Probability and statistics

(Lecture 2)

August 17

Repeat the random expt. R 'n' times

Define $f_A = \dfrac{\#\text{ heads in } n \text{ tosses}}{n}$

= relative frequency.

true??

let $n \to \infty$, $\underbrace{\text{see if } f_A \to \underline{p_A}\text{ ??}}$

Assign this number p_A as probability of A or probability of observing head.

observation: [R repeated n time, caculate f_A]

Repeat 'm' times

$n=10$, $f_A^{(1)} = 9/10$, $f_A^{(2)} = 3/10$, $f_A^{(3)} = 1/10$

Observation: Only for certain "key events" one needs to assign probabilities.

For instance: R: Rolling a die

We only assign $p(1)$, $p(2)$, $p(3)$, $p(4)$ and $p(5)$.

Then the probabilities of all other events are **automatically** assigned.

Q: How does this automatic assignment happen??

Because we intuitively assume certain laws of probability.

We want to formalize these laws into axiomatic definition of probability.

R: random expt. , Ω : sample space

<u>Event:</u> A subset of Ω.

<u>Expect:</u> union and intersection of events is also an event to which probability will be assigned. (complement of an event)

<u>Definition:</u> (σ- algebra)

A collection of sets $\mathcal{F}$ is called σ-algebra if

i) $A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$

ii) $A_1, A_2, \ldots \in \mathcal{F}$, then

$$\bigcup_{i=1}^{\infty} A_i \in \mathcal{F} \quad \text{and} \quad \bigcap_{i=1}^{\infty} A_i \in \mathcal{F}$$

$\mathcal{F}$ is "closed" under complementation & countable union & intersection.

<u>Ex:</u> R : Tossing a coin

$\Omega = \{H, T\}$

$\mathcal{F} = 2^{\Omega} = \{\phi, \{H\}, \{T\}, \{H, T\}\}$

<u>Ex:</u> R : Rolling a die

$\Omega = \{1, 2, \dots, 6\}$ $\qquad \# \mathcal{F} = 2^{6}$

$\mathcal{F} = 2^{\Omega}$

<u>Ex:</u> R : Random expt. with finite outcomes

$\Omega = \{1, 2, \dots, s\}$ $\qquad \# \mathcal{F} = 2^{s}$

$\mathcal{F} = 2^{\Omega}$

$\mathcal{F}$: σ-algebra

Ex: R: Random expt with $[0,1]$ as outcomes.

$\Omega \doteq [0,1]$

All the intervals in Ω of the type

$(a, b]$ for $a, b \in [0,1]$

Generate a σ-algebra with these intervals.

(Borel σ-algebra) $\mathcal{F}$

$\{0\} \in \mathcal{F}$

$A_1 = (1/2, 1]$

$A_1^c = [0, 1/2]$

$(0,1]$

Definition: Let Ω be the sample space for a random experiment R. Let $\mathcal{F}$ be the σ-algebra associated with Ω. Then a probability (measure) P is a function on $\mathcal{F}$ to $\mathbb{R}$ $(P: \mathcal{F} \to \mathbb{R})$ such that

i) $P(A) \geq 0 \quad \forall A \in \mathcal{F}$

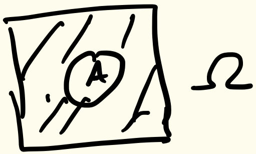
ii) $P(\Omega) = 1$

iii) If A_i are mutually disjoint for $i = 1, 2, \ldots$ $(A_i \cap A_j = \phi$ for $i \neq j$, $i, j = 1, 2, \ldots)$

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$$

Properties: $\mathbb{R}, \Omega, \mathcal{F}, P \quad , A, A_1, A_2, B, C \in \mathcal{F}$

1) $P(A^c) = 1 - P(A)$



observe: $A \cup A^c = \Omega$

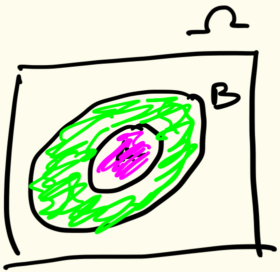
$\Rightarrow \quad P(A \cup A^c) = P(\Omega) = 1$ ← by second axiom

Also, $P(A \cup A^c) = P(A) + P(A^c)$ ← axiom(iii)

$P(A) + P(A^c) = P(\Omega) = 1 \quad \Rightarrow \quad \boxed{P(A^c) = 1 - P(A)}$

2) $P(\phi) = 0$

3) If $A \subseteq B$ then $P(A) \leq P(B)$



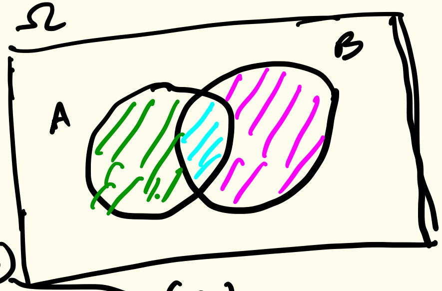
$B = A \cup (A^c \cap B)$

Axiom(iii) $\quad P(B) = P(A) + P(A^c \cap B) \geq P(A)$

$\blacksquare$

Axiom (i) $\quad P(A^c \cap B) \geq 0$

4) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$



$A \cup B = \boxed{(A \cap B^c)} \cup \boxed{(A^c \cap B)} \cup \boxed{(A \cap B)}$

$P(A \cup B) = P(A \cap B^c) + P(A^c \cap B) + P(A \cap B)$ —(1)

$A = (A \cap B^c) \cup (A \cap B) \Rightarrow P(A) = P(A \cap B^c) + P(A \cap B)$ —(2)

$B = (A^c \cap B) \cup (A \cap B) \Rightarrow P(B) = P(A^c \cap B) + P(A \cap B)$ —(3)

From (1), (2), (3)

$P(A \cup B) = P(A) - P(A \cap B) + P(B) - P(A \cap B) + P(A \cap B)$

$\blacksquare$

5) $A_1, A_2, A_3 \in \mathcal{F}$

$$P(\underbrace{A_1}_{A} \cup \underbrace{A_2 \cup A_3}_{B}) = P(A_1) + P(A_2) + P(A_3) - P(A_1 \cap A_2) - P(A_2 \cap A_3) - P(A_1 \cap A_3) + P(A_1 \cap A_2 \cap A_3)$$

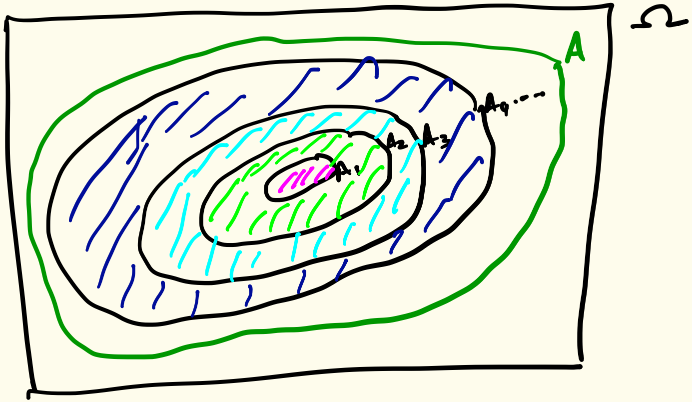
6) $P(A_1 \cup A_2 \cup \dots \cup A_n) =$??

Prove using induction !!

7) Let $A_1 \subset A_2 \subset A_3 \subset \cdots$ (sequence of increasing sets)

and $A = \bigcup_{i=1}^{\infty} A_i$. $(A_1, \dots, A_n, \dots \in \mathcal{F})$

Then $P(A) = \lim_{n \to \infty} P(A_n)$



Define:

$$B_1 = A_1$$

$$B_2 = A_2 \cap A_1^c$$

$$B_3 = A_3 \cap A_2^c$$

$$\vdots$$

$$B_n = A_n \cap A_{n-1}^c$$

$$\bigcup_{i=1}^{n} A_i = \bigcup_{i=1}^{n} B_i = A_n$$

$$P(A_n) = P\left(\bigcup_{i=1}^{n} A_i\right) = P\left(\bigcup_{i=1}^{n} B_i\right) = \sum_{i=1}^{n} P(B_i)$$

Observe: $A = \bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty} B_i = \lim_{n\to\infty} \bigcup_{i=1}^{n} B_i$ ∎

Disjoint cover of Ω

Let $\{B_1, \dots, B_n\}$ be a collection (subset of $\mathcal{F}$)

be such that

i) $B_i \cap B_j = \phi$ for $i \neq j$ $i, j = 1, 2, \dots, n,$

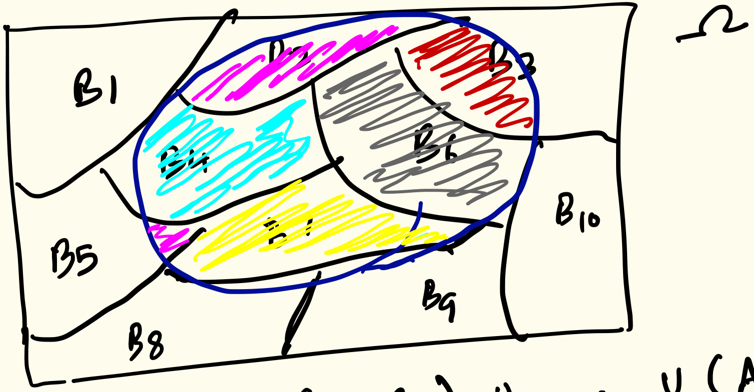
ii) $\bigcup_{i=1}^{n} B_i = \Omega.$

Then $\{B_1, \dots, B_n\}$ is called as disjoint cover of Ω.

Let $\{B_1, \dots, B_n\}$ be a disjoint cover of Ω.

For any $A \in \mathcal{F}$

$$P(A) = P(A \cap B_1) + P(A \cap B_2) + \dots + P(A \cap B_n)$$



$$\Omega = B_1 \cup \cdots \cup B_n$$

$$A \cap \Omega = A$$

$$= A \cap (B_1 \cup \cdots \cup B_n)$$

$$A = (A \cap B_1) \cup (A \cap B_2) \cup \cdots \cup (A \cap B_n)$$

$$P(A) = P(A \cap B_1) + P(A \cap B_2) + \cdots + P(A \cap B_n)$$

Uniform probability space:

Experiment: Picking up a number randomly from a given set.

case (i) Given set is finite and discrete

$\{1, 2, \ldots, s\} = \Omega$

Assignment of probabilities:

Uniform probability principle:

Probability of two subsets of Ω with equal "size" be equal.

Q: What do we mean by size??

Here, in case (i), when Ω is finite and discrete, we can associate size to the cardinality.

As a consequence of this, we can see that all the singletons are of same size (1) and hence according to the uniform probability principle $p(\{k\}) = \frac{1}{s}$ for $k = 1, 2, \dots, s$

This probability space $(\Omega, \mathcal{F}, p)$ is called as discrete uniform probability space.

case (ii) The given set Ω is finite and an interval of $\mathbb{R}$.

In particular, $\Omega = [0,1]$

Principle of uniformity: Pick two subsets of $\Omega = [0,1]$ of equal "size", then they shall have same probability.

Let $(a,b) \subseteq [0,1]$

Then size of (a,b) can be associated with the length of (a,b) given by $b-a$.

Probability assignment:

Take any intval of the form $(a,b]$

$$p((a,b]) = b-a \quad \text{for any } (a,b] \subseteq [0,1]$$

Sigma algebra is Borel σ-algebra.

This is continuous uniform probability space.

Remark:

i) $P(\{0\}) = 0$

In fact, $P(\{a\}) = 0 \quad \forall a \in [0,1]$

$$\bigcup_{a \in [0,1]} \{a\} = [0,1]$$

Is this union countable ??

$$P\underbrace{\{0, 1/2, 1/4, 1/8, 1/16, \dots\}}_{\displaystyle \bigcup_{i=0}^{\infty} \left\{ \frac{1}{2^i} \right\}} = 0$$

$$P\{1/n : n \in \mathbb{N}\} = 0$$