Transforms & basis

Consider $\mathbb{R}^n$ with basis: $\{e_1, \dots e_n\}$ → orthog.

$e_i = \begin{bmatrix}0\\ \vdots \\ 1 \\ \vdots \\ 0\end{bmatrix}$ → i^{th} pos^n

if $x \in \mathbb{R}^n$, then $x = x_1 e_1 + \cdots x_n e_n$

Consider a f^n space $\mathcal{F}$ of real valued f.^{n} on $\mathbb{R}$. e.g.

this space has a basis given by ~~dt~~ Dirac-delta f.^{n}s.

another basis: $\{e^{j\omega t}\}_{\omega \in \mathbb{R}}$ → orthog.

↓ repr. by Dirac-deltas in ω-domain.

another basis: $\{e^{st}\}_{s \in \mathbb{C}}$ → not orthog.

more bases such as wavelets.

$$e^{j\omega t} \longrightarrow \boxed{h} \longrightarrow H(j\omega)e^{j\omega t}$$

$$e^{st} \longrightarrow \boxed{h} \longrightarrow H(s)e^{st}$$

h : LTI

$e^{j\omega t} \rightarrow$ eigenf^{n}s of LTI sys. with eigenvalues $H(j\omega)$

$e^{st} \rightarrow$ eigenf^{n}s with e.val. $H(s)$

Fourier, Laplace, wavelet transforms are basically ~~ch~~ a change of basis.

diff. bases $\longleftrightarrow$ diff. building blocks ("atoms")

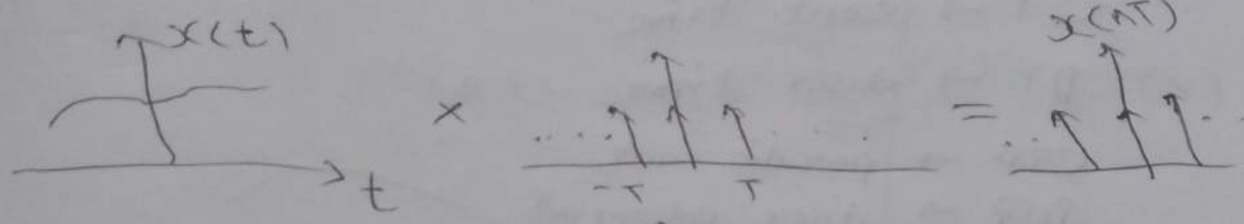
transforms : $\longrightarrow$ diff. rep^n using diff. building ~~b~~ blocks. (diff. decompos^{n}s)

$\downarrow$

reveals diff. features

similar analogues for discrete time ~~s~~ f^{n}s.

Contin. – discrete



(sampling)

$$\sum_{n=-\infty}^{\infty} \delta(t-nT)$$

$$x_d(t) = x(t) \times \sum_n \delta(t-nT) = \sum_n x(nT)\,\delta(t-nT)$$

$$\Rightarrow X_d(j\omega) = \sum_n x(nT)\,e^{-jnT\omega} \quad (\because \delta(t-nT) \leftrightarrow e^{-jnT\omega})$$

By convol.^{n} – mult.^{n} prop. of F.T. ∴

$$X_d(j\omega) = X(j\omega) * \frac{1\cancel{2\pi}}{T} \sum_n \delta\left(\omega - n\frac{2\pi}{T}\right)$$

$$= \frac{1\cancel{2\pi}}{T} \sum_n X(j\omega - n\omega_0)$$

$\Rightarrow$ DTFT is periodic ext.^{n} of CTFT

sampling in one domain $\longleftrightarrow$ periodic ext.^{n} in the other domain

Oppenh. 3.67 (heat eq^n)

$$\frac{\partial T(x,t)}{\partial t} = \frac{1}{2} k^2 \frac{\partial^2 T(x,t)}{\partial x^2} \quad —(1)$$

$T(0,t) = T(t) \rightarrow$ periodic with period $= 1$.

$$T(t) = \sum_n a_n e^{jn\omega_0 t}$$

$$T(x,t) = \sum_n b_n(x) e^{jn\omega_0 t}$$

$$\Rightarrow \frac{d^2 b_n}{dx^2} = \cancel{\tfrac{1}{2}} \frac{2 jn\omega_0 b_n}{k^2} \quad (\omega_0 = 2\pi) \quad —(2)$$

$$\left.\begin{aligned} b_n(0) &= a_n \\ \lim_{x\to\infty} b_n(x) &= \text{const.} \end{aligned}\right\} —(3)$$

$$\Rightarrow \frac{d^2 b_n}{dx^2} = \frac{4\pi jn}{k^2} b_n$$

$$\Rightarrow \left(D^2 - \frac{4\pi jn}{k^2}\right) b_n = 0$$

$$\Downarrow$$

roots: $\pm \frac{\sqrt{4\pi jn}}{k} = \pm \frac{2\sqrt{\pi n}\, e^{j\frac{\pi}{4}}}{k} = \pm s$

$\therefore$ ~~$\mathrm{Sol}^n = c_1 e^{st} + c_2 e^{-st}$~~

$$b_n(x) = c_1 e^{sx} + c_2 e^{-sx}$$

from (3),

$\cancel{c_1 = a_n}$ $c_{\cancel{1}} = 0$

if $n \geq 0$,

$$b_n(x) = a_n e^{-\frac{\sqrt{2\pi n}(1+j)x}{k}} \quad , \quad n \geq 0$$

if $n \leq 0$, $c_2 = 0$

$\cancel{b_n(x) = a_n e}$

$\cancel{So}$

$$b_n(x) = a_n e^{-\frac{\sqrt{2\pi |n|}(1-j)x}{k}} \quad , \quad n \leq 0$$

let $T(t) = d_0 + d_1 \sin 2\pi t$

$d_0 = 2$, $d_1 = 1$, $x = k\sqrt{\frac{\pi}{2}}$

$$\boxed{\begin{aligned} a_0 &= d_0 \\ a_1 &= \frac{d_1}{2j} \\ a_{-1} &= \frac{-d_1}{2j} \end{aligned}}$$

$$T(x,t) = \sum_n b_n(x) e^{jn2\pi t}$$

$\cancel{b_0 = 2}$

$$b_0(x) = d_0 = 2$$

$$b_1(x)\Big|_{x = k\sqrt{\frac{\pi}{2}}} = \frac{d_1}{2j} e^{-\frac{\sqrt{2\pi}}{k}(1+j)\cdot k\sqrt{\frac{\pi}{2}}} = \frac{1}{2j} e^{-\pi(1+j)}$$

$$b_{-1}(x)\Big|_{x = k\sqrt{\pi}} = \frac{-1}{2j} e^{-\pi(1-j)}$$

$$T\left(k\sqrt{\tfrac{\pi}{2}},\, t\right) = 2 + e^{-\pi}\sin(2\pi t - \pi)$$