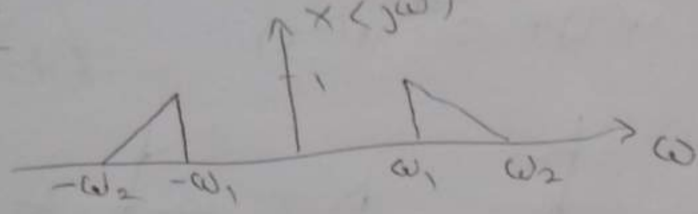
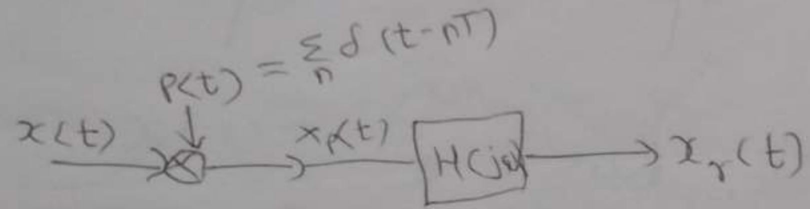
Opp. probl. (Sampling)

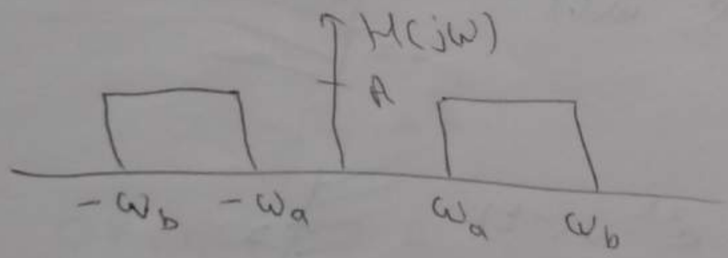
(Bandpass Sampling)

7.26.





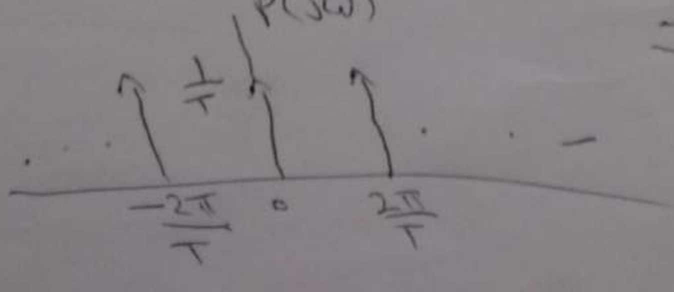
$2\omega_1 > \omega_2$



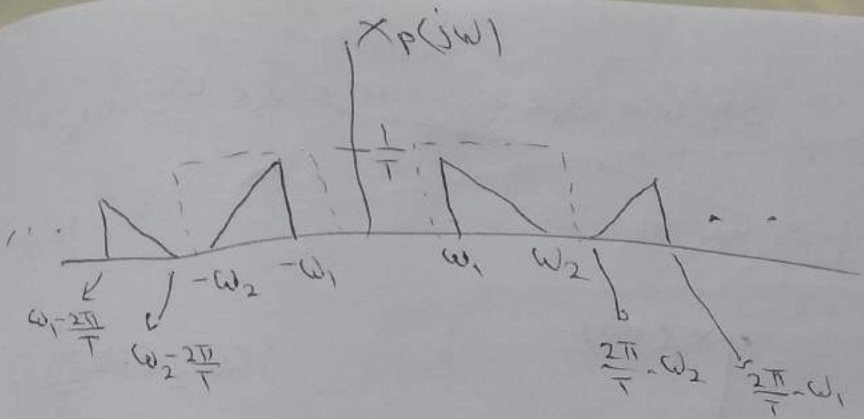
Find T, A, ω_a, ω_b s.t. $x_r(t) = x(t)$

$$P(j\omega) = \frac{2\pi}{T}\sum_k \delta\left(\omega - \frac{2\pi k}{T}\right)$$

$$X_p(j\omega) = \frac{1}{2\pi} X(j\omega) * P(j\omega) =$$



$$= \frac{1}{T}\sum_k X\left(\omega - \frac{2\pi k}{T}\right)$$



in the above case,

$$\frac{2\pi}{T} - \omega_2 > \omega_2 \Rightarrow \frac{2\pi}{T} > 2\omega_2$$

(Nyquist cd^n)

$\Rightarrow$ ~~2~~

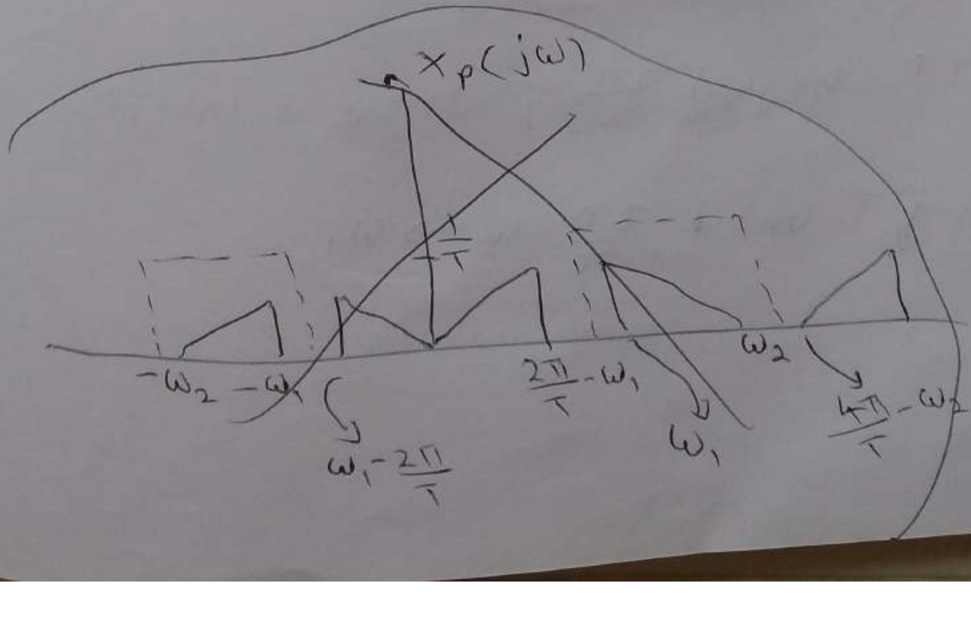
can take $A = T$, $0 < \omega_a < \omega_1$ &

$\omega_2 < \omega_b < \frac{2\pi}{T} - \omega_2$

~~T = 2~~ $T < \frac{\pi}{\omega_2}$ in this case.

consider the case when $\frac{2\pi}{T} - \omega_2 < \omega_2$

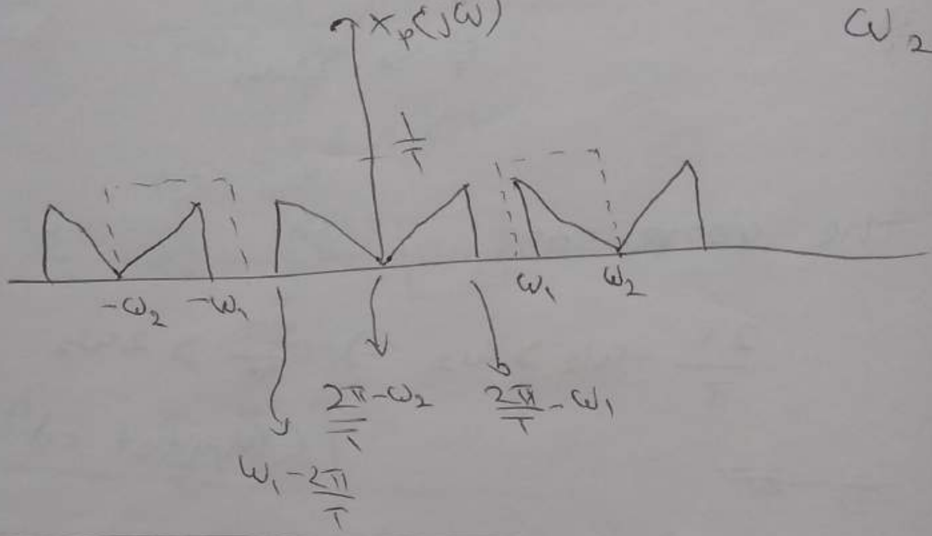
'' 0



$T = \frac{2\pi}{\omega}$, $\frac{2\pi}{T} - \omega_1 < \omega_a < \omega_1$, $\omega_2 < \omega_b < \frac{4\pi}{T} - \omega_2$

$\frac{2\pi}{T} - \omega_1 = \omega_2 - \omega_1$, $\frac{4\pi}{T} - \omega_2 = 2\omega_2 - \omega_2$

$\omega_2 = \frac{2\pi}{T}$



$T = \frac{2\pi}{\omega_2}$, $\omega_b = \omega_2$

$\frac{2\pi}{T} - \omega_1 < \omega_a \leq \omega_1$

$\underbrace{\omega_2}_{=\omega_b} - \omega_1 < \omega_a \leq \omega_1$

$\Rightarrow$ $\boxed{\omega_2 - \omega_1 < \omega_1 \Rightarrow \omega_2 < 2\omega_1}$ → hypoth.

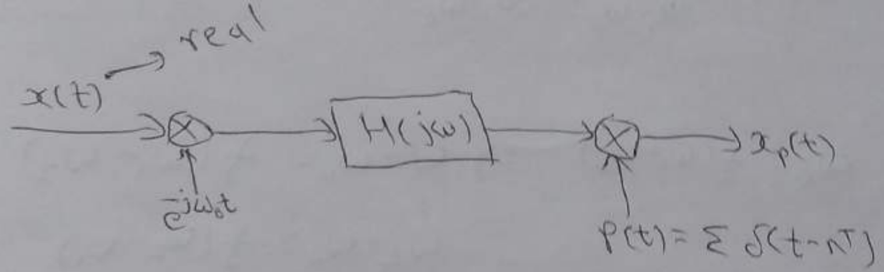
✓

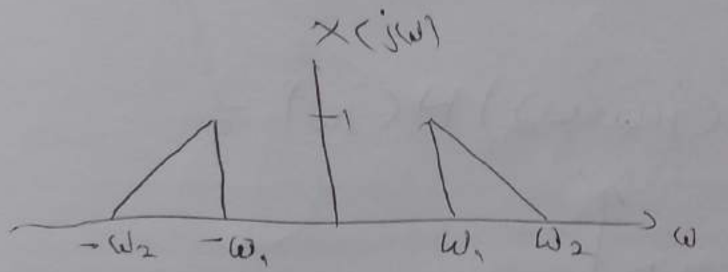
if $T \uparrow$ beyond $\frac{2\pi}{\omega_2}$, there is interf.

we can $\downarrow T$ until $\frac{2\pi}{T} - \omega_1 \to \omega_1$

($\therefore$ for $0 < \frac{2\pi}{T} - \omega_2 < \frac{2\pi}{T} - \omega_1 < \omega_1$,
there is no ~~a~~ alias.)

7.27





$\omega_0 = \frac{1}{2}(\omega_1 + \omega_2)$

$H \rightarrow$ LPF with cut off fr. $\frac{1}{2}(\omega_2 - \omega_1)$

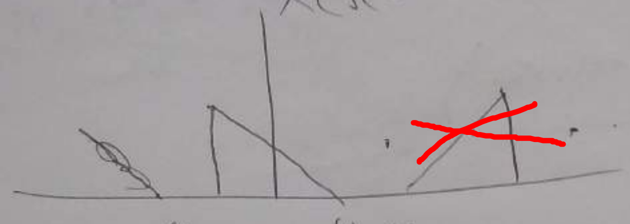
a) Sketch $X_p(j\omega)$

b) Find max. sampling period T s.t. $x(t)$ is recoverable from $x_p(t)$

c) Find a syst. to recover $x(t)$.

$$x(t) e^{-j\omega_0 t} \longleftrightarrow X(j(\omega + \omega_0))$$

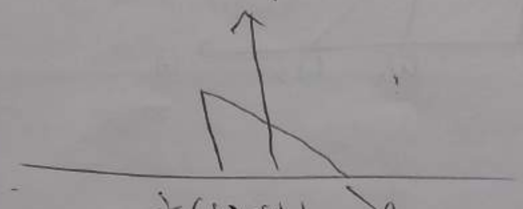
$X(j(\omega+\omega_0))$



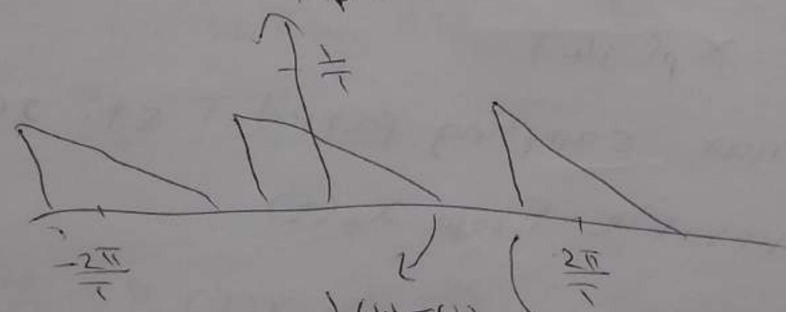
$$\omega_0 = \frac{1}{2}(\omega_1+\omega_2) \Rightarrow \omega_1-\omega_0 = \frac{1}{2}(\omega_1-\omega_2)$$

$$\omega_2-\omega_0 = \frac{1}{2}(\omega_2-\omega_1)$$

$\Rightarrow X(j(\omega+\omega_0))H(j\omega) =$



$X_p(j\omega)$



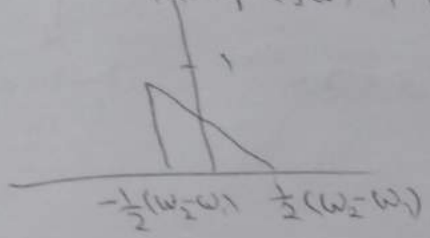
$\frac{2\pi}{T}$ → for recovery,

$$\frac{1}{2}(\omega_2-\omega_1) < \frac{2\pi}{T} - \frac{1}{2}(\omega_2-\omega_1)$$

$$\Rightarrow (\omega_2-\omega_1) < \frac{2\pi}{T} \Rightarrow T < \frac{2\pi}{\omega_2-\omega_1}$$

$H_1 \rightarrow$ LPF with cutoff at $\pm\frac{1}{2}(\omega_2-\omega_1)$

$X_p(j\omega)\, T H_1(j\omega) = Y(j\omega)$



~~$Y(j\omega)$~~ $Y(j(\omega-\omega_0)) = Z(j\omega) \leftrightarrow z(t) = e^{j\omega_0 t}\cdot y(t)$

~~$W(j\omega) = Z(j\omega) +$~~

Recall that $x(t)$ real $\Rightarrow X(j\omega) = X^*(-j\omega)$

let $W(j\omega) = Z(j\omega) + Z^*(-j\omega)$

$$\updownarrow \qquad \updownarrow \qquad \updownarrow$$

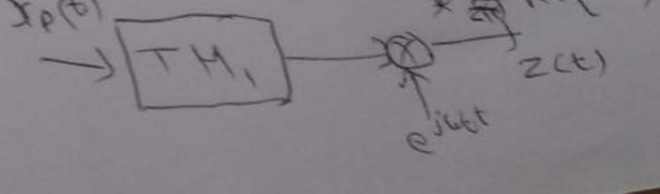
$$w(t) = z(t) + z^*(t)$$

$$\| \qquad\qquad \|$$

$$x(t) = 2\,\mathrm{Re}(z(t))$$

$$\|$$

$$2\,\mathrm{Re}(y(t)e^{j\omega_0 t})$$



$x(t) = 2\,\mathrm{Re}\{z(t)\}$

Bilinear transf^n : Rational f^n $\downarrow$ Rational f^n

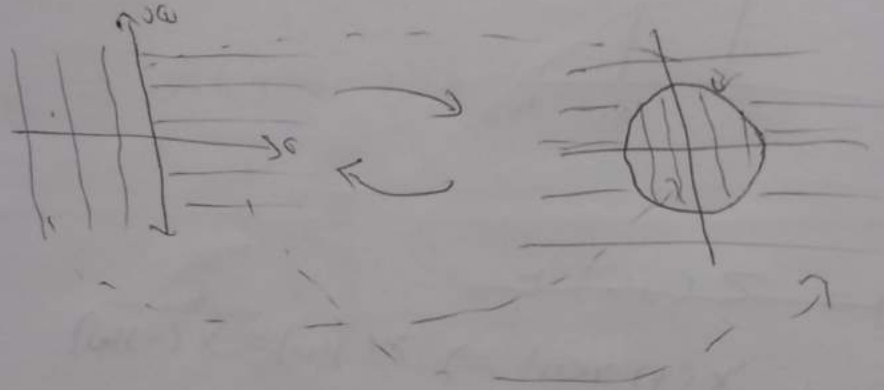
(BLT)

$s = \sigma + j\omega$ $\quad S \longrightarrow Z \quad z = re^{j\omega}$

$$\mathbb{C} \longrightarrow \mathbb{C}$$

$$s \longmapsto \frac{1 - z^{-1}}{1 + z^{-1}}$$

$$\frac{1+s}{1-s} \longleftarrow\!\!| \; z$$



$j\omega$ axis $\longleftrightarrow$ unit circle

LHP $\longleftrightarrow$ inside the unit circle

RHP $\longleftrightarrow$ outside the unit circle

$\cancel{j\omega \longmapsto t =}$

let $z = e^{jw} \rightarrow$ unit circle

$\cancel{z \longmapsto t + j}$

$$s = \cancel{1+e^{j\omega}} \quad \frac{1-e^{-j\omega}}{1+e^{-j\omega}} = \frac{e^{-j\frac{\omega}{2}}\, j\sin\frac{\omega}{2}}{e^{-j\frac{\omega}{2}}\, \cos\frac{\omega}{2}}$$

$$= j\tan\frac{\omega}{2} \quad \text{i.e. } s \to \text{purely imag.}$$

let $s = j\omega$, then

$$z = \frac{1+j\omega}{1-j\omega} = e^{j2\theta} \quad (\theta = \tan^{-1}\omega)$$

$$|z| = 1$$

thus the purely imag. axis & the unit circle are mapped to one another.

Bil. transf^n is used to transform fiters from cts $\leftrightarrow$ discrete. This is used a lot in filter design.

There are std. design tech. to design cts. time filters (Butterworth, Chebysh. etc.) as per specif^{n}s (cut off freq., passband/stopband gain/attenuat^n)

Discrete spec's & T.F. are transf. to cts. spec's & T.F. & the designed filter is conv. to discr. form by BLT.

Opp. prob. (Z trans. & filters) (BLT)

10.65. $H_c(s) \rightarrow$ causal, stable LTI

$\downarrow$ BLT

$H_d(z) \rightarrow$ " "

Some ch. of $|H_c(s)|$ are preserved in $|H_d(z)|$

a) $H_c(s) = \frac{a-s}{a+s}$ $a \in \mathbb{R}$, $a > 0$

s.t. $|H_c(j\omega)| = 1$ ✓

b) $H_d(z) = H_c(s)\Big|_{s = \frac{1-z^{-1}}{1+z^{-1}}}$

s.t. $H_d(z)$ has one pole inside the unit circle & one zero outside the unit circle.

c) s.t. $|H_d(e^{j\omega})| = 1$

$$H_d(z) = \frac{a - \frac{1-z^{-1}}{1+z^{-1}}}{a + \frac{1-z^{-1}}{1+z^{-1}}} = \frac{a(1+z^{-1}) - (1-z^{-1})}{a(1+z^{-1}) + (1-z^{-1})}$$

$$= \frac{(a-1) + z^{-1}(a+1)}{(a+1) + z^{-1}(a-1)}$$

$\Rightarrow$ ~~zero~~ pole at $-\frac{a-1}{a+1}$, ~~Pole~~ zero at $-\frac{a+1}{a-1}$

$\therefore$ pole & zeros are recip.

$$a>0 \Rightarrow a-1 < a+1$$

$$\Rightarrow \left|\frac{a-1}{a+1}\right| < 1$$

$$H_d(e^{j\omega}) = \frac{a-1+e^{-j\omega}(a+1)}{a+1+e^{-j\omega}(a-1)}$$

let $r = \frac{a-1}{a+1}$, $|r|<1$

$$\Rightarrow H_d(e^{j\omega}) = \frac{r+e^{-j\omega}}{1+re^{-j\omega}} = \frac{r+\cos\omega - j\sin\omega}{1+r\cos\omega - jr\sin\omega}$$

$$|H_d(e^{j\omega})| = \frac{\sqrt{(r+\cos\omega)^2+\sin^2\omega \cancel{-2(r+\cos\omega)\sin\omega}}}{\sqrt{(1+r\cos\omega)^2+r^2\sin^2\omega \cancel{-2(1+r\cos\omega)r\sin\omega}}}$$

$$= \frac{\sqrt{1+r^2+2r\cos\omega \cancel{-2r\sin\omega-2\sin\omega\cos\omega}}}{\sqrt{1+r^2+2r\cos\omega \cancel{-2r\sin\omega-2r^2\sin\omega\cos\omega}}}$$

$$= 1.$$

10.66. a) s.t. $H_d(e^{j\omega}) = H_c(j\tan\frac{\omega}{2})$

b) $H_c(s) = \frac{1}{(s+e^{j\frac{\pi}{4}})(s+e^{-j\pi/4})}$, $H_c \to$ causal

s.t. $H_c(0) = 1$, $|H_c(j\omega)| \downarrow$ as $\omega \uparrow$. &

$|H_c(j)|^2 = \frac{1}{2}$ & $H_c(\infty) = 0$.

c) s.t.

1. $H_d(z)$ has 2 poles, both inside the unit circle.
2. $H_d(e^{j0}) = 1$
3. $|H_d(e^{j\omega})| \downarrow$ as $\omega \uparrow$ from 0 to π.
4. Half power freq. of $H_d(e^{j\omega})$ is $\frac{\pi}{2}$.

ans. a) $H_d(\cancel{z}) = H_c(s)\Big|_{s=\frac{1-z^{-1}}{1+z^{-1}}}$

$\therefore H_d(e^{j\omega}) = H_c(s)\Big|_{s=\frac{1-e^{-j\omega}}{1+e^{-j\omega}}}$

$= H_c(s)\Big|_{s=j\tan\frac{\omega}{2}}$

b) easy verif^n

c)

$$H_d(z) = \frac{1}{\left(\frac{1-z^{-1}}{1+z^{-1}} + e^{j\frac{\pi}{4}}\right)\left(\frac{1-z^{-1}}{1+z^{-1}} + e^{-j\frac{\pi}{4}}\right)}$$

$$= \frac{1+z^{-1}}{\left(1-z^{-1} + e^{j\frac{\pi}{4}}(1+z^{-1})\right)\left(1-z^{-1} + e^{-j\frac{\pi}{4}}(1+z^{-1})\right)}$$

$$= \frac{1+z^{-1}}{\left(1+e^{j\frac{\pi}{4}} + z^{-1}(e^{j\frac{\pi}{4}}-1)\right)\left(1+e^{-j\frac{\pi}{4}} + z^{-1}(e^{-j\frac{\pi}{4}}-1)\right)}$$

$\Rightarrow$ poles at $-\frac{e^{j\frac{\pi}{4}}-1}{e^{j\frac{\pi}{4}}+1}$, $-\frac{e^{-j\frac{\pi}{4}}-1}{e^{-j\frac{\pi}{4}}+1}$

check that mag. < 1.

$$H_d(e^{j0}) = \frac{1}{(0+e^{j\frac{\pi}{4}})(0+e^{-j\frac{\pi}{4}})}$$

$$\cancel{= \frac{1}{1+1+e^{j\frac{\pi}{4}}+e^{-j\frac{\pi}{4}}}}$$

$$= 1$$

$$|H_d(e^{j\omega})| = |H_c(j\tan\tfrac{\omega}{2})| =$$

$$\frac{1}{\left|\frac{1}{\sqrt{2}} + j\left(\frac{1}{\sqrt{2}} + \tan\frac{\omega}{2}\right)\right| \left|\frac{1}{\sqrt{2}} + j\left(\tan\frac{\omega}{2} - \frac{1}{\sqrt{2}}\right)\right|}$$

$$= \frac{1}{\sqrt{1+\tan^4\frac{\omega}{2}}}$$