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**UNIVERSITÄT  
BERN**

# **3D Metric Fields**

## **Optimizing Frame Fields in a new Metric**

### **Bachelor Thesis**

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# Abstract

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# Chapter 1

## Introduction

In many applications of computer graphics or engineering, the geometry of objects has to be described in some way. Finite elements encode various information such as physical properties to be used in e.g. stress simulations. While it is possible for simple objects such as spheres or planes to describe them with analytical formulas, this is unfeasible for arbitrary objects of any shape. To describe any geometric object, it is discretized and encoded as a mesh. Volumetric meshes are used to encode the surface and the interior of an object. The most common types of volumetric meshes are tetrahedral and hexahedral meshes. Hexahedral meshes, that is a decomposition into cube-like elements, are usually preferred due to their superior numerical accuracy and faster compute time. However, automatically generating high-quality meshes remains a hard challenge in computer graphics. A promising research direction is through the use of frame fields [11].

A *frame field* can be seen as a generalization of vector fields. They prescribe a *frame* to each point on an object, that is, three linearly-independent vectors. Each frame locally represents the orientation and deformation of a cube. Consequently, frame fields must be so called boundary-aligned: At the boundary of an object, one vector of the frame must be aligned with the surface normal as hexahedral elements (*short: hex element*) should align at the boundary. Then, the objective is to find the “best” frame field. Usually, “best” is measured in the smoothness of the frame field. Smoothness is measured with the Dirichlet energy  $\int_{\Omega} \|\nabla \phi\|^2 dx$ , where  $\phi$  is a frame, and we measure how much these frames twist and rotate within the object.

To allow for non-uniform and anisotropic hex elements, we introduce *metric fields*. Optimizing the frame field in this new metric (instead of the usual euclidean metric) then results in elements that are uniform and isotropic in this new metric. To this end, the expression for the Dirichlet energy must be changed. We will show how to calculate a special rotation matrix  $R$ , which describes how frames are aligned in the new metric.

In this thesis, we rederive how to pull back the metric with the rotation coefficient  $R$  such that we can measure the Dirichlet energy in the usual cartesian coordinates as described in *Metric-Driven 3D Frame Field Generation*[3]. The necessary mathematical background is introduced in Section 2. In Section 3, by focusing on integrability of the frame field, we derive the connection  $\omega$  under our new metric field and how to evaluate it. Section 4 discusses how the metric field is piecewise-linearly discretized as a tetrahedral mesh and how the rotation coefficient  $R$  to align frames in the metric is calculated within a specific tetrahedron. Because in general,  $R$  is needed between arbitrary points in the metric field, a robust tetrahedron finder algorithm is presented in Section 5. We briefly explain how the actual frame field optimization works in Section 6. The whole frame field generation is tested with some experiments in Section 7. We conclude in Section 8.





## Chapter 2

# Mathematical Background

A solid mathematical introduction is key to understanding how and why the approach in this thesis work. This chapter serves as a primer to differential geometry and frame fields.

### 2.1 Differential Geometry

We will make heavy use of differential geometry in chapter 3. I introduce the basic concepts what we will use, but I refrain from giving any proofs. I will give definitions only as far as we need it. These definitions will by no means be exhaustive. The following is an incomplete summary of concepts that we need presented in *Introduction to smooth manifolds*[?].

#### 2.1.1 Manifold

A manifold  $\mathcal{M}$  is a space that locally looks like Euclidean space. More exactly, a  $n$ -manifold is a topological space, where each point on the manifold has an open neighborhood that is locally homeomorphic to an open subset of Euclidean space  $\mathbb{R}^n$ . A manifold can be equipped with additional structure. For example, we can work on *smooth manifolds*. In simple terms, a manifold is *smooth* if it is similar enough to  $\mathbb{R}^n$  that we can do Calculus like differentiation or integration on it. For this, each point on the manifold must be locally *diffeomorphic* to an open subset of  $\mathbb{R}^n$  space.

#### 2.1.2 Tangent space, Tangent bundle

There are many equivalent definitions for the tangent space. One definition is for each point  $p$  in the manifold  $\mathcal{M}$ , the tangent space  $T_p\mathcal{M}$  consists of  $\gamma'(0)$  for all differentiable paths  $\gamma : (-\varepsilon, \varepsilon) \rightarrow \mathcal{M}$  with  $p = \gamma(0)$ . The tangent space is a vector space which has the same dimension as its manifold, which is 3 in our case. These tangent spaces can be “glued” together to form the *tangent bundle*  $T\mathcal{M} = \sqcup_{p \in \mathcal{M}} T_p\mathcal{M}$ , which itself is a manifold of dimension  $2n$ . An element of  $T\mathcal{M}$  can be written as  $(p, v)$  with  $p \in \mathcal{M}$  and  $v \in T_p\mathcal{M}$ . This admits a natural projection  $\pi : T\mathcal{M} \rightarrow \mathcal{M}$ , which sends each vector  $v \in T_p\mathcal{M}$  to the point  $p$  where it is tangent:  $\pi(p, v) = p$ . A *section*  $\sigma : \mathcal{M} \rightarrow T\mathcal{M}$  is a continuous map, with  $\pi \circ \sigma = \text{Id}_{\mathcal{M}}$ . Sections of  $T\mathcal{M}$  are tangent vector fields on  $\mathcal{M}$ .

#### 2.1.3 Cotangent space, Cotangent bundle

The dual space  $V^*$  of a vector space  $V$  consists of all linear maps  $\omega : V \rightarrow \mathbb{R}$ . We call these functionals *covectors* on  $V$ .  $V^*$  is itself a vector space, with the same dimension as  $V$  and operations like addition and scalar multiplication can be performed on its elements. Any element in a vector space can be expressed as a finite linear combination of its basis. This basis is called the *dual basis*. Thus, we call the dual space of the vector space  $T_p\mathcal{M}$  its *cotangent space*, denoted by  $T_p^*\mathcal{M}$ . As before, the disjoint union of  $T_p^*\mathcal{M}$  forms the *cotangent bundle*:  $T^*\mathcal{M} = \sqcup_{p \in \mathcal{M}} T_p^*\mathcal{M}$ . Defined analogously from above, sections  $\sigma$  on  $T^*\mathcal{M}$  define *covector fields* or *1-forms*.

### 2.1.4 Tensors

Before we can introduce differential forms in the next paragraph, we need to go a little bit into *tensors*. In simple words, tensors are real-valued, multilinear functions. A map  $F : V_1 \times \dots \times V_k \rightarrow W$  is multilinear, if  $F$  is linear in each component. For example, the dot product in  $\mathbb{R}^n$  is a tensor. It takes two vectors and is linear in each component - bilinear. Another example is the *Tensor Product of Covectors*: Let  $V$  be a vector space and take two covectors  $\omega, \eta \in V^*$ . Define the new function  $\omega \otimes \eta : V \times V \rightarrow \mathbb{R}$  by  $\omega \otimes \eta(v_1, v_2) = \omega(v_1)\eta(v_2)$ . It is multilinear, because  $\omega$  and  $\eta$  are linear. We look at a special class of tensors, the *alternating tensors*. A tensor is alternating, if it changes sign whenever two arguments are interchanged, i.e.  $\omega(v_1, v_2) = -\omega(v_2, v_1)$ . A covariant tensor field over a manifold defines a covariant tensor at each point on the manifold, covariant because the tensor is over the cotangent space  $T_p^*\mathcal{M}$ . An alternating tensor field is called a *differential form*.

### 2.1.5 Differential Forms, Exterior Derivative

Recall that a section from  $T^*\mathcal{M}$  is called a differential 1-form, or just 1-form. Define the *wedge product* (or *exterior product*) between two 1-forms:

$$(\omega \wedge \eta)_p = \omega_p \wedge \eta_p$$

Notice the similarity to the *Tensor Product of Covectors*: We get a new map, (a 2-form):

$$\omega \wedge \eta : T\mathcal{M} \times T\mathcal{M} \rightarrow \mathbb{R}$$

The wedge product is antisymmetric, therefore  $\omega \wedge \eta = -\eta \wedge \omega$  for 1-forms  $\omega$  and  $\eta$ . We denote by  $\Omega^k(\mathcal{M})$  the space of differential  $k$ -forms on  $\mathcal{M}$ . There is a natural differential operator  $d$  on differential forms we call *exterior derivative*. It maps  $k$ -forms to  $(k+1)$ -forms, i.e.  $d : \Omega^k(\mathcal{M}) \rightarrow \Omega^{k+1}(\mathcal{M})$  and has the the following properties:

- $df$  is the ordinary differential of a smooth function  $f$ . Smooth functions are 0-forms.
- $d(d\alpha) = 0$
- $d(\alpha \wedge \beta) = (d\alpha \wedge \beta) + (-1)^k(\alpha \wedge d\beta)$  for a  $k$ -form  $\alpha$ . (Leibnitz Rule)

In section 3 we will need what  $d\omega$  is for some smooth 1-form  $\omega$ , the calculation will be done there. For now, just note that any arbitrary smooth 1-form can be written as  $\omega = Fdx + Gdy + Hdz$  for some appropriate smooth functions  $F, G, H$ .

### 2.1.6 Riemannian metric, $g$ -orthonormality

Inner products are examples of symmetric tensors. They allow us to define lengths and angles between vectors. We can apply this idea to manifolds. A Riemannian metric  $g$  is a symmetric positive-definite tensor field at each point. If  $\mathcal{M}$  is a manifold, the pair  $(\mathcal{M}, g)$  is called a *Riemannian manifold*. Let  $g$  be the Riemannian metric on  $\mathcal{M}$  and  $p \in \mathcal{M}$ , then  $g_p$  is an inner product on  $T_p\mathcal{M}$ . We write  $\langle \cdot, \cdot \rangle_g$  to denote this inner product. Any Riemannian metric can be written as positive-definite symmetric matrix, which allows for this simple form:  $\langle v, w \rangle_g = v^\top gw$ .

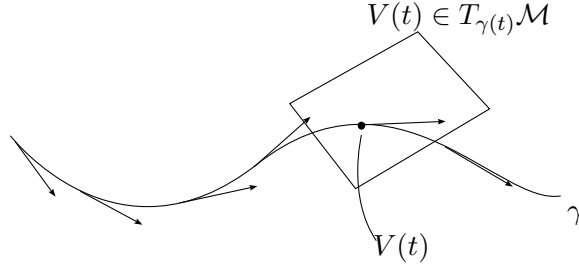
Such a new metric allows for the definition of  *$g$ -orthonormality*: A basis  $[e_1, e_2, e_3]$  of  $T_p\mathcal{M}$  is  *$g$ -orthonormal* if  $\langle e_i, e_j \rangle_g = \delta_{ij}$ .

### 2.1.7 Connection, Covariant derivative, Parallel transport

A connection defines how two different tangent spaces are connected to each other, such that tangent vector fields can be differentiated. There is an infinite amount of connections on a manifold. An *affine connection*  $\nabla$  is a bilinear map that takes two tangent vector fields  $X, Y$  and maps it to a new tangent vector field  $\nabla_X Y$  on  $\mathcal{M}$  such that

- $\nabla_{fX}Y = f\nabla_XY$ , where  $f \in C^\infty(\mathcal{M}, \mathbb{R})$
- $\nabla_X(fY) = f\nabla_XY + (Xf)Y$  for  $f \in C^\infty(\mathcal{M}, \mathbb{R})$ , it satisfies the Leibnitz rule in the second variable

We call  $\nabla_XY$  the *covariant derivative of  $Y$  in the direction of  $X$* . A connection  $\nabla$  defines the parallel transport of a vector along a curve. Given a curve  $\gamma : [0, 1] \rightarrow \mathcal{M}$  and a vector  $v_0 \in T_{\gamma(0)}\mathcal{M}$ , there exists a unique parallel vector field  $V : [0, 1] \rightarrow T\mathcal{M}$  along  $\gamma$  such that  $V(0) = v_0$  [5]. Recall: a vector field  $V$  along  $\gamma$  means  $\pi(V(t)) = \gamma(t)$ . The uniqueness and existence is a consequence of the vector field  $V$  being the solution of  $\nabla_{\dot{\gamma}(t)}V(t) = 0$  defining a linear ordinary differential equation with initial condition  $V(0) = v_0$ . This vector field  $V(t)$  is called the *parallel transport* of  $v_0$  along  $\gamma$ . It is “parallel” in the sense that the transported vector does not change within the tangent space. See figure 2.1 for an illustration in 2D.



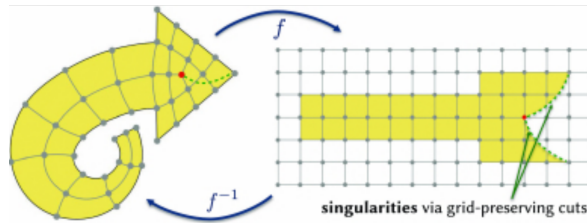
**Figure 2.1.** A parallel vector field  $V(t)$  along a curve  $\gamma$ . Each  $V(t) \in T_{\gamma(t)}\mathcal{M}$  and  $\nabla_{\dot{\gamma}(t)}V(t) = 0$ , so each vector that is drawn is parallel to each other.

## 2.2 Frame fields

Recall from the introduction that we are trying to find a hexahedral mesh for some geometric object. In the following sections, the key idea to tackle this problem through *integer-grid maps* is discussed and how the problem is relaxed through *frame fields*.

### 2.2.1 Integer-grid maps

In 2D, integer-grid maps have proven to be reliable in generating the analogue of hexahedral meshes, quad meshes [1]. The goal is to use the same approach, but in three dimensions. The objective is to find a parametrization  $\phi : \mathcal{M} \rightarrow \mathbb{R}^3$ , which embeds the 3-dimensional object to be meshed onto a 3-dimensional voxel grid, where the inverse  $\phi^{-1}$  maps the deformed grid back onto the object to recover a shape-aligned hexahedral mesh, see figure 2.2 for an illustration. This is called an integer-



**Figure 2.2.** Idea of quadrangulation with integer-grid maps in 2D

grid map and by minimizing distortion and satisfying boundary alignment, a hexahedral mesh could be extracted. However, this is a hard mixed-integer and non-convex optimization problem for which no current optimization technique is available that would result in an acceptable solution.

### 2.2.2 Frame fields as a relaxation

By taking the Jacobian of the parametrization,  $\nabla\phi$ , we get a mapping  $\nabla\phi : \mathcal{M} \rightarrow \mathbb{R}^{3 \times 3}$  of *frames*, a *frame field*. The idea of frame fields is then to search for an approximation of  $\nabla\phi$ . If  $F : \mathcal{M} \rightarrow \mathbb{R}^{3 \times 3}$  is the approximation of  $\nabla\phi$ , we can solve for  $\phi : \mathcal{M} \rightarrow \mathbb{R}^3$  with

$$\min_{\phi} \int_{\mathcal{M}} \|\nabla\phi F - \text{Id}\|^2$$

where  $\|\cdot\|$  is the Frobenius norm. If this is sufficiently small, then  $F^{-1} \approx \nabla\phi$  and the extracted hexahedral mesh closely follow the integer-grid isolines. A frame locally represents the edges of a deformed cube. We can think of a frame field as the composition of three vector fields and as a relaxation of the original problem. However, frame fields can contain singularities that the underlying vector fields do not contain [8] and contain types of singularities that cannot appear in hex meshes [7]. These singularities are said to be non-meshable. The problem of non-meshability is not discussed further here, but it is something to be aware of.

We treat a frame  $F$  as a set of 3 linearly independent vectors  $\{F_1, F_2, F_3\}$  which we can collect into a matrix  $F = (F_1, F_2, F_3) \in \mathbb{R}^{3 \times 3}$ . Notice that many of these frames are equivalent. There are  $2^3$  choices of the sign, and  $3!$  possible permutations, which gives  $2^3 \cdot 3! = 48$  equivalent frames. Since we want non-degenerate frames and the same orientation through the grid, the constraint  $\det(F) > 0$  is imposed, which leaves 24 equivalent frames. Equivalence of frames is then defined as

$$F_u \sim F_v \iff \exists R \in \mathcal{O} : F_u = RF_v$$

where  $\mathcal{O}$  is the *chiral cubical symmetry group* [8]. These symmetries make the optimization for frame fields more complicated. To handle these symmetries, we use the spherical harmonics based representation [4]. The idea is to use the polynomial  $x^4 + y^4 + z^4$  to express the frames as rotations. Under the restriction of the polynomial to the sphere, that is  $\mathbb{S}^2 \rightarrow \mathbb{R}$ , the octahedral frames are invariant under the chiral cubical symmetry group. However, the spherical harmonics manifold is 9-dimensional, so not all vectors in the spherical harmonics represent valid frames, since rotations only exhibit 3 degrees of freedom. While optimizing the frame field, some kind of “average” of two frames will be done. This averaging makes sense in the spherical harmonics space, but the result may be outside the valid space of frames. A projection from the spherical harmonics space back to the valid space of frames will be needed, see fig. 2.3 [12].

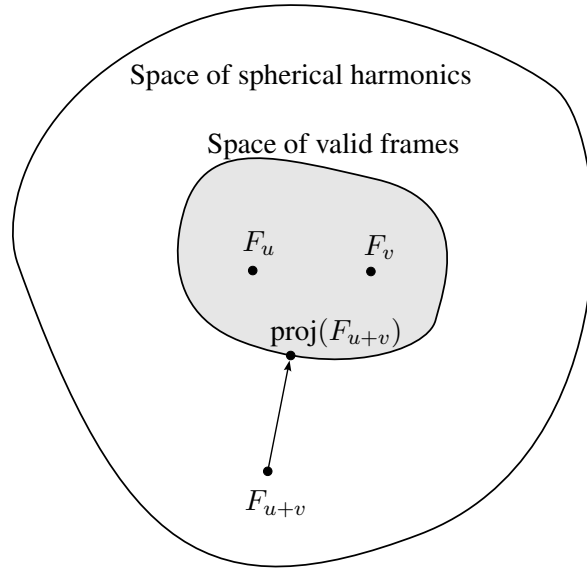
When a frame  $F$  has orthonormal columns, the resulting hex elements look like unit cubes. By introducing a metric  $g$ , we can control the size and shear of the cubes by relaxing the orthonormality constraint to  $g$ -orthonormality when

$$F^{-1}gF = \text{Id}$$

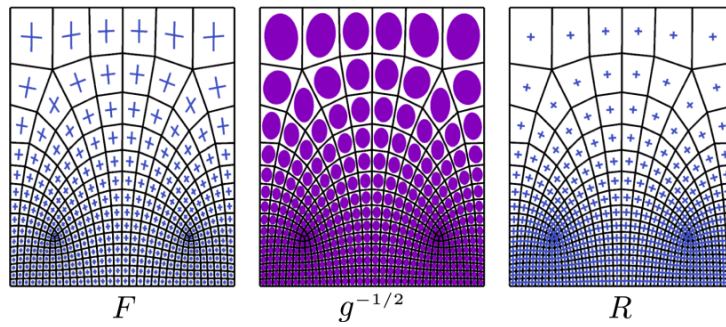
holds. We can decompose the frame field  $F$  into a rotational part  $R : \mathcal{M} \rightarrow \text{SO}(3)$  and a symmetric metric part  $g^{-1/2}$  (akin to the polar decomposition of linear transformations) [9], that is  $F = g^{-1/2}R$ , see figure 2.4.

TODO: Frame field goals

- Dirichlet energy, what are we optimizing
- Requirements for the frame field -  $\zeta$  integrability + boundary alignment



**Figure 2.3.** Combination of two valid frames leads to a resulting frame outside the valid frames. A projection to the nearest valid frame is done.



**Figure 2.4.** Factorization of a  $g$ -orthogonal frame field into a symmetric metric part  $g^{-1/2}$  and a rotational part  $R$  (Figure from [3]).



## Chapter 3

# Connection One-Form $\omega$

Recall that our goal is to find a frame field  $F : \mathcal{M} \rightarrow \mathbb{R}^{3 \times 3}$ , such that the target parametrization of  $\phi$  can sufficiently reduce the difference between  $\nabla\phi$  and  $F^{-1}$ . For this,  $F^{-1}$  needs to stay as close to being locally integrable as possible. In this chapter, by starting from the local integrability condition, we formulate a connection one-form  $\omega$  which is used to measure the Dirichlet energy in the new metric  $g$ , such that the frames stay close to being  $g$ -orthonormal ( $F^{-1}gF = \text{Id}$ ). We follow the steps in *Metric-Driven 3D Frame Field Generation*[3].

### 3.1 Local integrability

A vector field  $U$  is integrable if and only if  $\nabla \times U = 0$ , which means the vector field has vanishing curl [10]. Although in general it is more complicated [8], we can think of a frame field  $F$  as the composition of 3 vector fields

$$F = \begin{bmatrix} | & | & | \\ F_1 & F_2 & F_3 \\ | & | & | \end{bmatrix}$$

where  $F_i : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  are vector fields. To achieve local integrability for  $F^{-1}$ , we therefore want

$$\nabla \times F^{-1} \stackrel{!}{=} 0$$

where the curl is applied to each column. We can express this more naturally with the language of differential forms: The curl can be written as the exterior derivative  $d$  of a one-form  $\alpha$ . A one-form (more generally, a differential form) is closed, if  $d\alpha = 0$ . We construct a vector-valued one-form out of our frame field, given  $\mathbf{p} = (x, y, z)^\top$  in Euclidean coordinates

$$\alpha \triangleq F^{-1}d\mathbf{p} = R^\top g^{1/2}d\mathbf{p}$$

where  $d\mathbf{p} = (dx, dy, dz)^\top$  is the common orthonormal one-form basis. Local integrability of  $F^{-1}$  is then formulated as the closedness of  $\alpha$ , i.e.

$$F^{-1} \text{ locally integrable} \iff \mathbf{0} = d\alpha$$

Some reformulations yield:

$$\begin{aligned} \mathbf{0} &= d\alpha = d(F^{-1}d\mathbf{p}) = d(R^\top g^{1/2}d\mathbf{p}) \\ &= dR^\top \wedge (g^{1/2}d\mathbf{p}) + R^\top d(g^{1/2}d\mathbf{p}) \\ &= R^\top (\omega \wedge (g^{1/2}d\mathbf{p}) + d(g^{1/2}d\mathbf{p})) \end{aligned}$$

which we can simplify to

$$\mathbf{0} = \omega \wedge (g^{1/2}d\mathbf{p}) + d(g^{1/2}d\mathbf{p}) \tag{3.1}$$

where the Leibnitz Rule for the exterior derivative is applied and  $\wedge$  is the exterior product between a matrix-valued one-form and a vector-valued one-form, i.e. one-forms in each component and the matrix vector product uses  $\wedge$  as the multiplication, see section 3.3 for the evaluation. We further define

$$\omega = RdR^\top \in \mathfrak{so}(3)$$

which is an antisymmetric matrix-valued one-form. To see this, we differentiate the orthogonality condition of the rotation matrix  $R$  (we assume  $R$  rotates about the axis  $a$  with angle  $\theta$ ):

$$\begin{aligned} \text{Id} &= RR^\top \\ d(\text{Id}) &= d(RR^\top) \\ \mathbf{0} &= dRR^\top + RdR^\top \\ \mathbf{0} &= (RdR^\top)^\top + RdR^\top \\ -(RdR^\top)^\top &= RdR^\top \end{aligned}$$

The Lie algebra  $\mathfrak{so}(3)$  consists of all antisymmetric 3x3 matrices and has manifold structure. Elements of  $\mathfrak{so}(3)$  are infinitesimal rotations, that is, they are tangent to the manifold  $\text{SO}(3)$  at the element  $\text{Id}$ . Thus, we can use  $\omega$  as a connection one-form to do the alignment of frames in the new metric  $g$ , i.e. the parallel transport is done with  $\omega$  and then compared.

To solve for local integrability, we find  $\omega$  such that the one-form  $\alpha$  is closed, then try to match the  $\omega$  with  $R$ . This can be expressed as

$$\min_{R \in \text{SO}(3)} \|RdR^\top - \omega\|^2, \quad (3.2)$$

where  $\omega$  is determined by  $g$  through a system of linear equations (see 3.3). In general, Eq. 3.2 cannot be minimised to zero, therefore we solve for the nearly integrable 3D rotation field  $R$ .

### 3.2 Smoothness measure

Many frame field generation methods rely on maximising smoothness through the minimization of the Dirichlet energy. Here, we show that integrability through  $\omega$  is closely related to the usual Dirichlet energy. We can take Eq. 3.2 as a smoothness measure and reformulate, i.e.

$$\|RdR^\top - \omega\|^2 = \|-dRR^\top - \omega\|^2 = \|-dRR^\top R - \omega R\|^2 = \|dR + \omega R\|^2. \quad (3.3)$$

By defining

$$\mathcal{D}R \triangleq dR + \omega R, \quad (3.4)$$

we can measure the smoothness and the integrability with a single energy  $\int_{\mathcal{R}} \|\mathcal{D}R\|^2$ . Now, when  $g$  is constant in euclidean coordinates  $(x, y, z)$ , Eq. 3.1 tells us that  $\omega = 0$ , which reduces the smoothness measure  $\|\mathcal{D}R\|$  to  $\|dR\|$ , which corresponds to the usual Dirichlet energy. In fact,  $\mathcal{D}R$  corresponds to the covariant derivative of  $R$  under the connection  $\omega$ , which shows that local integrability is related to the covariant-based Dirichlet energy. Special care must be taken as we abuse the notation of the covariant derivative. The covariant derivative  $\mathcal{D}$  acts on each column of  $R$  separately. We define

$$\mathcal{D}R_i \triangleq \nabla_{\dot{\gamma}(t)} R_i \triangleq dR_i + \omega R_i$$

where  $dR_i$  is the derivative of each entry with respect to the angle  $\theta$  and  $\gamma$  is some curve on the manifold.

### 3.3 Connection evaluation

To find  $\omega$ , we use equation 3.1

$$\mathbf{0} = \omega \wedge (g^{1/2} d\mathbf{p}) + d(g^{1/2} d\mathbf{p})$$



and reformulate into a linear system. We represent the antisymmetric matrix-valued one-form  $\omega$

$$\omega = \begin{bmatrix} 0 & \omega_{12} & -\omega_{31} \\ -\omega_{12} & 0 & \omega_{23} \\ \omega_{31} & -\omega_{23} & 0 \end{bmatrix}$$

by  $[\omega_{23} \ \omega_{31} \ \omega_{12}] = [dx \ dy \ dz] W$ . We write  $W = [W_1, W_2, W_3]$ ,  $W_i \in \mathbb{R}^3$ . That is,  $W$  is the matrix with the coefficients for the one-forms, e.g.  $W_1 = [(\omega^{23})_1, (\omega^{23})_2, (\omega^{23})_3]^\top$ . Recall, a one-form can be expressed as

$$\omega_{ij} = (\omega^{ij})_1 dx + (\omega^{ij})_2 dy + (\omega^{ij})_3 dz$$

So e.g. for  $\omega_{23}$  we get

$$\omega_{23} = [dx \ dy \ dz] W_1 = (\omega^{23})_1 dx + (\omega^{23})_2 dy + (\omega^{23})_3 dz.$$

We also write  $A = g^{1/2} = [A^1, A^2, A^3]$ . Starting with the first part of eq. 3.1, we get

$$\begin{aligned} \omega \wedge g^{1/2} dp &= \begin{bmatrix} 0 & \omega_{12} & -\omega_{31} \\ -\omega_{12} & 0 & \omega_{23} \\ \omega_{31} & -\omega_{23} & 0 \end{bmatrix} \wedge \begin{bmatrix} A_1^1 & A_1^2 & A_1^3 \\ A_2^1 & A_2^2 & A_2^3 \\ A_3^1 & A_3^2 & A_3^3 \end{bmatrix} \begin{bmatrix} dx \\ dy \\ dz \end{bmatrix} \\ &= \begin{bmatrix} +\omega_{12} \wedge (A_2^1 dx + A_2^2 dy + A_2^3 dz) - \omega_{31} \wedge (A_3^1 dx + A_3^2 dy + A_3^3 dz) \\ -\omega_{12} \wedge (A_1^1 dx + A_1^2 dy + A_1^3 dz) + \omega_{23} \wedge (A_3^1 dx + A_3^2 dy + A_3^3 dz) \\ +\omega_{31} \wedge (A_1^1 dx + A_1^2 dy + A_1^3 dz) - \omega_{23} \wedge (A_2^1 dx + A_2^2 dy + A_2^3 dz) \end{bmatrix} \end{aligned}$$

It will get really messy if we calculate each component here, so let us calculate one component separately first:

$$\begin{aligned} \omega_{ij} \wedge (A_k^1 dx + A_k^2 dy + A_k^3 dz) &= ((\omega^{ij})_1 dx + (\omega^{ij})_2 dy + (\omega^{ij})_3 dz) \wedge (A_k^1 dx + A_k^2 dy + A_k^3 dz) \\ &= (\omega^{ij})_1 A_k^2 dx \wedge dy + (\omega^{ij})_1 A_k^3 dx \wedge dz \\ &\quad + (\omega^{ij})_2 A_k^1 dy \wedge dx + (\omega^{ij})_2 A_k^3 dy \wedge dz \\ &\quad + (\omega^{ij})_3 A_k^1 dz \wedge dx + (\omega^{ij})_3 A_k^2 dz \wedge dy \\ &= ((\omega^{ij})_1 A_k^2 - (\omega^{ij})_2 A_k^1) dx \wedge dy \\ &\quad + ((\omega^{ij})_2 A_k^3 - (\omega^{ij})_3 A_k^2) dy \wedge dz \\ &\quad + ((\omega^{ij})_3 A_k^1 - (\omega^{ij})_1 A_k^3) dz \wedge dx \end{aligned}$$

where we use the fact that  $dx \wedge dx = 0$  and  $dx \wedge dy = -dy \wedge dx$ . We can clean up the above expression using the cross product:

$$\begin{bmatrix} ((\omega^{ij})_2 A_k^3 - (\omega^{ij})_3 A_k^2) \\ ((\omega^{ij})_3 A_k^1 - (\omega^{ij})_1 A_k^3) \\ ((\omega^{ij})_1 A_k^2 - (\omega^{ij})_2 A_k^1) \end{bmatrix}^\top \begin{bmatrix} dy \wedge dz \\ dz \wedge dx \\ dx \wedge dy \end{bmatrix} = [W_i \times A^k]^\top \begin{bmatrix} dy \wedge dz \\ dz \wedge dx \\ dx \wedge dy \end{bmatrix}$$

We use the fact that  $[A_1, A_2, A_3] = [A^1, A^2, A^3]$  because  $A$  is symmetric, and  $W_1$  corresponds to  $\omega_{23}$ ,  $W_2$  to  $\omega_{31}$  and  $W_3$  to  $\omega_{12}$ . We continue with the second part of eq. 3.1:

$$d(g^{1/2} dp) = d(Adp) = d \left( \begin{bmatrix} A_1^1 & A_1^2 & A_1^3 \\ A_2^1 & A_2^2 & A_2^3 \\ A_3^1 & A_3^2 & A_3^3 \end{bmatrix} \begin{bmatrix} dx \\ dy \\ dz \end{bmatrix} \right)$$

Again, we can do this separately for each row (we use the fact that the exterior derivative  $d$  is the ordinary differential for a smooth function):

$$\begin{aligned}
& d(A_k^1 dx + A_k^2 dy + A_k^3 dz) \\
&= dA_k^1 \wedge dx + dA_k^2 \wedge dy + dA_k^3 \wedge dz \\
&= \frac{\partial A_k^1}{\partial x} dx \wedge dx + \frac{\partial A_k^1}{\partial y} dy \wedge dx + \frac{\partial A_k^1}{\partial z} dz \wedge dx \\
&+ \frac{\partial A_k^2}{\partial x} dx \wedge dy + \frac{\partial A_k^2}{\partial y} dy \wedge dy + \frac{\partial A_k^2}{\partial z} dz \wedge dy \\
&+ \frac{\partial A_k^3}{\partial x} dx \wedge dz + \frac{\partial A_k^3}{\partial y} dy \wedge dz + \frac{\partial A_k^3}{\partial z} dz \wedge dz \\
&= \left( \frac{\partial A_k^3}{\partial y} - \frac{\partial A_k^2}{\partial z} \right) dy \wedge dz + \left( \frac{\partial A_k^1}{\partial z} - \frac{\partial A_k^3}{\partial x} \right) dz \wedge dx + \left( \frac{\partial A_k^2}{\partial x} - \frac{\partial A_k^1}{\partial y} \right) dx \wedge dy \\
&= (\nabla \times A_k)^\top \begin{bmatrix} dy \wedge dz \\ dz \wedge dx \\ dx \wedge dy \end{bmatrix}
\end{aligned}$$

Finally, we can put everything together:

$$\begin{aligned}
\mathbf{0} &= \begin{bmatrix} (W_3 \times A^2 - W_2 \times A^3 + \nabla \times A^1)^\top \\ (W_1 \times A^3 - W_3 \times A^1 + \nabla \times A^2)^\top \\ (W_2 \times A^1 - W_1 \times A^2 + \nabla \times A^3)^\top \end{bmatrix} \begin{bmatrix} dy \wedge dz \\ dz \wedge dx \\ dx \wedge dy \end{bmatrix} \\
&\iff \begin{bmatrix} (W_2 \times A^3 - W_3 \times A^2)^\top \\ (W_3 \times A^1 - W_1 \times A^3)^\top \\ (W_1 \times A^2 - W_2 \times A^1)^\top \end{bmatrix} \begin{bmatrix} dy \wedge dz \\ dz \wedge dx \\ dx \wedge dy \end{bmatrix} = \begin{bmatrix} (\nabla \times A^1)^\top \\ (\nabla \times A^2)^\top \\ (\nabla \times A^3)^\top \end{bmatrix} \begin{bmatrix} dy \wedge dz \\ dz \wedge dx \\ dx \wedge dy \end{bmatrix}
\end{aligned}$$

Take the curl to the other side and switch order on the left-hand side to cancel the  $-1$ . As we are only interested in the 9 components of  $W$ , we omit the two-form basis and transform into a 9x9 linear system for  $W$ . We define  $A_\times$  and  $\text{vec}(\cdot)$  as

$$A_\times = \begin{bmatrix} 0 & -A_\times^3 & A_\times^2 \\ A_\times^3 & 0 & -A_\times^1 \\ -A_\times^2 & A_\times^1 & 0 \end{bmatrix}, \text{vec}(W) = \begin{bmatrix} W_1 \\ W_2 \\ W_3 \end{bmatrix}$$

with  $A_\times^i$  defined as

$$A_\times^i = [A_1^i \quad A_2^i \quad A_3^i]_\times = \begin{bmatrix} 0 & -A_3^i & A_2^i \\ A_3^i & 0 & -A_1^i \\ -A_2^i & A_1^i & 0 \end{bmatrix}$$

and  $\text{vec}(\cdot)$  turns a 3x3-matrix into a 9x1-vector by stacking the columns. With these two definitions, we can transform the above equality into a linear system

$$A_\times \text{vec}(W) = \text{vec}(\nabla \times A)$$

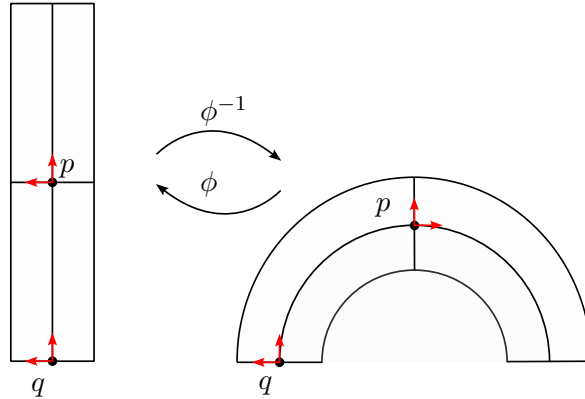
where  $\nabla \times A$  is just the curl applied to each column. This transformation can be checked by laboriously plugging in the definitions and comparing the coefficients. With tedious brute-force calculations, one can show that  $\det(A_\times) = -2 \det(A)^3 = -2 \det(g)^{3/2} < 0$ , which means this is a linear system that is solvable and can be used to calculate  $W$  at a point.

## Chapter 4

# Calculation of rotation coefficient

When the metric  $g$  is globally flat with no additional constraints like boundary alignment, a globally integrable frame field is achievable by parallel transport. No singularities in the frame field would be present. However, this is not possible in general. In this section, by starting from the assumption that the metric is flat, we recover the rotation happening by parallel transport under the connection  $\omega$  which allows us to pull back the covariant derivative based Dirichlet energy and minimize as if we were minimizing the Dirichlet energy in Euclidean coordinates.

The smoothness measure  $\mathcal{D}R$  is zero if all frames are parallel to each other under the connection  $\omega$ , i.e., if all frames are the same if parallel transported to the same point. This is deliberately kept vague as parallel transport is path dependent in general (parallel transport is path independent if the metric is flat). To get a feeling for the problem statement, see figure 4.1.



**Figure 4.1.** Under the metric induced by  $\phi$ , the frame at  $q$  is parallel to  $p$ . To compare them, we need to recover how  $p$  is rotated under  $\omega$ .

As we want parallelness under  $\omega$  along a path  $\gamma$ , we set  $\mathcal{D}R \stackrel{!}{=} 0$  and treat it as a set of differential equations. Again, as  $R$  essentially consists of three rotation fields, what is meant that  $\mathcal{D}R_i = 0$  for each field.

Let  $p, q$  be points on the manifold and  $\ell : [0, 1] \rightarrow \mathcal{M}$  a path connecting them, with  $\ell(0) = q$  and  $\ell(1) = p$ . Our differential equation is then given by

$$0 = dR_i(\ell(t)) + \omega(\ell(t))R_i(\ell(t)) \quad (4.1)$$

which we rearrange to

$$\underbrace{-\omega(\ell(t))}_{A(t)} \underbrace{R_i(\ell(t))}_{\alpha(t)} = \underbrace{dR_i(\ell(t))}_{\dot{\alpha}(t)} \quad (4.2)$$

where we explicitly indicate that  $\omega$  depends on the point evaluated and is not constant. Do not get

confused what we are differentiating: The differential equation consists of mappings

$$\omega : \mathcal{M} \rightarrow \mathbb{R}^{3 \times 3} \text{ and } R_i : \mathcal{M} \rightarrow \mathbb{R}^3.$$

Thus, the concatenation with  $\ell : [0, 1] \rightarrow \mathcal{M}$  gives us mappings  $A : [0, 1] \rightarrow \mathbb{R}^{3 \times 3}$  and  $\alpha : [0, 1] \rightarrow \mathbb{R}^3$  where it makes sense to take the derivative with respect only to  $t$ . This differential equation has the form of a *matrix differential equation*, for which a general closed form solution is hard to find. However, when  $\omega$  and its integral  $\int_0^t -\omega(\ell(s))ds$  commute, the general solution  $\alpha(t) = e^{\int_0^t A(s)ds} \alpha(0)$  exists, which is the case when  $\omega$  along  $\ell$  rotate around the same axis (the same way rotation matrices commute when they rotate around the same axis). Plugging in values 0 and 1 for  $t$  and assuming  $\omega$  commutes with its integral gives us the solution

$$\alpha(1) = R(\ell(1)) = R(p) = e^{\int_0^1 A(s)ds} R(\ell(0)) = e^{\int_0^1 A(s)ds} R(q),$$

which shows that we can align frames with a single rotation matrix  $R_{qp}$ , i.e.

$$R(p) = R_{qp} R(q)$$

which corresponds to the parallel transport under the connection  $\omega$  of the frame along  $\ell$ . Discretization of the covariant based Dirichlet energy is then given by

$$\|R_{qp} R(q) - R(p)\|^2. \quad (4.3)$$

We parametrize the path by  $\ell(0) = a, \ell(1) = b$ . We resort to numerical integration for  $R_{ab}$  and cut the path into  $n$  small segments, i.e.

$$R_{ab} = R_n R_{n-1} \cdots R_1$$

where  $R_i = \exp(-\omega(\dot{\ell}(i\gamma))\gamma)$ ,  $\gamma$  is the length of a segment and  $\dot{\ell}(s) = \frac{\partial \ell}{\partial s}(s)$ . By cutting up the path into small segments,  $\omega$  should rotate around the same axis along  $\ell$ , which as described above allows us to use the general solution  $R_i = \exp((-\int_\gamma \omega d\mathbf{p})_\times)$ . Calculating the exponential map of an antisymmetric matrix (which  $\omega \in \mathfrak{so}(3)$  is) yields a rotation matrix  $R \in \text{SO}(3)$  and can be done with Rodrigues' formula:

$$\exp(u_\times) = \text{Id} + \sin(\theta)\hat{u}_\times + (1 - \cos(\theta))\hat{u}_\times^2$$

where  $\theta = \|u\|_2$  is the rotation angle and  $\hat{u} = u/\theta$  is the rotation axis. We use the trapezoidal rule to evaluate the short interval of the integral of  $W$ , which is given by

$$R_{ab} = \exp\left(\left(\frac{1}{2}(W_a + W_b)^\top(b - a)\right)_\times\right)$$

where  $W_a, W_b$  is solved for by the 9x9 linear system given by  $A_\times$  and  $\nabla \times A$ .

## 4.1 Piecewise linear discretization

We discretize our metric field with a tetrahedral mesh  $\mathcal{T}$ . At each vertex, we attach a metric and linearly interpolate with barycentric coordinates within a tet.

Let  $A_i \in \mathbb{R}^{3 \times 3}, i \in \{1, 2, 3, 4\}$  be the square root metrics at the vertices  $v_i \in \mathbb{R}^3$  of a tet, such that  $A_i^2 = g(v_i)$ . We represent a tet given by its four vertices by a 3x4 matrix, i.e.

$$\begin{pmatrix} | & | & | & | \\ v_1 & v_2 & v_3 & v_4 \\ | & | & | & | \end{pmatrix} \in \mathbb{R}^{3 \times 4}.$$

Any point  $p$  within the tet can then be represented as

$$p = \alpha v_1 + \beta v_2 + \gamma v_3 + \delta v_4$$

with  $\alpha, \beta, \gamma, \delta \geq 0$  and  $\alpha + \beta + \gamma + \delta = 1$ . This is a linear transformation between two coordinate systems, which we can write in matrix form as

$$\underbrace{\begin{pmatrix} | & | & | & | \\ v_1 & v_2 & v_3 & v_4 \\ | & | & | & | \\ 1 & 1 & 1 & 1 \end{pmatrix}}_{T^{-1}} \underbrace{\begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix}}_{\lambda} = \underbrace{\begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}}_p \iff T^{-1}\lambda = p \iff \lambda = Tp$$

$T$  always exists because  $v_1, \dots, v_4$  are linearly independent, else it would not be a valid tetrahedron. By denoting  $T = \{t_{ij}\}_{i,j \in \{1, \dots, 4\}}$ , we can write our barycentric functions as

$$\begin{aligned} \alpha(x, y, z) &= t_{11}x + t_{12}y + t_{13}z + t_{14} \\ \beta(x, y, z) &= t_{21}x + t_{22}y + t_{23}z + t_{24} \\ \gamma(x, y, z) &= t_{31}x + t_{32}y + t_{33}z + t_{34} \\ \delta(x, y, z) &= t_{41}x + t_{42}y + t_{43}z + t_{44} \end{aligned}$$

The convex combination

$$A(x, y, z) = \alpha A_1 + \beta A_2 + \gamma A_3 + \delta A_4$$

is the metric prescribed in the tet. To find  $\nabla \times A$ , let  $A = (A^1, A^2, A^3)$ . We will need the derivatives for the curl, so let

$$(A_j^i)_x \triangleq \frac{\partial A_j^i}{\partial x}$$

be the derivative with respect to  $x$  of entry  $i, j$ . E.g.  $(A_j^i)_x$  is given by

$$(A_j^i)_x = \alpha_x (A_1)_j^i + \beta_x (A_2)_j^i + \gamma_x (A_3)_j^i + \delta_x (A_4)_j^i = t_{11} (A_1)_j^i + t_{21} (A_2)_j^i + t_{31} (A_3)_j^i + t_{41} (A_4)_j^i$$

If we write  $T = (T^1, T^2, T^3, T^4)$  and collect  $(A_k)_j^i$  into a vector

$$\bar{A}_j^i = \begin{pmatrix} (A_1)_j^i \\ (A_2)_j^i \\ (A_3)_j^i \\ (A_4)_j^i \end{pmatrix}$$

this can be shortened to  $\bar{A}_j^{i\top} T^1 = (A_j^i)_x$ . Analogously, we get

$$(A_j^i)_y = \bar{A}_j^{i\top} T^2 \text{ and } (A_j^i)_z = \bar{A}_j^{i\top} T^3$$

The curl is then given by

$$\nabla \times A^i = \begin{pmatrix} (A_3^i)_y - (A_2^i)_z \\ (A_1^i)_z - (A_3^i)_x \\ (A_2^i)_x - (A_1^i)_y \end{pmatrix} = \begin{pmatrix} \bar{A}_3^{i\top} T^2 - \bar{A}_2^{i\top} T^3 \\ \bar{A}_1^{i\top} T^3 - \bar{A}_3^{i\top} T^1 \\ \bar{A}_2^{i\top} T^1 - \bar{A}_1^{i\top} T^2 \end{pmatrix}$$

and  $\nabla \times A = \nabla \times (A^1, A^2, A^3)$ . Notice that the curl is constant within a tetrahedron.

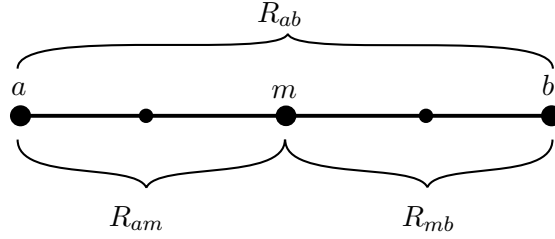
## 4.2 Recursive subdivision

Whenever the rotation coefficient  $R_{qp}$  is needed between some points  $p$  and  $q$ , it is unclear ahead of time how many sampling points on the line  $\ell$  are needed such that  $R_{qp}$  accurately describes how the frame rotates along  $\ell$ . We apply a recursive subdivision scheme to recursively sample more points on  $\ell$  only

where is needed until sampling more points leads to no noticeable improvement anymore (see figure 4.2). We begin by calculating  $R_{ab}$  with just the endpoints. This coefficient is then compared to the result if the midpoint was sampled as well, so i.e. if

$$\frac{||R_{ab} - R_{am} \cdot R_{mb}||_2^2}{\ell^2} < \varepsilon$$

then no measurable improvement happened. We divide by the length  $\ell = ||a - b||_2$  of the segment  $\vec{ab}$ , because the segments get smaller and we want the tolerance  $\varepsilon$  to remain the same. This approach has the advantage of only sampling points where there is improvement. See section 7 for results. The algorithm



**Figure 4.2.** Points are recursively sampled as midpoints between the line  $\vec{ab}$ . If sampling more points within some line segment leads to noticeable improvement, more points are sampled.

for this simple recursive scheme is shown in below.

---

**Algorithm 1** Recursive Subdivision

---

```

1: recursiveDivide ( $a, b$ )
2:    $R_1 = \text{Rotation}(a, b)$ 
3:    $\ell = \text{length}(a - b)$ 
4:    $\text{midpoint} = \frac{a+b}{2}$ 
5:   if  $\frac{||R_1 - \text{Rotation}(a, \text{midpoint}) \cdot \text{Rotation}(\text{midpoint}, b)||_2^2}{\ell^2} < \varepsilon$ 
6:     return  $R_1$ 
7:   else
8:     return  $\text{recursiveDivide}(a, \text{midpoint}) \cdot \text{recursiveDivide}(\text{midpoint}, b)$ 

```

---

## Chapter 5

# Algorithm for $R$ between two arbitrary points in a mesh

To measure the Dirichlet energy, we need to calculate the rotation coefficient  $R$  between two arbitrary points  $q$  and  $p$  that do not necessarily lie within the same tetrahedron. Since the metric and curl is different in each tet, we need to be able to efficiently determine all tets that get intersected by the straight line from  $q$  to  $p$ , and use the correct metric for each corresponding line segment. The calculation for the coefficient then works in the following way:

---

**Algorithm 2** Rotation coefficient  $R$  between  $q$  and  $p$

---

```
1: Input  $(q, p)$ 
2:   //returns all tets intersected by the line  $\vec{qp}$  with the line segments within them
3:    $\text{LINESEGMENTS} \leftarrow \text{tetFinder}(q, p)$ 
4:    $R \leftarrow \text{Id}$ 
5:   for each  $\text{SEGMENT}$  in  $\text{LINESEGMENTS}$ 
6:      $R \leftarrow R \cdot \text{calcCoeff}(\text{SEGMENT})$ 
7: return  $R$ 
```

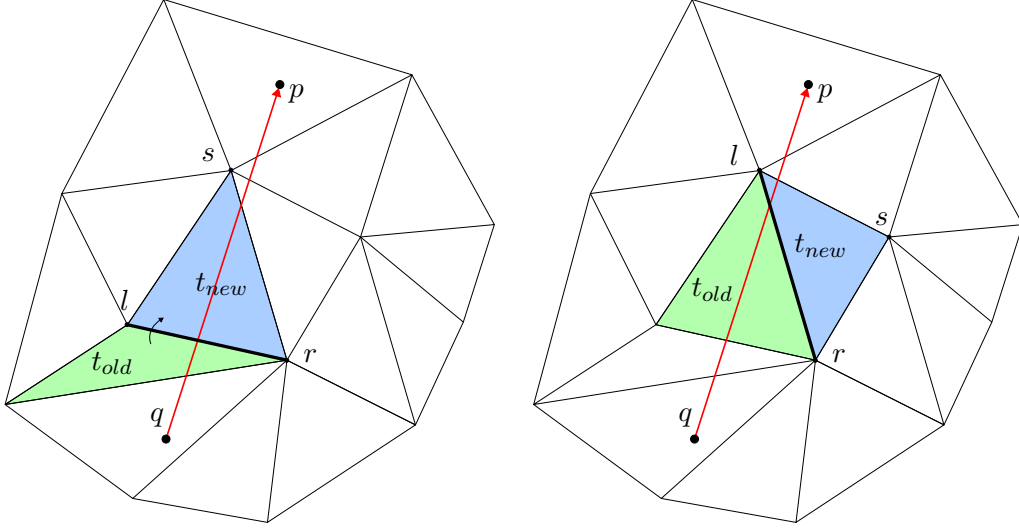
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The missing component here is how to efficiently find all tetrahedra that get intersected. One possibility would be to use ray-triangle intersection and test against the whole mesh, but this is not practical, as we have local information that we can exploit.

We use the idea of the straight walk from *Walking in a Triangulation*[2], which relies only on so called *orientation tests* to determine which triangles we traverse. This chapter covers how the *tetFinder* algorithm works, how it is made robust against degenerate cases and a full description of the algorithm is given, more than what is described in the reference.

### 5.1 Framework

Let  $\mathcal{T}$  be a triangulation of a domain  $\Omega$  that is convex. The straight walk traverses all triangles that get intersected by the line segment from  $q$  to  $p$ . The algorithm first makes an initialization step to get into a valid state, then the straight walk can start. To get a feeling how the algorithm works, let us go through an example in 2D. If the algorithm was in a valid state before, the line from  $q$  to  $p$  intersects with some edge  $\vec{lr}$ . Two triangles share this edge. We test on which side point  $p$  lies of this edge to decide whether the walk continues. If the walk continues, we jump through the edge to hop from the old triangle to a new one. This triangle is defined by three vertices  $(l, r, s)$ . We decide if the new candidate point  $s$  lies on the left side or right side of the line from  $q$  to  $p$ . If  $s$  lies on the left, point  $l$  is moved, else point  $r$  is moved. A new edge intersected with the ray  $\vec{qp}$  is found and the walk repeats. This process is illustrated in Figure 5.1



**Figure 5.1.** Straight walk step

Notice how the ray  $\vec{qp}$  always intersects the edge  $\vec{lr}$  at each update step. We can use this observation to add each edge at each update step to a list. When the algorithm terminates, we can just iterate over this list, find the intersection point of the ray  $\vec{qp}$  with the edge and calculate the rotation coefficient for this segment. The straight walk in 3d works similarly. The initialization step consists of finding a starting tet  $t$  where  $q$  is contained. Then, we find the face of the tet  $t$  that gets intersected by the ray  $\vec{qp}$ . Again, at each step, we know that the ray goes out of our current tet  $t$  through some face  $e$  defined by vertices  $uvw$ . We decide if the walk continues by checking on which side  $p$  lies relative to  $e$ . If the walk should continue, we hop through  $e$  to a new tet  $t_{new}$ . With two orientation tests we decide which of the vertices  $u, v, w$  gets moved to the new candidate point  $s$ . This defines a new face  $e_{new}$  where our ray intersects and the walk repeats. Degenerate cases such as when the ray  $\vec{qp}$  goes exactly through a vertex or when the ray lies within a face, the algorithm may get into an invalid state, where the algorithm may then traverse through cells that do not get intersected by the ray. We need to detect and escape those degenerate cases through some additional checks. These are not described in the paper[2]. In most cases, a simple reinitialization is enough when the invalid state is detected.

### 5.1.1 A note on orientation tests

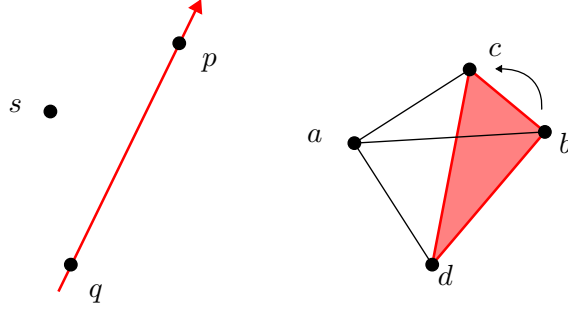
To determine on which side some point  $s$  lies relative to two other points  $q$  and  $p$  (that represent a line) in 2D, the geometric *orientation* predicate is used. It corresponds to evaluating the sign of a determinant. Analogously in 3D, the orientation predicate tests whether a fourth point lies above or below a plane defined by three other points. How “above” the plane is defined depends on the ordering within the determinant. In the case here, it is above if point  $a$  sees the triangle  $bcd$  when turning counterclockwise (see figure 5.2).

$$\text{orientation}(\alpha, \beta, \gamma) = \text{sign} \left( \begin{vmatrix} \beta_x - \alpha_x & \gamma_x - \alpha_x \\ \beta_y - \alpha_y & \gamma_y - \alpha_y \end{vmatrix} \right)$$

$$\text{orientation}(\alpha, \beta, \gamma, \delta) = \text{sign} \left( \begin{vmatrix} \beta_x - \alpha_x & \gamma_x - \alpha_x & \delta_x - \alpha_x \\ \beta_y - \alpha_y & \gamma_y - \alpha_y & \delta_y - \alpha_y \\ \beta_z - \alpha_z & \gamma_z - \alpha_z & \delta_z - \alpha_z \end{vmatrix} \right)$$

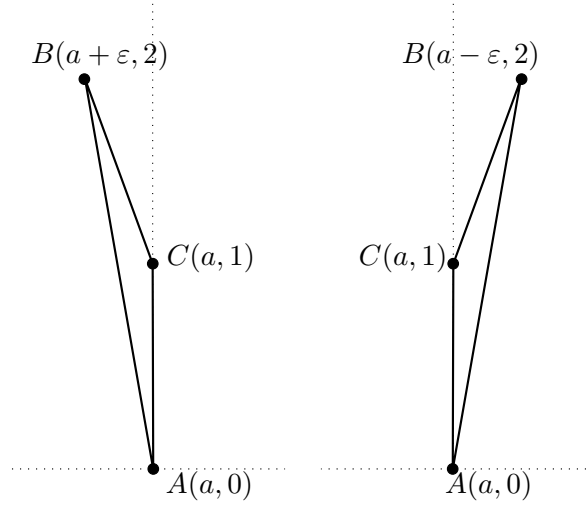
It is important that the sign of the determinant is evaluated exactly. If geometric predicates are implemented with floating point arithmetic, the answers may be inconsistent and wrong. Because of the finite mantissa in floating point representation, many numbers cannot be represented exactly. The machine needs to round a number  $x$  to its nearest number that is exactly representable. Many times, roundoff errors occur and the result is not exact. A famous example of this phenomenon is  $0.1 + 0.2 =$





**Figure 5.2.** Orientation predicate: Point  $a$  lies above the plane  $bcd$  because it sees the points in counter-clockwise order,  $\text{orientation}(a, b, c, d) > 0$

0.30000000000000004. We call the distance between two exactly representable floating point numbers *machine epsilon*  $\varepsilon$ . See figure 5.3 for how this machine epsilon can cause trouble (Example from [15]). Because of the imprecision of floating-point arithmetic (FP), logical decisions based on FP should be



**Figure 5.3.** The orientation predicate implemented with floating point arithmetic cannot distinguish these two cases. The orientation test calculates  $(a + \varepsilon) - a$ . If  $a = 0$ , the orientation test is strictly positive. If  $a = 1$ , then the result is 0, because during the calculation the machine had to round.

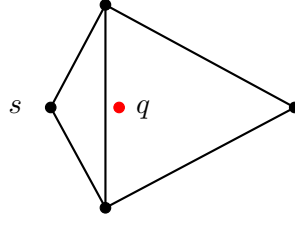
avoided. To fix this issue, we make use of exact predicates that are implemented with arbitrary precision [13]. It is assumed that inputs to the orientation test subroutines are exact, and during the calculation, precision is extended as needed such that we can be sure that the result has the correct sign.

## 5.2 Algorithm *tetFinder*

As described in section 5.1, the straight walk needs different components to work. These components are described in the following subsections. The final algorithm to find all intersected tetrahedra between two points is as follows.

### 5.2.1 Starting tet

To start, the algorithm needs to know in which tet the starting point  $q$  is located. With a simple linear nearest neighbor search of point  $q$ , we find a good heuristic starting point  $s$ . From  $s$ , every tet incident to  $s$  is checked if it contains  $q$ . Most of the times, this method works for locating which tet contains  $q$ . In case it fails, an exhaustive search through all the tets of the mesh is done to search which tet contains  $q$ . Figure 5.4 shows a constellation where the nearest neighbor approach fails. Checking if a point  $q$  is



**Figure 5.4.** The point  $s$  is the nearest neighbor to  $q$ . However, searching all incident triangles to  $s$ , point  $q$  cannot be located. A similar example can be constructed in 3D.

within some tet  $\mathcal{T} = [a, b, c, d]$  can be done with the orientation predicate, i.e.

$$q \in \mathcal{T} \iff \begin{aligned} & \text{sameSign}(\text{orientation}(a, b, c, d), \text{orientation}(a, b, c, q)) \&\& \\ & \text{sameSign}(\text{orientation}(b, c, d, a), \text{orientation}(b, c, d, q)) \&\& \\ & \text{sameSign}(\text{orientation}(c, d, a, b), \text{orientation}(c, d, a, q)) \&\& \\ & \text{sameSign}(\text{orientation}(d, a, b, c), \text{orientation}(d, a, b, q)) \end{aligned}$$

By checking for each plane of the tet, we check if the point  $q$  is on the same side of the plane (characterised by the same orientation sign) as the remaining vertex.

### 5.2.2 Valid state and initialization

To walk in the tet mesh, the algorithm needs to be in a state where everything is as expected. In 2D, we define this valid state as

- the line  $\vec{qp}$  intersects the edge defined by vertices  $\vec{lr}$
- vertex  $l$  lies on the left of the line defined by  $\vec{qp}$
- vertex  $r$  lies on the right of the line defined by  $\vec{qp}$

If the algorithm starts with these configurations, then in each step the valid configuration will be preserved and all edges intersected are traversed. Analogously in 3D, we define the valid state such that

- the line  $\vec{qp}$  intersects the triangle defined by vertices  $\triangle uvw$
- vertices of the triangle  $\triangle uvw$  are ordered in the way that

$$\text{orientation}(w, v, q, p) > 0, \text{orientation}(v, u, q, p) > 0, \text{orientation}(u, w, q, p) > 0$$

Again, by starting in this configuration, all triangles  $\triangle uvw$  traversed get intersected by  $\vec{qp}$ . Then, we find each intersection point with the ray, add the intersection points to a list with the current tet  $t$  and finally use this as input for calculating the rotation coefficient as described in chapter 4. To get into a valid state, an initialization needs to be performed. In *Walking in a triangulation*[2], they assume that  $q$  is a vertex in the tet mesh. Thus, their approach of turning around  $q$  is not applicable in our case.

We solve this problem with a brute-force approach. First, we find which tet  $t$  contains  $q$  as described in 5.2.1. From there, we go through the four faces of the tet and check which one gets intersected by the ray  $\vec{qp}$ . When the face which gets intersected was found, we check the 6 permutations of the three vertices  $(u, v, w)$  until the configuration is found, where the orientations are positive.

### 5.2.3 Robust ray-triangle intersection

[14]An algorithm for determining intersection segment-polygon in 3D

---

**Algorithm 3** tetFinder

---

```
1: Input  $(q, p)$ 
2:    $t = \text{LocateTet}(q)$ 
3:    $\text{initialization}(t)$ 

4:   //  $qp$  intersects triangle  $uvw$ 
5:   //  $wvqp, vuqp, uwqp$  are positively oriented
6:    $\text{LINESEGMENTS} = []$ 
7:    $\text{PREV} = q$ 
8:    $\text{CURR} = \text{intersection}(qp, uvw)$ 

9:    $\text{LINESEGMENTS.add}([t, \text{PREV}, \text{CURR}])$ 

10:  while  $\text{orientation}(u, w, v, p) > 0$  {
11:     $t = \text{neighbor}(t \text{ through } uvw)$ 
12:     $s = \text{vertex of } t, s \neq u, s \neq v, s \neq w$ 
13:     $\text{PREV} = \text{CURR}$ 
14:    if  $\text{orientation}(u, s, q, p) > 0$  //  $qp$  does not intersect triangle  $usw$ 
15:      if  $\text{orientation}(v, s, q, p) > 0$  //  $qp$  intersects triangle  $vsw$ 
16:         $u = s$ 
17:      else //  $qp$  intersects triangle  $usv$ 
18:         $w = s$ 
19:    else //  $qp$  does not intersect  $usv$ 
20:      if  $\text{orientation}(w, s, q, p) > 0$  //  $qp$  intersects triangle  $usw$ 
21:         $v = s$ 
22:      else //  $qp$  intersects triangle  $vsw$ 
23:         $u = s$ 
24:     $\text{CURR} = \text{intersection}(qp, uvw)$ 
25:     $\text{LINESEGMENTS.add}([t, \text{PREV}, \text{CURR}])$ 
26:  } //  $t$  contains  $p$ 
27:   $\text{LINESEGMENTS.add}([t, \text{PREV}, p])$ 
```

---

### 5.2.4 Robustness

The algorithm uses several subroutines which are described here:

- `neighbor(t through uvw)` returns the tetrahedron sharing face  $uvw$  with tetrahedron  $t$
- `vertex of  $t, s \neq u, s \neq v, s \neq w$`  returns the remaining vertex of a tetrahedron whose other three vertices are known
- `initialization()`

## Chapter 6

# Frame Field optimization

- Problem statement
- Simone algorithm from a high level
- Caching of coefficients

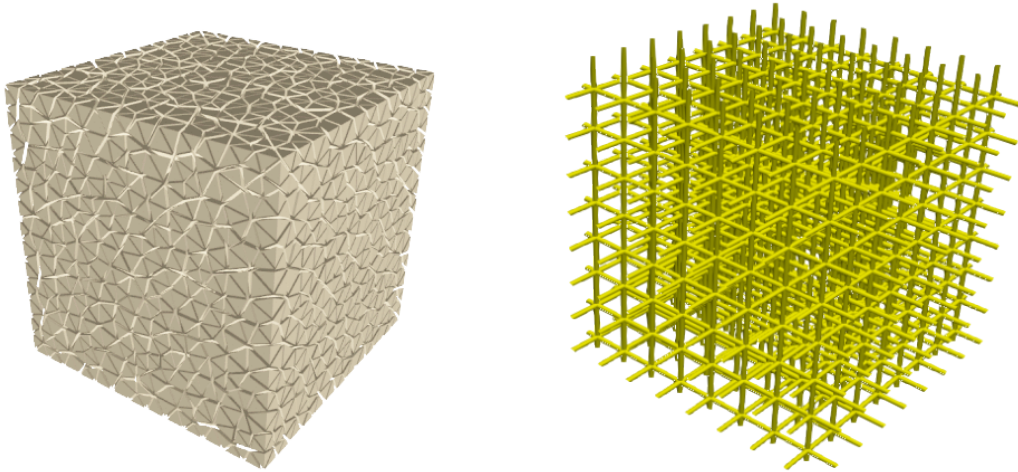


## Chapter 7

# Experiments

In this section, the frame field generation applied to a cube mesh with some different metrics is discussed. *Setup:* We store the associated matrices of the metric field at each vertex of the tet mesh. The cube has corners at  $(0, 0, 0)$  and  $(1, 1, 1)$ . The frame field optimization is run to depth 3.

To start, we apply the constant metric  $g^{1/2} = \text{diag}(1, 1, 1)$  everywhere. The rotation coefficients  $R$  are the identity matrix everywhere, as the metric does not twist or squish the space within the field. The result is the boundary aligned frame field with no singularities, as the energy can be minimised to zero if the frames are constant. Figure 7.1 shows the mesh used to store the metric and the resulting constant frame field.



**Figure 7.1.** Left: The mesh used to store the metric field. Right: Constant metric everywhere which gives the boundary aligned constant frame field with no singularities.

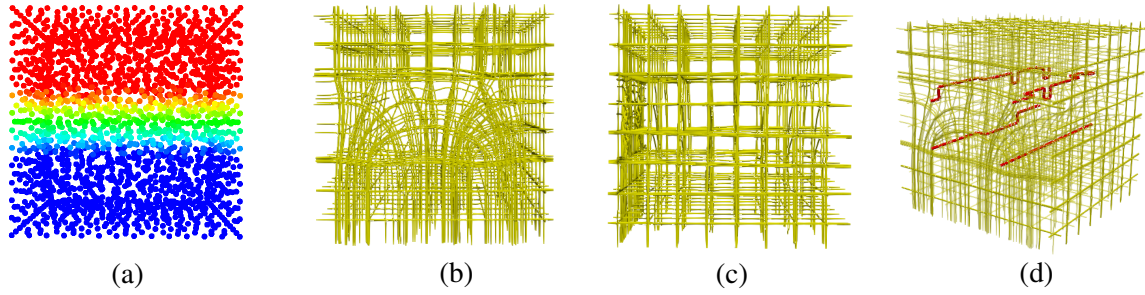
To further validate the frame field generation, we divide along the cube along the  $z$ -axis into three equally sized parts. The function to attach the metric to the vertices is then defined as

$$g^{1/2}(z) = \begin{cases} \text{diag}(1, 1, 1) & 0 < z < 1/3 \\ \text{diag}(27z - 8, 1, 27z - 8) & 1/3 < z < 2/3 \\ \text{diag}(10, 1, 10) & 2/3 < z < 1 \end{cases}$$

The metric is constant in the  $y$ -axis and constant-linear-constant in the  $x, z$ -axis. Thus, the rotation coefficients are only of the form

$$\begin{pmatrix} \cos(\alpha) & 0 & \sin(\alpha) \\ 0 & 1 & 0 \\ -\sin(\alpha) & 0 & \cos(\alpha) \end{pmatrix}$$

for some angle  $\alpha$ . The result is depicted in figure 7.2.



**Figure 7.2.** (a) The varying  $g_{00}^{1/2}$  component along the  $z$ -axis. Visible is how the streamlines of the frame field only change in the middle third (b) and how the frames do not change along the  $y$ -axis (c). The singularities (d) are only points when taking a slice along the  $y$ -axis.

- As a sanity check, we start with the constant metric everywhere.  
constant metric everywhere -> no singularities
- linearly increasing metric in  $z$ -axis, isotropic scaling
- 2d example anisotropic scaling, constant-linear-constant
- larger cubes at the edges of the cube, isotropic scaling



## **Chapter 8**

# **Conclusion**

The conclusion looks back at the entire work, gives a critical look, summarizes, and discusses extensions and future work.



## **Appendix A**

### **Extra material**

Extra material may be placed in an appendix that appears after the conclusion.



# Bibliography

- [1] D. Bommes, M. Campen, H.-C. Ebke, P. Alliez, and L. Kobbelt, “Integer-grid maps for reliable quad meshing,” *ACM Trans. Graph.*, vol. 32, jul 2013.
- [2] O. Devillers, S. Pion, M. Teillaud, T. Logiciel, and P. Prisme, “Walking in a triangulation,” *International Journal of Foundations of Computer Science*, 03 2001.
- [3] X. Fang, J. Huang, Y. Tong, and H. Bao, “Metric-driven 3d frame field generation,” *IEEE Transactions on Visualization and Computer Graphics*, vol. 29, no. 4, pp. 1964–1976, 2023.
- [4] J. Huang, Y. Tong, H. Wei, and H. Bao, “Boundary aligned smooth 3d cross-frame field,” *ACM Trans. Graph.*, vol. 30, pp. 1–8, dec 2011.
- [5] J. M. Lee, *Riemannian manifolds : : an introduction to curvature*. Graduate Texts in Mathematics ; 176, New York: Springer, 1st ed. 1997. ed., 1997.
- [6] J. M. Lee, *Introduction to Smooth Manifolds*. Springer, 2000.
- [7] H. Liu, P. Zhang, E. Chien, J. Solomon, and D. Bommes, “Singularity-constrained octahedral fields for hexahedral meshing,” *ACM Trans. Graph.*, vol. 37, jul 2018.
- [8] M. Nieser, U. Reitebuch, and K. Polthier, “Cubecover- parameterization of 3d volumes,” *Computer Graphics Forum*, vol. 30, no. 5, pp. 1397–1406, 2011.
- [9] D. Panozzo, E. Puppo, M. Tarini, and O. Sorkine-Hornung, “Frame fields: Anisotropic and non-orthogonal cross fields,” *ACM Trans. Graph.*, vol. 33, jul 2014.
- [10] C. Papachristou, *Aspects of Integrability of Differential Systems and Fields: A Mathematical Primer for Physicists*. Springer, 01 2020.
- [11] N. Pietroni, M. Campen, A. Sheffer, G. Cherchi, D. Bommes, X. Gao, R. Scateni, F. Ledoux, J. Remacle, and M. Livesu, “Hex-mesh generation and processing: A survey,” *ACM Trans. Graph.*, vol. 42, oct 2022.
- [12] N. Ray and D. Sokolov, “On smooth 3d frame field design,” *CoRR*, vol. abs/1507.03351, 2015.
- [13] J. Richard Shewchuk, “Adaptive precision floating-point arithmetic and fast robust geometric predicates,” *Discrete & Computational Geometry*, vol. 18, pp. 305–363, Oct 1997.
- [14] R. J. Segura and F. R. Feito, “An algorithm for determining intersection segment-polygon in 3d,” *Computers & Graphics*, vol. 22, no. 5, pp. 587–592, 1998.
- [15] M. Teichmann, “Robust predicates and degeneracy,” 2009.
- [16] A. Vaxman, M. Campen, O. Diamanti, D. Bommes, K. Hildebrandt, M. B.-C. Technion, and D. Panozzo, “Directional field synthesis, design, and processing,” in *ACM SIGGRAPH 2017 Courses*, SIGGRAPH ’17, (New York, NY, USA), Association for Computing Machinery, 2017.



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*Erklärung gemäss Art. 30 RSL Phil.-nat. 18*

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