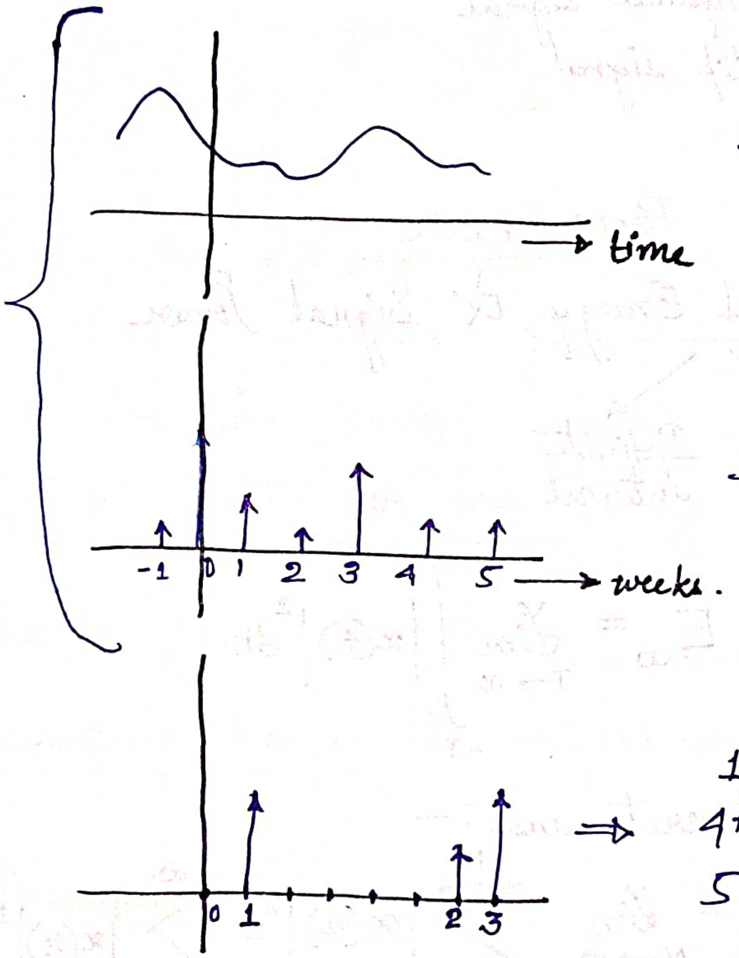
SIGNALS & SYSTEM

Signal $x(t)$ $\longrightarrow$ $\boxed{\text{System}}$ $\xrightarrow{y(t)}$ Signal out



Continuous

Digital (regular intervals)

$\Rightarrow$ 1st week $\rightarrow$ 20 mm
4th week $\rightarrow$ 10 mm
5th week $\rightarrow$ 20 mm.

Digital (irregular intervals)

Discrete time $\longrightarrow$ Independent variable time is discretized, but the signal value is not discretized.

SIGNAL TYPES

→ Periodic signal [sinusoidal]
→ Aperiodic signal [exponential]
→ Impulse signal
→ Step signal

PROPERTIES

Signal Energy & Signal Power.

Signal Energy → Finite interval, Infinite interval

$$\text{Energy } E_\infty = \lim_{T \to \infty} \int_{-T}^{T} |x(t)|^2 dt$$

For the discrete case:-

$$E_\infty = \lim_{N \to \infty} \sum_{n=-N}^{N} |x(n)|^2 = \sum_{n=-\infty}^{+\infty} |x(n)|^2$$

> If this summation / integration converges to a ~~final value~~ finite ~~energy~~ number, it is called Energy Signal

Power of a signal, $P_\infty = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{+T} |x(t)|^2 dt$

and for the discrete case:

$$P_\infty = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x(n)|^2$$

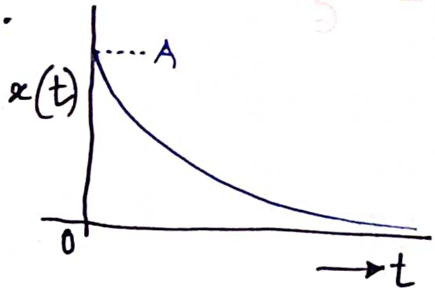
From the definition of power and energy of a signal, we can infer that —

For an energy signal $P_\infty = 0$, because

- E_∞ is finite for an energy signal
- For a "power signal" (P_∞ is finite), $E_\infty = \infty$, since essentially E_∞ is the integration of P_∞ over $-\infty$ to $+\infty$

Examples:—

1.



$x(t) = A\exp(-t)$

$$E_\infty = \int_{-\infty}^{+\infty} A^2 \exp(-2t)\,dt = \int_{0}^{\infty} A^2 \exp(-2t)\,dt = \frac{A^2}{2}$$

$$P_\infty = \lim_{T\to\infty}\left(\frac{1}{2T}\int_{-T}^{T} A^2 \exp(-2t)\,dt\right) = \lim_{T\to\infty} \frac{A^2}{4T} = 0$$

$\therefore$ The above signal is an energy signal

2. Take a sinusoidal signal

$$x(t) = A\sin(\omega_0 t + \phi)$$

This is periodic with period $\frac{2\pi}{\omega_0}$

$$P_\infty = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{+T} A^2 \sin^2(\omega_0 t + \phi)\, dt$$

$$= A^2 \lim_{T\to\infty} \frac{\omega_0}{4\pi}\int_{-\frac{2\pi}{\omega_0}}^{\frac{2\pi}{\omega_0}} \left[\frac{1}{2} - \frac{1}{2}\cos(2\omega_0 t + 2\phi)\right] dt$$

$$= \frac{A^2}{2}$$

[Can be done without substitution]

3. Discrete case

$$x(n) = \begin{cases} \frac{1}{n}, & n \geq 1 \\ 0, & n < 1 \end{cases}$$

$$\text{Energy} = \sum_{n=1}^{\infty} \left(\frac{1}{n}\right)^2 = \frac{\pi^2}{6}$$

4. $$x(n) = \begin{cases} 3(-1)^n, & n \geq 0 \\ 0, & n < 0 \end{cases}$$

$$E_\infty = \sum_{n=-\infty}^{+\infty} |x(n)|^2 = \sum_{n=0}^{\infty} 9 = \infty$$

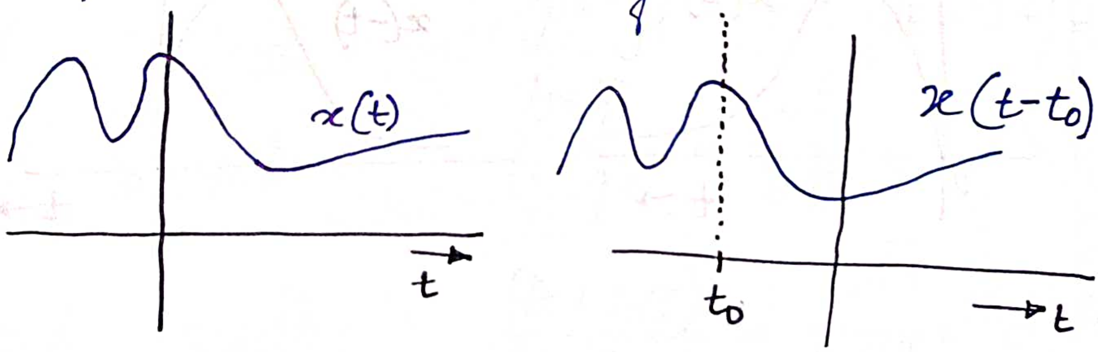
$$P_\infty = \lim_{N\to\infty} \frac{1}{2N+1}\left(9\sum_{n=0}^{N} 1\right) = \lim_{N\to\infty} \frac{9(N+1)}{2N+1} = 4.5$$

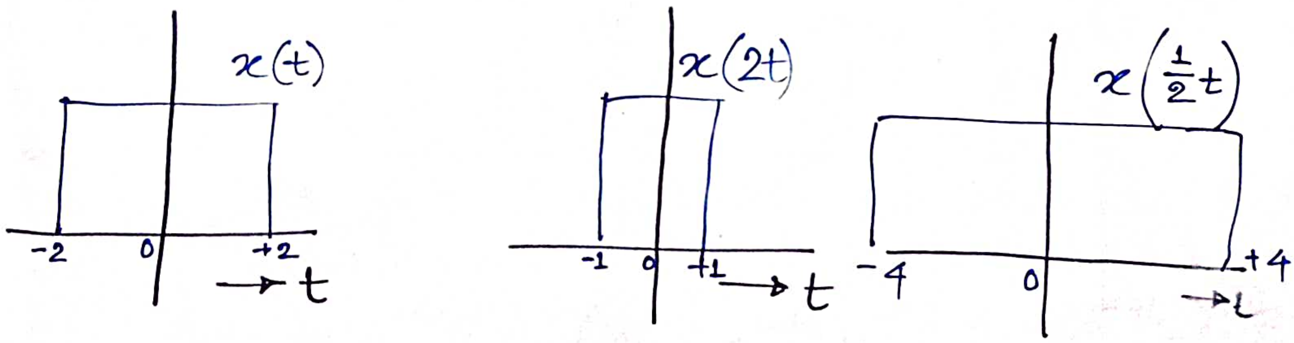
TRANSFORMATION OF INDEPENDENT VARIABLE

→ In one dimensional independent variable (like time), either by scaling or by shifting

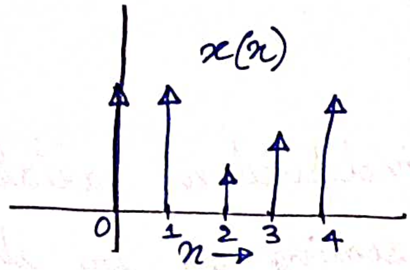
- For scaling, signals can be observed at a denser interval or a larger (rarer) interval by scaling time with a scale factor at α either $\alpha < 1$ or $\alpha > 1$

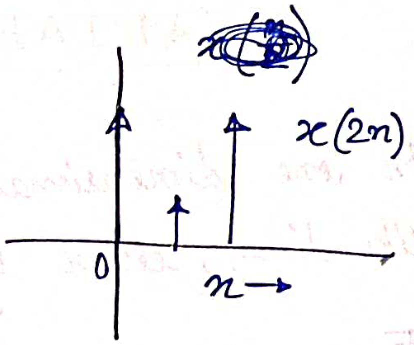
- Similarly the signal can be shifted by a positive value or a negative value.

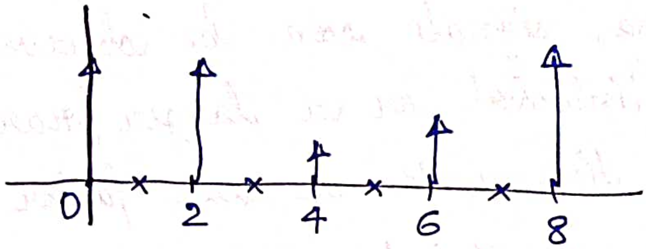




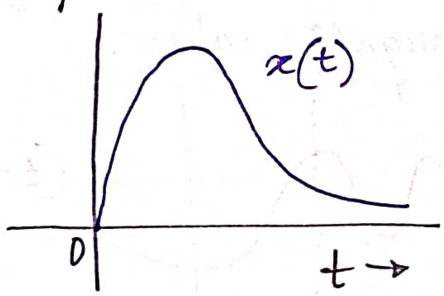
Discrete case :—

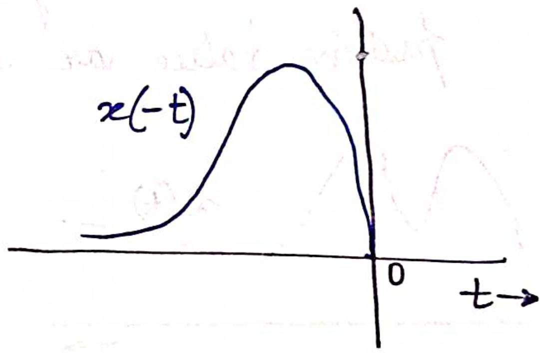






Reflection :—





Chapter 1 Problems :—

1.1) $\frac{1}{2}e^{j\pi} = -\frac{1}{2}$

$\frac{1}{2}e^{-j\pi} = -\frac{1}{2}$

$e^{j\frac{\pi}{2}} = j$

$e^{-j\frac{\pi}{2}} = -j$

$e^{j\frac{5\pi}{2}} = j$

$\sqrt{2}e^{j\frac{\pi}{4}} = 1+j$

$\sqrt{2}e^{j\frac{9\pi}{4}} = 1+j$

$\sqrt{2}e^{-j\frac{9\pi}{4}} = 1-j$

$\sqrt{2}e^{-j\frac{\pi}{4}} = 1-j$

1.2) $5 = 5e^{j0}$

$-2 = -2e^{j0} = 2e^{j\pi}$

$-3j = 3e^{-j\frac{\pi}{2}}$

$\frac{1}{2} - j\frac{\sqrt{3}}{2} = e^{-j\frac{\pi}{3}}$

$1+j = \sqrt{2}e^{j\frac{\pi}{4}}$

$(1-j)^2 = \left(\sqrt{2}e^{-j\frac{\pi}{4}}\right)^2 = 2e^{-j\frac{\pi}{2}}$

$j(1-j) = e^{j\frac{\pi}{2}}\left(\sqrt{2}e^{-j\frac{\pi}{4}}\right) = \sqrt{2}e^{j\frac{\pi}{4}}$

$\frac{(1+j)}{(1-j)} = \frac{e^{j\frac{\pi}{4}}}{e^{-j\frac{\pi}{4}}} = e^{j\frac{\pi}{2}}$

$\frac{(\sqrt{2}+j\sqrt{2})}{(1+j\sqrt{3})} = \frac{2e^{j\frac{\pi}{4}}}{2e^{j\frac{\pi}{3}}} = e^{-j\frac{\pi}{12}}$

1.3)

(a) $x_1(t) = e^{-2t}u(t)$

$$E_\infty = \lim_{t\to\infty}\int_{-t}^{t}|x(t)|^2dt = \lim_{t\to\infty}\int_{0}^{t}e^{-4t}dt = \lim_{t\to\infty}\frac{e^{-4t}-1}{-4}$$

$$= \frac{1}{4} \text{ (Ans)}$$

$P_\infty = 0$ [Energy Signal]

(b) $x_2(t) = e^{j\left(2t+\frac{\pi}{4}\right)}$

$$P_\infty = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} |x(t)|^2 dt = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} 1\, dt$$

$$= \lim_{T\to\infty} \frac{2T}{2T} = 1 \quad \left(\underline{\text{Ans}}\right)$$

$\Rightarrow E_\infty = \infty$ [Power signal]

(c) $x_3(t) = \cos(t)\, dt$

$$P_\infty = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} |x(t)|^2 dt$$

$$= \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \cos^2 t\, dt$$

$$= \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \frac{\cos 2t + 1}{2} dt$$

$$= \lim_{T\to\infty} \frac{1}{4\pi}\int_{-2\pi}^{2\pi} \left(\frac{1}{2}\cos 2t + \frac{1}{2}\right) dt$$

$$= \frac{1}{4\pi}\left[\frac{\sin 2t}{4} + \frac{t}{2}\right]_{-2\pi}^{2\pi}$$

$$= \frac{1}{4\pi} \cdot 2\pi = \frac{1}{2} \cdot$$ [Power signal]

$\Rightarrow E_\infty = \infty$

(d) $x_1[n] = \left(\frac{1}{2}\right)^n u[n]$

$$E_\infty = \lim_{N\to\infty} \cancel{\frac{1}{2N+1}} \sum_{n=-N}^{N} |x_1[n]|^2$$

$$= \lim_{N\to\infty} \cancel{\frac{1}{2N+1}} \sum_{n=0}^{N} \left|\left(\frac{1}{2}\right)^n\right|^2$$

$$= \lim_{N\to\infty} \cancel{\frac{1}{2N+1}} \sum_{n=0}^{N} \left(\frac{1}{2}\right)^{2n}$$

$$= \lim_{N\to\infty} \cancel{\frac{1}{2N+1}} \cdot \left(\frac{1-\left(\frac{1}{4}\right)^{N+1}}{1-\frac{1}{4}}\right)$$

$$= \lim_{N\to\infty} \frac{4}{3}\left(1-\left(\frac{1}{4}\right)^{N+1}\right) = \frac{4}{3} \quad \text{[Energy Signal]}$$

$P_\infty = 0$

(e) $x_2[n] = e^{j\left(\frac{\pi}{2n}+\frac{\pi}{8}\right)}$

$$P_\infty = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x_2[n]|^2 = \lim_{N\to\infty} \frac{\sum_{n=-N}^{N} 1}{2N+1} = 1$$

$E_\infty = \infty$ [~~Energy~~ Power Signal]

(f) $x_3[n] = \cos\left[\frac{\pi}{4}n\right]$

$$P_\infty = \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} \left|\cos\left(\frac{\pi}{4} n\right)\right|^2$$

$$= \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} \frac{1}{2} = \frac{1}{2} \text{ (Ans)}$$

$E_\infty = \infty$ (Power Signal)

1.4) $n < -2, \; n > 4, \Rightarrow x[n] = 0$

(a) $x[n-3] \longrightarrow n < 1, \; n > 7$

(b) $x[n+4] \longrightarrow n < -6, \; n > 0$

(c) $x[-n] \longrightarrow n < -4, \; n > 2$

(d) $x[-n+2] \rightarrow n < -2, \; n > 4$

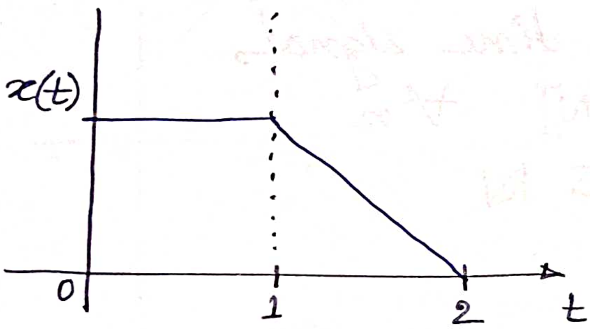
(e) $x[-n-2] \rightarrow n < -6, \; n > 0$

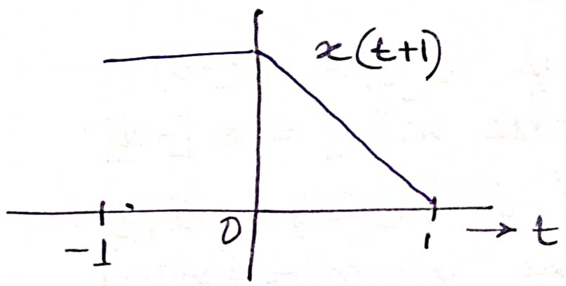
Combination of shift and time scaling

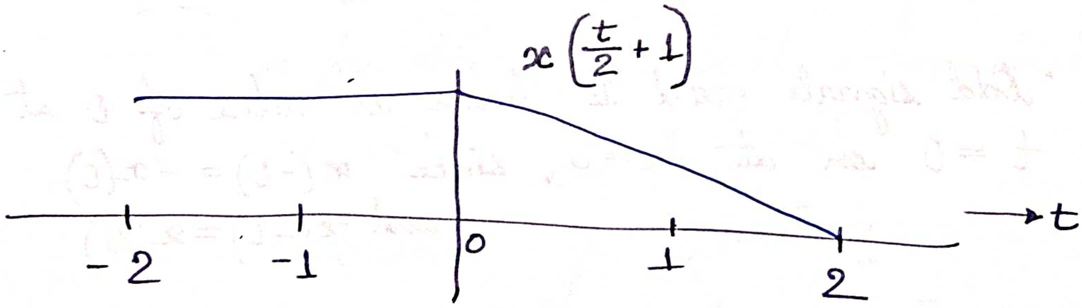
$$x(t) \Rightarrow x(\alpha t+\beta) \rightarrow x\left(\tfrac{t}{2}+1\right)$$

In this case, we first shift by 1 and then ~~compress~~ expand t by 2, since the shift is w.r.t. $x(t)$ and not $x(t/2)$.

To make it w.r.t. $x(t/2)$, shift needs to be multiplied by 2, or in general β scaled as $\frac{\beta}{\alpha}$







PERIODIC SIGNAL

A periodic signal if
$$x(t) = x(t+T) \quad \forall t,$$
for a fixed value of T.

The period is T and its integer multiples 2T, 3T also satisfy the condition of periodicity.

For a discrete time signal,
$$x[n] = x[n+N] \quad \forall n$$
for some $N \in \mathbb{N}$

EVEN & ODD SIGNAL

* A signal is even if $x(t) = x(-t)$ and in discrete case $x[n] = x[-n]$

* A signal is odd if $x(-t) = -x(t)$ and for discrete time $x[-n] = -x[n]$

- Odd signals need to have a value of 0 at $t=0$ or at $n=0$, since $x(-0) = -x(0)$ and $x(-0) = x(0)$

- A signal can be decomposed into even and odd component

$$\text{Even}[x(t)] = \frac{1}{2}[x(t) + x(-t)] \, ; \; \text{Odd}[x(t)] = \frac{1}{2}[x(t) - x(-t)]$$

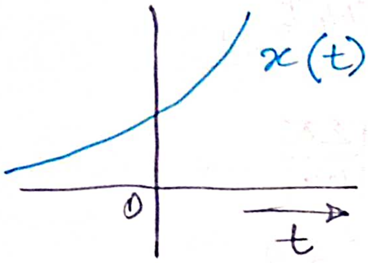
Similarly,

$$\text{Even}\left[x[n]\right] = \frac{1}{2}\left[x[n] + x[-n]\right]$$

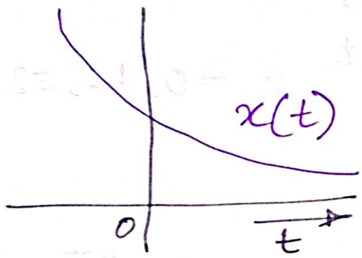
$$\text{Odd}\left[x[n]\right] = \frac{1}{2}\left[x[n] - x[-n]\right]$$

EXPONENTIAL & SINUSOIDAL SIGNALS

$x(t) = \cancel{ca}\ ce^{at}$ is an exponential signal where c and a are real/complex constants.



a, c are positive



a → negative, c → positive

Complex exponential

$$x(t) = e^{j\omega_0 t}$$

→ It is periodic unlike real exponential.

i.e. $e^{j\omega_0(t+T)} = e^{j\omega_0 t} \cdot e^{j\omega_0 T}$

This is periodic if $e^{j\omega_0 T} = 1$

[For $\omega_0 = 0$, $e^{j\omega_0 T} = 1$ for any T]

⇒ fundamental period is undefined.

If $\omega_0 \neq 0$, then $T_0 = \frac{2\pi}{|\omega_0|}$, T_0 = fundamental period

Power for one period

$$= \frac{1}{T_0}\int_0^{T_0} |e^{jw_0 t}|^2 dt$$

$$= \frac{1}{T_0}\int_0^{T_0} 1\, dt$$

$$= \frac{T_0}{T_0}$$

$$= 1.$$

Condition of periodicity $\rightarrow e^{jw_0 T_0} = 1$

$\therefore T_0$ is a multiple of $\left|\frac{2\pi}{\omega_0}\right|$

$\omega T_0 = 2\pi k, \quad k = 0, \pm 1, \pm 2$

In general, $\phi_k(t) = e^{jk\omega_0 t}, \quad k = 0, \pm 1, \pm 2$

for $k=0$, $\phi_k(t) = \text{constant}$

for $k \neq 0$, $\phi_k(t)$ is periodic with $T = \frac{2\pi}{|k\omega_0|}$

<u>General complex exponential</u>

$$c = |c|\, e^{j\theta}$$

and $a = r + j\omega_0$

$$c\, e^{at} = |c| e^{j\theta} e^{(r + j\omega_0)t}$$

$$= |c| e^{rt} e^{j(\omega_0 t + \theta)}$$

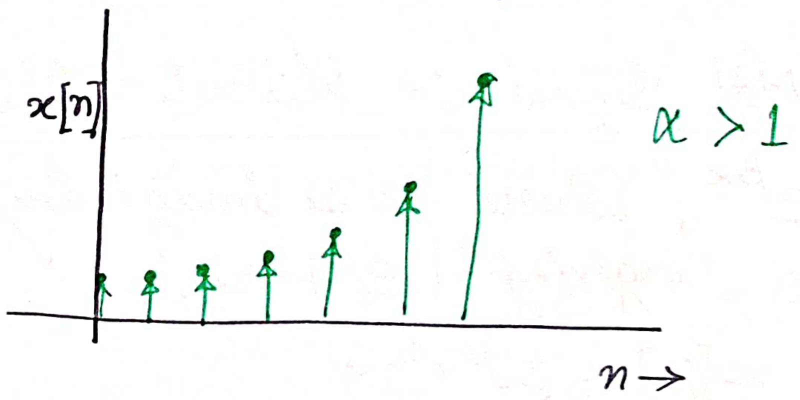
Discrete-Time Complex Exponential

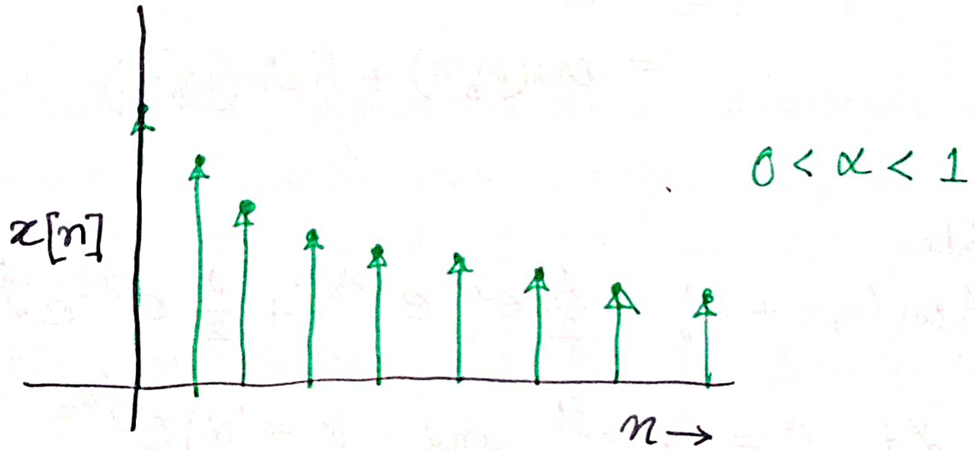
$$x[n] = c\alpha^n, \quad \text{where } c, \alpha \in \mathbb{C}$$

Alternatively, $x[n] = ce^{\beta n}$, $\alpha = e^{\beta}$.

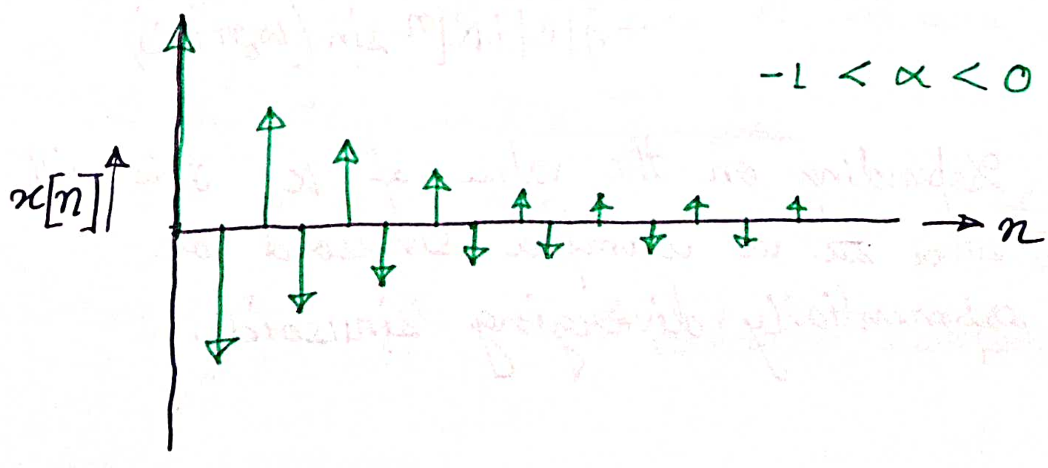
Behavior is similar to continuous one,

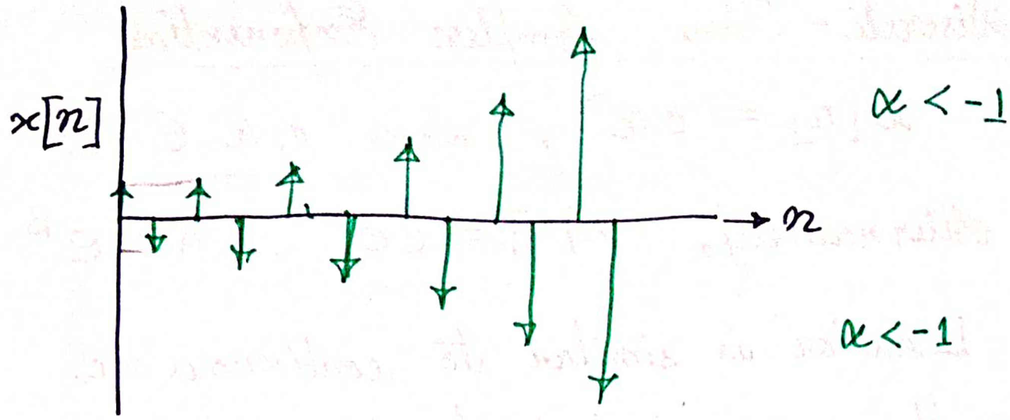
— if α is real & positive





— if α is negative, $x[n]$ oscillates.





Sinusoidal Signal in Discrete-Time

$c e^{\beta x}$, where β is imaginary giving $|e^{\beta}| = 1$.

Let $x[n] = e^{j\omega_0 n}$

$$= \cos(\omega_0 n) + j \sin(\omega_0 n)$$

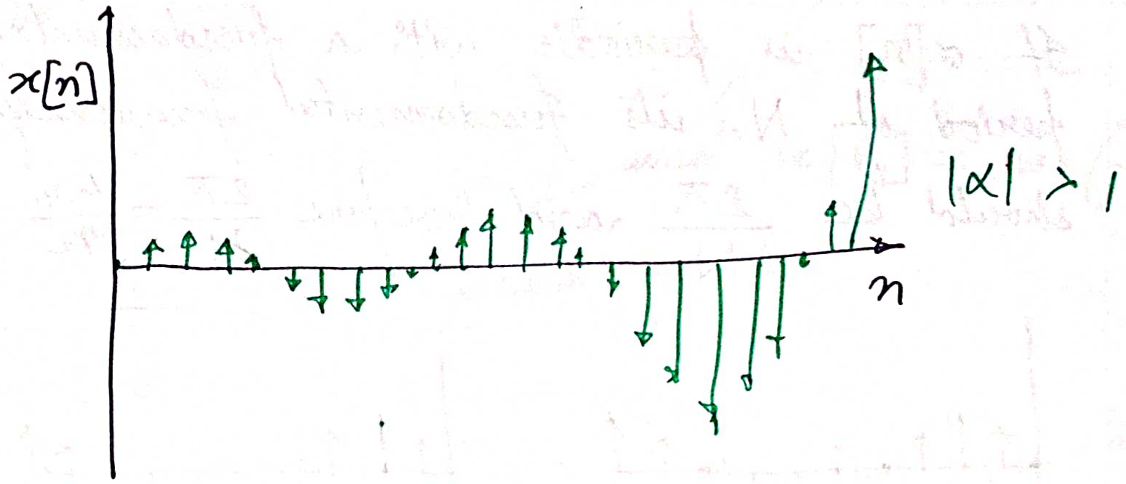
Take

$$A \cos(\omega_0 n + \phi) = \frac{A}{2} e^{j\phi} e^{j\omega_0 n} + \frac{A}{2} e^{-j\phi} e^{-j\omega_0 n}$$

Let $C = |c| e^{j\theta}$ and $\alpha = |\alpha| e^{j\omega_0}$

$$\Rightarrow x[n] = C\alpha^n = |c||\alpha|^n \cos(\omega_0 n + \theta) + j|c||\alpha|^n \sin(\omega_0 n + \theta)$$

Depending on the value of α, this will either be a damped sinusoid or exponentially diverging sinusoid.



Discrete - Time exponential periodicity

Take $x[n] = e^{j(\omega_0 + 2\pi)n}$

$$= e^{j2\pi n} e^{j\omega_0 n} = e^{j\omega_0 n}$$

$$\Rightarrow e^{j\pi n} = (-1)^n \;, \; e^{j2\pi n} = 1$$

Unlike the continuous time sinusoids, discrete time sinusoids are unique only in a principle interval of $0 \leq \omega_0 \leq 2\pi$ or $-\pi \leq \omega_0 \leq \pi$

Now, considering definition of periodicity,

$$e^{j\omega_0(N+n)} = e^{j\omega_0 n}, \text{ for periodic signal.}$$

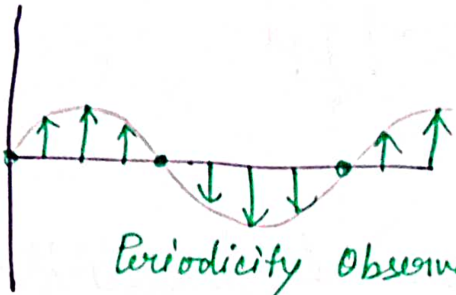
In this case $e^{j\omega_0 N} = 1$, or otherwise $\omega_0 = 0$

~~otherwise N is multiples of 2π~~

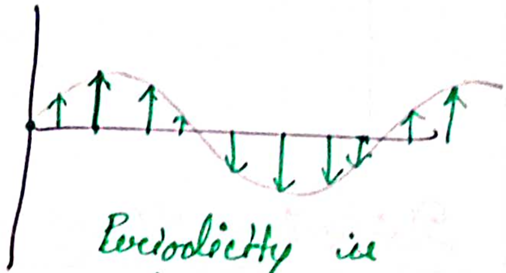
or, $\omega_0 N = 2\pi m$, m is an integer

or, $\frac{\omega_0}{2\pi} = \frac{m}{N} \Rightarrow$ A rational number.

If $x[n]$ is periodic with a fundamental period of N, its fundamental frequency should be $\frac{2\pi}{N}$ and therefore $\frac{2\pi}{N} = \frac{\omega_0}{m}$



Periodicity Observed



Periodicity is not observed.

Continuous-time	Discrete-time
• $e^{j\omega_0 t}$	• $e^{j\omega_0 n}$
• Distinct for distint ω_0	• Signal identical for values of ω_0 seperated by 2π
• Fundamental frequency ω_0	• Fundamental frequency $\frac{\omega_0}{m}$
• Periodic for a choice of ω_0	• Periodic only if $\omega_0 = \frac{2\pi m}{N}$
• Fundamental ~~frequency~~ period $\frac{2\pi}{\omega_0}$, $\omega_0 \neq 0$	• Fundamental periodic $m\frac{2\pi}{\omega_0}$, $\omega_0 \neq 0$.
• Period can be integer/real	• Period can only be integer.

Example :-

1. $x(t) = \cos\left(\frac{2\pi t}{12}\right)$ and $x[n] = \cos\left(\frac{2\pi n}{12}\right)$

Both have a period of 12.

~~$x[n]$ =~~

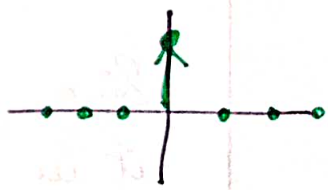
2. $x(t) = \cos\left(\frac{8\pi t}{31}\right)$ and $x[n] = \cos\left(\frac{8\pi n}{31}\right)$

$\Downarrow$ Period is $\frac{31}{4}$ $\qquad$ $\Downarrow$ Period is 31

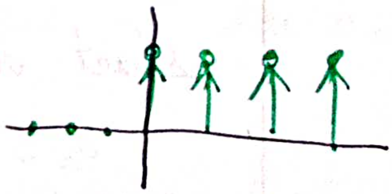
Unit Impulse and Step Function :-

A. Discrete - Time :-

$$\underset{\text{(Impulse)}}{\delta[n]} = \begin{cases} 0, & n \neq 0 \\ 1, & n = 0 \end{cases}$$



$$u[n] = \begin{cases} 0, & n < 0 \\ 1, & n \geq 0 \end{cases}$$



$$\delta[n] = u[n] - u[n-1]$$

$$u[n] = \sum_{m=-\infty}^{n} \delta[m]$$

$\delta[n]$ has the property of sampling or picking up a partictictcular n.

$$x[n]\,\delta[n] \longrightarrow x[0]\,\delta[0] = x[0]$$

$$x[n]\,\delta[n-n_0] \longrightarrow x[n_0]\,\delta[n_0-n_0] = x[n_0]$$

B: Continuous Domain

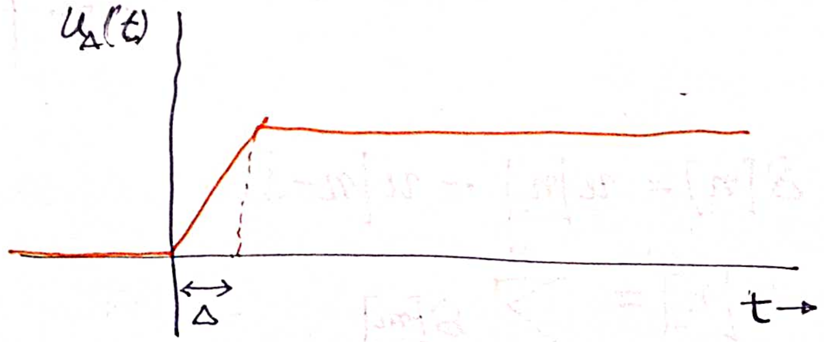
Step function:

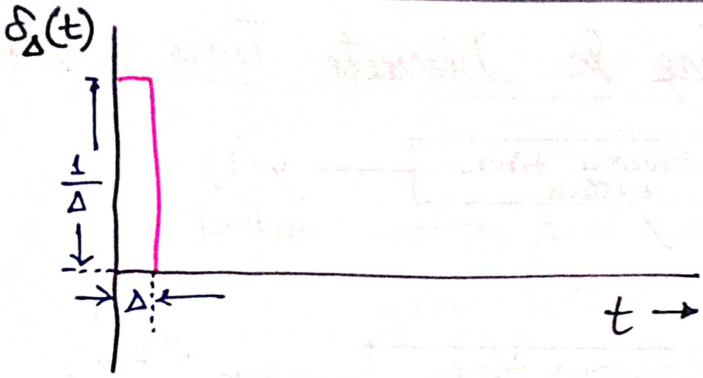
$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t > 0 \end{cases}$$

$$u(t) = \int_{-\infty}^{t} \delta(\tau)\, d\tau,$$ τ is also in time

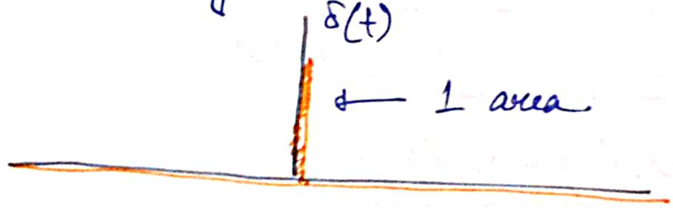
$$\delta(t) = \frac{d}{dt} u(t)$$ [This cannot be defined at $t=0$]

Since $u(t)$ is discontinuous at $t=0$, it is not differentiable at $t=0$. This can be done approximately over a short interval Δ, $\delta_\Delta(t) = \frac{d}{dt} u_\Delta(t)$





$\delta_\Delta(t)$ is a short pulse of duration Δ and unit area, $\delta(t)$, $\delta(t)$ can be defined in a limiting sense.



$$u(t) = \int_{-\infty}^{t} \delta(\tau)\, d\tau = \int_{-\infty}^{0} \delta(t-\sigma)\,(-d\sigma)$$

$$= \int_{0}^{\infty} \delta(t-\sigma)\, d\sigma$$

Similar to the discrete case it has a sampling property

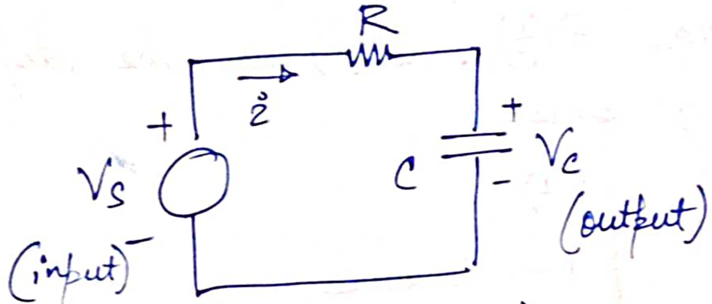
$x_1(t) = x(t)\,\delta_\Delta(t)$, assuming that $x(t)$ is constant over Δ duration.

As $\Delta \to 0$, $x(t)\cdot\delta(t) = x(0)\,\delta(t)$
and $x(t)\cdot\delta(t-t_0) = x(t_0)\,\delta(t-t_0)$

Continuous Time & Discrete Time System

$x(t)$ input $\longrightarrow$ continuous time system $\longrightarrow$ $y(t)$ output

$x[n]$ input $\longrightarrow$ continuous time system $\longrightarrow$ $y[n]$ output



$$i(t) = \frac{v_S(t) - v_C(t)}{R}$$

$$i(t) = C\frac{d}{dt} v_C(t)$$

$$\frac{d}{dt} v_C(t) + \frac{1}{RC} v_C(t) = \frac{1}{RC} v_S(t)$$

Discrete-time System

$$y[n] = \underbrace{1.01\,y[n-1] + x[n]}_{\text{Constant coefficient difference equation.}}$$

$$y[n] - 1.01\,y[n-1] = x[n]$$

$$\sum_{k} a_k\, y[n-k] = \sum_{k_1} b_{k_1}\, x[n-k_1]$$

$k, k_1 > 0$ gives past values

$k, k_1 < 0$ gives future values.

$$y[n] = x[n+1] - x[n-1]$$

$$y'[n] = 0.5\, x[n+1] - x[n-1]$$

System with and without memory

A system which requires storing / using past values (or both) ~~are a syste~~ is a system with memory.

$y[n] = 2x[n] - x^2[n]$ [System without memory]

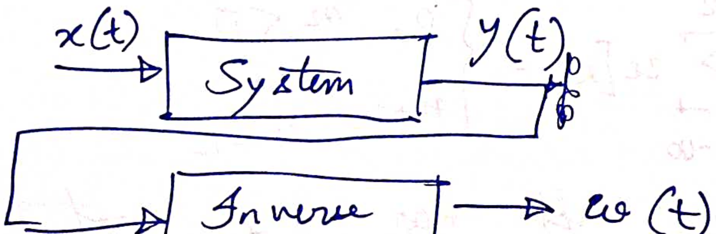
$y[n] = y[n-1] + x[n] \Rightarrow$ [with memory]

$y(t) = Rx(t) \rightarrow$ [System without memory]

Inverse System :-

$y(t) = 2x(t)$ (Original system)

$w(t) = \frac{1}{2} y(t) = x(t)$ [Inverse system]



Causality :-

A system is causal if the output at any time depends only on the values of present and past instances.

$$y[n] = \sum_{k=-\infty}^{n} x[n] \qquad \text{(Causal)}$$

$$y[n] = x[n] - x[n-1] \qquad \text{(Causal)}$$

$$y[n] = x[n+1] - x[n] \qquad \text{(Not Causal)}$$

$$y[n] = x[n]^2 \qquad \text{(Causal)}$$

Stability :-

If a system gives bounded output for bounded input the system is said to be stable. These are called Bounded Input Bounded Output (BIBO) stability.

For $|x(t)| < B$, $B < \infty$ for all t

or $-B < x(t) < B$.

If $|y(t)| < B_1$, $B_1 < \infty$ for all t

or, $-B_1 < y(t) < B_1$. Then the system is said to be stable.

$$y[n] = \sum_{k=-\infty}^{n} u[k] = \begin{cases} 0, & n < 0 \\ n+1, & n \geq 0 \end{cases}$$

As n can go upto $+\infty$, $y[n]$ is not bounded, hence not BIBO stable.

$$y[n] = x[n] - x[n-1]$$

$\Rightarrow$ If x is bounded, y is also bounded.

$\therefore$ This is BIBO stable.

Time Invariance :-

A system is time invariant if the behavior and characteristics of the system remain same over time.

$$x(t) \longrightarrow y(t)$$
$$x(t-t_0) \longrightarrow y(t-t_0)$$

Then the system is time invariant.

Given $y(t) = \sin(x(t))$.

Then $y_1(t) = \sin(x_1(t))$, for $x(t) = x_1(t)$

Let $x_2(t) = x_1(t-t_0)$

Then $y_2(t) = \sin(x_2(t)) = \sin(x_1(t-t_0))$

Again, $y_1(t-t_0) = \sin(x_1(t-t_0))$

$\Rightarrow$ The relation holds good irrespective of shift.
Therefore this is time invariant.

$$y(t) = x(2t)$$

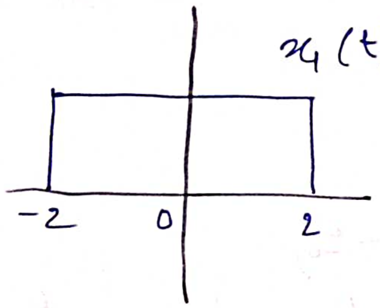
Let $x(t) = x_1(t)$

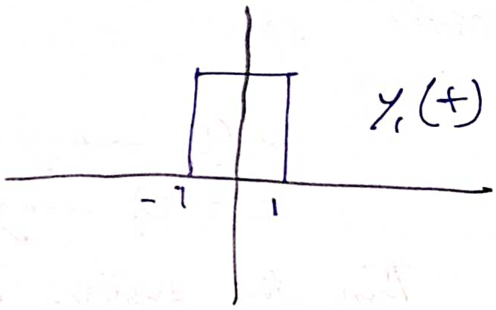
Then $y_1(t) = x_1(2t)$

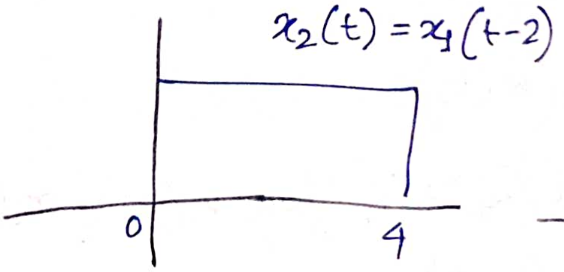
$$\cancel{x(t) =} x_2(t) = x_1(t - t_0)$$

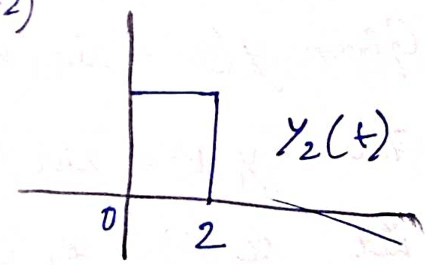
$$y_2(t) = x_2(2t) = x_1(2(t - t_0))$$

$$y_2'(t) = y_2(t - t_0) = x(2t - t_0)$$

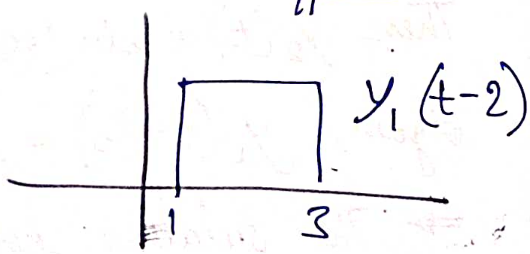








$\neq$



∴ The system is time variant.

Example :-

$$y[n] = \sum_{k=-\infty}^{n} x[k] \quad , \quad \text{Let } x_1[n] = x[n-n_0]$$

Let $$y_1[n] = \sum_{k=-\infty}^{n} x_1[k] = \sum_{k=-\infty}^{n} x[k-n_0]$$

Taking $k_1 = k - n_0$.

$$y_1[n] = \sum_{k_1=-\infty}^{n-n_0} x[k_1] \quad \ldots\ldots \text{(i)}$$

Next if we shift $y[n]$ by n_0,

$$y[n-n_0] = \sum_{k=-\infty}^{n-n_0} x[k] \quad \ldots\ldots \text{(ii)}$$

$\therefore y[n] = y[n-n_0] \Rightarrow$ Time Invariant.

$y[n] = x[Mn]$, M is an integer.

$x_1[n] = x[n-n_0]$ $\quad -\infty < n < \infty$

$\Rightarrow y_1[n] = x_1[Mn] = x[Mn - n_0]$

Again,

$$y[n-n_0] = x[M(n-n_0)] = x[Mn - Mn_0]$$

$\therefore y_1[n] \neq y[n-n_0]$

$\therefore$ It is time variant.

Linearity :-

A system is linear if, for a system with input $x(t)$ and output $y(t)$, satisfy the following relations.—

- $x_1(t) + x_2(t) \longrightarrow y_1(t) + y_2(t)$ [Superposition]
- $a x_1(t) \longrightarrow a y_1(t)$. [Scaling]

where a in general can be complex.

In general,

$$a x_1[n] + b x_2[n] \rightarrow a y_1[n] + b y_2[n]$$

or, $$a x_1(t) + b x_2(t) \rightarrow a y_1(t) + b y_2(t).$$

Let $$y[n] = 2x[n] + 3.$$

Let $x_1[n] = 2$ and $x_2[n] = 3$

then $y_1[n] = 7$ and $y_2[n] = 9$

Now, $x_3[n] = x_1[n] + x_2[n] = 5$

$$y_3[n] = 13 \neq \underbrace{[y_1[n] + y_2[n]]}_{16}$$

⇒ This is an equation of a straight line, Yet not-linear.

$$y(t) = t\,x(t)$$

$$x_1(t) \rightarrow y_1(t) = t\,x_1(t)$$
$$x_2(t) \rightarrow y_2(t) = t\,x_2(t)$$

Now, let $x_3(t) = a x_1(t) + b x_2(t)$.

then, $y_3(t) = t\,x_3(t)$

$$= t\left[a x_1(t) + b x_2(t)\right]$$
$$= a t x_1(t) + b t x_2(t)$$
$$= a y_1(t) + b y_2(t)$$

Therefore this system is linear

Linear - Time Invariant Systems

→ Consequence of linearity is that output can readily be inferred as a linear combination of a number of input.

→ Further if we apply time invariance, an arbitrary signal can be represented by ~~so~~ sum of weighted and delayed impulses, applying linearity and time invariance.

In general, $x[k] = \sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k]$

(Easily extendable to a system)

→ Let $h_k[n]$ denote the response of the linear system, due to shifted unit impulse responses $\delta[n-k]$

Then the output $y[n]$ of a linear system to an input $x[n]$ is —

$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]\, h_k[n]$$

Since $\delta[n-k]$ is the shifted version of $\delta[n]$, $h_k[n]$ is the shifted version of $h_0[n]$

(That is if $x[n] = \delta[n-k] \Rightarrow y[n] = h_k[n]$)

Let $h[n] = h_0[n]$

Then $$\boxed{y[n] = \sum_{k=-\infty}^{+\infty} x[k]\, h[n-k]}$$

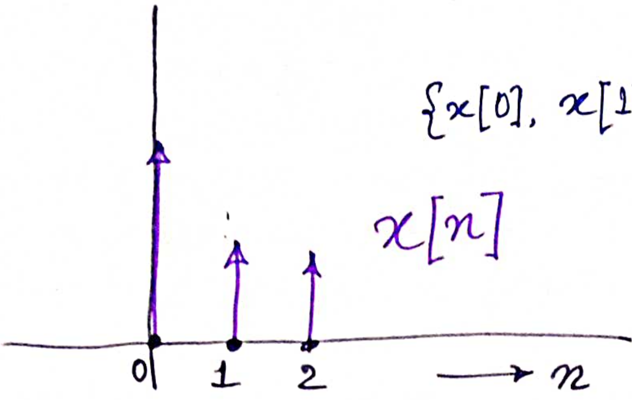
The convolution relation.

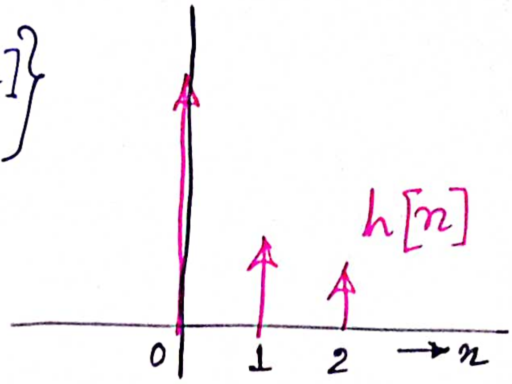
We write this as —

$$y[n] = x[n] * h[n]$$

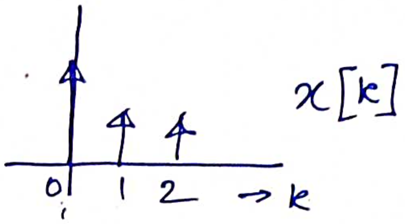
If $h[n]$ i.e. impulse response is known $y[n]$ can be found out for any arbitrary $x[n]$

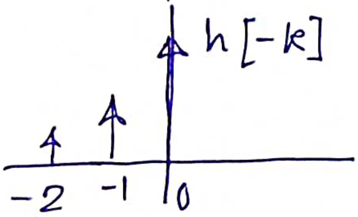
$\{x[0], x[1], x[2]\}$





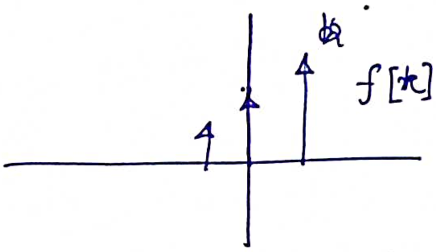
$$y[n] = \sum_k x[k]\, h[n-k]$$



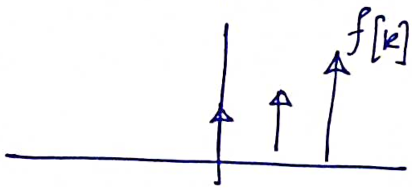


$y[0] = x[0]h[0]$

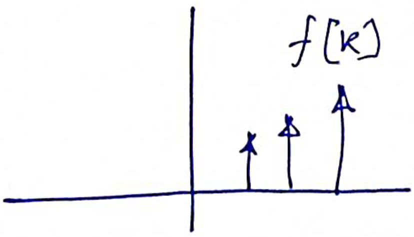
In general,
length (y)
= length (x) + length (h) - 1



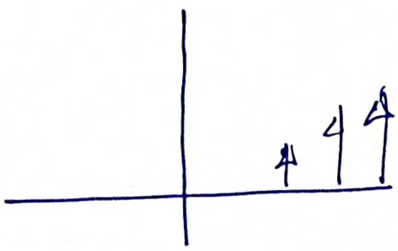
$y[1] = x[1]f[1] + x[0]f[0]$



$y[2] = x[0]f[0] + x[1]f[1] + x[2]f[2]$



$y[3] = x[1]f[1] + x[2]f[2]$



$y[4] = x[2]f[2]$

Correlation:

$$y_1(n) = \sum_{k=-\infty}^{\infty} x(k)\, h(n+k)$$

- Autocorrelation calculates the similarity between the same ~~signal~~ signal but different parts

- Cross corelation calculates the similarity between two different signals.

- Often correlation is quantified by correlation coefficient.

$$= \frac{1}{N} \sum_{k=0}^{N-1} x[k]\, h[n+k]$$

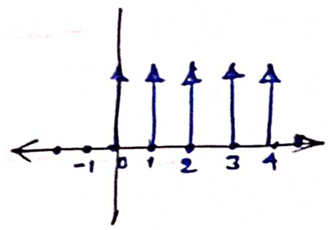
In vector form, correlation can be computed by taking dot product of X and H.

Convolution of finite and infinite sequence.

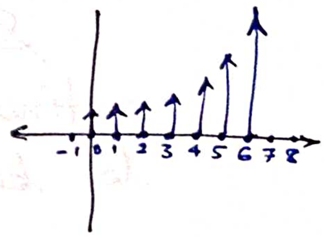
— Similar to the convolution of two finite discrete-time sequences, convolution of infinite sequences may be defined as well.

Example:

$$x[n] = \begin{cases} 1, & 0 \le n \le 4 \\ 0, & \text{otherwise.} \end{cases}$$



$$h[n] = \begin{cases} \alpha^n. & 0 \le n \le 6 \\ 0, & \text{otherwise.} \end{cases}$$



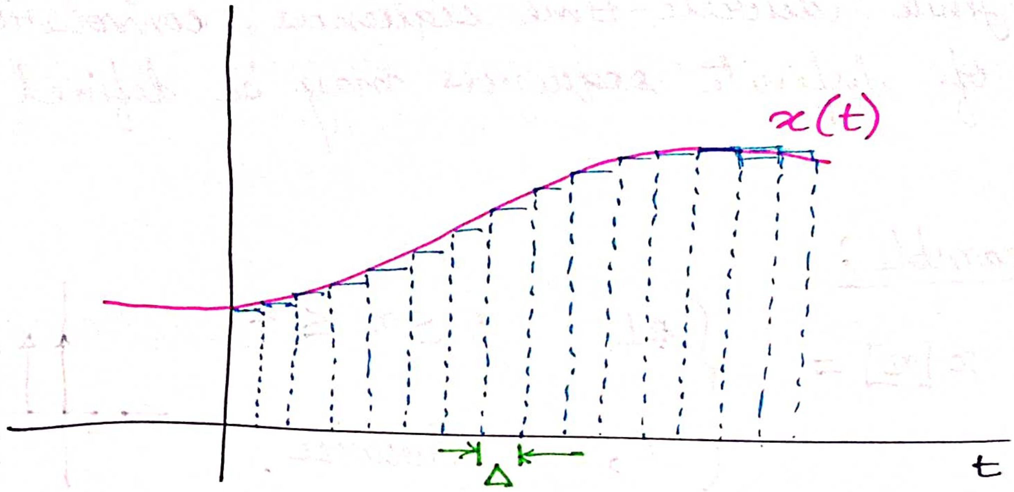
$$y[n] = \sum_{k=0}^{n} x[k]h[n-k]$$

$$= \sum_{k=0}^{n} \alpha^k = \frac{1-\alpha^{n+1}}{1-\alpha}, \quad \alpha \neq 1.$$

The value of $y[n]$ needs to be computed upto the non-overlapping portions of finite sequence.

In this case, for $n < 0$, there is no overlap, so $y[n] = 0$ for $n < 0$, and also for $n > 4$, there is no overlap of $x[n]$ and $h[n]$ (folded & shifted).

Continuous Time Linear Time Invariant System



Staircase approximation of a continuous time signal $x(t)$ is assumed to be constant for a small interval Δ

$$\delta_\Delta(t) = \begin{cases} \frac{1}{\Delta}, & 0 \le t \le \Delta \\ 0, & \text{otherwise} \end{cases}$$

$\frac{1}{\Delta} \times \Delta = 1.$

Approximated signal.

∴ We are multiplying with Δ.

$$\hat{x}(t) = \sum_{k=-\infty}^{+\infty} x(k\Delta) \boxed{\delta_\Delta(t - k\Delta)\,\Delta}$$

$$x(t) = \lim_{\Delta \to 0} \sum_{k=-\infty}^{+\infty} x(k\Delta)\, \delta_\Delta(t - k\Delta)\,\Delta$$

As $\Delta \to 0$, the summation may be replaced with integration.

$$\boxed{x(t) = \int_{-\infty}^{+\infty} x(\tau)\, \delta(t - \tau)\, d\tau}$$

Continuous time impulse response:

Let $\hat{h}_{k\Delta}(t)$ be the response of LTI system to the input of $\delta_\Delta(t-k\Delta)$

Then,

$$\hat{y}(t) = \sum_{k=-\infty}^{+\infty} x(k\Delta)\,\hat{h}_{k\Delta}(t)$$

The same staircase logic followed for $x(t)$.

Now, as $\Delta \to 0$, ~~$+\infty$~~

$$y(t) = \lim_{\Delta \to 0} \sum_{k=-\infty}^{\infty} x(k\Delta)\, h_{k\Delta}(t) \cdot \Delta$$

As $\Delta \to 0$, we replace the summation with integration and ~~fo~~ further replacing $k\Delta$ by a continuous time variable τ

$$y(t) = \int_{-\infty}^{+\infty} x(\tau)\, h_\tau(t)\, d\tau$$

Since τ is the shift in continuous time domain, and $h_\tau(t)$ is the shifted version of $h_0(t)$. This is due to the property of time invariance. Therefore $h_\tau(t)$ can be replaced by $h_0(t-\tau)$. For convinience, let $h(t) = h_0(t)$, we get —

$$\boxed{y(t) = \int_{-\infty}^{+\infty} x(\tau)\, h(t-\tau)\, d\tau}$$

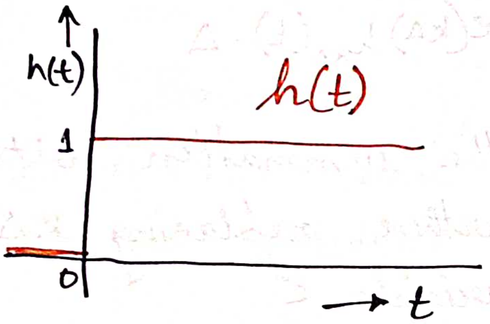
Similar to the discrete case, we can write $y(t)$ as a convolution of x and h.

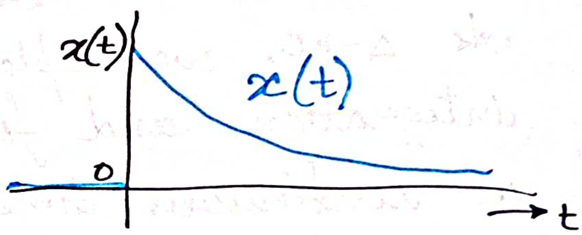
$$\therefore \quad y(t) = x(t) * h(t)$$

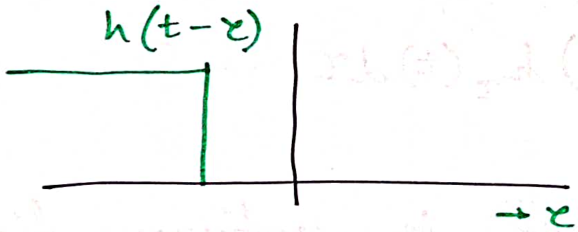
<u>Example:-</u>

$$x(t) = e^{-at} u(t), \qquad a > 0$$

and $h(t) = u(t)$







For $t < 0$, $y(t) = 0$,
for $t > 0$, $x(\tau) h(t-\tau) = e^{-a\tau}$

$$y(t) = \int_0^t e^{-a\tau} d\tau \qquad 0 < \tau < t$$

$$= \frac{1}{a} \left(1 - e^{-at}\right)$$

In general, $y(t) = \frac{1}{a} (1 - e^{-at}) \cdot u(t)$.

PROPERTIES OF LTI - system :—

1. Commutative property :

→ Holds good for both continuous and discrete LTI-systems.

$$x[n] * h[n] = h[n] * x[n] = \sum_{k=-\infty}^{\infty} h[k] \cdot x[n-k]$$

and, $$x(t) * h(t) = h(t) * x(t) = \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau .$$

Proof:—

$$x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$$

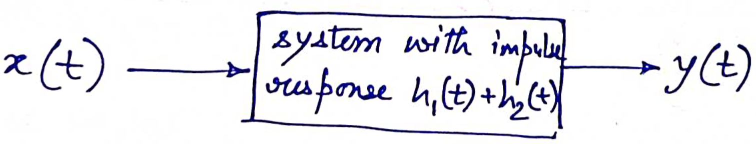
$$= \sum_{r=-\infty}^{\infty} x[n-r]\, h[r], \quad \text{where } r = n-k.$$

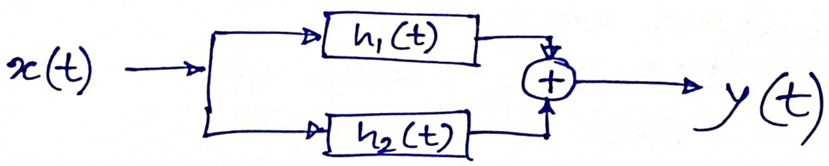
$$= h[n] * x[n]$$

2. Distributive Property:

$$x[n] * (h_1[n] + h_2[n]) = x[n] * h_1[n] + x[n] * h_2[n]$$

$$x(t) * [h_1(t) + h_2(t)] = x(t) * h_1(t) + x(t) * h_2(t)$$





"Connected in Parallel"

3. Associative Property:

$$x[n] * (h_1[n] * h_2[n]) = (x[n] * h_1[n]) * h_2[n]$$

$$x(t) * (h_1(t) * h_2(t)) = (x(t) * h_1(t)) * h_2(t)$$

$$x[n] \longrightarrow \boxed{h_1[n]} \longrightarrow \boxed{h_2[n]} \longrightarrow y[n]$$

"Connected in cascade/series"

$$x[n] \longrightarrow \boxed{h[n] = h_1[n] * h_2[n]} \longrightarrow y[n]$$

LTI system without memory:

LTI system without memory can be implemented only for imuplse response $h[n] = 0, \forall n \neq 0$, which means $h[n] = k\,\delta[n]$. We therefore get —

$$y[n] = k\,x[n].$$

In general LTI system is a system with memory.

Invertible condition for LTI system

$$x(t) \rightarrow \boxed{h(t)} \xrightarrow{y(t)} \boxed{h_I(t)} \rightarrow w(t)$$

LTI system is invertible if $w(t) = x(t)$.

$$x(t) \rightarrow \boxed{h(t) * h_I(t) = \delta(t)} \rightarrow x(t).$$

Similarly, for the discrete case, we need to have $h[n] * h_I[n] = \delta[n]$.

Example :-

$$y(t) = x(t-t_0); \quad h(t) = \delta(t-t_0)$$

or, $x(t-t_0) = x(t) * \delta(t-t_0)$

Then $h_I(t) = \delta(t+t_0)$

$$h(t) * h_I(t) = \delta(t-t_0) * \delta(t+t_0) = \delta(t)$$

Example :-

Given impulse response of a system S_1.

$$h[n] = \left(\frac{1}{5}\right)^n u[n]$$

a) Find integer A such that $h[n] - A\,h[n-1] = \delta[n]$

b) Determine the impulse response $g[n]$ of an LTI system S_2 which is inverse of S_1

a) $\left(\frac{1}{5}\right)^n u[n] - A \left(\frac{1}{5}\right)^{n-1} u[n-1] = \delta[n]$

This should satisfy at ~~n=1~~ $n > 1$.

$\therefore \frac{1}{5} - A = 0, \quad A = \frac{1}{5}$

b) $\therefore h[n] - \frac{1}{5} h[n-1] = \delta[n]$

We can manupulate this into a convolution —

$$(h[n] * \delta[n]) - \frac{1}{5}(h[n] * \delta[n-1]) = \delta[n]$$

or, $h[n] * \left(\delta[n] - \frac{1}{5}\delta[n-1]\right) = \delta[n]$

$\therefore g[n] = \delta[n] - \frac{1}{5}\delta[n-1]$.

$\therefore$ The impulse response ~~to the inve~~ of the inverse sequence is —

$$g[n] = \delta[n] - \frac{1}{5}\delta[n-1]$$

Can we have inverse of a finite sequence?

Let $h[n] = \{h_0, h_1\}$

$h_I[n] = \{h_{I_0}, h_{I_1}\}$

$$\begin{array}{rr}
 & h_0, \; h_1 \\
 & h_{I_0}, \; h_{I_1} \\
\hline
 & h_0 h_{I_1}, \; h_1 h_{I_1} \\
h_0 h_{I_0}, \; h_1 h_{I_0} & \\
\hline
\end{array}$$

$\{h_0 h_{I_0}, \; h_1 h_{I_0} + h_0 h_{I_1}, \; h_1 h_{I_1}\} = y[n]$

$$\delta[n] = 1, \quad n=0,$$

$$\Rightarrow h_{I_0} = \frac{1}{h_0} \; ; \; h_{I_1} = \frac{-h h_{I_0}}{h_0}$$

$\therefore$ The second term ~~is~~ can never be 0.

$\therefore$ Inverse of a finite impulse response can not be a finite impulse response. [other than the $\delta[n]$]

Causality of LTI system

→ Impulse response of a causal LTI system $h[n] = 0$ for $n < 0$.

$$\text{or, } y[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k] = \sum_{k=-\infty}^{n} x[k]\, h[n-k]$$

Similarly, for the continuous case,

$$h(t) = 0 \quad \text{for } t<0.$$

$$\text{or, } y(t) = \int_{\tau=-\infty}^{t} x(\tau)\, h(t-\tau)\, d\tau$$

$$= \int_{0}^{\infty} h(\tau)\, x(t-\tau)\, d\tau$$

[Using commutative property of convolution].

For example, $h(t) = \delta(t - t_0)$ is causal for $t_0 \geq 0$

Let $|x[n]| < B \quad \forall n, \; B < \infty$

Then $|y[n]| = \left| \sum_{k=-\infty}^{+\infty} h[k] \, x[n-k] \right|$

or, $|y[n]| \leq \sum_{k=-\infty}^{\infty} |h[k]| \, |x[n-k]|$

$\therefore |y[n]| \leq B \sum_{k=-\infty}^{\infty} |h[k]|$

$\therefore$ $y[n]$ is bounded if $\sum_{k=-\infty}^{\infty} |h[k]| < \infty$

$\therefore$ $y[n]$ is ~~abs~~ bounded if impulse response is absolutely summable.

In continuous domain, $\int_{-\infty}^{\infty} |h(t)| dt < \infty$

Eg: $h[n] = \delta[n]$

$$\sum_{-\infty}^{\infty} |h[n]| = 1 \Rightarrow \text{stable}.$$

if $h[n] = u[n]$

$$\sum_{-\infty}^{\infty} |h[n]| = \sum_{0}^{\infty} |u[n]| = \infty \Rightarrow \text{unstable}$$

Let $h(t) = e^{-t}u(t)$

$$\int_{-\infty}^{\infty} |h(\tau)| d\tau = \int_{0}^{\infty} e^{-t} d\tau = 1 \implies \text{Stable.}$$

Example 1:—

Given $h_2[n] = u[n] - u[n-2]$

and the system is given by ~~$y[n] = h_1[n] * h_2[n] * h_2[n]$~~

$$y[n] = \underbrace{\left(h_1[n] * h_2[n] * h_2[n]\right)}_{h[n]} * x[n]$$

and,

$$h[n] = \{\underset{\substack{\uparrow \\ 0}}{1},\ \underset{\substack{\uparrow \\ 1}}{5},\ \underset{\substack{\uparrow \\ 2}}{10},\ \underset{\substack{\uparrow \\ 3}}{11},\ \underset{\substack{\uparrow \\ 4}}{8},\ \underset{\substack{\uparrow \\ 5}}{4},\ \underset{\substack{\uparrow \\ 6}}{1}\}$$

(a) Find the impulse $h_1[n]$

(b) Find $y[n]$ for $x[n] = \delta[n] - \delta[n-1]$

$$h[n] = h_1[n] * h_2[n] * h_2[n]$$

Now, $h_2[n] = u[n] - u[n-2]$

$$= \delta[n] + \delta[n-1]$$

$\therefore$ $h_2[n] * h_2[n]$

$$= (\delta[n] + \delta[n-1]) * (\delta[n] + \delta[n-1])$$

$$= \delta[n] + \delta[n-1] + \delta[n-1] + \delta[n-2]$$

$$= \delta[n] + 2\delta[n-1] + \delta[n-2]$$

$$\therefore h_3[n] = h_1[n] * (\delta[n] + 2\delta[n-1] + \delta[n-2])$$

$$= h_1[n] + 2h_1[n-1] + h_1[n-2]$$

$\therefore h[0] = 1$

Now, we assume causality,

$\therefore h_1[0] = 1.$

$h_1[2] = 3$

Problem 2

Given

$$x[n] = \left(\frac{1}{2}\right)^{n-2} u[n-2]$$

and $h[n] = u[n+2]$

Determine $y[n] = x[n] * h[n]$

Let $x_1[n] = \left(\frac{1}{2}\right)^n u[n]$

and $h_1[n] = u[n]$

Then, $x[n] = x_1[n-2]$

and $h[n] = h_1[n+2]$

$$\therefore y[n] = x[n] * h[n]$$

$$= x_1[n-2] * h_1[n+2]$$

$$= \sum_{k=-\infty}^{\infty} x_1[k-2]\, h_1[n-k+2]$$

Taking $k = m + 2$

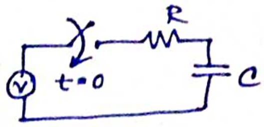
$$y[n] = \sum_{m=-\infty}^{\infty} x_1[m]\, h_1[n-m]$$

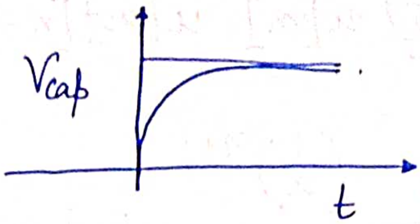
$$= \sum_{m=-\infty}^{+\infty} \left(\frac{1}{2}\right)^m u[m] \cdot u\,\cancel{[m]}\,[n-m]$$

$$\Rightarrow y[n] = \frac{1-\left(\frac{1}{2}\right)^{n+1}}{1-\frac{1}{2}}\, u[n] = 2\left[1-\left(\frac{1}{2}\right)^{n+1}\right] u[n]$$

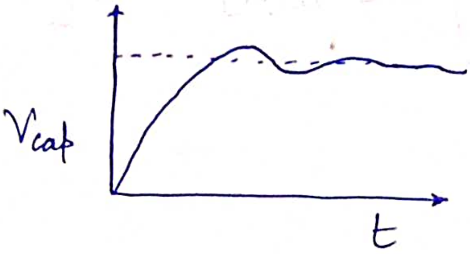
Step Response of LTI system

Step input pumps finite power and infinite energy.





(R-C circuit)



(R-L-C circuit)

Sinusoidal or any oscillatory input also pumps finite power along with infinite energy.

step response $s[n] = u[n] * h[n]$

$$= h[n] * u[n]$$

We also know that $u[n]$ is also the impulse response of the accumulator, i.e.

$$s[n] = \sum_{k=-\infty}^{n} h[k] \quad , \left[\because u[n] = \sum_{k=-\infty}^{n} \delta[n] \right]$$

Then $s[n-1] = \sum_{k=-\infty}^{n-1} h[k]$

$$\Rightarrow \boxed{s[n] - s[n-1] = h[n]}$$

[Discrete-time impulse response can be recovered from step response]

In continuous time —

$$s(t) = \int_{-\infty}^{t} h(\tau)\, d\tau = u(t) * h(t)$$

Therefore, $h(t) = \frac{d}{dt} s(t)$

Linear Constant Coefficient differential equation

This is used to characterize an arbitrary LTI system. It can be used to get impulse response.

output input.

$$\frac{dy(t)}{dt} + 2y(t) = x(t) \quad \text{——} \quad (1)$$

forcing function

Take $x(t) = ke^{3t}u(t)$

Solution $y(t) = y_p(t) + y_h(t)$

particular part

homogenous part.

Homogenous part

$$\frac{dy(t)}{dt} + 2y(t) = 0$$

$$\Rightarrow \frac{dy(t)}{y(t)} = -2dt \Rightarrow \log\{y(t)\} = -2t + C.$$

$$\Rightarrow y(t) = e^{-2t+C} = Ae^{-2t}$$

Particular part :-

Let us take $y_p(t) = Ye^{3t}$

for $t > 0$,

~~$3ye^{3t} +$~~

$$3Ye^{3t} + 2Ye^{3t} = ke^{3t}$$

$$\Rightarrow Y = \frac{k}{5}.$$

$$y_p(t) = \frac{k}{5} e^{3t}, \quad t > 0,$$

$$\therefore y(t) = Ae^{-2t} + \frac{k}{5} e^{3t}$$

We need an intial condition, to find A.

at $t = 0$, $y(0) = 0$. [System is relaxed]

$$\therefore A = -\frac{k}{5}$$

$$\therefore y(t) = \frac{k}{5}\left(e^{3t} - e^{-2t}\right)$$

Linear constant coefficient difference equation.

e.g. $y[n] = x[n] - x[n-1]$

$$y[n] - y[n-1] = x[n]$$

In general this can be represented as

$$\sum_{k=0}^{N} a_k \, y[n-k] = \sum_{k=0}^{M} b_k \, x[n-k] \quad \text{------} \quad (1)$$

The homogenous solution in this case is found by solving

$$\sum_{k}^{N} a_k y[n-k] = 0$$

$$\underline{\underline{\hspace{6cm}}}$$

homogenous solution

Equation ① can be written as —

$$y[n] = \frac{1}{a_0}\left\{ \sum_{k=0}^{M} b_k x[n-k] - \sum_{k=1}^{N} a_k y[n-k] \right\}$$

If we take $a_k = 0 \quad \forall \quad k \in \{1, 2, \dots, N\}$

$$y[n] = \sum_{k=0}^{M} \frac{b_k}{a_0} x[n-k] \rightarrow \text{Moving average (MA) systems.}$$

If $b_0 = 1$ and other b_k's are zero

$$\underbrace{y[n] + \sum_{k=1}^{M} \frac{a_k}{a_0} y[n-k]}_{\text{Auto regressive (AR)}} = \frac{x[n]}{a_0}.$$

Example: $y[n] - \frac{1}{2} y[n-1] = x[n]$

Taking initial rest condition $x[n] = k\delta[n]$

$x[n] = 0, \quad n \leq -1$.

and initial condition on $y[n]$ is $y[n] = 0$.

$y[0] = x[0] + \frac{1}{2}y[-1] = k$

$y[1] = x[1] + \frac{1}{2}y[0] = \frac{1}{2}k$

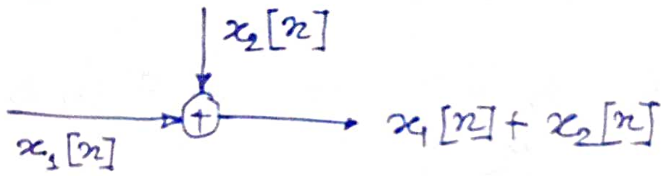
$y[2] = x[2] + \frac{1}{2}y[1] = \left(\frac{1}{2}\right)^2 k$

$y[3] = y[3] + \frac{1}{2}y[2] = \left(\frac{1}{2}\right)^3 k$

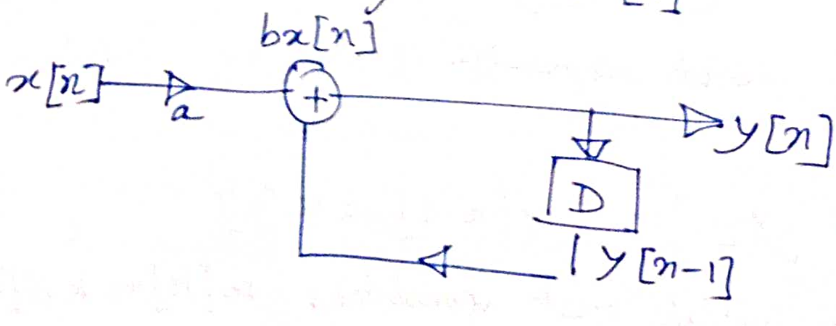
$y[n] = \left(\frac{1}{2}\right)^n k.$

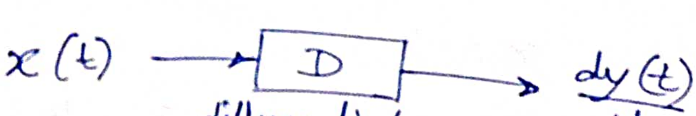
Impulse response can be found easily by putting $k = 1$

<u>Block Diagram</u>

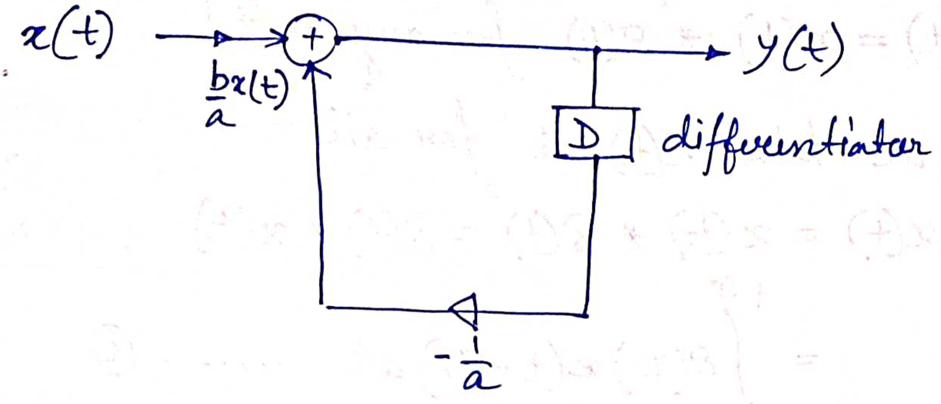


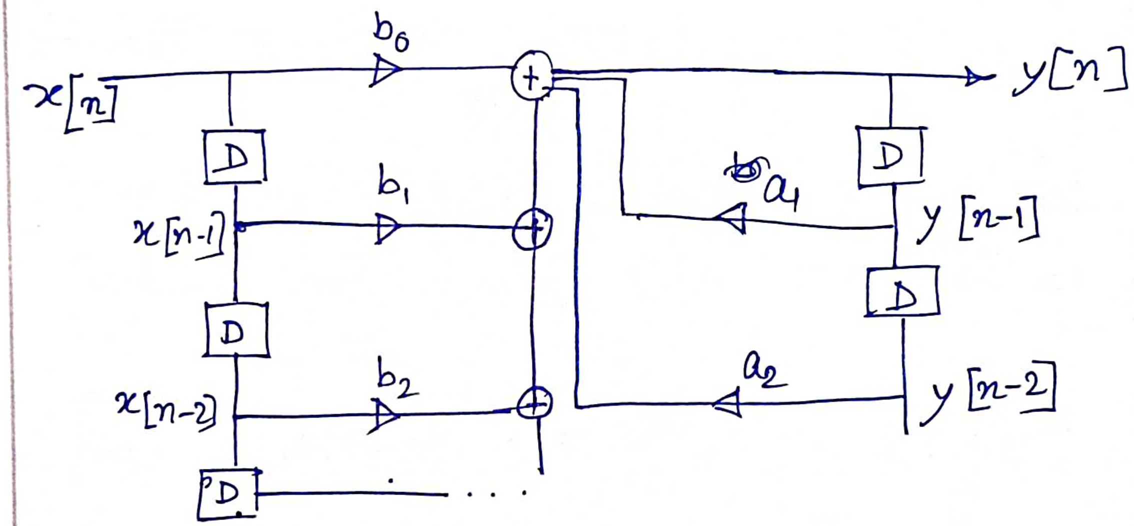
$y[n] = -ay[n-1] + bx[n]$





$$\frac{dy(t)}{dt} = bx(t) - ay(t)$$





→ In continuous domain, apart from differentiator integration blocks may be used.

Defining Unit Impulse Through Convolution

$x(t) = x(t) * \delta(t)$ for any $x(t)$

If we take $x(t) = 1$ for all t

$$1 = x(t) = x(t) * \delta(t) = \delta(t) * x(t)$$

$$= \int_{-\infty}^{+\infty} \delta(\tau)\, x(t-\tau)\, d\tau \quad \cdots\cdots \text{(1)}$$

$$= \int_{-\infty}^{+\infty} \delta(\tau)\, d\tau \longleftarrow \text{Unit area}$$

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Unit Doublet :-

Let us take a system $y(t) = \frac{d}{dt}(x(t))$

The unit impulse response of the system is the derivative of the unit impulse, which is known as the doublet.

In other words, $\frac{d}{dt}x(t) = x(t) * \underbrace{u_1(t)}_{\substack{\text{unit doublet} \\ \text{(impulse response of derivative)}}}$

Similarly, $\frac{d^2}{dt^2}x(t) = x(t) * u_2(t)$
$= x(t) * u_1(t) * u_1(t)$

$$\Rightarrow \boxed{u_2(t) = u_1(t) * u_1(t)}$$

$$\Rightarrow u_k(t) = \underbrace{u_1(t) * u_1(t) * \cdots * u_1(t)}_{k\text{-terms}}.$$

Let us take $x(t) = 1$.

$$\therefore 0 = \frac{d}{dt}x(t) = x(t) * u_1(t)$$
$$= \int_{-\infty}^{+\infty} u_1(\tau)\, x(t-\tau)\, d\tau = \int_{-\infty}^{+\infty} u_1(\tau)\, d\tau$$

$\boxed{\text{This shows that unit doublet has zero area.}}$

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To look into another property of unit doublet, let us take a signal $g(-t)$ and convolute it with $u_1(t)$

$$\int_{-\infty}^{+\infty} g(\tau - t)\, u_1(\tau)\, d\tau = g(-t) * u_1(t)$$

$$= \frac{d}{dt}\left(g(-t)\right)$$

$$= -g'(-t)$$

If we put $t = 0$,

$$\boxed{-g'(0) = \int_{-\infty}^{+\infty} g(\tau)\, u_1(\tau)\, d\tau}$$

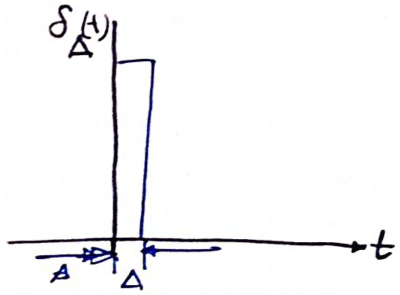
Let us check the doublet for δ function.

We know that $x(t) = x(t) * \delta(t)$

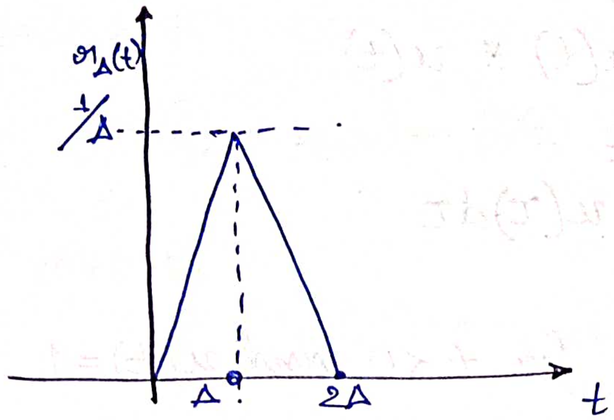
Now, if we take $x(t) = \delta(t)$, then

$$\delta(t) = \delta(t) * \delta(t)$$

To interpret the derivative of $\delta(t)$, we need to take the help of the approximate δ function, $\delta_\Delta(t)$.



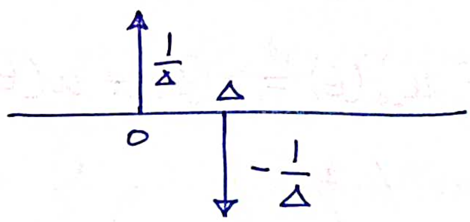
Then, $r_\Delta(t) = \delta_\Delta(t) * \delta_\Delta(t)$



Now, $\frac{d}{dt}\delta_\Delta(t) = \frac{1}{\Delta}\{\delta(t) - \delta(t-\Delta)\}$

Now, as $x(t-t_0) = x(t) * \delta(t-t_0)$,

$$x(t) * \frac{d}{dt}\{\delta_\Delta(t)\} = \frac{x(t) - x(t-\Delta)}{\Delta} \simeq \frac{dx(t)}{dt} \text{ as } \Delta \to 0$$



As unit step is the impulse response of an integrator, if we define

$$y(t) = \int_{-\infty}^{t} x(\tau)\,d\tau$$

$$\therefore \quad u(t) = \int_{-\infty}^{t} \delta(\tau)\,d\tau$$

and $$y(t) = x(t) * u(t) = \int_{-\infty}^{t} x(\tau)\,d\tau$$

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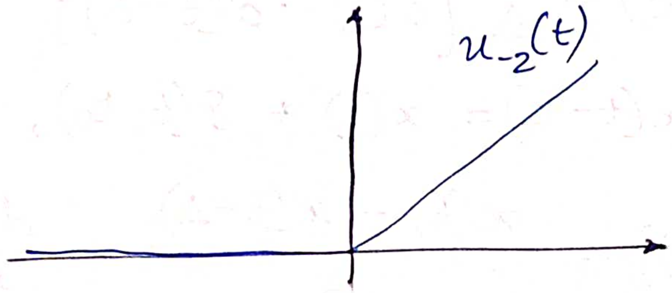
Double Integrator

$$u_{-2}(t) = u(t) * u(t)$$

$$= \int_{-\infty}^{t} u(\tau)\, d\tau$$

As $u(t)=0$ for $t<0$ and $u(t)=1$, for $t>0$



Take $x(t) * u_{-2}(t) = x(t) * u(t) * u(t)$

$$= \left(\int_{-\infty}^{t} x(\sigma)\, d\sigma \right) * u(t)$$

$$= \int_{-\infty}^{t} \left(\int_{-\infty}^{\tau} x(\sigma)\, d\sigma \right) d\tau$$

Similarly, if we convolute k times —

$$\boxed{\begin{aligned} u_{-k}(t) &= \underbrace{u(t) * u(t) * \cdots * u(t)}_{k\text{-terms}} \\ &= \int_{-\infty}^{t} u_{-(k-1)}(\tau)\, d\tau = \frac{t^{k-1}}{(k-1)!} u(t) \end{aligned}}$$

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Since derivative is an inverse operator of integrator.

$$u(t) * u_1(t) = \delta(t)$$

In general,

$$u_k(t) * u_r(t) = u_{(k+r)}(t)$$

FOURIER SERIES REPRESENTATION OF PERIODIC SIGNAL

→ An LTI system can be represented as a weighted sum of an elementary signal or a basic function.

→ The response of an LTI system to the set of functions need to be expressed in terms of the same input function. Such functions can be thought of as eigenfunction of the LTI system.

→ Complex exponential satisfy this condition of LTI systems.

In continuous time, if we take a complex

$$e^{st} \longrightarrow \boxed{} \longrightarrow H(s)e^{st}$$

Similarly in the discrete domain —

$$z^n \longrightarrow H(z)z^n$$

power series expansion of complex z

Let $x(t) = e^{st}$, then $y(t) = \int_{-\infty}^{+\infty} h(\tau)x(t-\tau)\,d\tau$

$$\therefore y(t) = \cancel{e^{st}} \int_{-\infty}^{+\infty} h(\tau)\, e^{s(t-\tau)}\, d\tau$$

$$= e^{st} \int_{-\infty}^{+\infty} h(\tau)\, e^{-st}\, d\tau$$

$$= \underbrace{H(s)}_{\text{complex constant as a function of } s.}\, e^{st}$$

In the discrete time case,

Let $x[n] = z^n \quad z \in \mathbb{C}$

$$\Rightarrow y[n] = \sum_{k=-\infty}^{+\infty} h[k]\, x[n-k]$$

$$= \sum_{k=-\infty}^{+\infty} h[k]\, z^{n-k}$$

$$= z^n \sum_{k=-\infty}^{+\infty} h[k]\, z^{-k}$$

$$= \underbrace{H(z)}_{\text{Eigenvalue}} \cdot z^n \quad \leftarrow \text{Eigenfunction.}$$

(Provided the summation converges)

Example:

Take $y(t) = x(t-3)$ and given $x(t) = e^{j2t}$

$$y(t) = e^{j2(t-3)}$$

$$= \underbrace{e^{-j6}}_{\text{eigenvalue}} \cdot \underbrace{e^{j2t}}_{\text{eigenfunction.}}$$

The impulse response of the system is $\delta(t-3)$

$$\therefore \ H(s) = \int_{-\infty}^{+\infty} \delta(\tau-3)\, e^{-s\tau}\, d\tau = e^{-3s}$$

[Here we had $s=j2 \Rightarrow H(s)= e^{-j6}$]

Linear Combination of Harmonically Related Complex Exponential

Let us start with a periodic signal,

$$x(t) = x(t+T) \quad , \forall t$$

where T is the time period.

Let

$$x(t) = e^{j\omega_0 t} \quad \text{and} \quad \underbrace{\Phi_k(t)}_{k\text{-th harmonic}} = e^{jk\omega_0 t} = e^{jk\left(\frac{2\pi}{T}\right)t}$$

$\Phi_k(t)$ is the harmonically related complex exponential with fundamental frequency ω_0.

$\therefore$ By superposition,

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k\, e^{jk\omega_0 t} = \sum_{k=-\infty}^{+\infty} a_k\, e^{jk\left(\frac{2\pi}{T}\right)t} \quad \text{—— (1)}$$

$\Rightarrow$ By superposition and linearity, sum of periodic signals which are harmonically distributed is also periodic with period T

This representation of a periodic signal in the form of (1) is referred to as "Fourier Series Representation"

Example:

$$x(t) = \sum_{k=-3}^{+3} a_k e^{j\cancel{\omega_0}k2\pi t} \quad \text{——} \quad (1)$$

given $a_0 = 1$;

$a_1 = a_{-1} = \frac{1}{4}$

$a_2 = a_{-2} = \frac{1}{2}$

$a_3 = a_{-3} = \frac{1}{3}$

$$\therefore x(t) = 1 + \frac{1}{4}\left(e^{j2\pi t} + e^{-j2\pi t}\right) + \frac{1}{2}\left(e^{j4\pi t} + e^{-j4\pi t}\right) + \frac{1}{3}\left(e^{j6\pi t} + e^{-j6\pi t}\right)$$

$$= 1 + \frac{1}{4}\cos(2\pi t) + \frac{1}{2}\cos(4\pi t) + \frac{2}{3}\cos(6\pi t)$$

Generally for real $x(t)$, $\quad x(t) = x^*(t)$

$$\therefore x(t) = x^*(t) = \sum_{k=-\infty}^{\infty} a_k^* e^{-jk2\pi t} \quad \text{——} \quad (2)$$

Replacing k by $-k$,

$$x(t) = \sum_{k=-\infty}^{\infty} a_{-k}^* e^{j\omega_0 t} \quad \text{——} \quad (3)$$

Comparing (1) and (3), we have

$$a_k^* = a_{-k} \Rightarrow \text{Conjugate Symmetry.}$$

or For real $x(t)$,

$$x(t) = a_0 + \sum_{k=1}^{\infty}\left[a_k e^{jk\omega_0 t} + a_{-k} e^{-jk\omega_0 t}\right]$$

$$\Rightarrow x(t) = a_0 + \sum_{k=\cancel{-\infty}\,1}^{\infty} \left[a_k e^{jk\omega_0 t} + a_k^{*} e^{-jk\omega_0 t} \right]$$

Complex conjugates

$$\Rightarrow x(t) = a_0 + 2 \sum_{k=1}^{\infty} \mathrm{Re}\left\{ a_k e^{jk\omega_0 t} \right\}$$

In polar form, if $a_k = A_k e^{j\theta_k}$

Then,

$$x(t) = a_0 + \sum_{k=1}^{\infty} 2\,\mathrm{Re}\left\{ A e^{j(k\omega_0 t + \theta_k)} \right\}$$

or, $$x(t) = a_0 + \sum_{k=1}^{\infty} 2 \cos(k\omega_0 t + \theta_k)$$

If we write a_k in the rectangular form, $a_k = B_k + jC_k$

Then,

$$x(t) = a_0 + 2 \sum_{k=1}^{\infty} \left[B_k \cos(k\omega_0 t) - C_k \sin(k\omega_0 t) \right]$$

If a_k is real, $a_k = A_k = B_k$ and $C_k = 0$.

Determination of Fourier Series Coefficients.

Let $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$

$\therefore\ x(t)\cdot e^{-jn\omega_0 t} = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} e^{-jn\omega_0 t}$

or, $\int_0^T x(t) e^{-jn\omega_0 t} dt = \int_0^T \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \cdot e^{-jn\omega_0 t} dt$

where $T = \frac{2\pi}{\omega_0} \longrightarrow$ Fundamental period

or $\int_0^T x(t) e^{-jn\omega_0 t} dt = \sum_{k=-\infty}^{\infty} a_k \int_0^T e^{j(k-n)\omega_0 t} dt$

Now, $\int_0^T e^{j(k-n)\omega_0 t} dt = \int_0^T \cos(k-n)\omega_0 t\, dt + j\int_0^T \sin(k-n)\omega_0 t\, dt$

If $k \neq n$, both terms are periodic sinusoidal function with period $\frac{T}{|k-n|}$ and becomes 0 when integrated over a period.

If $k = n$, $\int_0^T e^{j(k-n)\omega_0 t} dt = \int_0^T dt = T$

$$\therefore\ \int_T e^{j(k-n)\omega_0 t} dt = \begin{cases} T, & k = n \\ 0, & k \neq n \end{cases}$$

Putting this in ①, we get——

$$a_n = \frac{1}{T}\int_T x(t)\, e^{-jn\omega_0 t}\, dt$$

$\therefore$

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} a_k e^{jk\frac{2\pi}{T}t}$$

and $a_k = \frac{1}{T}\int_T x(t)\, e^{-jk\omega_0 t}\, dt = \frac{1}{T}\int_T x(t)\, e^{-jk\frac{2\pi}{T}t}\, dt$

Mapped to the harmonics of ω_0, starting from zero

__Example:__

$$x(t) = \sin(\omega_0 t)$$
$$= \frac{1}{2j}e^{j\omega_0 t} - \frac{1}{2j}e^{-j\omega_0 t}$$

$\therefore a_0 = 0;\quad a_1 = \frac{1}{2j};\quad a_{-1} = -\frac{1}{2j};\quad a_k = 0;$ for $|k| > 1$

__Example:__

$$x(t) = 1 + \sin(\omega_0 t) + 2\cos(\omega_0 t) + \cos\left(2\omega_0 t + \frac{\pi}{4}\right)$$

$$\therefore x(t) = 1 + \left(1 + \frac{1}{2j}\right)e^{j\omega_0 t} + \left(1 - \frac{1}{2j}\right)e^{-j\omega_0 t} + \frac{1}{2}e^{j\frac{\pi}{4}}e^{j2\omega_0 t} + \frac{1}{2}e^{-j\frac{\pi}{4}}e^{-j2\omega_0 t}$$

$\Rightarrow$ $a_0 = 1$

$a_1 = 1 + \frac{1}{2j}$

$a_{-1} = 1 - \frac{1}{2j}$

$a_2 = \frac{1}{2}e^{j\frac{\pi}{4}} = \frac{\sqrt{2}}{4}(1+j)$

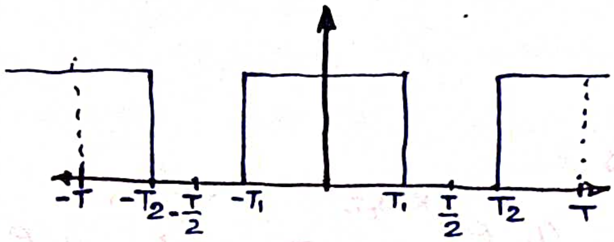
$a_{-2} = \frac{1}{2}e^{-j\frac{\pi}{4}} = \frac{\sqrt{2}}{4}(1-j)$

$a_k = 0$ for $|k| > 2$

Example :

$$x(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & T_1 < |t| < T_2 \end{cases} \Bigg\} \text{ Periodic}$$

over one period



$T_2 = T - T_1$

If we integrate over of a period,

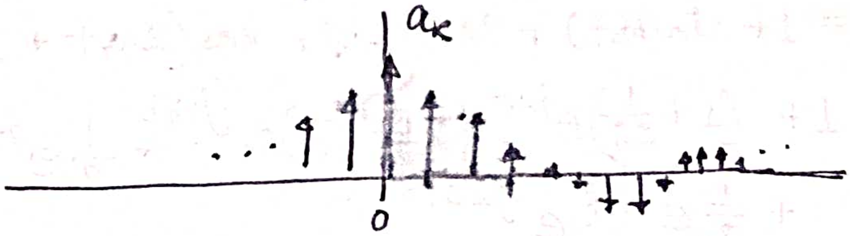
$$a_0 = \frac{1}{T}\int_{-T_1}^{+T_1} dt = \frac{2T_1}{T}$$

$$a_k = \frac{1}{T}\int_{-T_1}^{T_1} e^{-jk\omega_0 t}\,dt = \frac{\sin(k\omega_0 T_1)}{k\pi}, \quad k \neq 0$$

Put $\omega_0 T = 2\pi$, $a_k = \dfrac{\sin\left(2k\pi \frac{T_1}{T}\right)}{k\pi}$

If $T = 8T_1 \Rightarrow T_1 = \dfrac{T}{8} = \dfrac{2\pi}{8\omega_0} = \dfrac{\pi}{4\omega_0}$

then $a_k = \dfrac{\sin\left(\frac{\pi k}{4}\right)}{k\pi}$; $k \neq 0$



Sinc Function.

Though it is a valid ~~real~~ ~~for~~ Fourier Series expansion, a_k cannot be truncated at a finite k. If it is truncated, the reconstructed $x(t)$ from the Fourier Series coefficients will have a finite ~~erro~~ error.

Convergence of Fourier Series

Let $x(t)$ be the periodic signal, to be ~~for~~ represented by Fourier Series, as —

$$x_N(t) = \sum_{k=-N}^{N} a_k e^{jk\omega_0 t}$$

The approximation error $e_N(t) = x(t) - x_N(t)$

$$= x(t) - \sum_{k=-N}^{+N} a_k e^{jk\omega_0 t}$$

"One of the ways to find the quality of error is to find the energy of the error signal"

Now the energy of error over one period is —

$$E_N = \int_T |e_N(t)|^2 dt$$

The set of a_k that minimizes E_N is given by

$$a_k = \frac{1}{T}\int_T x(t) e^{-jk\omega_0 t} dt$$

that is $E_N \to 0$ as $N \to \infty$.

(This can be shown by taking the error in an interval)

$$E = \int_a^b \Big[x(t) - \sum_{k=-N}^{N} a_k \phi_k(t)\Big]\Big[x^*(t) - \sum_{k=-N}^{N} a_k^* \phi_k^*(t)\Big] dt$$

Complex exponential or in general any orthonormal function

Let $a_i = b_i + j c_i$

Then $\frac{\partial E}{\partial b_i} = 0 \Rightarrow$ Produce zero for all terms other than ith term

$$\frac{\partial E}{\partial b_i} = -\int_a^b \phi_i^*(t)\, x(t)\, dt + 2b_i - \int_a^b \phi_i(t)\, x^*(t)$$

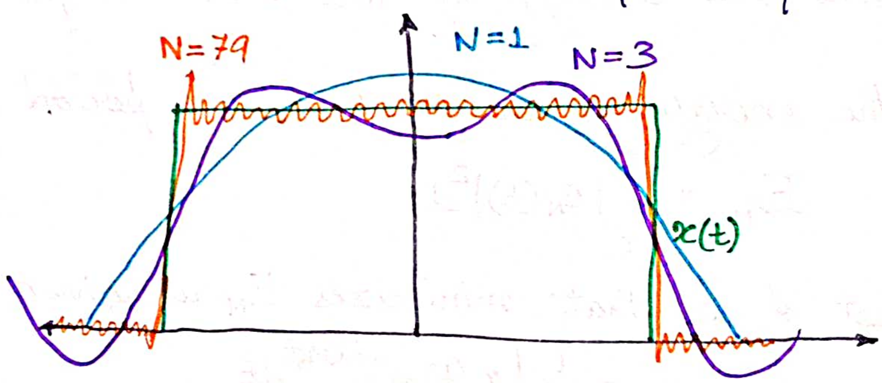
and $$\frac{\partial E}{\partial c_i} = 0 \Rightarrow j\int_a^b \phi_i(t)\, x^*(t)\, dt + 2c_i - j\int_a^b \phi_i(t)^*\, x(t)\, dt$$

$$\Rightarrow 2c_i = j\int_a^b \phi_i^*(t)\, x(t)\, dt - j\int_a^b \phi_i(t)\, x^*(t)\, dt$$

$$\therefore\ 2b_i + 2jc_i = 2\int_a^b \phi_i^*(t)\, x(t)\, dt$$

$$\Rightarrow b_i + jc_i = a_i = \int_a^b \phi_i^*(t)\, x(t)\, dt$$

Complex exponential



- This example shows that the error tends to zero in average sense but that does not mean $x(t) = x_N(t)$ at all t, for signal with discontinuity in particular.

- Further Drichlet found that as N increases $x(t)$ is equal at almost all points except the discontinuity. He also concluded that discontinuity Fourier Series converges in an average sense.

DRICHLET CONDITIONS:—

Condition 1: $x(t)$ must be absolutely integrable over one period

$$\int_T |x(t)|\,dt < \infty$$

$$|a_k| \leq \frac{1}{T}\int_T |x(t)\,e^{-jk\omega_0 t}|\,dt = \frac{1}{T}\int_T |x(t)|\,dt$$

$$\therefore \text{ if } \int_T |x(t)|\,dt < \infty, \quad a_k < \infty$$

Condition 2: In any finite interval of time, $x(t)$ is of bounded variation; that is there are no more than a finite number of maxima & minima during any single ~~for~~ period of signal.

Condition 3: In any finite interval of time, there are only a finite number of discontinuities and each of these are finite

In general however, Fourier representation of signal is exact for periodic signal without discontinuity.

Properties of Fourier Series

1. Linearity:

Let $x(t)$ and $y(t)$ be two periodic signals with period T having Fourier Series coefficients a_k and b_k respectively.

$$x(t) \xleftrightarrow{F.S.} a_k$$
$$y(t) \xleftrightarrow{F.S.} b_k$$

Now if $z(t) = Ax(t) + By(t)$

$$z(t) \xleftrightarrow{F.S.} c_k = Aa_k + Bb_k$$

2. Time Shifting:-

Let $y(t) = x(t-t_0)$

Then $b_k = \frac{1}{T}\int_T x(t-t_0)e^{-jk\omega_0 t}dt$

Let $\tau = t - t_0$

$$b_k = \frac{1}{T}\int_T x(\tau)e^{-jk\omega_0(\tau+t_0)}d\tau$$

$$\Rightarrow b_k = e^{-jk\omega_0 t_0} \cdot a_k = e^{-jk\left(\frac{2\pi}{T}\right)t_0} a_k$$

$$x(t-t_0) \xleftrightarrow{F.S.} e^{-jk\omega_0 t_0} a_k$$

3. Time Reversal:

Let $y(t) = x(-t)$, then $x(-t) = \sum_{k=-\infty}^{\infty} a_k e^{-jk\frac{2\pi}{T}t}$

Substituting $k = -m$,

$$y(t) = x(-t) = \sum_{m=-\infty}^{\infty} a_{-m} e^{jm\frac{2\pi}{T}t}$$

$\therefore$ $b_k = a_{-k}$ if $x(t) \xleftrightarrow{F.S.} a_k$

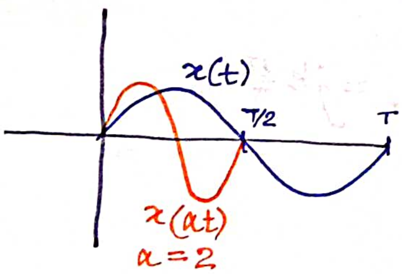
$\therefore$ If x is even, $a_k = a_{-k}$

& if x is odd, $a_k = -a_{-k}$

4. Time Scaling:

$$x(\alpha t) = \sum_{k=-\infty}^{+\infty} a_k e^{j\omega_0 k(\alpha t)} = \sum_{k=-\infty}^{+\infty} a_k e^{j(\alpha\omega_0)kt}$$

This means that the fundamental frequency value is also scaled by α.



5. Multiplication:

If $x(t) \xleftrightarrow{F.S.} a_k$ and $y(t) \xleftrightarrow{F.S.} b_k$

then,

$$x(t)\cdot y(t) \xleftrightarrow{F.S.} h_k = \sum_{l=-\infty}^{+\infty} a_l b_{k-l} \quad \text{(convolution)}$$

6. Convolution:

$$\int_T x(\tau)\, y(t-\tau)\, d\tau \xleftrightarrow{F.S.} T a_k b_k$$

$\quad \llcorner$ lcm (period($x(t)$), period($y(t)$))

7. Conjugate Symmetry:

If $x(t) \xleftrightarrow{F.S.} a_k$, then $x^*(t) \xleftrightarrow{F.S.} a_{-k}^*$

Now if $x(t)$ is real, $a_{-k} = a_k^*$

i.e. $|a_{-k}| = |a_k|$ for real $x(t)$

8. Parseval's Relation for continuous time Signal

$$\frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{+\infty} |a_k|^2$$

9. Frequency Shifting

$$x(t) \cdot e^{jM\omega_0 t} \longleftrightarrow a_{k-M}$$

10. Differentiation

$$\frac{d}{dt}\left[x(t)\right] \xleftrightarrow{F.S.} jk\omega_0 a_k = jk\frac{2\pi}{T} a_k$$

11. Integration

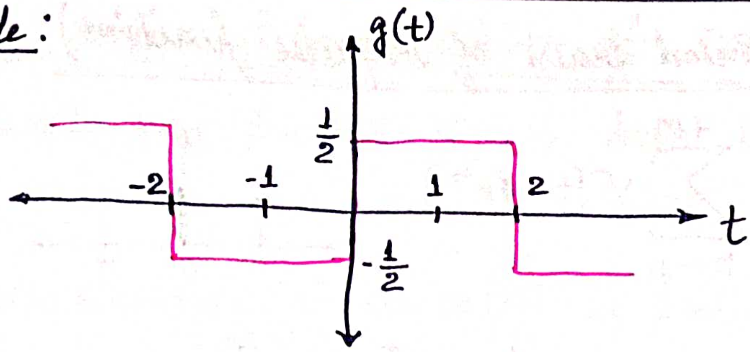
$$\int_{-\infty}^{t} x(\tau)\, d\tau \xleftrightarrow{F.S.} \frac{a_k}{jk\omega_0}$$

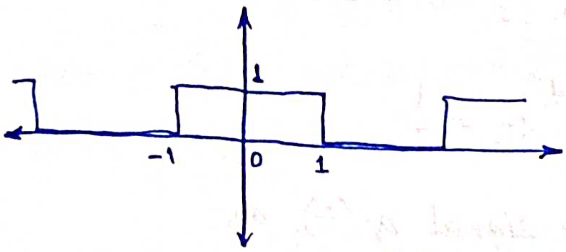
12. Even and odd signals

$x(t)$ is real & odd $\xleftrightarrow{F.S.}$ a_k is imaginary & odd.

$x(t)$ is real & even $\xleftrightarrow{F.S.}$ a_k is real & even.

Example:





Let $x(t) \xleftrightarrow{F.S.} a_k$

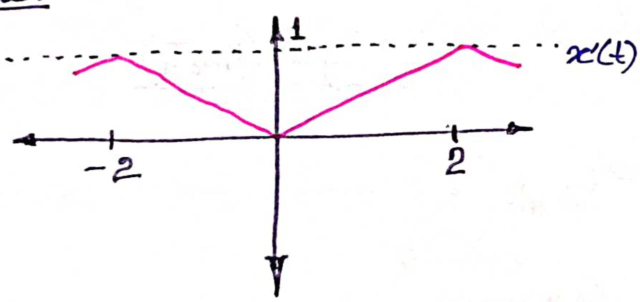
$g(t) = x(t-1) - \frac{1}{2}$

$$x(t-1) \xleftrightarrow{F.S.} b_k = a_k e^{-jk\frac{\pi}{2}}$$

$$-\frac{1}{2} \xleftrightarrow{F.S.} c_k = \begin{cases} 0 & \text{for } k \neq 0 \\ -\frac{1}{2} & \text{for } k = 0 \end{cases}$$

$$\therefore g(t) \Leftrightarrow \begin{cases} a_k e^{-jk\frac{\pi}{2}} & \text{for } k \neq 0 \\ a_0 - \frac{1}{2} & \text{for } k = 0 \end{cases} = \begin{cases} \frac{\sin\left(\frac{k\pi}{2}\right)}{k\pi} e^{-jk\frac{\pi}{2}} & , k \neq 0 \\ 0 & , k = 0 \end{cases}$$

Example:



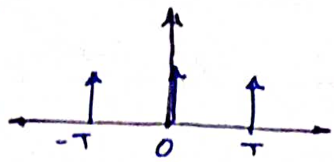
The derivative of $x'(t)$ is $g(t)$

$\therefore$ If $x'(t) \xleftrightarrow{F.S.} e_k$, then $d_k = jk\frac{\pi}{2} e_k$, where $g(t) \xleftrightarrow{F.S.} d_k$

or, $$e_k = \begin{cases} \frac{2d_k}{jk\pi} = \frac{2\sin\left(\frac{k\pi}{2}\right)}{j(k\pi)^2} e^{-jk\frac{\pi}{2}} & , k \neq 0 \\ 0 & , k = 0 \end{cases}$$

Example (Period train of impulse function)

$$x(t) = \sum_{k=-\infty}^{+\infty} \delta[t-kT]$$

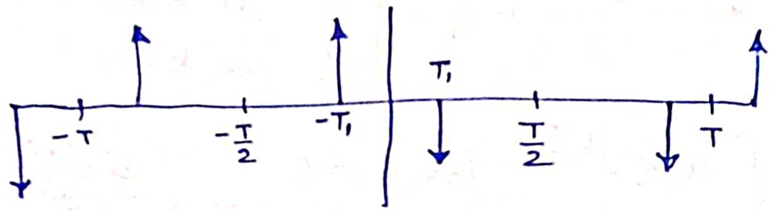


If we integrate from $-\frac{T}{2}$ to $+\frac{T}{2}$.

$$a_k = \frac{1}{T}\int_{-T/2}^{+T/2} \delta(t)\, e^{-jk\frac{2\pi}{T}t}\, dt = \frac{1}{T}$$

If we define another signal $q(t)$ as

$$q(t) = x(t+T_1) - x(t-T_1)$$



$$q(t) \xleftrightarrow{F.S.} b_k = e^{jk\omega_0 T_1} a_k - e^{-jk\omega_0 T_1} a_k$$

$$= \frac{1}{T}\left[e^{jk\omega_0 T_1} - e^{-jk\omega_0 T_1}\right]$$

$$= \frac{2j\sin(k\omega_0 T_1)}{T}$$

Fourier Series Representation of Discrete Time Signal

Discrete time periodic signals have some difference with respect to its continuous time counterparts.

A discrete-time signal $x[n]$ is periodic with period N if $x[n] = x[n+N]$, then fundamental frequency, $w_0 = \frac{2\pi}{N}$, while $e^{j\frac{2\pi}{N}n}$ and $e^{jk\frac{2\pi}{N}n}$ are both periodic with period N with $k \in \mathbb{Z}$

In $e^{jk\frac{2\pi}{N}n}$ there are only N number of distinct signals as k and n are integers, both varying as $0, \pm 1, \pm 2$

For any discrete-time function, e.g., complex exponential

$$\phi_N[n] = \phi_0[n], \quad \phi_{N+1}[n] = \phi_1[n]$$

or in general $\phi_{k+rN}[n] = \phi_k[n]$

Thus $x[n]$ is unique only in the principal interval of 0 to N-1 for a period N

$$\therefore \quad x[n] \xleftrightarrow{F.S.} \sum_{k=\langle N\rangle} a_k e^{jkw_0N} = \sum_{k=\langle N\rangle} a_k e^{jk\frac{2\pi}{N}n}$$

(any interval of N)
(N consecutive time interval)

Determination of Fourier Series Representation

Let $x[n]$ be periodic with fundamental frequency N, then

$$x[0] = \sum_{k=\langle N\rangle} a_k ,$$

$$x[1] = \sum_{k=\langle N\rangle} a_k e^{j\frac{2\pi}{N}k} ,$$

$$\vdots$$

$$x[N-1] = \sum_{k=\langle N\rangle} a_k e^{j\frac{2\pi}{N}(N-1)k} ,$$

We can say that

$$\sum_{n=\langle N\rangle} e^{jk\frac{2\pi}{N}n} = \begin{cases} N, & k = 0, \pm N, \pm 2N, \ldots \\ 0, & \text{otherwise} \end{cases}$$

We have $x[n] = \sum_{k=\langle N\rangle} a_k e^{jk\frac{2\pi}{N}n}$

Multiplying both sides with $e^{-jr\frac{2\pi}{N}\cdot n}$

$$\sum_{n=\langle N\rangle} x[n] e^{-jr\frac{2\pi}{N}n} = \sum_{n=\langle N\rangle} \sum_{k=\langle N\rangle} a_k e^{j(k-r)\frac{2\pi}{N}n}$$

or, $$\sum_{n=\langle N\rangle} x[n]\, e^{-jr\frac{2\pi}{N}n} = \sum_{k=\langle N\rangle} a_k \sum_{n=\langle N\rangle} e^{j(k-r)\frac{2\pi}{N}n}$$

For $k=r$, we have —

$$\sum_{n=\langle N\rangle} e^{j(k-r)\frac{2\pi}{N}n} = N$$

and for $k \neq r$,

$$\sum_{n=\langle N\rangle} e^{j(k-r)\frac{2\pi}{N}n} = 0$$

$$\therefore \sum_{n=\langle N\rangle} x[n]\, e^{-jr\frac{2\pi}{N}n} = N\, a_r$$

or, $$\boxed{a_r = \frac{1}{N}\sum_{n=\langle N\rangle} x[n]\, e^{-jr\frac{2\pi}{N}n}}$$

Thus,

$$x[n] = \sum_{k=\langle N\rangle} a_k e^{jk\omega_0 n} = \sum_{k=\langle N\rangle} a_k e^{jk\frac{2\pi}{N}n}$$

and, $\cancel{\overset{a_k}{x[n]} = \frac{1}{N}\sum_{n=\langle N\rangle}}$

$$a_k = \frac{1}{N}\sum_{n=\langle N\rangle} x[n]\, e^{-jk\omega_0 n} = \frac{1}{N}\sum_{n=\langle N\rangle} x[n]\, e^{-jk\frac{2\pi}{N}n}$$

Example of discrete sinusoid:

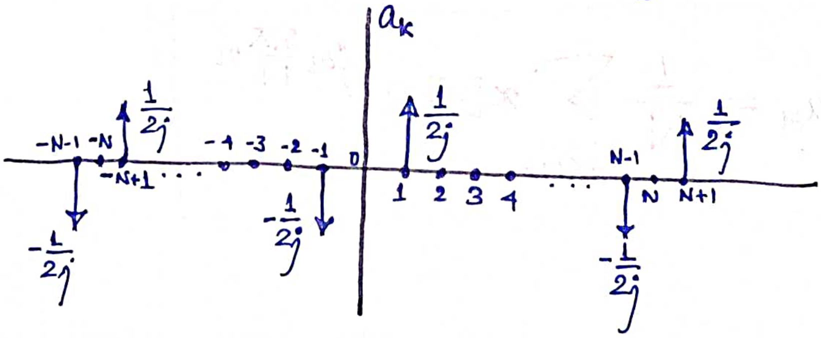
$$x[n] = \sin(\omega_0 n), \quad \omega_0 = \frac{2\pi}{N}$$

or, $$x[n] = \frac{1}{2j} e^{j\frac{2\pi}{N}n} - \frac{1}{2j} e^{-j\frac{2\pi}{N}n}$$

$$\therefore \; a_1 = \frac{1}{2j} \text{ and } a_{-1} = -\frac{1}{2j}$$

We may have also defined it for some other interval $\langle N \rangle$ of N consecutive integers.

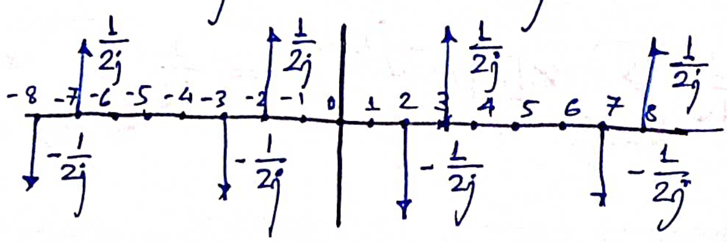
e.g. $a_{N+1} = \frac{1}{2j}$ and $a_{N-1} = -\frac{1}{2j}$



If $\omega_0 = \frac{2\pi M}{N}$; where M = 3 and N = 5

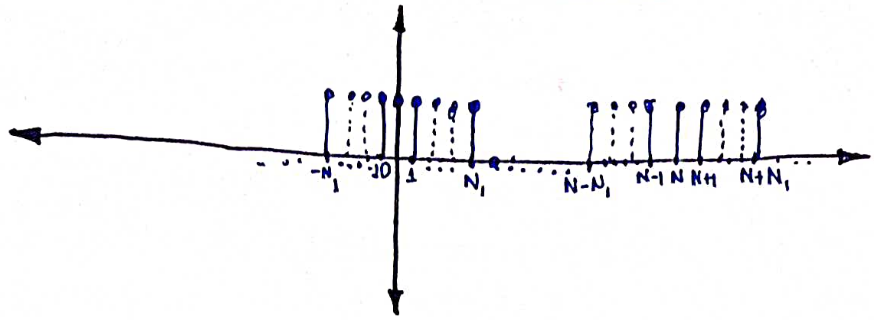
$$x[n] = \frac{1}{2j} e^{jM\left(\frac{2\pi}{N}\right)n} - \frac{1}{2j} e^{-jM\left(\frac{2\pi}{N}\right)n}$$

$\Rightarrow a_M = \frac{1}{2j}$ and $a_{-M} = -\frac{1}{2j}$, $\left[\because x[n] = \sin\left(\left(\frac{2\pi \cdot 3}{5}\right)n\right)\right]$



M = 3
N = 5

Example (Discrete square wave):—



$x[n] = 1 \quad$ for $\; -N_1 \le n \le N_1$

$$a_k = \frac{1}{N} \sum_{-N/2}^{N/2} e^{-jk\frac{2\pi}{N}\cdot n} \cdot x[n] = \frac{1}{N} \sum_{-N_1}^{N_1} e^{-jk\frac{2\pi}{N}\cdot n}$$

$\Rightarrow$ Let $m = N_1 + n$, then —

$$a_k = \frac{1}{N} \sum_{m=0}^{2N_1} e^{jk\left(\frac{2\pi}{N}\right)(N_1 - m)}$$

$$= \frac{1}{N} e^{jk\frac{2\pi}{N}N_1} \sum_{m=0}^{2N_1} e^{-jk\left(\frac{2\pi}{N}\right)m}$$

Applying finite summation rule, —

$$a_k = \frac{1}{N} e^{jk\frac{2\pi}{N}N_1} \left(\frac{1 - e^{-jk2\pi\left(\frac{2N_1+1}{N}\right)}}{1 - e^{-jk\left(\frac{2\pi}{N}\right)}} \right)$$

Finally, we get —

$$a_k = \frac{1}{N} \frac{\sin\left(2\pi\left(N_1 + \frac{1}{2}\right)/N\right)}{\sin\left(\frac{\pi k}{N}\right)}$$

Properties of Discrete-time Fourier Series:

Properties are similar to the properties of continuous time Fourier Series, the difference is that they need to be evaluated and interpreted over one period, say N points.

Linearity: $Ax[n] + By[n] \xleftrightarrow{F.S.} Aa_k + Bb_k$

Time Shift: $x[n-n_0] \xleftrightarrow{F.S.} a_k e^{-jk\left(\frac{2\pi}{N}\right)\cdot n_0}$

Frequency shift: $e^{jM\frac{2\pi}{N}\cdot n}\, x[n] \xleftrightarrow{F.S.} a_{k-M}$

Convolution (periodic) or circular: $\sum_{n=\langle N\rangle} x[r]\,y[n-r] \longleftrightarrow N a_k b_k$ both x & y have the same period.

$$\phi[n] = \sum_{k=0}^{N-1} x[k]\,y[n-k]$$

$x[n] = \{1, 2, 0, -1\}$ and $y[n] = \{1, 3, -1, -2\}$

then $\phi_L[n]$ (as linear convolution) $= \{-1, 5, 5, -5, -7, 1, 2\}$

Periodic or circular convolution $\phi[n] = \{-6, 6, 7, -5\}$

Multiplication: $x[n]\,y[n] \xleftrightarrow{F.S.} \sum_{l=\langle N\rangle} a_l\, b_{k-l}$

[Period of x & y is N]

Difference: $x[n] - x[n-1] \xleftrightarrow{F.S.} \left(1 - e^{-jk\frac{2\pi}{N}}\right) a_k$

Running Sum: $\sum_{k=-\infty}^{n} x[k] \xleftrightarrow{F.S.} \frac{1}{1-e^{-jk\frac{2\pi}{N}}} a_k$ [assumed $a_0 = 0$]

Periodic:

$x[n]$:	1	2	0	-1
$x[n-1]$:	-1	1	2	0
$x[n-2]$:	0	-1	1	2
$x[n-3]$:	2	0	-1	1

Parseval's Relation:

$$\frac{1}{N}\sum_{n=\langle N\rangle} |x[n]|^2 = \sum_{k=\langle N\rangle} |a_k|^2$$

Example:

$x[n]$ and $y[n]$ is periodic with period N.

$\omega[n] = \sum_{r=\langle N\rangle} x[r]\, y[n-r]$ is also periodic with period N.

Given Fourier series coefficients of $\omega[n]$ is —

$$c_k = \frac{\sin^2\left(\frac{3\pi k}{7}\right)}{7\sin^2\left(\frac{\pi k}{7}\right)} \text{ and } c_k = 7d_k^2,$$

d_k is the sinc function; fourier series of square wave $x[n]$, with $N_1 = 1$ and $N_2 = 7$

Using periodic convolution property,

$$\omega[n] = \sum_{r=\langle 7\rangle} x[r]\, x[n-r] = \sum_{r=-3}^{+3} x[r]\, x[n-r]$$

Fourier Series and LTI system

It is observed that for an input of a combination of complex exponentials. the output from an LTI system remain a combination of complex exponential that is weighted by a scale factor.

In other words if input $x(t) = e^{st}$ (s is complex) to an LTI system, output $y(t)$ is a weighted multiple of $x(t)$ —

$$y(t) = \underbrace{H(s)}_{\text{eigenvalue}} \underbrace{e^{st}}_{\text{eigenfunction}}.$$

where $H(s) = \int_{-\infty}^{+\infty} h(\tau)\, e^{-s\tau} d\tau$; $h(t)$ being the impulse response, and $s = \sigma + j\omega \in \mathbb{C}$

In discrete time, if $x[n] = z^n$ (where z is complex of the form $re^{j\omega}$).

$$y[n] = H(z)\, z^n$$

where $H(z) = \sum_{k=-\infty}^{+\infty} h[k] z^{-k}$, $h[n]$ is the discrete time function.

In general, s can be a complex number $s = \sigma + j\omega$. Now if s is taken as $j\omega$, making $\sigma = 0$, then $e^{j\omega t}$ become the input and $H(s)$ can be written as

$$H(j\omega) = \int_{-\infty}^{+\infty} h(t) e^{-j\omega t} dt$$ ← This is known as frequency response of the system.

Therefore the output becomes $y(t) = H(j\omega) e^{j\omega}$

Similarly for discrete time system, putting $r=1$

$$H(e^{j\omega}) \cancel{H(j\omega)} = \sum_{k=-\infty}^{+\infty} h[k]\, e^{-j\omega k}$$

Then the corresponding output becomes —

$$y[n] = H(e^{j\omega}) e^{j\omega n}$$

From the Fourier series expansion,

$$y(t) = \sum_{k=-\infty}^{\infty} a_k \underbrace{H(e^{jk\omega_0})}_{\text{Frequency response.}} e^{jk\omega_0 t}$$

where,

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k\, e^{jk\omega_0 t}$$

<u>Example:</u>

LTI system with $h[n] = \alpha^n u[n]$, $-1<\alpha<1$

and with input $x[n] = \cos\left(\frac{2\pi n}{N}\right)$

$$\Rightarrow x[n] = \frac{1}{2} e^{j\frac{2\pi}{N}n} + \frac{1}{2} e^{-j\frac{2\pi}{N}n}$$

Now, $$H(e^{j\omega}) = \sum_{n=0}^{\infty} \alpha^n e^{-j\omega n} = \sum_{n=0}^{\infty} (\alpha e^{-j\omega})^n$$

$$= \frac{1}{1-\alpha e^{-j\omega}}$$

Then,

$$y[n] = \frac{1}{2} H\left(e^{j\frac{2\pi}{N}}\right) e^{j\frac{2\pi}{N}n} + \frac{1}{2} H\left(e^{-j\frac{2\pi}{N}}\right) e^{-j\frac{2\pi}{N}n}$$

$$= \frac{1}{2}\, \frac{1}{1-\alpha e^{-j\frac{2\pi}{N}}} \cdot e^{j\frac{2\pi}{N}n} + \frac{1}{2}\, \frac{1}{1-\alpha e^{+j\frac{2\pi}{N}}} e^{-j\frac{2\pi}{N}n}$$

If $\dfrac{1}{1-\alpha e^{-j\frac{2\pi}{N}}} = r e^{j\theta}$, $y[n] = r\cos\left(\frac{2\pi}{N}n + \theta\right)$

Filtering :-

This concept of frequency response can be applied to design LTI system for pruning the frequency content of a signal.

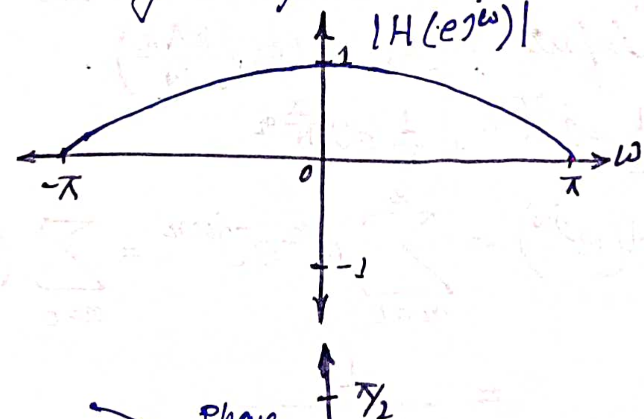
Generally these filters can be described by differential equation/difference equation in time; for example,

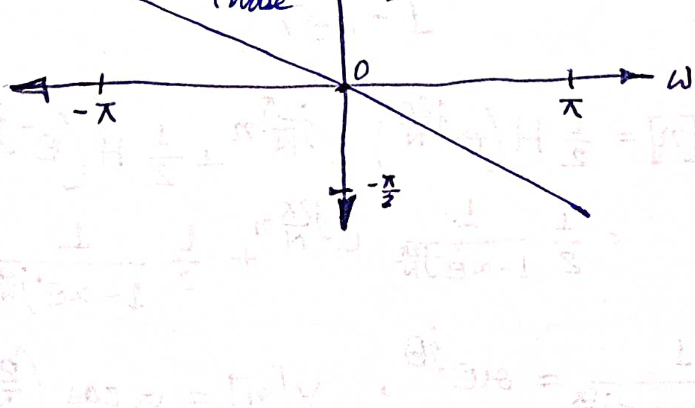
$$y[n] = \frac{1}{2}\{x[n] + x[n-1]\}$$

where $h[n] = \frac{1}{2}\{\delta[n] + \delta[n-1]\}$

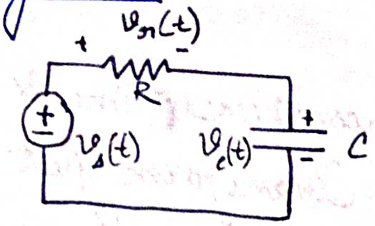
$$\therefore\ H(e^{j\omega}) = \frac{1}{2}[1 + e^{-j\omega}] = e^{-\frac{j\omega}{2}} \cos\left(\frac{\omega}{2}\right)$$

This is an averaging filter which passes lower half of the frequency component with graded weighted factors.





Analog Filter:

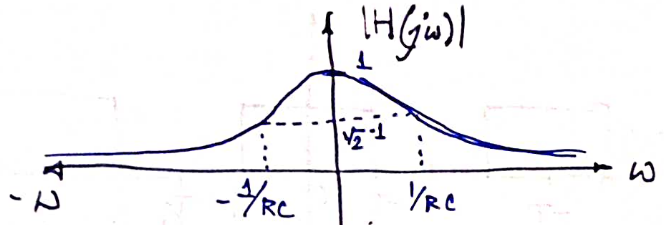


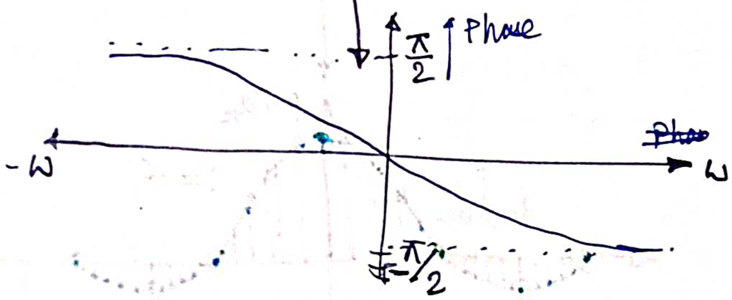
$$RC\frac{dv_c(t)}{dt} + v_c(t) = v_s(t)$$

If $x(t) = v_s(t) = e^{j\omega t}$

$$RC\frac{d}{dt}\left\{H(j\omega)e^{j\omega t}\right\} + H(j\omega)e^{j\omega t} = e^{j\omega t}$$

$$\therefore H(j\omega) = \frac{1}{1+RCj\omega} \text{ and } h(t) = \frac{1}{RC}e^{-\frac{t}{RS}}u(t)$$



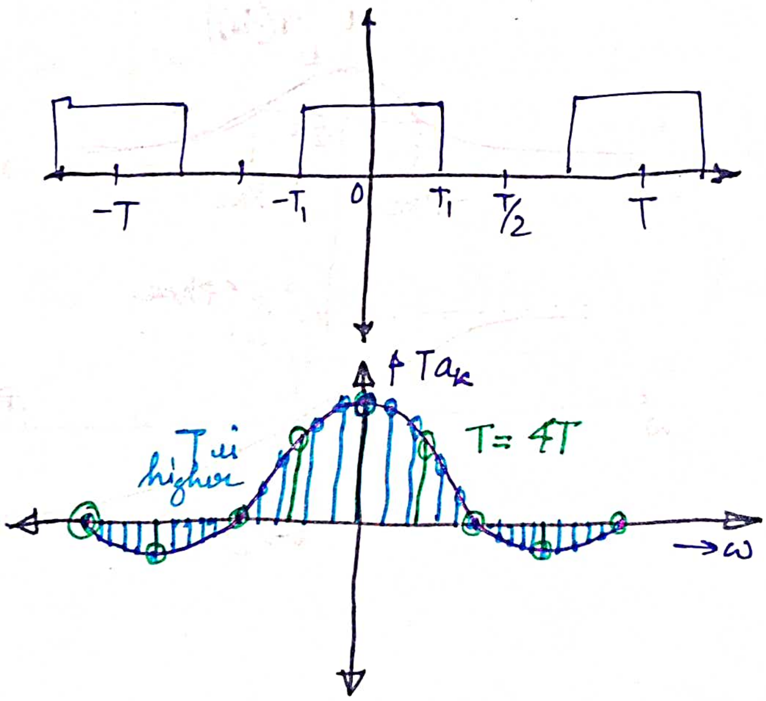


Fourier Transform

→ We have both continuous time & discrete time Fourier Transform just like Fourier Series.

Fourier series expansion as $x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jw_0kt}$

where $w_0 = \frac{2\pi}{T}$, now as $T \to \infty$, kw_0 becomes continuous.



$$a_k = \frac{2\sin(kw_0T_1)}{kw_0T} \; ; \text{ or } Ta_k = \frac{2\sin(kw_0T_1)}{kw_0}$$

$$\therefore Ta_k = \frac{2\sin(kw_0T_1)}{w} \Bigg|_{\text{at } w = kw_0}$$

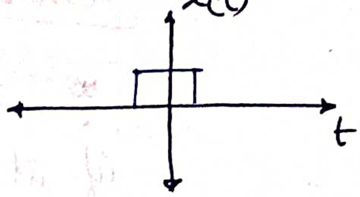
Varying T in terms ~~for~~ of T_1 as NT_1, shows that as N increases, the number of points on Ta_k increases.

For a single pulse —

$$\hat{x}(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$$

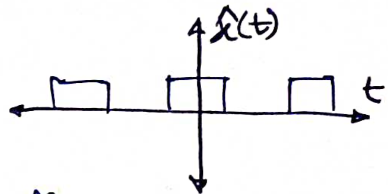
$$a_k = \frac{1}{T}\int_{-\frac{T}{2}}^{+\frac{T}{2}} \hat{x}(t) e^{-jk\omega_0 t} dt, \quad \omega_0 = \frac{2\pi}{T}$$

For a single pulse $\hat{x}(t) = x(t)$ if $|t| < \frac{T}{2}$



$$\Rightarrow a_k = \frac{1}{T}\int_{-\frac{T}{2}}^{+\frac{T}{2}} x(t) e^{-jk\omega_0 t} dt$$

$$= \frac{1}{T}\int_{-\infty}^{+\infty} x(t) e^{-jk\omega_0 t} dt$$



If we define $X(j\omega)$ as $X(j\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt$

then $a_k = \frac{1}{T} X(jk\omega)$

and $\hat{x}(t) = \sum_{k=-\infty}^{+\infty} \frac{1}{T} (jk\omega_0) e^{jk\omega_0 t}$

$$= \frac{1}{2\pi} \sum_{k=-\infty}^{+\infty} X(jk\omega) e^{jk\omega_0 t} \omega_0 \qquad \left[\omega_0 = \frac{2\pi}{T}\right]$$

As T increases and tends to ∞, $\omega_0 \rightarrow 0$, the summation may be replaced by integration in this limiting condition.

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega) e^{j\omega t} \cancel{dt}\, d\omega, \quad k\omega_0 = \omega$$

conversely, $X(j\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt$

* $$X(j\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt$$ [Forward Transform]

* $$x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega)\, e^{j\omega t}\, d\omega$$ [Inverse Transform]

Convergence of Fourier Transform

* If $x(t)$ has finite energy, that is if it is square integrable, so that

$$\int_{-\infty}^{+\infty} |x(t)|^2 dt < \infty$$

then it is guaranteed that the Fourier ~~Se~~ Transform $X(j\omega)$ converges.

* Other Drichlet conditions, on finite number of extrema and finite-numbered, finite-valued discontinuity also needs to be satisfied.

As we have considered $T \to \infty$, unlike Fourier Series, we can take in consideration the aperiodic signals as well, along withe periodic signals with the help of Fourier Transform.

Example:-

$x(t) = e^{-at}u(t)$, $a > 0$, an aperiodic signal.

$$X(j\omega) = \int_0^{\infty} e^{-at} e^{-j\omega t} dt = \int_0^{\infty} e^{-(a+j\omega)t} dt$$

$$= \frac{1}{a+j\omega}, \quad a > 0$$

Example:

$$x(t) = \delta(t)$$

$$X(j\omega) = \int_{-\infty}^{+\infty} \delta(t) e^{-j\omega t} dt = 1$$

For periodic signals, $\delta(\omega-\omega_0)$ appears at discrete values of ω i.e. at $\omega = \omega_0$.

For example, if we have $X(j\omega) = 2\pi\delta(\omega - \omega_0)$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} 2\pi \underbrace{\delta(\omega-\omega_0)}_{\text{non zero only at } \omega=\omega_0} e^{j\omega t} d\omega$$

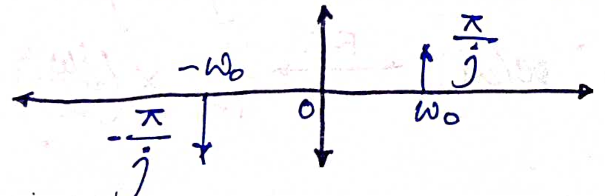
$$= e^{j\omega_0 t}$$

Example:- $x(t) = \sin(\omega_0 t)$

We have seen that Fourier Series coffient

$a_1 = \frac{1}{2j}$ and $a_{-1} = -\frac{1}{2j}$

Fourier Transform coefficients appear at $\omega = \pm\omega_0$



Example: $X(j\omega) = \delta(\omega) + \delta(\omega-\pi) + \delta(\omega-5)$

and $h(t) = u(t) - u(t-2)$

find $y(t) = x(t) * h(t)$

$$x(t) = \frac{1}{2\pi} + \frac{1}{2\pi}e^{j\pi t} + \frac{1}{2\pi}e^{j5t}$$ ← Non periodic

$$H(j\omega) = e^{-j\omega}\frac{2\sin\omega}{\omega},$$

$$Y(j\omega) = X(j\omega)H(j\omega)$$

$$= 2\delta(\omega) + \delta(\omega-5)H(j5) \quad \left[\begin{array}{l}\text{at } \omega = \pi \\ H(j\omega) = 0\end{array}\right]$$

$$\therefore y(t) = \frac{1}{2\pi}\cdot 2 + \frac{1}{2\pi}e^{-j5}\cdot\frac{2\sin 5}{5}\cdot e^{-j5t}$$

$$u(t) \xleftrightarrow{F.T.} \frac{1}{j\omega} + \pi\delta(\omega)$$

Properties of Fourier ~~Series~~ Transform

(Similar to Fourier Series)

Linearity: $ax(t) + by(t) \xleftrightarrow{F.T.} aX(j\omega) + bY(j\omega)$

Time shift: $x(t-t_0) \xleftrightarrow{F.T.} e^{-j\omega t_0}X(j\omega)$

Conjugate Symmetry: $x^*(t) \xleftrightarrow{F.T.} X^*(-j\omega)$

Differentiation: $\frac{d}{dt}x(t) \xleftrightarrow{F.T.} j\omega X(j\omega)$

Integration: $\int_{-\infty}^{t} x(t)\,dt \xleftrightarrow{F.T.} \frac{1}{j\omega}X(j\omega) + \pi X(0)\delta(\omega)$

Time scaling: $x(at) \xleftrightarrow{F.T.} \frac{1}{|a|}X\left(\frac{j\omega}{a}\right)$

__Convolution__: $y(t) = x(t) * h(t) \xleftrightarrow{F.T.} Y(j\omega) = X(j\omega)H(j\omega)$

__Multiplication__: $r(t) = s(t) * p(t) \xleftrightarrow{F.T.} \frac{1}{2\pi}\{S(j\omega) * P(j\omega)\}$

__Differential in frequency__: $tx(t) \xleftrightarrow{F.T.} j\frac{d}{d\omega}X(j\omega)$

$$\frac{d^2y(t)}{dt^2} + 4\frac{d}{dt}y(t) + 3y(t) = \frac{d}{dt}x(t) + 2x(t)$$

Taking Fourier Transform,

$$Y(j\omega)\{(j\omega)^2 + 4j\omega + 3\} = X(j\omega)\{j\omega + 2\}$$

$$\Rightarrow H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{j\omega + 2}{(j\omega)^2 + 4j\omega + 3} = \frac{\frac{1}{2}}{1 + j\omega} + \frac{\frac{1}{2}}{j\omega + 3}$$

$$\therefore h(t) = \frac{1}{2}e^{-t}u(t) + \frac{1}{2}e^{-3t}u(t)$$

$$\left[e^{-at}u(t) \xleftrightarrow{F.T.} \frac{1}{a + j\omega}\right]$$

$$\int_0^\infty e^{-at}e^{-j\omega t}dt$$

$$te^{-at}u(t) \xleftrightarrow{F.T.} \frac{1}{(a + j\omega)^2}$$

Discrete Time Fourier Transform

$$\hat{x}[n] = \sum_{k=\langle N\rangle} a_k \, e^{j\left(\frac{2\pi}{N}\right)n}$$

$\Leftarrow$ Discrete time Fourier series.

As $N \rightarrow \infty$, $\omega_0 \rightarrow 0$

$$\left\{ \begin{array}{l} x[n] = \dfrac{1}{2\pi} \displaystyle\int_{2\pi} X(e^{j\omega}) \, e^{j\omega n} d\omega \\ \text{and } X(e^{j\omega}) = \displaystyle\sum_{n=-\infty}^{+\infty} x[n] \, e^{-j\omega n} \end{array} \right\}$$

$$X\left(e^{j(\omega+2\pi)}\right) = \sum_{n=-\infty}^{\infty} x[n] \, e^{-j(\omega+2\pi)n}$$

$$= \sum_{n=-\infty}^{+\infty} x[n] \, e^{-j\omega n} \cdot e^{-j2\pi n}$$

Example: $x[n] = a^n u[n]$, $|a| < 1$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} a^n u[n] e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} a^n e^{-j\omega n} = \frac{1}{1 - ae^{-j\omega}}$$

Example: $x[n] = \begin{cases} 1, & |n| \leq N_1 \\ 0, & |n| > N_2 \end{cases}$

$$X(e^{j\omega}) = \sum_{n=N_1}^{+N_1} e^{-j\omega n} = \frac{\sin\left(N_1 + \frac{1}{2}\right)}{\sin\left(\frac{\omega}{2}\right)}$$

Example:

$x[n] = \cos(\omega_0 n)$

$x[n] = \frac{1}{2}e^{j\omega_0 n} + \frac{1}{2}e^{-j\omega_0 n}$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty}\left[\frac{1}{2}e^{j\omega_0 n} + \frac{1}{2}e^{-j\omega_0 n}\right]e^{-j\omega n}$$

$$= \sum_{l=-\infty}^{+\infty} \pi\,\delta(\omega-\omega_0-2\pi l) + \sum_{l=-\infty}^{+\infty} \pi\,\delta(\omega+\omega_0-2\pi l)$$

$$= \pi\,\delta(\omega-\omega_0) + \pi\,\delta(\omega+\omega_0), \quad -\pi \leq \omega \leq \pi$$

Let $S = \sum_{n=-\infty}^{+\infty} \frac{1}{2}e^{j(\omega-\omega_0)n}$

at $\omega = \omega_0$, $S = \frac{1}{2}$

$$\frac{1}{2}\sum_{n=-\infty}^{\infty}\left[\cos(\omega_0-\omega)n + j\sin(\omega_0-\omega)n\right]$$

Example: $x[n] = u[n-2] - u[n-6]$

$x[n] = \delta[n-2] + \delta[n-3] + \delta[n-4] + \delta[n-5]$

$X(e^{j\omega}) = e^{-2j\omega} + e^{-3j\omega} + e^{-4j\omega} + e^{-5j\omega}$.

Example: $x[n] = \left(\frac{1}{2}\right)^{-n} u[-n-1]$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{-1}\left(\frac{1}{2}\right)^{-n} e^{-j\omega n} = \sum_{m=1}^{\infty}\left(\frac{1}{2}e^{j\omega}\right)^{m} = \frac{e^{j\omega}}{2-e^{j\omega}}$$

Example:

$$x[n] = \underbrace{\frac{\sin\left(\frac{\pi n}{5}\right)}{\pi n}}_{x_1[n]} \underbrace{\cos\left(\frac{7\pi n}{2}\right)}_{x_2[n]}$$

$$X_1(e^{j\omega}) = \begin{cases} 1, & |\omega| < \frac{\pi}{5} \\ 0, & \frac{\pi}{5} \leq |\omega| < \pi \end{cases}$$

$$X_2(e^{j\omega}) = \pi\left[\delta\left(\omega - \frac{\pi}{2}\right) + \delta\left(\omega + \frac{\pi}{2}\right)\right]$$

$X(e^{j\omega})$ = Periodic convolution of $X_1(e^{j\omega})$ and $X_2(e^{j\omega})$

Example: $x[n] = \alpha^{|n|}$, $|\alpha| < 1$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} \alpha^{|n|} e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{-1} \alpha^{-n} e^{-j\omega n} + \sum_{n=0}^{\infty} \alpha^{n} e^{-j\omega n}$$

$$= \frac{\alpha e^{+j\omega}}{1 - \alpha e^{+j\omega}} + \frac{1}{1 - \alpha e^{-j\omega}}$$

$$= \frac{1 - \alpha^2}{1 - 2\alpha\cos\omega + \alpha^2}$$

Example: Given $h[n] = \left(\frac{1}{2}\right)^n u[n]$

Find output $y[n]$ for input $x[n] = (n+1)\left(\frac{1}{4}\right)^n u[n]$

$$X(e^{j\omega}) = \frac{1}{\left(1-\frac{1}{4}(e^{-j\omega})\right)^2};$$

$$H(e^{j\omega}) = \frac{1}{1-\frac{1}{2}e^{-j\omega}}$$

$$Y(e^{j\omega}) = X(e^{j\omega})\cdot H(e^{j\omega}) = \frac{1}{\left(1-\frac{1}{4}(e^{-j\omega})\right)^2\left(1-\frac{1}{2}e^{-j\omega}\right)}$$

$$= \frac{2}{1-\frac{1}{4}e^{-j\omega}} - \frac{3}{\left(1-\frac{1}{4}e^{-j\omega}\right)^2} + \frac{4}{1-\frac{1}{2}e^{-j\omega}}$$

$$y[n] = 2\left(\frac{1}{4}\right)^n u[n] - 3(n+1)\left(\frac{1}{4}\right)^n u[n] + 4\left(\frac{1}{2}\right)^n u[n]$$

Properties of DTFT:-

* **Periodic in frequency domain:** $X(e^{j(\omega+2\pi)}) = X(e^{j\omega})$
* **Time shifting:** $x[n-n_0] \xleftrightarrow{DTFT} e^{-j\omega n_0} X(e^{j\omega})$
* **Difference:** $x[n]-x[n-1] \xleftrightarrow{DTFT} (1-e^{-j\omega})X(e^{j\omega})$
* **Summation:** $\sum_{m=-\infty}^{+\infty} x[m] \xleftrightarrow{DTFT} \frac{1}{1-e^{-j\omega}} X(e^{j\omega}) \cancel{+\pi X(e}$

$$+\pi X(e^{j0})\sum_{k=-\infty}^{\infty}\delta(\omega-2k)$$

* **Convolution:** $x[n] * y[n] \xleftrightarrow{DTFT} X(e^{j\omega})\, Y(e^{j\omega})$
* **Multiplication:** $x[n]\cdot y[n] \xleftrightarrow{DTFT} \frac{1}{2\pi}\int_{2\pi} X(e^{j\theta})\cdot Y(e^{j(\omega-\theta)})\,d\theta$

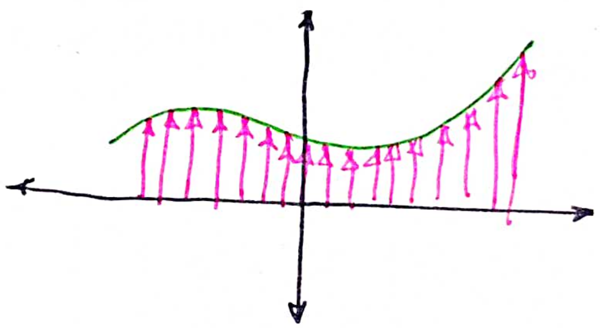
Transforms & Basis :-

Consider $\mathbb{R}^n$ with basis $\{e_1, \dots, e_i\}$

$$e_i = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \rightarrow i^{th} \text{ position.}$$

Consider a f^n space $\mathcal{F}$ of real valued functions in $\mathbb{R}$.

The basis is given by Dirac-delta functions —



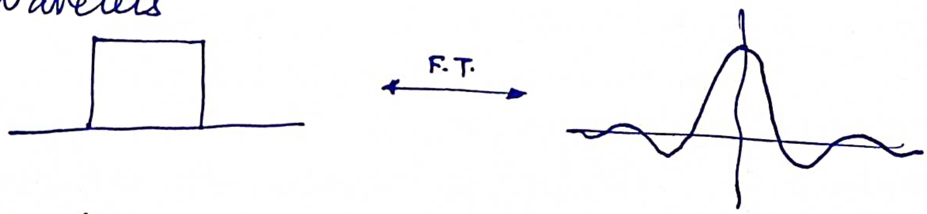
Inner product : $\displaystyle\int_{-\infty}^{+\infty} f_1(t)\cdot f_2(t)\,dt$

$\therefore$ The basis of Dirac-delta function train is orthogonal.

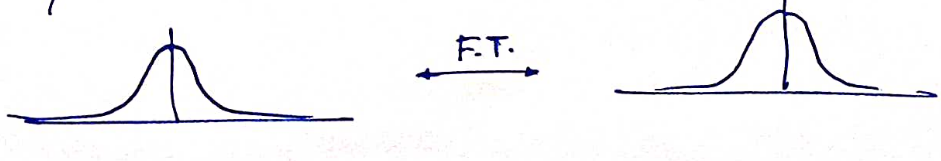
Another basis : Complex exponentials $\{e^{j\omega t}\}_{\omega}$

another basis : $\{e^{st}\}_{s\in\mathbb{C}}$ (orthogonal)

Wavelets



Gaussian



$x(t) = e^{-at}u(t)$, Show that $X(j\omega) = \dfrac{1}{a+j\omega}$

$$\mathcal{F}^{-1}\left\{\left(\frac{1}{a+j\omega}\right)\right\}$$

$$\mathcal{F}^{-1}\left\{\frac{1}{a+j\omega}\right\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{a-j\omega}{a^2+\omega^2} e^{j\omega t} d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{a\cos(\omega t) + aj\sin(\omega t) - j\omega\cos(\omega t) + \omega\sin(\omega t)}{a^2+\omega^2} d\omega$$

$$= \frac{1}{\pi}\int_{0}^{\infty} \frac{a\cos(\omega t) + \omega\sin(\omega t)}{a^2+\omega^2} d\omega$$

for $t \geq 0$,

$$\mathcal{F}^{-1}\left\{\frac{1}{a+j\omega}\right\} = \frac{1}{\pi}\left\{\frac{\pi}{2}e^{-at} + \frac{\pi}{2}e^{-at}\right\} = e^{-at}$$

$$A v_1 = \lambda_1 v_1 \quad , \quad A v_2 = \lambda_2 v_2$$

$$A \begin{bmatrix} v_1 & v_2 \end{bmatrix} = \begin{bmatrix} v_1 & v_2 \end{bmatrix} \cancel{\begin{bmatrix} \lambda_1 & 0 \\ & \lambda_2 \end{bmatrix}} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

$$A V_{n \times n} = V D \Rightarrow V^{-1} A V = D$$

For LTI systems,

$$e^{j\omega t} \longrightarrow \boxed{h} \longrightarrow H(j\omega)\, e^{j\omega t} .$$

Proof :-

$$y(t) = \int_{-\infty}^{+\infty} h(\tau) \cdot e^{j\omega(t-\tau)} d\tau = e^{j\omega t} \left\{ \int_{-\infty}^{+\infty} h(\tau) e^{-j\omega \tau} d\tau \right\}$$

$e^{j\omega t} \longrightarrow$ eigenfunction —
eigenvalues of $H(j\omega)$

Fourier, Laplace, Wavelet transforms are basically ~~building blocks of~~ change in basis.

* For unit step $\longrightarrow$

$$u(t) = \frac{1}{2} \left(sgn(x) + 1 \right) \longleftrightarrow \left\{ \frac{1}{j\omega} + \pi \delta(\omega) \right\}$$

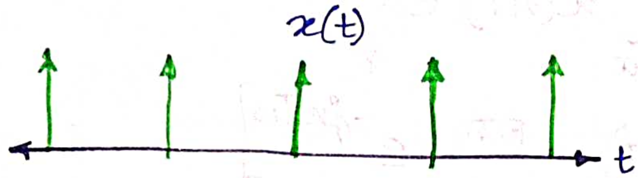
* For Tent function

$$F^{-1}\{ e^{-|a|t} \} \longrightarrow$$

Periodic Function :—

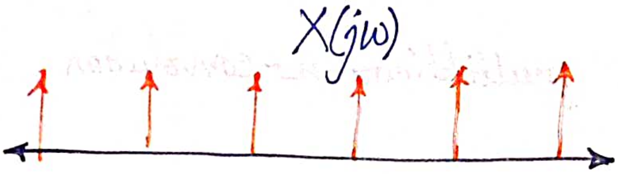
$$\mathcal{F}\left\{\sum_{k=-\infty}^{\infty} a_k e^{j\omega_0 kt}\right\} = 2\pi \sum_{k=-\infty}^{\infty} a_k \,\delta(\cancel{\omega_0}\,\omega - k\omega_0)$$

Train of Impulses :—



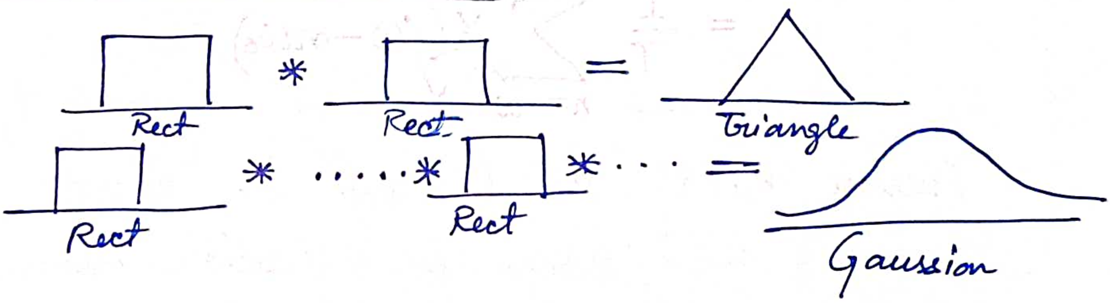
$$x(t) = \delta(t-kT)$$

where $a_k = \frac{1}{T}$, the Fourier Series Coefficients.



$$X(j\omega) = \mathcal{F}\{\delta(t-kT)\}$$
$$= \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta\left(\omega - k\frac{2\pi}{T}\right)$$

Central Limit Theorem :



If we have n i.i.d. random variables, $X_1, X_2, X_3, X_4, \dots, X_n$. Then

$Y = \sum_{k=1}^{n} X_k$ approaches Gaussian as $n \to \infty$

[Note: if $Y = X_1 + X_2$, then

$$f_Y(t) = f_{X_1}(t) * f_{X_2}(t)$$

where f_Y, f_{X_1}, f_{X_2} are p.d.f.s]

Sampled Signal:—

$$x_d(t) = x(t) \cdot \sum_{n=-\infty}^{\infty} \delta(t-nT) = \sum_{n=-\infty}^{+\infty} x(nT)\,\delta(t-nT)$$

$$\Rightarrow X_d(j\omega) = \mathcal{F}\left\{\sum_{n=-\infty}^{\infty} x(nT)\,\delta(t-nT)\right\}$$

$$= \sum_{n=-\infty}^{\infty} x(nT)\, e^{-jnT\omega}$$

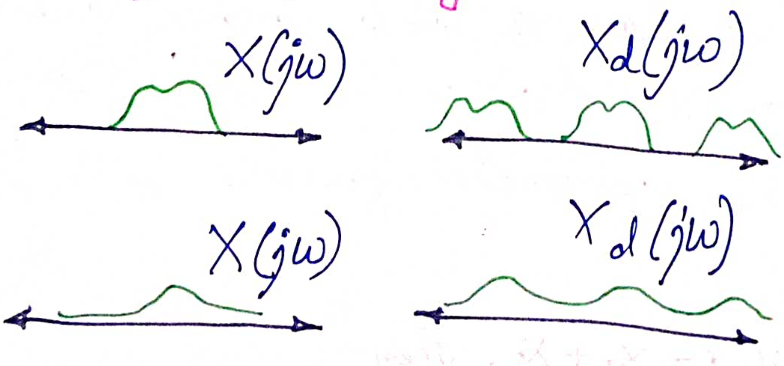
$$\left[\because\ \delta(t-nT) \xleftrightarrow{F.T.} e^{-jnT\omega}\right]$$

Again, if we use multiplication-convolution property of F.T.

$$X_d(j\omega) = X(j\omega) * \sum_{n=-\infty}^{\infty} \delta\left(\omega - n\frac{2\pi}{T}\right)$$

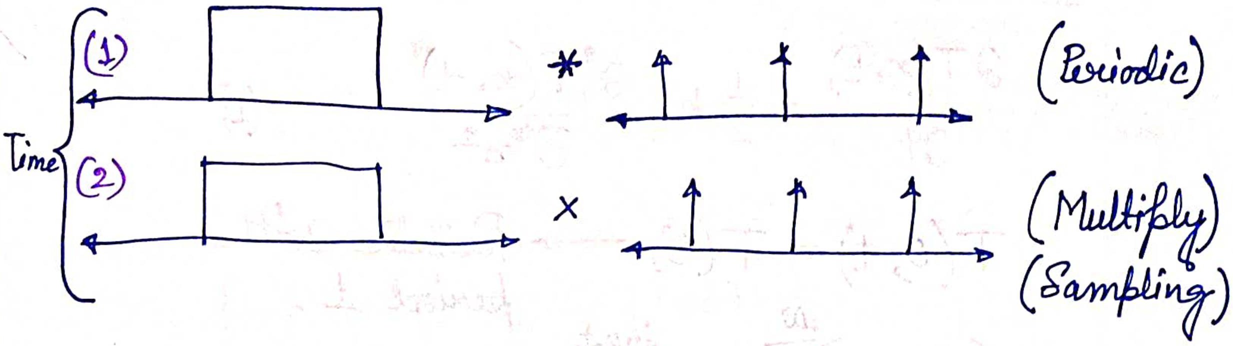
$$= \frac{1}{T}\sum_{n=-\infty}^{\infty} X(j\omega - n\omega_0)$$

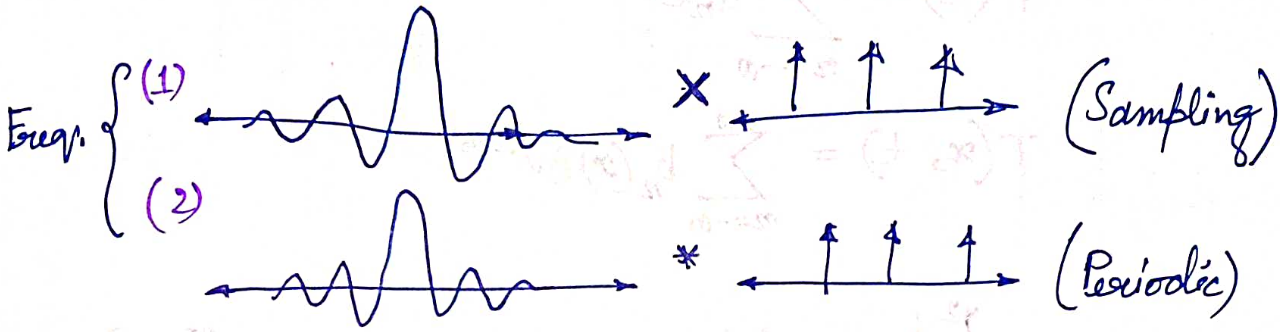
Observe $x_d(t)$ is basically the discrete version of $x(t)$; Hence discretization implies periodicity in frequency domain.



$$\boxed{\text{Sampling in time domain} \longleftrightarrow \text{Periodicity in frequency domain}}$$

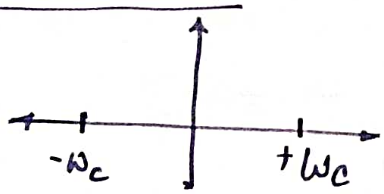
Informal Relation between Transforms :-



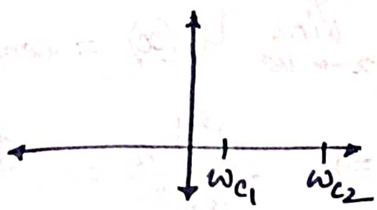


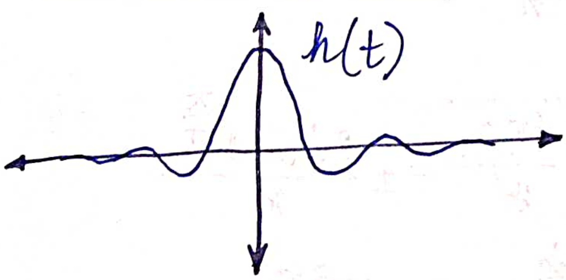
DTFT :-

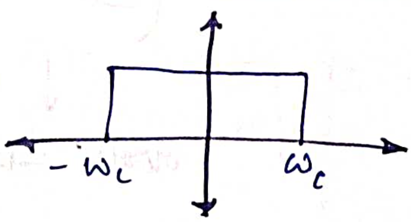
Band Limited



or,







$\int_{-\infty}^{+\infty} |h(t)|\, dt$ is not integrable, hence not stable

Heat Equation —

$$\frac{\partial T(x,t)}{\partial t} = \frac{1}{2}k^2 \frac{\partial^2 T(x,t)}{\partial x^2} \quad \text{——} (1)$$

$$T(0,t) = T(t) \longrightarrow \text{Periodic with period 1.}$$

$$T(t) = \sum_{n=-\infty}^{+\infty} a_n e^{jn\omega_0 t}$$

$$T(x,t) = \sum_{n=-\infty}^{+\infty} b_n(x) e^{jn\omega_0 t}$$

$$\Rightarrow \frac{d^2 b_n}{\partial x^2} = \frac{2jn\omega_0}{k^2} b_n \qquad \left[\text{where } \omega_0 = \frac{2\pi}{1} = 2\pi\right] \quad [\text{from } (1)]$$

$$b_n(0) = a_n$$

$$\lim_{x\to\infty} b_n(x) = \text{const.}$$

$$\Rightarrow \frac{d^2 b_n}{dx^2} = \frac{4\pi jn}{k^2} b_n$$

$$\Rightarrow \left(D^2 - \frac{4\pi jn}{k^2}\right) b_n = 0$$

$$\downarrow$$

$$\text{roots} \Rightarrow s = \pm\frac{\sqrt{4\pi jn}}{k} = \pm\frac{2\sqrt{\pi n}\, e^{j\frac{\pi}{4}}}{k}$$

$$b_n(x) = c_1 e^{sx} + c_2 e^{-sx}$$

Now if $n > 0$, $\lim_{x\to\infty} b_n(x)$ diverges if $c_1 \neq 0$

$\therefore \quad c_1 = 0$

$$\Rightarrow b_n(x) = a_n e^{-\frac{\sqrt{2\pi n}(1+j)}{k}x}$$

Similarly, for $n < 0$, $c_2 = 0$

Fast Fourier Transform

$$\text{DTFS} \approx \text{DFT}$$

$$\begin{bmatrix} \\ \\ \\ \end{bmatrix} = \begin{bmatrix} & F & \end{bmatrix} \begin{bmatrix} x(0) \\ \vdots \\ x(n) \end{bmatrix}$$

$$[\,- - - - -\,]\begin{bmatrix} \vdots \end{bmatrix} \to O(n) \qquad \begin{bmatrix} & & \end{bmatrix}\begin{bmatrix} \\ \end{bmatrix} \; O(n^2)$$

$$\underset{\text{matrix}}{F}\,\underset{\text{vector}}{x} = FPP^T x, \text{ where } P \text{ is a permutation matrix}$$

$$\cancel{F_8 x = F_8}$$

$$F_8 x = F_8(:, \text{cols})\, x(\text{cols}) = \begin{bmatrix} F_4 & \Omega_4 F_4 \\ F_4 & -\Omega_4 F_4 \end{bmatrix} \begin{bmatrix} x(1:2:8) \\ x(2:2:8) \end{bmatrix} = \begin{bmatrix} I_4 & \Omega_4 \\ I_4 & -\Omega_4 \end{bmatrix} \times \begin{bmatrix} F_4 x(1:2:8) \\ F_4 x(2:2:8) \end{bmatrix}$$

ALGORITHM:

function $y : fft(x, n)$

if $n = 1$

$y = x$

else

$m = \frac{n}{2}$

$y_T = fft(x(1:2:n), m)$

$y_B = fft(x(2:2:n), m)$

$\omega = e^{-\frac{2\pi j}{n}}$

$v = [1 \ \omega \ \ldots \ \omega^{m-1}]^T$

$z = v * y_B$

$y := \begin{bmatrix} y_T + z \\ y_T - z \end{bmatrix}$

end

$$f_n = 2 f_{\frac{n}{2}} + O(n) \Rightarrow f_n = O(n \log n)$$

LAPLACE TRANSFORM

The Laplace Transform of a signal is defined as :—

$$X(s) := \int_{-\infty}^{\infty} x(t)e^{-st}dt$$

where $s = \sigma + j\omega$, $s \in \mathbb{C}$

The region where $X(s)$ converges is called "Region of Convergenc

$$X(\sigma + j\omega) = \int_{-\infty}^{\infty} \left(x(t)e^{-\sigma t}\right) e^{-j\omega t}dt$$

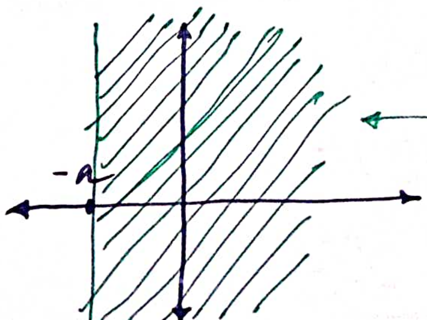
$$\Rightarrow x(t)e^{-\sigma t} = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\sigma + j\omega)\, e^{j\omega t}d\omega$$

$$\Rightarrow x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\sigma+j\omega)e^{(\sigma+j\omega)t}d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(s)e^{st}d\omega$$

$$\therefore X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}dt$$

$$\&,\ x(t) = \frac{1}{2\pi j}\int_{\sigma - j\infty}^{\sigma + j\infty} X(s)e^{-st}ds$$

Provided $\sigma \in$ Region of Convergence



Region of convergence for the signal $e^{-at}u(t)$

Example: $x(t) = e^{-at}u(t)$

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}\,dt$$

$$= \int_{\cancel{-\infty}\ 0}^{\infty} e^{-(s+a)t}\,dt = \frac{1}{s+a}$$

Region of Convergence (ROC) is where the integral $X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}\,dt$ converges

$u(t) \longleftrightarrow \frac{1}{s}$ [ROC $\rightarrow$ $Re(s) > 0$]

$\delta(t) \longleftrightarrow 1$ [ROC $\rightarrow$ $\mathbb{C}$]

ROC of $e^{-|a|t}$ is $\sigma \in (-a, a)$

Transfer

Transfer function is the impulse response of an LTI system.

$y = x * h$

If $H(s)$ is the transfer function,
that is $H(s) = \mathcal{X}(h(t))$

$\therefore\ y(t) = x(t) * h(t)$

$\Rightarrow Y(s) = X(s) \cdot H(s)$

if $H(s)$ is of the form $H(s) = \dfrac{P(s)}{Q(s)}$

$$\boxed{\begin{array}{l} \text{roots of } \{P(s)\} \longrightarrow \text{Zeros} \\ \text{roots of } \{Q(s)\} \longrightarrow \text{Poles} \end{array}}$$

$$e^{s_0 t} \longrightarrow \boxed{h} \longrightarrow H(s_0)e^{s_0 t}$$

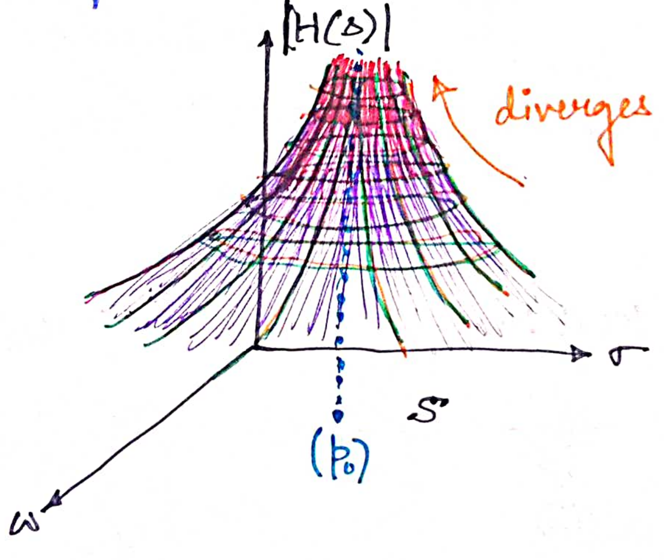
Now if s_0 is a zero of the transfer function

$\Rightarrow P(s_0) = 0$

$\Rightarrow H(s_0) = \dfrac{P(s_0)}{Q(s_0)} = 0$

$\Rightarrow H(s_0)e^{s_0 t} = 0$.

Similarly the ~~fo~~ output $H(s_0)e^{s_0 t}$ diverges at poles.

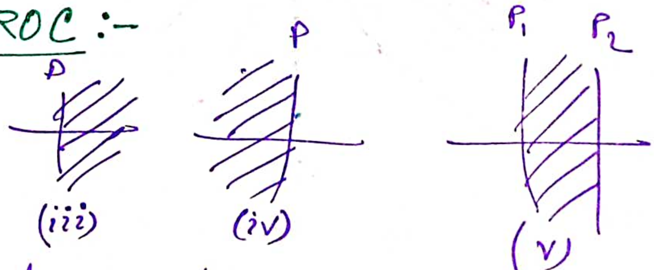


PROPERTIES OF ROC :-

1. ROC always consists of strips parallel to the $j\omega$ axis, in the s-plane
2. ROC doesn't contain any pole. (that is it doesn't contain the boundary)
3. If $x(t)$ is of finite duration, then ROC $= \mathbb{C}$ provided $x(t)$ is bounded.
4. The pole is in $\sigma > 0$ if causal, $\sigma < 0$ if anticausal.
5. If $x(t)$ is right sided ~~signal~~ signal & if $Re(s) < \sigma_0$, then ~~Re(s)~~ $s \in$ ROC (Same with left sided signal)
6. For both sided signals, it is either empty or bounded by poles (if it is rational)

Types of ROC :-

(a) Φ (i) $\mathbb{C}$ (ii)



(iii) (iv) (v)

(b) Causal signal may have ROC (i) & (iii)

(c) $x(t) \rightarrow$ finite duration, $\rightarrow$ ROC $= \mathbb{C}$

(ROC of $e^{-t^2} \rightarrow \mathbb{C}$)

PROPERTIES OF

1. Linearity: $a_1x_1(t) + a_2x(t) \longleftrightarrow a_1x_1(s) + a_2X_2(s)$

$$ROC \supseteq R_1 \cap R_2$$

2. Time shift: $x(t-t_0) \longleftrightarrow e^{-st_0}X(s)$

[ROC remains unchanged]

3. s-shift: $e^{s_0t}x(t) \longleftrightarrow X(s-s_0)$ $\quad ROC_{new} = ROC_{old} + Re(S_0)$

4. Time scale: $x(at) \longleftrightarrow \frac{1}{|a|}X\left(\frac{s}{a}\right)$ $\quad ROC_{new} = a\,ROC_{old}$

5. Conjugation: $x^*(t) \longleftrightarrow X^*(s^*)$ $\quad$ ROC unchanged.

6. Convolution: $x_1 * x_2 \longleftrightarrow X_1(s)\cdot X_2(s)$ $\quad ROC \supseteq R_1 \cap R_2$

7. Differentiation (time): $\frac{d}{dt}(x(t)) \longleftrightarrow sX(s)$ $\quad ROC_{new} \supseteq ROC_{old}$

8. Differentiation (freq): $-tx(t) \longleftrightarrow \frac{dX(s)}{ds}$ $\quad$ ROC unchanged

9. Integration (time): $\int_{-\infty}^{t} x(t)\,dt \longleftrightarrow \frac{X(s)}{s}$ $\quad ROC \supseteq R \cap \{Re\{s\} > 0\}$

Differentiation in time domain —

$$x(t) \longleftrightarrow X(s)$$

$$\therefore \frac{d}{dt}(x(t)) = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} sX(s)e^{st}\,ds$$

Conversely, $\int_{-\infty}^{\infty} \frac{dx}{dt}e^{-st}dt = x(t)(e^{-st})\Big|_{-\infty}^{\infty} + s\int_{-\infty}^{\infty} x(t)e^{-st}dt$

$$= \underbrace{x(t)(e^{-st})\Big|_{-\infty}^{\infty}}_{} + sX(s)$$

as Laplace Transform of $x(t)$ exists & its derivative exists, this must go to 0 at both extremeties

Initial Value Theorem:-

Consider a continuous time signal $x(t)$ with its Laplace Transform $X(s)$ such that $x(t) = 0$ for $t < 0$. Since it is causal, it contains the point ϕ in ROC & can be written as $x(t)u(t)$. Taking Taylor's expansion of $x(t)u(t)$ at $t = 0^+$

$$x(t) = \left[x(0^+) + x'(0^+)t + x''(0^+)\frac{t^2}{2}\right]u(t)$$

$$\Rightarrow X(s) = \frac{x(0^+)}{s} + \frac{x'(0^+)}{s^2} + \cdots$$

$$\Rightarrow sX(s) = x(0^+) + \frac{x'(0^+)}{s} + \cdots$$

Thus we have $\longrightarrow$ $\boxed{\lim_{s \to \infty} sX(s) = x(0^+)}$

Final Value Theorem:-

For a causal signal $x(t) \longrightarrow$

$$\int_0^\infty \frac{dx}{dt} e^{-st} dt = x(t)(e^{-st})\Big|_0^\infty + s\int_0^\infty x(t)e^{-st}dt$$

$$= -x(0) + sX(s)$$

$$\Rightarrow \lim_{s \to 0} \int_0^\infty \frac{dx}{dt} e^{-st} dt = \lim_{s \to 0} sX(s) - x(0)$$

$$\Rightarrow \lim_{t \to \infty} x(t) = \lim_{s \to 0} sX(s)$$

Unilateral Laplace Transform :-

It is defined as (only for causal signals)

$$X(s) := \int_{0^-}^{\infty} x(t)\, e^{-st}\, dt$$

Differentiation in time domain —

$$\int_{0^-}^{\infty} \frac{dx}{dt} e^{-st} dt = x(t)\, e^{-st}\Big|_{0^-}^{\infty} + s\int_{0^-}^{\infty} x(t) e^{-st} dt$$

$$= sX(s) - x(0^-)$$

Pro tip: Use Laplace Transform to convert differential equations into linear equations.

Example: $\dot{y} + ay = x$

$$\Rightarrow \quad sY(s) - y(0) + aY(s) = U(s)$$

$$\Rightarrow \quad Y(s) = \frac{U(s) + y(0)}{s+a}$$

We can deal with initial conditions using unilateral transform.

Example:-

Consider the pde

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x}, \quad \alpha \in \mathbb{R}$$

and $u(t, x)$ is the temperature distribution.

Let $u(0, x)$ be distribution at $t=0$.

$$\hat{u} = \mathcal{F}_x\{u\} = \int_{-\infty}^{\infty} u(t, x)\, e^{-j\omega x} dx$$

$$\therefore \hat{u}(t, \omega) = \int_{-\infty}^{\infty} u(t, x)\, e^{-j\omega x} dx.$$

$$U(s, x) = \int_{0^-}^{\infty} u(t, x)\, e^{-st} dt = \mathcal{L}\{(u)\}$$

Now, $\hat{U}(s, \omega) = \int_{0^-}^{\infty} \hat{u}(t, \omega)\, e^{-st} dt$

$$\therefore \frac{\partial \hat{u}}{\partial t} = -\alpha^2 \omega^2 \hat{u}$$

$$\Rightarrow s\hat{U}(s, \omega) - \hat{u}(0, \omega) = -\alpha^2 \omega^2 \hat{U}(s, \omega)$$

$$\Rightarrow \hat{U}(s, \omega) = \frac{\hat{u}(0, \omega)}{s + \alpha^2 \omega^2}$$

Taking inverse Laplace transform—

$$\hat{u}(t, \omega) = e^{-\alpha^2 \omega^2 t}\, \hat{u}(0, \omega)$$

Taking inverse Fourier Transform,

$$u(t,x) = \mathcal{F}^{-1}\{e^{-\alpha^2\omega^2 t}\} * u(0,x)$$

$$= \frac{e^{-\frac{x^2}{4\alpha^2 t}}}{2\alpha\sqrt{\pi t}} * u(0,x)$$

For example if $u(0,x) = \delta(x)$

$$\Rightarrow u(t,x) = \frac{e^{-\frac{x^2}{4\alpha^2 t}}}{2\alpha\sqrt{\pi t}}$$

Comment:-

Laplace Transform $\longleftrightarrow$ Systems

Fourier Transform $\longleftrightarrow$ Signals.

System properties via ROC

(1) Causality $\longrightarrow$ ROC is right half plane to the right most pole.

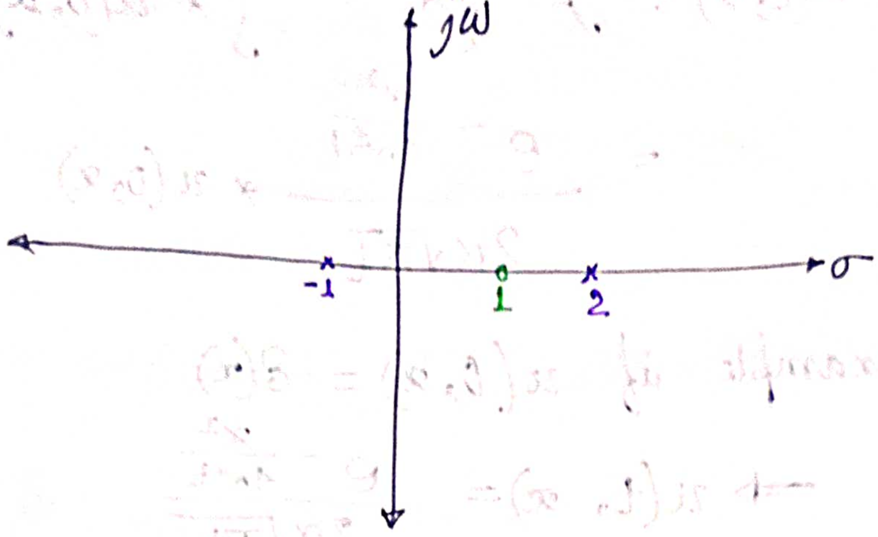
(2) Anticausal $\longrightarrow$ ROC is left half plane to the leftmost pole.

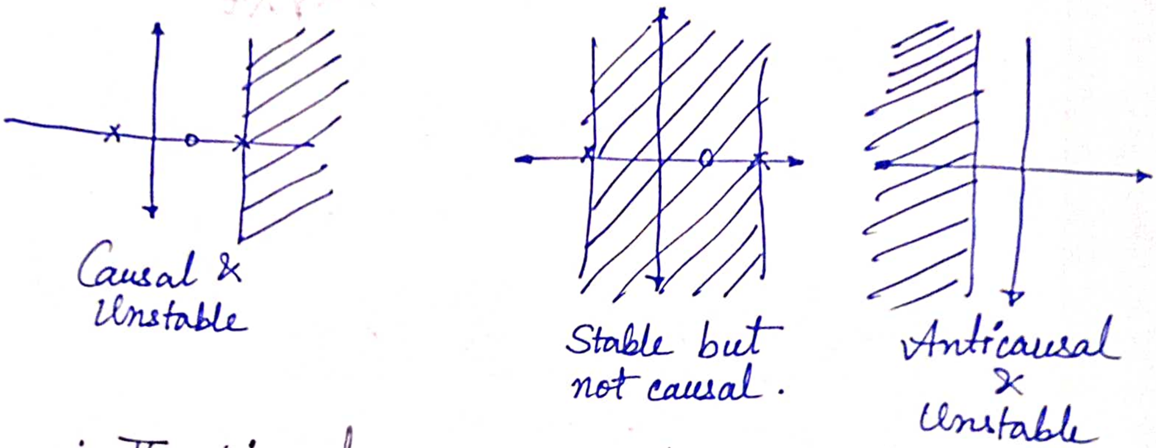
(3) Stability $\longrightarrow$ ROC must include the $j\omega$ axis.

$$\int_{-\infty}^{\infty} |h| dt < \infty \Rightarrow \int_{-\infty}^{\infty} h e^{-j\omega t} dt \Rightarrow H(j\omega) \Rightarrow \{\sigma = 0\} \in ROC$$

$\therefore$ Causal & Stable $\Rightarrow$ the rightmost pole must be less than 0.

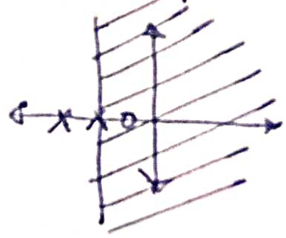
Example : $H(s) = \frac{s-1}{(s+1)(s-2)}$





∴ The signal can never be causal & stable simultaneously for this impulse response.

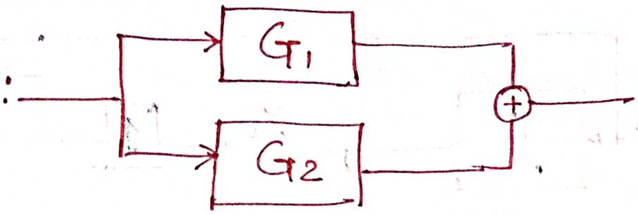
Example : $H(s) = \frac{s+1}{(s+2)(s+3)}$



Series:

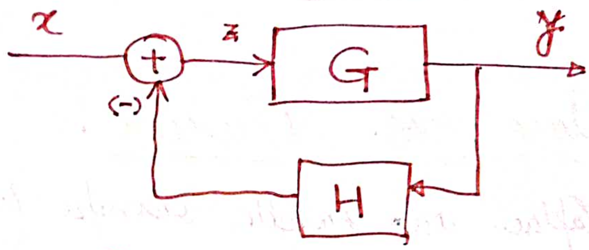
$G = G_1 G_2$

Parallel:



$G = G_1 + G_2$

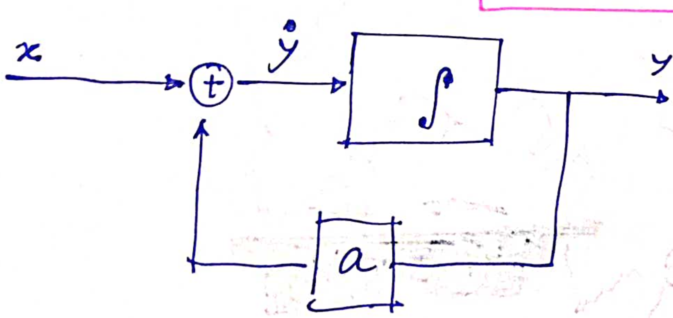
Closed Loop:



$$Y(s) = G \cdot Z$$
$$= G \cdot (-HY + X)$$
$$\Rightarrow Y(1+GH) = XG$$
$$\Rightarrow \frac{Y}{X} = \left(\frac{G}{1+GH}\right)$$

order: $\dot{y} + ay = x$

$\boxed{\text{Order} = \text{degree of denomenator polynomial}}$

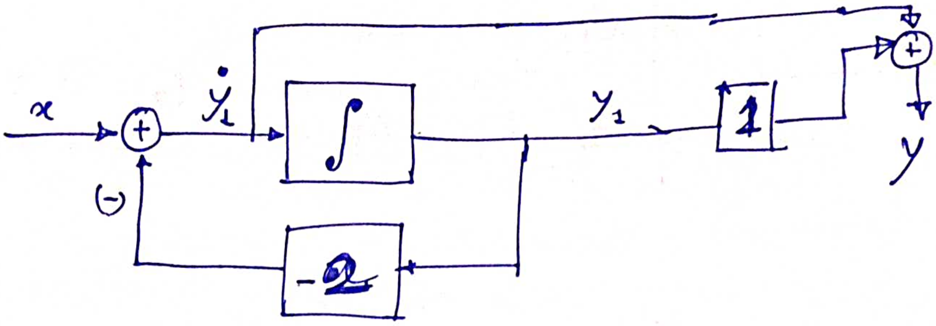


We will assum deg(Numerator) < deg(Denometer)

"Signals of rational transfer function are realizable"
(iff)

Eg: "Realize" $H(s) = \frac{s+1}{s+2}$

$$H(s) = \left(\frac{1}{s+2}\right) \cdot (s+1)$$



Laplace vs. Fourier:

* Laplace can handle ramps & Parabola.
* Laplace can handle DC along with AC.
* Laplace captures stuff about stability.
* ~~FFT~~ But FFT is computable much faster.

"Fourier Wins Special Diwali Battle"

Fig: Fourier

THE Z TRANSFORM

$$\boxed{X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}}$$

where $x(n)$ is a discrete signal, where $z \in \mathbb{C}$

With the polar representation — $z = re^{j\omega}$

$$X(re^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n) \cdot r^{n} e^{-nj\omega} .$$

$\therefore$ By inverse DTFT, $x(n) = \frac{r^n}{2\pi} \int_{2\pi} X(re^{j\omega}) e^{j\omega n} d\omega$

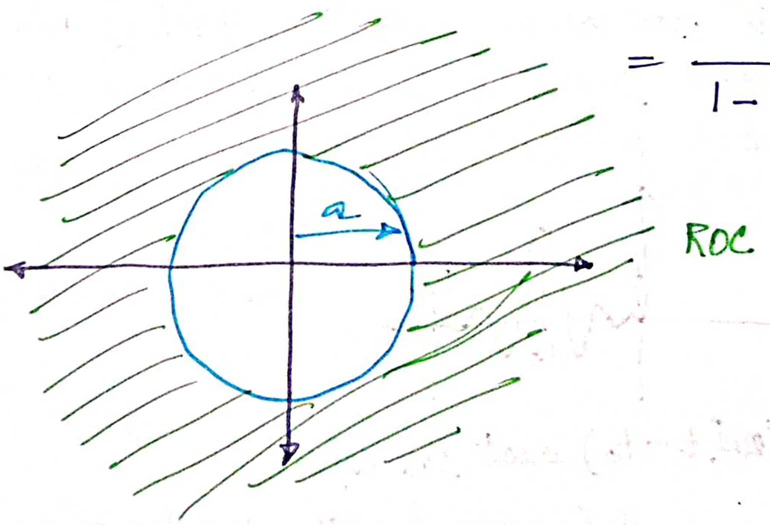
$\therefore$ Using the path $z = re^{j\omega}$.

$$\boxed{x(n) = \frac{1}{2\pi j} \int X(z)\, z^{n-1} dz}$$

Example:— $x(n) = u(n) \cdot a^n$

$x = a^n . u(n) \Rightarrow X(z) = \sum_{n=0}^{\infty} a^n z^{-n}$

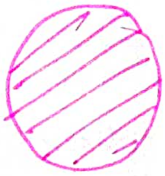
$= \frac{1}{1 - \frac{a}{z}}$, Provided $|a| < |z|$



Again, there are 5 types of ROC's —

Causality → ROC is outside pole of highest magnitude.

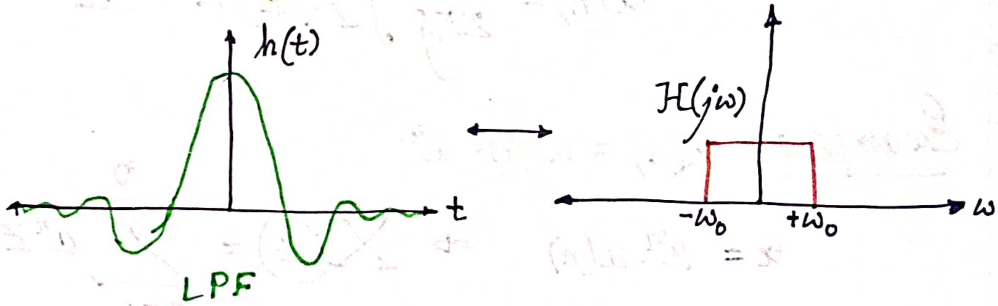
& Anticausal → ROC is inside pole of lowest magnitude



Stability → For stability, DTFT must exist, (which is $r=1$) hence Unit circle $\in$ ROC

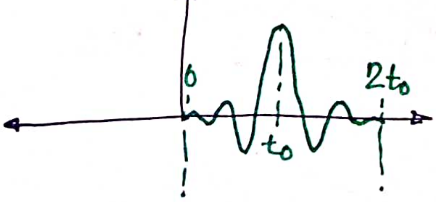
* Delays can be used analogously with Integrators.

FILTERS



LPF

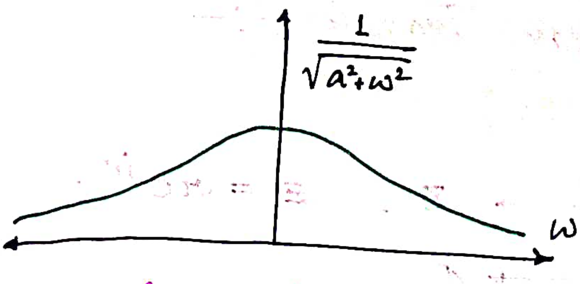
It is non causal (so not realizable) and it is not stable (something we would like to avoid).
So ideal low pass filter is not realizable

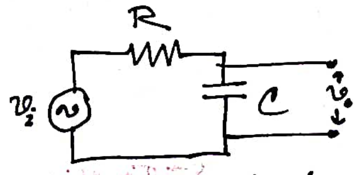


Causal & Stable

$\text{sinc}(t-t_0)\ \text{rect}(t_0, t_0)$

where $\text{rect}(\mu, x) = \begin{cases} 1 & \text{if } t \in [\mu - x, \mu + x] \\ 0 & \text{otherwise.} \end{cases}$





First order Low pass filter.

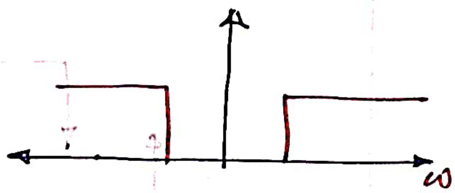
"Higher order Low pass filters are also possible, which are better & better approximation of the ideal low pass filter.

L.P.F.

$\mathrm{rect}(\omega)$

$\downarrow$

$\frac{\mathrm{sinc}(t)}{\pi}$



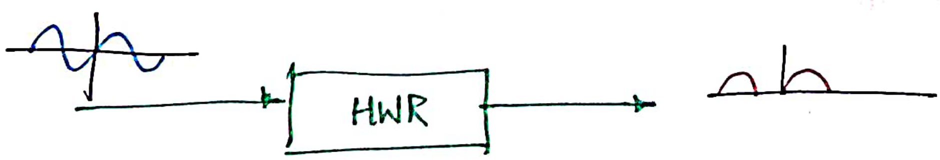
H.P.F.

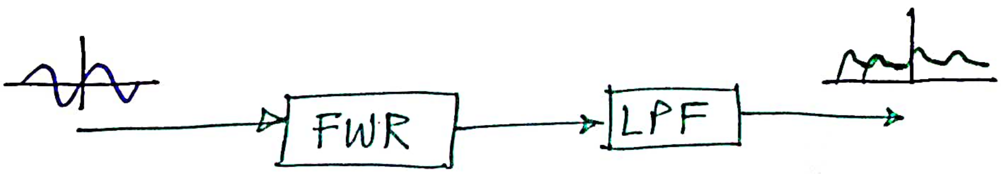
$1 - \mathrm{rect}(\omega)$

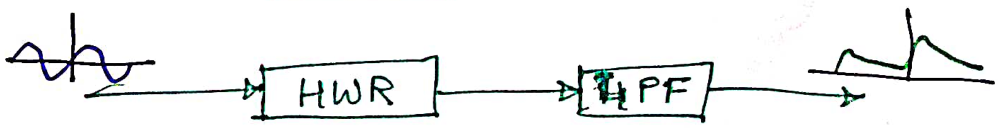
$\downarrow$

$\cancel{\delta(\omega)2\pi}\,\delta(t) - \frac{\mathrm{sinc}(t)}{\pi}$

"Similarly we can ~~d~~ frequency shift L.P.F. to get BPF and frequency shift H.P.F. to get BSF."





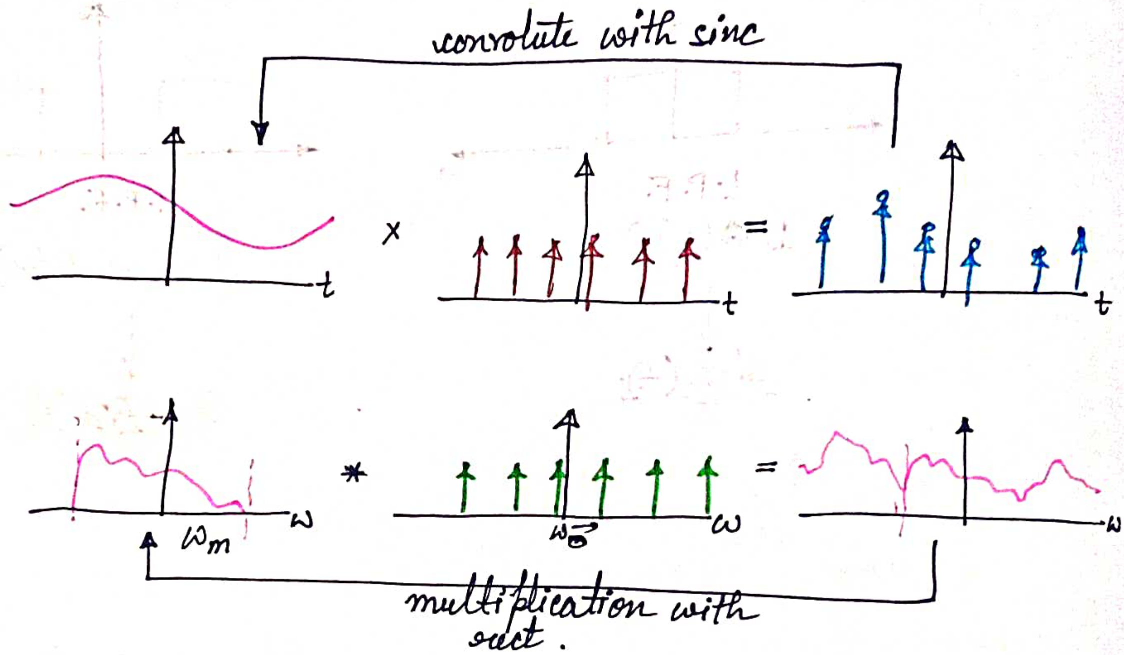


Bilinear Transform :-
(BLT)

$s = \sigma + j\omega$ $\quad S \longrightarrow Z, \quad z = re^{j\omega}.$

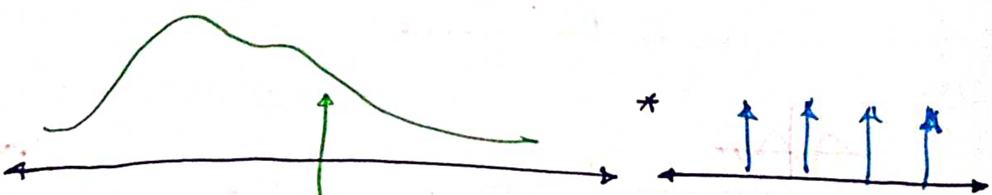
$\mathbb{C} \longrightarrow \mathbb{C}$

$\therefore$ Via Bilinear Transform, we can just convert the filters from continuous time to discrete time.



Uncertainty Principle:-

* Band limited → Time unlimited
* Time limited → Band unlimited



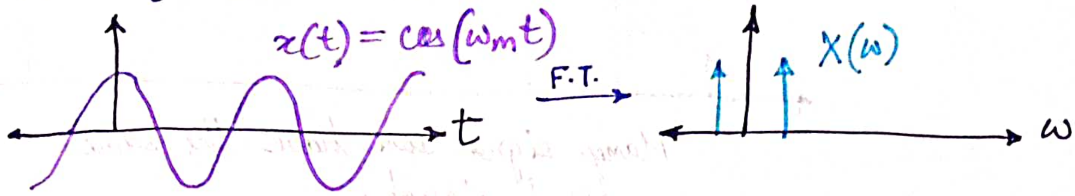
Doesn't have a band limited ~~sin~~ Fourier Transform. } Cannot be reconstructed.

* Sampling a signal having a band limited Fourier Transform fast enough enables recovery (with $\omega_0 > 2\omega_m$)

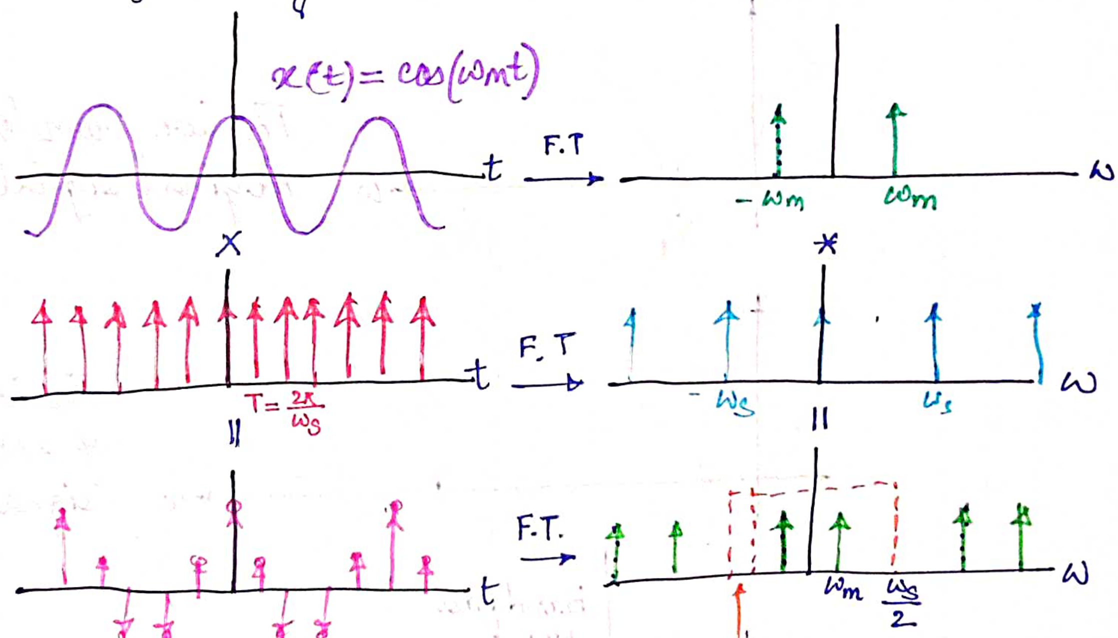
Aliasing :-

Start with a cosine wave, $x(t) = \cos(\omega_m t)$

$\therefore \mathcal{F}\{x(t)\} \rightarrow X(\omega) = \delta(\omega - \omega_m) + \delta(\omega + \omega_m)$.

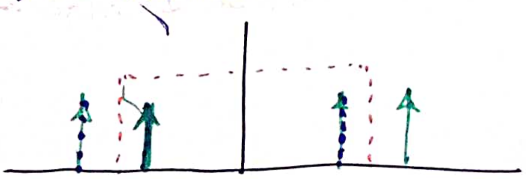


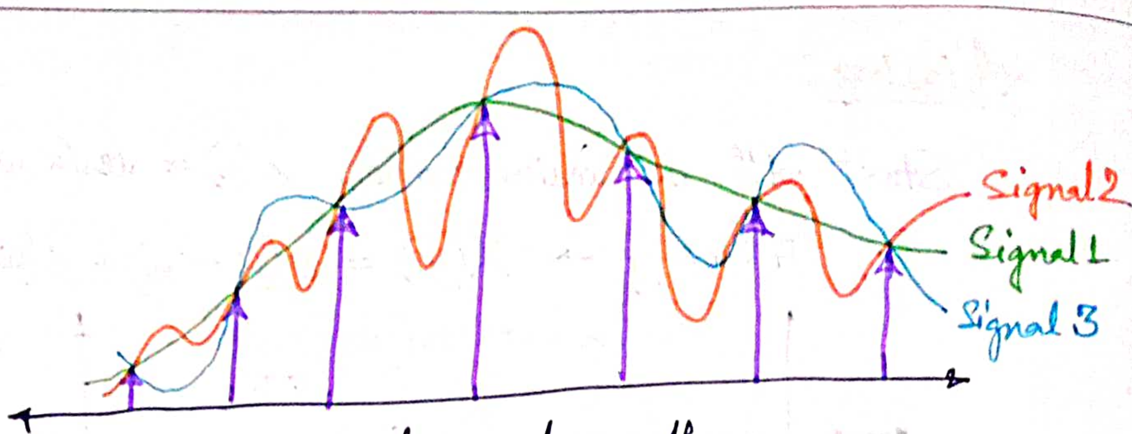
Now let us sample this using a train of impulse with frequency and $\omega_s > 2\omega_m$



We can use a low pass filter to get back the original signal. (Only if $\omega_s > 2\omega_m$)

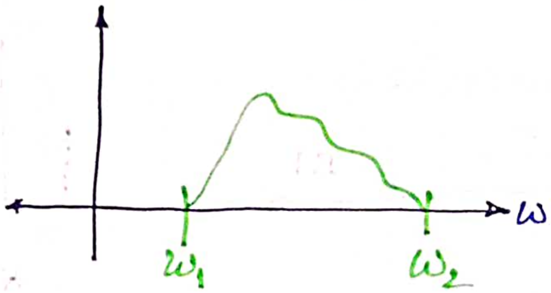
If we do not have $\omega_s > 2\omega_m$, we have the alias.



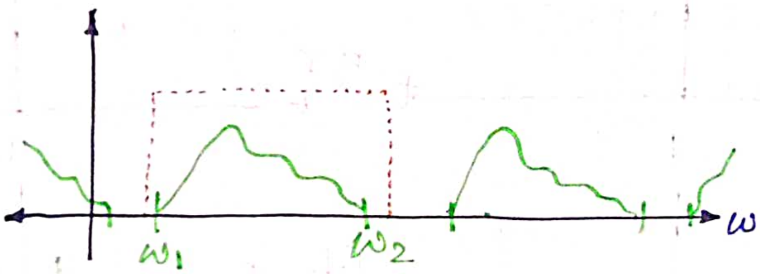


Many signal can have the same sampled signal.

Bandpass Sampling :-



Fourier transform of original signal

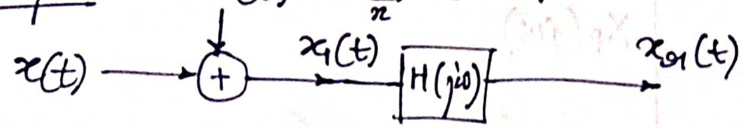


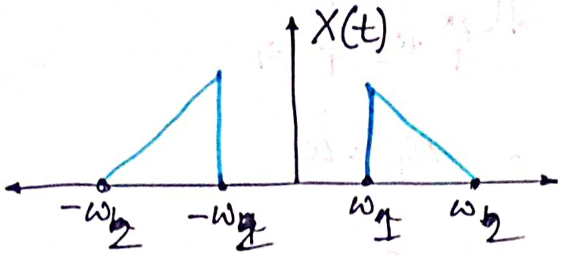
Fourier transform of sampled signal

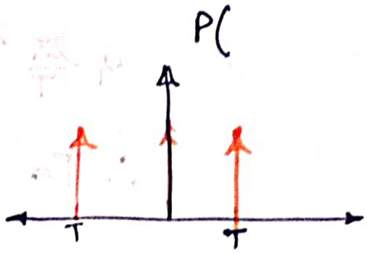
Bandpass Filtering.

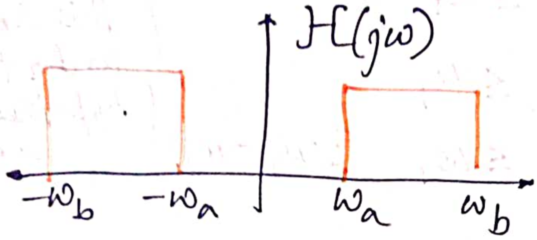
After passing it through a band pass filter, we can still get back the original signal.
[The frequency need to not be as large as $2\omega_2$]

Example: $P(t) = \sum_n \delta(t - nT)$ 　　　Find T, A, ω_a, ω_b



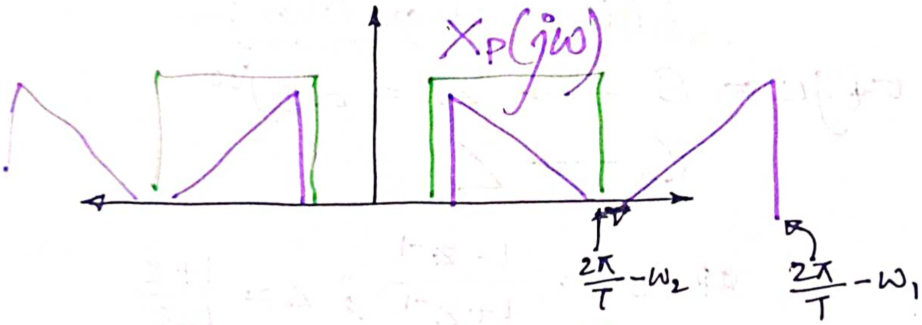






$$P(j\omega) = \frac{2\pi}{T} \sum_k \delta\left(\omega - \frac{2\pi}{T}k\right)$$

$$X_p(j\omega) = \frac{1}{2\pi} X(j\omega) * P(j\omega)$$



$$\therefore \frac{2\pi}{T} - \omega_2 > \omega_2 \Rightarrow \frac{2\pi}{T} > 2\omega_2$$

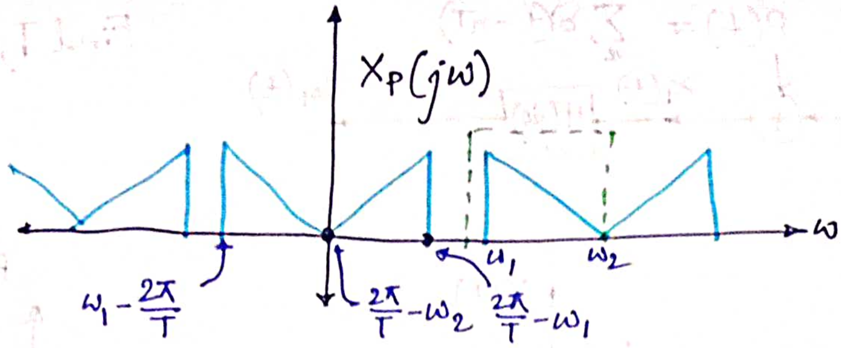
We can take $A = T$, $0 < \omega_a < \omega_1$ & $\omega_2 < \omega_b < \frac{2\pi}{T} - \omega_2$

$T < \frac{\pi}{\omega_2}$ in this case.

Consider the case, where $\frac{2\pi}{T} - \omega_2 = 0$

i.e. The first left slanted, positive triangle starts at origin.

(Consequently LPF won't suffice, we require BPF)



$\therefore T = \frac{2\pi}{\omega_2}$, $\omega_b = \omega_2$

and, $\frac{2\pi}{T} - \omega_1 < \omega_a \leq \omega_1$

$\Rightarrow \omega_2 - \omega_1 < \omega_a \leq \omega_1$

$\Rightarrow \omega_b - \omega_1 < \omega_a \leq \omega_1 \quad [\because \omega_b = \omega_2]$

This holds under the hypothesis $\omega_2 < 2\omega_1$

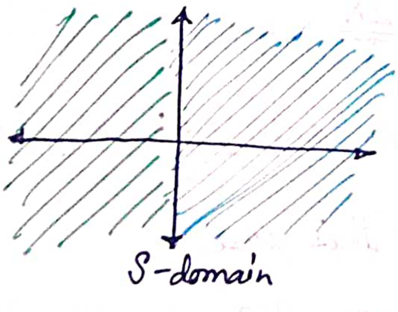
* If T increases beyond $\frac{2\pi}{T}$, there is interference.
* T can decrease as long as $\frac{2\pi}{T} - \omega_1 < \omega_1$

Bilinear Transform :-

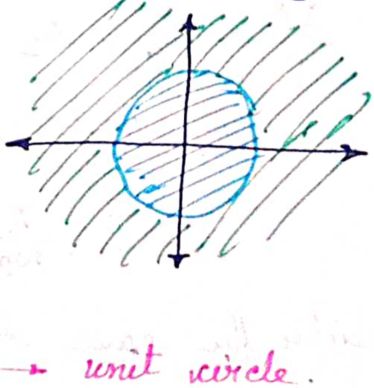
$$\sigma + j\omega = s \longrightarrow z = re^{j\omega}$$

$$\mathbb{C} \longrightarrow \mathbb{Z}$$

$$s = \frac{1 - z^{-1}}{1 + z^{-1}} \; ; \; z = \frac{1+s}{1-s}$$



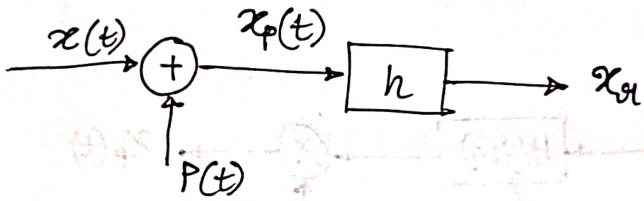
S-domain



$j\omega$ axis $\longleftrightarrow$ unit circle.

LHP $\longleftrightarrow$ inside unit circle.

RHP $\longleftrightarrow$ outside unit circle.



* Ideally we would have h as sinc in time domain (which becomes rect in frequency domain) (Sinc interpolati
* Instead of sinc, we can use rect in time domain.
(Rect Interpolation) (Realizable)

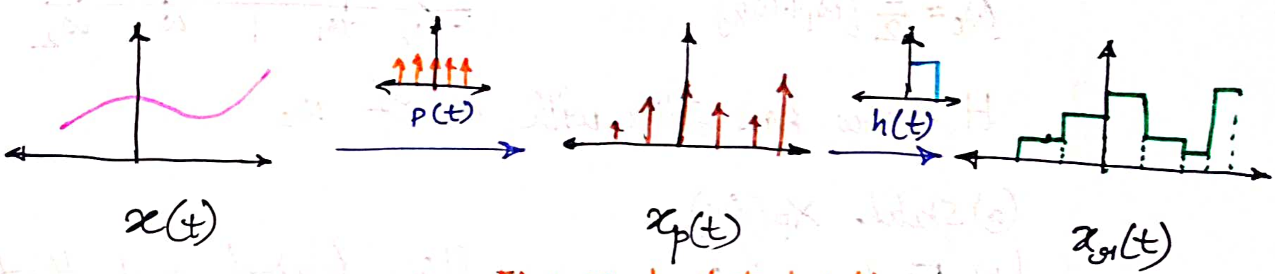
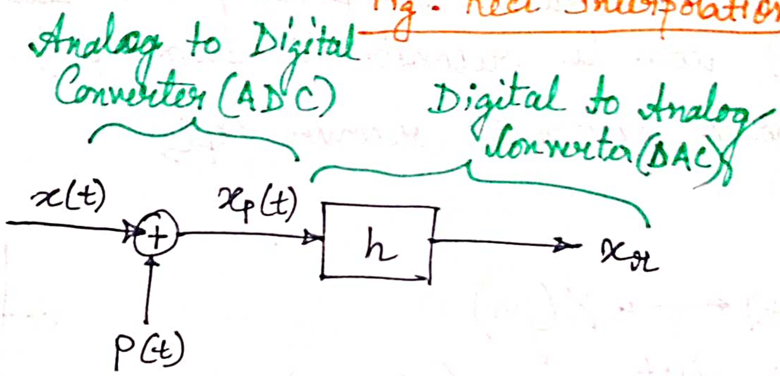
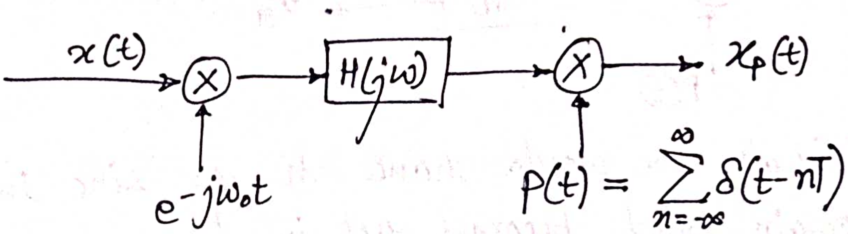


Fig: Rect Interpolation

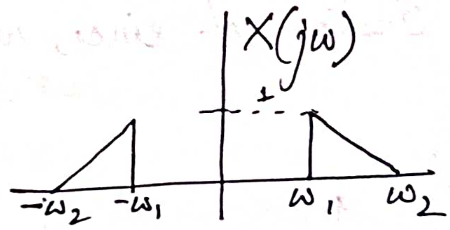


* If we use a triangle wave for h, then we would be doing triangular interpolation.

Example:—



$x(t)$ is real.



$\omega_0 = \frac{1}{2}(\omega_1+\omega_2)$

H : Low pass filter with cutoff ω_0

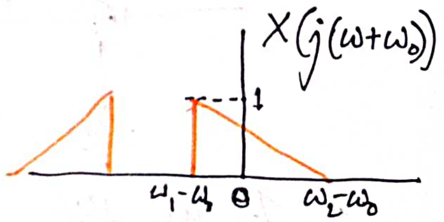
(a) Sketch $X_P(j\omega)$

(b) Find maximum sampling period such that $x(t)$ can be reconstructed from $x_p(t)$

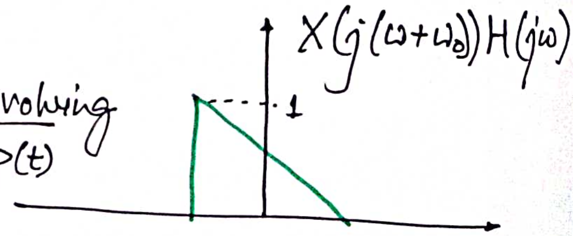
(c) Find a system to recover $x_p(t)$

$x(t) \longleftrightarrow X(j\omega)$

$x(t)e^{-j\omega_0 t} \longleftrightarrow X(j(\omega+\omega_0))$

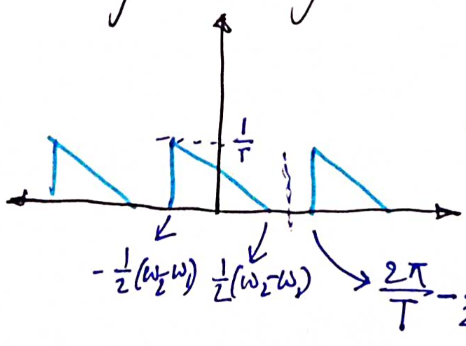


Passing through H



$\{X(j(\omega+\omega_0))H(j\omega)\} * P(j\omega)$

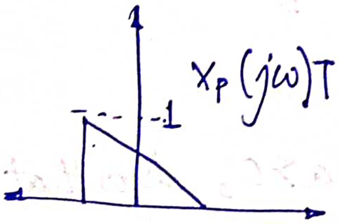
~~X(jω)~~ convolving with p(t)



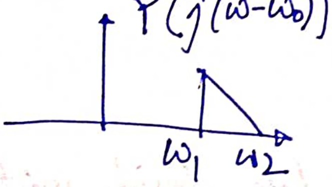
For recovery,

$\frac{1}{2}(\omega_2-\omega_1) < \frac{2\pi}{T} - \frac{1}{2}(\omega_2-\omega_1) \Rightarrow T < \frac{2\pi}{\omega_2-\omega_1}$

Let H_1 : L.P.F. with cutoff $\pm \frac{1}{2}(\omega_2-\omega_1)$



$$X_p(j\omega)TH_1(j\omega) = Y(j\omega)$$



Recall that $x(t) \in \mathbb{R}$.

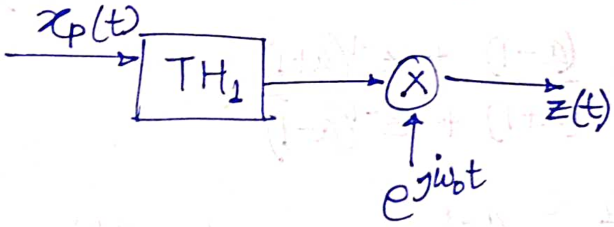
$$\Rightarrow X(j\omega) = X^*(-j\omega)$$

Let $W(j\omega) = Z(j\omega) + Z(-j\omega)$

$$\Rightarrow w(t) = z(t) + z^*(t)$$

$$\Rightarrow x(t) = 2\,Re\{z(t)\}$$

$$= 2\,Re\{y(t)\,e^{j\omega_0 t}\}$$



$x(t) = 2Re\{z(t)\}$

Example: $H_c(s) \longrightarrow$ Causal, stable, LTI

$H_d(z) \longrightarrow$ " "

BLT

(a) $H_c = \frac{a-s}{a+s}$ $a \in \mathbb{R}, a > 0$, show that $|H_c(j\omega)| = 1$

(b) $H_d(z) = H_c(s)\Big|_{s = \frac{1-z^{-1}}{1+z^{-1}}}$

show that $H_d(z)$ has one pole inside unit circle & one zero outside the unit circle.

(c) Show that $|H_d(e^{j\omega})| = 1$

$$H_d(z) = \frac{a - \frac{1-z^{-1}}{1+z^{-1}}}{a + \frac{1-z^{-1}}{1+z^{-1}}} = \frac{a(1+z^{-1}) - (1-z^{-1})}{a(1+z^{-1}) + (1-z^{-1})}$$

$$= \frac{(a-1) + z^{-1}(a+1)}{(a+1) + z^{-1}(a-1)}.$$

$\Rightarrow$ Pole at $-\frac{a-1}{a+1}$, zero at $-\frac{a+1}{a-1}$

Now $a > 0$, $a \in \mathbb{R}$ $\Rightarrow$ $a-1 < a+1$

$\Rightarrow$ $\left|\frac{a-1}{a+1}\right| < 1$ and $\left|\frac{a+1}{a-1}\right| > 1$

$$H_d(e^{j\omega}) = \frac{a-1+e^{-j\omega}(a+1)}{a+1+(e^{-j\omega})(a-1)}$$

Let $r = \frac{a-1}{a+1}$

$$\Rightarrow H_d(e^{j\omega}) = \frac{(r + e^{-j\omega})}{(1 + re^{-j\omega})} \xrightarrow{\text{Calc.!}} |H_d(e^{j\omega})| = 1$$

Example:

(a) Show that $H_d(e^{j\omega}) = H_c(j\tan\frac{\omega}{2})$

(b) $H_c(s) = \dfrac{1}{(s+e^{j\frac{\pi}{4}})(s+e^{-j\frac{\pi}{4}})}$, $H_c \rightarrow$ Causal.

show that $H_c(0)=1$, $|H_c(j\omega)| \downarrow$ with $\omega \uparrow$ &

$|H_c(j)|^2 = \frac{1}{2}$ & $H_c(\infty) = 0$

(c) Show that —

1. $H_d(z)$ has 2 poles, both inside unit circle.
2. $H_d(e^{j0}) = 1$
3. $|H_d(e^{j\omega})| \downarrow$ as $\omega \uparrow$ from 0 to π.
4. Half power frequency of $H_d(e^{j\omega})$ is $\frac{\pi}{2}$

(a) $H_d(e^{j\omega}) = H_c(s)\Big|_{s=\frac{1-z^{-1}}{1+z^{-1}}}$

$$H_d(e^{j\omega}) = H_c(s)\Big|_{s=\frac{1-e^{-j\omega}}{1+e^{-j\omega}} = j\tan\frac{\omega}{2}}$$

(b) Easy verification.

(c) 1. $H_d(z) = \dfrac{1}{\left(\frac{1-z^{-1}}{1+z^{-1}}+e^{j\frac{\pi}{4}}\right)\left(\frac{1-z^{-1}}{1+z^{-1}}+e^{-j\frac{\pi}{4}}\right)}$

$$= \frac{1+z^{-1}}{\left(1-z^{-1}+e^{j\frac{\pi}{4}}(1+z^{-1})\right)\left(1-z^{-1}+e^{-j\frac{\pi}{4}}(1+z^{-1})\right)}$$

$$= \frac{1+z^{-1}}{\left(1+e^{j\frac{\pi}{4}}+z^{-1}(e^{j\frac{\pi}{4}}-1)\right)\left(1+e^{-j\frac{\pi}{4}}+z^{-1}(e^{-j\frac{\pi}{4}}-1)\right)}$$

$\Rightarrow$ Poles at — $-\dfrac{e^{j\frac{\pi}{4}}-1}{e^{j\frac{\pi}{4}}+1}$, $-\dfrac{e^{-j\frac{\pi}{4}}-1}{e^{-j\frac{\pi}{4}}+1}$