

# Chapter 1

## Connection One-Form $\omega$

Recall that our goal is to find a frame field  $F : \mathcal{M} \rightarrow \mathbb{R}^{3 \times 3}$ , such that the target parametrization of  $\phi$  can sufficiently reduce the difference between  $\nabla\phi$  and  $F^{-1}$ . For this,  $F^{-1}$  needs to stay as close to being locally integrable as possible. In this chapter, by starting from the local integrability condition, we formulate a connection 1-form  $\omega$  which is used to measure the Dirichlet energy in the new metric  $g$ , such that the frames stay close to being  $g$ -orthonormal ( $F^{-1}gF = \text{Id}$ ). We follow the steps in *Metric-Driven 3D Frame Field Generation* [?].

### 1.1 Local integrability

A vector field  $U$  is integrable if and only if  $\nabla \times U = 0$ , which means the vector field has vanishing curl [?]. Although in general it is more complicated [?], we can think of a frame field  $F$  as the composition of 3 vector fields

$$F = \begin{bmatrix} | & | & | \\ F_1 & F_2 & F_3 \\ | & | & | \end{bmatrix}$$

where  $F_i : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  are vector fields. To achieve local integrability for  $F^{-1}$ , we therefore want

$$\nabla \times F^{-1} \stackrel{!}{=} 0$$

where the curl is applied to each column. We can express this more naturally with the language of differential forms by writing the curl as the exterior derivative  $d$  of a 1-form  $\alpha$ . A 1-form (more generally, a differential form) is closed, if  $d\alpha = 0$ . We construct a vector-valued 1-form out of our frame field, given  $\mathbf{p} = (x, y, z)^\top$  in Euclidean coordinates

$$\alpha \triangleq F^{-1}d\mathbf{p} = R^\top g^{1/2}d\mathbf{p}$$

where  $d\mathbf{p} = (dx, dy, dz)^\top$  is the common orthonormal 1-form basis  $\mathbb{E}^3$ . Local integrability of  $F^{-1}$  is then formulated as the closedness of  $\alpha$ , i.e.

$$F^{-1} \text{ locally integrable} \iff \mathbf{0} = d\alpha$$

Some reformulations yield:

$$\begin{aligned} \mathbf{0} &= d\alpha = d(F^{-1}d\mathbf{p}) = d(R^\top g^{1/2}d\mathbf{p}) \\ &= dR^\top \wedge (g^{1/2}d\mathbf{p}) + R^\top d(g^{1/2}d\mathbf{p}) \\ &= R^\top (\omega \wedge (g^{1/2}d\mathbf{p}) + d(g^{1/2}d\mathbf{p})) \end{aligned}$$

which we can simplify to

$$\mathbf{0} = \omega \wedge (g^{1/2}d\mathbf{p}) + d(g^{1/2}d\mathbf{p}) \tag{1.1}$$

where the Leibnitz Rule for the exterior derivative is applied and  $\wedge$  is the exterior product between a matrix-valued 1-form and a vector-valued 1-form, i.e. 1-forms in each component and the matrix vector product uses  $\wedge$  as the multiplication, see section 1.3 for the evaluation. We further define

$$\omega = RdR^\top \in \mathfrak{so}(3)$$

which is an antisymmetric matrix-valued 1-form. To see this, we differentiate the orthogonality condition of the rotation matrix  $R$  (we assume  $R$  rotates about the axis  $a$  with angle  $\theta$ ):

$$\begin{aligned} \text{Id} &= RR^\top \\ d(\text{Id}) &= d(RR^\top) \\ \mathbf{0} &= dRR^\top + RdR^\top \\ \mathbf{0} &= (RdR^\top)^\top + RdR^\top \\ -(RdR^\top)^\top &= RdR^\top \end{aligned}$$

The Lie algebra  $\mathfrak{so}(3)$  consists of all antisymmetric  $3 \times 3$  matrices. Elements of  $\mathfrak{so}(3)$  are infinitesimal rotations, that is, they are tangent to the manifold  $\text{SO}(3)$  at the element  $\text{Id}$ . Indeed, we characterised the elements of  $\mathfrak{so}(3)$  by taking the derivative at  $\text{Id}$ , which is one definition to get to the tangent space. Thus, we can use  $\omega$  as a connection 1-form to do the alignment of frames in the new metric  $g$ , i.e. the parallel transport is done with  $\omega$  and then compared.

To solve for local integrability, we find  $\omega$  such that the 1-form  $\alpha$  is closed, then try to match the  $\omega$  with  $R$ . This can be expressed as

$$\min_{R \in \text{SO}(3)} \|RdR^\top - \omega\|^2, \quad (1.2)$$

where  $\omega$  is determined by  $g$  through a system of linear equations (see 1.3). In general, equation 1.2 cannot be minimised to zero, therefore we solve for the nearly integrable 3D rotation field  $R$ .

## 1.2 Smoothness measure

Many frame field generation methods rely on maximising smoothness through the minimization of the Dirichlet energy. Here, we show that integrability through  $\omega$  is closely related to the usual Dirichlet energy. We can take equation 1.2 as a smoothness measure and reformulate, i.e.

$$\|RdR^\top - \omega\|^2 = \|-dRR^\top - \omega\|^2 = \|-dRR^\top R - \omega R\|^2 = \|dR + \omega R\|^2. \quad (1.3)$$

By defining

$$\mathcal{D}R \triangleq dR + \omega R, \quad (1.4)$$

we can measure the smoothness and the integrability with a single energy  $\int_{\mathcal{R}} \|\mathcal{D}R\|^2$ . Now, when  $g$  is constant in euclidean coordinates  $(x, y, z)$ , equation 1.1 tells us that  $\omega = 0$ , which reduces the smoothness measure  $\|\mathcal{D}R\|$  to  $\|dR\|$ , which corresponds to the usual Dirichlet energy. In fact,  $\mathcal{D}R$  corresponds to the covariant derivative of  $R$  under the connection  $\omega$ , which shows that local integrability is related to the covariant-based Dirichlet energy. Special care must be taken as we abuse the notation of the covariant derivative. The covariant derivative  $\mathcal{D}$  acts on each column of  $R$  separately. We define

$$\mathcal{D}R_i \triangleq \nabla_{\dot{\gamma}(t)} R_i \triangleq dR_i + \omega R_i$$

where  $dR_i$  is the derivative of each entry with respect to the angle  $\theta$  and  $\gamma$  is some curve on the manifold.

### 1.3 Connection evaluation

To find  $\omega$ , we use equation 1.1

$$0 = \omega \wedge (g^{1/2} d\mathbf{p}) + d(g^{1/2} d\mathbf{p})$$

and reformulate into a linear system. We represent the antisymmetric matrix-valued 1-form  $\omega$

$$\omega = \begin{bmatrix} 0 & \omega_{12} & -\omega_{31} \\ -\omega_{12} & 0 & \omega_{23} \\ \omega_{31} & -\omega_{23} & 0 \end{bmatrix}$$

by  $\begin{bmatrix} \omega_{23} & \omega_{31} & \omega_{12} \end{bmatrix} = \begin{bmatrix} dx & dy & dz \end{bmatrix} W$ . We write  $W = [W_1, W_2, W_3]$ ,  $W_i \in \mathbb{R}^3$ . That is,  $W$  is the matrix with the coefficients for the 1-forms, e.g.  $W_1 = [(\omega^{23})_1, (\omega^{23})_2, (\omega^{23})_3]^\top$ . Recall, a 1-form can be expressed as

$$\omega_{ij} = (\omega^{ij})_1 dx + (\omega^{ij})_2 dy + (\omega^{ij})_3 dz$$

So e.g. for  $\omega_{23}$  we get

$$\omega_{23} = \begin{bmatrix} dx & dy & dz \end{bmatrix} W_1 = (\omega^{23})_1 dx + (\omega^{23})_2 dy + (\omega^{23})_3 dz.$$

We also write  $A = g^{1/2} = [A^1, A^2, A^3]$ . Starting with the first part of equation 1.1, we get

$$\begin{aligned} \omega \wedge g^{1/2} d\mathbf{p} &= \begin{bmatrix} 0 & \omega_{12} & -\omega_{31} \\ -\omega_{12} & 0 & \omega_{23} \\ \omega_{31} & -\omega_{23} & 0 \end{bmatrix} \wedge \begin{bmatrix} A_1^1 & A_1^2 & A_1^3 \\ A_2^1 & A_2^2 & A_2^3 \\ A_3^1 & A_3^2 & A_3^3 \end{bmatrix} \begin{bmatrix} dx \\ dy \\ dz \end{bmatrix} \\ &= \begin{bmatrix} +\omega_{12} \wedge (A_2^1 dx + A_2^2 dy + A_2^3 dz) - \omega_{31} \wedge (A_3^1 dx + A_3^2 dy + A_3^3 dz) \\ -\omega_{12} \wedge (A_1^1 dx + A_1^2 dy + A_1^3 dz) + \omega_{23} \wedge (A_3^1 dx + A_3^2 dy + A_3^3 dz) \\ +\omega_{31} \wedge (A_1^1 dx + A_1^2 dy + A_1^3 dz) - \omega_{23} \wedge (A_2^1 dx + A_2^2 dy + A_2^3 dz) \end{bmatrix} \end{aligned}$$

It will get really messy if we calculate each component here, so let us calculate one component separately first:

$$\begin{aligned} \omega_{ij} \wedge (A_k^1 dx + A_k^2 dy + A_k^3 dz) &= ((\omega^{ij})_1 dx + (\omega^{ij})_2 dy + (\omega^{ij})_3 dz) \wedge (A_k^1 dx + A_k^2 dy + A_k^3 dz) \\ &= (\omega^{ij})_1 A_k^2 dx \wedge dy + (\omega^{ij})_1 A_k^3 dx \wedge dz \\ &\quad + (\omega^{ij})_2 A_k^1 dy \wedge dx + (\omega^{ij})_2 A_k^3 dy \wedge dz \\ &\quad + (\omega^{ij})_3 A_k^1 dz \wedge dx + (\omega^{ij})_3 A_k^2 dz \wedge dy \\ &= ((\omega^{ij})_1 A_k^2 - (\omega^{ij})_2 A_k^1) dx \wedge dy \\ &\quad + ((\omega^{ij})_2 A_k^3 - (\omega^{ij})_3 A_k^2) dy \wedge dz \\ &\quad + ((\omega^{ij})_3 A_k^1 - (\omega^{ij})_1 A_k^3) dz \wedge dx \end{aligned}$$

where we use the fact that  $dx \wedge dx = 0$  and  $dx \wedge dy = -dy \wedge dx$ . We can clean up the above expression using the cross product:

$$\begin{bmatrix} ((\omega^{ij})_2 A_k^3 - (\omega^{ij})_3 A_k^2) \\ ((\omega^{ij})_3 A_k^1 - (\omega^{ij})_1 A_k^3) \\ ((\omega^{ij})_1 A_k^2 - (\omega^{ij})_2 A_k^1) \end{bmatrix}^\top \begin{bmatrix} dy \wedge dz \\ dz \wedge dx \\ dx \wedge dy \end{bmatrix} = [W_i \times A^k]^\top \begin{bmatrix} dy \wedge dz \\ dz \wedge dx \\ dx \wedge dy \end{bmatrix}$$

We use the fact that  $[A_1, A_2, A_3] = [A^1, A^2, A^3]$  because  $A$  is symmetric, and  $W_1$  corresponds to  $\omega_{23}$ ,  $W_2$  to  $\omega_{31}$  and  $W_3$  to  $\omega_{12}$ . We continue with the second part of equation 1.1:

$$d(g^{1/2} d\mathbf{p}) = d(Ad\mathbf{p}) = d \left( \begin{bmatrix} A_1^1 & A_1^2 & A_1^3 \\ A_2^1 & A_2^2 & A_2^3 \\ A_3^1 & A_3^2 & A_3^3 \end{bmatrix} \begin{bmatrix} dx \\ dy \\ dz \end{bmatrix} \right)$$

Again, we can do this separately for each row (we use the fact that the exterior derivative  $d$  is the ordinary differential for a smooth function):

$$\begin{aligned}
& d(A_k^1 dx + A_k^2 dy + A_k^3 dz) \\
&= dA_k^1 \wedge dx + dA_k^2 \wedge dy + dA_k^3 \wedge dz \\
&= \frac{\partial A_k^1}{\partial x} dx \wedge dx + \frac{\partial A_k^1}{\partial y} dy \wedge dx + \frac{\partial A_k^1}{\partial z} dz \wedge dx \\
&+ \frac{\partial A_k^2}{\partial x} dx \wedge dy + \frac{\partial A_k^2}{\partial y} dy \wedge dy + \frac{\partial A_k^2}{\partial z} dz \wedge dy \\
&+ \frac{\partial A_k^3}{\partial x} dx \wedge dz + \frac{\partial A_k^3}{\partial y} dy \wedge dz + \frac{\partial A_k^3}{\partial z} dz \wedge dz \\
&= \left( \frac{\partial A_k^3}{\partial y} - \frac{\partial A_k^2}{\partial z} \right) dy \wedge dz + \left( \frac{\partial A_k^1}{\partial z} - \frac{\partial A_k^3}{\partial x} \right) dz \wedge dx + \left( \frac{\partial A_k^2}{\partial x} - \frac{\partial A_k^1}{\partial y} \right) dx \wedge dy \\
&= (\nabla \times A_k)^\top \begin{bmatrix} dy \wedge dz \\ dz \wedge dx \\ dx \wedge dy \end{bmatrix}
\end{aligned}$$

Finally, we can put everything together:

$$\begin{aligned}
\mathbf{0} &= \begin{bmatrix} (W_3 \times A^2 - W_2 \times A^3 + \nabla \times A^1)^\top \\ (W_1 \times A^3 - W_3 \times A^1 + \nabla \times A^2)^\top \\ (W_2 \times A^1 - W_1 \times A^2 + \nabla \times A^3)^\top \end{bmatrix} \begin{bmatrix} dy \wedge dz \\ dz \wedge dx \\ dx \wedge dy \end{bmatrix} \\
&\iff \begin{bmatrix} (W_2 \times A^3 - W_3 \times A^2)^\top \\ (W_3 \times A^1 - W_1 \times A^3)^\top \\ (W_1 \times A^2 - W_2 \times A^1)^\top \end{bmatrix} \begin{bmatrix} dy \wedge dz \\ dz \wedge dx \\ dx \wedge dy \end{bmatrix} = \begin{bmatrix} (\nabla \times A^1)^\top \\ (\nabla \times A^2)^\top \\ (\nabla \times A^3)^\top \end{bmatrix} \begin{bmatrix} dy \wedge dz \\ dz \wedge dx \\ dx \wedge dy \end{bmatrix}
\end{aligned}$$

We take the curl to the other side and switch order on the left-hand side to cancel the  $-1$ . As we are only interested in the 9 components of  $W$ , we omit the two-form basis and transform into a  $9 \times 9$  linear system for  $W$ . We define  $A_\times$  and  $\text{vec}(\cdot)$  as

$$A_\times = \begin{bmatrix} 0 & -A_\times^3 & A_\times^2 \\ A_\times^3 & 0 & -A_\times^1 \\ -A_\times^2 & A_\times^1 & 0 \end{bmatrix}, \text{vec}(W) = \begin{bmatrix} W_1 \\ W_2 \\ W_3 \end{bmatrix}$$

with  $A_\times^i$  defined as

$$A_\times^i = [A_1^i \ A_2^i \ A_3^i]_\times = \begin{bmatrix} 0 & -A_3^i & A_2^i \\ A_3^i & 0 & -A_1^i \\ -A_2^i & A_1^i & 0 \end{bmatrix}$$

and  $\text{vec}(\cdot)$  turns a  $3 \times 3$ -matrix into a  $9 \times 1$ -vector by stacking the columns. With these two definitions, we can transform the above equality into a linear system

$$A_\times \text{vec}(W) = \text{vec}(\nabla \times A) \tag{1.5}$$

where  $\nabla \times A$  is just the curl applied to each column. This transformation can be checked by laboriously plugging in the definitions and comparing the coefficients. With tedious brute-force calculations, one can show that  $\det(A_\times) = -2 \det(A)^3 = -2 \det(g)^{3/2} < 0$ , which means this is a linear system that is solvable and can be used to calculate  $W$  at a point.