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3D Metric Fields

A Novel Approach to a New Idea

Bachelor Thesis

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Abstract

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Contents

Chapter 1

Introduction

Every thesis should start with an introduction. This thesis is written in LaTeX [3].

Chapter 2

Background

This chapter sets the stage and introduces already existing material.

A frame F is a set of 6 vectors $\{\pm F_0, \pm F_1, \pm F_2\}$. We can represent such a frame F as a 3×3 matrix F, where the ith-column is F_i. A frame field then maps to every point in 3D-space such a frame, i.e. $F : \mathbb{R}^3 \to \mathbb{R}^{3\times3}$. Usually, we work on a 3-manifold $\mathcal{M}$ and a positively oriented frame field, i.e. $F|_{\mathcal{M}} : \mathcal{M} \to \mathbb{R}^{3\times3}$, where $\det(F) > 0$. To allow for anisotropic, nonuniform meshes, we generalize orthonormality of frames to g-orthonormal frames. Orthonormality is measured in some metric g, and a frame F satifisfies the condition $\langle F_i, F_j \rangle_g = \delta_{ij}$. Any frame field with $\det(F) > 0$ naturally defines a metric $g = (FF^\top)^{-1}$, where F is g-orthonormal

$$F^\top gF = Id.$$

We can factor the frame field F into a symmetric part $g^{1/2}$ and a rotational part R

$$F = g^{-1/2}R$$

The symmetric part $g^{-1/2}$ keeps F g-orthonormal

$$\implies F^\top gF = (g^{-1/2}R)^\top gg^{-1/2}R) = R^\top g^{-1/2}gg^{-1/2}R = Id.$$

and R represents a rotational field $R : \mathcal{M} \to SO(3)$. The requirements for our frame field are:

- Smoothness
- Integrability
- Metric consistency: $g = (FF^\top)^{-1}$

A vector field U is integrable, if and only if $\nabla \times U = 0$, which means the vector field has vanishing curl everywhere. We can express this more naturally with the language of differential forms: The curl can be written as the exterior derivative d of a one-form α. A one-form (more generally, a differential form) is closed, if $d\alpha = 0$. Therefore, the local integrability can be expressed as the closedness of a one-form. We want F^{-1} (TODO: why F^{-1}) to be integrable. To achieve local integrability for, it suffices to make R locally integrable. We can think of a rotation field R as the composition of 3 vector fields

$$R = \begin{bmatrix} | & | & | \\ R_1 & R_2 & R_3 \\ | & | & | \end{bmatrix}$$

where $R_i : \mathbb{R}^3 \to \mathbb{R}^3$ is a vector field. We can therefore construct a vector-valued one-form, given $p = (x, y, z)^\top$ in Euclidean coordinates

$$\alpha \triangleq F^{-1}dp = R^\top g^{1/2}dp$$

where $dp = (dx, dy, dz)^\top$ is the common orthonormal one-form basis of $\Omega^1(\mathcal{M})$.

$$R \text{ locally integrable} \iff \mathbf{0} = d\alpha$$

Some reformulations yield:

$$\begin{aligned}\mathbf{0} = d\alpha = d(R^\top g^{1/2} dp) &\overset{(1)}{=} dR^\top \wedge (g^{1/2}dp) + R^\top d(g^{1/2}dp) \\ &= R^\top(\omega \wedge (g^{1/2}dp) + d(g^{1/2}dp))\end{aligned}$$

where for (1), the Leibnitz Rule for the exterior derivative is applied, and we define

$$\omega = RdR^\top \in \mathfrak{so}(3).$$

Remark. *ω is an antisymmetric matrix-valued one-form (every element is a one-form).*

Proof. We differentiate the orthogonality condition of the rotation matrix R (taking the derivative w.r.t. every element):

$$\begin{aligned} Id &= RR^\top \\ d(Id) &= d(RR^\top) \\ \mathbf{0} &= dRR^\top + RdR^\top \\ \mathbf{0} &= (RdR^\top)^\top + RdR^\top \\ -(RdR^\top)^\top &= RdR^\top \end{aligned}$$

□

Elements of the Lie algebra $\mathfrak{so}(3)$ can be thought as infinitesimal rotations and ω can be used as a connection one-form. For our purposes, it suffices to know that elements are antisymmetric matrices. To make R locally integrable, we find ω such that the one-form α is closed ($d\alpha = 0$, curl-free), then try to match the ω with R, which can be expressed as

$$\min_{R \in SO(3)} ||RdR^T - \omega||^2.$$

Connection evaluation To find ω, we use the above equation

$$\mathbf{0} = R^\top(\omega \wedge (g^{1/2}dp) + d(g^{1/2}dp)) \iff \mathbf{0} = \omega \wedge (g^{1/2}dp) + d(g^{1/2}dp)$$

and reformulate into a linear system. We represent the antisymmetric matrix-valued one-form ω

$$\omega = \begin{bmatrix} 0 & \omega_{12} & -\omega_{31} \\ -\omega_{12} & 0 & \omega_{23} \\ \omega_{31} & -\omega_{23} & 0 \end{bmatrix}$$

by $\begin{bmatrix}\omega_{23} & \omega_{31} & \omega_{12}\end{bmatrix} = \begin{bmatrix}dx & dy & dz\end{bmatrix} W$. We write $W = \begin{bmatrix}W_1, W_2, W_3\end{bmatrix}$, $W_i \in \mathbb{R}^3$. That is, W is the matrix with the coefficients for the one-forms. Recall, a one-form can be expressed as

$$\omega_{ij} = (\omega^{ij})_1 dx + (\omega^{ij})_2 dy + (\omega^{ij})_3 dz$$

So e.g.

$$\omega_{23} = \begin{bmatrix}dx & dy & dz\end{bmatrix} W_1 = (\omega^{23})_1 dx + (\omega^{23})_2 dy + (\omega^{23})_3 dz$$

We write $A = g^{1/2} = \begin{bmatrix} A^1, A^2, A^3 \end{bmatrix}$. We use the equation $\mathbf{0} = \omega \wedge (g^{1/2}dp) + d(g^{1/2}dp)$. First part:

$$\begin{aligned}
\omega \wedge g^{1/2}dp &= \begin{bmatrix} 0 & \omega_{12} & -\omega_{31} \\ -\omega_{12} & 0 & \omega_{23} \\ \omega_{31} & -\omega_{23} & 0 \end{bmatrix} \bigwedge \begin{bmatrix} A_1^1 & A_1^2 & A_1^3 \\ A_2^1 & A_2^2 & A_2^3 \\ A_3^1 & A_3^2 & A_3^3 \end{bmatrix} \begin{bmatrix} dx \\ dy \\ dz \end{bmatrix} \\
&= \begin{bmatrix} +\omega_{12} \wedge (A_2^1 dx + A_2^2 dy + A_2^3 dz) - \omega_{31} \wedge (A_3^1 dx + A_3^2 dy + A_3^3 dz) \\ -\omega_{12} \wedge (A_1^1 dx + A_1^2 dy + A_1^3 dz) + \omega_{23} \wedge (A_3^1 dx + A_3^2 dy + A_3^3 dz) \\ +\omega_{31} \wedge (A_1^1 dx + A_1^2 dy + A_1^3 dz) - \omega_{23} \wedge (A_2^1 dx + A_2^2 dy + A_2^3 dz) \end{bmatrix}
\end{aligned}$$

It will get really messy if we calculate each component here, so let us calculate one component separately first:

$$\begin{aligned}
\omega_{ij} \wedge (A_k^1 dx + A_k^2 dy + A_k^3 dz) &= ((\omega^{ij})_1 dx + (\omega^{ij})_2 dy + (\omega^{ij})_3 dz) \wedge (A_k^1 dx + A_k^2 dy + A_k^3 dz) \\
&= (\omega^{ij})_1 A_k^2 dx \wedge dy + (\omega^{ij})_1 A_k^3 dx \wedge dz \\
&+ (\omega^{ij})_2 A_k^1 dy \wedge dx + (\omega^{ij})_2 A_k^3 dy \wedge dz \\
&+ (\omega^{ij})_3 A_k^1 dz \wedge dx + (\omega^{ij})_3 A_k^2 dz \wedge dy \\
&= ((\omega^{ij})_1 A_k^2 - (\omega^{ij})_2 A_k^1) dx \wedge dy \\
&+ ((\omega^{ij})_2 A_k^3 - (\omega^{ij})_3 A_k^2) dy \wedge dz \\
&+ ((\omega^{ij})_3 A_k^1 - (\omega^{ij})_1 A_k^3) dz \wedge dx
\end{aligned}$$

where we use the fact that $dx \wedge dx = 0$ and $dx \wedge dy = -dy \wedge dx$. We can clean up the above expression using the cross product:

$$\begin{bmatrix} ((\omega^{ij})_2 A_k^3 - (\omega^{ij})_3 A_k^2) \\ ((\omega^{ij})_3 A_k^1 - (\omega^{ij})_1 A_k^3) \\ ((\omega^{ij})_1 A_k^2 - (\omega^{ij})_2 A_k^1) \end{bmatrix}^\top \begin{bmatrix} dy \wedge dz \\ dz \wedge dx \\ dx \wedge dy \end{bmatrix} = \begin{bmatrix} W_i \times A^k \end{bmatrix}^\top \begin{bmatrix} dy \wedge dz \\ dz \wedge dx \\ dx \wedge dy \end{bmatrix}$$

We use the fact that $\begin{bmatrix} A_1, A_2, A_3 \end{bmatrix} = \begin{bmatrix} A^1, A^2, A^3 \end{bmatrix}$ because A is symmetric, and W_1 corresponds to ω_{23}, W_2 to ω_{31} and W_3 to ω_{12}. Second part:

$$d(g^{1/2}dp) = d(Adp) = d\left(\begin{bmatrix} A_1^1 & A_1^2 & A_1^3 \\ A_2^1 & A_2^2 & A_2^3 \\ A_3^1 & A_3^2 & A_3^3 \end{bmatrix} \begin{bmatrix} dx \\ dy \\ dz \end{bmatrix} \right)$$

Again, we can do this separately for each row:

$$\begin{aligned}
& d(A_k^1 dx + A_k^2 dy + A_k^3 dz) \\
&= dA_k^1 \wedge dx + dA_k^2 \wedge dy + dA_k^3 \wedge dz \\
&= \frac{\partial A_k^1}{\partial x} dx \wedge dx + \frac{\partial A_k^1}{\partial y} dy \wedge dx + \frac{\partial A_k^1}{\partial z} dz \wedge dx \\
&+ \frac{\partial A_k^2}{\partial x} dx \wedge dy + \frac{\partial A_k^2}{\partial y} dy \wedge dy + \frac{\partial A_k^2}{\partial z} dz \wedge dy \\
&+ \frac{\partial A_k^3}{\partial x} dx \wedge dz + \frac{\partial A_k^3}{\partial y} dy \wedge dz + \frac{\partial A_k^3}{\partial z} dz \wedge dz \\
&= \left(\frac{\partial A_k^3}{\partial y} - \frac{\partial A_k^2}{\partial z} \right) dy \wedge dz + \left(\frac{\partial A_k^1}{\partial z} - \frac{\partial A_k^3}{\partial x} \right) dz \wedge dx + \left(\frac{\partial A_k^2}{\partial x} - \frac{\partial A_k^1}{\partial y} \right) dx \wedge dy \\
&= (\nabla \times A_k)^\top \begin{bmatrix} dy \wedge dz \\ dz \wedge dx \\ dx \wedge dy \end{bmatrix}
\end{aligned}$$

Finally, we can put everything together:

$$
\begin{aligned}
\mathbf{0} &= \begin{bmatrix} (W_3 \times A^2 - W_2 \times A^3 + \nabla \times A^1)^\top \\ (W_1 \times A^3 - W_3 \times A^1 + \nabla \times A^2)^\top \\ (W_2 \times A^1 - W_1 \times A^2 + \nabla \times A^3)^\top \end{bmatrix} \begin{bmatrix} dy \wedge dz \\ dz \wedge dx \\ dx \wedge dy \end{bmatrix} \\
\iff & \begin{bmatrix} (W_2 \times A^3 - W_3 \times A^2)^\top \\ (W_3 \times A^1 - W_1 \times A^3)^\top \\ (W_1 \times A^2 - W_2 \times A^1)^\top \end{bmatrix} \begin{bmatrix} dy \wedge dz \\ dz \wedge dx \\ dx \wedge dy \end{bmatrix} = \begin{bmatrix} (\nabla \times A^1)^\top \\ (\nabla \times A^2)^\top \\ (\nabla \times A^3)^\top \end{bmatrix} \begin{bmatrix} dy \wedge dz \\ dz \wedge dx \\ dx \wedge dy \end{bmatrix}
\end{aligned}
$$

Take the curl to the other side and switch order on the left-hand side to cancel the -1.

Chapter 3

First main chapter

There are no fixed rules for the organization of the main chapters in a thesis. They should describe the project and its results according to the standard scientific approach in the field.

Software projects. For a software project, the report often has just three chapters:

1. Design;
2. Implementation;
3. Validation.

Theoretical projects. A report for a theoretical project should correspond to the organization of the material.

Chapter 4

Second main chapter

Here is an example for how to specify an algorithm in pseudo-code.

Algorithm 1 Byzantine Leader-Based Epoch-Change (process p_i).

State
    $\mathit{lastts} \leftarrow 0$: most recently started epoch
    $\mathit{nextts} \leftarrow 0$: timestamp of the next epoch
    $\mathit{newepoch} \leftarrow [\bot]^n$: list of NEWEPOCH messages

upon event $\mathit{complain}(p_\ell)$ **such that** $p_\ell = \mathit{leader}(\mathit{lastts})$ **do**
    if $\mathit{nextts} = \mathit{lastts}$ **then**
        $\mathit{nextts} \leftarrow \mathit{lastts} + 1$
        send message $[\text{NEWEPOCH}, \mathit{nextts}]$ to all $p_j \in \mathcal{P}$

upon receiving a message $[\text{NEWEPOCH}, \mathit{ts}]$ from p_j **such that** $\mathit{ts} = \mathit{lastts} + 1$ **do**
    $\mathit{newepoch}[j] \leftarrow \text{NEWEPOCH}$

upon exists ts **such that** $\{p_j \in \mathcal{P} | \ \mathit{newepoch}[j] = \mathit{ts}\} \in \mathcal{K}_i$ **and** $\mathit{nextts} = \mathit{lastts}$ **do**
    $\mathit{nextts} \leftarrow \mathit{lastts} + 1$
    send message $[\text{NEWEPOCH}, \mathit{nextts}]$ to all $p_j \in \mathcal{P}$

upon exists ts **such that** $\{p_j \in \mathcal{P} | \ \mathit{newepoch}[j] = \mathit{ts}\} \in \mathcal{Q}_i$ **and** $\mathit{nextts} > \mathit{lastts}$ **do**
    $\mathit{lastts} \leftarrow \mathit{nextts}$
    $\mathit{newepoch} \leftarrow [\bot]^n$
    output $\mathit{startepoch}(\mathit{lastts}, \mathit{leader}(\mathit{lastts}))$

Chapter 5

Conclusion

The conclusion looks back at the entire work, gives a critical look, summarizes, and discusses extensions and future work.

Appendix A

Extra material

Extra material may be placed in an appendix that appears after the conclusion.

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Erklärung

Erklärung gemäss Art. 30 RSL Phil.-nat. 18

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