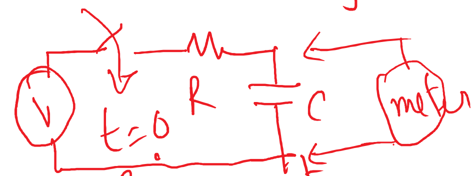
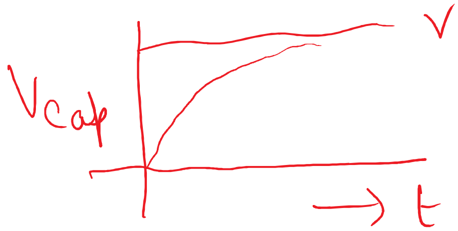
Step Response of LTI systems

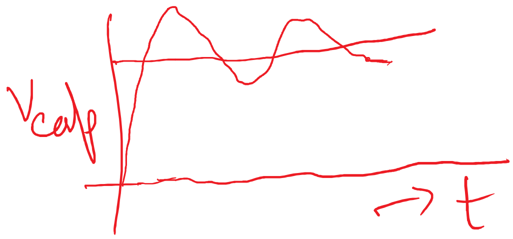
Wednesday, September 08, 2021 11:51 AM

→ Step input pumps finite power and infinite energy.





for R-C circuit



for R-L-C circuit

→ Also any periodic input pumps finite power and infinite energy.

→ The step response of a LTI system is the convolution of the unit step function with the impulse response.

Step response, $s[n] = u[n] * h[n]$
$= h[n] * u[n]$

→ We also know that $u[n]$ is the impulse response of the accumulator, i.e.,

$$s[n] = \sum_{k=-\infty}^{n} h[k] \quad -(1) \quad [u[n] = 0 \text{ for } n < 0]$$

Then $$s[n-1] = \sum_{k=-\infty}^{n-1} h[k] \quad -(2)$$

(1) − (2)

$$\therefore \; h[n] = s[n] - s[n-1]$$

for discrete time impulse response can be recovered from step response.

In continuous time,

$$s(t) = \int_{-\infty}^{t} h(\tau)\, d\tau$$

as, $s(t) = u(t) * h(t)$

and therefore $h(t) = \frac{d}{dt} s(t)$

Linear Constant Coefficient Differential or Difference equation

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→ This is used to characterize an arbitrary LTI system. It can be used to get impulse response and step response also.

$$\frac{dy(t)}{dt} + 2y(t) = x(t) \quad \text{—} (1)$$

forcing function

output input

Take $x(t) = K e^{3t} u(t)$

Solution $\Rightarrow y(t) = y_p(t) + y_h(t)$

particular part homogeneous part

Homogeneous part

$$\frac{dy(t)}{dt} + 2y(t) = 0$$

$$\frac{dy(t)}{y(t)} = -2\,dt$$

$$\int \frac{1}{y(t)}\,dy(t) = -2\int dt$$

$$\Rightarrow \log\{y(t)\} = -2t + C$$

or $y(t) = e^{-2t+c} = Ae^{-2t}$

For the particular part, let us take

$y_p(t) = Ye^{3t}$

$\uparrow$ constant

for $t > 0$,

$3Ye^{3t} + 2Ye^{3t} = Ke^{3t}$

$\Rightarrow 5Y = K$

or $Y = \frac{K}{5}$

$\therefore y_p(t) = \frac{K}{5}e^{3t}, \quad t > 0$

$\therefore y(t) = Ae^{-2t} + \frac{K}{5}e^{3t}, \quad t > 0$

→ We need a initial condition to find the value of A.,

initial condition: At $t = 0$, $y(0) = 0$

$\therefore A = -\frac{K}{5}$

and $y(t) = \frac{K}{5}(e^{3t} - e^{-2t})$

Linear constant coefficient difference equation is also similar. e.g. $y[n] = x[n] - x[n-1]$;

is also similar [e.g., $y[n] = x[n] - x[n-1]$;
$y[n] - y[n-1] = x[n]$]

In general this can be represented as

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k] \quad \text{--- (1)}$$

$$\underbrace{\sum_{k=0}^{N} a_k y[n-k] = 0}_{\text{homogeneous solution}}$$

Eqn (1) can be written as

$$y[n] = \frac{1}{a_0}\left\{ \sum_{k=0}^{M} b_k x[n-k] - \sum_{k=1}^{N} a_k y[n-k] \right\}$$

If we take a_k's to be zero.

$$y[n] = \sum_{k=0}^{M} \frac{b_k}{a_0} x[n-k] \Rightarrow \text{moving average (MA)}$$

If $b_0 = 1$ and other b_k's are zero

M $\rightarrow$ $x[n]$

If $b_0 = 1$

$$y[n] + \underbrace{\sum_{k=1}^{M} \frac{a_k}{a_0} y[n-k]}_{\text{auto-regressive (AR)}} = \frac{x[n]}{a_0}$$

Example

$$y[n] - \frac{1}{2} y[n-1] = x[n]$$

Taking initial rest condition

Given, $\underline{x[n]} = \underline{k\delta[n]}$

$x[n] = 0, \; n \leq -1$ ← initial rest condition

and initial condition on $y[n]$ is

$$y[n] = 0, \; n \leq -1, \; y[-1] = 0$$

$$\therefore \; y[0] = x[0] + \frac{1}{2} \overset{\Rightarrow 0}{y[-1]} = k$$

$$y[1] = \overset{\Rightarrow 0}{x[1]} + \frac{1}{2} y[0] = \frac{1}{2} k$$

$$y[2] = \overset{0}{x[2]} + \frac{1}{2} y[1] = \left(\frac{1}{2}\right)^2 k$$

$$\vdots$$

$$y[n] = x[n] + \frac{1}{2} y[n-1] = \left(\frac{1}{2}\right)^n k$$

for $n \geq 0$

→ Impulse response can be found out easily by putting $k = 1$,

$$h[n] = \left(\frac{1}{2}\right)^n u[n]$$

$$h[n] = \left(\frac{1}{2}\right)^n u[n]$$

$\downarrow$

takes account
the condition $n \geq 0$

Block Diagram Representation

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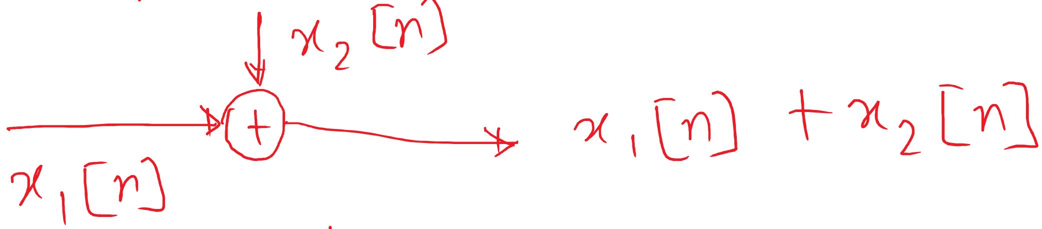
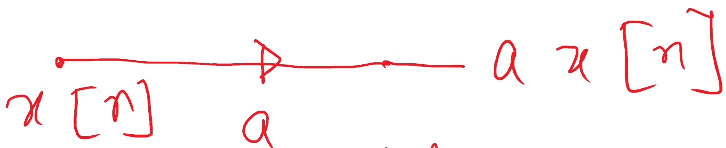
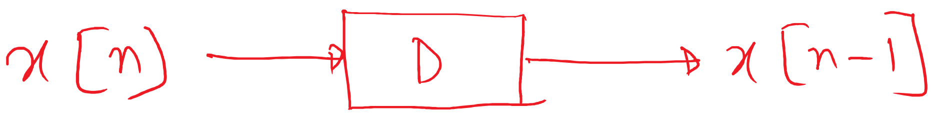


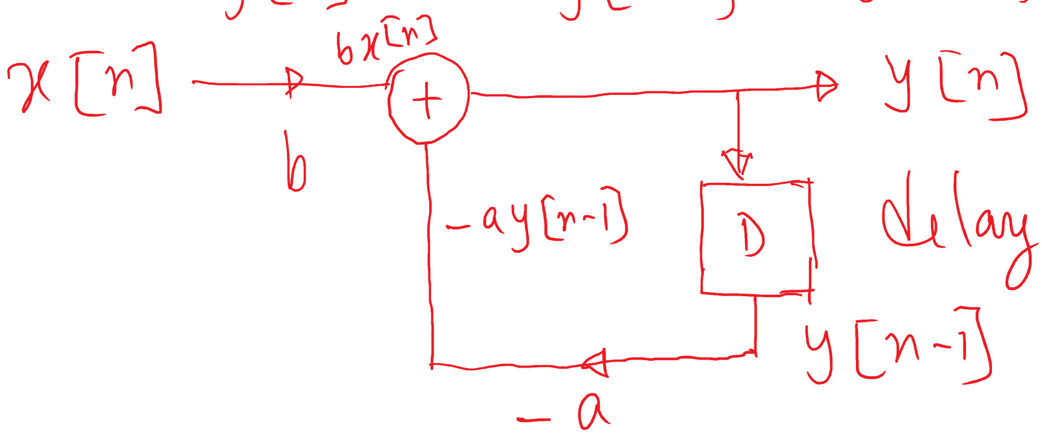
Adder



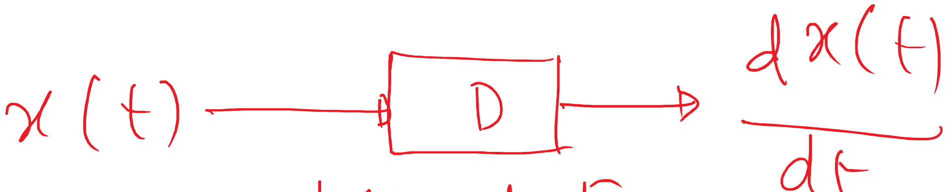
delay



$$y[n] = -a\,y[n-1] + b\,x[n]$$



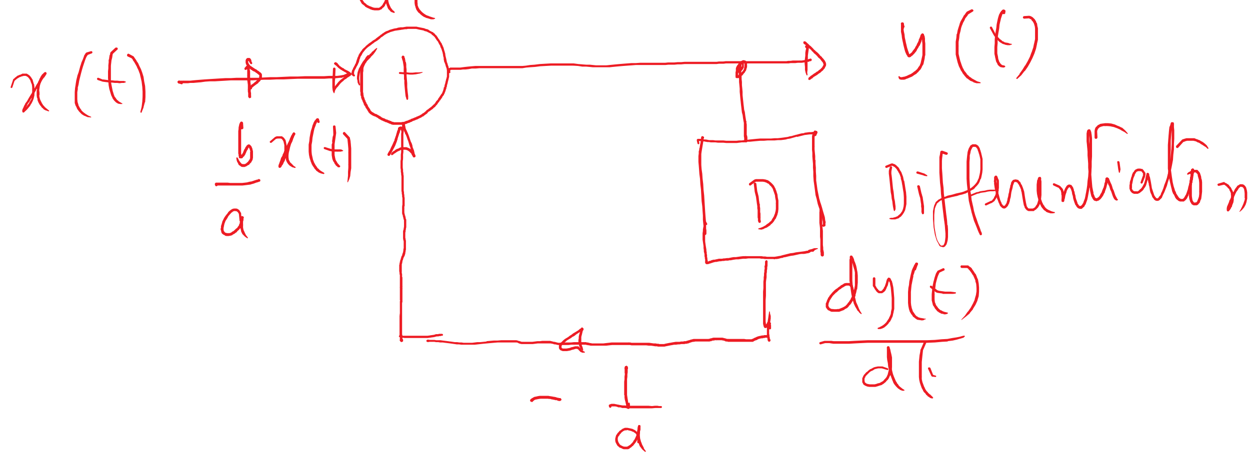
Continuous time



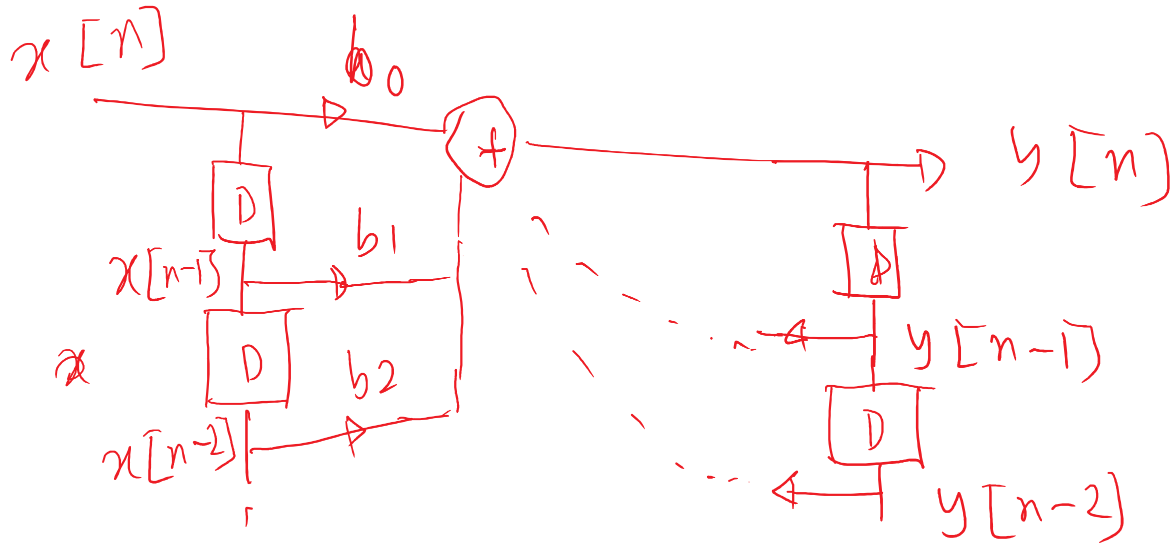
differentiator

$$\frac{dy(t)}{dt} = b\,x(t) - a\,y(t)$$

$$\frac{1}{a}\frac{dy(t)}{dt} = \frac{b}{a}x(t) - y(t)$$



$$y(t) = \frac{b}{a}x(t) - \frac{1}{a}\frac{dy(t)}{dt}$$



→ In the continuous domain, apart from differentiator integrator blocks can also be used.

Defining unit impulse through convolution

$x(t) = x(t) * \delta(t)$ for any $x(t)$

If we take $x(t) = 1$ for all t,

$$1 = x(t) = x(t) * \delta(t) = \delta(t) * x(t)$$

$$= \int_{-\infty}^{+\infty} \delta(\tau)\, x(t-\tau)\, d\tau \quad \text{—(1)}$$

$$= \int_{-\infty}^{+\infty} \delta(\tau)\, d\tau \longrightarrow$$ δ function has unit area

* Unit doublet

Let us take a system. $y(t) = \frac{d}{dt} x(t)$

→ The unit impulse response of this system is the derivative of the unit impulse, which is known as the doublet.

In other words, $\frac{d}{dt} x(t) = x(t) * u_1(t)$

$u_1(t)$ is the impulse response of the derivative operation } unit doublet

Similarly, $$\frac{d^2}{dt^2} x(t) = x(t) * u_2(t)$$

$$= \underbrace{x(t) * u_1(t)}_{\frac{d}{dt} x(t)} * u_1(t)$$

or, $u_2(t) = u_1(t) * u_1(t)$

or, $u_k(t) = \underbrace{u_1(t) * u_1(t) * \cdots * u_1(t)}_{k \text{ times}}$

Let us take $x(t) = 1$.

$$\therefore\ 0 = \frac{d x(t)}{dt} = x(t) * u_1(t)$$

$$= \int^{+\infty} u_1(\tau)\, x(t-\tau)\, d\tau$$

$$= \int_{-\infty}^{+\infty} u_1(\tau)\, x(t-\tau)\, d\tau$$

$$= \int_{-\infty}^{+\infty} u_1(\tau)\, d\tau$$

→ This shows that unit doublet has zero area.

To look into another property of unit doublet, let us take a signal $g(-t)$ and convolve with $u_1(t)$,

$$\int_{-\infty}^{+\infty} g(\tau - t)\, u_1(\tau)\, d\tau = g(-t) * u_1(t) = \frac{d}{dt} g(-t)$$

$$= -g'(-t)$$

If we put $t = 0$,

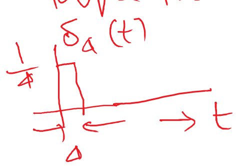
$$-g'(0) = \int_{-\infty}^{+\infty} g(\tau)\, u_1(\tau)\, d\tau$$

Let us check the doublet for δ function

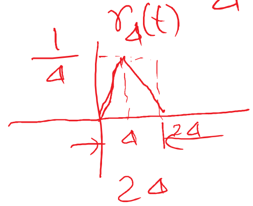
We know that $x(t) = x(t) * \delta(t)$

Now if we take $x(t) = \delta(t)$, then $\delta(t) = \delta(t) * \delta(t)$

→ Interpretation of derivative of $\delta(t)$ we need to take the help of approximate version of $\delta(t)$ as $\delta_\Delta(t)$



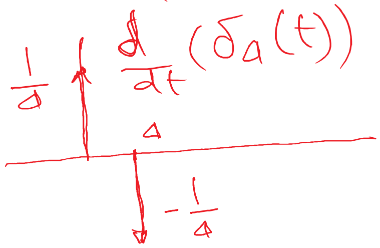
Then $r_\Delta(t) = \delta_\Delta(t) * \delta_\Delta(t)$



$$\frac{d}{dt}\delta_\Delta(t) = \frac{1}{\Delta}\left\{\delta(t) - \delta(t-\Delta)\right\}$$

$$\because\ x(t) * \delta(t - t_0) = x(t - t_0)$$

$$\therefore x(t) * \frac{d}{dt}\delta_\Delta(t) = \frac{x(t) - x(t-\Delta)}{\Delta} \simeq \frac{dx(t)}{dt} \text{ as } \Delta \to 0$$



As unit step is the impulse response of an integrator, if we define $y(t) = \int_{-\infty}^{t} x(\tau)\, d\tau$

$$\therefore u(t) = \int_{-\infty}^{t} \delta(\tau)\, d\tau$$

$$\text{and } y(t) = x(t) * u(t) = \int_{-\infty}^{t} x(\tau)\, d\tau$$

<u>Double integrator</u> $[u_{-2}(t)]$

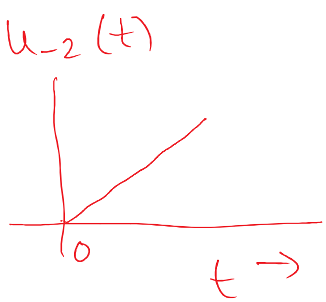
In other words, $u_{-2}(t) = u(t) * u(t)$

$$= \int_{-\infty}^{t} u(\tau)\, d\tau$$

$\rightarrow$ As $u(t) = 0$ for $t < 0$, and $u(t) = 1$ for $t > 0$ the integration will start from $t = 0$, and

$$u_{-2}(t) = t\,u(t)$$

unit ramp function starting from $t = 0$



Take $x(t) * u_{-2}(t) = x(t) * u(t) * u(t)$

$$= \left(\int_{-\infty}^{t} x(\sigma)\, d\sigma\right) * u(t)$$

$$= \int_{-\infty}^{t} \left(\int_{-\infty}^{\tau} x(\sigma)\, d\sigma\right) d\tau$$

If we convolute k times,

$$u_{-k}(t) = \underbrace{u(t) * u(t) * \cdots * u(t)}_{k \text{ times}}$$

$$= \int_{-\infty}^{t} u_{-(k-1)}(\tau)\, d\tau$$

or $$u_{-k}(t) = \frac{t^{k-1}}{(k-1)!} u(t)$$

→ Since derivative is an inverse operator of integrator,

$$u(t) * u_1(t) = \delta(t)$$

(or $u_{-1}(t)$)

In general, $u_k(t) * u_r(t) = u_{k+r}(t)$

Fourier Series Representation of Periodic Signals.

→ An LTI system can be represented as a weighted sum of an elementary signal or a basis function.

→ The response of an LTI system to the set of input functions need to be expressed in terms of the same input function. Such functions can be thought of as eigenfunctions of LTI system.

→ Complex exponential satisfy this conditions in case of LTI systems

In continuous time, if we take the complex exponential as e^{st},
[s is complex]

$$e^{st} \longrightarrow \boxed{\quad\quad} \longrightarrow \underbrace{H(s)}_{\text{eigenvalue}} e^{st}$$

Similarly, in the discrete domain,

$$z^n \longrightarrow H(z) \cdot z^n$$

↑ power series expansion of complex z

Let $x(t) = e^{st}$

Then $y(t) = \int_{-\infty}^{+\infty} h(\tau)\, x(t-\tau)\, d\tau$

$$= \int_{-\infty}^{+\infty} h(\tau)\, e^{s(t-\tau)}\, d\tau$$

$$= e^{st} \int_{-\infty}^{+\infty} h(\tau)\, e^{-s\tau}\, d\tau$$

$$= H(s)\, e^{st} \leftarrow \text{eigenfunction}$$

↑ complex constant as a function of s, eigenvalue

Discrete time case

Let $x[n] = z^n$, $z \rightarrow$ complex

Then $y[n] = \sum_{k=-\infty}^{+\infty} h[k] x[n-k]$

$= \sum_{k=-\infty}^{+\infty} h[k] z^{n-k}$

$= z^n \sum_{k=-\infty}^{+\infty} h[k] z^{-k}$

$= H(z) \cdot z^n$ ← eigenfunction

↑ eigenvalue, provided $\sum_{k=-\infty}^{+\infty} h[k] z^{-k}$ converges

Ex.

Take $y(t) = x(t-3)$ and given $x(t) = e^{j2t}$

Then, $y(t) = e^{j2(t-3)}$

$= e^{-j6} e^{j2t}$ ← eigenfunction

↑ eigenvalue

The impulse response of the system is $\delta(t-3)$

$H(s) = \int_{-\infty}^{+\infty} \delta(\tau - 3) e^{-s\tau} d\tau$

$= e^{-3s}$, if we sample it at $s = j2$.

$H(j2) = e^{-j6}$

Linear combination of Harmonically Related Complex exponential

→ Let us start with a periodic signal $x(t) = x(t+T)$ for all t where T is period.

Let $x(t) = e^{j\omega_0 t}$ and $\phi_k(t) = e^{jk\omega_0 t} = e^{jk\left(\frac{2\pi}{T}\right)t}$

↑ kth harmonic

∴ $\phi_k(t)$ is the harmonically related complex exponential with fundamental frequency ω_0

∴ By superposition,

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$$

$$= \sum_{k=-\infty}^{+\infty} a_k e^{jk\left(\frac{2\pi}{T}\right)t} \quad \text{— (1)}$$

→ By superposition and linearity, sum of periodic signal which is harmonically distributed is also periodic with T.

This representation of a periodic signal in form of eqn (1) is referred as Fourier Series representation.

Ex

$$x(t) = \sum_{k=-3}^{+3} a_k e^{jk2\pi t}$$

, given $a_0 = 1$,

$a_1 = a_{-1} = \frac{1}{4}$, $a_2 = a_{-2} = \frac{1}{2}$

$a_3 = a_{-3} = \frac{1}{3}$

$$x(t) = 1 + \frac{1}{4}\left(e^{j2\pi t} + e^{-j2\pi t}\right) + \frac{1}{2}\left(e^{j4\pi t} + e^{-j4\pi t}\right)$$

$$+ \frac{1}{3}\left(e^{j6\pi t} + e^{-j6\pi t}\right)$$

$$x(t) = 1 + \frac{1}{4}(e \quad + e \quad) \cdot 2(\quad + e \quad)$$

$$+ \frac{1}{3}\left(e^{j6\pi t} + e^{-j6\pi t}\right)$$

$$= 1 + \frac{1}{2}\cos(2\pi t) + \cos(4\pi t) + \frac{2}{3}\cos(6\pi t)$$

→ Generally for real $x(t)$, $x^*(t) = x(t)$

$$\therefore x(t) = \sum_{k=-\infty}^{+\infty} a_k^* e^{-jk\omega_0 t} \quad \text{—— (2)}$$

Replacing k by $-k$, $$x(t) = \sum_{k=-\infty}^{+\infty} a_{-k}^* e^{jk\omega_0 t} \quad \text{—— (3)}$$

Comparing (1) and (3), $a_k^* = a_{-k}$ ⇐ conjugate symmetry

For real a_k, $a_k = a_{-k}$ ⇒ symmetry

$$\text{or, } x(t) = a_0 + \sum_{k=1}^{\infty}\left[a_k e^{jk\omega_0 t} + a_{-k} e^{-jk\omega_0 t}\right]$$

$$= a_0 + \sum_{k=1}^{\infty}\left[a_k e^{jk\omega_0 t} + a_k^* e^{-jk\omega_0 t}\right]$$

↳ complex conjugates of each other

$$\text{or } x(t) = a_0 + \sum_{k=1}^{\infty} 2\,\mathrm{Re}\left\{a_k e^{jk\omega_0 t}\right\}$$

In polar form a_k is expressed as, $a_k = A_k e^{j\theta_k}$

$$\text{Then, } x(t) = a_0 + \sum_{k=1}^{\infty} 2\,\mathrm{Re}\left[A e^{j(k\omega_0 t + \theta_k)}\right]$$

or $x(t) = a_0 + \sum_{k=1}^{\infty} 2 \cos(k \omega_0 t + \theta_k)$

If we write a_k in rectangular form as

$$a_k = B_k + jC_k$$

Then,

$$x(t) = a_0 + 2 \sum_{k=1}^{\infty} \left[B_k \cos(k \omega_0 t) - C_k \sin(k \omega_0 t) \right]$$

If a_k is real, $a_k = A_k = B_k$, $C_k = 0$

Determination of Fourier Series Coefficients

Let $x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$

$\therefore x(t) \cdot e^{-jn\omega_0 t} = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} \cdot e^{-jn\omega_0 t}$

Integrating both sides over T,

or, $\int_0^T x(t) e^{-jn\omega_0 t} dt = \int_0^T \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} \cdot e^{-jn\omega_0 t} dt$

or, where $T = \frac{2\pi}{\omega_0} \rightarrow$ fundamental period.

or, $\int_0^T x(t) e^{-jn\omega_0 t} dt = \sum_{k=-\infty}^{+\infty} a_k \int_0^T e^{j(k-n)\omega_0 \cdot t} dt$ — ①

Now, $\int_0^T e^{j(k-n)\omega_0 \cdot t} dt = \int_0^T \cos(k-n)\omega_0 t \, dt + j\int_0^T \sin(k-n)\omega_0 t \, dt$

If $k \neq n$, they are periodic sinusoidal function with period $\frac{T}{|k-n|}$ and becomes 0 when integrated over a period.

If $k = n$, $\int_0^T e^{j(k-n)\omega_0 t} dt = T \leftarrow$ fundamental period

$\therefore \int_T e^{j(k-n)\omega_0 t} dt = \begin{cases} T, & k = n \\ 0, & k \neq n \end{cases}$

$\therefore$ Putting this in ①, $a_n = \frac{1}{T} \int_T x(t) e^{-jn\omega_0 t}$

periodic

$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{+\infty} a_k e^{j\left(k\frac{2\pi}{T}\right) \cdot t}$

and $a_k = \frac{1}{T} \int_T x(t) e^{-jk\omega_0 t} dt = \frac{1}{T} \int_T x(t) e^{-jk\frac{2\pi}{T} \cdot t} dt$

mapped to the harmonics of ω_0. starting from zero

Ex.

$$x(t) = \sin(\omega_0 t) \quad \leftarrow \text{odd signal}$$

$$= \frac{1}{2j} e^{j\omega_0 t} - \frac{1}{2j} e^{-j\omega_0 t}$$

$\therefore a_0 = 0, \quad a_1 = \frac{1}{2j}$ and $a_{-1} = -\frac{1}{2j}, \quad a_k = 0$ for $|k|>1$

Ex2

$$x(t) = 1 + \sin(\omega_0 t) + 2\cos(\omega_0 t) + \cos\left(2\omega_0 t + \frac{\pi}{4}\right)$$

$$\therefore x(t) = 1 + \left(1 + \frac{1}{2j}\right) e^{j\omega_0 t} + \left(1 - \frac{1}{2j}\right) e^{-j\omega_0 t} + \frac{1}{2} e^{j\pi/4} \cdot e^{j2\omega_0 t} + \frac{1}{2} e^{-j\pi/4} \cdot e^{-j2\omega_0 t}$$

$\Rightarrow a_0 = 1, \quad a_1 = 1 + \frac{1}{2j}, \quad a_{-1} = 1 - \frac{1}{2j}, \quad a_2 = \frac{1}{2} e^{j\pi/4} = \frac{\sqrt{2}}{4}(1+j)$

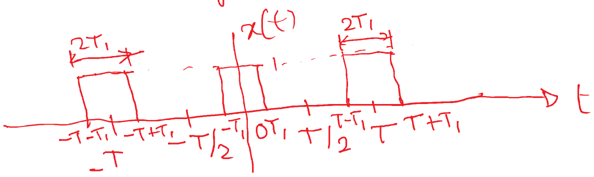
$a_{-2} = \frac{1}{2} e^{-j\pi/4} = \frac{\sqrt{2}}{4}(1-j), \quad a_k = 0, \ |k| > 2$

Ex

$$x(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & T_1 < |t| < T/2 \end{cases}$$

over one period

Given $x(t)$ is periodic, & $T_1 < T/2$



If we integrate over a period,

$$a_0 = \frac{1}{T}\int_{-T_1}^{+T_1} dt = \frac{2T_1}{T}$$

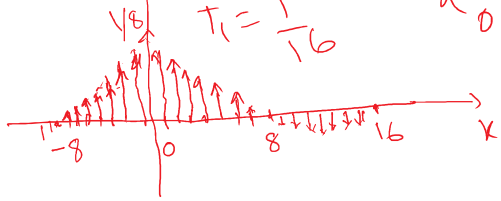
$$a_k = \frac{1}{T}\int_{-T_1}^{T_1} e^{-jk\omega_0 t}\, dt = \frac{2\sin(k\omega_0 T_1)}{k\omega_0 T}, \quad k \neq 0$$

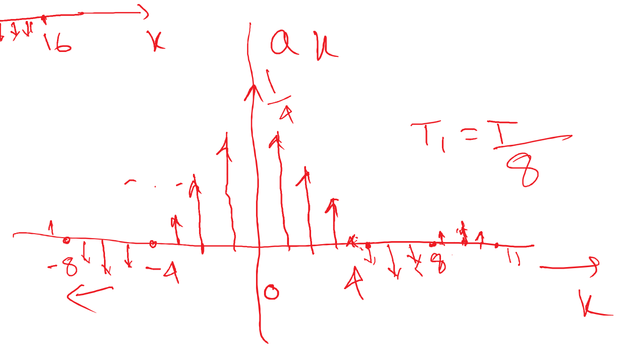
Put $\omega_0 T = 2\pi$ $\Longrightarrow$ $= \frac{\sin(k\omega_0 T_1)}{k\pi}, \quad k \neq 0$

If $T = 8T_1 \Rightarrow T_1 = \frac{T}{8} = \frac{2\pi}{8\omega_0} = \frac{\pi}{4\omega_0}$

Then $a_k = \frac{\sin(\pi k/4)}{k\pi}, \quad k \neq 0.$

$T_1 = \frac{T}{16}$ $\quad a_0 = \frac{1}{4}$





$T_1 = \frac{T}{8}$

Sinc function

→ Though it is a valid Fourier Series expansion but a_k can not be truncated at a finite k. If it is truncated, the reconstructed $x(t)$ from the Fourier series coefficient will have a finite error.

– Convergence of Fourier Series

Let $x(t)$ be the periodic signal to be represented by Fourier series, as $x_N(t) = \sum_{k=-N}^{N} a_k e^{jk\omega_0 t}$

the approximated $x(t)$.

The approximation error $e_N(t) = x(t) - x_N(t)$

$$= x(t) - \sum_{k=-N}^{+N} a_k e^{jk\omega_0 t}$$

Now, the energy of error over one period is

$$E_N = \int_T |e_N(t)|^2 \, dt$$

The set of a_k that minimizes E_N is given by $a_k = \frac{1}{T}\int_T x(t) e^{-jk\omega_0 t} dt$

i.e. $E_N \to 0$ as $N \to \infty$

This can be shown by taking the error in an interval.

$$E = \int^b \left[x(t) - \sum_{k=-N}^{+N} a_k \phi_k(t)\right]\left[x^*(t) - \sum_{k=-N}^{+N} a_k^* \phi_k^*(t)\right] dt$$

$$E = \int_a^b \left[x(t) - \sum_{k=-N}^{N} a_k \phi_k(t)\right]\left[x^*(t) - \sum_{k=-N}^{N} a_k^* \phi_k^*(t)\right] dt$$

$\phi_k(t)$ — complex exponential or in general any orthonormal function

Let $a_i = b_i + j c_i$,

Then $\frac{\partial E}{\partial b_i} = 0$, $\Rightarrow$ produce zero for all term other i^{th}

Check it

$$\frac{\partial E}{\partial b_i} = -\int_a^b \phi_i^*(t)x(t)\,dt + 2b_i - \int_a^b \phi_i(t)x^*(t)$$

and $\frac{\partial E}{\partial c_i} = 0 \Rightarrow j\int_a^b \phi_i(t)x^*(t)\,dt + 2c_i - j\int_a^b \phi_i^*(t)x(t)\,dt$

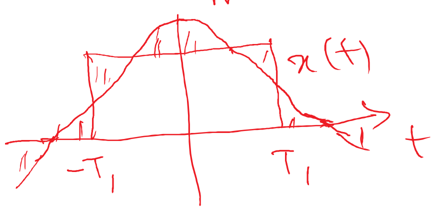
$$2c_i = j\int_a^b \phi_i^*(t)x(t)\,dt - j\int_a^b \phi_i(t)x^*(t)\,dt$$

$$\therefore 2b_i + 2jc_i = 2\int_a^b x(t)\,\phi^*(t)\,dt$$

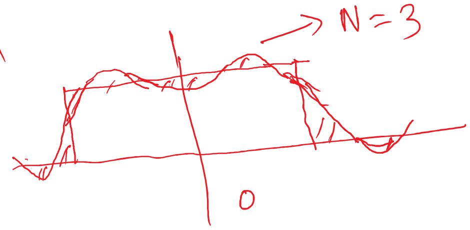
$$\Rightarrow b_i + jc_i = a_i = \int_a^b x(t)\,\phi^*(t)\,dt$$

$\phi^*(t)$ — complex exponential

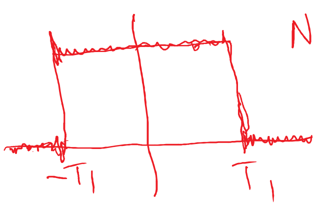
N = 1 — for one period



N = 3



N = 79



→ This example shows that the error tends to zero in average sense but that does not mean $x(t) = x_N(t)$ at all t, for signals with discontinuity in particular.

→ Further Dirichlet found that as N increases $x(t)$ is equal at almost all points except at the discontinuity. He also concluded that at the discontinuity Fourier expansion converges in the average sense.

Dirichlet Conditions

Condition 1: $x(t)$ must be absolutely integrable over one period.

$$\int_T |x(t)|\,dt < \infty$$

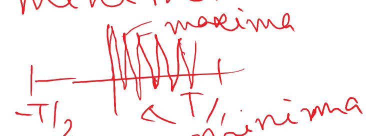
$$|a_k| \leq \frac{1}{T}\int_T \left|x(t)e^{-jk\omega_0 t}\right|dt = \frac{1}{T}\int_T |x(t)|\,dt$$

$$\left[\because \left|e^{-jk\omega_0 t}\right| = 1\right]$$

$$\therefore \text{ if } \int_T |x(t)|\,dt < \infty, \quad a_k < \infty$$

Condition 2

In any finite interval of time, $x(t)$ is of bounded variation; that is, there are no more than a finite number of maxima and minima during any single period of signal.



Condition 3: In any finite interval of time, there are only a finite number of discontinuities and each of these are finite.

→ In general, however, Fourier representation of signal is exact for periodic signal without discontinuity.

Properties of Fourier Series

Linearity: Let $x(t)$ and $y(t)$ are two periodic signals with period T, having Fourier series coefficients a_k and b_k

$$x(t) \xleftrightarrow{FS} a_k$$

$$y(t) \xleftrightarrow{FS} b_k$$

Now, if $z(t) = A\,x(t) + B\,y(t)$

$$z(t) \xleftrightarrow{FS} c_k = A a_k + B b_k$$

Time shifting: Let $y(t) = x(t-t_0)$

then, $b_k = \frac{1}{T}\int_T x(t-t_0)\, e^{-jk\omega_0 t}\, dt$

Take $\tau = t - t_0$,

$$\frac{1}{T}\int_T x(\tau)\, e^{-jk\omega_0(\tau+t_0)}\, d\tau$$

$$= e^{-jk\omega_0 t_0} \cdot a_k = e^{-jk\left(\frac{2\pi}{T}\right)\cdot t_0} \cdot a_k$$

$$x(t-t_0) \xleftrightarrow{FS} e^{-jk\omega_0 t_0} a_k$$

Time reversal

Let $y(t) = x(-t)$, then $x(-t) = \sum_{k=-\infty}^{+\infty} a_k e^{-jk\frac{2\pi}{T}\cdot t}$

Substituting $k = -m$,

$$y(t) = x(-t) = \sum_{m=-\infty}^{+\infty} a_{-m}\, e^{jm\frac{2\pi}{T}\cdot t}$$

$\therefore\ b_k = a_{-k}$, if $x(t) \xleftrightarrow{FS} a_k$

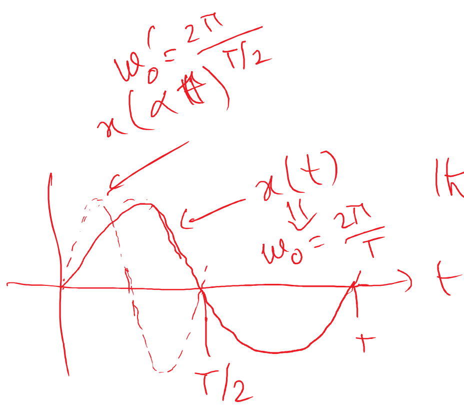
If $x(t)$ is even, $a_k = a_{-k}$ and for odd $x(t)$ $a_k = -a_{-k}$

Time scaling

$$x(\alpha t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\underline{(\alpha\omega_0)}t}$$

$\Downarrow$

$$x(\alpha t) = \sum_{k=-\infty} a_k e \quad \Downarrow$$



This means the fundamental frequency value change by α and the correspondence of the series coefficient

Multiplication

If $x(t) \xleftrightarrow{FS} a_k$, $y(t) \xleftrightarrow{FS} b_k$

then $x(t) \cdot y(t) \xleftrightarrow{FS} h_k = \sum_{l=-\infty}^{+\infty} a_l b_{k-l}$

Convolution

→ Convolution

$$\int_T x(\tau)\, y(t-\tau)\, d\tau \xleftrightarrow{FS} T a_k b_k$$

T ← period of $x(t)$ and $y(t)$, if they are of same frequency, otherwise T will be the period of the convoluted signal.

Conjugate Symmetry

If $x(t) \xleftrightarrow{FS} a_k$, then $x^*(t) \xleftrightarrow{FS} a^*_{-k}$

Now, if $x(t)$ is real, $a_{-k} = a^*_k$

i.e. $|a_{-k}| = |a_k|$ for real $x(t)$

Parseval's relation for continuous time signal

Parseval's relation for continuous time signal

$$\underbrace{\frac{1}{T}\int_T |x(t)|^2 dt}_{\text{average power}} = \sum_{k=-\infty}^{+\infty} |a_k|^2$$

$a_k(\cos(w_0 t) + j\sin(w_0 t))$
$a_k^*(\cos(w_0 t) - j\sin(w_0 t))$
$= 1 \cdot a_k a_k^* = |a_k|^2$

$$x(t) \xleftrightarrow{FS} \sum_{k=-\infty}^{+\infty} a_k e^{jk w_0 t}$$

Frequency shifting

$$x(t) e^{jM w_0 t} \Rightarrow a_{k-M}$$

$$x(t-t_0) \underset{\text{time}}{\longleftrightarrow} a_k e^{-jk w_0 t_0}$$

Differentiation

$$\frac{d}{dt} x(t) \xleftrightarrow{FS} jk w_0 a_k = jk\frac{2\pi}{T} \cdot a_k$$

Integration

$$\int_{-\infty}^{t} x(t)\, dt \xleftrightarrow{FS} \frac{1}{jk w_0} a_k$$

Even & Odd signal

$x(t)$ is real and even $\xleftrightarrow{FS}$ a_k real and even
$x(t)$ is real and odd $\xleftrightarrow{FS}$ a_k imaginary and odd

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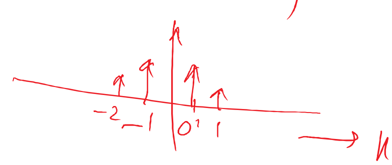
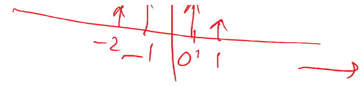
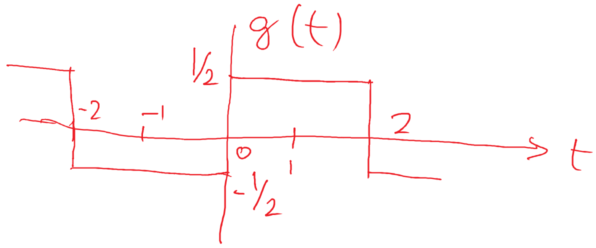
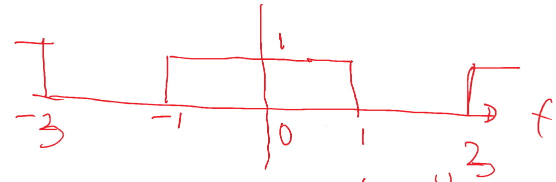


r. $x(t)$

Ex.







$x(t)$ ||

$a_0 = \frac{1}{2}$

$a_k = \text{sinc}(\cdot)$

Let $x(t) \xleftrightarrow{FS} a_k$

$g(t) = x(t-1) - \frac{1}{2}$

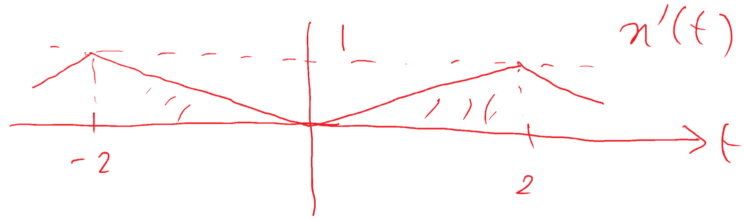
$x(t-1) \xleftrightarrow{FS} b_k = a_k e^{-jk\frac{\pi}{2}}$

$-\frac{1}{2} \xleftrightarrow{FS} c_k = 0$ for $k \neq 0$

$= -\frac{1}{2}$ for $k = 0$

$\therefore \quad g(t) \xleftrightarrow{FS} d_k = \begin{cases} a_k e^{-jk\pi/2} \\ a_0 - \frac{1}{2}, \text{ for } k = 0 \end{cases} = \begin{cases} \frac{\sin(k\pi/2)}{k\pi} e^{-jk\pi/2}, & k \neq 0 \\ 0, & k = 0 \end{cases}$

Ex.



The derivative of $x'(t)$ is $g(t)$ from the previous problem

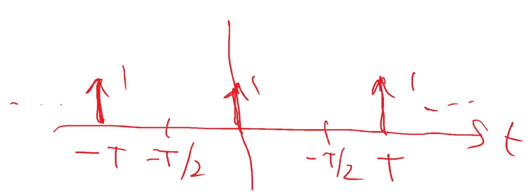
$\therefore$ If $x'(t) \xleftrightarrow{FS} e_k$, then $d_k = jk(\pi/2) \cdot e_k$

or $e_k = \frac{2 d_k}{jk\pi} = \frac{2\sin(k\pi/2)}{j(k\pi)^2} e^{-jk\pi/2}$, $k \neq 0$

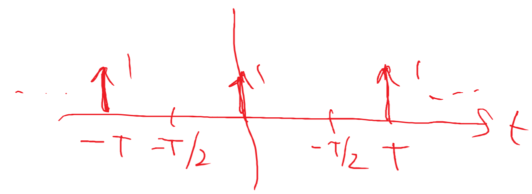
and $e_0 = \frac{1}{2}$

Example of periodic Train of impulses

$$x(t) = \sum_{k=-\infty}^{+\infty} \delta(t - kT)$$



$$x(t) = \sum_{k=-\infty}^{+\infty} \delta(t-kT)$$

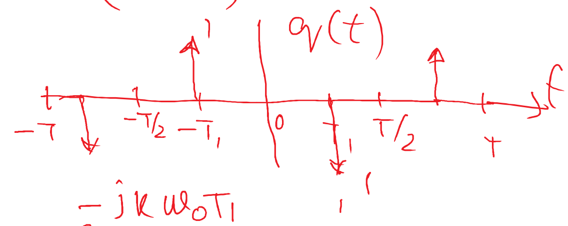


If we integrate from $-T/2$ to $+T/2$,

$$a_k = \frac{1}{T}\int_{-T/2}^{+T/2} \delta(t)\, e^{-jk\frac{2\pi}{T}\cdot t}\, dt = \frac{1}{T}$$

If we define another signal $q(t)$ as

$$q(t) = x(t+T_1) - x(t-T_1)$$



$$q(t) \xleftrightarrow{FS} b_k = e^{jk\omega_0 T_1} a_k - e^{-jk\omega_0 T_1} a_k$$

$$= \frac{1}{T}\left[e^{jk\omega_0 T_1} - e^{-jk\omega_0 T_1}\right]$$

$$= \frac{2j\sin(k\omega_0 T_1)}{T}$$

Fourier Series Representation of discrete-time periodic signal

Friday, September 24, 2021 10:24 AM

→ Discrete-time periodic signals have some difference with respect to its continuous time counterparts.

A discrete-time signal $x[n]$ is periodic with period N if $x[n] = x[n+N]$, Then fundamental frequency $w_0 = \frac{2\pi}{N}$, and $e^{j(\frac{2\pi}{N})n}$ and $e^{jk(\frac{2\pi}{N})n}$ are periodic with fundamental period N for any integer k.

In $e^{jk(\frac{2\pi}{N})n}$ there are only N number of distinct signals as k and n are integers, both varying as $0, \pm 1, \pm 2 \ldots$

For any discrete-time periodic function, e.g., complex exponential,

$$\phi_N[n] = \phi_0[n], \quad \phi_{N+1}[n] = \phi_1[n] \ldots$$

or in general $\phi_{k+rN}[n] = \phi_k[n]$

Thus,

$x[n]$ is unique only in principal interval of 0 to $N-1$ for a period N.

$$\therefore x[n] \xleftrightarrow{FS} \sum_{k=\langle N\rangle} a_k e^{jkw_0 n} = \sum_{k=\langle N\rangle} a_k e^{jk(\frac{2\pi}{N})n}$$

$k = \langle N \rangle$ ← any interval of N i.e. N consecutive number of points

Determination of Fourier Series Representation

Let $x[n]$ be periodic with fundamental period N

Then $x[0] = \sum_{k=\langle N\rangle} a_k$, $\quad x[1] = \sum_{k=\langle N\rangle} a_k e^{j\frac{2\pi}{N}\cdot k}$

$$\cdots \quad x[N-1] = \sum_{k=\langle N\rangle} a_k e^{j\frac{2\pi}{N}(N-1)\cdot k}$$

$$\sum_{n=\langle N\rangle} e^{jk\frac{2\pi}{N}\cdot n} = \begin{cases} N, & k=0, \pm N, \pm 2N \cdots \\ 0, & \text{otherwise} \end{cases}$$

We have $x[n] = \sum_{k=\langle N\rangle} a_k e^{jk\frac{2\pi}{N}\cdot n}$

Multiplying both sides by $e^{-jr\left(\frac{2\pi}{N}\right)\cdot n}$ and
r — any integer

summing N consecutive terms.

$$\sum_{n=\langle N\rangle} x[n] e^{-jr\frac{2\pi}{N}\cdot n} = \sum_{n=\langle N\rangle} \sum_{k=\langle N\rangle} a_k e^{j(k-r)\frac{2\pi}{N}\cdot n}$$

or, $$\sum_{\langle N\rangle} x[n] e^{-jr\frac{2\pi}{N}\cdot n} = \sum_{k=\langle N\rangle} a_k \sum_{n=\langle N\rangle} e^{j(k-r)\frac{2\pi}{N}\cdot n}$$

For $k=r$, $\sum_{n=\langle N\rangle} e^{j(k-r)\frac{2\pi}{N}\cdot n} = N$ and

for $k \neq r$, $\sum_{n=\langle N\rangle} e^{j(k-r)\frac{2\pi}{N}\cdot n} = 0$

$$\therefore \sum_{\langle N\rangle} x[n] e^{-jr\left(\frac{2\pi}{N}\right)n} = N a_r$$

or $$a_r = \frac{1}{N} \sum_{\langle N\rangle} x[n] e^{-jr\left(\frac{2\pi}{N}\right)n}$$

Thus, $x[n] = \sum a_k e^{jk\omega_0 n} = \sum a_k e^{jk\left(\frac{2\pi}{N}\right)n}$

Thus, $x[n] = \sum_{k=\langle N\rangle} a_k e^{jk\omega_0 n} = \sum_{k=\langle N\rangle} a_k e^{jk\left(\frac{2\pi}{N}\right)n}$

and

$$a_k = \frac{1}{N}\sum_{n=\langle N\rangle} x[n] e^{-jk\omega_0 n} = \frac{1}{N}\sum_{n=\langle N\rangle} x[n] e^{-jk\frac{2\pi}{N}\cdot n}$$

$$a_1 = \frac{1}{N}\sum_{\langle N\rangle} x[n] e^{-j\omega_0 n} = \Rightarrow e^{-j\frac{2\pi}{N}\cdot n}\left[\frac{1}{2j}e^{j\frac{2\pi}{N}\cdot n} - \frac{1}{2j}e^{-j\frac{2\pi}{N}n}\right]$$

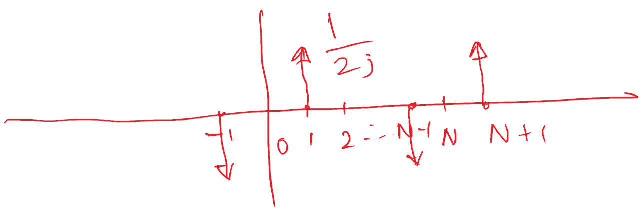
Ex. of discrete sinusoid: $x[n] = \sin(\omega_0 n)$, $\omega_0 = \frac{2\pi}{N}$

$x[n] = \sum_{k=\langle N\rangle} a_k e^{j\frac{2\pi}{N}\cdot n\cdot k}$ by identifying the terms

or, $x[n] = \frac{1}{2j} e^{j\frac{2\pi}{N}\cdot n} - \frac{1}{2j} e^{-j\frac{2\pi}{N}\cdot n}$

$\therefore\ a_1 = \frac{1}{2j}$ and $a_{-1} = -\frac{1}{2j}$

Also, $a_{N+1} = \frac{1}{2j}$ and $a_{N-1} = -\frac{1}{2j}$



$\omega_0 = \frac{2\pi M}{N}$, $M = 3$ and $N = 5$

$$x[n] = \frac{1}{2j} e^{jM\left(\frac{2\pi}{N}\right)n} - \frac{1}{2j} e^{-jM\left(\frac{2\pi}{N}\right)n}$$

$x[n] = \sum a_k e^{jk\omega_0 n}$

$\Downarrow$ $a_M = \frac{1}{2j}$ $\qquad$ $\Downarrow$ $a_{-M} = -\frac{1}{2j}$

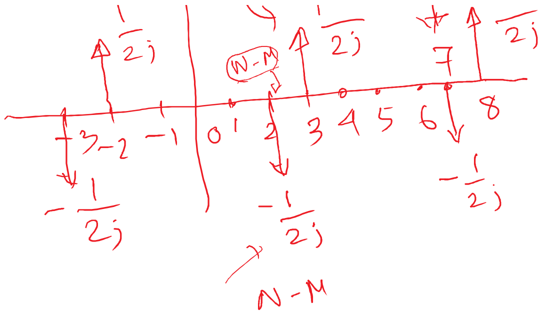
also at $N+M$ and $N-M$

$\Downarrow$ $5+3=8$ $\qquad$ $\Downarrow$ $5-3 = 2$

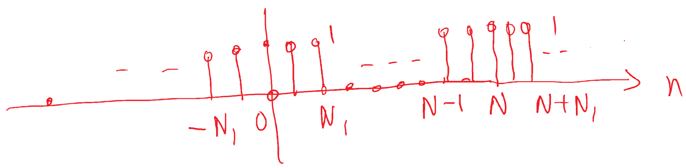
For $M = 3$, $N = 5$

$x[n] = \sin 3\left(\frac{2\pi}{5}\right)n$

$2N-M \Rightarrow 10-3 = 7$



Ex.



$x[n] = 1$ for $-N_1 \leq n \leq N_1$

$$a_k = \frac{1}{N} \sum_{-N/2}^{N/2} e^{-jk\frac{2\pi}{N}\cdot n}$$

$$= \frac{1}{N} \sum_{-N_1}^{N_1} e^{-jk\frac{2\pi}{N}\cdot n}$$

Let $m = N_1 + n$.

Then $$a_k = \frac{1}{N} \sum_{m=0}^{2N_1} e^{-jk\frac{2\pi}{N}(m-N_1)}$$

$$= \frac{1}{N} e^{jk\frac{2\pi}{N}\cdot N_1} \sum_{m=0}^{2N_1} e^{-jk\left(\frac{2\pi}{N}\right)m}$$

Applying finite summation rule,

$$a_k = \frac{1}{N} e^{jk\frac{2\pi}{N}\cdot N_1} \left(\frac{1 - e^{-jk2\pi\left((2N_1+1)/N\right)}}{1 - e^{-jk\left(\frac{2\pi}{N}\right)}} \right)$$

$$= \frac{1}{N} \frac{\sin\left(2\pi k\left(N_1 + \frac{1}{2}\right)/N\right)}{\sin\left(\frac{\pi k}{N}\right)}, \quad k \neq 0, \pm N, \pm 2N \cdots$$

$$a_k = \frac{2N_1+1}{N}, \quad k = 0, \pm N, \pm 2N \cdots$$

Properties of Discrete-time Fourier Series

Thursday, September 30, 2021 10:44 AM

→ properties are similar to the properties of continuous time Fourier Series, the difference is that they need to be evaluated and interpreted over one period, say N points.

Linearity $\quad A\,x[n] + B\,y[n] \xleftrightarrow{FS} A a_k + B b_k$

Time shift $\quad x[n-n_0] \xleftrightarrow{FS} a_k e^{-jk\left(\frac{2\pi}{N}\right)\cdot n_0}$ $\quad \left[\{1,0,-1\} \to 0,1,0,-1,0 \right.$ linear shift; $\{-1,1,0\}$ circular shift

Frequency shift (modulation) $\quad e^{jM\frac{2\pi}{N}\cdot n}\, x[n] \xleftrightarrow{FS} a_{k-M}$

Convolution (periodic or circular) $\quad \sum_{r=\langle N\rangle} x[r]\, y[n-r] \longleftrightarrow N a_k b_k$, both x & y have period N

$$\phi[n] = \sum_{k=0}^{N-1} x[k]\, y[n-k]$$, can also be defined from $-N/2$ to $N/2$

$N=4$

$x[n] = \{1, 2, 0, -1\}$ and $y[n] = \{1, 3, -1, -2\}$

$N=4$

Then $\phi_L[n]$ (as linear convolution) $= \{1, 5, 5, -5, -7, 1, 2\}$

$\{-7, 1, 2\}$

Periodic or circular convolution $\phi[n] = \{-6, 6, 7, -5\}$

Multiplication: $\quad x[n]\,y[n] \xleftrightarrow{FS} \sum_{l=\langle N\rangle} a_l b_{k-l}$ [periodic convolution and period of x & y is N]

Difference: $\quad x[n] - x[n-1] \xleftrightarrow{FS} \left(1 - e^{-jk\frac{2\pi}{N}}\right) a_k$

Running sum: $\quad \sum_{k=-\infty}^{n} x[k] \xleftrightarrow{FS} \frac{1}{1 - e^{-jk\frac{2\pi}{N}}} \cdot a_k$ [Assumed $a_0 = 0$]

$n \leftarrow$ current point

a_k ↓ FS of $x[n]$

N ↑ period of $x[n]$

Periodic $x[n]$	1	2	0	−1
$x[n-1]$	−1	1	2	0

$x[n-2]$	0	-1	1	2
$x[n-3]$	2	0	-1	1
$y[0]\,x[n] =$	1	2	0	-1
$y[1]\,x[n-1] =$	-3	3	6	0
$y[2]\,x[n-2] =$	0	1	-1	-2
$y[3]\,x[n-3] =$	-4	0	2	-2
	-6	6	7	-5

Parseval's Relation:

$$\frac{1}{N}\sum_{n=\langle N\rangle} |x[n]|^2 = \sum_{k=\langle N\rangle} |a_k|^2$$

Ex. $x[n]$ and $y[n]$ is periodic with period N

$$w[n] = \sum_{r=\langle N\rangle} x[r]\,y[n-r] \quad \text{is also periodic}$$

with period N

Given Fourier series coefficient of $w[n]$ is

$$d_k = \frac{\sin(3\pi k/7)}{7\sin(\pi k/7)} \qquad c_k = \frac{\sin^2(3\pi k/7)}{7\sin^2(\pi k/7)}$$

$\leftarrow$ multiplication of FS of a square wave and $c_k = 7 d_k \cdot d_k$

Fourier series coefficients of square wave $x[n]$, with $N_1 = 1$ and $N = 7$

Using periodic convolution property

$$w[n] = \sum_{r=\langle 7\rangle} x[r]\,x[n-r] = \sum_{r=-3}^{3} x[r]\,x[n-r]$$

$\mathfrak{p} = \langle 7 \rangle$ $r := -5$

Fourier Series and LTI system

Friday, October 01, 2021 8:46 AM

→ It is observed that for an input of a combination of complex exponentials, the output from an LTI system remain a combination of same set of ~~the~~ complex exponential that is weighted by a scale factor.

In other words, if input $x(t) = e^{st}$ (s is complex) to an LTI system, output $y(t) = H(s)e^{st}$, where $H(s) = \int_{-\infty}^{+\infty} h(\tau)e^{-s\tau}d\tau$

$h(t)$ is the impulse response.

In discrete time, if $x[n] = z^n$ (where z is complex of the form $re^{j\omega}$), $y[n] = H(z)z^n$. where

$$H(z) = \sum_{k=-\infty}^{+\infty} h[k]z^{-k},$$

$h[n]$ is the discrete time impulse function.

In general, s can be a complex number, e.g. $\sigma + j\omega$.

Now if s is taken as $j\omega$ making $\sigma = 0$, then the input $e^{j\omega t}$ is a complex exponential and $H(s)$ can be written in that case $H(j\omega) = \int_{-\infty}^{+\infty} h(t)e^{-j\omega t}dt$. ⇐ This is known as the frequency response of the system, the output being $y(t) = H(j\omega)e^{j\omega t}$

For discrete time system, $H(e^{j\omega}) = \sum_{k=-\infty}^{+\infty} h[n]e^{-j\omega n}$

↑ putting $r=1$

Then the corresponding output $y[n] = H(e^{j\omega})e^{j\omega n}$

From the Fourier series expansion,

$+\infty$ $(jk\omega_0)$ $e^{\underline{jk\omega_0 t}}$ $x(k)$

From the Fourier series expansion,

$$y(t) = \sum_{k=-\infty}^{+\infty} a_k \underbrace{H(e^{jk\omega_0})}_{\text{frequency response}} e^{jk\omega_0 t} \Rightarrow H(e^{jk\omega_0}) \cdot x(t) \Rightarrow h(t) * x(t)$$

for Complex exponentials

where $x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$

Ex.

LTI system with $h[n] = \alpha^n u[n]$, $-1 < \alpha < 1$

and with input $x[n] = \cos\left(\frac{2\pi n}{N}\right)$

or, $x[n] = \frac{1}{2} e^{j\frac{2\pi}{N} \cdot n} + \frac{1}{2} e^{-j\frac{2\pi}{N} \cdot n}$

Now, $H(e^{j\omega}) = \sum_{n=0}^{\infty} \alpha^n e^{-j\omega n} = \sum_{n=0}^{\infty} (\alpha e^{-j\omega})^n$

$$= \frac{1}{1-\alpha e^{-j\omega}} \quad \text{[infinite sum rule]}$$

Then, $y[n] = \frac{1}{2} H\left(e^{j\frac{2\pi}{N}}\right) e^{j\frac{2\pi}{N} \cdot n} + \frac{1}{2} H\left(e^{-j\frac{2\pi}{N}}\right) e^{-j\frac{2\pi}{N} \cdot n}$

$$= \frac{1}{2}\left(\frac{1}{1-\alpha e^{-j\frac{2\pi}{N}}}\right) \cdot e^{j\frac{2\pi}{N} \cdot n} + \frac{1}{2}\left(\frac{1}{1-\alpha e^{j\frac{2\pi}{N}}}\right) \cdot e^{-j\frac{2\pi}{N} \cdot n}$$

If $\frac{1}{1-\alpha e^{-j\frac{2\pi}{N}}} = r e^{j\theta}$

$$y[n] = r \cos\left(\frac{2\pi}{N} \cdot n + \theta\right)$$

→ This concept of frequency response can be applied

→ This concept of frequency response can be applied to design LTI systems for pruning the frequency content of a signal.

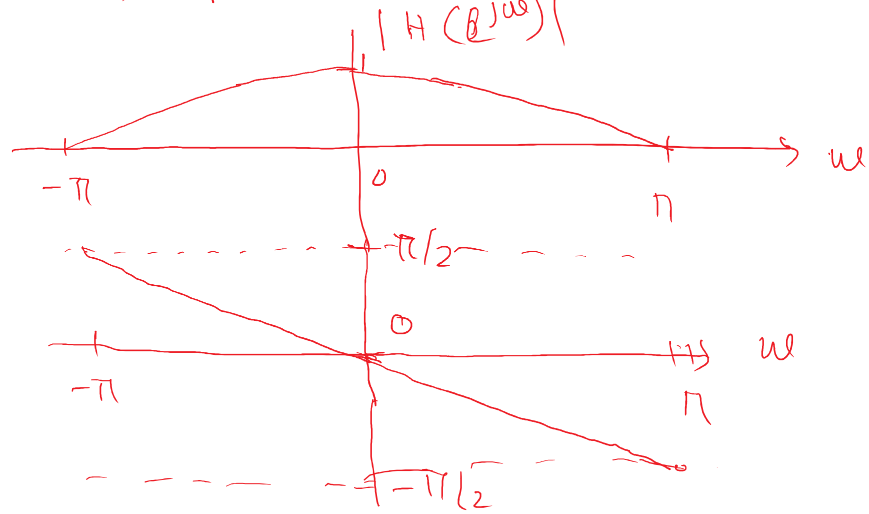
→ Generally, filters can be described by differential equations or difference equation in time.

For example, $y[n] = \frac{1}{2}\{x[n] + x[n-1]\}$ where

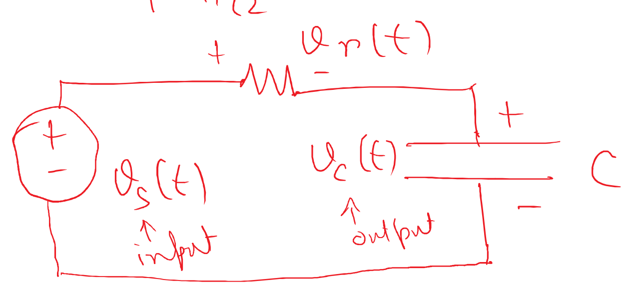
$$h[n] = \frac{1}{2}\{\delta[n] + \delta[n-1]\}$$

$$\therefore \; H(e^{j\omega}) = \frac{1}{2}\left[1 + e^{-j\omega}\right] = e^{-j\omega/2}\cos\left(\frac{\omega}{2}\right)$$

→ This is an averaging filter, which passes lower half of frequency component with graded weighting factor



Analog



$$RC\frac{dv_c(t)}{dt} + v_c(t) = v_s(t)$$

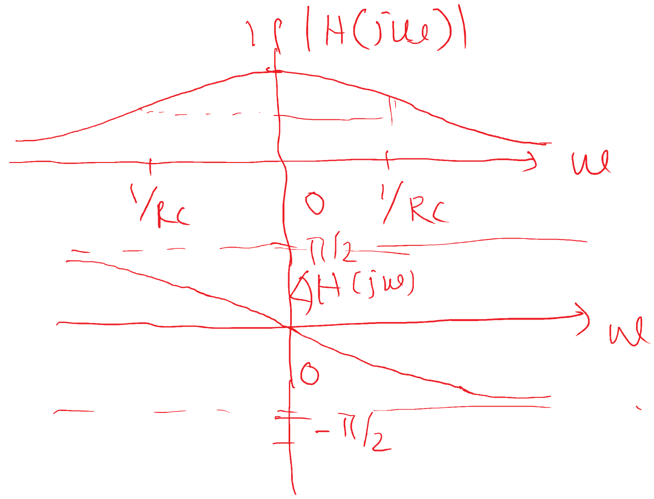
If $x(t) = v_s(t) = e^{j\omega t}$

$$RC\frac{d}{dt}\left\{H(j\omega)e^{j\omega t}\right\} + H(j\omega)e^{j\omega t} = e^{j\omega t}$$

$$\therefore H(j\omega) = \frac{1}{\;\;\;\;\;\;\;\;\;\;} \text{ and } h(t) = \frac{1}{RC}e^{-\frac{t}{RC}}u(t)$$

$RC \frac{d}{dt}\{ \quad \}$

$$\therefore \; H(j\omega) = \frac{1}{1 + RC \cdot j\omega} \text{ and } h(t) = \frac{1}{RC} e^{-\frac{t}{RC}} u(t)$$



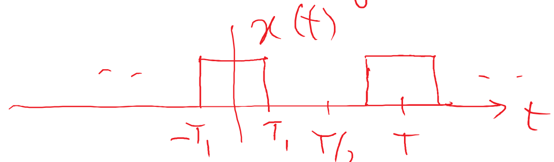
Fourier Transform

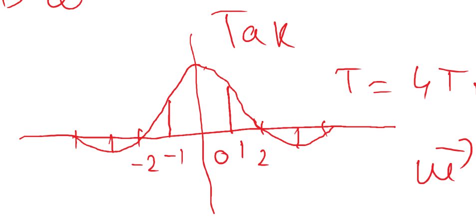
Friday, October 01, 2021 10:02 AM

→ We have both continuous time and discrete time Fourier Transforms just like Fourier series.

Fourier Series expansion as $x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$, $\omega_0 \to \frac{2\pi}{T}$

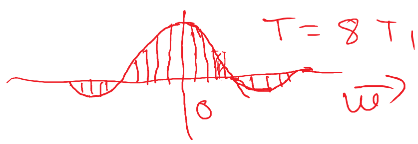
→ Fourier transform is ~~inp~~ interpreted as Fourier series for the limiting value $T \to \infty$



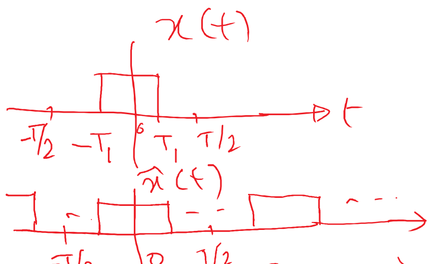


$$a_k = \frac{2\sin(k\omega_0 T_1)}{k\omega_0 T} \quad \text{or} \quad Ta_k = \frac{2\sin(k\omega_0 T_1)}{k\omega_0} = \left.\frac{2\sin(\omega T_1)}{\omega}\right|_{\text{at } \omega = k\omega_0}$$

→ Varying T in terms T_1 as NT_1 shows that as N increases the number of points on Ta_k increases.



$$\hat{x}(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} \quad \text{and}$$



$$a_k = \frac{1}{T}\int_{-T/2}^{+T/2} \hat{x}(t) e^{-jk\omega_0 t}\, dt, \quad \omega_0 = \frac{2\pi}{T}$$

For a single pulse $x(t)$, $\hat{x}(t) = x(t)$ for $|t| < \frac{T}{2}$

As $T \to \infty$, $\hat{x}(t) = x(t)$ and a_k can be evaluated from $-\infty$ to $+\infty$, a_k can be rewritten as, $a_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t) e^{-jk\omega_0 t}\, dt$

$$= \frac{1}{T}\int_{-\infty}^{+\infty} x(t) e^{-jk\omega_0 t}\, dt$$

If we define $X(j\omega)$ as $X(j\omega) = \int_{-\infty}^{+\infty}$ ... $e^{-j\omega t}$ ...

If we define $X(j\omega)$ as $X(j\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt$ as an envelope of Ta_k, Then $a_k = \frac{1}{T} X(jk\omega_0)$

and $\hat{x}(t) = \sum_{k=-\infty}^{+\infty} \frac{1}{T} X(jk\omega_0)\, e^{jk\omega_0 t}$

$$= \frac{1}{2\pi} \sum_{k=-\infty}^{+\infty} X(jk\omega_0)\, e^{jk\omega_0 t} \cdot \omega_0$$

$\left[\omega_0 = \frac{2\pi}{T}\right]$

As T increase and $\to \infty$, $\omega_0 \to 0$. In this limiting condition the summation can be replaced by integration. $\omega_0 \to d\omega$

Inverse Transform $x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega)\, e^{j\omega t}\, d\omega$, $k\omega_0 \Rightarrow \omega$

and conversely,

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt$$

Forward Transform

Convergence of Fourier Transform

If $x(t)$ has finite energy i.e. if it is square integrable, so that $\int_{-\infty}^{+\infty} |x(t)|^2 dt < \infty$, then it is guaranteed that $X(j\omega)$ is finite.

→ other Dirichlet conditions on finite number of extrema and finite number of discontinuities (finite value) also need to be satisfied.