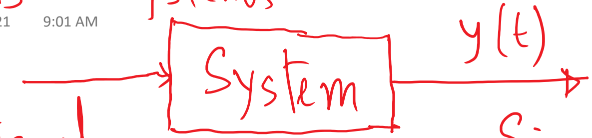
Signals and Systems

Thursday, August 12, 2021 9:01 AM

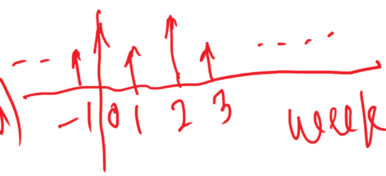


Signal $x(t)$ Signal out

Continuous

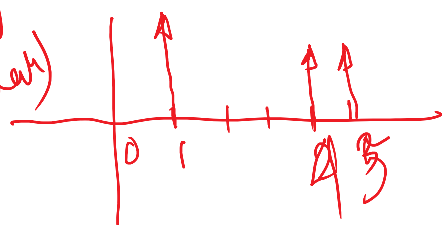
→ time

Digital (regular interval)



1st week - 20 mm
4th week - 10 mm
5th week - 15 mm

Digital (irregular)



Discrete time

→ independent variable time is discretized but the signal value is not discretized

Signal Types

examples ⇓

→ periodic signal [sinusoidal signal]
→ aperiodic signal [exponential signal]
→ impulse signal

→ step signal

property: Signal Energy and Signal power

Signal energy < finite interval / infinite interval

Energy $E_\infty = \underset{T\to\infty}{Lt} \int_{-T}^{T} |x(t)|^2 dt = \int_{-\infty}^{+\infty} |x(t)|^2 dt$

For the discrete case,

$$E_\infty = \underset{N\to\infty}{Lt} \sum_{n=-N}^{N} |x(n)|^2 = \sum_{n=-\infty}^{+\infty} |x(n)|^2$$

for real and complex valued signal

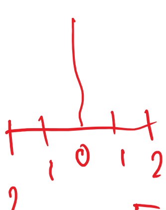
→ If this integration or summation converges to a finite value, Typically the signal an energy signal.

Similarly we can define power signal and subsequently power of a signal

$$P_\infty = \underset{T\to\infty}{Lt} \frac{1}{2T} \int_{-T}^{+T} |x(t)|^2 dt$$

and for the discrete case

$$P_\infty = \underset{N\to\infty}{Lt} \frac{1}{2N+1} \sum_{n=-N}^{+N} |x(n)|^2$$

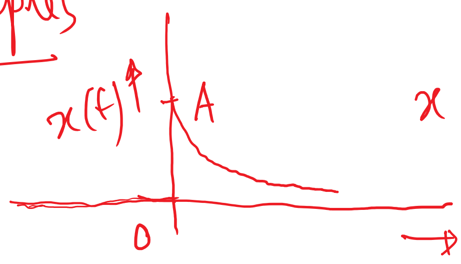


From the definition of power and energy of a signal, we can infer that

→ for an energy signal $P_\infty = 0$, because E_∞ is finite for a energy signal.

→ for a power signal, P_∞ = finite (non-zero), and therefore $E_\infty = \infty$, since essentially E_∞ is integration P_∞ over $-\infty$ to $+\infty$.

Examples



$$x(t) = A\exp(-t) \text{ for } t \geq 0$$
$$= 0 \text{ for } t < 0$$

$$E_\infty = \int_{-\infty}^{+\infty} A^2 \exp(-2t)\, dt = \int_0^\infty A^2 \exp(-2t)\, dt$$

$$= \frac{A^2}{2}$$

$$P_\infty = \lim_{T\to\infty} \left(\frac{1}{2T} \int_{-T}^{T} A^2 \exp(-2t)\, dt \right.$$

$$= \lim_{T\to\infty} \frac{A^2}{4T} = 0$$

ex2 Take a sinusoidal signal,

$$x(t) = A \sin(\omega_0 t + \phi)$$

This is periodic with period $\frac{2\pi}{\omega_0}$

$$P_\infty = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{+T} A^2 \sin^2(\omega_0 t + \phi)\, dt$$

$$P_\infty = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} A^2 \sin^2(\omega_0 t + \phi)\, dt$$

$$= \frac{A^2 \omega_0}{4\pi} \int_{-2\pi/\omega_0}^{+2\pi/\omega_0} \left[\frac{1}{2} - \frac{1}{2}\cos(2\omega_0 t + 2\phi)\right] dt$$

[Evaluating for one period]

$$= \frac{A^2}{2} \Rightarrow \text{This is a power signal}$$

$$\therefore E_\infty \to \infty$$

[Can be done without substitution]

Discrete case

$$x(n) = \begin{cases} \frac{1}{n}, & n \geq 1 \\ 0, & n < 1 \end{cases}$$

$$\text{Energy} = \sum_{n=1}^{\infty} \left(\frac{1}{n}\right)^2 = \frac{\pi^2}{6}$$

$$x(n) = \begin{cases} 3(-1)^n, & n \geq 0 \\ 0, & n < 0 \end{cases}$$

$$E_\infty = \sum_{n=-\infty}^{+\infty} |x(n)|^2 \Rightarrow \text{sum of positive numbers}$$

$$= \infty$$

$$P_\infty = \lim_{N\to\infty} \frac{1}{2N+1}\left(9\sum_{n=0}^{N} 1\right) = \lim_{N\to\infty} \frac{9(N+1)}{2N+1}$$

$$= 4.5$$

Transformation of Independent variable.

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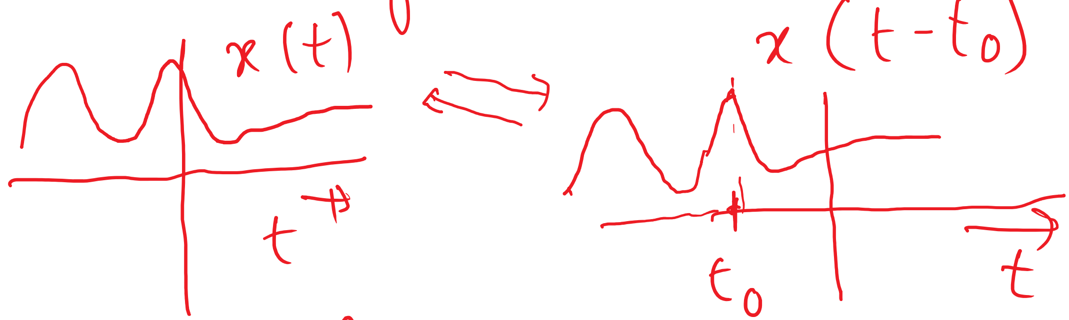
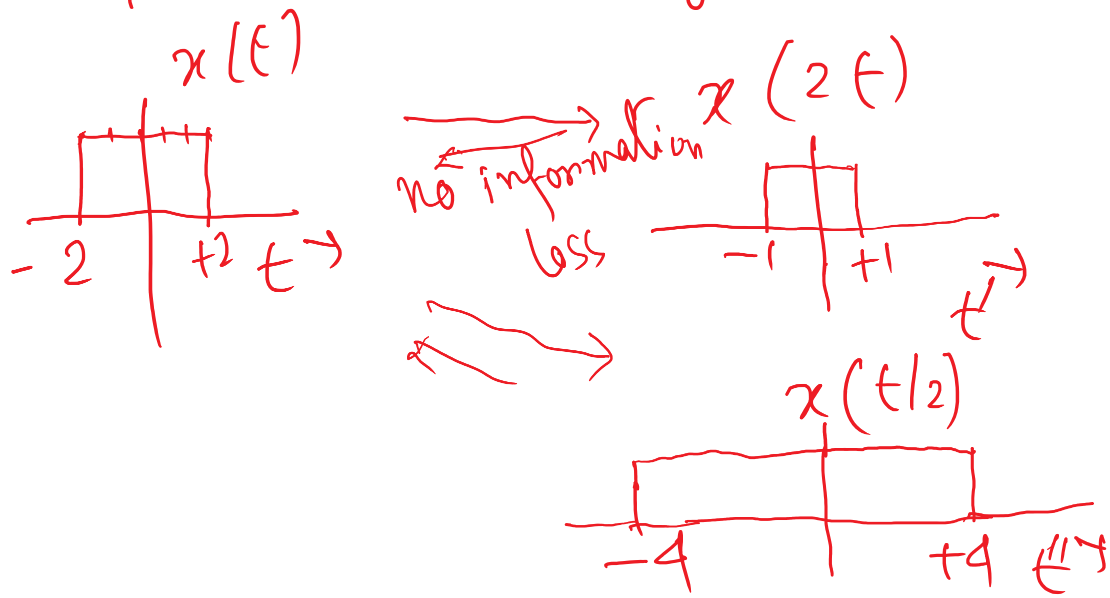
→ In 1-D independent variable like time we can manipulate by shifting or scaling.

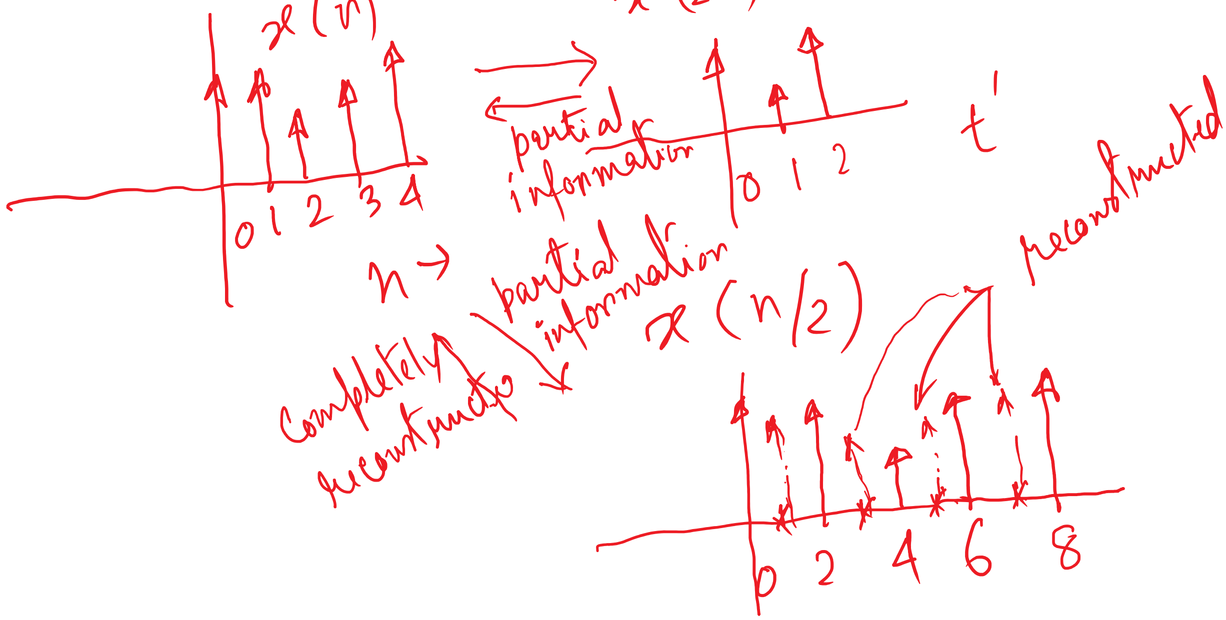
→ For example for scaling, signals can be observed at a denser interval or a rarer interval by scaling time with a scale factor of α either $\alpha < 1$ or $\alpha > 1$

→ similarly the signal can be shifted in time either by a positive value or a negative value.

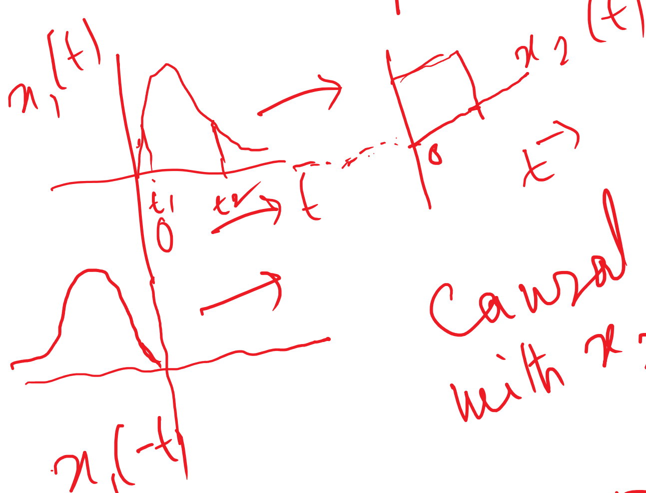




Discrete case



Reflection



Causal interaction with $x_2(t)$ is maintained

Combination of shift & time scaling

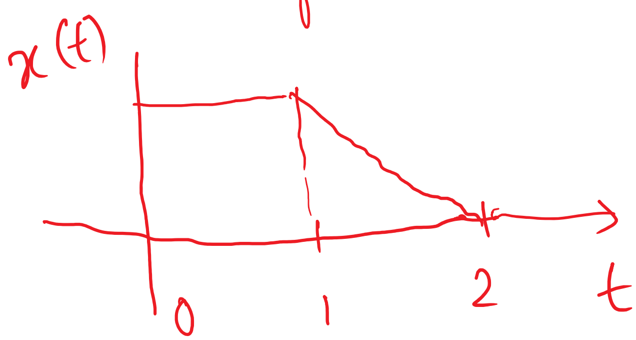
$x(t) \rightarrow x(\alpha t + \beta)$

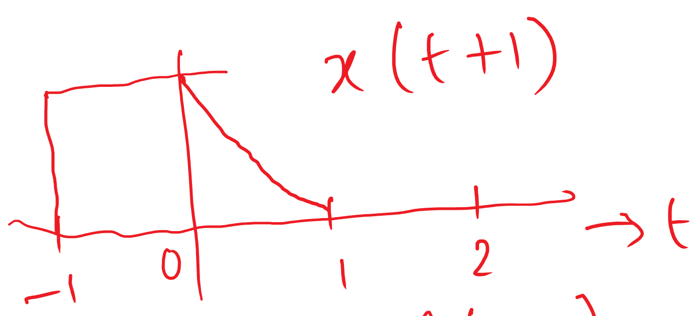
$$x(t) \Rightarrow x(\alpha t + \beta)$$

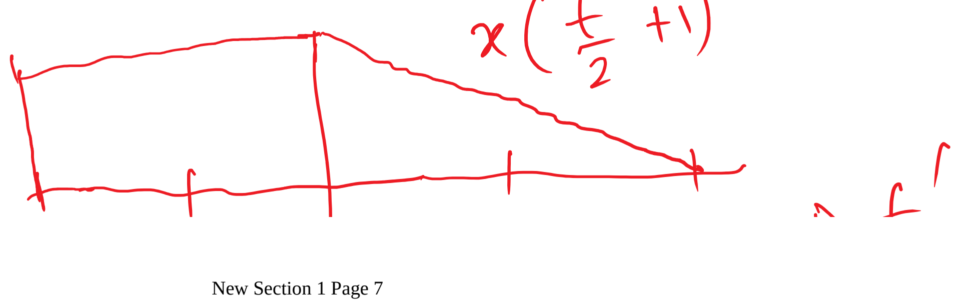
$$x\left(\frac{t}{2} + 1\right)$$

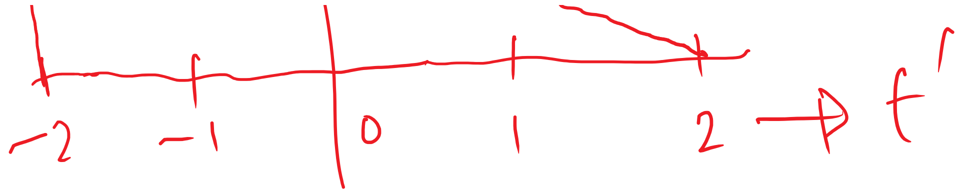
$\Rightarrow$ In this case first shift by 1 and then expand t by 2, since this shift is referred with respect to $x(t)$ and not with respect to $x\left(\frac{t}{2}\right)$

To make it with respect to $x\left(\frac{t}{2}\right)$, shift need to be multiplied by 2 or in general β scaled as $\frac{\beta}{\alpha}$








-2
-1
0
1
2
→ f'

Periodic Signal

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For a periodic signal, $x(t) = x(t+T)$ for all t for a fixed value of T

The period is T and its integer multiples $2T, 3T, \ldots$ also satisfy the condition of periodicity.

For a discrete-time signal, $x[n] = x[n+N]$ ← fixed number (N)

Even and odd signal

→ A signal is even if $x(-t) = x(t)$ and in discrete, time case $x[-n] = x[n]$

→ A signal is odd if
$$x(-t) = -x(t)$$
and for discrete-time, $x[-n] = -x[n]$

odd signal need to be 0 at $t=0$ or

at $n=0$, since $x(-0) = -x(0)$

→ A signal can be decomposed into even and odd components.

$$x_e(t) = \text{Even}\left[x(t)\right] = \frac{1}{2}\left[x(t) + x(-t)\right]$$

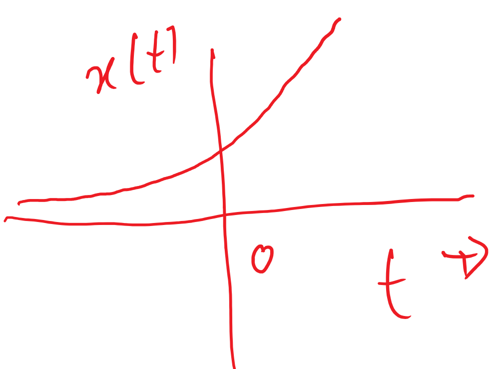
$$x_o(t) = \text{odd}\left[x(t)\right] = \frac{1}{2}\left[x(t) - x(-t)\right]$$

$$x_e[n] = \text{Even}\left[x[n]\right] = \frac{1}{2}\left[x[n] + x[-n]\right]$$

$$x_o[n] = \text{odd}\left[x[n]\right] = \frac{1}{2}\left[x[n] - x[-n]\right]$$

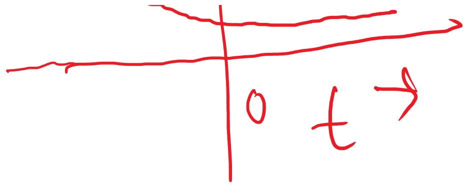
Exponential and sinusoidal signal

$x(t) = C e^{at}$, C & a can be real or complex.



a, c are +ve

a is −ve, c is +ve



Complex exponential

$$x(t) = e^{j\omega_0 t}$$

→ It is periodic unlike real exponential

i.e. $e^{j\omega_0(t+T)} = e^{j\omega_0 t} \cdot e^{j\omega_0 T}$

∴ This is periodic if $e^{j\omega_0 T} = 1$

[for $\omega_0 = 0$, $e^{j\omega_0 T} = 1$ for any T]

$\Rightarrow$ fundamental period is undefined for $\omega_0 = 0$

If $\omega_0 \neq 0$, then $T_0 = \dfrac{2\pi}{|\omega_0|}$,

$T_0 \rightarrow$ fundamental period.

Power for 1 period $= \dfrac{1}{T_0}\displaystyle\int_0^{T_0} \left|e^{j\omega_0 t}\right|^2 dt$

$$= \frac{1}{T_0}\int_0^{T_0} 1 \cdot dt = 1$$

$$P_\infty = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{+T} \left|e^{j\omega_0 t}\right|^2 dt = 1$$

$\Rightarrow$ condition of periodicity $\rightarrow e^{j\omega T_0} = 1$

$\therefore T_0$ is multiple of 2π, or

$\omega T_0 = 2\pi k, \; k = 0, \pm 1, \pm 2, \ldots$

In general, $\phi_k(t) = e^{jk\omega_0 t}, \; k = 0, \pm 1, \pm 2 \ldots$

for $k = 0$, $\phi_k(t) =$ constant

& for $k \neq 0$, $\phi_k(t)$ is periodic, $T = \dfrac{2\pi}{|k|\omega_0}$

$\omega_0 \rightarrow$ fundamental frequency with respect to T_0

General Complex exponential.

Take $\quad c = |c| e^{j\theta}$ and $a = r + j\omega_0$

$$c e^{at} = |c| e^{j\theta} e^{(r + j\omega_0)t}$$

$$= |c| e^{rt} e^{j(\omega_0 t + \theta)}$$

Discrete-time complex exponential

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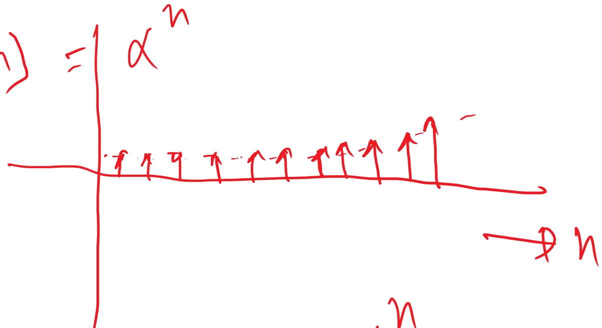
$x[n] = C\alpha^n$, C & α are both complex

This can be written as,

$x[n] = Ce^{\beta n}$, $\alpha = e^{\beta}$

→ Behaviour is similar to continuous one if α is real and positive

$x[n] = \alpha^n$

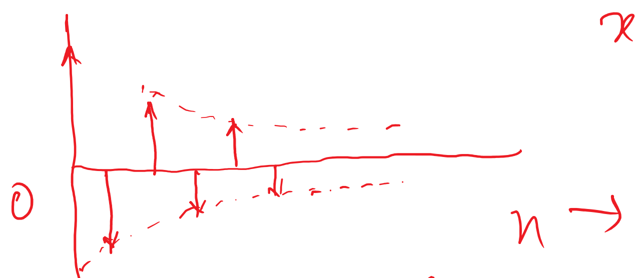


$\alpha > 1$

$x[n] = \alpha^n$

$0 < \alpha < 1$

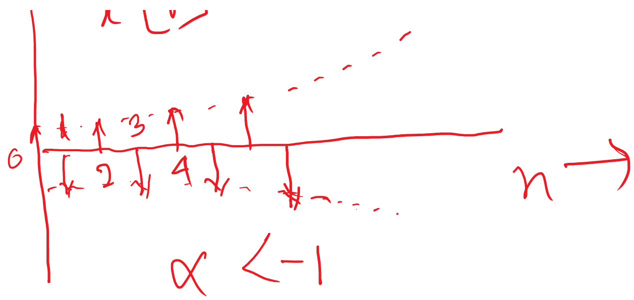
→ If α is -ve & real, $x[n]$ oscillates



$x[n] = \alpha^n$
$= (-0.5)^n$,
$n = 0, 1, 2 \dots$

$-1 < \alpha < 0$

$x[n] = \alpha^n$



$\alpha < -1$

Sinusoidal signal in discrete time

$Ce^{\beta n} \Rightarrow$ imaginary β gives $|e^{\beta}| = 1$

Let $x[n] = e^{j\omega_0 n}$

$$= \cos(\omega_0 n) + j\sin(\omega_0 n)$$

Take

$$A\cos(\omega_0 n + \phi) = \frac{A}{2} e^{j\phi} e^{j\omega_0 n} + \frac{A}{2} e^{-j\phi} e^{-j\omega_0 n}$$

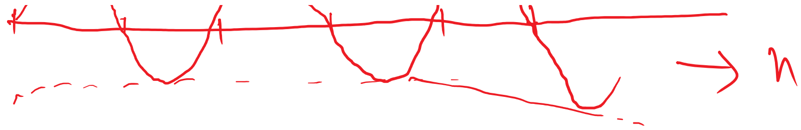
Let

$C = |c| e^{j\theta}$ and $\alpha = |\alpha| e^{j\omega_0}$

Then, $C\alpha^n = |c|\,|\alpha|^n \cos(\omega_0 n + \theta) + j\,|c|\,|\alpha|^n \sin(\omega_0 n + \theta)$

$\rightarrow$ Depending on the value of α, this will either a damped sinusoid or a exponentially diverging sinusoid

$|\alpha| > 1$



Periodicity of discrete-time exponential

Take $x[n] = e^{j(\omega_0 + 2\pi)\cdot n}$

$$= e^{j2\pi n} \, e^{j\omega_0 n} = e^{j\omega_0 n}$$

$$e^{j\pi n} = (-1)^n, \quad e^{j2\pi n} = 1$$

→ Unlike the continuous time sinusoids, discrete time sinusoids are unique only in a principal interval of $0 \le \omega_0 \le 2\pi$

or, $-\pi \le \omega_0 \le \pi$

Now considering definition of periodicity

$$e^{j\omega_0 (N+n)} = e^{j\omega_0 n} \text{ for periodic signal}$$

∴ $e^{j\omega_0 N} = 1$ or $\omega_0 = 0$ satisfies this relation.

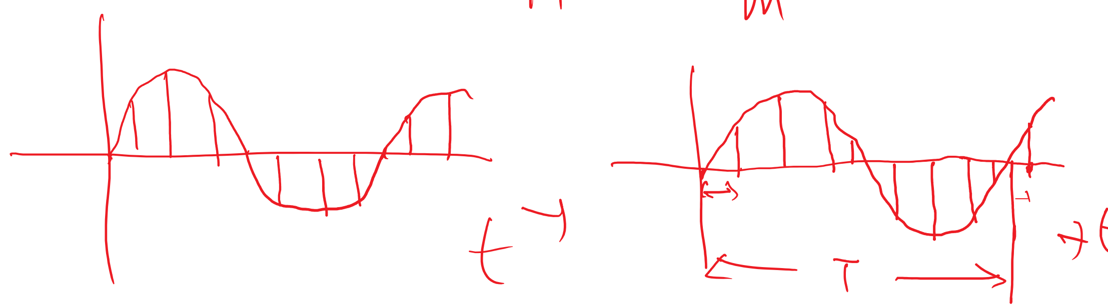
$e^{j\omega_0 N} = 1 \Rightarrow N$ is multiples of $\frac{2\pi}{\omega_0}$

or. $\omega_0 N = 2\pi m$, m is an integer

or $\frac{\omega_0}{} = \frac{m}{} \Rightarrow$ rational number.

or $\frac{\omega_0}{2\pi} = \frac{m}{N} \Rightarrow$ rational number.

If $x[n]$ is periodic with a fundamental period of N, it fundamental frequency is $\frac{2\pi}{N}$ and therefore $\frac{2\pi}{N} = \frac{\omega_0}{m}$



Continuous-time	Discrete-time
• $e^{j\omega_0 t}$	• $e^{j\omega_0 n}$
• Distinct for for distinct ω_0	• signal identical for values of ω_0 separated by multiples of 2π.
• fundamental frequency ω_0	• fundamental frequency $\frac{\omega_0}{m}$
• periodic for a choice of ω_0	• Periodic only if $\omega_0 = \frac{2\pi m}{N}$
• fundamental period $\frac{2\pi}{\omega_0} = T$	• fundamental period $m \cdot \frac{2\pi}{\omega_0}$, $\omega_0 \neq 0$

period $\frac{2\pi}{w_0} = T$, $w_0 \neq 0$	$m \cdot \frac{2\pi}{w_0}$, $w_0 \neq 0$
• period can be integer or real	• period can be only integer

Ex.

$$x(t) = \cos\left(\frac{2\pi t}{12}\right) \;\&\; x[n] = \cos\left(\frac{2\pi n}{12}\right)$$

$\Rightarrow$ both have period $= 12$

$$x(t) = \cos\left(\frac{8\pi t}{31}\right) \;\&\; x[n] = \cos\left(\frac{8\pi n}{31}\right)$$

$\Downarrow$ period $= \frac{31}{4}$ $\qquad$ $\Downarrow$ period $= 31$

Unit impulse & step function

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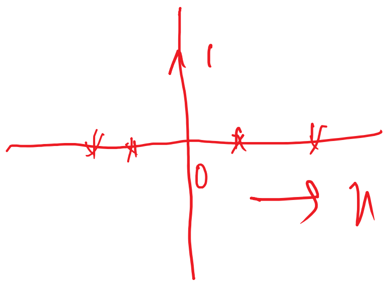
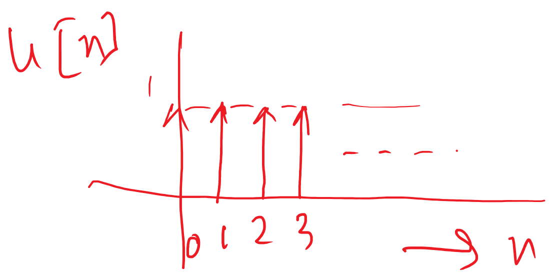
Discrete-time

$$\delta[n] = \begin{cases} 0, & n \neq 0 \\ 1, & n = 0 \end{cases}$$



$$u[n] = \begin{cases} 0, & n < 0 \\ 1, & n \geq 0 \end{cases}$$



$$\delta[n] = u[n] - u[n-1]$$

and $$u[n] = \sum_{m=-\infty}^{n} \delta[m]$$

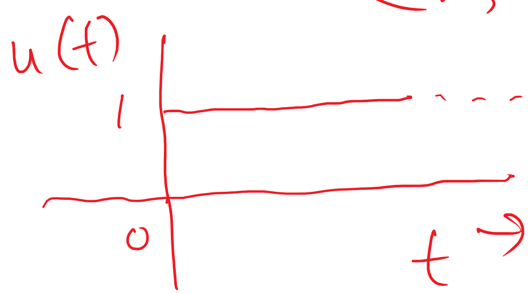
→ $\delta[n]$ has the property of sampling on picking up a particular n.

$$x[n]\,\delta[n] = x[0]\,\delta[n] = x[0]$$

$$x[n]\,\delta[n-n_0] = x[n_0]\,\delta[n-n_0] = x[n_0]$$

Continuous domain step function

$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t > 0 \end{cases}$$



→ Continuous time δ function is related to unit step function as

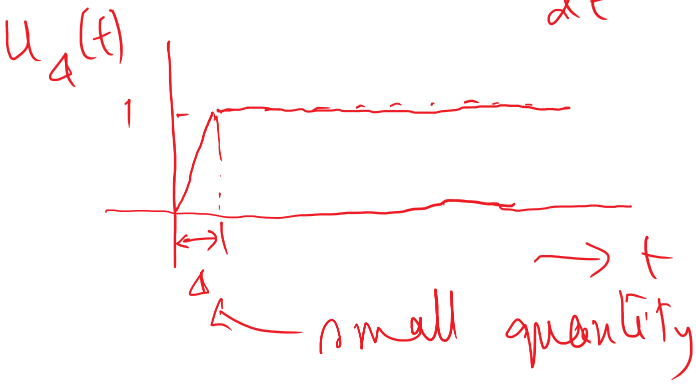
$$u(t) = \int_{-\infty}^{t} \delta(\tau)\, d\tau, \quad \tau \text{ is also in time}$$

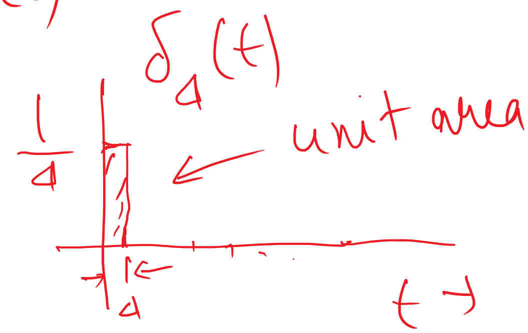
$$\delta(t) = \frac{d}{dt} u(t) \quad [\text{This can not be defined at } t=0]$$

→ Since $u(t)$ is discontinuous at $t=0$, it is not differentiable exactly at $t=0$. This can be done approximately over a short interval Δ.

... approximately over a short interval

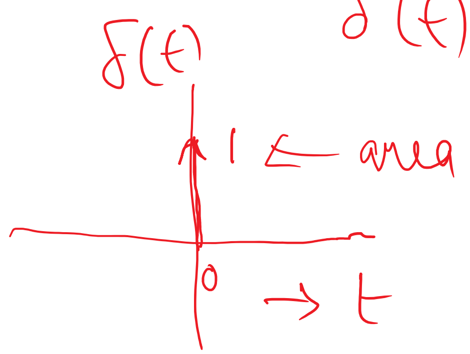
Δ, $\delta_\Delta(t) = \frac{d}{dt} u_\Delta(t)$





→ $\delta_\Delta(t)$ is short pulse of duration Δ and unit area. $\delta(t)$ can be defined in a limiting sense.

$$\delta(t) = \lim_{\Delta \to 0} \delta_\Delta(t)$$



$$u(t) = \int_{-\infty}^{t} \delta(\tau)\, d\tau = \int_{\infty}^{0} \delta(t-\sigma)(-d\sigma)$$

$$= \int_{0}^{\infty} \delta(t-\sigma)\, d\sigma$$

→ Similar to the discrete case, $\delta(t)$ has the sampling property.

Now, the sampling property.

$$x_1(t) = x(t)\,\delta_\Delta(t),$$ assuming that $x(t)$ is constant over Δ duration

As $\Delta \to 0$, $$x(t)\cdot\delta(t) = x(0)\,\delta(t)$$

and $$x(t)\,\delta(t-t_0) = x(t_0)\,\delta(t-t_0).$$

Continuous time and Discrete time System

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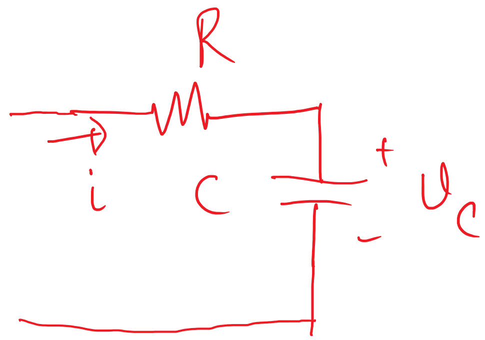
$x(t) \longrightarrow$ Continuous time system $\longrightarrow y(t)$

input output

$x[n] \longrightarrow$ Discrete-time System $\longrightarrow y[n]$

input output



$$i(t) = \frac{v_s(t) - v_c(t)}{R}$$

$$i(t) = C \frac{d}{dt} v_c(t)$$

$v_c(t) \rightarrow$ output

$v_s(t) \rightarrow$ input

$$\frac{d}{dt} v_c(t) + \frac{1}{RC} v_c(t) = \frac{1}{RC} v_s(t)$$

Discrete-time System

$$y[n] = 1.01\, y[n-1] + x[n]$$

$\Rightarrow$ Constant coefficient difference equation

$$y[n] - 1.01\, y[n-1] = x[n]$$

$$\sum a_k \, y[n-k] = \sum b_{k_1} x[n-k_1]$$

k, k_1 positive & >0 gives past value

k, k_1 negative gives future value

$$y[n] = x[n+1] - x[n-1]$$

$$y'[n] = 0.5\, x[n+1] - x[n-1]$$

Systems with & without memory

→ A system which require to store the past values or use past values of input or output or both is a system with memory.

→ A system which does not require past or future value is a system without memory

$$y[n] = 2x[n] - x^2[n]$$

$\Rightarrow$ memory less system

$y[n] = (x[n] + x[n-1])/2$

$y[n] = y[n-1] + x[n]$

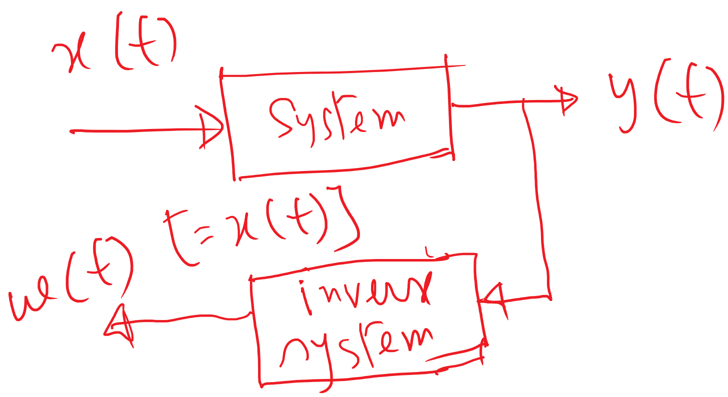
$\Rightarrow$ system with memory

$y(t) = R\,x(t) \Rightarrow$ system without memory

Inverse system

Original system $\Rightarrow$ $y(t) = 2\,x(t)$ (output: $y(t)$; input: $x(t)$)

Inverse system $\Rightarrow$ $w(t) = \frac{1}{2}\,y(t) = x(t)$ (input)



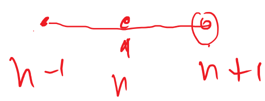
Causality

A system is causal if the output at any time depends only on the values input at present and/or past instants values of input at present and/or past instants.

$$y[n] = \sum_{k=-\infty}^{n} x[k] \Rightarrow \text{causal}$$

$$y'[n] = x[n] - x[n-1] \Rightarrow \text{causal}$$

$$y[n] = x[n+1] - x[n] \Rightarrow \text{non-causal}$$



$y[n] = x[n]^2 \Rightarrow$ causal

Stability

→ If a system gives bounded output for bounded input, the system is said to be stable (BIBO stability)

or
in discrete
domain it is
with $x[n]$ &
$y[n]$

For $|x(t)| < B$, $B < \infty$ for all t

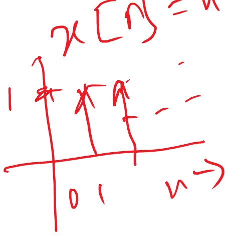
or $-B < x(t) < B$

if $|y(t)| < B_1$, $B_1 < \infty$ for all t

or $-B_1 < y(t) < B_1$

Then the system is said to be stable

$x[n] = u[n]$



$$y[n] = \sum_{k=-\infty}^{n} \underbrace{u[k]}_{\text{unit step}} = \begin{cases} 0, & n < 0 \\ n+1, & n \geq 0 \end{cases}$$

As n can go up to $+\infty$, therefore $y[n]$ is not bounded, the system is not BIBO stable.

$$y[n] = x[n] - x[n-1]$$

$\Rightarrow$ if x is bounded y is also bounded

$\therefore$ This is BIBO stable

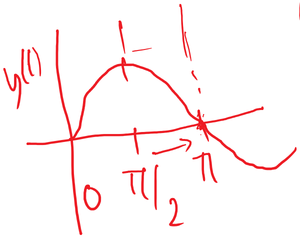
$$\boxed{\begin{aligned} y[n] &= 1, \text{ for } |x[n]| \geq B_2 \\ &= 0, \text{ otherwise} \\ \Rightarrow &\text{ not a linear system} \end{aligned}}$$

Time Invariance

→ A system is time invariant if the behaviour and characteristics of the system remain same over time.

$$\boxed{x(t) \rightarrow y(t)} \leftarrow \text{system}$$

$$x(t-t_0) \rightarrow y(t-t_0)$$



Given $y(t) = \sin(x(t))$

Then $y_1(t) = \sin(x_1(t))$, for $x(t) = x_1(t)$

Let $x_2(t) = x_1(t-t_0)$

$x_1(t-t_0)$

$\downarrow$

$$\boxed{y_2(t) = y_1(t-t_0)}$$

Then, $y_2(t) = \sin(x_2(t))$

$= \sin(x_1(t-t_0))$

~~A~~gain, $y_1(t-t_0) = \sin(x_1(t-t_0))$

$\Rightarrow$ The relation holds good irrespective of shift, thus it is time-invariant

$$y(t) = x(2t)$$

Let $x(t) = x_1(t)$, Then $y_1(t) = x_1(2t)$

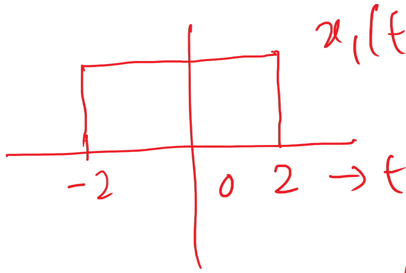
$x(t-t_0) = x_2(t) \Rightarrow y_2(t) = x_2(2t)$

$= x(2(t-t_0))$

$y_2(t) = y_2(t-t_0)$

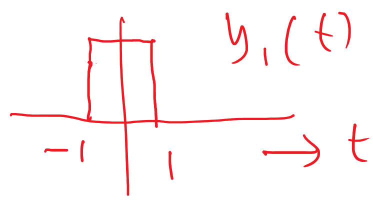
$= x(2t-t_0)$

$x_1(t)$

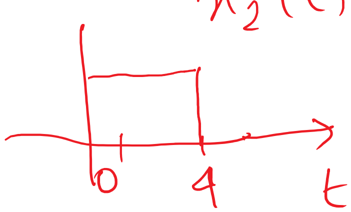


$\Rightarrow$

$y_1(t)$

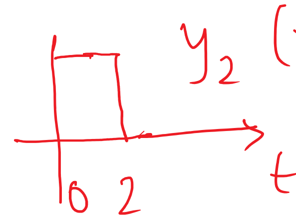


$x_2(t) = x_1(t-2)$

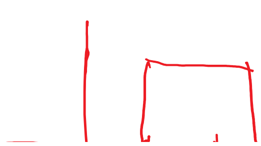


$\Rightarrow$

$y_2(t) = x_2(2t)$

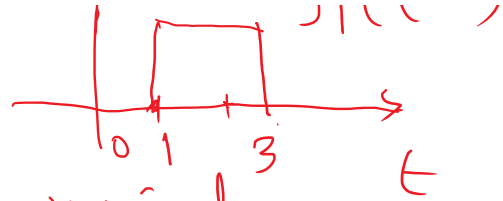


$y_2(t) \neq y_1(t-2)$



$y_1(t-2)$

$y_2(t) \neq y_1(t-2)$



∴ The system is time variant

Ex.

$$y[n] = \sum_{k=-\infty}^{n} x[k]$$

Let $x_1[n] = x[n-n_0]$

Then $$y_1[n] = \sum_{k=-\infty}^{n} x_1[k] = \sum_{k=-\infty}^{n} x[k-n_0]$$

Taking $k_1 = k - n_0$

$$y_1[n] = \sum_{k_1=-\infty}^{n-n_0} x[k_1] \quad \text{----} \; (1)$$

Next if we shift $y[n]$ by n_0,

$$y[n-n_0] = \sum_{k=-\infty}^{n-n_0} x[k] \quad \text{----} \; (2)$$

∴ $y_1[n] = y[n-n_0] \Rightarrow$ time invariant

$$y[n] = x[\underset{\substack{\uparrow \\ \text{integer}}}{M}n], \quad -\infty < n < +\infty$$

$$x_1[n] = x[n-n_0]$$

$$\Rightarrow y_1[n] = x_1[Mn] = x[Mn - n_0]$$

Again, $y[n-n_0] = x[M(n-n_0)]$

$$= x[Mn - Mn_0]$$

$$\therefore y_1[n] \neq y[n-n_0]$$

$\Downarrow$

time variant.

Linearity

→ A system is linear if, for a system with input $x(t)$ and output $y(t)$, satisfy the following relations.

superposition $\Leftarrow$ $x_1(t) + x_2(t) \rightarrow y_1(t) + y_2(t)$

scaling $\Leftarrow$ $a x_1(t) \longrightarrow a y_1(t)$, a in general can be complex

Or, in general,

$$a x_1(t) + b x_2(t) \rightarrow a y_1(t) + b y_2(t)$$

or in discrete domain

$$a x_1[n] + b x_2[n] \rightarrow a y_1[n] + b y_2[n].$$

$$y[n] = 2x[n] + 3$$

Let $x_1[n] = 2$ and $x_2[n] = 3$

Then, $y_1[n] = 7$, $y_2[n] = 9$

$$x_3[n] = x_1[n] + x_2[n] = 5$$

$$y_3[n] = 13 \neq \left[\underbrace{y_1[n] + y_2[n]}_{16}\right]$$

$\Rightarrow$ This is non-linear system.

Ex:

$$y(t) = t\,x(t)$$

$$x_1(t) \rightarrow y_1(t) = t \cdot x_1(t)$$

$$x_2(t) \rightarrow y_2(t) = t \cdot x_2(t)$$

$$x_3(t) = a x_1(t) + b x_2(t)$$

$$y_3(t) = t\,x_3(t) = a t x_1(t) + b t x_2(t)$$

$$= a y_1(t) + b y_2(t)$$

$\Rightarrow$ linear system

Linear Time Invariance System

Friday, August 27, 2021 10:17 AM

→ Consequence of linearity is that output can readily be inferred as a linear combination of a number of input.

→ Further, if we apply time invariance, an arbitrary signal can be represented by sum of weighted & delayed impulses, applying linearity and time invariance.

In general, $$x[n] = \sum_{k=-\infty}^{+\infty} x[k]\,\delta[n-k]$$

→ In fact it can very easily be extended to a system.

→ Let $h_k[n]$ denote the response of the linear system due to the shifted unit impulse response $\delta[n-k]$

Then the output $y[n]$ of a linear system to an input $x[n]$ is

$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]\, h_k[n]$$

→ Since $\delta[n-k]$ is the shifted version of $\delta[n]$, $h_k[n]$ is shifted version of $h_0[n]$; $h_k[n] = h_0[n-k]$

Let us represent $h[n] = h_0[n]$ for convenience

Then $$y[n] = \sum_{k=-\infty}^{+\infty} x[k]\, h[n-k]$$

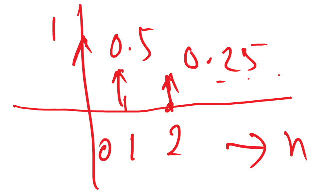
↑

Convolution relation

$$y[n] = x[n] * h[n]$$

→ If $h[n]$ i.e. impulse response is known, $y[n]$ can be found out for any arbitrary $x[n]$

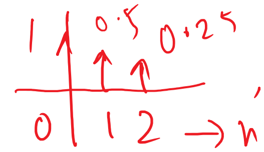
$x[n] \Rightarrow$



$, \{x[0], x[1], x[2]\}$

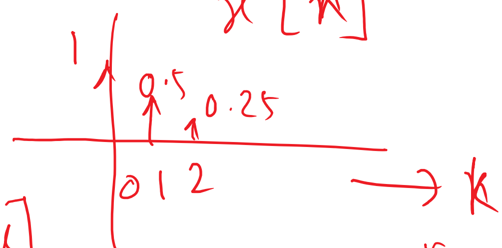
$\sum_k x[k]\, h[n-k]$

$h[n] =$



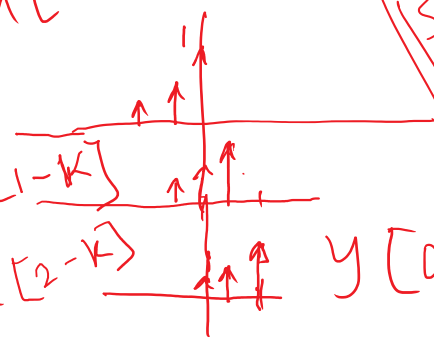
$, \{h[0], h[1], h[2]\}$

$x[k]$



reflecting or folding → $h[-k]$

$-\infty$ ⟶



$h[1-k]$

$h[2-k]$

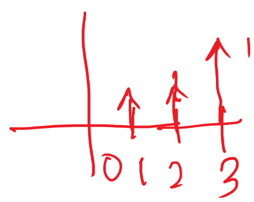
Shifting, multiplication & sum

$y[0] = x[0] \cdot h[0]$

$= 1$

$y[1] = 1 \times 0.5 + 0.5 \times 1$

$= 1$



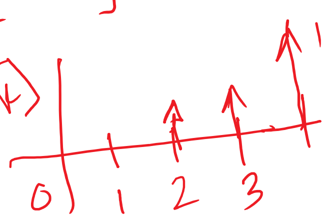
$h[-k+3]$

$y[2] = 1 \times 0.25 + 0.5 \times 0.5$

$+ 0.25 \times 1$

$= 0.75$

$h[4-k]$



$y[3] = 0.5 \times 0.25$

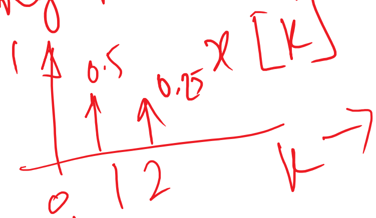
$+ 0.25 \times 0.5$

$= 0.25$

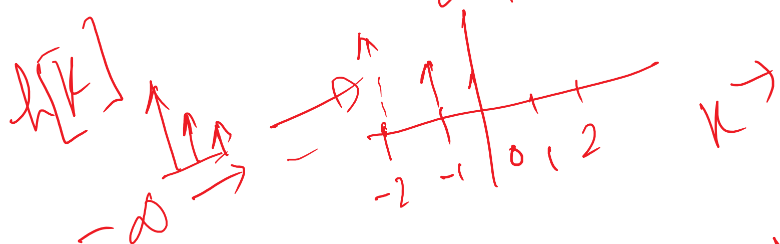
$y[4] = 0.25 \times 0.25$

$= 0.0625$

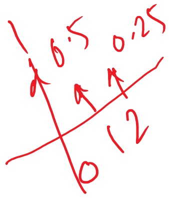
without folding $h[k]$



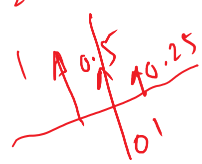
$h[k]$



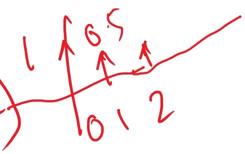
$h[-2+k]$



$h[-1+k]$



$h[0+k]$



$$y_1(n) = \sum_{k=-\infty}^{+\infty} x(k)\, h(n+k)$$

$y_1(-2) = 0.25 \times 1 = 0.25$

$y_1(-1) = 0.5 \times 1 + 0.25 \times 0.5$

$= 0.5 + 0.125$

$= 0.625$

$y_1(0) = 1 \times 1 + 0.5 \times 0.5$

$+ 0.25 \times 0.25$

$= 1 + 0.25 + 0.0625$

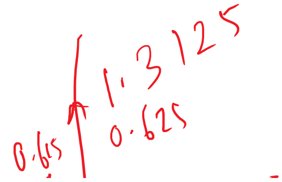
$= 1.3125$

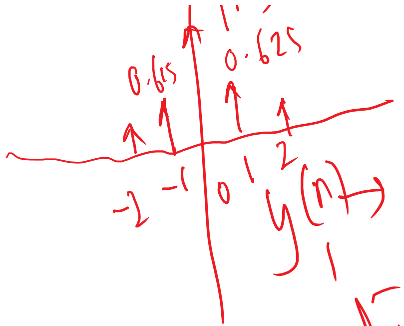
$y_1(1) = 1 \times 0.5$

$+ 0.5 \times 0.25$

$= 0.625$

$y_1(2) = 0.25$





<u>Correlation</u> calculates similarity between $x[k]$ and $h[k]$

→ Autocorrelation calculates the similarity between different parts of the same signal

→ Cross-correlation calculates the similarity between two different signals

→ Often correlation is quantified by correlation coefficient

$$\text{Correlation coefficient} = \frac{1}{N} \sum_{k=0}^{N-1} x[k]\, h[n+k]$$

→ In vector form, correlation can be computed by taking the dot product of x and H as $x^T H$.

Comp[illegible]

X and H as $X^T H$!

Convolution of finite and infinite sequence

Wednesday, September 01, 2021 12:37 PM

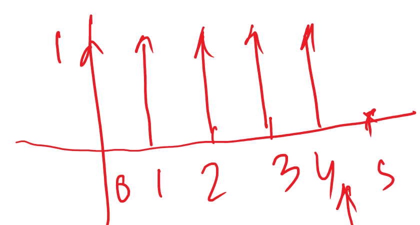
→ Similar to the convolution of two finite discrete-time sequences, convolution sum can be evaluated for infinite sequences as well.

Ex.

$$x[n] = \begin{cases} 1, & 0 \leq n \leq 4 \\ 0, & \text{otherwise} \end{cases}$$

$$h[n] = \begin{cases} \alpha^n, & 0 \leq n \leq 6 \\ 0, & \text{otherwise} \end{cases}$$

$x[n] \rightarrow$



$h[n] \rightarrow$

$\alpha > 1$

$$y[n] = \sum_{k=0}^{n} x[k]\, h[n-k] = \sum_{k=0}^{n} \alpha^k = \frac{1-\alpha^{n+1}}{1-\alpha}$$

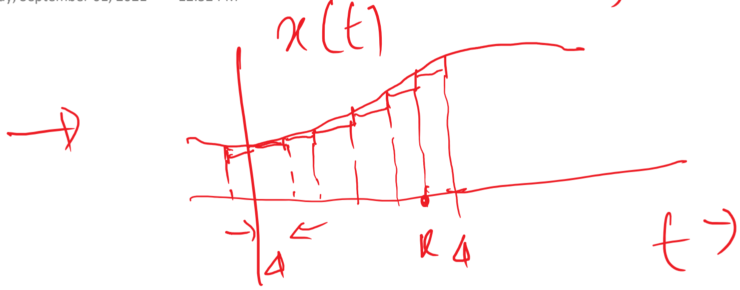
for $\alpha \neq 1$

The value of $y[n]$ need to be computed

upto the non-overlapping portion of the finite sequence i.e in this particular case, for $n < 0$ there is no overlap, so $y[n] = 0$, $n < 0$ and also for $n > 4$ there is no overlap of $x[n]$ and folded & shifted $h[n]$

Continuous time LTI system

Wednesday, September 01, 2021 12:52 PM



Staircase approximation of a continuous time signal $x(t)$ is assumed to be constant for ~~an~~ a small interval of Δ.

$$\delta_\Delta(t) = \begin{cases} \frac{1}{\Delta}, & 0 \leq t < \Delta \\ 0, & \text{otherwise} \end{cases}$$

$\Rightarrow \delta_\Delta(t) \cdot \Delta = 1, \; 0 \leq t < \Delta$

Approximated signal

$$\hat{x}(t) = \sum_{k=-\infty}^{+\infty} x(k\Delta)\, \delta_\Delta(t - k\Delta)\, \Delta$$

$$\therefore x(t) = \lim_{\Delta \to 0} \sum_{k=-\infty}^{+\infty} x(k\Delta)\, \delta_\Delta(t - k\Delta) \cdot \Delta$$

As $\Delta \to 0$, the summation can be approximated by integration.

Then,

$$x(t) = \int_{-\infty}^{+\infty} x(\tau)\, \delta(t - \tau)\, d\tau$$

Continuous-time impulse response

Let $\hat{h}_{k\Delta}(t)$ be the response of LTI system to the input of $\delta_\Delta(t - k\Delta)$

Then,

$$\hat{y}(t) = \sum_{k=-\infty}^{+\infty} x(k\Delta)\,\hat{h}_{k\Delta}(t)\cdot\Delta$$

, following the same staircase logic as followed for $x(t)$

Now, as $\Delta \to 0$,

$$y(t) = \lim_{\Delta \to 0} \sum_{k=-\infty}^{+\infty} x(k\Delta)\,h_{k\Delta}(t)\cdot\Delta$$

Since $\Delta \to 0$, the summation can be replaced by integration, & further replacing $k\Delta$ by a continuous time variable τ

$$y(t) = \int_{-\infty}^{+\infty} x(\tau)\,h_\tau(t)\,d\tau$$

→ Since τ is the shift in continuous time domain, and $h_\tau(t)$ is the shifted version of $h_0(t)$. This is due to the property of time invariance. Therefore,

$h_\tau(t)$ can be replaced by $h_0(t-\tau)$

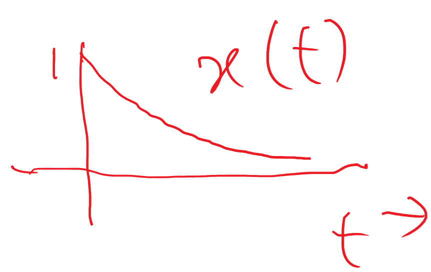
For convenience, if we drop the subscript and take $h(t) = h_0(t)$,

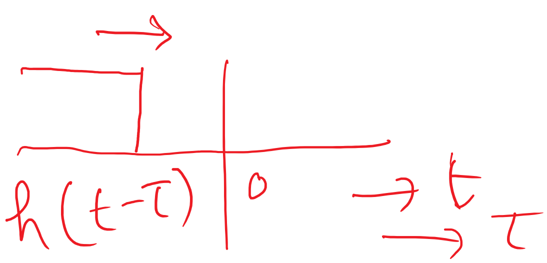
$$y(t) = \int_{-\infty}^{+\infty} x(\tau)\, h(t-\tau)\, d\tau$$

$$\boxed{y(t) = x(t) * h(t)}$$

Ex. $x(t) = e^{-at} u(t)$, $a > 0$

and $h(t) = u(t)$





For $t < 0$, $y(t) = 0$ and

for $t > 0$, $x(\tau)\, h(t-\tau) = e^{-a\tau}$, $0 < \tau < t$

$$y(t) = \int_0^t e^{-a\tau}\, d\tau$$

$$= \frac{1}{a}\left(1 - e^{-at}\right)$$

In general,

$$y(t) = \frac{1}{a}\left(1 - e^{-at}\right) u(t)$$

Properties of LTI system

Commutative property

→ This holds good for both continuous time and discrete time LTI system

$$x[n] * h[n] = h[n] * x[n] = \sum_{k=-\infty}^{+\infty} h[k] \cdot x[n-k]$$

$$x(t) * h(t) = h(t) * x(t)$$
$$= \int_{-\infty}^{+\infty} h(\tau) x(t-\tau) d\tau$$

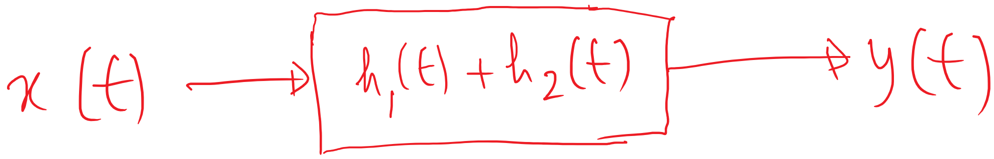
$$x[n] * h[n] = \sum_{k=-\infty}^{+\infty} x[k] \, h[n-k]$$

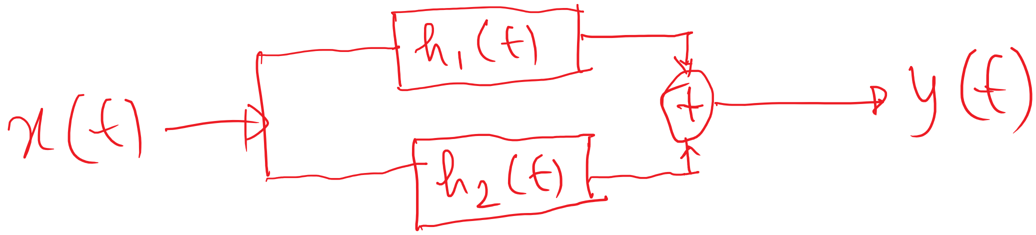
$$[r = n - k] \Rightarrow \sum_{r=-\infty}^{+\infty} x[n-r] \, h[r] = h[n] * x[n]$$

Distributive property

$$x[n] * \left(h_1[n] + h_2[n] \right) = x[n] * h_1[n] + x[n] * h_2[n]$$

$$x(t) * \left(h_1(t) + h_2(t) \right) = x(t) * h_1(t) + x(t) * h_2(t)$$

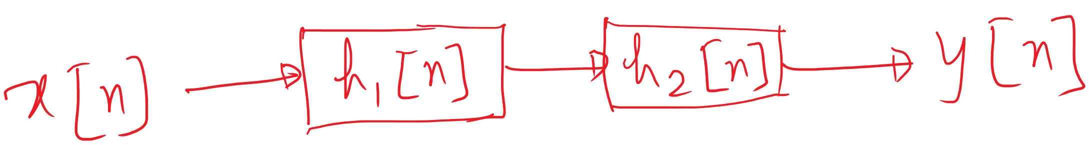


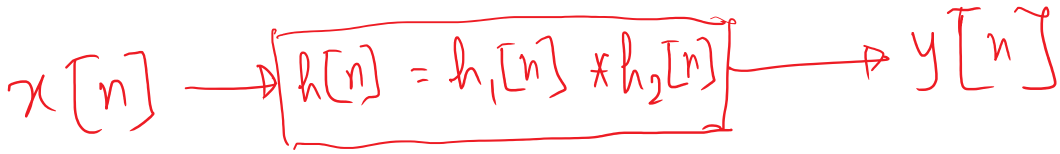


Associative property

$$x[n] * \left(h_1[n] * h_2[n]\right) = \left(x[n] * h_1[n]\right) * h_2[n]$$

$$x(t) * \left[h_1(t) * h_2(t)\right] = \left[x(t) * h_1(t)\right] * h_2(t)$$





→ LTI system without memory can be implemented only for impulse response $h[n] = 0$, for $n \neq 0$ or $h[n] = K\,\delta[n]$

or, $y[n] = k\,x[n]$

In general LTI system is a system with memory.

→ Invertibility condition for LTI system.

$$x(t) \longrightarrow \boxed{h(t)} \xrightarrow{y(t)} \boxed{h_I(t)} \longrightarrow w(t)$$

LTI system is invertible if $w(t) = x(t)$

$$x(t) \longrightarrow \boxed{h(t) * h_I(t) = \delta(t)} \longrightarrow x(t)$$

Similarly, $h[n] * h_I[n] = \delta[n]$

$\Downarrow$ inverse system response

Ex.

$$y(t) = x(t - t_0)$$

$$\therefore\ h(t) = \delta(t - t_0)$$

$$\text{or}\quad x(t - t_0) = x(t) * \delta(t - t_0)$$

$$\text{Then}\quad h_I(t) = \delta(t + t_0)$$

$$h(t) * h_I(t) = \delta(t - t_0) * \delta(t + t_0)$$

$$= \delta(t)$$