CS21201 Discrete Structures

Tutorial 1

Elementary Counting Techniques

1. You are given r red balls, g green balls, and b blue balls. Assume that r, g, b are positive integers. Your task is to arrange the balls on a line subject to the following conditions. Find the count of all possible arrangements in each case.

(a) All blue balls appear together.

(b) The arrangement must start with a green ball and end with a non-green ball.

(c) No two red balls appear together.

(d) No blue ball can appear after any red ball.

2. Let $A = \{a_1, a_2, \ldots, a_n\}$ be a set of $n \geqslant 1$ real numbers. The arithmetic mean (average) of A is denoted by

$$x = \frac{a_1 + a_2 + \cdots + a_n}{n}.$$

A has $2^n - 1$ non-empty subsets $A_1, A_2, \ldots, A_{2^n-1}$. Let y_i denote the arithmetic mean of the elements of A_i. Finally, denote the arithmetic mean of these $2^n - 1$ arithmetic means as

$$y = \frac{y_1 + y_2 + \cdots + y_{2^n-1}}{2^n - 1}.$$

Prove that $x = y$.

3. How many sorted arrays of size n are there if each element of the array is an integer in the range $1, 2, 3, \ldots, r$?

4. How many binary strings of length n are there with exactly k occurrences of the pattern 01? Assume that $n \geqslant 2k$.

5. Prove the following identity for any positive integer n.

$$2^n = \binom{n+1}{1} + \binom{n+1}{3} + \binom{n+1}{5} + \cdots + \begin{cases} \binom{n+1}{n+1} & \text{if } n \text{ is even,} \\ \binom{n+1}{n} & \text{if } n \text{ is odd.} \end{cases}$$

6. Consider paths from $(0,0)$ to (n,n) in an $n \times n$ grid, that never cross the diagonal. Impose an additional constraint that these paths are not allowed to touch the main diagonal except only at the beginning and at the end. How many such constrained paths are there?

7. Suppose that $m > n$. How many paths from $(0,0)$ to (m,n) with R and U movements are possible such that at no point of time, there are more U moves than R moves?

Additional Exercises

8. How many subsets of size k of $\{1,2,3,\ldots,n\}$ are there, that contain more odd numbers than even numbers?

9. **(a)** It is known that for all $i \geqslant 0$, the i-th Fibonacci number F_i is the integer closest to $\rho^i/\sqrt{5}$, where $\rho = (1+\sqrt{5})/2$ is the golden ratio. Assume that the math library calls `log` and `pow` take constant time per invocation. Propose an $O(1)$-time algorithm to determine how may Fibonacci numbers are there in the range $[1,n]$. Treat $F_1 = F_2 = 1$ as a single Fibonacci number.

(b) How many subsets of $\{1,2,3,\ldots,n\}$ contain exactly k Fibonacci numbers?

10. Take $A = \{a_1,a_2,\ldots,a_n\}$ and $x = (a_1+a_2+\cdots+a_n)/n$ as in Exercise 2. Fix an r in the range $1 \leqslant r \leqslant n$. There are $\binom{n}{r}$ subsets of A of size r. Let $z_{r,1}, z_{r,2}, \ldots, z_{r,\binom{n}{r}}$ denote the arithmetic means of these $\binom{n}{r}$ subsets. Take the arithmetic mean of these arithmetic means, that is, take

$$z_r = \frac{z_{r,1}+z_{r,2}+\cdots+z_{r,\binom{n}{r}}}{\binom{n}{r}}.$$

Finally, take the arithmetic mean of z_r, $r = 1,2,\ldots,n$, that is, take

$$z = \frac{z_1+z_2+\cdots+z_n}{n}.$$

Prove/Disprove: $z = x$.

* **11.** Let n be a positive integer. Expand n to the base 7 as

$$n = (d_l d_{l-1} d_{l-2} \ldots d_1 d_0)_7,$$

where each $d_i \in \{0,1,2,3,4,5,6\}$ is a 7-ary digit. Define

$$S_7(n) = d_l + d_{l-1} + d_{l-2} + \cdots + d_1 + d_0.$$

Notice that the sum $S_7(n)$ is not affected by leading zero digits. For example, the smallest 4-digit prime is

$$1009 = 2\times 7^3 + 6\times 7^2 + 4\times 7 + 1 = (2641)_7 = (02641)_7 = (002641)_7 = \cdots,$$

and so

$$S_7(1009) = 2+6+4+1 = 13.$$

Prove/Disprove: If $p \neq 7$ is a prime, then $S_7(p)$ is a prime too. (**Hint:** Think modulo 6.)

12. In how many ways you can express 100 as a sum

$$a_1 + a_2 + a_3 + \cdots + a_r = 100$$

for some r with each $a_i \in \{1,2,3\}$ and with $a_1 \leqslant a_2 \leqslant a_3 \leqslant \cdots \leqslant a_r$.

13. How many paths from $(0,0)$ to (n,n) are possible with only R and U movements such that the paths never go above the line $y = x+1$?

14. How many paths from $(0,0)$ to (n,n) are possible with only R and U movements such that the paths lie entirely within the two lines $y = x-1$ and $y = x+1$? Touching these two lines is allowed.

* **15.** Prove that the number of parenthesizations of the matrix product $A_0A_1A_2\cdots A_n$ is equal to the n-th Catalan number $C(n)$.