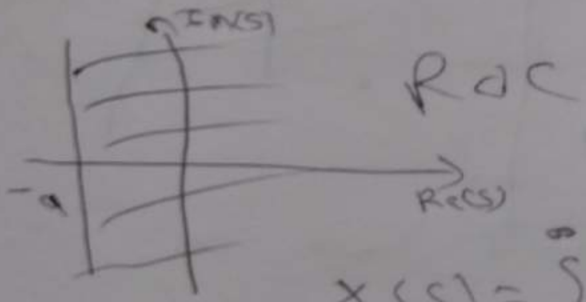
Laplace transf. & ROC

ROC : The subset of the complex plane $\mathbb{C}$ where the Laplace transf. integral conv. to a finite value

$$X(s) = \int_{-\infty}^{\infty} x(t) e^{-st}\, dt$$

ex.. $x_1(t) = e^{-at} u(t)$

$a>0$



$$X_1(s) = \int_0^{\infty} e^{-(s+a)t}\, dt = \frac{1}{s+a}$$

$Re(s+a) > 0$ for conv.

$\Rightarrow$ ROC: ~~$s \rightarrow Re(s) \geq -a$~~ $Re(s) \geq -a$

$x_2(t) = e^{-at} u(t)$

$a<0$

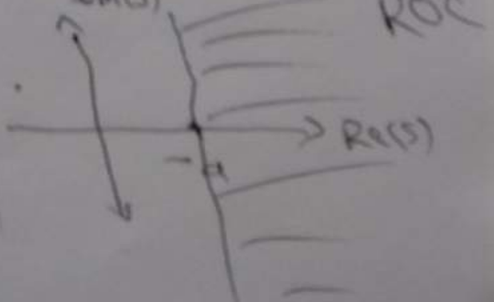
$$X_2(s) = \frac{1}{s+a}$$

let $b = -a > 0$

$$X_2(s) = \int_0^{\infty} e^{-(s-b)t}\, dt$$

$Re(s-b) > 0$

$\Rightarrow Re(s) > b = -a$



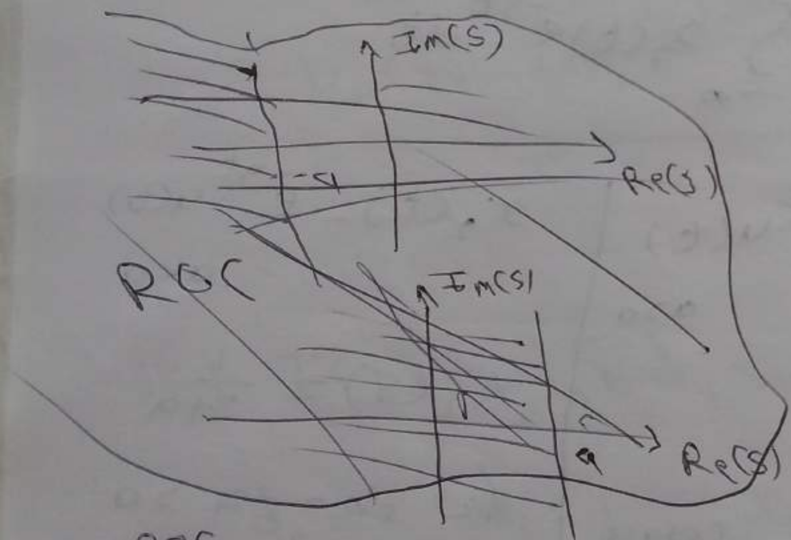
$x_3(t) = e^{-at}u(-t)$, $a > 0$

$$X_3(s) = \int_{-\infty}^{0} e^{-(s+a)t}\,dt$$

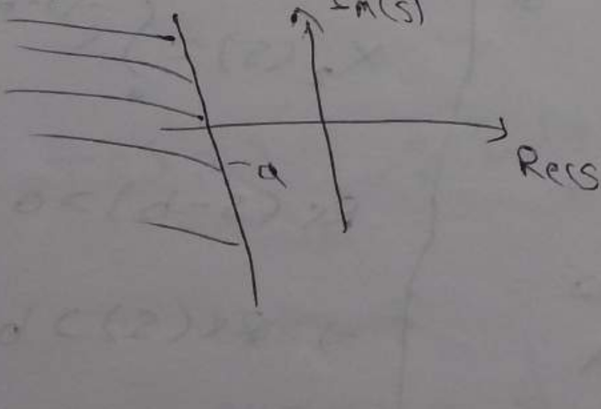
$\therefore$ ROC: $Re\{-(s+a)\} > 0$

$\Rightarrow Re(-s) > +a$

$\Rightarrow Re(s) < -a$



ROC



$x_4(t) = e^{-at}u(-t)$, $a < 0$

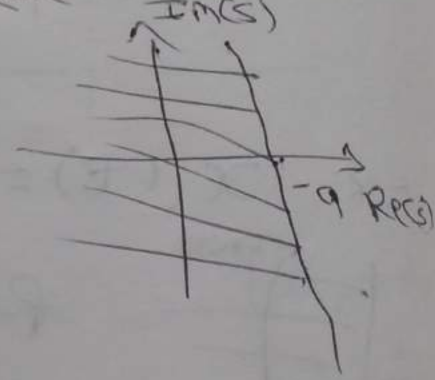
$$X_4(s) = \int_{-\infty}^{0} e^{-(s+a)t}\,dt$$

$\therefore$ ROC: $Re\{-(s+a)\} > 0$

$\Rightarrow Re\{-s\} > +a$

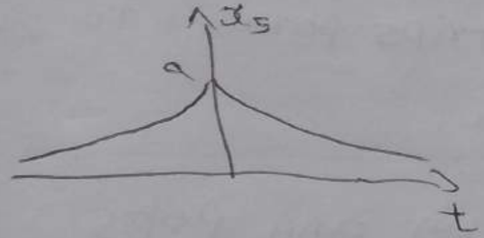
$\Rightarrow Re\{s\} < -a$

ROC



$x_5(t) = e^{-at} u(t) \quad t \geq 0 \qquad a > 0$

$\quad\;\; = e^{at} u(-t) \quad t < 0$

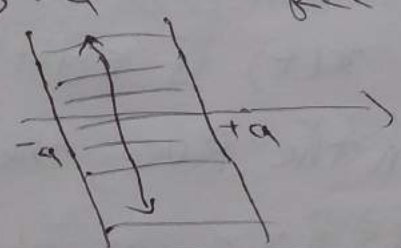


$$X_5(s) = \int_0^\infty e^{-(s+a)t}\,dt + \int_{-\infty}^{0} e^{-(s-a)t}\,dt$$

$\downarrow \qquad\qquad\qquad\qquad \downarrow$

~~ROC~~: $Re\{s\} > -a \qquad\qquad Re(s) < +a$

$\Rightarrow ROC = \cancel{\phi}$



~~$x_6(t) = e^{-at} u(t) - e^{+bt} u(-t)$ a, b > 0~~

~~↓ ROC Re{s} > -a Re{s} <~~

ROC: $-a < Re\{s\} < a$

Roc prop.

1. ROC consists of strips parallel to $j\omega$ axis in the s-plane.
2. ROC doesn't contain any poles
3. If $x(t)$ is of finite durat.^{n} & abs. integrable, ROC is entire s-plane
4. If $x(t)$ is right sided & if $Re(s) = \sigma_0$ is in the ROC, then ~~Re~~ $s \in \mathbb{C}$ such that $Re(s) > \sigma_0$ is also in the ROC.
5. If $x(t)$ is left sided . . . .

   . . . , , $Re(s) < \sigma_0$ ~~a~~ is also in the ROC.

6. If $x(t)$ is 2-sided & . . . , then the ROC consists of a strip in the s-plane that includes the line

7. If x

poles

8. If

the

rt

If

of

LT

1. li

2. tim

3. s-

4. t

5. c

6. c

7. d

8. d

9. i

7. If $X(s)$ is rational, then ROC is bdd. by poles or extends to ∞.

8. If $X(s)$ is rational & $x(t)$ is rt. sided, then the ROC is the region in the s-plane to the rt. of the rightmost pole.
   If $x(t)$ is left sided, the ROC is to the left of the leftmost pole.

LT prop. & ROC ($X(s)$ has ROC $= R$)

1. lin. $a_1 x_1(t) + a_2 x_2(t) \longleftrightarrow a_1 X_1(s) + a_2 X_2(s)$
   ROC $\supseteq R_1 \cap R_2$

2. time-shift: $x(t-t_0) \longleftrightarrow e^{-st_0} X(s)$ ROC $= R$

3. s-shift: $e^{s_0 t} x(t) \longleftrightarrow X(s-s_0)$ ROC: $R + Re(s_0)$

4. time scale: $x(at) \longleftrightarrow \frac{1}{|a|} X\left(\frac{s}{a}\right)$ ROC: aR

5. conjug^n: $x^*(t) \longleftrightarrow X^*(s^*)$ ROC $= R$

6. convol^n: $x_1 * x_2 \longleftrightarrow X_1(s) X_2(s)$ ROC $\supseteq R_1 \cap R_2$

7. diff. in time: $\frac{dx}{dt} \longleftrightarrow sX(s)$ ROC $\supseteq R$

8. diff. in freq. $-t\,x(t) \longleftrightarrow \frac{dX(s)}{ds}$ ROC $= R$

9. integr^n in time: $\int_{-\infty}^{t} x(\tau)\,d\tau \longleftrightarrow \frac{1}{s} X(s)$ ROC $\supseteq R \cap Re\{s\} > 0$

in prop. 1 (lin.) & 6 (convol'n),
there could be a pole zero cancell'n
leading to ⊇ instead of = in the R

ex. $X_1(s) = \frac{s+1}{s+2}$ $R_1 \to Re(s) > -2$

$X_2(s) = \frac{s+2}{s+1}$ $R_2 \to Re(s) > -1$

$R_1 \cap R_2 = R_2 \subset R =$ s-plane

Causality, stability & ROC

causality ⟷ ROC = rt. half plane to the rt. of the rightmost pole

anticausal ⟷ ROC = ? (left half ...)

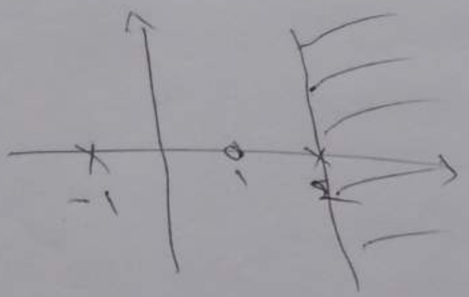
stability ⟷ ROC includes $j\omega$-axis

↓

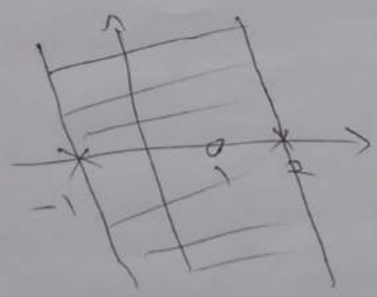
impulse resp. abs. integr. ⇒ Four. transf. converges ⇒ L.T. conv. on $j\omega$-axis

causal & stable $\longleftrightarrow$ poles in the LHP,
ROC includes $j\omega$-axis.

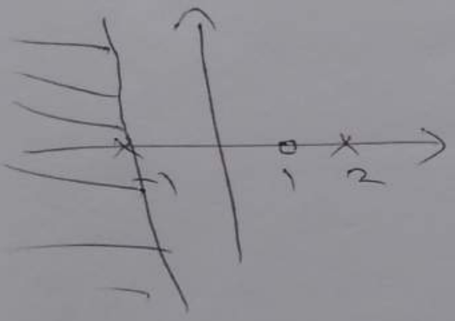
ex. $H(s) = \dfrac{s-1}{(s+1)(s-2)}$ when $h(t)$ is stable/causal?



R_1



R_2



R_3

$R_1 \rightarrow$ causal, unstable

$R_2 \rightarrow$ non causal, stable

$R_3 \rightarrow$ anti causal, unstable

ex. $H(s) = \dfrac{s+1}{(s+2)(s+3)}$

$\rightarrow$ stable & causal