Properties of LTI systems

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Ex. Given impulse response of a system S_1,

$$h[n] = \left(\frac{1}{5}\right)^n u[n], \quad u[n] \text{ is unit step}$$

a) Find A such that $h[n] - A\,h[n-1] = \delta[n]$

b) Determine the impulse response $g[n]$ of an LTI system S_2 which is inverse of S_1

a) $$\left(\frac{1}{5}\right)^n u[n] - A\left(\frac{1}{5}\right)^{n-1} u[n-1] = \delta[n]$$

This should satisfy at $n=1$ [for any $n \geq 1$]

$$\therefore \frac{1}{5} - A = 0, \quad A = \frac{1}{5}$$

b) $$\therefore h[n] - \frac{1}{5} h[n-1] = \delta[n]$$

We can manipulate this into a convolution.

$$h[n] * \delta[n] - \frac{1}{5} h[n] * \delta[n-1] = \delta[n]$$

or, $$h[n] * \left(\underline{\delta[n] - \frac{1}{5}\delta[n-1]}\right) = \delta[n]$$

$$\Downarrow$$

∴ The impulse response of the inverse system $\overset{g[n]}{}$ $g[n] = \delta[n] - \frac{1}{5}\delta[n-1]$

Ex

Can we have inverse of finite sequence?

non-zero non-zero

$h[n] = \{h_0, h_1\}$

Then take, $h_I[n] = \{h_{I_0}, h_{I_1}\}$

$$\begin{array}{cccc} & & h_0, & h_1 \\ & & h_{I_0}, & h_{I_1} \\ \hline & & h_0 h_{I_1} & h_1 h_{I_1} \\ & h_0 h_{I_0} & h_{I_0} h_1 & \\ \hline & h_0 h_{I_0}, & h_0 h_{I_1} + h_{I_0} h_1, & h_1 h_{I_1} \end{array}$$

non-zero

$y[n] = \{\underset{\overset{\uparrow}{0}}{h_0 h_{I_0}},\ \underset{\uparrow}{h_0 h_{I_1} + h_{I_0} h_1},\ \underline{\boxed{h_1 h_{I_1}}}\}$

Can not be zero

$\Rightarrow \delta[n]$?

non zero

$\delta[n] = 1,\ n = 0 \Rightarrow h_{I_0} = \frac{1}{\ }$, $h_{I_1} = \frac{-h_1 h_{I_0}}{\ }$ non zero

$\delta[n] = 1, n=0 \Rightarrow h_{I_0} = \frac{1}{h_0}$, $h_{I_1} = \frac{-h_1 h_{I_0}}{h_0}$ → non zero → non zero

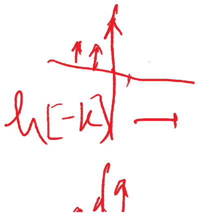
→ Inverse of a finite impulse response can not be a finite impulse response, other than the $\delta[n]$.

$$\begin{array}{r} 1,1 \\ 2,1 \\ \hline 1 \quad 1 \\ 2 \quad 2 \quad \\ \hline 2,3,1 \\ \uparrow \; \uparrow \; \uparrow \\ 0 \;\; 1 \;\; 2 \end{array} \qquad \begin{array}{r} 3,4 \\ 4,3 \\ \hline 9 \quad 12 \\ 12 \quad 16 \quad \\ \hline 12, 25, 12 \end{array}$$

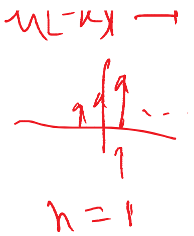
Causality of LTI system

→ Impulse response of a causal LTI system, $h[n] = 0$ for $n < 0$

or, $y[n] = \sum_{k=-\infty}^{n} x[k]\, h[n-k]$



$k = -\infty$



In continuous-time domain,

$$h(t) = 0 \text{ for } t < 0$$

or, $$y(t) = \int_{\tau=-\infty}^{t} x(\tau)\, h(t-\tau)\, d\tau$$

$$= \int_{0}^{\infty} h(\tau)\, x(t-\tau)\, d\tau$$

For example, $h(t) = \delta(t - t_0)$ is causal for $t_0 \geq 0$

Stability of LTI system

Let $|x[n]| < B$ for all n, $B < \infty$

Then $$|y[n]| = \left| \sum_{k=-\infty}^{+\infty} h[k]\, x[n-k] \right|$$

or, $$|y[n]| \leq \sum_{k=-\infty}^{+\infty} |h[k]|\, |x[n-k]|$$

$|x[n-k]| < B$, for all values of k and n

$$\sum^{+\infty} |h[k]|$$

$$|y[n]| \leq B \sum_{k=-\infty}^{+} |h[k]| ,$$

$\therefore$ $y[n]$ is bounded if $\sum_{k=-\infty}^{+\infty} |h[k]| < \infty$

i.e., the impulse response is absolutely summable.

In continuous domain, $\int_{\tau=-\infty}^{+\infty} |h(\tau)| d\tau < \infty$

for a stable system (BIBO stability)

Ex $h[n] = \delta[n-n_0]$

$$\sum_{n=-\infty}^{+\infty} |h[n]| = 1 \Rightarrow \text{stable}$$

$$h[n] = u[n]$$

$$\sum_{n=-\infty}^{+\infty} |u[n]| = \sum_{n=0}^{\infty} |u[n]| = \infty \Rightarrow \text{unstable.}$$

$$h(t) = e^{-t} u(t),$$

$$\int_{-\infty}^{+\infty} |h(\tau)| \, d\tau = \int_0^{\infty} e^{-t} \, d\tau = 1$$

$\Rightarrow$ stable

Problems

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Take $x[n] = (-1)^n u[n]$

$y(n) = x[n] * h[n]$

$y[n] \Rightarrow$ not bounded

$x[n] \Rightarrow$ bounded

$\Rightarrow$ not stable system

$$h[k] = \left\{ 1, -\frac{1}{2}, \frac{1}{3}, -1/4 \right\}$$

$$|h[k]| = \left\{ 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4} \cdots \right\}$$

$\Rightarrow$ not absolutely summable

Problem 1

Given $h_2[n] = \delta[n] + \delta[n-1]$

and the system is given by

$$y[n] = (\underbrace{h_1[n] * h_2[n] * h_2[n]}_{\Downarrow \; h[n]}) * x[n]$$

and

$$h[n] = \{ \underset{\substack{\uparrow \\ 0}}{1}, 5, 10, 11, 8, 4, 1 \}$$

a) Find the impulse response $h_1[n]$ [assume causal]

b) Find $y[n]$ for $x[n] = \delta[n] - \delta[n-1]$

$$h[n] = h_1[n] * h_2[n] * h_2[n]$$

$$= h_1[n] * (\delta[n] + 2\delta[n-1] + \delta[n-2])$$

$$\Big[\big(\delta[n] + \delta[n-1]\big) * \big(\delta[n] + \delta[n-1]\big)$$

$$= \delta[n] + \delta[n-1] + \delta[n-1] + \delta[n-2]\Big]$$

$$h[n] = h_1[n] + 2h_1[n-1] + h_1[n-2]$$

$$h[0] = 1$$

$h_1[n] = 0$, for $n < 0$

$$1 = h_1[0] \; \cancel{0} + 2\overset{\Rightarrow 0}{h_1[-1]} + \overset{\Rightarrow 0}{h_1[-2]}$$

$$h[1] = h_1[1] + 2h_1[0] \Rightarrow h_1[1] = 3$$

4.

$\vdots$

Prob2

Given

$$x[n] = \left(\frac{1}{2}\right)^{n-2} u[n-2]$$

and $h[n] = u[n+2]$

Determine $y[n] = x[n] * h[n]$

Let $x_1[n] = \left(\frac{1}{2}\right)^n u[n]$

and $h_1[n] = u[n]$

Then, $x[n] = x_1[n-2]$

and $h[n] = h_1[n+2]$

$$y[n] = x[n] * h[n]$$

$$= x_1[n-2] * h_1[n+2]$$

$$= \sum_{k=-\infty}^{+\infty} x_1[k-2]\, h_1[n-k+2]$$

Taking $k = m+2$,

$$y[n] = \sum_{m=-\infty}^{+\infty} x_1[m]\, h_1[n-m]$$

$$= \sum_{m=-\infty}^{+\infty} \frac{1}{2}^{m} u[m] \cdot \underset{\uparrow \ \text{unit step}}{u}[n-m]$$

$$\left[\ \overline{y[n] = \sum_{k=0}^{n}} \alpha^{k}, \quad n \geq 0, \quad \alpha < 1\right.$$

$1 \quad \alpha^{n+1}$ ⇐ Check

$$= \frac{1-\alpha^{n+1}}{1-\alpha} \Bigg] \Leftarrow \text{Check}$$

$$\therefore y[n] = \frac{1-\left(\frac{1}{2}\right)^{n+1}}{1-\frac{1}{2}} u[n]$$

$$= 2\left[1-\left(\frac{1}{2}\right)^{n+1}\right] u[n]$$