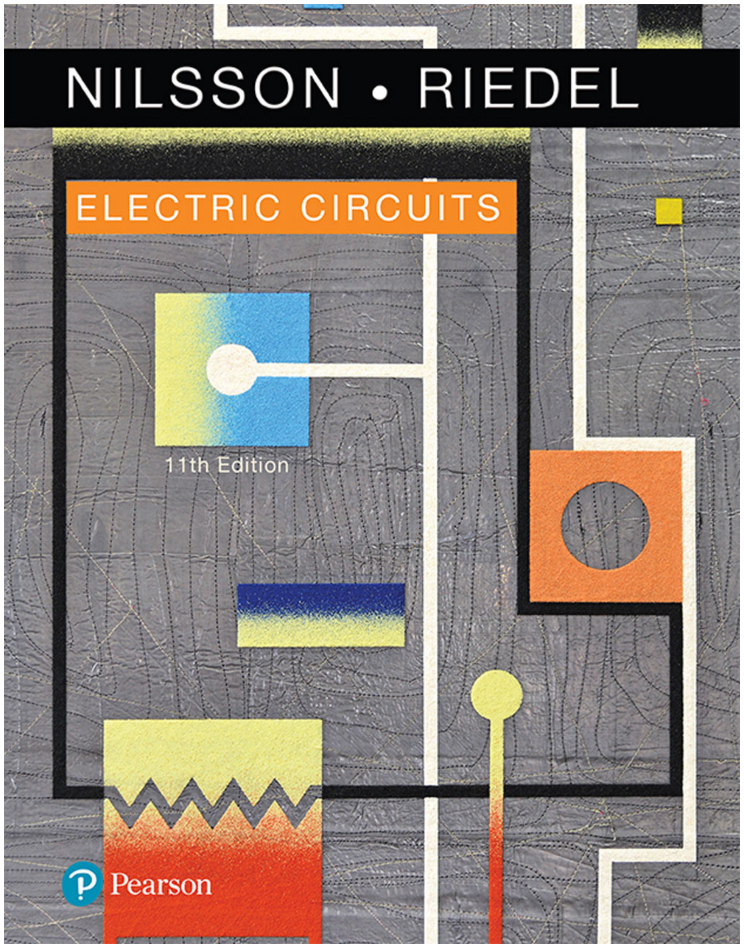
Electric Circuits

Eleventh Edition



Chapter 8

Natural and Step Responses of *RLC* Circuits

Pearson

Learning Objectives

- The Natural Response of a Parallel *RLC* Circuit
- The Forms of Natural Response of a Parallel *RLC* Circuit
- The Step Response of a Parallel *RLC* Circuit
- The Natural & Step Response of a Series *RLC* Circuit
- The Circuit with Two Integrating Amplifiers*

Practical Perspective - Clock for Computer Timing

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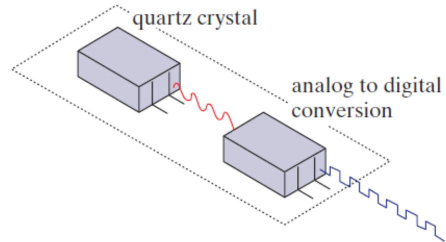

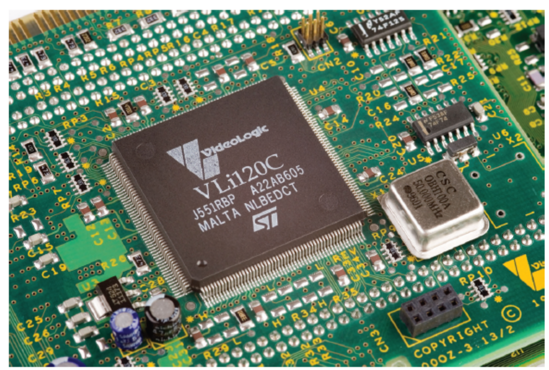

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8.1 Introduction to the Natural Response of a Parallel RLC Circuit

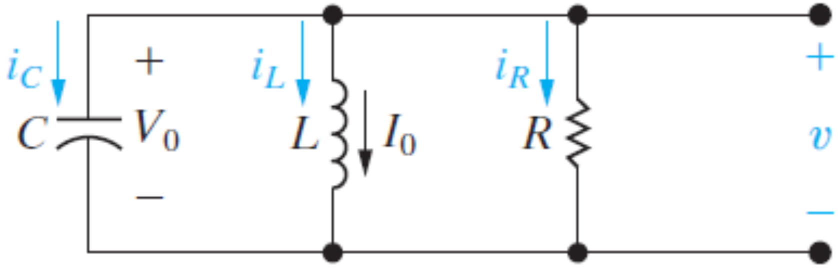


Figure 8.1: A circuit used to illustrate the natural response of a parallel *RLC* circuit.

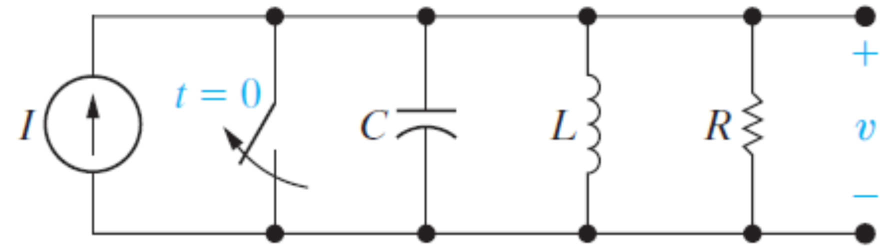


Figure 8.2: A circuit used to illustrate the step response of a parallel *RLC* circuit.

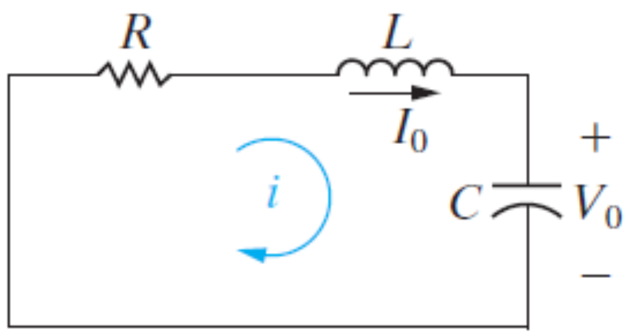


Figure 8.3: A circuit used to illustrate the natural response of a series *RLC* circuit.

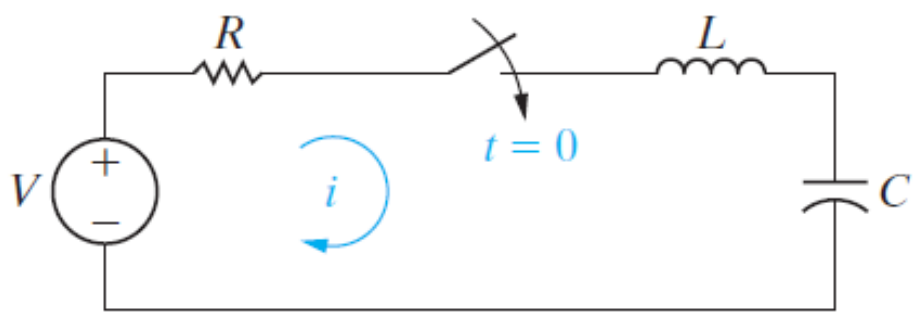
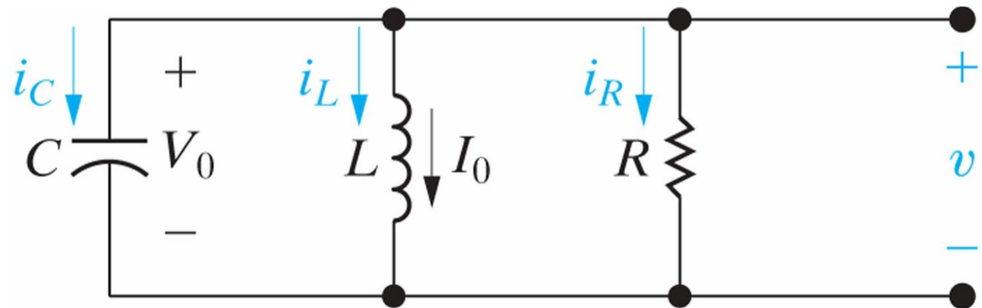


Figure 8.4: A circuit used to illustrate the step response of a series *RLC* circuit.

Natural Response of a Parallel RLC Circuit

The initial voltage on the capacitor, V_0 represents the initial energy stored in the capacitor. The initial current through the inductor, I_0 represents the initial energy stored in the inductor.



$$\frac{v}{R} + \frac{1}{L}\int_0^t v\,d\tau + I_0 + C\frac{dv}{dt} = 0$$

$$\frac{1}{R}\frac{dv}{dt} + \frac{v}{L} + C\frac{d^2v}{dt^2} = 0$$

Second-order circuit

$$\boxed{\frac{d^2v}{dt^2} + \frac{1}{RC}\frac{dv}{dt} + \frac{v}{LC} = 0}$$

General Solution for the Second-Order Differential Equation

$$\frac{d^2v}{dt^2} + \frac{1}{RC}\frac{dv}{dt} + \frac{v}{LC} = 0$$

$$v = Ae^{st}$$

$$\Rightarrow As^2e^{st} + \frac{As}{RC}e^{st} + \frac{Ae^{st}}{LC} = 0$$

$$\Rightarrow \boxed{s^2 + \frac{s}{RC} + \frac{1}{LC} = 0}$$

Characteristic Equation

Two roots:

$$s_1 = -\frac{1}{2RC} + \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}}$$

$$s_2 = -\frac{1}{2RC} - \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}}$$

TABLE 8.1 Natural-Response Parameters of the Parallel *RLC* Circuit

Parameter	Terminology	Value in Natural Response
s_1, s_2	$\boxed{\text{Characteristic roots}}$	$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2}$ $s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2}$
α	$\boxed{\text{Neper frequency}}$	$\alpha = \frac{1}{2RC}$
ω_0	$\boxed{\text{Resonant radian frequency}}$	$\omega_0 = \frac{1}{\sqrt{LC}}$

Denoting the two corresponding solutions v_1 and v_2, respectively, their sum also is a solution.

$$v = v_1 + v_2 = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

Only when $s_1 \neq s_2$

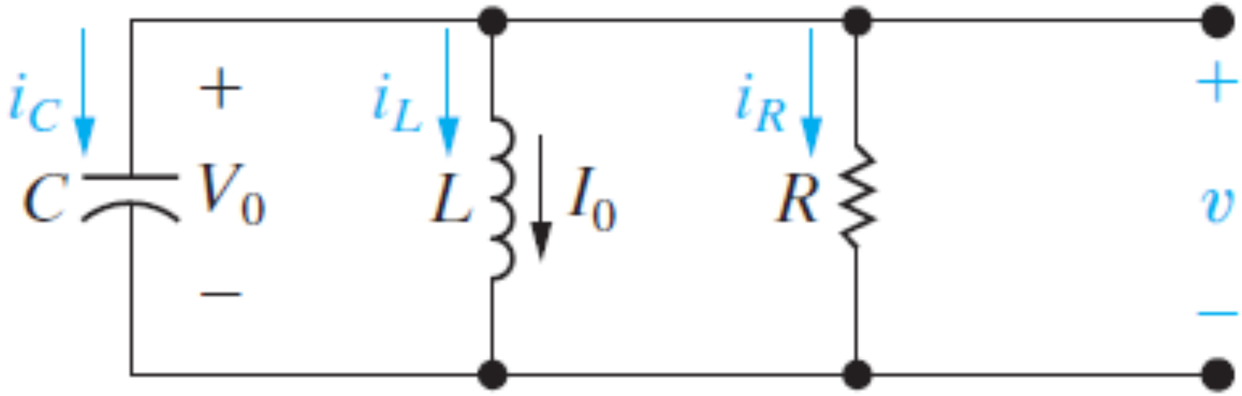
Classification of Solutions

Over damped: $\omega_0{}^2 < \alpha^2$, both roots are real and distinct

Under damped: $\omega_0{}^2 > \alpha^2$, two roots are complex conjugates

Critically damped: $\omega_0 = \alpha$, two roots are real and equal

Assessment Problems 8.1



For, R = 100 Ω and L = 20 mH:

a) What value of C makes the voltage response critically damped?

b) If C is adjusted to give a neper frequency of 5 krad/s, find that value of C and the roots of the characteristic equation.

c) If C is adjusted to give a resonant frequency of 20 krad/s, find that value of C and the roots of the characteristic equation.

8.2 The Forms of the Natural Response of a Parallel RLC Circuit

- Calculate the two roots s_1 and s_2, based on the given R, L and C
- Determine whether the response is over–, under–, or critically-damped
- Find the unknown coefficients, A_1 and A_2

Overdamped Voltage Response

$$\boxed{v = A_1 e^{s_1 t} + A_2 e^{s_2 t}}$$

$$\boxed{v(0^+) = A_1 + A_2} = V_0$$

$$\boxed{\frac{dv(0^+)}{dt} = s_1 A_1 + s_2 A_2}$$

$$\frac{dv(0^+)}{dt} = \frac{i_C(0^+)}{C} \qquad i_C(0^+) = \frac{-V_0}{R} - I_0$$

We summarize the process for finding the overdamped response, $v(t)$, as follows:

1. Find the roots of the characteristic equation, s_1 and s_2, using the values of R, L, and C.

2. Find $v(0^+)$ and $dv(0^+)/dt$ using circuit analysis.

3. Find the values of A_1 and A_2 by solving the following equations simultaneously:

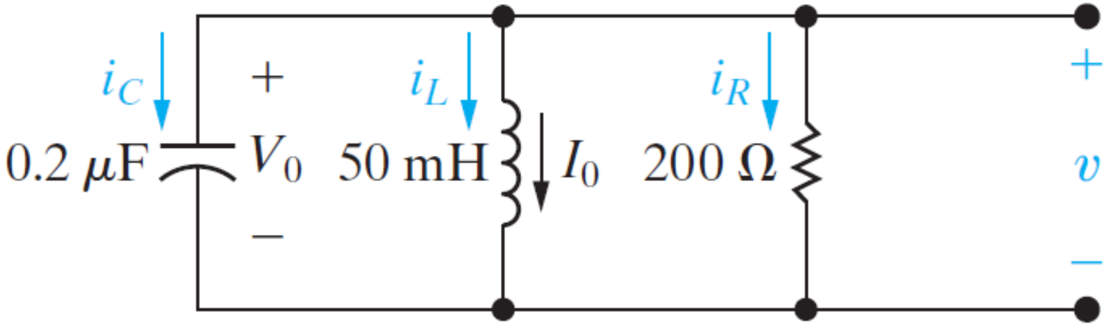
$$v(0^+) = A_1 + A_2$$

$$dv(0^+)/dt = i_C(0^+)/C = s_1A_1 + s_2A_2$$

4. Substitute the values for s_1, s_2, A_1, and A_2 into Eq. 8.18 to determine the expression for $v(t)$ for $t > 0$.

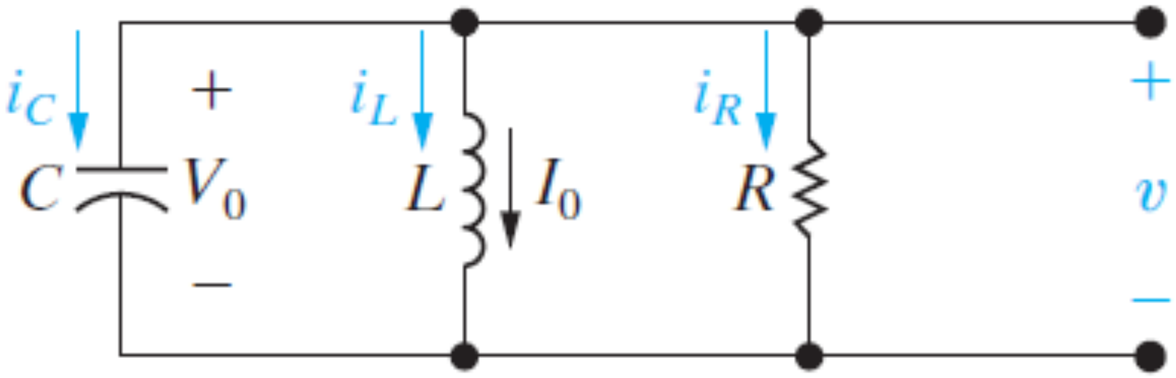
$$v = A_1e^{s_1t} + A_2e^{s_2t}$$

Assessment Problems 8.2



Use the integral relation between i_L and v to find an expression for i_L for the above circuit (from Example 8.2).

Assessment Problems 8.3



R = 2kΩ, L = 250 mH, and C = 10 nF. Also, the initial values: $V_0 =$ 0 and $I_0 =$ -4 mA. Find:

a) $i_R(0^+)$

b) $i_C(0^+)$

c) dv/dt at $t = 0^+$

d) A_1 and A_2

e) $v(t)$ for t ≥ 0

Underdamped Voltage Response

$$s_1 = -\alpha + \sqrt{-(\omega_0^2 - \alpha^2)} = -\alpha + j\sqrt{\omega_0^2 - \alpha^2} = -\alpha + j\omega_d$$

$$s_2 = -\alpha - j\omega_d \rightarrow \omega_d = \sqrt{\omega_0^2 - \alpha^2}$$

$$\begin{aligned} v(t) &= A_1 e^{(-\alpha + j\omega_d)t} + A_2 e^{-(\alpha + j\omega_d)t} \\ &= A_1 e^{-\alpha t} e^{j\omega_d t} + A_2 e^{-\alpha t} e^{-j\omega_d t} \rightarrow e^{\pm j\theta} = \cos\theta \pm j\sin\theta \\ &= e^{-\alpha t}(A_1 \cos\omega_d t + jA_1 \sin\omega_d t + A_2 \cos\omega_d t - jA_2 \sin\omega_d t) \\ &= e^{-\alpha t}[\underbrace{\boxed{(A_1 + A_2)}}_{B_1} \cos\omega_d t + \underbrace{\boxed{j(A_1 - A_2)}}_{B_2} \sin\omega_d t] \end{aligned}$$

Real!

$$v(t) = B_1 e^{-\alpha t} \cos \omega_d t + B_2 e^{-\alpha t} \sin \omega_d t$$

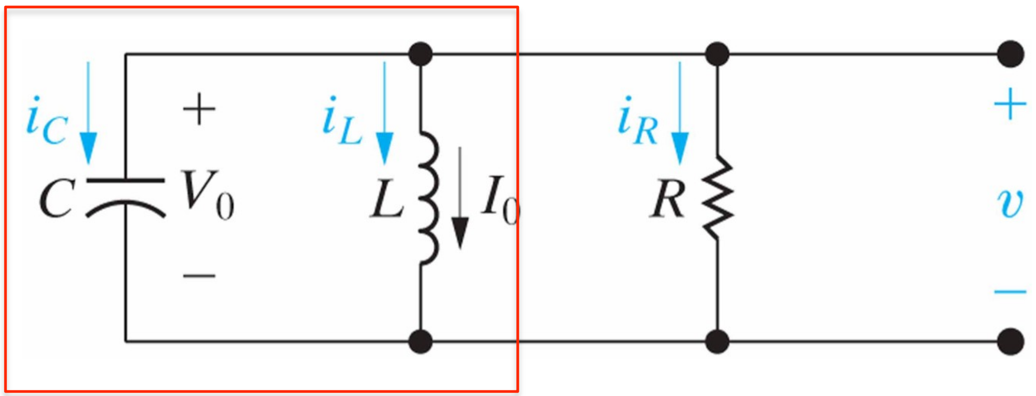
$$\boxed{\begin{aligned} v(0^+) &= V_0 = B_1 \\ \frac{dv(0^+)}{dt} &= \frac{i_c(0^+)}{C} = -\alpha B_1 + \omega_d B_2 \end{aligned}}$$

Because α determines how quickly the oscillations decay, it is also referred to as the damping factor (or damping coefficient).

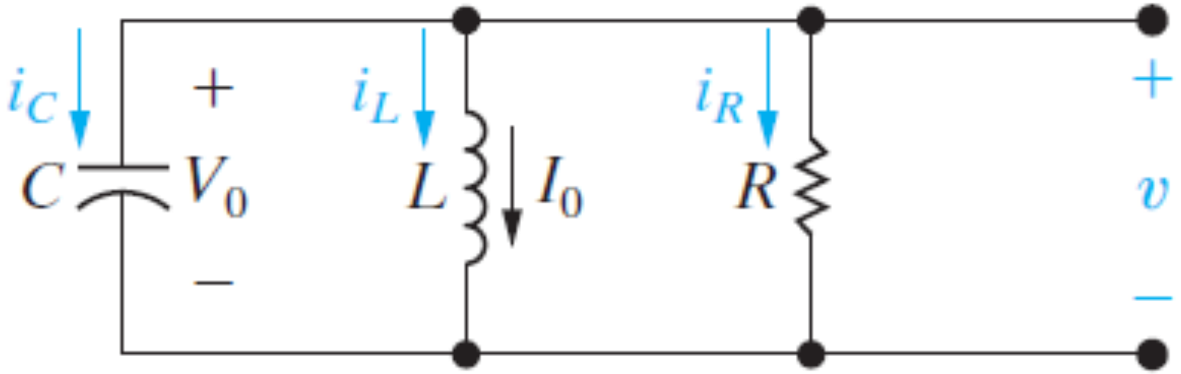
The oscillatory behavior is possible because of the two types of energy-storage elements in the circuit: the *inductor* and the *capacitor*.

Oscillations at $\omega_d = \omega_0$ sustained!

No Dissipation through R.



Assessment Problems 8.4



A 10 mH inductor, 1 μF capacitor, and a variable resistor are connected in parallel as shown. The resistor is adjusted to achieve: $s_{1,2}$ = -8000 ± j 6000 rad/s. $V_0 = 0$ and $I_0 = 80$ mA. Find:

a) R

b) dv/dt

c) B_1 and B_2

d) $i_L(t)$

Overdamped or Underdamped?

- When specifying the desired response of a second order system, you may want to *reach the final value in the shortest time possible*, and you *may not be concerned with small oscillations about that final value*. If so, you would design the system components to achieve an **underdamped response**.
- On the other hand, you *may be concerned that the response not exceed its final value*, perhaps to *ensure that components are not damaged*. In such a case, you would design the system components to achieve an **overdamped response**, and you would *have to accept a relatively slow rise to the final value*.

Critically Damped Voltage Response

$$s_1 = s_2 = -\alpha = -\frac{1}{2RC}$$

In the case of a repeated root, the solution involves a simple exponential term plus the product of a linear and an exponential term.

$$\boxed{v(t) = D_1 t e^{-\alpha t} + D_2 e^{-\alpha t}}$$

$$\boxed{v(0^+) = V_0 = D_2}$$

$$\boxed{\frac{dv(0^+)}{dt} = \frac{i_C(0^+)}{C} = D_1 - \alpha D_2}$$

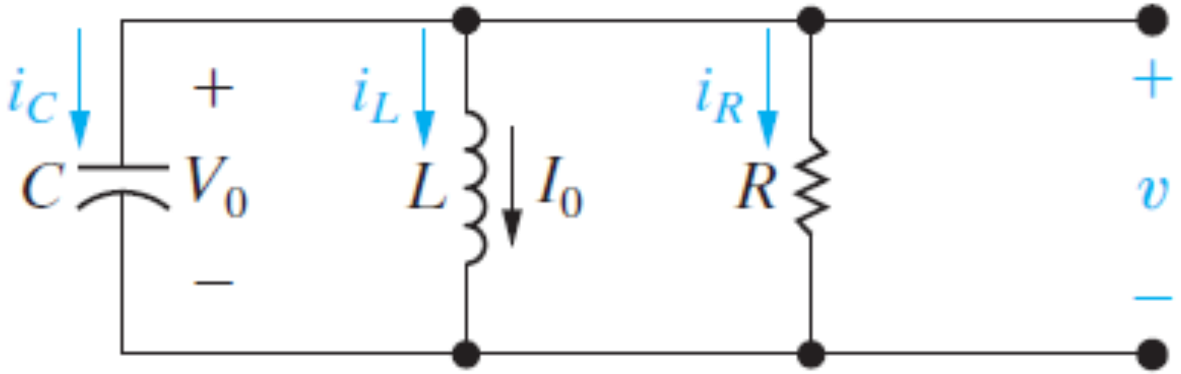
You rarely encounter critically damped systems in practice, because ω_0 must equal α exactly. Both of these quantities depend on circuit parameters, and in a real circuit it is very difficult to choose component values that satisfy an exact equality relationship.

TABLE 8.2 Equations for analyzing the natural response of parallel *RLC* circuits

Characteristic equation	$s^2 + \frac{1}{RC}s + \frac{1}{LC} = 0$
Neper, resonant, and damped frequencies	$\alpha = \frac{1}{2RC} \quad \omega_0 = \sqrt{\frac{1}{LC}} \quad \omega_d = \sqrt{\omega_0^2 - \alpha^2}$
Roots of the characteristic equation	$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2}, \quad s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2}$
$\alpha^2 > \omega_0^2$: overdamped	$v(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}, t \geq 0$ $v(0^+) = A_1 + A_2 = V_0$ $\frac{dv(0^+)}{dt} = s_1 A_1 + s_2 A_2 = \frac{1}{C}\left(\frac{-V_0}{R} - I_0\right)$
$\alpha^2 < \omega_0^2$: underdamped	$v(t) = B_1 e^{-\alpha t} \cos \omega_d t + B_2 e^{-\alpha t} \sin \omega_d t, t \geq 0$ $v(0^+) = B_1 = V_0$ $\frac{dv(0^+)}{dt} = -\alpha B_1 + \omega_d B_2 = \frac{1}{C}\left(\frac{-V_0}{R} - I_0\right)$
$\alpha^2 = \omega_0^2$: critically damped	$v(t) = D_1 t e^{-\alpha t} + D_2 e^{-\alpha t}, t \geq 0$ $v(0^+) = D_2 = V_0$ $\frac{dv(0^+)}{dt} = D_1 - \alpha D_2 = \frac{1}{C}\left(\frac{-V_0}{R} - I_0\right)$

(Note that the equations in the last three rows assume that the reference direction for the current in every component is in the direction of the reference voltage drop across that component.)

Assessment Problems 8.5



Here, L = 0.4 H and C = 10 μF, and R is adjusted to achieve critical damping. The initial energy stored in circuit is 25 mJ distributed equally between the inductor and capacitor. Find:

a) R

b) V_0

c) I_0

d) D_1 and D_2

e) $i_R(t)$

8.3 Step Response of a Parallel RLC Circuit

$$i_L + i_R + i_C = I$$

$$i_L + \frac{v}{R} + C\frac{dv}{dt} = I$$

$$\boxed{v = L\frac{di_L}{dt} \quad \frac{dv}{dt} = L\frac{d^2 i_L}{dt^2}}$$

$$i_L + \frac{L}{R}\frac{di_L}{dt} + LC\frac{d^2 i_L}{dt^2} = I$$

$$\boxed{\frac{d^2 i_L}{dt^2} + \frac{1}{RC}\frac{di_L}{dt} + \frac{i_L}{LC} = \frac{I}{LC}}$$

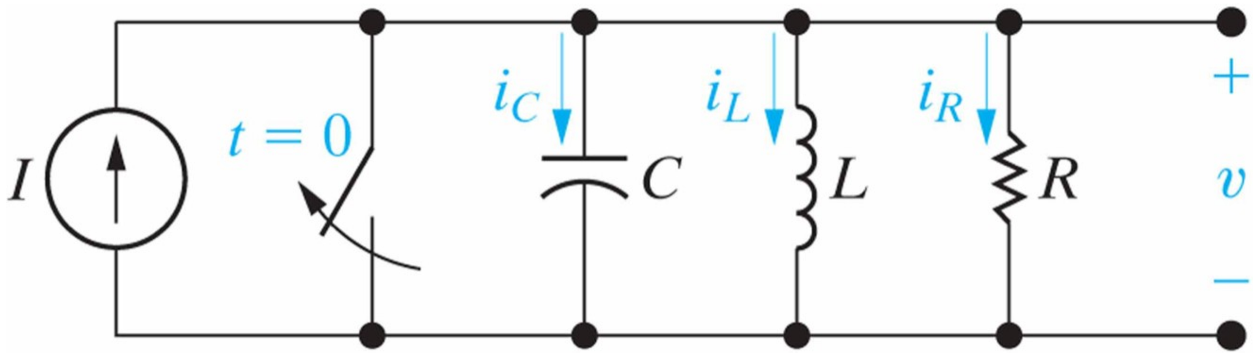


Figure 8.11: A circuit used to describe the step response of a parallel *RLC* circuit.

The Indirect Approach

$$i_L + \frac{v}{R} + C\frac{dv}{dt} = I$$

$$\frac{1}{L}\int_0^t v\,d\tau + \frac{v}{R} + C\frac{dv}{dt} = I$$

$$\frac{d^2v}{dt^2} + \frac{1}{RC}\frac{dv}{dt} + \frac{v}{LC} = 0$$

$$\boxed{\begin{aligned} v &= A_1e^{s_1t} + A_2e^{s_2t} \\ v &= B_1e^{-\alpha t}\cos\omega_d t + B_2e^{-\alpha t}\sin\omega_d t \\ v &= D_1te^{-\alpha t} + D_2e^{-\alpha t} \end{aligned}}$$

$$\boxed{\begin{aligned} i_L &= I + A_1'e^{s_1t} + A_2'e^{s_2t} \\ i_L &= I + B_1'e^{-\alpha t}\cos\omega_d t + B_2'e^{-\alpha t}\sin\omega_d t \\ i_L &= I + D_1'te^{-\alpha t} + D_2'e^{-\alpha t} \end{aligned}}$$

The Direct Approach

$$i = \underbrace{I_f}_{=I} + \left\{ \begin{array}{r} \text{function of the same form} \\ \text{as the natural response} \end{array} \right\}$$

$$v = \underbrace{V_f}_{=0} + \left\{ \begin{array}{r} \text{function of the same form} \\ \text{as the natural response} \end{array} \right\}$$

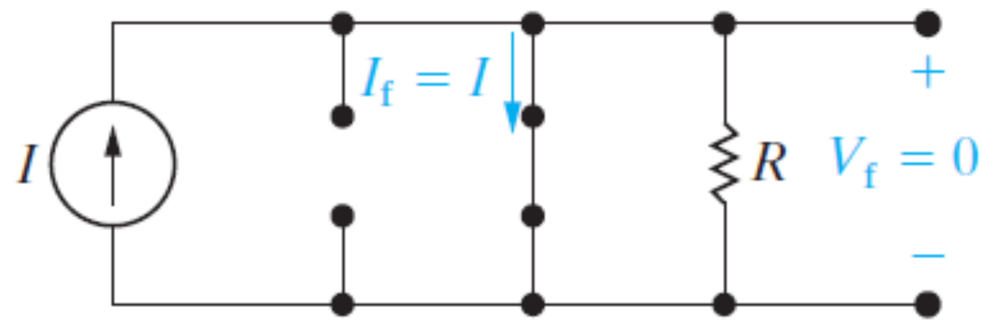


Figure 8.12: The circuit in Fig. 8.11 as $t \to \infty$.

TABLE 8.3 Equations for analyzing the step response of parallel *RLC* circuits

Characteristic equation	$s^2 + \frac{1}{RC}s + \frac{1}{LC} = \frac{I}{LC}$
Neper, resonant, and damped frequencies	$\alpha = \frac{1}{2RC} \quad \omega_0 = \sqrt{\frac{1}{LC}} \quad \omega_d = \sqrt{\omega_0^2 - \alpha^2}$
Roots of the characteristic equation	$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2}, \quad s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2}$
$\alpha^2 > \omega_0^2$: overdamped	$i_L(t) = I_f + A'_1e^{s_1t} + A'_2e^{s_2t}, \quad t \geq 0$ $i_L(0^+) = I_f + A'_1 + A'_2 = I_0$ $\frac{di_L(0^+)}{dt} = s_1A'_1 + s_2A'_2 = \frac{V_0}{L}$
$\alpha^2 < \omega_0^2$: underdamped	$i_L(t) = I_f + B'_1e^{-\alpha t}\cos\omega_dt + B'_2e^{-\alpha t}\sin\omega_dt, \quad t \geq 0$ $i_L(0^+) = I_f + B'_1 = I_0$ $\frac{di_L(0^+)}{dt} = -\alpha B'_1 + \omega_dB'_2 = \frac{V_0}{L}$
$\alpha^2 = \omega_0^2$: critically damped	$i_L(t) = I_f + D'_1te^{-\alpha t} + D'_2e^{-\alpha t}, \quad t \geq 0$ $i_L(0^+) = I_f + D'_2 = I_0$ $\frac{di_L(0^+)}{dt} = D'_1 - \alpha D'_2 = \frac{V_0}{L}$

(Note that the equations in the last three rows assume that the reference direction for the current in every component is in the direction of the reference voltage drop across that component.)

Comparing the Three-Step Response Functions

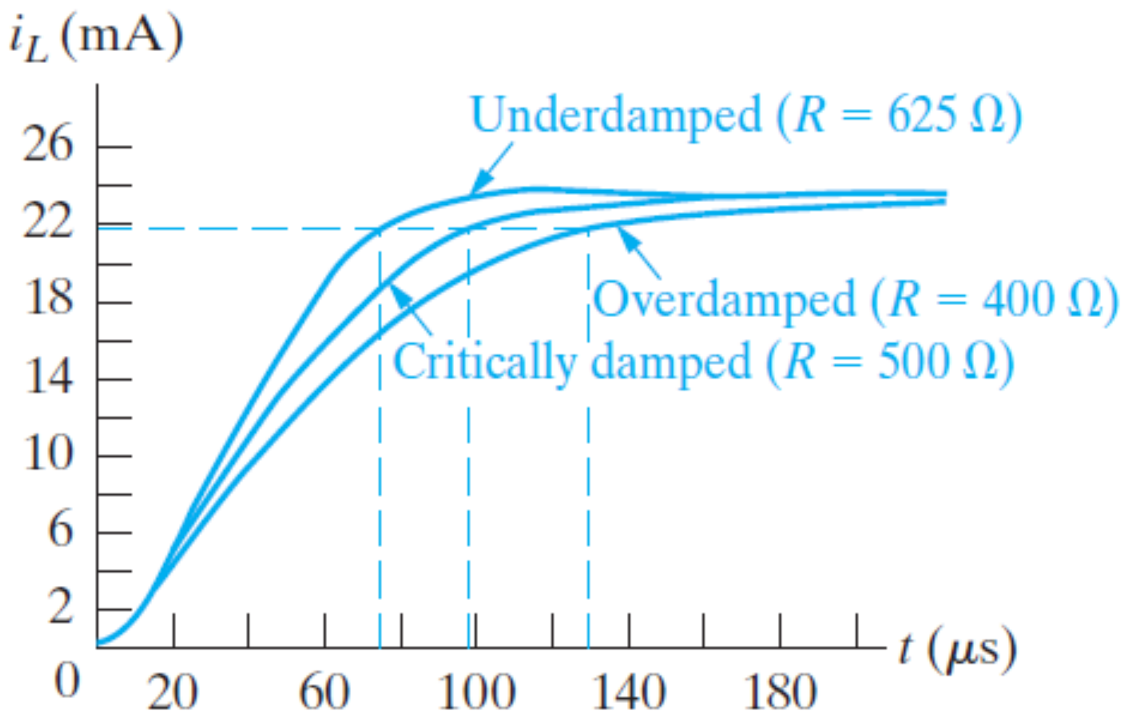
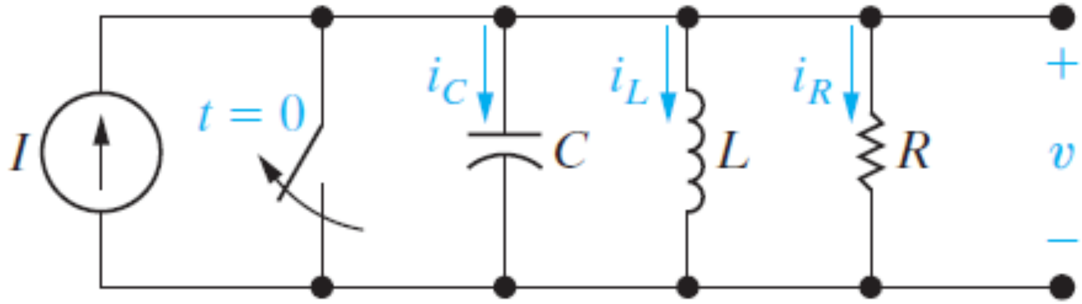


Figure 8.14: The current plots for Example 8.9.

Assessment Problems 8.6



For the circuit show, R = 500 Ω. L = 0.64 H, C = 1 μF, and I = -1 A. The initial voltage drop across the capacitor is 40 V and the initial inductor current is 0.5 A. Find:

a) $i_R(0^+)$

b) $i_C(0^+)$

c) $di_L(0^+)/dt$

d) What are the roots, s_1 and s_2, of the characteristic equation?

e) $i_L(t)$ for $t > 0$?

f) $v(t)$ for $t > 0$?

8.4 Natural & Step Response of a Series RLC Circuit

Natural Response of a Series RLC Circuit

$$Ri + L\frac{di}{dt} + \frac{1}{C}\int_0^t i\,d\tau + V_0 = 0$$

Differentiate

$$R\frac{di}{dt} + L\frac{d^2 i}{dt^2} + \frac{i}{C} = 0$$

Rearrange

$$\boxed{\frac{d^2 i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{i}{LC} = 0}$$

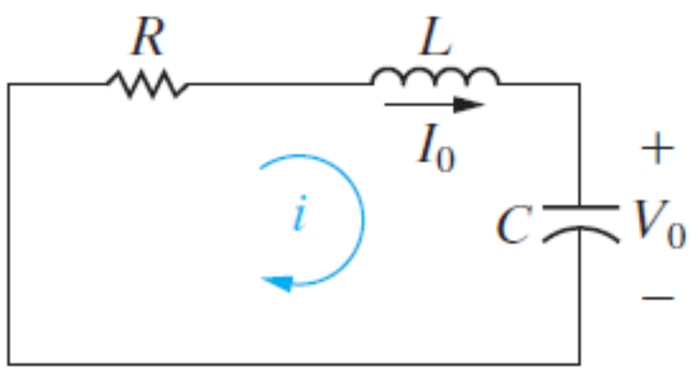


Figure 8.15: A circuit used to illustrate the natural response of a series *RLC* circuit.

$$\boxed{s^2 + \frac{R}{L}s + \frac{1}{LC} = 0}$$

$$s_{1,2} = -\frac{R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$$

$$\boxed{\begin{aligned} \omega_0 &= \frac{1}{\sqrt{LC}} \text{ rad/s} \\ \alpha &= \frac{R}{2L} \text{ rad/s} \end{aligned}}$$

Radian Resonant Frequency

Neper Frequency

$$\boxed{\begin{aligned} i(t) &= A_1 e^{s_1 t} + A_2 e^{s_2 t} \quad \text{(overdamped)}, \\ i(t) &= B_1 e^{-\alpha t} \cos \omega_d t + B_2 e^{-\alpha t} \sin \omega_d t \quad \text{(underdamped)}, \\ i(t) &= D_1 t e^{-\alpha t} + D_2 e^{-\alpha t} \quad \text{(critically damped)}. \end{aligned}}$$

TABLE 8.4 Equations for analyzing the natural response of series *RLC* circuits

Characteristic equation	$s^2 + \frac{R}{L}s + \frac{1}{LC} = 0$
Neper, resonant, and damped frequencies	$\alpha = \frac{R}{2L} \quad \omega_0 = \sqrt{\frac{1}{LC}} \quad \omega_d = \sqrt{\omega_0^2 - \alpha^2}$
Roots of the characteristic equation	$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2}, \quad s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2}$
$\alpha^2 > \omega_0^2$: overdamped	$i(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}, t \geq 0$ $i(0^+) = A_1 + A_2 = I_0$ $\frac{di(0^+)}{dt} = s_1 A_1 + s_2 A_2 = \frac{1}{L}(-RI_0 - V_0)$
$\alpha^2 < \omega_0^2$: underdamped	$i(t) = B_1 e^{-\alpha t} \cos \omega_d t + B_2 e^{-\alpha t} \sin \omega_d t, \quad t \geq 0$ $i(0^+) = B_1 = I_0$ $\frac{di(0^+)}{dt} = -\alpha B_1 + \omega_d B_2 = \frac{1}{L}(-RI_0 - V_0)$
$\alpha^2 = \omega_0^2$: critically damped	$i(t) = D_1 t e^{-\alpha t} + D_2 e^{-\alpha t}, \quad t \geq 0$ $i(0^+) = D_2 = I_0$ $\frac{di(0^+)}{dt} = D_1 - \alpha D_2 = \frac{1}{L}(-RI_0 - V_0)$

(Note that the equations in the last three rows assume that the reference direction for the current in every component is in the direction of the reference voltage drop across that component.)

Step Response of a Series RLC Circuit

$$V = Ri + L\frac{di}{dt} + v_C$$

$$i = C\frac{dv_C}{dt}$$

$$\frac{di}{dt} = C\frac{d^2v_C}{dt^2}$$

$$\boxed{\frac{d^2v_C}{dt^2} + \frac{R}{L}\frac{dv_C}{dt} + \frac{v_C}{LC} = \frac{V}{LC}}$$

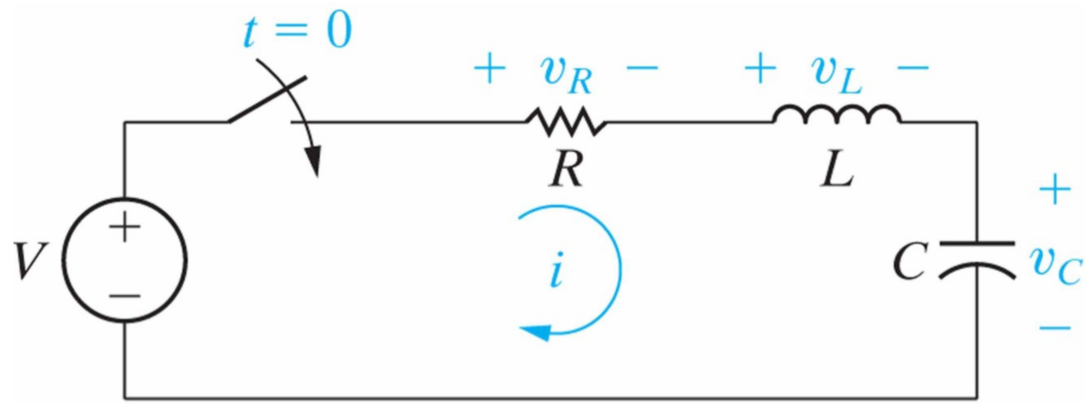


Figure 8.16: A circuit used to illustrate the step response of a series *RLC* circuit.

Compare Eq. 8.34 and Eq. 8.23:

the procedure for finding v_C similar that for finding i_L

$$v_C = V_f + A_1' e^{s_1 t} + A_2' e^{s_2 t} \quad \text{(overdamped)},$$

$$v_C = V_f + B_1' e^{-\alpha t}\cos\omega_d t + B_2' e^{-\alpha t}\sin\omega_d t \quad \text{(underdamped)},$$

$$v_C = V_f + D_1' t e^{-\alpha t} + D_2' e^{-\alpha t} \quad \text{(critically damped)},$$

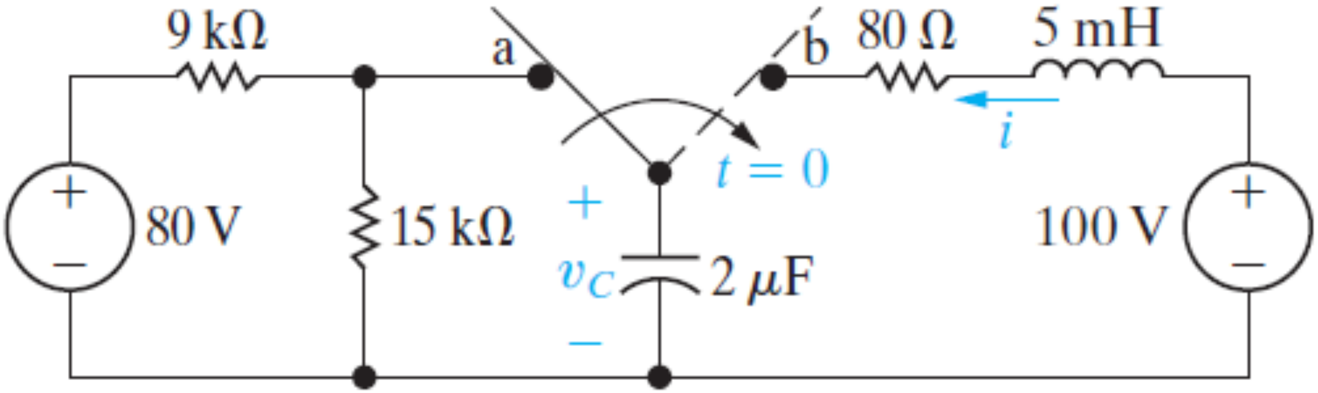
Pearson

TABLE 8.5 Equations for analyzing the step response of series *RLC* circuits

Characteristic equation	$s^2 + \frac{R}{L}s + \frac{1}{LC} = \frac{V}{LC}$
Neper, resonant, and damped frequencies	$\alpha = \frac{R}{2L} \quad \omega_0 = \sqrt{\frac{1}{LC}} \quad \omega_d = \sqrt{\omega_0^2 - \alpha^2}$
Roots of the characteristic equation	$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2}, \quad s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2}$
$\alpha^2 > \omega_0^2$: overdamped	$v_C(t) = V_f + A_1' e^{s_1 t} + A_2' e^{s_2 t}, t \geq 0$ $v_C(0^+) = V_f + A_1' + A_2' = V_0$ $\frac{dv_C(0^+)}{dt} = s_1 A_1' + s_2 A_2' = \frac{I_0}{C}$
$\alpha^2 < \omega_0^2$: underdamped	$v_C(t) = V_f + B_1' e^{-\alpha t} \cos \omega_d t + B_2' e^{-\alpha t} \sin \omega_d t, t \geq 0$ $v_C(0^+) = V_f + B_1' = V_0$ $\frac{dv_C(0^+)}{dt} = -\alpha B_1' + \omega_d B_2' = \frac{I_0}{C}$
$\alpha^2 = \omega_0^2$: critically damped	$v_C(t) = V_f + D_1' t e^{-\alpha t} + D_2' e^{-\alpha t}, t \geq 0$ $v_C(0^+) = V_f + D_2' = V_0$ $\frac{dv_C(0^+)}{dt} = D_1' - \alpha D_2' = \frac{I_0}{C}$

(Note that the equations in the last three rows assume that the reference direction for the current in every component is in the direction of the reference voltage drop across that component.)

Assessment Problems 8.7/8



The switch has been in position a for a long time. At t=0 it is moved to b. Find $v_C(t)$ and $i(t)$ for $t \geq 0$.

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