ECE 203

Circuits I

AC Steady-State Analysis

Lecture 12-1

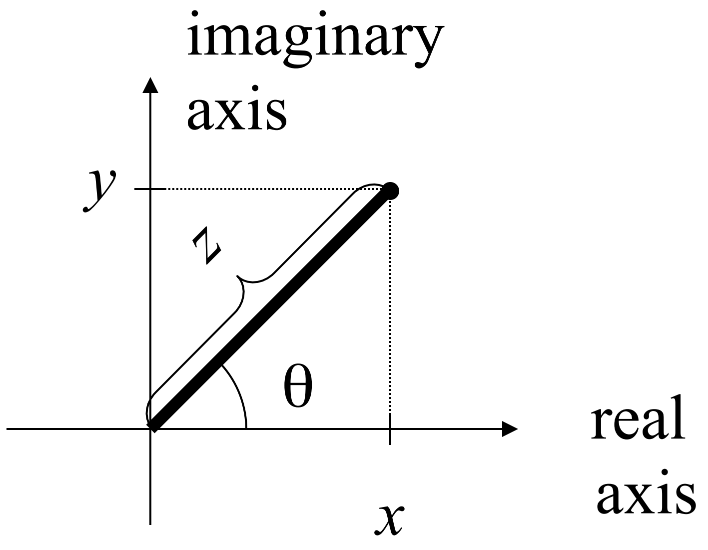
Sinusoidal forcing functions

Many circuits are excited by sinusoidal ac voltage or current sources

Steady-state means that although the voltage or current is oscillating, it continues to do so in the same way indefinitely

To solve circuit problems with these sorts of sources we need to review some properties of complex numbers

Complex Numbers



- *x* is the real part
- *y* is the imaginary part
- *z* is the magnitude
- θ is the phase

Complex Numbers

- Polar Coordinates: **A** = $z \angle \theta$
- Rectangular Coordinates: **A** = $x + jy$

$$x = z\cos\theta \qquad y = z\sin\theta$$

$$z = \sqrt{x^2 + y^2} \qquad \theta = \tan^{-1}\frac{y}{x}$$

Arithmetic with Complex Numbers

- To make computations using steady-state ac voltages and currents, we need to be able to perform computation with complex numbers.
  - Addition
  - Subtraction
  - Multiplication
  - Division

Addition

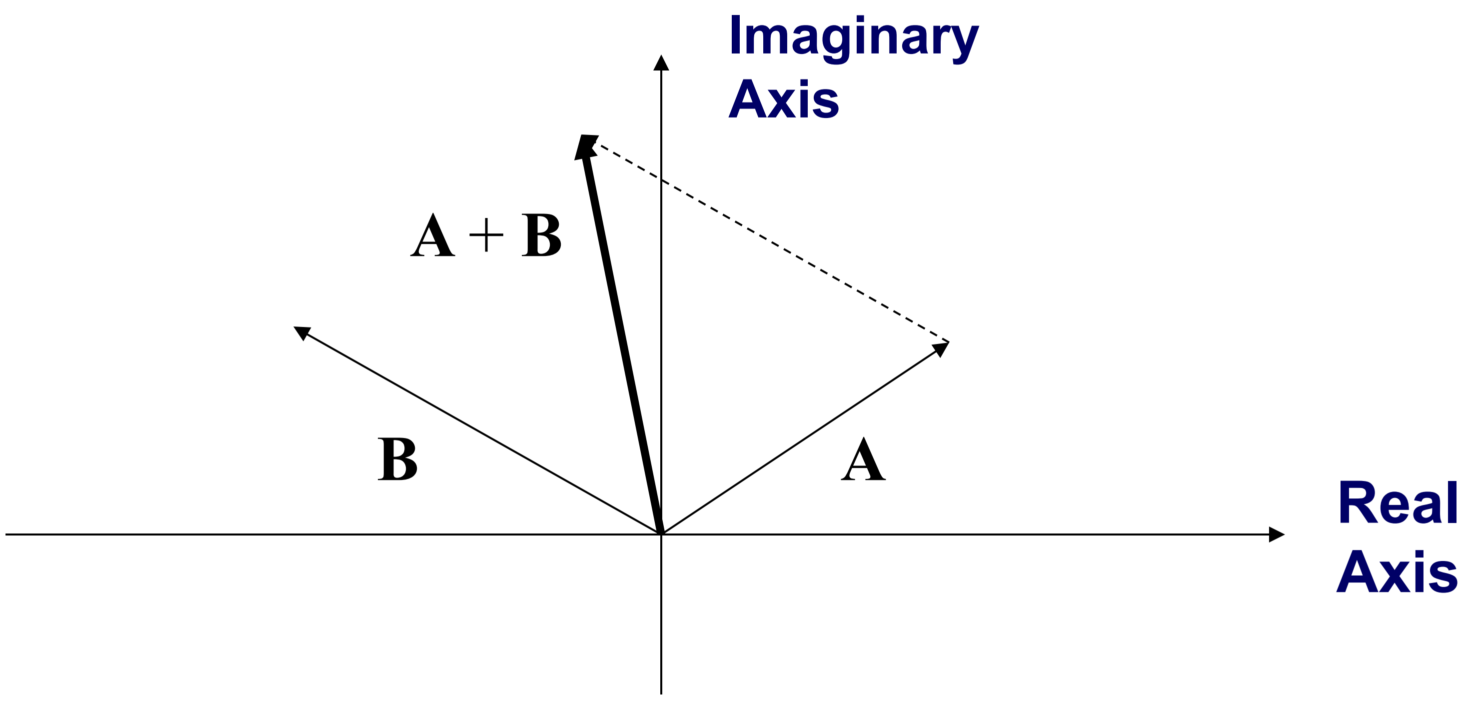
- Addition is most easily performed in rectangular coordinates:

$$\mathbf{A} = x + jy$$

$$\mathbf{B} = z + jw$$

$$\mathbf{A} + \mathbf{B} = (x + z) + j(y + w)$$

Addition



Subtraction

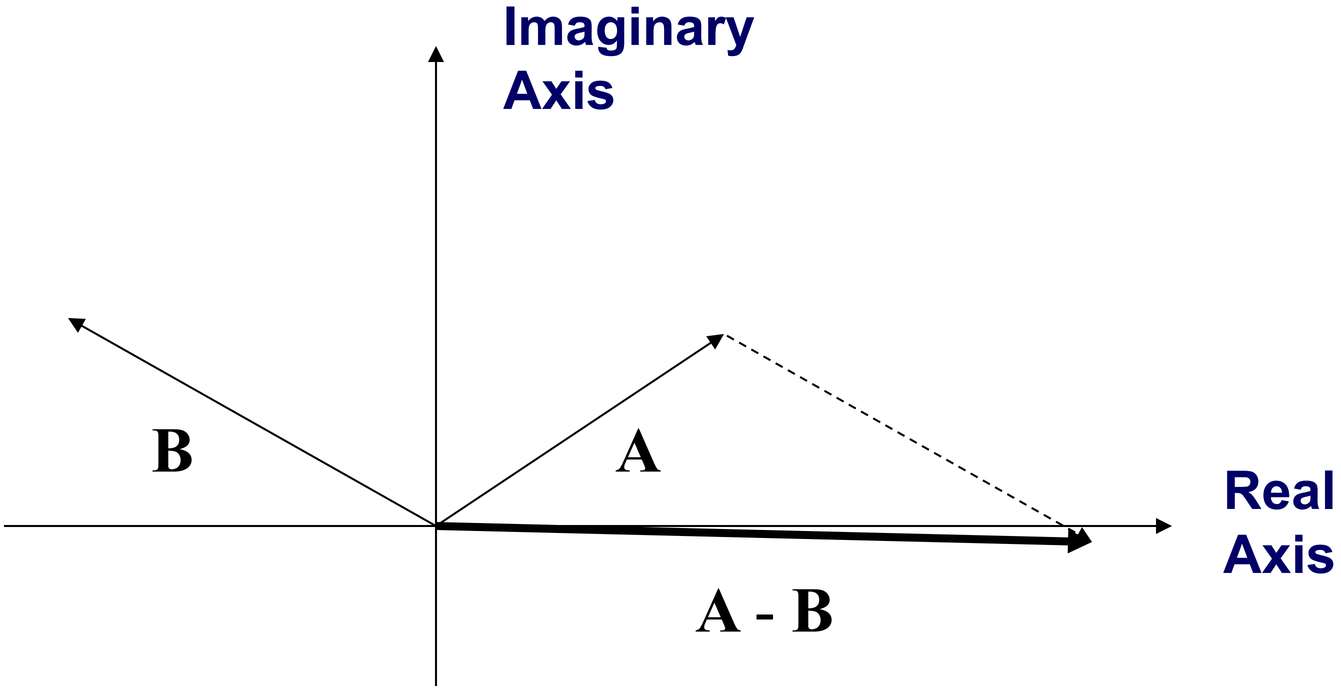
- Subtraction is most easily performed in rectangular coordinates:

$$\mathbf{A} = x + jy$$

$$\mathbf{B} = z + jw$$

$$\mathbf{A} - \mathbf{B} = (x - z) + j(y - w)$$

Subtraction



Multiplication

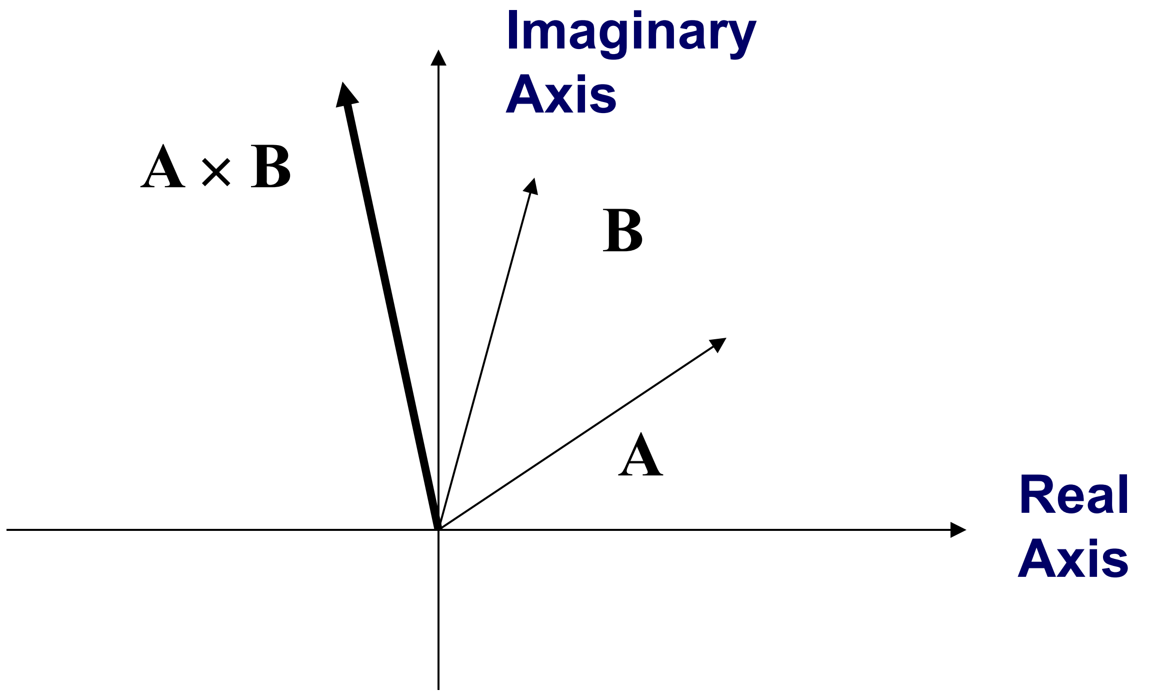
- Multiplication is most easily performed in polar coordinates:

$$\mathbf{A} = A_M \angle \theta$$

$$\mathbf{B} = B_M \angle \phi$$

$$\mathbf{A} \times \mathbf{B} = (A_M \times B_M) \angle (\theta + \phi)$$

Multiplication



Division

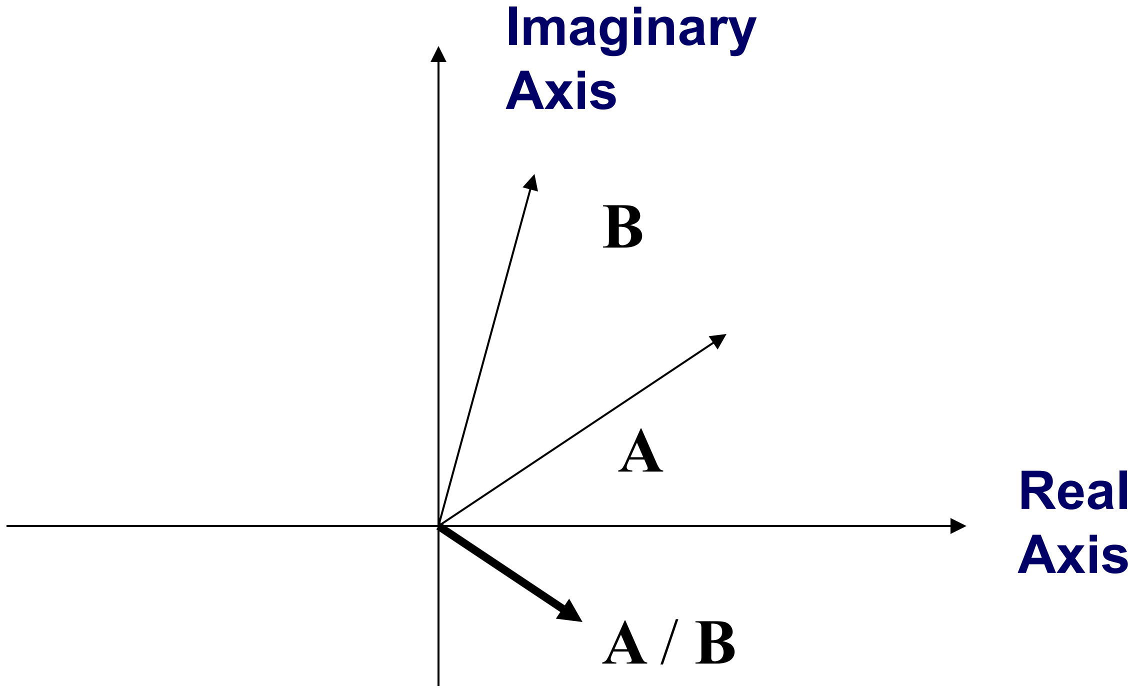
- Division is most easily performed in polar coordinates:

$$\mathbf{A} = A_M \angle \theta$$

$$\mathbf{B} = B_M \angle \phi$$

$$\mathbf{A} / \mathbf{B} = (A_M / B_M) \angle (\theta - \phi)$$

Division



Sinusoidal excitation

Assume that the voltage or current is of the form:

$v(t) = V_M \cos(\omega t + \theta)$

V_M is the amplitude

θ is the phase angle (more on this later)

Complex Exponentials

We can represent a real-valued sinusoid as the real part of a complex exponential.

$e^{j(\omega t + \theta)} = \cos(\omega t + \theta) + j \sin(\omega t + \theta)$

(Euler's equation)

So, $\cos(\omega t + \theta) = \mathrm{Re}[e^{j(\omega t + \theta)}]$

Where Re means "the real part"

Complex exponentials

For our purposes, we can assume that if a circuit is excited with a voltage or current at a given frequency, all of the currents or voltages in the circuit will also be at the same frequency (although the phase may be different).

With that assumption, all we need is the information regarding amplitude and phase

Complex Exponentials

So let's just represent our voltage as:

$$V = V_M \cos\theta + j\,V_M \sin\theta = \mathrm{V_M}\, e^{j\theta}$$

Or, more simply:

$$\mathrm{V} = \mathrm{V_M} \angle\theta$$

This is called phasor notation.

Sinusoids, Complex Exponentials, & Phasors

- Sinusoid:

$$v(t) = K\cos(\omega t+\theta)$$

- Complex exponential:

$$\mathbf{V} = Ae^{j\omega t} = K\,e^{j(\omega t+\theta)}$$

- Phasor:

$$\boxed{\mathbf{V} = K\angle\theta}$$

Phasor notation allows us to turn a problem involving differential equations into an algebraic problem.

Phasors

- Sinusoid is a time function:

$$x(t) = A \cos(\omega t + \theta)$$

- Phasor is a complex number that represents a sinusoid in the frequency domain:

$$\mathbf{X} = A \angle \theta = x + jy$$

What's missing in the phasor representation?

Some Examples

Find the time domain representations of:

X = -1 + *j*2

$$x(t) = \sqrt{5}\cos(\omega t - 63.4^o)$$

V = 104 - *j*60 V

$$v(t) = 120\cos(\omega t - 30^o)\ V$$

A = -1 - *j*3 mA

$$a(t) = \sqrt{10}\cos(\omega t + 71.6^o)\ mA$$

The Sinusoidal Function

Terms for describing sinusoids:

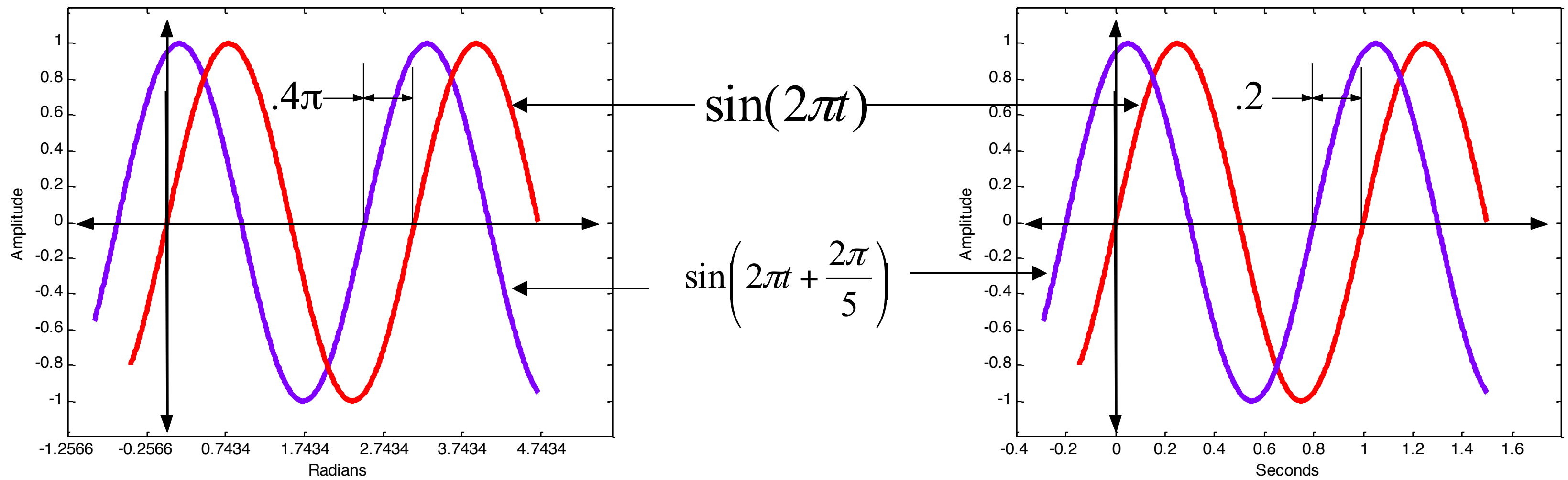
$$x(t) = X_m \sin(\omega t + \theta) = X_m \sin(2\pi f t + \theta)$$

Maximum Value, Amplitude, or Magnitude

Radian Frequency in Radian/second

Phase

Frequency in cycles/second or Hertz (Hz)



We build our techniques on a COSINE wave!

Sinusoidal Currents and Voltages

V_m is the **peak value**

ω is the **angular frequency** in radians per second

f = ω / 2π = 1/ T = frequency

θ is the **phase angle** in radians, although it is more often in degrees so watch for mixed units!

T is the **period**

Phasor Relationships for Circuit Elements

- Phasors allow us to express current-voltage relationships for inductors and capacitors much like we express the current-voltage relationship for a resistor.
- A complex exponential is the mathematical tool needed to obtain this relationship.

Phasor Definitions

$$\text{Time function:}\quad v_1(t) = V_1 \cos(\omega t + \theta_1)$$

$$\text{Phasor:}\quad \mathbf{V}_1 = V_1 \angle \theta_1$$

Definition

Sinusoids can be visualized as the real-axis projection of vectors rotating in the complex plane. The phasor for a sinusoid is a snapshot of the corresponding rotating vector at $t = 0$.

Trigonometric Identities

$$\cos(\omega t) = \sin\left(\omega t + \frac{\pi}{2}\right)$$

$$\sin(\omega t) = \cos\left(\omega t - \frac{\pi}{2}\right)$$

$$-\cos(\omega t) = \cos\left(\omega t \pm \pi(\text{or } 180^\circ)\right)$$

$$-\sin(\omega t) = \sin\left(\omega t \pm \pi(\text{or } 180^\circ)\right)$$

$$\sin(\alpha \pm \beta) = \sin(\alpha)\cos(\beta) \pm \cos(\alpha)\sin(\beta)$$

$$\cos(\alpha \pm \beta) = \cos(\alpha)\cos(\beta) \mp \sin(\alpha)\sin(\beta)$$

Radian to degree conversion

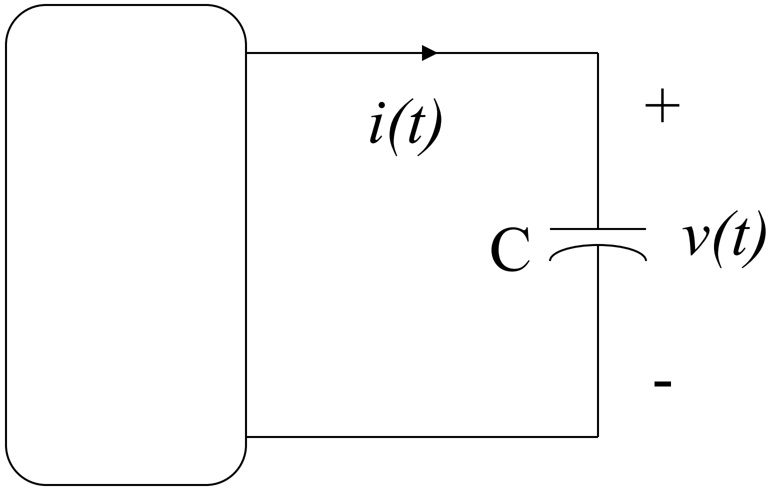
multiply by 180/π

Degree to radian conversion
multiply by π/180

$$X_m \sin\left(\omega t \pm \theta\right) = X_m \cos(\theta)\sin(\omega t) \pm X_m \sin(\theta)\cos(\omega t)$$

$$A\cos(\omega t) + B\sin(\omega t) = \sqrt{A^2 + B^2}\cos\left(\omega t + \tan^{-1}\left(\frac{-B}{A}\right)\right)$$

I-V Relationship for a Capacitor



$$i(t) = C\frac{dv(t)}{dt}$$

Suppose that *v(t)* is a sinusoid:

$$v(t) = V_M\, e^{j(\omega t+\theta)}$$

Find *i(t)*.

Computing the Current

$$i(t) = C\frac{dv(t)}{dt} = C\frac{dV_M e^{j\omega t + j\theta}}{dt}$$

$$i(t) = j\omega C V_M e^{j\omega t + j\theta} = \boxed{j\omega C v(t)}$$

$$v(t) = \frac{i(t)}{j\omega C}$$

This looks just like Ohm's law!

Except, R is replaced by $1/j\omega C$

Impedance

For an ac sinusoidal excitation, we replace the concept of a resistance with that of an impedance which is a complex number.

Denoted by **Z**

Capacitor Example

$$v(t) = 120\text{V} \cos(377t + 30°)$$

$$C = 2\mu\text{F}$$

$\mathbf{V} = 120 \angle 30^o$

1) What is **V**?

3) What is **I**?

2) What is *i(t)*?

i(t) = C dv/dt

$$= 120\ (2 \times 10^{-6})\ (377)\ (-1) \sin(377t + 30^o)$$

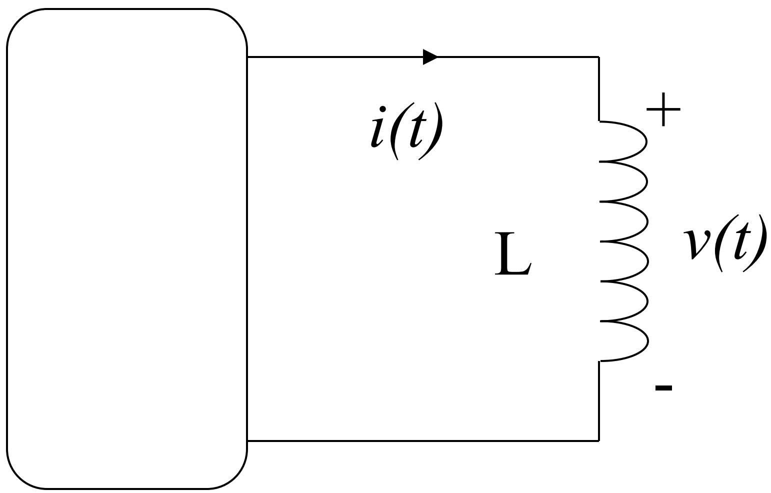
$$= 90.48 \text{ mA } (-1) \sin(377t + 30^o)$$

$$= 90.48 \text{ mA } \cos(377t + 120^o)$$

$\mathbf{I} = 90.48 \angle 120^o$ **mA**

Note that taking the derivative has led to a 90^o phase shift between V and I.

I-V Relationship for an Inductor



$$v(t) = L\frac{di(t)}{dt}$$

If $i(t) = I_M\, e^{j(\omega t+\theta)}$

Then, $\mathbf{V} = j\omega L\,\mathbf{I}$

Here, the impedance is jωL

Example

$$i(t) = 1\mu\text{A}\cos(2\pi\, 9.15\, 10^7 t + 30°)$$

$$L = 1\mu\text{H}$$

- What is **I**?
- What is **V**?
- What is *v(t)*?

Please give this a try!

You will also find a 90^o phase shift, but in the opposite direction.

Impedance of Circuit Elements

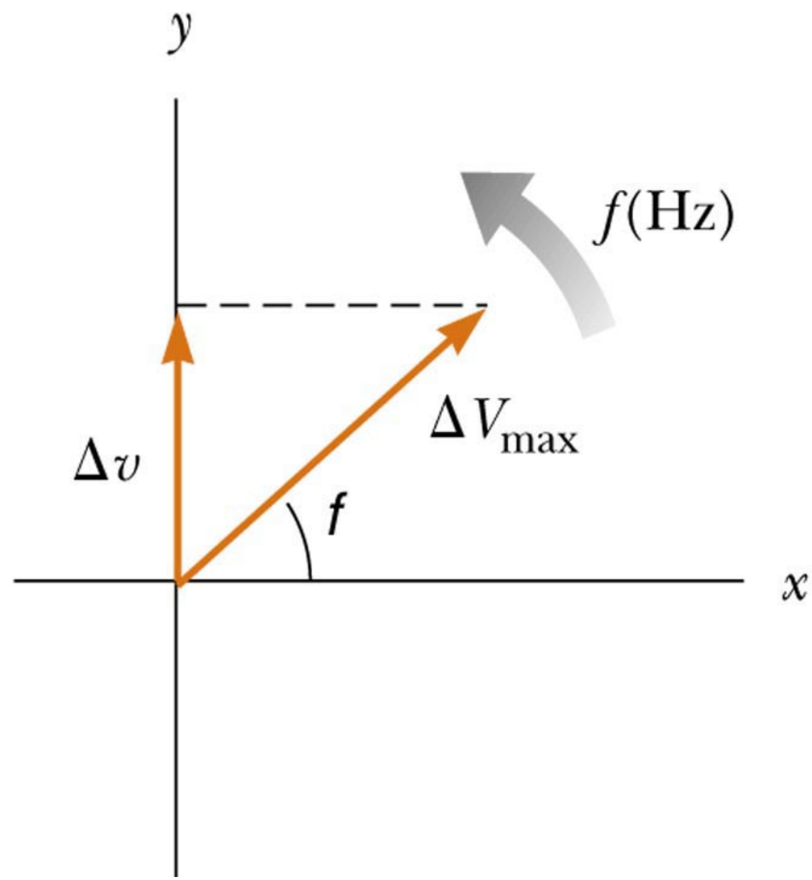
Element	Impedance	Phasor
Resistor	R	$R \angle 0^o$
Capacitor	$1/j\omega C$	$1/\omega C \angle -90^o$
Inductor	$j\omega L$	$\omega L \angle 90^o$

Phasor Diagrams

- A phasor diagram is just a graph of several phasors on the complex plane (using real and imaginary axes).
- A phasor diagram helps to visualize the relationships between currents and voltages.

Phasor Diagrams

- ***To account for the different phases of the voltage drops, vector techniques are used***
- ***Represent the voltage across each element as a rotating vector, called a phasor***
- ***The diagram is called a phasor diagram***



Impedance

- AC steady-state analysis using phasors allows us to express the relationship between current and voltage using a formula that looks likes Ohm's law:

$$\mathbf{V} = \mathbf{I}\,\mathbf{Z}$$

- **Z** is called impedance. Units of Ohms

FYI: Y = 1 / Z = Admittance (measured in Siemens)

Some Thoughts on Impedance

- Impedance depends on the frequency ω.
- Impedance is (often) a complex number.
- Impedance allows us to use the same solution techniques for AC steady state as we use for DC steady state.

Using Phasors to Add Sinusoids

$$v_1(t) = 20\cos\left(\omega t - 45^\circ\right)$$

$$v_2(t) = 10\cos\left(\omega t - 30^\circ\right)$$

$$\mathbf{V}_1 = 20\angle -45^\circ$$

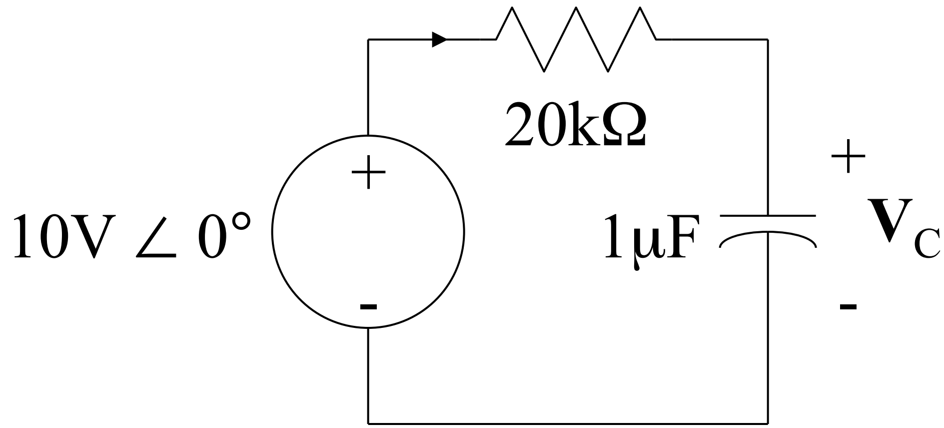
$$\mathbf{V}_2 = 10\angle -30^\circ$$

Vector Addition of Phasors

$$
\begin{aligned}
\mathbf{V}_\mathrm{s} &= \mathbf{V}_1 + \mathbf{V}_2 \\
&= 20\angle -45^\circ + 10\angle -30^\circ \\
&= 14.14 - j14.14 + 8.660 - j5 \\
&= 23.06 - j19.14 \\
&= 29.97\angle -39.7^\circ
\end{aligned}
$$

$$\boxed{v_s(t) = 29.97\cos\left(\omega t - 39.7^\circ\right)}$$

Example: Single Loop Circuit



$\omega = 377$

Find $\mathbf{V}_C$

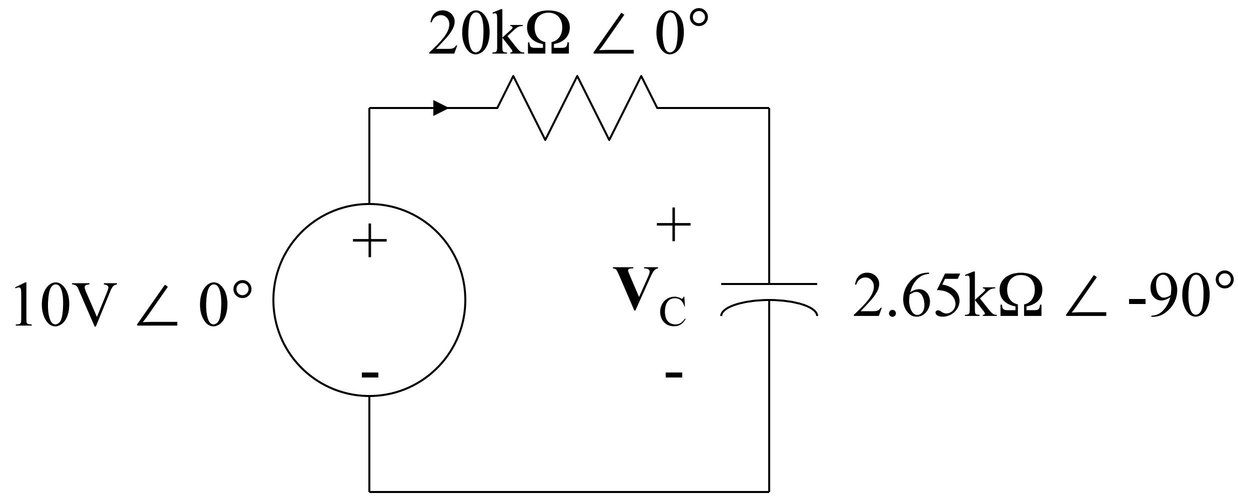
How do we find $\mathbf{V}_C$?

First compute impedances for resistor and capacitor:

$\mathbf{Z}_R = 20\text{k}\Omega = 20\text{k}\Omega \angle 0°$

$\mathbf{Z}_C = 1/j\,(377\ 1\mu\text{F}) = 2.65\text{k}\Omega \angle -90°$

Impedance Example



Now use the voltage divider to find $\mathbf{V}_C$:

$$\mathbf{V}_C = 10\text{V} \angle 0° \left(\frac{2.65\text{k}\Omega \angle -90°}{2.65\text{k}\Omega \angle -90° + 20\text{k}\Omega \angle 0°} \right)$$

$$\mathbf{V}_C = 1.31\text{V} \angle -82.4°$$