3D Metric Fields

Optimizing Frame Fields in a new Metric

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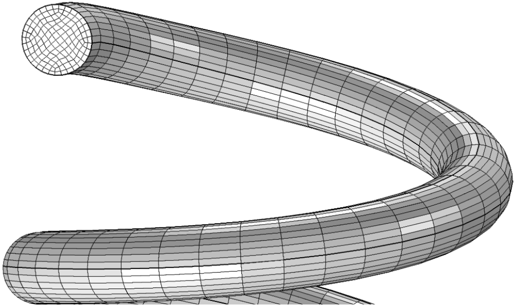
Presentation Overview

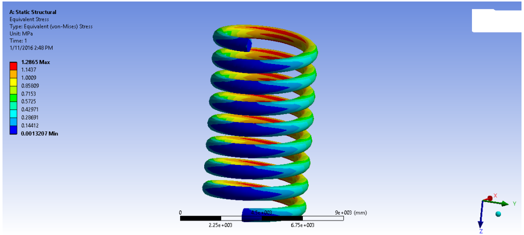
1. Problem Statement
   - Hexahedral Meshing
   - Integer-Grid Maps
   - Frame Fields
2. Frame Field Control
   - Metric Fields
   - Measuring Smoothness in new Metric
3. Discretization

Representation of geometry

Geometry needs to be represented in computers for applications



(a) Hexahedral mesh of spring ©COMSOL



(b) Stress simulation on spring ©ANSYS

Meshes out of hexahedral elements are prefered in practice

Integer-Grid Maps

Key Idea:
Search for map $\phi : \mathcal{M} \subset \mathbb{R}^3 \to \mathbb{R}^3$ which maps the object to a 3D grid

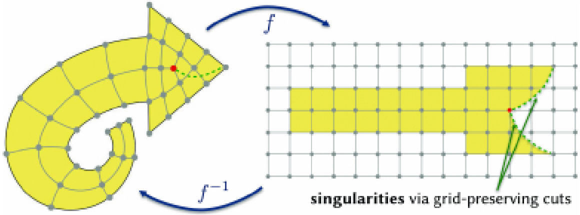


Figure: Integer-Grid map [Pietroni et al., 2022]

Direct search is infeasible $\to$ hard mixed-integer non-convex optimization problem

Search for Integer-Grid Maps

We want:

$$\phi : \mathcal{M} \to \mathbb{R}^3$$

Idea: Take the Jacobian

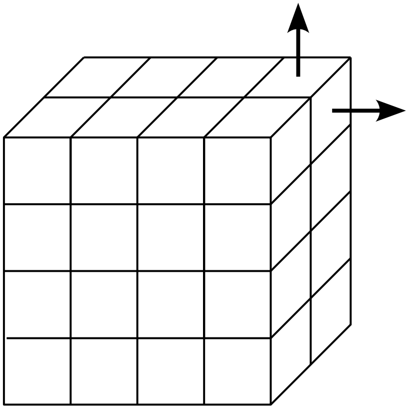
$$\nabla\phi : \mathcal{M} \to \mathbb{R}^{3\times 3}$$

and search for an approximation $F \approx \nabla\phi$. We call $F : \mathcal{M} \to \mathbb{R}^{3\times 3}$ a frame field. A frame locally represents the deformed edges of a cube.

Frame Fields

Think of F as three vector fields $F_i : \mathbb{R}^3 \to \mathbb{R}^3$

$$F = \begin{pmatrix} | & | & | \\ F_1 & F_2 & F_3 \\ | & | & | \end{pmatrix}$$



Goals for our frame field:

- Smoothness
- Boundary Alignment: One column should match with surface normal

Why?

Frame Fields

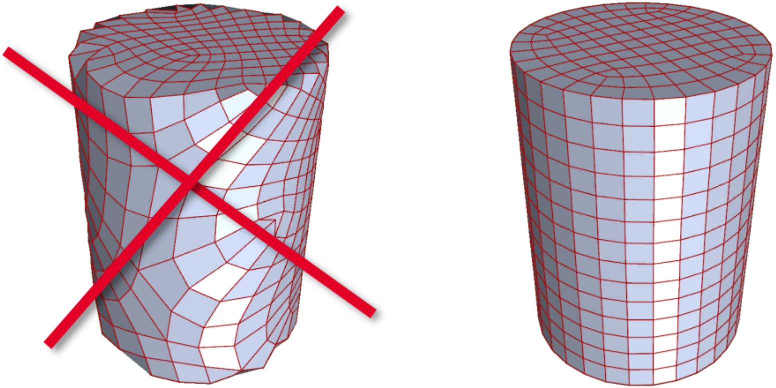


Figure: Left: Missing Boundary Alignment, not smooth. Right: Smooth + Boundary Aligned

Optimization Approach

- Goal: Smooth Frame Field + Boundary Aligned
- How to measure smoothness?

Dirichlet Energy

$$E(F) = \int_{\mathcal{M}} ||\nabla F||^2 \tag{1}$$

Controlling frames

- For orthonormal frame field $F^T F = \mathrm{Id}$, extracted elements are unit cubes
- Add metric field g for control of frames
- g-orthonormality $F^T g F = \mathrm{Id}$

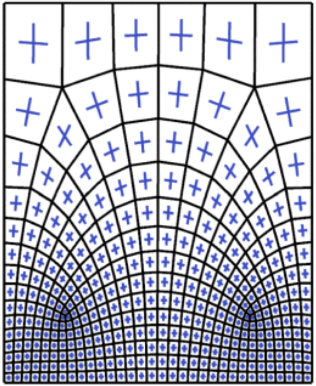


Figure: Under new metric g, lengths at the top are the same lengths as at the bottom [Fang et al., 2023]

Space deformation

- Metric g measures deformation of space
- Cannot use euclidean smoothness measure
- g defines infinitesimal rotation ω

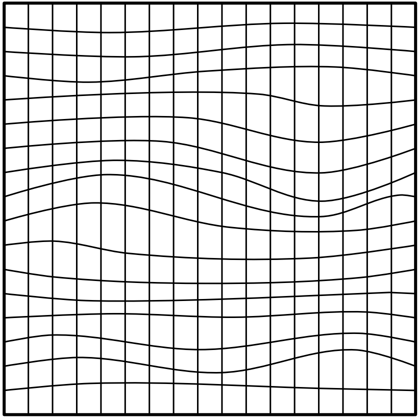


Figure: Metric g deforms the space, “straight” in g is not straight in euclidean metric

Measuring Smoothness in new Metric

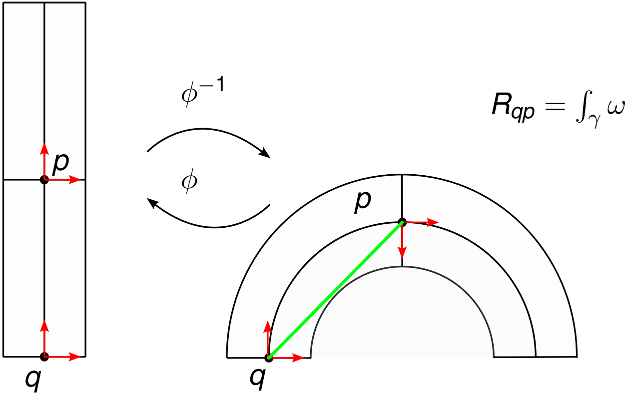
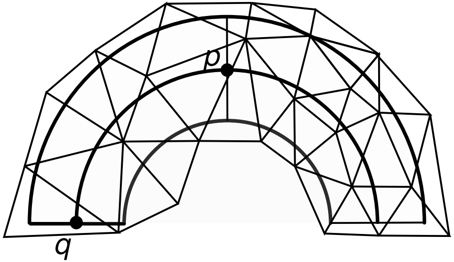
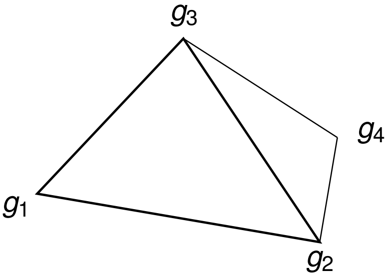


Figure: Under the metric induced by ϕ, the frame at q is parallel to p. To compare them, we need to recover how p is rotated under ω.

Discretizing metric field



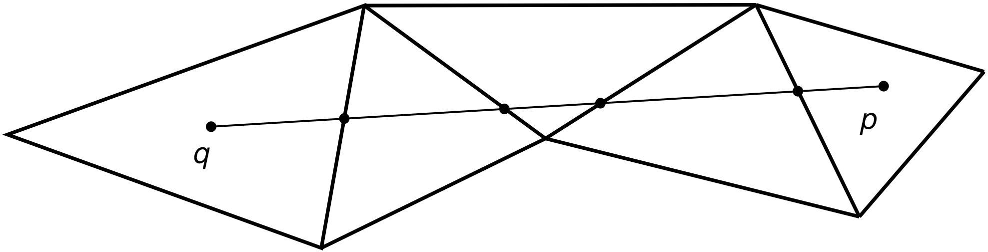


We cover the object with a tetrahedral mesh and store the metrics at the vertices. Within the tet, we linearly interpolate with barycentric coordinates.

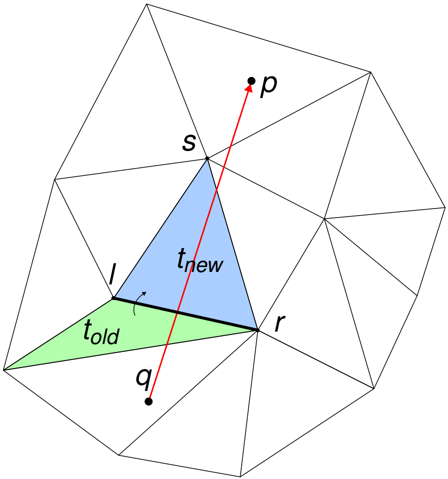
$g = \alpha g_1 + \beta g_2 + \gamma + g_3 + \delta g_4$
with $\alpha + \beta + \gamma + \delta = 1$ and
$\alpha, \beta, \gamma, \delta \geq 0$

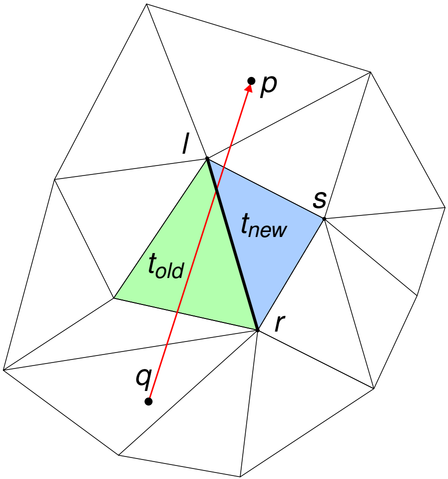
Walking on a triangulation

- Problem: When integrating from q to p, need to use appropriate metric field.
- Need to find all tets that the line segment from q to p lies in to do integration in correct field



Walking on a triangulation





Putting all together

1. Store metric field at vertices
2. Minimize discretized Dirichlet energy with minimizer

$$E(F) = \int_{\mathcal{M}} ||\nabla F||^2 \longrightarrow E(F) = \sum_{i,j} ||F_i - F_j||^2 \tag{2}$$

3. modify with rotation coefficient to align in new metric

$$E(F) = \sum_{q,p} ||[R_{qp}]F_q - F_p||^2 \tag{3}$$

Experiments

$$g = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

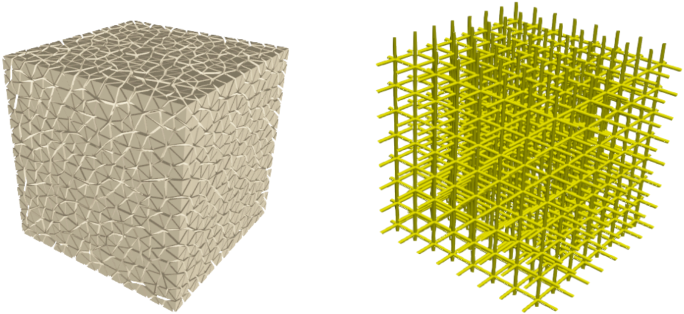
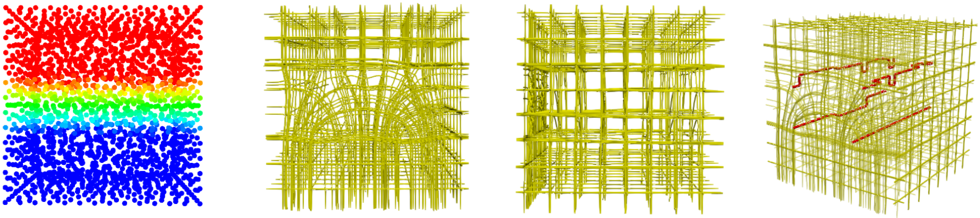


Figure: Constant metric everywhere, constant frame field

Experiments

$$g(z) = \begin{cases} \mathrm{diag}(1,1,1) & 0 < z < 1/3 \\ \mathrm{diag}(27z-8, 1, 27z-8) & 1/3 < z < 2/3 \\ \mathrm{diag}(10,1,10) & 2/3 < z < 1 \end{cases} \tag{4}$$



(a) (b) (c) (d)

Figure: Constant-linear-constant metric

References

Pietroni et al. (2022)
Hex-Mesh Generation and Processing: A Survey
ACM Transaction on Graphics

Fang et al. (2023)
Metric-Driven 3D Frame Field Generation
IEEE Transactions on Visualization and Computer Graphics.

The End

Questions? Comments?