Probability and statistics

(September-27)

$(\Omega, \mathfrak{F}, P)$, random variable, random vector

Random variable

- discrete: p.m.f.
- cts: pdf

CDF

Moments, Expectations,

MGFs

special distribution

<u>transformations</u>

Random vector

- discrete r.v.
- Continuous r.v.

joint pmf,

marginal pmfs,

independence,

eg $E(XY) = E(X)E(Y)$

$X_1, \ldots, X_n$ iid

$\bar{X}$

(WLLN)

joint CDF

$\boxed{\text{transformation}}$

Joint continuous random variables.

Let X and Y be continuous r.v.s. on the same probability space.

$$F(x,y) = \text{Prob}(X \le x, Y \le y) \qquad -\infty < x, y < \infty$$

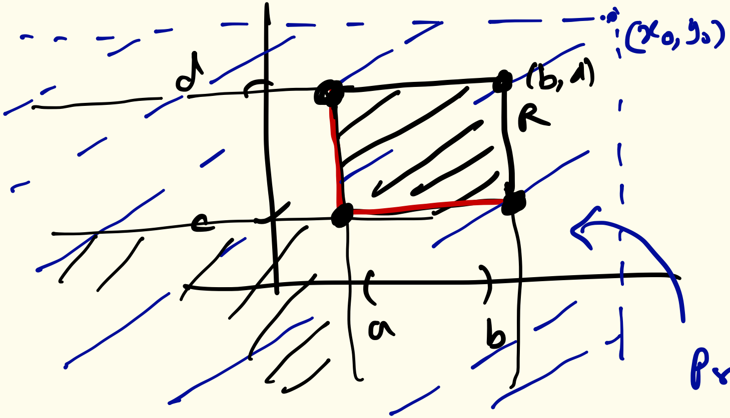
joint CDF

Rectangle $R = \{(x,y) \mid a < x \le b, \ c < y \le d\}$

$$P[(X,Y) \in R]$$

$$= P(a < X \le b, c < Y \le d)$$

$$= F(b,d) - F(a,d) - F(b,c) + F(a,c)$$



Probability of this blue shaded region is $F(x_0, y_0)$

Marginal CDFs.

Marginal CDF of X

$$F_X(x) = P(X \le x) = F(x, \infty) = \lim_{y \to \infty} F(x, y)$$

$$F_Y(y) = P(Y \le y) = F(\infty, y) = \lim_{x \to \infty} F(x, y)$$

If there exists a nonnegative function $f(x, y)$ over $\mathbb{R}^2$ such that

$$F(x, y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f(u, v)\, du\, dv ,$$

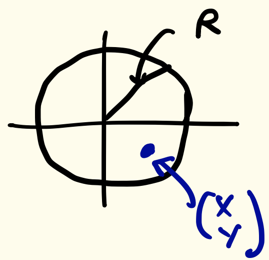
then $f(x, y)$ is called as joint pdf. $\begin{pmatrix} X \\ Y \end{pmatrix}$.

$$P\left(\begin{pmatrix} X \\ Y \end{pmatrix} \in A\right) = \iint_A f(x,y)\,dx\,dy$$

clearly, $\displaystyle\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1$ ✓

and $\displaystyle\frac{\partial^2}{\partial x \partial y} F(x,y) = f(x,y)$ ✓

<u>Ex:</u> Let (X, Y) denote the random vector representing the coordinates of randomly chosen point in circle with centre $(0,0)$ and radius R.



If

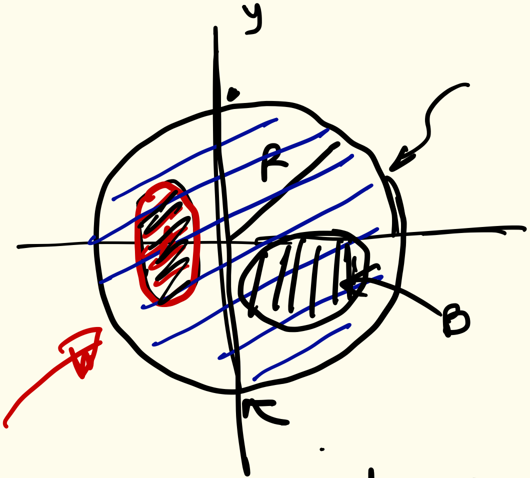
$$f(x,y) = c \quad \text{for } (x,y) \in \text{circle}$$
$$= 0 \quad \text{otherwise.}$$

Find c.

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1 \Rightarrow \iint_{\text{circle}} c \; dx\,dy = 1$$

$$\Rightarrow c = \frac{1}{\pi R^2}$$

$$\boxed{\begin{aligned} f(x,y) &= \frac{1}{\pi R^2} \quad \text{for } (x,y) \in \text{circle} \\ &= 0 \quad \text{otherwise} \end{aligned}}$$



$$f(x,y) = \frac{1}{\pi R^2}$$

uniform joint pdf.

$$\iint_A f(x,y)\,dx\,dy =$$

$$\shortparallel$$

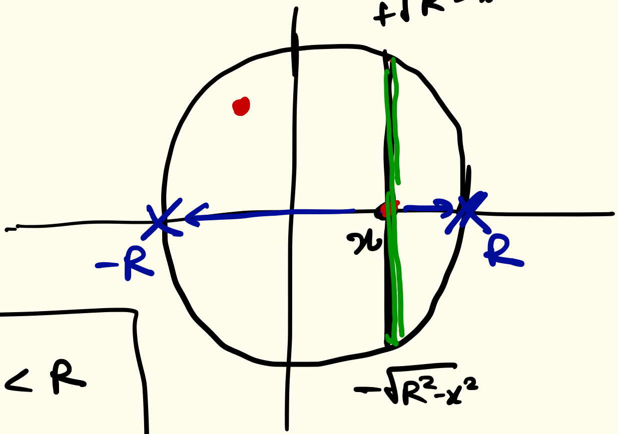
volume below the surface $z = f(x,y)$ above the subset A.

$$P((x,y)\in A) = \frac{\text{Area of } A}{\pi R^2} = \iint_A f(x,y)\,dx\,dy$$

$$F_X(x) = F(x,\infty) = \lim_{y\to\infty} F(x,y)$$

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$$ ---- marginal pdf of X

$$= \int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} \frac{1}{\pi R^2}\,dy$$

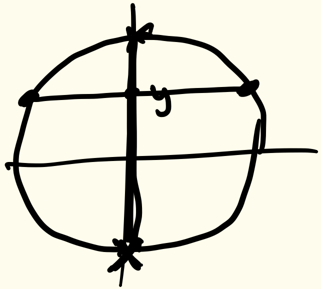


$$\boxed{\begin{aligned} f_X(x) &= \frac{2\sqrt{R^2-x^2}}{\pi R^2}, \quad -R<x<R \\ &= 0 \qquad \text{otherwise} \end{aligned}}$$

$$f_Y(y) = \int_{-\infty}^{\infty} f(x, y)\, dx$$

... Marginal pdf of Y.

$$= \int_{-\sqrt{R^2-y^2}}^{\sqrt{R^2-y^2}} \frac{1}{\pi R^2}\, dx$$



$$f_Y(y) = \frac{2\sqrt{R^2-y^2}}{\pi R^2}, \quad -R < y < R$$

$$= 0 \qquad \text{otherwise}$$

Independence: X and Y are called as independent random variable if and only if

$$f_{XY}(x, y) = f_X(x) f_Y(y)$$

(joint pdf is product of marginal pdfs).

In the previous example, X and Y are **NOT** independent.

Easy way to generate examples:

Let X and Y be two indep. continuous random variables with pdfs $f_X(x)$ and $f_Y(y)$ resp.

$$f_{XY}(x,y) = f_X(x)\, f_Y(y)$$

clearly, $\boxed{f_{XY}(x,y) \geq 0}$ $\quad \forall\ (x,y) \in \mathbb{R}^2$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y)\,dx\,dy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_X(x)\, f_Y(y)\,dx\,dy$$

$$= \int_{-\infty}^{\infty} f_X(x)\,dx \int_{-\infty}^{\infty} f_Y(y)\,dy$$

$$= 1 \times 1$$

$$\boxed{\int_{-\infty}^{\infty}\int_{\infty}^{\infty} f_{XY}(x,y)\,dx\,dy = 1}$$

∎

Ex: $X \sim N(0,1)$ and $Y \sim N(0,1)$

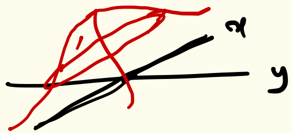
Let X & Y be independent.

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \qquad -\infty < x < \infty$$

$$f_Y(y) = \frac{1}{\sqrt{2\pi}} e^{-y^2/2} \qquad -\infty < y < \infty$$

$$f_{XY}(x,y) = f_X(x) f_Y(y)$$

$$\boxed{f_{XY}(x,y) = \frac{1}{2\pi} e^{-(x^2+y^2)/2}} \qquad -\infty < x, y < \infty$$



Ex: Let X and Y have joint density

$$f(x,y) = c\, e^{-(x^2 - xy + y^2)/2} \qquad -\infty < x, y < \infty$$

Find c??

Marginal of X.

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = c\int_{-\infty}^{\infty} e^{-(x^2 - xy + y^2)/2}\,dy$$

$$= c\int_{-\infty}^{\infty} e^{-\left[(y - x/2)^2 + 3x^2/4\right]/2}\,dy$$

$$= c\, e^{-3x^2/8} \int_{-\infty}^{\infty} \boxed{e^{-(y - x/2)^2/2}}\,dy$$

$\sqrt{2\pi} \times$ pdf of $N(\frac{x}{2}, 1)$

$$f_X(x) = \boxed{c\sqrt{2\pi}}\, e^{-3x^2/8} \quad \text{—— } (*) \qquad -\infty < x < \infty$$

$$\mu = 0, \ \sigma^2 = 4/3$$

pdf of $N(0, 4/3)$

$$\frac{\sqrt{3}}{\sqrt{4}\sqrt{2\pi}}\, e^{-\frac{1}{2}\frac{(x-0)^2}{4/3}}$$

$$= \boxed{\frac{\sqrt{3}}{2\sqrt{2\pi}}}\, e^{-3x^2/4} \quad \text{—— } (**)$$

$$\Rightarrow c = \frac{\sqrt{3}}{4\pi}$$

Distributions of sums and quotients.

Let X and Y be continuous random variables with joint pdf $f(x,y)$.

$\varphi : \mathbb{R}^2 \to \mathbb{R}$

Define $Z = \underbrace{\varphi(X,Y)}$

$\boxed{\begin{array}{l} \text{In particular,} \\ \varphi(X,Y) = X+Y \\ \varphi(X,Y) = X/Y \end{array}}$

For a fixed $z \in \mathbb{R}$, we are are interested in the event

$$\{ Z \leq z \}$$

By $A_z \subseteq \mathbb{R}^2$, define

$$A_z = \{ (x,y) \mid \varphi(x,y) \leq z \}$$

Let F_Z : CDF of Z.

$$
\begin{aligned}
F_Z(z) &= P(Z \leq z) \\
&= P((X, Y) \in A_z) \\
&= \iint_{A_z} f(x, y)\,dx\,dy
\end{aligned}
$$