, the ideal transformer's secondary coil reflects the load impedance back to the primary coil, with the scaling factor $1/a^2$.

Note that *the ideal transformer changes the magnitude of Z_L but does not affect its phase angle*. Whether Z_{IN} is greater or less than Z_L depends on the turns ratio a.