PHYS 1320 - Calculus-based Physics II

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Chapter 1

Module 1: Chapter 5 - Electric Charges and Fields

From Newton's Second Laws of Mechanics:

$$F = ma \tag{1.1}$$

A force can be recognized by the effect it has on an object. When studying gravitation, we examined the force of gravity that acts on all objects with mass. Similarly, the electric force acts on all objects with a property called charge. While gravity is an attractive force, the electric force can be either attractive or repulsive.

1.1 Section 5.1 Electric Charge

The ancient Greek philosopher Thales of Miletus (624-546 BCE)recorded that when amber was vigorously rubbed with a piece of fur, it created a force that attracted them to each other. They also attracted other non-metallic objects even when not touched.

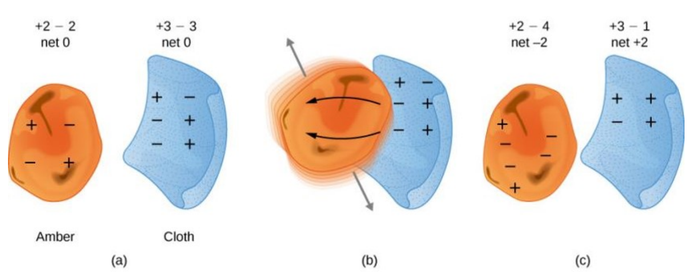


Figure 1.1: Materials rubbed together

The English physicist William Gilbert (1544-1603) also studied attractive forces. He worked with a variety of substances. His findings included:

- Metals never exhibited this force, whereas minerals did
- Two "electrified" amber rods would repel each other.

This suggested that were two types of electric properties: attractive and repulsive. This property came to be known as Electric Charge. The force is repulsive between between the same type of charge and attractive between the charges are of opposite types. Named after French physicist Charles Augustine de Coulomb (1736-1806), the unit of electric charge is the coulomb (C).

The American statesman and scientist Benjamin Franklin found that he could concentrate charge in Leyden jar [1] (a glass jar with two metal sheets one on the inside and one on the outside (essentially what we now call a capacitor). Franklin pointed out that the behavior could be explained by one type of charge remaining motionless and the other charge flowing from one piece of foil to the other. He had no way of determining which type of charge was moving, and unfortunately he guessed wrong: it was since learned that the charges that flow are the ones that Franklin named "negative" and the "positive" charges remain motionless.

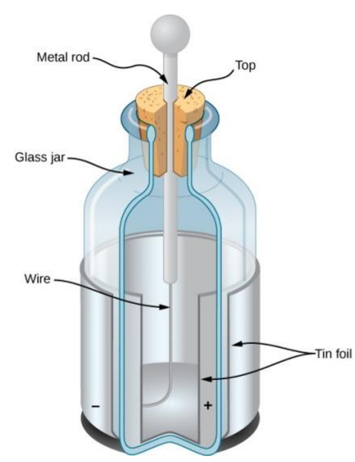


Figure 1.2: Leyden Jar

Observations about the Electric Force:

[1]Its invention was a discovery made independently by German cleric Ewald Georg von Kleist on 11 October 1745 and by Dutch scientist Pieter van Musschenbroek of Leiden (Leyden), Netherlands in 1745–1746. The invention was named after the city.

- The force acts without physical contact between objects
- The force is attractive or repulsive
- Not all objects are affected by this force
- The magnitude of the force decreases rapidly with distance between objects

Properties of Electric Charge

- Charge is quantized - The smallest amount of charge an object can have is $e = 1.602 * 10^{-19}C$. The charge on any object must be an integer multiple of e.
- The magnitude of a change is independent of the type. The smallest positive charge is $1.602 * 10^{-19}C$ and the smallest negative charge is $-1.602 * 10^{-19}C$; these values are exactly equal in magnitude.
- Charge is conserved. Change can not be created or destroyed. It can only be transferred. The net charge of the universe is constant.
- Charge is conserved in a closed system. Total charge in a closed system remains constant.

These last two items are referred to as the Law of Conservation of Charge.

The Sources of Charge: The Structure of the Atom

Atomic structure terminology

- Electron
- Proton
- Neutron
- Ion

This simplified model of a hydrogen atom shows a positively charged nucleus (consisting, in the case of hydrogen, of a single proton), surrounded by an electron "cloud." The charge of the electron cloud is equal (and opposite in sign) to the charge of the nucleus, but the electron does not have a definite location in space; hence, its representation here is as a cloud. Normal macroscopic amounts of matter contain immense numbers of atoms and molecules, and, hence, even greater numbers of individual negative and positive charges.

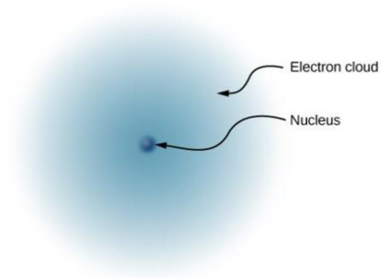


Figure 1.3: Simplified model of a hydrogen atom

The nucleus of a carbon atom is composed of six protons and six neutrons. As in hydrogen, the surrounding six electrons do not have definite locations and so can be considered to be a sort of cloud surrounding the nucleus.

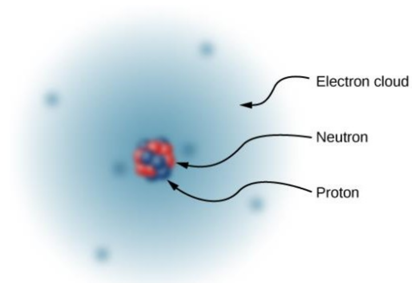


Figure 1.4: Carbon atom

1.2 Conductors, Insulators, and Charging by Induction

Conductors

Electrons surround the tiny nucleus in the form of a (comparatively) vast cloud of negative charge. However, this cloud does have a definite structure to it. If we consider an atom of copper, there is an outermost electron that is only loosely bound to the atom's nucleus. It can be easily dislodged; it then moves to a neighboring atom. In a large mass of copper atoms (such as a copper wire or a sheet of copper), these vast numbers of outermost electrons (one per atom) wander from atom to atom, and are the electrons that do the moving when electricity flows. These wandering, or "free," electrons are called conduction electrons, and copper is therefore an excellent conductor (of electric charge). All conducting elements have a similar arrangement of their electrons, with one or two conduction electrons. This includes most metals.

Insulators

Insulators, in contrast, are made from materials that lack conduction electrons; charge flows only with great difficulty, if at all. Even if excess charge is added to an insulating material, it cannot move, remaining indefinitely in place. This is why insulating materials exhibit the electrical attraction and repulsion forces described earlier, whereas conductors do not; any excess charge placed on a conductor would instantly flow away (due to mutual repulsion from existing charges), leaving no excess charge around to create forces. Charge cannot flow along or through an insulator, so its electric forces remain for long periods of time. (Charge will dissipate from an insulator, given enough time.) As it happens, amber, fur, and most semi-precious gems are insulators, as are materials like wood, glass, and plastic.

Charging by Induction

Induced polarization: A positively charged glass rod is brought near the left side of the conducting sphere, attracting negative charge and leaving the other side of the sphere positively charged. Although the sphere is overall still electrically neutral, it now has a charge distribution, so it can exert an electric force on other nearby charges. Furthermore, the distribution is such that it will be attracted to the glass rod.

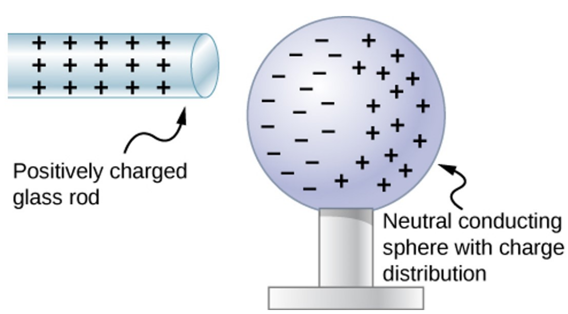


Figure 1.5: Induced Polarization

Both positive and negative objects attract a neutral object by polarizing its molecules.

- A positive object brought near a neutral insulator polarizes its molecules. There is a slight shift in the distribution of the electrons orbiting the molecule, with unlike charges being brought nearer and like charges moved away. Since the electrostatic force decreases with distance, there is a net attraction.
- A negative object produces the opposite polarization, but again attracts the neutral object.
- The same effect occurs for a conductor; since the unlike charges are closer, there is a net attraction.

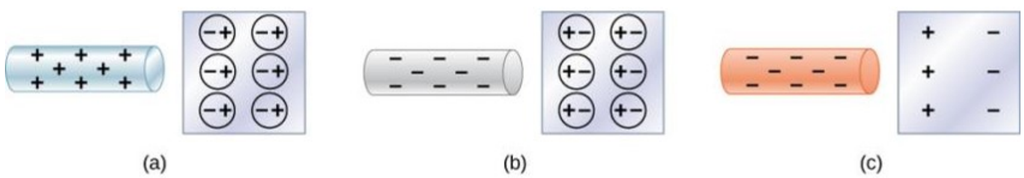


Figure 1.6: Attraction to neutral objects

Charging by induction.

- Two uncharged or neutral metal spheres are in contact with each other but insulated from the rest of the world.
- A positively charged glass rod is brought near the sphere on the left, attracting negative charge and leaving the other sphere positively charged.
- The spheres are separated before the rod is removed, thus separating negative and positive charges.
- The spheres retain net charges after the inducing rod is removed—without ever having been touched by a charged object.

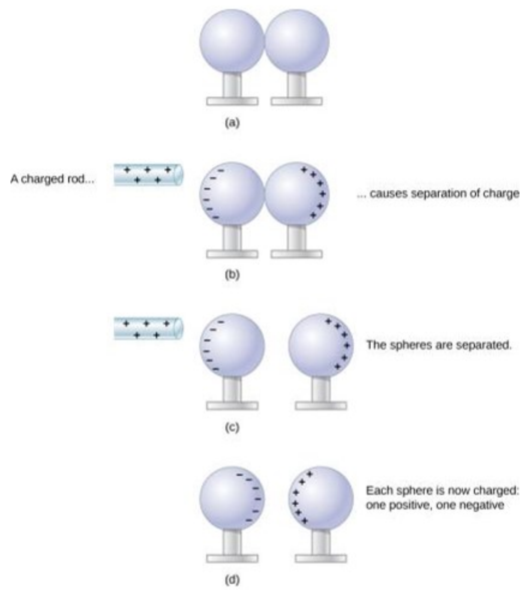


Figure 1.7: Charge by Induction

Similarly using a ground connection

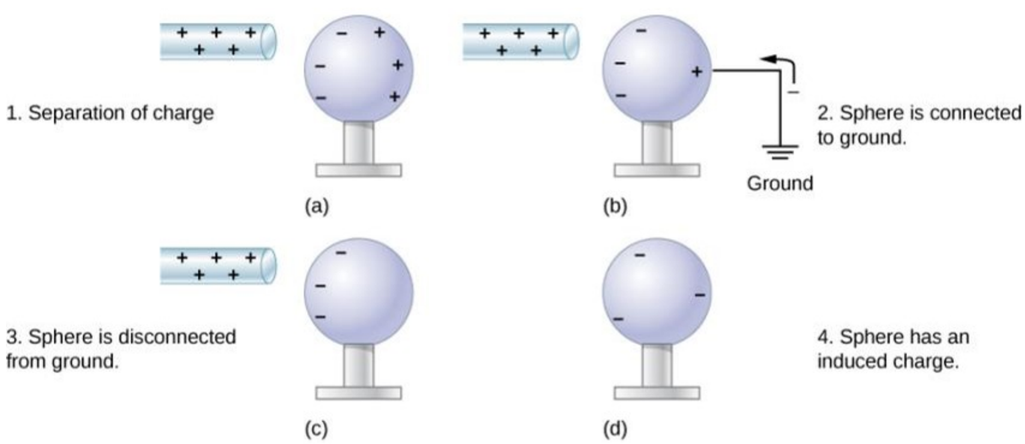


Figure 1.8: Charge by Induction with Ground Connection

1.3 Coulombs Law

Recall from Physics I the gravitational force equation

$$F_G = G\frac{m_1 m_2}{r^2} \tag{1.2}$$

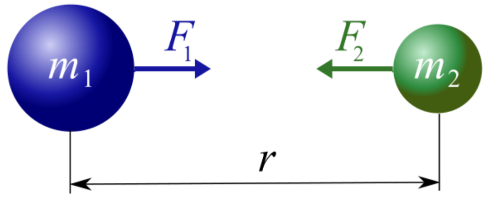


Figure 1.9: Newtons Law Gravitation

For the electric force, let

- q_1, q_2 = the net electric charges of two objects
- $\vec{r}_{12}$ = the vector displacement from q_1 to q_2.

$$F \propto \frac{q_1 q_2}{r_{12}^2} \tag{1.3}$$

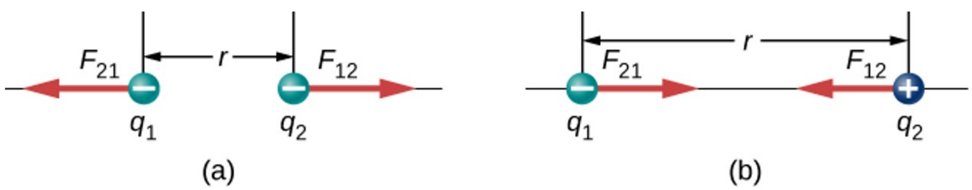


Figure 1.10: Electrostatic Force

Coulomb's Law: the electric force between two electrically charged particles is given by

$$\vec{F} = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r_{12}^2}\hat{r}_{12} \tag{1.4}$$

where $\vec{r}_{12}$ is the unit vector from particle 1 to particle 2, and where ϵ_0 is the permittivity of free space

$$\epsilon_0 = 8.85 * 10^{-12}\frac{C^2}{N \cdot m^2} \tag{1.5}$$

Which leads to Coulomb's constant (k):

$$k = \frac{1}{4\pi\epsilon_0} = 8.99 * 10^9 \frac{N \cdot m^2}{C^2} \tag{1.6}$$

Example: Force on the Electron in a Hydrogen Atom

Proton has a positive charge of $+e$, and an electron has a negative charge of $-e$. In the "ground state" of the atom, the electron orbits the proton at a probably distance of $5.29 * 10^{-11}m$.

$$q_1 = +e = +1.602 * 10^{-19}C \tag{1.7}$$

$$q_2 = -e = -1.602 * 10^{-19}C \tag{1.8}$$

$$r = 5.29 * 10^{-11}m \tag{1.9}$$

The magnitude of the force

$$F = \frac{1}{4\pi\epsilon_0}\frac{|e|^2}{r^2} = 8.99 * 10^9 \frac{N \cdot m^2}{C^2} * \frac{(1.602 * 10^{-19}C)^2}{(5.29 * 10^{-11}m)^2} = 8.25 * 10^{-8}N \tag{1.10}$$

The force is thus expressed as

$$\vec{F} = (8.25 * 10^{-8}N)\hat{r} \tag{1.11}$$

Multiple Sources of Charge

As with the forces encountered in Physics I, the net electric force is the vector sum of the individual forces.

$$\vec{F}(r) = \frac{1}{4\pi\epsilon_0} Q \sum_{i=1}^{N} \frac{q_i}{r_i^2}\hat{r}_i \tag{1.12}$$

where Q represents the charge of a particle that experiences the force $\vec{F}$ and is located at $\vec{r}$ from the origin; q_i are the N source charges, and the vectors $\vec{r}_i = r_i\hat{r}_i$ are the displacements from the position of the ith charge to the position of Q. All of this is with the simplifying assumption that the source charges are all fixed in place somehow. This is referred to as the electrostatic force.

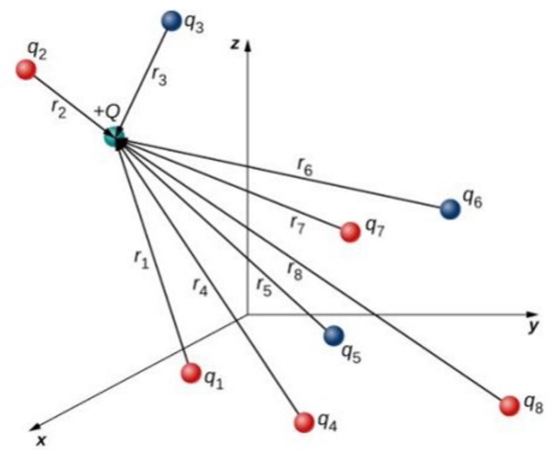


Figure 1.11: Multiple Source Charges

insert example 5.2

1.4 Electric Field

Next we define the Electric Field, which is independent of the test charge Q, and only depends on the configuration of the source charges.

$$\vec{F} = Q \cdot \vec{E} \tag{1.13}$$

where

$$\vec{E}(P) = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{N}\frac{q_i}{r_i^2}\hat{r}_i \tag{1.14}$$

expresses the Electric Field at position $P = P(x, y, z)$ of the N source charges.

This is analogous to the graviational field $\vec{g}$ of the Earth

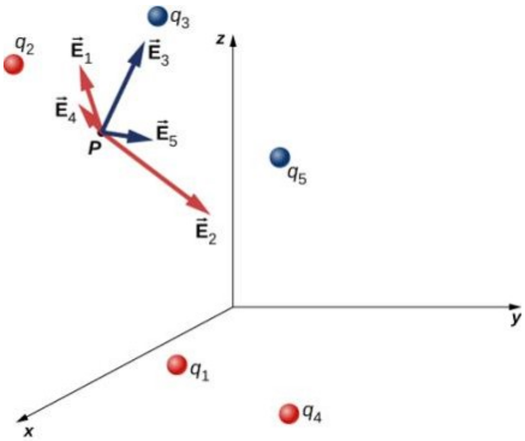


Figure 1.12: Electric Field - Multiple Source Charges

$$\vec{g} = G\frac{M}{r^2}\hat{r} \tag{1.15}$$

which gives us $9.81\frac{m}{s}$ near the Earth's surface.

The Electric Field is

- A vector field
- Obeys superposition
- By convention, the Electric Field points away from the positive charge.

1.5 Calculating Electric Field Charge Distributions

Definitions of charge density

- $\lambda \equiv$ charge per unit length (linear charge density) in $\frac{C}{m}$
- $\sigma \equiv$ charge per unit area (surface charge density) in $\frac{C}{m^2}$
- $\rho \equiv$ charge per unit volume (volume charge density) in $\frac{C}{m^3}$

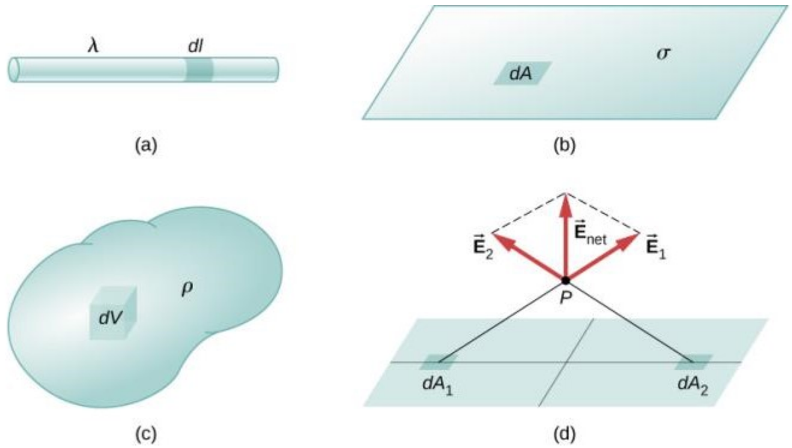


Figure 1.13: Configuration of Charge

Given these densities, the differential charge (dq) becomes λdl, σdA, and ρdV, respectively.

For these distributions, the summation becomes an integral

- Point Charge

$$\vec{E}(P) = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{N}\frac{q_i}{r^2}\hat{r} \tag{1.16}$$

- Line Charge

$$\vec{E}(P) = \frac{1}{4\pi\epsilon_0}\int_{line}(\frac{\lambda dl}{r^2})\hat{r} \tag{1.17}$$

- Surface Charge

$$\vec{E}(P) = \frac{1}{4\pi\epsilon_0}\int_{surface}(\frac{\sigma dA}{r^2})\hat{r} \tag{1.18}$$

- Volume Charge

$$\vec{E}(P) = \frac{1}{4\pi\epsilon_0}\int_{volume}(\frac{\rho dV}{r^2})\hat{r} \tag{1.19}$$

As $P = P(x, y, z)$, the integral is shorthand for three integrals (one in each direction)

$$\vec{E}_x(P) = \frac{1}{4\pi\epsilon_0}\int_{line}(\frac{\lambda dl}{r^2})_x, \vec{E}_y(P) = \frac{1}{4\pi\epsilon_0}\int_{line}(\frac{\lambda dl}{r^2})_y, \vec{E}_z(P) = \frac{1}{4\pi\epsilon_0}\int_{line}(\frac{\lambda dl}{r^2})_z \tag{1.20}$$

Electric Field of a Line Segment

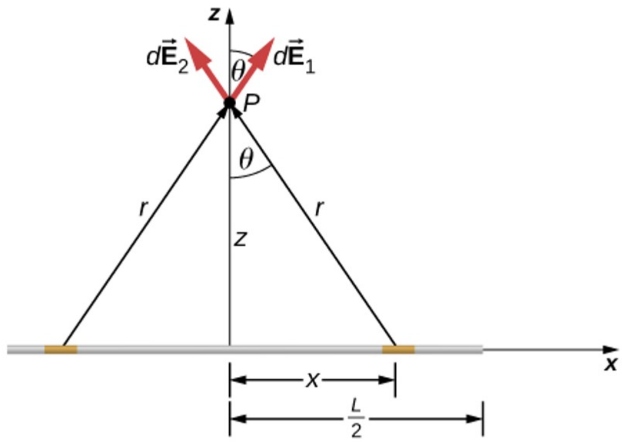


Figure 1.14: A uniformly charged segment of wire. The electric field at point P can be found by applying the superposition principle to symmetrically placed charge elements and integrating.

Start with

$$\vec{E}(P) = \frac{1}{4\pi\epsilon_0}\int_{line}(\frac{\lambda dl}{r^2})\hat{r} \tag{1.21}$$

The symmetry of the arrangement implies the horizontal components cancel.

$$\vec{E}(P) = \vec{E_1} + \vec{E_2} = E_{1,x}\hat{i} + E_{1,z}\hat{k} + E_{2,x}(-\hat{i}) + E_{2,z}\hat{k} \tag{1.22}$$

Due to symmetry, $E_{1,x} = E_{2,x}$, and as their directions are opposite, they cancel. So,

$$\vec{E}(P) = E_{1,z}\hat{k} + E_{2,z}\hat{k} = E_1\cos\theta\hat{k} + E_2\cos\theta\hat{k} \tag{1.23}$$

As these components are also equal, substituting into Equation 1.21 yields:

$$\vec{E}(P) = \frac{1}{4\pi\epsilon_0}\int_0^{\frac{L}{2}}(\frac{2\lambda dx}{r^2}\cos\theta)\hat{k} \tag{1.24}$$

To calculate the integral, we note that

$$r = \sqrt{(x^2 + z^2)} \tag{1.25}$$

and

$$\cos\theta = \frac{z}{r} = \frac{z}{\sqrt{(x^2 + z^2)}} = \frac{z}{(x^2 + z^2)^{\frac{1}{2}}} \tag{1.26}$$

Substituting for $\cos\theta$:

$$\vec{E}(P) = \frac{1}{4\pi\epsilon_0}\int_0^{\frac{L}{2}} (\frac{2\lambda dx}{(x^2 + z^2)}\frac{z}{(x^2 + z^2)^{\frac{1}{2}}})\hat{k} \tag{1.27}$$

$$\vec{E}(P) = \frac{1}{4\pi\epsilon_0}\int_0^{\frac{L}{2}} (\frac{2\lambda z}{(x^2 + z^2)^{\frac{3}{2}}})dx\hat{k} \tag{1.28}$$

Integrating utilizing Trig Substitution (See Appendix A):

$$\vec{E}(P) = \frac{2\lambda z}{4\pi\epsilon_0}[\frac{x}{z^2\sqrt{(x^2 + z^2)}}]\Big|_0^{\frac{L}{2}}\hat{k} \tag{1.29}$$

or

$$\vec{E}(z) = \frac{1}{4\pi\epsilon_0}\frac{\lambda L}{z\sqrt{(\frac{L}{4} + z^2)}}\hat{k} \tag{1.30}$$

For an infinite line: $L = \infty$

$$\vec{E}(z) = \frac{1}{4\pi\epsilon_0}\frac{2\lambda}{z}\hat{k} \tag{1.31}$$

noting that we lost the $\frac{1}{r^2}$ dependence

For a finite line of charge with $z >> L$

$$\vec{E}(z) = \frac{1}{4\pi\epsilon_0}\frac{\lambda L}{z^2}\hat{k} \tag{1.32}$$

Recalling that $q = \lambda L$, then we get the expression of the field of a point charge.

1.6 Electric Field Lines

Electric Field Lines allow us to visualize the electric field in space

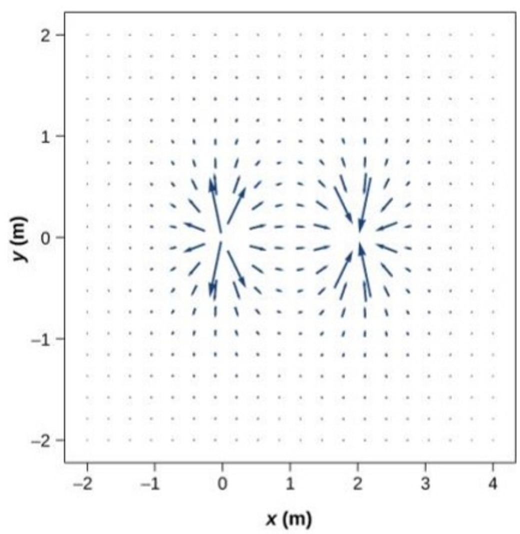


Figure 1.15: Vector field of dipole

Electric Field Line "Rules"

- Either begin at a positive charge or come in from infinity
- Either end at a negative charge or extend out to infinity
- The number of lines originating or terminating is proportional to the amount of charge.
- The density at any point in space is proportional to (and therefore is representative of) the magnitude of the field at that point in space

- The field lines never cross

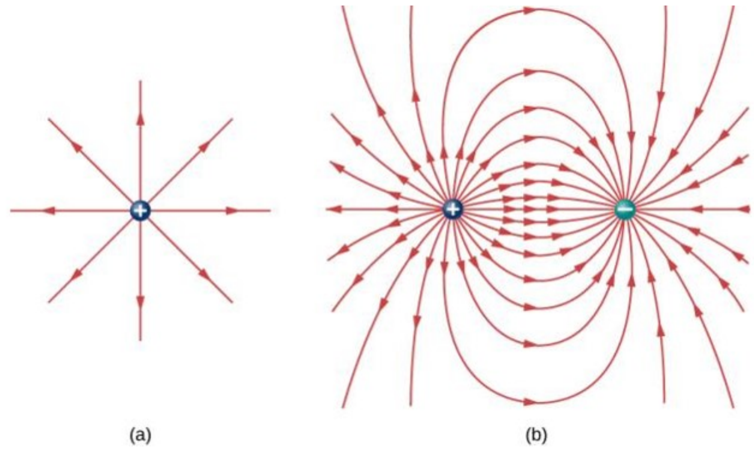


Figure 1.16: Electric Field of a dipole

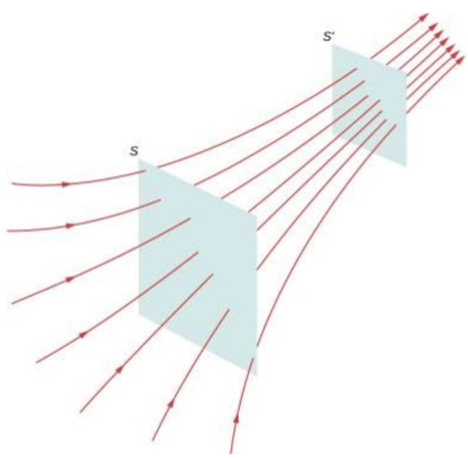


Figure 1.17: Field Line Density

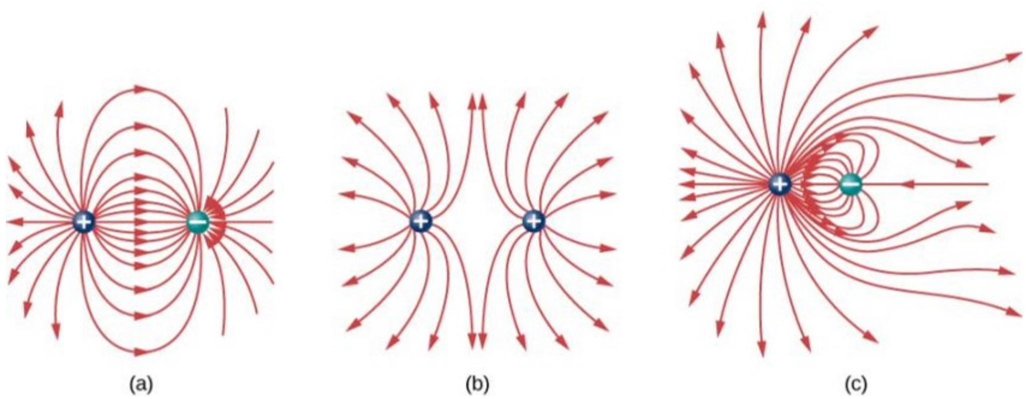


Figure 1.18: Typical Diagrams

Chapter 2

Module 2: Chapter 6 - Gauss's Law

Four main topics:

- Electric Flux
- Guass's Law
- Calculating Electric Field with Guass's Law
- Electric Field inside Conductors

2.1 Electric Flux

Flux describes how much of something goes through a given area. More formally, flux is the dot-product of a vector field (in our case, the electric field) with an area.

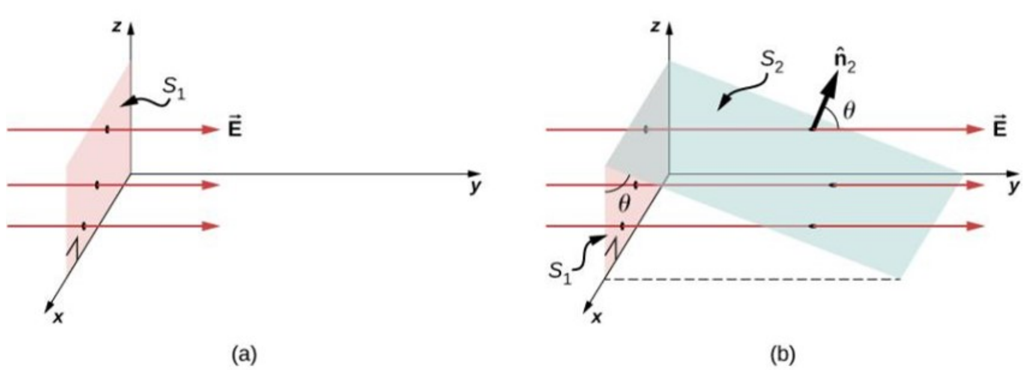


Figure 2.1: Flux through a plane

Electric Flux

For uniform field $\vec{E}$ and a flat surface:

$$\Phi = \vec{E} \cdot \vec{A} \tag{2.1}$$

$$\Phi = EA\cos\theta \tag{2.2}$$

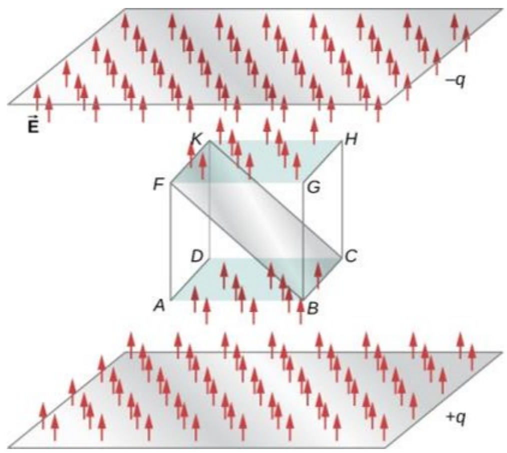


Figure 2.2: Flux through a cube

- Source and termination of electric field lines are outside the cube
- There is no charge inside the cube
- All electric field lines that enter the cube exit it, so the net flux through the cube is zero.
- By convention: if field lines are leaving a closed surface then Φ is positive.
- For field lines entering a closed surface, Φ is negative.

For a non-flat surface, we can take small patches that approximate a flat surface

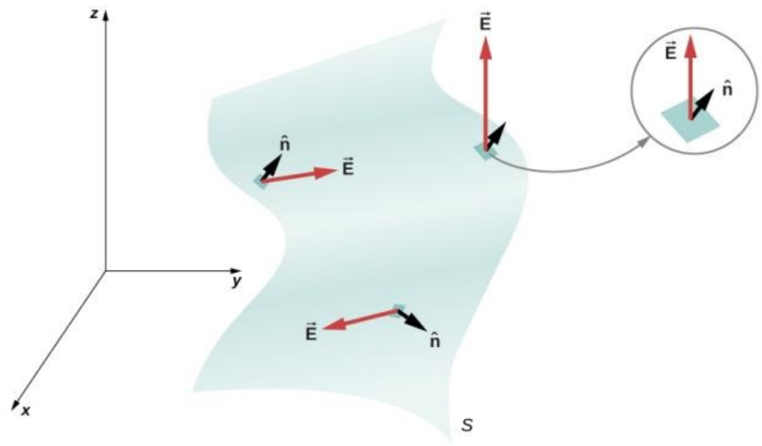


Figure 2.3: Surface divided into patches to find flux

Consider $\vec{E}_i$ to be the electric field over the i^{th} patch (δA_i), then

$$\Phi_i = \vec{E}_i \cdot \delta \vec{A}_i \tag{2.3}$$

then

$$\Phi = \sum_{i=1}^{N} \Phi_i = \sum_{i=1}^{N} \vec{E}_i \cdot \delta \vec{A}_i \tag{2.4}$$

As the patch gets infinitesimally small,

$$\delta \vec{A} \to \hat{n} dA \tag{2.5}$$

and the $\sum$ become an $\int_s$ over the entire surface

For an Open Surface:

$$\Phi = \int_s \vec{E} \cdot \hat{n} dA = \int_s \vec{E} \cdot d\vec{A} \tag{2.6}$$

For an Closed Surface:

$$\Phi = \oint_s \vec{E} \cdot \hat{n} dA = \oint_s \vec{E} \cdot d\vec{A} \tag{2.7}$$

2.2 Gauss's Law

Let's calculate the electric flux through a sphere that surrounds a point charge q.

Recall at Point P

$$\vec{E}_P = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \hat{r} \tag{2.8}$$

where $\hat{r}$ is the radial unit vector charge at the center to Point P.

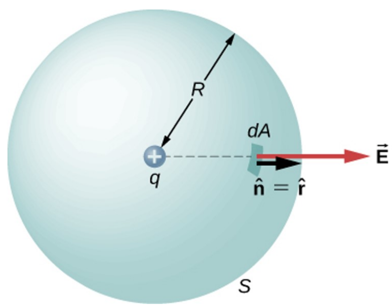


Figure 2.4: Closed sphere around point charge P

Applying the flux equation 2.7, where $\hat{n} = \hat{r}$ and $r = R$, for infinitesimal area dA:

$$d\Phi = \vec{E} \cdot \hat{n} dA = \frac{1}{4\pi\epsilon_0} \frac{q}{R^2} \hat{r} \cdot \hat{r} dA = \frac{1}{4\pi\epsilon_0} \frac{q}{R^2} dA \tag{2.9}$$

Integrating

$$\Phi = \frac{1}{4\pi\epsilon_0} \frac{q}{R^2} \oint_s dA = \frac{1}{4\pi\epsilon_0} \frac{q}{R^2} (4\pi R^2) = \frac{q}{\epsilon_0} \tag{2.10}$$

As r increases, there is a $\frac{1}{r^2}$ decrease in electric field counteracts the r^2 increase in the surface of the sphere and we find that $\Phi = \frac{q}{\epsilon_0}$ is independent of the size of the sphere.

Gauss's Law

$$\boxed{\Phi_{closedsurface} = \frac{q_{enc}}{\epsilon_0}} \tag{2.11}$$

where q_{enc} is the net charge enclosed by the surface

2.3 Applying Gauss's Law

Charge Distribution with Spherical Symmetry

Charge distribution is considered spherically symmetric if the density of charge depends only on the distance from a point in space and not direction. A spherically symmetric charge distribution does not change of you rotate the sphere.

$$\rho(r, \theta, \phi) = \rho(r) \tag{2.12}$$

For spherically symmetric:

$$\vec{E}_P = E_P(r)\hat{r} \tag{2.13}$$

The magnitude of the electric field $\vec{E}$ is the same everywhere on a spherical Gaussian surface concentric with the distribution. For a sphere of radius r:

$$\Phi = E_P 4\pi r^2 \tag{2.14}$$

From Gauss's Law

$$4\pi r^2 E = \frac{q_{enc}}{\epsilon_0} \tag{2.15}$$

Combining, the magnitude $E(r)$ is given by

$$E(r) = \frac{1}{4\pi\epsilon_0}\frac{q_{enc}}{r^2} \tag{2.16}$$

For a sphere of radius R

$$q_{enc} = \begin{cases} q_{tot}, & \text{if } r \geq R. \\ q_{(r<R)}, & \text{if } r < R. \end{cases} \tag{2.17}$$

Which leads to an Electric Field at point P (E_{out} for P outside the sphere, and E_{in} for P inside the sphere

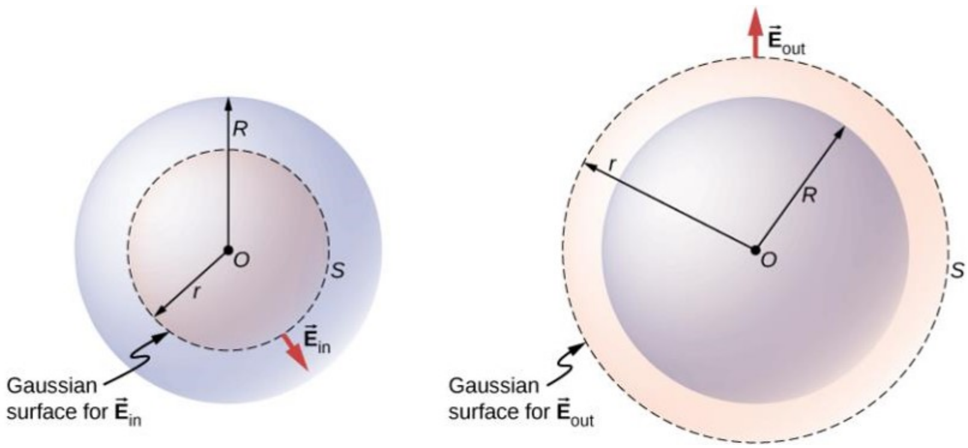


Figure 2.5: Outside and inside the sphere

$$E_{out} = \frac{1}{4\pi\epsilon_0}\frac{q_{tot}}{r^2} \tag{2.18}$$

$$E_{in} = \frac{1}{4\pi\epsilon_0}\frac{q_{(r<R)}}{r^2} \tag{2.19}$$

So, what is q_{enc} inside the sphere. Let's assume uniform distribution of charge ρ_o

$$q_{enc} = \int \rho_0 dV = \int_0^r \rho_0 4\pi r' dr' = \rho_0(\frac{4}{3}\pi r^3) \tag{2.20}$$

Therefore

$$E_{out} = \frac{1}{4\pi\epsilon_0}\frac{q_{tot}}{r^2}, \text{ with: } q_{tot} = \frac{4}{3}\pi R^3 \rho_0 \tag{2.21}$$

$$E_{in} = \frac{1}{4\pi\epsilon_0}\frac{q_{(r<R)}}{r^2} = \frac{\rho_0 r}{3\epsilon_0}, \text{ since: } q_{(r<R)} = \frac{4}{3}\pi r^3 \rho_0 \tag{2.22}$$

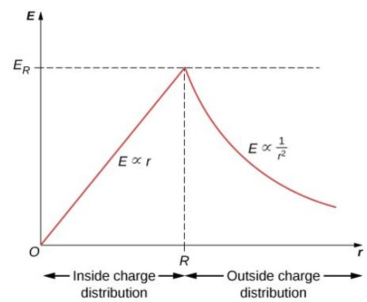


Figure 2.6: Electric Field of uniformly charged non-conducting sphere

Charge Distribution with Symmetric Cylinder

Charge distribution is considered cylindrical symmetric if the density of charge depends only on the distance r from the axis, and does not vary along or with direction of axis.

Consider an "infinitely[1] long" cylinder with cylindrical symmetric charge distribution.

$$\rho(r, \theta, z) = \rho(r) \tag{2.23}$$

$$\vec{E}_P = E_P(r)\hat{r} \tag{2.24}$$

[1]In the real world, cylinders are not finite, but if the length $L >> r$ then it can be approximated as infinitely long

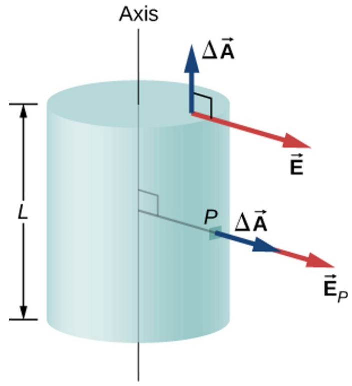


Figure 2.7: Cylindrically symmetric of length L

The electric field is perpendicular to the sides and parallel to the end caps.

The flux through the cylinder part is

$$\int_s \vec{E} \cdot \hat{n} dA = E \int_s dA = E(2\pi r L) \tag{2.25}$$

and the flux through the caps is zero, because

$$\vec{E} \cdot \hat{n} = 0 \tag{2.26}$$

The total flux is then

$$\int_s \vec{E} \cdot \hat{n} dA = 2\pi r L E + 0 + 0 = 2\pi r L E \tag{2.27}$$

Using Gauss's law and taking λ_{enc} to be the charge per unit length

$$q_{enc} = \lambda_{enc} L \tag{2.28}$$

$$\Phi = 2\pi r L E = \frac{q_{enc}}{\epsilon_0} \tag{2.29}$$

The magnitude of $\vec{E}$

$$E(r) = \frac{\lambda_{enc}}{2\pi\epsilon_0}\frac{1}{r} \tag{2.30}$$

$$\lambda_{enc}L = \begin{cases} q_{tot}, & \text{if } r \geq R. \\ q_{(r<R)}, & \text{if } r < R. \end{cases} \tag{2.31}$$

What happens of it is a cylindrical shell vs a solid cylinder?

Consider surface charge density of σ.

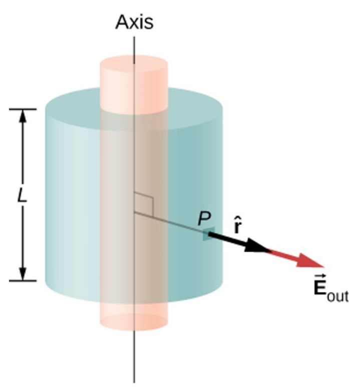


Figure 2.8: Guassian surface around a cylinderical shell

$$\lambda_{enc} = \frac{\sigma 2\pi RL}{L} = 2\pi R\sigma \tag{2.32}$$

For point P outside of the shell $r \geq R$:

$$\vec{E} = \frac{2\pi R\sigma}{2\pi\epsilon_0}\frac{1}{r}\hat{r} = \frac{R\sigma}{\epsilon_0}\frac{1}{r}\hat{r} \tag{2.33}$$

For point P inside of the shell $r < R$:

$$\lambda_{enc} = 0 \tag{2.34}$$

so

$$\vec{E} = 0 \tag{2.35}$$

Planar Symmetry

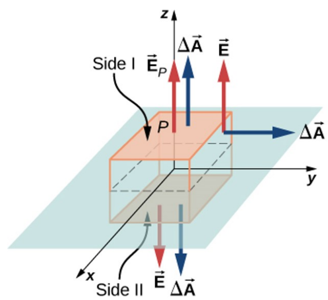


Figure 2.9: Thin Charged Sheet

Consider a large flat surface with uniform charge density. The charge density is the same at all points (x,y), so the electric field $\vec{E}$ at point P only depends on the z-coordinate.

$$\vec{E} = E(z)\hat{z} \tag{2.36}$$

Note that $E(z) = E(-z)$ thought the direction is opposite.

Let the electric field at point P be $E_P = E(z)$. If the charge is positive, the field lines point away from the sheet, so

$$\Phi = \oint_S \vec{E}_P \cdot \hat{n} dA = E_P A + E_P A + 0 + 0 + 0 + 0 = 2E_P A \tag{2.37}$$

The charge inside the gaussian box on the plane is

$$q_{enc} = \sigma A \tag{2.38}$$

Given Gauss's law ($\Phi = \frac{q_{enc}}{\epsilon_0}$) :

$$\frac{q_{enc}}{\epsilon_0} = \frac{\sigma A}{\epsilon_0} \tag{2.39}$$

$$E_P = \frac{\Phi}{2A} = \frac{\sigma}{2\epsilon_0} \tag{2.40}$$

$$\vec{E}_P = \frac{\sigma}{2\epsilon_0}\hat{n} \tag{2.41}$$

with $\hat{n} = +\hat{z}$ above the plane and $\hat{n} = -\hat{z}$ below the plane, and E_P doesn't depend on the distance above the plane for an infinite plane. Practically, this is a useful approximation for a finite plate near the center.

2.4 Conductors in Electrostatic Equilibrium

Moving from insulators to conductors. The electric field in conductors exerts a force on the free electrons (conduction electrons). As these electrons are not bound to an atom, they accelerate. However, moving charges are not static, so when electrostatic equilibrium is reached the charges are distributed in such a way that the electric field inside the conductor vanishes.

Electric Field inside a Conductor vanishes

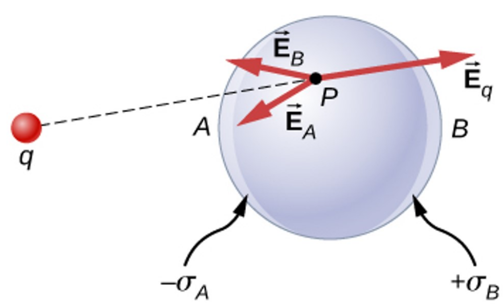


Figure 2.10: Metallic Sphere in presence of point charge

The redistribution of charges is such that

$$\vec{E}_P = \vec{E}_q + \vec{E}_A + \vec{E}_B = \vec{0} \tag{2.42}$$

for induced charge $-\sigma_A$ and $+\sigma_B$

There is no net charge enclosed by a Gaussian surface that is solely within the volume of a conductor at equilibrium. This gives us $q_{enc} = 0$ and

The redistribution of charges is such that

$$\vec{E}_{net} = \vec{0} \text{ at points inside a conductor} \tag{2.43}$$

Charge on a conductor

A consequence of a conductor in static equilibrium is that excess charge will end up on the outer surface of the conductor regardless of its origin.

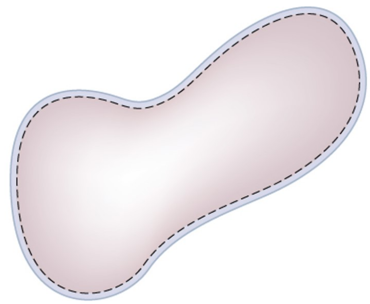


Figure 2.11: Gaussian surface just below conductor surface

Since $E = 0$ everywhere inside the conductor,

$$\oint_s = \vec{E} \cdot \hat{n} dA = 0 \tag{2.44}$$

As the Gaussian surface lies infinitesimally below the actual surface, and there is no charge within the Gaussian surface, then all the excess charge must be on the surface.

Electric Field at the Surface of a Conductor

If the electric field had a component parallel to the surface of the conductor, then the charges would move, which violates the electrostatic equilibrium assuming. Therefor the field must be normal to the surface.

Just above the surface the magnitude of the electric field (E) and the surface charge density (σ) are related by

$$E = \frac{\sigma}{\epsilon_0} \tag{2.45}$$

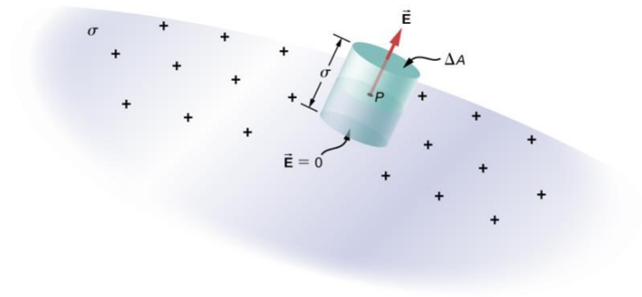


Figure 2.12: Field over the surface of a conductor

Consider an infinitesimally small cylinder on the surface of a conductor with one face inside the conductor and one face outside. The height is δ and the cross section ΔA.

Given that ΔA is infinitesimally small, the total charge in the cylinder is $\sigma \Delta A$.

The field is perpendicular to the surface, and thus the flux

$$\Phi = E\Delta A = \frac{\sigma \Delta A}{\epsilon_0} \tag{2.46}$$

yielding that charge right above the surface being given by:

$$E = \frac{\sigma}{\epsilon_0} \tag{2.47}$$

Electric Field of a Conducting Plate

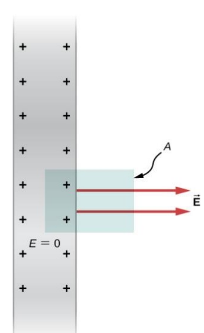


Figure 2.13: Conducting Plate

A conducting plate is similar to the thin conducting plate, however, with $E = 0$ inside the conductor,

$$\Phi = \oint_S \vec{E}_P \cdot \hat{n} dA = E_P A + 0 + 0 + 0 + 0 + 0 = E_P A \tag{2.48}$$

and thus

$$E = \frac{\sigma}{\epsilon_0} \tag{2.49}$$

Example: Parallel Plates

Now consider the electric field between two oppositely charged parallel plates.

If the surface charge density is $\sigma = 6.81 * 10^{-7} \frac{C}{m^2}$ and the distance between the plates is $l = 6.50mm$, what is the electric field?

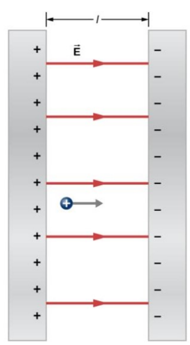


Figure 2.14: Between Conducting Plates

$$E = \frac{\sigma}{\epsilon_0} = \frac{6.81 * 10^{-7} \frac{C}{m^2}}{8.85 * 10^{-12}} \frac{C^2}{N \cdot m^2} = 7.69 * 10^4 \frac{N}{C} \tag{2.50}$$

Chapter 3

Module 3: Chapter 7 - Electric Potential

3.1 Electric Potential Energy

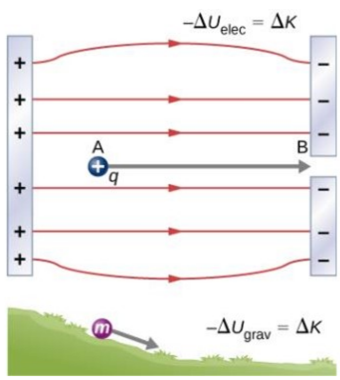


Figure 3.1: Charge and Mass in Gravity analogy

A charge accelerated by an electric field is analogous to a mass going down a hill. Thus, when a free charge is accelerated by an electric field it is also given kinetic energy. This kinetic energy exactly equals the decrease in potential energy

The electrostatic (or Coulomb) force is conservative. This means the work done on charge q is independent of the path taken.

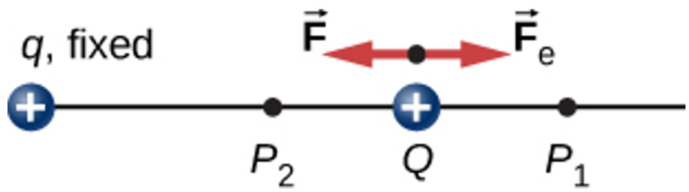


Figure 3.2: Displacement of "test" charge Q

Consider fixed charge $+q$ located at the origin. Push test charge $+Q$ towards $+q$ in such as way that the applied force $\vec{F}$ exactly balances out the electric force $\vec{F_e}$.

In mechanics, $W = mass * displacement$; similarly, for the electric field, the work done moving charge $+Q$ from P1 to P2 is found by

$$W_{12} = \int_{P1}^{P2} \vec{F} \cdot d\vec{l} \tag{3.1}$$

Because $\vec{F}$ balances out $\vec{F_e}$:

$$\vec{F} = -\vec{F_e} = -k\frac{qQ}{r^2}\hat{r} \tag{3.2}$$

Recalling Coulomb's constant: $k = \frac{1}{4\pi\epsilon_0} = 8.99 * 10^9 \frac{N \cdot m^2}{C^2}$

Finally, let U denote Potential Energy in Joules ($J = N \cdot m$). Negative work mean gain in potential energy

$$\Delta U = -W \tag{3.3}$$

Because the electrostatic force is conservative, the work W is independent of the path taken. Consider

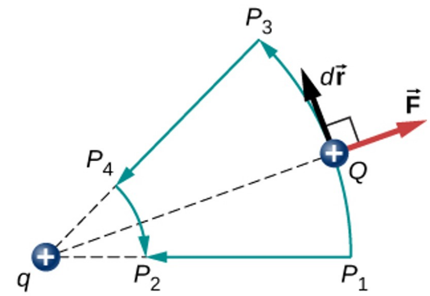


Figure 3.3: Work independent of path taken

The work on path P_1P_2 is equal to the work along path $P_1P_3P_4P_2$. And, in fact, the work on paths P_1P_3 and P_4P_2 are both zero as the $d\vec{r}$ is perpendicular to the electric field along these segments.

Example

How much work is required to assemble this charge configuration:

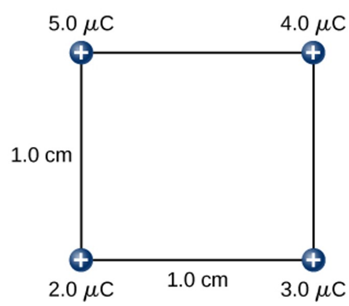


Figure 3.4: Work independent of path taken

1. Lower left: first bring this $2\mu C$ charge to the origin. This requires no work as there are no yet any other charges, and thus no electric field.
2. Lower right: bring $3\mu C$ charge to (x,y,z) coordinates (1 cm, 0, 0)

$$W_2 = k\frac{q_1q_2}{r_12} = (9.0 * 10^9\frac{N \cdot m^2}{C^2})\frac{(2 * 10^{-6}C)(3 * 10^{-6}C)}{1.0 * 10^2 m} = 5.4J \tag{3.4}$$

3. Upper right: $r_{23} = 1.0cm$ and $r_{13} = \sqrt{2.0}cm$

$$W_3 = 15.9J \tag{3.5}$$

4. Upper left: Similarly, $r_{14} = 1.0cm$, $r_{34} = 1.0cm$ and $r_{23} = \sqrt{2}cm$

$$W_4 = 36.5J \tag{3.6}$$

$$W_{total} = W_1 + W_2 + W_3 + W_4 = 0 + 5.4J + 15.9J + 36.5J = 57.8J \tag{3.7}$$

In general

$$W_{123\ldots N} = \frac{1}{2}k\sum_{i}^{N}\sum_{j}^{N}\frac{q_i q_j}{r_{ij}} \text{ for } i \neq j \tag{3.8}$$

The $\frac{1}{2}$ accounts for each pairing being counted twice.

3.2 Electric Potential and Potential Difference

The electric field is independent of the test charge. Calculating W directly can be difficult as direction and magnitude of $\vec{F}$ since $W = \vec{F} \cdot \vec{d}$ can be complex for multiple charges, odd shapes, and arbitrary paths. However, because we know $\vec{F} = q\vec{E}$ that the work and hence ΔU is proportional to the test charge q.

To have a physical quantity independent of the test charge, we define electric potential (V):

$$V = \frac{U}{q} \tag{3.9}$$

Since $U \propto q$, then V is independent of q. And, we have

$$\Delta V = V_A - V_B = \frac{\Delta U}{q} \tag{3.10}$$

or

$$\Delta U = q\Delta V \tag{3.11}$$

The units for potential difference ΔV are Joules per Coulomb, which is referred to as a Volt[1]. Whenever voltage is noted, it is understood to be the potential difference between two points.

Another fundamental constant is that over the electron volt (eV) which is the energy given to the fundamental charge accelerated through a potential difference of 1 volt.

$$1eV = (1.60 * 10^{-19}C)(1V) = (1.60 * 10^{-19}C)(1\frac{J}{C}) = 1.60 * 10^{-19}J \quad (3.12)$$

Due to conservation of energy

$$K + U = \text{constant} \quad (3.13)$$

where K is the Kinetic Energy and U is the Potential Energy

or

$$K_i + U_i = K_f + U_f \quad (3.14)$$

Example

Calculate the speed of an electron accelerated through 100V

$$K_i = 0, U_i = qV \quad (3.15)$$

$$K_f = \frac{1}{2}mv^2, U_f = 0 \quad (3.16)$$

Thus

$$qV = \frac{1}{2}mv^2 \quad (3.17)$$

[1]The volt was named after Alessandro Volta

$$v = \sqrt{\frac{2qV}{m}} \tag{3.18}$$

$$v = \sqrt{\frac{2(-1.6 * 10^{-19}C)(-100\frac{J/C}{)}}{9.11 * 10^{-31}kg}} \tag{3.19}$$

$$v = 5.93 * 10^6 \frac{m}{s} \tag{3.20}$$

Voltage and the Electric Field

Returning to the general formula for the potential energy of a point charge q at Point P relative to a reference Point R.

$$U_P = -\int_R^P \vec{F} \cdot d\vec{I} \tag{3.21}$$

given that $\vec{F} = q\vec{E}$:

$$U_P = -q\int_R^P \vec{E} \cdot d\vec{I} \tag{3.22}$$

as we defined $V = \frac{U}{q}$:

$$V_P = -\int_R^P \vec{E} \cdot d\vec{I} \tag{3.23}$$

From conservation of energy, the result is independent of the path chosen, so we can pick the path that is most convenient.

Special Case

Consider the special case with a positive charge at the origin. To calculate the potential at a point r relative to a potential of 0 at infinity. Let $P = r$, $R = \infty$, $d\vec{I} = d\vec{r} = \hat{r}dr$.

Using $\vec{E} = k\frac{q}{r^2}\hat{r}$:

$$V_r = -\int_{\infty}^{r} k\frac{q}{r^2}\hat{r}\cdot\hat{r}dr \tag{3.24}$$

$$V_r = -\int_{\infty}^{r} k\frac{q}{r^2}dr \tag{3.25}$$

$$V_r = k\frac{q}{r}\bigg|_{\infty}^{r} = k(\frac{q}{r} - \frac{q}{\infty}) \tag{3.26}$$

$$V_r = k\frac{q}{r} \tag{3.27}$$

Let's calculate the potential difference between two points equidistant for the reference point.

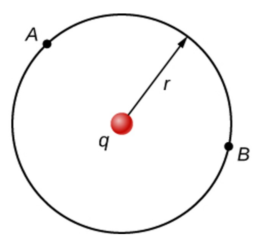


Figure 3.5: Potential difference between two equidistant points

$$\Delta V_{AB} = V_B - V_A = -\int_{A}^{B} \vec{E}\cdot d\vec{l} \tag{3.28}$$

$$\Delta V_{AB} = -\int_{A}^{B} \frac{kq}{r^2}\hat{r}\cdot r\hat{\phi}d\phi \tag{3.29}$$

as $\hat{r}\cdot\hat{\phi} = 0$:

$$\Delta V_{AB} = 0 \tag{3.30}$$

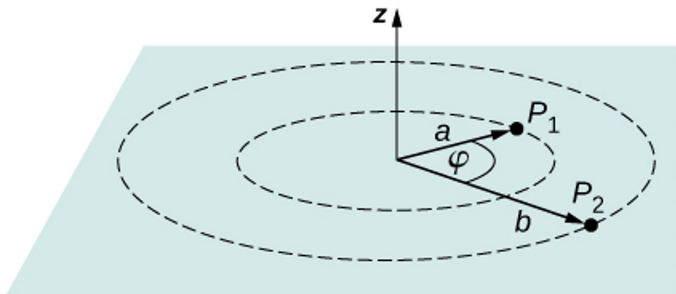


Figure 3.6: Potential difference between two points

3.3 Calculations of Electric Potential

- From a Point Charge

$$V = \frac{kq}{r} \tag{3.31}$$

- From multiple Point Charges

$$V = k \sum_{i=1}^{N} \frac{q_i}{i} \tag{3.32}$$

- From a continuous charge

$$V = k \int \frac{dq}{r} \tag{3.33}$$

where

$$dq = \begin{cases} \lambda dl \text{ (in one dimension)} \\ \sigma dA \text{ (in two dimensions)} \\ \rho dV \text{ (in three dimensions)} \end{cases} \tag{3.34}$$

Potential due to a ring

Consider a disk with uniform charge density λ (Coulombs per meter arc length). Let's find the electric potential at a point on the axis passing through the center of the ring.

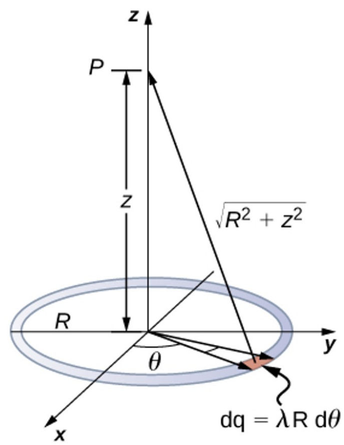


Figure 3.7: Ring of charge

The strategy is to divide the ring into infinitesimal elements shaped as arcs on the circle and to utility cylindrical coordinate system.

The element between angles θ and $\theta + d\theta$ is of length $Rd\theta$. It therefore contains charge $\lambda R d\theta$. The element is at distance $\sqrt{z^2 + R^2}$ from point P. Therefore:

$$V_P = k \int \frac{dq}{r} = k \int_0^{2\pi} \frac{\lambda R d\theta}{\sqrt{z^2 + R^2}} = \frac{k\lambda R}{\sqrt{z^2 + R^2}} \int_0^{2\pi} d\theta \tag{3.35}$$

$$= \frac{k\lambda R}{\sqrt{z^2 + R^2}} \theta \Big|_0^{2\pi} = \frac{2\pi k \lambda R}{\sqrt{z^2 + R^2}} = k \frac{q_{tot}}{\sqrt{z^2 + R^2}} \tag{3.36}$$

Potential due to a disk

Consider a disk with uniform charge density σ (Coulombs per meter squared). Let's find the electric potential at a point on the axis passing through the center of the disk.

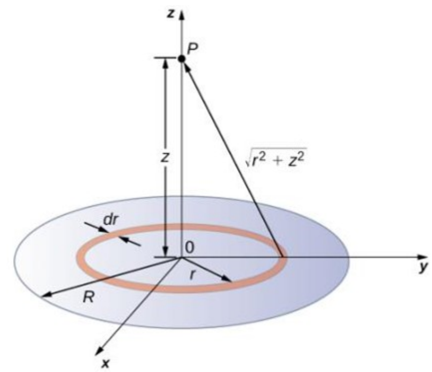


Figure 3.8: Disk of charge

The strategy is to divide the disk into ring-shaped elements, then taking the previous result integrate over r as well as θ.

The infinitesimal width ring elements between cylinder r and $r + dr$ will be a ring of charges whole electric potential dV_P at the field point that is

$$dV_P = k\frac{dq}{\sqrt{z^2 + R^2}} \tag{3.37}$$

where

$$dq = \sigma 2\pi r dr \tag{3.38}$$

Integrating from r = 0 to r = R

$$V_P = \int dV_P = k2\pi\sigma \int_0^R \frac{rdr}{\sqrt{z^2 + r^2}} \tag{3.39}$$

$$= k2\pi\sigma\sqrt{z^2 + r^2}\Big|_0^R = k2\pi\sigma(\sqrt{z^2 + R^2} - \sqrt{z^2}) \tag{3.40}$$

3.4 Determining Field From Potential

Recall that

$$U = V_B - V_A = \int_A^B \vec{E} \cdot ds \tag{3.41}$$

The electric field can be calculated by taking the derivative of U

For a uniform electric field

$$E = -\frac{\Delta V}{\Delta s} \tag{3.42}$$

where Δs is the distance over which the change in potential ΔV takes place.

For a continuously changing field, ΔV and Δs become infinitesimally small giving

$$E_s = -\frac{dV}{ds} \tag{3.43}$$

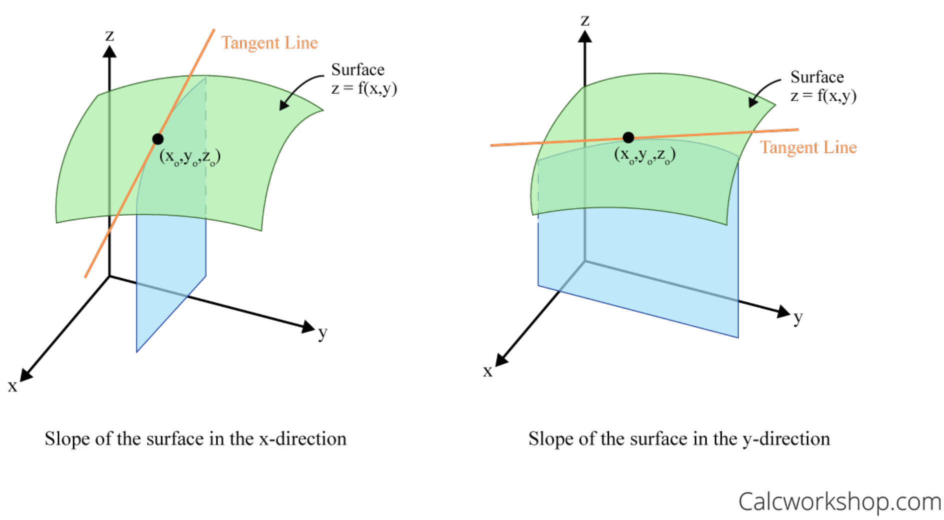


Figure 3.9: Partial Derivatives

Using Partial Derivatives, the electric field components in the Cartesian directions are given by

$$E_x = -\frac{\delta V}{\delta x}, E_y = -\frac{\delta V}{\delta y}, E_z = -\frac{\delta V}{\delta z} \tag{3.44}$$

Introducing the Gradient (grad) or Delta (del) vector operator:

$$\vec{\nabla} = \hat{i}\frac{\delta}{\delta x} + \hat{j}\frac{\delta}{\delta y} + \hat{k}\frac{\delta}{\delta k} \tag{3.45}$$

and thus

$$\vec{E} = -\vec{\nabla}V \tag{3.46}$$

For symmetric systems

- Cylindrical:
$$\vec{\nabla} = \hat{r}\frac{\delta}{\delta r} + \hat{\phi}\frac{1}{r}\frac{\delta}{\delta \phi} + \hat{z}\frac{\delta}{\delta z} \tag{3.47}$$
- Spherical:
$$\vec{\nabla} = \hat{r}\frac{\delta}{\delta r} + \hat{\theta}\frac{1}{r}\frac{\delta}{\delta \theta} + \hat{\phi}\frac{1}{r\sin\theta}\frac{\delta}{\delta \phi} \tag{3.48}$$

Example

Electric Field of a point charge

$$V = k\frac{q}{r}(\text{spherically symetric}) \tag{3.49}$$

$$\vec{E} = -\vec{\nabla}V \tag{3.50}$$

$$\vec{E} = -(\hat{r}\frac{\delta}{\delta r} + \hat{\theta}\frac{1}{r}\frac{\delta}{\delta \theta} + \hat{\phi}\frac{1}{r\sin\theta}\frac{\delta}{\delta \phi})k\frac{q}{r} \tag{3.51}$$

$$\vec{E} = -kq(\hat{r}\frac{\delta}{\delta r}(\frac{1}{r}) + \hat{\theta}\frac{1}{r}\frac{\delta}{\delta \theta}(\frac{1}{r}) + \hat{\phi}\frac{1}{r\sin\theta}\frac{\delta}{\delta \phi}(\frac{1}{r})) \tag{3.52}$$

$$\vec{E} = -kq(\hat{r}(\frac{-1}{r^2}) + \hat{\theta}(0) + \hat{\phi}(0)) \tag{3.53}$$

$$\vec{E} = +\frac{kq}{r^2}\hat{r} \tag{3.54}$$

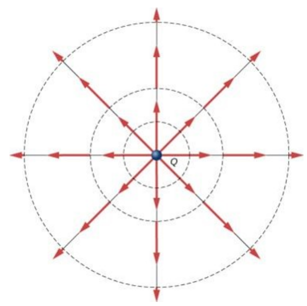


Figure 3.10: Electric Field from Point Charge

Electric Field from Ring of Charge

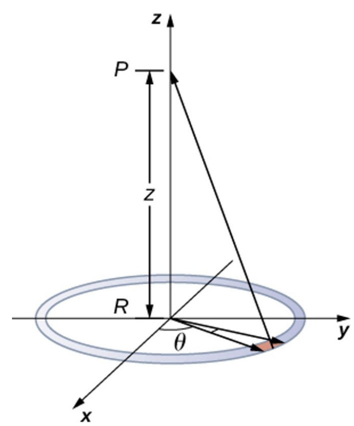


Figure 3.11: Electric Field from Ring Of Charge

$$E_z = -\frac{\delta V}{\delta z} \text{ with } V = k\frac{kq_{tot}}{\sqrt{z^2+R^2}} \tag{3.55}$$

$$E_z = -\frac{\delta}{\delta z}(k\frac{kq_{tot}}{\sqrt{z^2+R^2}}) \tag{3.56}$$

Given that

$$\frac{\delta}{\delta z}(\frac{1}{\sqrt{z^2+R^2}}) = \frac{\delta}{\delta z}(z^2+R^2)^{-\frac{1}{2}} = -\frac{1}{2}(z^2+R^2)^{-\frac{3}{2}}(2z) \tag{3.57}$$

We get

$$E_z = +k\frac{kq_{tot}z}{(z^2+R^2)^{\frac{3}{2}}} \tag{3.58}$$

3.5 Electropotential Surfaces and Conductors

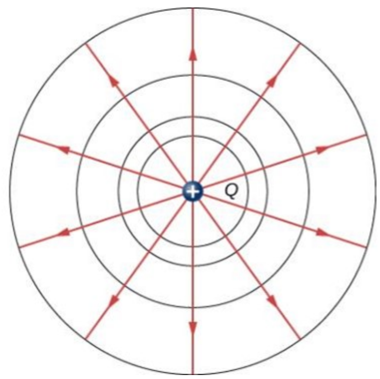


Figure 3.12: Red: Field Lines, Blue: Equipotential Lines

Equipotential Lines are always perpendicular to the electric field lines since along them $\Delta V = 0$.

$$W = -\Delta U = -q\Delta V = 0 \tag{3.59}$$

Work is zero when the direction of the force is perpendicular to displacement.

$$W = \vec{F}\cdot\vec{d} = q\vec{E}\cdot\vec{d} = qEd\cos\theta \text{ for } \theta = 90^\circ \text{ or } 270^\circ \tag{3.60}$$

Conductors

- One of the rules for static electric fields and conductors is that the $\vec{E}$ is perpendicular ($\perp$) to the surface.
- This implies that a conductor is an equipotential surface in static situations. There can be no voltage difference across the surface of a conductor, or charges will flow.

- One of the uses of this fact is that a conductor can be fixed at what we consider zero volts by connecting it to the earth with a good conductor — a process called grounding.

Because a conductor is an equipotential, it can replace any equipotential surface. For example, a charged spherical conductor can replace the point charge, and the electric field and potential surfaces outside of it will be unchanged, confirming the contention that a spherical charge distribution is equivalent to a point charge at its center.

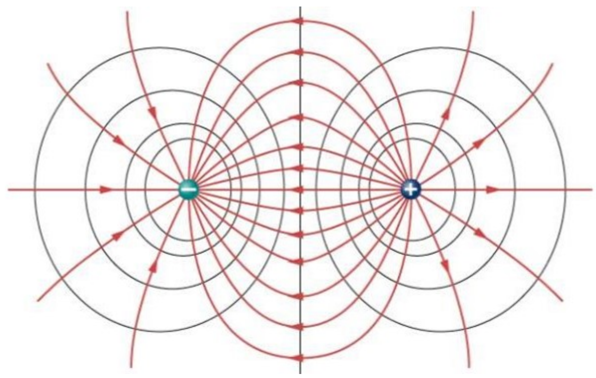


Figure 3.13: Potential lines, two opposite charges

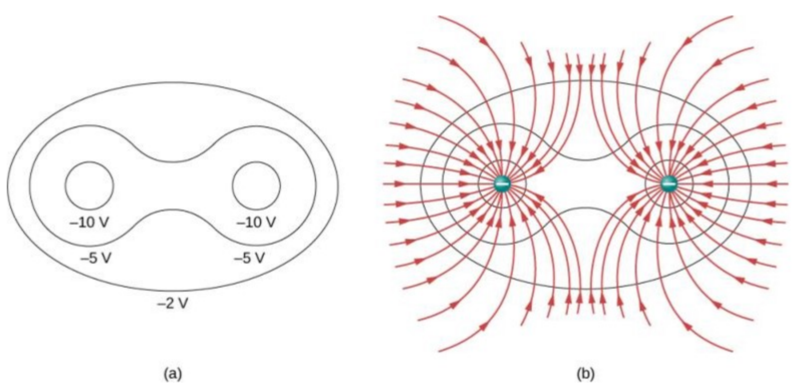


Figure 3.14: Equipotential lines (as might be measured by a voltmeter) and corresponding electric field

One of the most important cases is that of the familiar parallel conducting plates. Between the plates, the equipotentials are evenly spaced and parallel. The same field could be maintained by placing conducting plates at the equipotential lines at the potentials shown.

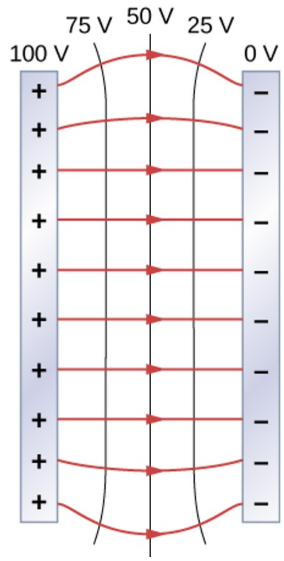


Figure 3.15: Two parallel plates - $\vec{E}$ and equipotential lines

Example - Calculating Equipotential Lines

Consider Figure 3.12 with $Q = +10nC$ at the Origin. What are the equipotential surfaces at which the potential is:

(a) 100V

(b) 50V

(c) 20V

(d) 10V

For $V = k\frac{q}{r}$, V is a constant. Thus, $r = k\frac{q}{V}$.

(a) 100V: $r = k\frac{q}{V} = (8.99 * 10^9 \frac{Nm^2}{C^2})\frac{10*10^{-9}C}{100V} = 0.90m$

(b) 50V $r = 1.8m$

(c) 20V $r = 4.5m$

(d) 10V $r = 9.0m$

Example - Potential Difference between Oppositely Charged Parallel Plates

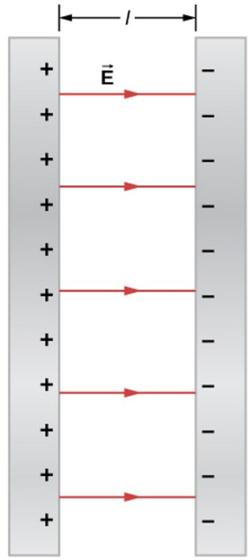


Figure 3.16: Oppositely charged parallel plates

Two large conducting plates carry equal and opposite charges, with a surface charge density $\sigma = 6.81 * 10^{-7} \frac{C}{m^2}$ of magnitude as shown in Figure 3.16. The separation between the plates is $l = 6.5mm$

(a) What is the electric field between the plates?

(b) What is the potential difference between the plates?

(c) What is the distance between equipotential planes which differ by 100 V?

Assume the size of the plates (L) is much larger than the distance l.

$$\Delta V_{AB} = -\int_A^B \vec{E} \cdot d\vec{l} \tag{3.61}$$

(a) Electric Field is directed from +plate to the -plate with magnitude

$$E = \frac{\sigma}{\epsilon_0} = \frac{6.81 * 10^{-7} \frac{C}{m^2}}{8.85 * 10^{-12} \frac{C^2}{Nm^2}} = 7.69 * 10^4 \frac{V}{m} \tag{3.62}$$

(b) The path from -plate to +plate is directed against $\vec{E}$

$$\vec{E} = d\vec{I} = -E \cdot dl \tag{3.63}$$

$$\Delta V = \int E \cdot dl = E \int dl = El = (7.69 * 10^4 \frac{V}{m})(6.5 * 10^{-3} m) = 500V \tag{3.64}$$

(c) Uniform electric field

$$\frac{500V}{6.5 * 10^{-3} m} = \frac{100V}{x} \tag{3.65}$$

$$x = 1.3 * 10^{-3} m \tag{3.66}$$

Distribution of Charges on Conductors

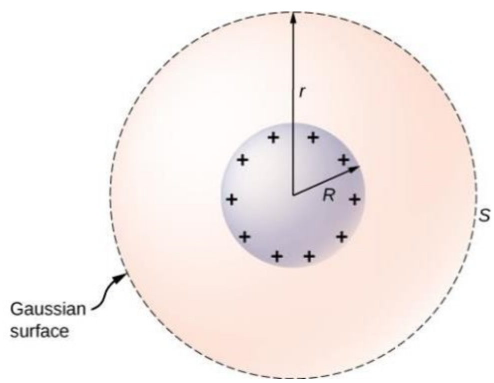


Figure 3.17: An isolated conducting sphere

$$\vec{E} = E(r)\hat{r} \tag{3.67}$$

Applying Gauss's Law over a closed surface s of radius r. As $r =$ a constant, $\hat{r} = \hat{n}$ on the surface of the sphere.

$$\oint_s \vec{E} \cdot \hat{n} = E(r) \oint_s da = E(r) 4\pi r^2 \tag{3.68}$$

For $r < R$: s in inside the conductor.

Magnitude

$$E(r) = \frac{1}{4\pi\epsilon_0}\frac{q_{enc}}{r^2} \tag{3.69}$$

$$q_{enc} = 0 \Rightarrow E(r) = 0 \tag{3.70}$$

For $r \geq R$: s encloses the conductor so $q_{enc} = q$

$$E(r) = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2} \tag{3.71}$$

Electric Field

$$\vec{E} = \begin{cases} 0 & (r < R) \\ \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{r} & (r \geq R) \end{cases} \tag{3.72}$$

Potential

- For $r < R$, $E = 0 \Rightarrow V(r)$ is constant, since $V(R) = \frac{1}{4\pi\epsilon_0}\frac{q}{R}$
- For $r \geq R$, potential is the same as a point charge: $V(R) = \frac{1}{4\pi\epsilon_0}\frac{q}{r}$

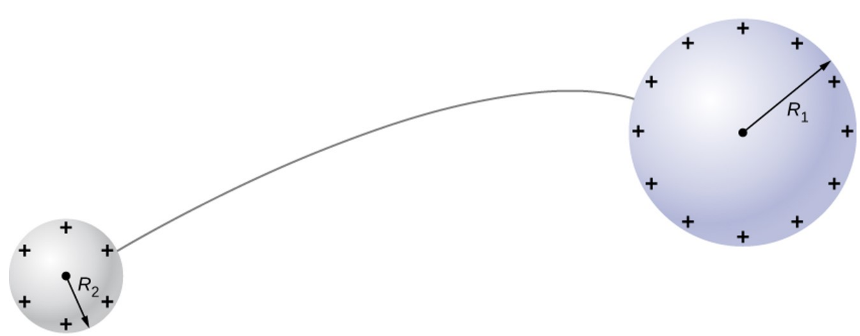


Figure 3.18: Two different radii

Now consider two conducting spheres of radii R_1 and R_2 connected by a thin conducting wire as shown in Figure 3.18

$$V = \frac{1}{4\pi\epsilon_0}\frac{q}{R} \tag{3.73}$$

Given that because of the conducting wire $V_1 = V_2$

$$\frac{1}{4\pi\epsilon_0}\frac{q_1}{R_1} = \frac{1}{4\pi\epsilon_0}\frac{q_2}{R_2} \tag{3.74}$$

$$\frac{q_1}{R_1} = \frac{q_2}{R_2} \tag{3.75}$$

Charge surface density σ:

$$q = \sigma(4\pi R^2) \tag{3.76}$$

Substituting in for q_1 and q_2 and simplifying

$$\sigma_1 R_1 = \sigma_2 R_2 \tag{3.77}$$

Different Radii of Curvature on the same object

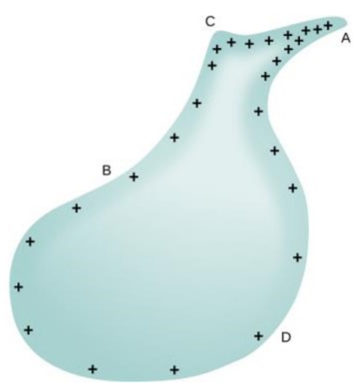


Figure 3.19: Radius of Curvature

Different Radii of Curvature on the same object

- Large Radius of Curvature $\Rightarrow$ σ and E are small
- Point $\Rightarrow$ σ and E are extremely large

Lightning Rod

On a very sharply curved surface, such as shown in 3.20, the charges are so concentrated at the point that the resulting electric field can be great enough to remove them from the surface. This can be useful.

Lightning rods work best when they are most pointed. The large charges created in storm clouds induce an opposite charge on a building that can result in a lightning bolt hitting the building. The induced charge is bled away continually by a lightning rod, preventing the more dramatic lightning strike.

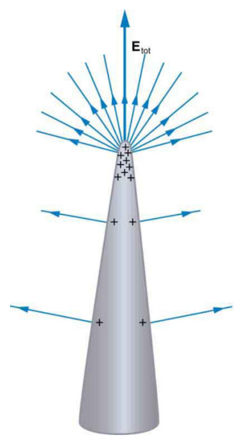


Figure 3.20: Lightning Rod

Chapter 4

Module 4 - Chapter 8 Capacitance

4.1 Capacitors and Capacitance

A capacitor is a device used to store electrical charge and electrical energy. Capacitors are generally with two electrical conductors separated by a distance. The space is usually filled with an insulating material known as a dielectric. The amount of storage in a capacitor is referred to as its capacitance.

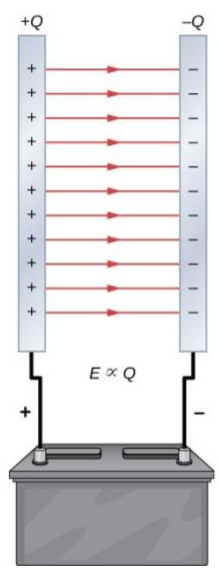


Figure 4.1: Parallel Plate Capacitor

$$C = \frac{Q}{V} \tag{4.1}$$

Capacitance is measured in the Farad[1] (F). One farad is one coulomb per volt: $1F = \frac{1C}{1V}$.

Calculation of Capacitance

1. Assume that the capacitor has charge Q
2. Determine the electric field $\vec{E}$
3. Find the potential between the conductors from

$$V_B - V_A = -\int_A^B \vec{E} \cdot d\vec{l} \tag{4.2}$$

   which leads to magnitude of the potential difference being $V = |V_B - V_A|$.

4. Find C from Equation 4.1

Parallel Plate Capacitor

Consider a parallel plate capacitor with each plate of area A separated by distance d. Assume when voltage V is applied, it stores charge Q.

Define the surface charge density σ as

$$\sigma = \frac{Q}{A} \tag{4.3}$$

When d is small compared to the size of the plates, the electric field between the plates is fairly uniform, with a magnitude give by:

$$E = \frac{\sigma}{\epsilon_0} \tag{4.4}$$

Since the electric field $\vec{E}$ between the plates is uniform

$$V = Ed = \frac{\sigma d}{\epsilon_0} = \frac{Qd}{\epsilon_0 A} \tag{4.5}$$

From Equation 4.1:

$$C = \frac{Q}{V} = \frac{Q}{\frac{Qd}{\epsilon_0 A}} = \epsilon_0 \frac{A}{d} \tag{4.6}$$

[1] Named after Michael Farday

Cylindrical Capacitor

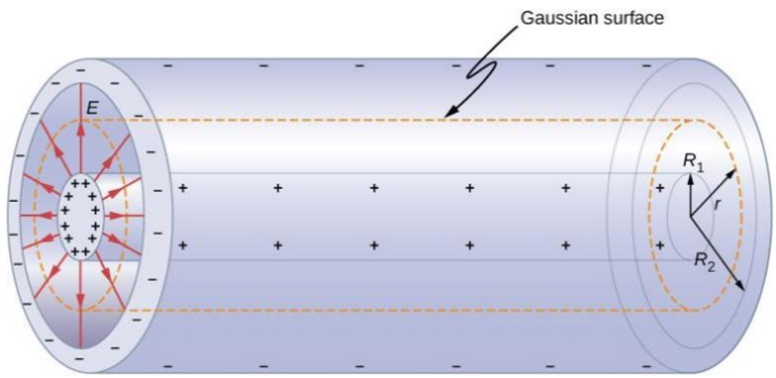


Figure 4.2: Cylindrical Capacitor

Consider the cylindrical capacitor with the inner cylinder (either solid or shell) of outer radius r_1 and out cylinder of inner radius r_2. Assume the length is l and the excess charges $+Q$ and $-Q$ are on the inner and outer cylinders, respectively.

Ignoring the edge effect, the electric field $\vec{E}$ radiates out radially from the common axis of the two cylinders. Using the Gaussian Surface shown in Figure 4.2, yields

$$\oint_S \vec{E} \cdot \vec{n} dA = E(2\pi r l) = \frac{Q}{\epsilon_0} \tag{4.7}$$

Therefore, the field between the two cylinders is

$$\vec{E} = \frac{1}{2\pi\epsilon_0} \frac{Q}{rl} \hat{r} \tag{4.8}$$

Substituting into Equation 4.2:

$$V = \int_{R_1}^{R_2} \vec{E} \cdot d\vec{l} = \frac{Q}{2\pi\epsilon_0 l} \int_{R_1}^{R_2} \frac{1}{r} \hat{r} \cdot (\hat{r} dr) = \frac{Q}{2\pi\epsilon_0 l} \int_{R_1}^{R_2} \frac{dr}{r} \tag{4.9}$$

$$= \frac{Q}{2\pi\epsilon_0 l} \ln{(r)} \Big|_{R_1}^{R_2} = \frac{Q}{2\pi\epsilon_0 l} \ln{(\frac{R_2}{R_1})} \tag{4.10}$$

Thus

$$C = \frac{Q}{V} = \frac{2\pi\epsilon_0 l}{\ln{(\frac{R_2}{R_1})}} \tag{4.11}$$

This is sometimes expressed at capacitance per unit length

$$\frac{C}{l} = \frac{2\pi\epsilon_0}{\ln\left(\frac{R_2}{R_1}\right)} \tag{4.12}$$

4.2 Capacitors in Series and Parallel

Capacitors in Series

When a series of capacitors are hooked up to a direct current voltage source (e.g., a battery) each acquires an identical charge Q.

- The plate connected to the positive terminal acquires a charge Q. And, the plate connected to the negative terminal acquires a charge $-Q$.
- A charge is then induced on each of the other plates so that the sum of the charge on all plates or on any pair of plates is zero.
- However, the potential drop on one pair of plates ($V_1 = \frac{Q}{C_1}$) may be different from a the drop across another pair of plates ($V_2 = \frac{Q}{C_2}$)

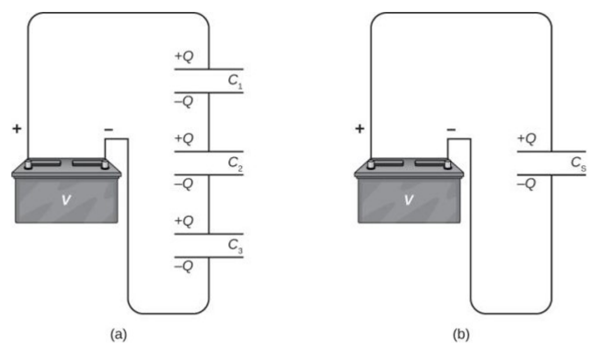


Figure 4.3: Capacitors in Series

The potentials must sum to the potential of the battery:

$$V = V_1 + V_2 + V_3 \tag{4.13}$$

The potential V is measured across a equivalent capacity holding charge Q, which is said to have an equivalent capacitance C_{EQ}. Leading to:

$$\frac{Q}{C_{EQ}} = \frac{Q}{C_1} + \frac{Q}{C_2} + \frac{Q}{C_3} \tag{4.14}$$

Thus:

$$\frac{1}{C_{EQ}} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} \tag{4.15}$$

Capacitors in Parallel

If, instead, the three capacitors are connected in parallel, they will have the same voltage across each pair of plates

$$V = V_1 + V_2 + V_3 \tag{4.16}$$

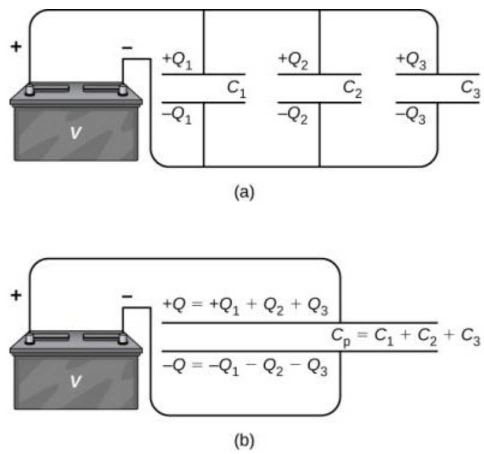


Figure 4.4: Capacitors in Parallel

The total charge stored across the plates Q is the equivalent of the sum of the charges on each plate

$$Q = Q_1 + Q_2 + Q_3 \tag{4.17}$$

Given that $Q = C * V$:

$$C_{EQ}V = C_1V + C_2V + C_3V \tag{4.18}$$

And, thus:

$$C_{EQ} = C_1 + C_2 + C_3 \tag{4.19}$$

Capacitors in Series and Parallel

Consider the below with $C_1 = 1.0pF$, $C_2 = 2.0pF$, $C_3 = 4.0pF$, and $C_4 = 5.0pF$. Assume the voltage potential across each circuit is $12.0V$:

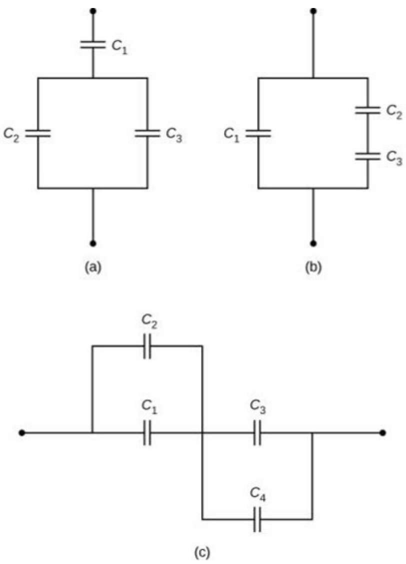


Figure 4.5: Capacitors in Series and Parallel

Energy Stored in a Capacitor

The energy U_C stored in a capacitor is electrostatic potential energy and is therefore related to the charge Q and the voltage V. The charged capacity stores energy in the electric field between the two plates.

The space between the plates has a volume Ad, where A is the area of the plates, and d is the distance between them. The energy density u_E is simply the total energy divided by the volume

$$U_C = u_E(Ad) \tag{4.20}$$

We will learn later (when we complete Maxwell's equations) that the energy density of free-space occupied by an electric field E only depends on the magnitude of the field

$$u_E = \frac{1}{2}\epsilon_0 E^2 \tag{4.21}$$

Thus

$$U_C = \frac{1}{2}\epsilon_0 E^2 Ad = \frac{1}{2}\epsilon_0 \frac{V^2}{d^2} Ad = \frac{1}{2} V^2 \epsilon_0 \frac{A}{d} = \frac{1}{2} V^2 C \tag{4.22}$$

Because $C = \frac{Q}{V}$:

$$U_C = \frac{1}{2}V^2C = \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2}QV \tag{4.23}$$

This expression for a parallel plate capacitor is generally valid for all types of capacitors. Consider an uncharged capacitor. At the instance that it is connected to the battery, the potential difference is between the the plates is $V = q/C$.

Initially

$$q = 0 \tag{4.24}$$

Gradually charge builds up on the plates. And, after some time it reaches the value $Q = CV$.

To move an infinitesimal charge dQ from the negative plate to the positive plate (from lower to higher potential) the amount of work dW that must be done on dq is

$$dW = V\,dQ = \frac{q}{C}dq \tag{4.25}$$

The work becomes the energy stored in the capacitor

$$W = \int_0^{W(Q)} dW = \int_0^Q \frac{q}{C}dq = \frac{1}{2}\frac{Q^2}{C} \tag{4.26}$$

As the geometry has not be specified, this is applicable to any type of capacitor. The total work W needed to charge a capacitor is the electrical potential energy, thus

$$U_E = W = \frac{1}{2}\frac{Q^2}{C} \tag{4.27}$$

which matches Equation 4.23.

Capacitor with a Dielectric

An insulating material placed between the plates of a capacitor is know as a dielectric.

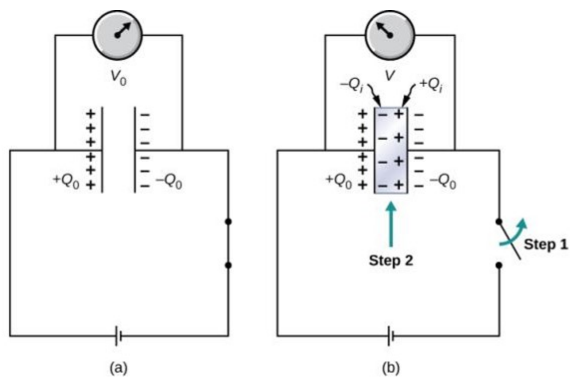


Figure 4.6: Adding a Dielectric to a Capacitor

Consider a capacitor of capacitance C_0 with air between the electrodes. When it is fully charged and then disconnected from the power source the potential between the plates is measured at V_0. Now suppose a dielectric is placed between the two plates. The new voltage is measured to be

$$V = \frac{1}{\kappa} V_0 \tag{4.28}$$

where κ is called the Dielectric Constant and $\kappa > 1$.

As the charge on the plate has not changed, this implies:

$$C = \frac{Q_0}{V} = \frac{Q_0}{\kappa V_0} = \kappa \frac{Q_0}{V_0} = \kappa C_0 \tag{4.29}$$

This, in turn, changes the potential energy stored in the capacitor

$$U = \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2}\frac{Q_0^2}{\kappa C_0} = \frac{1}{\kappa} U_0 \tag{4.30}$$

Molecular Model of a Dielectric

We can understand the effect of the dielectric on the capacitance by looking at the molecular level.

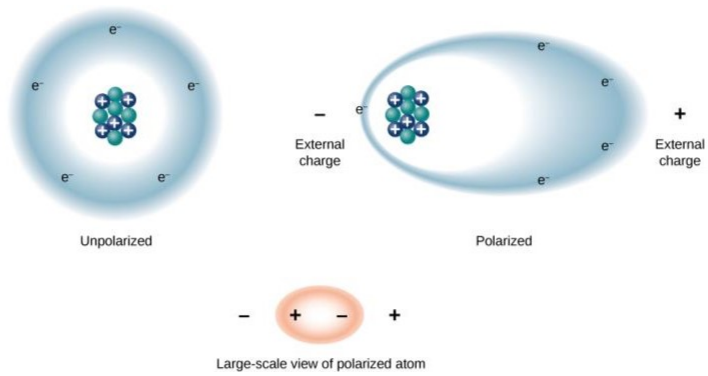


Figure 4.7: Polarized Atom

When an external electric field $\vec{E_0}$ is applied, the polar molecules align. The opposite charges in adjacent dipoles with the volume of the dielectric neutralize each other, so there is no net charge within the dielectric. However,at the edges of the dielectric that are perpendicular to the electric field, this is not the case. This creates an induced surface charge $+Q_i$ and Q_i. These produce an additional electric field, the induced electric field, $\vec{E_i}$ that opposes $\vec{E_0}$.

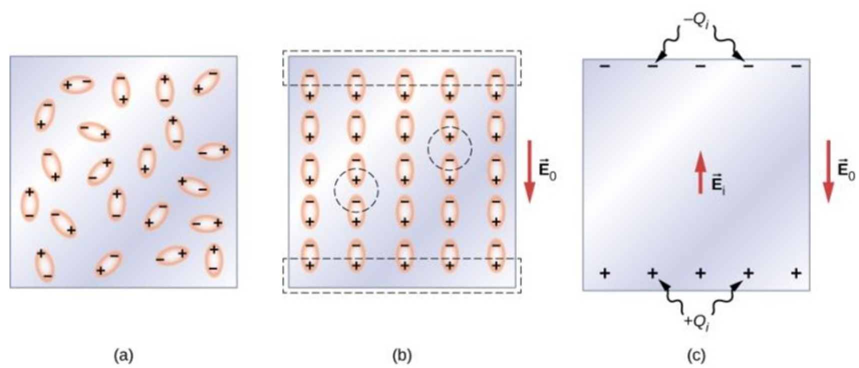


Figure 4.8: Dielectric Effect on Electric Field

Note: the same effect happens for non-polar molecules as they acquire an induced electric-dipole moment because of the external electric field.

The net electric field across the capacitor is

$$\vec{E} = \vec{E_0} + \vec{E_i} \tag{4.31}$$

and the effective charge becomes

$$Q = Q_0 - Q_i \tag{4.32}$$

This leads to the definition of the dielectric constant (κ) as

$$\kappa = \frac{E_0}{E} \tag{4.33}$$

As E_i always opposes E_0, E is always less than E_0 and $kappa > 1$. All of this leads to

$$\vec{E_i} = (\frac{1}{\kappa} - 1)\vec{E_0} \tag{4.34}$$

If the magnitude of the electric field becomes too large, the molecules in the dielectric become ionized creating free electrons and the material becomes conducting. This is known as dielectric breakdown.

The critical value of the electric field (E_C) at which molecules in an insulator become conducting is called the dielectric strength.

For air, $E_C = 3.0\frac{MV}{m}$, so for an air-filled capacitor with a plate separation of $d = 1.00mm$:

$$V = E_C d = (3.0 * 10^6 \frac{V}{m}(1.0x10^{-3}m) = 3.0kV. \tag{4.35}$$

Chapter 5

Module 5 - Chapter 9 Current and Resistance

Electric Current

Electric current is defined as the rate at which charge flows.

$$I = \frac{\Delta Q}{\Delta t} \tag{5.1}$$

where the unit for I is the ampere [1] (A) where

$$1A = 1\frac{C}{s} \tag{5.2}$$

Instantaneous electric current

$$I = \lim_{\Delta t \to 0} \frac{\Delta Q}{\Delta t} = \frac{dQ}{dt} \tag{5.3}$$

[1] named for French physicist Andrè-Marie Ampère

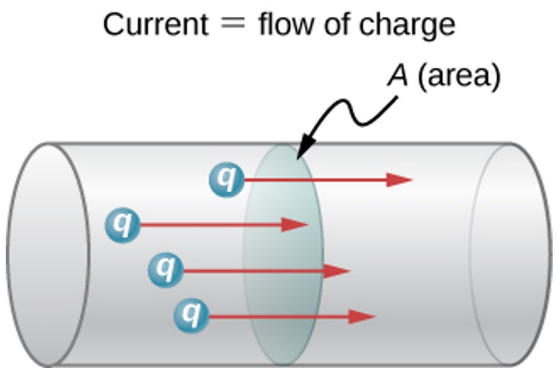


Figure 5.1: Rate of Flow

Example: Calculating Instantaneous Current

Consider a charge moving in a cross-section of wire where the charge is modeled as

$$Q(t) = Q_M(1 - e^{-\frac{t}{\tau}}) \tag{5.4}$$

where Q_M is the charge after a long time ($t \to \infty$)).

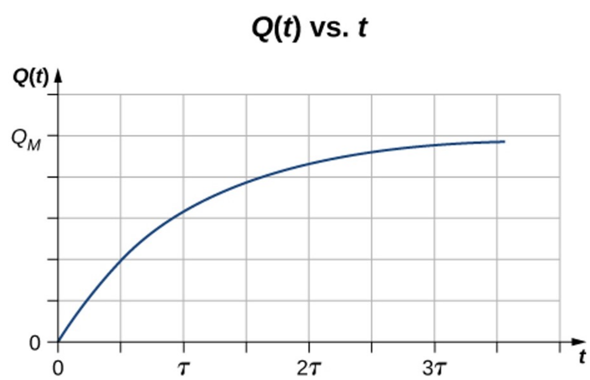


Figure 5.2: Charge moving through a wire over time

Using the equality:

$$\frac{d}{dx}e^u = e^u\frac{du}{dx} \tag{5.5}$$

$$I = \frac{dQ}{dt} = \frac{d}{dt}[Q_M(1 - e^{-\frac{t}{\tau}})] = \frac{Q_M}{\tau}e^{-\frac{t}{\tau}} \tag{5.6}$$

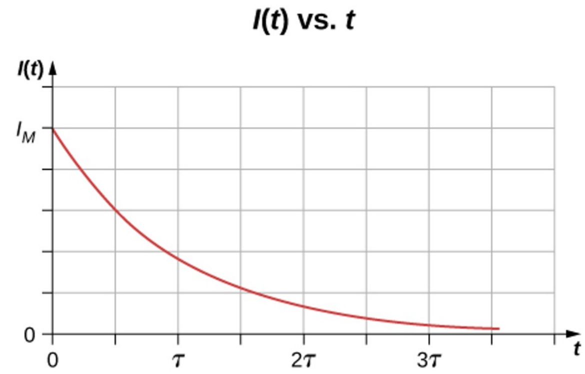


Figure 5.3: Current through a wire over time

Current in a Circuit

When the switch is closed, current flows

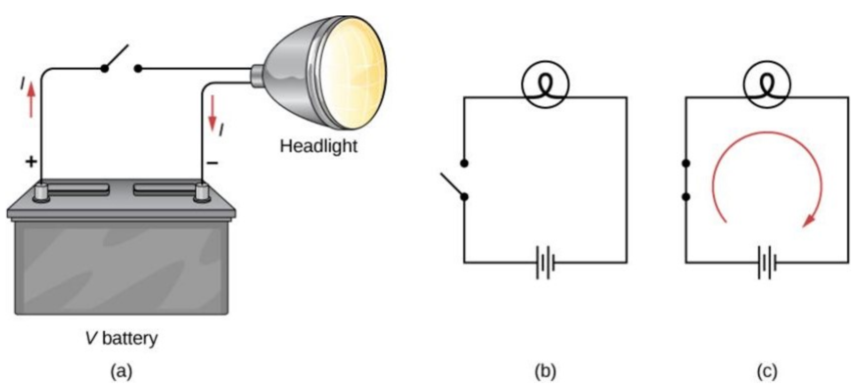


Figure 5.4: Current in a circuit

Conventionally, the direction of current represents the direction that positive charge would flow (positive terminal to negative terminal). However, we know that in fact, it is electrons (negative charge) that is flowing in the wire in the opposite direction of the current.

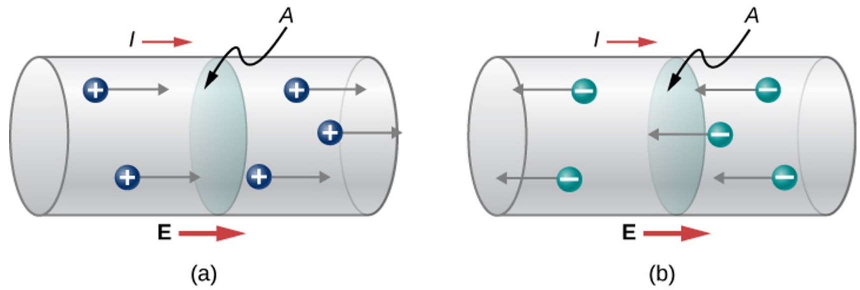


Figure 5.5: Direction of flow

5.1 Model of Conduction in Metals

Drift Velocity

The electrical signals carried by currents travel at a significant portion of the speed of light (of the magnitude of 10^8 m/s. However, individual electrons typically drift at speeds are around 10^{-4} m/s. How are these two different speeds reconciled?

The high speed electrical signals result from teh force between charges acting rapidly at a distance. When free charge is forced into a wire, the incoming charge pushes on the other charges ahead of it due to the repulsive force between like charges. The density of charge in the wire can't be readily increased, so the signal is passed rapidly. The resulting electrical "shock wave" moves through the system near the speed of light. To be precise, the fast-moving signal is a rapidly propagating change in the electric field.

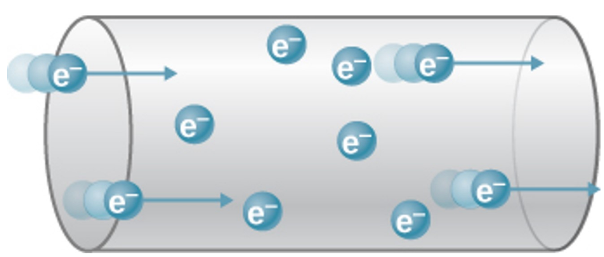


Figure 5.6: Charge in a volume

However, as the individual free charges (electrons) move, they collide with atoms in the conductor. The distance the electrons move before collision is quite small.

The electron paths appear to be random; however, there is an electric field in the conductor that causes the electrons to drift in the direction opposite the field. The drift velocity $\vec{V_d}$ is the average velocity of the free charges.

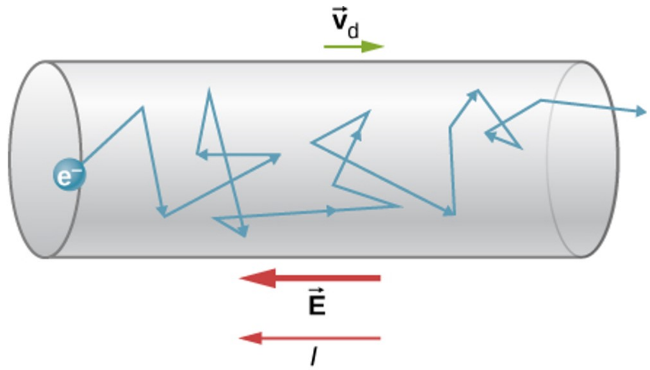


Figure 5.7: Drift

The relationship between current and drift velocity can be found by considering the number of free charges in a segment of wire. The number density of free charges (n) is given by

$$n = \frac{\text{number of charges}}{\text{volume}} \tag{5.7}$$

where the value of n depends on the material.

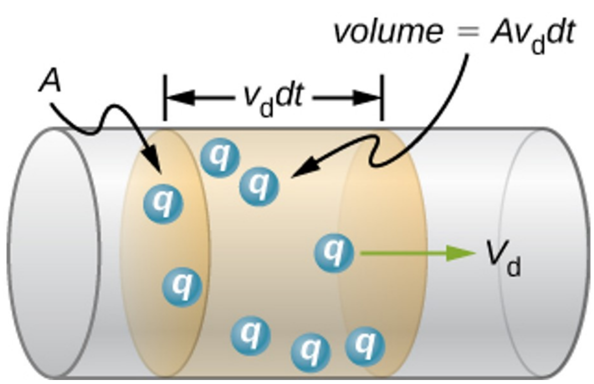


Figure 5.8: Drift Velocity

Consider the volume $Av_d dt$. The number of charges is thus $nAv_d dt$, and therefore $dQ = qnAv_d dt$.

From the current definition:

$$I = \frac{dQ}{dt} = qnAv_d \tag{5.8}$$

This leads to a drift velocity of

$$v_d = \frac{I}{nqA} \tag{5.9}$$

Current Density

While current is a scalar with a + or - to indicate direction, it is sometimes useful to consider the details of the motion of the charge.

We will consider the current density, $\vec{J}$, the flow of charge through an infinitesimal area, divided by the area. Consider the differential current through an area of $\vec{A}$

$$dI = \vec{J} \cdot d\vec{A} = JdA\cos\theta \tag{5.10}$$

Integrating

$$I = \iint_{area} \vec{J} \cdot d\vec{A} \tag{5.11}$$

Consider the magnitude of the current density

$$J = \frac{I}{A} = \frac{n|q|Av_d}{A} = n|q|v_d \tag{5.12}$$

Thus

$$\vec{J} = nq\vec{v_d} \tag{5.13}$$

Resistivity and Resistance

When a voltage is applied to a conductor an electric field $\vec{E}$ is created and charges in the conductor experience the force from the electric field. The resulting current density, $\vec{J}$, depends on the electric field and properties of the conductor:

$$\vec{J} = \sigma\vec{E} \tag{5.14}$$

where σ is the electrical conductivity of the material.

$$\sigma = \frac{|J|}{|E|}\left(\frac{A}{V \cdot m}\right) \tag{5.15}$$

Next we define the ohm[2] (Ω). The Ω is defined at $\frac{1V}{1A}$, given σ the units of $(\Omega \cdot m)^{-1}$.

The inverse of conductivity is resistivity (ρ):

$$\rho = \frac{1}{\sigma}(\Omega \cdot m) \tag{5.16}$$

We can define resistivity in terms of Electric Field and Current Density

$$\rho = \frac{E}{J} \tag{5.17}$$

Example - Copper Wire

Consider 5 meters of copper wire of diameter 2.053mm (12-gauge) carrying a current of 10mA.

First calculate the current density

$$J = \frac{I}{A} = \frac{10 * 10^{-3}}{(2.053 * 10^{-3})^2 \pi} \frac{A}{m^2} = \frac{10 * 10^{-3}}{3.31 * 10^{-6}} \frac{A}{m^2} = 3.0 * 10^3 \frac{A}{m^2} \tag{5.18}$$

Next, the resistance use $\rho_{copper} = 1.68 * 10^{-8}(\Omega \cdot m)$:

$$R = \rho \frac{L}{A} = 1.68 * 10^{-8}(\Omega \cdot m) \frac{5.00m}{3.31 * 10^{-6} m^2} = 0.025\Omega \tag{5.19}$$

Finally, the Electric Field

$$E = \rho J = 1.68 * 10^{-8}(\Omega \cdot m) \cdot 3.0 * 10^3 \frac{A}{m^2} = 5.07 * 10^{-5} \frac{V}{m} \tag{5.20}$$

This hints at Ohm's Law $A \cdot \Omega = V$ or $V = IR$.

Temperature Dependence of Resitivity

The Resistivity Table contains "Temperature Coefficient" as some materials the resistivity has a strong temperature dependence.

[2]Names for Georg Simon Ohm

$$\rho = \rho_0[1 + \alpha(T - T_0)] \tag{5.21}$$

where ρ_0 at T_0 are the resistivity at room temperature (usually assumed to be 20^oC.

Resistance

Resistance is a measure of how difficult it is for current to pass through a wire.

Consider this diagram:

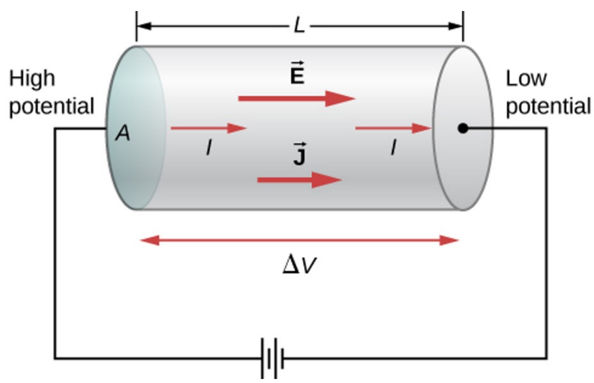


Figure 5.9: Resistance

$$E = \rho J \tag{5.22}$$

$$\frac{V}{L} = \rho\frac{I}{A} \tag{5.23}$$

$$V = (\rho\frac{L}{A})I \tag{5.24}$$

This leads us closer to Ohm's Law. The Resistance is defined as the ratio of Voltage to Current or

$$R = \frac{V}{I} \tag{5.25}$$

Application

Combining the concept of Resistance with Temperature Dependence, we get

$$R = R_0(1 + \alpha \Delta T) \tag{5.26}$$

where ΔT is the difference in temperature relative to room temperature.

This is how a digital thermometer works.

5.2 Ohm's Law

The relationship between Voltage and Current was empirically determined by Goerg Ohm[3]

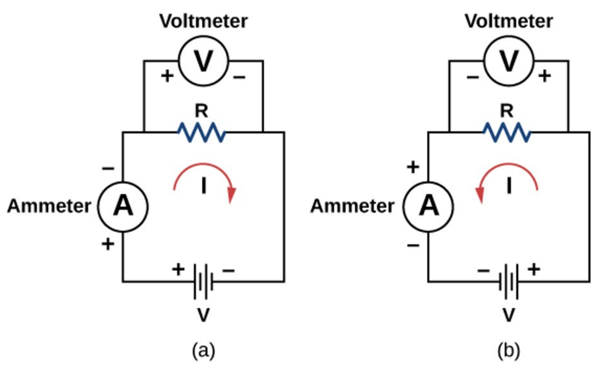
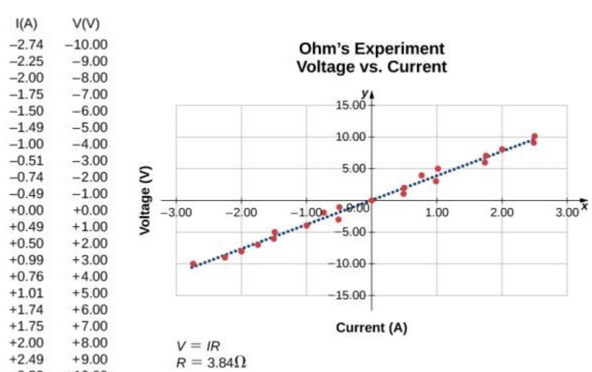


Figure 5.10: Goerg Ohm's Setup



I(A)	V(V)
−2.74	−10.00
−2.25	−9.00
−2.00	−8.00
−1.75	−7.00
−1.50	−6.00
−1.49	−5.00
−1.00	−4.00
−0.51	−3.00
−0.74	−2.00
−0.49	−1.00
+0.00	+0.00
+0.49	+1.00
+0.50	+2.00
+0.99	+3.00
+0.76	+4.00
+1.01	+5.00
+1.74	+6.00
+1.75	+7.00
+2.00	+8.00
+2.49	+9.00
+2.50	+10.00

Figure 5.11: Goerg Ohm's Data

Ohm's Law

$$V = I \cdot R \tag{5.27}$$

[3] A presented in a paper published in 1827

Diode

A diode allows current to flow in one direction. It has low resistance when forward biased and high resistance when reverse biased. Until breakdown.

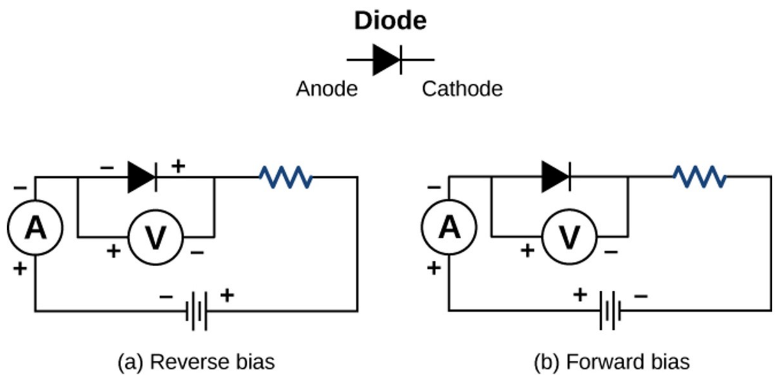


Figure 5.12: Diode Setup

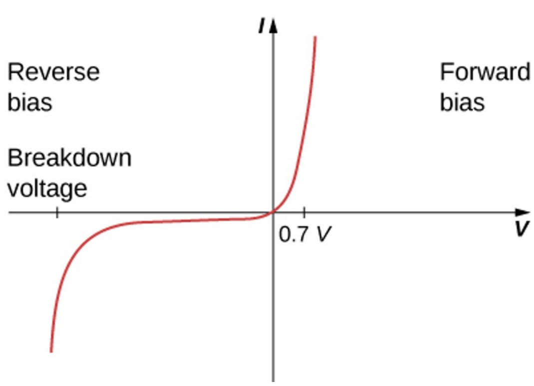


Figure 5.13: Diode Resistance

5.3 Electrical Energy and Power

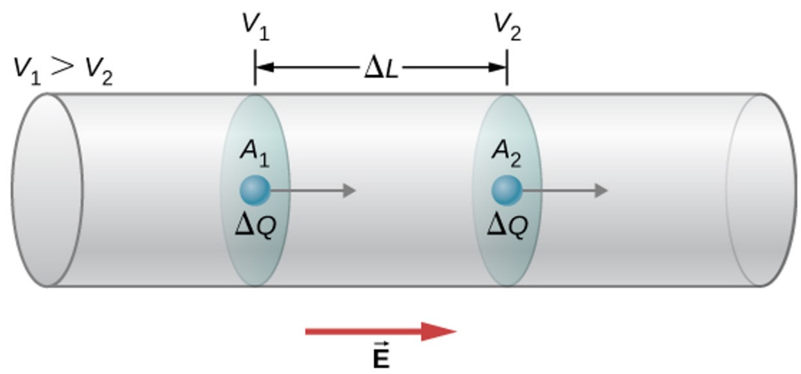


Figure 5.14: Power

To explore electric power, consider a voltage difference existing across a material. An electric potential difference exists with V_1 being higher than V_2 which is caused by an Electric field pointing from the higher to lower potential. Recall that $V = \frac{\Delta U}{q}$ and that a charge ΔQ loses potential energy moving through the potential difference.

$$\vec{F} = m\vec{a} = \Delta Q\vec{E} \tag{5.28}$$

The force does not accelerate the charge across the entire length ΔL as the charge interacts with the atoms and free electrons in the material. Thus, the charge has the same Drift Velocity v_d passing through A_2 as it does at A_1. However, there is work done on the charge.

$$E = \frac{(V_2 - V_1)}{\Delta L} = \frac{V}{\Delta L} \tag{5.29}$$

$$W = F\Delta L = (\Delta QE)\Delta L = (\Delta Q\frac{V}{\Delta L})\Delta L = \Delta QV = \Delta U \tag{5.30}$$

The lost potential energy appears as thermal energy in the material.

Power is work per unit time.

$$P = \frac{\Delta U}{\Delta t} = \frac{\Delta Q V}{\Delta t} = IV \tag{5.31}$$

For a resistor

$$P = IV = I(IR) = I^2 R \tag{5.32}$$

or

$$P = IV = \frac{V}{R} = \frac{V^2}{R} \tag{5.33}$$

The units for power is Watts (W):

$$1W = (1A)(1V) = (1\frac{C}{s})(1\frac{J}{C}) = 1\frac{J}{s} \tag{5.34}$$

Cost of Electricity

You pay for energy used. Power is change in Energy over time or $P = \frac{dE}{dt}$. This leads to

$$E = \int P dt \tag{5.35}$$

Cost Effectiveness of an LED Bulb

Consider that a $7.5W$ LED bulb has the same light output as a $100W$ incandescent bulb.

$$E_{incan} = Pt = 75W(\frac{1kW}{1000W})(\frac{3h}{day})365days = 81.125kW \cdot h \tag{5.36}$$

$$E_{LED} = Pt = 7.5W(\frac{1kW}{1000W})(\frac{3h}{day})365days = 8.113kW \cdot h \tag{5.37}$$

The average cost of electricity in Albuquerque is $14.60\frac{cents}{kWh}$

$$\Delta Cost = (81.125kWh - 8,113kWh) * \frac{\$0.146}{kWh} = \$10.66 \tag{5.38}$$

5.4 Superconductors

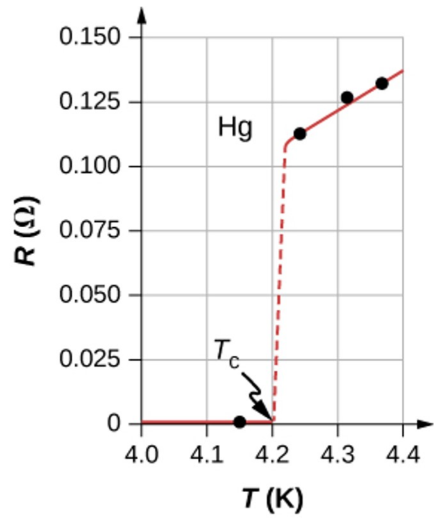


Figure 5.15: Superconductor

In 1911, Heike Kamerlingh Onnes found that when mercury was cooled before a critical temperature (4.2^oK) the resistance went to zero. Zero resistance means no power loss as $P = I^2R$

Additionally, there is a second effect, the exclusion of magnetic fields. Known as the Meissner effect allowing a light magnet to levitate above a superconducting surface.

Chapter 6

Module 6 - Chapter 10 Direct Current Circuits

6.1 Electromotive Force

A special type of potential difference is the Electromotive Force (EMF). It is not a force, but it is still called this for historical[1] reasons.

Where does EMF come from. A battery can be considered a two-terminal device with one terminal always at a higher potential than the other. The higher terminal is referred to as the positive terminal (+) and the low potential terminal referred to a the negative terminal (-).

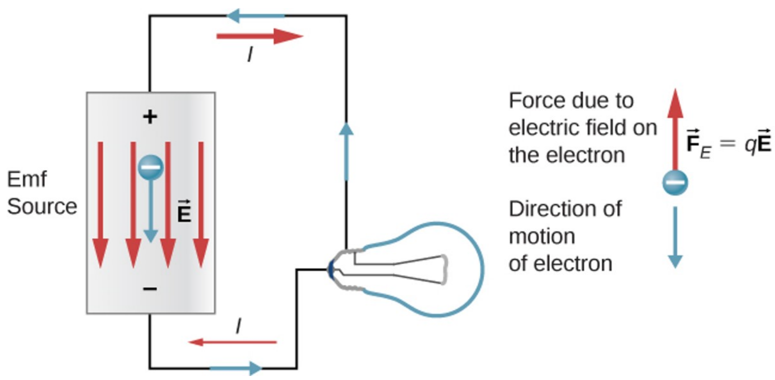


Figure 6.1: Electromotive Force

To maintain the higher potential on the positive terminal, negative charges must be moved to the negative terminal. This requires work which is supplied by the chemical reaction and this is what is referred to as the EMF. The EMF is equal to

[1]Alessandro Volta coined the term EMF in the 1800s

the work done on the charge per unit charge:

$$\epsilon = \frac{dW}{dq} \tag{6.1}$$

when there is no charge flowing. SInce the unit for work is the Joule and the unit for charge is the Coulomb, the unit for EMF is the volt ($1V = \frac{J}{C}$).

Internal Resistance and the Terminal Voltage

Batteries have internal resistance, so

$$V_{terminal} = \epsilon - Ir \tag{6.2}$$

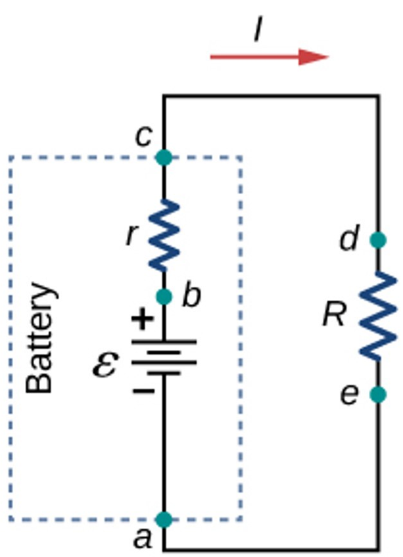


Figure 6.2: Voltage Source and Load

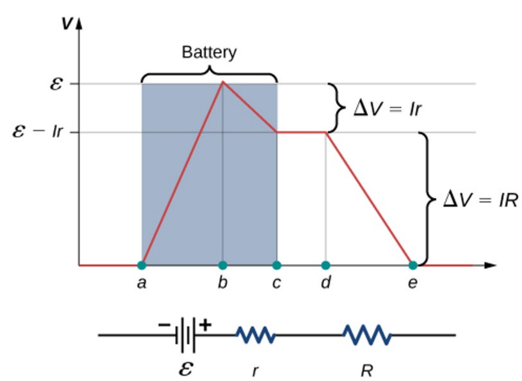


Figure 6.3: Voltage through the circuit

6.2 Resistors in Series and Parallel

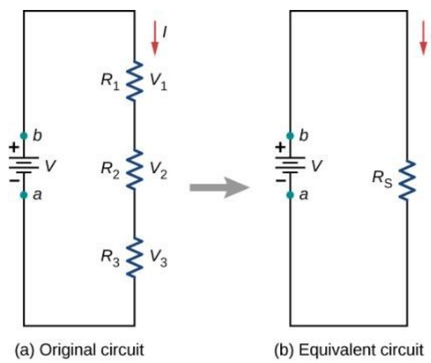


Figure 6.4: Resistors in Series

For resistors in Series

$$V = V_1 + V_2 + V_3 \tag{6.3}$$

$$V = 1R_1 + 1R_2 + 1R_3 \tag{6.4}$$

$$I = \frac{V}{R_1 + R_2 + R_3} \tag{6.5}$$

So

$$R_{eq} = \sum_{i=1}^{N} R_i \tag{6.6}$$

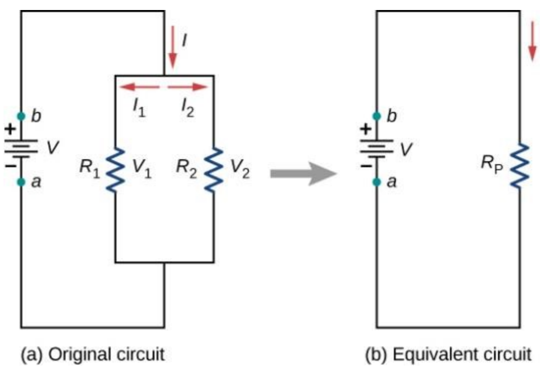


Figure 6.5: Resistors Parallel

$$V = V_1 = V_2 \tag{6.7}$$

$$I = I_1 + I_2 \tag{6.8}$$

$$\frac{V}{R_{eq}} = \frac{V_1}{R_1} + \frac{V_2}{R_2} \tag{6.9}$$

Because the Voltage is equal across the resistors

$$\frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2} \tag{6.10}$$

Or, more generically

$$R_{eq} = (\sum_{i=1} N \frac{1}{R_i})^{-1} \tag{6.11}$$

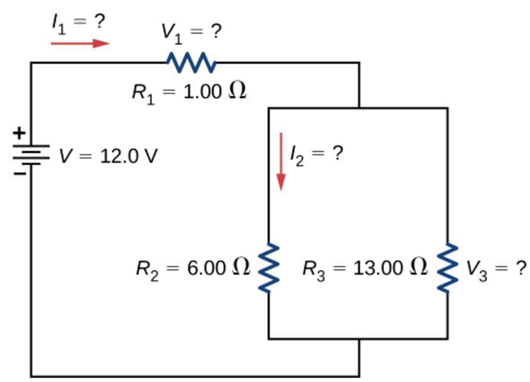


Figure 6.6: Resistors in Series and Parallel

6.3 Kirchhoff's Laws

Kirchhoff's First Law (the Node Law or the Junction Rule)

The sum of all currents entering a junction must equal the sum of all currents leaving the junction.

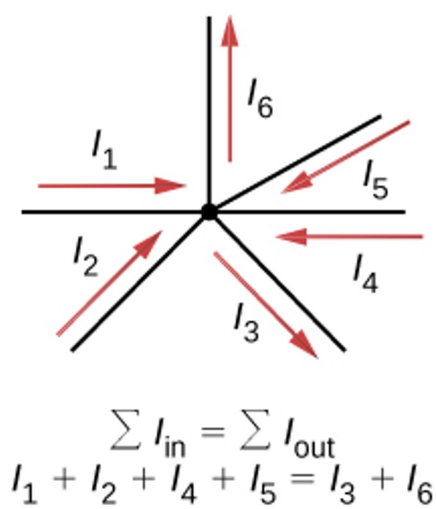


Figure 6.7: Kirchholff's Node Law

$$\sum I_{in} = \sum I_{out} \tag{6.12}$$

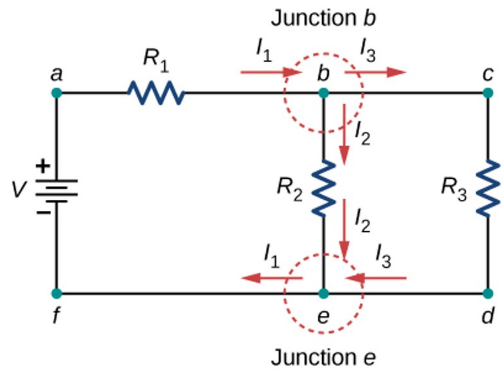


Figure 6.8: Kirchholff's Node Law Example

Kirchhoff's Second Law (the Loop Law or the Loop Rule)

The sum off all potential differences, including those supplied by voltage sources and resistive elements, around a closed loop equals zero.

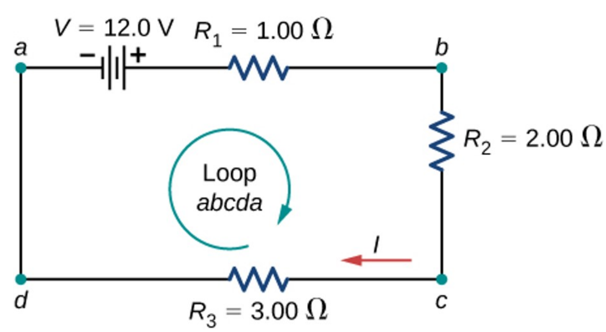


Figure 6.9: Kirchholff's Loop Law

$$\sum_{closedloop} V = 0 \tag{6.13}$$

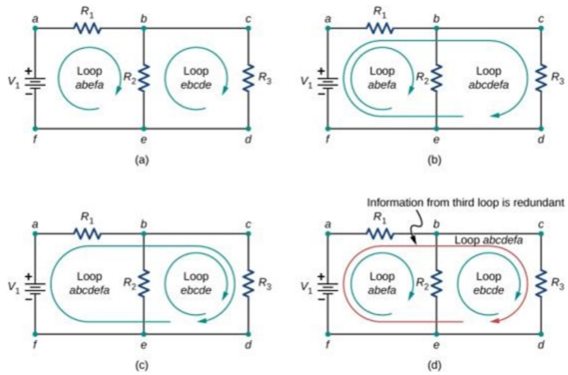


Figure 6.10: Kirchholff's Loop Law Example

Kirchhoff Examples

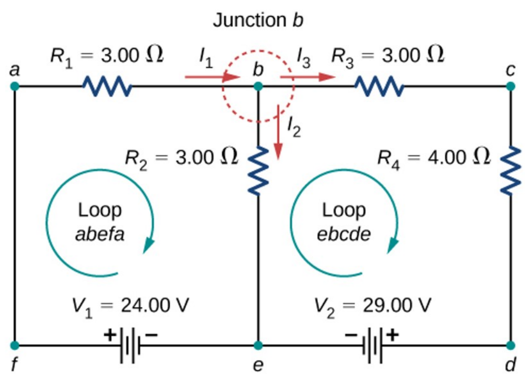


Figure 6.11: Kirchholff Example

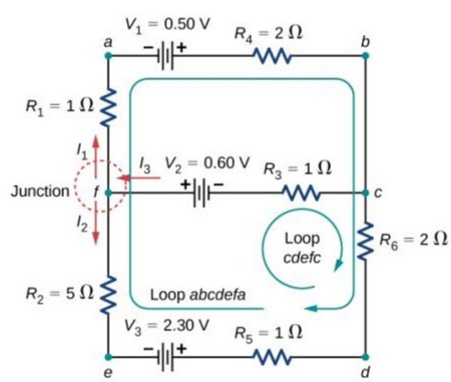


Figure 6.12: Kirchholff Examples

Examples - Circuits

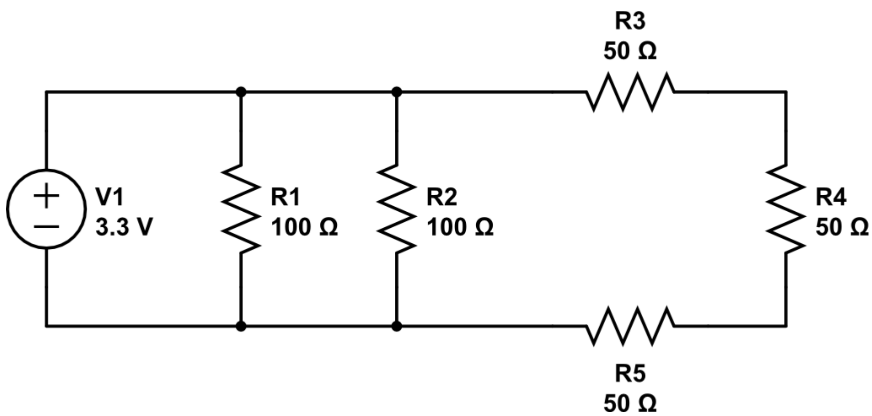


Figure 6.13: Example Circuit 1

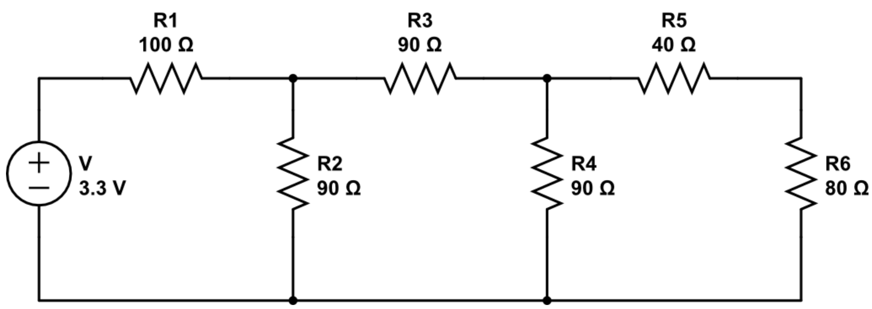


Figure 6.14: Example Circuit 2

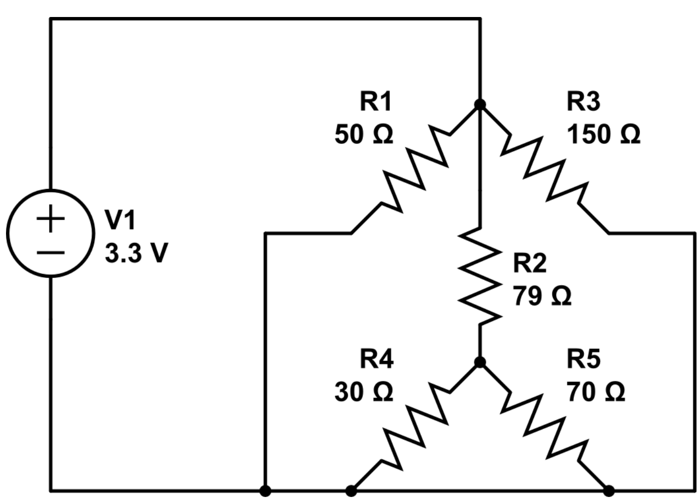


Figure 6.15: Example Circuit 3

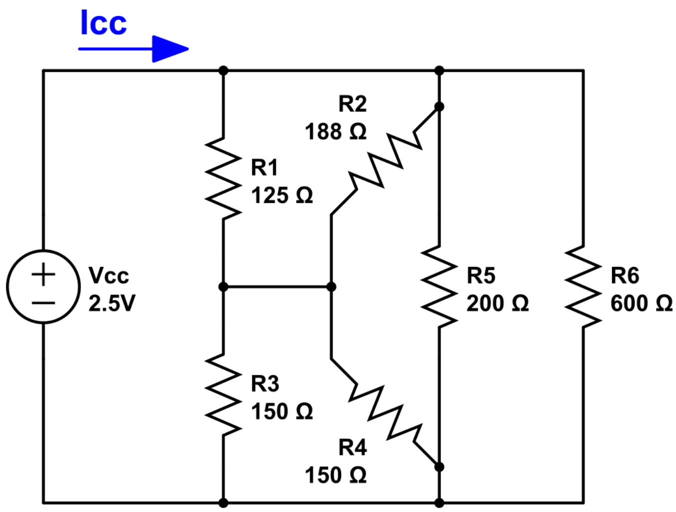


Figure 6.16: Example Circuit 4

6.4 RC Circuits

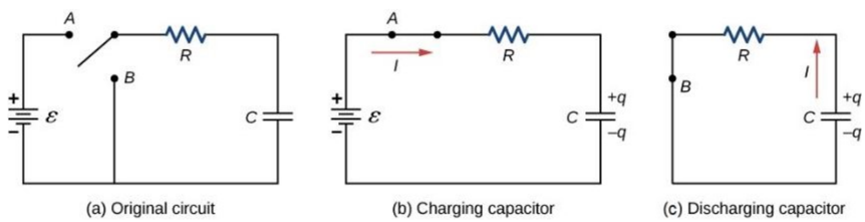


Figure 6.17: RC Circuit

When the switch is moved to Position A, from Kirchhoff's Loop Law

$$\epsilon - V_R - V_C = 0 \tag{6.14}$$

The voltage across C is $V_C = \frac{q}{C}$ and the voltage across R is $V_R = IR$ with the current $I = \frac{dq}{dt}$.

$$\epsilon - R\frac{dq}{dt} - \frac{q}{C} = 0 \tag{6.15}$$

This differential equation can be solved by rearranging

$$\frac{dq}{dt} = \frac{\epsilon C - q}{RC} \tag{6.16}$$

$$\int_0^q \frac{dq}{\epsilon C - q} = \frac{1}{RC}\int_0^t dt \tag{6.17}$$

By substitution if $u = \epsilon C - q$ then $du = -dq$

$$-\int_0^q \frac{du}{u} = \frac{1}{RC}\int_0^t dt \tag{6.18}$$

Integrating

$$\ln\frac{\epsilon C - q}{\epsilon C} = -\frac{1}{RC}t \tag{6.19}$$

$$\frac{\epsilon C - q}{\epsilon C} = e^{-\frac{t}{RC}} \tag{6.20}$$

Simplifying

$$q(t) = C\epsilon(1 - e^{\frac{t}{RC}}) = Q(1 - e^{-\frac{1}{\tau}}) \tag{6.21}$$

Where τ is the RC Time Constant ($\tau = RC$).

Note, as time approaches ∞, the exponential goes to zero, so the charge approaches the maximum charge $Q = C\epsilon$.

The current can likewise be calculated

$$I(t) = \frac{dq}{dt} = \frac{d}{dt}[C\epsilon(1 - e^{-\frac{t}{RC}}] \tag{6.22}$$

$$I(t) = C\epsilon(\frac{1}{RC})e^{-\frac{t}{RC}} = \frac{\epsilon}{R}e^{-\frac{t}{RC}} = I_0 e^{-\frac{1}{\tau}} \tag{6.23}$$

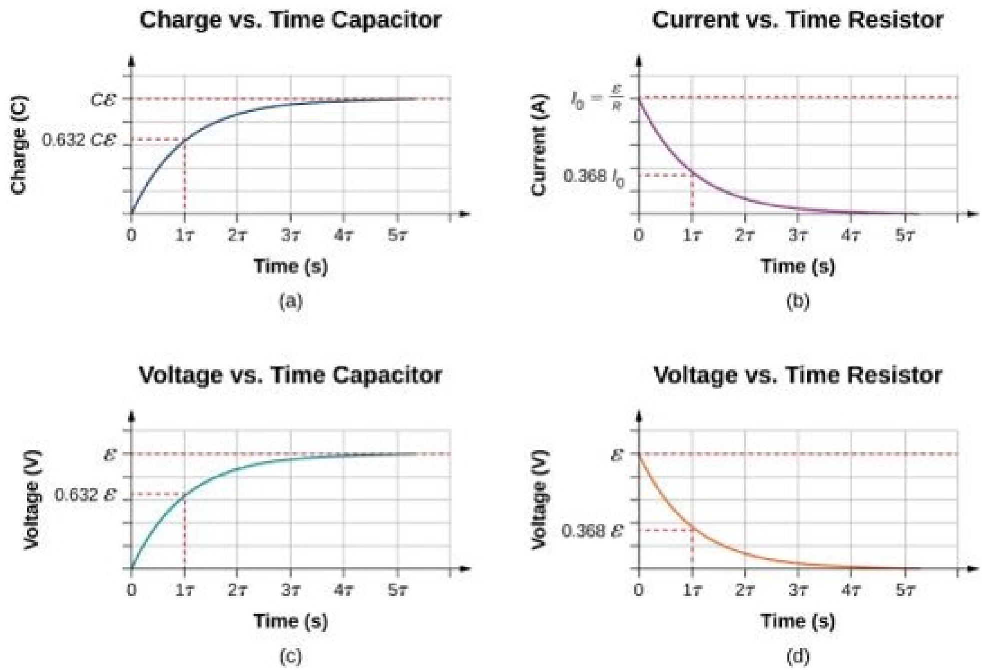


Figure 6.18: RC Circuit Charging

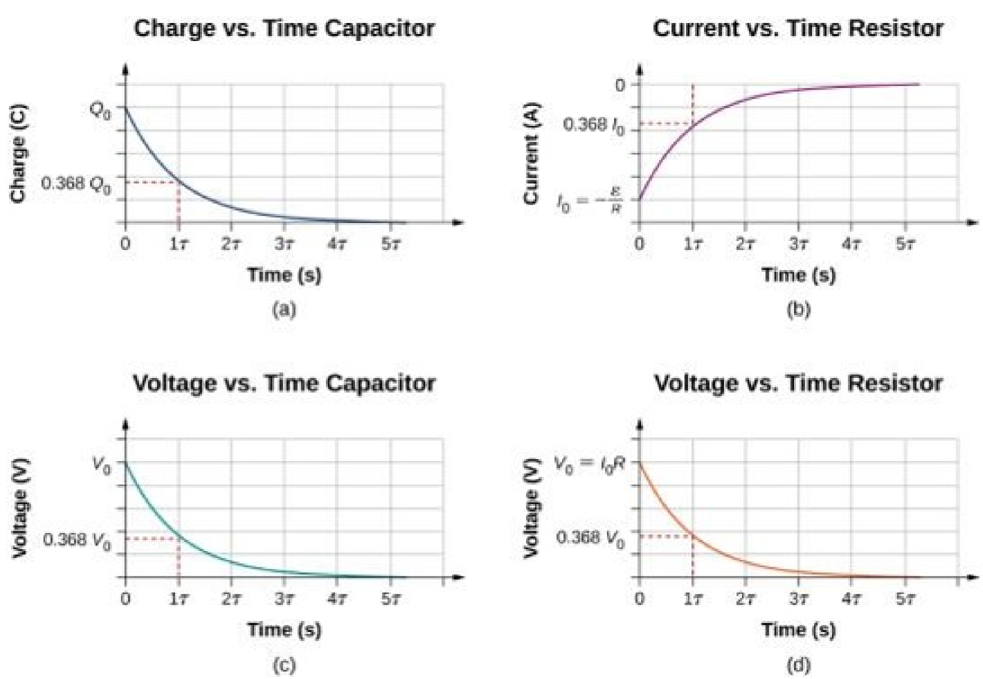


Figure 6.19: RC Circuit Discharging

Example - Intermittent Wiper

An intermittent works using a relaxation oscillator to control a pair of windshield wipers. Once the voltage on V_{out} reaches $10V$ the wiper motor engages and the capacitor discharges. After one cycle of the blades wiping, the capacitor starts charging again. The potentiometer can be adjusted from 0.00Ω to $10.00k\Omega$ to control the speed of charging.

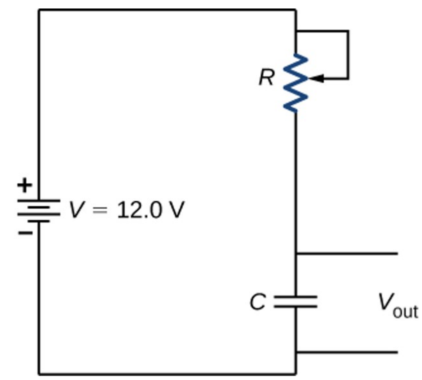


Figure 6.20: Intermittent Wiper

In Figure 6.20, assume that $C = 10mF$. What do we need to adjust the poteniometer R to in order to get the wiper frequency to be once every 10 seconds.

Start by noting that

$$V_out(t) = V(1 - e^{-\frac{t}{\tau}}) \tag{6.24}$$

This leads to

$$e^{-\frac{t}{\tau}} = 1 - \frac{V_{out}(t)}{V} \tag{6.25}$$

With $\tau = RC$ and taking the natural log of both sides:

$$-\frac{1}{RC} = ln(1 - \frac{V_{out}(t)}{V}) \tag{6.26}$$

Solving for R:

$$R = \frac{-t}{Cln(1 - \frac{V_{out}(t)}{V})} \tag{6.27}$$

Using the parameters from above

$$R = \frac{-10.00s}{(10 * 10^{-3}F)ln(1 - \frac{10V}{12V})} = 558.11\Omega \tag{6.28}$$

Chapter 7

Module 7 - Chapter 11 Magnetic Forces and Fields

We all know one big magnet. The North of the compass points towards the South pole of the magnet. So, the Geographic North Pole is actually near the Magnetic South Pole.

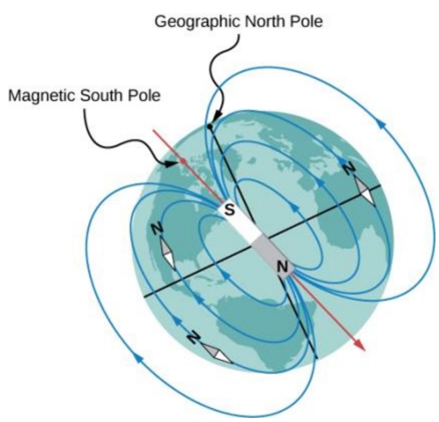


Figure 7.1: One Big Magnet

7.1 Magnetic Fields and Lines

A magnetic field is defined by the force of that a charged particle experiences moving in this field. The magnitude of the force is proportional to the amount of charge q, the speed of the charged particle v, and the magnitude of the applied magnetic field. The direction is perpendicular to both the direction of the moving charged particle and the direction of the applied magnetic field.

$$\vec{F}_m = q\vec{v} \times \vec{B} \tag{7.1}$$

where θ is the angle between the velocity and the magnetic field and the $\vec{B}$ is defined as the magnetic field.

This gives a magnitude of

$$F_m = qvBsin\theta \tag{7.2}$$

where the units for B are tesla (T).

$$1T = \frac{1N}{A \cdot m} \tag{7.3}$$

Sometimes it is useful to use a smaller unit, teh gauss (G), where $1G = 10^{-4}T$.

The direction of the Magnetic Force is given by the right hand rule:

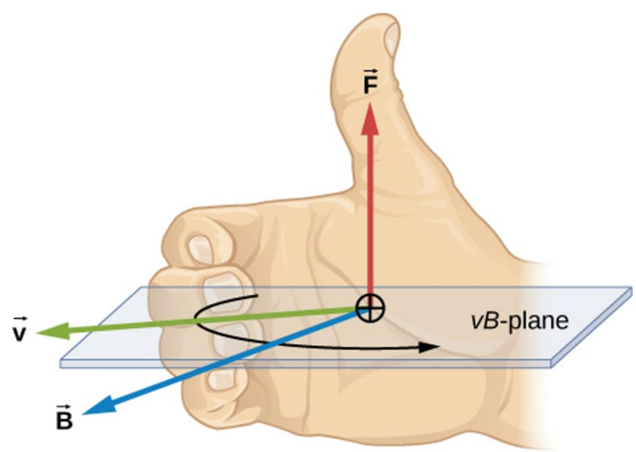


Figure 7.2: Right Hand Rule

Example

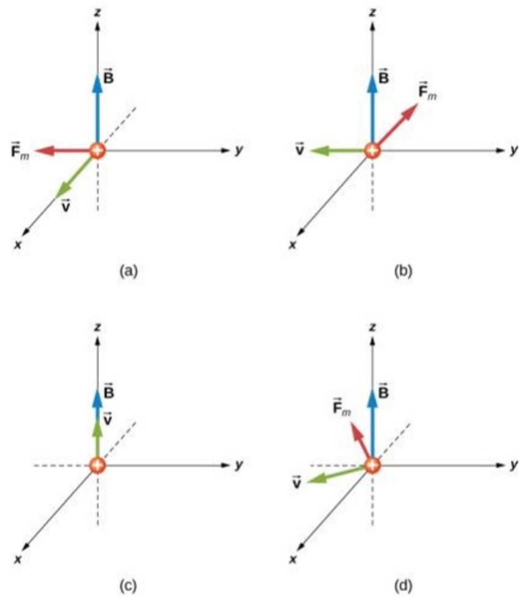


Figure 7.3: Magnetic Force on Alpha-Particle

Consider an alpha-particle ($q = 3.2 * 10^{-19}C$) moving through a uniform magnetic field of magnitude $1.5T$. The direction of the field is parallel to the positive z-axis. What is the magnetic force on the alpha-particle when it is moving:

(a) $5.0 * 10^4 \frac{m}{s}$ in the positive x-direction

$$\vec{F} = q\vec{v} \times \vec{B} \tag{7.4}$$

$$\vec{F} = (3.2 * 10^{-19}C)(5.0 * 10^4 \frac{m}{s}\hat{\imath}) \times (1.5T\hat{k}) = -2.4 * 10^{-14}N\hat{\jmath} \tag{7.5}$$

(b) $5.0 * 10^4 \frac{m}{s}$ in the negative y-direction

$$\vec{F} = (3.2 * 10^{-19}C)(-5.0 * 10^4 \frac{m}{s}\hat{\jmath}) \times (1.5T\hat{k}) = -2.4 * 10^{-14}N\hat{\imath} \tag{7.6}$$

(c) $5.0 * 10^4 \frac{m}{s}$ in the positive z-direction

$$\vec{F} = (3.2 * 10^{-19}C)(-5.0 * 10^4 \frac{m}{s}\hat{k}) \times (1.5T\hat{k}) = 0 \tag{7.7}$$

(d) with a velocity of $\vec{v} = (2.0\hat{\imath} - 3.0\hat{\jmath} + 1.0\hat{k}) * 10^4 \frac{m}{s}$

$$\vec{F} = (3.2 * 10^{-19} C)((2.0\hat{\imath} - 3.0\hat{\jmath} + 1.0\hat{k}) * 10^4 \frac{m}{s} \hat{\jmath}) \times (1.5T\hat{k}) \tag{7.8}$$

Solving, rearranging to the order we are use to $(\hat{\imath} + \hat{\jmath} + \hat{k})$:

$$\vec{F} = (-14.4\hat{\imath} - 9.6\hat{\jmath}) * 10^{-15} N \tag{7.9}$$

This can also be represented as a magnitude and angle:

$$|\vec{F}| = \sqrt{((-14.4)^2 + (-9.6)^2)} * 10^{-15} N = 1.7 * 10^{-14} N \tag{7.10}$$

and

$$\theta = \arctan \frac{-9.6 * 10^{-15} N}{-14.4 * 10^{-15} N} = 34^o \tag{7.11}$$

Representing Magnetic Fields

- The direction of the magnetic field is tangent to the field line at any point in space.
- The strength of the field is proportional to the closeness of the lines.
- Magnetic field lines can never cross
- Magnetic field lines are continuous, forming closed loops without a beginning or end. They are directed from the North Pole to the South Pole.

The last property is related to the fact hat North and South poles can not be separated. This makes then different from electric field lines.

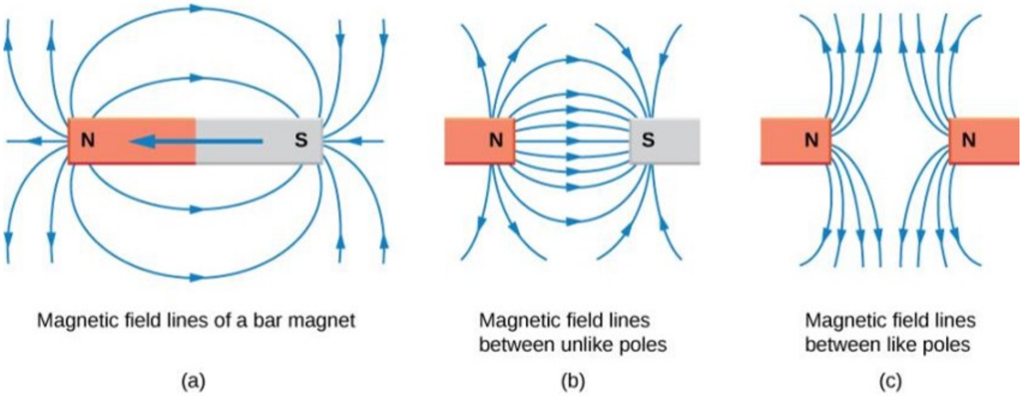


Figure 7.4: Magnetic Field Lines

7.2 Motion of Charged Particle In a Magnetic Field

Consider the case where the charged particle moves perpendicular to a uniform B-field.

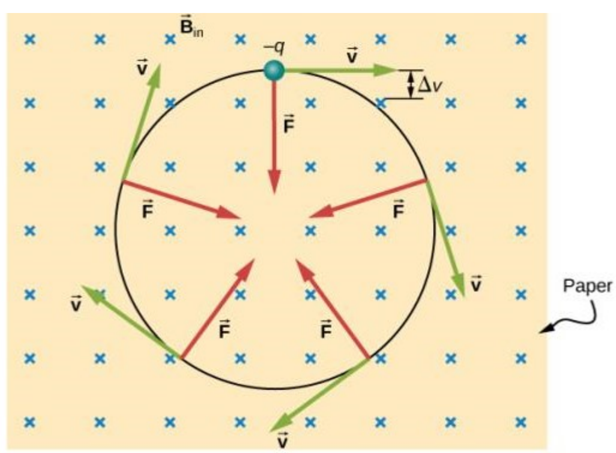


Figure 7.5: Motion in a Magnetic Field

The magnetic force supplies the centripetal force $F_c = \frac{mv^2}{r}$. Noting the velocity is perpendicular to the magnetic field, the magnetic force reduces to $F_m = qvB$. Because F_m supplies F_c:

$$qvB = \frac{mv^2}{r} \tag{7.12}$$

Solving for r yields:

$$r = \frac{mv}{qB} \tag{7.13}$$

The period can then be derived as

$$T = \frac{2\pi r}{v} = \frac{2\pi}{v}\frac{mv}{qB} = \frac{2\pi m}{qB} \tag{7.14}$$

If the velocity is not perpendicular to the magnetic field then the velocity can be broken into components

$$v_{perp} = v\sin\theta \tag{7.15}$$

and

$$v_{para} = v\cos\theta \tag{7.16}$$

The parallel motion determines the pitch (p) of the helix

$$p = v_{para}T \tag{7.17}$$

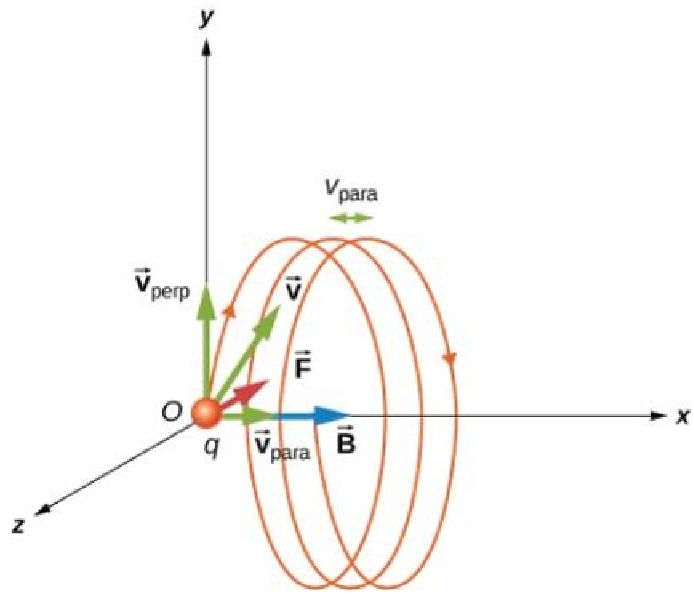


Figure 7.6: Helical Motion

7.3 Magnetic Force on a Current-Carrying Conductor

A moving charge experiences a force in a magnetic field. If the charges are moving in a wire, then the wire will also experience the force. Before we explore this, let's consider how a magnetic field is generated by an electric current.

Magnetic Fields Produced By Electrical Currents

In 1820, a Danish physicist, Hans Christian Oersted, discovered that there was a relationship between electricity and magnetism. By setting up a compass through a wire carrying an electric current, Oersted showed that moving electrons can create a magnetic field.

The compass needle near a wire experiences a force that aligns the needle tangent to a circle around the wire. Therefore, a current carrying wire produces circular loops of magnetic field. This leads us to the second right hand rule

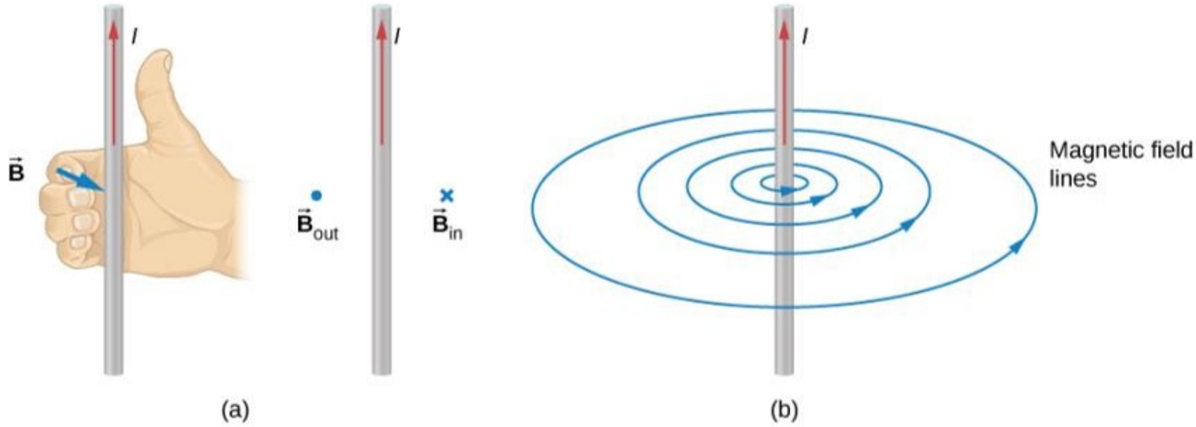


Figure 7.7: Right Hand Rule Revisited

Calculating the Magnetic Force

Recall that the current in a wire is given by

$$I = nqAv_d \tag{7.18}$$

where n is charge particle density, q is the charge, A is the cross sectional area of the wire, and v_d is the drift velocity.

Consider the magnetic force of a single charge: $q\vec{v} \times \vec{B}$. The magnetic force $d\vec{F}$ on the $nA \cdot dl$ segment of the wire is

$$d\vec{F} = (nA \cdot dl)q\vec{v}_d \times \vec{B} \tag{7.19}$$

We can define $\vec{dl}$ to by a vector of length dl pointing along $\vec{V}_d$:

$$d\vec{F} = nqAv_d\vec{dl} \times \vec{B} \tag{7.20}$$

which leads to

$$d\vec{F} = I\vec{dl} \times \vec{B} \tag{7.21}$$

Integrating

$$\vec{F} = I\vec{l} \times \vec{B} \tag{7.22}$$

Example - Force on a Circular Wire

Consider a circular loop of radius carrying a current I that is placed on the xy plane. A constant uniform magnetic field cuts through the loop parallel to the y-axis. Find the magnetic force on the upper half, lower half, and the total force on the loop.

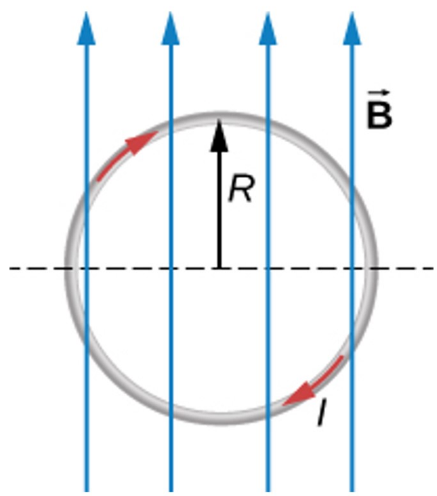


Figure 7.8: Magnetic Force on a Loop

The differential force on an arbitrary piece of wire located in the upper ring is

$$dF = IB\sin\theta dl \tag{7.23}$$

where θ is the angle between the magnetic field and the segment of wire. As $dl = Rd\theta$:

$$dF = IBR\sin\theta d\theta \tag{7.24}$$

Integrating over the top half of the circle

$$F = IRB \int_0^{\pi} \sin\theta d\theta = IBR\Big|_0^{\pi} = 2IBR \tag{7.25}$$

Similarly, for the lower loop

$$F = IRB \int_{\pi}^{2\pi} \sin\theta d\theta = IBR\Big|_{\pi}^{2\pi} = -2IBR \tag{7.26}$$

And the total force is thus equal to zero.

7.4 Force and Torque on a Current Loop

One application of the magnetic force on a current-carrying wire is a motor.

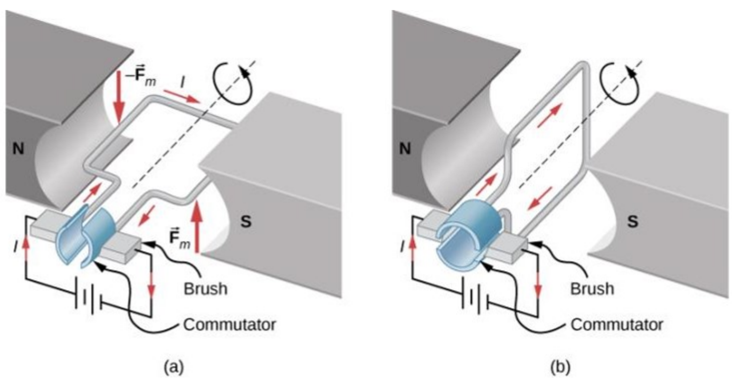


Figure 7.9: DC Electric Motor

In a uniform magnetic field, a current-carrying loop experiences both forces and torques on the loop. In the case of Figure 7.9, the rectangular loop has sides of lengths a and b, and is in a uniform magnetic field: $\vec{B} = B\hat{j}$.

The magnetic force on the wire is

$$\vec{F} = l\vec{I} \times \vec{B} \tag{7.27}$$

In order to find the total force on the loop, we look at the force on each of the sides.

$$\vec{F_1} = laB\sin(90^o - \theta)\hat{\imath} = laB\cos(\theta)\hat{\imath} \tag{7.28}$$

$$\vec{F}_3 = laB\sin(90^o + \theta)\hat{\imath} = -laB\cos(\theta)\hat{\imath} \tag{7.29}$$

Sides 2 and 4 are perpendicular to the magentic field $\vec{B}$:

$$\vec{F}_2 = lbB\hat{k} \tag{7.30}$$

$$\vec{F}_4 = -lbB\hat{k} \tag{7.31}$$

The net force on the loop is the sum of the four components

$$\sum \vec{F}_{net} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \vec{F}_4 = 0 \tag{7.32}$$

This holds generally for a current-carrying loop in a uniform magnetic field at the net force on it is zero.

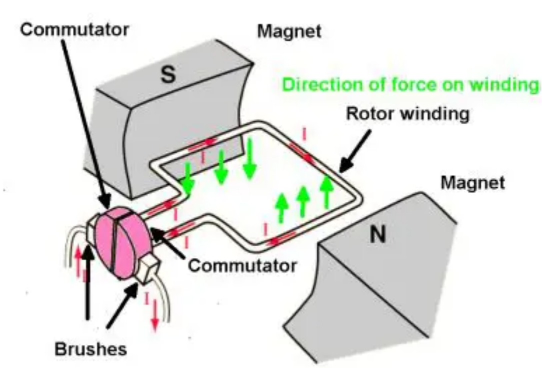


Figure 7.10: DC Motor with Brushes

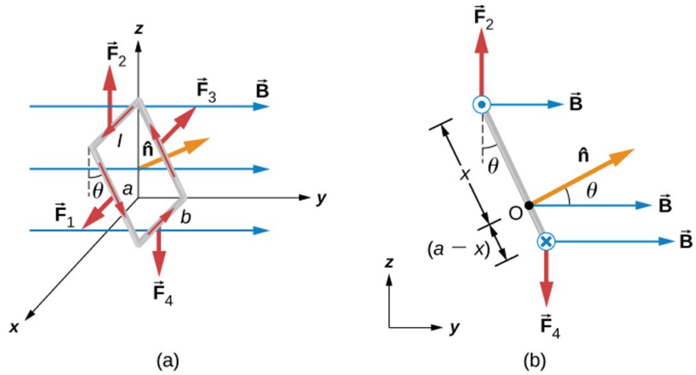


Figure 7.11: Torque but not Force

In Figure 7.11, consider first forces $\vec{F_1}$ and $\vec{F_3}$. Since they have the same line of action and are equal but opposite, the sum of their torques about any axis is zero. Thus, any torque on the loop must be furnished by $\vec{F_2}$ and $\vec{F_4}$.

Consider Point O which is a distance x from side 2 and a distance $(a - x)$ from side 4. The moment arms for F_2 and F_4 are $x \sin\theta$ and $(a - x)\sin\theta$, respectively. This gives the torque:

$$\sum \vec{\tau} = \vec{\tau}_2 + \vec{\tau}_4 = F_2 x \sin\theta(-\hat{\imath}) + F_4(a - x)\sin\theta(-\hat{\imath}) \tag{7.33}$$

where direction of Torgue $-\hat{\imath}$ comes from $\hat{r} \times \hat{k}$ for F_2 and $-\hat{r} \times -\hat{k}$ for F_4.

or

$$\sum \vec{\tau} = lbBx \sin\theta\hat{\imath} - lbB(a - x)\sin\theta\hat{\imath} \tag{7.34}$$

If we consider $A = a * b$, the area inside the loop, then this simplifies to

$$\vec{\tau} = -lAB \sin\theta\hat{\imath} \tag{7.35}$$

Notice that the torque is independent of x, and thus independent of point O. Consequently, the loop experiences the same torque from the magnetic field around any axis in the plane of the loop and parallel to the x-axis.

A closed-current loop is commonly referred to as a magnetic dipole and the term IA is known as the magnetic dipole moment μ.

$$\vec{\mu} = IA\hat{n} \tag{7.36}$$

where $\hat{n}$ is a unit vector perpendicular to the plane of the loop. The direction of $\hat{n}$ is obtained by the right-hand-rule.

If the loop contains N turns of the wire, then the magnetic dipole moment is given by

$$\vec{\mu} = NIA\hat{n} \tag{7.37}$$

In terms of magnetic dipole moment

$$\vec{\tau} = \vec{\mu} \times \vec{B} \tag{7.38}$$

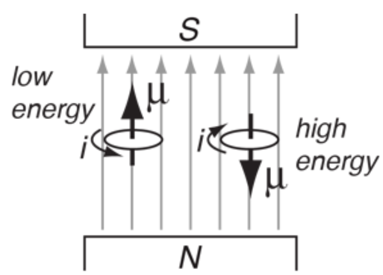


Figure 7.12: Magnetic Dipole Potential

Finally, in a way analogous the potential held in capacitance for electric dipoles, the potential energy of a magnetic dipole is

$$U = -\vec{\mu} \cdot \vec{B} \tag{7.39}$$

Example

A circular loop of radius 2.0cm carries a current of 2.0mA. What is the magnitude of the magnetic dipole moment. And, when the dipole is at 30^o to a uniform magnetic field of magnitude 0.50T, what is the torque and the potential energy?

Magnetic Moment:

$$\mu = IA = I(\pi R^2) = (2.0 * 10^{-3}A)(\pi(0.02m)^2) = 2.5 * 10^{-6}A \cdot m^2 \tag{7.40}$$

Torque:

$$\tau = \vec{u} \times \vec{B} = \mu B \sin\theta = (2.5 * 10^{-6}A \cdot m^2)(0.50T)\sin 30^o = 6.3 * 10^{-7}N \cdot m \tag{7.41}$$

Potential:

$$U = -\vec{u} \cdot \vec{B} = -\mu B \cos\theta - (-2.5 * 10^{-6}A \cdot m^2)(0.50T)\cos 30^o = -1.1 * 10^{-6}J \tag{7.42}$$

7.5 Hall Effect

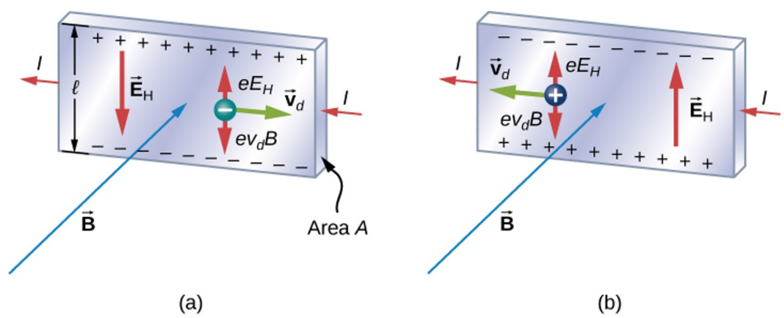


Figure 7.13: Hall Effect

Edwin Hall [1] investigated motion of electrons in a strip subject to a magnetic field.

Consider a magnetic strip of width l in a uniform magnetic field. As electrons move in the strip (i.e., current), the magnetic field pushes the electrons to one side of the strip. This leads to an excess of positive charge on the other side of the strip and leads to an electric field across the strip (perpendicular to the direction of the current). This electric field pulls the electrons against the magnetic force until an equilibrium is reached. Giving:

$$qE = qv_dB \tag{7.43}$$

or a drift velocity of

$$v_d = \frac{E}{B} \tag{7.44}$$

[1] in 1879 in the US

Recall that if the current in the strip is I, then

$$I = nqv_dA \tag{7.45}$$

where n is the density of charge carriers and A is the cross-sectional area of the strip.

Combining,

$$I = nq\frac{E}{B}A \tag{7.46}$$

Noting that $E = \frac{V}{l}$, we get a Hall Potential (V_{hall}) of

$$V_{hall} = \frac{IBl}{nqA} \tag{7.47}$$

Or, substituting $qE = qv_dB$ leads to

$$V = Blv_d \tag{7.48}$$

The Hall Effect can be used to "select" charges of a certain velocity and can be used to measure strength of a magnetic field.

7.6 Applications of Magnetic Forces and Fields

Mass Spectrometer

A mass spectrometer is a device that separates ions (charged particles) by their charge-to-mass ratio.

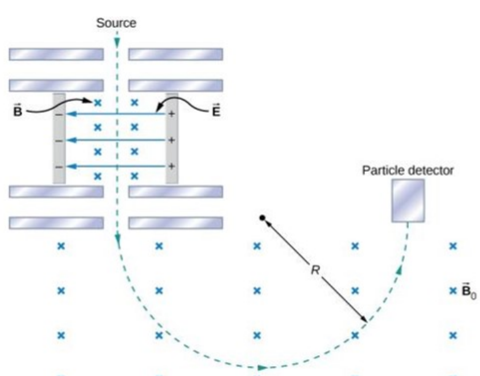


Figure 7.14: Mass Spectrometer

1. Ions are first sent through a velocity selector where all ions emerge with the speed $v = \frac{E}{B}$ as all other ions are deflected by either the electric or magnetic force.

2. The ions next enter the uniform magnetic field B_0 where they travel in a circular path R given by

$$R = \frac{mqv}{B_0} \tag{7.49}$$

3. Given that $v = \frac{E}{B}$, the ion moves in a circular path:

$$\frac{q}{m} = \frac{E}{BB_0R} \tag{7.50}$$

4. Ions of the appropriate charge-to-mass ratio are detected by the Particle detector at radius R.

Cyclotron

The cyclotron was developed by Ernest Lawrence [2] to accelerate charged particles. The particle moves between two charged containers called Dees. The dess are placed in a uniform mognetic field, $\vec{B}$.

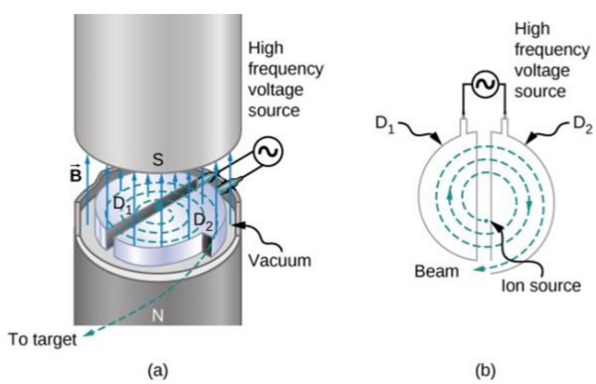


Figure 7.15: Cyclotron

A charged particle is injected into the gap and is accelerated from D2 to D1 (due to the electric field between the two of them). Once inside D1, the particle moves with a radius of

$$r = \frac{mv}{qB} \tag{7.51}$$

[2] Founder of both the Berkeley-Lawrence and the Lawrence-Livermore labs

with a period of

$$T = \frac{2\pi m}{qB} \tag{7.52}$$

The voltage between D1 and D2 is alternated with a period of $\frac{T}{2}$ so that when the particle enters the gap between D1 and D2 again, it is accelerated, gaining kinetic energy. The orbital period is independent of the radius and the kinetic energy, so the alternating voltage source only needs to be set to one value.

Eventually, the ion gets to the outer most radius and is ejected from the cyclotron with velocity

$$V_{max} = \frac{qBR}{m} \tag{7.53}$$

and thus with kinetic energy

$$\frac{1}{2}mv_{max} = \frac{q^2B^2R^2}{2m} \tag{7.54}$$

Example

A cyclotron used to accelerate alpha-particles ($m = 6.64 * 10^{-27}kg$ and $q = 3.2 * 10^{-19}C$) has a radius of $0.50m$ and a magnetic field of $1.8T$.

Period: Distance traveled in one revolution divided by the speed

$$T = \frac{2\pi R}{\frac{qBR}{m}} = \frac{2\pi m}{qB} = \frac{2\pi(6.64 * 10^{-27}kg)}{(3.2 * 10^{-19}C)1.8T} = 7.3 * 10^{-8}s \tag{7.55}$$

Kinetic Energy:

$$\frac{1}{2}mv^2 = \frac{q^2B^2R^2}{2m} = \frac{(3.2 * 10^{-19}C)^2(1.8T)^2(0.50m)^2}{2(6.64 * 10^{-27}kg)} = 6.2 * 10^{-12}J = 39MeV \tag{7.56}$$

Chapter 8

Module 8: Chapter 12 Sources of Magnetic Fields

8.1 The Biot-Savart Law

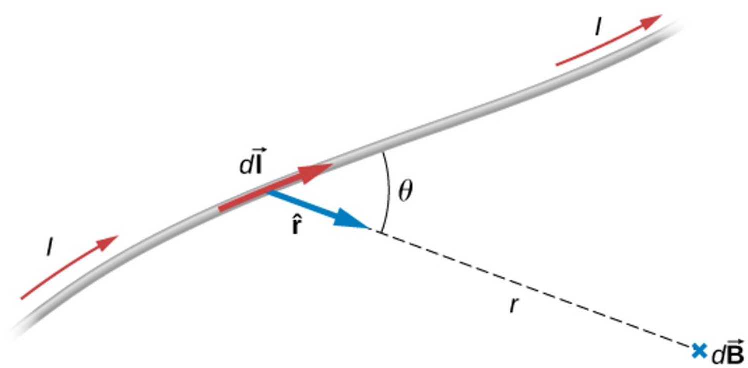


Figure 8.1: Biot-Savart Law

The Biot-Savart law[1] states that at any point P, the magnetic field $d\vec{B}$ due to the element $d\vec{I}$ of a current-carrying wire is given by

$$d\vec{B} = \frac{\mu_0}{4\pi}\frac{Id\vec{I}\times\hat{r}}{r^2} \tag{8.1}$$

where μ_0 is known as the permeability of free space and is exactly

$$\mu_0 = 4\pi * 10^{-7}T \cdot \frac{m}{A} \tag{8.2}$$

[1]Biot-Savart law was created by two French physicists, Jean Baptiste Biot and Felix Savart derived the mathematical expression for magnetic flux density at a point due to a nearby current-carrying conductor, in 1820.

The direction of $d\vec{B}$ is determined by applying the right-hand rule to the vector project $d\vec{l} \times \vec{r}$. And, the magnitude is given by

$$dB = \frac{\mu_0}{4\pi} \frac{I dl \sin\theta}{r^2} \tag{8.3}$$

where θ is the angle between $d\vec{l}$ and $\vec{r}$.

The magentic field due to a finite length of current-carrying wire is given by

$$\vec{B} = \frac{\mu_0}{4\pi} \int_{wire} \frac{I d\vec{I} \times \hat{r}}{r^2} \tag{8.4}$$

8.2 Magnetic Field due to a Thin Straight Wire

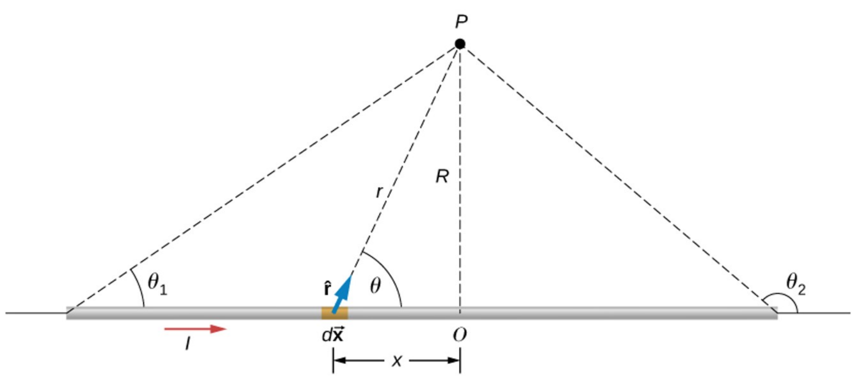


Figure 8.2: Magnetic Field: Straight Thing Wire

Start by considering the magnetic field due to the current Idx alogn a infinitely long wire as seen in Figure 8.2. $d\vec{x} \times \vec{r}$ points out of the page at Point P. As the magnetic field is in the same direction for all dx, the net field is the scalar sum of all the contributing elements.

Given that

$$|d\vec{x} \times \vec{r}| = I dx \sin\theta \tag{8.5}$$

We get

$$B = \frac{\mu_0}{4\pi} \int_{wire} \frac{I dl \sin\theta}{r^2} \tag{8.6}$$

From the geometry of Figure 8.2

$$r = \sqrt{x^2 + R^2} \tag{8.7}$$

and

$$\sin\theta = \frac{R}{\sqrt{x^2 + R^2}} \tag{8.8}$$

This leads to

$$B = \frac{\mu_0}{4\pi} \int_0^\infty \frac{R dx}{(x^2 + R^2)^{\frac{3}{2}}} \tag{8.9}$$

or

$$B = \frac{\mu_0 I}{4\pi R} [\frac{x}{(x^2 + R^2)^{\frac{1}{2}}}] \Big|_0^\infty \tag{8.10}$$

Which leads to

$$B = \frac{\mu_0 I}{4\pi R} \tag{8.11}$$

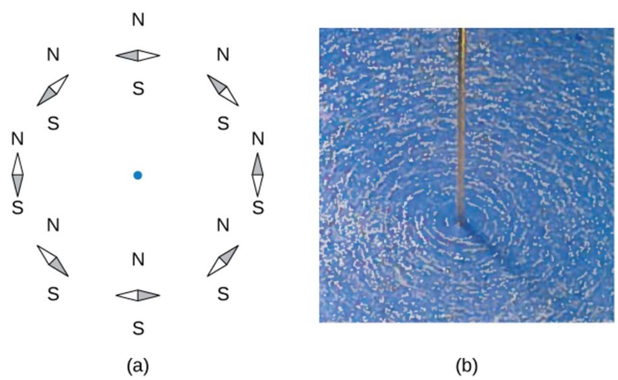


Figure 8.3: Shape of Magnetic Field Around A Wire

Magnetic Field Due to Three Wire

Consider three wires sitting on the corners of a square with a side of $1cm$. Each wire cares $2A$. What is the magnetic field at the forth corner?

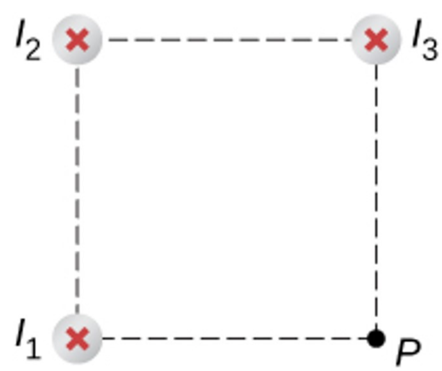


Figure 8.4: Magnetic Field from Three Wires

The magnitude of the magnetic field from Wires 1 and 3 are the same:

$$B_1 = B_3 = \frac{\mu_0 I}{4\pi R} = \frac{(4\pi * 10^{-7} T\frac{m}{A})(2A)}{2\pi(0.01m)} = 4 * 10^{-5} T \tag{8.12}$$

As Wire 2 has a longer distance:

$$B_1 = B_3 = \frac{\mu_0 I}{4\pi R} = \frac{(4\pi * 10^{-7} T\frac{m}{A})(2A)}{2\pi(0.01414m)} = 2.83 * 10^{-5} T \tag{8.13}$$

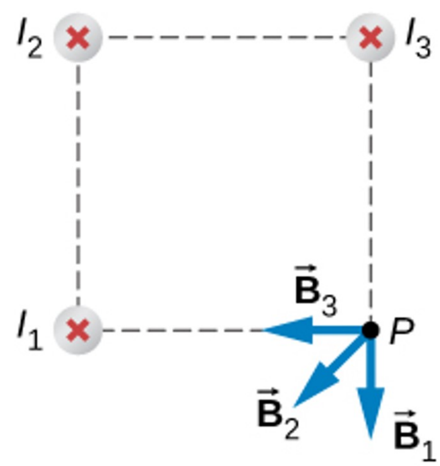


Figure 8.5: Magnetic Field Direction from Three Wires

The direction of each component is given by the tangent to a circle centered on each wire as seen in Figure 8.5.

$$B_x = -B_3 - B_2 \cos\theta = -(4 * 10^{-5}T) - (2.83 * 10^{-5}T)(0.707) = -6 * 10^{-5}T \quad (8.14)$$

and

$$B_y = -B_3 - B_1 \cos\theta = -(4 * 10^{-5}T) - (2.83 * 10^{-5}T)(0.707) = -6 * 10^{-5}T \quad (8.15)$$

Therefore

$$B = \sqrt{B_x^2 + B_y^2} = \sqrt{(-6 * 10^{-5}T)^2 + (-6 * 10^{-5}T)^2} = 8.49 * 10^{-5}T \quad (8.16)$$

with

$$\theta = 180^o + \arctan\frac{B_y}{B_x} = 225^o \quad (8.17)$$

8.3 Magnetic Force between Two Parallel Currents

Two current-carrying wires generate significant forces between them.

- A current produces a magnetic field, and
- a magnetic field exerts a force on a current.

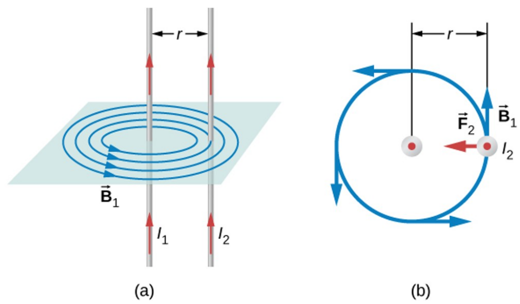


Figure 8.6: Magnetic Field from two parallel wires

Consider two parallel conductors as shown in Figure 8.6 separated by a distance r. Consider the field produced by wire 1 and the force exerted on wire 2.

The field created by the current in wire 1:

$$B_1 = \frac{\mu_0 I_1}{2\pi r} \tag{8.18}$$

and the force on wire 2 is given by $F = IlB\sin\theta$ where $\sin\theta = 1$:

$$F_2 = I_2 l B_1 \tag{8.19}$$

The force on wire 1 is equal and opposite $\vec{F_1} = -\vec{F_2}$. And, it is conveinent to think in terms of $\frac{F}{l}$:

$$\frac{F}{l} = \frac{\mu_0 I_1 I_2}{2\pi r} \tag{8.20}$$

and is attractive between the two wires.

8.4 Magnetic Field of a Current Loop

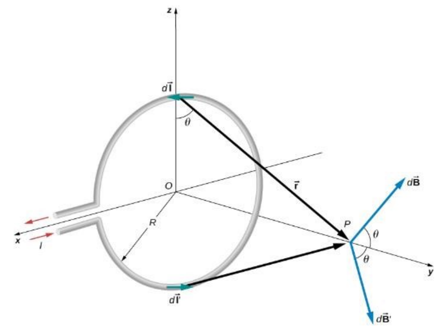


Figure 8.7: Magnetic Field current-carrying loop

Consider the current-carrying loop (as shown in Figure 8.7, with radius R, current I, and lying in the xz-plane.

From the right-hand rule, there is a magnetic field $d\vec{B}$ at Point P produced by current element $Id\vec{l}$.

$$dB = \frac{\mu_0}{4\pi}\frac{Idl\sin\theta}{r^2} = \frac{\mu_0}{4\pi}\frac{Idl}{y^2+R^2} \tag{8.21}$$

where $r^2 = y^2 + R^2$.

Now consider the current element $Id\vec{l}$ directly opposite the above element. While the first element has creates a magnetic field in the direction θ above the y-axis, the second element creates a magnetic field of the same strengths with θ below the y-axis.

This leads to

$$\vec{B} = \hat{j}\int_{loop} dB\cos\theta = \hat{j}\frac{\mu_0 I}{4\pi}\int_{loop}\frac{\cos\theta dl}{y^2+R^2} \tag{8.22}$$

For all elements $d\vec{l}$, y,R,$\cos\theta$ are constant and

$$\cos\theta = \frac{R}{\sqrt{y^2+R^2}} \tag{8.23}$$

Therefore

$$\vec{B} = \hat{j}\frac{\mu_0 IR}{4\pi(y^2+R^2)^{\frac{3}{2}}}\int_{loop} dl = \frac{\mu_0 IR^2}{2(y^2+R^2)^{\frac{3}{2}}}\hat{j} \tag{8.24}$$

The closed current loop is a magnetic dipoloe of moment

$$\vec{\mu} = IA\hat{n} \tag{8.25}$$

in this case $A = \pi R^2$ and $\hat{n} = \hat{j}$ so we can rewrite $|vecB$ as:

$$B = \frac{\mu_0 \mu}{2\pi(y^2+R^2)^{\frac{3}{2}}}\hat{j} \tag{8.26}$$

At the center of the loop $(y = 0)$:

$$\vec{B} = \frac{\mu_0 I}{2R}\hat{j} \tag{8.27}$$

or

$$\vec{B} = \frac{\mu_0 \vec{\mu}}{2\pi R^3} \tag{8.28}$$

When $y >> R$:

$$\vec{B} = \frac{\mu_0 \vec{\mu}}{2\pi y^3} \tag{8.29}$$

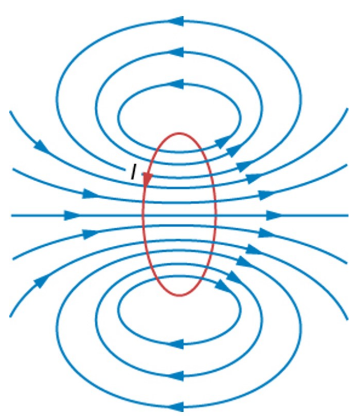


Figure 8.8: Magnetic Field lines around a current-carrying loop

8.5 Ampere's Law

A fundamental property of a static magnetic field, unlike an electrostatic field, is that it is not conservative.

Conservative field: does the same amount of work on a particle moving between two different points regardless of the path taken.

The static magnetic field is NOT conservative. Instead there is a relationship between the magnetic field and its source, the electric current. It is expressed as a line integral of $\vec{B}$ and is known as Ampere's Law.

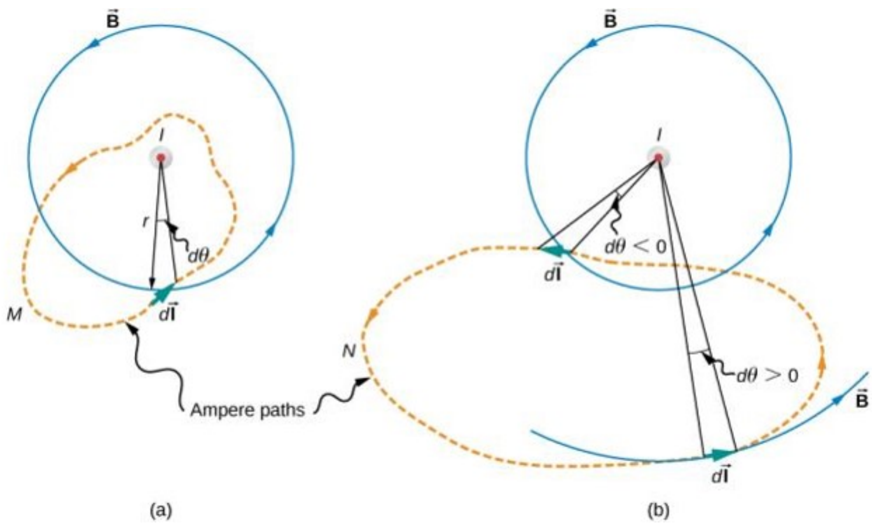


Figure 8.9: Non conservative static magnetic field

In Figure 8.9 shows the magnetic field generated by a infinitely long wire coming straight out of the page. Consider $\oint \vec{B} \cdot \vec{l}$ over the closed paths M and N. Notice that M encloses the wire and N does not. Since the field lines are circular, $\vec{B} \cdot \vec{l}$ is a product of B and the projection of dl onto the circle passing through $d\vec{l}$. If the radius is r, then the projection is $rd\theta$ and

$$\vec{B} \cdot d\vec{l} = Brd\theta. \tag{8.30}$$

This leads to

$$\oint \vec{B} \cdot d\vec{l} = (\frac{\mu_0 I}{2\pi r})rd\theta. \tag{8.31}$$

For path M, which circulates around the wire, $\oint_M d\theta = 2\pi$ and

$$\oint_M \vec{B} \cdot d\vec{l} = \mu_0 I \tag{8.32}$$

For path N, which circulates both positive (counterclockwise) and negative (clockwise) $d\theta$ and since it is closed $\oint_N d\theta = 0$ and

$$\oint_N \vec{B} \cdot d\vec{l} = 0 \tag{8.33}$$

Example: Using Ampere's Law to Calculate the Magnetic Field due to a Wire

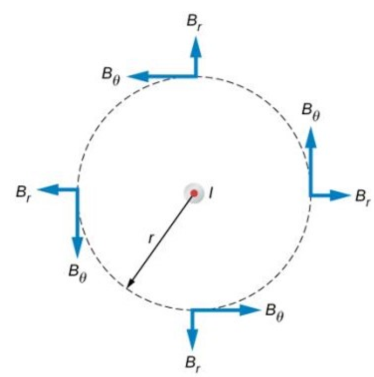


Figure 8.10: Magnetic Field Components due to Current I

Consider an arbitrary plane perpendicular to a wire with current directed out of the page. The possible magnetic field components are B_r and B_θ. Since the field is cylindrical symmetric, neither B_r nor B_θ vary with position.

The radial lines, if they exist, must be directed either all inward or all outward from the wire. This means that there must be a net magnetic flux across cylinders that are concentric with the wire. This means the radial magnetic component must be zero because $\vec{B_r} \cdot d\vec{l} = 0$. Therefore applying ampere's law to the circular path as shown in Figure 8.10:

$$\oint \vec{B} \cdot d\vec{l} = B_\theta \oint dl = B_\theta(2\pi r) \tag{8.34}$$

Thus, Ampere's Law reduces to

$$B_\theta(2\pi r) = \mu_0 I \tag{8.35}$$

And, since the only subscript that remains is *theta*, it can be dropped giving Biot-Savart

$$B = \frac{\mu_0 I}{2\pi r} \tag{8.36}$$

Example: Magnetic Field of a Thick Wire with Ampere's Law

Consider a long straight wire of radius a carrying a current I_0.

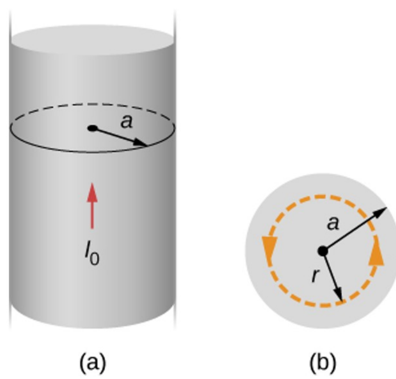


Figure 8.11: Think Wire and Ampere's Law

For any circular path of radius r that is centered on the wire:

$$\oint \vec{B} \cdot d\vec{l} = \oint Bdl = B \oint dl = B(2\pi r) \tag{8.37}$$

By Ampere's law, this equals μ_0 times the current passing through any surface bounded by the path of integration.

First consider $r < a$. The current passing through area enclosed by the path of integration is given by the current density J times the area. Since the current density is uniform:

$$J = \frac{I_0}{\pi a^2} \tag{8.38}$$

and therefore

$$I = \frac{\pi r^2}{\pi a^2} I_0 = \frac{r^2}{a^2} I_0 \tag{8.39}$$

From Ampere's Law

$$B(2\pi r) = \mu_0 (\frac{r^2}{a^2}) I_0 \tag{8.40}$$

and thus the Magnetic Field inside the wire ($r < a$) is

$$B = \frac{\mu_0 I_0}{2\pi} (\frac{r}{a^2}) \tag{8.41}$$

as seen in the left side of Figure 8.12.

Outside of the wire, the $I = I_0$ leading to

$$B = \frac{\mu_0 I_0}{2\pi r} \tag{8.42}$$

as seens in the right aide of Figure 8.12.

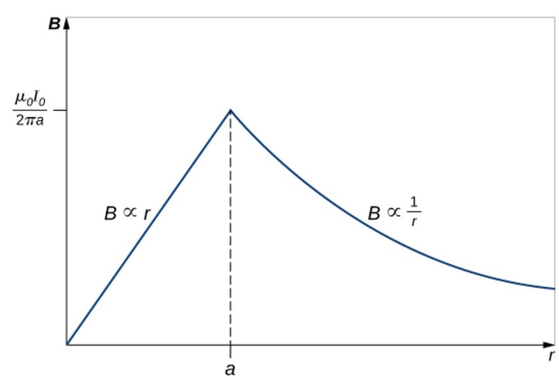


Figure 8.12: Magnetic Field inside and outside of thick wire

8.6 Solenoids and Toroids

A long wire that forms a helical coil is known as a Solenoid.

Consider the Solenoid shown in Figure 8.13 consisting of N turns of tightly wound wire of length L.

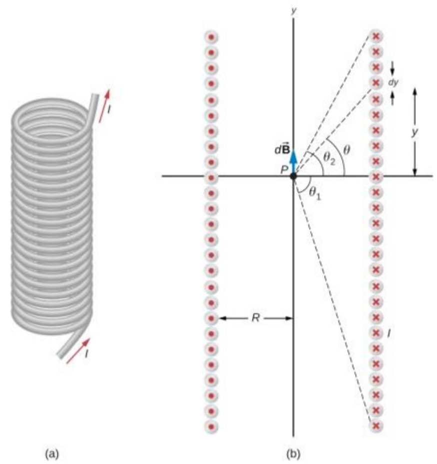


Figure 8.13: Solenoid

The number of turns per unit length is $\frac{N}{L}$ and therefore the number of turns over an infinitesimal length dy are $\frac{N}{L}dy$ turns.

$$dl = \frac{N}{L}dy \tag{8.43}$$

To calculate the magnetic field at point P on the center axis of the solenoid.

$$d\vec{B} = \frac{\mu_0 r^2 dl}{2(y^2+R^2)^{\frac{3}{2}}}\hat{j} = [\frac{\mu_0 I R^2 N}{2L}\hat{j}]\frac{dy}{2(y^2+R^2)^{\frac{3}{2}}} \tag{8.44}$$

Note that, from inspection,

$$\sin\theta = \frac{y}{\sqrt{y^2+R^2}} \tag{8.45}$$

taking the differential of both sides:

$$\cos\theta d\theta = [-\frac{y^2}{2(y^2+R^2)^{\frac{3}{2}}} + \frac{1}{\sqrt{y^2+R^2}}]dy = \frac{R^2 dy}{2(y^2+R^2)^{\frac{3}{2}}} \tag{8.46}$$

This leads to

$$\vec{B} = \frac{\mu_0 I N}{2L}\hat{j}\int_{\theta_1}^{\theta_2}\cos\theta d\theta = \frac{\mu_0 I N}{2L}(\sin\theta_2 - \sin\theta_1)\hat{j} \tag{8.47}$$

Consider the special case where $L >> R$, in which $\theta_1 = -\frac{\pi}{2}$ and $\theta_2 = +\frac{\pi}{2}$:

$$\vec{B} = \frac{\mu_0 IN}{2L}(\sin(\frac{\pi}{2}) - \sin(-\frac{\pi}{2}))\hat{j} = \frac{\mu_0 IN}{L}\hat{j} \tag{8.48}$$

or

$$\vec{B} = \mu_0 nI\hat{j} \tag{8.49}$$

where n is the number of turns per unit length

Toroids

A similar method can be used to calculate the magnetic field in a toroid (Figure 8.14.

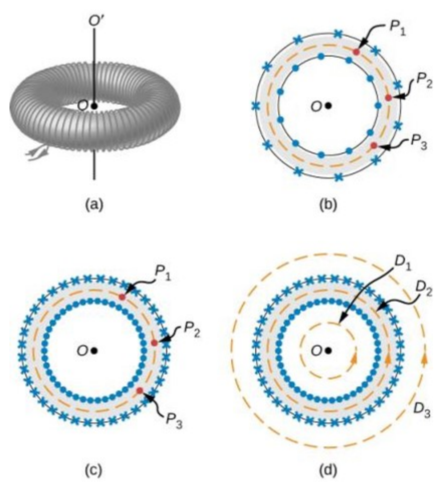


Figure 8.14: Toroid

$$\vec{B} = \frac{\mu_0 NI}{2\pi r} \tag{8.50}$$

Chapter 9

Module 11 - Chapter 13 Electromagnetic Induction

Some of the first experiments concerning time-varying magnetic fields was done by Michael Faraday[1] in 1831. In his early experiments, he moved a permanent magnet into and out of a coil of wire. The emf is produced by the moving magnetic field. If the magnet is reverse, then the emf change direction as well. And, the same effect is seen by moving the coil rather than the magnet.

Faraday also discovered that a similar effect can be seen in two circuits. A changing current in one of the circuits induces a a current in the second, nearby circuit.

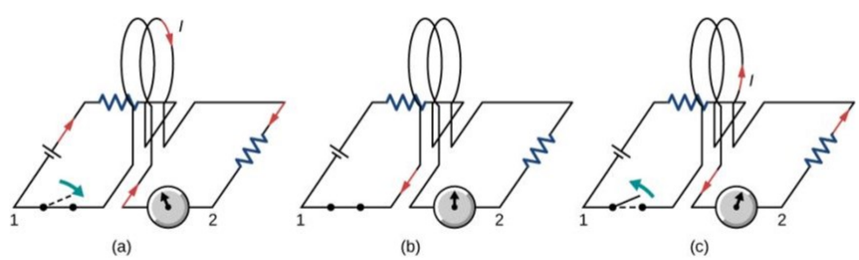


Figure 9.1: Current induced from one circuit to another

This led to the definition of Magnetic Flux, the amount of magnetic field lines through a given surface.

$$\Phi_M = \int_S \vec{B} \cdot \hat{n} dA \tag{9.1}$$

[1]English Scientist

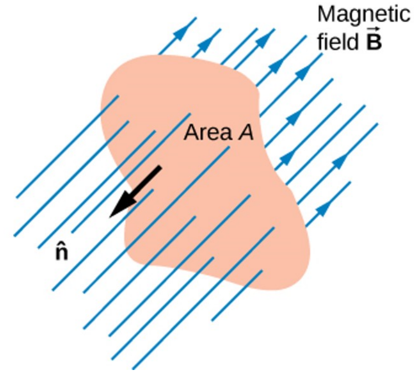


Figure 9.2: Magnetic Flux

then the induced emf (or the voltage generated by a conductor or coil moving in a magnetic field is

$$\epsilon = -\frac{d}{dt}\int_S \vec{B}\cdot\hat{n}dA = -\frac{d\Phi_M}{dt} \tag{9.2}$$

The units of Magnetic Flux is the weber: $1Wb = 1T \cdot m^2$.

In practice the ciruit often consists of a number N of tightly wound terns, so Faraday's law is written as

$$\epsilon = -\frac{d}{dt}(N\Phi_M) = -N\frac{d\Phi_M}{dt} \tag{9.3}$$

Example: A square coil in a changing magnetic field

Consider a square coil that has sides $l = 0.25m$ and is tightly wound with $N = 200$ turns. The resistance of the coil is 5.0Ω. The coil is placed in a spatially uniform magnetic field that is directed perpendicular to the face of the coil and whose magnitude is decreasing at a rate $\frac{dB}{dt} = -0.040\frac{T}{s}$.

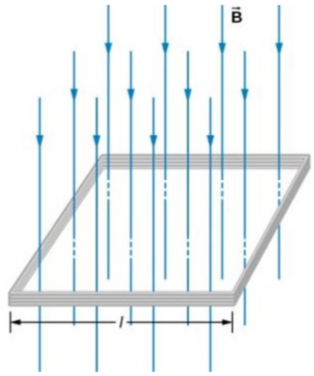


Figure 9.3: Square coil in a changing magnetic field

The flux through one turn is

$$\Phi_M = BA = Bl^2 \tag{9.4}$$

so, the magnitude of the emf is

$$|e| = -N\frac{d\Phi_M}{dt} = Nl^2\frac{dB}{dt} = (200)(0.25m^2)(0.040\frac{T}{s} = 0.50V \tag{9.5}$$

which gives a current of

$$I = \frac{e}{R} = \frac{0.50V}{5.0\Omega} = 0.10A \tag{9.6}$$

9.1 Lenz's Law

Lenz's Law[2]: The direction of the induced emf drives a current around the a wire loop to always oppose the change in magnetic flux that cuases the emf.

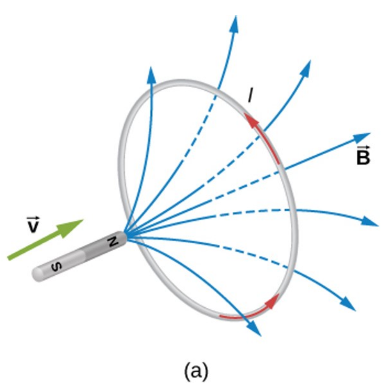


Figure 9.4: Direction of induced current from Lenz's Law

[2]named in honor of Heinrich Lenz (1804-1865), but also discovered independently by Michael Faraday

9.2 Motional EMF

Magnetic flux depends on three factors:

- Strength of the magnetic field
- Area through which the field lines pass
- The orientation of the field with the surface area

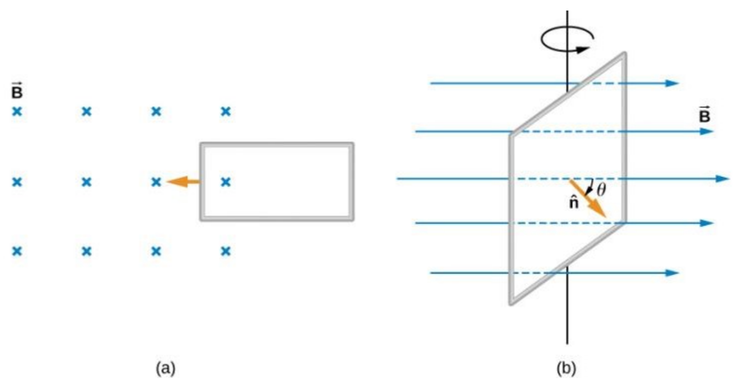


Figure 9.5: Examples of Flux Changes

Consider the circuit shown in Figure 9.6. As the rod is pulled through the circuit this causes changing magnetic flux.

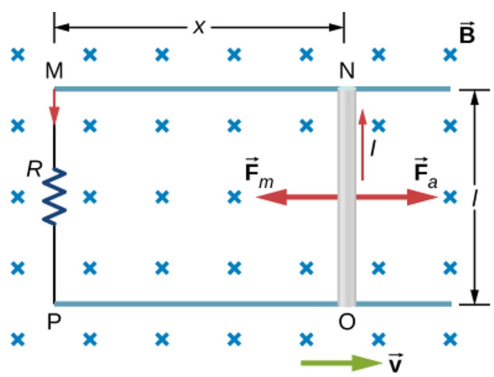


Figure 9.6: Flux in circuit with moving rod

The area enclosed by the circuit "MNOP" is $A = lx$ and is perpendicular to the magnetic field. This leads to

$$\Phi_M = Blx \tag{9.7}$$

Since B and l are constants, the velocity of the rod is

$$v = \frac{dx}{dt} \tag{9.8}$$

From Faraday's Law

$$\epsilon = \frac{d\Phi_M}{dt} = Bl\frac{dx}{dt} = Blv \tag{9.9}$$

And the current induced in the circuit

$$I = \frac{Blv}{R} \tag{9.10}$$

Faraday's law holds for all flux changes, whether they are caused by a time-varying magnetic field, or by motion, or by a combination of the two.

From an energy perspective, power is equal to force times velocity, so in Figure 9.6 the force $\vec{F_a}$ produces power $F_a v$. Since the rod is moving a constant velocity, the applied force F_a must balance the magnetic force $F_m = IlB$.

Thus:

$$F_a v = Ilbv = \frac{Blv}{R} \cdot lBv = \frac{l^2B^2v^2}{R} \tag{9.11}$$

And, the power dissipated is

$$P = I^2R = (\frac{Blv}{R})^2R = \frac{l^2B^2v^2}{R} \tag{9.12}$$

Thus, the power produced and the power dissipated, satisfying of the law of conservation of energy.

Example: A Rectangular Coil Rotating in a Magnetic Field

Consider a rectangular coil of area A and N turns placed in a uniform magnetic field $\vec{B} = B\hat{j}$ as shown in Figure 9.7.

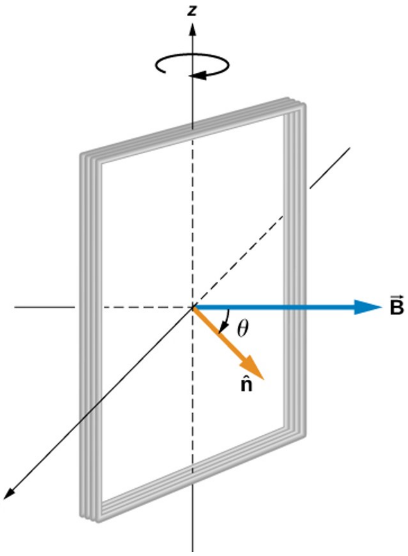


Figure 9.7: A Rectangular Coil Rotating in a Magnetic Field

$$\Phi_M = \int_S \vec{B} \cdot \hat{n} dA = BA\cos\theta \tag{9.13}$$

From Faraday's Law

$$\epsilon = -N\frac{d\Phi_M}{dt} = NBA\sin\theta\frac{d\theta}{dt} \tag{9.14}$$

The constant angular velocity is $\omega = \frac{d\theta}{dt}$. Where the angle θ represents the time evolution of the angular velocity $\theta = \omega t$.

Which leads to

$$\epsilon = \epsilon_\omega \sin(\omega t) \tag{9.15}$$

where $\epsilon_\omega = BNA\omega$

9.3 Induced Electric Fields

The work done my an electric field $\vec{E}$ in moving a unit charge completely around a circuit is the induced emf ϵ:

$$\epsilon = \oint \vec{E} \cdot d\vec{l} \tag{9.16}$$

Leading to Faraday's law being written in terms of the induced electric field

$$\epsilon = \oint \vec{E} \cdot d\vec{l} = \frac{d\Phi_M}{dt} \tag{9.17}$$

Example: Electric Field Induced by the Changing Magnetic Field of a Solenoid

Consider a long solenoid with radius R and n turns per unit length. The current decreases with time according to $I = I_0 e^{-\alpha t}$ as shown in Figure

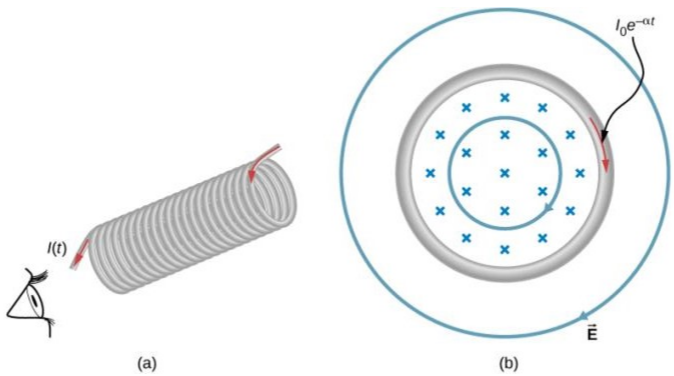


Figure 9.8: Electric Field Induced in Solenoid

Using the formulas for the a magnetic field inside of a solenoid and Faraday's law:

$$B = \mu_0 n I = \mu_0 n I_0 e^{-\alpha t} \tag{9.18}$$

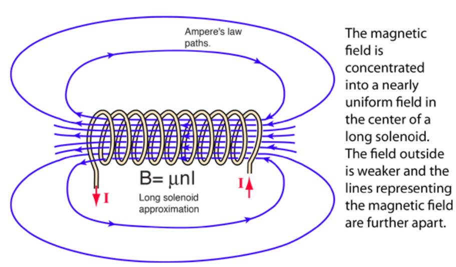


Figure 9.9: Magnetic field in Solenoid

The magnetic field is much weaker outside of the solenoid. This all leads to the magnetic flux through a circular path whole radius r is greater than R (the solenoid radius):

$$\Phi_M = BA = \mu_0 n I_0 \pi R^2 e^{-\alpha t} \tag{9.19}$$

The induced field $\vec{E}$ is tangent to this path, and because of cylindrical symmetry, its magnitude is constant on the path. Hence:

$$|\oint \vec{E} \cdot d\vec{l}| = |\frac{d\Phi_M}{dt}| \tag{9.20}$$

$$E(2\pi r) = |\frac{d}{dt}(\mu_0 n I_0 \pi R^2 e^{-\alpha t})| = \alpha \mu_0 n I_0 \pi R^2 e^{-\alpha t} \tag{9.21}$$

or for $r > R$:

$$E = \frac{\alpha \mu_0 n I_0 R^2}{2r} e^{-\alpha t} \tag{9.22}$$

Now, for a path of radius r inside ($r < R$) the solenoid, $\Phi_M = B\pi r^2$, leading to

$$E(2\pi r) = |\frac{d}{dt}(\mu_0 n I_0 \pi r^2 e^{-\alpha t})| = \alpha \mu_0 n I_0 \pi r^2 e^{-\alpha t} \tag{9.23}$$

And the induced field

$$E = \frac{\alpha \mu_0 n I_0 r}{2} e^{-\alpha t} \tag{9.24}$$

Note, the electric field increase linearly inside the solenoid and drops off with $\frac{1}{r}$ outside the solenoid.

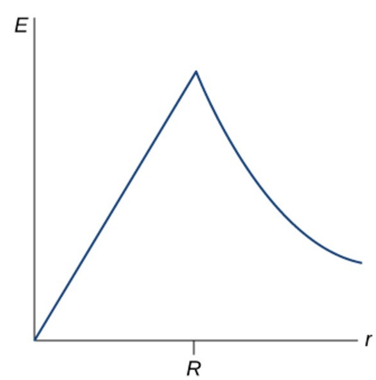


Figure 9.10: Electric Field inside and outside of a Solenoid

9.4 Electrical Generators

Electric generators induce an emf by rotating a coil in a magnetic field. Consider Figure 9.11 where a square loop of width w and length l is rotating in a uniform magnetic field.

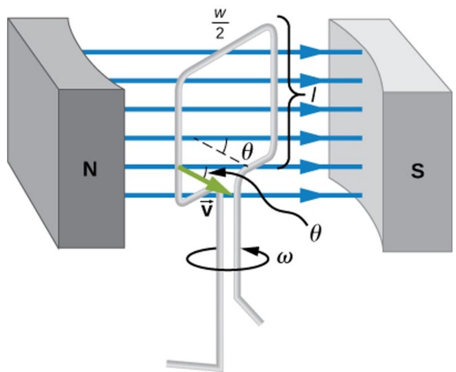


Figure 9.11: Generator

Charges in the wires of the loop experience the magentic force as they are moving in a magnetic field.

- Charges in the Vertical Wires experience forces parallel to the wire, causing current.
- Charges in the Top/Bottom Wires experience forces perpendicular to the wire, which does not cause a current.

Find the induced emf by first considering on side of the loop.

The motional emf is given by:

$$\epsilon = Blv \tag{9.25}$$

where the velocity v is perpendicular to the magnetic field. For the velocity at angle θ the component perpendicular to $\vec{B}$ is $v \sin\theta$.

Thus, the total emf around the loop is

$$\epsilon = 2Blv \sin\theta \tag{9.26}$$

To find the emf as a function of time, we assume the coil rotates at a constant angular velocity ω. The angle θ is related t angular velocity by $\theta = \omega t$. Leading to

$$\epsilon = 2Blv \sin(\omega t) \tag{9.27}$$

The linear velocity v is related to the angular velocity ω by $v = r\omega$, where $r = \frac{w}{2}$

$$\epsilon = 2Bl\frac{w}{2}\omega \sin(\omega t) = B(lw)\omega \sin(\omega t) \tag{9.28}$$

Noting that the area of the loop $A = lw$ and allowing for N loops, we find that

$$\epsilon = NBA\omega \sin(\omega t) \tag{9.29}$$

That can also be expressed as

$$\epsilon = \epsilon_0 \sin(\omega t) \tag{9.30}$$

where

$$\epsilon_0 = NBA\omega \tag{9.31}$$

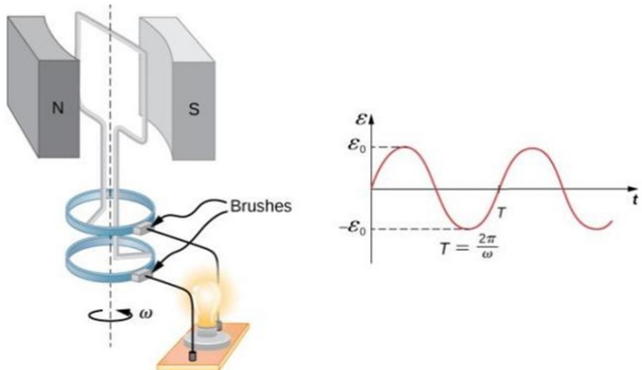


Figure 9.12: Generator creating ac emf

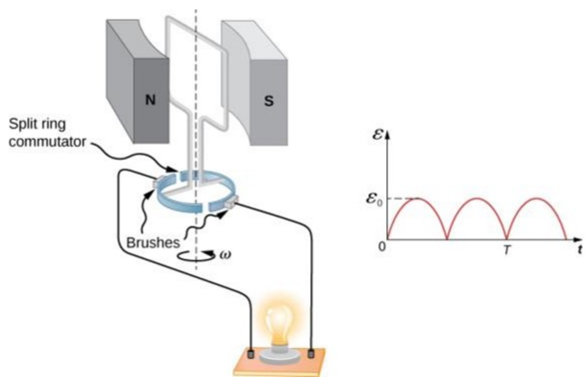


Figure 9.13: Generator with split rings (commutators) produce pulsed dc

9.5 Back emf

When the coil of a motor is turned, magnetic flux changes through the coil, and an emf (consistent with Faraday's law) is induced. The motor thus acts as a generator whenever its coil rotates. This happens whether the shaft is turned by an external input, like a belt drive, or by the action of the motor itself. That is, when a motor is doing work and its shaft is turning, an emf is generated. Lenz's law tells us the emf opposes any change, so that the input emf that powers the motor is opposed by the motor's self-generated emf, called the back emf of the motor as shown in 9.14.

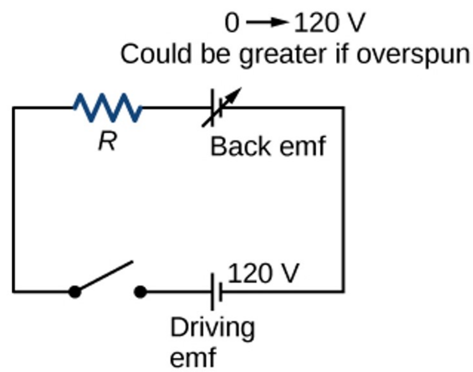


Figure 9.14: Back emf

Chapter 10

Module 12 - Chapter 16 Electromagnetic Waves

Recall from Ampere's law, the integral of a magnetic field around a closed loop C is proportional to the current I passing through any surface whose boundary is loop C itself:

$$\oint_C \vec{B} \cdot d\vec{s} = \mu_0 I \tag{10.1}$$

Consider the setup in Figure 10.1.

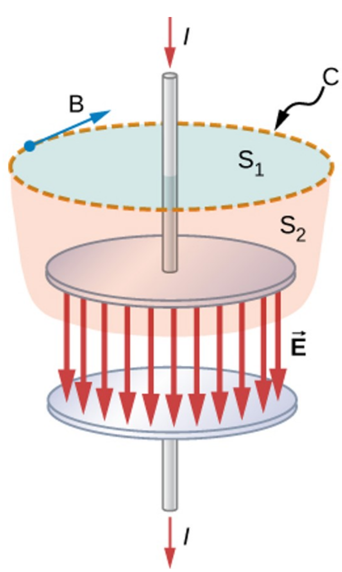


Figure 10.1: Current through unequal surfaces

$$\oint_C \vec{B} \cdot d\vec{s} = \begin{cases} \mu_0 I, & \text{if surface S1 is used} \\ 0, & \text{if surface S2 is used} \end{cases} \tag{10.2}$$

Ampere's law in its usual form doesn't work here. Ampere's law previosly was applied to steady charge, which is not the case in this example. Maxwell suggested adding a displacement current I_d,

$$\oint_C \vec{B} \cdot d\vec{s} = \mu_0 (I + I_0) \tag{10.3}$$

where the displacement current is defined as

$$I_d = \epsilon_0 \frac{d\Phi_E}{dt} \tag{10.4}$$

where ϵ_0 is the permittivity of free space and Φ is the electric flux:

$$\Phi_E = \int_S \vec{E} \cdot d\vec{A} \tag{10.5}$$

Maxwell's Equations

Gauss's Law

The electric flux out of any closed surface is proportional to the total charge enclosed within the surface.

$$\oint \vec{E} \cdot d\vec{A} = \frac{Q_{enc}}{\epsilon_0} \tag{10.6}$$

Gauss's Law for Magnetism

The net magnetic flux out of any closed surface is zero.

$$\oint \vec{B} \cdot d\vec{A} = 0 \tag{10.7}$$

Faraday's Law

The line integral of the electric field around a closed loop is equal to the negative of the rate of change of the magnetic flux through the area enclosed by the loop.

$$\oint \vec{E} \cdot d\vec{s} = -\frac{\Phi_M}{dt} \tag{10.8}$$

Ampere-Maxwell Law

In the case of static electric field, the line integral of the magnetic field around a closed loop is proportional to the electric current flowing through the loop.

$$\oint \vec{B} \cdot d\vec{s} = \mu_0 I + \epsilon_0 \mu_0 \frac{\Phi_E}{dt} \tag{10.9}$$

Once the fields have been calculated using these four equations, the Lorentz force equation becomes

$$F = q\vec{E} + q\vec{v} \times \vec{B} \tag{10.10}$$

The Mechanism of Electromagnetic Wave Propagation

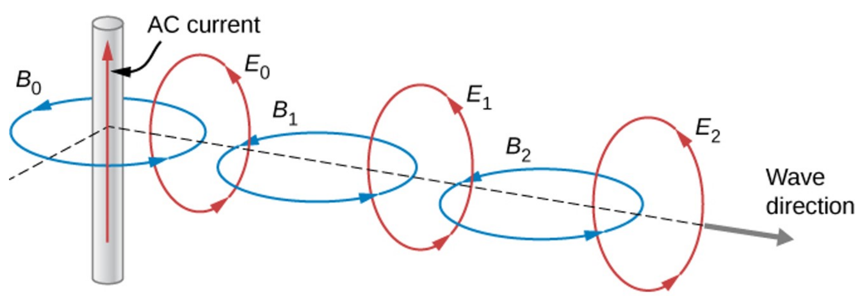


Figure 10.2: How changing $\vec{E}$ and $\vec{B}$ fields propagate

Consider a time-varying magnetic field $\vec{B}_0(t)$ produced by the high-frequency alternating current seen in Figure 10.2. ($\vec{B}_0(t)$ in the diagram is represented by one of it's field lines). From Faraday's Law, the changing magnetic field through a surface induces a time-varying electric field $\vec{E}_0(t)$ at the boundary of the surface. The displacement current source for the electric field, like the Faraday's law source for the magnetic field, produces only closed loops. The changing field $\vec{E}_0(t)$ creates a magnetic feild $\vec{B}_1(t)$ according to the modified Ampere's Law. The changing field induces $\vec{E}_1(t)$, which induces $\vec{B}_2(t)$, and so on.

Hertz Observations

The German physicist Heinrich Hertz (1857–1894) was the first to generate and detect certain types of electromagnetic waves in the laboratory. Starting in 1887, he performed a series of experiments that not only confirmed the existence of electromagnetic waves but also verified that they travel at the speed of light.

Hertz used an alternating-current RLC (resistor-inductor-capacitor) circuit that resonates at a known frequency $f_0 = \frac{1}{2\pi\sqrt{LC}}$ and connected it to a loop of wire, as shown in Figure 10.3. High voltages induced across the gap in the loop produced sparks that were visible evidence of the current in the circuit and helped generate electromagnetic waves.

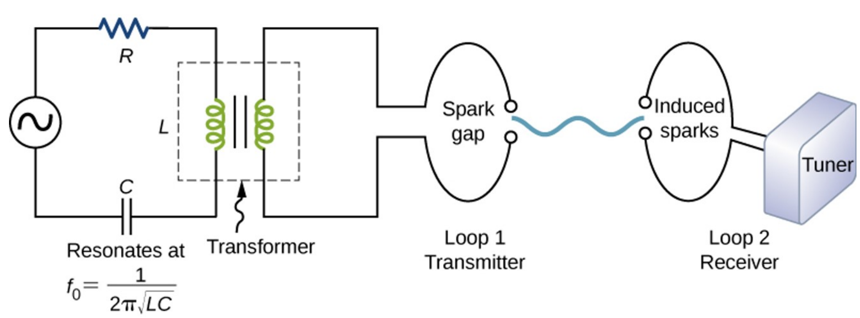


Figure 10.3: Apparatus used to generate and detect electromagnetic wave

10.1 Plane Electromagnetic Waves

We will consider the propagation of electromagnetic waves along the x-axis of a given coordinate system.

Electromagnetic waves in one direction

Start by examining Gauss's law for electric fields and the implications on the relative direction of the electric field and propagation direction in a electromagnetic wave. Consider the Gaussian surface of a box showing in Figure 10.4.

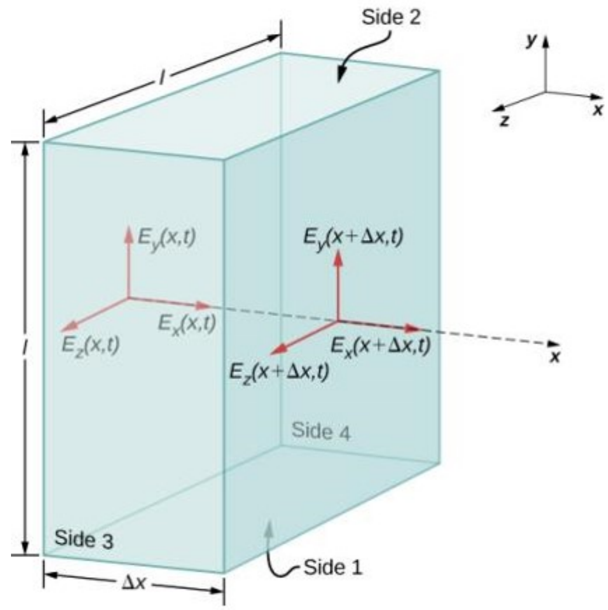


Figure 10.4: Gaussian Surface of l x l x Δt

Because the electric field is only a function a x and t, the y-component of the electric field is the same on both the top and bottom of the box, so these two contributions cancel. Likewise of the z-axis. Therefore $E_x(x,t)$ is constant over the face of the box with area A and possibly has a different value $E_x(x+\Delta x,t)$

$$\text{Net Flux} = -E_x(x,t)A + E_x(x+\Delta x,t)A = \frac{Q_{enc}}{\epsilon_0} \tag{10.11}$$

where $A = l * l$.

Given that $Q_{enc} = 0$ this leads to for any Δx.

$$E_x(x,t) + E_x(x+\Delta x,t) \tag{10.12}$$

Therefore, if there is an x-component to the electric field, it can not vary with x. A field of this kind would be superimposed on the traveling wave for example by having a pair of charged plates. And, thus, such a component $E_x(x,t)$ would not be part of the traveling wave. Therefore

$$E_x(x,t) = 0 \tag{10.13}$$

and the only non-zero components of the traveling wave are $E_y(x,t)$ and $E_z(x,t)$.

A similar argument can be made substituting $\vec{B}$ for $\vec{E}$ and applying Gauss's Law for Magnetism. The electromagnetic wave is therefore a transverse wave, with its oscillating electric and magnetic fields perpendicular to its direction of propagation.

The speed of propagation of electromagnetic waves

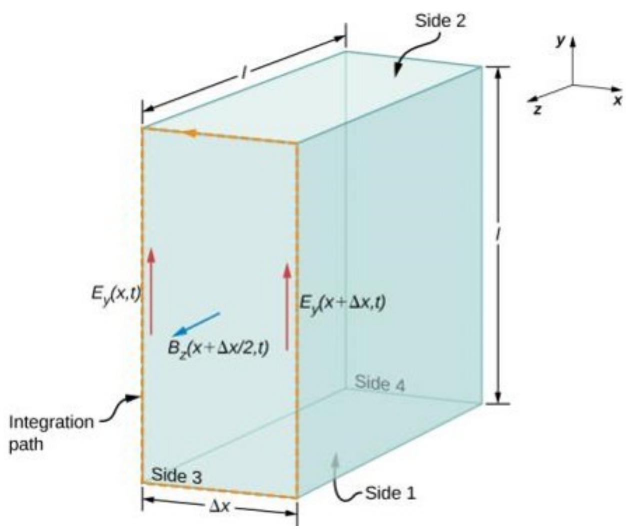


Figure 10.5: Faraday's law applied to propagation of EM wave

Applying Faraday's law over Side 3 fof the Gaussian surface, using the path shown in 10.5, and because $E_x(x,t) = 0$, we have

$$\oint \vec{E} \cdot d\vec{s} = -E_y(x,t)l + E_y(x+\Delta x,t)l \tag{10.14}$$

Assuming that Δx is small and approximating $E_y(x+\Delta x,t)$ by

$$E_y(x+\Delta x,t) = E_y(x,t) + \frac{\delta E_y(x,t)}{\delta x}\Delta x \tag{10.15}$$

and we obtain

$$\oint \vec{E} \cdot d\vec{s} = \frac{\delta E_y(x,t)}{\delta x}(l\Delta x) \tag{10.16}$$

Additionally, because Δx is small, the magnetic flux through the face can be approximated by its value in the center of the area traversed, specifically $B_z(x+\frac{\Delta x}{2}, t)$.

The flux on Side 3 is therefore

$$\oint_S \vec{B} \cdot \hat{n} dA = B_z(x + \frac{\Delta x}{2}, t)(l\Delta x) \tag{10.17}$$

From Faraday's Law

$$\oint \vec{E} \cdot \vec{s} = -\frac{d}{dt} \int_S \vec{B} \cdot \hat{n} dA \tag{10.18}$$

This leads to:

$$\frac{\delta E_y(x,t)}{\delta x}(l\Delta x) = -\frac{\delta}{\delta t}[B_z(x + \frac{\Delta x}{2}, t)](l\Delta x) \tag{10.19}$$

Taking the limit $\Delta x \rightarrow 0$, we are left with

$$\frac{\delta E_y(x,t)}{\delta x} = -\frac{\delta B_z(x,t)}{\delta t} \tag{10.20}$$

Likewise, we can apply Faraday's law to the top surface which yields:

$$\frac{\delta E_z(x,t)}{\delta x} = -\frac{\delta B_y(x,t)}{\delta t} \tag{10.21}$$

The equations are describing the spatially dependent E field produced by the time-dependent B field.

Next, apply the Ampere-Maxwell law (with $I = 0$) over the same two surfaces

$$\oint \vec{B} \cdot d\vec{s} = \mu_0 \epsilon_0 \frac{d}{dt} \int_S \vec{E} \cdot \hat{n} dA \tag{10.22}$$

This yields

$$\frac{\delta B_y(x,t)}{\delta x} = -\mu_0 \epsilon_0 \frac{\delta E_z(x,t)}{\delta t} \tag{10.23}$$

and

$$\frac{\delta B_z(x,t)}{\delta x} = -\mu_0\epsilon_0\frac{\delta E_y(x,t)}{\delta t} \tag{10.24}$$

These equations are describing the spatially dependent B field produced by the time-dependent E field.

Combining the equations showing a changing B field producing an E field with the equations showing a changing E field producing an B field, and taking the derivative with respect to x we get

$$\frac{\delta^2 E_y}{\delta x^2} = \frac{\delta}{\delta x}(\frac{\delta E_y}{\delta x}) = -\frac{\delta}{\delta x}(\frac{\delta B_z}{\delta t}) = -\frac{\delta}{\delta t}(\frac{\delta B_z}{\delta x}) = \frac{\delta}{\delta t}(\mu_0\epsilon_0\frac{\delta E_y}{\delta t}) \tag{10.25}$$

or, finally,

$$\frac{\delta^2 E_y}{\delta x^2} = \mu_0\epsilon_0\frac{\delta^2 E_y}{\delta t^2} \tag{10.26}$$

A general solution to this equation is the arbitrary functions $f(x-vt)$ and $g(x+vt)$ for waves traveling in the positive and negative directions. Since this is quite general, we can opt for a sinusoidal function of wavelength λ. Which allows us to write

$$E = E_0 \sin(\frac{2\pi}{\lambda}(x-vt)) \tag{10.27}$$

Which leads to

$$\frac{\delta^2 E_y}{\delta x^2} = -E_0(\frac{2\pi}{\lambda})^2 \sin(\frac{2\pi}{\lambda}(x-vt)) \tag{10.28}$$

and

$$\frac{\delta^2 E_y}{\delta t^2} = -E_0(\frac{2\pi v}{\lambda})^2 \sin(\frac{2\pi}{\lambda}(x-vt)) \tag{10.29}$$

Combining into Equation 10.26:

$$-E_0(\frac{2\pi}{\lambda})^2 \sin(\frac{2\pi}{\lambda}(x-vt)) = \mu_0\epsilon_0[-E_0(\frac{2\pi v}{\lambda})^2 \sin(\frac{2\pi}{\lambda}(x-vt))] \tag{10.30}$$

simplifying to

$$1 = \mu_0 \epsilon_0 v^2 \tag{10.31}$$

or

$$v = \frac{1}{\sqrt{\mu_0 \epsilon_0}} \tag{10.32}$$

Solving we obtain the speed of light

$$v = \frac{1}{\sqrt{\mu_0 \epsilon_0}} = \frac{1}{\sqrt{(4\pi * 10^{-7} \frac{Tm}{A})(8.85 * 10^{-12} \frac{C^2}{Nm^2})}} = 2.99 * 10^8 \frac{m}{s} = c \tag{10.33}$$

How are E and B fields related

The E field and B field are in phase. And, it can be shown that

$$\frac{E_y}{B_z} = \frac{E_0}{B_0} = c \tag{10.34}$$

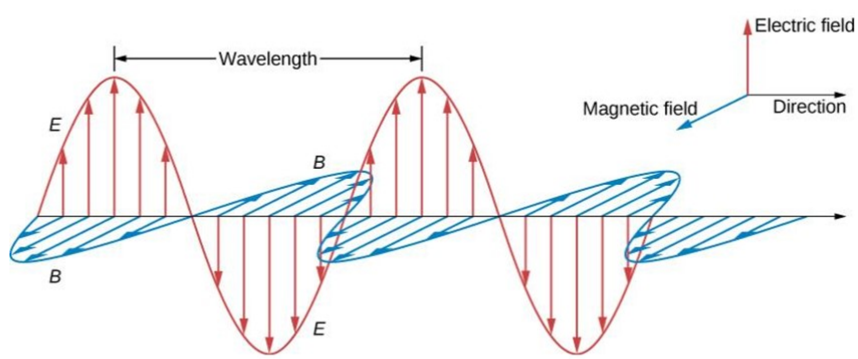


Figure 10.6: Plane wave solution to Maxwell's equation

10.2 Energy Carried by Electromagnetic Waves

Consider a plane-wave moving in the positive x direction, so that the maximum is at the origin at t = 0, the electric and magnetic fields obey

$$E_y(x,t) = E_0 \cos(kx - \omega t) \tag{10.35}$$

and

$$B_z(x,t) = B_0 \cos(kx - \omega t) \tag{10.36}$$

The energy density u is the sum of the energy density of the E-field, U_E, and the B-field, u_B. This leads to:

$$u(x,t) = u_E + u_B = \frac{\epsilon_0}{2}E^2 + \frac{1}{2\mu_0}B^2 \tag{10.37}$$

Given that $\frac{E_0}{B_0} = c$ or $E = cB$ or

$$E = \frac{1}{\sqrt{\mu_0 \epsilon_0}}B \tag{10.38}$$

this leads to the energy densities being equal.

This equality leads to

$$u(x,t) = \epsilon_0 E^2 = \frac{1}{\mu_0}B^2 \tag{10.39}$$

The reate of transport of energy can be found by considering a small time interval Δt and looking at the energy contained in a cylinder of length $c\Delta t$ with a cross-sectional area A.

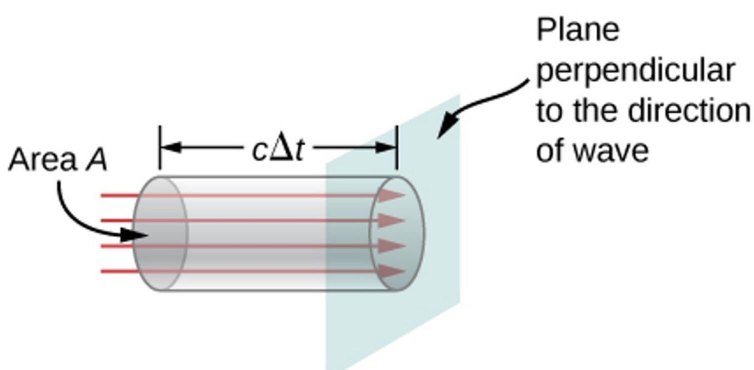


Figure 10.7: The energy $uAc\Delta t$

The energy passing through the area A in time Δt is

$$u * \text{ volume} = uAc\Delta t \tag{10.40}$$

The energy per unit area per unit time through a plane perpendicular to the wave is called the energy flux (S)

More generally, when dependent on orientation

$$\vec{S} = \frac{1}{\mu_0}\vec{E} \times \vec{B} \tag{10.41}$$

that is the cross-product of $\vec{E}$ and $\vec{B}$ is the direction perpendicular to both vectors.

The energy flux varies with time

$$S(x,t) = c\epsilon_0 E_0^2 \cos^2{(kx - \omega t)} \tag{10.42}$$

The time average of the energy flux is the intensity I of the electromagnetic wave

$$I = S_{avg} = c\epsilon_0 E^2 \frac{1}{T}\int_0^T \cos^2{(2\pi\frac{t}{T})}dt \tag{10.43}$$

If we evaluate the average of either cos() or sin() we get equivalence as they only differ in phase. Meaning

$$< \cos^2 \xi >= \frac{1}{2}[< \cos^2 \xi > + < \sin^2 \xi >= \frac{1}{2} < 1 >= \frac{1}{2} \tag{10.44}$$

Where the triangle brackets $< ... >$ stand for the time-averaging operation .

$$I = S_{avg} = \frac{1}{2}c\epsilon_0 E_0^2 \tag{10.45}$$

Appendix A

Integration by Trig Substitution

Find the integral of

$$\int \frac{1}{(a^2+x^2)^{\frac{3}{2}}} \tag{A.1}$$

Integrate by trig substitution by setting $x = a\tan u$ which leads to

$$\frac{dx}{du} = \frac{a\tan u}{du} = \frac{a}{\cos^2 u} \tag{A.2}$$

Which leads to

$$dx = (\frac{a}{\cos^2 u})du \tag{A.3}$$

Thus

$$\int \frac{1}{(a^2+x^2)^{\frac{3}{2}}} = \int \frac{1}{(a^2+(a\tan(u))^2)^{\frac{3}{2}}}(\frac{a}{\cos^2(u)})du \tag{A.4}$$

$$= \int \frac{1}{(a^2)^{\frac{3}{2}}(1+\tan^2(u))^{\frac{3}{2}}}(\frac{a}{\cos^2(u)})du \tag{A.5}$$

$$= \int \frac{1}{(a^3)(\frac{1}{\cos^2(u)})^{\frac{3}{2}}}(\frac{a}{\cos^2(u)})du \tag{A.6}$$

$$= \frac{1}{a^2}\int \frac{1}{(\frac{1}{\cos^2(u)})^{\frac{3}{2}}}(\frac{1}{\cos^2(u)})du \tag{A.7}$$

$$= \frac{1}{a^2}\int \frac{1}{(\frac{1}{\cos^3(u)})}(\frac{1}{\cos^2(u)})du \tag{A.8}$$

$$= \frac{1}{a^2}\int \cos(u)du \tag{A.9}$$

$$= \frac{1}{a^2}\sin(u) + C \tag{A.10}$$

From the above $\arctan(\frac{x}{a}) = u$ so

$$= \frac{1}{a^2}\sin(\arctan(\frac{x}{a})) + C \tag{A.11}$$

$$= \frac{1}{a^2}[\frac{\frac{x}{a}}{\sqrt{1+(\frac{x}{a})^2}}] + C \tag{A.12}$$

$$= \frac{x}{a^3}[\frac{1}{\frac{1}{a}\sqrt{a^2+x^2}}] + C \tag{A.13}$$

Which finally leads to

$$\int \frac{1}{(a^2+x^2)^{\frac{3}{2}}} = \frac{x}{a^2}[\frac{1}{\sqrt{a^2+x^2}}] + C \tag{A.14}$$

Appendix B

Chain Rule

$$\frac{d}{dx}\left(\frac{1}{\sqrt{x^2+R^2}}\right) = \frac{d}{dx}(x^2+R^2)^{-\frac{1}{2}} \tag{B.1}$$

The chain rule $f(g(x))' = f'(g(x) \cdot g'(x)$. In this case $g(x) = x^2 + R^2$. From this

$$f(g(x)) = g(x)^{-\frac{1}{2}} \tag{B.2}$$

thus

$$f'(g(x)) = -\frac{1}{2}g(x)^{-\frac{3}{2}} \tag{B.3}$$

and

$$g'(x) = 2x \tag{B.4}$$

Thus

$$f(g(x))' = -\frac{1}{2}g(x)^{-\frac{3}{2}} \cdot 2x \tag{B.5}$$

$$\frac{d}{dx}\left(\frac{1}{\sqrt{x^2+R^2}}\right) = \frac{-x}{(x^2+R^2)^{\frac{3}{2}}} \tag{B.6}$$