

INDIAN INSTITUTE OF TECHNOLOGY KHARAGPUR

Stamp / Signature of the Invigilator

EXAMINATION (End Semester)												SEMESTER (Autumn)				
Roll Number										Section		Name				
Subject Numbe	r	С	S	2	1	0	0	1	Su	bject Nan	ne	Discrete Structures				
Department / Center of the Student														Additional sheets		

Important Instructions and Guidelines for Students

- 1. You must occupy your seat as per the Examination Schedule/Sitting Plan.
- 2. Do not keep mobile phones or any similar electronic gadgets with you even in the switched off mode.
- 3. Loose papers, class notes, books or any such materials must not be in your possession, even if they are irrelevant to the subject you are taking examination.
- 4. Data book, codes, graph papers, relevant standard tables/charts or any other materials are allowed only when instructed by the paper-setter.
- 5. Use of instrument box, pencil box and non-programmable calculator is allowed during the examination. However, exchange of these items or any other papers (including question papers) is not permitted.
- 6. Write on both sides of the answer script and do not tear off any page. **Use last page(s) of the answer script for rough work.** Report to the invigilator if the answer script has torn or distorted page(s).
- 7. It is your responsibility to ensure that you have signed the Attendance Sheet. Keep your Admit Card/Identity Card on the desk for checking by the invigilator.
- 8. You may leave the examination hall for wash room or for drinking water for a very short period. Record your absence from the Examination Hall in the register provided. Smoking and the consumption of any kind of beverages are strictly prohibited inside the Examination Hall.
- 9. Do not leave the Examination Hall without submitting your answer script to the invigilator. In any case, you are not allowed to take away the answer script with you. After the completion of the examination, do not leave the seat until the invigilators collect all the answer scripts.
- 10. During the examination, either inside or outside the Examination Hall, gathering information from any kind of sources or exchanging information with others or any such attempt will be treated as 'unfair means'. Do not adopt unfair means and do not indulge in unseemly behavior.

Violation of any of the above instructions may lead to severe punishment.

Signature of the Student

			Τ	o be filled	in by the	e examine	er				
Question Number	1	2	3	4	5	6	7	8	9	10	Total
Marks Obtained											
Marks ob	tained (in	words)		Sigr	nature of	the Exam	iner	Signature of the Scrutineer			

CS21201/CS21001 Discrete Structures, Autumn 2022–2023

21-Nov-2022, 02:00pm-05:00pm

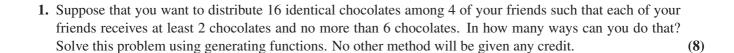
End-Semester Test

Maximum marks: 60

Instructions

- Write your answers in the question paper itself. Be brief and precise. Answer <u>all</u> questions.
- Write the answers only in the respective spaces provided. The last three blank pages may be used for rough work.
- If you use any theorem/result/formula covered in the class, just mention it, do not elaborate.
- Write all the proofs in mathematically precise language. Unclear and/or dubious statements would be severely penalized.

Do not write anything on this page.



Solution The relevant generating function is

$$\left(x^2 + x^3 + x^4 + x^5 + x^6 \right)^4$$

$$= x^8 \left(1 + x + x^2 + x^3 + x^4 \right)^4$$

$$= x^8 \left(\frac{1 - x^5}{1 - x} \right)^4$$

$$= x^8 \left(1 - 4x^5 + 6x^{10} - 4x^{15} + x^{20} \right) \left(\sum_{n \ge 0} {n+3 \choose 3} x^n \right).$$

The desired count is the coefficient of x^{16} in this expansion, that is,

$$\binom{8+3}{3} - 4 \times \binom{3+3}{3} = \binom{11}{3} - 4 \times \binom{6}{3} = 165 - 4 \times 20 = 85.$$

- **2.** A string of decimal digits is considered to be a valid codeword if it contains an even number of 0 digits. For example, 02310023089 and 7254193776 are valid codewords, but 060796007620 is not valid. Let c_n denote the number of valid n-digit codewords.
 - (a) Derive, with clear justifications, a recurrence relation for c_n . Also supply the required number of initial conditions. (6)

Solution The initial condition is $c_0 = 1$ (the empty string contains zero (an even number) 0 digits). One may instead supply the initial condition $c_1 = 9$ (all length-one strings except 0 are valid codewords).

Now, take $n \ge 1$. A valid codeword W of length n can be obtained in two mutually exclusive ways.

- (1) W starts with 0. Then the remaining n-1 digits form an invalid codeword of length n-1. The count of valid codewords W in this case is $10^{n-1} c_{n-1}$.
- (2) W starts with a digit other than 0 (there are nine possibilities). In this case, the remaining n-1 digits of W form a valid codeword of length n-1. The count of valid codewords W is this case is therefore $9c_{n-1}$.

Summing these counts gives

$$c_n = (10^{n-1} - c_{n-1}) + 9c_{n-1} = 8c_{n-1} + 10^{n-1}.$$

(b) Solve the recurrence relation of Part (a) to obtain a closed-form expression for c_n . For your derivation, you may use any method covered in the class. Clearly show all the steps. (6)

Solution This is an order-one recurrence, so we can simply unwind it to get

$$c_{n} = 8c_{n-1} + 10^{n-1}$$

$$= 8(8c_{n-2} + 10^{n-2}) + 10^{n-1}$$

$$= 8^{2}c_{n-2} + \left[8 \times 10^{n-2} + 10^{n-1}\right]$$

$$= 8^{2}(8c_{n-3} + 10^{n-3}) + 8 \times 10^{n-2} + 10^{n-1}$$

$$= 8^{3}c_{n-3} + \left[8^{2} \times 10^{n-3} + 8 \times 10^{n-2} + 10^{n-1}\right]$$

$$= \cdots$$

$$= 8^{n}c_{0} + \left[8^{n-1} + 8^{n-2} \times 10 + \cdots + 8^{2} \times 10^{n-3} + 8 \times 10^{n-2} + 10^{n-1}\right]$$

$$= 8^{n} + 10^{n-1} \left[1 + \frac{8}{10} + \left(\frac{8}{10}\right)^{2} + \cdots + \left(\frac{8}{10}\right)^{n-1}\right]$$

$$= 8^{n} + 10^{n-1} \left[\frac{1 - \left(\frac{8}{10}\right)^{n}}{1 - \frac{8}{10}}\right]$$

$$= 8^{n} + \frac{10^{n} - 8^{n}}{10 - 8}$$

$$= \frac{10^{n} + 8^{n}}{2}.$$

3. Consider the following recursively defined sequence:

$$a_0 = 2,$$

 $a_1 = 8,$
 $a_n = 4a_{n-1} - 4a_{n-2} + n^2 - 5n + 2 \text{ for } n \ge 2.$

Derive a closed-form expression for a_n for all $n \ge 0$. You are not allowed to use generating functions in this exercise. (10)

Solution The characteristic equation of the recurrence is $r^2 - 4r + 4 = 0$, that is, $(r-2)^2 = 0$. Therefore the homogeneous solution will be of the form

$$a_n^{(h)} = (An + B)2^n.$$

The non-homogeneous part of the recurrence is $f(n) = n^2 - 5n + 2 = (n^2 - 5n + 2) \times 1^n$. Since 1 is not a characteristic root, the particular solution will be of the form

$$a_n^{(p)} = Un^2 + Vn + W.$$

Putting this in the recurrence gives

$$Un^{2} + Vn + W = 4\left(U(n-1)^{2} + V(n-1) + W\right) - 4\left(U(n-2)^{2} + V(n-2) + W\right) + n^{2} - 5n + 2.$$

Equating the coefficients of n^2 , n^1 , n^0 from the two sides gives

$$U = 4U - 4U + 1$$
, that is, $U = 1$,
 $V = -8U + 4V + 16U - 4V - 5$, that is, $V = 8U - 5$, that is, $V = 3$,
 $W = 4U - 4V + 4W - 16U + 8V - 4W + 2$, that is, $W = -12U + 4V + 2$, that is, $W = 2$.

We have therefore determined

$$a_n^{(p)} = n^2 + 3n + 2$$
.

and so

$$a_n = a_n^{(h)} + a_n^{(p)} = (An + B)2^n + n^2 + 3n + 2.$$

The initial conditions give $a_0 = 2 = B + 2$, that is, B = 0, and $a_1 = 8 = 2(A + B) + 6 = 2A + 6$, that is, A = 1. Therefore

$$a_n = n2^n + n^2 + 3n + 2$$
 for all $n \ge 0$.

4. Consider the following C function. The parameters consist of a non-negative integer n and a bit $a \in \{0,1\}$.

```
int f ( int n, int a )
{
   if (n == 0) return (1 - a);
   if (a == 0) return f(n - 1, 0) + f(n - 1, 1);
   return f(n - 1, 0);
}
```

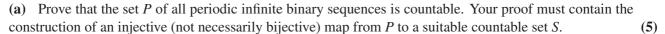
(a) Work out what f(n,0) and f(n,1) return for n = 0,1,2,3,4,5. Show your calculations. (3)

Solution These values are listed in the following table.

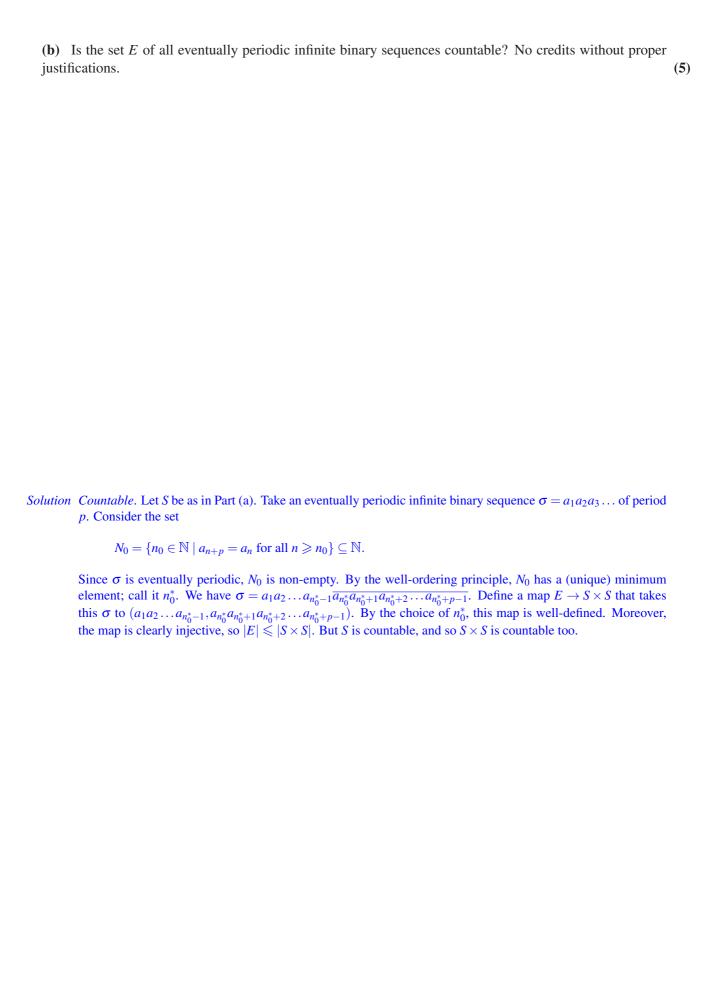
```
 \begin{array}{|c|c|c|c|c|} \hline n & f(n,0) & f(n,1) \\ \hline 0 & f(0,0) = 1 - 0 = 1 & f(0,1) = 1 - 1 = 0 \\ 1 & f(1,0) = f(0,0) + f(0,1) = 1 & f(1,1) = f(0,0) = 1 \\ 2 & f(2,0) = f(1,0) + f(1,1) = 2 & f(2,1) = f(1,0) = 1 \\ 3 & f(3,0) = f(2,0) + f(2,1) = 3 & f(3,1) = f(2,0) = 2 \\ 4 & f(4,0) = f(3,0) + f(3,1) = 5 & f(4,1) = f(3,0) = 3 \\ 5 & f(5,0) = f(4,0) + f(4,1) = 8 & f(5,1) = f(4,0) = 5 \\ \hline \end{array}
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Solution	An inspection of the return values for small n tends to indicate that $f(n,0) = F_{n+1}$ and $f(n,1) = F_n$ for all $n \ge 0$,	
	where F_n is the <i>n</i> -th Fibonacci number. Let us prove this hunch by induction on <i>n</i> . [Basis] For $n = 0$, we have $f(n,0) = 1 = F_1$ and $f(n,1) = 0 = F_0$.	
	[Induction] Take $n \ge 1$, and assume that $f(n-1,0) = F_n$ and $f(n-1,1) = F_{n-1}$. Then, $f(n,0) = f(n-1,0) + f(n-1,0) = f(n-1,0)$	
	$f(n-1,1) = F_n + F_{n-1} = F_{n+1}$, whereas $f(n,1) = f(n-1,0) = F_n$.	

itive integer <i>p</i>
For example,
ntal line at the
nce $a_1 a_2 a_3$
For all $n \ge n_0$.
1011011 =
]



Solution We know that the set S of all binary strings (finite sequences) is countable. Define the map $f: P \to S$ as follows. Take any periodic sequence $\overline{a_1a_2\dots a_p}$ with p the period, and map this sequence to the (non-empty) string $a_1a_2\dots a_p$. The map is clearly injective, so $|P| \leqslant |S|$, that is, P is countable.



6. Let \mathbb{R} be the set of real numbers, and $\pi = 3.1415926535...$ (the ratio of the circumference to the diameter of any circle). Define two operations on \mathbb{R} as

$$a \oplus b = a + b + \pi,$$

 $a \odot b = a + b + \frac{ab}{\pi},$

where the expressions on the right-hand sides use the standard arithmetic of real numbers.

(a) Show that $(\mathbb{R}, \oplus, \odot)$ is a commutative ring. Verify all the axioms that define a commutative ring. Show all the steps clearly (despite that similar calculations were done in the class). (6)

Solution [Closure under \oplus] Obvious.

[Associativity of \oplus] $a \oplus (b \oplus c) = a \oplus (b+c+\pi) = a+(b+c+\pi)+\pi = a+b+c+2\pi$, whereas $(a \oplus b) \oplus c = (a+b+\pi) \oplus c = (a+b+\pi)+c+\pi = a+b+c+2\pi$.

[Commutativity of \oplus] Obvious.

[Additive identity] $a \oplus (-\pi) = a - \pi + \pi = a$, and so also $(-\pi) \oplus a = -\pi + a + \pi = a$. Therefore $-\pi$ is the additive identity.

[Additive inverse] $a \oplus b = -\pi$ requires $a + b + \pi = -\pi$, that is, $b = -(a + 2\pi)$, that is, $-(a + 2\pi)$ is the additive inverse of a.

[Closure under \odot] Obvious (division by π is legal in \mathbb{R}).

 $[\text{Associativity of} \odot] \quad a\odot(b\odot c) = a\odot(b+c+\frac{bc}{\pi}) = a+(b+c+\frac{bc}{\pi}) + \frac{a(b+c+\frac{bc}{\pi})}{\pi} = a+b+c+\frac{ab+bc+ca}{\pi} + \frac{abc}{\pi^2}, \\ \text{whereas } (a\odot b)\odot c = (a+b+\frac{ab}{\pi})\odot c = (a+b+\frac{ab}{\pi})+c+\frac{(a+b+\frac{ab}{\pi})c}{\pi} = a+b+c+\frac{ab+bc+ca}{\pi} + \frac{abc}{\pi^2}.$

[Commutativity of ⊙] Obvious.

[Distributivity of \odot over \oplus] $a \odot (b \oplus c) = a \odot (b+c+\pi) = a+(b+c+\pi) + \frac{a(b+c+\pi)}{\pi} = 2a+b+c+\frac{ab+ac}{\pi} + \pi$. On the other hand, $(a \odot b) \oplus (a \odot c) = (a+b+\frac{ab}{\pi}) \oplus (a+c+\frac{ac}{\pi}) = (a+b+\frac{ab}{\pi}) + (a+c+\frac{ac}{\pi}) + \pi = 2a+b+c+\frac{ab+ac}{\pi} + \pi$.



