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3D Metric Fields
A Novel Approach to a New Idea

Bachelor Thesis

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Abstract

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Contents

Chapter 1

Introduction

Every thesis should start with an introduction. This thesis is written in LaTeX [3].

Chapter 2

Background

This chapter sets the stage and introduces already existing material.

A frame F is a set of 6 vectors $\{\pm F_0, \pm F_1, \pm F_2\}$. We can represent such a frame F as a 3×3 matrix F, where the ith-column is F_i. A frame field then maps to every point in 3D-space such a frame, i.e. $F : \mathbb{R}^3 \to \mathbb{R}^{3\times3}$. Usually, we work on a 3-manifold $\mathcal{M}$ and a positively oriented frame field, i.e. $F|_{\mathcal{M}} : \mathcal{M} \to \mathbb{R}^{3\times3}$, where $\det(F) > 0$. To allow for anisotropic, nonuniform meshes, we generalize orthonormality of frames to g-orthonormal frames. Orthonormality is measured in some metric g, and a frame F satifisfies the condition $\langle F_i, F_j \rangle_g = \delta_{ij}$. Any frame field with $\det(F) > 0$ naturally defines a metric $g = (FF^T)^{-1}$, where F is g-orthonormal

$$F^T g F = Id.$$

We can factor the frame field F into a symmetric part $g^{1/2}$ and a rotational part R

$$F = g^{-1/2} R$$

The symmetric part $g^{-1/2}$ keeps F g-orthonormal

$$\implies F^T g F = (g^{-1/2}R)^T g g^{-1/2} R) = R^T g^{-1/2} g g^{-1/2} R = Id.$$

and R represents a rotational field $R : \mathcal{M} \to SO(3)$. The requirements for our frame field are:

- Smoothness
- Integrability
- Metric consistency: $g = (FF^T)^{-1}$

A vector field U is integrable, if and only if $\nabla \times U = 0$, which means the vector field has vanishing curl everywhere. We can express this more naturally with the language of differential forms: The curl can be written as the exterior derivative d of a one-form α. A one-form (more generally, a differential form) is closed, if $d\alpha = 0$. Therefore, the local integrability can be expressed as the closedness of a one-form. We want F^{-1} (TODO: why F^{-1}) to be integrable. To achieve local integrability for, it suffices to make R locally integrable. We can think of a rotation field R as the composition of 3 vector fields

$$R = \begin{bmatrix} | & | & | \\ R_1 & R_2 & R_3 \\ | & | & | \end{bmatrix}$$

where $R_i : \mathbb{R}^3 \to \mathbb{R}^3$ is a vector field. We can therefore construct a vector-valued one-form, given $p = (x, y, z)^T$ in Euclidean coordinates

$$\alpha \triangleq F^{-1} dp = R^T g^{1/2} dp$$

where $dp = (dx, dy, dz)^T$ is the common orthonormal one-form basis of $\Omega^1(\mathcal{M})$.

$$R \text{ locally integrable} \iff 0 = d\alpha$$

Chapter 3

First main chapter

There are no fixed rules for the organization of the main chapters in a thesis. They should describe the project and its results according to the standard scientific approach in the field.

Software projects. For a software project, the report often has just three chapters:

1. Design;
2. Implementation;
3. Validation.

Theoretical projects. A report for a theoretical project should correspond to the organization of the material.

Chapter 4

Second main chapter

Here is an example for how to specify an algorithm in pseudo-code.

Algorithm 1 Byzantine Leader-Based Epoch-Change (process p_i).

State
    $\textit{laststs} \leftarrow 0$: most recently started epoch
    $\textit{nextts} \leftarrow 0$: timestamp of the next epoch
    $\textit{newepoch} \leftarrow [\perp]^n$: list of NEWEPOCH messages

upon event $\textit{complain}(p_\ell)$ **such that** $p_\ell = \textit{leader}(\textit{lastts})$ **do**
    if $\textit{nextts} = \textit{lastts}$ **then**
        $\textit{nextts} \leftarrow \textit{lastts} + 1$
        send message $[\text{NEWEPOCH}, \textit{nextts}]$ to all $p_j \in \mathcal{P}$

upon receiving a message $[\text{NEWEPOCH}, \textit{ts}]$ from p_j **such that** $\textit{ts} = \textit{lastts} + 1$ **do**
    $\textit{newepoch}[j] \leftarrow \text{NEWEPOCH}$

upon exists $\textit{ts}$ **such that** $\{p_j \in \mathcal{P} | \textit{ newepoch}[j] = \textit{ts}\} \in \mathcal{K}_i$ **and** $\textit{nextts} = \textit{lastts}$ **do**
    $\textit{nextts} \leftarrow \textit{lastts} + 1$
    send message $[\text{NEWEPOCH}, \textit{nextts}]$ to all $p_j \in \mathcal{P}$

upon exists $\textit{ts}$ **such that** $\{p_j \in \mathcal{P} | \textit{ newepoch}[j] = \textit{ts}\} \in \mathcal{Q}_i$ **and** $\textit{nextts} > \textit{lastts}$ **do**
    $\textit{lastts} \leftarrow \textit{nextts}$
    $\textit{newepoch} \leftarrow [\perp]^n$
    output $\textit{startepoch}(\textit{lastts}, \textit{leader}(\textit{lastts}))$

Chapter 5

Conclusion

The conclusion looks back at the entire work, gives a critical look, summarizes, and discusses extensions and future work.

Appendix A

Extra material

Extra material may be placed in an appendix that appears after the conclusion.

Bibliography

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[2] C. Cachin, R. Guerraoui, and L. E. T. Rodrigues, *Introduction to Reliable and Secure Distributed Programming (2. ed.)*. Springer, 2011.

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Erklärung

Erklärung gemäss Art. 30 RSL Phil.-nat. 18

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