UNIVERSITÄT BERN

3D Metric Fields

A Novel Approach to a New Idea

Bachelor Thesis

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Abstract

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Chapter 1

Introduction

- What are Frame Fields?
- Why are they important?
- How do *we* generate them?

A frame field

A vector field is locally integrable, if and only if $\nabla \times A = 0$. [3]

Chapter 2

Mathematical Background

We will make heavy use of differential geometry in the following sections. To get us all on the same page, I introduce the basic concepts what we will use, but I refrain from giving any proofs. I will give definitions only as far as we need it. These definitions will by no means be exhaustive. The following is an incomplete summary of what we need presented in "Introduction to smooth manifolds" [2]

Manifold A manifold $\mathcal{M}$ is a space that locally looks like Euclidean space. More exactly, a n-manifold is a topological space, where each point on the manifold has an open neighborhood that is locally homeomorphic to an open subset of Euclidean space $\mathbb{R}^n$. A manifold can be equipped with additional structure. For example, we can work on *smooth manifolds*. In simple terms, a manifold is *smooth* if it is similar enough to $\mathbb{R}^n$ that we can do Calculus like differentiation or integration on it. For this, each point on the manifold must be locally *diffeomorphic* to an open subset of $\mathbb{R}^n$ space.

A *Riemannian manifold* $(\mathcal{M}, g)$ is a real, smooth manifold, which is additionally equipped with a metric g at each point p on the manifold.

Tangent space, Tangent bundle There are many equivalent definitions for the tangent space. One definition is for each point p in the manifold $\mathcal{M}$, the tangent space $T_p\mathcal{M}$ consists of $\gamma'(0)$ for all differentiable paths $\gamma : (-\varepsilon, \varepsilon) \to \mathcal{M}$ with $p = \gamma(0)$. The tangent space is a vector space which has the same dimension as it's manifold, which is 3 in our case. These tangent spaces can be "glued" together to form the *tangent bundle* $T\mathcal{M} = \sqcup_{p\in\mathcal{M}} T_p\mathcal{M}$ which itself is a manifold of dimension $2n$. An element of $T\mathcal{M}$ can be written as (p, v) with $p \in \mathcal{M}$ and $v \in T_p\mathcal{M}$. This admits a natural projection $\pi : T\mathcal{M} \to \mathcal{M}$, which sends each vector $v \in T_p\mathcal{M}$ to the point p where it is tangent: $\pi(p, v) = p$. A *section* $\sigma : \to T\mathcal{M}$ is a continuos map, with $\pi \circ \sigma = Id_{\mathcal{M}}$

A frame F is a set of 6 vectors $\{\pm F_0, \pm F_1, \pm F_2\}$. We can represent such a frame F as a 3×3 matrix F, where the ith-column is F_i. A frame field then maps to every point in 3D-space such a frame, i.e. $F : \mathbb{R}^3 \to \mathbb{R}^{3\times 3}$. Usually, we work on a 3-manifold $\mathcal{M}$ and a positively oriented frame field, i.e. $F|_{\mathcal{M}} : \mathcal{M} \to \mathbb{R}^{3\times 3}$, where $\det(F) > 0$. To allow for anisotropic, nonuniform meshes, we generalize orthonormality of frames to g-orthonormal frames. Orthonormality is measured in some metric g, and a frame F satifisfies the condition $\langle F_i, F_j \rangle_g = \delta_{ij}$. Any frame field with $\det(F) > 0$ naturally defines a metric $g = (FF^\top)^{-1}$, where F is g-orthonormal

$$F^\top g F = Id.$$

We can factor the frame field F into a symmetric part $g^{1/2}$ and a rotational part R

$$F = g^{-1/2} R$$

The symmetric part $g^{-1/2}$ keeps F g-orthonormal

$$\implies F^\top g F = (g^{-1/2} R)^\top g g^{-1/2} R) = R^\top g^{-1/2} g g^{-1/2} R = Id.$$

and R represents a rotational field $R : \mathcal{M} \to SO(3)$. The requirements for our frame field are:

- Smoothness
- Integrability
- Metric consistency: $g = (FF^\top)^{-1}$

Chapter 3

Connection One-Form ω

A vector field U is integrable, if and only if $\nabla \times U = 0$, which means the vector field has vanishing curl everywhere. We can express this more naturally with the language of differential forms: The curl can be written as the exterior derivative d of a one-form α. A one-form (more generally, a differential form) is closed, if $d\alpha = 0$. Therefore, the local integrability can be expressed as the closedness of a one-form. We want F^{-1} (TODO: why F^{-1}) to be integrable. To achieve local integrability for, it suffices to make R locally integrable. We can think of a rotation field R as the composition of 3 vector fields

$$R = \begin{bmatrix} | & | & | \\ R_1 & R_2 & R_3 \\ | & | & | \end{bmatrix}$$

where $R_i : \mathbb{R}^3 \to \mathbb{R}^3$ is a vector field. We can therefore construct a vector-valued one-form, given $p = (x, y, z)^\top$ in Euclidean coordinates

$$\alpha \triangleq F^{-1}dp = R^\top g^{1/2} dp$$

where $dp = (dx, dy, dz)^\top$ is the common orthonormal one-form basis of $\Omega^1(\mathcal{M})$.

$$R \text{ locally integrable} \iff \mathbf{0} = d\alpha$$

Some reformulations yield:

$$\begin{aligned} \mathbf{0} = d\alpha = d(R^\top g^{1/2} dp) &\overset{(1)}{=} dR^\top \wedge (g^{1/2} dp) + R^\top d(g^{1/2} dp) \\ &= R^\top (\omega \wedge (g^{1/2} dp) + d(g^{1/2} dp)) \end{aligned}$$

where for (1), the Leibnitz Rule for the exterior derivative is applied, and we define

$$\omega = R dR^\top \in \mathfrak{so}(3).$$

Remark. *ω is an antisymmetric matrix-valued one-form (every element is a one-form).*

Proof. We differentiate the orthogonality condition of the rotation matrix R (taking the derivative w.r.t. every element):

$$\begin{aligned} Id &= RR^\top \\ d(Id) &= d(RR^\top) \\ \mathbf{0} &= dRR^\top + RdR^\top \\ \mathbf{0} &= (RdR^\top)^\top + RdR^\top \\ -(RdR^\top)^\top &= RdR^\top \end{aligned}$$

$\square$

Elements of the Lie algebra $\mathfrak{so}(3)$ can be thought as infinitesimal rotations and ω can be used as a connection one-form. For our purposes, it suffices to know that elements are antisymmetric matrices. To make R locally integrable, we find ω such that the one-form α is closed ($d\alpha = 0$, curl-free), then try to match the ω with R, which can be expressed as

$$\min_{R \in SO(3)} ||RdR^T - \omega||^2.$$

Connection evaluation To find ω, we use the above equation

$$\mathbf{0} = R^\top(\omega \wedge (g^{1/2}dp) + d(g^{1/2}dp)) \iff \mathbf{0} = \omega \wedge (g^{1/2}dp) + d(g^{1/2}dp)$$

and reformulate into a linear system. We represent the antisymmetric matrix-valued one-form ω

$$\omega = \begin{bmatrix} 0 & \omega_{12} & -\omega_{31} \\ -\omega_{12} & 0 & \omega_{23} \\ \omega_{31} & -\omega_{23} & 0 \end{bmatrix}$$

by $\begin{bmatrix} \omega_{23} & \omega_{31} & \omega_{12} \end{bmatrix} = \begin{bmatrix} dx & dy & dz \end{bmatrix} W$. We write $W = \begin{bmatrix} W_1, W_2, W_3 \end{bmatrix}$, $W_i \in \mathbb{R}^3$. That is, W is the matrix with the coefficients for the one-forms, e.g. $W_1 = \begin{bmatrix} (\omega^{23})_1, (\omega^{23})_2, (\omega^{23})_3 \end{bmatrix}^\top$. Recall, a one-form can be expressed as

$$\omega_{ij} = (\omega^{ij})_1 dx + (\omega^{ij})_2 dy + (\omega^{ij})_3 dz$$

So e.g.

$$\omega_{23} = \begin{bmatrix} dx & dy & dz \end{bmatrix} W_1 = (\omega^{23})_1 dx + (\omega^{23})_2 dy + (\omega^{23})_3 dz$$

We write $A = g^{1/2} = \begin{bmatrix} A^1, A^2, A^3 \end{bmatrix}$. We use the equation $\mathbf{0} = \omega \wedge (g^{1/2}dp) + d(g^{1/2}dp)$. First part:

$$\begin{aligned}
\omega \wedge g^{1/2}dp &= \begin{bmatrix} 0 & \omega_{12} & -\omega_{31} \\ -\omega_{12} & 0 & \omega_{23} \\ \omega_{31} & -\omega_{23} & 0 \end{bmatrix} \bigwedge \begin{bmatrix} A_1^1 & A_1^2 & A_1^3 \\ A_2^1 & A_2^2 & A_2^3 \\ A_3^1 & A_3^2 & A_3^3 \end{bmatrix} \begin{bmatrix} dx \\ dy \\ dz \end{bmatrix} \\
&= \begin{bmatrix} +\omega_{12} \wedge (A_2^1 dx + A_2^2 dy + A_2^3 dz) - \omega_{31} \wedge (A_3^1 dx + A_3^2 dy + A_3^3 dz) \\ -\omega_{12} \wedge (A_1^1 dx + A_1^2 dy + A_1^3 dz) + \omega_{23} \wedge (A_3^1 dx + A_3^2 dy + A_3^3 dz) \\ +\omega_{31} \wedge (A_1^1 dx + A_1^2 dy + A_1^3 dz) - \omega_{23} \wedge (A_2^1 dx + A_2^2 dy + A_2^3 dz) \end{bmatrix}
\end{aligned}$$

It will get really messy if we calculate each component here, so let us calculate one component separately first:

$$\begin{aligned}
\omega_{ij} \wedge (A_k^1 dx + A_k^2 dy + A_k^3 dz) &= ((\omega^{ij})_1 dx + (\omega^{ij})_2 dy + (\omega^{ij})_3 dz) \wedge (A_k^1 dx + A_k^2 dy + A_k^3 dz) \\
&= (\omega^{ij})_1 A_k^2 dx \wedge dy + (\omega^{ij})_1 A_k^3 dx \wedge dz \\
&+ (\omega^{ij})_2 A_k^1 dy \wedge dx + (\omega^{ij})_2 A_k^3 dy \wedge dz \\
&+ (\omega^{ij})_3 A_k^1 dz \wedge dx + (\omega^{ij})_3 A_k^2 dz \wedge dy \\
&= ((\omega^{ij})_1 A_k^2 - (\omega^{ij})_2 A_k^1) dx \wedge dy \\
&+ ((\omega^{ij})_2 A_k^3 - (\omega^{ij})_3 A_k^2) dy \wedge dz \\
&+ ((\omega^{ij})_3 A_k^1 - (\omega^{ij})_1 A_k^3) dz \wedge dx
\end{aligned}$$

where we use the fact that $dx \wedge dx = 0$ and $dx \wedge dy = -dy \wedge dx$. We can clean up the above expression using the cross product:

$$\begin{bmatrix} ((\omega^{ij})_2 A_k^3 - (\omega^{ij})_3 A_k^2) \\ ((\omega^{ij})_3 A_k^1 - (\omega^{ij})_1 A_k^3) \\ ((\omega^{ij})_1 A_k^2 - (\omega^{ij})_2 A_k^1) \end{bmatrix}^\top \begin{bmatrix} dy \wedge dz \\ dz \wedge dx \\ dx \wedge dy \end{bmatrix} = \begin{bmatrix} W_i \times A^k \end{bmatrix}^\top \begin{bmatrix} dy \wedge dz \\ dz \wedge dx \\ dx \wedge dy \end{bmatrix}$$

We use the fact that $\begin{bmatrix}A_1, A_2, A_3\end{bmatrix} = \begin{bmatrix}A^1, A^2, A^3\end{bmatrix}$ because A is symmetric, and W_1 corresponds to ω_{23}, W_2 to ω_{31} and W_3 to ω_{12}. Second part:

$$d(g^{1/2}dp) = d(Adp) = d\left(\begin{bmatrix}A_1^1 & A_1^2 & A_1^3\\ A_2^1 & A_2^2 & A_2^3\\ A_3^1 & A_3^2 & A_3^3\end{bmatrix}\begin{bmatrix}dx\\ dy\\ dz\end{bmatrix}\right)$$

Again, we can do this separately for each row:

$$\begin{aligned}
&d(A_k^1 dx + A_k^2 dy + A_k^3 dz)\\
&= dA_k^1 \wedge dx + dA_k^2 \wedge dy + dA_k^3 \wedge dz\\
&= \frac{\partial A_k^1}{\partial x} dx \wedge dx + \frac{\partial A_k^1}{\partial y} dy \wedge dx + \frac{\partial A_k^1}{\partial z} dz \wedge dx\\
&+ \frac{\partial A_k^2}{\partial x} dx \wedge dy + \frac{\partial A_k^2}{\partial y} dy \wedge dy + \frac{\partial A_k^2}{\partial z} dz \wedge dy\\
&+ \frac{\partial A_k^3}{\partial x} dx \wedge dz + \frac{\partial A_k^3}{\partial y} dy \wedge dz + \frac{\partial A_k^3}{\partial z} dz \wedge dz\\
&= \left(\frac{\partial A_k^3}{\partial y} - \frac{\partial A_k^2}{\partial z}\right) dy \wedge dz + \left(\frac{\partial A_k^1}{\partial z} - \frac{\partial A_k^3}{\partial x}\right) dz \wedge dx + \left(\frac{\partial A_k^2}{\partial x} - \frac{\partial A_k^1}{\partial y}\right) dx \wedge dy\\
&= (\nabla \times A_k)^\top \begin{bmatrix}dy \wedge dz\\ dz \wedge dx\\ dx \wedge dy\end{bmatrix}
\end{aligned}$$

Finally, we can put everything together:

$$\begin{aligned}
\mathbf{0} &= \begin{bmatrix}(W_3 \times A^2 - W_2 \times A^3 + \nabla \times A^1)^\top\\ (W_1 \times A^3 - W_3 \times A^1 + \nabla \times A^2)^\top\\ (W_2 \times A^1 - W_1 \times A^2 + \nabla \times A^3)^\top\end{bmatrix}\begin{bmatrix}dy \wedge dz\\ dz \wedge dx\\ dx \wedge dy\end{bmatrix}\\
\iff &\begin{bmatrix}(W_2 \times A^3 - W_3 \times A^2)^\top\\ (W_3 \times A^1 - W_1 \times A^3)^\top\\ (W_1 \times A^2 - W_2 \times A^1)^\top\end{bmatrix}\begin{bmatrix}dy \wedge dz\\ dz \wedge dx\\ dx \wedge dy\end{bmatrix} = \begin{bmatrix}(\nabla \times A^1)^\top\\ (\nabla \times A^2)^\top\\ (\nabla \times A^3)^\top\end{bmatrix}\begin{bmatrix}dy \wedge dz\\ dz \wedge dx\\ dx \wedge dy\end{bmatrix}
\end{aligned}$$

Take the curl to the other side and switch order on the left-hand side to cancel the -1. As we are only interested in the 9 components of W, we omit the two-form basis and transform into a 9x9 linear system for W. We define $A_\times$ and $\mathrm{vec}(\cdot)$ as

$$A_\times = \begin{bmatrix}0 & -A_\times^3 & A_\times^2\\ A_\times^3 & 0 & -A_\times^1\\ -A_\times^2 & A_\times^1 & 0\end{bmatrix}, \mathrm{vec}(W) = \begin{bmatrix}W_1\\ W_2\\ W_3\end{bmatrix}$$

with $A_\times^i$ defined as

$$A_\times^i = \begin{bmatrix}A_1^i & A_2^i & A_3^i\end{bmatrix}_\times = \begin{bmatrix}0 & -A_3^i & A_2^i\\ A_3^i & 0 & -A_1^i\\ -A_2^i & A_1^i & 0\end{bmatrix}$$

and $\mathrm{vec}(\cdot)$ turns a 3x3-matrix into a 9x1-vector by stacking the columns. With these two definitions, we can transform the above equality into a linear system

$$A_\times \, \mathrm{vec}(W) = \mathrm{vec}(\nabla \times A)$$

where $\nabla \times A$ is just the curl applied to each column. This transformation can be checked by laboriously plugging in the definitions and comparing the coefficients. With tedious calculations, one can show that $\det(A_\times) = -2\det(A)^3 = -2\det(g)^{3/2} < 0$, which means this is a linear system that can be used to solve for W at a point.

Chapter 4

Discretized connection evaluation

- Calculation of R
- Piecewise linear discretization

Chapter 5

Algorithm for R between two arbitrary points in a mesh

[1] Here is an example for how to specify an algorithm in pseudo-code.

Algorithm 1 Byzantine Leader-Based Epoch-Change (process p_i).

State
  lastts $\leftarrow 0$: most recently started epoch
  nextts $\leftarrow 0$: timestamp of the next epoch
  newepoch $\leftarrow [\perp]^n$: list of NEWEPOCH messages

upon event *complain*(p_ℓ) **such that** $p_\ell = \textit{leader}(\textit{lastts})$ **do**
  if *nextts* $=$ *lastts* **then**
    nextts $\leftarrow$ *lastts* $+ 1$
    send message $[\text{NEWEPOCH}, \textit{nextts}]$ to all $p_j \in \mathcal{P}$

upon receiving a message $[\text{NEWEPOCH}, \textit{ts}]$ from p_j **such that** $\textit{ts} = \textit{lastts} + 1$ **do**
  newepoch$[j] \leftarrow$ NEWEPOCH

upon exists *ts* **such that** $\{p_j \in \mathcal{P} \mid \textit{newepoch}[j] = \textit{ts}\} \in \mathcal{K}_i$ **and** *nextts* $=$ *lastts* **do**
  nextts $\leftarrow$ *lastts* $+ 1$
  send message $[\text{NEWEPOCH}, \textit{nextts}]$ to all $p_j \in \mathcal{P}$

upon exists *ts* **such that** $\{p_j \in \mathcal{P} \mid \textit{newepoch}[j] = \textit{ts}\} \in \mathcal{Q}_i$ **and** *nextts* $>$ *lastts* **do**
  lastts $\leftarrow$ *nextts*
  newepoch $\leftarrow [\perp]^n$
  output *startepoch*$(\textit{lastts}, \textit{leader}(\textit{lastts}))$

Chapter 6

Conclusion

The conclusion looks back at the entire work, gives a critical look, summarizes, and discusses extensions and future work.

Appendix A

Extra material

Extra material may be placed in an appendix that appears after the conclusion.

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Erklärung

Erklärung gemäss Art. 30 RSL Phil.-nat. 18

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