Quantum Math

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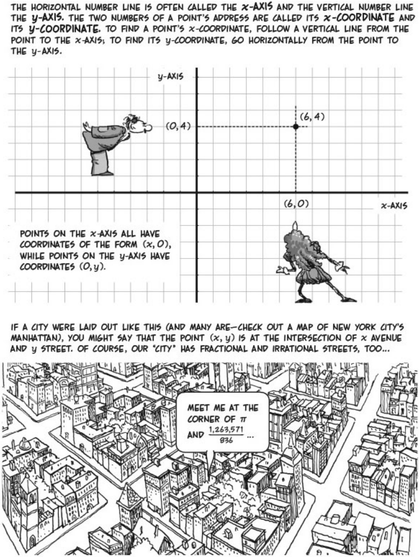
Algebra

Algebra Overview

- Functions
- Transformations
- Polynomials
- Rational Functions
- Exponentials and Logarithms

Cartesian Coordinates

Some text



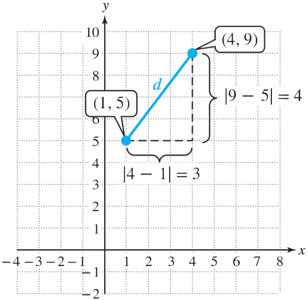
Measuring Distance - Pythagorean Theorem

Pythagorean Theorem:

$$a^2 + b^2 = c^2$$

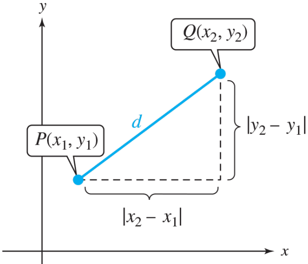
For example:

$$d^2 = 3^2 + 4^2$$
$$d^2 = 9 + 16 = 25$$
$$d = \sqrt{25} = 5$$



More generally for two points $P(x_1, y_1)$ and $Q(x_2, y_2)$

$$d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$$
$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$



Noting that $\mid a \mid = (a)^2$:

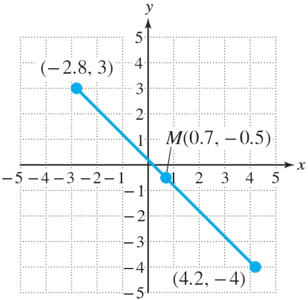
Midpoints and Intercepts

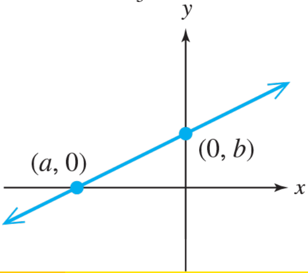
Midpoint:

$$M = \left(\frac{x_1+x_2}{2}, \frac{y_2+y_1}{2}\right)$$

Intercepts:

Two key features of a graph are where the graph intersects the x and y axes, the x-intercept and y-intercept, respectively.

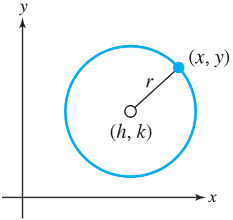




The Circle

A circle is a set of all points that are equidistant from a fixed point called the center (h, k). The distance from any point on the circle to the center is called the radius (r)

$r = \sqrt{(x-h)^2 + (y-k)^2}$



Equation of a circle:
Standard form: $(x-h)^2 + (y-k)^2 = r^2$

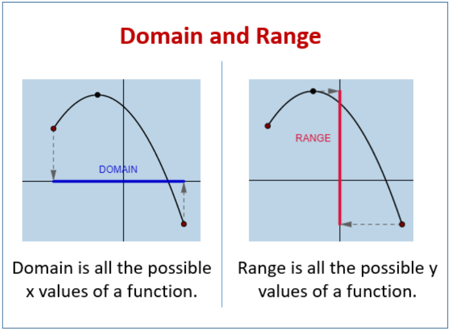
Expand binomials:
$x^2 - hx + h^2 + y^2 - ky + k^2 - r^2 = 0$

General form:
$x^2 + y^2 - hx - ky + (h^2 + k^2 - r^2) = 0$

or

$x^2 + y^2 + Ax + By + C = 0$

Domain and Range



A set of ordered pairs (x, y) is called a relation in x and y.

- The set of x-values in the ordered pairs is called the domain of the relations.
- The set of y-values in the ordered pairs is called the range of the relations.

Linear Equations with Two Variables

A linear equation in variables x and y can be written in the standard form:

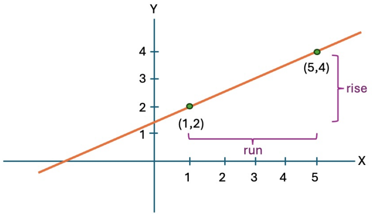
$$Ax + By = C \tag{1}$$

However, it is more common to see it in slope-intercept form:

$$y = mx + b \tag{2}$$

where, m is the slope and b is the y-intercept

Linear Conversion - Slope and Y-Intercept



$$y = mx + b$$

where m is slope and b y-intercept.

For example, given two points:

- $(x_1, y_1) = (1, 2)$
- $(x_2, y_2) = (5, 4)$

Find slope

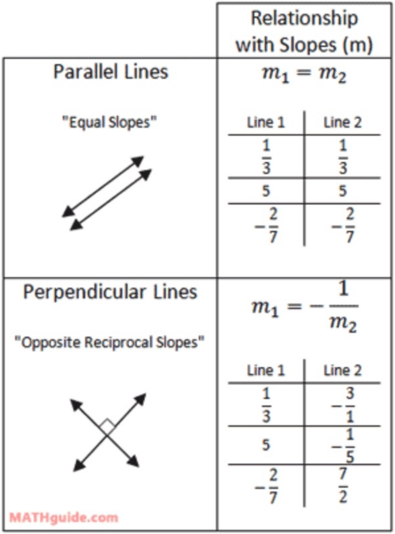
- $m = \frac{rise}{run} = \frac{4-2}{5-1} = \frac{1}{2}$

Find y-intercept

- $y_1 = m * x_1 + b$
- $b = y_1 - (m * x_1)$
- $b = 2 - (\frac{1}{2} * 1) = 1\frac{1}{2}$

Use this to find the conversion from Celsius to Fahrenheit.

Parallel and Perpendicular Lines



	Relationship with Slopes (m)
Parallel Lines "Equal Slopes"	$m_1 = m_2$
Perpendicular Lines "Opposite Reciprocal Slopes"	$m_1 = -\frac{1}{m_2}$

Parallel Lines:

Line 1	Line 2
$\frac{1}{3}$	$\frac{1}{3}$
5	5
$-\frac{2}{7}$	$-\frac{2}{7}$

Perpendicular Lines:

Line 1	Line 2
$\frac{1}{3}$	$-\frac{3}{1}$
5	$-\frac{1}{5}$
$-\frac{2}{7}$	$\frac{7}{2}$

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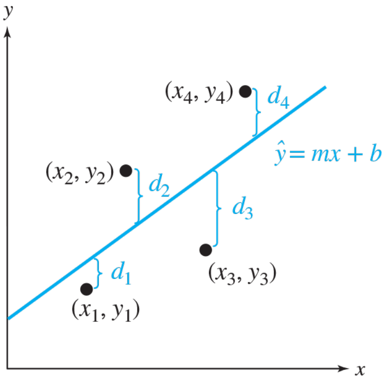
Linear Regression



Consider a set of data:
$(x_1, y_1), (x_2, y_2), (x_3, y_3), \ldots, (x_n, y_n)$

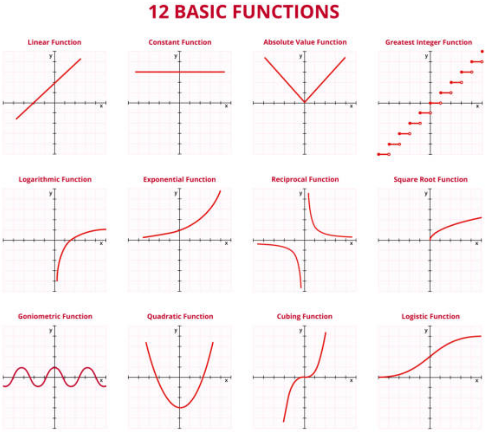
- The least-squares regression line $\hat{y} = mx + b$, is a unique line that minimizes the sum of the squared vertical deviations from the the observed data points to the line.

Use this to find the conversion from Celsius to Fahrenheit.

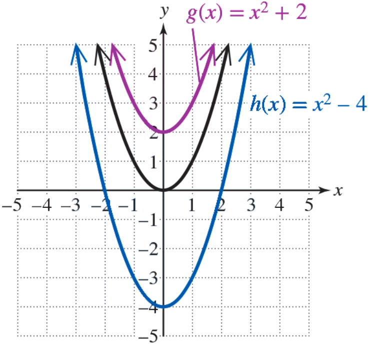
Recognizing Functions

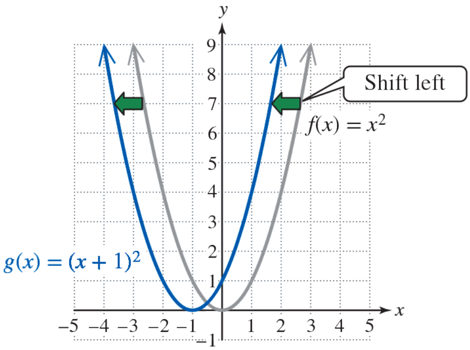
An algebraic function provides a "y-value" for every "x-value"

- Linear: $y = x + 2$
- Quadratic: $y = x^2$
- Periodic: $y = sin(x)$

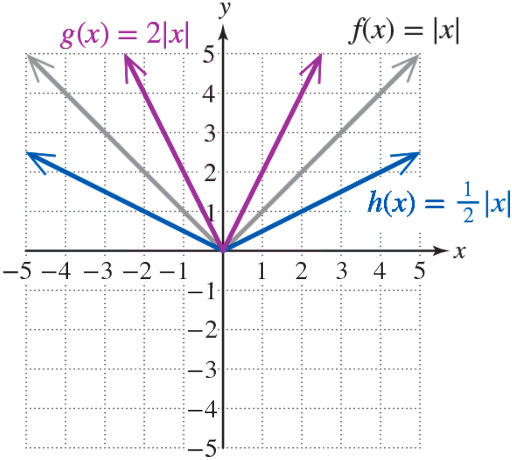


Vertical and Horizontal Shifts

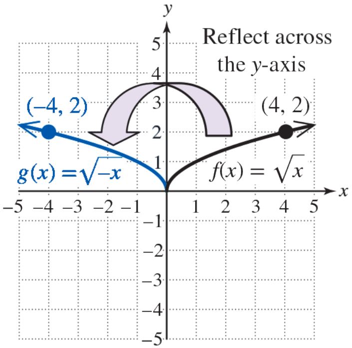


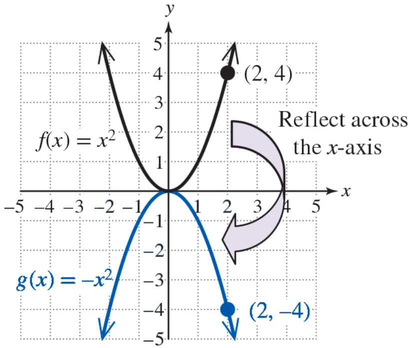


Shrink and Expand



X and Y Reflections

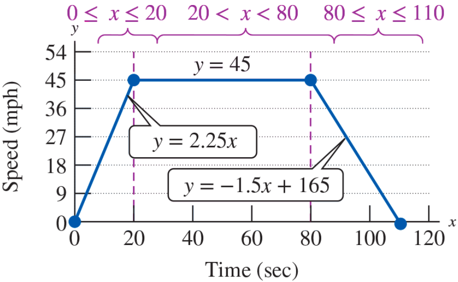




Summary - Transformations of Functions

Transformation	Effect on the Graph of f	Changes to Points on f
Vertical translation (shift) $y = f(x) + k$ $y = f(x) - k$	Shift upward k units Shift downward k units	Replace (x, y) by $(x, y + k)$. Replace (x, y) by $(x, y - k)$.
Horizontal translation (shift) $y = f(x - h)$ $y = f(x + h)$	Shift to the right h units Shift to the left h units	Replace (x, y) by $(x + h, y)$. Replace (x, y) by $(x - h, y)$.
Vertical stretch/shrink $y = a[f(x)]$	Vertical stretch (if $a > 1$) Vertical shrink (if $0 < a < 1$) Graph is stretched/shrunk vertically by a factor of a.	Replace (x, y) by (x, ay).
Horizontal stretch/shrink $y = f(a \cdot x)$	Horizontal shrink (if $a > 1$) Horizontal stretch (if $0 < a < 1$) Graph is shrunk/stretched horizontally by a factor of $\frac{1}{a}$.	Replace (x, y) by $\left(\frac{x}{a}, y\right)$.
Reflection $y = -f(x)$ $y = f(-x)$	Reflection across the x-axis Reflection across the y-axis	Replace (x, y) by $(x, -y)$. Replace (x, y) by $(-x, y)$.

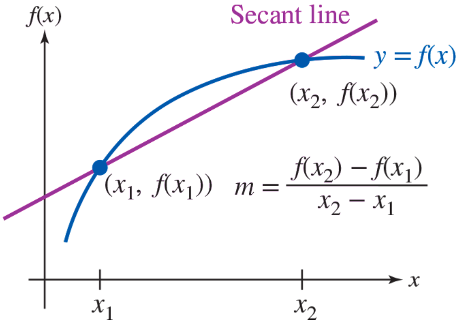
Piece-Wise Functions



$$f(x) = \begin{cases} 2.25x & \text{for } 0 \le x \le 20 \\ 45 & \text{for } 20 < x < 80 \\ -1.5x + 165 & \text{for } 80 \le x \le 100 \end{cases}$$

Rate of Change

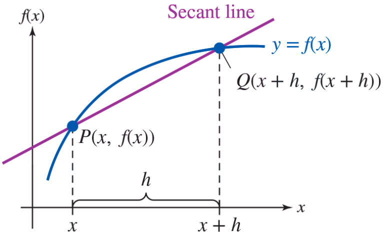
Given points (x_1, y_1) and (x_2, y_2) as points on the graph of a function $f()$, if $f()$ is defined on the interval $[x_1, x_2]$, then the average rate of change is the slope of the secant[1] line containing $(x_1, f(x_1))$ and $(x_2, f(x_2))$.



[1] Secante comes from the latin secare meaning "to cut."

Difference Quotient

Suppose we choose a value x from the domain of $f()$ and a second value $x + h$, where $h \neq 0$, but very small.



The difference quotient[2].

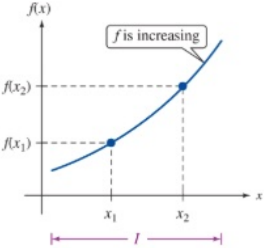
$$m = \frac{f(x+h) - f(x)}{(x+h) - x} = \frac{f(x+h) - f(x)}{h} \qquad (3)$$

[2] The difference quotient is important to calculus, where the exact rate of change at a point is given by $\lim_{h\to 0}(m)$

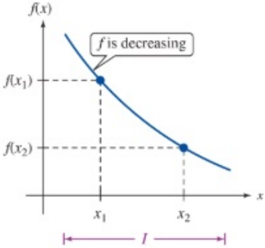
Increaing, Decreaing, Constant

Suppose that I is an interval contained within the domain of a function f.

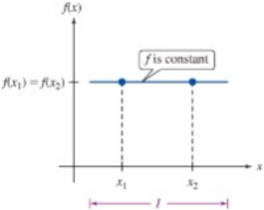
- f is increasing on I if $f(x_1) < f(x_2)$ for all $x_1 < x_2$ on I.
- f is decreasing on I if $f(x_1) > f(x_2)$ for all $x_1 < x_2$ on I.
- f is constant on I if $f(x_1) = f(x_2)$ for all x_1 and x_2 on I.



For all $x_1 < x_2$ on I,
$f(x_1) < f(x_2)$



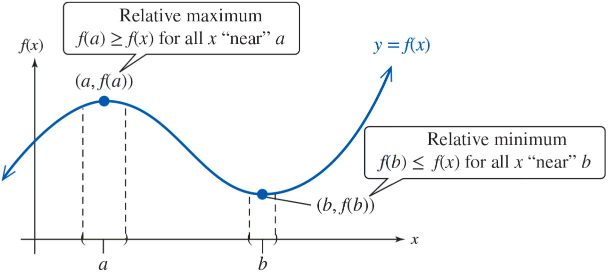
For all $x_1 < x_2$ on I,
$f(x_1) > f(x_2)$



For all x_1 and x_2 on I,
$f(x_1) = f(x_2)$

Local Minima and Maxima

- $f(a)$ is a relative maximum of f if there exists an open interval[3] containing a such that $f(a) \geq f(x)$ for all x in the interval.
- $f(b)$ is a relative minimum of f if there exists an open interval[4] containing a such that $f(b) \leq f(x)$ for all x in the interval.



[3] An open interval is an interval in which the endpoints are not included.
[4] An open interval is an interval in which the endpoints are not included.

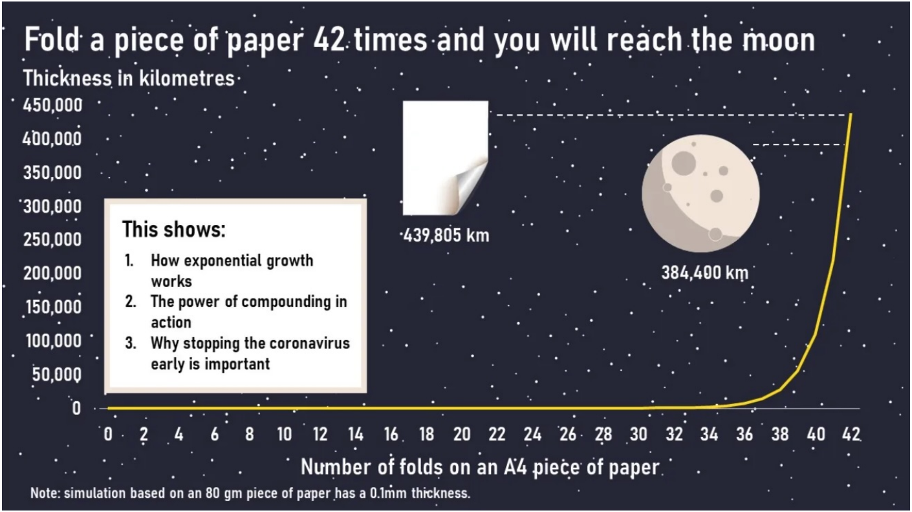
Operations on Functions

Exponential Functions

- Linear growth - a constant rate of change, that is,a constant number by which the output increased for each unit increase in input.
- Exponential growth - increase based on a constant multiplicative rate of change over equal increments of time, that is, a percent increase of the original amount over time.

x	$f(x) = 2^x$	$g(x) = 2x$
0	1	0
1	2	2
2	4	4
3	8	6
4	16	8
5	32	10
6	64	12

Origami to the Moon

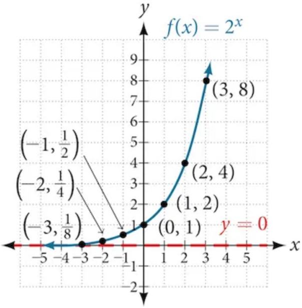


What about Negative Exponents

For example, $f(x) = 2^x$:

The general form of an exponential function is $f(x) = ab^x$, where a is any non-zero number and b is an positive number not equal to 1.

- If $b > 1$ the function grows at a rate proportional to its size.
- If $0 < b < 1$ the function decays at a rate proportional to its size.



x	-3	-2	-1	0	1	2	3
$f(x) = 2^x$	$2^{-3} = \frac{1}{8}$	$2^{-2} = \frac{1}{4}$	$2^{-1} = \frac{1}{2}$	$2^0 = 1$	$2^1 = 2$	$2^2 = 4$	$2^3 = 8$

Scientific (SI) Prefixes

The Metric System Prefixes				
Prefix	**Label**	**Decimal Value**	**Scientific**	**Colloquial**
yocto	y	0.000 000 000 000 000 000 000 001	10^{-24}	septillionth
zepto	z	0.000 000 000 000 000 000 001	10^{-21}	sextillionth
atto	a	0.000 000 000 000 000 001	10^{-18}	quintillionth
femto	f	0.000 000 000 000 001	10^{-15}	quadrillionth
pico	p	0.000 000 000 001	10^{-12}	trillionth
nano	n	0.000 000 001	10^{-9}	billionth
micro	µ	0.000 001	10^{-6}	millionth
milli	m	0.001	10^{-3}	thousandth
centi	c	0.01	10^{-2}	hundredth
deci	d	0.1	10^{-1}	tenth
---	---	1	10^{0}	one
deka	da	10	10^{1}	ten
hecto	h	100	10^{2}	hundred
kilo	k	1 000	10^{3}	thousand
mega	M	1 000 000	10^{6}	million
giga	G	1 000 000 000	10^{9}	billion
tera	T	1 000 000 000 000	10^{12}	trillion
peta	P	1 000 000 000 000 000	10^{15}	quadrillion
exa	E	1 000 000 000 000 000 000	10^{18}	quintillion
zetta	Z	1 000 000 000 000 000 000 000	10^{21}	sextillion
yotta	Y	1 000 000 000 000 000 000 000 000	10^{24}	septillion

e - an interesting aside

The letter e represents the irrational number:

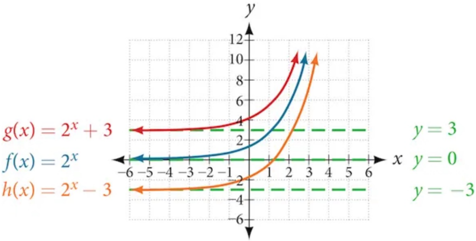
$$e = (1 + \frac{1}{n})^n \tag{4}$$

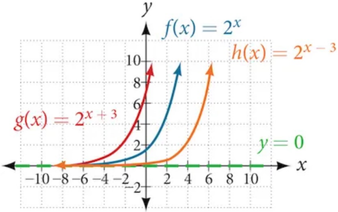
as n increases without bound.

The number e is used as a base for many real-world exponential models. To work with base e, we use the approximation, $e \approx 2.718282$. The constant was named by the Swiss mathematician Leonhard Euler (1707–1783) who first investigated and discovered many of its properties.

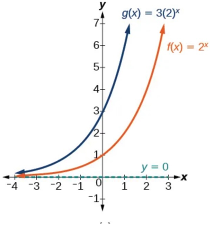
Graphing Exponentials

Shifts:

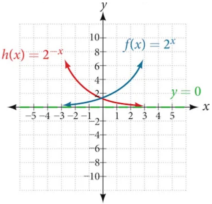




Stretch:



Flip:

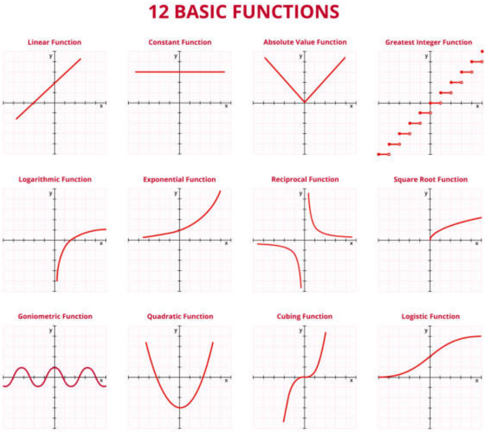


Trigonometry

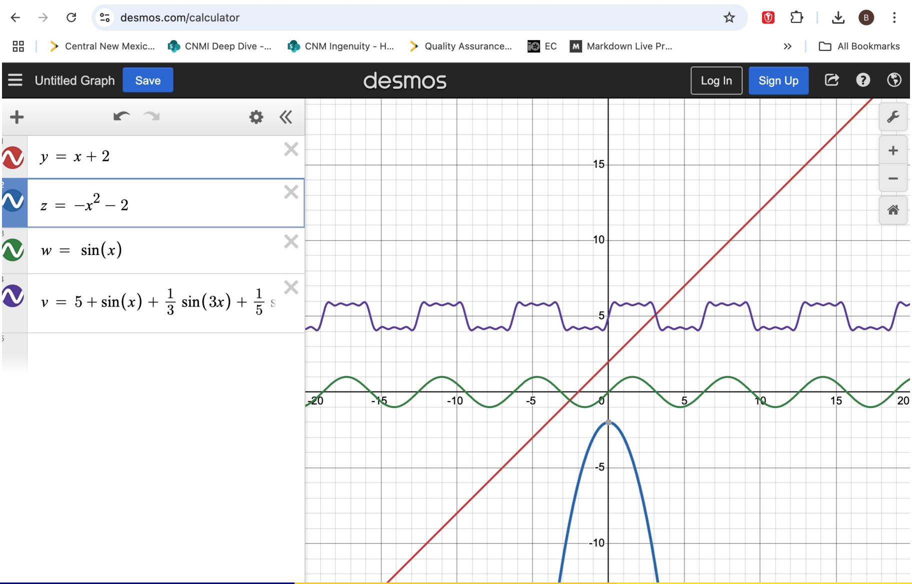
Algebraic Functions

An algebraic function provides a "y-value" for every "x-value"

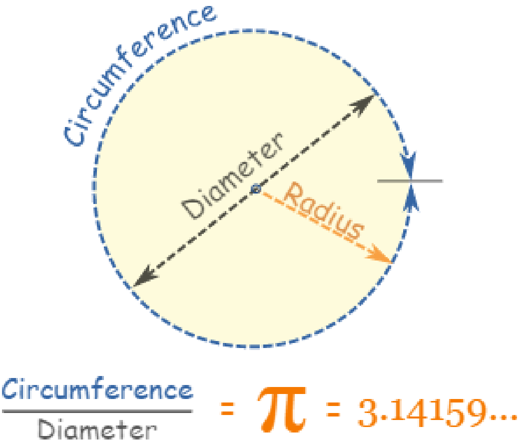
- Linear: $y = x + 2$
- Quadratic: $y = x^2$
- Periodic: $y = sin(x)$



More Desmos Fun

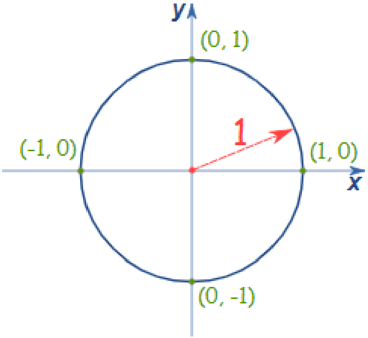


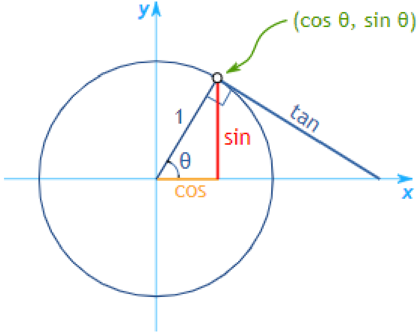
Pi (π)


Circumference
Diameter
Radius
Circumference / Diameter = π = 3.14159...

Unit Circle and Trigonometric Functions

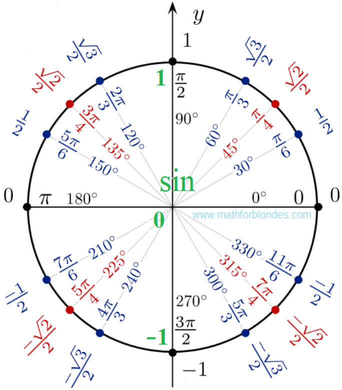
The Unit Circle is a circle with a radius of 1.

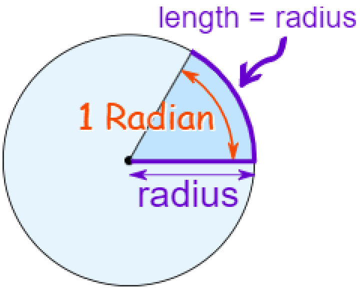




The Unit Circle can be used to map out the trigonometric values of sine, cosine, and tangent.

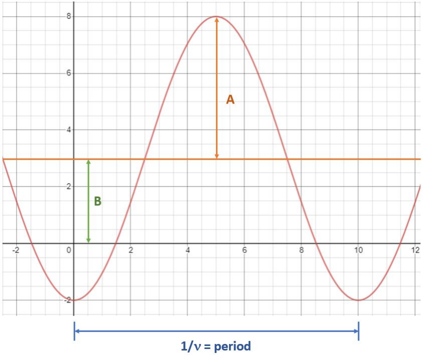
Unit Circle and the Value of *sin*(θ)





- $sin(\theta)$ is the y-value of the point on the Unit Circle at angle θ.
- In our trig functions, θ is measured in radians (rad), not degrees.
- 360 degrees $= 2\pi$ radians.

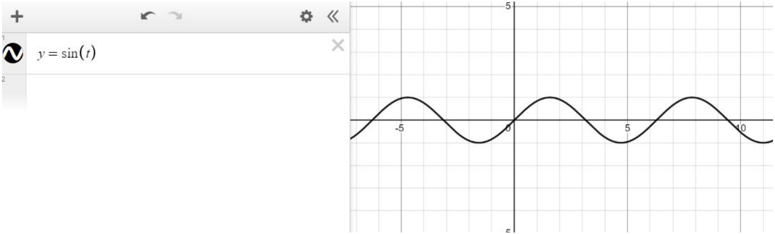
Sine Waves

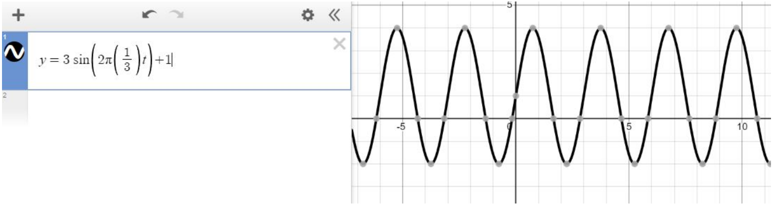


$$y = A * sin(2 * \pi * \nu * t) + B$$

where A = amplitude, B = offset, ν = frequency = $\frac{1}{period}$, and t = time in seconds.

Using Desmos (desmos.com/calculator)

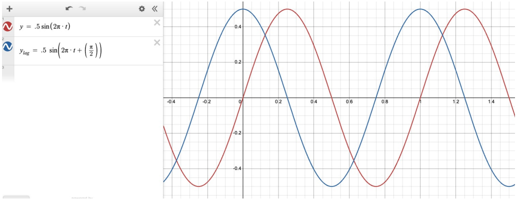




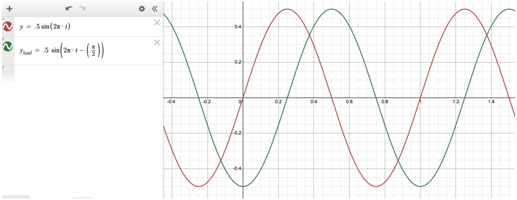
Phase Shift

The sine wave can be shifted relative to each other by adding in a phase shift (ϕ), which will shift the wave to the left or right.

Blue lags Red:

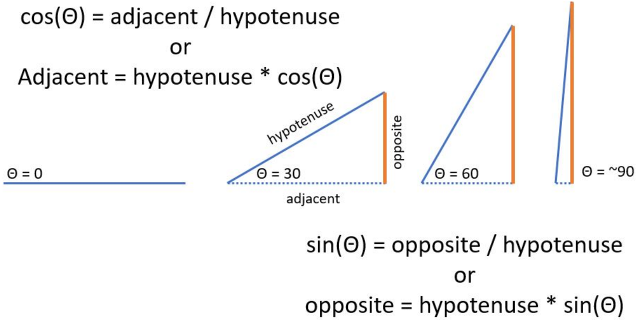


Green leads Red:



SOH CAH TOA

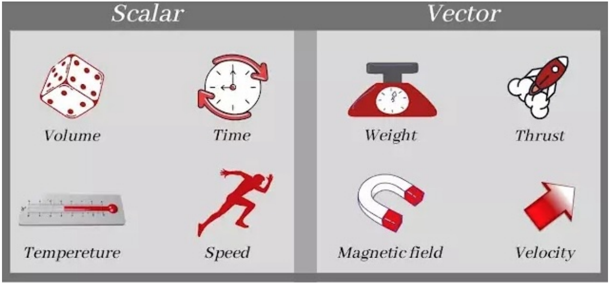
- sin = opposite over hypotenuse
- cos = adjacent over hypotenuse
- tan = opposite over adjacent



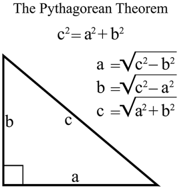
Vectors

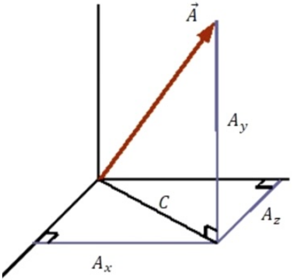
Scalars and Vectors

Scalars are quantities that are fully described by a magnitude (or numerical value) alone.
Vectors are quantities that are fully described by both a magnitude and a direction.



Pythagorean Theorem in 3 Dimensions





To add orthogonal (at right angles to each other) vectors in 3 Dimensions:

- $C = \sqrt{A_x^2 + A_y^2}$
- $A_{total} = \sqrt{C^2 + A_z^2}$
- $A_{total} = \sqrt{A_x^2 + A_y^2 + A_z^2}$