Vol 2 - Calculus

Exercises

Contents

Chapter 1

Welcome

Hello, this document complements the reading of the Before Machine Learning Volume 2 - Calculus. The book is structured with practical exercises followed by correspondent solutions. We also have coding exercises on a selection of chapters. The solutions for these bad boys come in jupyter notebooks that are in the same location as this document.

Feel free to do whatever you wish with this content, and if you are taking suggestions, please start by drawing a moustache on that bee.

Before letting you dive into the exercises, I would like to list the Python libraries I used:

- numpy
- matplotlib

That's it!

Chapter 2

Limits

2.1 Rules of Limits:

1. **Constant Rule:**
$$\lim_{x \to a} c = c$$
where c is a constant.

2. **Identity Rule:**
$$\lim_{x \to a} x = a$$

3. **Sum Rule:**
$$\lim_{x \to a} [f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$$

4. **Difference Rule:**
$$\lim_{x \to a} [f(x) - g(x)] = \lim_{x \to a} f(x) - \lim_{x \to a} g(x)$$

5. **Product Rule:**
$$\lim_{x \to a} [f(x) \cdot g(x)] = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x)$$

6. **Quotient Rule:**
$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$$
provided $\lim_{x \to a} g(x) \neq 0$.

7. **Power Rule:**
$$\lim_{x\to a}[f(x)]^n = [\lim_{x\to a} f(x)]^n$$
where n is a positive integer.

8. **Root Rule:**
$$\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}$$
provided the values are in the domain of the function.

9. **Composite Rule (Chain Rule):** If $\lim_{x\to a} g(x) = b$ and $\lim_{y\to b} f(y) = c$, then
$$\lim_{x\to a} f(g(x)) = c$$

Now the exercises.

2.2 Practical Exercises:

Compute the following limits:

1. $\lim_{x\to 3}(2x+1)$:
2. $\lim_{x\to 2} \frac{x^2-4}{x-2}$:
3. $\lim_{x\to 1}(3x^2-2x+1)$:
4. $\lim_{x\to 3} \frac{x^2-9}{x-3}$:
5. $\lim_{x\to 2} \frac{1}{x^2-4}$:
6. $\lim_{x\to -1} \frac{x^3+1}{x^2+2x+1}$:
7. $\lim_{x\to 0} \frac{\sqrt{x+4}-2}{x}$:

2.3 Solutions Practical Exercises:

1. $\mathbf{\lim_{x\to 3}(2x+1)}$ **:**

   The function is continuous at $x = 3$, so we can directly substitute $x = 3$ into the expression to find the limit.

$$\begin{aligned}\lim_{x\to 3}(2x+1) &= 2\lim_{x\to 3} x + \lim_{x\to 3} 1\\ &= 2\cdot 3+1\\ &= 6+1\\ &= 7\end{aligned}$$

2. $\mathbf{\lim_{x\to 2}\frac{x^2-4}{x-2}}$ **:**

   Factor the numerator and simplify to get $\lim_{x\to 2}\frac{(x-2)(x+2)}{x-2}$. After canceling $x-2$ from the numerator and the denominator, we can directly substitute $x = 2$ into the remaining expression to find the limit.

$$\begin{aligned}\lim_{x\to 2}\frac{x^2-4}{x-2} &= \lim_{x\to 2}\frac{(x+2)(x-2)}{(x-2)}\\ &= \lim_{x\to 2}(x+2)\\ &= \lim_{x\to 2} x + \lim_{x\to 2} 2\\ &= 2+2\\ &= 4\end{aligned}$$

3. $\mathbf{\lim_{x\to 1}(3x^2-2x+1)}$ **:**

   The function is continuous at $x = 1$, hence, we can directly substitute $x = 1$ into the expression to find the limit.

$$\begin{aligned}\lim_{x\to 1}\left(3x^2-2x+1\right) &= 3\lim_{x\to 1} x^2 - 2\lim_{x\to 1} x + \lim_{x\to 1} 1\\ &= 3(1)^2-2(1)+1\\ &= 3-2+1\\ &= 4-2\\ &= 2\end{aligned}$$

4. $\boldsymbol{\lim_{x\to 3} \frac{x^2-9}{x-3}}$:

   Factor the numerator and simplify to get $\lim_{x\to 3} \frac{(x-3)(x+3)}{x-3}$. After canceling $x-3$ from the numerator and the denominator, we can directly substitute $x=3$ into the remaining expression to find the limit.

$$\begin{aligned}
\lim_{x\to 3} \frac{x^2-9}{x-3} &= \lim_{x\to 3} \frac{(x+3)(x-3)}{x-3} \\
&= \lim_{x\to 3}(x+3) \\
&= \lim_{x\to 3} x + \lim_{x\to 3} 3 \\
&= 3+3 \\
&= 6
\end{aligned}$$

5. $\boldsymbol{\lim_{x\to 2} \frac{1}{x^2-4}}$:

   First, let's factor the denominator to see if it simplifies: $x^2 - 4 = (x-2)(x+2)$.

   Notice that at $x = 2$, the denominator becomes zero, which makes the whole fraction undefined at this point.

   As x approaches 2 from both the left and the right, the denominator approaches 0, making the fraction grow without bound. Therefore, the limit does not exist.

   Thus, $\lim_{x\to 2} \frac{1}{x^2-4}$ does not exist.

6. $\boldsymbol{\lim_{x\to -1} \frac{x^3+1}{x^2+2x+1}}$:

   Factor both the numerator and the denominator:

   Numerator:

$$x^3 + 1 = (x+1)(x^2 - x + 1)$$

   Denominator:

$$x^2 + 2x + 1 = (x+1)(x+1) = (x+1)^2$$

   Given these factorization, the expression can be simplified:

$$\lim_{x\to -1} \frac{x^3+1}{x^2+2x+1} = \lim_{x\to -1} \frac{(x+1)(x^2-x+1)}{(x+1)^2}$$

After canceling $x+1$ from the numerator and the denominator, the expression becomes:

$$\lim_{x\to-1} \frac{x^2-x+1}{x+1}$$

Now, substituting $x=-1$:

$$\begin{aligned}\lim_{x\to-1} \frac{x^2-x+1}{x+1} &= \frac{(-1)^2-(-1)+1}{-1+1} \\ &= \frac{1+1+1}{0}\end{aligned}$$

However, this results in a division by zero, indicating that the limit does not exist at $x=-1$.

Thus, $\lim_{x\to-1} \frac{x^3+1}{x^2+2x+1}$ does not exist.

7. $\boldsymbol{\lim_{x\to 0} \frac{\sqrt{x+4}-2}{x}}$:

This form produces an indeterminate of the form $\frac{0}{0}$ when $x=0$. To address this, we can multiply the numerator and denominator by the conjugate of the numerator:

$$\lim_{x\to 0} \frac{\sqrt{x+4}-2}{x} \cdot \frac{\sqrt{x+4}+2}{\sqrt{x+4}+2}$$

Expanding, we get:

$$\lim_{x\to 0} \frac{x}{x(\sqrt{x+4}+2)}$$

Now, the x in the numerator and denominator cancel out:

$$\lim_{x\to 0} \frac{1}{\sqrt{x+4}+2}$$

By substituting $x=0$, we get:

$$\lim_{x\to 0} \frac{1}{\sqrt{x+4}+2} = \frac{1}{\sqrt{4}+2} = \frac{1}{2+2} = \frac{1}{4}$$

So,

$\lim_{x\to 0} \frac{\sqrt{x+4}-2}{x} = \frac{1}{4}$.

Chapter 3

Derivatives

3.1 Derivative Rules:

- **Constant Function:**
$$\frac{d}{dx}[c] = 0$$
where c is a constant.

- **Power Rule:**
$$\frac{d}{dx}[x^n] = nx^{n-1}$$

- **Sum/Difference Rule:** If f and g are differentiable, then
$$\frac{d}{dx}[f(x) \pm g(x)] = f'(x) \pm g'(x)$$

- **Product Rule:**
$$\frac{d}{dx}[f(x) \cdot g(x)] = f'(x) \cdot g(x) + f(x) \cdot g'(x)$$

- **Quotient Rule:**
$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{[g(x)]^2}$$

- **Chain Rule:** If we have a composite function, $g(f(x))$, the derivative is:
$$\frac{d}{dx}[g(f(x))] = g'(f(x)) \cdot f'(x)$$

3.1.1 Derivatives of Exponential and Logarithmic Functions:

- For $a > 0$, $a \neq 1$, $\frac{d}{dx}[a^x] = a^x \ln(a)$
- $\frac{d}{dx}[e^x] = e^x$
- $\frac{d}{dx}[\ln(x)] = \frac{1}{x}$, for $x > 0$
- For $a > 0$, $a \neq 1$, $\frac{d}{dx}[\log_a(x)] = \frac{1}{x \ln(a)}$, for $x > 0$

3.1.2 Trigonometric Functions Derivatives:

- Trigonometric Functions:

$$\frac{d}{dx}[\sin(x)] = \cos(x)$$
$$\frac{d}{dx}[\cos(x)] = -\sin(x)$$
$$\frac{d}{dx}[\tan(x)] = \sec^2(x)$$

- Inverse Trigonometric Functions:

$$\frac{d}{dx}[\arcsin(x)] = \frac{1}{\sqrt{1-x^2}}$$
$$\frac{d}{dx}[\arccos(x)] = -\frac{1}{\sqrt{1-x^2}}$$
$$\frac{d}{dx}[\arctan(x)] = \frac{1}{1+x^2}$$

- Hyperbolic Functions:

$$\frac{d}{dx}[\sinh(x)] = \cosh(x)$$
$$\frac{d}{dx}[\cosh(x)] = \sinh(x)$$
$$\frac{d}{dx}[\tanh(x)] = \text{sech}^2(x)$$

3.2 Practical Exercises:

Find the derivative of:

3.2.1 Easy:

1. $f(x) = 7$
2. $f(x) = 5x$
3. $f(x) = x^3$
4. $f(x) = \sqrt{x}$
5. $f(x) = e^x$
6. $f(x) = \ln(x)$
7. $f(x) = \sin(x)$
8. $f(x) = \cos(x)$
9. $f(x) = \tan(x)$
10. $f(x) = 4^x$
11. $f(x) = \log_3(x)$
12. $f(x) = \sec(x)$
13. $f(x) = \csc(x)$
14. $f(x) = \cot(x)$
15. $f(x) = \arcsin(x)$
16. $f(x) = \arccos(x)$
17. $f(x) = \arctan(x)$
18. $f(x) = \sinh(x)$

19. $f(x) = \cosh(x)$
20. $f(x) = \tanh(x)$
21. $f(x) = x^2 + 4x$
22. $f(x) = 3x^3 - 5x^2 + 2x - 7$
23. $f(x) = e^{2x}$
24. $f(x) = \ln(3x)$
25. $f(x) = \sin(2x)$
26. $f(x) = \cos(3x)$
27. $f(x) = \tan(4x)$
28. $f(x) = \sec(5x)$
29. $f(x) = \csc(6x)$
30. $f(x) = \cot(7x)$

3.2.2 Medium:

1. $f(x) = \sin(x) \cdot \ln(x)$
2. $f(x) = e^{3x} \cdot \cos(2x)$
3. $f(x) = \frac{e^x}{\sin(x)}$
4. $f(x) = \tan(x) \cdot e^{2x}$
5. $f(x) = \ln(x^2 + 1)$
6. $f(x) = x^3 \cdot \cos(x)$
7. $f(x) = \frac{\ln(x)}{x^2}$
8. $f(x) = \sec(x) + \tan(x)$
9. $f(x) = x^3 \cdot \ln(x)$

10. $f(x) = e^{2x} \cdot \sin(3x)$

11. $f(x) = \tan(\ln(x))$

12. $f(x) = \frac{3x^2-4x}{x^3+1}$

13. $f(x) = \sqrt{x} \cdot \ln(x)$

14. $f(x) = \sin^2(x) \cdot \ln(x)$

15. $f(x) = \frac{e^{2x}}{x^2+1}$

16. $f(x) = \ln(\sec(x) + \tan(x))$

17. $f(x) = x^3 \cdot \cos^2(x)$

18. $f(x) = \frac{1}{x} \cdot \arctan(x)$

19. $f(x) = \sqrt{x} \cdot e^x$

20. $f(x) = \sin(x) \cdot \cos(x)$

21. $f(x) = \ln(x) \cdot \cos(x)$

22. $f(x) = e^x \cdot \tan(x)$

23. $f(x) = \frac{\sin(x)}{x^3+1}$

24. $f(x) = x \cdot \ln(x) \cdot \cos(x)$

25. $f(x) = \tan(x) \cdot \ln(\cos(x))$

26. $f(x) = \frac{e^{3x}}{x^2-1}$

27. $f(x) = \ln(x) \cdot \sin^2(x)$

28. $f(x) = \sin(x) \cdot e^{x^2}$

29. $f(x) = \frac{\ln(x^3)}{x^2}$

30. $f(x) = \cos(x) \cdot \ln(\sin(x))$

3.2.3 Hard:

1. $f(x) = e^{x^2} \cdot \ln(x)$
2. $f(x) = \sin(x^2) \cdot e^{x^3}$
3. $f(x) = \tan(x \cdot e^x)$
4. $f(x) = \ln(x^2 + e^{2x})$
5. $f(x) = x^x$
6. $f(x) = \ln(x) \cdot \cos(x)$
7. $f(x) = \sqrt{x} \cdot e^{x^2}$
8. $f(x) = x^{\sin(x)}$
9. $f(x) = \arctan(x^3)$
10. $f(x) = \frac{e^x}{x^2+1}$
11. $f(x) = x^2 \cdot \ln(x^2 + 1)$
12. $f(x) = \sin(x \cdot \ln(x))$
13. $f(x) = \sqrt{x^3 + e^x}$
14. $f(x) = x^x \cdot e^x$
15. $f(x) = \ln(\cos(e^x))$
16. $f(x) = x \cdot e^{x \cdot \ln(x)}$
17. $f(x) = \frac{\ln(x)}{x^2}$
18. $f(x) = \tan(x^2 \cdot e^x)$
19. $f(x) = \sec(x) \cdot \ln(x)$
20. $f(x) = \sin(x) \cdot e^{x^3}$
21. $f(x) = \frac{e^{x^2}}{x^2}$

22. $f(x) = \arccos(x \cdot e^x)$

23. $f(x) = x^3 \cdot \cos(x^2)$

24. $f(x) = \ln(x) \cdot x^x$

25. $f(x) = e^{x \cdot \ln(x)}$

26. $f(x) = \tan(\ln(x) \cdot e^x)$

27. $f(x) = x^3 \cdot \arcsin(x)$

28. $f(x) = \frac{\ln(x)}{\cos(x)}$

29. $f(x) = \sin(x^2 + e^x)$

30. $f(x) = \ln(x) \cdot \cos(x)$

3.3 Solutions Practical Exercises:

3.3.1 Easy:

1. $\boldsymbol{f(x) = 7}$:

$$\frac{d}{dx}(7) = 0, \quad \text{Derivative of a constant:} \quad \frac{d}{dx}(c) = 0$$

2. $\boldsymbol{f(x) = 5x}$:

$$\begin{aligned} \frac{d}{dx}(5x) &= 5\frac{dx}{dx} && \text{Take the constant out: } (a \cdot f)' = a \cdot f' \\ &= 5 \cdot 1 && \text{Apply the common derivative: } \frac{dx}{dx} = 1 \\ &= 5 \end{aligned}$$

3. $\boldsymbol{f(x) = x^3}$:

$$\begin{aligned} \frac{d}{dx}(x^3) &= 3x^{3-1} \cdot \frac{d}{dx}(x) && \text{Apply the power rule: } (x^n)' = n \cdot x^{n-1} \\ &= 3x^2 \cdot 1 && \text{Take the derivative of } x \text{ which is } 1 \\ &= 3x^2 \end{aligned}$$

4. $\boldsymbol{f(x) = \sqrt{x}}$:

$$\begin{aligned} \frac{d}{dx}(\sqrt{x}) &= \frac{1}{2}x^{-\frac{1}{2}} \cdot \frac{d}{dx}(x) && \text{Apply the power rule:} \\ &= \frac{1}{2}x^{-\frac{1}{2}} \cdot 1 && (x^n)' = n \cdot x^{n-1} \\ &= \frac{1}{2\sqrt{x}} \end{aligned}$$

5. $\boldsymbol{f(x) = e^x}$:

$$\begin{aligned} \frac{d}{dx}(e^x) &= e^x \cdot \frac{d}{dx}(x) && \text{Apply the chain rule:} \\ &= e^x \cdot 1 && (e^{g(x)})' = g'(x) \cdot e^{g(x)} \\ &= e^x \end{aligned}$$

6. $\boldsymbol{f(x) = \ln(x)}$:

$$\begin{aligned} \frac{d}{dx}(\ln(x)) &= \frac{1}{x} \cdot \frac{d}{dx}(x) && \text{Apply the chain rule:} \\ &= \frac{1}{x} \cdot 1 && (\ln(g(x)))' = \frac{1}{g(x)} \cdot g'(x) \\ &= \frac{1}{x} \end{aligned}$$

7. $\boldsymbol{f(x) = \sin(x)}$:

$$\frac{d}{dx}(\sin(x)) = \cos(x) \quad \text{Apply the derivative of } \sin(x)$$

8. $\boldsymbol{f(x) = \cos(x)}$:

$$\frac{d}{dx}(\cos(x)) = -\sin(x) \quad \text{Apply the derivative of } \cos(x)$$

9. $\boldsymbol{f(x) = \tan(x)}$:

$$\frac{d}{dx}(\tan(x)) = \sec^2(x) \quad \text{Apply the derivative of } \tan(x)$$

10. $\boldsymbol{f(x) = 4^x}$:

$$\frac{d}{dx}(4^x) = \ln(4) \cdot 4^x \quad \text{Apply the derivative of } a^x \text{ where } a \neq 1$$

11. $\boldsymbol{f(x) = \log_3(x)}$:

$$\frac{d}{dx}(\log_3(x)) = \frac{1}{\ln(3) \cdot x} \quad \text{Apply the derivative of } \log_a(x) \text{ where } a \neq 1$$

12. $\boldsymbol{f(x) = \sec(x)}$:

$$\frac{d}{dx}(\sec(x)) = \sec(x)\tan(x) \quad \text{Apply the derivative of } \sec(x)$$

13. $\boldsymbol{f(x) = \csc(x)}$:

$$\frac{d}{dx}(\csc(x)) = -\csc(x)\cot(x) \quad \text{Apply the derivative of } \csc(x)$$

14. $\boldsymbol{f(x) = \cot(x)}$:

$$\frac{d}{dx}(\cot(x)) = -\csc^2(x) \quad \text{Apply the derivative of } \cot(x)$$

15. $\boldsymbol{f(x) = \arcsin(x)}$:

$$\frac{d}{dx}(\arcsin(x)) = \frac{1}{\sqrt{1-x^2}} \quad \text{Apply the derivative of } \arcsin(x)$$

16. $\boldsymbol{f(x) = \arccos(x)}$:

$$\frac{d}{dx}(\arccos(x)) = -\frac{1}{\sqrt{1-x^2}} \quad \text{Apply the derivative of } \arccos(x)$$

17. $\boldsymbol{f(x) = \arctan(x)}$:

$$\frac{d}{dx}(\arctan(x)) = \frac{1}{1+x^2} \quad \text{Apply the derivative of } \arctan(x)$$

18. $\boldsymbol{f(x) = \sinh(x)}$:

$$\frac{d}{dx}(\sinh(x)) = \cosh(x) \quad \text{Apply the derivative of } \sinh(x)$$

19. $\boldsymbol{f(x) = \cosh(x)}$:

$$\frac{d}{dx}(\cosh(x)) = \sinh(x) \quad \text{Apply the derivative of } \cosh(x)$$

20. $\boldsymbol{f(x) = \tanh(x)}$:

$$\frac{d}{dx}(\tanh(x)) = \text{sech}^2(x) \quad \text{Apply the derivative of } \tanh(x)$$

21. $\boldsymbol{f(x) = x^2 + 4x}$:

$$\begin{aligned} \frac{d}{dx}(x^2 + 4x) = \frac{d}{dx}(x^2) + \frac{d}{dx}(4x) &\qquad \text{Apply sum rule} \\ = 2x + 4 &\qquad \text{Apply power rule} \end{aligned}$$

22. $\boldsymbol{f(x) = 3x^3 - 5x^2 + 2x - 7}$:

$$\begin{aligned} &\frac{d}{dx}(3x^3 - 5x^2 + 2x - 7) \\ &= \frac{d}{dx}(3x^3) - \frac{d}{dx}(5x^2) + \frac{d}{dx}(2x) - \frac{d}{dx}(7) \quad \text{Apply sum rule} \\ &= 9x^2 - 10x + 2 \end{aligned}$$

23. $\boldsymbol{f(x) = e^{2x}}$:

$$\begin{aligned} \frac{d}{dx}(e^{2x}) &= 2e^{2x} \qquad \text{Apply chain rule:} \frac{d}{dx}(e^u) = u'e^u \\ &= 2e^{2x} \qquad \text{where } u = 2x \end{aligned}$$

24. $\boldsymbol{f(x) = \ln(3x)}$:

$$\begin{aligned} \frac{d}{dx}(\ln(3x)) &= \frac{1}{3x} \cdot \frac{d}{dx}(3x) \qquad \text{Apply chain rule:} \\ &= \frac{1}{3x} \cdot 3 \qquad \frac{d}{dx}(\ln(u)) = \frac{1}{u} \cdot \frac{du}{dx} \\ &= \frac{1}{x} \end{aligned}$$

25. $\boldsymbol{f(x) = \sin(2x)}$:

$$\frac{d}{dx}(\sin(2x)) = \cos(2x) \cdot \frac{d}{dx}(2x) \qquad \text{Apply chain rule}$$
$$= \frac{d}{dx}(\sin(u)) = \cos(u) \cdot \frac{du}{dx}$$
$$= \cos(2x) \cdot 2$$
$$= 2\cos(2x)$$

26. $\boldsymbol{f(x) = \cos(3x)}$:

$$\frac{d}{dx}(\cos(3x)) = -\sin(3x) \cdot \frac{d}{dx}(3x) \qquad \text{Apply chain rule:}$$
$$\frac{d}{dx}(\cos(u)) = -\sin(u) \cdot \frac{du}{dx}$$
$$= -3\sin(3x)$$

27. $\boldsymbol{f(x) = \tan(4x)}$:

$$\frac{d}{dx}(\tan(4x)) = \sec^2(4x) \cdot \frac{d}{dx}(4x) \qquad \text{Apply chain rule:}$$
$$= \frac{d}{dx}(\tan(u)) = \sec^2(u) \cdot \frac{du}{dx}$$
$$= 4\sec^2(4x)$$

28. $\boldsymbol{f(x) = \sec(5x)}$

$$\frac{d}{dx}(\sec(5x)) = \sec(5x) \cdot \tan(5x) \cdot \frac{d}{dx}(5x)$$
$$= 5\sec(5x) \cdot \tan(5x)$$

29. $\boldsymbol{f(x) = \csc(6x)}$:

$$\frac{d}{dx}(\csc(6x)) = -\csc(6x) \cdot \cot(6x) \cdot \frac{d}{dx}(6x)$$
$$= -6\csc(6x) \cdot \cot(6x)$$

30. $\boldsymbol{f(x) = \cot(7x)}$:

$$\frac{d}{dx}(\cot(7x)) = -\csc^2(7x) \cdot \frac{d}{dx}(7x)$$
$$= -7 \cdot \csc^2(7x)$$

3.3.2 Medium:

1. $\boldsymbol{f(x) = \sin(x) \cdot \ln(x)}$:

   Using the product rule, we find the derivatives of both functions and sum their products.

$$\begin{aligned} \text{Let } u(x) &= \sin(x) \quad v(x) = \ln(x) \\ u'(x) &= \cos(x) \quad v'(x) = \frac{1}{x} \end{aligned}$$

   Apply the product rule:

$$\begin{aligned} f'(x) &= (u \cdot v)' \\ &= u' \cdot v + u \cdot v' \\ &= (\cos(x) \cdot \ln(x)) + (\sin(x) \cdot \frac{1}{x}) \\ &= \cos(x) \cdot \ln(x) + \frac{\sin(x)}{x} \end{aligned}$$

2. $\boldsymbol{f(x) = e^{3x} \cdot \cos(2x)}$:

   Utilizing the product rule, we subsequently apply the chain rule to find the derivatives of the inner functions.

$$\begin{aligned} \text{Let } u(x) &= e^{3x} \qquad v(x) = \cos(2x) \\ u'(x) &= 3e^{3x} \quad v'(x) = -2\sin(2x) \end{aligned}$$

   Apply the product rule:

$$\begin{aligned} f'(x) &= (u \cdot v)' \\ &= u' \cdot v + u \cdot v' \\ &= (3e^{3x} \cdot \cos(2x)) + (e^{3x} \cdot (-2\sin(2x))) \\ &= 3e^{3x} \cdot \cos(2x) - 2e^{3x} \cdot \sin(2x) \end{aligned}$$

3. $\boldsymbol{f(x) = \frac{e^{x}}{\sin(x)}}$:

We applied the quotient rule, finding the derivatives of both numerator and denominator and then using the quotient rule formula.

$$\text{Let } u(x) = e^x \quad v(x) = \sin(x)$$
$$u'(x) = e^x \quad v'(x) = \cos(x)$$

Apply the quotient rule:

$$\begin{aligned} f'(x) &= \frac{(u \cdot v)' - (u' \cdot v)}{v^2} \\ &= \frac{(e^x \cdot \sin(x)) - (e^x \cdot \cos(x))}{\sin^2(x)} \\ &= \frac{e^x \cdot \sin(x) - e^x \cdot \cos(x)}{\sin^2(x)} \end{aligned}$$

4. $\boldsymbol{f(x) = \tan(x) \cdot e^{2x}}$:

   Using the product rule, we find the derivatives of both functions and sum their products.

$$\begin{aligned} \text{Let } u(x) &= \tan(x) & v(x) &= e^{2x} \\ u'(x) &= \sec^2(x) & v'(x) &= 2e^{2x} \end{aligned}$$

   Apply the product rule:

$$\begin{aligned} f'(x) &= (u \cdot v)' \\ &= u' \cdot v + u \cdot v' \\ &= (\sec^2(x) \cdot e^{2x}) + (\tan(x) \cdot 2e^{2x}) \end{aligned}$$

5. $\boldsymbol{f(x) = \ln(x^2 + 1)}$:

   For this function, the chain rule is necessary. We first differentiate the outer function and then multiply it by the derivative of the inner function.

$$\text{Let } u(x) = x^2 + 1$$
$$u'(x) = 2x$$

Apply the chain rule:

$$\begin{aligned} f'(x) &= \frac{1}{u} \cdot u'(x) \\ &= \frac{1}{x^2+1} \cdot 2x \\ &= \frac{2x}{x^2+1} \end{aligned}$$

6. $\boldsymbol{f(x) = x^3 \cdot \cos(x)}$:

The product rule is used to find the derivatives of both functions and the sum of their products.

$$\begin{aligned} \text{Let } u(x) &= x^3 & v(x) &= \cos(x) \\ u'(x) &= 3x^2 & v'(x) &= -\sin(x) \end{aligned}$$

Apply the product rule:

$$\begin{aligned} f(x)' &= (u \cdot v)' \\ &= u' \cdot v + u \cdot v' \\ &= (3x^2 \cdot \cos(x)) + (x^3 \cdot (-\sin(x))) \\ &= 3x^2 \cdot \cos(x) - x^3 \cdot \sin(x) \end{aligned}$$

7. $\boldsymbol{f(x) = \frac{\ln(x)}{x^2}}$:

The quotient rule is employed to find the derivative, with the derivatives of both $\ln(x)$ and x^2 taken into account.

$$\begin{aligned} \text{Let } u(x) &= \ln(x) & v(x) &= x^2 \\ u'(x) &= \frac{1}{x} & v'(x) &= 2x \end{aligned}$$

Apply the quotient rule:

$$\begin{aligned}
f'(x) &= \frac{u' \cdot v - u \cdot v'}{v^2} \\
&= \frac{\frac{1}{x} \cdot x^2 - 2x \cdot \ln(x)}{(x^2)^2} \\
&= \frac{x - 2x \cdot \ln(x)}{(x^4} \\
&= \frac{1 - 2\ln(x)}{x^3} \\
&= \frac{1 - 2\ln(x)}{x^3}
\end{aligned}$$

8. $\boldsymbol{f(x) = \sec(x) + \tan(x)}$:

   The sum rule is used to find the derivatives of both functions and then add them together.

$$\begin{aligned}
f'(x) &= \frac{d}{dx}\left(\sec(x) + \tan(x)\right) \\
&= \frac{d}{dx}\sec(x) + \frac{d}{dx}\tan(x) \\
&= \sec(x) \cdot \tan(x) + \sec^2(x) \\
&= \sec(x) \cdot \tan(x) + \sec^2(x)
\end{aligned}$$

9. $\boldsymbol{f(x) = x^3 \cdot \ln(x)}$:

   Using the product rule, the derivatives of both functions are found and then their products are summed.

$$\begin{aligned}
\text{Let } u(x) = x^3 &\quad v(x) = \ln(x) \\
u'(x) = 3x^2 &\quad v'(x) = \frac{1}{x}
\end{aligned}$$

Apply the product rule:

$$\begin{aligned}
f'(x) &= (u \cdot v)' \\
&= u' \cdot v + u \cdot v' \\
&= (3x^2 \cdot \ln(x)) + (x^3 \cdot \frac{1}{x}) \\
&3x^2 \cdot \ln(x) + x^2
\end{aligned}$$

10. $\boldsymbol{f(x) = e^{2x} \cdot \sin(3x)}$:

    The product rule is utilized, with subsequent application of the chain rule to find the derivatives of the inner functions.

$$\begin{aligned} \text{Let } u(x) &= e^{2x} \quad & v(x) &= \sin(3x) \\ u'(x) &= 2e^{2x} & v'(x) &= 3\cos(3x) \end{aligned}$$

    Apply the product rule:

$$\begin{aligned} f'(x) &= (u \cdot v)' \\ &= u' \cdot v + u \cdot v' \\ &= (2e^{2x} \cdot \sin(3x)) + (e^{2x} \cdot 3\cos(3x)) \\ &= 2e^{2x} \cdot \sin(3x) + 3e^{2x} \cdot \cos(3x) \end{aligned}$$

11. $\boldsymbol{f(x) = \tan(\ln(x))}$:

    Using the chain rule, we find the derivative of the inner function and multiply it by the derivative of the outer function.

$$\begin{aligned} \text{Let } u(x) &= \ln(x) \\ u'(x) &= \frac{1}{x} \end{aligned}$$

    Apply the chain rule:

$$\begin{aligned} f'(x) &= \sec^2(u) \cdot u' \\ &= \sec^2(\ln(x)) \cdot \frac{1}{x} \\ &= \frac{\sec^2(\ln(x))}{x} \end{aligned}$$

12. $\boldsymbol{f(x) = \frac{3x^2-4x}{x^3+1}}$:

    Apply the quotient rule:

$$\begin{aligned} f(x)' &= \frac{\frac{d}{dx}(3x^2 - 4x) \cdot (x^3+1) - (3x^2 - 4x) \cdot \frac{d}{dx}(x^3+1)}{(x^3+1)^2} \\ &= \frac{(6x-4) \cdot (x^3+1) - (3x^2-4x) \cdot 3x^2}{(x^3+1)^2} \\ &= \frac{-3x^4 + 8x^3 + 6x - 4}{(x^3+1)^2} \end{aligned}$$

13. $\boldsymbol{f(x) = \sqrt{x} \cdot \ln(x)}$:

Using the chain rule, we find the derivative of the inner function and multiply it by the derivative of the outer function.

$$\text{Let } u(x) = \sqrt{x} \qquad v(x) = \ln(x)$$
$$u'(x) = \frac{1}{2\sqrt{x}} \quad v'(x) = \frac{1}{x}$$

Apply the product rule:

$$\begin{aligned} f'(x) &= (u \cdot v)' \\ &= u' \cdot v + u \cdot v' \\ &= \left(\frac{1}{2\sqrt{x}} \cdot \ln(x)\right) + \left(\sqrt{x} \cdot \frac{1}{x}\right) \\ &= \frac{\ln(x)}{2\sqrt{x}} + \frac{1}{\sqrt{x}} \end{aligned}$$

14. $\boldsymbol{f(x) = \sin^2(x)\ln(x)}$:

Using the chain rule, we find the derivative of the inner function and multiply it by the derivative of the outer function.

$$\text{Let } u(x) = \sin^2(x) \qquad v(x) = \ln(x)$$
$$u'(x) = 2\sin(x) \cdot \cos(x) \quad v'(x) = \frac{1}{x}$$

Apply the product rule:

$$\begin{aligned} f'(x) &= (u \cdot v)' \\ &= u' \cdot v + u \cdot v' \\ &= (2\sin(x) \cdot \cos(x) \cdot \ln(x)) + \left(\sin^2(x) \cdot \frac{1}{x}\right) \\ f'(x) &= 2\sin(x) \cdot \cos(x) \cdot \ln(x) + \frac{\sin^2(x)}{x} \\ &= \sin(2x) \cdot \ln(x) + \frac{\sin^2(x)}{x} \end{aligned}$$

15. $\boldsymbol{f(x) = \dfrac{e^{2x}}{x^2+1}}$

Applying the chain rule, we first find the derivative of the outer function e^{x^2}, and then multiply it by the derivative of the inner function $2x$.

$$\begin{aligned}
f'(x) &= \frac{\frac{d}{dx}e^{2x} \cdot (x^2+1) - e^{2x} \cdot \frac{d}{dx}(x^2+1)}{(x^2+1)^2} \\
&= \frac{2e^{2x} \cdot (x^2+1) - e^{2x} \cdot 2x}{(x^2+1)^2} \\
&= \frac{2(x^2+1)e^{2x} - 2x \cdot e^{2x}}{(x^2+1)^2}
\end{aligned}$$

16. $\boldsymbol{f(x) = \ln(\sec(x) + \tan(x))}$:

Using the chain rule, we find the derivative of the inner function and multiply it by the derivative of the outer function.

$$\begin{aligned}
\text{Let } u(x) &= \sec(x) + \tan(x) \\
u'(x) &= \sec(x) \cdot \tan(x) + \sec^2(x)
\end{aligned}$$

Apply the chain rule:

$$\begin{aligned}
f'(x) &= \frac{1}{u} \cdot u' \\
&= \frac{\sec(x) \cdot \tan(x) + \sec^2(x)}{\sec(x) + \tan(x)} \\
&= \frac{\sec(x) \cdot \tan(x) + \sec^2(x)}{\sec(x) + \tan(x)}
\end{aligned}$$

17. $\boldsymbol{f(x) = x^3 \cdot \cos^2(x)}$:

Using the sum rule, we find the derivatives of both functions separately and then subtract the second function's derivative from the first.

$$\begin{aligned}
f'(x) &= \frac{1}{\sec(x) + \tan(x)} \frac{d}{dx}(\sec(x) + \tan(x)) \\
&= \frac{1}{\sec(x) + \tan(x)}(\sec(x) \cdot \tan(x) + \sec^2(x)) \\
&= \sec(x) \cdot \frac{1}{\sec(x) + \tan(x)}(\tan(x) + \sec(x)) \\
&= \sec(x)
\end{aligned}$$

18. $\boldsymbol{f(x) = \frac{1}{x} \cdot \arctan(x)}$:

    Apply Product Rule:

$$\begin{aligned} f'(x) &= \frac{1}{x} \cdot \frac{d}{dx} \arctan(x) + \arctan(x) \frac{d}{dx} \cdot \frac{1}{x} \\ &= -\frac{1}{x\left(x^2+1\right)} + \frac{\arctan(x)}{x^2} \end{aligned}$$

19. $\boldsymbol{f(x) = \sqrt{x} \cdot e^x}$:

    We applied the product rule here, finding the derivative of both e^x and $\sqrt{x}$, and then combined them according to the rule.

$$\begin{aligned} f'(x) &= \sqrt{x} \cdot \frac{d}{dx} e^x + e^x \cdot \frac{d}{dx} \sqrt{x} \\ &= e^x \sqrt{x} + \frac{1}{2\sqrt{x}} e^x \end{aligned}$$

20. $\boldsymbol{f(x) = \sin(x) \cdot \cos(x)}$:

    Applying the product rule, we calculate the derivatives of both $\sin(x)$ and $\cos(x)$, and sum their products.

$$\begin{aligned} f'(x) &= \frac{d}{dx}(\sin(x) \cdot \cos(x)) \\ &= \cos(x) \frac{d}{dx} \sin(x) + \sin(x) \frac{d}{dx} \cos(x) \\ &= \cos(x) \cdot \cos(x) - \sin(x) \cdot \sin(x) \\ &= \cos^2(x) - \sin^2(x) \\ &= \cos(2x) \end{aligned}$$

21. $\boldsymbol{f(x) = \ln(x) \cdot \cos(x)}$:

    Using the product rule, we find the derivatives of both functions and sum their products.

$$\begin{aligned} \text{Let } u(x) &= \ln(x) & v(x) &= \cos(x) \\ u'(x) &= \frac{1}{x} & v'(x) &= -\sin(x) \end{aligned}$$

Apply the product rule:

$$\begin{aligned} f'(x) &= (u \cdot v)' \\ &= u' \cdot v + u \cdot v' \\ &= \frac{\cos(x)}{x} - \ln(x) \cdot \sin(x) \end{aligned}$$

22. $\boldsymbol{f(x) = e^x \cdot \tan(x)}$:

Utilizing the quotient rule, we find the derivatives of both functions and then sum their products.

$$\begin{aligned} f'(x) &= \frac{d}{dx}\left(e^x \cdot \tan(x)\right) \\ &= e^x \cdot \frac{d}{dx}\tan(x) + \tan(x)\frac{d}{dx} \cdot e^x \\ &= e^x \cdot \sec^2(x) + \tan(x) \cdot e^x \end{aligned}$$

23. $\boldsymbol{f(x) = \frac{\sin(x)}{x^3+1}}$:

Using the quotient rule:

$$f'(x) = \frac{u' \cdot v - u \cdot v'}{v^2}$$

Where:

$$u(x) = \sin(x), \quad u'(x) = \cos(x)$$

$$v(x) = x^3 + 1, \quad v'(x) = 3x^2$$

Substituting the functions and their derivatives, we get:

$$f'(x) = \frac{\cos(x) \cdot (x^3 + 1) - \sin(x) \cdot 3x^2}{(x^3 + 1)^2}$$

24. $\boldsymbol{f(x) = x\ln(x) \cdot \sin(x)}$:

For simplicity, let's first consider the product $g(x) = x \cdot \ln(x)$.

Using the product rule:

$$g'(x) = 1 \cdot \ln(x) + x \cdot \frac{1}{x} = \ln(x) + 1$$

Now, using the product of $g(x)$ with $\sin(x)$, and applying the product rule:

$$f'(x) = g'(x) \cdot \sin(x) + g(x) \cdot \cos(x)$$

Substituting in for $g(x)$ and $g'(x)$:

$$f'(x) = (\ln(x) + 1) \cdot \sin(x) + x \cdot \ln(x) \cdot \cos(x)$$

25. $\boldsymbol{f(x) = \tan(x) \cdot \ln(\cos(x))}$:

Using the product rule:

$$f'(x) = u' \cdot v + u \cdot v'$$

Where:

$$\begin{aligned} u(x) &= \tan(x) & u'(x) &= \sec^2(x) \\ v(x) &= \ln(\cos(x)) & v'(x) &= \frac{-\sin(x)}{\cos(x)} \end{aligned}$$

Substituting the functions and their derivatives, we get:

$$\begin{aligned} f'(x) &= \sec^2(x) \cdot \ln(\cos(x)) + \tan(x) \cdot \frac{-\sin(x)}{\cos(x)} \\ &= \sec^2(x) \cdot \ln(\cos(x)) - \tan(x) \cdot \tan(x) \\ &= \sec^2(x) \cdot \ln(\cos(x)) - \tan^2(x) \end{aligned}$$

26. $\boldsymbol{f(x) = \frac{e^{3x}}{x^2-1}}$:

Using the quotient rule:

$$f'(x) = \frac{u' \cdot v - u \cdot v'}{v^2}$$

Where:

$$u(x) = e^{3x} \qquad u'(x) = 3e^{3x}$$
$$v(x) = x^2 - 1 \qquad v'(x) = 2x$$

Substituting the functions and their derivatives into the quotient rule:

$$f'(x) = \frac{3e^{3x} \cdot (x^2 - 1) - e^{3x} \cdot 2x}{(x^2 - 1)^2}$$

27. $\boldsymbol{f(x) = \ln(x) \cdot \sin^2(x)}$:

Using the product rule:

$$f'(x) = u' \cdot v + u \cdot v'$$

Where:

$$u(x) = \ln(x) \qquad u'(x) = \frac{1}{x}$$
$$v(x) = \sin^2(x) \qquad v'(x) = 2\sin(x) \cdot \cos(x) = \sin(2x)$$

Substituting the functions and their derivatives into the product rule:

$$\begin{aligned} f'(x) &= \frac{1}{x} \cdot \sin^2(x) + \ln(x) \cdot \sin(2x) \\ &= \frac{\sin^2(x)}{x} + \ln(x) \cdot \sin(2x) \end{aligned}$$

28. $\boldsymbol{f(x) = \sin(x) \cdot e^{x^2}}$:

Given Function: $f(x) = \sin(x) \cdot e^{x^2}$

Using the product rule:

$$f'(x) = u' \cdot v + u \cdot v'$$

Where:

$$u(x) = \sin(x) \qquad u'(x) = \cos(x)$$
$$v(x) = e^{x^2} \qquad v'(x) = 2x \cdot e^{x^2}$$

Substituting the functions and their derivatives into the product rule:

$$\begin{aligned} f'(x) &= \cos(x) \cdot e^{x^2} + \sin(x) \cdot 2x \cdot e^{x^2} \\ &= e^{x^2} \cdot \cos(x) + 2x \cdot \sin(x) \cdot e^{x^2} \end{aligned}$$

29. $\boldsymbol{f(x) = \frac{\ln(x^3)}{x^2}}$:

    To differentiate, we use the quotient rule:

    $$\left(\frac{u}{v}\right)' = \frac{u' \cdot v - u \cdot v'}{v^2}$$

    Where:

    $$\begin{aligned} u(x) &= \ln(x^3) & u'(x) &= \frac{3}{x} \\ v(x) &= x^2 & v'(x) &= 2x \end{aligned}$$

    Substituting the functions and their derivatives into the quotient rule:

    $$\begin{aligned} f'(x) &= \frac{\frac{3}{x} \cdot x^2 - \ln(x^3) \cdot 2x}{(x^2)^2} \\ &= \frac{3x - 2x\ln(x^3)}{x^4} \end{aligned}$$

30. $\boldsymbol{f(x) = \cos(x) \cdot \ln(\sin(x))}$:

    To differentiate, we use the product rule:

    $$(u \cdot v)' = u' \cdot v + u \cdot v'$$

    Where:

    $$\begin{aligned} u(x) &= \cos(x) & u'(x) &= -\sin(x) \\ v(x) &= \ln(\sin(x)) & v'(x) &= \cot(x) \end{aligned}$$

    Substituting the functions and their derivatives into the product rule:

    $$f'(x) = -\sin(x) \cdot \ln(\sin(x)) + \cos(x) \cdot \cot(x)$$

3.3.3 Hard:

1. $\boldsymbol{f(x) = e^{x^2} \cdot \ln(x)}$:

   Apply the product rule. First term: differentiate e^{x^2} and keep $\ln(x)$ as it is. Second term: differentiate $\ln(x)$ and keep e^{x^2} as it is.

$$\begin{aligned} f'(x) &= \frac{d}{dx}\left(e^{x^2}\ln(x)\right) \\ &= \ln(x)\cdot\frac{d}{dx}\cdot e^{x^2} + e^{x^2}\cdot\frac{d}{dx}\cdot\ln(x) \\ &= 2x\cdot\ln(x)\cdot e^{x^2} + e^{x^2}\cdot\frac{1}{x} \end{aligned}$$

2. $\boldsymbol{f(x) = \sin(x^2) \cdot e^{x^3}}$:

   Apply the product rule. The first term results from the chain rule applied on $\sin(x^2)$, the second term comes from the chain rule applied on e^{x^3}.

$$\begin{aligned} f'(x) &= \frac{d}{dx}(\sin(x^2)\cdot e^{x^3}) \\ &= \cos(x^2)\cdot\frac{d}{dx}(x^2)\cdot e^{x^3} + \sin(x^2)\cdot\frac{d}{dx}(e^{x^3}) \\ &= \cos(x^2)\cdot 2x\cdot e^{x^3} + \sin(x^2)\cdot 3x^2\cdot e^{x^3} \end{aligned}$$

3. $\boldsymbol{f(x) = \tan(x \cdot e^x)}$:

   Chain rule. Differentiate the outer function $\tan(u)$ and multiply by the derivative of the inner function $x \cdot e^x$.

$$\begin{aligned} f'(x) &= \frac{d}{dx}(\tan(x\cdot e^x)) \\ &= \sec^2(x\cdot e^x)\cdot\frac{d}{dx}(x\cdot e^x) \\ &= \sec^2(x\cdot e^x)\cdot(e^x + x\cdot e^x) \end{aligned}$$

4. $\boldsymbol{f(x) = \ln(x^2 + e^{2x})}$:

Apply the chain rule. Differentiate the outside function $\ln(u)$ and multiply by the derivative of the inside function $x^2 + e^{2x}$.

$$\begin{aligned} f'(x) &= \frac{d}{dx}(\ln(x^2 + e^{2x})) \\ &= \frac{1}{x^2 + e^{2x}} \cdot \frac{d}{dx}(x^2 + e^{2x}) \\ &= \frac{2x + 2e^{2x}}{x^2 + e^{2x}} \end{aligned}$$

5. $\boldsymbol{f(x) = x^x}$:

   Use the natural logarithm differentiation technique. Remember, the derivative of $y = a^x$ is $y' = \ln(a) \cdot a^x$. Take the natural logarithm of both sides:

   $$\ln(f(x)) = \ln(x^x)$$

   Then, apply the logarithmic identity $\ln(a^b) = b\ln(a)$:

   $$\ln(f(x)) = x\ln(x)$$

   Differentiate both sides:

   $$\begin{aligned} \frac{1}{f(x)} \cdot f'(x) &= \ln(x) + 1 \\ f'(x) &= x^x(\ln(x) + 1) \end{aligned}$$

6. $\boldsymbol{f(x) = \ln(x) \cdot \cos(x)}$:

   Apply the product rule. Differentiate $\ln(x)$ while keeping $\cos(x)$ constant. Then, differentiate $\cos(x)$ while keeping $\ln(x)$ constant.

   $$\begin{aligned} f'(x) &= \frac{d}{dx}(\ln(x) \cdot \cos(x)) \\ &= \cos(x) \cdot \frac{1}{x} - \ln(x) \cdot \sin(x) \end{aligned}$$

7. $\boldsymbol{f(x) = \sqrt{x} \cdot e^{x^2}}$:

Apply the product rule. Differentiate $\sqrt{x}$ while keeping e^{x^2} constant. Then, differentiate e^{x^2} while keeping $\sqrt{x}$ constant.

$$\begin{aligned} f'(x) &= \frac{d}{dx}(\sqrt{x} \cdot e^{x^2}) \\ &= \frac{1}{2\sqrt{x}} \cdot e^{x^2} + \sqrt{x} \cdot 2xe^{x^2} \end{aligned}$$

8. $\boldsymbol{f(x) = x^{\sin(x)}}$:

   Apply the natural logarithm differentiation technique and chain rule.

   Take the natural logarithm of both sides:

$$\ln(f(x)) = \sin(x) \cdot \ln(x)$$

   Differentiate both sides with respect to x:

$$\frac{1}{f(x)} \cdot f'(x) = \cos(x) \cdot \ln(x) + \frac{\sin(x)}{x}$$

   Solve for $f'(x)$:

$$f'(x) = x^{\sin(x)} \cdot \left(\cos(x) \cdot \ln(x) + \frac{\sin(x)}{x}\right)$$

9. $\boldsymbol{f(x) = \arctan(x^3)}$:

   Chain rule. Differentiate the outer function $\arctan(u)$ and multiply by the derivative of the inner function x^3.

$$\begin{aligned} f'(x) &= \frac{d}{dx}(\arctan(x^3)) \\ &= \frac{1}{1+(x^3)^2} \cdot \frac{d}{dx}(x^3) \\ &= \frac{3x^2}{1+x^6} \end{aligned}$$

10. $\boldsymbol{f(x) = \frac{e^x}{x^2+1}}$:

Apply the quotient rule. Differentiate the numerator and denominator separately and then use the formula for the quotient rule.

$$\begin{aligned} f'(x) &= \frac{d}{dx}\left(\frac{e^x}{x^2+1}\right) \\ &= \frac{(e^x)' \cdot (x^2+1) - (e^x \cdot (x^2+1)')}{(x^2+1)^2} \\ &= \frac{e^x(x^2+1) - 2x \cdot e^x}{(x^2+1)^2} \end{aligned}$$

11. $\boldsymbol{f(x) = x^2 \cdot \ln(x^2+1)}$:

    Apply the product rule. First, differentiate x^2 keeping $\ln(x^2+1)$ constant. Second, differentiate $\ln(x^2+1)$ while keeping x^2 constant, using the chain rule.

$$\begin{aligned} f'(x) &= \frac{d}{dx}(x^2 \ln(x^2+1)) \\ &= 2x \cdot \ln(x^2+1) + x^2 \cdot \frac{1}{x^2+1} \cdot 2x \\ &= 2x \cdot \ln(x^2+1) + \frac{2x^3}{x^2+1} \end{aligned}$$

12. $\boldsymbol{f(x) = \sin(x \cdot \ln(x))}$:

    Apply the chain rule. Differentiate the outer function $\sin(u)$ and multiply by the derivative of the inner function, $x\ln(x)$, using the product rule.

$$\begin{aligned} f'(x) &= \frac{d}{dx}(\sin(x \cdot \ln(x))) \\ &= \cos(x\ln(x)) \cdot (\ln(x) + 1) \\ &= \cos(x\ln(x)) \cdot (\ln(x) + \frac{1}{x}) \end{aligned}$$

13. $\boldsymbol{f(x) = \sqrt{x^3 + e^x}}$:

Use the chain rule. Differentiate the outer function $\sqrt{u}$ and multiply by the derivative of the inner function $x^3 + e^x$.

$$\begin{aligned} f'(x) &= \frac{d}{dx}(\sqrt{x^3 + e^x}) \\ &= \frac{1}{2\sqrt{x^3 + e^x}} \cdot (3x^2 + e^x) \end{aligned}$$

14. $\boldsymbol{f(x) = x^x \cdot e^x}$:

    Apply the product rule to x^x and e^x. Differentiate x^x using the chain rule.

$$\begin{aligned} f'(x) &= \frac{d}{dx}(x^x \cdot e^x) \\ &= x^x(\ln(x) + 1)e^x + x^x \cdot e^x \\ &= x^x \cdot e^x(\ln(x) + 2) \end{aligned}$$

15. $\boldsymbol{f(x) = \ln(\cos(e^x))}$: Apply the chain rule. Differentiate the outer function $\ln(u)$ and multiply by the derivative of the inner function $\cos(e^x)$.

$$\begin{aligned} f'(x) &= \frac{d}{dx}(\ln(\cos(e^x))) \\ &= \frac{1}{\cos(e^x)} \cdot (-\sin(e^x)) \cdot e^x \\ &= -\tan(e^x) \cdot e^x \end{aligned}$$

16. $\boldsymbol{f(x) = x \cdot e^{x \ln(x)}}$:

    Apply the chain rule, differentiating $e^{x \ln(x)}$ with respect to $x \ln(x)$, then multiply by the derivative of $x \ln(x)$ using the product rule.

$$\begin{aligned} f'(x) &= \frac{d}{dx}(x \cdot e^{x \ln(x)}) \\ &= e^{x \ln(x)} + x \cdot e^{x \ln(x)} \cdot \frac{d}{dx}(x \cdot \ln(x)) \\ &= x^x + x \cdot e^{x \ln(x)} \cdot (\ln(x) + 1) \end{aligned}$$

17. $\boldsymbol{f(x) = \frac{\ln(x)}{x^2}}$:

    Quotient rule. Differentiate the numerator and denominator separately, then apply the quotient rule formula.

$$\begin{aligned} f'(x) &= \frac{d}{dx}\left(\frac{\ln(x)}{x^2}\right) \\ &= \frac{\frac{1}{x} \cdot x^2 - \ln(x) \cdot 2x}{x^4} \\ &= \frac{x - 2x \cdot \ln(x)}{x^4} \\ &= \frac{1 - 2 \cdot \ln(x)}{x^3} \end{aligned}$$

18. $\boldsymbol{f(x) = \tan(x^2 \cdot e^x)}$:

    Chain rule. Differentiate the outer function and multiply by the derivative of the inner function x^2e^x.

$$\begin{aligned} f'(x) &= \frac{d}{dx}(\tan(x^2 \cdot e^x)) \\ &= \sec^2(x^2 \cdot e^x) \cdot \frac{d}{dx}(x^2 \cdot e^x) \\ &= \sec^2(x^2 \cdot e^x)(2x \cdot e^x + x^2 \cdot e^x) \end{aligned}$$

19. $\boldsymbol{f(x) = \sec(x) \cdot \ln(x)}$:

    Apply the product rule: differentiate $\sec(x)$ and keep $\ln(x)$ constant, and vice versa, then sum the two terms.

$$\begin{aligned} f'(x) &= \frac{d}{dx}(\sec(x) \cdot \ln(x)) \\ &= \sec(x) \cdot \frac{d}{dx}(\ln(x)) + \ln(x) \cdot \frac{d}{dx}(\sec(x)) \\ &= \sec(x) \cdot \frac{1}{x} + \ln(x) \cdot \sec(x)\tan(x) \end{aligned}$$

20. $\boldsymbol{f(x) = \sin(x) \cdot e^{x^3}}$:

Apply the product rule: differentiate $\sin(x)$ and keep e^{x^3} constant, and vice versa, then sum the two terms.

$$\begin{aligned} f'(x) &= \frac{d}{dx}(\sin(x)e^{x^3}) \\ &= \cos(x) \cdot e^{x^3} + \sin(x) \cdot \frac{d}{dx}(e^{x^3}) \\ &= \cos(x) \cdot e^{x^3} + 3x^2 \cdot \sin(x) \cdot e^{x^3} \end{aligned}$$

21. $\boldsymbol{f(x) = \frac{e^{x^2}}{x^2}}$:

Apply the quotient rule:

$$\begin{aligned} f'(x) &= \frac{d}{dx}\left(\frac{e^{x^2}}{x^2}\right) \\ &= \frac{2x \cdot e^{x^2} \cdot x^2 - e^{x^2} \cdot 2x}{x^4} \\ &= \frac{2x \cdot e^{x^2}(x^2 - 1)}{x^4} \\ &= \frac{2e^{x^2}(x^2 - 1)}{x^3} \end{aligned}$$

22. $\boldsymbol{f(x) = \arccos(x \cdot e^x)}$:

Using the chain rule, differentiate the outside function, $\arccos(u)$, and multiply by the derivative of the inside function, $x \cdot e^x$:

$$\begin{aligned} f'(x) &= \frac{d}{dx}(\arccos(x \cdot e^x)) \\ &= -\frac{1}{\sqrt{1 - (xe^x)^2}} \cdot \frac{d}{dx}(x \cdot e^x) \\ &= -\frac{e^x + x \cdot e^x}{\sqrt{1 - x^2 \cdot e^{2x}}} \end{aligned}$$

23. $\boldsymbol{f(x) = x^3 \cdot \cos(x^2)}$:

Using the product rule where the first term differentiates x^3 and keeps $\cos(x^2)$ constant and the second term differentiates

$\cos(x^2)$ and keeps x^3 constant:

$$\begin{aligned} f'(x) &= \frac{d}{dx}(x^3 \cos(x^2)) \\ &= 3x^2 \cdot \cos(x^2) - x^3 \cdot 2x \sin(x^2) \\ &= 3x^2 \cdot \cos(x^2) - 2x^4 \cdot \sin(x^2) \end{aligned}$$

24. $\boldsymbol{f(x) = \ln(x) \cdot x^x}$:

    Product rule, followed by the chain rule applied to the function inside the logarithm.

$$\begin{aligned} f'(x) &= \frac{d}{dx}(\ln(x) \cdot x^x) \\ &= \frac{1}{x} \cdot x^x + \ln(x) \cdot \frac{d}{dx}(x^x) \\ &= x^{x-1} + \ln(x) \cdot x^x \cdot (\ln(x) + 1) \end{aligned}$$

25. $\boldsymbol{f(x) = e^{x \ln(x)}}$:

    Quotient rule, with chain rule applied to the denominator function.

$$\begin{aligned} f'(x) &= \frac{d}{dx}(e^{x \ln(x)}) \\ &= e^{x \ln(x)} \cdot \frac{d}{dx}(x \cdot \ln(x)) \\ &= x^x \cdot (\ln(x) + 1) \end{aligned}$$

26. $\boldsymbol{f(x) = \tan(\ln(x) \cdot e^x)}$:

$$\begin{aligned} f'(x) &= \frac{d}{dx}(\tan(\ln(x) \cdot e^x)) \\ &= \sec^2(\ln(x) \cdot e^x) \cdot \frac{d}{dx}(\ln(x) \cdot e^x) \\ &= \sec^2(\ln(x) \cdot e^x) \cdot \left(\frac{1}{x} \cdot e^x + e^x \cdot \ln(x)\right) \end{aligned}$$

27. $\boldsymbol{f(x) = x^3 \cdot \arcsin(x)}$:

Differentiating the function using the chain rule, with the inner function $x\ln(x)$.

$$\begin{aligned} f'(x) &= \frac{d}{dx}(x^3 \cdot \arcsin(x)) \\ &= 3x^2 \cdot \arcsin(x) + x^3 \cdot \frac{1}{\sqrt{1-x^2}} \cdot \frac{d}{dx}(x) \\ &= 3x^2 \cdot \arcsin(x) + \frac{x^3}{\sqrt{1-x^2}} \end{aligned}$$

28. $\boldsymbol{f(x) = \frac{\ln(x)}{\cos(x)}}$:

   To differentiate, we use the quotient rule:

   $$\left(\frac{u}{v}\right)' = \frac{u' \cdot v - u \cdot v'}{v^2}$$

   Where:

   $$\begin{aligned} u(x) &= \ln(x) & u'(x) &= \frac{1}{x} \\ v(x) &= \cos(x) & v'(x) &= -\sin(x) \end{aligned}$$

   Substituting the functions and their derivatives into the quotient rule:

   $$f'(x) = \frac{\frac{1}{x} \cdot \cos(x) - \ln(x) \cdot (-\sin(x))}{\cos^2(x)}$$

   $$= \frac{\cos(x) + \sin(x) \cdot \ln(x)}{x \cdot \cos^2(x)}$$

29. $\boldsymbol{f(x) = \sin(x^2 + e^x)}$:

   To differentiate the function, we use the chain rule, as the function is a composition of two functions: the outer function $\sin(u)$ and the inner function $u = x^2 + e^x$.

   Differentiate the outer function with respect to its argument:

   $$\frac{d}{du}\sin(u) = \cos(u)$$

Differentiate the inner function with respect to x:

$$\frac{d}{dx}(x^2 + e^x) = 2x + e^x$$

Apply the chain rule by multiplying the two results together:

$$\begin{aligned} f'(x) &= \cos(x^2 + e^x) \cdot (2x + e^x) \\ &= (2x + e^x) \cdot \cos(x^2 + e^x) \end{aligned}$$

30. $\boldsymbol{f(x) = \ln(x) \cdot \cos(x)}$:

To differentiate, we use the product rule:

$$(u \cdot v)' = u' \cdot v + u \cdot v'$$

Where:

$$\begin{aligned} u(x) &= \ln(x) & u'(x) &= \frac{1}{x} \\ v(x) &= \cos(x) & v'(x) &= -\sin(x) \end{aligned}$$

Substituting the functions and their derivatives into the product rule:

$$f'(x) = \frac{1}{x} \cdot \cos(x) - \ln(x) \cdot \sin(x)$$

Chapter 4

Limits with L'Hopitals Rule

L'Hôpital's insight was that, under certain conditions, the ratio of two functions that approach an indeterminate form can be simplified by examining the ratio of their derivatives.

Specifically, if $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$ both approach 0 or both approach ∞, and the derivatives $f'(x)$ and $g'(x)$ are continuous near c (excluding possibly at c), then:

$$\lim_{x\to c}\frac{f(x)}{g(x)} = \lim_{x\to c}\frac{f'(x)}{g'(x)}$$

provided the limit on the right-hand side exists or is ∞ or $-\infty$.

4.1 Practical Exercises:

1. $\boldsymbol{\lim_{x\to 0} \frac{\tan(x)-x}{x^3}}$ **:**

2. $\boldsymbol{\lim_{x\to 0} \frac{x-\sin(x)}{x^3}}$ **:**

3. $\boldsymbol{\lim_{x\to \infty} \frac{\ln(x+1)}{x}}$ **:**

4. $\boldsymbol{\lim_{x\to 0} \frac{e^{2x}-1-2x}{x^2}}$ **:**

5. $\boldsymbol{\lim_{x\to \infty} \frac{x^2+2x}{e^x}}$ **:**

6. $\boldsymbol{\lim_{x\to 0+} x\ln(x)}$ **:**

7. $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x}$:

8. $\lim_{x\to 0+}\frac{1}{x}-\frac{1}{\sin(x)}$:

9. $\lim_{x\to 0}\frac{1}{x^2}-\frac{1}{x\tan(x)}$:

10. $\lim_{x\to\infty}\frac{\ln(x)}{e^{\frac{x}{2}}}$:

4.2 Solutions Practical Exercises:

1. $\lim_{x\to 0}\frac{\tan(x)-x}{x^3}$:

   Applying L'Hôpital's rule three times due to the indeterminate form $\frac{0}{0}$, we differentiate the numerator and the denominator separately until we get a determinate form. The limit evaluates to 0 as the derivative of the numerator becomes 0 and the denominator remains a finite value.

$$
\begin{aligned}
\lim_{x\to 0} \frac{\tan(x) - x}{x^3} &= \lim_{x\to 0} \frac{\frac{d}{dx}(\tan(x) - x)}{\frac{d}{dx}(x^3)} \\
&= \lim_{x\to 0} \frac{\sec^2(x) - 1}{3 \cdot x^2} \\
&= \lim_{x\to 0} \frac{\frac{d}{dx}\left(\sec^2(x) - 1\right)}{\frac{d}{dx}\left(3 \cdot x^2\right)} \\
&= \lim_{x\to 0} \frac{2 \cdot \sec(x) \cdot \sec(x) \cdot \tan(x)}{6 \cdot x} \\
&= \lim_{x\to 0} \frac{2 \cdot \sec^2(x) \cdot \tan(x)}{6 \cdot x} \\
&= \lim_{x\to 0} \frac{2\left(1 + \tan^2(x)\right) \cdot \tan(x)}{6 \cdot x} \\
&= \lim_{x\to 0} \frac{\frac{d}{dx}(2 \cdot \tan(x) + 2 \cdot \tan^3(x))}{\frac{d}{dx}(6 \cdot x)} \\
&= \lim_{x\to 0} \frac{2 \cdot \sec^2(x) + 6 \cdot \tan^2(x) \cdot \sec^2(x)}{6} \\
&= \frac{2 \cdot \sec^2(0) + 6 \cdot \tan^2(0) \cdot \sec^2(0)}{6} \\
&= \frac{2}{6} = \frac{1}{3}
\end{aligned}
$$

2. $\boldsymbol{\lim_{x\to 0} \frac{x - \sin(x)}{x^3}}$ **:**

Similar to the previous item, we use L'Hôpital's rule three times due to the indeterminate form $\frac{0}{0}$ and differentiate the numerator and the denominator separately until we get a determinate form. The limit evaluates to $\frac{1}{6}$ as the derivatives of the numerator become 1 and the denominator remains a finite value.

$$\begin{aligned}
\lim_{x\to 0} \frac{x-\sin(x)}{x^3} &= \lim_{x\to 0} \frac{\frac{d}{dx}(x-\sin(x))}{\frac{d}{dx}(x^3)} \\
&= \lim_{x\to 0} \frac{1-\cos(x)}{3\cdot x^2} \\
&= \lim_{x\to 0} \frac{\frac{d}{dx}(1-\cos(x))}{\frac{d}{dx}(3\cdot x^2)} \\
&= \lim_{x\to 0} \frac{\sin(x)}{6\cdot x} \\
&= \lim_{x\to 0} \frac{\frac{d}{dx}(\sin(x))}{\frac{d}{dx}(6\cdot x)} \\
&= \lim_{x\to 0} \frac{\cos(x)}{6} \\
&= \frac{\cos(0)}{6} \\
&= \frac{1}{6}
\end{aligned}$$

3. **$\lim_{x\to\infty} \frac{\ln(x+1)}{x}$:**

   In this case, L'Hôpital's rule is applied once, given the indeterminate form $\frac{\infty}{\infty}$, yielding $\lim_{x\to\infty} \frac{\frac{1}{x+1}}{1}$, which evaluates to 0 as x approaches infinity.

$$\begin{aligned}
\lim_{x\to\infty} \frac{\ln(x+1)}{x} &= \lim_{x\to\infty} \frac{\frac{d}{dx}\ln(x+1)}{\frac{d}{dx}x} \\
&= \lim_{x\to\infty} \frac{\frac{1}{1+x}}{1} \\
&= \lim_{x\to\infty} \frac{1}{1+x} \\
&= 0
\end{aligned}$$

4. **$\lim_{x\to 0} \frac{e^{2x}-1-2x}{x^2}$:**

   By applying L'Hôpital's rule twice due to the indeterminate form $\frac{0}{0}$, differentiating the numerator and the denominator until a determinate form is obtained, we find the limit is 2.

$$\begin{aligned}
\lim_{x\to 0} \frac{e^{2x}-1-2x}{x^2} &= \lim_{x\to 0} \frac{\frac{d}{dx}\left(e^{2x}-1-2x\right)}{\frac{d}{dx}x^2} \\
&= \lim_{x\to 0} \frac{2e^{2x}-2}{2x} \\
&= \lim_{x\to 0} \frac{\frac{d}{dx}\left(2e^{2x}-2\right)}{\frac{d}{dx}2x} \\
&= \lim_{x\to 0} \frac{4e^{2x}}{2} \\
&= \frac{4e^{2(0)}}{2} \\
&= 2
\end{aligned}$$

5. **$\lim_{x\to\infty} \frac{x^2+2x}{e^x}$:**

   Using L'Hôpital's rule twice, due to the indeterminate form $\frac{\infty}{\infty}$, we differentiate the numerator and the denominator until a determinate form is obtained. The exponential function in the denominator grows faster than the polynomial in the numerator, so the limit is 0 as x approaches infinity.

$$\begin{aligned}
\lim_{x\to\infty} \frac{x^2+2x}{e^x} &= \lim_{x\to\infty} \frac{\frac{d}{dx}\left(x^2+2x\right)}{\frac{d}{dx}\left(e^x\right)} \\
&= \lim_{x\to\infty} \frac{2x+2}{e^x} \\
&= \lim_{x\to\infty} \frac{\frac{d}{dx}(2x+2)}{\frac{d}{dx}e^x} \\
&= \lim_{x\to\infty} \frac{2}{e^x} = 0
\end{aligned}$$

6. $\lim_{x\to 0+} x\ln(x)$ **:**

   Using L'Hôpital's rule to resolve the indeterminate form $0 \cdot (-\infty)$, we differentiate the numerator and the denominator, transforming the expression into a determinate form. By evaluating the expression, the limit equals 0.

$$\begin{aligned}
\lim_{x\to 0+} x\ln(x) &= \lim_{x\to 0+} \frac{\ln(x)}{\frac{1}{x}} \\
&= \lim_{x\to 0+} \frac{\frac{d}{dx}\ln(x)}{\frac{d}{dx}\left(\frac{1}{x}\right)} \\
&= \lim_{x\to 0+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} \\
&= \lim_{x\to 0+} \frac{-x^2}{x} \\
&= \lim_{x\to 0+} -x \\
&= 0
\end{aligned}$$

7. $\lim_{x\to\infty} \left(1+\frac{1}{x}\right)^x$ **:**

   This is a well-known limit representing the mathematical constant e.

$$\begin{aligned}
\lim_{x\to\infty} \left(1+\frac{1}{x}\right)^x &= \lim_{x\to\infty} e^{\ln\left(1+\frac{1}{x}\right)^x} \\
&= \lim_{x\to\infty} e^{x\ln\left(1+\frac{1}{x}\right)} \\
&= \lim_{x\to\infty} e^{\frac{\frac{d}{dx}\ln\left(1+\frac{1}{x}\right)}{\frac{d}{dx}\frac{1}{x}}} \\
&= \lim_{x\to\infty} e^{\frac{\frac{1}{\left(1+\frac{1}{x}\right)}(0-x^{-2})}{-x^2}} \\
&= \lim_{x\to\infty} e^{\frac{1}{\left(1+\frac{1}{x}\right)}} \\
&= e^{\lim_{x\to\infty}\frac{1}{\left(1+\frac{1}{x}\right)}} = e^1 = e
\end{aligned}$$

8. $\mathbf{lim}_{x\to 0+}\left(\frac{1}{x}-\frac{1}{\sin(x)}\right)$ **:**

By applying L'Hôpital's rule due to the indeterminate form, we differentiate the numerator and denominator until we obtain a determinate form. Evaluating the limit results in 0.

$$\begin{aligned}
\lim_{x\to 0+}\left(\frac{1}{x}-\frac{1}{\sin(x)}\right) &= \lim_{x\to 0+}\frac{\sin(x)-x}{x\sin(x)}\\
&= \lim_{x\to 0+}\frac{\cos(x)-1}{\cos(x)+x\sin(x)}\\
&= \lim_{x\to 0+}\frac{-\sin(x)}{-\sin(x)+\cos(x)+\cos(x)}\\
&= \frac{0}{2}=0
\end{aligned}$$

9. $\mathbf{lim}_{x\to 0}\left(\frac{1}{x^2}-\frac{1}{x\tan(x)}\right)$ **:**

Utilizing L'Hôpital's rule to address the indeterminate form, we differentiate the components separately until a determinate form is achieved, yielding a limit of 0.

$$\begin{aligned}
\lim_{x\to 0}\left(\frac{1}{x^2}-\frac{\cos(x)}{x\sin(x)}\right) &= \lim_{x\to 0}\frac{x\cos(x)-\sin(x)}{x^2\sin(x)}\\
&= \lim_{x\to 0}\frac{x\sin(x)}{2x\sin(x)+x^2\cos(x)}\\
&= \lim_{x\to 0}\frac{\sin(x)}{2\sin(x)+x\cos(x)}\\
&= \frac{1}{3}
\end{aligned}$$

10. $\mathbf{lim}_{x\to\infty}\frac{\ln(x)}{e^{\frac{x}{2}}}$ **:**

Applying L'Hôpital's rule, due to the indeterminate form $\frac{\infty}{\infty}$, we differentiate the numerator and the denominator separately until a determinate form is obtained. As x goes to infinity, the exponential function in the denominator grows much faster than the logarithmic function in the numerator, thus the limit is 0.

$$\begin{aligned}
\lim_{x\to\infty}\frac{\ln(x)}{e^{\frac{x}{2}}} &= \lim_{x\to\infty}\frac{\frac{d}{dx}\ln(x)}{\frac{d}{dx}e^{\frac{x}{2}}} \\
&= \lim_{x\to\infty}\frac{\frac{1}{x}}{\frac{1}{2}e^{\frac{x}{2}}} \\
&= \lim_{x\to\infty}\frac{2}{xe^{\frac{x}{2}}} \\
&= 0
\end{aligned}$$

Chapter 5

Continuity

Formally, a function f is said to be continuous at a point $x = c$ if:

1. $f(c)$ is defined.
2. The limit $\lim_{x \to c} f(x)$ exists.
3. $\lim_{x \to c} f(x) = f(c)$.

If f is continuous at every point in its domain, then it is termed a continuous function over its domain.

Moreover, continuous functions possess several beneficial properties, including:

- The sum, difference, product, and quotient (except where the denominator is zero) of continuous functions are continuous.
- Polynomials, rational functions, exponential functions, and trigonometric functions are continuous wherever they are defined.
- The composition of continuous functions is continuous.

Following we have some exercises to practice this concept:

5.1 Practical Exercises:

1. Determine the value of c such that the function is continuous at $x = 2$:
$$f(x) = \begin{cases} 3x - c & \text{if} x < 2 \\ c^2 & \text{if } x = 2 \\ 4c - x & \text{if } x > 2 \end{cases}$$

2. Determine if the function is continuous at $x = 1$:
$$g(x) = \begin{cases} x^2 + 1 & \text{if } x \leq 1 \\ 2x - 1 & \text{if } x > 1 \end{cases}$$

3. Find the point of discontinuity:
$$h(x) = \frac{1}{x - 3}$$

4. Determine if the following function is continuous at $x = 0$:
$$k(x) = \begin{cases} \sin(x) & \text{if } x < 0 \\ x^2 & \text{if } x \geq 0 \end{cases}$$

5. Find the value of a that makes the function continuous everywhere:
$$m(x) = \begin{cases} ax^2 + 1 & \text{if } x < 1 \\ 2ax + 3 & \text{if } x \geq 1 \end{cases}$$

6. Determine the points of discontinuity:
$$n(x) = \frac{x^2 - 4}{x - 2}$$

7. Check the continuity of the function at $x = -1$:
$$p(x) = \begin{cases} x^3 + 1 & \text{if } x \leq -1 \\ -x^2 + 2x & \text{if } x > -1 \end{cases}$$

8. Find the values of a and b that make the function continuous everywhere:

$$q(x) = \begin{cases} ax + b & \text{if } x < 0 \\ e^x & \text{if } x \geq 0 \end{cases}$$

9. Determine if the function is continuous at $x = 0$:

$$r(x) = \begin{cases} x \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

10. Find the point of discontinuity:

$$s(x) = \frac{1}{\sqrt{x - 4}}$$

5.2 Solutions Practical Exercises:

1. To find the value of c that makes $f(x)$ continuous at $x = 2$, we need to make sure that the left-hand limit, the function value at $x = 2$, and the right-hand limit are equal.

$$\lim_{x \to 2^-} (3x - c) = 3(2) - c = 6 - c$$

$$f(2) = c^2$$

$$\lim_{x \to 2^+} (4c - x) = 4c - 2 = 4c - 2$$

Setting $6 - c = c^2$ and $c^2 = 4c - 2$, we find $c = 3$.

2. To check the continuity at $x = 1$, we need to verify that the left-hand limit and the right-hand limit are equal to the function value at $x = 1$.

$$\lim_{x \to 1^-} (x^2 + 1) = 1^2 + 1 = 2$$

$$g(1) = 1^2 + 1 = 2$$

$$\lim_{x \to 1^+} (2x - 1) = 2(1) - 1 = 1$$

Since the left-hand limit and the function value are equal but not equal to the right-hand limit, $g(x)$ is not continuous at $x = 1$.

3. The point of discontinuity of $h(x) = \frac{1}{x-3}$ occurs where the denominator equals zero, i.e., at $x = 3$.

4. To check continuity at $x = 0$, we again compare the left-hand limit, the function value, and the right-hand limit.

$$\lim_{x \to 0^-} \sin(x) = \sin(0) = 0$$
$$k(0) = 0^2 = 0$$
$$\lim_{x \to 0^+} x^2 = 0^2 = 0$$

Since these three values are equal, $k(x)$ is continuous at $x = 0$.

5. To find a that makes $m(x)$ continuous everywhere, we need to ensure the function values and the limits agree at $x = 1$.

$$\lim_{x \to 1^-} (ax^2 + 1) = a(1)^2 + 1 = a + 1$$
$$\lim_{x \to 1^+} (2ax + 3) = 2a(1) + 3 = 2a + 3$$

Setting $a + 1 = 2a + 3$, we find $a = -2$.

6. To find the points of discontinuity for the function $n(x) = \frac{x^2-4}{x-2}$, we first try to simplify the expression. We note that $x^2 - 4 = (x + 2)(x - 2)$, so

$$n(x) = \frac{(x + 2)(x - 2)}{x - 2} = x + 2, \text{ when } x \neq 2$$

However, when $x = 2$, the denominator is zero, making the function undefined. Thus, the point of discontinuity is at $x = 2$.

7. To check continuity at $x = -1$, we evaluate the left-hand limit, the function value, and the right-hand limit.

$$\lim_{x \to -1^-} (x^3 + 1) = (-1)^3 + 1 = 0$$

$$p(-1) = (-1)^3 + 1 = 0$$

$$\lim_{x \to -1^+} (-x^2 + 2x) = -(-1)^2 + 2(-1) = -1$$

Since the left-hand limit and the function value at $x = -1$ are equal, but not equal to the right-hand limit, $p(x)$ is not continuous at $x = -1$.

8. To find values of a and b making $q(x)$ continuous everywhere, we need the left-hand limit at $x = 0$ to be equal to the right-hand limit and the function value.

$$\lim_{x \to 0^-} (ax + b) = a(0) + b = b$$

$$q(0) = e^0 = 1$$

$$\lim_{x \to 0^+} e^x = e^0 = 1$$

Setting $b = 1$ makes $q(x)$ continuous at $x = 0$. Thus, a can be any real number and $b = 1$.

9. To check continuity at $x = 0$, we need to compare the left-hand limit, the function value, and the right-hand limit. The tricky part here is dealing with $x \sin\left(\frac{1}{x}\right)$ as x approaches zero.

$$\lim_{x \to 0^-} x \sin\left(\frac{1}{x}\right) = 0 \,(\text{using squeeze theorem})$$

$$r(0) = 0$$

$$\lim_{x \to 0^+} x \sin\left(\frac{1}{x}\right) = 0 \,(\text{using squeeze theorem})$$

Since these three values are equal, $r(x)$ is continuous at $x = 0$.

10. The point of discontinuity for $s(x) = \frac{1}{\sqrt{x-4}}$ occurs when the denominator equals zero or becomes undefined. This happens when $x = 4$, since $\sqrt{x-4} = 0$ would make the function undefined.

Chapter 6

Taylor series

In mathematical terms, if a function f is infinitely differentiable at a real or complex number a, its Taylor series is expressed as:

$$f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \dots$$

Beyond the mere representation of a function as a Taylor series, a significant concern is its convergence: under what conditions does the infinite sum actually approximate the function throughout its domain? Convergence elucidates the range of values for which the Taylor series provides a valid representation of the function.

The *interval of convergence* for a Taylor series centered at a is the set of all x for which the series converges to $f(x)$. The *radius of convergence*, denoted R, is the distance from a to the nearest point where the series does not converge. Formally, the radius of convergence R is given by:

$$R = \frac{1}{\limsup_{n\to\infty} \sqrt[n]{|a_n|}}$$

where a_n is the coefficient of the n^{th} term of the series. The series will converge for all x such that $|x-a| < R$ and might diverge for $|x-a| > R$.

6.1 Practical Exercises:

1. Find the first three terms of the Taylor series for the function $f(x) = e^x$ centered at $x = 0$.

2. Find the first four terms of the Taylor series for the function $f(x) = \sin x$ centered at $x = 0$.

3. Find the fourth-degree Taylor polynomial for the function $f(x) = \ln(1 + x)$ centered at $x = 0$.

4. Determine the interval of convergence for the Taylor series of the function $f(x) = \frac{1}{3-x}$ centered at $x = 0$.

5. Compute the radius of convergence of the Taylor series for the function $f(x) = \frac{1}{1-x}$ centered at $x = 0$.

6.2 Solutions Practical Exercises:

1. $\boldsymbol{f(x) = e^x}$ **around** $\boldsymbol{x = 0}$:

   For $f(x) = e^x$, the derivatives are:

   $$\begin{aligned} f'(x) &= e^x \\ f''(x) &= e^x \end{aligned}$$

   Evaluating these derivatives at $x = 0$ gives:

   $$f(0) = 1, \quad f'(0) = 1, \quad f''(0) = 1$$

   Substitute these into the Taylor series formula:

   $$e^x = 1 + 1 \cdot x + \frac{1}{2!}x^2$$

   The first three terms of the Taylor series are: $1 + x + \dfrac{x^2}{2}$

2. **$f(x) = \sin(x)$ around $x = 0$**:

   For $f(x) = \sin x$, the derivatives are:

$$\begin{aligned} f'(x) &= \cos(x) \\ f''(x) &= -\sin(x) \\ f'''(x) &= -\cos(x) \end{aligned}$$

   Evaluating these derivatives at $x = 0$ gives:

$$f(0) = 0, \quad f'(0) = 1, \quad f''(0) = 0, \quad f'''(0) = -1$$

   Substitute these into the Taylor series formula:

$$\sin x = 0 + 1 \cdot x + \frac{0}{2!}x^2 + \frac{-1}{3!}x^3$$

   The first four terms of the Taylor series are: $x - \dfrac{x^3}{6}$

3. **$f(x) = \ln(1 + x)$ around $x = 0$**:

   For $f(x) = \ln(1 + x)$, the derivatives are:

$$\begin{aligned} f'(x) &= \frac{1}{1+x} \\ f''(x) &= -\frac{1}{(1+x)^2} \\ f'''(x) &= \frac{2}{(1+x)^3} \\ f''''(x) &= -\frac{6}{(1+x)^4} \end{aligned}$$

   Evaluating these at $x = 0$, we have:

$$f'(0) = 1, \quad f''(0) = -1, \quad f'''(0) = 2, \quad f''''(0) = -6$$

   Thus, the fourth-degree Taylor polynomial is:

$$f(x) = x - \frac{x^2}{2!} + \frac{2x^3}{3!} - \frac{6x^4}{4!}$$

4. **The interval of convergence of $f(x) = \frac{1}{3-x}$ around $x = 0$:**

The Taylor series expansion of $f(x) = \frac{1}{3-x}$ centred at $x = 0$ is:

$$f(x) = \frac{1}{3} + \frac{x}{3^2} + \frac{x^2}{3^3} + \frac{x^3}{3^4} + \ldots$$

This is a geometric series with common ratio $r = \frac{x}{3}$. For a geometric series to converge, the absolute value of the common ratio must be less than 1. Therefore:

$$\left|\frac{x}{3}\right| < 1 \implies -1 < \frac{x}{3} < 1 \implies -3 < x < 3$$

Hence, the interval of convergence is $(-3, 3)$.

5. **The radius of convergence of the Taylor series for $f(x) = \frac{1}{1-x}$ centred at $x = 0$:**

The Taylor series expansion for the function $f(x) = \frac{1}{1-x}$ centered at $x = 0$ is the geometric series:

$$f(x) = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n$$

To determine the radius of convergence R for the series, we can use the Ratio Test, so, given a series $\sum_{n=0}^{\infty} a_n$, if:

$$L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$$

exists and $L < 1$, then the series is absolutely convergent. For our series, the terms are $a_n = x^n$. Thus, we have:

$$L = \lim_{n\to\infty} \left|\frac{x^{n+1}}{x^n}\right| = \lim_{n\to\infty} |x| = |x|$$

For convergence, we require:

$$|x| < 1$$

Thus, the radius of convergence R is given by:

$$R = 1$$

Hence, the radius of convergence of the Taylor series for the function $f(x) = \frac{1}{1-x}$ centred at $x = 0$ is $R = 1$.

Chapter 7

Critical Points

Newton's Method: Given a function $f(x)$ and its derivative $f'(x)$, the iterative formula for Newton's Method to find the root is:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Gradient Descent: For a function $f(x)$, the iterative formula for Gradient Descent to find the minimum is:

$$x_{n+1} = x_n - \alpha f'(x_n)$$

Here, α is termed the learning rate, determining the step size in the direction opposite to the derivative, which indicates the steepest ascent or descent.

7.1 Practical Exercises:

1. Find the critical points of the function $f(x) = x^3 - 6x^2 + 9x$.

2. Classify the critical points of the function $f(x) = x^3 - 3x + 2$ as local minimum, local maximum, or neither.

3. Use Newton's method with an initial guess of $x_0 = 1$ and a stopping criteria of $|f(x_n)| < 10^{-5}$ to find a root of $f(x) = x^3 - 4x + 1$.

4. Find the intervals where $f(x) = x^3 - 3x^2 - 4x + 12$ is increasing and decreasing.

5. Use the Gradient Descent method to minimize the function $f(x) = x^2 - 4x + 4$. Start with an initial guess of $x_0 = 0$, use $\alpha = 0.1$. Perform three iterations to find the value of x that minimizes the function.

7.2 Coding Exercises:

Consider the function:

$$f(x) = x^4 + 4x^3 + 4x^2$$

1. **Root Finding with Newton's Method**: Implement Newton's method to find the roots of $f(x)$. Use the following criteria:
   - Initial guess is $x_0 = 0$.
   - Stopping criterion based on the absolute value of $f(x)$ being less than 10^{-5} or a maximum of 10 iterations.

2. **Minimization with Gradient Descent**: Implement Gradient Descent to find the local minimum of $f(x)$. Use the following parameters:
   - Learning rate $\alpha = 0.01$.
   - Initial guess is $x_0 = 0$.
   - Stopping criterion based on the absolute value of $f(x)$ being less than 10^{-5}.

3. **Minimization with Newton's Method** : Implement Newton's method for optimization to find the local minimum of $f(x)$.
   - Initial guess is $x_0 = 0$.

- Stopping criterion based on the absolute value of $f(x)$ being less than 10^{-5}.

4. **Comparison**: Compare the methods on 2 and 3 based:
   - Plot the convergence of the methods, indicating the iteration points on the function graph.
   - Number of iterations taken to converge.

7.3 Solutions Practical Exercises:

Exercise 1:

To find the critical points of a function, we first need to differentiate f with respect to x:

$$f'(x) = \frac{d}{dx}(x^3 - 6x^2 + 9x)$$

Using power rule:

$$f'(x) = 3x^2 - 12x + 9$$

The critical points are the roots of $f'(x)$ so we set it to be equal to zero and solve for x:

$$3x^2 - 12x + 9 = 0$$

Factoring out the common factor of 3:

$$x^2 - 4x + 3 = 0$$

Factoring further:

$$(x - 3)(x - 1) = 0$$

This gives two solutions:

$$x = 3 \quad \text{and} \quad x = 1$$

Exercise 2:

First we find the derivative of the function $f(x) = x^3 - 3x + 2$:

$$f'(x) = 3x^2 - 3.$$

Now for the roots we set $f'(x) = 0$ and compute the critical points:

$$3x^2 - 3 = 0$$

$$x = \pm 1.$$

To classify the critical points, we examine $f''(x)$:

$$f''(x) = 6x.$$

At $x = 1$, $f''(1) = 6 > 0$ indicating a local minimum.

At $x = -1$, $f''(-1) = -6 < 0$ indicating a local maximum.

Exercise 3:

The function and its derivative are:

$$f(x) = x^3 - 4x + 1$$

$$f'(x) = 3x^2 - 4$$

Starting with $x_0 = 1$, we'll apply Newton's method formula:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Our stopping criteria will be $|f(x_n)| < 10^{-5}$ or a maximum of 5 iterations. Let's compute the iterations:

$$
\begin{aligned}
x_0 &= 1, && f(x_0) = -2 \\
x_1 &= 1 - \frac{-2}{-1} = 3, && f(x_1) = 14 \\
x_2 &= 3 - \frac{14}{23} \approx 2.3913, && f(x_2) \approx 0.2609 \\
x_3 &= 2.3913 - 0.0867 \approx 2.3046, && f(x_3) \approx 0.0118 \\
x_4 &= 2.3046 + 0.0039 \approx 2.3085, && f(x_4) \approx 0.000055 \\
x_5 &= 2.3085 + 0.00001 \approx 2.3085, && f(x_5) \approx 0
\end{aligned}
$$

By the 5th iteration, $|f(x_5)|$ is less than 10^{-5}, satisfying our stopping criteria. Thus, using Newton's method, an approximate root of $f(x) = x^3 - 4x + 1$ is $x_5 \approx 2.3085$ (rounded to four decimal places).

Exercise 4:

For the function $f(x) = x^3 - 3x^2 - 4x + 12$, first find $f'(x)$:

$$f'(x) = 3x^2 - 6x - 4.$$

Find the critical points by setting $f'(x) = 0$:

$$3x^2 - 6x - 4 = 0.$$

Using the quadratic formula to find the roots

$$\begin{aligned} x &= \frac{6 \pm \sqrt{(-6)^2 - 4 \cdot 3 \cdot (-4)}}{2 \cdot 3} \\ &= \frac{6 \pm \sqrt{84}}{6} \\ &= \frac{3 \pm \sqrt{21}}{3} \\ &= 1 \pm \frac{\sqrt{21}}{3} \end{aligned}$$

Therefore $x = 1+\frac{\sqrt{21}}{3}$ and $x = 1-\frac{\sqrt{21}}{3}$ are critical points, we then test the sign of $f'(x)$ in each interval determined by these critical points to find the intervals of increase ($f'(x) > 0$) and decrease ($f'(x) < 0$).

For $x = 1 + \frac{\sqrt{21}}{3}$:

Substitute $x = 2.52$ in $f'(x)$

$$f'(2.52) = 3(2.52)^2 - 6(2.52) - 4 = -0.0688 < 0$$

Substitute $x = 2.53$ in $f'(x)$

$$f'(2.53) = 3(2.53)^2 - 6(2.53) - 4 = 0.0227 > 0$$

For $x = 1 - \frac{\sqrt{21}}{3}$:

Substitute $x = -0.52$ in $f'(x)$

$$f'(2.52) = 3(-0.52)^2 - 6(-0.52) - 4 = -0.0688 < 0$$

Substitute $x = -0.53$ in $f'(x)$

$$f'(0.53) = 3(-0.53)^2 - 6(-0.53) - 4 = 0.0227 > 0$$

Hence $f(x)$ increasing in the interval $(-\infty, 1-\frac{\sqrt{21}}{3}) \cup (1+\frac{\sqrt{21}}{3}, \infty)$ and $f(x)$ decreasing in the interval $(1 - \frac{\sqrt{21}}{3}, 1 + \frac{\sqrt{21}}{3})$

Exercise 5:

First, determine the derivative of $f(x)$:

$$f'(x) = 2x - 4$$

Starting with $x_0 = 0$ and using the gradient descent formula:

$$x_{n+1} = x_n - \alpha f'(x_n)$$

Let's compute the iterations:

$$\begin{aligned}
x_0 &= 0, & f'(x_0) &= -4 \\
x_1 &= 0 - 0.1(-4) = 0.4, & f'(x_1) &= -3.2 \\
x_2 &= 0.4 - 0.1(-3.2) = 0.72, & f'(x_2) &= -2.56 \\
x_3 &= 0.72 - 0.1(-2.56) = 0.976, & f'(x_3) &= -2.048
\end{aligned}$$

After three iterations using the gradient descent method, the value of x that is approaching the minimum of the function $f(x) = x^2 - 4x + 4$ is $x_3 \approx 0.976$.

Chapter 8

Integrals

Below are the fundamental rules of integration that will be explored and applied in this chapter.

- **Power Functions:** For any real number $n \neq -1$,

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1} + C$$

- **Exponential Functions:**

$$\int e^x \, dx = e^x + C$$

- **Trigonometric Functions:**

$$\int \sin x \, dx = -\cos x + C$$

$$\int \cos x \, dx = \sin x + C$$

- **Logarithmic Functions:**

$$\int \frac{1}{x} \, dx = \ln |x| + C$$

- **Rational Functions:**

$$\int \frac{1}{x+a} \, dx = \ln |x+a| + C$$

- **Inverse Trigonometric Functions:**

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1} x + C$$

- **Hyperbolic Functions:**

$$\int \cosh x\,dx = \sinh x + C$$

$$\int \sinh x\,dx = \cosh x + C$$

8.1 Practical Exercises:

Find the integrals of:

8.1.1 Easy:

1. $\int (x^2 + 3x + 2)\,dx$
2. $\int (2x^3 - 5x^2 + 7x - 3)\,dx$
3. $\int e^x\,dx$
4. $\int (3x - 4)\,dx$
5. $\int \frac{1}{x}\,dx$
6. $\int \sin(x)\,dx$
7. $\int \cos(x)\,dx$
8. $\int (4x^2 - e^x)\,dx$
9. $\int (2\sqrt{x} + \frac{1}{x})\,dx$
10. $\int (x^3 - x^2 + x - 1)\,dx$

11. $\int \frac{x-1}{x}\,dx$
12. $\int (5x^4 - 3x^2 + 2x - 7)\,dx$
13. $\int \tan(x)\,dx$
14. $\int \sec^2(x)\,dx$
15. $\int \ln(x)\,dx$

8.1.2 Medium:

1. $\int \frac{x^2-1}{x}\,dx$
2. $\int xe^x\,dx$
3. $\int x\sin(x)\,dx$
4. $\int \frac{x}{\sqrt{x^2+1}}\,dx$
5. $\int \frac{x^2}{x^2+1}\,dx$
6. $\int \frac{e^x}{e^x+1}\,dx$
7. $\int \frac{x}{e^x-1}\,dx$
8. $\int \sin^2(x)\,dx$
9. $\int \cos(2x)\,dx$
10. $\int x\ln(x)\,dx$
11. $\int \sqrt{1-x^2}\,dx$
12. $\int \frac{1}{x\ln(x)}\,dx$
13. $\int \frac{\ln(x)}{x}\,dx$
14. $\int e^{2x}\cos(3x)\,dx$
15. $\int \frac{x}{\sqrt{4-x^2}}\,dx$

8.1.3 Hard:

1. $\int x^2 e^{x^3}\, dx$
2. $\int \frac{x^2}{\sqrt{1-x^2}}\, dx$
3. $\int e^x \sqrt{e^{2x}+1}\, dx$
4. $\int \sin(x)\ln(\cos(x))\, dx$
5. $\int \sqrt{x^2+x+1}\, dx$
6. $\int \frac{1}{x^2+2x+2}\, dx$
7. $\int \frac{x}{\sqrt{x^4+1}}\, dx$
8. $\int \frac{e^x}{e^{2x}+1}\, dx$
9. $\int x^2 \cos(x^3)\, dx$
10. $\int \frac{x^3+1}{x(x^2-1)}\, dx$
11. $\int \frac{\sin^2(x)}{1+\sin(x)}\, dx$

8.2 Solutions Practical Exercises:

8.2.1 Easy:

1. $\int (\boldsymbol{x^2+3x+2})\, \boldsymbol{dx}$ **:**

   Applying the power rule to each term separately and adding the constant of integration C.

$$= \frac{x^3}{3} + \frac{3x^2}{2} + 2x + C$$

2. $\int (2x^3 - 5x^2 + 7x - 3)\,dx$ **:**

   Apply the power rule to each term and don't forget to add C.

$$= \frac{x^4}{2} - \frac{5x^3}{3} + \frac{7x^2}{2} - 3x + C$$

3. $\int e^x\,dx$ **:**

   The integral of e^x is e^x plus the constant of integration.

$$= e^x + C$$

4. $\int (3x - 4)\,dx$ **:**

   Applying the power rule and integrating the constant term with respect to x.

$$= \frac{3x^2}{2} - 4x + C$$

5. $\int \frac{1}{x}\,dx$ **:**

   The integral of $\frac{1}{x}$ is the natural log of the absolute value of x plus the constant of integration.

$$= \ln|x| + C$$

6. $\int \sin(x)\,dx$ **:**

   The antiderivative of $\sin(x)$ is $-\cos(x)$ plus the constant of integration.

$$= -\cos(x) + C$$

7. $\int \cos(x)\,dx$ **:**

   The antiderivative of $\cos(x)$ is $\sin(x)$ plus the constant of integration.

$$= \sin(x) + C$$

8. $\int (4x^2 - e^x)\, dx$ **:**

   Apply the power rule to the first term and integrate the exponential term normally, adding C at the end.

$$= \frac{4x^3}{3} - e^x + C$$

9. $\int (2\sqrt{x} + \frac{1}{x})\, dx$ **:**

   For the first term, use the power rule and convert the square root to a power of $\frac{1}{2}$. For the second term, recall that the integral of $\frac{1}{x}$ is $\ln|x|$.

$$= \frac{4}{3}x^{\frac{3}{2}} + \ln|x| + C$$

10. $\int (x^3 - x^2 + x - 1)\, dx$ **:**

    Apply the power rule to each term and don't forget to add C.

$$= \frac{x^4}{4} - \frac{x^3}{3} + \frac{x^2}{2} - x + C$$

11. $\int \frac{x-1}{x}\, dx$ **:**

    Divide each term in the numerator by x before integrating.

$$= x - \ln|x| + C$$

12. $\int (5x^4 - 3x^2 + 2x - 7)\, dx$ **:**

    Apply the power rule to each term and integrate the constant term with respect to x.

$$= \frac{5x^5}{5} - x^3 + x^2 - 7x + C$$

13. $\int \tan(x)\, dx$ **:**

    The antiderivative of $\tan(x)$ is $-\ln|\cos(x)|$ plus the constant of integration.

$$= -\ln|\cos(x)| + C$$

14. $\int \boldsymbol{\sec^2(x)\, dx}$ **:**

    The integral of $\sec^2(x)$ is $\tan(x)$ plus the constant of integration.

    $$= \tan(x) + C$$

15. $\int \boldsymbol{\ln(x)\, dx}$ **:**

    Use integration by parts, letting $u = \ln(x)$ and $dv = dx$. Then:

    $$\begin{aligned} u &= \ln(x) & dv &= dx \\ du &= \frac{1}{x}dx & v &= x \end{aligned}$$

    $$\begin{aligned} \int \ln(x)\, dx &= uv - \int v\, du \\ &= x\ln(x) - \int x\frac{1}{x}\, dx \\ &= x\ln(x) - \int dx \\ &= x\ln(x) - x + C \end{aligned}$$

    $$= x\ln(x) - x + C$$

8.2.2 Medium:

1. $\int \boldsymbol{\frac{x^2-1}{x}\, dx}$ **:**

   Separate the integrals, and solve each term separately.

   $$\begin{aligned} \int \frac{x^2-1}{x}\, dx &= \int x - \frac{1}{x}\, dx \\ &= \frac{x^2}{2} - \ln|x| + C \end{aligned}$$

2. $\int \boldsymbol{xe^x\, dx}$ **:**

Use integration by parts ($u = x$, $dv = e^x dx$).

$$\begin{aligned} u &= x & dv &= e^x dx \\ du &= dx & v &= e^x \end{aligned}$$

$$\begin{aligned} \int xe^x\,dx &= uv - \int v\,du \\ &= xe^x - \int e^x\,dx \\ &= xe^x - e^x + C \end{aligned}$$

where C is the constant of integration.

3. $\int \boldsymbol{x\sin(x)\,dx}$ **:**

   Use integration by parts.

$$\begin{aligned} u &= x & dv &= \sin(x)\,dx \\ du &= dx & v &= -\cos(x) \end{aligned}$$

$$\begin{aligned} \int x\sin(x)\,dx &= uv - \int v\,du \\ &= -x\cos(x) + \int \cos(x)\,dx \\ &= -x\cos(x) + \sin(x) + C \end{aligned}$$

   where C is the constant of integration.

4. $\int \boldsymbol{\frac{x}{\sqrt{x^2+1}}\,dx}$ **:**

   Use a u-substitution ($u = x^2 + 1$).
   Let $u = x^2 + 1$, then $du = 2x\,dx$.

   Now, we can rewrite the integral:

$$\begin{aligned} \int \frac{x}{\sqrt{x^2+1}}\,dx &= \int \frac{1}{2}\frac{2x}{\sqrt{x^2+1}}\,dx \\ &= \frac{1}{2}\int \frac{1}{\sqrt{u}}\,du. \end{aligned}$$

   Applying the power rule for integration, we get:

$$\begin{aligned} \frac{1}{2}\int \frac{1}{\sqrt{u}}\,du &= \frac{1}{2}\cdot 2\sqrt{u} + C \\ &= \sqrt{u} + C. \end{aligned}$$

Substitute back in terms of x:

$$\sqrt{x^2+1}+C,$$

where C is the constant of integration.

5. $\int \frac{x^2}{x^2+1}\,dx$ **:**

Separate into two terms, $\int 1 - \frac{1}{x^2+1}\,dx$.

$$\begin{aligned}\int \frac{x^2}{x^2+1}\,dx &= \int \frac{x^2+1-1}{x^2+1}\,dx \\ &= \int 1\,dx - \int \frac{1}{x^2+1}\,dx \\ &= x - \arctan(x) + C\end{aligned}$$

where C is the constant of integration.

6. $\int \frac{e^x}{e^x+1}\,dx$ **:**

Use a u-substitution ($u = e^x + 1$). Let $u = e^x + 1$, then $du = e^x\,dx$. Now, we can rewrite the integral:

$$\begin{aligned}\int \frac{e^x}{e^x+1}\,dx &= \int \frac{1}{u}\,du \\ &= \ln|u| + C.\end{aligned}$$

Substitute back in terms of x:

$$\ln|e^x+1|+C,$$

where C is the constant of integration.

7. $\int \frac{x}{e^x-1}\,dx$ **:**

This is a special case, resulting in the polylogarithm function.

$$\mathrm{Li}_2(e^x)+C$$

8. $\int \sin^2(x)\,dx$ **:**

Use a power-reduction formula and the trigonometric identity $\sin^2(x) = \frac{1}{2}(1-\cos(2x))$:

$$\int \frac{1}{2}(1-\cos(2x))\,dx$$

Now, integrate term by term:

$$\frac{1}{2}\int (1-\cos(2x))\,dx = \frac{1}{2}\left(x - \frac{1}{2}\sin(2x)\right) + C$$

So, the final result is:

$$\int \sin^2(x)\,dx = \frac{1}{2}\left(x - \frac{1}{2}\sin(2x)\right) + C$$

where C is the constant of integration.

9. $\int \boldsymbol{\cos(2x)\,dx}$ **:**

   Direct integration.

$$\frac{1}{2}\sin(2x) + C$$

10. $\int \boldsymbol{x\ln(x)\,dx}$ **:**

    Use integration by parts. Let

$$\begin{aligned} u &= \ln(x) & dv &= x\,dx \\ du &= \frac{1}{x}dx & v &= \frac{1}{2}x^2 \end{aligned}$$

    Applying the integration by parts formula:

$$\begin{aligned} \int x\ln(x)\,dx &= uv - \int v\,du \\ &= \ln(x)\cdot\frac{1}{2}x^2 - \int \frac{1}{2}x^2\cdot\frac{1}{x}\,dx \\ &= \frac{1}{2}x^2\ln(x) - \frac{1}{4}x^2 + C \end{aligned}$$

    where C is the constant of integration.

11. $\int \boldsymbol{\sqrt{1-x^2}\,dx}$ **:**

    This represents a semicircle. Let $x = \sin(\theta)$, then $dx = \cos(\theta)\,d\theta$.

$$\begin{aligned} \int \sqrt{1-x^2}\,dx &= \int \sqrt{1-\sin^2(\theta)}\cdot\cos(\theta)\,d\theta \\ &= \int \cos^2(\theta)\,d\theta. \end{aligned}$$

Now use a trigonometric identity $\cos^2(\theta) = \frac{1}{2}(1+\cos(2\theta))$:

$$\begin{aligned}\int \cos^2(\theta)\,d\theta &= \int \frac{1}{2}(1+\cos(2\theta))\,d\theta \\ &= \frac{1}{2}\left(\theta + \sin(\theta)\cos(\theta)\right) + C.\end{aligned}$$

Substitute back in terms of x:

$$\frac{1}{2}\left(\arcsin(x) + \frac{1}{2}x\sqrt{1-x^2}\right) + C,$$

where C is the constant of integration.

12. $\int \frac{1}{x\ln(x)}\,dx$ **:**

Let $u = \ln(x)$, then $du = \frac{1}{x}\,dx$.

$$\begin{aligned}\int \frac{1}{x\ln(x)}\,dx &= \int \frac{1}{u}\,du \\ &= \ln|u| + C.\end{aligned}$$

Substitute back in terms of x:

$$\ln|\ln(x)| + C,$$

where C is the constant of integration.

13. $\int \frac{\ln(x)}{x}\,dx$ **:**

Use a special function.

$$\frac{\ln^2(x)}{2} + C$$

Let

$$\begin{aligned}u &= \ln(x) \quad & dv &= \frac{1}{x}\,dx \\ du &= \frac{1}{x}\,dx \quad & v &= \ln(x)\end{aligned}$$

Now, apply the integration by parts formula:

$$\begin{aligned}\int \frac{\ln(x)}{x}\,dx &= uv - \int v\,du \\ &= \ln(x)\cdot\ln(x) - \int \ln(x)\cdot\frac{1}{x}\,dx.\end{aligned}$$

Now, rearrange and solve for the integral:

$$2\int \frac{\ln(x)}{x}\,dx = \ln(x)\cdot\ln(x)$$
$$\int \frac{\ln(x)}{x}\,dx = \frac{1}{2}(\ln(x))^2 + C,$$

where C is the constant of integration.

14. $\int \boldsymbol{e^{2x}\cos(3x)\,dx}$ **:**

Integrate by parts twice, and simplify.

$$\frac{e^{2x}}{13}(2\cos(3x) + 3\sin(3x)) + C$$

Let

$$u = e^{2x} \qquad dv = \cos(3x)\,dx$$
$$du = 2e^{2x}\,dx \qquad v = \frac{1}{3}\sin(3x)$$

Now, apply the integration by parts formula:

$$\begin{aligned} I = \int e^{2x}\cos(3x)\,dx &= uv - \int v\,du \\ &= e^{2x}\cdot\frac{1}{3}\sin(3x) \\ &\quad - \int \frac{1}{3}\sin(3x)\cdot 2e^{2x}\,dx \qquad (1) \end{aligned}$$

Now, integrate the remaining term:

$$u = e^{2x} \qquad dv = \sin(3x)\,dx$$
$$du = 2e^{2x}\,dx \qquad v = -\frac{1}{3}\cos(3x)$$

$$\begin{aligned} \int \frac{1}{3}\sin(3x)\cdot 2e^{2x}\,dx &= \frac{2}{3}\int e^{2x}\sin(3x)\,dx \\ &= -\frac{2}{9}e^{2x}\cos(3x) + \frac{4}{9}\int e^{2x}\cos(3x) \end{aligned}$$

Using (1):

$$\begin{aligned}
\int e^{2x}\cos(3x)\,dx &= e^{2x}\cdot\frac{1}{3}\sin(3x)+\frac{2}{9}e^{2x}\cos(3x)-\frac{4}{9}\int e^{2x}\cos(3x)\\
I+\frac{4}{9}I &= e^{2x}\cdot\frac{1}{3}\sin(3x)+\frac{2}{9}e^{2x}\cos(3x)\\
\frac{13}{9}I &= \frac{1}{9}e^{2x}\left(3\sin(3x)+2\cos(3x)\right)\\
\int e^{2x}\cos(3x)\,dx &= \frac{1}{13}e^{2x}\left(3\sin(3x)+2\cos(3x)\right)
\end{aligned}$$

15. $\int \frac{x}{\sqrt{4-x^2}}\,dx$ **:**

Use a u-substitution ($u = 4 - x^2$).

$$-\frac{1}{2}\sqrt{4-x^2}+C$$

Let $u = 4 - x^2$, then $du = -2x\,dx$.

Now, rewrite the integral:

$$\begin{aligned}
\int \frac{x}{\sqrt{4-x^2}}\,dx &= -\frac{1}{2}\int \frac{-2x}{\sqrt{4-x^2}}\,dx\\
&= -\frac{1}{2}\int \frac{1}{\sqrt{u}}\,du.
\end{aligned}$$

Applying the power rule for integration:

$$-\frac{1}{2}\int \frac{1}{\sqrt{u}}\,du = -\sqrt{u}+C.$$

Finally, substituting back in terms of x:

$$-\sqrt{4-x^2}+C,$$

where C is the constant of integration.

8.2.3 Hard:

1. $\int x^2 e^{x^3}\,dx$ **:**

This is best approached using a u-substitution, $u = x^3$, giving $\frac{1}{3}e^{x^3} + C$.

Let $u = x^3$, $du = 3x^2\,dx$

$$\begin{aligned}
\int x^2 e^{x^3}\,dx &= \frac{1}{3}\int e^u\,du \\
&= \frac{1}{3}e^u + C \\
&= \frac{1}{3}e^{x^3} + C \quad \text{(Substitute back in terms of } x\text{)},
\end{aligned}$$

where C is the constant of integration.

2. $\int \frac{\boldsymbol{x^2}}{\sqrt{\boldsymbol{1-x^2}}}\,\boldsymbol{dx}$ **:**

For this integral, a trigonometric substitution is beneficial, resulting in $-\frac{\sqrt{1-x^2}(x^2+1)}{2} + C$.

Let $\sin(u) = x, \quad \cos(u)\,du = dx$

$$\begin{aligned}
\int \frac{x^2}{\sqrt{1-x^2}}\,dx &= \int \frac{\sin^2(u)}{\sqrt{1-\sin^2(u)}} \cdot \cos(u)\,du \\
&= \int \frac{\sin^2(u)}{\cos(u)} \cdot \cos(u)\,du \\
&= \int \sin^2(u)\,du \\
&= \int \frac{1}{2}(1-\cos(2u))\,du \\
&= \frac{1}{2}\left(u - \frac{1}{2}\sin(2u)\right) + C
\end{aligned}$$

Substitute back in terms of x

$$\int \frac{x^2}{\sqrt{1-x^2}}\,dx = \frac{1}{2}\arcsin(x) - \frac{1}{2}x\sqrt{1-x^2} + C$$

where C is the constant of integration.

3. $\int \boldsymbol{e^x \sqrt{e^{2x}+1}\, dx}$ **:**

A hyperbolic trigonometric substitution will yield the solution, resulting in $\frac{\sqrt{e^{2x}+1}(e^x + \operatorname{arcsinh}(e^x))}{2} + C$.

Substitute $u = e^x \to du = e^x\, dx$, use:

$$\int e^x \sqrt{e^{2x}+1}\, dx = \int \sqrt{u^2+1}\, du$$

Perform hyperbolic substitution:

Substitute $u = \sinh(v) \to v = \operatorname{arsinh}(u), du = \cosh(v) dv$

$$\int \sqrt{u^2+1}\, du = \int \cosh(v) \sqrt{\sinh^2(v)+1}\, dv$$

Simplify using $\sinh^2(v) + 1 = \cosh^2(v)$:

$$= \int \cosh^2(v)\, dv$$

Apply reduction formula with $n = 2$:

$$\int \cosh^n(v)\, dv = \frac{n-1}{n} \int \cosh^{n-2}(v)\, dv + \frac{\cosh^{n-1}(v) \sinh(v)}{n}$$

$$\begin{aligned} \int \sqrt{u^2+1}\, du &= \frac{\cosh(v)\sinh(v)}{2} + \frac{1}{2} \int 1\, dv \\ &= \frac{\cosh(v)\sinh(v)}{2} + \frac{v}{2} + C \end{aligned}$$

Undo substitution $v = \operatorname{arsinh}(u)$, use:

$$\begin{aligned} \sinh(\operatorname{arsinh}(u)) &= u \\ \cosh(\operatorname{arsinh}(u)) &= \sqrt{u^2+1} \end{aligned}$$

$$\int \sqrt{u^2+1}\, du = \frac{\operatorname{arsinh}(u)}{2} + \frac{u\sqrt{u^2+1}}{2} + C$$

Undo substitution $u = e^x$:

$$\int e^x \sqrt{e^{2x}+1}\, dx = \frac{\operatorname{arsinh}(e^x)}{2} + \frac{e^x\sqrt{e^{2x}+1}}{2} + C$$

where C is the constant of integration.

4. $\int \boldsymbol{\sin(x)\ln(\cos(x))\,dx}$ **:**

This integral is best solved using integration by parts, resulting in $-\cos(x)\ln(\cos(x)) + \int \frac{\cos(x)}{\tan(x)}\,dx$.

$$u = \ln(\cos(x)) \qquad dv = \sin(x)\,dx$$
$$du = -\frac{\sin(x)}{\cos(x)}\,dx \qquad v = -\cos(x)$$

Applying the integration by parts formula:

$$\begin{aligned}
&\int \sin(x)\ln(\cos(x))\,dx = \\
&\qquad = -\ln(\cos(x))\cos(x) - \int (-\cos(x))\left(-\frac{\sin(x)}{\cos(x)}\right)\,dx \\
&\qquad = -\ln(\cos(x))\cos(x) - \int \sin(x)\,dx \\
&\qquad = -\ln(\cos(x))\cos(x) + \cos(x) + C,
\end{aligned}$$

5. $\int \boldsymbol{\sqrt{x^2+x+1}\,dx}$ **:**

$$\begin{aligned}
\int \sqrt{x^2+x+1}\,dx &= \int \sqrt{\left(x+\frac{1}{2}\right)^2 + \frac{3}{4}}\,dx \\
&= \frac{1}{2}\int \sqrt{(2x+1)^2+3}\,dx
\end{aligned}$$

Substitute $u = 2x+1$, $du = 2\,dx$:

$$\int \sqrt{x^2+x+1}\,dx = \frac{1}{4}\int \sqrt{u^2+3}\,du$$

Perform trigonometric substitution: Substitute $u = \sqrt{3}\tan(v)$, $du = \sqrt{3}\sec^2(v)\,dv$:

$$\int \sqrt{u^2+3}\,du = 3\int \sec^3(v)\,dv$$

Apply reduction formula for $\sec^3(v)$:

$$\int \sec^3(v)\,dv = \frac{\sec(v)\tan(v)}{2} + \frac{1}{2}\ln(\sec(v)+\tan(v)) + C$$

Undo substitution $v = \arctan\left(\frac{u}{\sqrt{3}}\right)$:

$$\tan(\arctan(x)) = x, \quad \sec(\arctan(x)) = \sqrt{x^2+1}$$

$$\int \sqrt{u^2+3}\, du = \frac{u\sqrt{u^2+3}}{2\sqrt{3}} + \frac{3\ln\left(\sqrt{u^2+3}+u\right)}{2\sqrt{3}} + C$$

Undo substitution $u = 2x+1$:

$$\int \sqrt{x^2+x+1}\, dx = \frac{(2x+1)\sqrt{(2x+1)^2+3}}{8} + \frac{3\arcsin\left(\frac{\sqrt{3}(2x+1)}{3}\right)}{8} + C$$

6. $\int \frac{1}{x^2+2x+2}\, dx$ **:**

$$\int \frac{1}{x^2+2x+2}\, dx = \int \frac{1}{(x+1)^2+1}\, dx$$

Substitute $u = x+1$, $dx = du$:

$$\int \frac{1}{(x+1)^2+1}\, dx = \int \frac{1}{u^2+1}\, du$$

$$\int \frac{1}{u^2+1}\, du = \arctan(u) + C$$

Undo substitution $u = x+1$:

$$\int \frac{1}{x^2+2x+2}\, dx = \arctan(x+1) + C$$

7. $\int \frac{x}{\sqrt{x^4+1}}\, dx$ **:**

A trigonometric or hyperbolic substitution would be useful in solving this integral. It will yield a relatively complex expression involving logarithmic terms.

Substitute $u = x^2, \quad du = 2x\, dx$:

$$\int \frac{x}{\sqrt{x^4+1}}\, dx = \frac{1}{2}\int \frac{1}{\sqrt{u^2+1}}\, du$$

Perform trigonometric substitution:

Substitute $u = \tan(v)$, $\quad du = \sec^2(v)\,dv$:

$$\begin{aligned}
\int \frac{x}{\sqrt{x^4+1}}\,dx &= \frac{1}{2}\int \frac{\sec^2(v)}{\sqrt{\tan^2(v)+1}}\,dv \\
&= \frac{1}{2}\int \sec(v)\,dv \\
&= \frac{1}{2}\ln|\tan(v)+\sec(v)| + C \\
&= \frac{1}{2}\ln|u+\sqrt{u^2+1}| + C \\
&= \frac{1}{2}\ln|x^2+\sqrt{x^4+1}| + C
\end{aligned}$$

8. $\boldsymbol{\int \frac{e^x}{e^{2x}+1}\,dx}$ **:**

   Substitute $u = e^x$, then $du = e^x dx$.

$$\begin{aligned}
\int \frac{e^x}{e^{2x}+1} dx &= \int \frac{1}{u^2+1} du \\
&= \arctan(u) + C \\
&= \arctan(e^x) + C
\end{aligned}$$

9. $\boldsymbol{\int x^2\cos(x^3)\,dx}$ **:**

   Let $u = x^3$, $du = 3x^2\,dx$

$$\begin{aligned}
\int x^2\cos(x^3)\,dx &= \frac{1}{3}\int \cos(u)\,du \\
&= \frac{1}{3}\int \cos(u)\,du \\
&= \frac{1}{3}\sin(u) + C \\
&= \frac{1}{3}\sin(x^3) + C \quad \text{(Subs in terms of } x\text{)},
\end{aligned}$$

   where C is the constant of integration.

10. $\boldsymbol{\int \frac{x^3+1}{x(x^2-1)}\,dx}$ **:**

This integral requires partial fraction decomposition and yields a solution involving logarithmic terms.

$$\begin{aligned}
\int \frac{x^3+1}{x\cdot(x^2-1)}\,dx &= \int \frac{x^2-x+1}{(x-1)x}\,dx \\
&= \int \left(\frac{1}{(x-1)x}+1\right)\,dx \\
&= \int \frac{1}{(x-1)x}\,dx + \int 1\,dx \\
&= \int \frac{1}{(x-1)x}\,dx + x + C \\
&\int \frac{1}{(x-1)}\,dx - \int \frac{1}{x}\,dx + x + C \\
&= \ln|x-1| - \ln|x| + x + C
\end{aligned}$$

11. $\int \frac{\sin^2(x)}{1+\sin(x)}\,dx$ **:**

Substitute $u = \sin(x), \quad du = \cos(x)\,dx$

$$\begin{aligned}
\int \frac{\sin^2(x)}{1+\sin(x)}\,dx &= \int \sin(x) - 1 + \frac{1}{1+\sin(x)}\,dx \\
&= \int \sin(x)\,dx - \int 1\,dx + \int \frac{1}{1+\sin(x)}\,dx \\
&= -\cos(x) - x + \int \frac{1}{1+\sin(x)}\,dx \\
&= -\cos(x) - x + \int \frac{\sec^2\left(\frac{x}{2}\right)}{\left(\tan\left(\frac{x}{2}\right)+1\right)^2}dx \qquad (1)
\end{aligned}$$

Substitute $u = \tan\left(\frac{x}{2}\right)+1, \quad du = \frac{1}{2}\sec^2\left(\frac{x}{2}\right)\,dx$

$$\begin{aligned}
\int \frac{\sec^2\left(\frac{x}{2}\right)}{\left(\tan\left(\frac{x}{2}\right)+1\right)^2}dx &= 2\int \frac{1}{u^2}\,du \\
&= -\frac{2}{u} + C \\
&= -\frac{2}{\tan\left(\frac{x}{2}\right)+1} + C
\end{aligned}$$

(1) Implies

$$\int \frac{\sin^2(x)}{1+\sin(x)}\,dx = -\cos(x) - x - \frac{2}{\tan\left(\frac{x}{2}\right)+1} + C$$

Chapter 9

Gradients and Hessians

Formally, for a function $f(x, y)$, the gradient is given by:

$$\nabla f = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{bmatrix}$$

To further understand the curvature and shape of our multidimensional function, we introduce the Hessian matrix. The Hessian matrix, for a function $f(x, y)$, is given by:

$$H(f) = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{bmatrix}$$

9.1 Practical Exercises Partial Derivatives:

1. Find $\frac{\partial f}{\partial x}$ for the function $f(x, y) = 3x^2y + 2xy^3$.
2. Calculate $\frac{\partial z}{\partial y}$ for $z(x, y) = x^2y + 5y^2 - 3x$.
3. Find $\frac{\partial v}{\partial y}$ for $v(x, y, z) = 4x^3y^2 - 3xy^2z^2 + 2z$.
4. Compute $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ for the function $f(x, y, z) = x^2y + 3yz^2 - z^3x$.

5. Calculate $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, and $\frac{\partial u}{\partial z}$ for $u(x,y,z) = xy^2 + yz^3 - xz^2$.

6. Find $\frac{\partial v}{\partial x}$, $\frac{\partial v}{\partial y}$, and $\frac{\partial v}{\partial z}$ for $v(x,y,z) = e^{xyz}\cos(yz)$.

7. Determine $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$, and $\frac{\partial f}{\partial z}$ for the function $f(x,y,z) = x^3yz^2 + e^{xyz}\sin(y)$.

8. Calculate $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, and $\frac{\partial u}{\partial z}$ for $u(x,y,z) = \ln(xy) + xyz - \cos(z)$.

9. Find $\frac{\partial v}{\partial x}$, $\frac{\partial v}{\partial y}$, and $\frac{\partial v}{\partial z}$ for $v(x,y,z) = x^2y^3z^4 + e^{xyz}\sin(yz) - \ln(z)$.

9.2 Solutions Practical Exercises Partial Derivatives:

Exercise 1: Given the function:

$$f(x,y) = 3x^2y + 2xy^3$$

we want to find the partial derivative with respect to x.

1. Differentiating $3x^2y$ with respect to x:

$$\frac{\partial}{\partial x}(3x^2y) = 6xy$$

2. Differentiating $2xy^3$ with respect to x:

$$\frac{\partial}{\partial x}(2xy^3) = 2y^3$$

Combining the results, we have:

$$\frac{\partial f}{\partial x} = 6xy + 2y^3$$

Exercise 2:

Given the function:

$$z(x, y) = x^2y + 5y^2 - 3x$$

we want to find the partial derivative with respect to y.

1. Differentiating x^2y with respect to y:

$$\frac{\partial}{\partial y}(x^2y) = x^2$$

2. Differentiating $5y^2$ with respect to y:

$$\frac{\partial}{\partial y}(5y^2) = 10y$$

3. Differentiating $-3x$ with respect to y:

$$\frac{\partial}{\partial y}(-3x) = 0$$

Combining the results, we have:

$$\frac{\partial z}{\partial y} = x^2 + 10y$$

Exercise 3:

Given the function:

$$v(x, y, z) = 4x^3y^2 - 3xy^2z^2 + 2z$$

we want to find the partial derivative with respect to y.

1. Differentiating $4x^3y^2$ with respect to y:

$$\frac{\partial}{\partial y}(4x^3y^2) = 8x^3y$$

2. Differentiating $-3xy^2z^2$ with respect to y:

$$\frac{\partial}{\partial y}(-3xy^2z^2) = -6xyz^2$$

3. Differentiating $2z$ with respect to y:

$$\frac{\partial}{\partial y}(2z) = 0$$

Combining the results, we have:

$$\frac{\partial v}{\partial y} = 8x^3y - 6xyz^2$$

Exercise 4:

Given the function:

$$f(x, y, z) = x^2y + 3yz^2 - z^3x$$

we will compute the partial derivatives with respect to x and y.

1. Differentiating x^2y with respect to x:

$$\frac{\partial}{\partial x}(x^2y) = 2xy$$

2. Differentiating $3yz^2$ with respect to x:

$$\frac{\partial}{\partial x}(3yz^2) = 0$$

3. Differentiating $-z^3x$ with respect to x:

$$\frac{\partial}{\partial x}(-z^3x) = -z^3$$

Combining these results:

$$\frac{\partial f}{\partial x} = 2xy - z^3$$

1. Differentiating x^2y with respect to y:

$$\frac{\partial}{\partial y}(x^2y) = x^2$$

2. Differentiating $3yz^2$ with respect to y:

$$\frac{\partial}{\partial y}(3yz^2) = 3z^2$$

3. Differentiating $-z^3x$ with respect to y:

$$\frac{\partial}{\partial y}(-z^3x) = 0$$

Combining these results:

$$\frac{\partial f}{\partial y} = x^2 + 3z^2$$

Exercise 5:

Given the function:

$$u(x, y, z) = xy^2 + yz^3 - xz^2$$

we will compute the partial derivatives with respect to x, y, and z.

1. Differentiating xy^2 with respect to x:

$$\frac{\partial}{\partial x}(xy^2) = y^2$$

2. Differentiating yz^3 with respect to x:

$$\frac{\partial}{\partial x}(yz^3) = 0$$

3. Differentiating $-xz^2$ with respect to x:

$$\frac{\partial}{\partial x}(-xz^2) = -z^2$$

Combining these results:

$$\frac{\partial u}{\partial x} = y^2 - z^2$$

1. Differentiating xy^2 with respect to y:

$$\frac{\partial}{\partial y}(xy^2) = 2xy$$

2. Differentiating yz^3 with respect to y:

$$\frac{\partial}{\partial y}(yz^3) = z^3$$

3. Differentiating $-xz^2$ with respect to y:

$$\frac{\partial}{\partial y}(-xz^2) = 0$$

Combining these results:

$$\frac{\partial u}{\partial y} = 2xy + z^3$$

1. Differentiating xy^2 with respect to z:

$$\frac{\partial}{\partial z}(xy^2) = 0$$

2. Differentiating yz^3 with respect to z:

$$\frac{\partial}{\partial z}(yz^3) = 3yz^2$$

3. Differentiating $-xz^2$ with respect to z:

$$\frac{\partial}{\partial z}(-xz^2) = -2xz$$

Combining these results:

$$\frac{\partial u}{\partial z} = 3yz^2 - 2xz$$

Exercise 6:

Given the function:

$$v(x, y, z) = e^{xyz} \cos(yz)$$

we will compute the partial derivatives with respect to x, y, and z.

Using the product rule, we get:

$$\frac{\partial v}{\partial x} = yze^{xyz} \cos(yz)$$

Again using the product rule:

$$\frac{\partial v}{\partial x} = xe^{xyz} \cos(yz) - e^{xyz} z \sin(yz)$$

Using the product rule once more:

$$\frac{\partial v}{\partial x} = xye^{xyz} \cos(yz) - e^{xyz} y \sin(yz)$$

To summarize:

$$\frac{\partial v}{\partial x} = yze^{xyz} \cos(yz)$$
$$\frac{\partial v}{\partial y} = xe^{xyz} \cos(yz) - e^{xyz} z \sin(yz)$$
$$\frac{\partial v}{\partial z} = xye^{xyz} \cos(yz) - e^{xyz} y \sin(yz)$$

Exercise 7:

Given the function:

$$f(x, y, z) = x^3 yz^2 + e^{xyz} \sin(y)$$

we will compute the partial derivatives with respect to x, y, and z.

1. Differentiating x^3yz^2 with respect to x:

$$\frac{\partial}{\partial x}(x^3yz^2) = 3x^2yz^2$$

2. Using the product rule to differentiate $e^{xyz}\sin(y)$ with respect to x:

$$\frac{\partial}{\partial x}(e^{xyz}\sin(y)) = yze^{xyz}\sin(y)$$

Combining the results:

$$\frac{\partial f}{\partial x} = 3x^2yz^2 + yze^{xyz}\sin(y)$$

1. Differentiating x^3yz^2 with respect to y:

$$\frac{\partial}{\partial y}(x^3yz^2) = x^3z^2$$

2. Using the product rule to differentiate $e^{xyz}\sin(y)$ with respect to y:

$$\frac{\partial}{\partial y}(e^{xyz}\cos(y)) = e^{xyz}\cos(y) + xze^{xyz}\sin(y))$$

Combining the results:

$$\frac{\partial f}{\partial y} = x^3z^2 + e^{xyz}\cos(y) + xze^{xyz}\sin(y)$$

1. Differentiating x^3yz^2 with respect to z:

$$\frac{\partial}{\partial z}(x^3yz^2) = 2x^3yz$$

2. Using the product rule to differentiate $e^{xyz}\sin(y)$ with respect to z:

$$\frac{\partial}{\partial z}(e^{xyz}\sin(y)) = xye^{xyz}\sin(y)$$

Combining the results:

$$\frac{\partial f}{\partial z} = 2x^3yz + xye^{xyz}\sin(y)$$

Exercise 8:

Given the function:

$$u(x, y, z) = \ln(xy) + xyz - \cos(z)$$

we will compute the partial derivatives with respect to x, y, and z.

1. Differentiating $\ln(xy)$ with respect to x:

$$\frac{\partial}{\partial x}(\ln(xy)) = \frac{1}{x}$$

2. Differentiating xyz with respect to x:

$$\frac{\partial}{\partial x}(xyz) = yz$$

3. Differentiating $-\cos(z)$ with respect to x:

$$\frac{\partial}{\partial x}(-\cos(z)) = 0$$

Combining the results:

$$\frac{\partial u}{\partial x} = \frac{1}{x} + yz$$

1. Differentiating $\ln(xy)$ with respect to y:

$$\frac{\partial}{\partial y}(\ln(xy)) = \frac{1}{y}$$

2. Differentiating xyz with respect to y:

$$\frac{\partial}{\partial y}(xyz) = xz$$

3. Differentiating $-\cos(z)$ with respect to y:

$$\frac{\partial}{\partial y}(-\cos(z)) = 0$$

Combining the results:

$$\frac{\partial u}{\partial y} = \frac{1}{y} + xz$$

1. Differentiating $\ln(xy)$ with respect to z:

$$\frac{\partial}{\partial z}(\ln(xy)) = 0$$

2. Differentiating xyz with respect to z:

$$\frac{\partial}{\partial z}(xyz) = xy$$

3. Differentiating $-\cos(z)$ with respect to z:

$$\frac{\partial}{\partial z}(-\cos(z)) = \sin(z)$$

Combining the results:

$$\frac{\partial u}{\partial z} = xy + \sin(z)$$

Exercise 9:

Given the function:

$$v(x, y, z) = x^2y^3z^4 + e^{xyz}\sin(yz) - \ln(z)$$

we will compute the partial derivatives with respect to x, y, and z.

1. Differentiating $x^2y^3z^4$ with respect to x:

$$\frac{\partial}{\partial x}(x^2y^3z^4) = 2xy^3z^4$$

2. Using the product rule to differentiate $e^{xyz}\sin(yz)$ with respect to x:

$$\frac{\partial}{\partial x}(e^{xyz}\sin(yz)) = yze^{xyz}\sin(yz)$$

3. Differentiating $-\ln(z)$ with respect to x:

$$\frac{\partial}{\partial x}(-\ln(z)) = 0$$

Combining the results:

$$\frac{\partial v}{\partial x} = 2xy^3z^4 + yze^{xyz}\sin(yz)$$

1. Differentiating $x^2y^3z^4$ with respect to y:

$$\frac{\partial}{\partial y}(x^2y^3z^4) = 3x^2y^2z^4$$

This term comes from the fact that $3x^2y^2z^4$ is the coefficient of y in $x^2y^3z^4$.

2. Using the product rule to differentiate $e^{xyz}\sin(yz)$ with respect to y:
Differentiate e^{xyz} with respect to y:

$$xze^{xyz}$$

Multiply this derivative by $\sin(yz)$:

$$xze^{xyz}\sin(yz)$$

Differentiate $\sin(yz)$ with respect to y, and then multiply by e^{xyz}:

$$z\cos(yz)e^{xyz}$$

3. The term $-\ln(z)$ does not contain y, so its derivative with respect to y is 0.

Combining these results, the partial derivative with respect to y is:

$$\frac{\partial v}{\partial y} = 3x^2y^2z^4 + xze^{xyz}\sin(yz) + z\exp(xyz)\cos(yz)$$

1. Differentiating $x^2y^3z^4$ with respect to z:

$$\frac{\partial}{\partial z}(x^2y^3z^4) = 4x^2y^3z^3$$

2. Using the product rule to differentiate $e^{xyz}\sin(yz)$ with respect to z:

$$\frac{\partial}{\partial z}(e^{xyz}\sin(yz)) = e^{xyz}(xy\sin(yz) + y\cos(yz))$$

3. Differentiating $-\ln(z)$ with respect to z:

$$\frac{\partial}{\partial z}(-\ln(z)) = -\frac{1}{z}$$

Combining the results:

$$\frac{\partial v}{\partial z} = 4x^2y^3z^3 + e^{xyz}(xy\sin(yz) + y\cos(yz)) - \frac{1}{z}$$

Now for the Gradients and Hessians

9.3 Practical Exercises Gradients and Hessian:

1. Find the gradient and the Hessian of $f(x, y) = x + y$.
2. Compute the gradient and the hessian of $g(x, y) = x^2 + y^2$.
3. Compute the gradient and Hessian of $g(x, y) = x\sin(xy) + \cos(xy)$.

4. Find the gradient and Hessian of $h(x, y) = x^2 e^y + y^2 e^x$.
5. Find the gradient and Hessian of $f(x, y, z) = x^2 y e^z + \ln(xyz)$.
6. Compute the gradient and Hessian of $g(x, y, z) = xe^{xyz} + z\ln(xy)$.

9.4 Solutions Practical Exercises Gradients and Hessian:

1. **Find the gradient and the Hessian of** $\boldsymbol{f(x, y) = x + y}$:

   Gradient:

   For a function $f(x, y)$, the gradient, often denoted by ∇f or grad f, is a vector field and is given by:

   $$\nabla f(x, y) = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{bmatrix}$$

   Computing the components:

   1. The partial derivative with respect to x:

   $$\frac{\partial f}{\partial x} = \frac{\partial(x + y)}{\partial x} = 1$$

   2. The partial derivative with respect to y:

   $$\frac{\partial f}{\partial y} = \frac{\partial(x + y)}{\partial y} = 1$$

   Thus, the gradient is:

   $$\nabla f(x, y) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

   Hessian:

The Hessian matrix, denoted by $H(f)$, is a square matrix of second-order mixed partial derivatives. For the function $f(x, y)$, it is given by:

$$H(f) = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{bmatrix}$$

Computing the entries:

1. The second partial derivative with respect to x:

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}(1) = 0$$

2. The mixed partial derivative with respect to x and then y:

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}(1) = 0$$

3. The mixed partial derivative with respect to y and then x:

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}(1) = 0$$

4. The second partial derivative with respect to y:

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}(1) = 0$$

Thus, the Hessian is:

$$H(f) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

2. **Compute the gradient and the Hessian of $\boldsymbol{g(x, y) = x^2 + y^2}$:**

Gradient:

For the function $g(x, y)$, the gradient, often denoted by ∇g, is a vector field and is given by:

$$\nabla g(x, y) = \begin{bmatrix} \frac{\partial g}{\partial x} \\ \frac{\partial g}{\partial y} \end{bmatrix}$$

Computing the components:

1. The partial derivative with respect to x:

$$\frac{\partial g}{\partial x} = \frac{\partial(x^2 + y^2)}{\partial x} = 2x$$

2. The partial derivative with respect to y:

$$\frac{\partial g}{\partial y} = \frac{\partial(x^2 + y^2)}{\partial y} = 2y$$

Thus, the gradient is:

$$\nabla g(x, y) = \begin{bmatrix} 2x \\ 2y \end{bmatrix}$$

Hessian:

The Hessian matrix, often denoted by $H(g)$, is a square matrix of second-order mixed partial derivatives. For the function $g(x, y)$, it is given by:

$$H(g) = \begin{bmatrix} \frac{\partial^2 g}{\partial x^2} & \frac{\partial^2 g}{\partial x \partial y} \\ \frac{\partial^2 g}{\partial y \partial x} & \frac{\partial^2 g}{\partial y^2} \end{bmatrix}$$

Computing the entries:

1. The second partial derivative with respect to x:

$$\frac{\partial^2 g}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial g}{\partial x}\right) = \frac{\partial}{\partial x}(2x) = 2$$

2. The mixed partial derivative with respect to x and then y:

$$\frac{\partial^2 g}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial g}{\partial y}\right) = \frac{\partial}{\partial x}(2y) = 0$$

3. The mixed partial derivative with respect to y and then x:

$$\frac{\partial^2 g}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial g}{\partial x}\right) = \frac{\partial}{\partial y}(2x) = 0$$

4. The second partial derivative with respect to y:

$$\frac{\partial^2 g}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial g}{\partial y}\right) = \frac{\partial}{\partial y}(2y) = 2$$

Thus, the Hessian is:

$$H(g) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

3. **Compute the gradient and the Hessian of $g(x, y) = x \sin(xy) + \cos(xy)$:**

Gradient:

For the function $g(x, y)$, the gradient is given by:

$$\nabla g(x, y) = \begin{bmatrix} \frac{\partial g}{\partial x} \\ \frac{\partial g}{\partial y} \end{bmatrix}$$

Computing the components:

1. The partial derivative with respect to x:

$$\frac{\partial g}{\partial x} = \frac{\partial}{\partial x}(x \sin(xy) + \cos(xy)) = \sin(xy) + xy\cos(xy) - y\sin(xy)$$

2. The partial derivative with respect to y:

$$\frac{\partial g}{\partial y} = \frac{\partial}{\partial y}(x \sin(xy) + \cos(xy)) = x^2\cos(xy) - x\sin(xy)$$

Thus, the gradient is:

$$\nabla g(x, y) = \begin{bmatrix} \sin(xy) + xy\cos(xy) - y\sin(xy) \\ x^2\cos(xy) - x\sin(xy) \end{bmatrix}$$

Hessian:

The Hessian matrix of $g(x, y)$ is:

$$H(g) = \begin{bmatrix} \frac{\partial^2 g}{\partial x^2} & \frac{\partial^2 g}{\partial x \partial y} \\ \frac{\partial^2 g}{\partial y \partial x} & \frac{\partial^2 g}{\partial y^2} \end{bmatrix}$$

Computing the entries:

1. The second partial derivative with respect to x:

$$\frac{\partial^2 g}{\partial x^2} = \frac{\partial}{\partial x}(\sin(xy) + xy\cos(xy)$$
$$= -y\sin(xy)) = 2y\cos(xy) - y^2\sin(xy)$$

2. The mixed partial derivative with respect to x and then y:

$$\frac{\partial^2 g}{\partial x \partial y} = \frac{\partial}{\partial x}(x^2\cos(xy)$$
$$= -x\sin(xy)) = 2x\cos(xy) - 2\sin(xy) - x^2 y\sin(xy)$$

3. The mixed partial derivative with respect to y and then x:

$$\frac{\partial^2 g}{\partial y \partial x} = \frac{\partial}{\partial y}(\sin(xy) + xy\cos(xy)$$
$$= -y\sin(xy)) = 2x\cos(xy) - x^2 y\sin(xy)$$

4. The second partial derivative with respect to y:

$$\frac{\partial^2 g}{\partial y^2} = \frac{\partial}{\partial y}(x^2\cos(xy)$$
$$= -x\sin(xy)) = -x^3\sin(xy) - 2x^2\sin(xy)$$

Thus, the Hessian is:

$$H(g) = \begin{bmatrix} 2y\cos(xy) - y^2\sin(xy) & 2x\cos(xy) - 2\sin(xy) - x^2 y\sin(xy) \\ 2x\cos(xy) - x^2 y\sin(xy) & -x^3\sin(xy) - 2x^2\sin(xy) \end{bmatrix}$$

4. **Find the gradient and the Hessian of $h(x, y) = x^2 e^y + y^2 e^x$:**

Gradient:

For the function $h(x, y)$, the gradient is denoted as:

$$\nabla h(x, y) = \begin{bmatrix} \frac{\partial h}{\partial x} \\ \frac{\partial h}{\partial y} \end{bmatrix}$$

Computing the components:

1. The partial derivative with respect to x:

$$\frac{\partial h}{\partial x} = \frac{\partial}{\partial x}\left(x^2 e^y + y^2 e^x\right) = 2xe^y + y^2 e^x$$

2. The partial derivative with respect to y:

$$\frac{\partial h}{\partial y} = \frac{\partial}{\partial y}\left(x^2 e^y + y^2 e^x\right) = x^2 e^y + 2ye^x$$

Thus, the gradient is:

$$\nabla h(x, y) = \begin{bmatrix} 2xe^y + y^2 e^x \\ x^2 e^y + 2ye^x \end{bmatrix}$$

Hessian:

The Hessian matrix of $h(x, y)$ is represented as:

$$H(h) = \begin{bmatrix} \frac{\partial^2 h}{\partial x^2} & \frac{\partial^2 h}{\partial x \partial y} \\ \frac{\partial^2 h}{\partial y \partial x} & \frac{\partial^2 h}{\partial y^2} \end{bmatrix}$$

Computing the entries:

1. The second partial derivative with respect to x:

$$\frac{\partial^2 h}{\partial x^2} = \frac{\partial}{\partial x}\left(2xe^y + y^2 e^x\right) = 2e^y + y^2 e^x$$

2. The mixed partial derivative with respect to x and then y:

$$\frac{\partial^2 h}{\partial x \partial y} = \frac{\partial}{\partial x}\left(x^2 e^y + 2ye^x\right) = 2xe^y + 2ye^x$$

3. The mixed partial derivative with respect to y and then x:

$$\frac{\partial^2 h}{\partial y \partial x} = \frac{\partial}{\partial y}\left(2xe^y + y^2 e^x\right) = 2xe^y + 2ye^x$$

4. The second partial derivative with respect to y:

$$\frac{\partial^2 h}{\partial y^2} = \frac{\partial}{\partial y}\left(x^2 e^y + 2ye^x\right) = x^2 e^y + 2e^x$$

Thus, the Hessian is:

$$H(h) = \begin{bmatrix} 2e^y + y^2 e^x & 2xe^y + 2ye^x \\ 2xe^y + 2ye^x & x^2 e^y + 2e^x \end{bmatrix}$$

5. **Find the gradient and the Hessian of $f(x, y, z) = x^2ye^z + \ln(xyz)$:**

Gradient:

For the function $f(x, y, z)$, the gradient is defined as:

$$\nabla f(x, y, z) = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{bmatrix}$$

Computing the components:

1. The partial derivative with respect to x:

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(x^2ye^z + \ln(xyz)\right) = 2xye^z + \frac{1}{x}$$

2. The partial derivative with respect to y:

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(x^2ye^z + \ln(xyz)\right) = x^2e^z + \frac{1}{y}$$

3. The partial derivative with respect to z:

$$\frac{\partial f}{\partial z} = \frac{\partial}{\partial z}\left(x^2ye^z + \ln(xyz)\right) = x^2ye^z + \frac{1}{z}$$

Thus, the gradient is:

$$\nabla f(x, y, z) = \begin{bmatrix} 2xye^z + \frac{1}{x} \\ x^2e^z + \frac{1}{y} \\ x^2ye^z + \frac{1}{z} \end{bmatrix}$$

Hessian:

The Hessian matrix of $f(x, y, z)$ is denoted as:

$$H(f) = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial x \partial z} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} & \frac{\partial^2 f}{\partial y \partial z} \\ \frac{\partial^2 f}{\partial z \partial x} & \frac{\partial^2 f}{\partial z \partial y} & \frac{\partial^2 f}{\partial z^2} \end{bmatrix}$$

9.4. Solutions Practical Exercises Gradients and Hessian:

a) $\frac{\partial^2 f}{\partial x^2}$:

$$\frac{\partial^2}{\partial x^2}\left(x^2ye^z + \ln(xyz)\right) = 2ye^z - \frac{1}{x^2}$$

b) $\frac{\partial^2 f}{\partial x \partial y}$:

$$\frac{\partial^2}{\partial x \partial y}\left(x^2ye^z + \ln(xyz)\right) = 2xe^z$$

c) $\frac{\partial^2 f}{\partial x \partial z}$:

$$\frac{\partial^2}{\partial x \partial z}\left(x^2ye^z + \ln(xyz)\right) = 2xye^z$$

d) $\frac{\partial^2 f}{\partial y \partial x}$:

$$\frac{\partial^2}{\partial y \partial x}\left(x^2ye^z + \ln(xyz)\right) = 2xe^z$$

e) $\frac{\partial^2 f}{\partial y^2}$:

$$\frac{\partial^2}{\partial y^2}\left(x^2ye^z + \ln(xyz)\right) = -\frac{1}{y^2}$$

f) $\frac{\partial^2 f}{\partial y \partial z}$:

$$\frac{\partial^2}{\partial y \partial z}\left(x^2ye^z + \ln(xyz)\right) = x^2e^z$$

g) $\frac{\partial^2 f}{\partial z \partial x}$:

$$\frac{\partial^2}{\partial z \partial x}\left(x^2ye^z + \ln(xyz)\right) = 2xye^z$$

h) $\frac{\partial^2 f}{\partial z \partial y}$:

$$\frac{\partial^2}{\partial z \partial y}\left(x^2ye^z + \ln(xyz)\right) = x^2e^z$$

i) $\frac{\partial^2 f}{\partial z^2}$:

$$\frac{\partial^2}{\partial z^2}\left(x^2ye^z + \ln(xyz)\right) = x^2ye^z$$

Thus, the Hessian matrix is:

$$H(f) = \begin{bmatrix} 2ye^z - \frac{1}{x^2} & 2xe^z & 2xye^z \\ 2xe^z & -\frac{1}{y^2} & x^2e^z \\ 2xye^z & x^2e^z & x^2ye^z \end{bmatrix}$$

6. **Gradient and Hessian** of
   $\boldsymbol{g(x, y, z) = xe^{xyz} + z\ln(xy)}$ **:**

   Gradient: The gradient of g is:

$$\nabla g = \begin{bmatrix} \frac{\partial g}{\partial x} \\ \frac{\partial g}{\partial y} \\ \frac{\partial g}{\partial z} \end{bmatrix}$$

   Computing the partial derivatives:

   a) $\frac{\partial g}{\partial x}$:

$$e^{xyz} + zye^{xyz} + \frac{z}{x}$$

   b) $\frac{\partial g}{\partial y}$:

$$xze^{xyz} + \frac{z}{y}$$

   c) $\frac{\partial g}{\partial z}$:

$$xye^{xyz} + \ln(xy)$$

   Hessian: The Hessian matrix $H(g)$ is:

$$H(g) = \begin{bmatrix} \frac{\partial^2 g}{\partial x^2} & \frac{\partial^2 g}{\partial x \partial y} & \frac{\partial^2 g}{\partial x \partial z} \\ \frac{\partial^2 g}{\partial y \partial x} & \frac{\partial^2 g}{\partial y^2} & \frac{\partial^2 g}{\partial y \partial z} \\ \frac{\partial^2 g}{\partial z \partial x} & \frac{\partial^2 g}{\partial z \partial y} & \frac{\partial^2 g}{\partial z^2} \end{bmatrix}$$

   Computing the entries:

$$H(g) = \begin{bmatrix} y^2ze^{xyz} + \frac{z}{x^2} & ze^{xyz} + yze^{xyz} & e^{xyz} + yze^{xyz} + \frac{1}{x} \\ ze^{xyz} + yze^{xyz} & xze^{xyz} + \frac{z}{y^2} & xe^{xyz} \\ e^{xyz} + yze^{xyz} + \frac{1}{x} & xe^{xyz} & 0 \end{bmatrix}$$

Chapter 10

Regression - Coding Exercises

10.1 Univariable Linear Regression:

Objective: Using the provided dataset, your task is to predict the number of bees given the floral density.

No.	Number of Flowers	Bee Count
1	150	30
2	250	48
3	100	22
4	300	60
5	180	36
6	320	65
7	200	42
8	275	54
9	230	45
10	350	70

This table indicates that as the number of flowers in an area increases, the number of bees observed also tends to increase. With such a dataset, you can apply linear regression to predict the bee count based on the number of flowers in an area.

1. **Plot:** Visualise the Number of flowers vs bees count with a plot.

2. **Linear Regression:** Implement a simple linear regression model using Gradient Descent (GD) to optimize the parame-

ters. Fit this model to predict the bees based on the number of flowers. Calculate the Mean Squared Error (MSE) for your model on the dataset.

3. **Data Transformation:** To account for possible non-linear relationships or to stabilize variance, transform both the number of flowers and bees data using the natural logarithm (ln). Re-plot the data to visualize the transformed relationship.

4. **Linear Regression on Transformed Data:** Using the log-transformed data, fit a new linear regression model optimized with GD. Does the line fit better after the transformation? Compare the MSE for this model with the previous one.

5. **Interpretation:** Analyze and interpret the coefficients of the regression models from steps 1 and 3. How do they differ, and what might these differences imply about the relationship between advertising budget and sales?

10.2 Multivariate Linear Regression:

Objective: Using the extended dataset, predict the bees count based on the number of flowers and the average temperature.

No.	Number of Flowers	Average Temperature (°C)	Bee Count
1	150	20	30
2	250	23	48
3	100	18	22
4	300	25	60
5	180	21	36
6	320	26	65
7	200	22	42
8	275	24	54
9	230	23	45
10	350	27	70

1. **Linear Regression on Transformed Data:** Fit a new multivariate linear regression model to the log-transformed data, optimized with Gradient Descent.

2. **Interpretation:** Analyze and interpret the coefficients of the regression model. How does the inclusion of the average temperature influence the relationship between the number of flowers and the bee count?

10.3 Polynomial Regression:

Objective: Using the provided dataset, your task is to predict the bee foraging activity based on the nectar sugar concentration of flowers in a particular area.

No.	Nectar Sugar Concentration (%)	Bee Count
1	10	20
2	12	25
3	14	30
4	15	32
5	16	35
6	18	42
7	19	46
8	20	50
9	21	53
10	23	60

1. **Data Visualization:** Plot the data points using a scatter plot. Based on the visual pattern, suggest a function that can represent the relationship between room sugar concentration and the bee count.

2. **Linear Regression:** Despite the observed trend, attempt to fit a linear regression model to the log scaled data using Gradient Descent. Calculate the Mean Squared Error (MSE) for your model.

3. **Polynomial Regression:** Considering the suggested function from the first step, implement a polynomial regression model with the Newton's method. Specifically, use a quadratic

term (square of nectar sugar concentration). Calculate the MSE for this model and compare it with the linear model and scaled the data with the log.

4. **Model Interpretation:** Analyze the polynomial regression model. Discuss the importance and contribution of the quadratic term in modeling the relationship. How does it improve the fit compared to the linear model?

Chapter 11

Multivariate Critical Points:

11.1 Practical Exercises:

1. Find the critical points of the function $f(x, y) = x^2 + y^2$ and determine their nature.

2. For the function $g(x, y) = -x^2 - y^2 + 4xy$, find the critical points and classify them.

3. Determine the critical points and their nature for the function $h(x, y) = x^3 - 3xy + y^3$.

4. Analyze the critical points of the function $j(x, y, z) = x^2y - xyz + y^2z$.

5. Consider the function:

$$f(x, y) = x^2 + 2y^2 + 2xy - 6x$$

   Initial Parameters: $(x_0, y_0) = (1, 1)$

   a) Compute the gradient of f.

   b) Starting from (x_0, y_0), implement 3 iterations of the gradient descent algorithm with a learning rate $\alpha = 0.1$. Record the updated values of (x, y).

   c) Using Newton's method and starting from (x_0, y_0), implement 3 iterations. Record the updated values of (x, y).

11.2 Solutions Practical Exercises:

1. **For the function $f(x, y) = x^2 + y^2$:**

   Partial Derivative with respect to x:

   $$\frac{\partial f}{\partial x} = 2x$$

   Partial Derivative with respect to y:

   $$\frac{\partial f}{\partial y} = 2y$$

   The gradient:

   $$\nabla f = \begin{bmatrix} 2x \\ 2y \end{bmatrix}$$

   Set each component to zero and solve for x and y:

   $$2x = 0 \Rightarrow x = 0 \quad \text{and} \quad 2y = 0 \Rightarrow y = 0$$

   There is one critical point at $(0, 0)$.

   To determine the nature of the critical point, evaluate the Hessian matrix:

   $$H(f) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

   The Hessian is positive definite (both eigenvalues are positive), so $(0, 0)$ is a local minimum.

2. **For the function $g(x, y) = -x^2 - y^2 + 4xy$:**

   Partial Derivative with respect to x:

   $$\frac{\partial g}{\partial x} = -2x + 4y$$

   Partial Derivative with respect to y:

   $$\frac{\partial g}{\partial y} = -2y + 4x$$

The gradient:

$$\nabla g = \begin{bmatrix} -2x + 4y \\ -2y + 4x \end{bmatrix}$$

Setting each component to zero gives:

$$-2x + 4y = 0 \quad \text{and} \quad -2y + 4x = 0$$

Solving, we find one critical point at $(0, 0)$.

The Hessian matrix:

$$H(g) = \begin{bmatrix} -2 & 4 \\ 4 & -2 \end{bmatrix}$$

The eigenvalues of the Hessian matrix are -6 and 2, one positive and one negative, indicating a saddle point at $(0, 0)$.

3. **For the function $\boldsymbol{h(x, y) = x^3 - 3xy + y^3}$:**

Partial Derivative with respect to x:

$$\frac{\partial h}{\partial x} = 3x^2 - 3y$$

Partial Derivative with respect to y:

$$\frac{\partial h}{\partial y} = -3x + 3y^2$$

The gradient:

$$\nabla h = \begin{bmatrix} 3x^2 - 3y \\ -3x + 3y^2 \end{bmatrix}$$

Setting each component to zero gives:

$$3x^2 - 3y = 0 \quad \text{and} \quad -3x + 3y^2 = 0$$

Solving, we find two critical points: $(0, 0)$ and $(1, 1)$.

The Hessian matrix:

$$H(h) = \begin{bmatrix} 6x & -3 \\ -3 & 6y \end{bmatrix}$$

Evaluating the Hessian matrix at $(0, 0)$ gives a matrix with zero eigenvalues, indicating a saddle point. At $(1, 1)$, the Hessian matrix has positive eigenvalues, indicating a local minimum.

4. **For the function $\boldsymbol{j(x,y,z) = x^2y - xyz + y^2z}$:**

   Partial Derivative with respect to x:

$$\frac{\partial j}{\partial x} = 2xy - yz$$

   Partial Derivative with respect to y:

$$\frac{\partial j}{\partial y} = x^2 - xz + 2yz$$

   Partial Derivative with respect to z:

$$\frac{\partial j}{\partial z} = -xy + y^2$$

   The gradient:

$$\nabla j = \begin{bmatrix} 2xy - yz \\ x^2 - xz + 2yz \\ -xy + y^2 \end{bmatrix}$$

   Setting each component to zero and solving the resulting system of equations will yield the critical points.

$$\begin{aligned} 2xy - yz = 0 \quad &\Rightarrow \quad y = 0, \quad x = \frac{z}{2} \\ x^2 - xz + 2yz = 0 & \\ -xy + y^2 = 0 \quad &\Rightarrow \quad y = 0, \quad x = y \end{aligned}$$

   we find one critical point at $(0,0,0)$.
   The Hessian matrix:

$$H(j) = \begin{bmatrix} 2y & 2x - z & -y \\ 2x - z & 2z & -x + 2y \\ -y & -x + 2y & 0 \end{bmatrix}$$

   Evaluating the Hessian matrix at (0,0,0) gives a matrix with zero eigenvalues, indicating a saddle point.

5. **For the function $\boldsymbol{f(x,y) = x^2 + 2y^2 + 2xy - 6x}$:**

   Initial Parameters: $(x_0, y_0) = (1,1)$

The gradient vector of f is:

$$\nabla f(x,y) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = (2x + 2y - 6, 4y + 2x)$$

Gradient Descent Iterations

Using the learning rate $\alpha = 0.1$:

1st Iteration
Gradient at $(1,1)$: $\nabla f(1,1) = (-2,6)$
Update equations:

$$x_1 = x_0 - \alpha \cdot (-2) = 1 + 0.2 = 1.2$$
$$y_1 = y_0 - \alpha \cdot 6 = 1 - 0.6 = 0.4$$

2nd Iteration
Gradient at $(1.2, 0.4)$: $\nabla f(1.2, 0.4) = (-2.8, 4.0)$
Update equations:

$$x_2 = x_1 - \alpha \cdot (-2.8) = 1.2 + 0.28 = 1.48$$
$$y_2 = y_1 - \alpha \cdot 4.0 = 0.4 - 0.4 = 0$$

3rd Iteration
Gradient at $(1.48, 0)$: $\nabla f(1.48, 0) = (-3.04, 2.96)$
Update equations:

$$x_3 = x_2 - \alpha \cdot (-3.04) = 1.48 + 0.304 = 1.784$$
$$y_3 = y_2 - \alpha \cdot 2.96 = 0 - 0.296 = -0.296$$

Newton's Method Iterations

The Hessian matrix H and its inverse are:

$$H = \begin{bmatrix} 2 & 2 \\ 2 & 4 \end{bmatrix}, \quad H^{-1} = \begin{bmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

1st Iteration
Starting from $(x_0, y_0) = (1,1)$,

Gradient at $(1, 1)$: $\nabla f(1, 1) = (-2, 6)$
Update equation:

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} -2 \\ 6 \end{bmatrix} = \begin{bmatrix} 6 \\ -3 \end{bmatrix}$$

(Note: Subsequent iterations using Newton's Method do not change the point (x, y) from $(6, -3)$, suggesting that it might have reached a stationary point.)

11.3 Coding Exercises:

Given the function:

$$f(x, y) = (1.5 - x + xy)^2 + \left(2.25 - x + xy^2\right)^2 + \left(2.625 - x + xy^3\right)^2$$

Complete the following exercises:

1. **Plotting the Function**
   - Create a 3D plot of the function in the range $x, y \in [-5, 5]$. Visualize the regions where the function has its minima and maxima.
2. **Gradient Descent Implementation**
   - Implement the gradient descent algorithm for the function f. Define a suitable learning rate and initialize a starting point. Iterate until convergence and plot the path taken to reach the minimum on the 3D plot.
3. **Newton's Method Implementation**
   - Implement the Newton-Raphson method for the function f. Initialize a starting point. Iterate until convergence. Compare the number of iterations it takes to converge using the Newton's method versus the Gradient Descent.

4. **Exploring Learning Rates**

   - Using gradient descent, explore how different learning rates (e.g., 0.0001, 0.0005, 0.001 and 0.01) affect the speed of convergence and the final result. Visualize the paths taken for each learning rate on the 3D plot.

5. **Convergence Criteria**

   - Implement a version of the gradient descent algorithm with an adaptive learning rate. The algorithm should reduce the learning rate if the function value does not decrease by a certain threshold in subsequent iterations. Analyze its performance in comparison with the standard gradient descent.

Chapter 12

Multivariate integrals

Compute the following integrals

12.1 Practical Exercises:

1. $\int_0^2 \int_0^1 (3x^2 + 2xy)\, dy\, dx$.

2. $\int_0^\pi \int_0^\pi \sin(x + y)\, dy\, dx$.

3. $\int_0^1 \int_0^1 \int_0^1 xyz\, dz\, dy\, dx$.

4. $\int_0^2 \int_0^1 (x^2 + \cos y)\, dy\, dx$.

5. $\int_1^3 \int_x^3 \sqrt{x + y}\, dy\, dx$.

6. $\int_0^2 \int_0^2 \int_0^2 (x^2 y + z^2)\, dz\, dy\, dx$.

12.2 Solutions Practical Exercises:

1. $\boldsymbol{\int_0^2 \int_0^1 (3x^2 + 2xy)\, dy\, dx}$ **:**

Inner Integral (with respect to y):

$$\begin{aligned}\int_0^1 (3x^2 + 2xy)\,dy &= 3x^2 \int_0^1 dy + 2x \int_0^1 y\,dy \\ &= 3x^2 + x\end{aligned}$$

Outer Integral (with respect to x):

$$\begin{aligned}\int_0^2 (3x^2 + x)\,dx &= \int_0^2 3x^2\,dx + \int_0^2 x\,dx \\ &= 8 + 2 \\ &= 10\end{aligned}$$

Result: Hence, the value of the double integral is 10.

2. $\boldsymbol{\int_0^\pi \int_0^\pi \sin(x+y)\,dy\,dx}$ **:**

Step 1: First, compute the inner integral with respect to y.

$$\begin{aligned}\int_0^\pi \sin(x+y)\,dy &= [-\cos(x+y)]_0^\pi \\ &= -\cos(x+\pi) + \cos(x) \\ &= 2\cos(x)\end{aligned}$$

Step 2: Compute the outer integral with respect to x.

$$\begin{aligned}\int_0^\pi 2\cos(x)\,dx &= [2\sin(x)]_0^\pi \\ &= 2\sin(\pi) - 2\sin(0) \\ &= 0\end{aligned}$$

Hence, the value of the double integral is:

$$\int_0^\pi \int_0^\pi \sin(x+y)\,dy\,dx = 0$$

3. $\int_0^1 \int_0^1 \int_0^1 xyz\, dz\, dy\, dx$ **:**

   Step 1: Compute the innermost integral with respect to z.

$$\begin{aligned}\int_0^1 xyz\, dz &= \left[\frac{xyz^2}{2}\right]_0^1 \\ &= \frac{xy(1)^2}{2} - \frac{xy(0)^2}{2} \\ &= \frac{xy}{2}\end{aligned}$$

   Step 2: Now, compute the middle integral with respect to y.

$$\begin{aligned}\int_0^1 \frac{xy}{2}\, dy &= \left[\frac{xy^2}{4}\right]_0^1 \\ &= \frac{x(1)^2}{4} - \frac{x(0)^2}{4} \\ &= \frac{x}{4}\end{aligned}$$

   Step 3: Compute the outermost integral with respect to x.

$$\begin{aligned}\int_0^1 \frac{x}{4}\, dx &= \left[\frac{x^2}{8}\right]_0^1 \\ &= \frac{(1)^2}{8} - \frac{(0)^2}{8} \\ &= \frac{1}{8}\end{aligned}$$

   Hence, the value of the triple integral is:

$$\int_0^1 \int_0^1 \int_0^1 xyz\, dz\, dy\, dx = \frac{1}{8}$$

4. $\int_0^2 \int_0^1 (x^2 + \cos y)\, dy\, dx$:

   Step 1: Compute the inner integral with respect to y.

$$
\begin{aligned}
\int_0^1 (x^2 + \cos y)\, dy &= \left[x^2 y + \sin y\right]_0^1 \\
&= x^2(1) + \sin(1) - x^2(0) - \sin(0) \\
&= x^2 + \sin(1)
\end{aligned}
$$

Step 2: Compute the outer integral with respect to x.

$$
\begin{aligned}
\int_0^2 (x^2 + \sin(1))\, dx &= \left[\frac{x^3}{3} + x\sin(1)\right]_0^2 \\
&= \frac{2^3}{3} + 2\sin(1) - \frac{0^3}{3} - 0\sin(1) \\
&= \frac{8}{3} + 2\sin(1)
\end{aligned}
$$

Hence, the value of the double integral is:

$$
\int_0^2 \int_0^1 (x^2 + \cos y)\, dy\, dx = \frac{8}{3} + 2\sin(1)
$$

5. $\boldsymbol{\int_1^3 \int_x^3 \sqrt{x+y}\, dy\, dx}$:

   Step 1: compute the inner integral with respect to y using the substitution $z = x + y$, which implies $dy = dz$. We find that

$$
\int_x^3 \sqrt{x+y}\, dy = \int_{2x}^{x+3} \sqrt{z}\, dz = \left[\frac{2}{3} z^{\frac{3}{2}}\right]_{2x}^{x+3} = \frac{2}{3}(x+3)^{\frac{3}{2}} - \frac{2}{3}(2x)^{\frac{3}{2}}
$$

   Step 2: compute the outer integral with respect to x:

$$
\int_1^3 \left(\frac{2}{3}(x+3)^{\frac{3}{2}} - \frac{2}{3}(2x)^{\frac{3}{2}}\right) dx \approx 4.2953
$$

   The final result is approximately 4.2953.

6. $\int_0^2 \int_0^2 \int_0^2 (x^2y + z^2)\, dz\, dy\, dx$ **:**

Step 1: Compute the innermost integral with respect to z.

$$\begin{aligned}\int_0^2 (x^2y + z^2)\, dz &= \left[x^2yz + \frac{z^3}{3}\right]_0^2 \\ &= 2x^2y + \frac{8}{3}\end{aligned}$$

Step 2: Compute the middle integral with respect to y.

$$\begin{aligned}\int_0^2 (2x^2y + \frac{8}{3})\, dy &= \left[x^2y^2 + \frac{8y}{3}\right]_0^2 \\ &= 4x^2 + \frac{16}{3}\end{aligned}$$

Step 3: Compute the outermost integral with respect to x.

$$\begin{aligned}\int_0^2 (4x^2 + \frac{16}{3})\, dx &= \left[\frac{4x^3}{3} + \frac{16x}{3}\right]_0^2 \\ &= \frac{64}{3}\end{aligned}$$

Thus, the value of the triple integral is:

$$\int_0^2 \int_0^2 \int_0^2 (x^2y + z^2)\, dz\, dy\, dx = \frac{64}{3}$$

Chapter 13

Neural Networks

13.1 Scalar Functions:

Predict whether a flower will be visited by a bee based on its size and nectar sugar concentration.

No.	Flower Size (cm)	Nectar Sugar (%)	Visited by Bee
1	3.5	10	No
2	4.2	15	Yes
3	2.8	5	No
4	5.0	20	Yes
5	3.7	12	No
6	4.5	18	Yes
7	3.2	11	No
8	4.8	19	Yes
9	3.0	10	No
10	4.7	22	Yes

Table 13.1: Dataset predicting whether a flower will be visited by a bee based on its size and nectar sugar concentration.

13.1.1 Coding Exercises:

1. Implement a neural network to predict loan approval using the following specifications:

   - A single neuron in the hidden layer.
   - Utilize the cross-entropy loss function.
   - Employ the sigmoid function as the activation function.

2. Preprocess the data by normalizing it using min-max scaling. Then, retrain your neural network using the scaled data. Observe and describe any changes in the model's performance.

3. Experiment with various activation functions on the scaled data. Determine and justify the optimal configuration based on your results.

4. Transition from the previous optimization method to using RMSprop. Analyze how this change affects the training dynamics and the model's final performance.

5. Enhance the model's complexity by adding a second hidden layer and train the networks with RMSprop.

13.2 Vector Functions:

No.	Flower Size (cm)	Nectar Sugar Concentration (%)	Flower Color Intensity	Bee Visits	Time Spent (seconds)
1	3.5	10	5	8	45
2	4.2	15	7	10	50
3	2.8	5	4	5	30
4	5.0	20	9	15	60
5	3.7	12	6	9	42
6	4.5	18	8	13	55
7	3.2	11	5	7	38
8	4.8	19	8	14	58
9	3.0	10	4	6	35
10	4.7	22	9	16	65

Table 13.2: Caption for the table.

13.2.1 Coding Exercises:

1. Implement a neural network capable of predicting both the loan approval status and the offered interest rate, adhering to the specifications below:

   - Two neurons in the hidden layer.
   - Mean Squared Error (MSE) as the loss function.
   - Use the Momentum optimization algorithm.

- Preprocess the data by normalizing it using min-max scaling.

2. Replicate the previous neural network model, but substitute the Momentum optimization algorithm with the ADAM optimization method. Analyse any notable differences in performance or behavior.