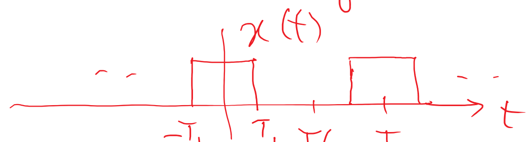
Fourier Transform

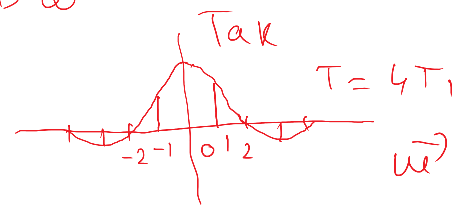
Friday, October 01, 2021 10:02 AM

→ We have both continuous time and discrete time Fourier Transforms just like Fourier series.

Fourier Series expansion as $x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$, $\omega_0 \to \frac{2\pi}{T}$

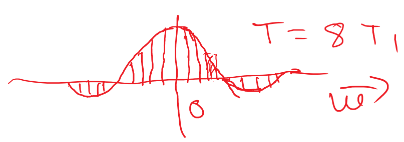
→ Fourier transform is ~~inp~~ interpreted as Fourier series for the limiting value $T \to \infty$



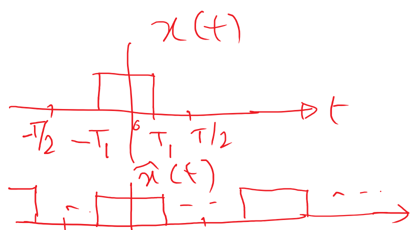


$$a_k = \frac{2\sin(k\omega_0 T_1)}{k\omega_0 T} \quad \text{or} \quad Ta_k = \frac{2\sin(k\omega_0 T_1)}{k\omega_0} = \left.\frac{2\sin(\omega T_1)}{\omega}\right|_{\text{at } \omega = k\omega_0}$$

→ Varying T in terms T_1 as NT_1 shows that as N increases the number of points on Ta_k increases.



$$\hat{x}(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} \quad \text{and}$$



$$a_k = \frac{1}{T}\int_{-T/2}^{+T/2} \hat{x}(t) e^{-jk\omega_0 t}\, dt, \quad \omega_0 = \frac{2\pi}{T}$$

For a single pulse $x(t)$, $\hat{x}(t) = x(t)$ for $|t| < \frac{T}{2}$

As $T \to \infty$, $\hat{x}(t) = x(t)$ and a_k can be evaluated from $-\infty$ to $+\infty$, a_k can be rewritten as, $a_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t) e^{-jk\omega_0 t}\, dt$

$$= \frac{1}{T}\int_{-\infty}^{+\infty} x(t) e^{-jk\omega_0 t}\, dt$$

If we define $X(j\omega)$ as $X(j\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t}\, dt$

as an envelope of Ta_k, Then $a_k = \frac{1}{T} X(jk\omega_0)$

and $\hat{x}(t) = \sum_{k=-\infty}^{+\infty} \frac{1}{T} X(jk\omega_0) e^{jk\omega_0 t}$

$$= \frac{1}{2\pi} \sum_{k=-\infty}^{+\infty} X(jk\omega_0) e^{jk\omega_0 t} \cdot \omega_0$$

$\left[\omega_0 = \frac{2\pi}{T}\right]$

As T increase and $\rightarrow \infty$, $\omega_0 \rightarrow 0$. In this limiting condition the summation can be replaced by integration. $\omega_0 \rightarrow d\omega$

Inverse Transform $x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega) e^{j\omega t} d\omega$, $k\omega_0 \Rightarrow \omega$

and conversely,

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt$$

Forward Transform

Convergence of Fourier Transform

If $x(t)$ has finite energy i.e. if it is square integrable, so that $\int_{-\infty}^{+\infty} |x(t)|^2 dt < \infty$.

Then it is guaranteed that $X(j\omega)$ is finite.

→ other Drichilet conditions on finite number of extrema and finite number of discontinuities (finite value) also need to be satisfied.

Ex. → Fourier Transform exists for both periodic and aperiodic signal; For Fourier series $w = k w_0$, where w_0 has a finite value k is an integer.

Ex.

$$x(t) = e^{-at} u(t), \quad a > 0, \quad \text{aperiodic signal}$$

$$X(jw) = \int_0^{\infty} e^{-at} e^{-jwt} dt = \int_0^{\infty} e^{-(a+jw)t} dt$$

$$= -\frac{1}{a+jw} e^{-(a+jw)t} \Big|_0^{\infty} = \frac{1}{a+jw}, \quad a > 0$$

Ex

$$x(t) = \delta(t)$$

$$X(jw) = \int_{-\infty}^{+\infty} \delta(t) e^{-jwt} dt = 1$$

→ For periodic signals $\delta(w - w_0)$ appears at discrete values of w i.e. at $w = w_0$

For example, if we have $X(jw) = 2\pi \delta(w - w_0)$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} 2\pi \underbrace{\delta(w - w_0)}_{\text{non-zero only at } w = w_0} e^{jwt} dw$$

$$= e^{jw_0 t} \Leftarrow \text{which is a periodic function}$$

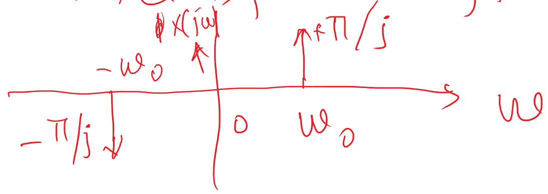
Ex.

$$x(t) = \sin(w_0 t)$$

We have seen that Fourier series coefficient,

$$a_1 = \frac{1}{2j} \quad \text{and} \quad a_{-1} = -\frac{1}{2j}$$

Fourier Transform coefficients appear at $w = \pm w_0$



Ex.

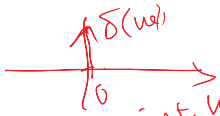
$$X(jw) = \delta(w) + \delta(w - \pi) + \delta(w - 5)$$

$\delta(w)$ | $h(t) = u(t) - u(t-2)$

Ex. $X(j\omega) = \delta(\omega) + \delta(\omega - \pi) + \delta(\omega - 5)$

and $h(t) = u(t) - u(t-2)$

Find $y(t) = x(t) * h(t)$



$\frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(\omega) e^{j\omega t} d\omega = \frac{1}{2\pi}$

$$x(t) = \frac{1}{2\pi} + \frac{1}{2\pi} e^{j\pi t} + \frac{1}{2\pi} e^{j5t} \Leftarrow \text{non-periodic}$$

$$H(j\omega) = e^{-j\omega} \frac{2\sin\omega}{\omega}, \quad \text{At } \omega \to 0, \ H(j\omega) \Rightarrow 2$$

$$y(j\omega) = X(j\omega)\, H(j\omega)$$

$H(j5) \leftarrow$ complex value

$$= 2\delta(\omega) + \delta(\omega - 5). \quad \left[\begin{array}{l}\text{At } \omega = \pi, \\ H(j\omega) = 0\end{array}\right]$$

$$\therefore y(t) = \frac{1}{2\pi}\cdot 2 + \frac{1}{2\pi} e^{j5t} \cdot H(j5)$$ // check

$$u(t) \xleftrightarrow{FT} \frac{1}{j\omega} + \pi\delta(\omega)$$

Properties

→ All properties are similar to Fourier Series for periodic (and aperiodic function as well)

Linearity $\quad a x(t) + b y(t) \xleftrightarrow{FT} a X(j\omega) + b Y(j\omega)$

Time shift $\quad x(t - t_0) \xleftrightarrow{FT} e^{-j\omega t_0} X(j\omega)$

Conjugate Symmetry $\quad x^*(t) \xleftrightarrow{FT} X^*(-j\omega)$

Differentiation $\quad \frac{d}{dt} x(t) \xleftrightarrow{FT} j\omega X(j\omega)$

Integration $\quad \int_{-\infty}^{t} x(t)\, dt \xleftrightarrow{FT} \frac{1}{j\omega} X(j\omega) + \pi X(0)\delta(\omega)$

Time scaling $\quad x(at) \xleftrightarrow{FT} \frac{1}{|a|} X\left(\frac{j\omega}{a}\right)$

Convolution $\quad y(t) = x(t) * h(t) \xleftrightarrow{FT} Y(j\omega) = X(j\omega).$

$H(j\omega)$

Multiplication $r(t) = s(t)\cdot p(t) \xleftrightarrow{FT} \frac{1}{2\pi}\{S(j\omega) * P(j\omega)\}$

Differential in frequency $\quad t\,x(t) \xleftrightarrow{FT} j\frac{d}{d\omega}X(j\omega)$

$$\frac{d^2y(t)}{dt^2} + 4\frac{d}{dt}y(t) + 3y(t) = \frac{d}{dt}x(t) + 2x(t)$$

Taking Fourier Transform,

$$Y(j\omega)\{(j\omega)^2 + 4j\omega + 3\} = X(j\omega)\{j\omega + 2\}$$

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{j\omega + 2}{(j\omega)^2 + 4j\omega + 3}$$

$$= \frac{\frac{1}{2}}{1 + j\omega} + \frac{\frac{1}{2}}{j\omega + 3}$$

$$\therefore h(t) = \frac{1}{2}e^{-t}u(t) + \frac{1}{2}e^{-3t}u(t)$$

$$\left[\ e^{-at}u(t) \xleftrightarrow{FT} \frac{1}{a + j\omega},\ \mathrm{Re}[a] > 0 \right.$$

$$\int_{-\infty}^{\infty} e^{-at}e^{-j\omega t}\,dt$$

$$t\,e^{-at}u(t) \xleftrightarrow{FT} \frac{1}{(a + j\omega)^2},\ \mathrm{Re}[a] > 0$$

Discrete-Time Fourier Transform

$$\tilde{x}[n] = \sum_{k=\langle N\rangle} a_k e^{j\left(\frac{2\pi}{N}\right)n} \quad \Leftarrow \text{Discrete time Fourier Series}$$

($\frac{2\pi}{N} \leftarrow \omega_0$)

As $N \to \infty$, $\omega_0 \to 0$

Discrete /

$$x[n] = \frac{1}{2\pi}\int X(e^{j\omega})e^{j\omega n}\,d\omega$$

Discrete Time Fourier Transform

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})\, e^{j\omega n}\, d\omega$$

and

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n]\, e^{-j\omega n}$$

$$X\left(e^{j(\omega+2\pi)}\right) = \sum_{n=-\infty}^{+\infty} x[n]\, e^{-j(\omega+2\pi)n}$$

$$= \sum_{n=-\infty}^{+\infty} x[n]\, e^{-j\omega n} \cdot e^{-j2\pi n}$$

$$|e^{-j2\pi n}| = 1$$

Ex.

$$x[n] = a^n u[n], \quad |a| < 1$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} a^n u[n]\, e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} (a e^{-j\omega})^n = \frac{1}{1 - a e^{-j\omega}}$$

Ex.

$$x[n] = \begin{cases} 1, & |n| \le N_1 \\ 0, & |n| > N_1 \end{cases}$$

$$X(e^{j\omega}) = \sum_{n=-N_1}^{N_1} e^{-j\omega n} = \frac{\sin\left(N_1 + \frac{1}{2}\right)}{\sin\left(\frac{\omega}{2}\right)}$$

Ex.

$$x[n] = \cos(\omega_0 n)$$

$$= \frac{1}{2} e^{j\omega_0 n} + \frac{1}{2} e^{-j\omega_0 n}$$

$$\Leftarrow X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} \left[\frac{1}{2} e^{j\omega_0 n} + \frac{1}{2} e^{-j\omega_0 n}\right] e^{-j\omega n}$$

$$= \sum_{l=-\infty}^{+\infty} \pi\,\delta(\omega - \omega_0 - 2\pi l) + \sum_{l=-\infty}^{+\infty} \pi\,\delta(\omega + \omega_0 - 2\pi l)$$

$$S = \sum_{n=-\infty}^{+\infty} \frac{1}{2} e^{j(\omega_0 - \omega)n}$$

at $\omega = \omega_0$, $S = \frac{1}{2}\pi\delta(\cdot)$

$$\frac{1}{2}\sum_{n=-\infty}^{+\infty}\left(\cos(\omega_0 - \omega)n + j\sin(\omega_0 - \omega)n\right)$$

$$\pi\,\delta(\omega - \omega_0)$$

$\frac{1}{2}\sum_{n=-\infty}$ $\qquad l=-\infty$ $\qquad l=-\infty$

[Let them $X'(e^{j\omega}) = \pi\delta(\omega-\omega_0) + \pi\delta(\omega+\omega_0)$]

Then evaluate $x[n]$ to get $\cos(\omega_0 n)$.

$$= \pi\,\delta(\omega-\omega_0) + \pi\,\delta(\omega+\omega_0), \quad -\pi \le \omega \le \pi$$

Ex. $x[n] = u[n-2] - u[n-6]$

$$= \delta[n-2] + \delta[n-3] + \delta[n-4] + \delta[n-5]$$

$$X(e^{j\omega}) = e^{-2j\omega} + e^{-3j\omega} + e^{-4j\omega} + e^{-5j\omega}$$

Ex. $x[n] = \left(\frac{1}{2}\right)^{-n} u[-n-1]$

$$X(e^{j\omega}) = \sum_{-\infty}^{-1} \left(\frac{1}{2}\right)^{-n} e^{-j\omega n} = \sum_{m=1}^{\infty} \left(\frac{1}{2} e^{j\omega}\right)^m$$

$$\left[\frac{1}{1-\frac{1}{2}e^{j\omega}} - 1\right] \qquad = \frac{e^{j\omega}}{2}\,\frac{1}{1-\frac{1}{2}e^{j\omega}}$$

Ex. $x[n] = \dfrac{\sin\left(\frac{\pi n}{5}\right)}{\pi n} \cos\left(\dfrac{7\pi n}{2}\right)$

$$\Downarrow \qquad\qquad \Downarrow$$

$$x_1[n] \qquad\qquad x_2[n]$$

$$X_1(e^{j\omega}) = \begin{cases} 1, & |\omega| < \frac{\pi}{5} \\ 0, & \frac{\pi}{5} \le |\omega| < \pi \end{cases}$$

$$X_2(e^{j\omega}) = \pi\left[\delta\left(\omega - \frac{\pi}{2}\right) + \delta\left(\omega + \frac{\pi}{2}\right)\right]$$ as $\frac{7\pi}{2} = 3\pi + \frac{\pi}{2}$

$X(e^{j\omega})$ = Periodic convolution $X_1(e^{j\omega})$ and $X_2(e^{j\omega}) \Rightarrow$ train of $X_1(e^{j\omega})$

$\Downarrow$

due to convolution of box function and the delta function

Ex.

$$x[n] = \alpha^{|n|}, \quad |\alpha| < 1$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} \alpha^{|n|} e^{-j\omega n}$$

equivalent to $\alpha^{|n|}$ in the negative half

$$= \sum_{n=-\infty}^{-1} \alpha^{-n} e^{-j\omega n} + \sum_{n=0}^{\infty} \alpha^{n} e^{-j\omega n}$$

$$= \frac{1}{1-\alpha^{-1}e^{-j\omega}} + \frac{1}{1-\alpha e^{-j\omega}} = \frac{1-\alpha^2}{1-2\alpha\cos\omega+\alpha^2}$$

Ex.

Given $h[n] = \left(\frac{1}{2}\right)^n u[n]$

Find output $y[n]$ for input $x[n] = (n+1)\left(\frac{1}{4}\right)^n u[n]$

$$X(e^{j\omega}) = \frac{1}{\left(1-\frac{1}{4}e^{-j\omega}\right)^2}$$ [due to derivative rule]

$$H(e^{j\omega}) = \frac{1}{1-\frac{1}{2}e^{-j\omega}}$$

$$\therefore Y(e^{j\omega}) = X(e^{j\omega}) \cdot H(e^{j\omega})$$

$$= \frac{}{(1-\frac{1}{4}e^{-j\omega})^2} \cdot \frac{}{1-\frac{1}{2}e^{-j\omega}}$$

$$= \frac{2}{1-\frac{1}{4}e^{-j\omega}} - \frac{3}{(1-\frac{1}{4}e^{-j\omega})^2} + \frac{4}{1-\frac{1}{2}e^{-j\omega}}$$

Check

$$y[n] = 2\left(\frac{1}{4}\right)^n u[n] - 3(n+1)\left(\frac{1}{4}\right)^n u[n] + 4\left(\frac{1}{2}\right)^n u[n]$$

Properties of DTFT

Periodic in frequency domain : $X\left(e^{j(\omega+2\pi)}\right) = X(e^{j\omega})$

Time shifting: $x[n-n_0] \xleftrightarrow{DTFT} e^{-j\omega n_0} X(e^{j\omega})$

Difference : $x[n] - x[n-1] \xleftrightarrow{DTFT} (1-e^{-j\omega}) X(e^{j\omega})$

Summation : $\sum_{m=-\infty}^{+\infty} x[m] \xleftrightarrow{DTFT} \frac{1}{1-e^{-j\omega}} X(e^{j\omega}) + \pi X(e^{j0}) \sum_{k=-\infty}^{+\infty} \delta(\omega - 2\pi k)$

Convolution : $x[n] * y[n] \xleftrightarrow{DTFT} X(e^{j\omega}) Y(j\omega)$

Multiplication : $x[n] \cdot y[n] \xleftrightarrow{DTFT} \frac{1}{2\pi} \int_{2\pi} X(e^{j\theta}) \cdot Y(e^{j(\omega-\theta)}) \, d\theta$

$\Downarrow$

periodic convolution