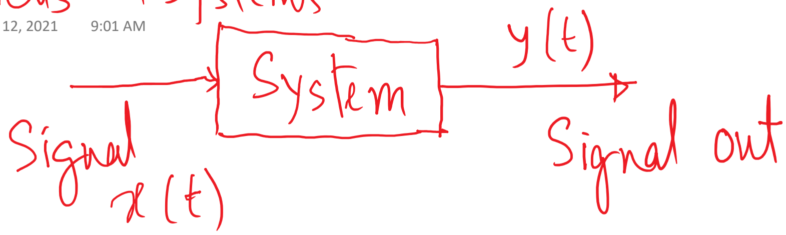
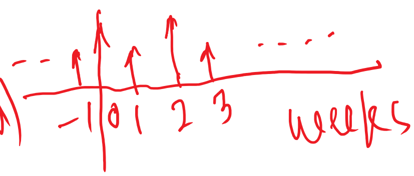
Signals and Systems

Thursday, August 12, 2021 9:01 AM



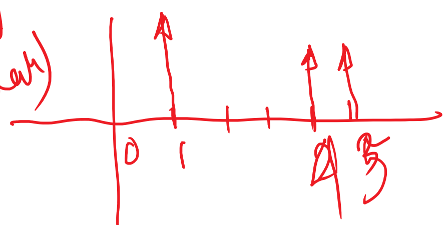
Continuous

Digital (regular interval)



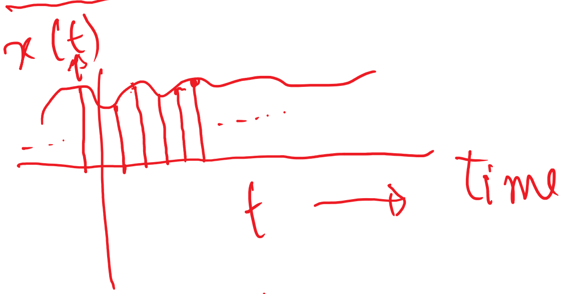
1st week - 20 mm
4th week - 10 mm
5th week - 15 mm

Digital (irregular)



Discrete time

→ independent variable time is discretized but the signal value is not discretized



Signal Types

examples ⇓

- → periodic signal [sinusoidal signal]
- → aperiodic signal [exponential signal]
- → impulse signal

→ step signal

property: Signal Energy and Signal power

Signal energy < finite interval / infinite interval

Energy $E_\infty = \underset{T\to\infty}{Lt} \int_{-T}^{T} |x(t)|^2 dt = \int_{-\infty}^{+\infty} |x(t)|^2 dt$

For the discrete case,

$$E_\infty = \underset{N\to\infty}{Lt} \sum_{n=-N}^{N} |x(n)|^2 = \sum_{n=-\infty}^{+\infty} |x(n)|^2$$

for real and complex valued signal

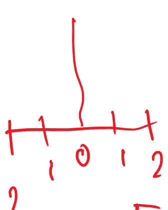
→ If this integration or summation converges to a finite value, Typically the signal an energy signal.

Similarly we can define power signal and subsequently power of a signal

$$P_\infty = \underset{T\to\infty}{Lt} \frac{1}{2T} \int_{-T}^{+T} |x(t)|^2 dt$$

and for the discrete case

$$P_\infty = \underset{N\to\infty}{Lt} \frac{1}{2N+1} \sum_{n=-N}^{+N} |x(n)|^2$$



From the definition of power and energy of a signal, we can infer that

→ for an energy signal $P_\infty = 0$, because E_∞ is finite for a energy signal.

→ for a power signal, P_∞ = finite (non-zero), and therefore $E_\infty = \infty$, since essentially E_∞ is integration P_∞ over $-\infty$ to $+\infty$.

Examples

$$x(t) = A\exp(-t) \text{ for } t \geq 0$$
$$= 0 \text{ for } t < 0$$

$$E_\infty = \int_{-\infty}^{+\infty} A^2 \exp(-2t)\, dt = \int_0^\infty A^2 \exp(-2t)\, dt$$

$$= \frac{A^2}{2}$$

$$P_\infty = \lim_{T\to\infty} \left(\frac{1}{2T} \int_{-T}^{T} A^2 \exp(-2t)\, dt \right.$$

$$= \lim_{T\to\infty} \frac{A^2}{4T} = 0$$

ex2 Take a sinusoidal signal,

$$x(t) = A\sin(w_0 t + \phi)$$

This is periodic with period $\frac{2\pi}{w_0}$

$$P_\infty = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{+T} A^2 \sin^2(w_0 t + \phi)\, dt$$

$$P_\infty = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} A^2 \sin^2(\omega_0 t + \phi)\, dt$$

$$= \frac{A^2\,\omega_0}{4\pi}\int_{-2\pi/\omega_0}^{+2\pi/\omega_0}\left[\frac{1}{2} - \frac{1}{2}\cos(2\omega_0 t + 2\phi)\right] dt$$

[Evaluating for one period]

$$= \frac{A^2}{2} \Rightarrow \text{This is a power signal}$$

$$\therefore E_\infty \to \infty$$

[Can be done without substitution]

Discrete case

$$x(n) = \begin{cases} \frac{1}{n}, & n \geq 1 \\ 0, & n < 1 \end{cases}$$

$$\text{Energy} = \sum_{n=1}^{\infty}\left(\frac{1}{n}\right)^2 = \frac{\pi^2}{6}$$

$$x(n) = \begin{cases} 3(-1)^n, & n \geq 0 \\ 0, & n < 0 \end{cases}$$

$$E_\infty = \sum_{n=-\infty}^{+\infty} |x(n)|^2 \Rightarrow \text{sum of positive numbers}$$

$$= \infty$$

$$P_\infty = \lim_{N\to\infty} \frac{1}{2N+1}\left(9\sum_{n=0}^{N} 1\right) = \lim_{N\to\infty}\frac{9(N+1)}{2N+1}$$

$$= 4.5$$

Transformation of Independent variable.

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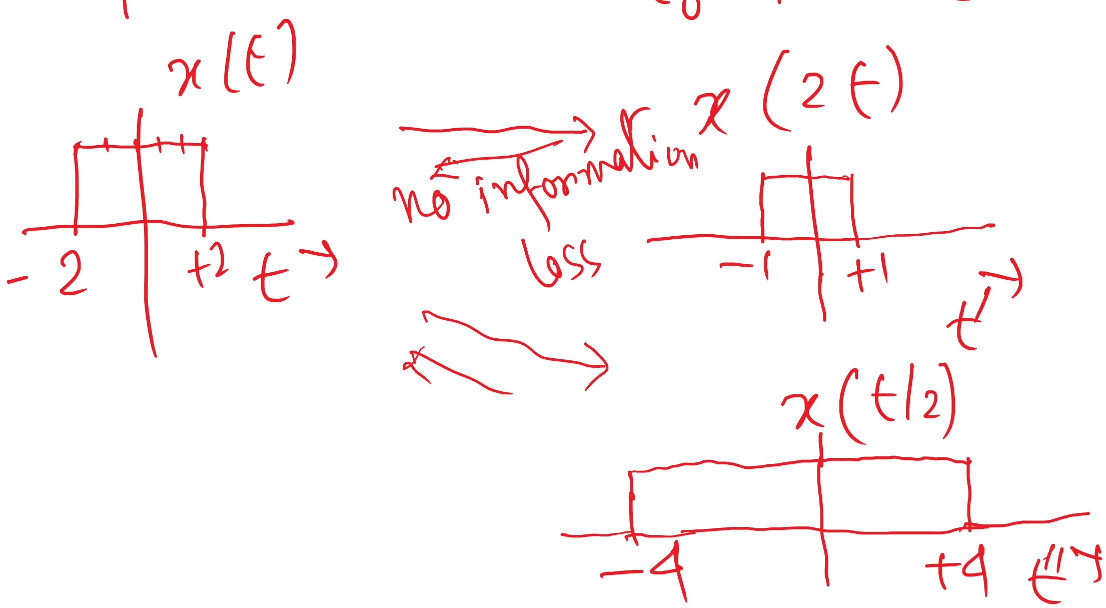
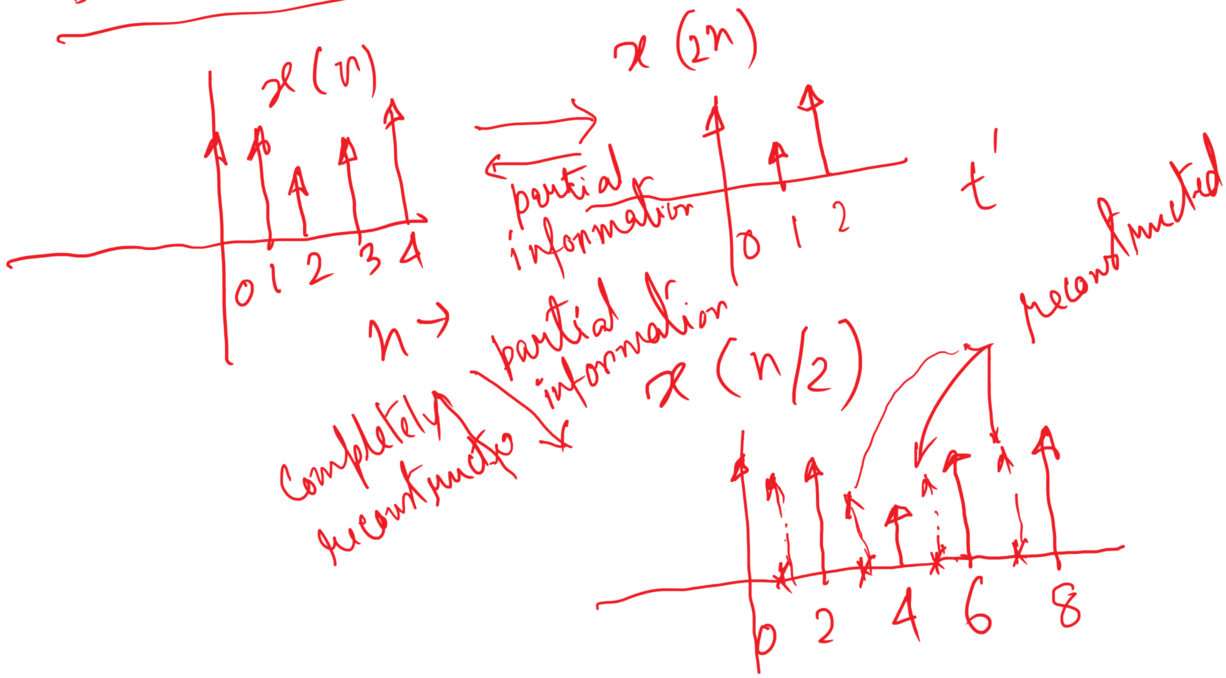
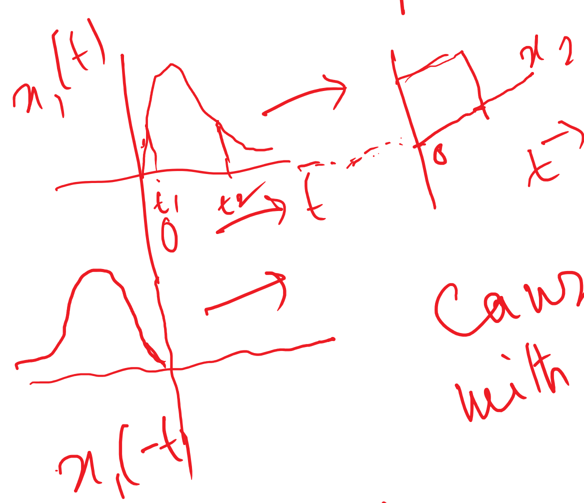
- In 1-D independent variable like time we can manipulate by shifting or scaling.
- For example for scaling, signals can be observed at a denser interval or a rarer interval by scaling time with a scale factor of α eigther $\alpha < 1$ or $\alpha > 1$
- similarly the signal can be shifted in time eigther by a positive value or a negative value.



Discrete case



Reflection



Causal interaction with $x_2(t)$ is maintained

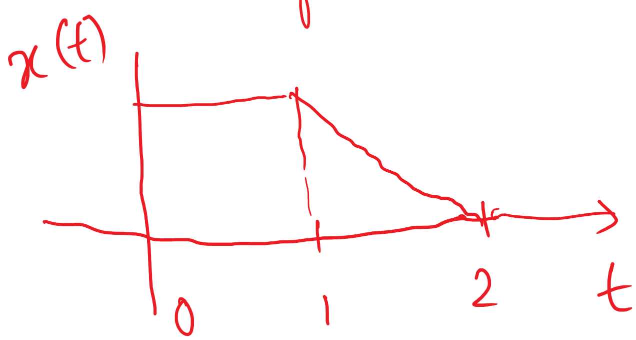
Combination of shift & time scaling

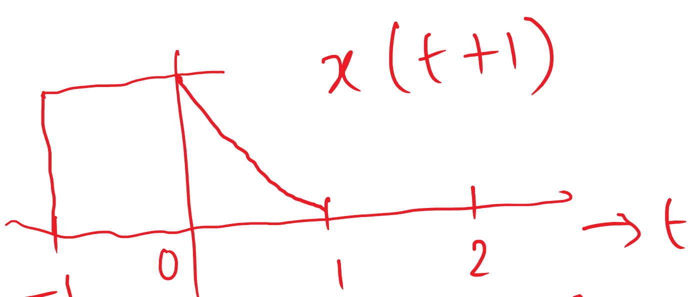
$x(t) \rightarrow x(\alpha t + \beta)$

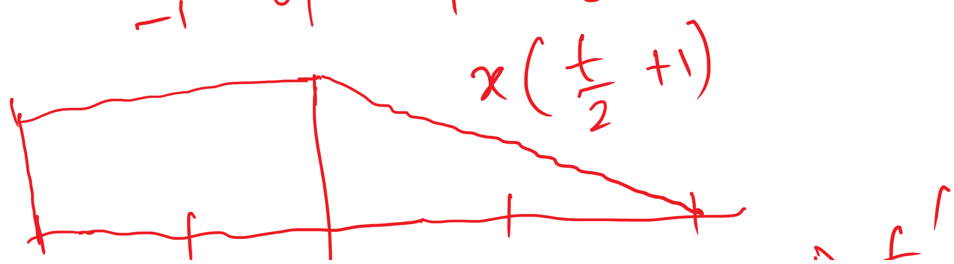
$$x(t) \Rightarrow x(\alpha t + \beta)$$

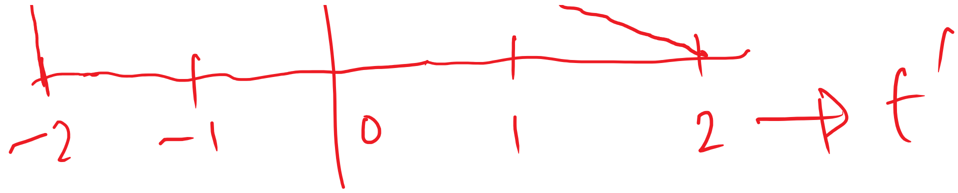
$$x\left(\frac{t}{2} + 1\right)$$

$\Rightarrow$ In this case first shift by 1 and then expand t by 2, since this shift is referred with respect to $x(t)$ and not with respect to $x\left(\frac{t}{2}\right)$. To make it with respect to $x\left(\frac{t}{2}\right)$, shift need to be multiplied by 2 or in general β scaled as $\frac{\beta}{\alpha}$








-2
-1
0
1
2
→ f'

Periodic Signal

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For a periodic signal, $x(t) = x(t+T)$ for all t for a fixed value of T

The period is T and its integer multiples $2T, 3T, \ldots$ also satisfy the condition of periodicity.

For a discrete-time signal,

$$x[n] = x[n+N]$$

N ← fixed number

Even and odd signal

→ A signal is even if $x(-t) = x(t)$ and in discrete time case $x[-n] = x[n]$

→ A signal is odd if

$$x(-t) = -x(t)$$

and for discrete-time, $x[-n] = -x[n]$

odd signal need to be 0 at $t=0$ or

at $n=0$, since $x(-0) = -x(0)$

→ A signal can be decomposed into even and odd components.

$$x_e(t) = \text{Even}[x(t)] = \frac{1}{2}[x(t) + x(-t)]$$

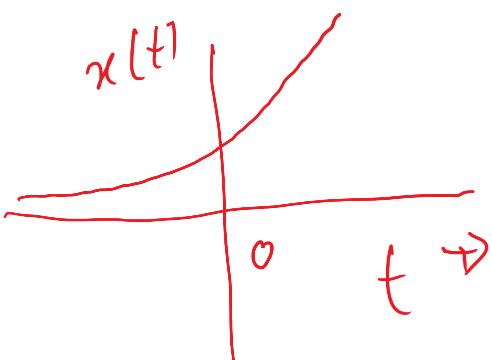
$$x_o(t) = \text{odd}[x(t)] = \frac{1}{2}[x(t) - x(-t)]$$

$$x_e[n] = \text{Even}[x[n]] = \frac{1}{2}[x[n] + x[-n]]$$

$$x_o[n] = \text{odd}[x[n]] = \frac{1}{2}[x[n] - x[-n]]$$

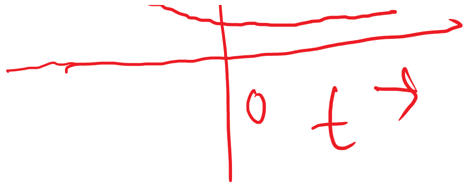
Exponential and sinusoidal signal

$x(t) = Ce^{at}$, C & a can be real or complex.



a, c are +ve

a is −ve, c is +ve



Complex exponential

$$x(t) = e^{j\omega_0 t}$$

→ It is periodic unlike real exponential

i.e. $e^{j\omega_0(t+T)} = e^{j\omega_0 t} \cdot e^{j\omega_0 T}$

$\therefore$ This is periodic if $e^{j\omega_0 T} = 1$

[for $\omega_0 = 0$, $e^{j\omega_0 T} = 1$ for any T]

$\Rightarrow$ fundamental period is undefined for $\omega_0 = 0$

If $\omega_0 \neq 0$, then $T_0 = \dfrac{2\pi}{|\omega_0|}$,

$T_0 \rightarrow$ fundamental period.

$$\text{Power for 1 period} = \frac{1}{T_0}\int_0^{T_0} \left|e^{j\omega_0 t}\right|^2 dt$$

$$= \frac{1}{T_0}\int_0^{T_0} 1 \cdot dt = 1$$

$$P_\infty = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{+T} \left|e^{j\omega_0 t}\right|^2 dt = 1$$

$\Rightarrow$ condition of periodicity $\rightarrow e^{j\omega T_0} = 1$

$\therefore T_0$ is multiple of 2π, or

$\omega T_0 = 2\pi k, \; k = 0, \pm 1, \pm 2, \cdots$

In general, $\phi_k(t) = e^{jk\omega_0 t}, \; k = 0, \pm 1, \pm 2 \cdots$

for $k = 0$, $\phi_k(t) =$ constant

& for $k \neq 0$, $\phi_k(t)$ is periodic, $T = \dfrac{2\pi}{|k|\omega_0}$

$\omega_0 \rightarrow$ fundamental frequency with respect to T_0

General Complex exponential.

Take $\quad c = |c| e^{j\theta}$ and $a = r + j\omega_0$

$$c e^{at} = |c| e^{j\theta} e^{(r + j\omega_0)t}$$
$$= |c| e^{rt} e^{j(\omega_0 t + \theta)}$$

Discrete-time complex exponential

Wednesday, August 25, 2021 11:51 AM

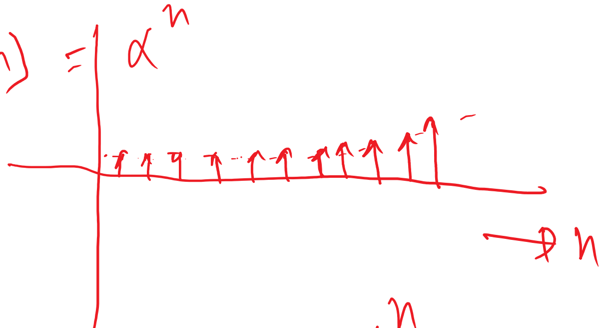
$x[n] = C\alpha^n$, C & α are both complex

This can be written as,

$$x[n] = Ce^{\beta n}, \quad \alpha = e^{\beta}$$

→ Behaviour is similar to continuous one if α is real and positive

$x[n] = \alpha^n$

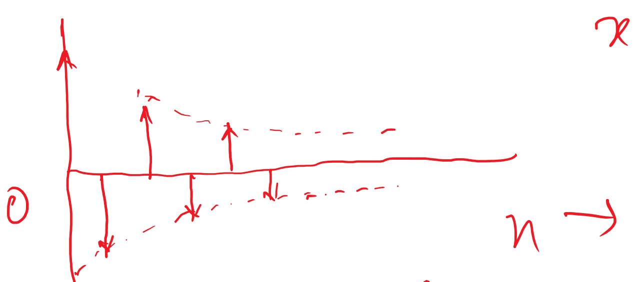


$\alpha > 1$

$x[n] = \alpha^n$

$0 < \alpha < 1$

→ If α is −ve & real, $x[n]$ oscillates



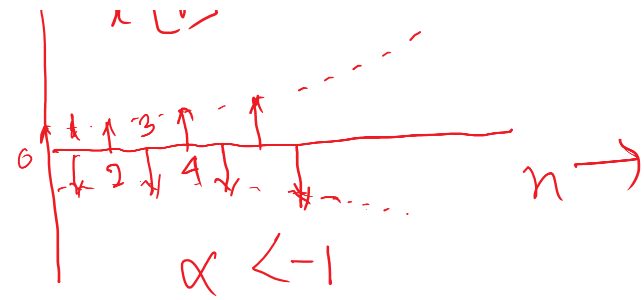
$x[n] = \alpha^n$

$= (-0.5)^n$,

$n = 0, 1, 2 \ldots$

$-1 < \alpha < 0$

$x[n] = \alpha^n$



Sinusoidal signal in discrete time

$Ce^{\beta n} \Rightarrow$ imaginary β gives $|e^{\beta}| = 1$

Let $x[n] = e^{j\omega_0 n}$

$= \cos(\omega_0 n) + j \sin(\omega_0 n)$

Take

$$A\cos(\omega_0 n + \phi) = \frac{A}{2} e^{j\phi} e^{j\omega_0 n} + \frac{A}{2} e^{-j\phi} e^{-j\omega_0 n}$$

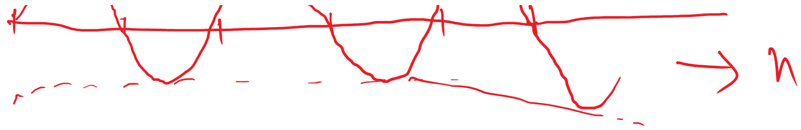
Let

$C = |c| e^{j\theta}$ and $\alpha = |\alpha| e^{j\omega_0}$

Then, $C\alpha^n = |c|\,|\alpha|^n \cos(\omega_0 n + \theta)$

$+ j\,|c|\,|\alpha|^n \sin(\omega_0 n + \theta)$

$\rightarrow$ Depending on the value of α, this will either a damped sinusoid or a exponentially diverging sinusoid



Periodicity of discrete-time exponential

Take $x[n] = e^{j(\omega_0 + 2\pi)\cdot n}$

$$= e^{j2\pi n}\, e^{j\omega_0 n} = e^{j\omega_0 n}$$

$$e^{j\pi n} = (-1)^n, \quad e^{j2\pi n} = 1$$

→ Unlike the continuous time sinusoids, discrete time sinusoids are unique only in a principal interval of $0 \le \omega_0 \le 2\pi$

or, $-\pi \le \omega_0 \le \pi$

Now considering definition of periodicity

$$e^{j\omega_0 (N+n)} = e^{j\omega_0 n} \text{ for periodic signal}$$

$\therefore$ $e^{j\omega_0 N} = 1$ or $\omega_0 = 0$ satisfies this relation.

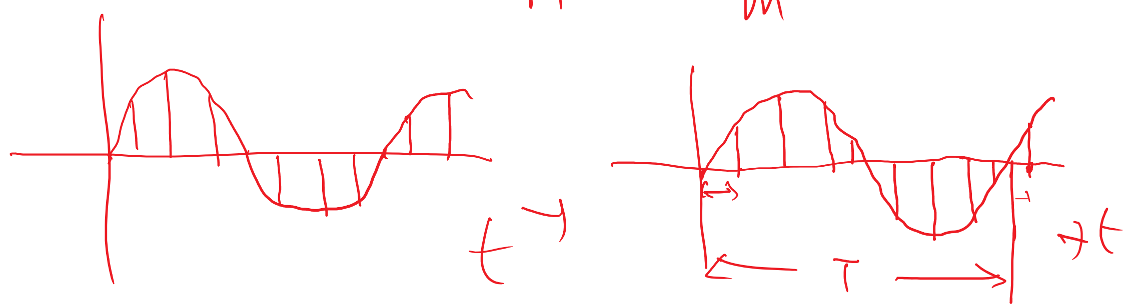
$$e^{j\omega_0 N} = 1 \Rightarrow N \text{ is multiples of } \frac{2\pi}{\omega_0}$$

or $\omega_0 N = 2\pi m$, m is an integer

or $\frac{\omega_0}{\ } = \frac{m}{\ } \Rightarrow$ rational number.

or $\frac{\omega_0}{2\pi} = \frac{m}{N} \Rightarrow$ rational number.

If $x[n]$ is periodic with a fundamental period of N, it fundamental frequency is $\frac{2\pi}{N}$ and therefore $\frac{2\pi}{N} = \frac{\omega_0}{m}$



Continuous-time	Discrete-time
• $e^{j\omega_0 t}$	• $e^{j\omega_0 n}$
• Distinct for for distinct ω_0	• signal identical for values of ω_0 separated by multiples of 2π.
• fundamental frequency ω_0	• fundamental frequency $\frac{\omega_0}{m}$
• periodic for a choice of ω_0	• Periodic only if $\omega_0 = \frac{2\pi m}{N}$
• fundamental period $\frac{2\pi}{\omega_0} = T$	• fundamental period $m \cdot \frac{2\pi}{\omega_0}$, $\omega_0 \neq 0$

period $\frac{2\pi}{w_0} = T$, $w_0 \neq 0$

- period can be integer or real

$m \cdot \frac{2\pi}{w_0}$, $w_0 \neq 0$

- period can be only integer

Ex.

$x(t) = \cos\left(\frac{2\pi t}{12}\right)$ & $x[n] = \cos\left(\frac{2\pi n}{12}\right)$

$\Rightarrow$ both have period $= 12$

$x(t) = \cos\left(\frac{8\pi t}{31}\right)$ & $x[n] = \cos\left(\frac{8\pi n}{31}\right)$

$\Downarrow$ period $= \frac{31}{4}$

$\Downarrow$ period $= 31$

Unit impulse & step function

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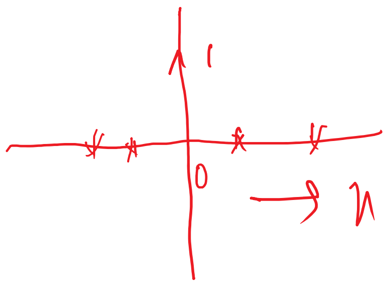
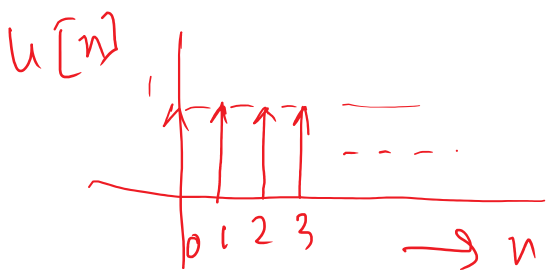
Discrete-time

$$\delta[n] = \begin{cases} 0, & n \neq 0 \\ 1, & n = 0 \end{cases}$$



$$u[n] = \begin{cases} 0, & n < 0 \\ 1, & n \geq 0 \end{cases}$$



$$\delta[n] = u[n] - u[n-1]$$

and $$u[n] = \sum_{m=-\infty}^{n} \delta[m]$$

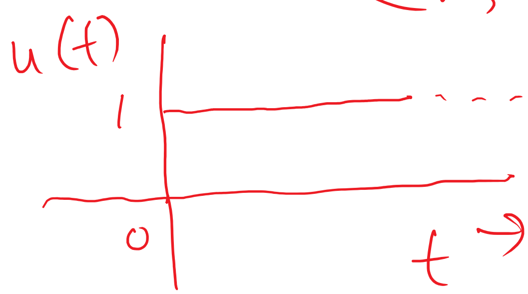
→ $\delta[n]$ has the property of sampling on picking up a particular n.

$$x[n]\,\delta[n] = x[0]\,\delta[n] = x[0]$$

$$x[n]\,\delta[n-n_0] = x[n_0]\,\delta[n-n_0] = x[n_0]$$

Continuous domain step function

$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t > 0 \end{cases}$$



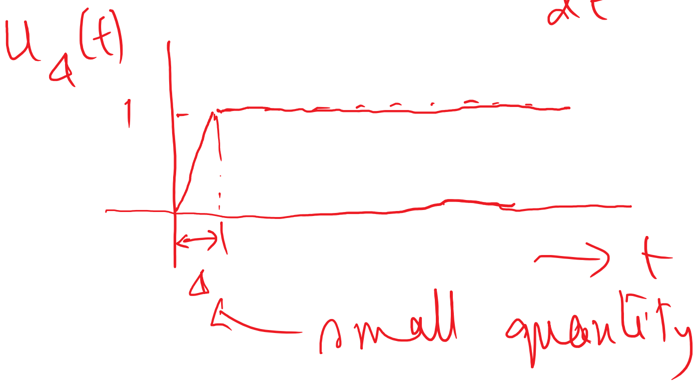
→ Continuous time δ function is related to unit step function as

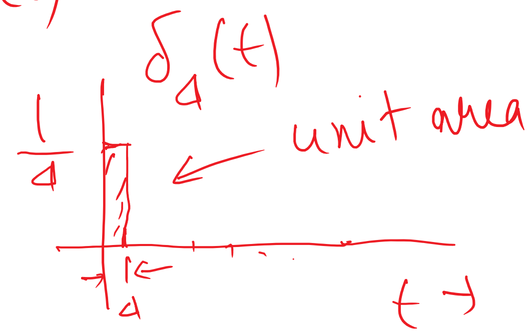
$$u(t) = \int_{-\infty}^{t} \delta(\tau)\, d\tau, \quad \tau \text{ is also in time}$$

$$\delta(t) = \frac{d}{dt} u(t) \quad [\text{This can not be defined at } t=0]$$

→ Since $u(t)$ is discontinuous at $t=0$, it is not differentiable exactly at $t=0$. This can be done approximately over a short interval Δ.

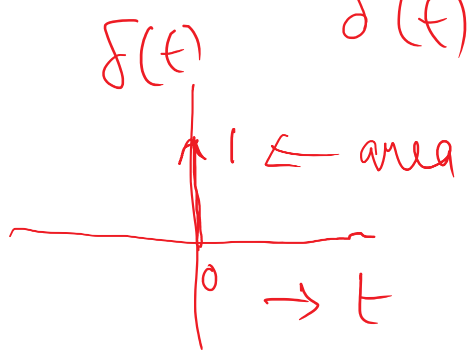
... approximately over a short interval Δ, $\delta_\Delta(t) = \frac{d}{dt} u_\Delta(t)$





→ $\delta_\Delta(t)$ is short pulse of duration Δ and unit area. $\delta(t)$ can be defined in a limiting sense.

$$\delta(t) = \lim_{\Delta \to 0} \delta_\Delta(t)$$



$$u(t) = \int_{-\infty}^{t} \delta(\tau)\, d\tau = \int_{\infty}^{0} \delta(t-\sigma)(-d\sigma)$$

$$= \int_{0}^{\infty} \delta(t-\sigma)\, d\sigma$$

→ Similar to the discrete case, $\delta(t)$ has the sampling property.

... assuming that

Now, the sampling property.

$$x_1(t) = x(t)\,\delta_\Delta(t),$$ assuming that $x(t)$ is constant over Δ duration

As $\Delta \to 0$, $x(t)\cdot\delta(t) = x(0)\,\delta(t)$

and $x(t)\,\delta(t-t_0) = x(t_0)\,\delta(t-t_0)$.

Continuous time and Discrete time System

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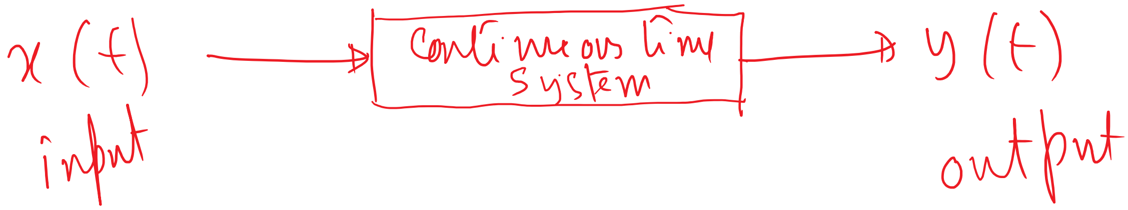
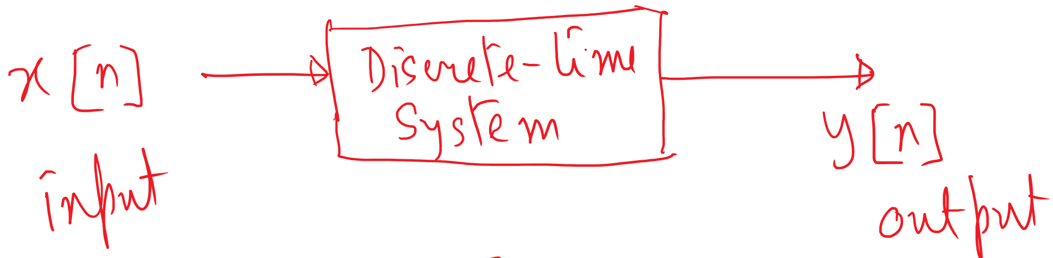
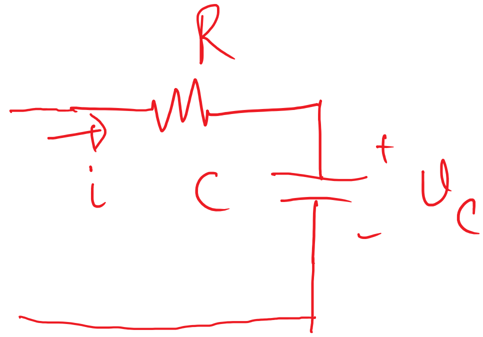






$$i(t) = \frac{v_s(t) - v_c(t)}{R}$$

$$i(t) = C\frac{d}{dt}v_c(t)$$

$v_c(t) \rightarrow$ output

$v_s(t) \rightarrow$ input

$$\frac{d}{dt}v_c(t) + \frac{1}{RC}v_c(t) = \frac{1}{RC}v_s(t)$$

Discrete-time System

$$y[n] = 1.01\,y[n-1] + x[n]$$

$\Rightarrow$ Constant coefficient difference equation

$$y[n] - 1.01\,y[n-1] = x[n]$$

$$\sum a_k \, y[n-k] = \sum b_{k_1} \, x[n-k_1]$$

k, k_1 positive & >0 gives past value

k, k_1 negative gives future value

$$y[n] = x[n+1] - x[n-1]$$

$$y'[n] = 0.5\,x[n+1] - x[n-1]$$

Systems with & without memory

→ A system which require to store the past values or use past values of input or output or both is a system with memory.

→ A system which does not require past or future value is a system without memory

$$y[n] = 2x[n] - x^2[n]$$

$\Rightarrow$ memory less system

$$y[n] = (x[n] + x[n-1])/2$$
$$y[n] = y[n-1] + x[n]$$

$\Rightarrow$ system with memory

$y(t) = R\,x(t) \Rightarrow$ system without memory

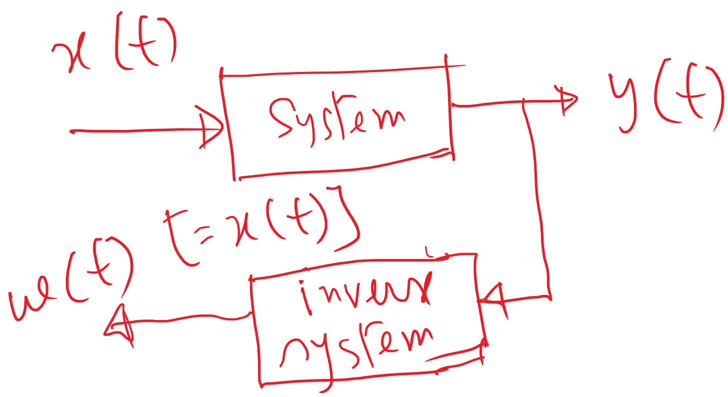
Inverse system

output $\rightarrow$ $y(t)$, input $\rightarrow$ $x(t)$

Original system $\Rightarrow$ $y(t) = 2\,x(t)$

Inverse system $\Rightarrow$ $w(t) = \frac{1}{2} y(t) = x(t)$

input



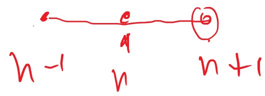
Causality

A system is causal if the output at any time depends only on the values input at at present and/or past instants values of input at present and/or past instants.

$$y[n] = \sum_{k=-\infty}^{n} x[k] \Rightarrow \text{causal}$$

$$y'[n] = x[n] - x[n-1] \Rightarrow \text{causal}$$

$$y[n] = x[n+1] - x[n] \Rightarrow \text{non-causal}$$



$$y[n] = x[n]^2 \Rightarrow \text{causal}$$

Stability

→ If a system gives bounded output for bounded input, the system is said to be stable (BIBO stability)

or in discrete domain it is with $x[n]$ & $y[n]$

For $|x(t)| < B$, $B < \infty$ for all t

or $-B < x(t) < B$

if $|y(t)| < B_1$, $B_1 < \infty$ for all t

or $-B_1 < y(t) < B_1$

Then the system is said to be stable

$x[n] = u[n]$

unit step

$$y[n] = \sum_{k=-\infty}^{n} u[k] = \begin{cases} 0, & n < 0 \\ n+1, & n \geq 0 \end{cases}$$

As n can go up to $+\infty$, therefore $y[n]$ is not bounded, the system is not BIBO stable.

$$y[n] = x[n] - x[n-1]$$

$\Rightarrow$ if x is bounded y is also bounded

$\therefore$ This is BIBO stable

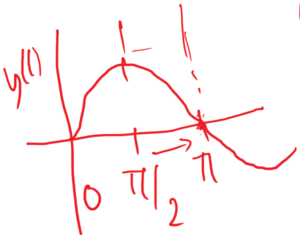
$$\boxed{\begin{aligned} y[n] &= 1, \text{ for } |x[n]| \geq B_2 \\ &= 0, \text{ otherwise} \\ &\Rightarrow \text{not a linear system} \end{aligned}}$$

Time Invariance

$\rightarrow$ A system is time invariant if the behaviour and characteristics of the system remain same over time.

$$\boxed{x(t) \rightarrow y(t)} \leftarrow \text{system}$$

$$x(t-t_0) \rightarrow y(t-t_0)$$



Given $y(t) = \sin(x(t))$

Then $y_1(t) = \sin(x_1(t))$, for $x(t) = x_1(t)$

Let $x_2(t) = x_1(t-t_0)$

$x_1(t-t_0)$ Then, $y_2(t) = \sin(x_2(t))$

$\boxed{y_2(t) = y_1(t-t_0)}$ $= \sin(x_1(t-t_0))$

Again, $y_1(t-t_0) = \sin(x_1(t-t_0))$

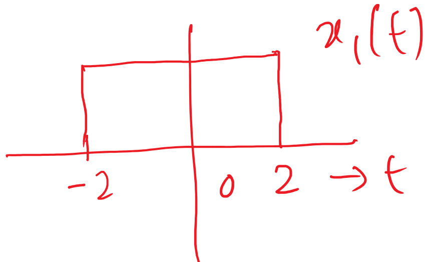
$\Rightarrow$ The relation holds good irrespective of shift., Thus it is time-invariant

$$y(t) = x(2t)$$

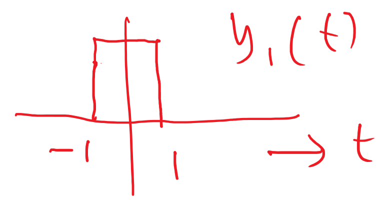
Let $x(t) = x_1(t)$, Then $y_1(t) = x_1(2t)$

$$x(t-t_0) = x_2(t) \Rightarrow y_2(t) = x_2(2t)$$
$$= x(2(t-t_0))$$

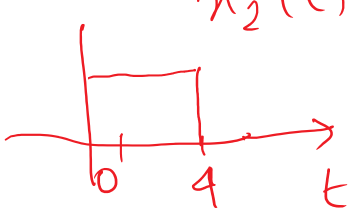
$$y_2(t) = y_2(t-t_0)$$
$$= x(2t-t_0)$$



$\Rightarrow$

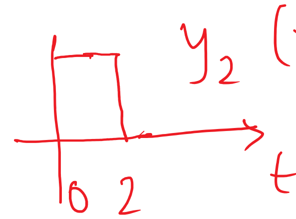


$x_2(t) = x_1(t-2)$



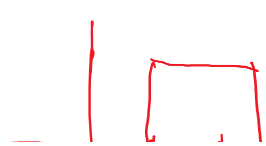
$\Rightarrow$

$y_2(t) = x_2(2t)$

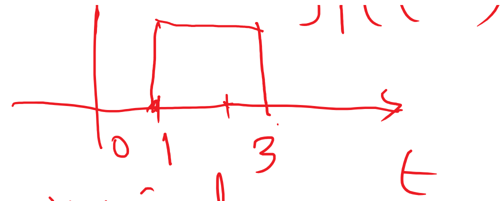


$y_2(t) \neq y_1(t-2)$

$y_1(t-2)$



$y_2(t) \neq y_1(t-2)$



$\therefore$ The system is time variant

Ex.

$$y[n] = \sum_{k=-\infty}^{n} x[k]$$

Let $x_1[n] = x[n-n_0]$

Then $y_1[n] = \sum_{k=-\infty}^{n} x_1[k] = \sum_{k=-\infty}^{n} x[k-n_0]$

Taking $k_1 = k - n_0$

$$y_1[n] = \sum_{k_1=-\infty}^{n-n_0} x[k_1] \quad \text{----- (1)}$$

Next if we shift $y[n]$ by n_0,

$$y[n-n_0] = \sum_{k=-\infty}^{n-n_0} x[k] \quad \text{----- (2)}$$

$\therefore \quad y_1[n] = y[n-n_0] \Rightarrow$ time invariant

---

$$y[n] = x[Mn], \quad -\infty < n < +\infty$$

(M ↑ integer)

$$x_1[n] = x[n-n_0]$$

$$\Rightarrow y_1[n] = x_1[Mn] = x[Mn - n_0]$$

Again, $$y[n-n_0] = x[M(n-n_0)]$$

$$= x[Mn - Mn_0]$$

$$\therefore y_1[n] \neq y[n-n_0]$$

$\Downarrow$

time variant.

Linearity

$\rightarrow$ A system is linear if, for a system with input $x(t)$ and output $y(t)$, satisfy the following relations.

superposition $\Leftarrow$ $x_1(t) + x_2(t) \rightarrow y_1(t) + y_2(t)$

scaling $\Leftarrow$ $a x_1(t) \longrightarrow a y_1(t)$, a in general can be complex

Or, in general,

$$a x_1(t) + b x_2(t) \rightarrow a y_1(t) + b y_2(t)$$

or in discrete domain

$$a x_1[n] + b x_2[n] \rightarrow a y_1[n] + b y_2[n].$$

$$y[n] = 2x[n] + 3$$

Let $x_1[n] = 2$ and $x_2[n] = 3$

Then, $y_1[n] = 7$, $y_2[n] = 9$

$$x_3[n] = x_1[n] + x_2[n] = 5$$

$$y_3[n] = 13 \neq [\underbrace{y_1[n] + y_2[n]}_{16}]$$

$\Rightarrow$ This is non-linear system.

Ex: $y(t) = t\,x(t)$

$$x_1(t) \rightarrow y_1(t) = t \cdot x_1(t)$$

$$x_2(t) \rightarrow y_2(t) = t \cdot x_2(t)$$

$$x_3(t) = a x_1(t) + b x_2(t)$$

$$y_3(t) = t\,x_3(t) = a t\,x_1(t) + b t\,x_2(t)$$

$$= a y_1(t) + b y_2(t)$$

$\Rightarrow$ linear system

Linear Time Invariance System

Friday, August 27, 2021 10:17 AM

→ Consequence of linearity is that output can readily be inferred as a linear combination of a number of input.

→ Further, if we apply time invariance, an arbitrary signal can be represented by sum of weighted & delayed impulses, applying linearity and time invariance.

In general, $x[n] = \sum_{k=-\infty}^{+\infty} x[k]\,\delta[n-k]$

→ In fact it can very easily be extended to a system.

→ Let $h_k[n]$ denote the response of the linear system due to the shifted unit impulse response $\delta[n-k]$

Then the output $y[n]$ of a linear system to an input $x[n]$ is

$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]\, h_k[n]$$

→ Since $\delta[n-k]$ is the shifted version of $\delta[n]$, $h_k[n]$ is shifted version of $h_0[n]$; $h_k[n] = h_0[n-k]$

Let us represent $h[n] = h_0[n]$ for convenience

Then $$y[n] = \sum_{k=-\infty}^{+\infty} x[k]\, h[n-k]$$

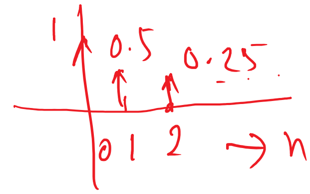
↑

Convolution relation

$$y[n] = x[n] * h[n]$$

→ If $h[n]$ i.e. impulse response is known, $y[n]$ can be found out for any arbitrary $x[n]$

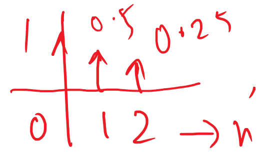
$x[n] \Rightarrow$



$, \{x[0], x[1], x[2]\}$

$\sum_k x[k]\, h[n-k]$

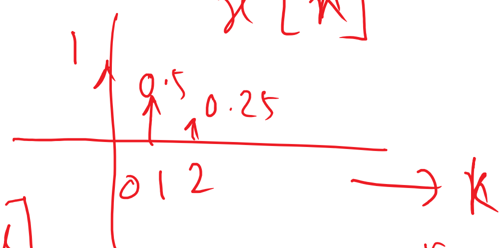
$h[n] =$



$, \{h[0], h[1], h[2]\}$

reflecting or folding → $h[-k]$

$x[k]$

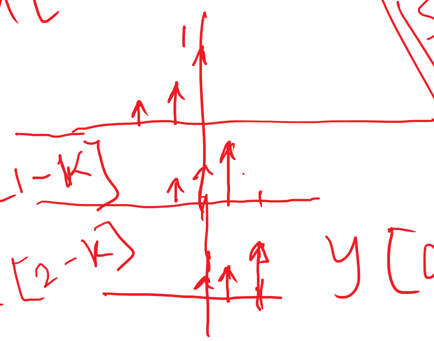


$-\infty \longrightarrow$

$h[-k]$

$h[1-k]$

$h[2-k]$



Shifting, multiplication & sum

$y[0] = x[0] \cdot h[0]$
$= 1$

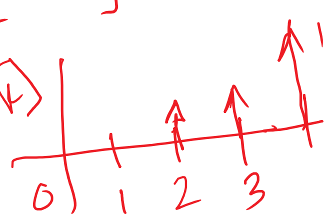
$y[1] = 1 \times 0.5 + 0.5 \times 1$
$= 1$

$h[-k+3]$

$y[2] = 1 \times 0.25 + 0.5 \times 0.5$
$+ 0.25 \times 1$

$= 0.75$

$h[4-k]$



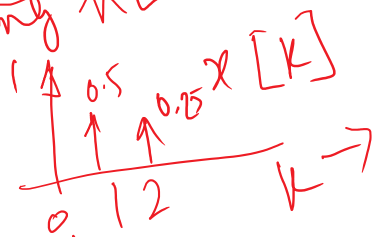
$y[3] = 0.5 \times 0.25$
$+ 0.25 \times 0.5$

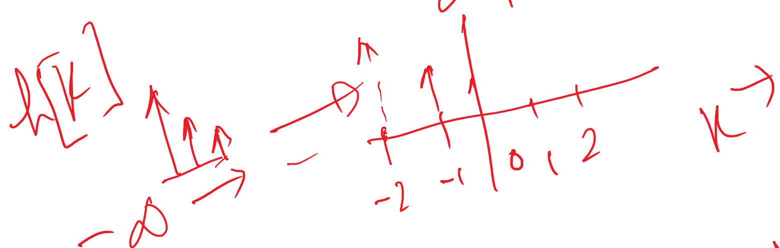
$= 0.25$

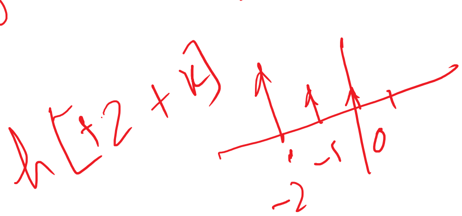
$y[4] = 0.25 \times 0.25$

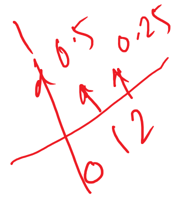
$= 0.0625$

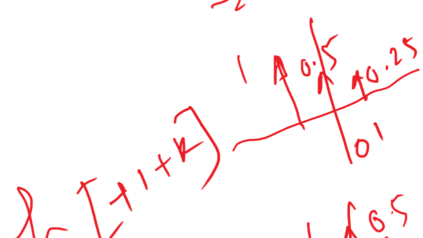
without folding $h[k]$











$$y_1(n) = \sum_{k=-\infty}^{+\infty} x(k)\, h(n+k)$$

$y_1(-2) = 0.25 \times 1 = 0.25$

$y_1(-1) = 0.5 \times 1 + 0.25 \times 0.5$

$= 0.5 + 0.125$

$= 0.625$

$y_1(0) = 1 \times 1 + 0.5 \times 0.5$

$+ 0.25 \times 0.25$

$= 1 + 0.25 + 0.0625$

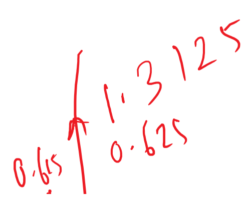
$= 1.3125$

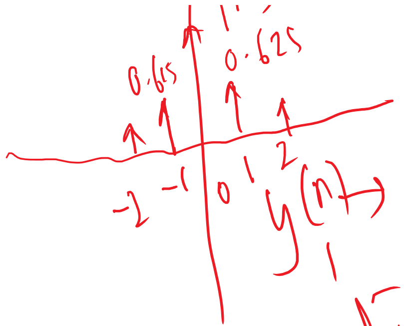
$y_1(1) = 1 \times 0.5$

$+ 0.5 \times 0.25$

$= 0.625$

$y_1(2) = 0.25$





<u>Correlation</u> calculates similarity between $x[k]$ and $h[k]$

→ Autocorrelation calculates the similarity between different parts of the same signal

→ Cross-correlation calculates the similarity between two different signals

→ Often correlation is quantified by correlation coefficient

$$\text{Correlation coefficient} = \frac{1}{N}\sum_{k=0}^{N-1} x[k]\,h[n+k]$$

→ In vector form, correlation can be computed by taking the dot product of x and H as $X^T H$.

compute

X and H as $X^T H$

Convolution of finite and infinite sequence

Wednesday, September 01, 2021 12:37 PM

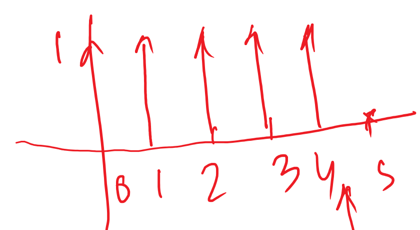
→ Similar to the convolution of two finit discrete-time sequences, convolution sum can be evaluated for infinite sequences as well.

Ex.

$$x[n] = \begin{cases} 1, & 0 \le n \le 4 \\ 0, & \text{otherwise} \end{cases}$$

$$h[n] = \begin{cases} \alpha^n, & 0 \le n \le 6 \\ 0, & \text{otherwise} \end{cases}$$

$x[n] \rightarrow$



$h[n] \rightarrow$

$\alpha > 1$

$$y[n] = \sum_{k=0}^{n} x[k]\, h[n-k] = \sum_{k=0}^{n} \alpha^k = \frac{1-\alpha^{n+1}}{1-\alpha}$$

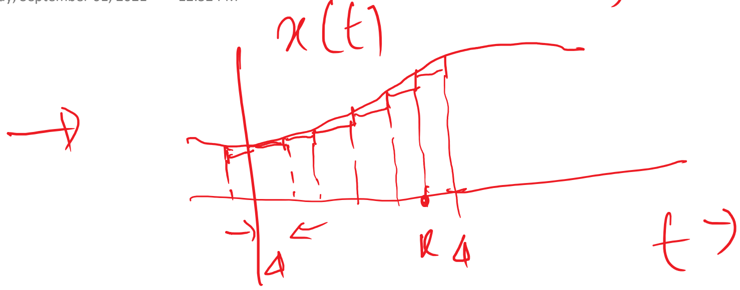
for $\alpha \neq 1$

The value of $y[n]$ need to be computed

upto the non-overlapping portion of the finite sequence i.e in this particular case, for $n < 0$ there is no overlap, so $y[n] = 0$, $n < 0$ and also for $n > 4$ there is no overlap of $x[n]$ and folded & shifted $h[n]$

Continuous time LTI system

Wednesday, September 01, 2021 12:52 PM



Staircase approximation of a continuous time signal $x(t)$ is assumed to be constant for ~~an~~ a small interval of Δ.

$$\delta_\Delta(t) = \begin{cases} \frac{1}{\Delta}, & 0 \le t < \Delta \\ 0, & \text{otherwise} \end{cases}$$

$\Rightarrow \delta_\Delta(t)\cdot\Delta = 1, \ 0 \le t < \Delta$

Approximated signal

$$\hat{x}(t) = \sum_{k=-\infty}^{+\infty} x(k\Delta)\, \delta_\Delta(t - k\Delta)\, \Delta$$

$$\therefore x(t) = \lim_{\Delta \to 0} \sum_{k=-\infty}^{+\infty} x(k\Delta)\, \delta_\Delta(t - k\Delta)\cdot\Delta$$

As $\Delta \to 0$, the summation can be approximated by integration.

Then, $$x(t) = \int_{-\infty}^{+\infty} x(\tau)\, \delta(t-\tau)\, d\tau$$

Continuous-time impulse response

Let $\hat{h}_{k\Delta}(t)$ be the response of LTI system to the input of $\delta_\Delta(t - k\Delta)$

Then,

$$\hat{y}(t) = \sum_{k=-\infty}^{+\infty} x(k\Delta)\,\hat{h}_{k\Delta}(t)\cdot\Delta$$

, following the same staircase logic as followed for $x(t)$

Now, as $\Delta \to 0$,

$$y(t) = \lim_{\Delta \to 0} \sum_{k=-\infty}^{+\infty} x(k\Delta)\,h_{k\Delta}(t)\cdot\Delta$$

Since $\Delta \to 0$, the summation can be replaced by integration, & further replacing $k\Delta$ by a continuous time variable τ

$$y(t) = \int_{-\infty}^{+\infty} x(\tau)\,h_\tau(t)\,d\tau$$

$\to$ Since τ is the shift in continuous time domain, and $h_\tau(t)$ is the shifted version of $h_0(t)$. This is due to the property of time invariance. Therefore,

$h_\tau(t)$ can be replaced by $h_0(t-\tau)$

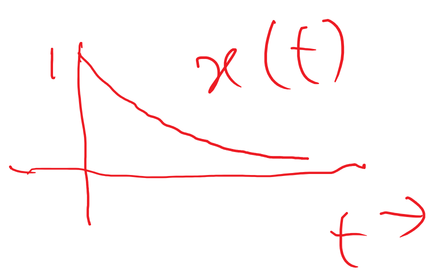
For convenience, if we drop the subscript and take $h(t) = h_0(t)$,

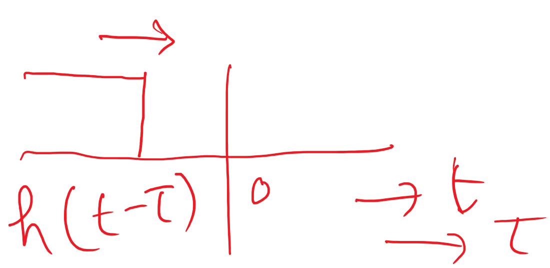
$$y(t) = \int_{-\infty}^{+\infty} x(\tau)\, h(t-\tau)\, d\tau$$

$$\boxed{y(t) = x(t) * h(t)}$$

Ex. $x(t) = e^{-at} u(t)$, $a > 0$

and $h(t) = u(t)$





For $t < 0$, $y(t) = 0$ and

for $t > 0$, $x(\tau)\, h(t-\tau) = e^{-a\tau}$, $0 < \tau < t$

$$y(t) = \int_0^t e^{-a\tau}\, d\tau$$

$$= \frac{1}{a}\left(1 - e^{-at}\right)$$

In general,

$$y(t) = \frac{1}{a}\left(1 - e^{-at}\right) u(t)$$

Properties of LTI system

Commutative property

→ This holds good for both continuous time and discrete time LTI system

$$x[n] * h[n] = h[n] * x[n] = \sum_{k=-\infty}^{+\infty} h[k] \cdot x[n-k]$$

$$x(t) * h(t) = h(t) * x(t)$$
$$= \int_{-\infty}^{+\infty} h(\tau)\, x(t-\tau)\, d\tau$$

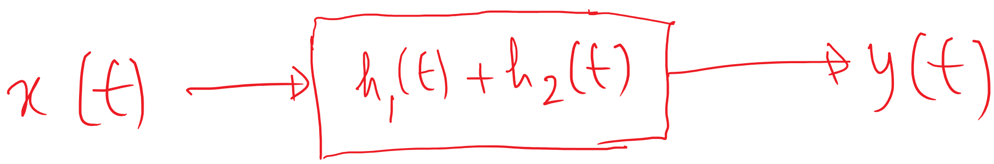
$$x[n] * h[n] = \sum_{k=-\infty}^{+\infty} x[k]\, h[n-k]$$

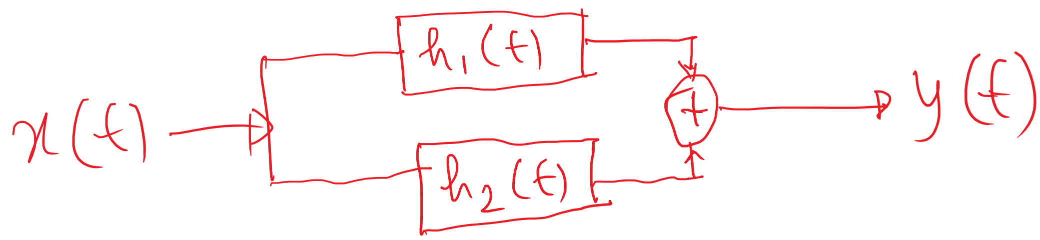
$$[r = n-k] \Rightarrow \sum_{r=-\infty}^{+\infty} x[n-r]\, h[r] = h[n] * x[n]$$

Distributive property

$$x[n] * (h_1[n] + h_2[n]) = x[n] * h_1[n] + x[n] * h_2[n]$$

$$x(t) * (h_1(t) + h_2(t)) = x(t) * h_1(t) + x(t) * h_2(t)$$

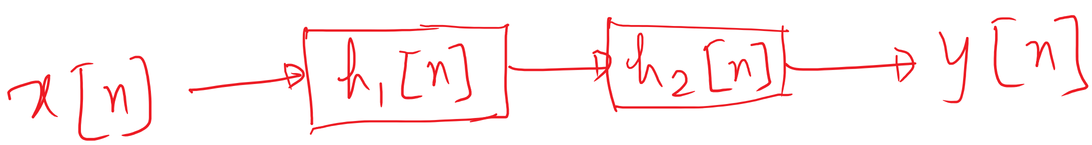


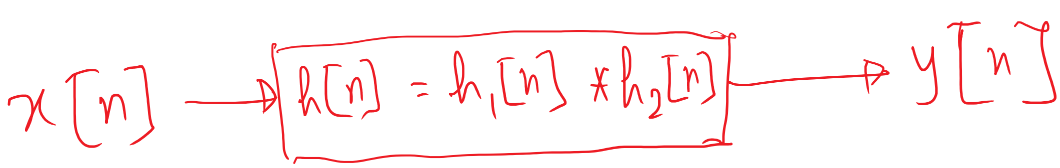


Associative property

$$x[n] * \left(h_1[n] * h_2[n]\right) = \left(x[n] * h_1[n]\right) * h_2[n]$$

$$x(t) * \left[h_1(t) * h_2(t)\right] = \left[x(t) * h_1(t)\right] * h_2(t)$$





→ LTI system without memory can be implemented only for impulse response $h[n] = 0$, for $n \neq 0$ or $h[n] = K\,\delta[n]$

or, $y[n] = k\,x[n]$

In general LTI system is a system with memory.

→ Invertibility condition for LTI system.

$$x(t) \longrightarrow \boxed{h(t)} \xrightarrow{y(t)} \boxed{h_I(t)} \longrightarrow w(t)$$

LTI system is invertible if $w(t) = x(t)$

$$x(t) \longrightarrow \boxed{h(t) * h_I(t) = \delta(t)} \longrightarrow x(t)$$

Similarly, $h[n] * h_I[n] = \delta[n]$

$\Downarrow$ inverse system response

Ex.

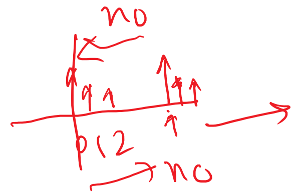
$$y(t) = x(t - t_0)$$

$$\therefore\ h(t) = \delta(t - t_0)$$

or $x(t - t_0) = x(t) * \delta(t - t_0)$

Then $h_I(t) = \delta(t + t_0)$

$$h(t) * h_I(t) = \delta(t - t_0) * \delta(t + t_0)$$
$$= \delta(t)$$



Properties of LTI systems

Friday, September 03, 2021 9:04 AM

Ex: Given impulse response of a system S_1,

$h[n] = \left(\frac{1}{5}\right)^n u[n]$, $u[n]$ is unit step

a) Find A such that $h[n] - A\,h[n-1] = \delta[n]$

b) Determine the impulse response $g[n]$ of an LTI system S_2 which is inverse of S_1

---

a) $\left(\frac{1}{5}\right)^n u[n] - A\left(\frac{1}{5}\right)^{n-1} u[n-1] = \delta[n]$

This should satisfy at $n=1$ [for any $n \geq 1$]

$\therefore \frac{1}{5} - A = 0$, $A = \frac{1}{5}$

b) $\therefore h[n] - \frac{1}{5} h[n-1] = \delta[n]$

We can manipulate this into a convolution.

$h[n] * \delta[n] - \frac{1}{5} h[n] * \delta[n-1] = \delta[n]$

or, $h[n] * \underbrace{\left(\delta[n] - \frac{1}{5}\delta[n-1]\right)}_{\Downarrow} = \delta[n]$

$\therefore$ The impulse response of the inverse system $g[n] = \delta[n] - \frac{1}{5}\delta[n-1]$

Ex

Can we have inverse of finite sequence?

$h[n] = \{h_0, h_1\}$ (non-zero, non-zero)

Then take, $h_I[n] = \{h_{I_0}, h_{I_1}\}$

$$\begin{array}{ccc} & h_0, & h_1 \\ & h_{I_0}, & h_{I_1} \\ \hline & h_0 h_{I_1} & h_1 h_{I_1} \\ h_0 h_{I_0} & h_{I_0} h_1 & \\ \hline h_0 h_{I_0}, & h_0 h_{I_1} + h_{I_0} h_1, & h_1 h_{I_1} \end{array}$$

$y[n] = \{h_0 h_{I_0}, \; h_0 h_{I_1} + h_{I_0} h_1, \; h_1 h_{I_1}\}$ (non-zero; cannot be zero)

$\Rightarrow \delta[n]$?

$\delta[n] = 1, n=0 \Rightarrow h_{I_0} = \frac{1}{h_0}$, $\quad h_{I_1} = \frac{-h_1 h_{I_0}}{h_0}$ (non zero)

$\delta[n] = 1, n = 0 \Rightarrow h_{I_0} = \frac{1}{h_0}$, $h_{I_1} = \frac{-h_1 h_{I_0}}{h_0}$ → non zero

h_0 → non zero

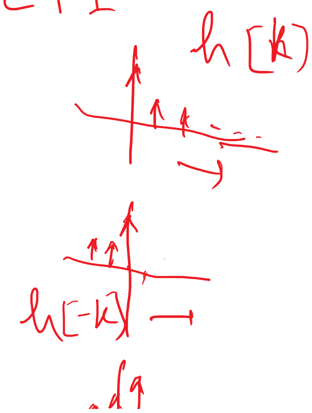
→ Inverse of a finite impulse response can not be a finite impulse response, other than the $\delta[n]$.

1, 1	3, 4
2, 1	4, 3
1 1	9 12
2 2	12 16
2, 3, 1	12, 25, 12
↑ ↑ ↑	
0 1 2	

Causality of LTI system

→ Impulse response of a causal LTI system, $h[n] = 0$ for $n < 0$

or, $y[n] = \sum_{k=-\infty}^{n} x[k]\, h[n-k]$



$k = -\infty$

In continuous-time domain,

$h(t) = 0$ for $t < 0$

or, $y(t) = \int_{\tau=-\infty}^{t} x(\tau)\, h(t-\tau)\, d\tau$

$= \int_{0}^{\infty} h(\tau)\, x(t-\tau)\, d\tau$

For example, $h(t) = \delta(t - t_0)$ is causal for $t_0 \geq 0$

Stability of LTI system

Let $|x[n]| < B$ for all n, $B < \infty$

Then $|y[n]| = \left| \sum_{k=-\infty}^{+\infty} h[k]\, x[n-k] \right|$

or, $|y[n]| \leq \sum_{k=-\infty}^{+\infty} |h[k]|\, |x[n-k]|$

$|x[n-k]| < B$, for all values of k and n

$\sum^{+\infty} |h[k]|$

$$|y[n]| \leq B \sum_{k=-\infty}^{+\infty} |h[k]| ,$$

$\therefore$ $y[n]$ is bounded if $\displaystyle\sum_{k=-\infty}^{+\infty} |h[k]| < \infty$

i.e., the impulse response is absolutely summable.

In continuous domain, $\displaystyle\int_{\tau=-\infty}^{+\infty} |h(\tau)| \, d\tau < \infty$

for a stable system (BIBO stability)

<u>Ex</u>

$$h[n] = \delta[n - n_0]$$

$$\sum_{n=-\infty}^{+\infty} |h[n]| = 1 \implies \text{stable}$$

$$h[n] = u[n]$$

$$\sum_{n=-\infty}^{+\infty} |u[n]| = \sum_{n=0}^{\infty} |u[n]| = \infty \implies \text{unstable.}$$

$$h(t) = e^{-t}u(t),$$

$$\int_{-\infty}^{+\infty} |h(\tau)|\, d\tau = \int_{0}^{\infty} e^{-t}\, d\tau = 1$$

$\Rightarrow$ stable

Problems

Friday, September 03, 2021 10:01 AM

Take $x[n] = (-1)^n u[n]$
$y[n] = x[n] * h[n]$
$y[n] \Rightarrow$ not bounded
$x[n] \Rightarrow$ bounded
$\Rightarrow$ not stable system

$$h[k] = \left\{ 1, -\frac{1}{2}, \frac{1}{3}, -1/4 \right\}$$

$$|h[k]| = \left\{ 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4} \cdots \right\}$$

$\Rightarrow$ not absolutely summable

Problem

Given $h_2[n] = \delta[n] + \delta[n-1]$

and the system is given by

$$y[n] = \left(\underbrace{h_1[n] * h_2[n] * h_2[n]}_{\Downarrow \; h[n]}\right) * x[n]$$

and

$$h[n] = \left\{ \underset{\uparrow \atop 0}{1}, 5, 10, 11, 8, 4, 1 \right\}$$

a) Find the impulse response $h_1[n]$ [assume causal]

b) Find $y[n]$ for $x[n] = \delta[n] - \delta[n-1]$

$$h[n] = h_1[n] * h_2[n] * h_2[n]$$
$$= h_1[n] * \left(\delta[n] + 2\delta[n-1] + \delta[n-2]\right)$$

$$\left[\left(\delta[n] + \delta[n-1] \right) * \left(\delta[n] + \delta[n-1] \right) = \delta[n] + \delta[n-1] + \delta[n-1] + \delta[n-2] \right]$$

$$h[n] = h_1[n] + 2h_1[n-1] + h_1[n-2]$$

$$h[0] = 1$$

$h_1[n] = 0$, for $n < 0$

$$1 = h_1[0] + 2h_1[-1] + h_1[-2]$$

(with $h_1[-1] \Rightarrow 0$ and $h_1[-2] \Rightarrow 0$)

$$h[1] = h_1[1] + 2h_1[0] \Rightarrow h_1[1] = 3$$

$$4$$

$$\vdots$$

Prob2 Given

$$x[n] = \left(\frac{1}{2}\right)^{n-2} u[n-2]$$

and $h[n] = u[n+2]$

Determine $y[n] = x[n] * h[n]$

Let $x_1[n] = \left(\frac{1}{2}\right)^n u[n]$

and $h_1[n] = u[n]$

Then, $x[n] = x_1[n-2]$

and $h[n] = h_1[n+2]$

$$y[n] = x[n] * h[n]$$

$$= x_1[n-2] * h_1[n+2]$$

$$= \sum_{k=-\infty}^{+\infty} x_1[k-2]\, h_1[n-k+2]$$

Taking $k = m+2$,

$$y[n] = \sum_{m=-\infty}^{+\infty} x_1[m]\, h_1[n-m]$$

$$= \sum_{m=-\infty}^{+\infty} \frac{1}{2}^{m} u[m] \cdot u[n-m]$$

unit step

$$\left[\; y[n] = \sum_{k=0}^{n} \alpha^{k} \right., \quad n \geq 0 \quad , \alpha < 1$$

$1 - \alpha^{n+1}$ ⇐ check

$$= \frac{1-\alpha^{n+1}}{1-\alpha} \Big] \Leftarrow \text{Check}$$

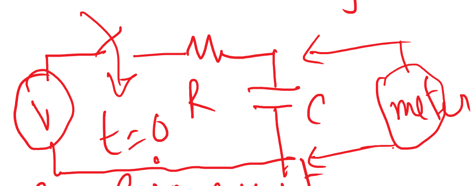
$$\therefore y[n] = \frac{1-\left(\frac{1}{2}\right)^{n+1}}{1-\frac{1}{2}} u[n]$$

$$= 2\left[1-\left(\frac{1}{2}\right)^{n+1}\right] u[n]$$

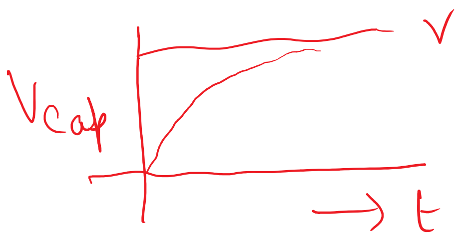
Step Response of LTI systems

Wednesday, September 08, 2021 11:51 AM

→ Step input pumps finite power and infinite energy.



for R-C circuit



for R-L-C circuit

→ Also any periodic input pumps finite power and infinite energy.

→ The step response of a LTI system is the convolution of the unit step function with the impulse response.

Step response, $s[n] = u[n] * h[n]$
$= h[n] * u[n]$

→ We also know that $u[n]$ is the impulse response of the accumulator, i.e.,

$$s[n] = \sum_{k=-\infty}^{n} h[k] \quad \text{---} (1) \qquad [u[n] = 0 \text{ for } n < 0]$$

Then $$s[n-1] = \sum_{k=-\infty}^{n-1} h[k] \quad \text{---} (2)$$

(1) - (2)

$$\therefore \; h[n] = s[n] - s[n-1]$$

for discrete time impulse response can be recovered from step response.

In continuous time,

$$s(t) = \int_{-\infty}^{t} h(\tau)\, d\tau$$

as, $s(t) = u(t) * h(t)$

and therefore $h(t) = \frac{d}{dt} s(t)$

Linear Constant Coefficient Differential or Difference equation

Wednesday, September 08, 2021 12:29 PM

→ This is used to characterize an arbitrary LTI system. It can be used to get impulse response and step response also.

$$\frac{dy(t)}{dt} + 2y(t) = x(t) \quad \text{— (1)}$$

forcing function

$y(t)$: output $x(t)$: input

Take $x(t) = K e^{3t} u(t)$

Solution $\Rightarrow y(t) = y_p(t) + y_h(t)$

$y_p(t)$: particular part $y_h(t)$: homogeneous part

Homogeneous part

$$\frac{dy(t)}{dt} + 2y(t) = 0$$

$$\frac{dy(t)}{y(t)} = -2\,dt$$

$$\int \frac{1}{y(t)}\,dy(t) = -2\int dt$$

$$\Rightarrow \log\{y(t)\} = -2t + C$$

or $y(t) = e^{-2t + c} = A e^{-2t}$

For the particular part, let us take

$$y_p(t) = Y e^{3t}$$

$Y \uparrow$ constant

for $t > 0$,

$$3Y e^{3t} + 2Y e^{3t} = K e^{3t}$$

$$\Rightarrow 5Y = K$$

$$\text{or } Y = \frac{K}{5}$$

$$\therefore y_p(t) = \frac{K}{5} e^{3t}, \quad t > 0$$

$$\therefore y(t) = A e^{-2t} + \frac{K}{5} e^{3t}, \quad t > 0$$

→ We need a initial condition to find the value of A.,

initial condition ⇒ At $t = 0$, $y(0) = 0$

$$\therefore A = -\frac{K}{5}$$

$$\text{and } y(t) = \frac{K}{5}\left(e^{3t} - e^{-2t}\right)$$

Linear constant coefficient difference equation is also similar. [e.g. $y[n] = x[n] - x[n-1]$;

is also similar. [e.g., $y[n] = x[n] - x[n-1]$;
$y[n] - y[n-1] = x[n]$]

In general this can be represented as

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k] \quad (1)$$

$$\underbrace{\sum_{k=0}^{N} a_k y[n-k] = 0}_{\text{homogeneous solution}}$$

Eqn (1) can be written as

$$y[n] = \frac{1}{a_0}\left\{ \sum_{k=0}^{M} b_k x[n-k] - \sum_{k=1}^{N} a_k y[n-k] \right\}$$

If we take a_k's to be zero.

$$y[n] = \sum_{k=0}^{M} \frac{b_k}{a_0} x[n-k] \Rightarrow \text{moving average (MA)}$$

If $b_0 = 1$ and other b_k's are zero.

M $x[n]$

If $b_0 = 1$

$$y[n] + \underbrace{\sum_{k=1}^{M} \frac{a_k}{a_0}\, y[n-k]}_{\text{auto-regressive (AR)}} = \frac{x[n]}{a_0}$$

<u>Example</u>

$$y[n] - \frac{1}{2} y[n-1] = x[n]$$

Taking initial rest condition

Given, $\underline{x[n]} = \underline{k\,\delta[n]}$

$x[n] = 0, \; n \le -1 \leftarrow$ initial rest condition

and initial condition on $y[n]$ is

$$y[n] = 0, \; n \le -1, \; y[-1] = 0$$

$$\therefore \quad y[0] = x[0] + \frac{1}{2} \overset{\Rightarrow 0}{y[-1]} = k$$

$$y[1] = \overset{\Rightarrow 0}{x[1]} + \frac{1}{2} y[0] = \frac{1}{2} k$$

$$y[2] = \overset{0}{x[2]} + \frac{1}{2} y[1] = \left(\frac{1}{2}\right)^2 k$$

$$\vdots$$

$$y[n] = x[n] + \frac{1}{2} y[n-1] = \left(\frac{1}{2}\right)^n k$$

for $n \ge 0$

→ Impulse response can be found out easily by putting $k = 1$,

$$h[n] = \left(\frac{1}{2}\right)^n u[n]$$

$$h[n] = \left(\frac{1}{2}\right)^n u[n]$$

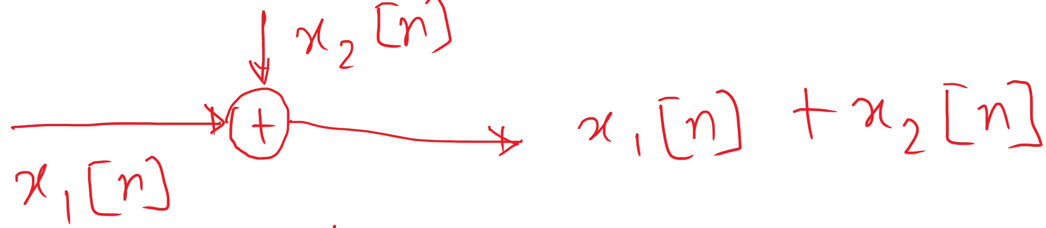
$\downarrow$

takes account

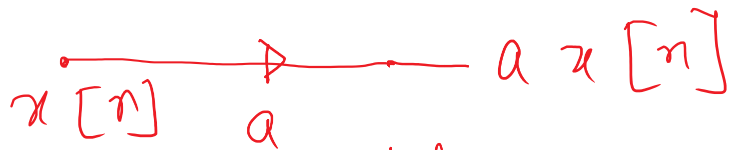
the condition $n \geq 0$

Block Diagram Representation

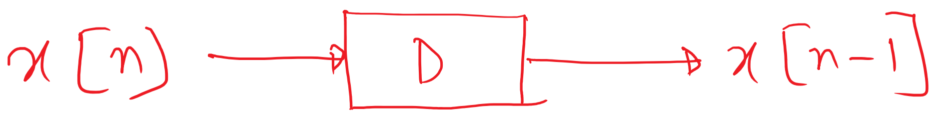
Thursday, September 09, 2021 11:12 AM



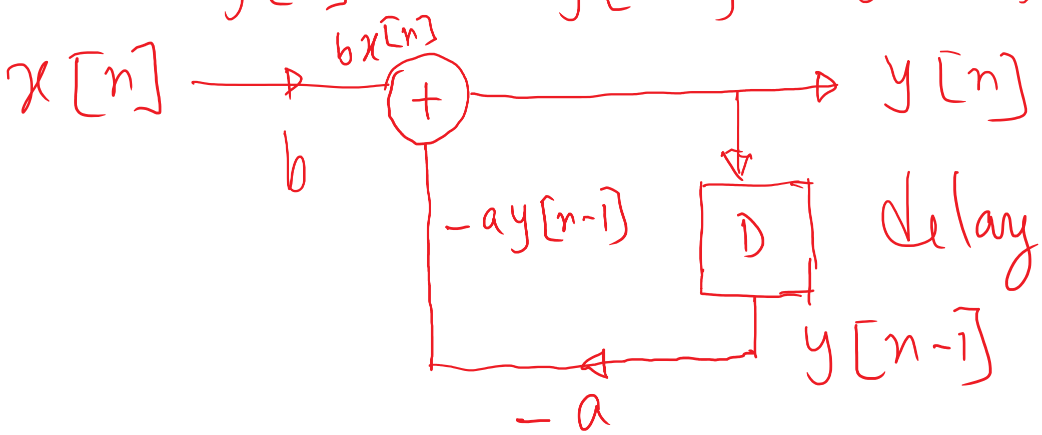
Adder



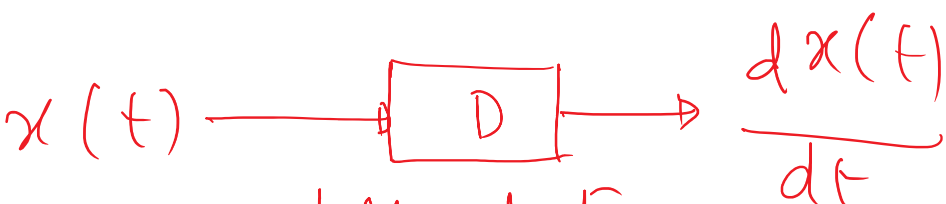
delay



$$y[n] = -a\,y[n-1] + b\,x[n]$$



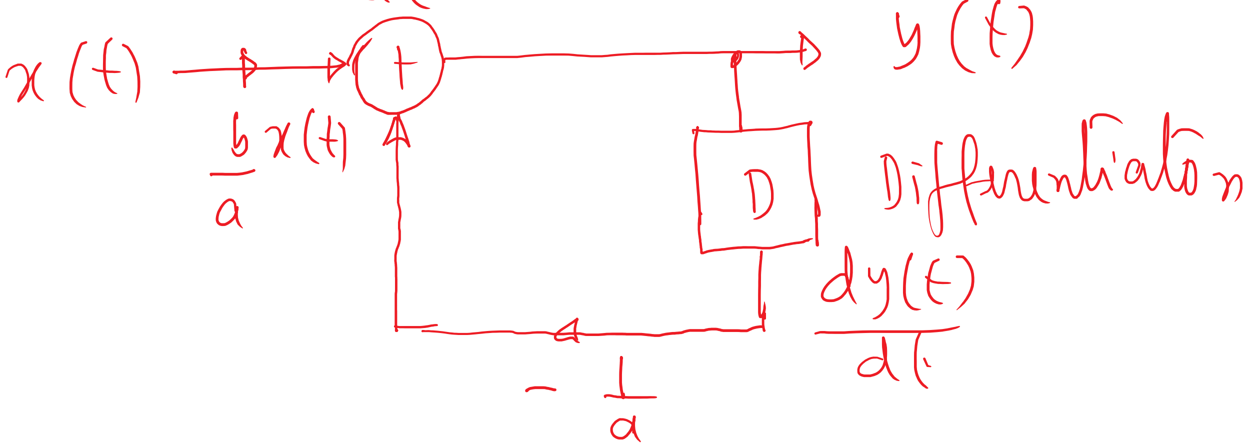
Continuous time



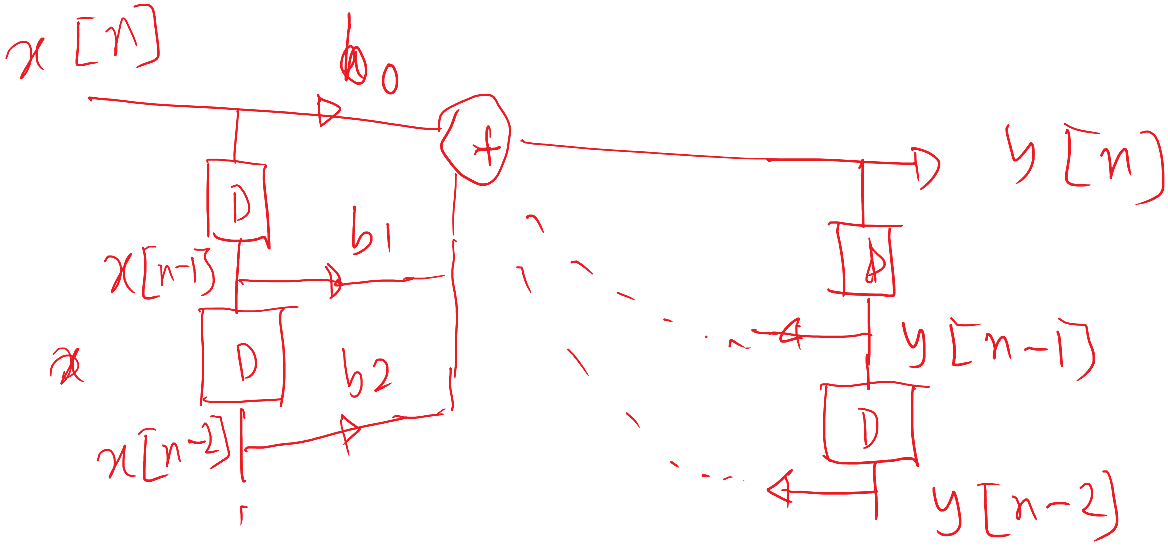
differentiator

$$\frac{dy(t)}{dt} = b\,x(t) - a\,y(t)$$

$$\frac{1}{a}\frac{dy(t)}{dt} = \frac{b}{a}x(t) - y(t)$$



$$y(t) = \frac{b}{a}x(t) - \frac{1}{a}\frac{dy(t)}{dt}$$



→ In the continuous domain, apart from differentiator integrator blocks can also be used.

Defining unit impulse through convolution

$x(t) = x(t) * \delta(t)$ for any $x(t)$

If we take $x(t) = 1$ for all t,

$$1 = x(t) = x(t) * \delta(t) = \delta(t) * x(t)$$

$$= \int_{-\infty}^{+\infty} \delta(\tau)\, x(t-\tau)\, d\tau \quad \text{— (1)}$$

$$= \int_{-\infty}^{+\infty} \delta(\tau)\, d\tau \longrightarrow \text{unit area}$$

δ function has unit area

* Unit doublet

Let us take a system, $y(t) = \frac{d}{dt} x(t)$

→ The unit impulse response of this system is the derivative of the unit impulse, which is known as the doublet.

In other words, $\frac{d}{dt} x(t) = x(t) * u_1(t)$

$u_1(t)$ is the impulse response of the derivative operation } unit doublet

Similarly, $\frac{d^2}{dt^2} x(t) = x(t) * u_2(t)$

$$= \underbrace{x(t) * u_1(t)}_{\frac{d}{dt} x(t)} * u_1(t)$$

or, $u_2(t) = u_1(t) * u_1(t)$

or, $u_k(t) = \underbrace{u_1(t) * u_1(t) * \cdots * u_1(t)}_{k \text{ times}}$

Let us take $x(t) = 1$.

$$\therefore 0 = \frac{d x(t)}{dt} = x(t) * u_1(t)$$

$$= \int^{+\infty} u_1(\tau)\, x(t-\tau)\, d\tau$$

$$= \int_{-\infty}^{+\infty} u_1(\tau)\, x(t-\tau)\, d\tau$$

$$= \int_{-\infty}^{+\infty} u_1(\tau)\, d\tau$$

→ This shows that unit doublet has zero area.

To look into another property of unit doublet, let us take a signal $g(-t)$ and convolve with $u_1(t)$,

$$\int_{-\infty}^{+\infty} g(\tau - t)\, u_1(\tau)\, d\tau = g(-t) * u_1(t) = \frac{d}{dt} g(-t)$$
$$= -g'(-t)$$

If we put $t=0$,

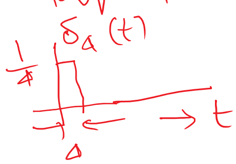
$$-g'(0) = \int_{-\infty}^{+\infty} g(\tau)\, u_1(\tau)\, d\tau$$

Let us check the doublet for δ function

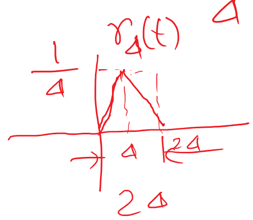
We know that $x(t) = x(t) * \delta(t)$

Now if we take $x(t) = \delta(t)$, then $\delta(t) = \delta(t) * \delta(t)$

→ Interpretation of derivative of $\delta(t)$ we need to take the help of approximate version of $\delta(t)$ as $\delta_\Delta(t)$



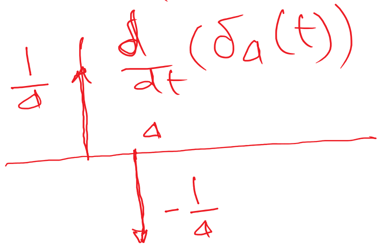
Then $r_\Delta(t) = \delta_\Delta(t) * \delta_\Delta(t)$



$$\frac{d}{dt}\delta_\Delta(t) = \frac{1}{\Delta}\left\{\delta(t) - \delta(t-\Delta)\right\}$$

$$\therefore\ x(t) * \delta(t - t_0) = x(t - t_0)$$

$$\therefore x(t) * \frac{d}{dt}\delta_\Delta(t) = \frac{x(t) - x(t-\Delta)}{\Delta} \simeq \frac{dx(t)}{dt} \text{ as } \Delta \to 0$$



As unit step is the impulse response of an integrator, if we define $y(t) = \int_{-\infty}^{t} x(\tau)\,d\tau$

$$\therefore u(t) = \int_{-\infty}^{t} \delta(\tau)\,d\tau$$

$$\text{and } y(t) = x(t) * u(t) = \int_{-\infty}^{t} x(\tau)\,d\tau$$

Double integrator $[u_{-2}(t)]$

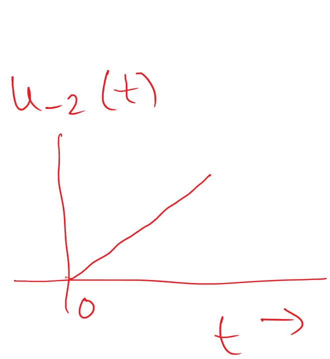
In other words, $u_{-2}(t) = u(t) * u(t)$

$$= \int_{-\infty}^{t} u(\tau)\,d\tau$$

→ As $u(t) = 0$ for $t < 0$, and $u(t) = 1$ for $t > 0$ the integration will start from $t = 0$, and

$$u_{-2}(t) = t\,u(t)$$

unit ramp function starting from $t = 0$



Take $x(t) * u_{-2}(t) = x(t) * u(t) * u(t)$

$$= \left(\int_{-\infty}^{t} x(\sigma)\, d\sigma \right) * u(t)$$

$$= \int_{-\infty}^{t} \left(\int_{-\infty}^{\tau} x(\sigma)\, d\sigma \right) d\tau$$

If we convolute k times,

$$u_{-k}(t) = \underbrace{u(t) * u(t) * \cdots * u(t)}_{k \text{ times}}$$

$$= \int_{-\infty}^{t} u_{-(k-1)}(\tau)\, d\tau$$

or $u_{-k}(t) = \dfrac{t^{k-1}}{(k-1)!} u(t)$

→ Since derivative is an inverse operator of integrator,

$u(t) * u_1(t) = \delta(t)$

(or $u_{-1}(t)$)

In general, $u_k(t) * u_r(t) = u_{k+r}(t)$

Fourier Series Representation of Periodic Signals.

→ An LTI system can be represented as a weighted sum of an elementary signal or a basis function.

→ The response of an LTI system to the set of input functions need to be expressed in terms of the same input function. Such functions can be thought of as eigenfunctions of LTI system.

→ Complex exponential satisfy this conditions in case of LTI systems

In continuous time, if we take the complex exponential as e^{st}, [s is complex]

$$e^{st} \longrightarrow \boxed{} \longrightarrow \underbrace{H(s)}_{\text{eigenvalue}} e^{st}$$

Similarly, in the discrete domain,

$$z^n \longrightarrow H(z) \cdot z^n$$

z ← power series expansion of complex z

Let $x(t) = e^{st}$

Then $y(t) = \int_{-\infty}^{+\infty} h(\tau)\, x(t-\tau)\, d\tau$

$$= \int_{-\infty}^{+\infty} h(\tau)\, e^{s(t-\tau)}\, d\tau$$

$$= e^{st} \int_{-\infty}^{+\infty} h(\tau)\, e^{-s\tau}\, d\tau$$

$$= H(s)\, e^{st}$$ ← eigenfunction

complex constant as a function of s, eigenvalue

Discrete time case

Let $x[n] = z^n$, $z \rightarrow$ complex

Then $y[n] = \sum_{k=-\infty}^{+\infty} h[k]\, x[n-k]$

$= \sum_{k=-\infty}^{+\infty} h[k]\, z^{n-k}$

$= z^n \sum_{k=-\infty}^{+\infty} h[k]\, z^{-k}$

$= H(z) \cdot z^n$ ← eigenfunction

eigenvalue, provided $\sum_{k=-\infty}^{+\infty} h[k]\, z^{-k}$ converges

Ex.

Take $y(t) = x(t-3)$ and given $x(t) = e^{j2t}$

Then, $y(t) = e^{j2(t-3)}$

$= e^{-j6}\, e^{j2t}$ ← eigenfunction

eigenvalue

The impulse response of the system is $\delta(t-3)$

$H(s) = \int_{-\infty}^{+\infty} \delta(\tau - 3)\, e^{-s\tau}\, d\tau$

$= e^{-3s}$, if we sample it at $s = j2$.

$H(j2) = e^{-j6}$

Linear combination of Harmonically Related Complex exponential

→ Let us start with a periodic signal $x(t) = x(t+T)$ for all t where T is period.

Let $x(t) = e^{jw_0 t}$ and $\phi_k(t) = e^{jkw_0 t} = e^{jk\left(\frac{2\pi}{T}\right)t}$

kth harmonic

∴ $\phi_k(t)$ is the harmonically related complex exponential with fundamental frequency w_0

∴ By superposition,

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jkw_0 t}$$

$$= \sum_{k=-\infty}^{+\infty} a_k e^{jk\left(\frac{2\pi}{T}\right)t} \quad \text{— (1)}$$

→ By superposition and linearity, sum of periodic signal which is harmonically distributed is also periodic with T.

This representation of a periodic signal in form of eqn (1) is referred as Fourier Series representation.

Ex

$$x(t) = \sum_{k=-3}^{+3} a_k e^{jk2\pi t}$$

, given $a_0 = 1$, $a_1 = a_{-1} = \frac{1}{4}$, $a_2 = a_{-2} = \frac{1}{2}$, $a_3 = a_{-3} = \frac{1}{3}$

$$x(t) = 1 + \frac{1}{4}\left(e^{j2\pi t} + e^{-j2\pi t}\right) + \frac{1}{2}\left(e^{j4\pi t} + e^{-j4\pi t}\right)$$

$$+ \left(e^{j6\pi t} \; e^{-j6\pi t}\right)$$

$$x(t) = 1 + \frac{1}{4}\left(e \quad + e \quad\right) + \frac{1}{2}\left(\quad + e \quad \right)$$

$$+ \frac{1}{3}\left(e^{j6\pi t} + e^{-j6\pi t}\right)$$

$$= 1 + \frac{1}{2}\cos(2\pi t) + \cos(4\pi t) + \frac{2}{3}\cos(6\pi t)$$

→ Generally for real $x(t)$, $x^*(t) = x(t)$

$$\therefore x(t) = \sum_{k=-\infty}^{+\infty} a_k^* e^{-jk\omega_0 t} \quad \text{——} \ (2)$$

Replacing k by $-k$, $x(t) = \sum_{k=-\infty}^{+\infty} a_{-k}^* e^{jk\omega_0 t}$ —— (3)

Comparing (1) and (3), $a_k^* = a_{-k}$ ⇐ conjugate symmetry

For real a_k, $a_k = a_{-k}$ ⇒ symmetry

or, $x(t) = a_0 + \sum_{k=1}^{\infty}\left[a_k e^{jk\omega_0 t} + a_{-k} e^{-jk\omega_0 t}\right]$

$$= a_0 + \sum_{k=1}^{\infty}\left[a_k e^{jk\omega_0 t} + a_k^* e^{-jk\omega_0 t}\right]$$

↳ complex conjugates of each other

or $x(t) = a_0 + \sum_{k=1}^{\infty} 2\,\mathrm{Re}\left\{a_k e^{jk\omega_0 t}\right\}$

In polar form a_k is expressed as, $a_k = A_k e^{j\theta_k}$

Then, $x(t) = a_0 + \sum_{k=1}^{\infty} 2\,\mathrm{Re}\left[A e^{j(k\omega_0 t + \theta_k)}\right]$

or $x(t) = a_0 + \sum_{k=1}^{\infty} 2 \cos(k\omega_0 t + \theta_k)$

If we write a_k in rectangular form as

$$a_k = B_k + jC_k$$

Then,

$$x(t) = a_0 + 2\sum_{k=1}^{\infty} \left[B_k \cos(k\omega_0 t) - C_k \sin(k\omega_0 t)\right]$$

If a_k is real, $a_k = A_k = B_k$, $C_k = 0$

Determination of Fourier Series Coefficients

Let $x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$

$\therefore\ x(t)\cdot e^{-jn\omega_0 t} = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}\cdot e^{-jn\omega_0 t}$

Integrating both sides over T,

or, $\int_0^T x(t) e^{-jn\omega_0 t} dt = \int_0^T \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}\cdot e^{-jn\omega_0 t}\, dt$

or, where $T = \frac{2\pi}{\omega_0} \rightarrow$ fundamental period.

or, $\int_0^T x(t) e^{-jn\omega_0 t} dt = \sum_{k=-\infty}^{+\infty} a_k \int_0^T e^{j(k-n)\omega_0\cdot t}\, dt$ — (1)

Now, $\int_0^T e^{j(k-n)\omega_0\cdot t} dt = \int_0^T \cos(k-n)\omega_0 t\, dt + j\int_0^T \sin(k-n)\omega_0 t\, dt$

If $k \neq n$, they are periodic sinusoidal function with period $\frac{T}{|k-n|}$ and becomes 0 when integrated over a period.

If $k = n$, $\int_0^T e^{j(k-n)\omega_0 t} dt = T \leftarrow$ fundamental period

$$\therefore \int_T e^{j(k-n)\omega_0 t} dt = \begin{cases} T, & k = n \\ 0, & k \neq n \end{cases}$$

$\therefore$ Putting this in (1), $a_n = \frac{1}{T}\int_T x(t) e^{-jn\omega_0 t}$

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{+\infty} a_k e^{j\left(k\frac{2\pi}{T}\right)\cdot t}$$

and $a_k = \frac{1}{T}\int_T x(t) e^{-jk\omega_0 t} dt = \frac{1}{T}\int_T x(t) e^{-jk\frac{2\pi}{T}\cdot t} dt$

k mapped to the harmonics of ω_0, starting from zero

Ex.

$$x(t) = \sin(\omega_0 t) \leftarrow \text{odd signal}$$

$$= \frac{1}{2j} e^{j\omega_0 t} - \frac{1}{2j} e^{-j\omega_0 t}$$

$\therefore a_0 = 0, \quad a_1 = \frac{1}{2j}$ and $a_{-1} = -\frac{1}{2j}, \quad a_k = 0$ for $|k|>1$

Ex2

$$x(t) = 1 + \sin(\omega_0 t) + 2\cos(\omega_0 t) + \cos\left(2\omega_0 t + \frac{\pi}{4}\right)$$

$$\therefore x(t) = 1 + \left(1 + \frac{1}{2j}\right) e^{j\omega_0 t} + \left(1 - \frac{1}{2j}\right) e^{-j\omega_0 t} + \frac{1}{2} e^{j\pi/4} \cdot e^{j2\omega_0 t} + \frac{1}{2} e^{-j\pi/4} \cdot e^{-j2\omega_0 t}$$

$\Rightarrow a_0 = 1, \quad a_1 = 1 + \frac{1}{2j}, \quad a_{-1} = 1 - \frac{1}{2j}, \quad a_2 = \frac{1}{2} e^{j\pi/4} = \frac{\sqrt{2}}{4}(1+j)$

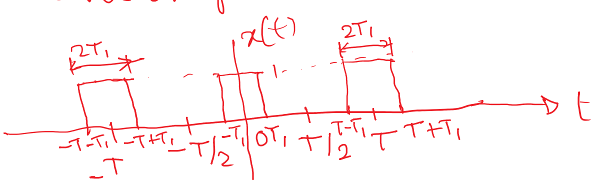
$a_{-2} = \frac{1}{2} e^{-j\pi/4} = \frac{\sqrt{2}}{4}(1-j), \quad a_k = 0, \ |k| > 2$

Ex

$$x(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & T_1 < |t| < T/2 \end{cases}$$

over one period

Given $x(t)$ is periodic & $T_1 < T/2$



If we integrate over a period,

$$a_0 = \frac{1}{T}\int_{-T_1}^{+T_1} dt = \frac{2T_1}{T}$$

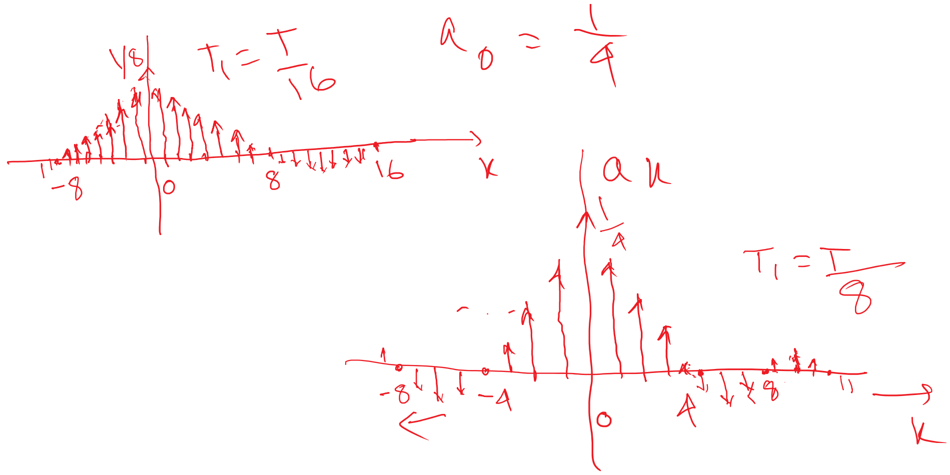
$$a_k = \frac{1}{T}\int_{-T_1}^{T_1} e^{-jk\omega_0 t}\, dt = \frac{2\sin(k\omega_0 T_1)}{k\omega_0 T}, \quad k \neq 0$$

Put $\omega_0 T = 2\pi \Longrightarrow \quad = \frac{\sin(k\omega_0 T_1)}{k\pi}, \quad k \neq 0$

If $T = 8T_1 \Rightarrow T_1 = \frac{T}{8} = \frac{2\pi}{8\omega_0} = \frac{\pi}{4\omega_0}$

Then $a_k = \frac{\sin(\pi k/4)}{k\pi}, \quad k \neq 0.$

$a_0 = \frac{1}{4}$



Sinc function

→ Though it is a valid Fourier Series expansion but a_k can not be truncated at a finite k. If it is truncated, the reconstructed $x(t)$ from the Fourier series coefficient will have a finite error.

Convergence of Fourier Series

Let $x(t)$ be the periodic signal to be represented by Fourier series, as $x_N(t) = \sum_{k=-N}^{N} a_k e^{jk\omega_0 t}$

the approximated $x(t)$.

The approximation error $e_N(t) = x(t) - x_N(t)$

$$= x(t) - \sum_{k=-N}^{+N} a_k e^{jk\omega_0 t}$$

Now, the energy of error over one period is

$$E_N = \int_T |e_N(t)|^2 dt$$

The set of a_k that minimizes E_N is given by $a_k = \frac{1}{T}\int_T x(t) e^{-jk\omega_0 t} dt$

i.e. $E_N \to 0$ as $N \to \infty$

This can be shown by taking the error in an interval.

$$|E = \int^b \left[x(t) - \sum_{k=-N}^{+N} a_k \phi_k(t)\right]\left[x^*(t) - \sum_{k=-N}^{+N} a_k^* \phi_k^*(t)\right] dt$$

$$E = \int_a^b \left[x(t) - \sum_{k=-N}^{N} a_k \phi_k(t)\right]\left[x^*(t) - \sum_{k=-N}^{N} a_k^* \phi_k^*(t)\right] dt$$

ϕ_k: complex exponential or in general any orthonormal function

Let $a_i = b_i + j c_i$,

Then $\frac{\partial E}{\partial b_i} = 0$, $\Rightarrow$ produce zero for all term other i^{th}

Check it

$$\frac{\partial E}{\partial b_i} = -\int_a^b \phi_i^*(t) x(t)\, dt + 2b_i - \int_a^b \phi_i(t) x^*(t)$$

and $\frac{\partial E}{\partial c_i} = 0 \Rightarrow j\int_a^b \phi_i(t) x^*(t)\, dt + 2c_i - j\int_a^b \phi_i^*(t) x(t)\, dt$

$$2c_i = j\int_a^b \phi_i^*(t) x(t)\, dt - j\int_a^b \phi_i(t) x^*(t)\, dt$$

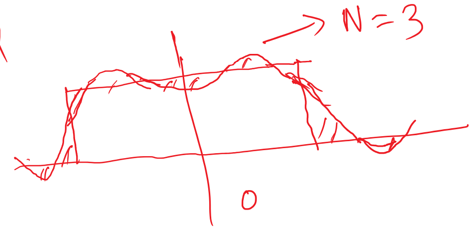
$$\therefore 2b_i + 2jc_i = 2\int_a^b x(t)\, \phi^*(t)\, dt$$

$$\Rightarrow b_i + jc_i = a_i = \int_a^b x(t)\, \phi^*(t)\, dt$$

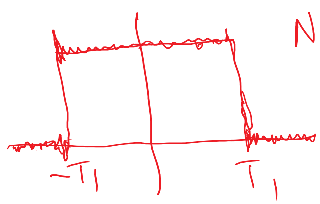
ϕ^*: complex exponential

N = 1 — for one period

N = 3



N = 79



→ This example shows that the error tends to zero in average sense but that does not mean $x(t) = x_N(t)$ at all t, for signals with discontinuity in particular.

→ Further Dirichlet found that as N increases $x(t)$ is equal at almost all points except at the discontinuity. He also concluded that at the discontinuity Fourier expansion converges in the average sense.

Dirichlet Conditions

Condition 1: $x(t)$ must be absolutely integrable over one period.

$$\int_T |x(t)|\,dt < \infty$$

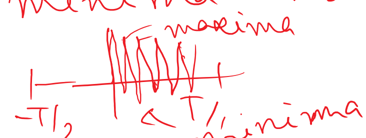
$$|a_k| \le \frac{1}{T}\int_T \left|x(t)e^{-jk\omega_0 t}\right| dt = \frac{1}{T}\int_T |x(t)|\,dt$$

$$\left[\because \left|e^{-jk\omega_0 t}\right| = 1\right]$$

$\therefore$ if $\int_T |x(t)|\,dt < \infty$, $a_k < \infty$

Condition 2

In any finite interval of time, $x(t)$ is of bounded variation; that is, there are no more than a finite number of maxima and minima during any single period of signal.



Condition 3: In any finite interval of time, there are only a finite number of discontinuities and each of these are finite.

→ In general, however, Fourier representation of signal is exact for periodic signal without discontinuity.