Quantum Math

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August 2025

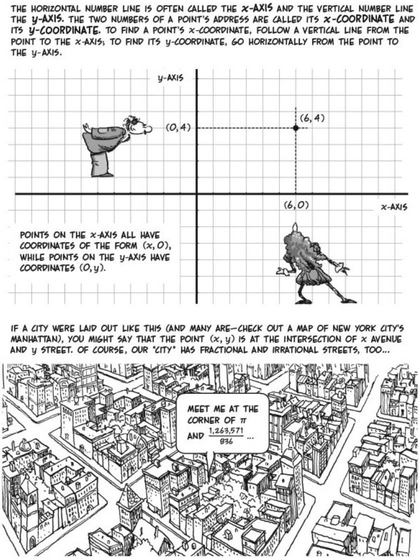
Algebra

Algebra Overview

- Functions
- Transformations
- Polynomials
- Rational Functions
- Exponentials and Logarithms

Cartesian Coordinates

Some text



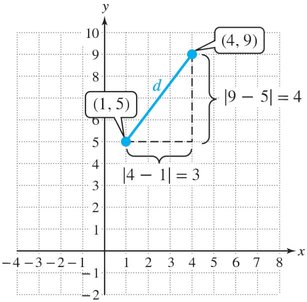
Measuring Distance - Pythagorean Theorem

Pythagorean Theorem:

$$a^2 + b^2 = c^2$$

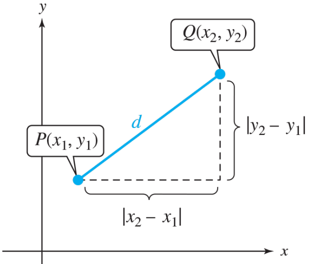
For example:

$$d^2 = 3^2 + 4^2$$
$$d^2 = 9 + 16 = 25$$
$$d = \sqrt{25} = 5$$



More generally for two points $P(x_1, y_1)$ and $Q(x_2, y_2)$

$$d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$$
$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

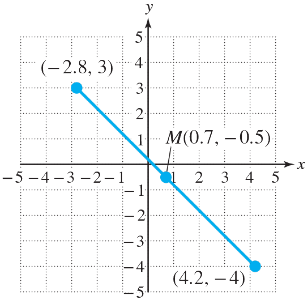


Noting that $\mid a \mid = (a)^2$:

Midpoints and Intercepts

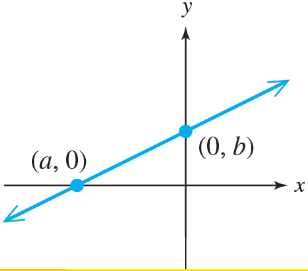
Midpoint:

$$M = \left(\frac{x_1 + x_2}{2}, \frac{y_2 + y_1}{2}\right)$$



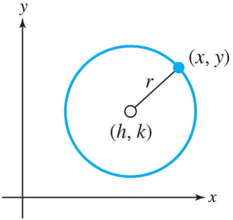
Intercepts:

Two key features of a graph are where the graph intersects the x and y axes, the x-intercept and y-intercept, respectively.



The Circle

A circle is a set of all points that are equidistant from a fixed point called the center (h, k). The distance from any point on the cirecle to the center is called the radius (r)
$r = \sqrt{(x-h)^2 + (y-k)^2}$

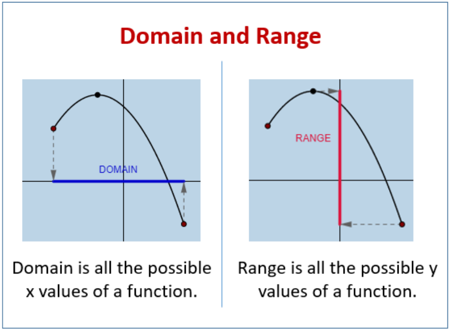


Equation of a circle:
Standard form: $(x-h)^2 + (y-k)^2 = r^2$

Expand binomials:
$x^2 - hx + h^2 + y^2 - ky + k^2 - r^2 = 0$

General form:
$x^2 + y^2 - hx - ky + (h^2 + k^2 - r^2) = 0$
or
$x^2 + y^2 + Ax + By + C = 0$

Domain and Range



A set of ordered pairs (x, y) is called a relation in x and y.

- The set of x-values in the ordered pairs is called the domain of the relations.
- The set of y-values in the ordered pairs is called the range of the relations.

Linear Equations with Two Variables

A linear equation in variables x and y can be written in the standard form:

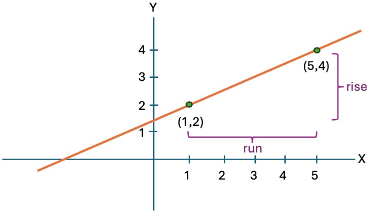
$$Ax + By = C \tag{1}$$

However, it is more common to see it in slope-intercept form:

$$y = mx + b \tag{2}$$

where, m is the slope and b is the y-intercept

Linear Conversion - Slope and Y-Intercept



$$y = mx + b$$

where m is slope and b y-intercept.

For example, given two points:

- $(x_1, y_1) = (1, 2)$
- $(x_2, y_2) = (5, 4)$

Find slope

- $m = \frac{rise}{run} = \frac{4-2}{5-1} = \frac{1}{2}$

Find y-intercept

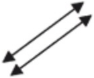
- $y_1 = m * x_1 + b$
- $b = y_1 - (m * x_1)$
- $b = 2 - (\frac{1}{2} * 1) = 1\frac{1}{2}$

Use this to find the conversion from Celsius to Fahrenheit.

Parallel and Perpendicular Lines

<table>
<tr><th></th><th>Relationship with Slopes (m)</th></tr>
<tr><td>Parallel Lines

"Equal Slopes"

</td><td>$m_1 = m_2$

<table><tr><th>Line 1</th><th>Line 2</th></tr><tr><td>$\frac{1}{3}$</td><td>$\frac{1}{3}$</td></tr><tr><td>5</td><td>5</td></tr><tr><td>$-\frac{2}{7}$</td><td>$-\frac{2}{7}$</td></tr></table></td></tr>
<tr><td>Perpendicular Lines

"Opposite Reciprocal Slopes"

</td><td>$m_1 = -\frac{1}{m_2}$

<table><tr><th>Line 1</th><th>Line 2</th></tr><tr><td>$\frac{1}{3}$</td><td>$-\frac{3}{1}$</td></tr><tr><td>5</td><td>$-\frac{1}{5}$</td></tr><tr><td>$-\frac{2}{7}$</td><td>$\frac{7}{2}$</td></tr></table></td></tr>
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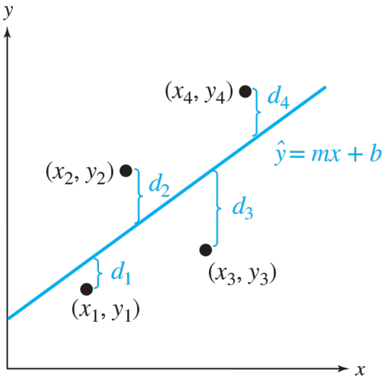
Linear Regression



Consider a set of data:
$(x_1, y_1), (x_2, y_2), (x_3, y_3), \ldots, (x_n, y_n)$

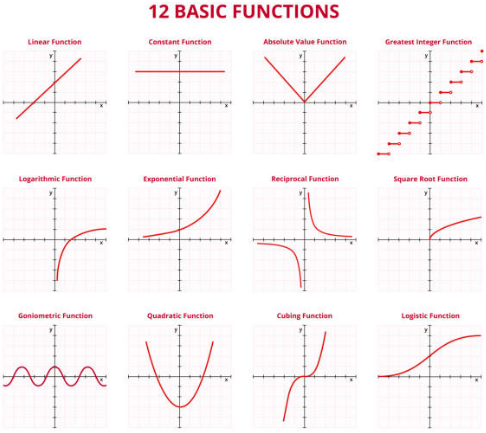
- The least-squares regression line $\hat{y} = mx + b$, is a unique line that minimizes the sum of the squared vertical deviations from the the observed data points to the line.

Use this to find the conversion from Celsius to Fahrenheit.

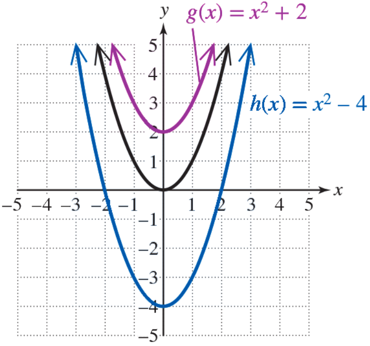
Recognizing Functions

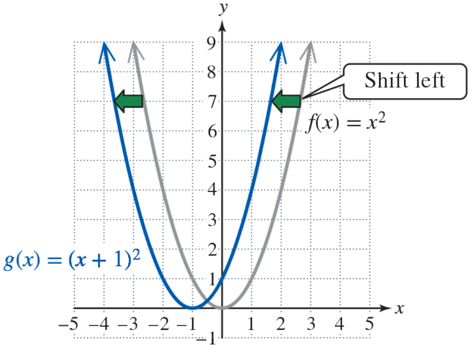
An algebraic function provides a "y-value" for every "x-value"

- Linear: $y = x + 2$
- Quadratic: $y = x^2$
- Periodic: $y = sin(x)$

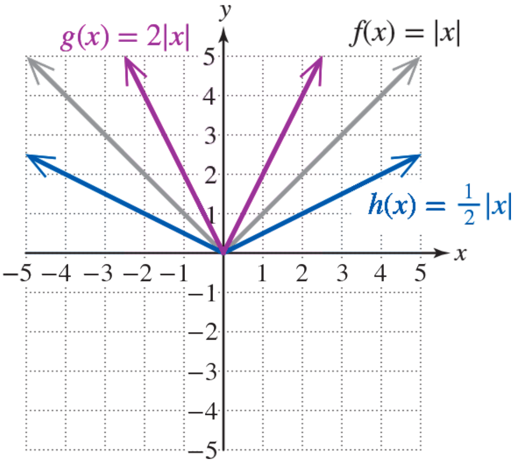


Vertical and Horizontal Shifts

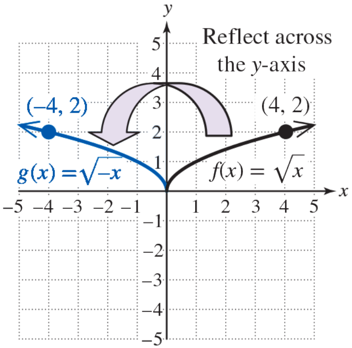


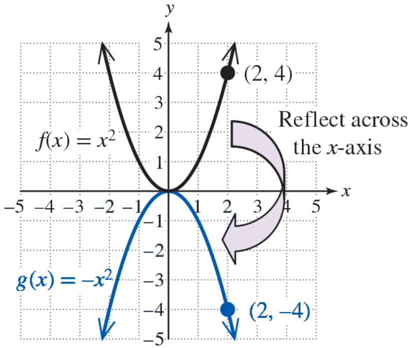


Shrink and Expand



X and Y Reflections

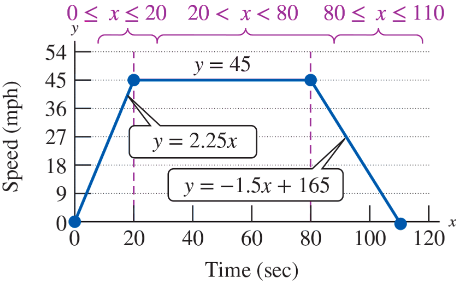




Summary - Transformations of Functions

Transformation	Effect on the Graph of f	Changes to Points on f
Vertical translation (shift) $y = f(x) + k$ $y = f(x) - k$	Shift upward k units Shift downward k units	Replace (x, y) by $(x, y + k)$. Replace (x, y) by $(x, y - k)$.
Horizontal translation (shift) $y = f(x - h)$ $y = f(x + h)$	Shift to the right h units Shift to the left h units	Replace (x, y) by $(x + h, y)$. Replace (x, y) by $(x - h, y)$.
Vertical stretch/shrink $y = a[f(x)]$	Vertical stretch (if $a > 1$) Vertical shrink (if $0 < a < 1$) Graph is stretched/shrunk vertically by a factor of a.	Replace (x, y) by (x, ay).
Horizontal stretch/shrink $y = f(a \cdot x)$	Horizontal shrink (if $a > 1$) Horizontal stretch (if $0 < a < 1$) Graph is shrunk/stretched horizontally by a factor of $\frac{1}{a}$.	Replace (x, y) by $\left(\frac{x}{a}, y\right)$.
Reflection $y = -f(x)$ $y = f(-x)$	Reflection across the x-axis Reflection across the y-axis	Replace (x, y) by $(x, -y)$. Replace (x, y) by $(-x, y)$.

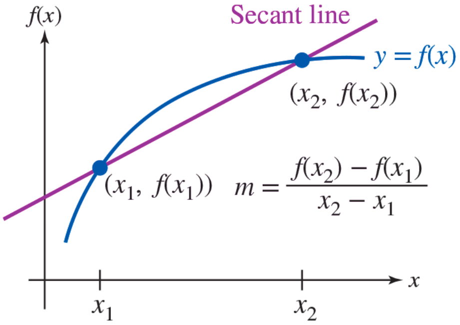
Piece-Wise Functions



$$f(x) = \begin{cases} 2.25x & \text{for } 0 \le x \le 20 \\ 45 & \text{for } 20 < x < 80 \\ -1.5x + 165 & \text{for } 80 \le x \le 100 \end{cases}$$

Rate of Change

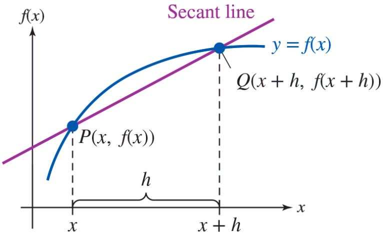
Given points (x_1, y_1) and (x_2, y_2) as points on the graph of a function $f()$, if $f()$ is defined on the interval $[x_1, x_2]$, then the average rate of change is the slope of the secant[1] line containing $(x_1, f(x_1))$ and $(x_2, f(x_2))$.



[1] Secante comes from the latin secare meaning "to cut."

Difference Quotient

Suppose we choose a value x from the domain of $f()$ and a second value $x+h$, where $h \neq 0$, but very small.



The difference quotient[2].

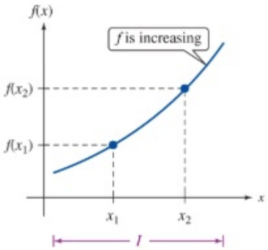
$$m = \frac{f(x+h)-f(x)}{(x+h)-x} = \frac{f(x+h)-f(x)}{h} \tag{3}$$

[2] The difference quotient is important to calculus, where the exact rate of change at a point is given by $\lim_{h\to 0}(m)$

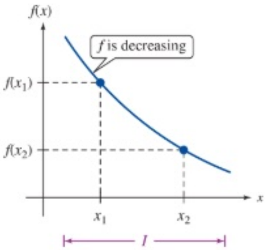
Increaing, Decreaing, Constant

Suppose that I is an interval contained within the domain of a function f.

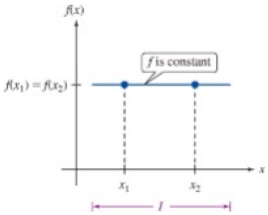
- f is increasing on I if $f(x_1) < f(x_2)$ for all $x_1 < x_2$ on I.
- f is decreasing on I if $f(x_1) > f(x_2)$ for all $x_1 < x_2$ on I.
- f is constant on I if $f(x_1) = f(x_2)$ for all x_1 and x_2 on I.



For all $x_1 < x_2$ on I,
$f(x_1) < f(x_2)$



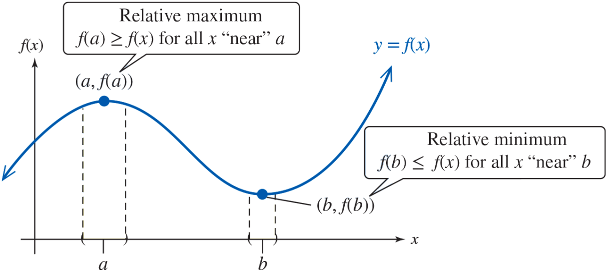
For all $x_1 < x_2$ on I,
$f(x_1) > f(x_2)$



For all x_1 and x_2 on I,
$f(x_1) = f(x_2)$

Local Minima and Maxima

- $f(a)$ is a relative maximum of f if there exists an open interval[3] containing a such that $f(a) \geq f(x)$ for all x in the interval.
- $f(b)$ is a relative minimum of f if there exists an open interval[4] containing a such that $f(b) \leq f(x)$ for all x in the interval.

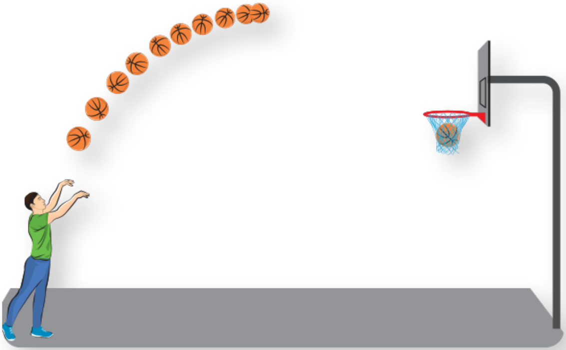


[3] An open interval is an interval in which the endpoints are not included.
[4] An open interval is an interval in which the endpoints are not included.

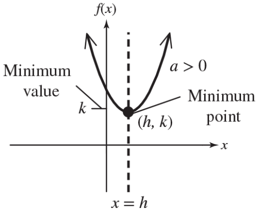
Operations on Functions

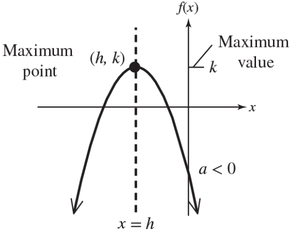
Quadratic Function

A quadratic function is often used as a model for the projectile motion. This is the motion followed by an object influenced by an initial force and by the force of gravity.



Quadratic Function - Vertex Form





Quadratic Function:
$f(x) = ax_2 + bx + c(a \neq 0)$
By completing the square, it can be expressed in vertex form: $f(x) = a(x-h)^2 + k$

- The graft of f(x) is a parabola with vertex (h, k)
- If $a > 0$ the parabola opens upward and minimum value is k.
- If $a > 0$ the parabola opens downward and maximum value is k.
- The axis of symmetry is $x = h$.

Polynomial Functions

Definition of a Polynomial Function

Let $a_n, a_{n-1}, a_{n-2}, \ldots, a_0$ represent real numbers and $n, n-1, n-2, \ldots, 0$ represent whole numbers. Then a function defined by

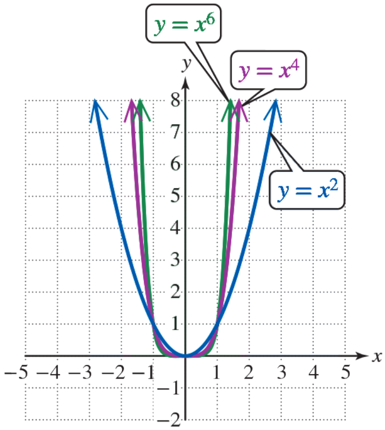
$$f(x) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \cdots + a_1 x + a_0$$

is called a **polynomial function**.

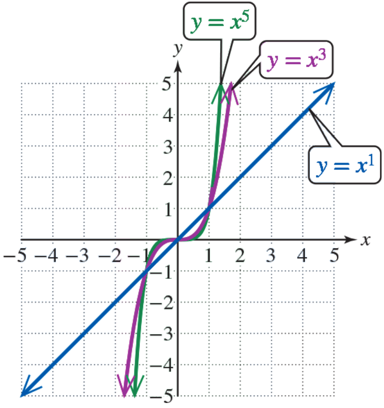
The term $a_n x^n$ is called the **leading term**, the coefficient a_n is the **leading coefficient**, and the exponent n is the **degree** of the polynomial function.

Even and Odd Exponents

Even:



Odd:



Polynomial End Behavior

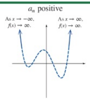
The Leading Term Test

Consider a polynomial function given by

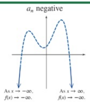
$$f(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1 x + a_0.$$

As $x \to \infty$ or as $x \to -\infty$, the graph of f will eventually have no more turns and will become forever increasing or forever decreasing. Thus, the graph of f far to the left and far to the right will follow the general behavior of $y = a_n x^n$.

n is even

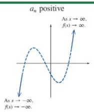


End behavior: up left/up right

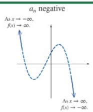


End behavior: down left/down right

n is odd



End behavior: down left/up right



End behavior: up left/down right

Polynomial Finding Zeros

Rational Functions

Definition of a Rational Function

Let $p(x)$ and $q(x)$ be polynomials where $q(x) \neq 0$. A function f defined by

$$f(x) = \frac{p(x)}{q(x)}$$ is called a **rational function**.

Note: The domain of a rational function is all real numbers excluding the real zeros of $q(x)$.

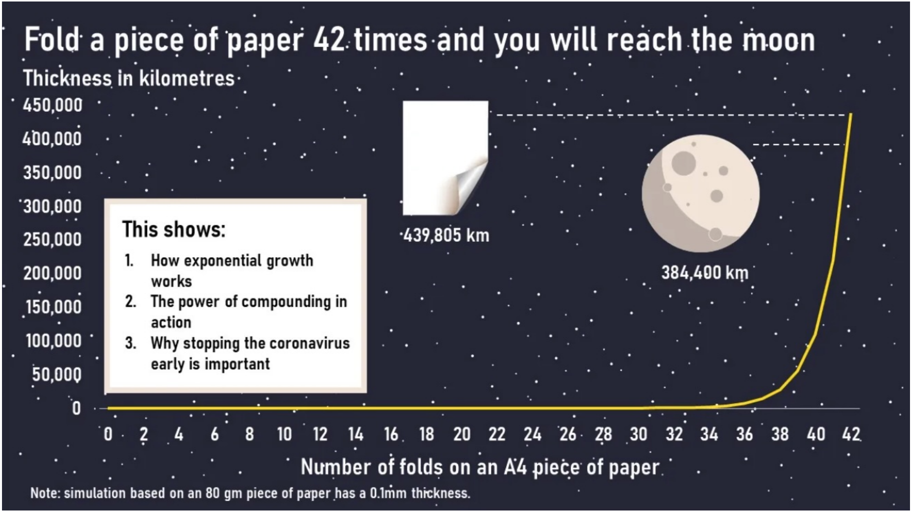
Function	**Factored Form**	**Domain**
$f(x) = \frac{1}{x}$	$f(x) = \frac{1}{x}$	$\{x \mid x \neq 0\}$ $(-\infty, 0) \cup (0, \infty)$
$g(x) = \frac{5x^2}{2x^2 + 5x - 12}$	$g(x) = \frac{5x^2}{(2x-3)(x+4)}$	$\left\{x \middle\vert x \neq \frac{3}{2}, x \neq -4\right\}$ $(-\infty, -4) \cup \left(-4, \frac{3}{2}\right) \cup \left(\frac{3}{2}, \infty\right)$
$k(x) = \frac{x+3}{x^2+4}$	$k(x) = \frac{x+3}{x^2+4}$	$\mathbb{R}$ $(-\infty, \infty)$

Exponential Functions

- Linear growth - a constant rate of change, that is,a constant number by which the output increased for each unit increase in input.
- Exponential growth - increase based on a constant multiplicative rate of change over equal increments of time, that is, a percent increase of the original amount over time.

x	$f(x) = 2^x$	$g(x) = 2x$
0	1	0
1	2	2
2	4	4
3	8	6
4	16	8
5	32	10
6	64	12

Origami to the Moon

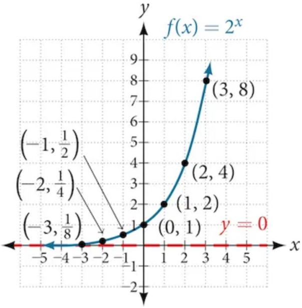


What about Negative Exponents

The general form of an exponential function is $f(x) = ab^x$, where a is any non-zero number and b is an positive number not equal to 1.

- If $b > 1$ the function grows at a rate proportional to its size.
- If $0 < b < 1$ the function decays at a rate proportional to its size.

For example, $f(x) = 2^x$:



x	-3	-2	-1	0	1	2	3
$f(x) = 2^x$	$2^{-3} = \frac{1}{8}$	$2^{-2} = \frac{1}{4}$	$2^{-1} = \frac{1}{2}$	$2^0 = 1$	$2^1 = 2$	$2^2 = 4$	$2^3 = 8$

Scientific (SI) Prefixes

The Metric System Prefixes

Prefix	Label	Decimal Value	Scientific	Colloquial
yocto	y	0.000 000 000 000 000 000 000 001	10^{-24}	septillionth
zepto	z	0.000 000 000 000 000 000 001	10^{-21}	sextillionth
atto	a	0.000 000 000 000 000 001	10^{-18}	quintillionth
femto	f	0.000 000 000 000 001	10^{-15}	quadrillionth
pico	p	0.000 000 000 001	10^{-12}	trillionth
nano	n	0.000 000 001	10^{-9}	billionth
micro	μ	0.000 001	10^{-6}	millionth
milli	m	0.001	10^{-3}	thousandth
centi	c	0.01	10^{-2}	hundredth
deci	d	0.1	10^{-1}	tenth
---	---	1	10^{0}	one
deka	da	10	10^{1}	ten
hecto	h	100	10^{2}	hundred
kilo	k	1 000	10^{3}	thousand
mega	M	1 000 000	10^{6}	million
giga	G	1 000 000 000	10^{9}	billion
tera	T	1 000 000 000 000	10^{12}	trillion
peta	P	1 000 000 000 000 000	10^{15}	quadrillion
exa	E	1 000 000 000 000 000 000	10^{18}	quintillion
zetta	Z	1 000 000 000 000 000 000 000	10^{21}	sextillion
yotta	Y	1 000 000 000 000 000 000 000 000	10^{24}	septillion

e - an interesting aside

The letter e represents the irrational number:

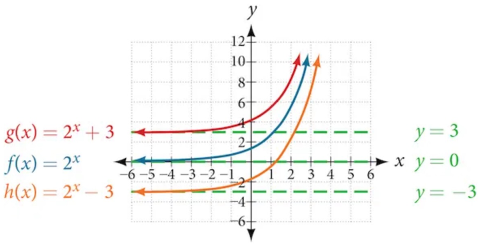
$$e = (1 + \frac{1}{n})^n \tag{4}$$

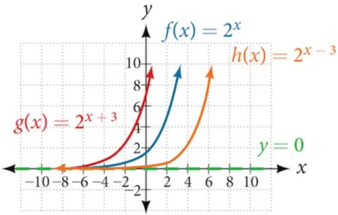
as n increases without bound.

The number e is used as a base for many real-world exponential models. To work with base e, we use the approximation, $e \approx 2.718282$. The constant was named by the Swiss mathematician Leonhard Euler (1707–1783) who first investigated and discovered many of its properties.

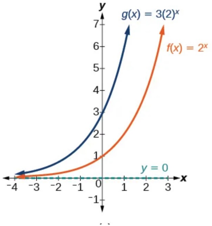
Graphing Exponentials

Shifts:

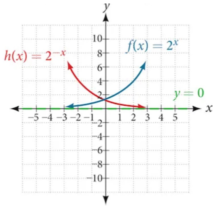




Stretch:



Flip:



Logarithmic Functions

Definition of a Logarithmic Function

If x and b are positive real numbers such that $b \neq 1$, then $y = \log_b x$ is called the **logarithmic function base *b***, where

$$y = \log_b x \text{ is equivalent to } b^y = x$$

Notes:

- Given $y = \log_b x$, the value y is the exponent to which b must be raised to obtain x.
- The value of y is called the **logarithm**, b is called the **base**, and x is called the **argument**.
- The equations $y = \log_b x$ and $b^y = x$ both define the same relationship between x and y. The expression $y = \log_b x$ is called the **logarithmic form**, and $b^y = x$ is called the **exponential form**.

Natural Log

Definition of Common and Natural Logarithmic Functions

- The logarithmic function base 10 is called the **common logarithmic function**. The common logarithmic function is denoted by $y = \log x$. Notice that the base 10 is not explicitly written; that is, $y = \log_{10} x$ is written simply as $y = \log\ x$.
- The logarithmic function base e is called the **natural logarithmic function**. The natural logarithmic function is denoted by $y = \ln x$; that is, $y = \log_e x$ is written as $y = \ln x$.

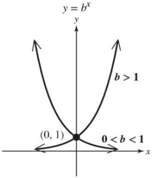
Logarithmic Functions

Logarithmic form: $y = log_b x$		Exponential form: $b^y = x$
$log_2 16 = 4$	$\Longleftrightarrow$	$2^4 = 16$
$log_1 0 \frac{1}{100} = -2$	$\Longleftrightarrow$	$10^{-2} = \frac{1}{100}$
$log_7 1 = 0$	$\Longleftrightarrow$	$7^0 = 1$

Graphing Exponential and Logarithmic Functions

Graphs of Exponential and Logarithmic Functions

Exponential Functions



Domain: $(-\infty, \infty)$

Range: $(0, \infty)$

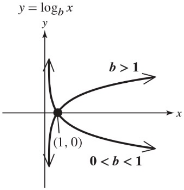
Horizontal asymptote: $y = 0$

Passes through $(0,\ 1)$

If $b > 1$, the function is increasing.

If $0 < b < 1$, the function is decreasing.

Logarithmic Functions



Domain: $(0, \infty)$

Range: $(-\infty, \infty)$

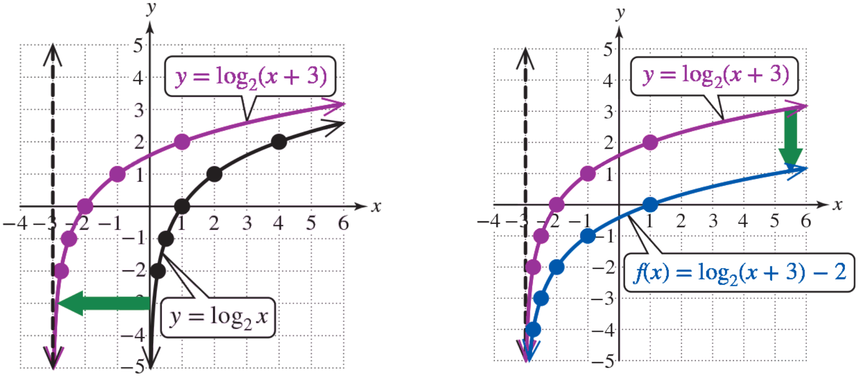
Vertical asymptote: $x = 0$

Passes through $(1, 0)$

If $b > 1$, the function is increasing.

If $0 < b < 1$, the function is decreasing.

Logarithmic Transforms



Logarithm Rules

- Product Property

$$log_b(xy) = log_b(x) + log_b(y) \tag{5}$$

- Quotient Property

$$log_b(\frac{x}{y}) = log_b(x) - log_b(y) \tag{6}$$

- Power Property

$$log_b x^p = p \cdot log_b x \tag{7}$$

Properties of Logarithms

Properties of Logarithms

Let b, x, and y be positive real numbers where $b \neq 1$, and let p be a real number. Then the following properties of logarithms are true.

1. $\log_b 1 = 0$

2. $\log_b b = 1$

3. $\log_b b^p = p$

4. $b^{\log_b x} = x$

5. $\log_b (xy) = \log_b x + \log_b y$ **Product property**

6. $\log_b \left(\frac{x}{y}\right) = \log_b x - \log_b y$ **Quotient property**

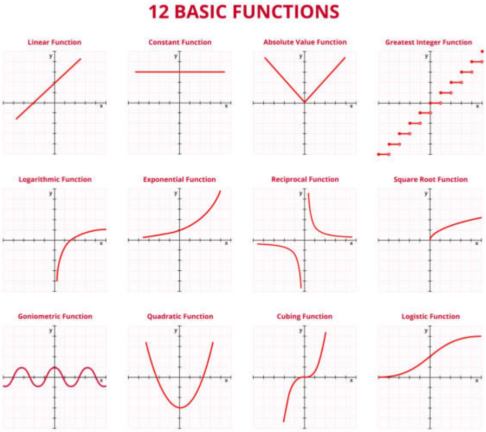
7. $\log_b x^p = p\log_b x$ **Power property**

Trigonometry

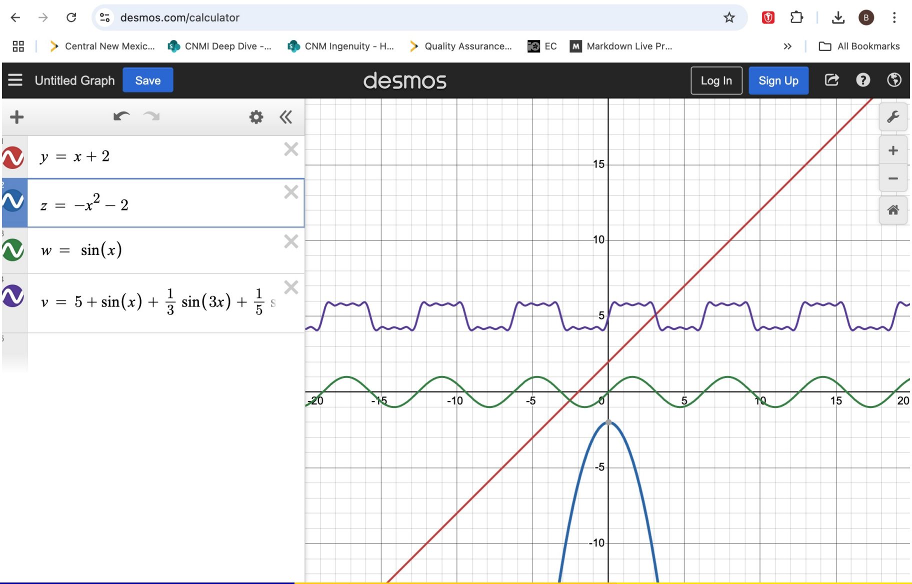
Algebraic Functions

An algebraic function provides a "y-value" for every "x-value"

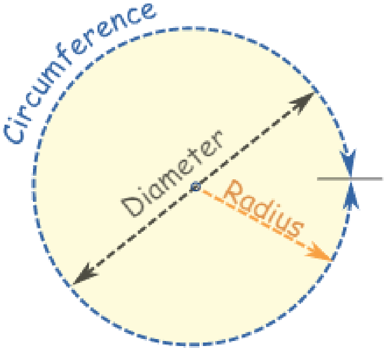
- Linear: $y = x + 2$
- Quadratic: $y = x^2$
- Periodic: $y = sin(x)$



More Desmos Fun



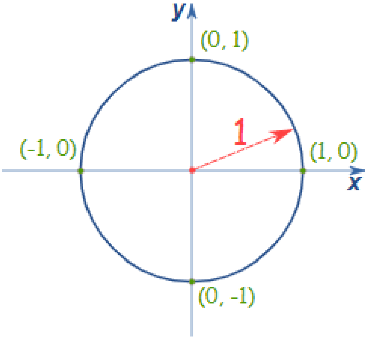
Pi (π)

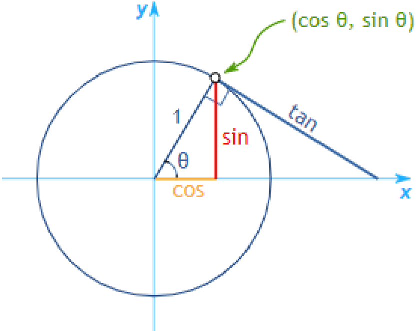


$$\frac{\text{Circumference}}{\text{Diameter}} = \pi = 3.14159\ldots$$

Unit Circle and Trigonometric Functions

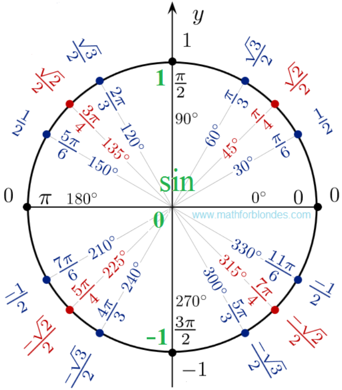
The Unit Circle is a circle with a radius of 1.

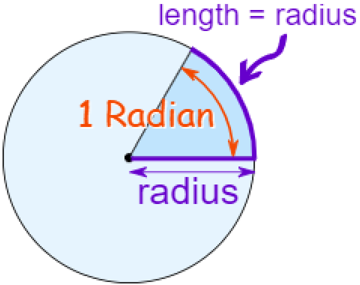




The Unit Circle can be used to map out the trigonometric values of sine, cosine, and tangent.

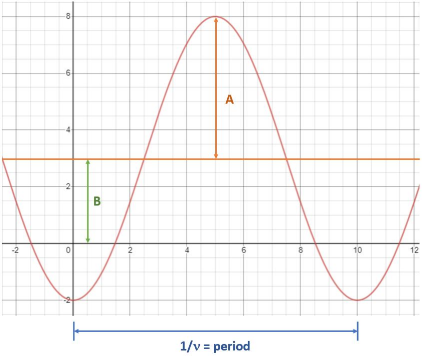
Unit Circle and the Value of $sin(\theta)$





- $sin(\theta)$ is the y-value of the point on the Unit Circle at angle θ.
- In our trig functions, θ is measured in radians (rad), not degrees.
- 360 degrees $= 2\pi$ radians.

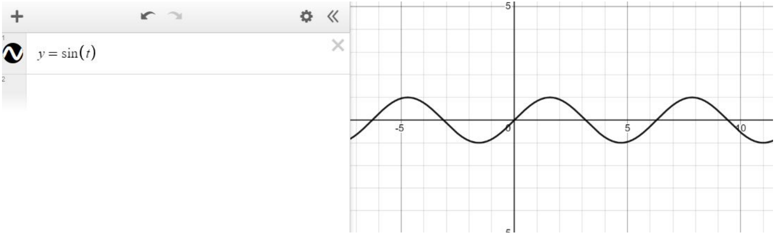
Sine Waves

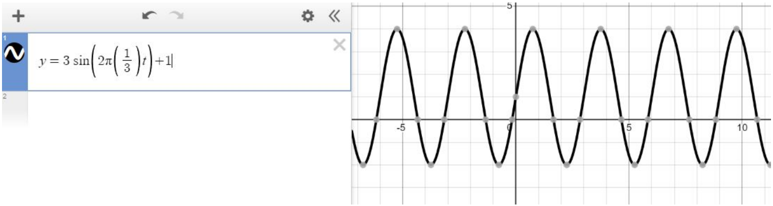


$$y = A * sin(2 * \pi * \nu * t) + B$$

where A = amplitude, B = offset, ν = frequency = $\frac{1}{period}$, and t = time in seconds.

Using Desmos (desmos.com/calculator)

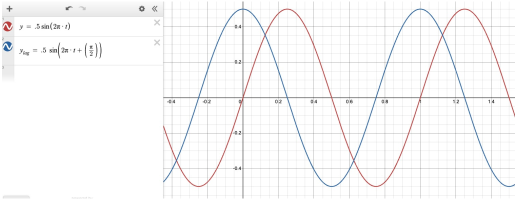




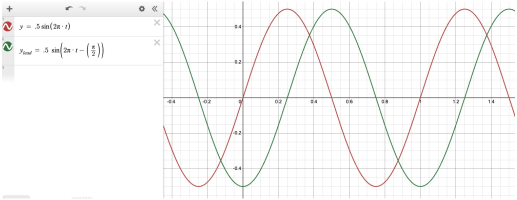
Phase Shift

The sine wave can be shifted relative to each other by adding in a phase shift (ϕ), which will shift the wave to the left or right.

Blue lags Red:

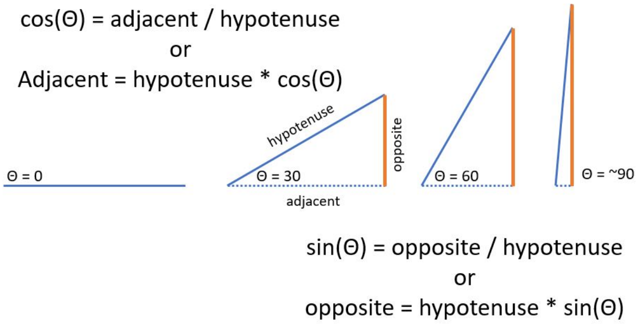


Green leads Red:



SOH CAH TOA

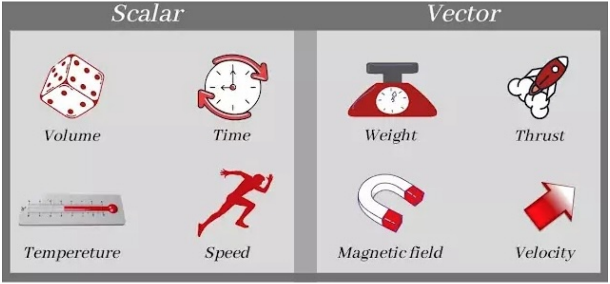
- sin = opposite over hypotenuse
- cos = adjacent over hypotenuse
- tan = opposite over adjacent



Vectors

Scalars and Vectors

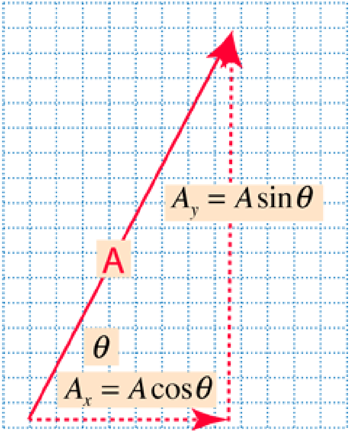
Scalars are quantities that are fully described by a magnitude (or numerical value) alone.
Vectors are quantities that are fully described by both a magnitude and a direction.



Vector Components

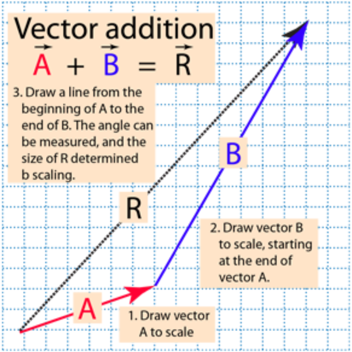
Finding the components of a vector involves forming a right triangle and using trigonometry's SOH-CAH-TOA

- $A_x = Acos(\theta)$
- $A_y = Asin(\theta)$



Graphical Vector Addition

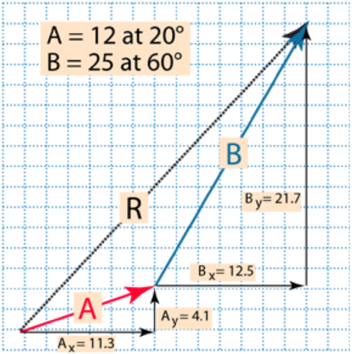
Adding two vectors A and B graphically can be visualized like two successive walks, with the vector sum being the vector distance from the beginning to the end point. Representing the vectors by arrows drawn to scale, the beginning of vector B is placed at the end of vector A. The vector sum R can be drawn as the vector from the beginning to the end point.



Vector Components

Finding the components of a vector involves forming a right triangle and using trigonometry's SOH-CAH-TOA

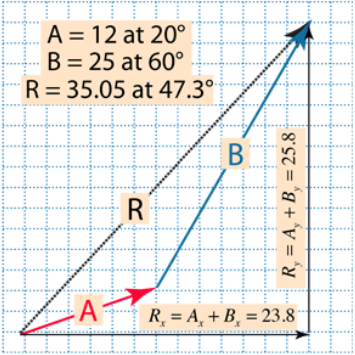
- Add the X components
  - $A_x = 12cos(20^\circ) = 11.3$
  - $B_x = 25cos(60^\circ) = 12.5$
  - $R_x = A_x + B_x = 23.8$
- Add the Y components
  - $A_y = 12sin(20^\circ) = 4.1$
  - $B_y = 25sin(60^\circ) = 21.7$
  - $R_y = A_y + B_y = 25.8$



Polar Form

After finding the components, the result can be placed in Polar Form:

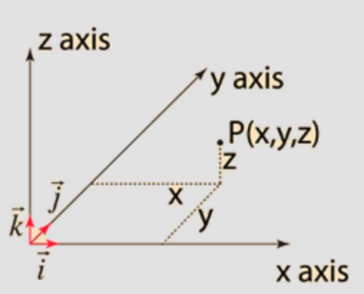
- $R_x = 23.8$
- $R_y = 25.8$
- $R = \sqrt{R_x^2 + R_y^2} = 35.05$
- $\theta_R = tan^{-1}(\frac{R-y}{R_x}) = 47.3^\circ$



Unit Vectors in 3 Dimensions

Vectors of unit length (i.e., length equals 1) can be used to specify the direction of vector quantities in various coordinate systems.
In Cartesian coordinates, it is typical to use i, j, and k to represent the unit vectors in the x, y, and z directions, respectively:

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k} \tag{8}$$



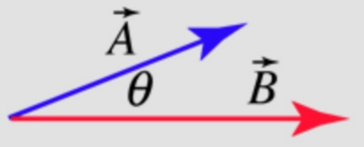
Dot (Scalar) Product

The dot (or inner or scalar) product of two vectors can be constructed by taking the component of the first vector in the direction of the second vector and by multiplying it by the second vector's magnitude.

$$\vec{A} \cdot \vec{B} = ABcos(\theta) \tag{9}$$

or

$$\vec{A} \cdot \vec{B} = A_x B_x + A_y B_y + A_z B_y \tag{10}$$

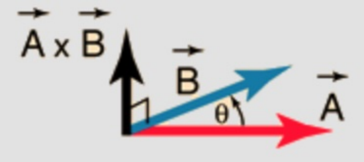


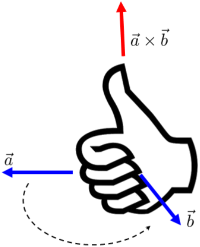
Cross (Vector) Product

The magnitude of the cross (or outer or vector) product of vectors can be constructed by taking the product of the magnitude of the vectors times the sine of the angle between them.

$$\vec{A} \times \vec{B}_{magnitude} = ABsin(\theta) \quad (11)$$

and the direction is given by the right hand rule.





In terms of unit vectors:

$$\vec{A} \times \vec{B} = \vec{i}(A_y B_z - A_z B_y) + \vec{j}(A_z B_x - A_x B_z) + \vec{k}(A_x B_y - A_y B_x) \quad (12)$$

Bonus - Cross Product - Determinant Form

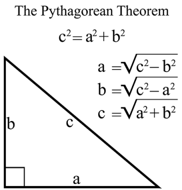
THe cross product can be compactly stated in teh form of a determinant:

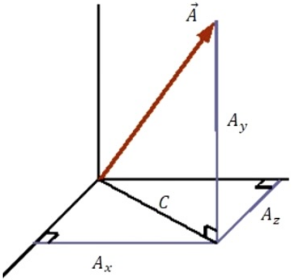
$$\vec{A} \times \vec{B} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

Which can be expanded to

$$\vec{A} \times \vec{B} = \vec{i}(A_y B_z - A_z B_y) + \vec{j}(A_z B_x - A_x B_z) + \vec{k}(A_x B_y - A_y B_x)$$

Pythagorean Theorem in 3 Dimensions



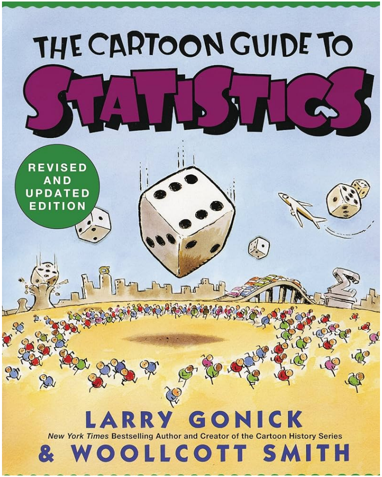


To add orthogonal (at right angles to each other) vectors in 3 Dimensions:

- $C = \sqrt{A_x^2 + A_y^2}$
- $A_{total} = \sqrt{C^2 + A_z^2}$
- $A_{total} = \sqrt{A_x^2 + A_y^2 + A_z^2}$

Probability and Statistics

Statistics



- Data Analysis: The gathering, display, and summary of data
- Probability: The laws of chance (inside and outside of a casino)
- Statistical Inference: the science of drawing statistical conclusions from specific data using the laws of probability

Summary Statistics

- Mean
- Median
- Spread
- Interquartile Range (IQR)
- Standard Deviation

Probability