Prob-Stat(MA20205)/CT/2

Fill in the blanks (Numerical)

Date of Exam : 06th Nov, 2022

Time : SLOT A

Duration : 40min

No of questions: 5 out of 8 questions

Type: Random-sequential (navigation allowed)

Each question carries 2 marks

October 29, 2022

A Q21 . Let continuous random variables (X, Y) have joint PDF given by

$$f(x,y) = \begin{cases} 2, & 0 < x < y < 1 \\ 0, & \text{otherwise.} \end{cases}$$

Then $P\left(X \geq \frac{1}{3} \mid y = \frac{2}{3}\right)$ is equal to ——
(answer should be correct up to three decimal places, error range: 0.005)

Ans: 0.5.

ERROR RANGE: 0.005

Solution: The marginal density function of Y is

$$f_Y(y) = \int_0^y 2dx = \begin{cases} 2y, & 0 < y < 1 \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$f_{X|Y}(x|y) = \frac{1}{y}, \; 0 < x < y$$

which is uniform on the interval $(0, y)$. Therefore

$$P\left(X \geq \frac{1}{3} \mid y = \frac{2}{3}\right) = \int_{1/3}^{2/3} \frac{1}{2/3} dx = \frac{1}{2}$$

A Q65. Let U and V be two independent random variables each having the standard normal distribution. Define the random variables X and Y by $X = U + 3V$ and $Y = U + V$. What is the correlation coefficient of X and Y? (answer should be correct up to three decimal places, error range: 0.005)

ANSWER: 0.894427

ERROR RANGE: 0.005

Solution:- The Jacobian of the transformation is $\frac{1}{2}$. The density function of (X, Y) is $\frac{1}{4\pi}e^{-\ldots}$. So, $\mu_i = 0, \sigma_2 = \frac{1}{\sqrt{5}}$ and $\rho = \frac{2}{\sqrt{5}}$

[ALTERNATIVE SOLUTION $(1+3)/\sqrt{(1+9)*(1+1)} = 4/2\sqrt{5} = 2/\sqrt{5}$]

A Q66. Let X and Y have the joint pdf

$$f(x, y) = cx^2y, \quad -y < x < 1, 0 < y < 1,$$

where c is a constant. Find the value of c.(answer should be correct up to three decimal places, error range: 0.005)

ANSWER : 30/7= 4.286

ERROR RANGE: 0.005

Solution: $1 = \int_0^1 \int_{-y}^1 cx^2y\,dx\,dy = 7c/30$. **Hence** $c = \frac{30}{7}$

A Q61. Let X follows Uniform[0, 1] distribution and $Y|X = x$ be Binomial$(10, x)$. What the expected value of Y? (answer should be correct up to three decimal places, error range: 0.005)

ANS: 5

ERROR RANGE: 0.005

ANSWER : Y has discrete uniform $\{0, \ldots, 10\}$ distribution. So expected value is 5

A Q67. Let (X, Y) be Bivariate Normal$(\mu_x = 1, \sigma_x^2 = 4, \mu_y = 1, \sigma_y^2 = 4, \rho = 1/2)$ random variables. Find $Var(Y|X = 1)$. (answer should be correct up to three decimal places, error range: 0.005)

ANS: 3

ERROR RANGE: 0.005

ANSWER : $Var(Y|X = 1) = \sqrt{1 - \rho^2}\sigma_y^2 = 3$

[CORRECTED TO $Var(Y|X = 1) = (1 - \rho^2)\sigma_y^2 = 3$]

A Q71. Let $X_1, X_2 \ldots$ be i.i.d. Poisson random variables with mean 0.5. Define $Y_k = k$ if $\sum_{i=1}^{k} X_i \leq k \in \mathbb{N}$ and $Y_k = 0$ otherwise. Find $E(Y_4)$. (answer should be correct up to three decimal places, error range: 0.005)

ANS: 0.9473

ERROR RANGE: 0.005

Soln. $Z = \sum_{i=1}^{4} X_i$ follows Poisson distribution with mean 2. So $E(Y_4) = 4 * P(Z \leq 4) = 0.9473$

A Q76. Let X_1 and X_2 be Poisson random variables with mean 2 and 3 respectively. Find the conditional expectation of X_1 when it is given that $X_1 + X_2 = 10$.

ANSWER : 4

ERROR RANGE: 0.005

Soln: $X_1 | X_1 + X_2 = k \sim bin(k, 2/5)$ Hence the answer is 4.

A Q80. Let X and Y have the joint pmf

$$f(x, y) = \frac{k\,x}{y}, \quad x = 1, 2; \; y = 1, 2,$$

where k is suitable constant. Find $P(X + Y = 3 | X = 1)$.(answer should be correct up to three decimal places, error range: 0.005)

ANSWER : 0.33333

ERROR RANGE: 0.005

Solution:

$P(X + Y = 3 | X = 1) = P(X = 1, Y = 2)/P(X = 1) = \frac{k/2}{k + k/2} = 1/3.$

Prob-Stat(MA20205)/CT/2

Fill in the blanks (Numerical)

Date of Exam : 06th Nov, 2022

Time : SLOT B

Duration : 40min

No of questions: 5 out of 8 questions

Type: Random-sequential (navigation allowed)

Each question carries 2 marks

October 29, 2022

B Q22. Suppose two (six faced) fair dice are rolled independently and the numbers observed on two upper faces are the random variables X and Y. Let $Z = X + Y$. Then $P(X = 4 \mid Z = 8) =$ ——. (answer should be correct up to three decimal places, error range: 0.005)

Answer: 0.2.

ERROR RANGE: 0.005

Solution: Z takes the values between 2 and 12 with $P(Z = 2) = P(X = 1, Y = 1) = P(X = 1)P(Y = 1) = \frac{1}{6}\frac{1}{6} = 1/36$, $P(Z = 3) = P(X = 1, Y = 2) + P(X = 2, Y = 1) = P(X = 1)P(Y = 2)P(Y = 1) = 1/36 + 1/36 = 1/18$, $P(Z = 4) = P(X = 1, Y = 3) + P(X = 2, Y = 2) + P(X = 3, Y = 1) = 3/36 = 1/12$. Similarly we get $P(Z = 8) = \frac{5}{36}$. Then $P(X = 4|Z = 8) = \frac{P(X=4,Z=8)}{P(Z=8)} = \frac{P(X=4,Y=4)}{P(Z=8)} = \frac{1/36}{5/36} = 1/5$.

[ALTERNATIVE SOLUTION $\frac{\#\{(4,4)\}}{\#\{(2,6),(3,5),(4,4),(5,3),(6,2)\}} = 1/5$]

B Q44. Let X and Y have the joint density function

$$f(x, y) = \begin{cases} 3x, & 0 < y < x < 1 \\ 0, & \text{elsewhere.} \end{cases}$$

What is $Cov(X, Y)$ where $Cov(X, Y)$ is the covariance of X and Y? (answer should be correct up to three decimal places, error range: 0.005)

ANSWER: 0.1875

ERROR RANGE: 0.005

Solution:- For $0 < x < 1$, $f_X(x) = \int_0^x 3x\ dy = 3x^2$. For $0 < y < 1$, $f_Y(y) = \frac{3}{2}(1 - y^2)$. So, $E(X) = \frac{3}{4}$, $E(Y) = \frac{3}{8}$, $E(XY) = \frac{3}{10}$. So, $Cov(X, Y) = \frac{3}{160} = 0.1875$

B Q56. Let X_1 and X_2 have the joint pdf

$$f(x_1, x_2) = e^{-x_1 - x_2}, \quad 0 < x_1 < \infty, 0 < x_2 < \infty.$$

Consider the transformation $Y_1 = X_1 - X_2$ and $Y_2 = X_1 + X_2$. Ley $g(y_1, y_2)$ be the joint pdf of Y_1 and Y_2. If $g(y_1, y_2) = |J| f(x_1, x_2)$, then find the value of $|J|$ where $|J|$ is the absolute value of the the determinant of Jacobian matrix of transformation. (answer should be correct up to three decimal places, error range: 0.005)

ANSWER : 0.5

ERROR RANGE: 0.005

Solution: t is easy to see that $x_1 = \frac{y_1 + y_2}{2}$ and $x_2 = \frac{y_2 - y_1}{2}$. $J = \begin{bmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{bmatrix}$ Therefore $|J| = \frac{1}{2}$

B Q57. Let X and Y have the joint pmf

$$f(x, y) = \frac{x + y}{21}, \ x = 1, 2, 3 \ y = 1, 2.$$

Calculate the probability $P(X = 2 | Y = 2)$.(answer should be correct up to three decimal places, error range: 0.005)

ANSWER : 0.333

ERROR RANGE: 0.005

Solution: The conditional pmf $g(x|y)$ of X, given $Y = y$ is equal to $\frac{f(x,y)}{f_2(y)}$, where $f_2(y)$ marginal pmf for Y.

$f_2(y) = \frac{3y+6}{21}$, $y = 1, 2$.

So $g(x|y) = \frac{x+y}{3y+6}$, $x = 1, 2, 3$, when $y = 1$ or $y = 2$.

$P(X = 2|Y = 2) = g(2|2) = 4/12 = 0.333$

[ALTERNATIVE SOLN $(2 + 2)/((1 + 2) + (2 + 2) + (3 + 2)) = 4/12 = 0.333$]

B Q62. Let X and Y be random variables such that $Y|X = x$ follows Normal$(1, x^2)$ distribution. What is $E(Y)$? (answer should be correct up to three decimal places, error range: 0.005)

ANS: 1

ERROR RANGE: 0.005

ANSWER : $E(E((Y|X))) = E(Y)$. Now $E(Y|X = x) = 1$ for all x. So as a function of x, $E(Y|X = x)$ is the constant function 1.So expected value is 1

B Q64. Let X be a continuous random variable with pdf $f_X(x) = 2x, 0 < x \leq 1; f_X(x) = 0,$ *otherwise.* Let Y be a random variable such that the distribution of $Y|X = x$ is Uniform$[-x, x]$. Find $P(|Y| < X^3)$.(answer should be correct up to three decimal places, error range: 0.005)

ANS: 0.5

ERROR RANGE: 0.005

ANSWER : $P(|Y| < X^3) = \int_0^1 P(|Y| < X^3|X = x)f_X(x)dx$. Now $f_{Y|X}(y|x) = \frac{1}{2x}, -x \leq y \leq x$ and $f(x) = 2x, 0 \leq x \leq 1$. So $\int_0^1 P(|Y| < X^3|X = x)f_X(x)dx = \int_0^1 \frac{2x^3}{2x}2xdx = \frac{1}{2}$. So answer is 0.5.

B Q66. Let X and Y be independent Uniform$[0, 1]$ random variables and $Z = e^{X+Y}$. Find $E(Z)$. (answer should be correct up to three decimal places, error range: 0.005)

ANS: 2.952492

ERROR RANGE: 0.005

ANSWER : $E(e^{X+Y}) = E(e^X)E(e^Y)$ as X and Y are independent. Now $\int_0^1 e^x dx = e - 1$.Hence the answer is $(e-1)^2$ which is 2.952492.

[CORRECTED TO $(e-1)^2$=2.952492]

B Q72. Let (X, Y) follow a bivariate normal distribution with $(\mu_x = 2, \mu_y = 3, \sigma_x^2 = 4, \sigma_y^2 = 9, \rho = 1/3)$. Find $P(|3X - 2Y| \leq \sqrt{3})$. NOTE $\Phi(0.25) = 0.5987063$.(answer should be correct up to three decimal places, error range: 0.005)

ANS : 0.1974127

ERROR RANGE: 0.005

Soln: $3X - 2Y \sim N(0, 48)$. Hence $P(|3X - 2Y| \leq \sqrt{3}) = P(|Z| \leq 0.25) = 2\Phi(0.25) - 1 = 0.1974127$

Prob.Stat(MA20203)/CT/2

Fill in the blanks (Numerical)

Date of Exam : 06th Nov. 2022

Time : SLOT C

Duration : 40min

No of questions: 3 out of 8 questions

Type: Random-sequential (navigation allowed)

Each question carries 2 marks

October 30, 2022

C-Q8. Let the joint pdf of continuous random variables X, Y be given by

$$f(x,y) = \begin{cases} x+y, & 0 < x, y < 1 \\ 0, & \text{otherwise} \end{cases}$$

Then $E(Y \mid X = \frac{1}{3}) =$ ______
(answer should be correct up to three decimal places, error range: 0.005)

Answer: 0.6
ERROR RANGE: 0.005

Solution: $f_X(x) = \int_0^1 (x+y)dy = x + \frac{1}{2}$. Then

$$f_{Y|X}(y|x) = \frac{x+y}{x+\frac{1}{2}}$$

Then

$$E\left(Y \mid X = \frac{1}{3}\right) = \int_0^1 y f(y|x) dy = \int_0^1 y \frac{x+y}{x+\frac{1}{2}} dy = \int_0^1 y \frac{\frac{1}{3}+y}{\frac{5}{6}} = \frac{3}{5}$$

C-Q3. Suppose there are two fair coins. The first coin is tossed five times. Let the random variable X be the number of heads in these five tosses. The second coin is tossed X times. Let Y be the number of heads in the tosses of the second coin. What is the conditional probability $P(X = 4|Y = 4)$? (answer should be correct up to three decimal places, error range: 0.005)

ANSWER: 0.667

ERROR RANGE: 0.005

Solution:- $p_Y(4) = \sum_{x=4}^{5} P(X = x, Y = 4) = 5.\frac{3}{2}.(\frac{1}{2})^8$. So, $P(X = 4|Y = 4) = \frac{\binom{5}{4}\binom{4}{4}(\frac{1}{2})^8}{5.\frac{3}{2}.(\frac{1}{2})^8} = 2/3$.

C Q47. The joint density function of the continuous random variables X and Y is

$$f(x, y) = \begin{cases} cxy, & 0 < x < 4, 1 < y < 5 \\ 0, & \text{elsewhere.} \end{cases}$$

What is $10P(X + Y < 3)$?

(answer should be correct up to three decimal places, error range: 0.005)

ANSWER: 0.2083

ERROR RANGE: 0.005

Solution:- $\int_{x=0}^{4} \int_{y=1}^{5} cxy \ dydx = 1$ gives $c = \frac{1}{96}$. So, $P(X + Y < 3) = \int_{x=0}^{2} \int_{y=1}^{3-x} \frac{1}{96} xy \ dydx = \frac{1}{48}$.

C Q51. Let the continuous random variables X and Y have the joint pdf

$$f(x, y) = \frac{3}{2} x^2 (1 - |y|), -1 < x < 1, -1 < y < 1.$$

Calculate the probability that (X, Y) falls in A, where $A = \{(x, y) : 0 < x < 1, 0 < y < x)\}$.(answer should be correct up to three decimal places, error range: 0.005)

ANSWER : 0.225

ERROR RANGE: 0.005

Solution: $P((X, Y) \in A) = \int_0^1 \int_0^x \frac{3}{2} x^2 (1 - y) dydx = \frac{9}{40}$

C Q63. Let X and Y be independent Normal(0, 1) random variables. Find the correlation between $X + Y$ and $X - Y$. (answer should be correct up to three decimal places, error range: 0.005)

ANS: 0

ERROR RANGE: 0.005

ANSWER : They are independent. So answer is 0.

C Q73. Let (X, Y) follow a bivariate normal distribution with ($\mu_x = 2, \mu_y = 3, \sigma_x^2 = 4, \sigma_y^2 = 9, \rho = 1/3$). Find $P(Y > 6|X = 4)$.NOTE: $\Phi(1/\sqrt{2}) = 0.7602499$.(answer should be correct up to three decimal places, error range: 0.005)

ANS :0.2397501

ERROR RANGE: 0.005

Soln: $Y|X = 4$ follows $N(4, 8)$ distribution. So $P(Y > 6|X = 4) = P(Y > 1/\sqrt{2}) = 1 - 0.7602499$

C Q75. Let X be a continuous random variable with the c.d.f. $F(x) = \frac{e^x}{1+e^x}, \quad -\infty < x < \infty$. Let $Y = e^X$. Find $P(Y < 4)$ (answer should be correct up to three decimal places, error range: 0.005)

ANSWER : 0.8

ERROR RANGE: 0.005

ANS: $f_X(x) = \frac{e^X}{(1+e^X)^2}$ so $f_Y(y) = \frac{1}{(1+y)^2}$ and $F_Y(y) = \frac{y}{(1+y)}$ when $Y > 0$ Hence $F_y(4) = 4/5$.
Aliter: $P(e^X < 4) = P(X < log_e 4) = F(log_e 4) = 0.8$.

C Q77. Let X_1 and X_2 be independent random variables with $bin(10, p)$ and $bin(20, p)$ distributions respectively. Find the conditional expectation of X_1 when it is given that $X_1 + X_2 = 3$.

ANSWER : 1.

ERROR RANGE: 0.005

Soln: $X_1|X_1 + X_2 = 3 \sim Hypergeometric(M = 10, N = 30, n = 3)$ Hence the $E(X_1|X_1 + X_2 = 3) = \frac{3*10}{30} = 1$.

Prob-Stat(MA20205)/CT/2

Fill in the blanks (Numerical)

Date of Exam : 06th Nov, 2022

Time : SLOT D

Duration : 40min

No of questions: 5 out of 8 questions

Type: Random-sequential (navigation allowed)

Each question carries 2 marks

October 29, 2022

D Q23. Let X, Y be two random variables with $E(XY) = 3$ and mean of X and Y are both equal to 2. Then the covariance of the random variables $2X + 10$ and $-\frac{5}{2}Y + 3$ is ———.
(answer should be correct up to three decimal places, error range: 0.005)

Answer: 5

ERROR RANGE: 0.005

Solution: Since $E(XY) = 3$ and $E(X) = 2 = E(Y)$, the

$$Cov(X, Y) = E(XY) - E(X)E(Y) = 3 - 4 = -1.$$

Then

$$Cov(2X + 10, -\frac{5}{2}Y + 3) = 2\left(-\frac{5}{2}\right) Cov(X, Y) = (-5)(-1) = 5.$$

Solution:

$$Var(X - Y) = Var(X) + Var(Y) - 2Cov(X, Y) \Rightarrow 16 = 4 + 9 - 2Cov(X, Y).$$

D Q42. Let X and Y be independent standard normal random variables. Consider the circle centred at the origin and passing through the point (X, Y). What is the expected value of its area? (answer should be correct up to three decimal places, error range: 0.005)

ANSWER: 6.283

ERROR RANGE: 0.005

Solution:- $E(\textbf{Area}) = \pi E(X^2 + Y^2) = 2\pi$

D Q46. The random variables X and Y have a joint density function. The random variable Y takes positive values with $E(Y) = 1$. The conditional distribution of X given that $Y = y$ is the uniform distribution on $(1-y, 1+y)$. What is $E(X)$? (answer should be correct up to three decimal places, error range: 0.005)

ANSWER: 1

ERROR RANGE: 0.005

Solution:- $f_{X|Y}(X|Y=y) = \begin{cases} \frac{1}{2y}, & x \in (1-y, 1+y) \\ 0, & \text{elsewhere.} \end{cases}$

$E(X|Y=y) = 1, \;\; E(X) = E_Y \; E(X|Y=y) = \int f_Y(y)dy = 1.$

D Q52. Let the continuous random variables X and Y have the joint pdf

$$f(x,y) = \begin{cases} 2e^{-(x+2y)}; & 0 < x < \infty, 0 < y < \infty \\ 0; \text{ otherwise.} \end{cases}$$

Calculate $P\{X < Y\}$.(answer should be correct up to three decimal places, error range: 0.005)

ANSWER : 0.333

ERROR RANGE: 0.005

Solution: $P(X < Y) = \int_0^\infty \int_0^y 2e^{-x}e^{-2y}dxdy = \frac{1}{3}$

D Q74. Let $(X, Y) \sim$ Bivariate Normal($\mu_x = 0, \mu_y = 0, \sigma_x^2 = 1, \sigma_y^2 = 1, \rho = 0.5$). Find $E(Y^2|X = 1)$. (answer should be correct up to three decimal places, error range: 0.005)

ANSWER : 1.0

ERROR RANGE: 0.005

ANS: $Y|x \sim$ Normal distribution with

$E(Y|x) = \mu_y + \rho\frac{\sigma_y}{\sigma_x}(x - \mu_x) = 0.5$

$V(Y|x) = (1 - \rho^2)\sigma_y^2 = 0.75$

$E(Y^2|x) = 1$

D Q78. Let $X \sim exp(1)$ and $Y|X = \lambda \sim$ Poisson(λ). Find $P(Y = 3)$

ANSWER : 0.625

ERROR RANGE: 0.005

Soln: $Y \sim geo(0.5)$, hence $P(Y = 3) = 0.5^4 = 0.625$

D Q79. Two (six faced) fair dice are rolled independently and X and Y are random variables denoting the upper face values. Find $E(XY|X+Y=8)$. (answer should be correct up to three decimal places, error range: 0.005)

Answer: 14.

ERROR RANGE: 0.005

Soln. $(X,Y) \in \{(2,6),(3,5),(4,4),(5,3),(6,2)\}$ such that $X+Y=8$. Hence $E(XY) = (12+15+16+15+12)/5 = 14$

D Q81. Let (X,Y) be continuous with joint density

$$f(x,y) = \begin{cases} c(x^2+y^2), & \text{if } 0<x<1,\ 0<y<1 \\ 0, & \text{otherwise.} \end{cases}$$

Find $P(X^2+Y^2<1)$.(answer should be correct up to three decimal places, error range: 0.005)

ANSWER : 0.5890486

ERROR RANGE: 0.005

Soln: $c = 1.5$ hence $P(X^2+Y^2<1) = \int_0^1\int_0^1 f(x,y)dxdy = 1.5\int_0^{\pi/2}\int_0^1 r^3 dr d\theta = 1.5\pi/8 = 0.5890486$