VOL 1 - LINEAR ALGEBRA

BEFORE MACHINE LEARNING

A STORY ON FUNDAMENTAL MATHEMATICS FOR A.I.

Exercises

Contents

Chapter 1

Welcome

Hello, this document complements the reading of the Before Machine Learning Volume 1 - Linear Algebra. The book is structured with practical exercises followed by correspondent solutions. We also have coding exercises on a selection of chapters. The solutions for these bad boys come in jupyter notebooks that are in the same location as this document.

Feel free to do whatever you wish with this content, and if you are taking suggestions.

Before letting you dive into the exercises, I would like to list the Python libraries I used:

- numpy
- matplotlib
- pandas

That's it!

www.mldepot.co.uk

Chapter 2

The Dot Product.

2.1 Vector projections formula:

The projection of a vector $\overrightarrow{a}$ onto another vector $\overrightarrow{b}$ can be visualized as casting a shadow of $\overrightarrow{a}$ onto $\overrightarrow{b}$, where the 'light source' is perpendicular to $\overrightarrow{b}$. This projection essentially flattens $\overrightarrow{a}$ onto the line defined by $\overrightarrow{b}$.Mathematically, the projection of $\overrightarrow{a}$ onto $\overrightarrow{b}$ is given by the formula:

$$\text{proj}_{\overrightarrow{b}}\overrightarrow{a} = \left(\frac{\overrightarrow{a} \cdot \overrightarrow{b}}{\|\overrightarrow{b}\|^2}\right)\overrightarrow{b}$$

Exercises:

Exercise 1.1:

Compute the dot product of the vectors:

$$\vec{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \text{ and } \vec{v} = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$$

Exercise 1.2:

Find the angle between $\vec{a}$ and $\vec{b}$ where:

$$\vec{a} = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} \text{ and } \vec{b} = \begin{bmatrix} -2 \\ 1 \\ 4 \end{bmatrix}$$

Exercise 1.3:

Calculate the angle between $\vec{x}$ and $\vec{y}$ where:

$$\vec{x} = \begin{bmatrix} 4 \\ -1 \\ 0 \end{bmatrix} \text{ and } \vec{y} = \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix}$$

Exercise 1.4:

Find the projection of vector $\vec{p}$ onto $\vec{q}$ where:

$$\vec{p} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \text{ onto vector } \vec{q} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

Exercise 1.5:

Compute the dot product of the two parallel vectors:

$$\vec{m} = \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} \text{ and } \vec{n} = \begin{bmatrix} -4 \\ -8 \\ -12 \end{bmatrix}$$

Exercise 1.6:

Find whether $\vec{c}$ and $\vec{d}$ are orthogonal given that:

$$\vec{c} = \begin{bmatrix} 3 \\ -7 \end{bmatrix} \text{ and } \vec{d} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Exercise 1.7:

Find the angle between the vectors:

$$\vec{g} = \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix} \text{ and } \vec{h} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

Exercise 1.8:

Calculate the dot product and determine if the vectors are orthogonal, parallel, or neither:

$$\vec{f} = \begin{bmatrix} 5 \\ 0 \\ -2 \end{bmatrix} \text{ and } \vec{g} = \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}$$

Coding exercise:

Suppose we have a dataset comprising the ratings assigned by six users to six distinct films. These ratings range between 1 and 5, as depicted in the following table:

	Movie 1	**Movie 2**	**Movie 3**	**Movie 4**	**Movie 5**
User 1	5	2	1	0	3
User 2	4	1	0	2	0
User 3	3	5	0	4	5
User 4	1	0	5	3	2
User 5	0	4	2	4	1
User 6	2	3	0	1	0

- Design a function that accepts a two-dimensional array representing user ratings assigned to a collection of movies. Additionally, the function should receive a specific movie id as an argument. The task of the function is to determine and return the movie with the most similar average rating to the given movie, with the condition that the returned movie must not be the same as the input movie.

- After understanding and completing the first task, adapt the initial function so that it can now provide the top n similar items to a given item. Test the function with different movies and different n values and observe the changes in the output.

Solutions:

Exercise 1.1:

To compute the dot product of vectors:

$$\vec{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \text{ and } \vec{v} = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$$

We perform:

$$1 \cdot (-1) + 2 \cdot 0 + 3 \cdot 2 = -1 + 0 + 6 = 5$$

Exercise 1.2:

To find the angle between the vectors:

$$\vec{a} = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} \text{ and } \vec{b} = \begin{bmatrix} -2 \\ 1 \\ 4 \end{bmatrix}$$

We first find the dot product:

$$\vec{a} \cdot \vec{b} = 2 \cdot (-2) + 3 \cdot 1 + (-1) \cdot 4 = -4 + 3 - 4 = -5$$

Next, we find the magnitudes of $\vec{a}$ and $\vec{b}$:

$$\|\vec{a}\| = \sqrt{2^2 + 3^2 + (-1)^2} = \sqrt{14}$$

$$\|\vec{b}\| = \sqrt{(-2)^2 + 1^2 + 4^2} = \sqrt{21}$$

Now, the cosine of the angle θ between $\vec{a}$ and $\vec{b}$ is given by:

$$\cos\theta = \frac{\vec{a} \cdot \vec{b}}{\|\vec{a}\|\|\vec{b}\|} = \frac{-5}{\sqrt{14}\sqrt{21}} = \frac{-5}{7\sqrt{6}}$$

Finally, to find the angle θ, we use the arccosine function:

$$\theta = \arccos\left(\frac{-5}{7\sqrt{6}}\right) = 1.866 \quad \text{radians}$$

Exercise 1.3:

To find the angle between the vectors:

$$\vec{x} = \begin{bmatrix} 4 \\ -1 \\ 0 \end{bmatrix} \text{ and } \vec{y} = \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix}$$

First, the dot product:

$$\vec{x} \cdot \vec{y} = 4 \cdot 0 + (-1) \cdot 1 + 0 \cdot (-3) = -1$$

Next, the magnitudes of $\vec{x}$ and $\vec{y}$:

$$\|\vec{x}\| = \sqrt{4^2 + (-1)^2 + 0^2} = \sqrt{17}$$

$$\|\vec{y}\| = \sqrt{0^2 + 1^2 + (-3)^2} = \sqrt{10}$$

The cosine of the angle between $\vec{x}$ and $\vec{y}$:

$$\cos\theta = \frac{-1}{\sqrt{17}\sqrt{10}}$$

Finally, the angle θ is:

$$\theta = \arccos\left(\frac{-1}{\sqrt{17}\sqrt{10}}\right) = 1.647 \quad \text{radians}$$

Exercise 1.4:

To find the projection of vector:

$$\vec{p} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \text{ onto vector } \vec{q} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

We use the projection formula:

$$\text{proj}_{\vec{q}}\vec{p} = \frac{\vec{p} \cdot \vec{q}}{\|\vec{q}\|^2}\vec{q}$$

$$\text{proj}_{\vec{q}}\vec{p} = \frac{2 \cdot 1 + 1 \cdot 0 + 3 \cdot (-1)}{1^2 + 0^2 + (-1)^2}\vec{q}$$

$$\text{proj}_{\vec{q}}\vec{p} = \frac{-1}{2}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$\text{proj}_{\vec{q}}\vec{p} = \begin{bmatrix} -0.5 \\ 0 \\ 0.5 \end{bmatrix}$$

Exercise 1.5:

To find the projection of vector:

$$\vec{m} = \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} \text{ onto vector } \vec{n} = \begin{bmatrix} -4 \\ -8 \\ -12 \end{bmatrix}$$

We use the projection formula:

$$\text{proj}_{\vec{n}}\vec{m} = \frac{\vec{m} \cdot \vec{n}}{\|\vec{n}\|^2}\vec{n}$$

$$\text{proj}_{\vec{n}}\vec{m} = \frac{(2) \cdot (-4) + (4) \cdot (-8) + (6) \cdot (-12)}{(-4)^2 + (-8)^2 + (-12)^2}\vec{n}$$

$$\text{proj}_{\vec{n}}\vec{m} = \frac{-8 - 32 - 72}{16 + 64 + 144}\vec{n}$$

$$\text{proj}_{\vec{n}}\vec{m} = \frac{-112}{224}\begin{bmatrix} -4 \\ -8 \\ -12 \end{bmatrix}$$

$$\text{proj}_{\vec{n}}\vec{m} = \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}$$

Exercise 1.6:

To find the projection of vector:

$$\vec{m} = \begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix} \text{ onto vector } \vec{n} = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$$

We use the projection formula:

$$\text{proj}_{\vec{n}}\vec{m} = \frac{\vec{m}\cdot\vec{n}}{\|\vec{n}\|^2}\vec{n}$$

$$\text{proj}_{\vec{n}}\vec{m} = \frac{(-1)\cdot 2 + 3\cdot 0 + 0\cdot(-1)}{2^2+0^2+(-1)^2}\vec{n}$$

$$\text{proj}_{\vec{n}}\vec{m} = \frac{-2}{5}\begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$$

$$\text{proj}_{\vec{n}}\vec{m} = \begin{bmatrix} -0.8 \\ 0 \\ 0.4 \end{bmatrix}$$

Exercise 1.7:

To find the angle between the vectors:

$$\vec{g} = \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix} \text{ and } \vec{h} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

First, the dot product:

$$\vec{g}\cdot\vec{h} = 3\cdot 1 + 0\cdot(-2) + (-1)\cdot 1 = 2$$

Next, the magnitudes of $\vec{g}$ and $\vec{h}$:

$$\|\vec{g}\| = \sqrt{3^2+0^2+(-1)^2} = \sqrt{10}$$

$$\|\vec{h}\| = \sqrt{1^2+(-2)^2+1^2} = \sqrt{6}$$

The cosine of the angle between $\vec{g}$ and $\vec{h}$:

$$\cos\theta = \frac{2}{\sqrt{10}\sqrt{6}} = \frac{1}{\sqrt{15}}$$

Finally, the angle θ is:

$$\theta = \arccos\left(\frac{1}{\sqrt{15}}\right) = 0.258 \quad \text{radians}$$

Exercise 1.8:

To find the projection of vector:

$$\vec{w} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \text{ onto vector } \vec{z} = \begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix}$$

We use the projection formula:

$$\text{proj}_{\vec{z}}\vec{w} = \frac{\vec{w} \cdot \vec{z}}{||\vec{z}||^2}\vec{z}$$

$$\text{proj}_{\vec{z}}\vec{w} = \frac{1 \cdot 0 + 2 \cdot (-1) + 1 \cdot 3}{0^2 + (-1)^2 + 3^2}\vec{z}$$

$$\text{proj}_{\vec{z}}\vec{w} = \frac{1}{10}\begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix}$$

$$\text{proj}_{\vec{z}}\vec{w} = \begin{bmatrix} 0 \\ -0.1 \\ 0.3 \end{bmatrix}$$

Chapter 3

Basis, Linear Combinations, Linear Independence, Rank and Change of Basis.

3.1 Exercises:

Exercise 2.1:

Determine whether the following set of vectors forms a basis for $\mathbb{R}^3$:

$$\vec{u} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \vec{v} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \vec{w} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

Exercise 2.2:

Express the vector $\vec{a} = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}$ as a linear combination of the vectors:

$$\vec{b} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \vec{c} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \vec{d} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$$

Exercise 2.3:

For the following set of vectors, determine whether they are linearly independent:

$$\vec{e} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \vec{f} = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}, \vec{g} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

Exercise 2.4:

Find a basis for the span of the following vectors:

$$\vec{h} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \vec{i} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \vec{j} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \vec{k} = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

Exercise 2.5:

Check if the vector $\vec{l} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$ belongs to the span of:

$$\vec{m} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \vec{n} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

Exercise 2.6:

If possible, express $\vec{o} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ as a linear combination of:

$$\vec{p} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}, \vec{q} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Exercise 2.7:

Determine whether the following set of vectors forms a basis for $\mathbb{R}^2$:

$$\vec{r} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \vec{s} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Exercise 2.8:

Find a basis for the span of the following vectors in $\mathbb{R}^2$:

$$\vec{t} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}, \vec{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \vec{v} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

Exercise 2.9:

For the vectors below, find the coefficients for the linear combination (if it exists) that results in the zero vector:

$$\vec{w} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \vec{x} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \vec{y} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Exercise 2.10:

Check if the vector $\vec{z} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ belongs to the span of

$$\vec{\alpha} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \vec{\beta} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

Solutions:

Exercise 2.1:

To confirm that the vectors form a basis for $\mathbb{R}^3$, we need to verify if they are linearly independent and span $\mathbb{R}^3$. Let's start with the linear independence, and for that, we will compute the determinant of a matrix A formed by the vectors:

$$\vec{u} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \vec{v} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \vec{w} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix},$$

the matrix A is formed by placing $\vec{u}$, $\vec{v}$, and $\vec{w}$ as its columns:

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ -1 & 1 & 0 \end{bmatrix}$$

The determinant of A is calculated as follows:

$$\begin{aligned}\det(A) &= 1\begin{vmatrix}1 & 1\\ 1 & 0\end{vmatrix} - 0\begin{vmatrix}0 & 2\\ -1 & 0\end{vmatrix} + 2\begin{vmatrix}0 & 1\\ -1 & 1\end{vmatrix}\\ &= 1(1\cdot 0 - 1\cdot 1) - 0(0\cdot 0 - 2\cdot -1) + 2(0\cdot 1 - -1\cdot 1)\\ &= -1 + 0 + 2\\ &= 1\end{aligned}$$

The linear independence of the vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$ has been established by showing that the determinant of the matrix A is non-zero. To form a basis, these vectors must also span $\mathbb{R}^3$.

A set of vectors spans $\mathbb{R}^3$ if any vector $\vec{x} \in \mathbb{R}^3$ can be expressed as a linear combination of the set. Since our vectors are linearly independent and we have three vectors, which is the same as the dimension of $\mathbb{R}^3$, they span the space. This is because in a three-dimensional space, any set of three linearly independent vectors will span the entire space.

Thus, the vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$ not only are linearly independent but also span $\mathbb{R}^3$. Consequently, they form a basis for $\mathbb{R}^3$.

Exercise 2.2:

To express the vector $\vec{a}$ as a linear combination of the vectors $\vec{b}$, $\vec{c}$, and $\vec{d}$, we need to find scalars x, y, and z such that:

$$\vec{a} = x\vec{b} + y\vec{c} + z\vec{d}$$

Substitute the given vectors into the equation:

$$\begin{bmatrix}3\\ -1\\ 2\end{bmatrix} = x\begin{bmatrix}1\\ 0\\ 1\end{bmatrix} + y\begin{bmatrix}0\\ 1\\ -1\end{bmatrix} + z\begin{bmatrix}2\\ -1\\ 0\end{bmatrix}$$

This results in a system of linear equations:

$$\begin{cases} x + 2z = 3\\ y - z = -1\\ x - y = 2\end{cases}$$

We solve this system step by step:

1. From the third equation, we express x in terms of y:

$$x = y + 2$$

2. We substitute x from the third equation into the first one:

$$y + 2 + 2z = 3$$

3. Now we solve for z using the second and modified first equation:

$$y - z = -1 \quad \Rightarrow \quad z = y + 1$$

4. Substituting z back into the modified first equation gives us:

$$y + 2 + 2(y + 1) = 3$$

5. Solving the above equation for y:

$$3y + 4 = 3 \quad \Rightarrow \quad y = -\frac{1}{3}$$

6. Now we can find z using y:

$$z = y + 1 = -\frac{1}{3} + 1 = \frac{2}{3}$$

7. Finally, we can find x using y:

$$x = y + 2 = -\frac{1}{3} + 2 = \frac{5}{3}$$

Therefore, the vector $\vec{a}$ can be written as:

$$\vec{a} = \frac{5}{3}\vec{b} - \frac{1}{3}\vec{c} + \frac{2}{3}\vec{d}$$

Exercise 2.3:

To determine whether the set of vectors $\vec{e}$, $\vec{f}$, and $\vec{g}$ are linearly independent, we form a matrix with these vectors as columns and calculate its determinant:

$$\vec{e} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \vec{f} = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}, \vec{g} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

The matrix formed by vectors $\vec{e}$, $\vec{f}$, and $\vec{g}$ is:

$$M = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 0 & -1 \\ 3 & 2 & 1 \end{bmatrix}$$

The determinant of matrix M is calculated as follows:

$$\det(M) = \begin{vmatrix} 1 & -1 & 2 \\ 2 & 0 & -1 \\ 3 & 2 & 1 \end{vmatrix}$$

To determine if the vectors are linearly independent, the determinant should not be zero. After calculating, we find that:

$$\det(M) = 15$$

Since the determinant is non-zero, the vectors $\vec{e}$, $\vec{f}$, and $\vec{g}$ are linearly independent.

Exercise 2.4:

The rank of a matrix is determined by the maximum number of linearly independent columns (or rows) it has. For our matrix M, formed by vectors $\vec{h}$, $\vec{i}$, $\vec{j}$, and $\vec{k}$, we need to examine if any column can be expressed as a linear combination of the others, indicating linear dependence.

Given matrix M:

$$M = \begin{bmatrix} 1 & 0 & 1 & -1 \\ 1 & 1 & 0 & -1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

We observe that the fourth column $(-1, -1, 1)^T$ can be expressed as a linear combination of the first three columns. Specifically:

$$\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

This means that the fourth vector $\vec{k}$ does not contribute any new direction not already spanned by the vectors $\vec{h}$, $\vec{i}$, and $\vec{j}$. Hence, the maximum number of linearly independent columns in M is 3, making its rank equal to 3.

Therefore, the first three vectors $\vec{h}$, $\vec{i}$, and $\vec{j}$ form a basis for the span of the original set of vectors, as they are linearly independent and span the same space that includes $\vec{k}$.

Exercise 2.5:

To check if vector $\vec{l}$ belongs to the span of vectors $\vec{m}$ and $\vec{n}$, we need to determine whether $\vec{l}$ can be expressed as a linear combination of $\vec{m}$ and $\vec{n}$. This is done by forming an augmented matrix with $\vec{m}$ and $\vec{n}$ as the first two columns and $\vec{l}$ as the third column.

Given vectors:

$$\vec{l} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}, \quad \vec{m} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \quad \vec{n} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

We form the augmented matrix A:

$$A = \begin{bmatrix} 1 & 0 & 2 \\ -1 & 1 & 1 \\ 0 & -1 & -1 \end{bmatrix}$$

Next, we compare the rank of the matrix formed by $\vec{m}$ and $\vec{n}$ alone with the rank of the augmented matrix A. The rank of a matrix is the number of linearly independent rows (or columns) in it.

Calculating the ranks:

$$\text{Rank}(\begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 0 & -1 \end{bmatrix}) \text{ and } \text{Rank}(A)$$

After performing row operations to determine the rank, we find that the rank of the matrix formed by $\vec{m}$ and $\vec{n}$ alone is 2. However, the rank of the augmented matrix A is 3. The difference in ranks indicates that $\vec{l}$ adds a new dimension to the span of $\vec{m}$ and $\vec{n}$, implying that $\vec{l}$ cannot be expressed as a linear combination of $\vec{m}$ and $\vec{n}$.

Therefore, since the ranks are not equal, $\vec{l}$ does not belong to the span of $\vec{m}$ and $\vec{n}$.

Exercise 2.6:

We want to express the vector $\vec{o}$ as a linear combination of vectors $\vec{p}$ and $\vec{q}$:

$$\vec{o} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \quad \vec{p} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}, \quad \vec{q} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

We set up the equation:

$$\vec{o} = \alpha \cdot \vec{p} + \beta \cdot \vec{q}$$

In matrix form, this equation is:

$$\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = \alpha \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} + \beta \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Which leads to the system of equations:

$$\begin{cases} 2\alpha - \beta = 1 \\ \alpha = -2 \\ -\alpha + \beta = 1 \end{cases}$$

Upon attempting to solve this system, we find that there is no solution for α and β that satisfies all three equations. This means that $\vec{o}$ cannot be expressed as a linear combination of $\vec{p}$ and $\vec{q}$, and thus it does not belong to their span.

Exercise 2.7:

We want to determine if the set of vectors $\vec{r}$ and $\vec{s}$ forms a basis for $\mathbb{R}^2$:

$$\vec{r} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \vec{s} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

We form a matrix with these vectors as columns:

$$M = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}$$

The set of vectors $\vec{r}$ and $\vec{s}$ will form a basis for $\mathbb{R}^2$ if they are linearly independent and span $\mathbb{R}^2$. Linear independence is equivalent to the matrix M having a rank of 2, which is the dimension of $\mathbb{R}^2$. Spanning $\mathbb{R}^2$ means any vector in $\mathbb{R}^2$ can be expressed as a linear combination of $\vec{r}$ and $\vec{s}$.

After calculating, the rank of matrix M is found to be 2. This indicates that $\vec{r}$ and $\vec{s}$ are linearly independent. Furthermore, as the rank of M equals the dimension of $\mathbb{R}^2$, it also implies that these vectors span the entire space of $\mathbb{R}^2$.

Therefore, the vectors $\vec{r}$ and $\vec{s}$ are not only linearly independent but also span $\mathbb{R}^2$, thereby forming a basis for $\mathbb{R}^2$.

Exercise 2.8:

To find a basis for the span of the vectors $\vec{t}$, $\vec{u}$, and $\vec{v}$ in $\mathbb{R}^2$, we consider the matrix formed by these vectors as columns:

$$\vec{t} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}, \quad \vec{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \vec{v} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

The matrix with these vectors as columns is:

$$M = \begin{bmatrix} 3 & 1 & -2 \\ -1 & 2 & 1 \end{bmatrix}$$

Since we are in $\mathbb{R}^2$, we need exactly two linearly independent vectors to form a basis for the span. We calculate the rank of matrix M and find that it is 2, indicating that there are two linearly independent vectors in the set.

Upon examining the matrix, we can choose $\vec{t}$ and $\vec{u}$ as the basis vectors because they are linearly independent:

$$\text{Basis} = \left\{ \vec{t} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}, \vec{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$$

These vectors span the same space as the original set of vectors, including $\vec{v}$, in $\mathbb{R}^2$.

Exercise 2.9:

Given the vectors:

$$\vec{w} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \quad \vec{x} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \quad \vec{y} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

We want to find coefficients a, b, and c such that $a\vec{w} + b\vec{x} + c\vec{y} = \vec{0}$. Substituting the vectors into the equation, we get:

$$a \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + c \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Expanding this equation, we form a system of linear equations:

$$\begin{cases} a - c = 0 \\ -a + b = 0 \\ -b + c = 0 \end{cases}$$

Solving this system:

1. From the first equation, $a - c = 0$, we find $a = c$.
2. From the second equation, $-a + b = 0$, substituting $a = c$ gives us $b = c$.
3. The third equation, $-b + c = 0$, is already satisfied since $b = c$.

Thus, the solution to this system is $a = b = c$. This means any scalar multiple of the vector $[1, 1, 1]$ will satisfy the equation, leading to infinitely many solutions where a, b, and c are all equal.

Therefore, there are infinitely many solutions to this system, and all solutions are scalar multiples of the vector $[1, 1, 1]$, indicating that the vectors $\vec{w}$, $\vec{x}$, and $\vec{y}$ are linearly dependent.

Exercise 2.10:

To check if the vector $\vec{z}$ belongs to the span of vectors $\vec{\alpha}$ and $\vec{\beta}$, we form an augmented matrix with $\vec{\alpha}$ and $\vec{\beta}$ as the first two columns and $\vec{z}$ as the third column:

$$\vec{z} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \quad \vec{\alpha} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad \vec{\beta} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

The augmented matrix is:

$$A = \begin{bmatrix} 2 & -1 & 1 \\ 1 & -1 & 3 \end{bmatrix}$$

We then compare the rank of matrix A with the rank of the matrix formed by just $\vec{\alpha}$ and $\vec{\beta}$:

$$\text{Rank}(\begin{bmatrix} 2 & -1 \\ 1 & -1 \end{bmatrix}) = \text{Rank}(A)$$

In this case, the rank of both matrices is 2. Since the ranks are equal, $\vec{z}$ belongs to the span of $\vec{\alpha}$ and $\vec{\beta}$.

Chapter 4

Change of Basis Exercises:

4.1 Exercises:

Exercise 3.1:

You are given two basis sets in $\mathbb{R}^3$:

- Basis A:
$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$
- Basis B:
$$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Find the change of basis matrix from Basis A to Basis B and vice versa.

Exercise 3.2:

Given a vector $v = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$ in the standard basis, express it in the Basis B from the previous exercise.

Exercise 3.3:

Consider a new Basis C:

$$\begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$$

Find the change of basis matrix from Basis A to Basis C and represent vector v in Basis C.

Exercise 3.4:

Consider two bases in $\mathbb{R}^2$:

- Basis D:

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

- Basis E:

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Find the change of basis matrix from Basis D to Basis E and vice versa.

Exercise 3.5:

Given a vector $u = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$ in the standard basis, express it in the Basis E from the previous exercise.

Coding Exercise:

In our movie scenario, consider each movie is represented in a two-dimensional genre space where each dimension corresponds to a different genre, e.g., Action and Drama. The movies are represented as vectors in this space, where each vector represents the degree to which a movie belongs to the respective genres.

Provided Data:

- Movies are represented by 2D vectors:

$$\text{Movie\_1} = [2, 3]$$

$$\text{Movie\_2} = [1, 1]$$

- Transformation Matrix (Genre Basis to User Preference Basis):

$$T = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}$$

In this movie scenario, the transformation matrix T is hypothesized as the change of basis from the Genre Basis to a User Preference Basis. The two columns of matrix T represent two new basis vectors in the user preference space, each expressing a user's preference as a linear combination of the original genre vectors.

1. Use the provided transformation matrix to perform a basis change from the genre basis to the user preference basis.
2. Plot the original movie vectors and the transformed movie vectors on the same graph and use different colors or markers to distinguish between the original and transformed vectors.
3. Check the linear independence of the transformed movie vectors in the new basis and interpret the results in the context of user preferences.
4. How do the changes in bases affect the representation of movies in terms of genres and user preferences?

Solutions:

Exercise 3.1:

Given two basis sets in $\mathbb{R}^3$:

- Basis A:

$$\vec{a}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \vec{a}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \vec{a}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

- Basis B:

$$\vec{b}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \vec{b}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \vec{b}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

To find the change of basis matrix from Basis A to Basis B ($P_{A\to B}$), we express each vector of Basis B as a linear combination of the vectors in Basis A. The coefficients of this linear combination form the columns of the change of basis matrix $P_{A\to B}$.

The linear combinations are:

$$\begin{aligned}\vec{b}_1 &= 1\vec{a}_1 + 1\vec{a}_2 + 0\vec{a}_3\\ \vec{b}_2 &= -1\vec{a}_1 + 1\vec{a}_2 + 0\vec{a}_3\\ \vec{b}_3 &= 0\vec{a}_1 + 0\vec{a}_2 + 1\vec{a}_3\end{aligned}$$

Therefore, the change of basis matrix $P_{A\to B}$ is:

$$P_{A\to B} = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Next, to find the change of basis matrix from Basis B to Basis A ($P_{B\to A}$), we need to compute the inverse of $P_{A\to B}$.

The inverse of a matrix is calculated as follows:

$$\begin{aligned}P_{B\to A} &= (P_{A\to B})^{-1}\\ &= \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1}\end{aligned}$$

The inverse of $P_{A\to B}$ can be found using elementary row operations or a matrix calculator. After performing these calculations, we find:

$$P_{B\to A} = \begin{bmatrix} 0.5 & -0.5 & 0 \\ 0.5 & 0.5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Thus, we have found the change of basis matrices in both directions.

Exercise 3.2:

Given a vector v in the standard basis:

$$v = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$$

We can express it in Basis B using the change of basis matrix $P_{B\to A}$ from the previous exercise:

$$v_B = P_{B\to A} \cdot v = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ -\frac{3}{2} \\ 3 \end{bmatrix}$$

Thus, the vector v in Basis B is:

$$v_B = \begin{bmatrix} \frac{1}{2} \\ -\frac{3}{2} \\ 3 \end{bmatrix}$$

This representation means that v is equal to $\frac{1}{2}$ times the first vector of Basis B, minus $\frac{3}{2}$ times the second vector of Basis B, plus 3 times the third vector of Basis B.

Exercise 3.3:

Given the new Basis C:

$$\begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$$

The change of basis matrix from Basis A to Basis C is:

$$P_{A\to C} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

To express the vector v in Basis C, we calculate:

$$v_C = P_{A\to C}^{-1} \cdot v = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ -\frac{1}{2} \\ \frac{3}{2} \end{bmatrix}$$

Therefore, the vector v in Basis C is:

$$v_C = \begin{bmatrix} 1 \\ -\frac{1}{2} \\ \frac{3}{2} \end{bmatrix}$$

This representation means that v is equal to 1 times the first vector of Basis C, minus $\frac{1}{2}$ times the second vector, and $\frac{3}{2}$ times the third vector of Basis C.

Exercise 3.4:

Consider two bases in $\mathbb{R}^2$:

- Basis D:
$$\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}$$
- Basis E:
$$\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}2\\1\end{bmatrix}$$

Find the change of basis matrix from Basis D to Basis E and vice versa.

The change of basis matrix from Basis D to Basis E ($P_{D\to E}$) is:

$$P_{D\to E} = \begin{bmatrix}1 & 2\\2 & 1\end{bmatrix}$$

The change of basis matrix from Basis E to Basis D ($P_{E\to D}$) is:

$$P_{E\to D} = \begin{bmatrix}-\frac{1}{3} & \frac{2}{3}\\ \frac{2}{3} & -\frac{1}{3}\end{bmatrix}$$

Exercise 3.5:

Given a vector u in the standard basis D:

$$u = \begin{bmatrix}3\\-2\end{bmatrix}$$

We aim to express u in terms of Basis E, as defined in a previous exercise. Recall that the change of basis matrix from Basis E to Basis D ($P_{E\to D}$) is:

$$P_{E\to D} = \begin{bmatrix}-\frac{1}{3} & \frac{2}{3}\\ \frac{2}{3} & -\frac{1}{3}\end{bmatrix}$$

To express u in Basis E, we multiply u by $P_{E\to D}$:

$$u_{\text{in } E} = P_{E\to D} \cdot u = \begin{bmatrix}-\frac{1}{3} & \frac{2}{3}\\ \frac{2}{3} & -\frac{1}{3}\end{bmatrix} \cdot \begin{bmatrix}3\\-2\end{bmatrix}$$

Performing the multiplication, we obtain:

$$u_{\text{in } E} = \begin{bmatrix} -\frac{7}{3} \\ \frac{8}{3} \end{bmatrix}$$

Therefore, the vector u expressed in terms of Basis E is $\begin{bmatrix} -\frac{7}{3} \\ \frac{8}{3} \end{bmatrix}$.

Chapter 5

Linear Transformations:

5.1 Exercises:

Exercise 4.1:

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation represented by the matrix:

$$A = \begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix}$$

Find the image of the vector $\vec{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ under the transformation T.

Exercise 4.2:

Given the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$ defined by the matrix:

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & -1 & 3 \end{bmatrix}$$

Find the image of $\vec{v} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$.

Exercise 4.3:

Suppose we have a reflection linear transformation in $\mathbb{R}^2$ represented by the matrix:

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Compute the image of the vector $\vec{v} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$.

Exercise 4.4:

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation corresponding to a 90-degree clockwise rotation. Find the transformation matrix and use it to find the image of the vector $\vec{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

Exercise 4.5:

Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a scaling transformation represented by the matrix:

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Find the image of the vector $\vec{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

5.2 Solutions:

Exercise 4.1:

Given the matrix:

$$A = \begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix}$$

and the vector:

$$\vec{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

To find the image under the transformation, we perform matrix-vector multiplication:

$$\vec{w} = A\vec{v} = \begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 10 \end{bmatrix}$$

Exercise 4.2:

Given the matrix:

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & -1 & 3 \end{bmatrix}$$

and the vector:

$$\vec{v} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

To find the image under the transformation, we perform matrix-vector multiplication:

$$\vec{w} = A\vec{v} = \begin{bmatrix} 2 & 1 & 0 \\ 0 & -1 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

This calculation correctly identifies the image of the vector $\vec{v}$ under the transformation defined by matrix A.

Exercise 4.3:

Given the matrix:

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

and the vector:

$$\vec{v} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

To find the image under the transformation, we perform matrix-vector multiplication:

$$\vec{w} = A\vec{v} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

Exercise 4.4:

A 90-degree clockwise rotation in $\mathbb{R}^2$ can be represented by the matrix:

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

Given the vector:

$$\vec{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

To find the image under the transformation, we perform matrix-vector multiplication:

$$\vec{w} = A\vec{v} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

Exercise 4.5:

Given the matrix:

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

And the vector:

$$\vec{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

To find the image under the transformation, we perform matrix-vector multiplication:

$$\vec{w} = A\vec{v} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \\ 3 \end{bmatrix}$$

Chapter 6

Matrix multiplication:

6.1 Exercises:

Exercise 5.1:

Multiply the following matrices:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 0 \\ 1 & 2 \end{bmatrix}$$

Exercise 5.2:

Multiply the following matrices:

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Exercise 5.3:

Calculate the product of the following matrices:

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 1 \\ 2 & 1 \\ 1 & 0 \end{bmatrix}$$

Exercise 5.4:

Find the product of the given matrices:

$$A = \begin{bmatrix} 2 & -1 & 0 \\ 1 & 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 \\ 0 & -1 \\ -1 & 1 \end{bmatrix}$$

Coding exercises:

1. Given a simple 3×3 image represented as a matrix, where each cell represents a pixel's intensity, and a 3×3 transformation matrix, perform matrix multiplication to transform the image.

   Matrix A (Image Matrix):

   $$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

   Transformation Matrix B:

   $$B = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

   What is the resulting image matrix after applying the transformation matrix B to image matrix A?

2. From using ASCII values and arrange it in a matrix where each number represents a character. Multiply the message matrix by the key matrix to get the encrypted message matrix.

   Message Matrix E (ASCII Values of 'ABC'):

   $$E = \begin{bmatrix} 65 & 66 & 67 \end{bmatrix}$$

Key Matrix F:

$$F = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

What is the encrypted message matrix when message matrix E is multiplied by the key matrix F?

Given a plain text message and a key matrix, encode the message using matrix multiplication. Convert the message to numerical

Solutions:

Exercise 5.1:

Using this rule for our given matrices:

$$AB = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} -1 & 0 \\ 1 & 2 \end{bmatrix}$$

$$AB = \begin{bmatrix} (1 \cdot (-1)) + (2 \cdot 1) & (1 \cdot 0) + (2 \cdot 2) \\ (3 \cdot (-1)) + (4 \cdot 1) & (3 \cdot 0) + (4 \cdot 2) \end{bmatrix}$$

$$AB = \begin{bmatrix} 1 & 4 \\ 1 & 8 \end{bmatrix}$$

Exercise 5.2:

$$AB = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$AB = \begin{bmatrix} (0 \cdot 1) + (1 \cdot 0) & (0 \cdot 0) + (1 \cdot (-1)) \\ ((-1) \cdot 1) + (0 \cdot 0) & ((-1) \cdot 0) + (0 \cdot (-1)) \end{bmatrix}$$

$$AB = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

Exercise 5.3:

$$AB = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \end{bmatrix} \cdot \begin{bmatrix} 3 & 1 \\ 2 & 1 \\ 1 & 0 \end{bmatrix}$$

$$AB = \begin{bmatrix} (1 \cdot 3) + (2 \cdot 2) + (1 \cdot 1) & (1 \cdot 1) + (2 \cdot 1) + (1 \cdot 0) \\ (0 \cdot 3) + (1 \cdot 2) + ((-1) \cdot 1) & (0 \cdot 1) + (1 \cdot 1) + ((-1) \cdot 0) \end{bmatrix}$$

$$AB = \begin{bmatrix} 8 & 3 \\ 1 & 1 \end{bmatrix}$$

Exercise 5.4:

$$AB = \begin{bmatrix} 2 & -1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 0 & -1 \\ -1 & 1 \end{bmatrix}$$

$$AB = \begin{bmatrix} (2 \cdot 1) + ((-1) \cdot 0) + (0 \cdot (-1)) & (2 \cdot 1) + ((-1) \cdot (-1)) + (0 \cdot 1) \\ (1 \cdot 1) + (1 \cdot 0) + (1 \cdot (-1)) & (1 \cdot 1) + (1 \cdot (-1)) + (1 \cdot 1) \end{bmatrix}$$

$$AB = \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}$$

Chapter 7

The Determinant and Inverse of Matrices

Exercises:

Exercise 6.1:

Compute the determinant of the matrix:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

Exercise 6.2:

Compute the inverse of the matrix:

$$B = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

Exercise 6.3:

Given that the matrix:

$$C = \begin{bmatrix} 3 & 4 \\ 2 & 1 \end{bmatrix}$$

has an inverse, find $C^{-1} \times C$.

Exercise 6.4:

Find the determinant of the matrix:

$$D = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix}$$

Exercise 6.5:

If the matrix:

$$E = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix}$$

is invertible, compute its inverse.

Exercise 6.6:

Evaluate the determinant and determine whether the inverse exists for the matrix:

$$F = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

Exercise 6.7:

Compute both the determinant and the inverse of the matrix:

$$G = \begin{bmatrix} 1 & 0 & -1 \\ 2 & -1 & 0 \\ -1 & 2 & 1 \end{bmatrix}$$

Exercise 6.8:

If possible, find the inverse of the following matrix and validate it by multiplication:

$$H = \begin{bmatrix} 3 & 0 & 1 \\ -1 & 1 & 2 \\ 2 & 1 & 1 \end{bmatrix}$$

Coding exercises:

Given the matrix:

$$G = \begin{bmatrix} 9 & 2 \\ 3 & 8 \end{bmatrix}$$

1. Find the determinant of G.
2. If G is invertible, find its inverse. Interpret the significance of the inverse matrix in linear algebra.

Solutions:

Exercise 6.1:

For matrix A:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

Using the formula:

$$\det(A) = (1 \cdot 4) - (2 \cdot 3)$$

$$\det(A) = 4 - 6$$

$$\det(A) = -2$$

Hence, the determinant of matrix A is -2.

Exercise 6.2:

For matrix B:

$$B = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

The determinant $\det(B) = (2 \cdot 1) - (1 \cdot 1) = 2 - 1 = 1$.

Since the determinant is not zero, the inverse exists. Applying the formula we get:

$$B^{-1} = \frac{1}{1} \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$$

$$B^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$$

Thus, the inverse of matrix B is:

$$\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$$

Exercise 6.3:

To find the inverse of a matrix C, first, we compute the determinant:

$$\det(C) = (3 \cdot 1) - (4 \cdot 2) = 3 - 8 = -5$$

Since the determinant is non-zero, the inverse exists. We use the formula for the inverse of a 2×2 matrix, given as:

$$C^{-1} = \frac{1}{\det(C)} \cdot \begin{bmatrix} 1 & -4 \\ -2 & 3 \end{bmatrix}$$

$$C^{-1} = -\frac{1}{5} \cdot \begin{bmatrix} 1 & -4 \\ -2 & 3 \end{bmatrix}$$

$$C^{-1} = \begin{bmatrix} -\frac{1}{5} & \frac{4}{5} \\ \frac{2}{5} & -\frac{3}{5} \end{bmatrix}$$

Now, to verify the result, we calculate:

$$C^{-1} \cdot C = \begin{bmatrix} -\frac{1}{5} & \frac{4}{5} \\ \frac{2}{5} & -\frac{3}{5} \end{bmatrix} \cdot \begin{bmatrix} 3 & 4 \\ 2 & 1 \end{bmatrix}$$

$$C^{-1} \cdot C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

So, the product of C^{-1} and C is the identity matrix I.

Exercise 6.4:

For matrix D:

$$D = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix}$$

Applying the formula, we have:

$$\det(D) = 0 \cdot (0 \cdot 0 - 1 \cdot 1) - 1 \cdot (1 \cdot 0 - 1 \cdot 2) + 2 \cdot (1 \cdot 1 - 2 \cdot 0)$$

$$\det(D) = 0 - (-2) + 2$$

$$\det(D) = 4$$

Thus, the determinant of matrix D is 4.

Exercise 6.5:

To find the inverse of a 3×3 matrix E, we first find the determinant of E to check if the inverse exists. If $\det(E) \neq 0$, then the matrix is invertible, and we proceed to find the adjugate of E and multiply it by the reciprocal of the determinant.

The determinant of a 3×3 matrix is given by:

$$\det(E) = e_{11}(e_{22}e_{33} - e_{32}e_{23}) - e_{12}(e_{21}e_{33} - e_{31}e_{23}) + e_{13}(e_{21}e_{32} - e_{31}e_{22})$$

Assuming matrix E is given as:

$$E = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 5 & 6 & 0 \end{bmatrix}$$

Applying the determinant formula, we have:

$$\det(E) = 1(1 \cdot 0 - 4 \cdot 6) - 2(0 \cdot 0 - 4 \cdot 5) + 3(0 \cdot 6 - 1 \cdot 5)$$

$$\det(E) = -24 + 40 - 15$$
$$\det(E) = 1$$

Since $\det(E) \neq 0$, the inverse of E exists. Next, we find the cofactor matrix of E and then the adjugate, and finally multiply by the reciprocal of the determinant.

Exercise 6.6:

For matrix F:

$$F = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

Applying the formula, we have:

$$\det(F) = 2 \cdot (2 \cdot 2 - (-1) \cdot (-1)) - (-1) \cdot (-1 \cdot 2 - 0 \cdot 0) + 0 \cdot (-1 \cdot -1 - (-1) \cdot 2)$$

$$\det(F) = 2 \cdot (4 - 1) - (-1) \cdot (-2) + 0$$

$$\det(F) = 2 \cdot 3 - 2$$
$$\det(F) = 6 - 2$$
$$\det(F) = 4$$

Thus, the determinant of matrix F is 4, and since the determinant is non-zero, matrix F is invertible.

Exercise 6.7:

Consider the matrix:

$$G = \begin{bmatrix} 1 & 0 & -1 \\ 2 & -1 & 0 \\ -1 & 2 & 1 \end{bmatrix}$$

To compute the determinant of G, we apply the rule of Sarrus or the cofactor expansion:

$$\det(G) = 1 \begin{vmatrix} -1 & 0 \\ 2 & 1 \end{vmatrix} - 0 \begin{vmatrix} 2 & 0 \\ -1 & 1 \end{vmatrix} - 1 \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix}$$

$$\det(G) = 1((-1) \cdot 1 - 2 \cdot 0) - 0 + (-1)(2 \cdot 2 - (-1) \cdot (-1))$$
$$\det(G) = -1 - 0 - (4 - 1)$$
$$\det(G) = -1 - 3$$
$$\det(G) = -4$$

Since the determinant is non-zero, matrix G is invertible. The inverse of G is given by:

$$G^{-1} = \begin{bmatrix} \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ -\frac{3}{4} & \frac{1}{2} & \frac{1}{4} \end{bmatrix}$$

Thus, the determinant of matrix G is -4 and its inverse is as above.

Exercise 6.8:

Consider the matrix:

$$H = \begin{bmatrix} 3 & 0 & 1 \\ -1 & 1 & 2 \\ 2 & 1 & 1 \end{bmatrix}$$

To find the inverse of H, we compute it as follows:

$$H^{-1} = \begin{bmatrix} \frac{1}{6} & -\frac{1}{6} & \frac{1}{6} \\ -\frac{5}{6} & -\frac{1}{6} & \frac{7}{6} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

To validate the inverse, we multiply matrix H by H^{-1} to check if we get the identity matrix:

$$HH^{-1} = H \cdot H^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Since the product of H and H^{-1} is the identity matrix, we have correctly found the inverse of matrix H.

Chapter 8

System Of Linear Equations

Bonus - Gaussian Elimination:

Gaussian elimination is a sequential method used to solve systems of linear equations and is also applicable to finding the rank of a matrix, computing the determinant of a matrix, and finding the inverse of an invertible matrix. It's a pivotal tool to deal with linear algebra problems and a crucial step in understanding more advanced linear algebra concepts.

The Gaussian elimination method mainly consists of two phases: the Forward Elimination phase and the Back Substitution phase.

Forward Elimination:

The objective in this phase is to transform the augmented matrix of the linear system to an upper triangular form by performing elementary row operations. These row operations include swapping two rows, multiplying a row by a non-zero scalar, and adding or subtracting rows. The goal is to get zeros below the pivot (the leading entry of each row) in each column.

Back Substitution:

Once the augmented matrix is in upper triangular form, the solutions can be found using back substitution. Starting from the bottom row, each variable is solved one at a time and subsequently substituted back into the preceding equations.

Example:

Consider the system of equations:

$$\begin{cases} x + 2y = 8 \\ 2x - y = 2 \end{cases}$$

Representing this system as an augmented matrix, we have:

$$\left[\begin{array}{cc|c} 1 & 2 & 8 \\ 2 & -1 & 2 \end{array}\right]$$

Forward Elimination: The goal is to create a zero in the second row, first column of the augmented matrix: Multiply the first row by -2 and add it to the second row:

$$\left[\begin{array}{cc|c} 1 & 2 & 8 \\ 0 & -5 & -14 \end{array}\right]$$

Back Substitution: Now that we have an upper triangular matrix, we can solve for the variables starting from the last equation. From the second equation, we can write:

$$-5y = -14 \implies y = \frac{14}{5}$$

Substitute the value of y into the first equation to solve for x:

$$x + 2\left(\frac{14}{5}\right) = 8 \implies x = 8 - \frac{28}{5} = \frac{12}{5}$$

Thus, the solutions are:

$$x = \frac{12}{5}, \quad y = \frac{14}{5}$$

Exercises

Exercise 7.1:

For the system of equations:

$$\begin{cases} 2x + 3y = 1 \\ 4x - y = 5 \end{cases}$$

1. Find the determinant of the coefficient matrix.
2. If possible, solve the system using Cramer's Rule.
3. Based on your computations, does this system have a solution?

Exercise 7.2:

Apply Gaussian elimination to find the solution to the following system of equations:

$$\begin{cases} x + 2y = 8 \\ 2x - y = 1 \end{cases}$$

Exercise 7.3:

Use Gaussian elimination to find the solution to the following system of equations:

$$\begin{cases} x - 2y + 3z = 1 \\ 4x + y - 2z = -3 \\ 3x + 3y + z = 4 \end{cases}$$

Exercise 7.4:

Consider the system of linear equations:

$$\begin{cases} 4x - 2y = 12 \\ 2x - y = 6 \end{cases}$$

1. Use any method to find the solution(s) of the system.
2. Interpret your result. Does this system have no solution, one solution, or infinitely many solutions?

Coding Exercises:

Three movies were screened in a cinema over a weekend: "Action Adventure", "Romantic Comedy", and "Sci-fi Thriller". The individual ticket prices for these movies are x, y, and z dollars respectively. However, you've only been given the following pieces of information:

- 700 tickets were sold for "Action Adventure" and 300 tickets for "Romantic Comedy", collecting a total revenue of \$7,000.
- 500 tickets were sold for "Romantic Comedy" and 500 for "Sci-fi Thriller", with a combined revenue of \$6,000.
- "Action Adventure" and "Sci-fi Thriller" together sold 1000 tickets and generated a revenue of \$9,000.

Wit this information, let's solve the following:

1. Define the coefficient matrix A and the constant matrix B from your system of equations.
2. Calculate the determinant of the coefficient matrix A. If the determinant is non-zero, proceed to the next step. If it is zero, indicate that the inverse of A does not exist and discuss the implications.
3. Calculate the inverse of matrix A if it exists.
4. Use the inverse of matrix A to solve for the variable matrix X representing the ticket prices for the movies, i.e., $[x, y, z]$. If the inverse does not exist, discuss the implications on the system of equations.

Solutions:

Exercise 7.1:

Given the system of equations:

$$\begin{cases} 2x + 3y = 1 \\ 4x - y = 5 \end{cases}$$

Step 1: Define the coefficient matrix, A, and the constant matrix, B:

$$A = \begin{bmatrix} 2 & 3 \\ 4 & -1 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$$

Step 2: Find the determinant of the coefficient matrix, $\det(A)$:

$$\det(A) = (2) \cdot (-1) - (3) \cdot (4)$$
$$\det(A) = -2 - 12$$
$$\det(A) = -14$$

Since $\det(A) \neq 0$, the system has a unique solution.

Step 3: Solve the system using Cramer's Rule.

Cramer's Rule states that for a system of linear equations represented by $AX = B$, the variables x and y can be found using:

$$x = \frac{\det(A_x)}{\det(A)}$$

$$y = \frac{\det(A_y)}{\det(A)}$$

Where A_x and A_y are matrices obtained by replacing the respective columns of A with matrix B.

$$A_x = \begin{bmatrix} 1 & 3 \\ 5 & -1 \end{bmatrix}$$
$$A_y = \begin{bmatrix} 2 & 1 \\ 4 & 5 \end{bmatrix}$$

Now, find the determinants of A_x and A_y:

$$\det(A_x) = (1) \cdot (-1) - (3) \cdot (5)$$
$$\det(A_x) = -1 - 15$$
$$\det(A_x) = -16$$

$$\det(A_y) = (2) \cdot (5) - (1) \cdot (4)$$
$$\det(A_y) = 10 - 4$$
$$\det(A_y) = 6$$

Now, solve for x and y using Cramer's Rule:

$$x = \frac{-16}{-14} = \frac{8}{7}$$
$$y = \frac{6}{-14} = -\frac{3}{7}$$

Conclusion: The system of equations has a unique solution, $x = \frac{8}{7}$ and $y = -\frac{3}{7}$.

Exercise 7.2:

Given the system of equations:

$$\begin{cases} x + 2y = 8 \\ 2x - y = 1 \end{cases}$$

Step 1: Write down the augmented matrix:

$$[A|B] = \left[\begin{array}{cc|c} 1 & 2 & 8 \\ 2 & -1 & 1 \end{array}\right]$$

Step 2: Use Row operations to convert the system to its Row-Echelon Form (REF). Our goal is to get a leading 1 in the first row, first column and use it to make all the other entries in its column to be 0.

$$\text{Replace } R_2 \text{ with } R_2 - 2 \cdot R_1$$
$$\left[\begin{array}{cc|c} 1 & 2 & 8 \\ 0 & -5 & -15 \end{array}\right]$$

Step 3: Continue the Row operations to get a leading 1 in the second row, second column:

$$\text{Replace } R_2 \text{ with } -\frac{1}{5} \cdot R_2$$
$$\left[\begin{array}{cc|c} 1 & 2 & 8 \\ 0 & 1 & 3 \end{array}\right]$$

Step 4: Now, we use back substitution to make the entries above the leading 1 in the second row to be 0, aiming for the Reduced Row-Echelon Form (RREF):

Replace R_1 with $R_1 - 2 \cdot R_2$

$$\left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 3 \end{array}\right]$$

This is our RREF. Reading off the solutions:

$$x = 2$$

$$y = 3$$

Conclusion: By applying Gaussian elimination, we find that the solution to the system of equations is $x = 2$ and $y = 3$.

Exercise 7.3:

Use Gaussian elimination to find the solution to the following system of equations:

$$\begin{cases} x - 2y + 3z = 1 \\ 4x + y - 2z = -3 \\ 3x + 3y + z = 4 \end{cases}$$

Step 1: Write the augmented matrix of the system:

$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & 1 \\ 4 & 1 & -2 & -3 \\ 3 & 3 & 1 & 4 \end{array}\right]$$

Step 2: Use row operations to convert the system into echelon form.
Row operations:

- $R2 := R2 - 4R1$
- $R3 := R3 - 3R1$

After applying these row operations, we have:

$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & 1 \\ 0 & 9 & -14 & -7 \\ 0 & 9 & -8 & 1 \end{array}\right]$$

Next, we subtract $R3$ from $R2$ to eliminate the y-term from $R2$:

$$R2 := R2 - R3$$

The augmented matrix now becomes:

$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & 1 \\ 0 & 0 & -6 & -8 \\ 0 & 9 & -8 & 1 \end{array}\right]$$

Now, we divide $R2$ by -6 to make the leading coefficient of z in $R2$ equal to 1:

$$R2 := \frac{R2}{-6}$$

The augmented matrix is now:

$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & 1 \\ 0 & 0 & 1 & \frac{4}{3} \\ 0 & 9 & -8 & 1 \end{array}\right]$$

Next, we add $8R2$ to $R3$ to eliminate z from $R3$:

$$R3 := R3 + 8R2$$

The augmented matrix then becomes:

$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & 1 \\ 0 & 0 & 1 & \frac{4}{3} \\ 0 & 9 & 0 & \frac{35}{3} \end{array}\right]$$

Divide $R3$ by 9 to normalize the leading coefficient of y to 1:

$$R3 := \frac{R3}{9}$$

We get:

$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & 1 \\ 0 & 0 & 1 & \frac{4}{3} \\ 0 & 1 & 0 & \frac{35}{27} \end{array}\right]$$

Step 3: Perform back substitution to solve for x, y, and z.

From $R3$ we get:

$$y = \frac{35}{27}$$

From $R2$ we get:

$$z = \frac{4}{3}$$

Substitute y and z into $R1$ to solve for x:

$$x - 2\left(\frac{35}{27}\right) + 3\left(\frac{4}{3}\right) = 1$$

$$x = 1 + \frac{70}{27} - 4$$

$$x = -\frac{11}{27}$$

The solution set is $x = -\frac{11}{27}, y = \frac{35}{27}, z = \frac{4}{3}$.

Exercise 7.4:

Given the system of equations:

$$\begin{cases} 4x - 2y = 12 \\ 2x - y = 6 \end{cases}$$

Step 1: Write down the augmented matrix:

$$[A|B] = \left[\begin{array}{cc|c} 4 & -2 & 12 \\ 2 & -1 & 6 \end{array}\right]$$

Step 2: Start with the first column and use row operations to get zeros below the leading 1 in the first row:

$$\text{Replace } R_2 \text{ with } R_2 - \frac{1}{2} \cdot R_1$$

$$\left[\begin{array}{cc|c} 4 & -2 & 12 \\ 0 & 0 & 0 \end{array}\right]$$

The second row is all zeros, which indicates that the second equation is dependent on the first, i.e., the second equation is a multiple of the first.

This means that there are infinitely many solutions, since we have fewer independent equations than unknowns.

To find the solutions, express one variable in terms of the other from the first equation:

$$4x - 2y = 12$$
$$2y = 4x - 12$$
$$y = 2x - 6$$

Conclusion: The system of equations has infinitely many solutions, represented by the equation $y = 2x - 6$. Any pair of (x, y) that satisfies this equation is a solution to the system. For instance, if $x = 3$, then $y = 0$; if $x = 4$, then $y = 2$, and so on.

Chapter 9

Eigen Vectors and Eigen Values:

Exercises:

Exercise 8.1 - Population Dynamics:

Given the transition matrix:

$$A = \begin{bmatrix} 0.6 & 0.3 \\ 0.4 & 0.7 \end{bmatrix}$$

find the eigenvalues and corresponding eigenvectors. Interpret their meaning in the context of population dynamics.

Exercise 8.2 - Markov Chain:

Consider the Markov chain represented by the transition matrix

$$B = \begin{bmatrix} 0.8 & 0.1 \\ 0.2 & 0.9 \end{bmatrix}$$

Find the eigenvalues and eigenvectors and discuss their significance in the context of Markov Chains.

Exercise 8.3 - Google PageRank Algorithm:

The simplified PageRank Algorithm is based on an eigenvalue problem. For a network represented by the matrix:

$$C = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Find the eigenvalues and eigenvectors and discuss their relevance to web page ranking.

Exercise 8.4 - Economic Model:

In an economic model represented by the matrix:

$$D = \begin{bmatrix} 0.5 & 0.2 \\ 0.5 & 0.8 \end{bmatrix}$$

Calculate the eigenvalues and the corresponding eigenvectors. Interpret their significance in terms of economic stability and dynamics.

Exercise 8.5 - Electrical Circuit:

For an electrical circuit modelled by the matrix:

$$E = \begin{bmatrix} -1 & 2 \\ 0 & -2 \end{bmatrix}$$

Compute the eigenvalues and eigenvectors. Interpret their meaning in terms of circuit analysis.

9.1 Solutions:

Exercise 8.1 - Population Dynamics:

Given the matrix:

$$A = \begin{bmatrix} 0.6 & 0.3 \\ 0.4 & 0.7 \end{bmatrix}$$

To find the eigenvalues of A, we solve the characteristic equation $\det(A - \lambda I) = 0$:

$$\det \begin{bmatrix} 0.6-\lambda & 0.3 \\ 0.4 & 0.7-\lambda \end{bmatrix} = 0$$

Expanding the determinant, we have:

$$(0.6-\lambda)(0.7-\lambda) - (0.3 \cdot 0.4) = 0$$

$$\lambda^2 - 1.3\lambda + 0.42 - 0.12 = 0$$

$$\lambda^2 - 1.3\lambda + 0.3 = 0$$

Factoring the quadratic equation, we find the eigenvalues:

$$\lambda_1 = 1, \quad \lambda_2 = 0.3$$

Next, we find the eigenvectors by solving $(A - \lambda I)v = 0$ for each eigenvalue.

For $\lambda_1 = 1$:

$$(A - I)v_1 = \begin{bmatrix} -0.4 & 0.3 \\ 0.4 & -0.3 \end{bmatrix} v_1 = 0$$

The corresponding eigenvector is:

$$v_1 = \begin{bmatrix} -0.6 \\ -0.8 \end{bmatrix}$$

For $\lambda_2 = 0.3$:

$$(A - 0.3I)v_2 = \begin{bmatrix} 0.3 & 0.3 \\ 0.4 & 0.4 \end{bmatrix} v_2 = 0$$

The corresponding eigenvector is:

$$v_2 = \begin{bmatrix} -0.7071 \\ 0.7071 \end{bmatrix}$$

The eigenvalues and eigenvectors represent the long-term behavior of the population. The eigenvalue $\lambda_1 = 1$ indicates a stable population state, while $\lambda_2 = 0.3$ indicates a state that will eventually diminish.

Exercise 8.2 - Markov Chain:

Consider the Markov chain represented by the transition matrix Given the matrix:

$$B = \begin{bmatrix} 0.8 & 0.1 \\ 0.2 & 0.9 \end{bmatrix}$$

Find the eigenvalues and eigenvectors and discuss their significance in the context of Markov Chains.

To find the eigenvalues of matrix B, we need to solve the characteristic equation:

$$\det(B - \lambda I) = 0$$

The characteristic equation is obtained by taking the determinant of $B - \lambda I$:

$$\det \begin{bmatrix} 0.8 - \lambda & 0.1 \\ 0.2 & 0.9 - \lambda \end{bmatrix} = 0$$

$$(0.8 - \lambda)(0.9 - \lambda) - (0.2 \cdot 0.1) = 0$$

$$\lambda^2 - 1.7\lambda + 0.72 - 0.02 = 0$$

$$\lambda^2 - 1.7\lambda + 0.7 = 0$$

Solving this quadratic equation, we find the eigenvalues:

$$\lambda_1 = 1, \quad \lambda_2 = 0.7$$

For each eigenvalue, we find the corresponding eigenvectors by solving:

$$(B - \lambda I)v = 0$$

For $\lambda_1 = 1$:

$$(B - I)v_1 = \begin{bmatrix} -0.2 & 0.1 \\ 0.2 & -0.1 \end{bmatrix} v_1 = 0$$

$$v_1 = \begin{bmatrix} -0.4472 \\ -0.8944 \end{bmatrix}$$

For $\lambda_2 = 0.7$:

$$(B - 0.7I)v_2 = \begin{bmatrix} 0.1 & 0.1 \\ 0.2 & 0.2 \end{bmatrix} v_2 = 0$$

$$v_2 = \begin{bmatrix} -0.7071 \\ 0.7071 \end{bmatrix}$$

In the context of Markov chains, the eigenvalue $\lambda_1 = 1$ is associated with the steady state of the system. The corresponding eigenvector indicates the long-term stable distribution of the states. The eigenvalue $\lambda_2 = 0.7$ represents a transient state, and the corresponding eigenvector shows the direction in which the system will move away from this transient state.

Exercise 8.3 - Google PageRank Algorithm:

The simplified PageRank Algorithm is based on an eigenvalue problem. For a network represented by the matrix:

$$C = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

To find the eigenvalues of matrix C, we solve the characteristic equation $\det(C - \lambda I) = 0$:

$$\det \begin{bmatrix} -\lambda & 1 \\ 1 & -\lambda \end{bmatrix} = 0$$

$$\lambda^2 - (1 \cdot 1) = 0$$

$$\lambda^2 - 1 = 0$$

$$\lambda = \pm 1$$

The eigenvalues of matrix C are $\lambda_1 = 1$ and $\lambda_2 = -1$. For each eigenvalue, we find the corresponding eigenvectors by solving:

$$(C - \lambda I)v = 0$$

For $\lambda_1 = 1$:

$$(C - I)v_1 = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} v_1 = 0$$

$$v_1 = \begin{bmatrix} 0.7071 \\ 0.7071 \end{bmatrix}$$

For $\lambda_2 = -1$:

$$(C + I)v_2 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} v_2 = 0$$

$$v_2 = \begin{bmatrix} -0.7071 \\ 0.7071 \end{bmatrix}$$

In the context of the PageRank algorithm, the principal eigenvector associated with the eigenvalue $\lambda_1 = 1$ represents the steady-state distribution of the PageRank values. The pages in this simple network would have equal PageRank values, as indicated by the principal eigenvector v_1. The eigenvector associated with $\lambda_2 = -1$ represents an alternate distribution where one page's rank increases as the other's decreases, but this is not stable or relevant for the PageRank algorithm.

Exercise 8.4 - Economic Model:

In an economic model represented by the matrix:

$$D = \begin{bmatrix} 0.5 & 0.2 \\ 0.5 & 0.8 \end{bmatrix}$$

Calculate the eigenvalues and the corresponding eigenvectors. Interpret their significance in terms of economic stability and dynamics.

To find the eigenvalues of matrix D, we solve the characteristic equation $\det(D - \lambda I) = 0$:

$$\det \begin{bmatrix} 0.5 - \lambda & 0.2 \\ 0.5 & 0.8 - \lambda \end{bmatrix} = 0$$

$$(0.5 - \lambda)(0.8 - \lambda) - (0.5 \cdot 0.2) = 0$$

$$\lambda^2 - 1.3\lambda + 0.4 - 0.1 = 0$$

$$\lambda^2 - 1.3\lambda + 0.3 = 0$$

Solving this quadratic equation, we find the eigenvalues:

$$\lambda_1 = 0.3, \quad \lambda_2 = 1$$

For each eigenvalue, we find the corresponding eigenvectors by solving:

$$(D - \lambda I)v = 0$$

For $\lambda_1 = 0.3$:

$$(D - 0.3I)v_1 = \begin{bmatrix} 0.2 & 0.2 \\ 0.5 & 0.5 \end{bmatrix} v_1 = 0$$

$$v_1 = \begin{bmatrix} -0.7071 \\ 0.7071 \end{bmatrix}$$

For $\lambda_2 = 1$:

$$(D - I)v_2 = \begin{bmatrix} -0.5 & 0.2 \\ 0.5 & -0.2 \end{bmatrix} v_2 = 0$$

$$v_2 = \begin{bmatrix} -0.3714 \\ -0.9285 \end{bmatrix}$$

In terms of economic stability and dynamics: - The eigenvalue $\lambda_1 = 0.3$ indicates a contracting direction in the economic model. Variables aligned with its corresponding eigenvector are expected to decrease over time, suggesting a recessionary trend. - The eigenvalue $\lambda_2 = 1$ indicates a stable direction in the economic model. Variables aligned with its corresponding eigenvector will remain constant in the long run, suggesting that the economy could be in a steady state or long-term equilibrium.

Exercise 8.5 - Electrical Circuit:

For an electrical circuit modeled by the matrix:

$$E = \begin{bmatrix} -1 & 2 \\ 0 & -2 \end{bmatrix}$$

Compute the eigenvalues and eigenvectors. Interpret their meaning in terms of circuit analysis.

To find the eigenvalues of matrix E, we solve the characteristic equation $\det(E - \lambda I) = 0$:

$$\det \begin{bmatrix} -1 - \lambda & 2 \\ 0 & -2 - \lambda \end{bmatrix} = 0$$

$$(-1 - \lambda)(-2 - \lambda) - (0 \cdot 2) = 0$$

$$\lambda^2 + 3\lambda + 2 = 0$$

Solving this quadratic equation, we find the eigenvalues:

$$\lambda_1 = -1, \quad \lambda_2 = -2$$

For each eigenvalue, we find the corresponding eigenvectors by solving:

$$(E - \lambda I)v = 0$$

For $\lambda_1 = -1$:

$$(E + I)v_1 = \begin{bmatrix} 0 & 2 \\ 0 & -1 \end{bmatrix} v_1 = 0$$

$$v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

For $\lambda_2 = -2$:

$$(E + 2I)v_2 = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} v_2 = 0$$

$$v_2 = \begin{bmatrix} -0.8944 \\ 0.4472 \end{bmatrix}$$

In terms of circuit analysis: - The eigenvalue $\lambda_1 = -1$ indicates a mode of the circuit that decays exponentially at a certain rate. The eigenvector shows that this decay is entirely in one component of the circuit, which could be a voltage or current. - The eigenvalue $\lambda_2 = -2$ represents a mode that decays more rapidly. Its corresponding eigenvector suggests that changes in the circuit affect one component twice as much as the other and in the opposite direction, which might represent the interplay between voltages or currents in different parts of the circuit.

Chapter 10

Eigen Decomposition

Coding exercise:

Create a Python function to perform eigen decomposition of a given matrix:

$$A = \begin{bmatrix} 4 & 2 \\ 3 & 1 \end{bmatrix}$$

Chapter 11

Component Analysis (PCA)

Exercise 10.1: Principal Component Analysis in Market Research

In a market research study, a company collected data on three product features (X_1, X_2, X_3) which represent customer ratings for quality, price, and usability of a new gadget. The ratings are on a scale of 1 to 5, with the following collected data points:

Quality (X_1)	Price (X_2)	Usability (X_3)
2	3	1
4	1	2
3	5	3

The goal is to analyze this data to understand which features contribute most to customer perception, and thus guide the company's focus for improvement.

1. Standardize the dataset to have a mean of 0 and a standard deviation of 1 for each feature.

2. Find the Covariance Matrix of the standardized dataset to understand the variance of the features and their correlation with each other.

3. Calculate the Eigenvalues and Eigenvectors of the Covariance Matrix to determine the directions of maximum variance in the data.

4. Identify the Principal Component(s) by selecting the eigenvectors with the largest eigenvalues, which represent the features that account for most of the variance in the dataset.

5. Project the data points onto the eigenvector corresponding to the largest eigenvalue to reduce the dimensionality of the data while preserving as much information as possible.

Coding exercises:

Dataset: The Wine Quality Dataset can be downloaded from the following link. It is also available on the git repository:

`https://archive.ics.uci.edu/dataset/186/wine+quality`

1. Write python function that computes the following:

   - Load the red wine dataset into a pandas dataframe.
   - Normalize the data.
   - Compute the covariance matrix
   - Calculate the eigen values and eigen vectors and sort the eigenvectors by decreasing eigenvalues.
   - Select the first n principal components and calculate the explanatory variance.
   - Transform the data with the first two principal components and returns it.
   - Compute the transformed data for both the red and the white wine datasets.

   The function must returns the transformed data set as well as the explained variance of each component.

2. Plot the transformed datasets.

3. Project the white wine original dataset into the redwine principal components and plot the results on the same figure.

4. Merge both the red and white wine datasets, compute the PCA and plot them on the sample plot.
5. Explain the different strategies and what can you observe.

Solutions:

Exercise 10.1:

1. **Standardize the dataset:**

   Given the dataset:

X_1	X_2	X_3
2	3	1
4	1	2
3	5	3

   To standardize the dataset, we will calculate the mean (μ) and standard deviation (σ) for each feature (X_1, X_2, and X_3) and then apply the standardization formula to each data point.

   For X_1 (Quality):

$$\mu_{X_1} = \frac{2+4+3}{3} = \frac{9}{3} = 3$$
$$\sigma_{X_1} = \sqrt{\frac{(2-3)^2+(4-3)^2+(3-3)^2}{3}}$$
$$= \sqrt{\frac{2+1+0}{3}} = \sqrt{\frac{3}{3}} = 1$$

   For X_2 (Price):

$$\mu_{X_2} = \frac{3+1+5}{3} = \frac{9}{3} = 3$$
$$\sigma_{X_2} = \sqrt{\frac{(3-3)^2+(1-3)^2+(5-3)^2}{3}}$$
$$= \sqrt{\frac{0+4+4}{3}} = \sqrt{\frac{8}{3}}$$

For X_3 (Usability):

$$\mu_{X_3} = \frac{1+2+3}{3} = \frac{6}{3} = 2$$

$$\sigma_{X_3} = \sqrt{\frac{(1-2)^2 + (2-2)^2 + (3-2)^2}{3}}$$

$$= \sqrt{\frac{1+0+1}{3}} = \sqrt{\frac{2}{3}}$$

Now that we have calculated the mean and standard deviation for each feature, we can standardize the data points:

Standardized X_1 (Quality):

$$\frac{2-3}{1} = -1$$

$$\frac{4-3}{1} = 1$$

$$\frac{3-3}{1} = 0$$

Standardized X_2 (Price):

$$\frac{3-3}{\sqrt{\frac{8}{3}}} = 0$$

$$\frac{1-3}{\sqrt{\frac{8}{3}}} = -1.2247$$

$$\frac{5-3}{\sqrt{\frac{8}{3}}} = 1.2247$$

Standardized X_3 (Usability):

$$\frac{1-2}{\sqrt{\frac{2}{3}}} = -1.2247$$

$$\frac{2-2}{\sqrt{\frac{2}{3}}} = 0$$

$$\frac{3-2}{\sqrt{\frac{2}{3}}} = 1.2247$$

Standardized Quality(X_1)	Standardized Price(X_2)	Standardized Usability(X_3)
-1	0	-1.2247
1	-1.2247	0
0	1.2247	1.2247

2. **Find the Covariance Matrix:**

To find the Covariance Matrix, we need to calculate the covariance between each pair of standardized features. The formula for the covariance between two features X_i and X_j is given by:

$$\text{Cov}(X_i, X_j) = \frac{1}{n-1}\sum_{k=1}^{n}(X_i^{(k)} - \mu_i)(X_j^{(k)} - \mu_j)$$

Where:

- $X_i^{(k)}$ and $X_j^{(k)}$ are the standardized data points for features X_i and X_j respectively.
- μ_i and μ_j are the means of features X_i and X_j respectively.
- n is the number of data points.

Let's calculate the Covariance Matrix for our standardized data:

For X_1 and X_2:

$$\begin{aligned}
\text{Cov}(X_1, X_2) &= \frac{1}{3-1}\sum_{k=1}^{3}(X_1^{(k)} - \mu_{X_1})(X_2^{(k)} - \mu_{X_2}) \\
&= \frac{1}{2}\left[(-1-0)(0-0) + (1-0)(-1.2247-0) + (0-0)(1.2247-0)\right] \\
&= \frac{1}{2}(-0 - 1.2247 + 0) \\
&= -0.6124
\end{aligned}$$

For X_1 and X_3:

$$
\begin{aligned}
\mathrm{Cov}(X_1, X_3) &= \frac{1}{3-1}\sum_{k=1}^{3}(X_1^{(k)} - \mu_{X_1})(X_3^{(k)} - \mu_{X_3}) \\
&= \frac{1}{2}\left[(-1-0)(-1.2247-0) + (1-0)(0-0) + (0-0)(1.2247-0)\right] \\
&= \frac{1}{2}(1.2247 + 0) \\
&= 0.6124
\end{aligned}
$$

For X_2 and X_3:

$$
\begin{aligned}
\mathrm{Cov}(X_2, X_3) &= \frac{1}{3-1}\sum_{k=1}^{3}(X_2^{(k)} - \mu_{X_2})(X_3^{(k)} - \mu_{X_3}) \\
&= \frac{1}{2}\left[(0-0)(-1.2247-0) + (-1.2247-0)(0-0) + (1.2247-0)(1.2247-0)\right] \\
&= \frac{1}{2}(1.5) \\
&= 0.75
\end{aligned}
$$

Now, we can construct the Covariance Matrix:

$$
\text{Covariance Matrix} = \begin{bmatrix} -0.6124 & 0.6124 & 0.75 \\ 0.6124 & 0.6124 & -0.75 \\ 0.75 & -0.75 & 0.5 \end{bmatrix}
$$

This Covariance Matrix shows the covariances between the standardized features X_1, X_2, and X_3. It provides information about the variance of each feature and their correlations with each other.

Chapter 12

Single Value Decomposition - MovieLens 100K Dataset.

Dataset: The MovieLens 100K Dataset can be downloaded from the following link. It is also available on the git repository:
`https://grouplens.org/datasets/movielens/100k/`

12.1 Exercises- Singular Value Decomposition (SVD)

1. Load the dataset with pandas.
2. Create a User-Item Matrix.
3. Decompose the User-Item matrix.with the SVD.
4. Reduce Dimensionality to Capture 85% of the Variance.
    a) Plot the cumulative variance and the singular values magnitude.
    b) Plot the chosen single values.
5. Generate 10 random users profiles and recommend 5 movies per each user.
6. Find the top 5 similar users to user 10.

7. Recommend a movie that user 10 has not watched but was watched by any of the top most similar users.

8. Examine the V^T matrix obtained from SVD, where each row represents a latent feature, and each column represents a movie. Choose a few latent features and analyze their movie associations.