Chapter 1

Mathematical Background

We will make heavy use of differential geometry in the following sections. To get us all on the same page, I introduce the basic concepts what we will use, but I refrain from giving any proofs. I will give definitions only as far as we need it. These definitions will by no means be exhaustive. The following is an incomplete summary of what we need presented in "Introduction to smooth manifolds" [**?**]

Manifold A manifold $\mathcal{M}$ is a space that locally looks like Euclidean space. More exactly, a n-manifold is a topological space, where each point on the manifold has an open neighborhood that is locally homeomorphic to an open subset of Euclidean space $\mathbb{R}^n$. A manifold can be equipped with additional structure. For example, we can work on *smooth manifolds*. In simple terms, a manifold is *smooth* if it is similar enough to $\mathbb{R}^n$ that we can do Calculus like differentiation or integration on it. For this, each point on the manifold must be locally *diffeomorphic* to an open subset of $\mathbb{R}^n$ space.

Tangent space, Tangent bundle There are many equivalent definitions for the tangent space. One definition is for each point p in the manifold $\mathcal{M}$, the tangent space $T_p\mathcal{M}$ consists of $\gamma'(0)$ for all differentiable paths $\gamma : (-\varepsilon, \varepsilon) \to \mathcal{M}$ with $p = \gamma(0)$. The tangent space is a vector space which has the same dimension as its manifold, which is 3 in our case. These tangent spaces can be "glued" together to form the *tangent bundle* $T\mathcal{M} = \sqcup_{p \in \mathcal{M}} T_p\mathcal{M}$, which itself is a manifold of dimension $2n$. An element of $T\mathcal{M}$ can be written as (p, v) with $p \in \mathcal{M}$ and $v \in T_p\mathcal{M}$. This admits a natural projection $\pi : T\mathcal{M} \to \mathcal{M}$, which sends each vector $v \in T_p\mathcal{M}$ to the point p where it is tangent: $\pi(p, v) = p$. A *section* $\sigma : \mathcal{M} \to T\mathcal{M}$ is a continuos map, with $\pi \circ \sigma = \mathrm{Id}_{\mathcal{M}}$. Sections of $T\mathcal{M}$ are vector fields on $\mathcal{M}$.

Cotangent space, Cotangent bundle The dual space V^* of a vector space V consists of all linear maps $\omega : V \to \mathbb{R}$. We call these functionals *covectors* on V. V^* is itself a vector space, with the same dimension as V and operations like addition and scalar multiplication can be performed on its elements. Any element in a vector space can be expressed as a finite linear combination of its basis. This basis is called the *dual basis*. Thus, we call the dual space of the vector space $T_p\mathcal{M}$ its *cotangent space*, denoted by $T_p^*\mathcal{M}$. As before, the disjoint union of $T_p^*\mathcal{M}$ forms the *cotangent bundle*: $T^*\mathcal{M} = \sqcup_{p \in \mathcal{M}} T_p^*\mathcal{M}$. Defined analogously from above, sections σ on $T^*\mathcal{M}$ define *covector fields* or *1-forms*.

Tensors Before we can introduce differential forms in the next paragraph, we need to go a little bit into *tensors*. In simple words, tensors are real-valued, multilinear functions. A map $F : V_1 \times \ldots V_k \to W$ is multilinear, if F is linear in each component. For example, the dot product in $\mathbb{R}^n$ is a tensor. It takes two vectors and is linear in each component - bilinear. Another example is the *Tensor Product of Covectors*: Let V be a vector space and take two covectors $\omega, \eta \in V^*$. Define the new function $\omega \otimes \eta : V \times V \to \mathbb{R}$ by $\omega \otimes \eta(v_1, v_2) = \omega(v_1)\eta(v_2)$. It is multilinear, because ω and η are linear. We look at a special class of tensors, the *alternating tensors*. A tensor is alternating, if it changes sign whenever two arguments are interchanged, i.e. $\omega(v_1, v_2) = -\omega(v_2, v_1)$. A covariant tensor field over a manifold defines a covariant tensor at each point on the manifold, covariant because the tensor is over the cotangent space $T_p^*\mathcal{M}$. An alternating tensor field is called a *differential form*.

Differential Forms, Exterior Derivative Recall that a section from $T^*\mathcal{M}$ is called a differential 1-form, or just 1-form. Define the *wedge product* (or *exterior product*) between two 1-forms:

$$(\omega \wedge \eta)_p = \omega_p \wedge \eta_p$$

Notice the similarity to the *Tensor Product of Covectors*: We get a new map, (a 2-form):

$$\omega \wedge \eta : T\mathcal{M} \times T\mathcal{M} \to \mathbb{R}$$

The wedge product is antisymmetric, therefore $\omega \wedge \eta = -\eta \wedge \omega$ for 1-forms ω and η. There is a natural differential operator d on differential forms we call *exterior derivative*. The exterior derivative is a generalization of the differential of a function. In particular, a smooth function f (a 0-form) has the derivative df which is a 1-form. The exact definition is not important for us, so let us just look at some properties so we can work with it. If ω is a k-form, $d\omega$ is a (k+1)-form. TODO

Riemannian metric, g-orthonormality Inner products are examples of symmetric tensors. They allow us to define lengths and angles between vectors. We can apply this idea to manifolds. A Riemannian metric g is a symmetric positive-definite tensor field at each point. If $\mathcal{M}$ is a manifold, the pair $(\mathcal{M}, g)$ is called a *Riemannian manifold*. Let g be the Riemannian metric on $\mathcal{M}$ and $p \in \mathcal{M}$, then g_p is an inner product on $T_p\mathcal{M}$. We write $\langle \cdot, \cdot \rangle_g$ to denote this inner product. Any Riemannian metric can be written as positive-definite symmetric matrix, which allows for this simple form: $\langle v, w \rangle_g = v^\top g w$.

Such a new metric allows for the definition of *g-orthonormality*: A basis $[e_1, e_2, e_3]$ of $T_p\mathcal{M}$ is g-orthonormal if $\langle e_i, e_j \rangle_g = \delta_{ij}$.

connection, covariant derivative

lie algebra so(3)

Frame field, vector field, integrability

A frame F is a set of 6 vectors $\{\pm F_0, \pm F_1, \pm F_2\}$. We can represent such a frame F as a 3×3 matrix F, where the ith-column is F_i. A frame field then maps to every point in 3D-space such a frame, i.e. $F : \mathbb{R}^3 \to \mathbb{R}^{3\times 3}$. Usually, we work on a 3-manifold $\mathcal{M}$ and a positively oriented frame field, i.e. $F|_{\mathcal{M}} : \mathcal{M} \to \mathbb{R}^{3\times 3}$, where $\det(F) > 0$. To allow for anisotropic, nonuniform meshes, we generalize orthonormality of frames to g-orthonormal frames. Orthonormality is measured in some metric g, and a frame F satifisfies the condition $\langle F_i, F_j\rangle_g = \delta_{ij}$. Any frame field with $\det(F) > 0$ naturally defines a metric $g = (FF^\top)^{-1}$, where F is g-orthonormal

$$F^\top g F = Id.$$

We can factor the frame field F into a symmetric part $g^{1/2}$ and a rotational part R

$$F = g^{-1/2} R$$

The symmetric part $g^{-1/2}$ keeps F g-orthonormal

$$\implies F^\top g F = (g^{-1/2}R)^\top g g^{-1/2}R) = R^\top g^{-1/2} g g^{-1/2} R = Id.$$

and R represents a rotational field $R : \mathcal{M} \to SO(3)$. The requirements for our frame field are:

- Smoothness
- Integrability
- Metric consistency: $g = (FF^\top)^{-1}$