1. Let X, Y be i.i.d. $N(0,1)$ random variables. Define $U = 3X + 4Y$, $V = 4X - 3Y$ and $W = \frac{U}{V}$. Find the followings

(a) $E(U) =$ $\boxed{0}$

[1 marks]

(b) $Var(V) =$ $\boxed{25}$

[2 marks]

(c) $cov(U, V) =$ $\boxed{0}$

[2 marks]

(d) $P(|W| < 1) =$ $\boxed{0.5}$

[2 marks]

(answer should be correct up to three decimal places, error range: 0.005)

Answer : (a) $E(U) = 0$
(b) $Var(V) = 16 + 9 = 25$
(c) $cov(U, V) = 12 - 12 = 0$
(d) $|U|$, $|V|$ be i.i.d. random variables. That implies $P(|U| < |V|) = 0.5$. Hence $P(|W| < 1) = 0.5$

$|V|$

(a) $E(U) = 3E(X) + 4E(Y) = 0$ ——— 1

(b) $Var(V) = 9\,Var(X) + 16\,Var(Y) + 3\cdot4\cdot2\cdot Cov(X,Y)$
$= 9 + 16 = 25$ ——— 2

(c) 1 $cov(U,V) = cov(3X + 4Y, 4X - 3Y)$
$= 12\,cov(X,X) - 9\,cov(X,Y) + 16\,cov(X,Y) - 12\,cov(Y,Y)$
$= 12 - 12 = 0.$ ——— 2

OR 11 show by transforation of variables with jacobian.

(d) If c is correctly done either by 1 OR 11
U and V are independat and identically distribut.
|U| and |V| are iid. ——— 1

$\Rightarrow P(|U| > |V|) = 1/2$ ——— 1

OR $U/V \sim$ cauchy(0,1) do the rest cancutation.

2. (i) In 2021, the amount of a particular food item consumed by a person residing in Norway had a mean value of 147 pounds with a standard deviation of 62 pounds. If a random samples 25 peoples is chosen, then approximate the probability that the average amount of that food item consumed by the members of the group in 2021 exceeded 150 pounds. Given $\Phi(0.242) = 0.596$.

[4 marks]

(ii) For discrete random variables X and Y with the joint probability distribution is provided as $P(X = 0, Y = 0) = 0.2$, $P(X = 1, Y = 1) = 0.1$, $P(X = 1, Y = 2) = 0.1$, $P(X = 2, Y = 1) = 0.1$, $P(X = 2, Y = 2) = 0.1$, $P(X = 3, Y = 3) = 0.4$. Determine the value of correlation coefficient $\rho(X, Y)$.

[3 marks]

(answer should be correct up to three decimal places, error range: 0.005)

ANSWER : (a.) 0.404, (b.) 0.926

Solution: (a.) Let X_i be the amount consumed by the i-th member of the sample, where $i = 1, 2, \ldots, 25$. The desired probability is

$$P(\frac{X_1 + X_2 + \cdots + X_{25}}{25}) > 150 = P(\overline{X} > 150)$$

where $\overline{X}$ is the sample mean of the 25 sample values. Since we can regard the X_i as being independent random variables with mean 147 and standard deviation 62. It follows from central limit theorem, that their sample mean will be approximately normal with mean 147 and standard deviation $\frac{62}{5}$. Thus Z being a standard normal variable, we have

$$P(\overline{X} > 150) = P(\frac{\overline{X} - 147}{12.4} > \frac{150 - 147}{12.4}) \approx P(Z > 0.242) = 0.404$$

(b.) $E(X) = 1.8 = E(Y)$, $E(XY) = 4.5$, $Var(X) = 1.36 = Var(Y)$ $\sigma_{XY} = E(XY) - E(X)E(Y) = 4.5 - 1.8 * 1.8 = 1.26$; $\sigma_X = \sqrt{1.36}$, $\sigma_Y = \sqrt{1.36}$ $\rho_{XY} = \sigma_{XY}/\sigma_X\sigma_Y = 0.926$

For Part (a): Correct answers with proper justifications using central limit theorem, 4 marks are awarded.

If One reaches till the central limit theorem correctly afterward wrong calculations, then they will be awarded one mark.

In every other cases zero mark is awarded.

For Part (b): With all the intermediate correct calculations, if the final answer is correct, then three marks.

If any two of E(XY), Var(X), Var(Y), σ_{XY} are correct, then one mark.

In all other cases, zero mark is awarded.

For both part (a) and (b): If only the answers are provided in the boxes but no justifications, zero mark is awarded even though the answer is correct.

3. (i) Let X and Y be independent and identically distributed random variables with the density function

$$f(x) = \begin{cases} 3e^{-3x} & \text{for } 0 < x < \infty \\ 0 & \text{otherwise.} \end{cases}$$

Let $U = X + 2Y$ and $V = 2X + Y$. Then determine the joint density of U and V.

[3 marks]

(ii) Suppose $X_1, X_2, \ldots, X_n$ be a random sample of size $n = 25$ from a population which has mean $\mu = 71.43$ and variance $\sigma^2 = 56.25$. Let $\overline{X}$ denote the sample mean. Then determine the probability that the sample mean is between 68.91 and 71.97.
(Given $\Phi(1.68) = 0.9535, \Phi(0.37) = .6443$)

[4 marks]

(answer should be correct up to three decimal places, error range: 0.005)

ANSWER(i) : $g(u,v) = \begin{cases} 3e^{-(u+v)} & \text{for } 0 < u, v < \infty \\ 0 & \text{otherwise.} \end{cases}$

No partial for one to be correct.

ANS: Solving $U = X + 2Y$, $V = 2X + Y$ we obtain

$$X = -\frac{1}{3}U + \frac{2}{3}V = R(U,V), \; Y = \frac{2}{3}U - \frac{1}{3}V = S(U,V). \quad 1$$

Hence the Jacobian of the transformation is given by

$$J = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} = -\frac{1}{3}. \quad 1$$

Then the joint density function of U and V is

$$\begin{aligned} g(u,v) &= |J| f(R(u,v), S(u,v)) && (1) \\ &= \frac{1}{3} 9 e^{R(u,v)} e^{S(u,v)} && (2) \\ &= 3e^{-(u+v)} \quad , 0 < \frac{v}{2} < u < v. \quad 1 && (3) \end{aligned}$$

ANSWER(ii) : 0.5941
ERROR RANGE: 0.005
ANS: The mean of $\overline{X}$ is given by 71.43 and the variance of $\overline{(X)}$ is

$$Var(\overline{X}) = \frac{\sigma^2}{n} = \frac{56.25}{25} = 2.25 \quad 1$$

Now from central limit theorem,

$$\frac{\overline{X} - \mu}{\frac{\sigma}{\sqrt{n}}} \sim N(0,1) \quad 1$$

as n approaches to ∞. Therefore

$$\begin{aligned} P(68.91 \leq \overline{X} \leq 71.97) &= \left(\frac{68.91 - 71.43}{\sqrt{2.25}} \leq \frac{\overline{X} - 71.43}{\sqrt{2.25}} \leq \frac{71.97 - 71.43}{\sqrt{2.25}} \right) \\ &= P(-1.68 \leq W \leq 0.37) \\ &= P(W \leq 0.36) + P(W \leq 1.68) - 1 \quad 2 \\ &= 0.5978. \end{aligned}$$

4. (i) Let $X_1, X_2, \ldots, X_{64}$ be a random sample of size 64 from a normal distribution $\mathcal{N}(50, 16)$. Then determine $P(49 < X_8 < 51)$. (Given $\Phi(0.25) = 0.5987$.)

[2 marks]

(ii) Let $Y = -\ln X$, where X is uniformly distributed over the interval $(0,\ 1)$. Then determine the pdf of Y where it is nonzero.

[2 marks]

(iii) Let X, Y be two random variables. If $Var(X+Y) = 3$, $Var(X-Y) = 1$, $E(X) = 1$, and $E(Y) = 2$ then determine $E(XY)$.

[3 marks]

(answer should be correct up to three decimal places, error range: 0.005)

ANSWER : (i) 0.1974 (ii) $Y \sim Exp(1)$ (ii) 2.5

ANS (i): Since $X_8 \sim \mathcal{N}(50, 16)$, we have

$$\begin{aligned} P(49 < X_8 < 51) &= P\left(\frac{49-50}{4} < \frac{X_8 - 50}{4} < \frac{51-50}{4}\right) \\ &= P\left(-\frac{1}{4} < \frac{X_8 - 50}{4} < \frac{1}{4}\right) \\ &= P\left(-\frac{1}{4} < Z < \frac{1}{4}\right), Z \sim \mathcal{N}(0,1) \\ &= 2P\left(Z < \frac{1}{4}\right) - 1 \\ &= 0.1974 \text{ (from normal table)} \end{aligned}$$

1

1

ANS (ii) Since $y = T(x) = -\ln x$, we have $x = W(y) = e^{-y}$. Therefore $\frac{dx}{dy} = -e^{-y}$. Then the pdf of Y is

$$g(y) = \left|\frac{dx}{dy}\right| f(W(y)) = e^{-y}, \quad y > 0.$$

1

1

ANS: (iii)

$$\begin{aligned} Var(X+Y) &= \sigma_X^2 + \sigma_Y^2 + 2Cov(X,Y) \\ Var(X-Y) &= \sigma_X^2 + \sigma_Y^2 - 2Cov(X,Y). \end{aligned}$$

1

Hence

$$Cov(X,Y) = \frac{1}{4}\left[Var(X+Y) - Var(X-Y)\right] = \frac{1}{2}.$$

1

Then

$$E(XY) = Cov(X,Y) + E(X)E(Y) = \frac{5}{2}.$$

1

① Answer to Q. 5 !

(a) $$c\int_0^\infty e^{-\frac{1}{2}x}\left(\int_0^\infty e^{-\frac{1}{2}xy^2}\,dy\right)dx = 1.$$

Now, we know, $$\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}\;\frac{1}{\sqrt{x}}}\,e^{-\frac{1}{2}y^2/\left(\frac{1}{\sqrt{x}}\right)^2}\,dy = 1.$$

$$\Rightarrow \int_0^\infty e^{-\frac{1}{2}y^2/\frac{1}{x}}\,dy = \frac{1}{2}\cdot\frac{\sqrt{2\pi}}{\sqrt{x}}.$$

$$\therefore c\int_0^\infty \frac{1}{2}\cdot\frac{\sqrt{2\pi}}{\sqrt{x}}\,e^{-\frac{1}{2}x}\,dx = 1$$

$$\Rightarrow \frac{c\sqrt{2\pi}}{2}\cdot\frac{\Gamma(\frac{1}{2})}{(\frac{1}{2})^{\frac{1}{2}}}\int_0^\infty \frac{(\frac{1}{2})^{\frac{1}{2}}}{\Gamma(\frac{1}{2})}\,\frac{e^{-\frac{1}{2}x}}{\sqrt{x}}\,dx = 1$$

$$\Rightarrow c = \frac{2\,(\frac{1}{2})^{\frac{1}{2}}}{\sqrt{2\pi}\,\Gamma(\frac{1}{2})} = \frac{1}{\pi} = \cdot 3183$$

Marking scheme: Right answer will give 2 marks. Answers close to the correct answer will give 1 or $\frac{1}{2}$ mark.

(b) $$f_X(x) = \frac{1}{\pi}\int_0^\infty e^{-\frac{1}{2}x}\cdot e^{-\frac{1}{2}xy^2}\,dy$$

$$= \frac{1}{\pi}e^{-\frac{1}{2}x}\int_0^\infty e^{-\frac{1}{2}xy^2}\,dy$$

$$= \frac{1}{\pi}\sqrt{\frac{\pi}{2}}\,\frac{1}{\sqrt{x}}\cdot e^{-\frac{1}{2}x} = \Gamma\left(\tfrac{1}{2},\tfrac{1}{2}\right)$$

② $\therefore$ Standard deviation $= \sqrt{2} = 1.4142$

Marking scheme:— Correct answer will give 2 marks. Close to correct answer will give $1\frac{1}{2}$ marks. Correct expression of the marginal pdf of X will give 1 mark. Correct formula will give $\frac{1}{2}$ mark.

(c)

$$V := Y\sqrt{X}, \quad W := X.$$

$$\therefore J = \begin{vmatrix} 0 & 1 \\ \frac{1}{\sqrt{w}} & -\frac{3}{2}\frac{v}{w^{3/2}} \end{vmatrix} = -\frac{1}{\sqrt{w}}$$

$$\therefore f_{V,W}(v,w) = \frac{1}{\pi} \cdot e^{-\frac{1}{2}w(1+v^2/w)} \frac{1}{\sqrt{w}}$$

$$= \frac{1}{\pi} \cdot \frac{1}{\sqrt{w}} \cdot e^{-\frac{1}{2}w} \cdot e^{-\frac{1}{2}v^2}, \quad v>0, w>0.$$

$$\therefore f_V(v) = \frac{1}{\pi} e^{-\frac{1}{2}v^2} \int_0^\infty \frac{1}{\sqrt{w}} \cdot e^{-\frac{1}{2}w}\, dw$$

$$\alpha = \tfrac{1}{2}, \lambda = \tfrac{1}{2}$$

$$= \frac{\sqrt{2\pi}}{\pi} e^{-\frac{1}{2}v^2} = \sqrt{\frac{2}{\pi}}\, e^{-\frac{1}{2}v^2}, \quad v>0$$

$$\therefore E(V) = \int_0^\infty \sqrt{\frac{2}{\pi}}\, v\, e^{-\frac{1}{2}v^2}\, dv$$

$$\tfrac{1}{2}v^2 = z \quad \therefore v\,dv = dz$$

$$= \int_0^\infty \sqrt{\frac{2}{\pi}}\, e^{-z}\, dz = \sqrt{\frac{2}{\pi}} = .7979.$$

(3) Marking scheme:- Correct answer will give 2 marks. Correct expression of marginal pdf of $y\sqrt{x}$ will give 1 mark. Only correct formula will give $1/2$ mark.

(d)

$$f_W(w) = \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{w}} \cdot e^{-\frac{1}{2}w}, \quad w > 0$$

$$\therefore f_{V,W}(v, w) = f_V(v) \cdot f_W(v)$$

$\therefore$ $y\sqrt{x}$ and x are independent.

$Var(y\sqrt{x}) \neq 0$, $Var(X) \neq 0$

$\therefore$ $Cor(Y\sqrt{X}, X) = 0$

Marking scheme:- Correct ~~ans~~ answer or simply showing that $Y\sqrt{X}$ and X are independent, will give 2 marks. Only correct formula will give 1 mark.

6. Let (X, Y) be Bivariate normal with $\mu_x = 0, \mu_y = -1, \sigma_x^2 = 1, \sigma_y^2 = 4, \rho = -1/2$. Find the value of the following. [Given $\Phi(0.5773) = 0.7181$, $\Phi(0.6488) = 0.7481$.]

SK

(a) $P(X + Y > 0) =$ $\boxed{0.2819}$ [2 marks]

(b) $P(2X + 3Y > 1) =$ $\boxed{0.2236 / 0.2519}$ [2 marks]

(c) $E((5X + 6Y)^2 | Y = 2) =$ $\boxed{86.8125}$ [3 marks]

(answer should be correct up to three decimal places, error range: 0.005)

$(X,Y) \sim BVN(0, -1, 1; 4, -\frac{1}{2})$

$X+Y \sim BVN(-1, 3)$ —(1M)

$P(X+Y > 0) = P\left(\frac{X+Y+1}{\sqrt{3}} > \frac{0+1}{\sqrt{3}}\right) = \Phi(-0.5773)$

(1M) $= 1 - 0.7181 = 0.2819$

b) $2X+3Y \sim N(-3, 28)$ } 2M

$P(2X+3Y > 1) = P\left(\frac{2X+3Y+3}{\sqrt{28}} > \frac{1+3}{\sqrt{28}}\right) = P(Z > \frac{4}{\sqrt{28}})$

$= \Phi(-0.7559) = 0.2236$ (0.2519)

(c) $X|_{Y=2} \sim N(-\frac{3}{4}, \frac{3}{4})$

$(5X+12)|_{Y=2} \sim N(\frac{33}{4}, \frac{75}{4})$ } (2M)

$E((5X+12)^2|Y=2) = \left(\frac{33}{4}\right)^2 + \frac{75}{4} = 86.8125$

$= \frac{1389}{16}$

(1M)

7. Let $X_1, \ldots, X_9$ be independent random variables with Uniform$[0,1]$ distribution.

(a)Let $Y = \max\{X_1, \ldots, X_9\}$. Find $P(Y < 0.5)$? [3 marks]

(b)Let $Z = \min\{X_1, \ldots, X_9\}$. Find $P(1 - Z < 0.8)$? [4 marks]

(answer should be correct up to three decimal places, error range: 0.005)

ANSWER : 0.00195 and 0.1342

ERROR RANGE:

ANS: $P(Y < 0.5) = P(X_i < 0.5, 1 \leq i \leq 9) = (1/2)^9 = 0.00195$. Also $P(1 - Z < 0.8) = P(Z > 0.2) = P(X_i > 0.2, 1 \leq i \leq 9) = (0.8)^9 = 0.1342$

7.a) (i) Correct answer + No logic : 1

(ii) Correct answer + ~~wrong~~ Incomplete partial logic : 2

(iii) Complete logic + No/wrong answer : 2

(iv) Correct answer + Complete logic : 3

7.b) i) Correct answer : 1 + No logic

(ii) Correct answer + Incomplete partial logic

iii) complete logic + No/wrong ans : 3

: 2 or 3 based on the reasoning

(iv) correct answer + complete logic : 4