Chapter 1

Connection One-Form ω

Recall that our goal is to find a frame field $F : \mathcal{M} \to \mathbb{R}^{3\times 3}$, such that the target parametrization of ϕ can sufficiently reduce the difference between $\nabla\phi$ and F^{-1}. For this, F^{-1} needs to stay as close to being locally integrable as possible. In this chapter, by starting from the local integrability condition, we formulate a connection 1-form ω which is used to measure the Dirichlet energy in the new metric g, such that the frames stay close to being g-orthonormal ($F^{-1}gF = \mathrm{Id}$). We follow the steps in *Metric-Driven 3D Frame Field Generation* [**?**].

1.1 Local integrability

A vector field U is integrable if and only if $\nabla \times U = 0$, which means the vector field has vanishing curl [**?**]. Although in general it is more complicated [**?**], we can think of a frame field F as the composition of 3 vector fields

$$F = \begin{bmatrix} | & | & | \\ F_1 & F_2 & F_3 \\ | & | & | \end{bmatrix}$$

where $F_i : \mathbb{R}^3 \to \mathbb{R}^3$ are vector fields. To achieve local integrability for F^{-1}, we therefore want

$$\nabla \times F^{-1} \stackrel{!}{=} 0$$

where the curl is applied to each column. We can express this more naturally with the language of differential forms by writing the curl as the exterior derivative d of a 1-form α. A 1-form (more generally, a differential form) is closed, if $d\alpha = 0$. We construct a vector-valued 1-form out of our frame field, given $\boldsymbol{p} = (x, y, z)^\top$ in Euclidean coordinates

$$\alpha \triangleq F^{-1} d\boldsymbol{p} = R^\top g^{1/2} d\boldsymbol{p}$$

where $d\boldsymbol{p} = (dx, dy, dz)^\top$ is the common orthonormal 1-form basis $\mathbb{E}^3$. Local integrability of F^{-1} is then formulated as the closedness of α, i.e.

$$F^{-1} \text{ locally integrable} \iff \mathbf{0} = d\alpha$$

Some reformulations yield:

$$\begin{aligned} \mathbf{0} = d\alpha &= d(F^{-1} d\boldsymbol{p}) = d(R^\top g^{1/2} d\boldsymbol{p}) \\ &= dR^\top \wedge (g^{1/2} d\boldsymbol{p}) + R^\top d(g^{1/2} d\boldsymbol{p}) \\ &= R^\top (\omega \wedge (g^{1/2} d\boldsymbol{p}) + d(g^{1/2} d\boldsymbol{p})) \end{aligned}$$

which we can simplify to

$$\mathbf{0} = \omega \wedge (g^{1/2} d\boldsymbol{p}) + d(g^{1/2} d\boldsymbol{p}) \tag{1.1}$$

where the Leibnitz Rule for the exterior derivative is applied and $\wedge$ is the exterior product between a matrix-valued 1-form and a vector-valued 1-form, i.e. 1-forms in each component and the matrix vector product uses $\wedge$ as the multiplication, see Section 1.3 for the evaluation. We further define

$$\omega = RdR^\top \in \mathfrak{so}(3)$$

which is an antisymmetric matrix-valued 1-form. To see this, we differentiate the orthogonality condition of the rotation matrix R (we assume R rotates about the axis a with angle θ):

$$\begin{aligned}
\mathrm{Id} &= RR^\top \\
d(\mathrm{Id}) &= d(RR^\top) \\
\mathbf{0} &= dRR^\top + RdR^\top \\
\mathbf{0} &= (RdR^\top)^\top + RdR^\top \\
-(RdR^\top)^\top &= RdR^\top
\end{aligned}$$

The Lie algebra $\mathfrak{so}(3)$ consists of all antisymmetric 3×3 matrices. Elements of $\mathfrak{so}(3)$ are infinitesimal rotations, that is, they are tangent to the manifold $\mathrm{SO}(3)$ at the element Id. Indeed, we characterised the elements of $\mathfrak{so}(3)$ by taking the derivative at Id, which is one definition to get to the tangent space. Thus, we can use ω as a connection 1-form to do the alignment of frames in the new metric g, i.e. the parallel transport is done with ω and then compared.

To solve for local integrability, we find ω such that the 1-form α is closed, then try to match the ω with R. This can be expressed as

$$\min_{R \in \mathrm{SO}(3)} ||RdR^\top - \omega||^2, \tag{1.2}$$

where ω is determined by g through a system of linear equations (see 1.3). In general, Equation 1.2 cannot be minimised to zero, therefore we solve for the nearly integrable 3D rotation field R.

1.2 Smoothness measure

Many frame field generation methods rely on maximising smoothness through the minimization of the Dirichlet energy. Here, we show that integrability through ω is closely related to the usual Dirichlet energy. We can take Equation 1.2 as a smoothness measure and reformulate, i.e.

$$||RdR^\top - \omega||^2 = || - dRR^\top - \omega||^2 = || - dRR^\top R - \omega R||^2 = ||dR + \omega R||^2. \tag{1.3}$$

By defining

$$\mathcal{D}R \triangleq dR + \omega R, \tag{1.4}$$

we can measure the smoothness and the integrability with a single energy $\int_{\mathcal{R}} ||\mathcal{D}R||^2$. Now, when g is constant in euclidean coordinates (x, y, z), Equation 1.1 tells us that $\omega = 0$, which reduces the smoothness measure $||\mathcal{D}R||$ to $||dR||^2$, which corresponds to the usual Dirichlet energy. In fact, $\mathcal{D}R$ corresponds to the covariant derivative of R under the connection ω, which shows that local integrability is related to the covariant-based Dirichlet energy. Special care must be taken as we abuse the notation of the covariant derivative. The covariant derivative $\mathcal{D}$ acts on each column of R separately. We define

$$\mathcal{D}R_i \triangleq \nabla_{\dot{\gamma}(t)} R_i \triangleq dR_i + \omega R_i$$

where dR_i is the derivative of each entry with respect to the angle θ and γ is some curve on the manifold (again, we assume R rotates about an axis a with angle θ).

1.3 Connection evaluation

To find ω, we use Equation 1.1

$$\mathbf{0} = \omega \wedge (g^{1/2} d\boldsymbol{p}) + d(g^{1/2} d\boldsymbol{p})$$

and reformulate into a linear system. We represent the antisymmetric matrix-valued 1-form ω

$$\omega = \begin{bmatrix} 0 & \omega_{12} & -\omega_{31} \\ -\omega_{12} & 0 & \omega_{23} \\ \omega_{31} & -\omega_{23} & 0 \end{bmatrix}$$

by $\begin{bmatrix} \omega_{23} & \omega_{31} & \omega_{12} \end{bmatrix} = \begin{bmatrix} dx & dy & dz \end{bmatrix} W$. We write $W = \begin{bmatrix} W_1, W_2, W_3 \end{bmatrix}$, $W_i \in \mathbb{R}^3$. That is, W is the matrix with the coefficients for the 1-forms, e.g. $W_1 = \begin{bmatrix} (\omega^{23})_1, (\omega^{23})_2, (\omega^{23})_3 \end{bmatrix}^\top$. Recall, a 1-form can be expressed as

$$\omega_{ij} = (\omega^{ij})_1 dx + (\omega^{ij})_2 dy + (\omega^{ij})_3 dz$$

So e.g. for ω_{23} we get

$$\omega_{23} = \begin{bmatrix} dx & dy & dz \end{bmatrix} W_1 = (\omega^{23})_1 dx + (\omega^{23})_2 dy + (\omega^{23})_3 dz.$$

We also write $A = g^{1/2} = \begin{bmatrix} A^1, A^2, A^3 \end{bmatrix}$. Starting with the first part of Equation 1.1, we get

$$\begin{aligned} \omega \wedge g^{1/2} d\boldsymbol{p} &= \begin{bmatrix} 0 & \omega_{12} & -\omega_{31} \\ -\omega_{12} & 0 & \omega_{23} \\ \omega_{31} & -\omega_{23} & 0 \end{bmatrix} \bigwedge \begin{bmatrix} A_1^1 & A_1^2 & A_1^3 \\ A_2^1 & A_2^2 & A_2^3 \\ A_3^1 & A_3^2 & A_3^3 \end{bmatrix} \begin{bmatrix} dx \\ dy \\ dz \end{bmatrix} \\ &= \begin{bmatrix} +\omega_{12} \wedge (A_2^1 dx + A_2^2 dy + A_2^3 dz) - \omega_{31} \wedge (A_3^1 dx + A_3^2 dy + A_3^3 dz) \\ -\omega_{12} \wedge (A_1^1 dx + A_1^2 dy + A_1^3 dz) + \omega_{23} \wedge (A_3^1 dx + A_3^2 dy + A_3^3 dz) \\ +\omega_{31} \wedge (A_1^1 dx + A_1^2 dy + A_1^3 dz) - \omega_{23} \wedge (A_2^1 dx + A_2^2 dy + A_2^3 dz) \end{bmatrix} \end{aligned}$$

It will get really messy if we calculate each component here, so let us calculate one component separately first:

$$\begin{aligned} \omega_{ij} \wedge (A_k^1 dx + A_k^2 dy + A_k^3 dz) &= ((\omega^{ij})_1 dx + (\omega^{ij})_2 dy + (\omega^{ij})_3 dz) \wedge (A_k^1 dx + A_k^2 dy + A_k^3 dz) \\ &= (\omega^{ij})_1 A_k^2 dx \wedge dy + (\omega^{ij})_1 A_k^3 dx \wedge dz \\ &+ (\omega^{ij})_2 A_k^1 dy \wedge dx + (\omega^{ij})_2 A_k^3 dy \wedge dz \\ &+ (\omega^{ij})_3 A_k^1 dz \wedge dx + (\omega^{ij})_3 A_k^2 dz \wedge dy \\ &= ((\omega^{ij})_1 A_k^2 - (\omega^{ij})_2 A_k^1) dx \wedge dy \\ &+ ((\omega^{ij})_2 A_k^3 - (\omega^{ij})_3 A_k^2) dy \wedge dz \\ &+ ((\omega^{ij})_3 A_k^1 - (\omega^{ij})_1 A_k^3) dz \wedge dx \end{aligned}$$

where we use the fact that $dx \wedge dx = 0$ and $dx \wedge dy = -dy \wedge dx$. We can clean up the above expression using the cross product:

$$\begin{bmatrix} ((\omega^{ij})_2 A_k^3 - (\omega^{ij})_3 A_k^2) \\ ((\omega^{ij})_3 A_k^1 - (\omega^{ij})_1 A_k^3) \\ ((\omega^{ij})_1 A_k^2 - (\omega^{ij})_2 A_k^1) \end{bmatrix}^\top \begin{bmatrix} dy \wedge dz \\ dz \wedge dx \\ dx \wedge dy \end{bmatrix} = \begin{bmatrix} W_i \times A^k \end{bmatrix}^\top \begin{bmatrix} dy \wedge dz \\ dz \wedge dx \\ dx \wedge dy \end{bmatrix}$$

We use the fact that $\begin{bmatrix} A_1, A_2, A_3 \end{bmatrix} = \begin{bmatrix} A^1, A^2, A^3 \end{bmatrix}$ because A is symmetric, and W_1 corresponds to ω_{23}, W_2 to ω_{31} and W_3 to ω_{12}. We continue with the second part of Equation 1.1:

$$d(g^{1/2} d\boldsymbol{p}) = d(A d\boldsymbol{p}) = d\left(\begin{bmatrix} A_1^1 & A_1^2 & A_1^3 \\ A_2^1 & A_2^2 & A_2^3 \\ A_3^1 & A_3^2 & A_3^3 \end{bmatrix} \begin{bmatrix} dx \\ dy \\ dz \end{bmatrix} \right)$$

Again, we can do this separately for each row (we use the fact that the exterior derivative d is the ordinary differential for a smooth function):

$$
\begin{aligned}
& d(A_k^1 dx + A_k^2 dy + A_k^3 dz) \\
={}& dA_k^1 \wedge dx + dA_k^2 \wedge dy + dA_k^3 \wedge dz \\
={}& \frac{\partial A_k^1}{\partial x} dx \wedge dx + \frac{\partial A_k^1}{\partial y} dy \wedge dx + \frac{\partial A_k^1}{\partial z} dz \wedge dx \\
&+ \frac{\partial A_k^2}{\partial x} dx \wedge dy + \frac{\partial A_k^2}{\partial y} dy \wedge dy + \frac{\partial A_k^2}{\partial z} dz \wedge dy \\
&+ \frac{\partial A_k^3}{\partial x} dx \wedge dz + \frac{\partial A_k^3}{\partial y} dy \wedge dz + \frac{\partial A_k^3}{\partial z} dz \wedge dz \\
={}& \left(\frac{\partial A_k^3}{\partial y} - \frac{\partial A_k^2}{\partial z} \right) dy \wedge dz + \left(\frac{\partial A_k^1}{\partial z} - \frac{\partial A_k^3}{\partial x} \right) dz \wedge dx + \left(\frac{\partial A_k^2}{\partial x} - \frac{\partial A_k^1}{\partial y} \right) dx \wedge dy \\
={}& (\nabla \times A_k)^\top \begin{bmatrix} dy \wedge dz \\ dz \wedge dx \\ dx \wedge dy \end{bmatrix}
\end{aligned}
$$

Finally, we can put everything together:

$$
\begin{aligned}
\mathbf{0} &= \begin{bmatrix} (W_3 \times A^2 - W_2 \times A^3 + \nabla \times A^1)^\top \\ (W_1 \times A^3 - W_3 \times A^1 + \nabla \times A^2)^\top \\ (W_2 \times A^1 - W_1 \times A^2 + \nabla \times A^3)^\top \end{bmatrix} \begin{bmatrix} dy \wedge dz \\ dz \wedge dx \\ dx \wedge dy \end{bmatrix} \\
\iff & \begin{bmatrix} (W_2 \times A^3 - W_3 \times A^2)^\top \\ (W_3 \times A^1 - W_1 \times A^3)^\top \\ (W_1 \times A^2 - W_2 \times A^1)^\top \end{bmatrix} \begin{bmatrix} dy \wedge dz \\ dz \wedge dx \\ dx \wedge dy \end{bmatrix} = \begin{bmatrix} (\nabla \times A^1)^\top \\ (\nabla \times A^2)^\top \\ (\nabla \times A^3)^\top \end{bmatrix} \begin{bmatrix} dy \wedge dz \\ dz \wedge dx \\ dx \wedge dy \end{bmatrix}
\end{aligned}
$$

We take the curl to the other side and switch order on the left-hand side to cancel the -1. As we are only interested in the 9 components of W, we omit the two-form basis and transform into a 9×9 linear system for W. We define $A_\times$ and $\mathrm{vec}(\cdot)$ as

$$
A_\times = \begin{bmatrix} 0 & -A_\times^3 & A_\times^2 \\ A_\times^3 & 0 & -A_\times^1 \\ -A_\times^2 & A_\times^1 & 0 \end{bmatrix}, \mathrm{vec}(W) = \begin{bmatrix} W_1 \\ W_2 \\ W_3 \end{bmatrix}
$$

with $A_\times^i$ defined as

$$
A_\times^i = \begin{bmatrix} A_1^i & A_2^i & A_3^i \end{bmatrix}_\times = \begin{bmatrix} 0 & -A_3^i & A_2^i \\ A_3^i & 0 & -A_1^i \\ -A_2^i & A_1^i & 0 \end{bmatrix}
$$

and $\mathrm{vec}(\cdot)$ turns a 3×3-matrix into a 9×1-vector by stacking the columns. With these two definitions, we can transform the above equality into a linear system

$$
A_\times \, \mathrm{vec}(W) = \mathrm{vec}(\nabla \times A) \tag{1.5}
$$

where $\nabla \times A$ is just the curl applied to each column. This transformation can be checked by laboriously plugging in the definitions and comparing the coefficients. With tedious brute-force calculations, one can show that $\det(A_\times) = -2\det(A)^3 = -2\det(g)^{3/2} < 0$, which means this is a linear system that is solvable and can be used to calculate W at a point.