

Chapter 1

Algorithm for R between two arbitrary points in a mesh

To measure the Dirichlet energy, we need to calculate the rotation coefficient R between two arbitrary points q and p that do not necessarily lie within the same tetrahedron. Since the metric and curl is different in each tet, we need to be able to efficiently determine all tets that get intersected by the straight line from q to p , and use the correct metric for each corresponding line segment. The calculation for the coefficient then works in the following way:

Algorithm 1 Rotation coefficient R between q and p

```
1: Input  $(q, p)$ 
2:   //returns all tets intersected by the line  $\vec{qp}$  with the line segments within them
3:    $\text{LINESEGMENTS} \leftarrow \text{tetFinder}(q, p)$ 
4:    $R \leftarrow \text{Id}$ 
5:   for each  $\text{SEGMENT}$  in  $\text{LINESEGMENTS}$ 
6:      $R \leftarrow R \cdot \text{calcCoeff}(\text{SEGMENT})$ 
7: return  $R$ 
```

The missing component here is how to efficiently find all tetrahedra that get intersected. One possibility would be to use ray-triangle intersection and test against the whole mesh, but this is not practical, as we have local information that we can exploit.

We use the idea of the straight walk from *Walking in a Triangulation*[], which relies only on so called *orientation tests* to determine which triangles we traverse. This chapter covers how the *tetFinder* algorithm works, how it is made robust against degenerate cases and a full description of the algorithm is given, more than what is described in the reference.

1.1 Framework

Let \mathcal{T} be a triangulation of a domain Ω that is convex. The straight walk traverses all triangles that get intersected by the line segment from q to p . The algorithm first makes an initialization step to get into a valid state, then the straight walk can start. To get a feeling how the algorithm works, let us go through an example in 2D. If the algorithm was in a valid state before, the line from q to p intersects with some edge \vec{lr} . Two triangles share this edge. We test on which side point p lies of this edge to decide whether the walk continues. If the walk continues, we jump through the edge to hop from the old triangle to a new one. This triangle is defined by three vertices (l, r, s) . We decide if the new candidate point s lies on the left side or right side of the line from q to p . If s lies on the left, point l is moved, else point r is moved. A new edge intersected with the ray \vec{qp} is found and the walk repeats. This process is illustrated in Figure 1.1

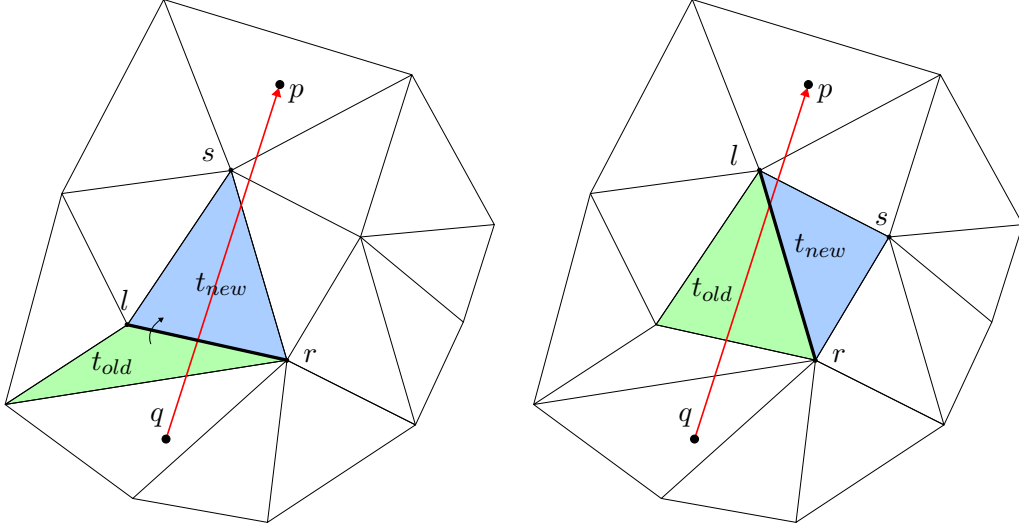


Figure 1.1. Straight walk step

Notice how the ray \vec{qp} always intersects the edge \vec{lr} at each update step. We can use this observation to add each edge at each update step to a list. When the algorithm terminates, we can just iterate over this list, find the intersection point of the ray \vec{qp} with the edge and calculate the rotation coefficient for this segment. The straight walk in 3d works similarly. The initialization step consists of finding a starting tet t where q is contained. Then, we find the face of the tet t that gets intersected by the ray \vec{qp} . Again, at each step, we know that the ray goes out of our current tet t through some face e defined by vertices uvw . We decide if the walk continues by checking on which side p lies relative to e . If the walk should continue, we hop through e to a new tet t_{new} . With two orientation tests we decide which of the vertices u, v, w gets moved to the new candidate point s . This defines a new face e_{new} where our ray intersects and the walk repeats. Degenerate cases such as when the ray \vec{qp} goes exactly through a vertex or when the ray lies within a face, the algorithm may get into an invalid state, where the algorithm may then traverse through cells that do not get intersected by the ray. We need to detect and escape those degenerate cases through some additional checks. These are not described in the paper[?]. In most cases, a simple reinitialization is enough when the invalid state is detected.

1.1.1 A note on orientation tests

To determine on which side some point s lies relative to two other points q and p (that represent a line) in 2D, the geometric *orientation* predicate is used. It corresponds to evaluating the sign of a determinant. Analogously in 3D, the orientation predicate tests whether a fourth point lies above or below a plane defined by three other points. How “above” the plane is defined depends on the ordering within the determinant. In the case here, it is above if point a sees the triangle bcd when turning counterclockwise (see figure 1.2).

$$\text{orientation}(\alpha, \beta, \gamma) = \text{sign} \left(\begin{vmatrix} \beta_x - \alpha_x & \gamma_x - \alpha_x \\ \beta_y - \alpha_y & \gamma_y - \alpha_y \end{vmatrix} \right)$$

$$\text{orientation}(\alpha, \beta, \gamma, \delta) = \text{sign} \left(\begin{vmatrix} \beta_x - \alpha_x & \gamma_x - \alpha_x & \delta_x - \alpha_x \\ \beta_y - \alpha_y & \gamma_y - \alpha_y & \delta_y - \alpha_y \\ \beta_z - \alpha_z & \gamma_z - \alpha_z & \delta_z - \alpha_z \end{vmatrix} \right)$$

It is important that the sign of the determinant is evaluated exactly. If geometric predicates are implemented with floating point arithmetic, the answers may be inconsistent and wrong. Because of the finite mantissa in floating point representation, many numbers cannot be represented exactly. The machine needs to round a number x to its nearest number that is exactly representable. Many times, roundoff errors occur and the result is not exact. A famous example of this phenomenon is $0.1 + 0.2 =$

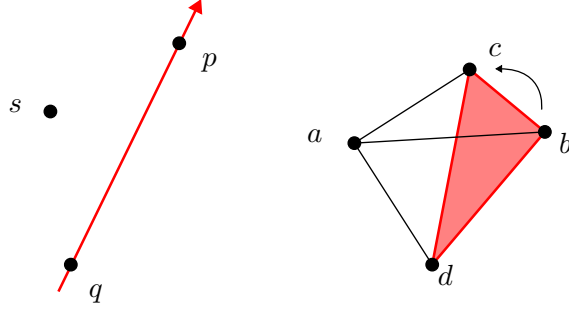


Figure 1.2. Orientation predicate: Point a lies above the plane bcd because it sees the points in counter-clockwise order, $\text{orientation}(a, b, c, d) > 0$

0.30000000000000004. We call the distance between two exactly representable floating point numbers *machine epsilon* ε . See figure 1.3 for how this machine epsilon can cause trouble (Example from [?]). Because of the imprecision of floating-point arithmetic (FP), logical decisions based on FP should be

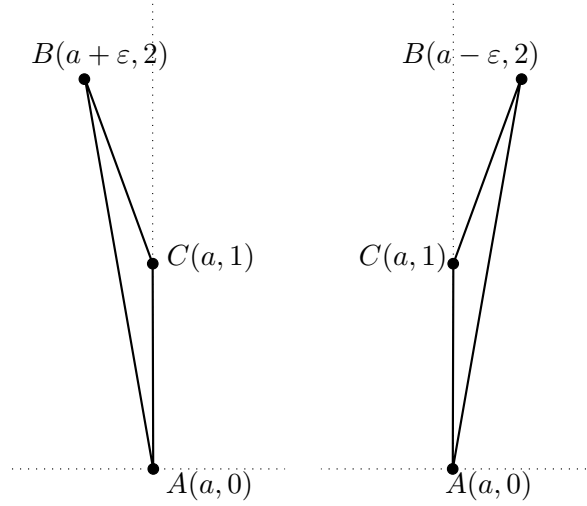


Figure 1.3. The orientation predicate implemented with floating point arithmetic cannot distinguish these two cases. The orientation test calculates $(a + \varepsilon) - a$. If $a = 0$, the orientation test is strictly positive. If $a = 1$, then the result is 0, because during the calculation the machine had to round.

avoided. To fix this issue, we make use of exact predicates that are implemented with arbitrary precision [?]. It is assumed that inputs to the orientation test subroutines are exact, and during the calculation, precision is extended as needed such that we can be sure that the result has the correct sign.

1.2 Algorithm *tetFinder*

As described in section 1.1, the straight walk needs different components to work. These components are described in the following subsections. The final algorithm to find all intersected tetrahedra between two points is as follows.

1.2.1 Starting point

To start, the algorithm needs to know in which tet the starting point q is located. With a simple linear nearest neighbor search of point q , we find a good heuristic starting point s . From s , every tet incident to s is checked if it contains q . Most of the times, this method works for locating which tet contains q . In case it fails, an exhaustive search through all the tets of the mesh is done to search which tet contains q . Figure 1.4 shows a constellation where the nearest neighbor approach fails. Checking if a point q is

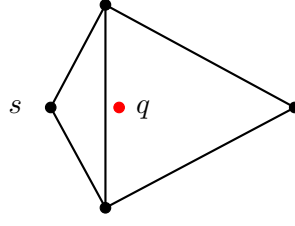


Figure 1.4. The point s is the nearest neighbor to q . However, searching all incident triangles to s , point q cannot be located. A similar example can be constructed in 3D.

within some tet $\mathcal{T} = [a, b, c, d]$ can be done with the orientation predicate, i.e.

$$q \in \mathcal{T} \iff \begin{aligned} & \text{sameSign}(\text{orientation}(a, b, c, d), \text{orientation}(a, b, c, q)) \&\& \\ & \text{sameSign}(\text{orientation}(b, c, d, a), \text{orientation}(b, c, d, q)) \&\& \\ & \text{sameSign}(\text{orientation}(c, d, a, b), \text{orientation}(c, d, a, q)) \&\& \\ & \text{sameSign}(\text{orientation}(d, a, b, c), \text{orientation}(d, a, b, q)) \end{aligned}$$

By checking for each plane of the tet, we check if the point q is on the same side of the plane (characterised by the same orientation sign) as the remaining vertex.

1.2.2 Initialization and valid state

To walk in the tet mesh, the algorithm needs to be in a state where everything is as expected. In 2D, we define this valid state as

- the line \vec{qp} intersects the edge defined by vertices \vec{lr}
- vertex l lies on the left of the line defined by \vec{qp}
- vertex r lies on the right of the line defined by \vec{qp}

If the algorithm starts with these configurations, then in each step the valid configuration will be preserved and all edges intersected are traversed. Analogously in 3D, we define the valid state such that

- the line \vec{qp} intersects the triangle defined by vertices $\triangle uvw$
- vertices of the triangle $\triangle uvw$ are ordered in the way that

$$\text{orientation}(w, v, q, p) > 0, \text{orientation}(v, u, q, p) > 0, \text{orientation}(u, w, q, p) > 0$$

1.2.3 Exact ray-triangle intersection

1.2.4 Robustness

The algorithm uses several subroutines which are described here:

- `neighbor(t through uvw)` returns the tetrahedron sharing face uvw with tetrahedron t
- `vertex of t, $s \neq u, s \neq v, s \neq w$` returns the remaining vertex of a tetrahedron whose other three vertices are known
- `initialization()`

Algorithm 2 tetFinder

```
1: Input  $(q, p)$ 
2:    $t = \text{LocateTet}(q)$ 
3:    $\text{initialization}(t)$ 

4:   //  $qp$  intersects triangle  $uvw$ 
5:   //  $wvqp, vuqp, uwqp$  are positively oriented
6:    $\text{LINESEGMENTS} = []$ 
7:    $\text{PREV} = q$ 
8:    $\text{CURR} = \text{intersection}(qp, uvw)$ 

9:    $\text{LINESEGMENTS.add}([t, \text{PREV}, \text{CURR}])$ 

10:  while  $\text{orientation}(u, w, v, p) > 0$  {
11:     $t = \text{neighbor}(t \text{ through } uvw)$ 
12:     $s = \text{vertex of } t, s \neq u, s \neq v, s \neq w$ 
13:     $\text{PREV} = \text{CURR}$ 
14:    if  $\text{orientation}(u, s, q, p) > 0$  //  $qp$  does not intersect triangle  $usw$ 
15:      if  $\text{orientation}(v, s, q, p) > 0$  //  $qp$  intersects triangle  $vsw$ 
16:         $u = s$ 
17:      else //  $qp$  intersects triangle  $usv$ 
18:         $w = s$ 
19:    else //  $qp$  does not intersect  $usv$ 
20:      if  $\text{orientation}(w, s, q, p) > 0$  //  $qp$  intersects triangle  $usw$ 
21:         $v = s$ 
22:      else //  $qp$  intersects triangle  $vsw$ 
23:         $u = s$ 
24:     $\text{CURR} = \text{intersection}(qp, uvw)$ 
25:     $\text{LINESEGMENTS.add}([t, \text{PREV}, \text{CURR}])$ 
26:  } //  $t$  contains  $p$ 
27:   $\text{LINESEGMENTS.add}([t, \text{PREV}, p])$ 
```
