Probability and statistics

August 30

Lecture #5

Random expt. $(\Omega, \mathcal{F}, P)$

Ω — sample space

$\mathcal{F}$ — σ-algebra

P — Probability (measure)

E: Tossing a coin

$\Omega = \{\text{Heads}, \text{Tails}\}$

E: Tossing n. coins independently.

$\Omega = \{HH\ldots H, TH\ldots H, HTH\ldots H, \ldots\} \leftarrow 2^n$

<u>Random variable:</u> A random variable X is a function from Ω to $\mathbb{R}$ if $\{\omega \in \Omega \mid X(\omega) = x\} \in \mathcal{F}$ for every $x \in \mathbb{R}$.

$\{\omega \in \Omega \mid X(\omega) = x\} = X^{-1}(x) \subseteq \Omega$ — inverse image of x

$$P(X = x) = P\underbrace{\{\omega \in \Omega \mid X(\omega) = x\}}_{X^{-1}(x)}$$

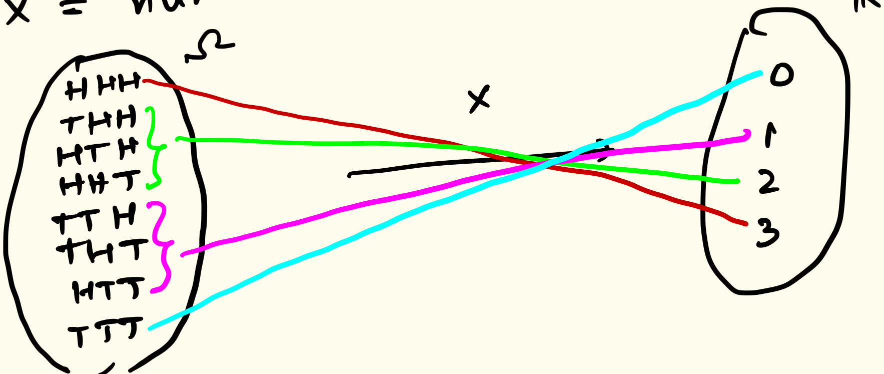
<u>Expt:</u> 3 fair coins are tossed independently.

$$\Omega = \{HHH, THH, HTH, HHT, TTH, THT, HTT, TTT\}$$

$$\mathcal{F} = 2^{\Omega}$$

$$p(\omega) = \frac{1}{8} = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \quad \forall\, \omega \in \Omega$$

Define X = number of Heads in 3 tosses.



$x \in \mathbb{R}, \quad x = 1$

X is random variable.

$X^{-1}(1) = \{ \omega \in \Omega \mid X(\omega) = 1 \}$

$= \{TTH, THT, HTT\} \in \mathcal{F}$

$P(X = 1) = P\{ \omega \in \Omega \mid X(\omega) = 1 \} = P(X^{-1}(1))$

$= 3/8$

$X^{-1}(2) = \{HHT, HTH, THH\} \in \mathcal{F}$

$P(X=2) = 3/8$

$X^{-1}(0) = \{TTT\} \in \mathcal{F}$

$P(X=0) = 1/8$

$X^{-1}(3) = \{HHH\} \in \mathcal{F}$

$P(X=3) = 1/8$

$X^{-1}(3.7) = \phi \in \mathcal{F}$

$P(X=3.7) = P(\Phi) = 0$

$X^{-1}(-0.6) = \phi \in \mathcal{F}$

$P(X=-0.6) = 0$

Range of random variable $\{ x \in \mathbb{R} \mid Prob(X = x) > 0 \}$ — (*) will modify slightly.

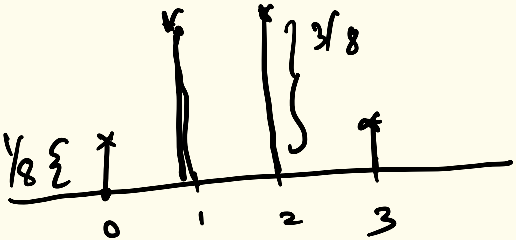
$R_X = \{0, 1, 2, 3\}$ in the previous example.

<u>Defn</u>: A random variable is called as discrete random variable if the range of X is countably finite or countably infinite.

<u>Defn</u>: A real valued function $f(x) = P(X = x)$ is called as probability mass function (pmf) of discrete random variable X.

$$f(x) = \begin{cases} 1/8 & \text{for } x = 0 \\ 3/8 & \text{for } x = 1 \\ 3/8 & \text{for } x = 2 \\ 1/8 & \text{for } x = 3 \\ 0 & \text{otherwise (for all other values of } x\text{)} \end{cases}$$

(formula)



← graphical representation of f(x)

x	0	1	2	3
$f(x)$	1/8	3/8	3/8	1/8

– tabluar representation

Ex: Bernoulli

Expt: Tossing a coin (Bernoulli trial)

$\Omega = \{H, T\}$, $P(H) = p$; $p(T) = 1-p$

X : r.v.

$X(H) = 1$; $X(T) = 0$

$$P(X=0) = P\{\omega \in \Omega \mid X(\omega) = 0\}$$
$$= P\{T\}$$
$$= 1-p$$

$R_x = \{0, 1\}$

$$P(X=1) = p$$

$$f(x) = \left.\begin{array}{ll} p & \text{for } x = 1 \\ = 1-p & \text{for } x = 0 \\ = 0 & \text{o.w.} \end{array}\right\}$$

Bernoulli pmf

$X \sim$ Bernoulli (p)

parameter

Ex: Binomial

Expt: n - independent Bernoulli trials (p) are performed.

Define X = number of successes in these n trials.

$$R_X = \{0, 1, 2, \dots, n\}$$

SFSF .. S

$w =$ 1010 . . . 1

$$P(X=1) = P\{w \in \Omega \mid X(w) = 1\}$$

$$= P\{SFF..F, FSF..F, \dots, FF..FS\}$$

$$= \binom{n}{1} p (1-p)^{n-1}$$

$$f(x) = P(X = x)$$

$$f(x) = \begin{cases} \binom{n}{x} p^x (1-p)^{n-x} & x = 0, 1, 2, \dots, n \\ 0 & \text{o. w.} \end{cases}$$

$$X \sim \text{Binomial}(n, p)$$

no. of indep. Bernoulli trials

probability of success.

Ex: Geometric

Expt: Perform indep. Bernoulli(p) trials until the first success.

$$\Omega = \{ S, \; FS, \; FFS, \; FFFS, \; \cdots \}$$

$$p \qquad (1-p)p \qquad (1-p)^2 p \qquad (1-p)^3 p$$

Define X = The number failures before the first success.

$R_X = \{0, 1, 2, 3, \cdots\}$ discrete r.v.

$$f(x) = p(1-p)^x \quad \text{for } x = 0, 1, 2, \cdots$$
$$= 0 \quad \text{o.w.}$$

$X \sim \text{geometric}(p)$

Understanding pmf:

$\underbrace{(\Omega, \mathcal{F}, P)}$

X : r.v. on Ω

Let R_X : Range of X be discrete

$R_X = \{x_1, x_2, \ldots\}$

by defn of pmf

$$\sum_{x_i \in R_X} f(x_i) = \sum_{x_i \in R_X} P(X = x_i)$$

by probability assignment to a r.v.

$$= \underbrace{\sum_{x_i \in R_X} P\{w \in \Omega \mid X(w) = x_i\}}$$

$$= 1$$

$$\bigcup_{x_i \in R_X} \{w \in \Omega \mid X(w) = x_i\} = \text{disjoint union}$$

$$= \Omega$$

So far,

Random expt $\rightarrow$ $\Omega, \mathcal{F}, P$ $\longrightarrow$ X $\longrightarrow$ pmf.

$\mathbb{R}$

Properties of pmf.

i) $f(x) \geq 0$ $\quad \forall x \in \mathbb{R}$

ii) $\{x \mid f(x) > 0\}$ is a countably finite or countably infinite set.

$R_X = \{x_1, x_2, \dots, x_n, \dots\}$

iii) $$\sum_{x_i \in R_X} f(x_i) = 1$$

Any function f which satisfies these three properties is called as probability mass function (pmf) or $\boxed{\text{discrete density}}$.

Ex: Uniform pmf

$$f(x) = \frac{1}{s} \qquad x \in \{x_1, x_2, \dots, x_s\}$$

$$= 0 \qquad \text{otherwise.}$$

Ex: Negative Binomial pmf [for $\alpha \in \mathbb{N}$ (no. of successes), $p \in (0,1)$ (probability of success in Bernoulli trial).

$$f(x) = p^{\alpha} \binom{\alpha + x - 1}{x} (1-p)^{x} \qquad x = 0, 1, 2, \dots$$

$$= 0 \qquad \text{otherwise}$$

[For $\alpha = 1$, NB $\rightarrow$ geometric)

Special examples of r.v.s.

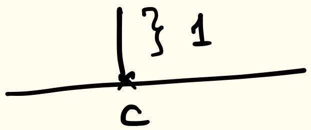
Ex: Constant r.v

$\Omega, \mathcal{F}, P$

$X : \Omega \longrightarrow \mathbb{R}$

$X(\omega) = c \quad \forall\, \omega \in \Omega$ for some const. $c \in \mathbb{R}$.

$P(X = c) = 1 \quad ; \quad P(X \neq c) = 0$



Ex: Indicator random variable. $(\Omega, \mathcal{F}, P)$

Let $A \in \mathcal{F}$ Define $\chi_A : \Omega \longrightarrow \mathbb{R}$

$$\chi_A : \Omega \longrightarrow \mathbb{R}$$

$$\chi_A(\omega) = 1 \qquad \forall\, \omega \in A \qquad p = P(A)$$

$$= 0 \qquad \forall\, \omega \in A^c \qquad 1 - p = P(A^c)$$