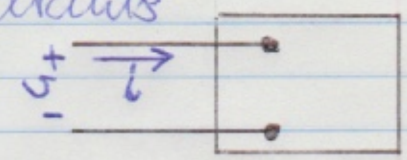
Chapter 10 Sinusoidal Steady-State Power

Power associated with ac circuits



10.1 Instantaneous Power

In ac circuits the voltage and current are given as time-dependent sinusoidal functions with relative phases:-

$$v(t) = V_m \cos(\omega t + \theta_v)$$
$$i(t) = I_m \cos(\omega t + \theta_i)$$

Where, $\theta_{v,i}$ are the voltage and current phase angles (see previous chapter).

Thus, at any instant in time, the <u>instantaneous</u> power:

$$p(t) = v(t)\, i(t)$$

For convenience, we shift the current so that its phase is zero, i.e. it passes through a maximum at $t=0$. To do this we shift the phase of voltage by θ_i,

$$v(t) = V_m \cos(\omega t + \theta_v - \theta_i) \quad 10.1$$
$$i(t) = I_m \cos \omega t \quad 10.2$$

$$p(t) = V_m I_m \cos(\omega t + \theta_v - \theta_i) \cos \omega t$$

We can re-write this as using $\cos\alpha\cos\beta = \frac{1}{2}\cos(\alpha-\beta) + \frac{1}{2}\cos(\alpha+\beta)$

$$p(t) = \frac{1}{2} V_m I_m \cos(\theta_v - \theta_i) + \frac{1}{2} V_m I_m \cos(2\omega t + \theta_v - \theta_i)$$

Taking this further, using $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

we can therefore write the instantaneous power as

$$p(t) = \overbrace{\tfrac{1}{2}V_m I_m \cos(\theta_v - \theta_i)}^{\text{constant}} + \tfrac{1}{2}V_m I_m \cos(\theta_v - \theta_i)\cos \overbrace{2\omega t}^{\text{double frequency}}$$
$$- \tfrac{1}{2}V_m I_m \sin(\theta_v - \theta_i)\sin \overbrace{2\omega t}^{\text{double}} \qquad 10.3$$

Notice: instantaneous power is negative during each cycle... remember using the passive sign convention that negative power means that the circuit element is delivering power to the circuit - energy is being extracted from inductors and/or capacitors.

10.2 Average & Reactive Power

We assign different symbols to the terms in Eq. 10.3:

$$p(t) = P + P\cos 2\omega t - Q\sin 2\omega t \qquad 10.4$$

which defines the following quantities:

Average (Real) Power : $P = \frac{1}{2}V_m I_m \cos(\theta_v - \theta_i)$ 10.5

which we can also write in terms of the rms or effective value

$$P = V_{rms} I_{rms} \cos(\theta_v - \theta_i) \qquad 10.9$$

Where, $V_{rms} = \frac{V_m}{\sqrt{2}}$ and $I_{rms} = \frac{I_m}{\sqrt{2}}$

Similarly, we define

Reactive Power : $Q = \frac{1}{2}V_m I_m \sin(\theta_v - \theta_i) = V_{rms} I_{rms} \sin(\theta_v - \theta_i)$ 10.6/10.10

P is known both as the average and real power because it is the 'useful' power developed in the circuit. To see this, evaluate the total power over one cycle. The sin2ωt, terms result in zero contributions.

What do we mean by 'useful' power? Well, the point of a circuit is to take electrical energy provided by the source and use it to drive something, e.g. lightbulb. In other words transform electrical energy into some other form of energy, such as ~~heat~~ thermal energy to light the bulb.

So what does it mean to have both positive and negative power. Let's take our basic circuit elements in turn:

Purely Resistive Circuit

If the circuit between terminals is purely resistive, v and i are in phase $\Rightarrow \theta_v = \theta_i$:

$$p(t) = P + P\cos 2\omega t \quad \text{(purely resistive)}$$

This is the instantaneous real power, the average is given by P, and recognise that p can never be negative for a purely resistive network, i.e. power can never be extracted from a purely resistive network - resistors only dissipate energy.

Purely Inductive Circuit

Recall that for inductors, the current lags the voltage by 90°, i.e. $\theta_v - \theta_i = 90°$, and since $\sin 90° = 1$:

$$p(t) = -Q\sin 2\omega t \quad \text{(purely inductive)}$$

Thus, the average power is zero and no net power is transformed into 'useful' energy. Power in purely inductive circuits continually bounces between the source and the inductor as it charges and discharges at a rate of 2ω.

Hence, the phrase reactive power. Inductors respond or react to power supplied to the circuit. Thus, although P & Q have the same dimensions (units of power), to distinguish between average or real power from reactive power, we denote Q to have the units of var (volt-amp reactive).

Purely Capacitive Circuit

Conversely to inductors, for capacitors, the current leads the voltage by 90° → $\theta_v - \theta_i = -90°$. Nevertheless,

$$p = -Q \sin 2\omega t \quad \text{(purely capacitive)}$$

Again, the average power is zero and is measured in units of var. The energy supplied by source bounces between electrical energy stored between capacitor plates at it charges, and discharging.

Note: by convention Q is $\begin{cases}\text{positive for inductors}\\ \text{negative for capacitors}\end{cases}$

Power Factor : $pf = \cos(\theta_v - \theta_i)$ 10.8

Obviously the relative phase difference between the voltage and current it an important parameter determining both average and reactive powers. We denote this difference

$$\underline{\theta_v - \theta_i = \text{power factor angle}}$$

which defines the power factor above. Similarly, the

$$\text{reactive (power) factor} : rf = \sin(\theta_v - \theta_i)$$

But to fully specify the angle, it is described as lagging power factor (current lags voltage) or leading power factor (current leads voltage)

inductive load capacitive load

10.4 Complex Power

Let's take stock of where we've arrived at. We have found that there are 3 principal parameters that allows to compute the power in an ac circuit: (i) average power P, (ii) reactive power Q, and (iii) the power factor angle $(\theta_v - \theta_i)$.

Thus, at this stage it becomes convenient to combine them into a single entity:

complex power $$\underset{\Re\{S\}}{S = P} + j\underset{\Im\{S\}}{Q} \qquad 10.11$$

Recall: units P = W; Q = var, so to distinguish between these,

units of complex power : volt-amps (VA)

Table 10.2

Power and Units

Quantity	Units
Complex Power S	VA
Average Power P	W
Reactive Power Q	VAR

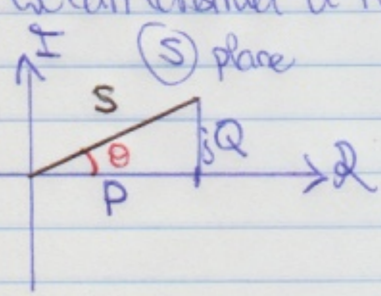
Definitions

$$P = V_{rms} I_{rms} \cos(\theta_v - \theta_i)$$

$$Q = V_{rms} I_{rms} \sin(\theta_v - \theta_i)$$

Thus, we can construct a right angle triangle or complex plane vector:

*Note: when designing the requirements of device operation need to take into account $|S| \geq |P|$



Apparent Power*

$$|S| = \sqrt{P^2 + Q^2} \qquad 10.12$$

Magnitude of complex power is referred to as the apparent power, measured in VA.

This follows from realising, $$\frac{Q}{P} = \frac{V_{rms} I_{rms} \sin(\theta_v - \theta_i)}{V_{rms} I_{rms} \cos(\theta_v - \theta_i)} = \tan\theta$$

pf angle $\theta = \theta_v - \theta_i$

10.5 Power Calculations

Finally, in this section we develop a set of complimentary ~~eqn~~ relations to enable power calculations.

We start with the basic definition of complex power: $S = P + jQ$, which we explicitly write as,

$$S = V_{rms} I_{rms} \cos(\theta_v - \theta_i) + j V_{rms} I_{rms} \sin(\theta_v - \theta_i).$$

Naturally, we can write this in exponential form (Euler's theorem):

$$S = V_{rms} I_{rms} e^{j(\theta_v - \theta_i)}$$

which in angle notation becomes

$$S = V_{rms} I_{rms} \underline{/(\theta_v - \theta_i)}$$

Also recognize, that the exponential form is a product of two exponentials:

$$S = (V_{rms} e^{j\theta_v})(I_{rms} e^{-j\theta_i})$$

and since V_{rms} and I_{rms} are just magnitudes and realising that if $z = re^{j\theta} \rightarrow z^* = re^{-j\theta}$:

$$\text{complex power: } S = \underline{V}_{rms} \underline{I}^*_{rms} \qquad 10.13$$

$$= \tfrac{1}{2} \underline{V}\, \underline{I}^* \qquad 10.14$$

Alternative (Explicit) Expressions

We can reexpress these equations in more explicit form in terms of the impedances by utilizing Ohm's Law : $\underline{V}_{rms} = Z\,\underline{I}_{rms}$.

equivalent inductance/capacitance (+) (−)

Hence, $S = (Z\,\underline{I}_{rms})\,\underline{I}^*_{rms} = |\underline{I}_{rms}|^2 Z = I^2_{rms}(R + jX)$

If we equate $S = P + jQ$:

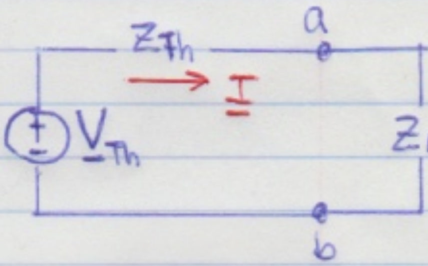
$$I^2_{rms} R + j I^2_{rms} X = P + jQ$$

$$\Rightarrow \quad P = I^2_{rms} R = \tfrac{1}{2} I_m^2 R = \frac{V^2_{rms}}{R} = \frac{1}{2}\frac{V_m^2}{R} \qquad 10.16/19$$

$$\Rightarrow Q = I^2_{rms} X = \tfrac{1}{2} I_m^2 X = \frac{V^2_{rms}}{X} = \frac{1}{2}\frac{V_m^2}{X} \qquad 10.17/20$$

[Recall: X can be positive (inductive) or negative (capacitive)]

10.6 Maximum Power Transfer



Recall from Ch. 4 we briefly analyzed the maximum power in a Thevénin equivalent. We found that $R_{Load} = R_{Th}$. What is the analogous result for ac circuits that have complex impedances?

Here, we have the Thevénin equivalent circuit with its corresponding Thevénin impedance Z_{Th} in series with the load impedance Z_L. The average power delivered: $P = |\underline{I}|^2 R_L = \dfrac{V^2_{Th} R_L}{(R_{Th}+R_L)^2 + (X_{Th}+X_L)^2}$ 10.22

To find maximum, we have R_{Th}, X_{Th} fixed, but now both R_L, X_L variables. Need to find when partials $= 0$:

$$\frac{\partial P}{\partial R_L} = 0 = \frac{\partial P}{\partial X_L} \quad \text{when} \quad X_L = -X_{Th} \ \text{and} \ R_L = R_{Th}$$

Thus, condition for maximum power transfer : $Z_L = Z^*_{Th}$ 10.21

$$\rightarrow \ P_{max} = \frac{|\underline{V}_{Th}|^2}{4R_L} = \frac{|\underline{V}_m|^2}{8R_L} \qquad 10.23/24$$