Chapter 1

Connection One-Form ω

A vector field U is integrable, if and only if $\nabla \times U = 0$, which means the vector field has vanishing curl everywhere. We can express this more naturally with the language of differential forms: The curl can be written as the exterior derivative d of a one-form α. A one-form (more generally, a differential form) is closed, if $d\alpha = 0$. Therefore, the local integrability can be expressed as the closedness of a one-form. We want F^{-1} (TODO: why F^{-1}) to be integrable. To achieve local integrability for, it suffices to make R locally integrable. We can think of a rotation field R as the composition of 3 vector fields

$$R = \begin{bmatrix} | & | & | \\ R_1 & R_2 & R_3 \\ | & | & | \end{bmatrix}$$

where $R_i : \mathbb{R}^3 \to \mathbb{R}^3$ is a vector field. We can therefore construct a vector-valued one-form, given $p = (x, y, z)^\top$ in Euclidean coordinates

$$\alpha \triangleq F^{-1} dp = R^\top g^{1/2} dp$$

where $dp = (dx, dy, dz)^\top$ is the common orthonormal one-form basis of $\Omega^1(\mathcal{M})$.

$$R \text{ locally integrable} \iff \mathbf{0} = d\alpha$$

Some reformulations yield:

$$\begin{aligned} \mathbf{0} = d\alpha = d(R^\top g^{1/2} dp) &\overset{(1)}{=} dR^\top \wedge (g^{1/2} dp) + R^\top d(g^{1/2} dp) \\ &= R^\top (\omega \wedge (g^{1/2} dp) + d(g^{1/2} dp)) \end{aligned}$$

where for (1), the Leibnitz Rule for the exterior derivative is applied, and we define

$$\omega = R dR^\top \in \mathfrak{so}(3).$$

Remark. *ω is an antisymmetric matrix-valued one-form (every element is a one-form).*

Proof. We differentiate the orthogonality condition of the rotation matrix R (taking the derivative w.r.t. every element):

$$\begin{aligned} Id &= RR^\top \\ d(Id) &= d(RR^\top) \\ \mathbf{0} &= dRR^\top + RdR^\top \\ \mathbf{0} &= (RdR^\top)^\top + RdR^\top \\ -(RdR^\top)^\top &= RdR^\top \end{aligned}$$

□

Elements of the Lie algebra $\mathfrak{so}(3)$ can be thought as infinitesimal rotations and ω can be used as a connection one-form. The Lie algebra $\mathfrak{so}(3)$ has manifold structure, but for our purposes, it suffices to know that elements are antisymmetric matrices. To make R locally integrable, we find ω such that the one-form α is closed ($d\alpha = 0$, curl-free), then try to match the ω with R, which can be expressed as

$$\min_{R \in SO(3)} ||RdR^T - \omega||^2.$$

Connection evaluation To find ω, we use the above equation

$$\mathbf{0} = R^\top(\omega \wedge (g^{1/2}dp) + d(g^{1/2}dp)) \iff \mathbf{0} = \omega \wedge (g^{1/2}dp) + d(g^{1/2}dp)$$

and reformulate into a linear system. We represent the antisymmetric matrix-valued one-form ω

$$\omega = \begin{bmatrix} 0 & \omega_{12} & -\omega_{31} \\ -\omega_{12} & 0 & \omega_{23} \\ \omega_{31} & -\omega_{23} & 0 \end{bmatrix}$$

by $\begin{bmatrix}\omega_{23} & \omega_{31} & \omega_{12}\end{bmatrix} = \begin{bmatrix}dx & dy & dz\end{bmatrix} W$. We write $W = \begin{bmatrix}W_1, W_2, W_3\end{bmatrix}$, $W_i \in \mathbb{R}^3$. That is, W is the matrix with the coefficients for the one-forms, e.g. $W_1 = \begin{bmatrix}(\omega^{23})_1, (\omega^{23})_2, (\omega^{23})_3\end{bmatrix}^\top$. Recall, a one-form can be expressed as

$$\omega_{ij} = (\omega^{ij})_1 dx + (\omega^{ij})_2 dy + (\omega^{ij})_3 dz$$

So e.g.

$$\omega_{23} = \begin{bmatrix}dx & dy & dz\end{bmatrix} W_1 = (\omega^{23})_1 dx + (\omega^{23})_2 dy + (\omega^{23})_3 dz$$

We also write $A = g^{1/2} = \begin{bmatrix}A^1, A^2, A^3\end{bmatrix}$. We use the equation $\mathbf{0} = \omega \wedge (g^{1/2}dp) + d(g^{1/2}dp)$. Let us start with the first part:

$$\begin{aligned}
\omega \wedge g^{1/2}dp &= \begin{bmatrix} 0 & \omega_{12} & -\omega_{31} \\ -\omega_{12} & 0 & \omega_{23} \\ \omega_{31} & -\omega_{23} & 0 \end{bmatrix} \bigwedge \begin{bmatrix} A_1^1 & A_1^2 & A_1^3 \\ A_2^1 & A_2^2 & A_2^3 \\ A_3^1 & A_3^2 & A_3^3 \end{bmatrix} \begin{bmatrix} dx \\ dy \\ dz \end{bmatrix} \\
&= \begin{bmatrix} +\omega_{12} \wedge (A_2^1 dx + A_2^2 dy + A_2^3 dz) - \omega_{31} \wedge (A_3^1 dx + A_3^2 dy + A_3^3 dz) \\ -\omega_{12} \wedge (A_1^1 dx + A_1^2 dy + A_1^3 dz) + \omega_{23} \wedge (A_3^1 dx + A_3^2 dy + A_3^3 dz) \\ +\omega_{31} \wedge (A_1^1 dx + A_1^2 dy + A_1^3 dz) - \omega_{23} \wedge (A_2^1 dx + A_2^2 dy + A_2^3 dz) \end{bmatrix}
\end{aligned}$$

It will get really messy if we calculate each component here, so let us calculate one component separately first:

$$\begin{aligned}
\omega_{ij} \wedge (A_k^1 dx + A_k^2 dy + A_k^3 dz) &= ((\omega^{ij})_1 dx + (\omega^{ij})_2 dy + (\omega^{ij})_3 dz) \wedge (A_k^1 dx + A_k^2 dy + A_k^3 dz) \\
&= (\omega^{ij})_1 A_k^2 dx \wedge dy + (\omega^{ij})_1 A_k^3 dx \wedge dz \\
&+ (\omega^{ij})_2 A_k^1 dy \wedge dx + (\omega^{ij})_2 A_k^3 dy \wedge dz \\
&+ (\omega^{ij})_3 A_k^1 dz \wedge dx + (\omega^{ij})_3 A_k^2 dz \wedge dy \\
&= ((\omega^{ij})_1 A_k^2 - (\omega^{ij})_2 A_k^1) dx \wedge dy \\
&+ ((\omega^{ij})_2 A_k^3 - (\omega^{ij})_3 A_k^2) dy \wedge dz \\
&+ ((\omega^{ij})_3 A_k^1 - (\omega^{ij})_1 A_k^3) dz \wedge dx
\end{aligned}$$

where we use the fact that $dx \wedge dx = 0$ and $dx \wedge dy = -dy \wedge dx$. We can clean up the above expression using the cross product:

$$\begin{bmatrix} ((\omega^{ij})_2 A_k^3 - (\omega^{ij})_3 A_k^2) \\ ((\omega^{ij})_3 A_k^1 - (\omega^{ij})_1 A_k^3) \\ ((\omega^{ij})_1 A_k^2 - (\omega^{ij})_2 A_k^1) \end{bmatrix}^\top \begin{bmatrix} dy \wedge dz \\ dz \wedge dx \\ dx \wedge dy \end{bmatrix} = \begin{bmatrix} W_i \times A^k \end{bmatrix}^\top \begin{bmatrix} dy \wedge dz \\ dz \wedge dx \\ dx \wedge dy \end{bmatrix}$$

We use the fact that $\begin{bmatrix}A_1, A_2, A_3\end{bmatrix} = \begin{bmatrix}A^1, A^2, A^3\end{bmatrix}$ because A is symmetric, and W_1 corresponds to ω_{23}, W_2 to ω_{31} and W_3 to ω_{12}. Second part:

$$d(g^{1/2}dp) = d(Adp) = d\left(\begin{bmatrix}A_1^1 & A_1^2 & A_1^3\\ A_2^1 & A_2^2 & A_2^3\\ A_3^1 & A_3^2 & A_3^3\end{bmatrix}\begin{bmatrix}dx\\ dy\\ dz\end{bmatrix}\right)$$

Again, we can do this separately for each row:

$$\begin{aligned}
&d(A_k^1dx + A_k^2dy + A_k^3dz)\\
&= dA_k^1 \wedge dx + dA_k^2 \wedge dy + dA_k^3 \wedge dz\\
&= \frac{\partial A_k^1}{\partial x}dx \wedge dx + \frac{\partial A_k^1}{\partial y}dy \wedge dx + \frac{\partial A_k^1}{\partial z}dz \wedge dx\\
&+ \frac{\partial A_k^2}{\partial x}dx \wedge dy + \frac{\partial A_k^2}{\partial y}dy \wedge dy + \frac{\partial A_k^2}{\partial z}dz \wedge dy\\
&+ \frac{\partial A_k^3}{\partial x}dx \wedge dz + \frac{\partial A_k^3}{\partial y}dy \wedge dz + \frac{\partial A_k^3}{\partial z}dz \wedge dz\\
&= \left(\frac{\partial A_k^3}{\partial y} - \frac{\partial A_k^2}{\partial z}\right)dy \wedge dz + \left(\frac{\partial A_k^1}{\partial z} - \frac{\partial A_k^3}{\partial x}\right)dz \wedge dx + \left(\frac{\partial A_k^2}{\partial x} - \frac{\partial A_k^1}{\partial y}\right)dx \wedge dy\\
&= (\nabla \times A_k)^\top \begin{bmatrix}dy \wedge dz\\ dz \wedge dx\\ dx \wedge dy\end{bmatrix}
\end{aligned}$$

Finally, we can put everything together:

$$\begin{aligned}
\mathbf{0} &= \begin{bmatrix}(W_3 \times A^2 - W_2 \times A^3 + \nabla \times A^1)^\top\\ (W_1 \times A^3 - W_3 \times A^1 + \nabla \times A^2)^\top\\ (W_2 \times A^1 - W_1 \times A^2 + \nabla \times A^3)^\top\end{bmatrix}\begin{bmatrix}dy \wedge dz\\ dz \wedge dx\\ dx \wedge dy\end{bmatrix}\\
\iff &\begin{bmatrix}(W_2 \times A^3 - W_3 \times A^2)^\top\\ (W_3 \times A^1 - W_1 \times A^3)^\top\\ (W_1 \times A^2 - W_2 \times A^1)^\top\end{bmatrix}\begin{bmatrix}dy \wedge dz\\ dz \wedge dx\\ dx \wedge dy\end{bmatrix} = \begin{bmatrix}(\nabla \times A^1)^\top\\ (\nabla \times A^2)^\top\\ (\nabla \times A^3)^\top\end{bmatrix}\begin{bmatrix}dy \wedge dz\\ dz \wedge dx\\ dx \wedge dy\end{bmatrix}
\end{aligned}$$

Take the curl to the other side and switch order on the left-hand side to cancel the -1. As we are only interested in the 9 components of W, we omit the two-form basis and transform into a 9x9 linear system for W. We define $A_\times$ and $\mathrm{vec}(\cdot)$ as

$$A_\times = \begin{bmatrix}0 & -A_\times^3 & A_\times^2\\ A_\times^3 & 0 & -A_\times^1\\ -A_\times^2 & A_\times^1 & 0\end{bmatrix}, \mathrm{vec}(W) = \begin{bmatrix}W_1\\ W_2\\ W_3\end{bmatrix}$$

with $A_\times^i$ defined as

$$A_\times^i = \begin{bmatrix}A_1^i & A_2^i & A_3^i\end{bmatrix}_\times = \begin{bmatrix}0 & -A_3^i & A_2^i\\ A_3^i & 0 & -A_1^i\\ -A_2^i & A_1^i & 0\end{bmatrix}$$

and $\mathrm{vec}(\cdot)$ turns a 3x3-matrix into a 9x1-vector by stacking the columns. With these two definitions, we can transform the above equality into a linear system

$$A_\times \,\mathrm{vec}(W) = \mathrm{vec}(\nabla \times A)$$

where $\nabla \times A$ is just the curl applied to each column. This transformation can be checked by laboriously plugging in the definitions and comparing the coefficients. With tedious calculations, one can show that $\det(A_\times) = -2\det(A)^3 = -2\det(g)^{3/2} < 0$, which means this is a linear system that is solvable and can be used to calculate W at a point.