ENGR2910 - Circuit Analysis I

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Chapter 1

Circuit Variables

1.1 Electrical Engineering: An Overview

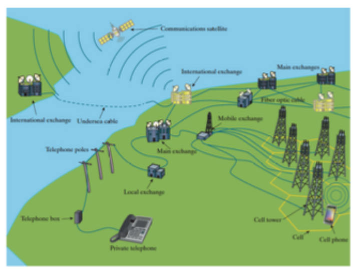


Figure 1.1: Telephone System

Five classifications of electrical systems:

1. Communication Systems
2. Computer Systems
3. Control Systems
4. Power Systems
5. Signal-Processing Systems

Circuit Theory

Three assumptions:

Quantity	Basic unit	Symbol
Length	meter	m
Mass	kilogram	Kg
Time	second	s
Electric current	ampere	A
Thermodynamic temperature	kelvin	K
Amount of substance	mole	mol
Luminous intensity	candela	cd

Figure 1.2: Scientific Units

Quantity	Unit Name (Symbol)	Formula
Frequency	hertz (Hz)	s^{-1}
Force	newton (N)	$kg \cdot m/s^2$
Energy or work	joule (J)	$N \cdot m$
Power	watt (W)	J/s
Electric charge	coulomb (C)	$A \cdot s$
Electric potential	volt (V)	J/C
Electric resistance	ohm (Ω)	V/A
Electric conductance	siemens (S)	A/V
Electric capacitance	farad (F)	C/V
Magnetic flux	weber (Wb)	$V \cdot s$
Inductance	henry (H)	Wb/A

Figure 1.3: Derived Units

Prefix	Symbol	Power
atto	a	10^{-18}
femto	f	10^{-15}
pico	p	10^{-12}
nano	n	10^{-9}
micro	μ	10^{-6}
milli	m	10^{-3}
centi	c	10^{-2}
deci	d	10^{-1}
deka	da	10
hecto	h	10^{2}
kilo	k	10^{3}
mega	M	10^{6}
giga	G	10^{9}
tera	T	10^{12}

Figure 1.4: Powers of 10

1.2 International System of Units

Circuit Analysis: An Overview

All engineering designs begin with a need that may include a Circuit Model before a physical prototype:

Figure 1.4: A conceptual model for electrical engineering design.

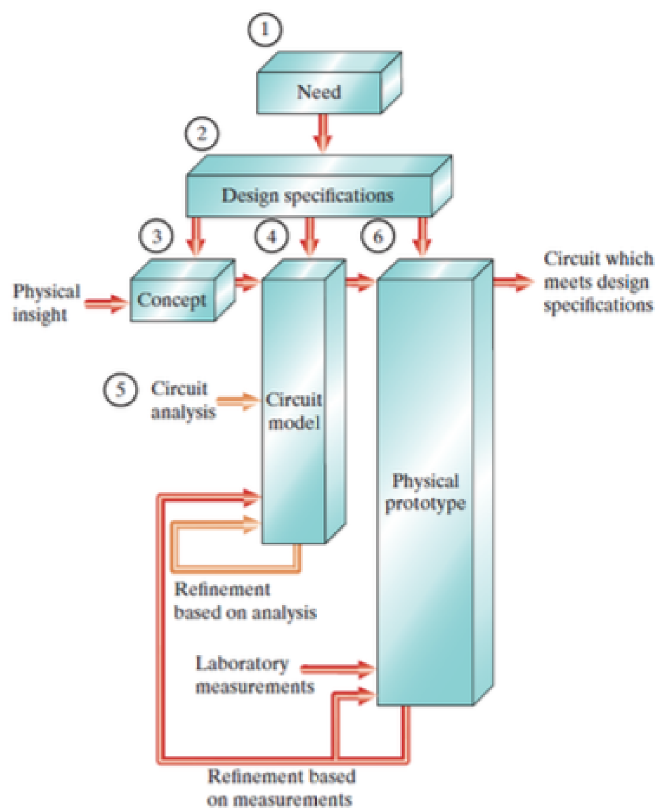


Figure 1.5: Conceptual Model for Electrical Engineering Design

Voltage and Current

Definition of Voltage (v)

$$v = \frac{dw}{dq} \tag{1.1}$$

where w is energy in joules and q is the charge in coulombs. Note that the charge of one electron (e is

$$e = 1.60022 \times 10^{-19} C \tag{1.2}$$

Definition of Current (i)

$$i = \frac{dq}{dt}, \tag{1.3}$$

where q is charge in coulombs and t is time in seconds.

Note, the direction of current is defined by the direction of flow of positive charge.

DC vs AC

Direct current is constant with time. Alternating current varies (sinusoidally) with time and reverses direction

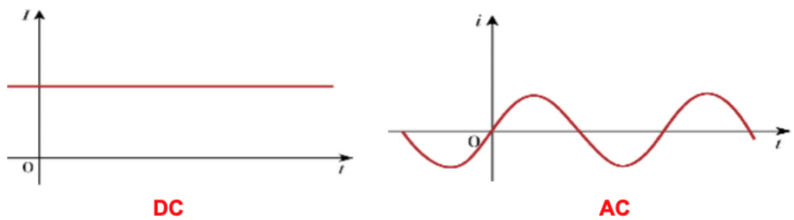


Figure 1.6: DC vs AC

Ideal Basic Circuit Element

The ideal circuit element has three attributes:

1. It has only two terminals
2. It is described mathematically in terms of current and/or voltage
3. It can not be subdivided to make other elements

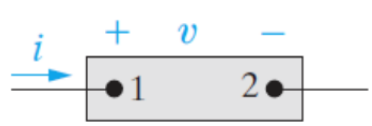


Figure 1.7: Ideal basic circuit element

WARNING: Positive Sign Convention

Whenever the reference direction for the current in an element is in the direction of the reference voltage drop across the element, use a positive sign in any expression that relates he voltage to current.

1.3 Power and Energy

Power is the energy per unit time

$$p = \frac{dw}{dt}, \tag{1.4}$$

where p s power in watts, w is energy in joules, and t is time in seconds. And, where $1W = 1\frac{J}{s}$.

$$p = \frac{dw}{dt} = (\frac{dw}{dq})(\frac{dq}{dt}) \tag{1.5}$$

therefore,

$$p = vi \tag{1.6}$$

Note, by convention, power is positive ($p > 0$) if power is being delivered, and power is negative if power is being extracted from the circuit.

Law of Conservation of Energy

$$\sum p = 0 \tag{1.7}$$

Energy is the capacity to do work (measured in J)

$$w = \int_{t_0}^{t} pdt = \int_{t_0}^{t} vidt \tag{1.8}$$

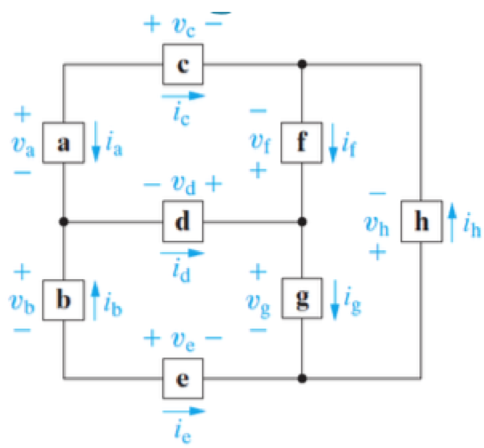


Figure 1.7: Circuit model for power distribution in a home, with voltages and currents defined.

Component	v(V)	i(A)
a	120	−10
b	120	9
c	10	10
d	10	1
e	−10	−9
f	−100	5
g	120	4
h	−220	−5

$$p_a = v_a i_a = (120)(-10) = -1200 \text{ W} \qquad p_b = -v_b i_b = -(120)(9) = -1080 \text{ W}$$

$$p_c = v_c i_c = (10)(10) = 100 \text{ W} \qquad p_d = -v_d i_d = -(10)(1) = -10 \text{ W}$$

$$p_e = v_e i_e = (-10)(-9) = 90 \text{ W} \qquad p_f = -v_f i_f = -(-100)(5) = 500 \text{ W}$$

$$p_g = v_g i_g = (120)(4) = 480 \text{ W} \qquad p_h = v_h i_h = (-220)(-5) = 1100 \text{ W}$$

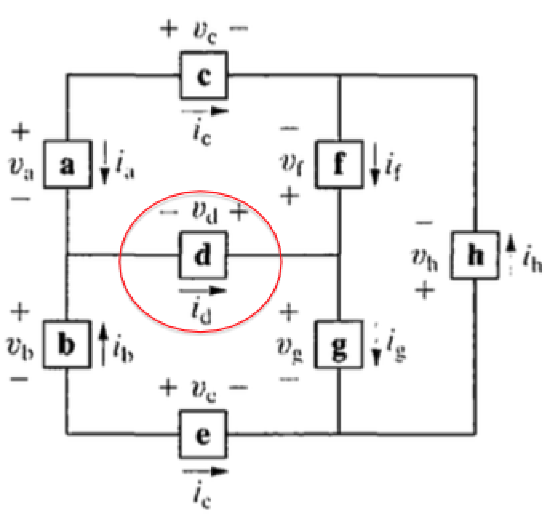
Figure 1.8: Balancing Power Example

$$p_{\text{supplied}} = p_a + p_b + p_d = -1200 - 1080 - 10 = -2290\ \text{W}$$

$$p_{\text{absorbed}} = p_c + p_e + p_f + p_g + p_h$$
$$= 100 + 90 + 500 + 480 + 1100 = 2270\ \text{W}$$

$$p_{\text{supplied}} + p_{\text{absorbed}} = -2290 + 2270 = -20\ \text{W}$$

Something is wrong—if the values for voltage and current in this circuit are correct, the total power should be zero!



Component	v(V)	i(A)
a	120	−10
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e	−10	−9
f	−100	5
g	120	4
h	−220	−5

Figure 1.9: Balancing Power Correction

Chapter 2

Circuit Elements

2.1 Voltage and Current Sources

When we speak of circuit elements, it is important to differentiate between the physical device itself and the mathematical model which we will use to analyze its behavior in a circuit. We will use the expression circuit element to refer to the mathematical model. All the simple circuit elements that we will consider can be classified according to the relationship between current through the element to the voltage across the element.

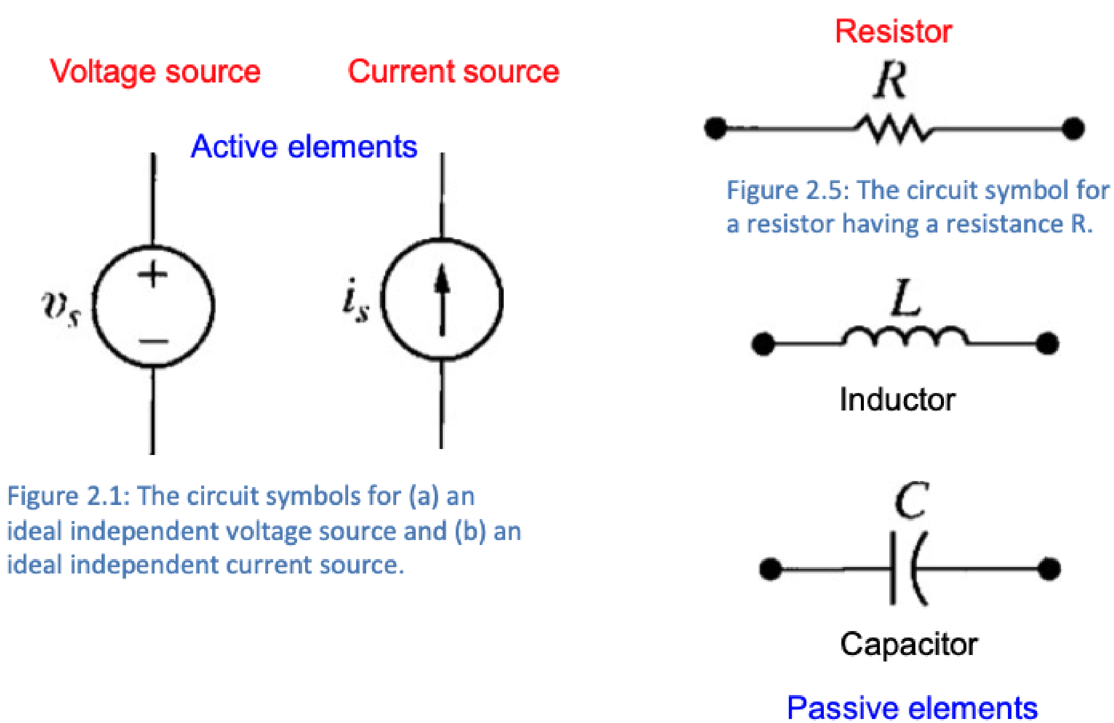


Figure 2.1: Five Basic Circuit Elements

A dependent source establishes a voltage or current whose value depends on the value of a voltage or current elsewhere in the circuit. You cannot specify the value of a dependent source unless you know the value of the voltage or current on which it depends.

There are four kinds of controlled sources:

- current-controlled current source (CCCS)
- voltage-controlled current source (VCCS)
- voltage-controlled voltage source, (VCVS)
- current-controlled voltage source, (CCVS)

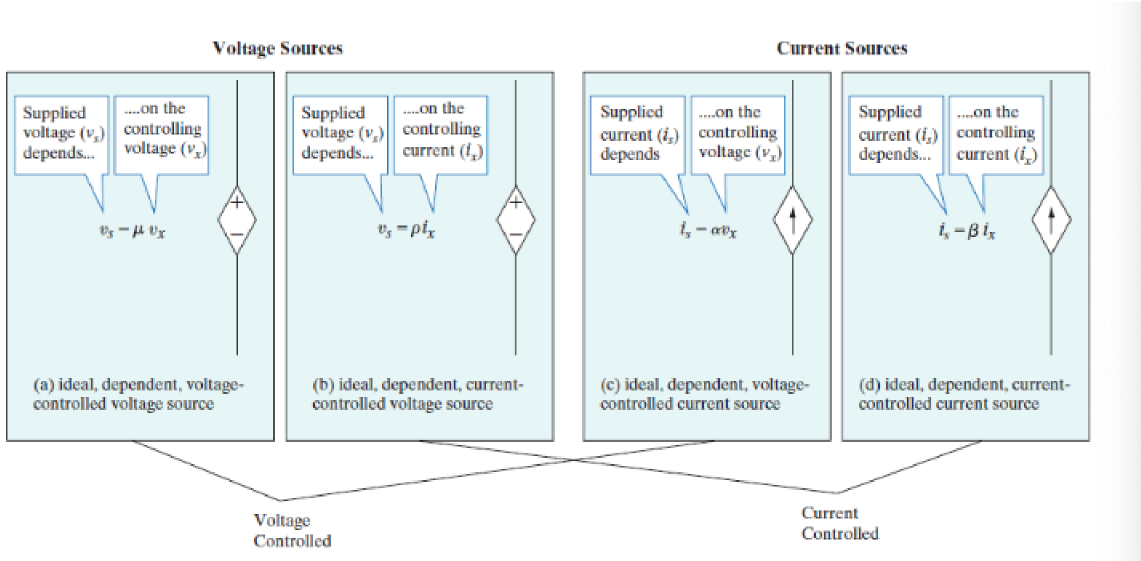


Figure 2.2: Four Controlled Sources

2.2 Electrical Resistance (Ohm's Law)

Resistance is the capacity of materials to impede the flow of current or, more specifically, the flow of electric charge. The circuit element used to model this behavior is the resistor. The linear resistor is the simplest passive element.

Ohm's Law

The relationship between Voltage and Current was empirically determined by Goerg Ohm[1]

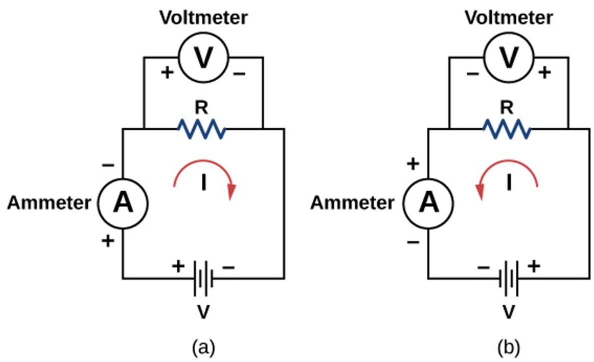
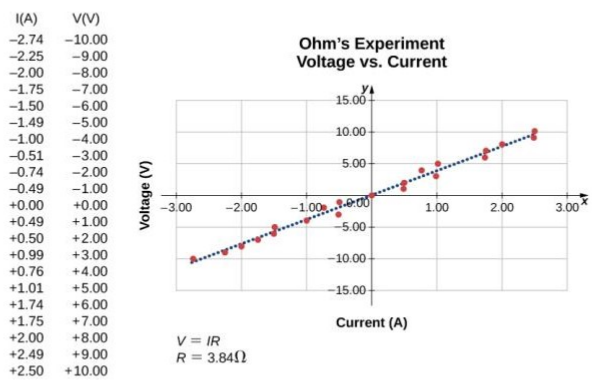


Figure 2.3: Goerg Ohm's Setup



I(A)	V(V)
−2.74	−10.00
−2.25	−9.00
−2.00	−8.00
−1.75	−7.00
−1.50	−6.00
−1.49	−5.00
−1.00	−4.00
−0.51	−3.00
−0.74	−2.00
−0.49	−1.00
+0.00	+0.00
+0.49	+1.00
+0.50	+2.00
+0.99	+3.00
+0.76	+4.00
+1.01	+5.00
+1.74	+6.00
+1.75	+7.00
+2.00	+8.00
+2.49	+9.00
+2.50	+10.00

Figure 2.4: Goerg Ohm's Data

Ohm's Law

$$V = I \cdot R \tag{2.1}$$

2.3 Kirchhoff's Laws

Kirchhoff's First Law (the Node Law or the Junction Rule)

The sum of all currents entering a junction must equal the sum of all currents leaving the junction.

[1] A presented in a paper published in 1827

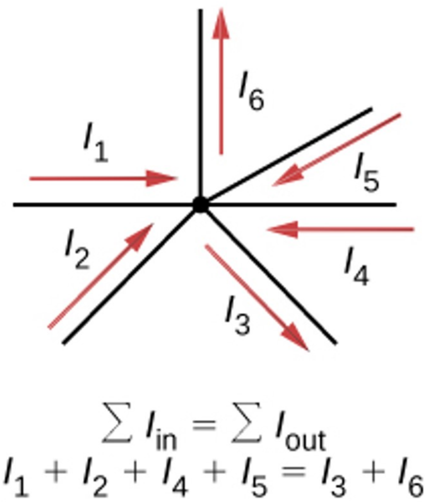


Figure 2.5: Kirchholff's Node Law

$$\sum I_{in} = \sum I_{out} \tag{2.2}$$

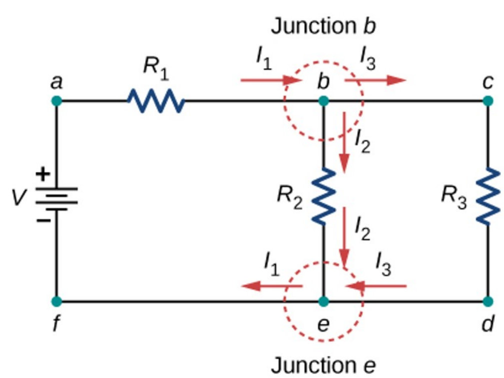


Figure 2.6: Kirchholff's Node Law Example

Kirchhoff's Second Law (the Loop Law or the Loop Rule)

The sum off all potential differences, including those supplied by voltage sources and resistive elements, around a closed loop equals zero.

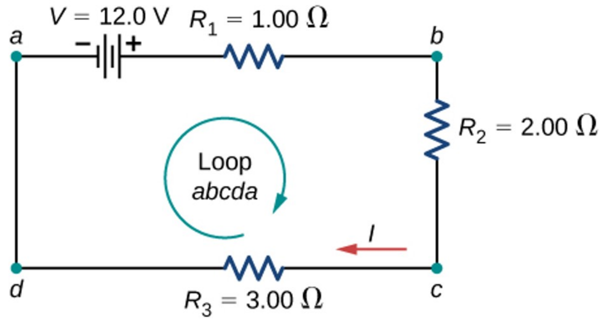


Figure 2.7: Kirchholff's Loop Law

$$\sum_{closedloop} V = 0 \tag{2.3}$$

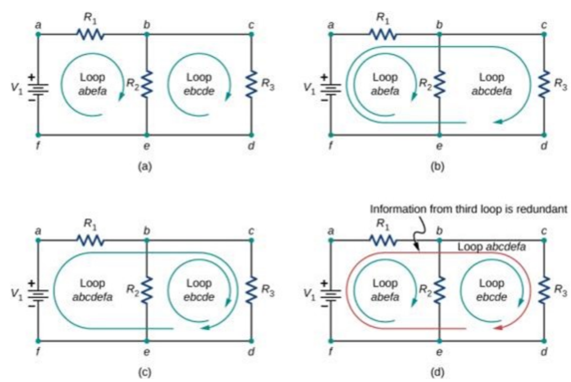


Figure 2.8: Kirchholff's Loop Law Example

2.4 Kirchhoff Examples

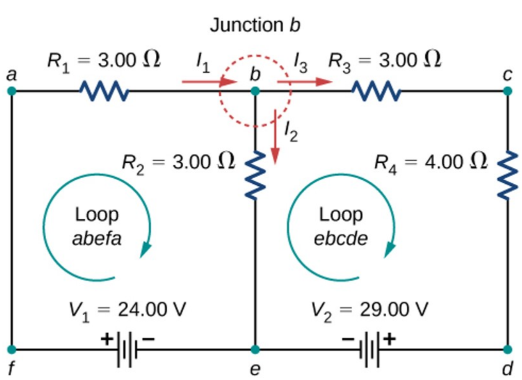


Figure 2.9: Kirchholff Example

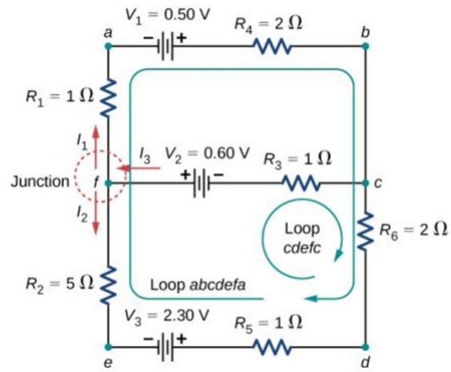


Figure 2.10: Kirchholff Examples

2.5 Circuits Containing Dependent Sources

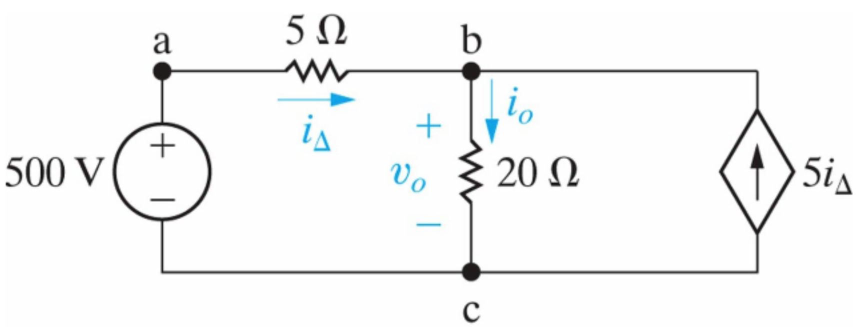


Figure 2.22: A circuit with a dependent source.

KCL $i_o = i_\Delta + 5i_\Delta = 6i_\Delta$

KVL $500 = 5i_\Delta + 20i_o$

$i_\Delta = 4$ A,

$i_o = 24$ A.

$v_o = 20i_o = 480$ V

Figure 2.11: Controlled Sources

Chapter 3

Simple Resistive Circuits

3.1 Resistors in Series

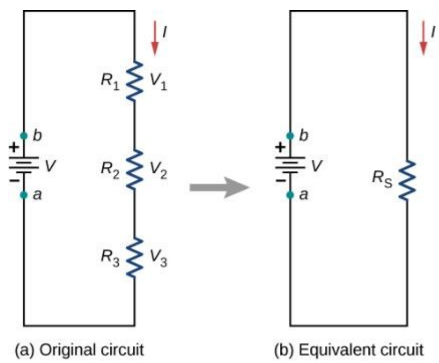


Figure 3.1: Resistors in Series

For resistors in Series

$$V = V_1 + V_2 + V_3 \tag{3.1}$$

$$V = 1R_1 + 1R_2 + 1R_3 \tag{3.2}$$

$$I = \frac{V}{R_1 + R_2 + R_3} \tag{3.3}$$

So

$$R_{eq} = \sum_{i=1}^{N} R_i \tag{3.4}$$

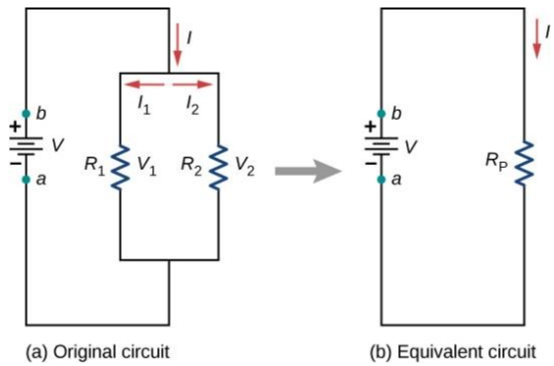


Figure 3.2: Resistors Parallel

3.2 Resistors in Parallel

$$V = V_1 = V_2 \tag{3.5}$$

$$I = I_1 + I_2 \tag{3.6}$$

$$\frac{V}{R_{eq}} = \frac{V_1}{R_1} + \frac{V_2}{R_2} \tag{3.7}$$

Because the Voltage is equal across the resistors

$$\frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2} \tag{3.8}$$

Or, more generically

$$R_{eq} = (\sum_{i=1} N \frac{1}{R_i})^{-1} \tag{3.9}$$

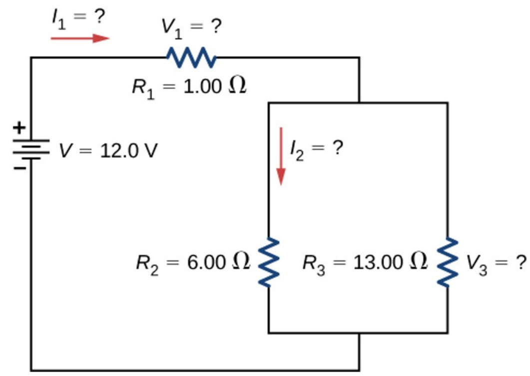
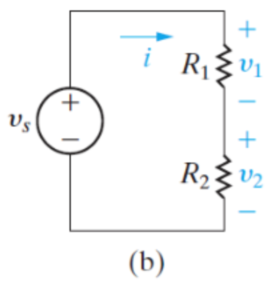


Figure 3.3: Resistors in Series and Parallel

3.3 Divider Circuits



$$i = \frac{v_s}{R_1 + R_2}$$

$$v_1 = iR_1 = v_s \frac{R_1}{R_1 + R_2}$$

$$v_2 = iR_2 = v_s \frac{R_2}{R_1 + R_2}$$

Figure 3.4: Voltage Divider

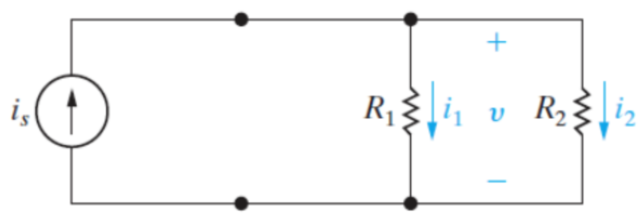


Figure 3.19: The current-divider circuit.

$$v = i_1 R_1 = i_2 R_2 = \frac{R_1 R_2}{R_1 + R_2} i_s$$

$$i_1 = \frac{R_2}{R_1 + R_2} i_s$$

$$i_2 = \frac{R_1}{R_1 + R_2} i_s$$

Figure 3.5: Current Divider

With a Load

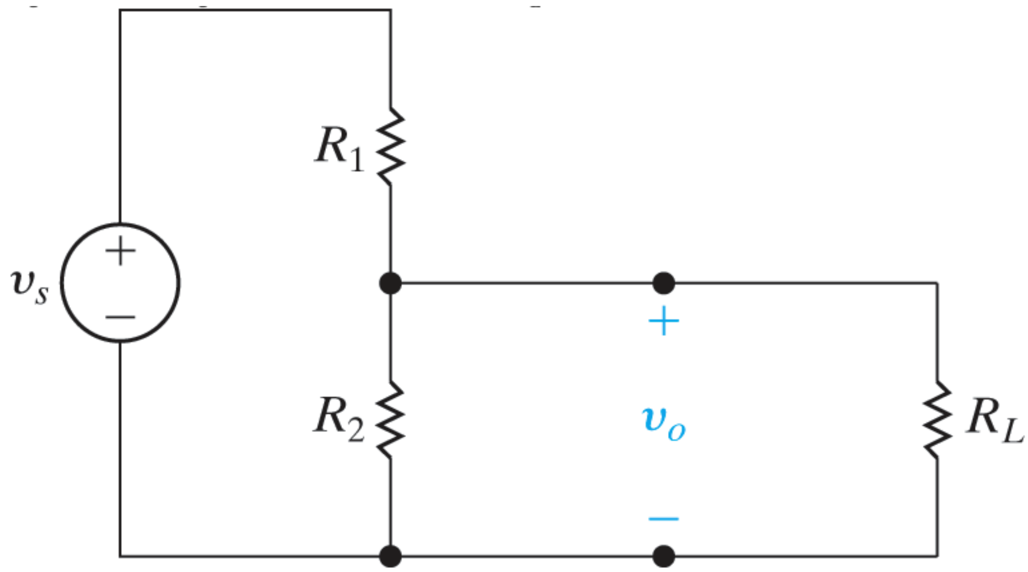


Figure 3.6: Voltage Divider with Load

$$v_0 = \frac{R_{eq}}{R_1 + R_{eq}} v_s \tag{3.10}$$

where

$$R_{eq} = \frac{R_2 R_L}{R_2 + R_L} \tag{3.11}$$

substituting

$$v_0 = \frac{R_2}{R_1[1 + \frac{R_2}{R_L}] + R_2} v_s \tag{3.12}$$

3.4 Measuring Voltage and Current

- An ammeter is an instrument designed to measure current; it is placed in series with the circuit element whose current is being measured.
- A voltmeter is an instrument designed to measure voltage; it is placed in parallel with the circuit element whose current is being measured.

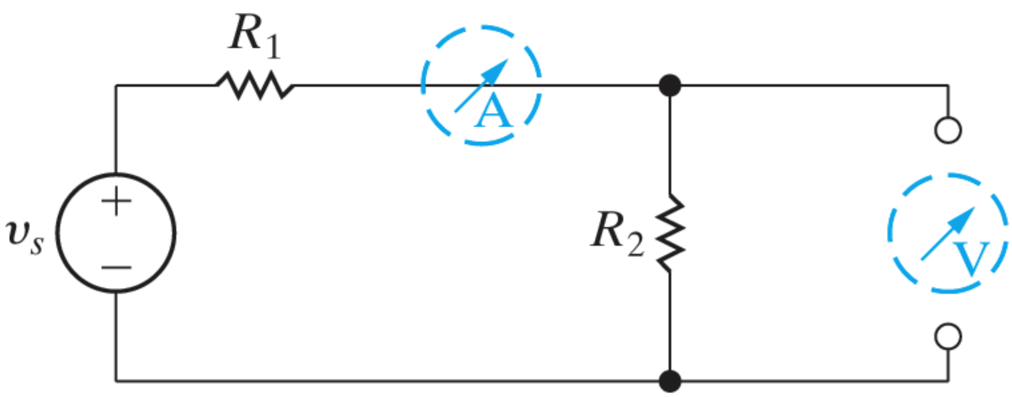


Figure 3.7: Short-circuit model for ideal ammeter, and open-circuit model for ideal volt meter

d'Arsonval meter

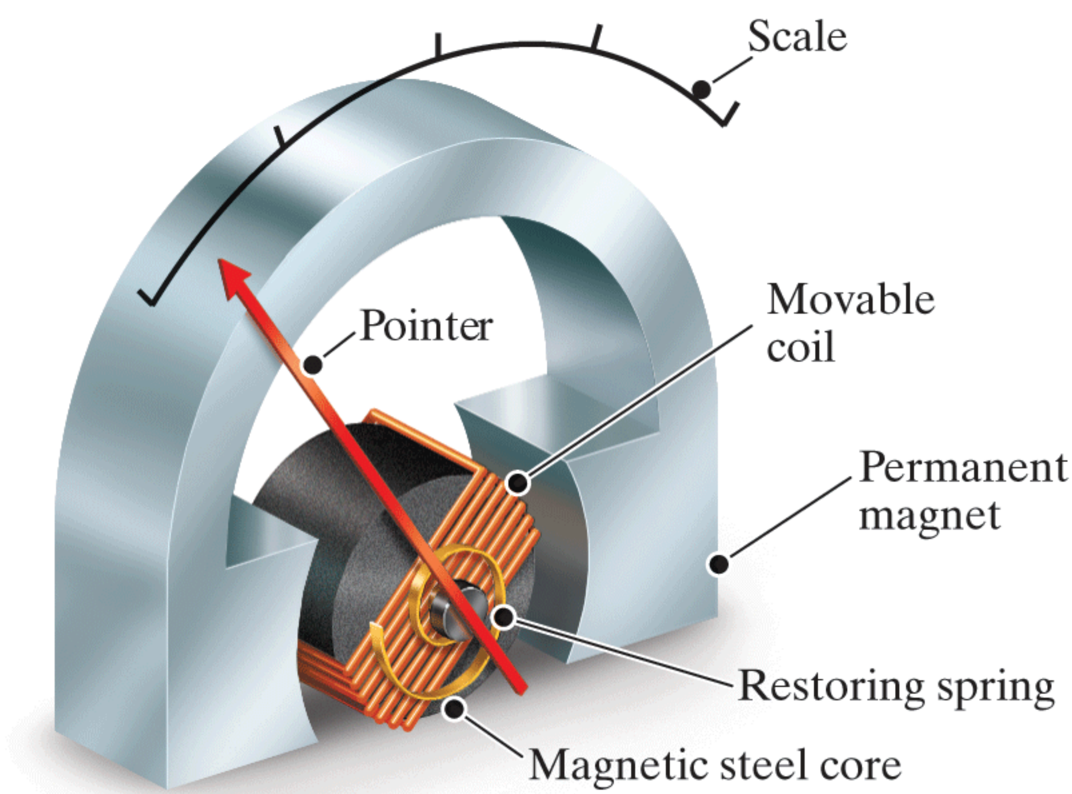


Figure 3.8: d'Arsonval meter movement

Non-ideal meters

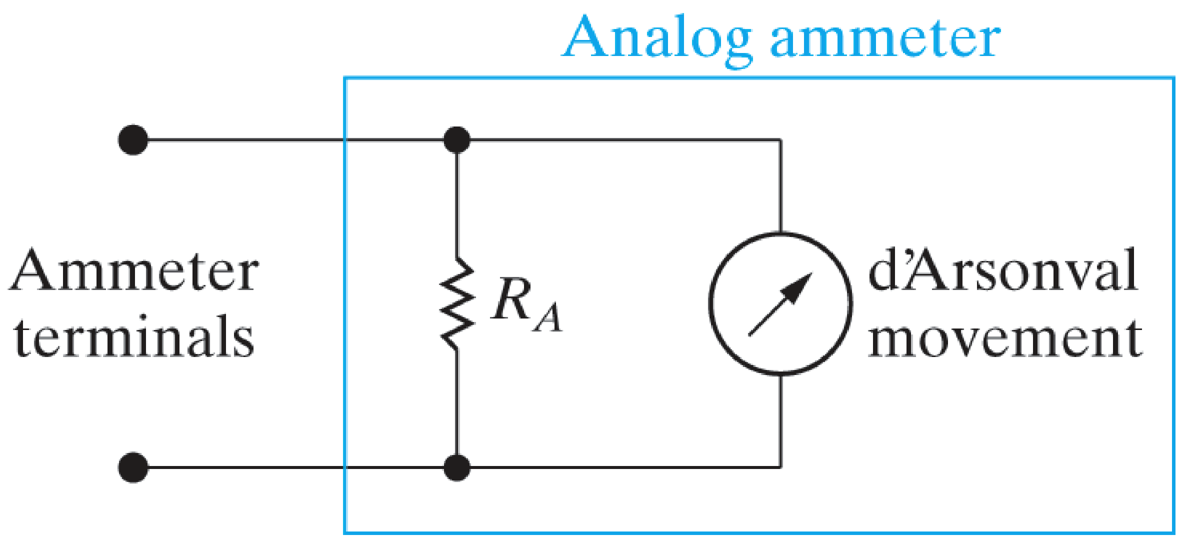


Figure 3.9: Non-Ideal Ammeter

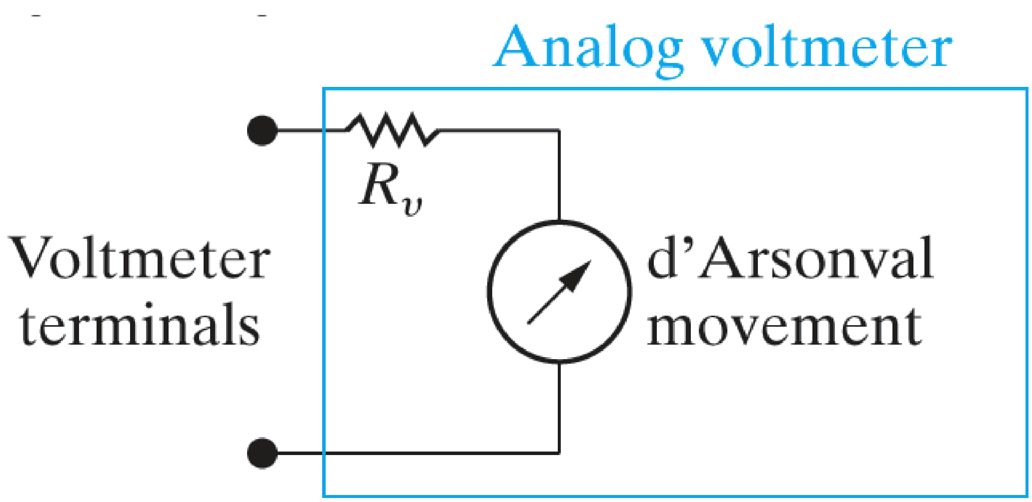


Figure 3.10: Non-ideal Voltmeter

The Wheatstone Bridge

The Wheatstone Bridge[1] is one, of many, configurations that can be used to measure resistance.

[1]Sir Charles Wheatstone (6 February 1802 – 19 October 1875), was an English scientist and inventor of many scientific breakthroughs of the Victorian era. Wheatstone is best known for his contributions in the development of the Wheatstone bridge, originally invented by Samuel Hunter Christie, which is used to measure an unknown electrical resistance

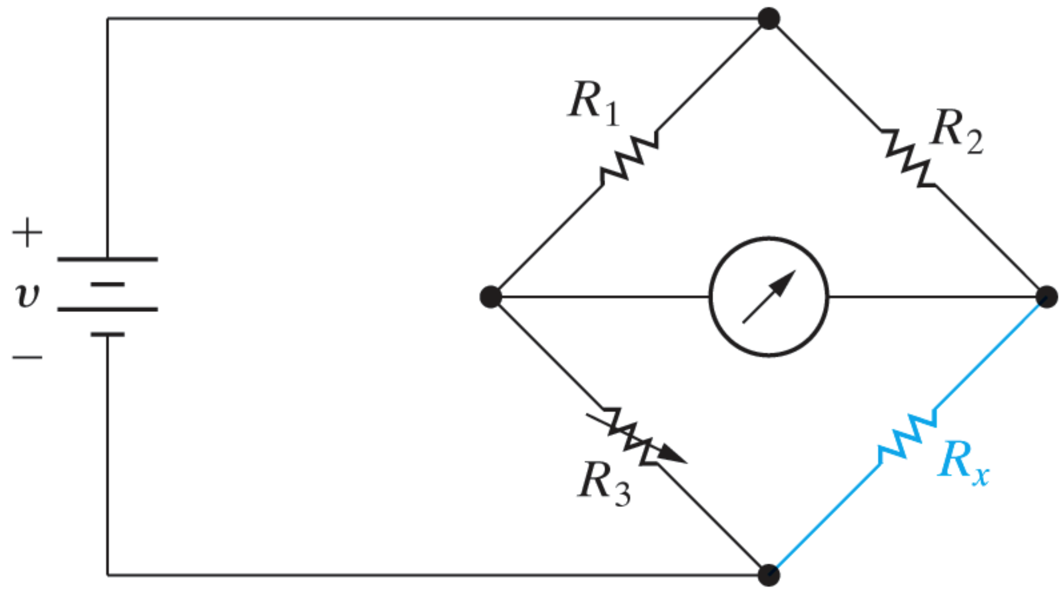


Figure 3.11: Wheatstone Bridge

To find R_x, the variable resistor R_3 is adjusted until t here is no current in the galvanometer. Then the value of unknown resistor can be found by

$$R_x = \frac{R_2}{R_1} R_3 \tag{3.13}$$

Derivation using Kirchhoff's Laws

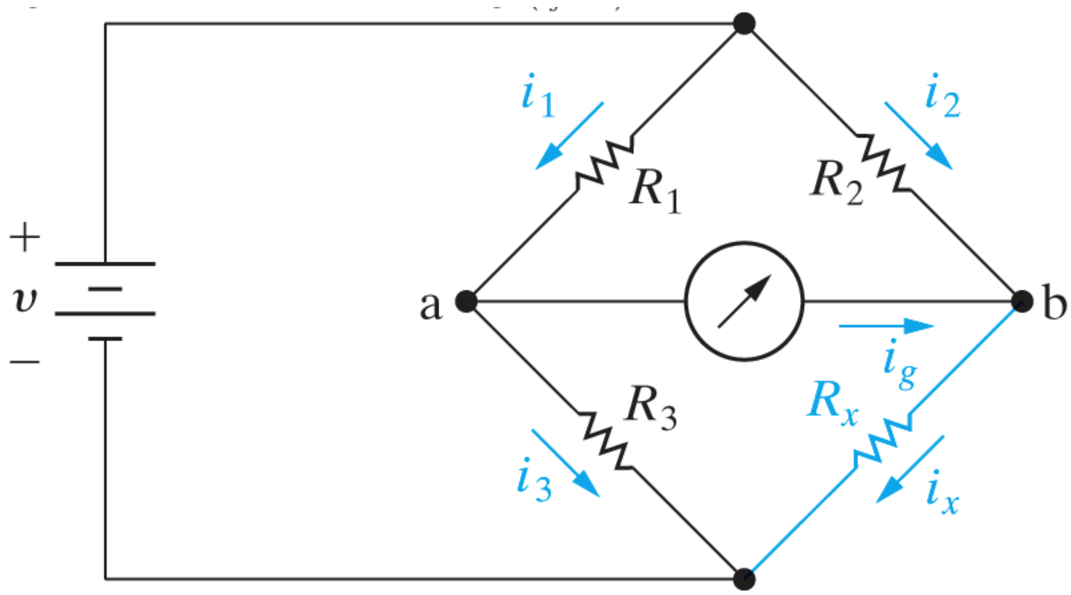


Figure 3.12: Balanced Wheatstone Bridge

Using KCL, since $i_g = 0$, at Node A:

$$i_1 = i_3 \tag{3.14}$$

And, at Node B:

$$i_2 = i_x \tag{3.15}$$

Because $i_g = 0$, this also implies the voltage drop across the detector is also zero, and thus Nodes A and B are at the same potential.

From KVL:

$$i_3R_3 - i_xR_x = 0 \tag{3.16}$$

or

$$i_3R_3 = i_xR_x \tag{3.17}$$

Likewise,

$$i_1 R_1 = i_2 R_2 \tag{3.18}$$

Divide the first KVL equation by the second

$$\frac{i_3 R_3}{i_1 R_1} = \frac{i_x R_x}{i_2 R_2} \tag{3.19}$$

Since $i_1 = i_3$ and $i_2 = i_x$, solving for R_x:

$$R_x = \frac{R_2}{R_1} R_3 \tag{3.20}$$

3.5 Delta-to-Wye (Pi-to-Tee) Equivalent Circuits

The Δ and Y structures are present in a variety of useful circuits. Hence, the $\Delta - Y$ transformation is helpful in circuit analysis.

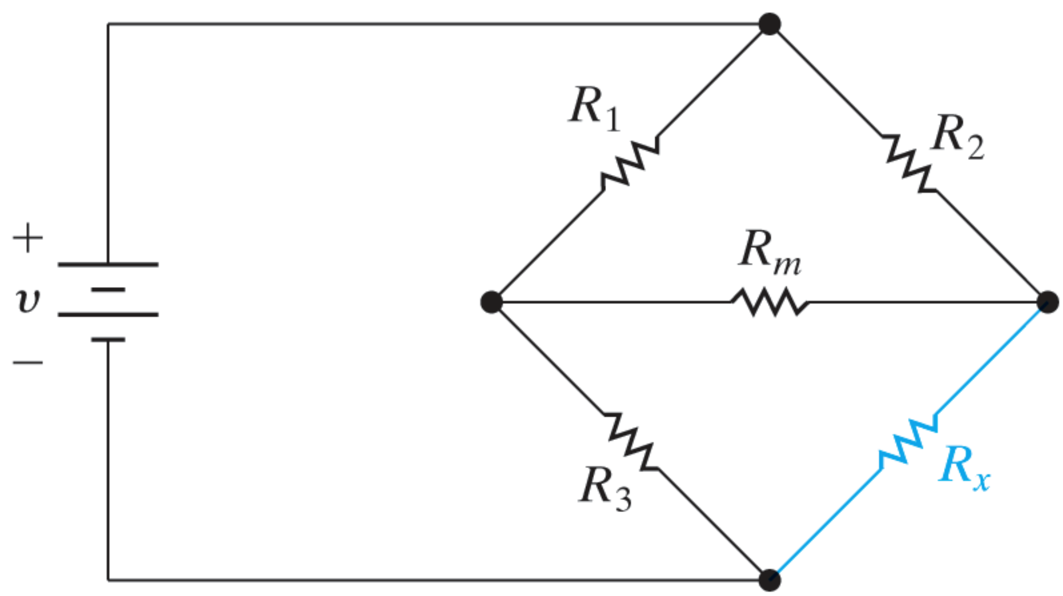


Figure 3.13: Resistive Network generated by a Wheatstone Bridge circuit

Resistors R_1, R_2, and R_m (or R_3, R_x, and R_m) form a delta (Δ) interconnect.

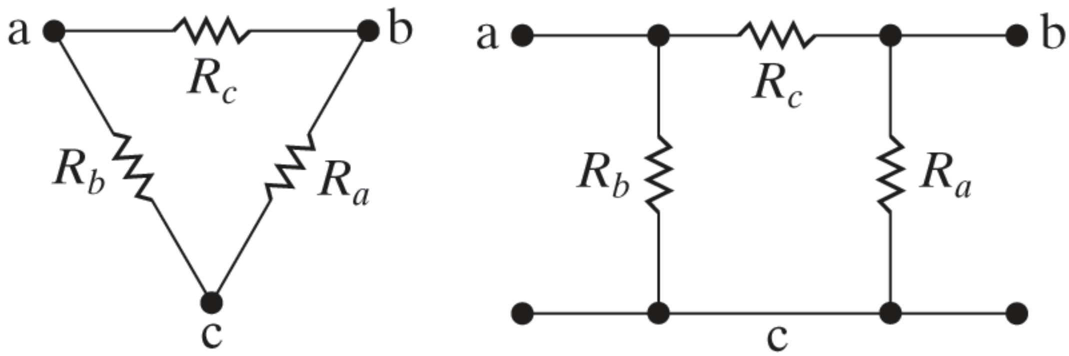


Figure 3.14: Δ configuration viewed as a π configuration

Another configuration is the wye (Y) or the electrically equivalent tee (T) interconnection

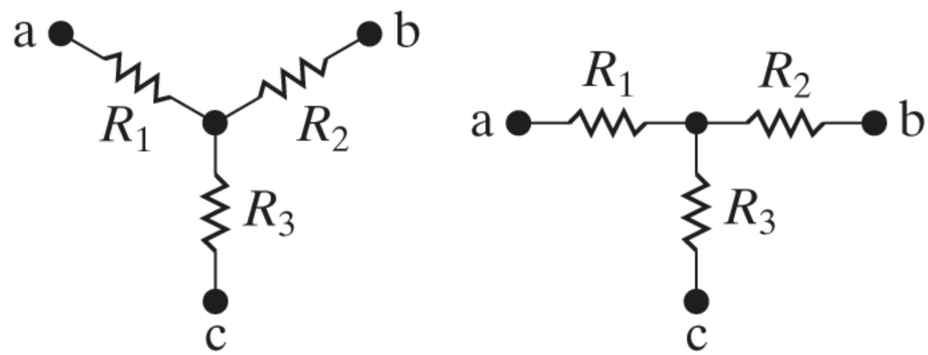


Figure 3.15: Y configuration viewed as a T configuration

The Delta to Y transformation

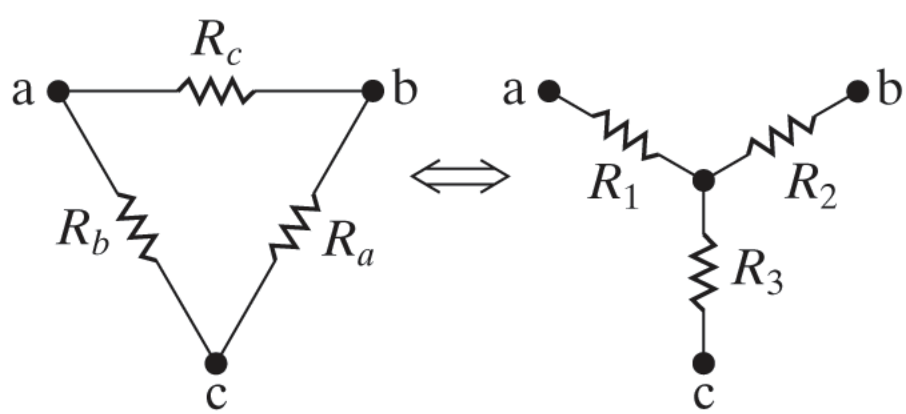


Figure 3.16: Δ to Y Transformation

The resistance between terminals needs to be the same whether they are in series or parallel. The three equivalent resistance equations are:

$$R_{ab} = \frac{R_c(R_a + R_b)}{R_a + R_b + R_c} = R_1 + R_2 \tag{3.21}$$

$$R_{bc} = \frac{R_a(R_b + R_c)}{R_a + R_b + R_c} = R_2 + R_3 \tag{3.22}$$

$$R_{ca} = \frac{R_b(R_c + R_a)}{R_a + R_b + R_c} = R_3 + R_1 \tag{3.23}$$

Through algebraic manipulation

$$R_1 = \frac{R_b R_c}{R_a + R_b + R_c} \tag{3.24}$$

$$R_2 = \frac{R_c R_a}{R_a + R_b + R_c} \tag{3.25}$$

$$R_3 = \frac{R_a R_b}{R_a + R_b + R_c} \tag{3.26}$$

Or this can be reversed

$$R_a = \frac{R_1 R_2 + R_2 R_3 + R_3 R_1}{R_1} \tag{3.27}$$

$$R_b = \frac{R_1 R_2 + R_2 R_3 + R_3 R_1}{R_2} \tag{3.28}$$

$$R_c = \frac{R_1 R_2 + R_2 R_3 + R_3 R_1}{R_3} \tag{3.29}$$

Chapter 4

Techniques of Circuit Analysis

Appendix A

Integration by Trig Substitution

Find the integral of

$$\int \frac{1}{(a^2+x^2)^{\frac{3}{2}}} \tag{A.1}$$

Integrate by trig substitution by setting $x = a\tan u$ which leads to

$$\frac{dx}{du} = \frac{a\tan u}{du} = \frac{a}{\cos^2 u} \tag{A.2}$$

Which leads to

$$dx = (\frac{a}{\cos^2 u})du \tag{A.3}$$

Thus

$$\int \frac{1}{(a^2+x^2)^{\frac{3}{2}}} = \int \frac{1}{(a^2+(a\tan(u))^2)^{\frac{3}{2}}}(\frac{a}{\cos^2(u)})du \tag{A.4}$$

$$= \int \frac{1}{(a^2)^{\frac{3}{2}}(1+\tan^2(u))^{\frac{3}{2}}}(\frac{a}{\cos^2(u)})du \tag{A.5}$$

$$= \int \frac{1}{(a^3)(\frac{1}{\cos^2(u)})^{\frac{3}{2}}}(\frac{a}{\cos^2(u)})du \tag{A.6}$$

$$= \frac{1}{a^2} \int \frac{1}{(\frac{1}{\cos^2(u)})^{\frac{3}{2}}} (\frac{1}{\cos^2(u)}) du \tag{A.7}$$

$$= \frac{1}{a^2} \int \frac{1}{(\frac{1}{\cos^3(u)})} (\frac{1}{\cos^2(u)}) du \tag{A.8}$$

$$= \frac{1}{a^2} \int \cos(u) du \tag{A.9}$$

$$= \frac{1}{a^2} \sin(u) + C \tag{A.10}$$

From the above $\arctan(\frac{x}{a}) = u$ so

$$= \frac{1}{a^2} \sin(\arctan(\frac{x}{a})) + C \tag{A.11}$$

$$= \frac{1}{a^2} [\frac{\frac{x}{a}}{\sqrt{1 + (\frac{x}{a})^2}}] + C \tag{A.12}$$

$$= \frac{x}{a^3} [\frac{1}{\frac{1}{a}\sqrt{a^2 + x^2}}] + C \tag{A.13}$$

Which finally leads to

$$\int \frac{1}{(a^2 + x^2)^{\frac{3}{2}}} = \frac{x}{a^2} [\frac{1}{\sqrt{a^2 + x^2}}] + C \tag{A.14}$$

Appendix B

Chain Rule

$$\frac{d}{dx}(\frac{1}{\sqrt{x^2+R^2}}) = \frac{d}{dx}(x^2+R^2)^{-\frac{1}{2}} \tag{B.1}$$

The chain rule $f(g(x))' = f'(g(x) \cdot g'(x)$. In this case $g(x) = x^2 + R^2$. From this

$$f(g(x)) = g(x)^{-\frac{1}{2}} \tag{B.2}$$

thus

$$f'(g(x)) = -\frac{1}{2}g(x)^{-\frac{3}{2}} \tag{B.3}$$

and

$$g'(x) = 2x \tag{B.4}$$

Thus

$$f(g(x))' = -\frac{1}{2}g(x)^{-\frac{3}{2}} \cdot 2x \tag{B.5}$$

$$\frac{d}{dx}(\frac{1}{\sqrt{x^2+R^2}}) = \frac{-x}{(x^2+R^2)^{\frac{3}{2}}} \tag{B.6}$$