

# Chapter 1

## Connection One-Form $\omega$

A vector field  $U$  is integrable, if and only if  $\nabla \times U = 0$ , which means the vector field has vanishing curl everywhere. We can express this more naturally with the language of differential forms: The curl can be written as the exterior derivative  $d$  of a one-form  $\alpha$ . A one-form (more generally, a differential form) is closed, if  $d\alpha = 0$ . Therefore, the local integrability can be expressed as the closedness of a one-form. We want  $F^{-1}$  (TODO: why  $F^{-1}$ ) to be integrable. To achieve local integrability for, it suffices to make  $R$  locally integrable. We can think of a rotation field  $R$  as the composition of 3 vector fields

$$R = \begin{bmatrix} | & | & | \\ R_1 & R_2 & R_3 \\ | & | & | \end{bmatrix}$$

where  $R_i : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  is a vector field. We can therefore construct a vector-valued one-form, given  $p = (x, y, z)^\top$  in Euclidean coordinates

$$\alpha \triangleq F^{-1}dp = R^\top g^{1/2}dp$$

where  $dp = (dx, dy, dz)^\top$  is the common orthonormal one-form basis of  $\Omega^1(\mathcal{M})$ .

$$R \text{ locally integrable} \iff \mathbf{0} = d\alpha$$

Some reformulations yield:

$$\begin{aligned} \mathbf{0} = d\alpha &= d(R^\top g^{1/2}dp) \stackrel{(1)}{=} dR^\top \wedge (g^{1/2}dp) + R^\top d(g^{1/2}dp) \\ &= R^\top (\omega \wedge (g^{1/2}dp) + d(g^{1/2}dp)) \end{aligned}$$

where for (1), the Leibnitz Rule for the exterior derivative is applied, and we define

$$\omega = RdR^\top \in \mathfrak{so}(3).$$

**Remark.**  $\omega$  is an antisymmetric matrix-valued one-form (every element is a one-form).

*Proof.* We differentiate the orthogonality condition of the rotation matrix  $R$  (taking the derivative w.r.t. every element):

$$\begin{aligned} Id &= RR^\top \\ d(Id) &= d(RR^\top) \\ \mathbf{0} &= dRR^\top + RdR^\top \\ \mathbf{0} &= (RdR^\top)^\top + RdR^\top \\ -(RdR^\top)^\top &= RdR^\top \end{aligned}$$

□

Elements of the Lie algebra  $\mathfrak{so}(3)$  can be thought as infinitesimal rotations and  $\omega$  can be used as a connection one-form. The Lie algebra  $\mathfrak{so}(3)$  has manifold structure, but for our purposes, it suffices to know that elements are antisymmetric matrices. To make  $R$  locally integrable, we find  $\omega$  such that the one-form  $\alpha$  is closed ( $d\alpha = 0$ , curl-free), then try to match the  $\omega$  with  $R$ , which can be expressed as

$$\min_{R \in SO(3)} \|RdR^T - \omega\|^2.$$

**Connection evaluation** To find  $\omega$ , we use the above equation

$$\mathbf{0} = R^\top (\omega \wedge (g^{1/2} dp) + d(g^{1/2} dp)) \iff \mathbf{0} = \omega \wedge (g^{1/2} dp) + d(g^{1/2} dp)$$

and reformulate into a linear system. We represent the antisymmetric matrix-valued one-form  $\omega$

$$\omega = \begin{bmatrix} 0 & \omega_{12} & -\omega_{31} \\ -\omega_{12} & 0 & \omega_{23} \\ \omega_{31} & -\omega_{23} & 0 \end{bmatrix}$$

by  $\begin{bmatrix} \omega_{23} & \omega_{31} & \omega_{12} \end{bmatrix} = \begin{bmatrix} dx & dy & dz \end{bmatrix} W$ . We write  $W = [W_1, W_2, W_3]$ ,  $W_i \in \mathbb{R}^3$ . That is,  $W$  is the matrix with the coefficients for the one-forms, e.g.  $W_1 = [(\omega^{23})_1, (\omega^{23})_2, (\omega^{23})_3]^\top$ . Recall, a one-form can be expressed as

$$\omega_{ij} = (\omega^{ij})_1 dx + (\omega^{ij})_2 dy + (\omega^{ij})_3 dz$$

So e.g.

$$\omega_{23} = \begin{bmatrix} dx & dy & dz \end{bmatrix} W_1 = (\omega^{23})_1 dx + (\omega^{23})_2 dy + (\omega^{23})_3 dz$$

We also write  $A = g^{1/2} = [A^1, A^2, A^3]$ . We use the equation  $\mathbf{0} = \omega \wedge (g^{1/2} dp) + d(g^{1/2} dp)$ . Let us start with the first part:

$$\begin{aligned} \omega \wedge g^{1/2} dp &= \begin{bmatrix} 0 & \omega_{12} & -\omega_{31} \\ -\omega_{12} & 0 & \omega_{23} \\ \omega_{31} & -\omega_{23} & 0 \end{bmatrix} \wedge \begin{bmatrix} A_1^1 & A_1^2 & A_1^3 \\ A_2^1 & A_2^2 & A_2^3 \\ A_3^1 & A_3^2 & A_3^3 \end{bmatrix} \begin{bmatrix} dx \\ dy \\ dz \end{bmatrix} \\ &= \begin{bmatrix} +\omega_{12} \wedge (A_2^1 dx + A_2^2 dy + A_2^3 dz) - \omega_{31} \wedge (A_3^1 dx + A_3^2 dy + A_3^3 dz) \\ -\omega_{12} \wedge (A_1^1 dx + A_1^2 dy + A_1^3 dz) + \omega_{23} \wedge (A_3^1 dx + A_3^2 dy + A_3^3 dz) \\ +\omega_{31} \wedge (A_1^1 dx + A_1^2 dy + A_1^3 dz) - \omega_{23} \wedge (A_2^1 dx + A_2^2 dy + A_2^3 dz) \end{bmatrix} \end{aligned}$$

It will get really messy if we calculate each component here, so let us calculate one component separately first:

$$\begin{aligned} \omega_{ij} \wedge (A_k^1 dx + A_k^2 dy + A_k^3 dz) &= ((\omega^{ij})_1 dx + (\omega^{ij})_2 dy + (\omega^{ij})_3 dz) \wedge (A_k^1 dx + A_k^2 dy + A_k^3 dz) \\ &= (\omega^{ij})_1 A_k^2 dx \wedge dy + (\omega^{ij})_1 A_k^3 dx \wedge dz \\ &\quad + (\omega^{ij})_2 A_k^1 dy \wedge dx + (\omega^{ij})_2 A_k^3 dy \wedge dz \\ &\quad + (\omega^{ij})_3 A_k^1 dz \wedge dx + (\omega^{ij})_3 A_k^2 dz \wedge dy \\ &= ((\omega^{ij})_1 A_k^2 - (\omega^{ij})_2 A_k^1) dx \wedge dy \\ &\quad + ((\omega^{ij})_2 A_k^3 - (\omega^{ij})_3 A_k^2) dy \wedge dz \\ &\quad + ((\omega^{ij})_3 A_k^1 - (\omega^{ij})_1 A_k^3) dz \wedge dx \end{aligned}$$

where we use the fact that  $dx \wedge dx = 0$  and  $dx \wedge dy = -dy \wedge dx$ . We can clean up the above expression using the cross product:

$$\begin{bmatrix} ((\omega^{ij})_2 A_k^3 - (\omega^{ij})_3 A_k^2) \\ ((\omega^{ij})_3 A_k^1 - (\omega^{ij})_1 A_k^3) \\ ((\omega^{ij})_1 A_k^2 - (\omega^{ij})_2 A_k^1) \end{bmatrix}^\top \begin{bmatrix} dy \wedge dz \\ dz \wedge dx \\ dx \wedge dy \end{bmatrix} = [W_i \times A^k]^\top \begin{bmatrix} dy \wedge dz \\ dz \wedge dx \\ dx \wedge dy \end{bmatrix}$$

We use the fact that  $[A_1, A_2, A_3] = [A^1, A^2, A^3]$  because  $A$  is symmetric, and  $W_1$  corresponds to  $\omega_{23}$ ,  $W_2$  to  $\omega_{31}$  and  $W_3$  to  $\omega_{12}$ . Second part:

$$d(g^{1/2}dp) = d(Adp) = d\left(\begin{bmatrix} A_1^1 & A_1^2 & A_1^3 \\ A_2^1 & A_2^2 & A_2^3 \\ A_3^1 & A_3^2 & A_3^3 \end{bmatrix} \begin{bmatrix} dx \\ dy \\ dz \end{bmatrix}\right)$$

Again, we can do this separately for each row:

$$\begin{aligned} & d(A_k^1 dx + A_k^2 dy + A_k^3 dz) \\ &= dA_k^1 \wedge dx + dA_k^2 \wedge dy + dA_k^3 \wedge dz \\ &= \frac{\partial A_k^1}{\partial x} dx \wedge dx + \frac{\partial A_k^1}{\partial y} dy \wedge dx + \frac{\partial A_k^1}{\partial z} dz \wedge dx \\ &+ \frac{\partial A_k^2}{\partial x} dx \wedge dy + \frac{\partial A_k^2}{\partial y} dy \wedge dy + \frac{\partial A_k^2}{\partial z} dz \wedge dy \\ &+ \frac{\partial A_k^3}{\partial x} dx \wedge dz + \frac{\partial A_k^3}{\partial y} dy \wedge dz + \frac{\partial A_k^3}{\partial z} dz \wedge dz \\ &= \left(\frac{\partial A_k^3}{\partial y} - \frac{\partial A_k^2}{\partial z}\right) dy \wedge dz + \left(\frac{\partial A_k^1}{\partial z} - \frac{\partial A_k^3}{\partial x}\right) dz \wedge dx + \left(\frac{\partial A_k^2}{\partial x} - \frac{\partial A_k^1}{\partial y}\right) dx \wedge dy \\ &= (\nabla \times A_k)^\top \begin{bmatrix} dy \wedge dz \\ dz \wedge dx \\ dx \wedge dy \end{bmatrix} \end{aligned}$$

Finally, we can put everything together:

$$\begin{aligned} \mathbf{0} &= \begin{bmatrix} (W_3 \times A^2 - W_2 \times A^3 + \nabla \times A^1)^\top \\ (W_1 \times A^3 - W_3 \times A^1 + \nabla \times A^2)^\top \\ (W_2 \times A^1 - W_1 \times A^2 + \nabla \times A^3)^\top \end{bmatrix} \begin{bmatrix} dy \wedge dz \\ dz \wedge dx \\ dx \wedge dy \end{bmatrix} \\ &\iff \begin{bmatrix} (W_2 \times A^3 - W_3 \times A^2)^\top \\ (W_3 \times A^1 - W_1 \times A^3)^\top \\ (W_1 \times A^2 - W_2 \times A^1)^\top \end{bmatrix} \begin{bmatrix} dy \wedge dz \\ dz \wedge dx \\ dx \wedge dy \end{bmatrix} = \begin{bmatrix} (\nabla \times A^1)^\top \\ (\nabla \times A^2)^\top \\ (\nabla \times A^3)^\top \end{bmatrix} \begin{bmatrix} dy \wedge dz \\ dz \wedge dx \\ dx \wedge dy \end{bmatrix} \end{aligned}$$

Take the curl to the other side and switch order on the left-hand side to cancel the  $-1$ . As we are only interested in the 9 components of  $W$ , we omit the two-form basis and transform into a 9x9 linear system for  $W$ . We define  $A_\times$  and  $\text{vec}(\cdot)$  as

$$A_\times = \begin{bmatrix} 0 & -A_\times^3 & A_\times^2 \\ A_\times^3 & 0 & -A_\times^1 \\ -A_\times^2 & A_\times^1 & 0 \end{bmatrix}, \text{vec}(W) = \begin{bmatrix} W_1 \\ W_2 \\ W_3 \end{bmatrix}$$

with  $A_\times^i$  defined as

$$A_\times^i = [A_1^i \quad A_2^i \quad A_3^i]_\times = \begin{bmatrix} 0 & -A_3^i & A_2^i \\ A_3^i & 0 & -A_1^i \\ -A_2^i & A_1^i & 0 \end{bmatrix}$$

and  $\text{vec}(\cdot)$  turns a 3x3-matrix into a 9x1-vector by stacking the columns. With these two definitions, we can transform the above equality into a linear system

$$A_\times \text{vec}(W) = \text{vec}(\nabla \times A)$$

where  $\nabla \times A$  is just the curl applied to each column. This transformation can be checked by laboriously plugging in the definitions and comparing the coefficients. With tedious calculations, one can show that  $\det(A_\times) = -2 \det(A)^3 = -2 \det(g)^{3/2} < 0$ , which means this is a linear system that is solvable and can be used to calculate  $W$  at a point.