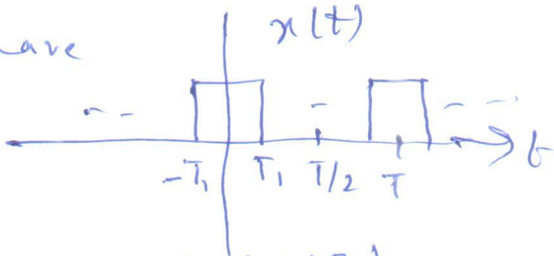
Fourier Transform of Aperiodic signal

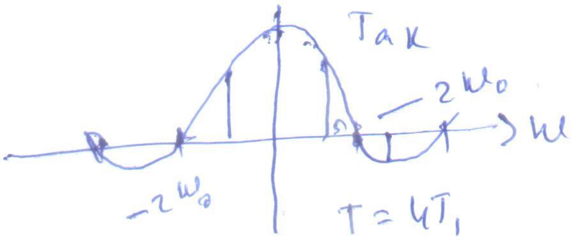
Let us take a periodic square wave

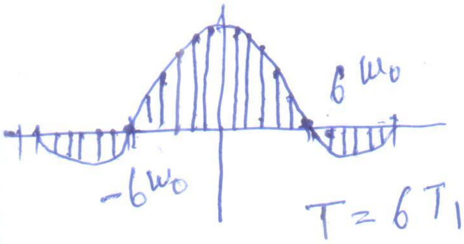
$$x(t) = \begin{cases} 1, & |t| < T_1 \\ 0 & T_1 < |t| < T/2 \end{cases}$$



Fourier Series coefficient $a_k = \dfrac{2 \sin(k\omega_0 T_1)}{k\omega_0 T}$

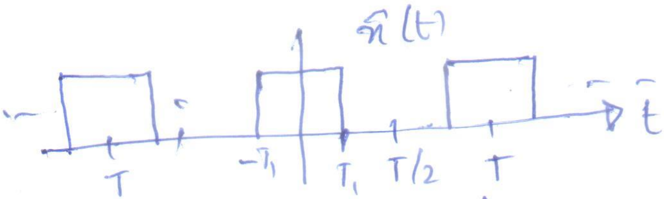
It can be interpreted as samples of an envelope function $Ta_k = \left.\dfrac{2\sin \omega T_1}{\omega}\right|_{\omega = k\omega_0}$

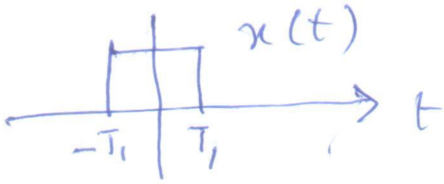




→ As T/T_1 increases Ta_k is sampled more densely and therefore in the limiting case when $T \to \infty$, $\omega_0 \to 0$ Ta_k is sampled continuously.

As $T \to \infty$, $\hat{x}(t)$ becomes $x(t)$





$$\hat{x}(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \quad \text{and} \quad a_k = \frac{1}{T}\int_{-T/2}^{T/2} \hat{x}(t) e^{-jk\omega_0 t} dt$$

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t) e^{-jk\omega_0 t} dt = \frac{1}{T}\int_{-\infty}^{+\infty} x(t) e^{-k\omega_0 t} dt$$

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt$$

The envelope of $T a_k$ is can be written as

$$X(jw) = \int_{-\infty}^{+\infty} x(t) e^{-jwt} dt$$

$$a_k = \frac{1}{T} X(jw)$$

$$\therefore \tilde{x}(t) = \sum_{k=-\infty}^{+\infty} \frac{1}{T} X(jkw_0) e^{jkw_0 t}$$

$$2\pi/T = w_0 \Rightarrow \tilde{x}(t) = \frac{1}{2\pi} \sum_{k=-\infty}^{+\infty} X(jw_0) e^{jkw_0 t} w_0$$

As $T \to \infty$, $w_0 \Rightarrow dw$ and summation replaced by integration for $\tilde{x} = x(t)$

$$\therefore x(t) = \frac{1}{2\pi} \int X(jw_0) e^{j\cancel{kw_0 t}\,wt} dw$$

inverse transform $\rightarrow$

Forward transform $\rightarrow$ $$X(jw) = \int_{-\infty}^{+\infty} x(t) e^{-jwt} dt$$

$$a_k = \frac{1}{T} X(jw)\Big|_{w = kw_0}$$

Convergence is similar to Fourier series given by Dirichlet conditions.

Ex.

$$x(t) = e^{-at} u(t), \quad a > 0$$

$$X(jw) = \int_0^{\infty} e^{-at} e^{-jwt} dt = -\frac{1}{a+jw} e^{-(a+jw)t}\Big|_0^{\infty}$$

$$= \frac{1}{a+jw}, \quad a > 0$$

Ex. $x(t) = \delta(t)$

$$X(j\omega) = \int_{-\infty}^{+\infty} \delta(t) e^{-j\omega t} dt = 1$$

Fourier Transform of periodic signal:

Let $X(j\omega) = 2\pi\delta(\omega - \omega_0)$

$$\therefore x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} 2\pi\delta(\omega-\omega_0) e^{j\omega t} d\omega$$

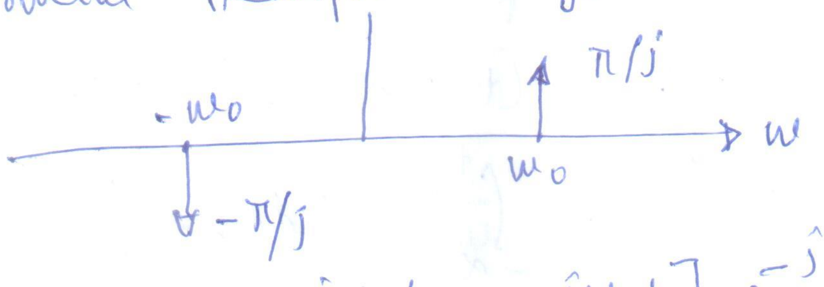
$$= e^{j\omega_0 t}$$

Ex. $x(t) = \sin(\omega_0 t)$

Fourier Series coefficients $a_1 = \frac{1}{2j}$, $a_{-1} = -\frac{1}{2j}$

$a_k = 0, \ |k| \neq 1$

Fourier Transform coefficient,



$$\frac{1}{2j}\int \left[e^{+j\omega_0 t} - e^{-j\omega_0 t}\right] e^{-j\omega t} dt$$

$$\frac{1}{2j}\int e^{j(\omega_0-\omega)t} dt - \frac{1}{2j}\int e^{-j(\omega+\omega_0)t} dt$$

Properties : same as Fourier series mostly

Linearity $\quad a\,x(t) + b\,y(t) \xleftrightarrow{FT} a\,X(j\omega) + b\,Y(j\omega)$

Time shift $\quad x(t-t_0) \xleftrightarrow{FT} e^{-j\omega t_0} X(j\omega)$

Conjugate symmetry: $\quad x^*(t) \xleftrightarrow{FT} X^*(-j\omega)$

Differentiation. $\quad \frac{d}{dt}x(t) \xleftrightarrow{FT} j\omega\, X(j\omega)$

Integration $\quad \int_{-\infty}^{t} x(t)\,dt \xleftrightarrow{FT} \frac{1}{j\omega} X(j\omega) + \pi X(0)\,\delta(\omega)$

Time Scaling $\quad x(at) \xleftrightarrow{FT} \frac{1}{|a|} X\left(\frac{j\omega}{a}\right)$

Duality

$$x_1(t) = \begin{cases} 1, & |t| < T_1 \\ 0 & |t| > T_1 \end{cases} \xleftrightarrow{FT} \frac{2\sin\omega T_1}{\omega}$$

$$x_2(t) = \frac{\sin Wt}{\pi t} \xleftrightarrow{FT} X_2(j\omega) = \begin{cases} 1, & |\omega| < W \\ 0 & |\omega| > W \end{cases}$$

Convolution $\quad y(t) = h(t) * x(t) \xleftrightarrow{FT} Y(j\omega) = H(j\omega)X(j\omega)$

Parseval's Relation. $\quad \int_{-\infty}^{+\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} |X(j\omega)|^2 d\omega$

Multiplication $\quad r(t) = s(t)\,p(t) \xleftrightarrow{FT} \frac{1}{2\pi}\left\{ S(j\omega) * P(j\omega) \right\}$

Differentiation in frequency $\quad t\,x(t) \xleftrightarrow{FT} j\frac{d}{d\omega} X(j\omega)$

$\delta(t) \xleftrightarrow{FT} 1 \Rightarrow$ FS does not exist

not periodic

$x(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & |t| > T_1 \end{cases} \xleftrightarrow{FT} \dfrac{2\sin \omega T_1}{\omega}$, FS does not exist

$u(t) \xleftrightarrow{FT} \dfrac{1}{j\omega} + \pi\delta(\omega)$ "

$e^{-at}u(t) \xleftrightarrow{FT} \dfrac{1}{a+j\omega}$ "

$t e^{-at}u(t) \xleftrightarrow{FT} \dfrac{1}{(a+j\omega)^2}$, FS does not exist

---

Ex. $\delta(t) \xleftrightarrow{FT} 1$

$$u(t) = x(t) = \int_{-\infty}^{t} g(\tau)\,d\tau$$

$$X(j\omega) = \frac{G(j\omega)}{j\omega} + \pi G(0)\,\delta(\omega)$$

$$= \frac{1}{j\omega} + \pi\delta(\omega)$$

Ex $g(t) = \dfrac{2}{1+t^2}$

$x(t) = e^{-a|t|}$ $\quad a > 0$

$$X(j\omega) = \int_{-\infty}^{+\infty} e^{-|a|t} e^{-j\omega t}\,dt$$

$$= \int_{-\infty}^{0} e^{at} e^{-j\omega t}\,dt + \int_{0}^{\infty} e^{-at} e^{-j\omega t}\,dt$$

$$= \frac{1}{a-j\omega} + \frac{1}{a+j\omega} = \frac{2a}{a^2+\omega^2}$$

$$x(t) = e^{-2|t|} \xleftrightarrow{FT} X(j\omega) = \frac{2}{1+\omega^2}$$

$$e^{-2|t|} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{2}{1+\omega^2} e^{j\omega t}\, dt$$

Putting $t = -t$

$$2\pi e^{-2|t|} = \int_{-\infty}^{+\infty} \frac{2}{1+\omega^2} e^{-j\omega t}\, dt$$

Using duality property ω & t can be exchanged

$$2\pi e^{-2|\omega|} = \int_{-\infty}^{\infty} \frac{2}{1+t^2} e^{-j\omega t}\, dt$$

$$FT\left\{\frac{2}{1+t^2}\right\} = 2\pi e^{-2|\omega|}$$

Ex. $h(t) = \delta(t - t_0)$,

$H(j\omega) = e^{-j\omega t_0}$

For any input $x(t)$, $FT\{x(t)\} = X(j\omega)$

output $FT\{y(t)\} = Y(j\omega) = H(j\omega) X(j\omega)$

$= e^{-j\omega t_0} X(j\omega)$

Using time shifting property, $y(t) = x(t - t_0)$

Ex. $x(t) = \dfrac{\sin(\omega_i t)}{\pi t}$ and $h(t) = \dfrac{\sin \omega_c t}{\pi t}$

$y(t) = x(t) * h(t)$.

It can be done easily by using convolution property.

$Y(j\omega) = X(j\omega) H(j\omega)$

$$X(j\omega) = \begin{cases} 1, & |\omega| \leq \omega_i \\ 0, & \text{otherwise} \end{cases}$$

and $$H(j\omega) = \begin{cases} 1, & |\omega| \leq \omega_c \\ 0 & \text{otherwise} \end{cases}$$

$$\therefore Y(j\omega) = \begin{cases} 1, & |\omega| \leq \omega_0 \\ 0, & \text{otherwise.} \end{cases}$$

where ω_0 is the minimum of ω_i and ω_c

$$\therefore y(t) = \begin{cases} \dfrac{\sin \omega_c t}{\pi t}, & \text{if } \omega_c \leq \omega_i \\ \dfrac{\sin \omega_i t}{\pi t}, & \text{if } \omega_i \leq \omega_c \end{cases}$$

Differentiation property.

$$\frac{d^2 y(t)}{dt^2} + 4\frac{d}{dt}y(t) + 3y(t) = \frac{dx(t)}{dt} + 2x(t)$$

$$Y(j\omega)\{(j\omega)^2 + 4j\omega + 3\} = j\omega X(j\omega) + 2X(j\omega)$$

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{j\omega + 2}{(j\omega)^2 + 4j\omega + 3}$$

$$= \frac{\frac{1}{2}}{1 + j\omega} + \frac{\frac{1}{2}}{j\omega + 3}$$

$$\therefore h(t) = \frac{1}{2}e^{-t}u(t) + \frac{1}{2}e^{-3t}u(t)$$

Discrete-time Fourier Transform

$$\hat{x}[n] = \sum_{k=\langle N\rangle} a_k e^{jk\left(\frac{2\pi}{N}\right)n}$$

and $$a_k = \frac{1}{N}\sum_{n=\langle N\rangle} \hat{x}[n] e^{-jk\frac{2\pi}{N}\cdot n}$$

In the interval $-N_1 \le n \le N_2$, $\hat{x}[n] = x[n]$

$$a_k = \frac{1}{N}\sum_{n=-N_1}^{N_2} x[n] e^{-jk\left(\frac{2\pi}{N}\right)n} = \frac{1}{N}\sum_{n=-\infty}^{+\infty} x[n] e^{-jk\frac{2\pi}{N}\cdot n}$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\omega n}$$

$$a_k = \frac{1}{N} X(e^{jk\omega_0}), \quad \omega_0 = \frac{2\pi}{N}$$

$$\hat{x}[n] = \sum_{k=\langle N\rangle} \frac{1}{N} X(e^{jk\omega_0}) e^{j\omega_0\cdot n}$$

$\because \omega_0 = \frac{2\pi}{N}$, $\frac{1}{N} = \frac{\omega_0}{2\pi}$. $\hat{x}[n] = \frac{1}{2\pi}\sum_{k=\langle N\rangle} X(e^{j\omega_0 k}) e^{j\omega_0 k n}$

As, $N \to \infty$, $\omega_0 \to 0$

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega}) e^{j\omega n} d\omega$$

and $$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\omega n}$$

Thus $X(j\omega)$ is a continuous function of ω.

On the other hand, Discrete Fourier Transform (DFT) is given by,

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}\cdot n\cdot k}$$

and $$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k] e^{j\frac{2\pi}{N}\cdot n k}$$

which is representation of finite sequence $x[n]$

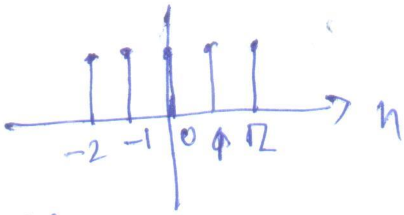
Ex. $a^n u[n]$, $|a| < 1$.

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} a^n u[n] e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} (a e^{-j\omega})^n = \frac{1}{1 - a e^{-j\omega}}$$ [infinite sum rule]

Ex. $x[n] = \begin{cases} 1, & |n| \leq N_1 \\ 0, & |n| > N_1 \end{cases}$

$$X(e^{j\omega}) = \sum_{n=-N_1}^{N_1} e^{-j\omega n}$$



$$X(e^{j\omega}) = \frac{\sin \omega (N_1 + \frac{1}{2})}{\sin(\omega/2)}$$ [Finite sum rule]

Ex. $x[n] = \delta[n]$,

$$X(e^{j\omega}) = 1 ,$$

Ex. $x[n] = \cos(\omega_0 n) = \frac{1}{2} e^{j\omega_0 n} + \frac{1}{2} e^{-j\omega_0 n}$

$$X(e^{j\omega}) = \sum_{l=-\infty}^{+\infty} \pi \delta(\omega - \omega_0 - 2\pi l) + \sum_{l=-\infty}^{+\infty} \pi \delta(\omega + \omega_0 - 2\pi l)$$

$$= \pi \delta(\omega - \omega_0) + \pi \delta(\omega + \omega_0)$$

$$-\pi \leq \omega \leq \pi$$

Properties: Similar to Fourier Transform

Linearity: $a x_1[n] + b x_2[n] \xleftrightarrow{FT} a X(e^{j\omega}) + b X_2(e^{j\omega})$

Time shifting: $x[n-n_0] \xleftrightarrow{FT} e^{-j\omega n_0} X(e^{j\omega})$

Freq. shifting: $e^{j\omega_0 n} x[n] \xleftrightarrow{FT} X(e^{j(\omega-\omega_0)})$

Difference: $x[n] - x[n-1] \xleftrightarrow{FT} (1-e^{-j\omega}) X(e^{j\omega})$

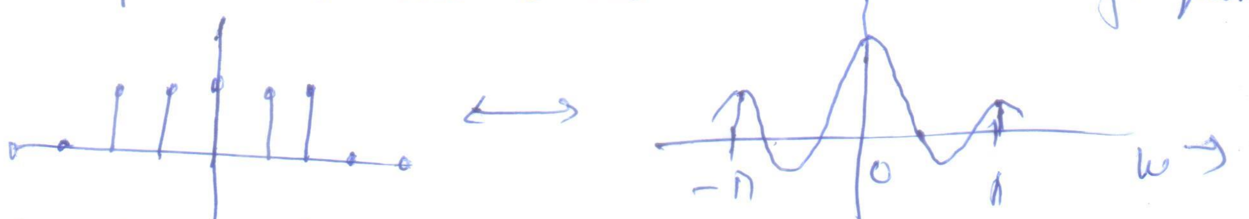
Summation: $\sum_{m=-\infty}^{+\infty} x[m] \xleftrightarrow{FT} \frac{1}{1-e^{-j\omega}} X(e^{j\omega})$
$+ \pi X(e^{j0}) \sum_{k=-\infty}^{+\infty} \delta(\omega - 2\pi k)$

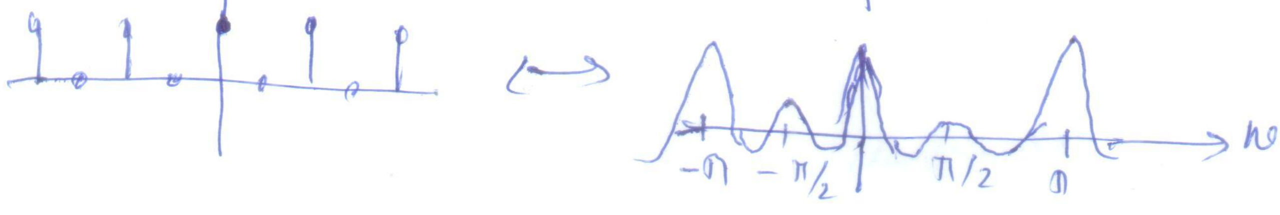
Time reversal: $x[-n] \xleftrightarrow{FT} X(e^{-j\omega})$

Time scaling: $x_k[n] = \begin{cases} x[n/k], & \text{if } n \text{ is multiple of } k \\ 0 & \text{if } n \text{ is not multiple of } k \end{cases}$

$$X_k(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x_k[n] e^{-j\omega n}$$
$$= \sum_{r=-\infty}^{+\infty} x_k[rk] e^{-j\omega r k}$$
$$= \sum_{r=-\infty}^{+\infty} x[r] e^{-j(k\omega)\cdot r}$$
$$= X(e^{jk\omega})$$

Expansion in time $\longleftrightarrow$ contraction in frequency





Convolution (circular) : $x[n] * y[n] \xleftrightarrow{FT} X(e^{j\omega})Y(e^{j\omega})$

Multiplication : $x[n]\,y[n] \xleftrightarrow{FT} \frac{1}{2\pi}\int_{2\pi} X(e^{j\theta})Y(e^{j(\omega-\theta)})\,d\theta$

~~conjugation~~ : $\cancel{x^*[n] \xleftrightarrow{FT} X^*(e^{j(\omega)})}$

Conjugation : $x^*[n] \xleftrightarrow{FT} X^*(e^{j\omega})$

Symmetry : $x[n]$ real $\longleftrightarrow$ $X(e^{j\omega}) = X^*(e^{-j\omega})$

---

not periodic

$\delta[n] \xleftrightarrow{FT} 1$ Fourier Series does not exist

$u[n] \xleftrightarrow{FT} \frac{1}{1-e^{-j\omega}}$ "

$(n+1)a^n u[n],\ |a|<1 \xleftrightarrow{FT} \frac{1}{(1-ae^{-j\omega})^2}$ "

$\frac{(n+r-1)!}{n!\,(n-r)!}a^n u[n],\ |a|<1 \xleftrightarrow{FT} \frac{1}{(1-ae^{-j\omega})^r}$ "

Ex.

$$y[n] - \frac{3}{4}\, y[n-1] + \frac{1}{8}\, y[n-2] = 2x[n]$$

$$Y(e^{j\omega}) - \frac{3}{4} e^{-j\omega} Y(e^{j\omega}) + \frac{1}{8} e^{-j2\omega} Y(e^{j\omega}) = 2X(e^{j\omega})$$

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{2}{1 - \frac{3}{4} e^{-j\omega} + \frac{1}{8} e^{-j2\omega}}$$

$$= \frac{4}{1 - \frac{1}{2} e^{-j\omega}} - \frac{2}{1 - \frac{1}{4} e^{-j\omega}}$$

$$\therefore h[n] = 4\left(\frac{1}{2}\right)^n u[n] - 2\left(\frac{1}{4}\right)^n u[n]$$