Probability and Statistics

(September - 28)

Continuous random vectors.

Joint CDF $\longrightarrow$ joint pdf $\to$ marginal pdfs $\to$
independence of random variables.

Let X and Y be continuous random variables with joint pdf $f(x,y)$

$\boxed{\varphi(x,y) : \mathbb{R}^2 \longrightarrow \mathbb{R}}$

Define $Z = \underline{\varphi(X,Y)}$

for a fixed $z \in \mathbb{R}$, we are interested in the event $\{Z \le z\}$.

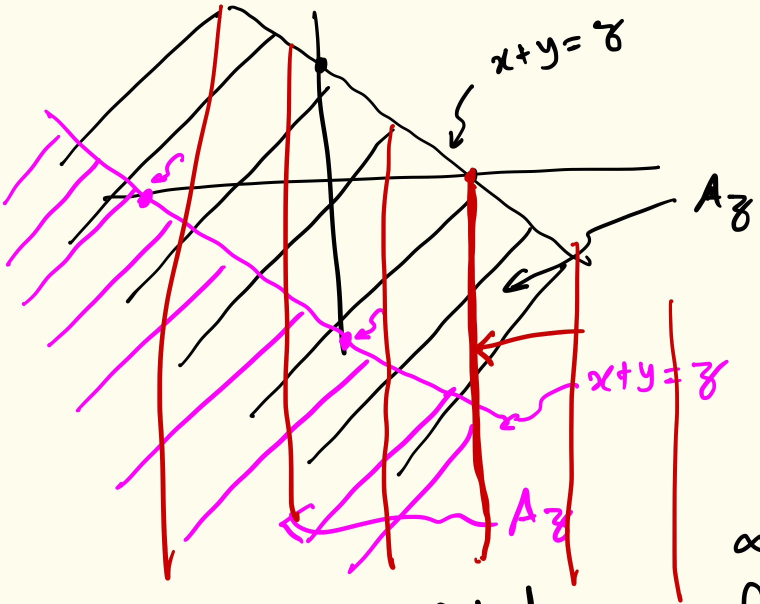
By $A_z \subseteq \mathbb{R}^2$ define $A_z = \{(x,y) \mid \varphi(x,y) \le z\}$

$F_Z(z) = \text{Prob}\{Z \le z\} = \text{Prob}((X,Y) \in A_z) \leftarrow$

$$\boxed{F_Z(z) = \iint_{A_z} f(x,y)\,dx\,dy}$$

Distribution of sum $(f(X,Y) = X+Y)$.

$A_z = \{(x,y) \mid x+y \le z\}$ for a fixed $z \in \mathbb{R}$.



We are interested in computing

$$P((X,Y) \in A_z)$$

$$= \iint_{A_z} f(x,y)\,dx\,dy$$

$$F_Z(z) = \iint_{A_z} f(x,y)\,dx\,dy = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{z-x} f(x,y)\,dy \right] dx$$

Substitute $y = v - x$ in inner integral.

$$F_Z(z) = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{z} f(x, z-x)\, dv \right] dx$$

$$F_Z(z) = \int_{-\infty}^{z} \boxed{\left[\int_{-\infty}^{\infty} f(x, z-x)\, dx \right]} dv$$

convolution integral.

PDF of Z is obtaind as $f_Z(z) = \frac{d}{dz} F_Z(z)$

$$\boxed{f_{X+Y}(z) = \int_{-\infty}^{\infty} f(x, z-x)\, dx \quad ; \quad -\infty < z < \infty}$$

If X and Y are independent,

$$\boxed{f_{X+Y}(z) = \int_{-\infty}^{\infty} f_X(x)\, f_Y(z-x)\, dx \quad ; \quad -\infty < x < \infty}$$

If X and Y are both non-negative r.v.s.

$$\boxed{\begin{aligned} f_{X+Y}(z) &= \int_0^z f_X(x)\, f_Y(z-x)\, dx \quad ; \quad 0<z<\infty \\ &= 0 \qquad\qquad\qquad\qquad\qquad \text{otherwise.} \end{aligned}}$$

Ex: Let $X, Y \sim \exp(\lambda)$ are independent.

$$\begin{aligned} f_X(x) &= \lambda e^{-\lambda x} \qquad x>0 \\ &= 0 \qquad\qquad \text{o.w.} \end{aligned}$$

$$\begin{aligned} f_Y(y) &= \lambda e^{-\lambda y} \qquad y>0 \\ &= 0 \qquad\qquad \text{o.w.} \end{aligned}$$

Find the density of $X+Y$.

$$f_{X+Y}(z) = \int_0^z f_X(x)\, f_Y(z-x)\, dx \qquad z \geq 0$$

$$= 0 \qquad \text{o.w.}$$

For $z \geq 0$

$$f_{X+Y}(z) = \int_0^z f_X(x)\, f_Y(z-x)\, dx$$

$$= \int_0^z \lambda e^{-\lambda x} \quad \lambda e^{-\lambda(z-x)} \quad dx$$

$$= \lambda^2 e^{-\lambda z} \int_0^z dx$$

$$f_{X+Y}(z) = \lambda^2 e^{-\lambda z} \cdot z \qquad z \geq 0$$

$$f_{X+Y}(z) = \lambda^2 z e^{-\lambda z} \qquad z \geq 0$$
$$= 0 \qquad \text{o.w.}$$

Thus $X+Y \sim \text{gamma}(2, \lambda)$

Generalizing: $X_1, X_2, \ldots, X_n \sim \exp(\lambda)$ are iid.

$$\sum_{i=1}^{n} X_i \sim \text{gamma}(n, \lambda)$$

<u>Ex:</u> $X, Y \sim U(0,1)$ are iid

$f_{X+Y} = ??$

$$\int_0^z \boxed{f_X(x) f_Y(z-x)} dx \qquad \boxed{z \geq 0}$$

$\leftarrow 0 \qquad \text{o.w.}$

$f_X(x) = 1 \qquad 0 \le x \le 1 \qquad\qquad f_Y(y) = 1 \qquad 0 \le y \le 1$

$\quad\;\; = 0 \qquad\;\; \text{o.w.} \qquad\qquad\qquad\quad\;\; = 0 \qquad\;\; \text{o.w.}$

The integrand $f_X(x) f_Y(z-x)$ takes only two values 0 or 1.

$f_X(x) f_Y(z-x)$ takes value 1 when

$0 \le x \le 1$ and $0 \le z-x \le 1$

If $0 \le z \le 1$, then integrand has value 1 on the set $0 \le x \le z$.

$$f_{X+Y}(z) = \int_0^z 1 \, dz = z \qquad 0 \le z \le 1$$

If $1 < z \leq 2$, then the integrand has value 1

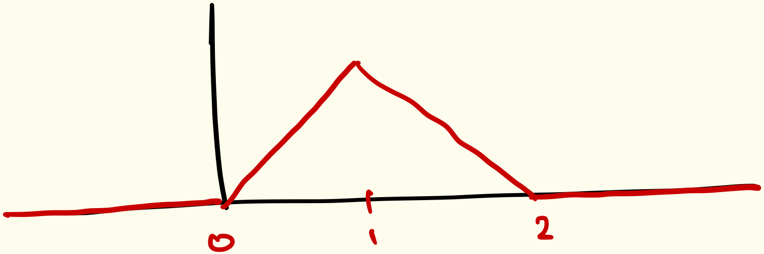
on $z-1 \leq x \leq 1$

$$f_{x+y}(z) = 2-z \quad ; \quad 1 < z \leq 2$$

for $z > 2$, $f_{x+y}(z) = 0$

Thus,

$$f_{x+y}(z) = \begin{cases} 0 & z < 0 \\ z & 0 \leq z \leq 1 \\ 2-z & 1 < z \leq 2 \\ 0 & z > 0 \end{cases}$$



Ex: Let $X \sim \Gamma(\alpha_1, \lambda)$ and $Y \sim \Gamma(\alpha_2, \lambda)$ be independent random variables. Then

$$X + Y \sim \Gamma(\alpha_1 + \alpha_2, \lambda)$$

$$f_X(x) = \frac{\lambda^{\alpha_1}}{\Gamma(\alpha_1)} x^{\alpha_1 - 1} e^{-\lambda x} \qquad x > 0$$

$$= 0 \qquad \text{o.w.}$$

$$f_Y(y) = \frac{\lambda^{\alpha_2}}{\Gamma(\alpha_2)} x^{\alpha_2 - 1} e^{-\lambda y} \qquad y > 0$$

$$= 0 \qquad \text{o.w.}$$

Note X and Y are non-negative.

$$f_{X+Y}(z) = \int_0^z f_X(x)\, f_Y(z-x)\, dx \qquad z \geq 0$$

$$= 0 \qquad \text{o.w.}$$

For $z > 0$

$$f_{X+Y}(z) = \frac{\lambda^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^z e^{-\lambda z}\, x^{\alpha_1-1}\, (z-x)^{\alpha_2-1}\, dx$$

$$\Rightarrow X+Y \sim \Gamma(\alpha_1+\alpha_2, \lambda)$$

If X and Y are continuous r.v.s. with joint pdf $f_{XY}(x,y)$. Then

$$E(h(X,Y)) = \iint h(x,y)\, f_{XY}(x,y)\, dx\, dy$$

Suppose $h(x,y) = \varphi_1(x)\varphi_2(y)$ and X & Y are independent.

$$E(h(X,Y)) = \iint \varphi_1(x)\varphi_2(y)\, \underline{f_{XY}(x,y)}\, dx\, dy$$

independ. of X & Y

$$= \iint \varphi_1(x)\varphi_2(y)\, \underline{f_X(x) f_Y(y)}\, dx\, dy$$

$$= \int \varphi_1(x) f_X(x)\, dx \cdot \int \varphi_2(y) f_Y(y)\, dy$$

$$= E(\varphi_1(X))\, E(\varphi_2(Y))$$

Let X & Y be independent r.v.s.

MGF of $X+Y$ $= M_{X+Y}(t) = E(e^{t(X+Y)})$

$$= E(e^{tX} \cdot e^{tY}) = E(e^{tX}) \cdot E(e^{tY}) = M_X(t)\, M_Y(t)$$

Ex: let $X_i \sim \Gamma(\alpha_i, \lambda)$ for $i=1,2$ are indep.

$$M_{X_i}(t) = \left(\frac{\lambda}{\lambda - t}\right)^{\alpha_i}$$

$$E(e^{tX})$$

$$\int e^{tx} f(x)\,dx$$

$$M_{X_1+X_2}(t) = \boxed{M_{X_1}(t)\, M_{X_2}(t)}$$

$$= \left(\frac{\lambda}{\lambda - t}\right)^{\alpha_1} \left(\frac{\lambda}{\lambda - t}\right)^{\alpha_2}$$

$$= \left(\frac{\lambda}{\lambda - t}\right)^{\alpha_1 + \alpha_2}$$

$$\Rightarrow X_1 + X_2 \sim \Gamma(\alpha_1 + \alpha_2, \lambda)$$

Ex: Let X, Y be independent r.v.s with

$$X \sim N(\mu_1, \sigma_1^2) \quad , \quad Y \sim N(\mu_2, \sigma_2^2)$$

pdf of $X+Y$??

$$M_{X+Y}(t) = M_X(t)\, M_Y(t)$$

$$= e^{\mu_1 t + \frac{1}{2}\sigma_1^2 t^2} \cdot e^{\mu_2 t + \frac{1}{2}\sigma_2^2 t^2}$$

$$= e^{(\mu_1+\mu_2)t + \frac{1}{2}(\sigma_1^2+\sigma_2^2)t}$$

$$\Rightarrow \quad X+Y \sim N(\mu_1+\mu_2 \,, \sigma_1^2+\sigma_2^2)$$

Generalization:

Let $X_1, X_2, \ldots, X_n$ are independent r.v.s.

$$M_{X_1+\cdots+X_n}(t) = M_{X_1}(t)\, M_{X_2}(t) \cdots M_{X_n}(t)$$

If $X_1, \ldots, X_n$ are iid. r.v.s.

$$M_{X_1+\cdots+X_n}(t) = \left[M_{X_1}(t)\right]^n$$

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1. $X_i \sim \exp(\lambda)$ iid for $i = 1, 2, \ldots, n$

$$\sum_{i=1}^{n} X_i \sim \text{gamma}(n, \lambda)$$

2. $X_i \sim gamma(\alpha_i, \lambda)$ are independent.

for $i = 1, 2, \dots, n$

$$\sum_{i=1}^{n} X_i \sim gamma\left(\sum_{i=1}^{n} \alpha_i , \lambda \right)$$

$\boxed{3.}$ $X_i \sim normal(\mu_i, \sigma_i^2)$ are independent

for $i = 1, 2, \dots, n$

$$\sum_{i=1}^{n} X_i \sim normal\left(\sum_{i=1}^{n} \mu_i , \sum_{i=1}^{n} \sigma_i^2 \right)$$

Let $x_1, \ldots, x_n \overset{iid}{\sim} \underline{\underline{N}}(\mu, \sigma^2)$

$$x_1 + \cdots + x_n \sim N(n\mu, n\sigma^2)$$

$$\bar{x} \sim \underline{\underline{N}}(\mu, \sigma^2/n)$$

What we had seen earlier is independent with $x_1, \ldots, x_n$ are mean μ and variance σ^2.

then $\text{mean}(\bar{x}) = \mu$

$\text{var}(\bar{x}) = \sigma^2/n$

Let $X_1, \dots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$

Prove : $\bar{X} \sim N(\mu, \sigma^2/n)$

<u>Soln!</u>

$$M_{\frac{X_1 + \dots + X_n}{n}}(t) = M_{\bar{X}}(t)$$

$$= M_{X_1 + \dots + X_n}\left(\frac{t}{n}\right)$$

$$= M_{X_1}\left(\frac{t}{n}\right) \cdots M_{X_n}\left(\frac{t}{n}\right)$$

$$= \left[M_{X_1}\left(\frac{t}{n}\right)\right]^n$$

$$= \left[e^{\mu \frac{t}{n} + \sigma^2 t^2 / 2n^2}\right]^n = e^{\mu t + \sigma^2 t^2 / 2n}$$

$$\bar{X} \sim N(\mu, \sigma^2/n)$$

$$M_{aX+b}(t)$$

$$= E\left(e^{taX+b}\right)$$

$$= E\left(e^b \cdot e^{taX}\right)$$

$$= e^b \cdot E\left(e^{atX}\right)$$

$$= e^b \cdot M_X(at)$$

$X = X_1 + \dots + X_n$

$a = \frac{1}{n}, \quad b = 0$

$e^{(\text{mean})t + (\text{variance})t^2/2}$ ← normal MGF

$e^{\mu t + \sigma^2 t^2/2n}$

$= e^{\boxed{\mu}t + \sigma^2/n \cdot t^2/2}$

$\Rightarrow \bar{X} \sim N(\mu, \sigma^2/n)$ ∎

Some topics I had left (as easy exercises!)

Y (is cts. random variable) is said to follow Weibull density with parameters γ and β if its pdf is given as:

$$f_Y(y) = \frac{\gamma}{\beta} y^{\gamma-1} e^{-y^{\gamma}/\beta} \quad , \; 0<y<\infty, \; \gamma>0, \; \beta>0$$

$$= 0 \quad \text{o.w.}$$

If $\gamma = 1$, then we get exponential density $(1/\beta)$.

Ex: show that if $X \sim$ exponential $(1/\beta)$, then $Y = X^{1/\gamma}$ has Weibull density.

Beta distribution:

A random variable X is said to follow Beta density with parameters α & β if it has the pdf

$$f_X(x) = \frac{1}{B(\alpha,\beta)} \boxed{x^{\alpha-1} (1-x)^{\beta-1}}, \quad 0 < x < 1$$

$$= 0 \quad \text{o.w.}$$

$\alpha > 0, \beta > 0$

Here, $B(\alpha, \beta) = \int_0^1 x^{\alpha-1} (1-x)^{\beta-1} \, dx$

$$= \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

$$E(x^n) = \frac{1}{B(\alpha,\beta)} \int_0^1 x^n \cdot x^{\alpha-1} (1-x)^{\beta-1} \, dx$$

$$= \frac{1}{B(\alpha,\beta)} \cdot B(\alpha+n, \beta)$$

For $n=1$

$$E(x) = \frac{\alpha}{\alpha+\beta}$$

$$\mathrm{var}(x) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$$

Lognormal distribution:

If X is a random variable whose logarithm is normally distributed (i.e. $\ln X \sim N(\mu, \sigma^2)$), then X has lognormal density. The pdf is given by:

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} \frac{1}{x} e^{-(\log x - \mu)/2\sigma^2} \qquad 0 < x < \infty$$

$$E(X) = E(e^{\log X}) = E(e^{Y}) \qquad \text{where } Y = \underbrace{\log X \sim N(\mu, \sigma^2)}$$

$$= e^{\mu + \sigma^2/2}$$

$$\mathrm{var}(x) = e^{2(\mu+\sigma^2)} - e^{2\mu+\sigma^2}$$

(This density is popular in modeling skew-data (right. skewed data)).

<u>Examples:</u> Let $X_1, X_2, \cdots, X_n$ be iid r.v.s. with CDF $F(x)$.

$$Y = \max\{X_1, \ldots, X_n\}$$

Find pdf of Y??

$G(y)$: CDF of Y.

$$G(y) = P(Y \le y)$$

$$= P(X_1 \le y, X_2 \le y, \dots, X_n \le y)$$ independence

$$= P(X_1 \le y)\, P(X_2 \le y) \dots P(X_n \le y)$$

$$= \prod_{i=1}^{n} F(y)$$

$$\boxed{G(y) = (F(y))^n}$$

Special case: $X_i \overset{iid}{\sim} U(0,1)$

$$G(y) = y^n$$

$$g(y) = \frac{d}{dy} G(y) = ny^{n-1} \qquad y \in [0,1]$$

$$= 0 \qquad \text{otherwise.}$$

$$g(y) = \begin{cases} n y^{n-1} & y \in [0,1] \\ 0 & \text{o.w.} \end{cases}$$

This is Beta$(n, 1)$

<u>Ex:</u> $X_1, X_2, \ldots, X_n$ are all iid with CDF $F(x)$ and pdf $f(x)$

$$Z = \min\{X_1, \ldots, X_n\}$$

Let $G(z)$ be the CDF of Z.

$$\begin{aligned} G(z) &= P(Z \leq z) \\ &= P\{\min\{X_1, \ldots, X_n\} \leq z\} \\ &= 1 - P(\min\{X_1, \ldots, X_n\} > z\} \end{aligned}$$

$$G(z) = 1 - P(X_1 > z, X_2 > z, \dots, X_n > z)$$ } indep. of X_i's

$$= 1 - P(X_1 > z)\, P(X_2 > z) \cdots P(X_n > z)$$

$$= 1 - \prod_{i=1}^{n} (1 - F(z))$$

$$\boxed{G(z) = 1 - (1 - F(z))^n}$$

In particular, $X_i \overset{iid}{\sim} U[0,1]$ for $i = 1, 2, \dots, n$

$$G(z) = 1 - (1 - z)^n \qquad z \in [0,1]$$

$g(z)$: pdf of Z

$$g(z) = \frac{d}{dz} G(z) = n(1-z)^{n-1} \qquad z \in [0,1]$$

$$= 0 \qquad \text{o.w.}$$

Note: $X_i \overset{iid}{\sim} U[0,1]$ for $i = 1, 2, \cdots, n$

$$\max X_i \sim \text{Beta}(n, 1)$$

$$\min X_i \sim \text{Beta}(1, n)$$