Chapter 1

Mathematical Background

We will make heavy use of differential geometry in the following chapters. To get us all on the same page, I introduce the basic concepts what we will use, but I refrain from giving any proofs. I will give definitions only as far as we need it. These definitions will by no means be exhaustive. The following is an incomplete summary of what we need presented in *Introduction to smooth manifolds*[**?**].

Manifold A manifold $\mathcal{M}$ is a space that locally looks like Euclidean space. More exactly, a n-manifold is a topological space, where each point on the manifold has an open neighborhood that is locally homeomorphic to an open subset of Euclidean space $\mathbb{R}^n$. A manifold can be equipped with additional structure. For example, we can work on *smooth manifolds*. In simple terms, a manifold is *smooth* if it is similar enough to $\mathbb{R}^n$ that we can do Calculus like differentiation or integration on it. For this, each point on the manifold must be locally *diffeomorphic* to an open subset of $\mathbb{R}^n$ space.

Tangent space, Tangent bundle There are many equivalent definitions for the tangent space. One definition is for each point p in the manifold $\mathcal{M}$, the tangent space $T_p\mathcal{M}$ consists of $\gamma'(0)$ for all differentiable paths $\gamma : (-\varepsilon, \varepsilon) \to \mathcal{M}$ with $p = \gamma(0)$. The tangent space is a vector space which has the same dimension as its manifold, which is 3 in our case. These tangent spaces can be "glued" together to form the *tangent bundle* $T\mathcal{M} = \sqcup_{p\in\mathcal{M}} T_p\mathcal{M}$, which itself is a manifold of dimension $2n$. An element of $T\mathcal{M}$ can be written as (p, v) with $p \in \mathcal{M}$ and $v \in T_p\mathcal{M}$. This admits a natural projection $\pi : T\mathcal{M} \to \mathcal{M}$, which sends each vector $v \in T_p\mathcal{M}$ to the point p where it is tangent: $\pi(p, v) = p$. A *section* $\sigma : \mathcal{M} \to T\mathcal{M}$ is a continuos map, with $\pi \circ \sigma = \mathrm{Id}_{\mathcal{M}}$. Sections of $T\mathcal{M}$ are tangent vector fields on $\mathcal{M}$.

Cotangent space, Cotangent bundle The dual space V^* of a vector space V consists of all linear maps $\omega : V \to \mathbb{R}$. We call these functionals *covectors* on V. V^* is itself a vector space, with the same dimension as V and operations like addition and scalar multiplication can be performed on its elements. Any element in a vector space can be expressed as a finite linear combination of its basis. This basis is called the *dual basis*. Thus, we call the dual space of the vector space $T_p\mathcal{M}$ its *cotangent space*, denoted by $T_p^*\mathcal{M}$. As before, the disjoint union of $T_p^*\mathcal{M}$ forms the *cotangent bundle*: $T^*\mathcal{M} = \sqcup_{p\in\mathcal{M}} T_p^*\mathcal{M}$. Defined analogously from above, sections σ on $T^*\mathcal{M}$ define *covector fields* or *1-forms*.

Tensors Before we can introduce differential forms in the next paragraph, we need to go a little bit into *tensors*. In simple words, tensors are real-valued, multilinear functions. A map $F : V_1 \times \ldots V_k \to W$ is multilinear, if F is linear in each component. For example, the dot product in $\mathbb{R}^n$ is a tensor. It takes two vectors and is linear in each component - bilinear. Another example is the *Tensor Product of Covectors*: Let V be a vector space and take two covectors $\omega, \eta \in V^*$. Define the new function $\omega \otimes \eta : V \times V \to \mathbb{R}$ by $\omega \otimes \eta(v_1, v_2) = \omega(v_1)\eta(v_2)$. It is multilinear, because ω and η are linear. We look at a special class of tensors, the *alternating tensors*. A tensor is alternating, if it changes sign whenever two arguments are interchanged, i.e. $\omega(v_1, v_2) = -\omega(v_2, v_1)$. A covariant tensor field over a manifold defines a covariant tensor at each point on the manifold, covariant because the tensor is over the cotangent space $T_p^*\mathcal{M}$. An alternating tensor field is called a *differential form*.

Differential Forms, Exterior Derivative Recall that a section from $T^*\mathcal{M}$ is called a differential 1-form, or just 1-form. Define the *wedge product* (or *exterior product*) between two 1-forms:

$$(\omega \wedge \eta)_p = \omega_p \wedge \eta_p$$

Notice the similarity to the *Tensor Product of Covectors*: We get a new map, (a 2-form):

$$\omega \wedge \eta : T\mathcal{M} \times T\mathcal{M} \to \mathbb{R}$$

The wedge product is antisymmetric, therefore $\omega \wedge \eta = -\eta \wedge \omega$ for 1-forms ω and η. We denote by $\Omega^k(\mathcal{M})$ the space of differential k-forms on $\mathcal{M}$. There is a natural differential operator d on differential forms we call *exterior derivative*. It maps k-forms to $(k+1)$-forms, i.e. $d : \Omega^k(\mathcal{M}) \to \Omega^{k+1}(\mathcal{M})$ and has the the following properties:

- df is the ordinary differential of a smooth function f. Smooth functions are 0-forms.
- $d(d\alpha) = 0$
- $d(\alpha \wedge \beta) = (d\alpha \wedge \beta) + (-1)^k(\alpha \wedge d\beta)$ for a k-form α. (Leibnitz Rule)

In section **??** we will need what $d\omega$ is for some smooth 1-form ω, the calculation will be done there. For now, just note that any arbitrary smooth 1-form can be written as $\omega = Fdx + Gdy + Hdz$ for some appropriate smooth functions F, G, H.

Riemannian metric, g-orthonormality Inner products are examples of symmetric tensors. They allow us to define lengths and angles between vectors. We can apply this idea to manifolds. A Riemannian metric g is a symmetric positive-definite tensor field at each point. If $\mathcal{M}$ is a manifold, the pair $(\mathcal{M}, g)$ is called a *Riemannian manifold*. Let g be the Riemannian metric on $\mathcal{M}$ and $p \in \mathcal{M}$, then g_p is an inner product on $T_p\mathcal{M}$. We write $\langle \cdot, \cdot \rangle_g$ to denote this inner product. Any Riemannian metric can be written as positive-definite symmetric matrix, which allows for this simple form: $\langle v, w \rangle_g = v^\top g w$.

Such a new metric allows for the definition of *g-orthonormality*: A basis $[e_1, e_2, e_3]$ of $T_p\mathcal{M}$ is g-orthonormal if $\langle e_i, e_j \rangle_g = \delta_{ij}$.

Connection, Covariant derivative, Parallel transport A connection defines how two different tangent spaces are connected to each other, such that tangent vector fields can be differentiated. There is an infinite amount of connections on a manifold. An *affine connection* ∇ is a bilinear map that takes two tangent vector fields X, Y and maps it to a new tangent vector field $\nabla_X Y$ on $\mathcal{M}$ such that

- $\nabla_{fX} Y = f \nabla_X Y$, where $f \in C^\infty(\mathcal{M}, \mathbb{R})$
- $\nabla_X(fY) = f\nabla_X Y + (Xf)Y$ for $f \in C^\infty(\mathcal{M}, \mathbb{R})$, it satisfies the Leibnitz rule in the second variable

We call $\nabla_X Y$ the *covariant derivative of Y in the direction of X*. A connection ∇ defines the parallel transport of a vector along a curve. Given a curve $\gamma : [0, 1] \to \mathcal{M}$ and a vector $v_0 \in T_{\gamma(0)}\mathcal{M}$, there exists a unique parallel vector field $V : [0, 1] \to T\mathcal{M}$ along γ such that $V(0) = v_0$ [**?**]. Recall: a vector field V along γ means $\pi(V(t)) = \gamma(t)$. The uniqueness and existence is a consequence of the vector field V being the solution of $\nabla_{\dot{\gamma}(t)} V(t) = 0$ defining a linear ordinary differential equation with initial condition $V(0) = v_0$. This vector field $V(t)$ is called the *parallel transport* of v_0 along γ. It is "parallel" in the sense that the transported vector does not change within the tangent space. See figure 1.1

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Frame field, vector field, integrability

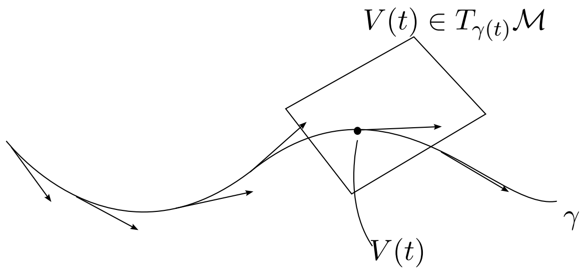

Figure 1.1. A parallel vector field $V(t)$ along a curve γ. Each $V(t) \in T_{\gamma(t)}\mathcal{M}$ and $\nabla_{\dot\gamma(t)}V(t) = 0$, so each vector that is drawn is parallel to each other.

A frame F is a set of 6 vectors $\{\pm F_0, \pm F_1, \pm F_2\}$. We can represent such a frame F as a 3×3 matrix F, where the ith-column is F_i. A frame field then maps to every point in 3D-space such a frame, i.e. $F : \mathbb{R}^3 \to \mathbb{R}^{3\times 3}$. Usually, we work on a 3-manifold $\mathcal{M}$ and a positively oriented frame field, i.e. $F|_{\mathcal{M}} : \mathcal{M} \to \mathbb{R}^{3\times 3}$, where $\det(F) > 0$. To allow for anisotropic, nonuniform meshes, we generalize orthonormality of frames to g-orthonormal frames. Orthonormality is measured in some metric g, and a frame F satifisfies the condition $\langle F_i, F_j\rangle_g = \delta_{ij}$. Any frame field with $\det(F) > 0$ naturally defines a metric $g = (FF^\top)^{-1}$, where F is g-orthonormal

$$F^\top g F = Id.$$

We can factor the frame field F into a symmetric part $g^{1/2}$ and a rotational part R

$$F = g^{-1/2}R$$

The symmetric part $g^{-1/2}$ keeps F g-orthonormal

$$\implies F^\top g F = (g^{-1/2}R)^\top g g^{-1/2}R) = R^\top g^{-1/2} g g^{-1/2} R = Id.$$

and R represents a rotational field $R : \mathcal{M} \to SO(3)$. The requirements for our frame field are:

- Smoothness
- Integrability
- Metric consistency: $g = (FF^\top)^{-1}$