

# Chapter 1

## Calculation of rotation coefficient

Recall that we want to measure the smoothness of the frame field by the Dirichlet energy.

We are interested in the rotation. Let  $p, q$  be points on the manifold and  $\ell$  a path connecting them. Let  $R : \mathcal{M} \rightarrow SO(3)$  be a rotation field, i.e.  $R(p), R(q)$  are orthogonal frames. Under the initial condition that  $R(p) = \text{Id}$ , the equation

$$R(q) = \exp(R_{pq}(\omega))R(p)$$

is a differential equation that corresponds to the parallel transport under the connection  $\omega$  of the frame along  $\ell$ . To recover this rotation  $R_{pq}$ , we integrate  $\omega$  along  $\ell$ .

**Setup** We parametrize the path by  $\ell(0) = a, \ell(1) = b$ . We resort to numerical integration for  $R_{ab}$  and cut the path into  $n$  small segments, i.e.

$$R_{ab} = R_n R_{n-1} \cdots R_1$$

where  $R_i = \exp(-\omega(\dot{\ell}(i\gamma))\gamma) = \exp((\int W^\top dp)_\times)$ , and  $\gamma$  is the length of a segment and  $\dot{\ell}(s) = \frac{\partial \ell}{\partial s}(s)$ . Calculating the exponential map of an antisymmetric matrix (which  $\omega \in \mathfrak{so}(3)$  is) can be done with Rodrigues' formula:

$$\exp(u_\times) = \text{Id} + \sin(\theta)\hat{u}_\times + (1 - \cos(\theta))\hat{u}_\times^2$$

where  $\theta = \|u\|_2$  is the rotation angle and  $\hat{u} = u/\theta$  is the rotation axis. We use the trapezoidal rule to evaluate the a short interval of the integral of  $W$ , which is given by

$$R_{ab} = \exp\left(\left(\frac{1}{2}(W_a + W_b)^\top(b - a)\right)_\times\right)$$

where  $W_a, W_b$  is solved for by the 9x9 linear system given by  $A_\times$  and  $\nabla \times A$ .

### 1.1 Piecewise linear discretization

We discretize our metric field with a tetrahedral mesh  $\mathcal{T}$ . At each vertex, we attach a metric and linearly interpolate with barycentric coordinates within a tet.

Let  $A_i \in \mathbb{R}^{3 \times 3}, i \in \{1, 2, 3, 4\}$  be the square root metrics at the vertices  $v_i \in \mathbb{R}^3$  of a tet, such that  $A_i^2 = g(v_i)$ . We represent a tet given by its four vertices by a 3x4 matrix, i.e.

$$\begin{pmatrix} | & | & | & | \\ v_1 & v_2 & v_3 & v_4 \\ | & | & | & | \end{pmatrix} \in \mathbb{R}^{3 \times 4}.$$

Any point  $p$  within the tet can then be represented as

$$p = \alpha v_1 + \beta v_2 + \gamma v_3 + \delta v_4$$

with  $\alpha, \beta, \gamma, \delta \geq 0$  and  $\alpha + \beta + \gamma + \delta = 1$ . This is a linear transformation between two coordinate systems, which we can write in matrix form as

$$\underbrace{\begin{pmatrix} | & | & | & | \\ v_1 & v_2 & v_3 & v_4 \\ | & | & | & | \\ 1 & 1 & 1 & 1 \end{pmatrix}}_{T^{-1}} \underbrace{\begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix}}_{\lambda} = \underbrace{\begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}}_p \iff T^{-1}\lambda = p \iff \lambda = Tp$$

$T$  always exists because  $v_1, \dots, v_4$  are linearly independent, else it would not be a tetrahedron. By denoting  $T = \{t_{ij}\}_{i,j \in \{1, \dots, 4\}}$ , we can write our barycentric functions as

$$\begin{aligned} \alpha(x, y, z) &= t_{11}x + t_{12}y + t_{13}z + t_{14} \\ \beta(x, y, z) &= t_{21}x + t_{22}y + t_{23}z + t_{24} \\ \gamma(x, y, z) &= t_{31}x + t_{32}y + t_{33}z + t_{34} \\ \delta(x, y, z) &= t_{41}x + t_{42}y + t_{43}z + t_{44} \end{aligned}$$

The convex combination

$$A(x, y, z) = \alpha A_1 + \beta A_2 + \gamma A_3 + \delta A_4$$

is the metric prescribed in the tet. To find  $\nabla \times A$ , let  $A = (A^1, A^2, A^3)$ . We will need the derivatives for the curl, so let

$$(A_j^i)_x \triangleq \frac{\partial A_j^i}{\partial x}$$

be the derivative with respect to  $x$  of entry  $i, j$ . E.g.  $(A_j^i)_x$  is given by

$$(A_j^i)_x = \alpha_x (A_1)_j^i + \beta_x (A_2)_j^i + \gamma_x (A_3)_j^i + \delta_x (A_4)_j^i = t_{11} (A_1)_j^i + t_{21} (A_2)_j^i + t_{31} (A_3)_j^i + t_{41} (A_4)_j^i$$

If we write  $T = (T^1, T^2, T^3, T^4)$  and collect  $(A_k)_j^i$  into a vector

$$\bar{A}_j^i = \begin{pmatrix} (A_1)_j^i \\ (A_2)_j^i \\ (A_3)_j^i \\ (A_4)_j^i \end{pmatrix}$$

this can be shortened to  $\bar{A}_j^{i\top} T^1 = (A_j^i)_x$ . Analogously, we get

$$(A_j^i)_y = \bar{A}_j^{i\top} T^2 \text{ and } (A_j^i)_z = \bar{A}_j^{i\top} T^3$$

The curl is then given by

$$\nabla \times A^i = \begin{pmatrix} (A_3^i)_y - (A_2^i)_z \\ (A_1^i)_z - (A_3^i)_x \\ (A_2^i)_x - (A_1^i)_y \end{pmatrix} = \begin{pmatrix} \bar{A}_3^{i\top} T^2 - \bar{A}_2^{i\top} T^3 \\ \bar{A}_1^{i\top} T^3 - \bar{A}_3^{i\top} T^1 \\ \bar{A}_2^{i\top} T^1 - \bar{A}_1^{i\top} T^2 \end{pmatrix}$$

and  $\nabla \times A = \nabla \times (A^1, A^2, A^3)$ . Notice that the curl is constant within a tetrahedron.

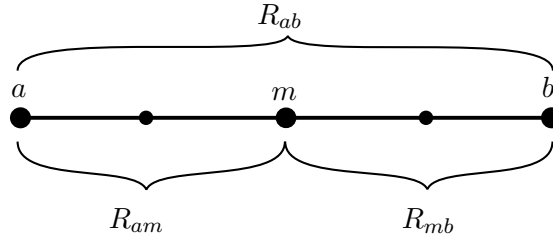
## 1.2 Recursive subdivision

Whenever the rotation coefficient  $R_{pq}$  is needed between some points  $p$  and  $q$ , it is unclear ahead of time how many sampling points on the line  $\ell$  are needed such that  $R_{pq}$  accurately describes how the frame rotates along  $\ell$ . We apply a recursive subdivision scheme to recursively sample more points on  $\ell$  only

where is needed until sampling more points leads to no noticeable improvement anymore (see figure 1.1). We begin by calculating  $R_{ab}$  with just the endpoints. This coefficient is then compared to the result if the midpoint was sampled as well, so i.e. if

$$\frac{||R_{ab} - R_{am} \cdot R_{mb}||_2^2}{\ell^2} < \varepsilon$$

then no measurable improvement happened. We divide by the length of the segment  $\vec{ab}$ , because the segments get smaller and we want the tolerance  $\varepsilon$  to remain the same. This approach has the advantage of only sampling points where there is improvement. See section ?? for results.



**Figure 1.1.** Points are recursively sampled as midpoints between the line  $\vec{ab}$ . If sampling more points within some line segment leads to noticeable improvement, more points are sampled.

The following code does what is described above.

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**Algorithm 1** Recursive Subdivision

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1: recursiveDivide ( $a, b$ )
2:    $R_1 = \text{Rotation}(a, b)$ 
3:    $\ell = \text{length}(a - b)$ 
4:    $\text{midpoint} = \frac{a+b}{2}$ 
5:   if  $\frac{||R_1 - \text{Rotation}(a, \text{midpoint}) \cdot \text{Rotation}(\text{midpoint}, b)||_2^2}{\ell^2} < \varepsilon$ 
6:     return  $R_1$ 
7:   else
8:     return  $\text{recursiveDivide}(a, \text{midpoint}) \cdot \text{recursiveDivide}(\text{midpoint}, b)$ 

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