Probability and Statistics

(october- 5)

$(X,Y) \to$ continuous random vector

$f_{XY}(x,y)$: joint density of (X,Y).

$\varphi : \mathbb{R}^2 \to \mathbb{R}$

Define $Z = \varphi(X,Y)$.

We are interested in pdf Z.

$P(Z \le z)$: CDF of Z and then differentiate.

$$\begin{aligned} &P(Z \le z) \\ &\quad \| \\ &P(A_z) = P\{(x,y) \mid \varphi(x,y) \le z\} \end{aligned}$$

Two important transformations: $\varphi(X,Y) = X+Y$

$\boxed{\varphi(X,Y) = \frac{X}{Y}}$

$X_1, \dots, X_n$ are independent.

i) $X_i \sim \exp(\lambda)$; $\sum_{i=1}^{n} X_i \sim \text{gamma}(n,\lambda)$

ii) $X_i \sim \text{gamma}(\alpha_i,\lambda)$; $\sum_{i=1}^{n} X_i \sim \text{gamma}\left(\sum_{i=1}^{n} \alpha_i , \lambda\right)$

iii) $X_i \sim N(\mu_i,\sigma_i^2)$; $\sum X_i \sim N\left(\sum_{i=1}^{n} \mu_i , \sum_{i=1}^{n} \sigma_i^2\right)$

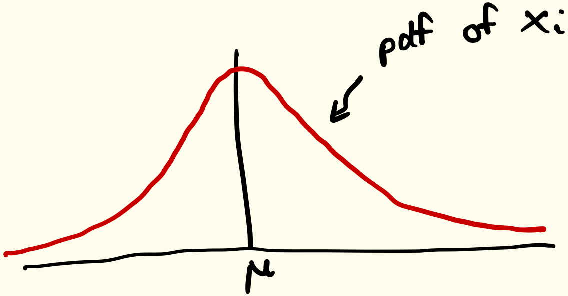
If $X_1, \dots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$

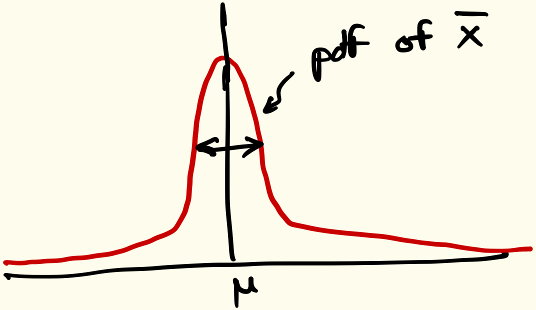
$$\bar{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$$

Previously, $X_1, X_2, \dots, X_n$ are independent r.v.s. with mean μ and variance σ^2.

$$\left[\begin{aligned} E(\bar{X}) &= \mu \\ \operatorname{var}(\bar{X}) &= \frac{\sigma^2}{n} \end{aligned}\right]$$

$n \uparrow \infty ; \frac{\sigma^2}{n} \downarrow 0$





Distribution on quotients.

Let X and Y be continuous r.v.s. with joint pdf $f_{XY}(x,y)$.

Q: What is the density of $Z = \frac{Y}{X}$.

$$A_z = \{(x,y) \mid \frac{y}{x} \le z\} \quad \text{for any real } z \in \mathbb{R}.$$

If $x < 0$, then $\frac{y}{x} \le z \iff y \ge z x$

$$A_z = \{(x,y) \mid x < 0 \text{ and } y \ge xz\} \cup \{(x,y) \mid x > 0 \text{ and } y \le xz\}$$

CDF of Z is given by $(z \in \mathbb{R})$

$$F_Z(z) = \text{Prob}(Z \le z)$$

$$= \text{Prob}((X,Y) \in A_z)$$

$$= \iint_{A_z} f_{XY}(x,y)\,dx\,dy$$

$$= \int_{-\infty}^{0} \left[\int_{xz}^{\infty} f_{XY}(x,y)\,dy \right] dx$$

$$+ \int_{0}^{\infty} \left[\int_{-\infty}^{xz} f_{XY}(x,y)\,dy \right] dx$$

Substitute: $y = xv$ $(dy = x\,dv)$ in the inner integral

$$F_{\frac{Y}{X}}(z) = \int_{-\infty}^{0} \left[\int_{z}^{-\infty} x\, f(x, xv)\, dv \right] dx + \int_{0}^{\infty} \left[\int_{-\infty}^{z} x\, f(x, xv)\, dv \right] dx$$

$$F_{Y/X}(z) = \int_{-\infty}^{z} \boxed{\left[\int_{-\infty}^{\infty} |x|\, f(x, xv)\, dx \right]} dv$$

$$\boxed{f_{\frac{Y}{X}}(z) = \int_{-\infty}^{\infty} |x|\, f(x, xv)\, dx}$$

For non-negative and independent r.v.s.

$$f_{Y/X}(z) = \int_0^\infty x \, f_X(x) f_Y(xz) \, dx \qquad -\infty < z < \infty$$

<u>Example:</u> Let X and Y be independent r.v.s with densities gamma (α_1, λ) and gamma (α_2, λ) respectively.

Prove: $$f_{Y/X}(z) = \frac{\Gamma(\alpha_1+\alpha_2)}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \frac{z^{\alpha_2-1}}{(z+1)^{\alpha_1+\alpha_2}}, \quad 0 < z < \infty$$

$$= 0 \qquad \text{o.w.}$$

Recall: $f_X(x) = \dfrac{\lambda^{\alpha_1}}{\Gamma(\alpha_1)} x^{\alpha_1 - 1} e^{-\lambda x} \quad x > 0$

$$f_Y(y) = \frac{\lambda^{\alpha_2}}{\Gamma(\alpha_2)} y^{\alpha_2 - 1} e^{-\lambda y} \quad y > 0$$

$$f_{Y/X}(z) = \frac{\lambda^{\alpha_1 + \alpha_2}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} z^{\alpha_2 - 1} \int_0^\infty x^{\alpha_1 + \alpha_2 - 1} e^{-x\lambda(z+1)} \, dx$$

$$= \frac{\lambda^{\alpha_1 + \alpha_2} \, z^{\alpha_2 - 1}}{\Gamma(\alpha_1) \, \Gamma(\alpha_2)} \frac{\Gamma(\alpha_1 + \alpha_2)}{(\lambda(z+1))^{\alpha_1 + \alpha_2}}$$

Ex: Let X and Y be independent $N(0, \sigma^2)$ r.v.s.

$$X^2, Y^2 \sim \text{gamma}\left(\frac{1}{2}, \frac{1}{2\sigma^2}\right)$$

$$f_{Y^2/X^2}(z) = \begin{cases} \dfrac{1}{\pi(z+1)\sqrt{z}} & 0 < z < \infty \\ 0 & \text{o.w.} \end{cases}$$

Conditional densities

Let (X, Y) be a discrete random vector.

Conditional probability

$$P(Y=y \mid X=x) = \frac{P(X=x, Y=y)}{P(X=x)}$$

$$= \frac{f(x,y)}{f_X(x)}$$

where $f(x,y)$ is joint p.m.f. of (X,Y) and $f_X(x)$ is the marginal pmf of X.

Definition: Let X and Y be continuous r.v.s. with joint pdf $f_{XY}(x,y)$. Then the conditional density of Y given X, denoted as $f_{Y|X}(y|x)$, is defined as:

$$f_{Y|X}(y|x) = \begin{cases} \dfrac{f_{XY}(x,y)}{f_X(x)} & 0 < f_X(x) < \infty \\ 0 & \text{otherwise} \end{cases}$$

$$P(a \le Y \le b \mid X = x) = \int_a^b f_{Y|X}(y|x)\, dy$$

Observe:

$$f_{Y|X}(y|x) = \frac{f_{XY}(x,y)}{f_X(x)}$$

$$\Rightarrow \boxed{f_{XY}(x,y) = f_{Y|X}(y|x)\, f_X(x)}$$

If X and Y are independent r.v.s., then

$$f_{XY}(x,y) = f_X(x)\, f_Y(y)$$

$$\Rightarrow f_{Y|X}(y|x) = f_Y(y)$$

Example:

$$f(x,y) = \frac{\sqrt{3}}{4\pi} e^{-(x^2 - xy + y^2)/2} \qquad -\infty < x, y < \infty$$

Recall: $X \sim N(0, 4/3)$

$f_X(x)$ is normal pdf with parameters 0, and 4/3.

Compute $f_{Y|X}(y|x) = \dfrac{f(x,y)}{f_X(x)}$

$$= \frac{\frac{\sqrt{3}}{4\pi} e^{-(x^2 - xy + y^2)/2}}{\frac{\sqrt{3}}{2\sqrt{2\pi}} e^{-3x^2/8}}$$

$$f_{Y|X}(y|x) = \frac{1}{\sqrt{2\pi}} e^{-(y - x/2)^2/2} \quad ; \quad -\infty < y < \infty$$

$$\tilde{Y} = \Rightarrow \boxed{Y|X = x} \sim N\left(\frac{x}{2}, 1\right)$$

$f_{Y|X}(y|x)$ is $N\left(\frac{x}{2}, 1\right)$

$$\text{Prob}(0 \leq Y \leq 2 \mid X = 0) =$$

observe: $\tilde{Y} = \underbrace{Y|X=0} \sim N(0,1)$

$$0 \leq Y \leq 2 \mid X = 0 = 0 \leq \tilde{Y} \leq 2 = \Phi(2) - \Phi(0)$$

where Φ is the CDF of $N(0,1)$.

$$\text{Prob}(0 \le Y \le 2 \mid X = 2) = \Phi(1) - \Phi(-1) = 2\Phi(1) - 1$$

observe: $\tilde{Y} = Y \mid X = \underbrace{2}_{x} \sim N(\underbrace{1}_{\frac{x}{2}}, 1)$

$$\text{Prob}(0 \le \tilde{Y} \le 2) = \text{Prob}(0 \le Y \le 2 \mid X = 2)$$

$$= \text{Prob}\left(\frac{0-1}{1} \le \frac{\tilde{Y}-1}{1} \le \frac{2-1}{1}\right)$$

$$= \text{Prob}(-1 \le Z \le 1) \quad \text{where } Z \sim N(0,1)$$

$$= \Phi(1) - \Phi(-1)$$

$$= 2\Phi(1) - 1$$

Ex: Let $X \sim U[0,1]$ and r.v. $\underbrace{Y \sim U[0, X]}_{}$ and

Find joint density of X and Y and the marginal density of Y.

$$f_X(x) = \begin{cases} 1 & 0 < x < 1 \\ 0 & \text{o.w.} \end{cases}$$

$$f_{Y|X}(y|x) = \begin{cases} \frac{1}{x} & 0 < y < x < 1 \\ 0 & \text{o.w.} \end{cases}$$

$$f_{XY}(x,y) = f_{Y|X}(y|x) \, F_X(x)$$

$$= \begin{cases} \frac{1}{x} & 0 < y < x < 1 \\ 0 & \text{o.w.} \end{cases}$$

Marginal density of Y

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$$

$$= \int_{y}^{1} \frac{1}{x}\,dx$$

$$\boxed{\begin{aligned} f_Y(y) &= -\log y && 0<y<1 \\ &= 0 && \text{o.w.} \end{aligned}}$$

Baye's rule for densities:

$$f_{X|Y}(x|y) = \frac{f_{XY}(x,y)}{f_Y(y)}$$

$$= \frac{f_X(x)\, f_{Y|X}(y|x)}{\int_{-\infty}^{\infty} f_{XY}(x,y)\,dx}$$

$$\boxed{f_{X|Y}(x|y) = \frac{f_X(x)\, f_{Y|X}(y|x)}{\int_{-\infty}^{\infty} f_X(x)\, f_{Y|X}(y|x)\,dx}}$$

Bivariate Normal density:

Random vector (X_1, X_2) is said to follow bivariate normal density if its joint pdf is given by:

$$f(x_1, x_2) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\left\{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x_1-\mu}{\sigma_1}\right)^2 - 2\rho\left(\frac{x_1-\mu_1}{\sigma_1}\right)\left(\frac{x_2-\mu_2}{\sigma_2}\right) + \left(\frac{x_2-\mu_2}{\sigma_2}\right)^2\right]\right\}$$

$$-\infty < x_1 < \infty, \quad -\infty < x_2 < \infty$$

Five parameters: $-\infty < \mu_1 < \infty$; $-\infty < \mu_2 < \infty$, $\sigma_1 > 0$, $\sigma_2 > 0$, $-1 < \rho < 1$

Probability computation:

$$p(a \le X_1 \le b_1,\ a_2 \le X_2 \le b_2)$$

$$= \int_{a_1}^{b_1} \int_{a_2}^{b_1} f(x_1, x_2)\, dx_1\, dx_2$$

The marginal densities

$$f_{X_1}(x_1) = \int_{-\infty}^{\infty} f(x_1, x_2)\, dx_2$$

$$= \frac{1}{\sigma_1 \sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x_1 - \mu_1}{\sigma_1}\right)^2} \qquad -\infty < x_1 < \infty$$

$$\Rightarrow X_1 \sim N(\mu_1, \sigma_1^2)$$

Similarly, marginal density of X_2

$$f_{X_2}(x_2) = \int_{-\infty}^{\infty} f(x_1, x_2)\, dx_1$$

$$= \frac{1}{\sigma_2\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x_2-\mu_2}{\sigma_2}\right)^2} \qquad -\infty < x_2 < \infty$$

$$X_2 \sim N(\mu_2, \sigma_2^2)$$

$E(X_1) = \mu_1$, $\quad E(X_2) = \mu_2$

$var(X_1) = \sigma_1^2$ $\quad var(X_2) = \sigma_2^2$

$$cov(X_1, X_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x_1 - \mu_1)(x_2 - \mu_2)\, f(x_1, x_2)\, dx_1\, dx_2$$

$$= E(X_1 - \mu_1)(X_2 - \mu_2)$$

correlation coefficient (X_1, X_2)

$$= \frac{\operatorname{cov}(X_1, X_2)}{\sigma_1 \cdot \sigma_2}$$

$= \rho$ (can be shown)

ρ is the 5th parameter

Note in general,

X_1, X_2 are independent $\Rightarrow$ correlation coefficient between X_1 & $X_2 = 0$

$\not\Leftarrow$

In case of bivariate normal random variable, $\rho = 0 \Rightarrow X_1$ and X_2 are indep.

the converse is true.

$$\rho = 0 \Rightarrow f_{X_1 X_2}(x_1, x_2) = f_{X_1}(x_1)\, f_{X_2}(x_2)$$

(joint pdf = product of marginals)

Conditional densities:

$$f_{X_1 | X_2}(x_1 | x_2) = \frac{f(x_1, x_2)}{f_{X_2}(x_2)}$$

$$= \frac{1}{\underbrace{\sigma_1}\sqrt{2\pi}\,\underbrace{\sqrt{1-\rho^2}}} \exp\left[-\frac{1}{\underbrace{2\sigma_1^2(1-\rho^2)}_{\text{variance}}}\left\{x_1 - \underbrace{\left(\mu_1 + \rho\left(\frac{\sigma_1}{\sigma_2}\right)(x_2-\mu_2)\right)}_{\text{mean}}\right\}^2\right]$$

$$\boxed{X_1 | X_2 = x_2 \sim N\left(\mu_1 + \rho\left(\frac{\sigma_1}{\sigma_2}\right)(x_2-\mu_2),\ \sigma_1^2(1-\rho^2)\right)}$$

$$f_{X_2|X_1}(x_2|x_1) = \frac{f(x_1, x_2)}{f_{X_1}(x_1)}$$

$$= \frac{1}{\sigma_2\sqrt{2\pi}\sqrt{1-\rho^2}} \exp\left[\underbrace{\frac{-1}{2\sigma_2^2(1-\rho^2)}}_{\text{variance}} \left\{x_2 - \underbrace{\left[\mu_2 + \rho\left(\frac{\sigma_2}{\sigma_1}\right)(x_1-\mu_1)\right]}_{\text{mean}}\right\}^2\right]$$

$$\boxed{X_2|X_1 = x_1 \sim N\left(\mu_2 + \rho\left(\frac{\sigma_2}{\sigma_1}\right)(x_1-\mu_1),\ \sigma_2^2(1-\rho^2)\right)}$$

$X_2|X_1 = x_1$ and $X_1|X_2 = x_2$ are both normal.

Ex: Let (X_1, X_2) be a bivariate random vector with parameters $\mu_1 = 0.2$, $\mu_2 = 1100$, $\sigma_1^2 = 0.02$,

$\sigma_2^2 = 525$, $\rho = 0.9$

Compute:

1) $E(X_2 \mid x_1)$ = mean of $X_2 \mid X_1 = x_1$

$$= \mu_2 + \rho\left(\frac{\sigma_2}{\sigma_1}\right)(x_1 - \mu_1)$$

$$= 1100 + 0.9\left(\frac{\sqrt{525}}{\sqrt{0.02}}\right)(x_1 - 0.2)$$

$$= 145.8\, x_1 + 1070.84$$

$$E(X_2 \mid X_1 = 1) = 145.8(1) + 1070.84 = \underline{1216.64}$$

2) $\mathrm{Var}(X_2 \mid X_1 = 1) = \sigma_2^2 (1 - \rho^2)$

$$= 525(1 - 0.81)$$

$\mathrm{var}(X_2 \mid X_1 = 1) = \underline{99.75}$

3) Compute $P(X_2 \geq 1080 \mid X_1 = 1)$

<u>Observe</u>: $Y = X_2 \mid X_1 = 1$

$$Y \sim \underline{N}(\mu_Y, \sigma_Y^2)$$

$\mu_Y = E(X_2 \mid X_1 = 1) = 1216.64$

$\sigma_Y^2 = \mathrm{var}(X_2 \mid X_1 = 1) = 99.75$

$$P(X_2 \geq 1080 \mid X_1 = 1) = P(Y \geq 1080)$$

$$= P\left(\frac{Y - \mu_Y}{\sigma_Y} \geq \frac{1080 - \mu_Y}{\sigma_Y}\right)$$

Here $Z \sim N(0,1)$

$$= P\left(Z \geq \frac{1080 - 1216.64}{\sqrt{99.75}}\right)$$

$$= P(Z \geq -13.6)$$

$$= 1 - \Phi(-13.6)$$

$$= \Phi(13.6)$$

$$\approx 1$$

Change of variable formula:

Let $X_1, X_2, \dots, X_n$ be random variables with joint pdf $f(x_1, \dots, x_n)$

Define $Y_i = \sum_{j=1}^{n} a_{ij} X_j \qquad i = 1, 2, \dots, n$

$$A = [a_{ij}]$$

$$= \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

$$\begin{bmatrix} Y_1 \\ \vdots \\ Y_n \end{bmatrix} = A \begin{bmatrix} X_1 \\ \vdots \\ X_n \end{bmatrix}$$

If $\det(A) \neq 0$,

Define $B = A^{-1}$

$$\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = B \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$$

$$x_i = \sum_{j=1}^{n} b_{ij} \, y_j \qquad \text{for } i = 1, 2, \ldots, n$$

$$f(y_1, \ldots, y_n) = \frac{1}{|\det(A)|} f(x_1, \ldots, x_n)$$

Non-linear transformation case:

$$y_i = g_i(x_1, \ldots, x_n) \qquad \text{for } i = 1, 2, \ldots, n$$

$$J(x_1, \ldots, x_n) = \begin{bmatrix} \frac{\partial y_1}{\partial x_1} & \cdots & \frac{\partial y_1}{\partial x_n} \\ \vdots & & \\ \frac{\partial y_n}{\partial x_1} & \cdots & \frac{\partial y_n}{\partial x_n} \end{bmatrix}$$

$$f(y_1, \dots, y_n) = \frac{1}{|\det(J)|} f(x_1, \dots, x_n)$$ ▪

Ex: Let $X_1, \dots, X_n$ be iid $\exp(\lambda)$

Define: $Y_i = X_1 + X_2 + \dots + X_i$ for $i = 1, 2, \dots, n$

$Y_1 = X_1$; $Y_2 = X_1 + X_2$, $\dots$, $Y_n = X_1 + X_2 + \dots + X_n$

$$A = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 1 & 1 & 0 \cdots & 0 \\ \vdots & & & \\ 1 & 1 & \cdots & 1 \end{bmatrix} \Rightarrow \det(A) = 1$$

$\Rightarrow$ Joint density of $Y_1, \dots, Y_n$ is same as the joint density of $X_1, X_2, \dots, X_n$ in this case.

Summary:

joint r.v.s.

discrete

$\downarrow$

joint p.m.f.

marginal p.m.f.s

conditional p.m.f.s

cts

$\downarrow$

joint CDFs

$\downarrow$

joint pdfs

$\downarrow$

marginal pdfs

$\downarrow$

conditional pdfs

(Bivariate normal)

moments, Independence of r.v.s.

covariance, correlation, independence $\Rightarrow E(X_1 X_2) = E(X_1)E(X_2)$

$\not\Leftarrow \quad \mathrm{cov}(X_1, X_2) = 0$

$\rho(X_1, X_2) = 0$

transformations: X, Y : $\varphi(X, Y) = X + Y$ $\leftarrow$ addition

$= X/Y$ $\leftarrow$ quotient

- Addition/Sum of r.v.s.

convolution sum / convolution integral $\equiv$ product of mgfs

pmf pdf

- distribution of quotients

$\varphi(x,y) = x+y$

$\varphi(x,y) = x/y$

$\Bigg\}$ $\varphi : \mathbb{R}^2 \to \mathbb{R}$ / $\mathbb{R}^n \to \mathbb{R}$

Transformations $\mathbb{R}^n \to \mathbb{R}^n$

- linear

- non-linear