



^b
**UNIVERSITÄT
BERN**

3D Metric Fields

A Novel Approach to a New Idea

Bachelor Thesis

Florin Achermann
from
Bern, Switzerland

Faculty of Science, University of Bern

15. September 2023

Prof. David Bommes
Denis Kalmykov
Computer Graphics Group
Institute of Computer Science
University of Bern, Switzerland

Abstract

Lorem ipsum dolor sit amet, consectetur adipiscing elit, sed do eiusmod tempor incididunt ut labore et dolore magna aliqua. Ut enim ad minim veniam, quis nostrud exercitation ullamco laboris nisi ut aliquip ex ea commodo consequat. Duis aute irure dolor in reprehenderit in voluptate velit esse cillum dolore eu fugiat nulla pariatur. Excepteur sint occaecat cupidatat non proident, sunt in culpa qui officia deserunt mollit anim id est laborum.

Contents

1	Introduction	1
2	Mathematical Background	3
3	Connection One-Form ω	5
4	Discretized connection evaluation	9
5	Algorithm for R between two arbitrary points in a mesh	11
6	Conclusion	13
A	Extra material	15

Chapter 1

Introduction

- What are Frame Fields?
- Why are they important?
- How do *we* generate them?

A frame field

A vector field is locally integrable, if and only if $\nabla \times A = 0$. [2]

Chapter 2

Mathematical Background

This chapter sets the stage and introduces already existing material.

A frame F is a set of 6 vectors $\{\pm F_0, \pm F_1, \pm F_2\}$. We can represent such a frame F as a 3×3 matrix F , where the i th-column is F_i . A frame field then maps to every point in 3D-space such a frame, i.e. $F : \mathbb{R}^3 \rightarrow \mathbb{R}^{3 \times 3}$. Usually, we work on a 3-manifold \mathcal{M} and a positively oriented frame field, i.e. $F|_{\mathcal{M}} : \mathcal{M} \rightarrow \mathbb{R}^{3 \times 3}$, where $\det(F) > 0$. To allow for anisotropic, nonuniform meshes, we generalize orthonormality of frames to g -orthonormal frames. Orthonormality is measured in some metric g , and a frame F satisfies the condition $\langle F_i, F_j \rangle_g = \delta_{ij}$. Any frame field with $\det(F) > 0$ naturally defines a metric $g = (FF^\top)^{-1}$, where F is g -orthonormal

$$F^\top g F = Id.$$

We can factor the frame field F into a symmetric part $g^{1/2}$ and a rotational part R

$$F = g^{-1/2} R$$

The symmetric part $g^{-1/2}$ keeps F g -orthonormal

$$\implies F^\top g F = (g^{-1/2} R)^\top g g^{-1/2} R = R^\top g^{-1/2} g g^{-1/2} R = Id.$$

and R represents a rotational field $R : \mathcal{M} \rightarrow SO(3)$. The requirements for our frame field are:

- Smoothness
- Integrability
- Metric consistency: $g = (FF^\top)^{-1}$

Chapter 3

Connection One-Form ω

A vector field U is integrable, if and only if $\nabla \times U = 0$, which means the vector field has vanishing curl everywhere. We can express this more naturally with the language of differential forms: The curl can be written as the exterior derivative d of a one-form α . A one-form (more generally, a differential form) is closed, if $d\alpha = 0$. Therefore, the local integrability can be expressed as the closedness of a one-form. We want F^{-1} (TODO: why F^{-1}) to be integrable. To achieve local integrability for, it suffices to make R locally integrable. We can think of a rotation field R as the composition of 3 vector fields

$$R = \begin{bmatrix} | & | & | \\ R_1 & R_2 & R_3 \\ | & | & | \end{bmatrix}$$

where $R_i : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ is a vector field. We can therefore construct a vector-valued one-form, given $p = (x, y, z)^\top$ in Euclidean coordinates

$$\alpha \triangleq F^{-1}dp = R^\top g^{1/2}dp$$

where $dp = (dx, dy, dz)^\top$ is the common orthonormal one-form basis of $\Omega^1(\mathcal{M})$.

$$R \text{ locally integrable} \iff \mathbf{0} = d\alpha$$

Some reformulations yield:

$$\begin{aligned} \mathbf{0} = d\alpha &= d(R^\top g^{1/2}dp) \stackrel{(1)}{=} dR^\top \wedge (g^{1/2}dp) + R^\top d(g^{1/2}dp) \\ &= R^\top (\omega \wedge (g^{1/2}dp) + d(g^{1/2}dp)) \end{aligned}$$

where for (1), the Leibnitz Rule for the exterior derivative is applied, and we define

$$\omega = RdR^\top \in \mathfrak{so}(3).$$

Remark. ω is an antisymmetric matrix-valued one-form (every element is a one-form).

Proof. We differentiate the orthogonality condition of the rotation matrix R (taking the derivative w.r.t. every element):

$$\begin{aligned} Id &= RR^\top \\ d(Id) &= d(RR^\top) \\ \mathbf{0} &= dRR^\top + RdR^\top \\ \mathbf{0} &= (RdR^\top)^\top + RdR^\top \\ -(RdR^\top)^\top &= RdR^\top \end{aligned}$$

□

Elements of the Lie algebra $\mathfrak{so}(3)$ can be thought as infinitesimal rotations and ω can be used as a connection one-form. For our purposes, it suffices to know that elements are antisymmetric matrices. To make R locally integrable, we find ω such that the one-form α is closed ($d\alpha = 0$, curl-free), then try to match the ω with R , which can be expressed as

$$\min_{R \in SO(3)} \|RdR^T - \omega\|^2.$$

Connection evaluation To find ω , we use the above equation

$$\mathbf{0} = R^\top (\omega \wedge (g^{1/2} dp) + d(g^{1/2} dp)) \iff \mathbf{0} = \omega \wedge (g^{1/2} dp) + d(g^{1/2} dp)$$

and reformulate into a linear system. We represent the antisymmetric matrix-valued one-form ω

$$\omega = \begin{bmatrix} 0 & \omega_{12} & -\omega_{31} \\ -\omega_{12} & 0 & \omega_{23} \\ \omega_{31} & -\omega_{23} & 0 \end{bmatrix}$$

by $\begin{bmatrix} \omega_{23} & \omega_{31} & \omega_{12} \end{bmatrix} = \begin{bmatrix} dx & dy & dz \end{bmatrix} W$. We write $W = [W_1, W_2, W_3]$, $W_i \in \mathbb{R}^3$. That is, W is the matrix with the coefficients for the one-forms, e.g. $W_1 = [(\omega^{23})_1, (\omega^{23})_2, (\omega^{23})_3]^\top$. Recall, a one-form can be expressed as

$$\omega_{ij} = (\omega^{ij})_1 dx + (\omega^{ij})_2 dy + (\omega^{ij})_3 dz$$

So e.g.

$$\omega_{23} = \begin{bmatrix} dx & dy & dz \end{bmatrix} W_1 = (\omega^{23})_1 dx + (\omega^{23})_2 dy + (\omega^{23})_3 dz$$

We write $A = g^{1/2} = [A^1, A^2, A^3]$. We use the equation $\mathbf{0} = \omega \wedge (g^{1/2} dp) + d(g^{1/2} dp)$. First part:

$$\begin{aligned} \omega \wedge g^{1/2} dp &= \begin{bmatrix} 0 & \omega_{12} & -\omega_{31} \\ -\omega_{12} & 0 & \omega_{23} \\ \omega_{31} & -\omega_{23} & 0 \end{bmatrix} \wedge \begin{bmatrix} A_1^1 & A_1^2 & A_1^3 \\ A_2^1 & A_2^2 & A_2^3 \\ A_3^1 & A_3^2 & A_3^3 \end{bmatrix} \begin{bmatrix} dx \\ dy \\ dz \end{bmatrix} \\ &= \begin{bmatrix} +\omega_{12} \wedge (A_2^1 dx + A_2^2 dy + A_2^3 dz) - \omega_{31} \wedge (A_3^1 dx + A_3^2 dy + A_3^3 dz) \\ -\omega_{12} \wedge (A_1^1 dx + A_1^2 dy + A_1^3 dz) + \omega_{23} \wedge (A_3^1 dx + A_3^2 dy + A_3^3 dz) \\ +\omega_{31} \wedge (A_1^1 dx + A_1^2 dy + A_1^3 dz) - \omega_{23} \wedge (A_2^1 dx + A_2^2 dy + A_2^3 dz) \end{bmatrix} \end{aligned}$$

It will get really messy if we calculate each component here, so let us calculate one component separately first:

$$\begin{aligned} \omega_{ij} \wedge (A_k^1 dx + A_k^2 dy + A_k^3 dz) &= ((\omega^{ij})_1 dx + (\omega^{ij})_2 dy + (\omega^{ij})_3 dz) \wedge (A_k^1 dx + A_k^2 dy + A_k^3 dz) \\ &= (\omega^{ij})_1 A_k^2 dx \wedge dy + (\omega^{ij})_1 A_k^3 dx \wedge dz \\ &\quad + (\omega^{ij})_2 A_k^1 dy \wedge dx + (\omega^{ij})_2 A_k^3 dy \wedge dz \\ &\quad + (\omega^{ij})_3 A_k^1 dz \wedge dx + (\omega^{ij})_3 A_k^2 dz \wedge dy \\ &= ((\omega^{ij})_1 A_k^2 - (\omega^{ij})_2 A_k^1) dx \wedge dy \\ &\quad + ((\omega^{ij})_2 A_k^3 - (\omega^{ij})_3 A_k^2) dy \wedge dz \\ &\quad + ((\omega^{ij})_3 A_k^1 - (\omega^{ij})_1 A_k^3) dz \wedge dx \end{aligned}$$

where we use the fact that $dx \wedge dx = 0$ and $dx \wedge dy = -dy \wedge dx$. We can clean up the above expression using the cross product:

$$\begin{bmatrix} ((\omega^{ij})_2 A_k^3 - (\omega^{ij})_3 A_k^2) \\ ((\omega^{ij})_3 A_k^1 - (\omega^{ij})_1 A_k^3) \\ ((\omega^{ij})_1 A_k^2 - (\omega^{ij})_2 A_k^1) \end{bmatrix}^\top \begin{bmatrix} dy \wedge dz \\ dz \wedge dx \\ dx \wedge dy \end{bmatrix} = [W_i \times A^k]^\top \begin{bmatrix} dy \wedge dz \\ dz \wedge dx \\ dx \wedge dy \end{bmatrix}$$

We use the fact that $[A_1, A_2, A_3] = [A^1, A^2, A^3]$ because A is symmetric, and W_1 corresponds to ω_{23} , W_2 to ω_{31} and W_3 to ω_{12} . Second part:

$$d(g^{1/2}dp) = d(Adp) = d\left(\begin{bmatrix} A_1^1 & A_1^2 & A_1^3 \\ A_2^1 & A_2^2 & A_2^3 \\ A_3^1 & A_3^2 & A_3^3 \end{bmatrix} \begin{bmatrix} dx \\ dy \\ dz \end{bmatrix}\right)$$

Again, we can do this separately for each row:

$$\begin{aligned} & d(A_k^1 dx + A_k^2 dy + A_k^3 dz) \\ &= dA_k^1 \wedge dx + dA_k^2 \wedge dy + dA_k^3 \wedge dz \\ &= \frac{\partial A_k^1}{\partial x} dx \wedge dx + \frac{\partial A_k^1}{\partial y} dy \wedge dx + \frac{\partial A_k^1}{\partial z} dz \wedge dx \\ &+ \frac{\partial A_k^2}{\partial x} dx \wedge dy + \frac{\partial A_k^2}{\partial y} dy \wedge dy + \frac{\partial A_k^2}{\partial z} dz \wedge dy \\ &+ \frac{\partial A_k^3}{\partial x} dx \wedge dz + \frac{\partial A_k^3}{\partial y} dy \wedge dz + \frac{\partial A_k^3}{\partial z} dz \wedge dz \\ &= \left(\frac{\partial A_k^3}{\partial y} - \frac{\partial A_k^2}{\partial z}\right) dy \wedge dz + \left(\frac{\partial A_k^1}{\partial z} - \frac{\partial A_k^3}{\partial x}\right) dz \wedge dx + \left(\frac{\partial A_k^2}{\partial x} - \frac{\partial A_k^1}{\partial y}\right) dx \wedge dy \\ &= (\nabla \times A_k)^\top \begin{bmatrix} dy \wedge dz \\ dz \wedge dx \\ dx \wedge dy \end{bmatrix} \end{aligned}$$

Finally, we can put everything together:

$$\begin{aligned} \mathbf{0} &= \begin{bmatrix} (W_3 \times A^2 - W_2 \times A^3 + \nabla \times A^1)^\top \\ (W_1 \times A^3 - W_3 \times A^1 + \nabla \times A^2)^\top \\ (W_2 \times A^1 - W_1 \times A^2 + \nabla \times A^3)^\top \end{bmatrix} \begin{bmatrix} dy \wedge dz \\ dz \wedge dx \\ dx \wedge dy \end{bmatrix} \\ &\iff \begin{bmatrix} (W_2 \times A^3 - W_3 \times A^2)^\top \\ (W_3 \times A^1 - W_1 \times A^3)^\top \\ (W_1 \times A^2 - W_2 \times A^1)^\top \end{bmatrix} \begin{bmatrix} dy \wedge dz \\ dz \wedge dx \\ dx \wedge dy \end{bmatrix} = \begin{bmatrix} (\nabla \times A^1)^\top \\ (\nabla \times A^2)^\top \\ (\nabla \times A^3)^\top \end{bmatrix} \begin{bmatrix} dy \wedge dz \\ dz \wedge dx \\ dx \wedge dy \end{bmatrix} \end{aligned}$$

Take the curl to the other side and switch order on the left-hand side to cancel the -1 . As we are only interested in the 9 components of W , we omit the two-form basis and transform into a 9x9 linear system for W . We define A_\times and $\text{vec}(\cdot)$ as

$$A_\times = \begin{bmatrix} 0 & -A_\times^3 & A_\times^2 \\ A_\times^3 & 0 & -A_\times^1 \\ -A_\times^2 & A_\times^1 & 0 \end{bmatrix}, \text{vec}(W) = \begin{bmatrix} W_1 \\ W_2 \\ W_3 \end{bmatrix}$$

with A_\times^i defined as

$$A_\times^i = [A_1^i \quad A_2^i \quad A_3^i]_\times = \begin{bmatrix} 0 & -A_3^i & A_2^i \\ A_3^i & 0 & -A_1^i \\ -A_2^i & A_1^i & 0 \end{bmatrix}$$

and $\text{vec}(\cdot)$ turns a 3x3-matrix into a 9x1-vector by stacking the columns. With these two definitions, we can transform the above equality into a linear system

$$A_\times \text{vec}(W) = \text{vec}(\nabla \times A)$$

where $\nabla \times A$ is just the curl applied to each column. This transformation can be checked by laboriously plugging in the definitions and comparing the coefficients. With tedious calculations, one can show that $\det(A_\times) = -2 \det(A)^3 = -2 \det(g)^{3/2} < 0$, which means this is a linear system that can be used to solve for W at a point.

Chapter 4

Discretized connection evaluation

- Calculation of R
- Piecewise linear discretization

Chapter 5

Algorithm for R between two arbitrary points in a mesh

[1] Here is an example for how to specify an algorithm in pseudo-code.

Algorithm 1 Byzantine Leader-Based Epoch-Change (process p_i).

```
1: State
2:    $lastts \leftarrow 0$ : most recently started epoch
3:    $nextts \leftarrow 0$ : timestamp of the next epoch
4:    $newepoch \leftarrow [\perp]^n$ : list of NEWEPOCH messages

5: upon event  $complain(p_\ell)$  such that  $p_\ell = leader(lastts)$  do
6:   if  $nextts = lastts$  then
7:      $nextts \leftarrow lastts + 1$ 
8:     send message  $[NEWPOCH, nextts]$  to all  $p_j \in \mathcal{P}$ 

9: upon receiving a message  $[NEWPOCH, ts]$  from  $p_j$  such that  $ts = lastts + 1$  do
10:    $newepoch[j] \leftarrow NEWPOCH$ 

11: upon exists  $ts$  such that  $\{p_j \in \mathcal{P} \mid newepoch[j] = ts\} \in \mathcal{K}_i$  and  $nextts = lastts$  do
12:    $nextts \leftarrow lastts + 1$ 
13:   send message  $[NEWPOCH, nextts]$  to all  $p_j \in \mathcal{P}$ 

14: upon exists  $ts$  such that  $\{p_j \in \mathcal{P} \mid newepoch[j] = ts\} \in \mathcal{Q}_i$  and  $nextts > lastts$  do
15:    $lastts \leftarrow nextts$ 
16:    $newepoch \leftarrow [\perp]^n$ 
17:   output  $startepoch(lastts, leader(lastts))$ 
```

Chapter 6

Conclusion

The conclusion looks back at the entire work, gives a critical look, summarizes, and discusses extensions and future work.

Appendix A

Extra material

Extra material may be placed in an appendix that appears after the conclusion.

Bibliography

- [1] X. Fang, J. Huang, Y. Tong, and H. Bao, “Metric-driven 3d frame field generation,” *IEEE Transactions on Visualization and Computer Graphics*, vol. 29, no. 4, pp. 1964–1976, 2023.
- [2] C. Papachristou, *Aspects of Integrability of Differential Systems and Fields: A Mathematical Primer for Physicists*. 01 2020.

Erklärung

Erklärung gemäss Art. 30 RSL Phil.-nat. 18

Ich erkläre hiermit, dass ich diese Arbeit selbstständig verfasst und keine anderen als die angegebenen Quellen benutzt habe. Alle Stellen, die wörtlich oder sinngemäss aus Quellen entnommen wurden, habe ich als solche gekennzeichnet. Mir ist bekannt, dass andernfalls der Senat gemäss Artikel 36 Absatz 1 Buchstabe r des Gesetzes vom 5. September 1996 über die Universität zum Entzug des auf Grund dieser Arbeit verliehenen Titels berechtigt ist.

Für die Zwecke der Begutachtung und der Überprüfung der Einhaltung der Selbständigkeitserklärung bzw. der Reglemente betreffend Plagiate erteile ich der Universität Bern das Recht, die dazu erforderlichen Personendaten zu bearbeiten und Nutzungshandlungen vorzunehmen, insbesondere die schriftliche Arbeit zu vervielfältigen und dauerhaft in einer Datenbank zu speichern sowie diese zur Überprüfung von Arbeiten Dritter zu verwenden oder hierzu zur Verfügung zu stellen.

Ort/Datum

Unterschrift