Prob-Stat(MA20205)/CT/1

Fill in the blanks (Numerical)

Date of Exam : 04th Sep, 2022

Time : SLOT A

Duration : 40min

No of questions: 5 out of 10 questions

Type: Random-sequential (navigation allowed)

Each question carries 2 marks

September 2, 2022

A Q41. A box contains 2 white balls and 1 red ball. A person draws 2 balls at random with replacement. What is the probability that both the drawn balls are white given that at least one them is white? (Answer should be correct up to three decimal places. error range: 0.005)

ANSWER: 0.5

ERROR RANGE: 0.005

`ANS: Conditional probability is` $\dfrac{\frac{2}{3} \times \frac{2}{3}}{(\frac{2}{3})^2 + 2 \times \frac{2}{3} \times \frac{1}{3}} = \dfrac{1}{2}$.

A Q43. In a penalty shoot-out of a football match, suppose 5 penalties will be taken independently. If the probability of shooting successfully is 0.4, then what is the probability that at least 3 shootings will be successful? (Answer should be correct up to three decimal places. error range: 0.005)

ANSWER: 0.31744

ERROR RANGE: 0.005

`The probability is` $\binom{5}{3} \times (.4)^3 \times (.6)^2 + \binom{5}{4} \times (.4)^4 \times .6 + (.4)^5$

$= (.4)^3 \times 4.96$

$= .317$

A Q46. Suppose a person is at a bus stand at 10 p.m., waiting for the last bus. Let the amount of time (in minutes) the person has to wait for bus, be a random variable with the probability density function as follows

$$f(x) = \begin{cases} \dfrac{e^x}{e^3-1}, & 0 < x < 3, \\ 0, & \text{otherwise} . \end{cases}$$

What is the expected amount of waiting time (in minutes) the person has to wait? (Answer should be correct up to three decimal places. error range: 0.005)

ANS: 2.157187

ERROR RANGE: 0.005

ANSWER: $\dfrac{2e^3+1}{e^3-1} = (2 * exp(3) + 1)/(exp(3) - 1) = 2.157187$

A Q47. Two integers are chosen at random from $1, 2, \ldots, 6$ without replacement. Let X be the absolute value of the difference of those two numbers. What is the expected value of X? (Answer should be correct up to three decimal places. error range: 0.005)

ANSWER: 2.3333

ERROR RANGE: 0.005

ANS: $P(X=1) = \frac{10}{30}$, $P(X=2) = \frac{8}{30}$, $P(X=3) = \frac{6}{30}$, $P(X=4) = \frac{4}{30}$, $P(X=5) = \frac{2}{30}$.

Then $EX = \sum_{i=1}^{5} iP(X=i) = \frac{7}{3} = 2.3333$.

A Q62. Suppose there are four letters and four labelled envelopes. If someone randomly assigns letters to the envelopes, such that each envelop contains only one letter. Then what is the probability of getting all letters into wrong envelopes?(answer should be correct up to three decimal places, error range: 0.005)

ANSWER : 0.375

ERROR RANGE: 0.005

ANS: Let A_i denotes the match at ith envelop. Then the required probability is

$1 - P(\cup_{i=1}^{4} A_i) = 1 - 1/1! + 1/2! - 1/3! + 1/4! = 3/8 = 0.375$

A Q65. Let $f(x) = cx^2$ for $0 < x < 3$ and $f(x) = 0$ otherwise be the p.d.f. of a random variable X. Then find the value of $Var(X)$. (answer should be correct up to three decimal places, error range: 0.005)

ANSWER : 0.3375

ERROR RANGE: 0.005

ANS: $c = 1/9$. $(3^3)/5 - (3^4)/4^2 = 0.3375$

A Q71. Let $Xu^2 + Yu + Z = 0$ be a an equation in u with random coefficients X, Y, Z which are independently and identically distributed random variables taking the face value of a die with equal probability for each side. Find the the probability that the equation will have no common roots. (answer should be correct up to three decimal places, error range: 0.005)

ANSWER : 0.9768519

ERROR RANGE: 0.005

```
ANS: Non-favourable cases $(Y,X,Z)=\{(2,1,1), (4,4,1), (4,1,4),(4,2,2),(6,3,3)\} $
 and count of all possible cases is $6^3$. Hence, the probability is $(1-5/6^3)= 0.9768519$
```

A Q73. Let the probability density function of X is given by

$$f(x) = \begin{cases} ax + bx^2 & \text{if } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

If $E(X) = 1/9$, find $6a - 5b$. (answer should be correct up to three decimal places, error range: 0.005)

ANSWER : 192

ERROR RANGE: 0.005

```
ANS: Solve for  a/2+b/3=1 and a/3+b/4=1/9 to get  a=46/3 and b=-20. Hence  6a-5b=192
```

Prob-Stat(MA20205)/CT/1

Fill in the blanks (Numerical)

Date of Exam : 04th Sep, 2022

Time : SLOT B

Duration : 40min

No of questions: 5 out of 10 questions

Type: Random-sequential (navigation allowed)

Each question carries 2 marks

September 2, 2022

B Q44. Suppose A and B are two independent events and P denotes the probability function. If $P(A) = 0.6$ and $P(B) = 0.5$, then what is $P(B|A^c)$? (Here A^c is the complement of the event A and $P(B|A^c)$ is the conditional probability of B given A^c). (Answer should be correct up to three decimal places. Error range 0.005)

ANSWER: .5

ERROR RANGE: .005

A `and` B `are indepemdent means` A^c `and` B `are independent. Then`

$$P(B|A^c) = \frac{P(B \cap A^c)}{P(A^c)} = P(B) = .5$$

B Q68. Let the number of customers arriving at a grocery store in an hour be a random variable X with p.m.f.

$$f(x) = \begin{cases} \frac{e^{-10}10^x}{x!} & \text{if } x = 0, 1, 2, \ldots \\ 0 & \text{otherwise} \end{cases}$$

Find the value of $E(X^2 - X)$. (answer should be correct up to three decimal places, error range: 0.005)

ANS: 100

ERROR RANGE: 0.005

ANSWER: $E(X(X-1)) = \lambda^2$

B Q72. Suppose a random number N is chosen between $1, 2, \ldots 20$ with $P(N = n) = C\ 0.2(0.8)^n$ for $n = 1, 2, \ldots 20$. If it is given that the chosen number is divisible by 4, compute the probability that the number is either divisible by 3 or divisible by 5 too. (answer should be correct up to three decimal places, error range: 0.005)

ANSWER : 0.1170201

ERROR RANGE: 0.005

```
ANS:(0.2*0.8^12+0.2*0.8^20)/(0.2*0.8^4+0.2*0.8^8+0.2*0.8^12+0.2*0.8^16+0.2*0.8^20)=0.1170201
```

B Q74. Consider a random variable X with c.d.f. $F(x)$, where

$$F(x) = \begin{cases} 1 - e^{-x} - xe^{-x} & \text{if } 0 < x < \infty, \\ 0 & \text{otherwise} \end{cases}$$

If $M = E(X)$, find the value of $F(M)$. (answer should be correct up to three decimal places, error range: 0.005)

ANSWER : 0.5939942

ERROR RANGE: 0.005

```
ANS: Note that M=2 , Hence F(M)=1-3*exp(-2)= 0.5939942
```

B Q75. Let the lifetime of an electric bulb is denoted by a random variable X with p.d.f. $f(x)$. If the mean of the lifetime is M_1 and $P(X \leq M_2) = 0.5$, find $M_1 - M_2$, where

$$f(x) = \begin{cases} \frac{1}{4}e^{-x/4} & \text{if } 0 < x < \infty, \\ 0 & \text{otherwise} \end{cases}$$

(answer should be correct up to three decimal places, error range: 0.005)

ANSWER : 1.227411

ERROR RANGE: 0.005

```
ANS: Note that M_1=4  and M_2= 4 log(2). Hence 4-4*log(2)=1.227411
```

B Q76. Let $A_{5\times5} = ((a_{ij}))$ be a random matrix. Each a_{ij} independently takes value 1 or 0 with equal probability 0.5. What is the probability that both the row-sum of the second row and column-sum of the third column of $A_{5\times5}$ are 2 ?(answer should be correct up to three decimal places, error range: 0.005)

ANSWER: 0.1015625

ERROR RANGE: 0.005

```
ANS: Given a_{23}=0 then prob is  ( 4 choose 2)*(1/2)^4 for the rest row or column
Given a_{23}=1  then prob is  ( 4 choose 1)*(1/2)^4 for the rest row and column
dbinom(2,4,0.5)^2*0.5+dbinom(1,4,0.5)^2*0.5=0.1015625
```

B Q78. Let X be the face value of a fair die when rolled first time. If $X = k$ where $k \in \{1,2,3,4,5,6\}$ the the die rolled k times more independently. Find the probability that the first roll will have face value greater than or equal to the face values observed in all subsequent roll(s).
(answer should be correct up to three decimal places, error range: 0.005)

ANSWER : 0.3336977

ERROR RANGE: ∓ 0.005

ANS: $1/6 * (1/6 + (2/6)^2 + (3/6)^3 + (4/6)^4 + (5/6)^5 + 1) = 0.3336977$

B: Q79. Consider a continuous random variable X with the probability density function $f(x) = ae^{-2|x|}$, $-\infty < x < \infty$, where a is a positive constant. Find the variance of X.

(answer should be correct up to three decimal places, error range: 0.005)

Answer: 0.5

ERROR RANGE: 0.005

Prob-Stat/Question

SLOT C

CT1

CQ31. Let $\mathbb{Z}$ is the set of integers. For what value of $k \in \mathbb{R}$

$$f(x) = \frac{k}{|x|^2} \quad \text{for } x \in \mathbb{Z} \setminus \{0\}$$

will be a probability mass function?

(Answer should be correct up to three decimal places, error range: 0.005)

ANSWER : 0.3040

ERROR RANGE: 0.005

`ANS:`

$$\sum_{x=-\infty, x\neq 0}^{\infty} \frac{1}{|x|^2} = 2 \times \frac{\pi^2}{6}$$
$$\Rightarrow k = \frac{3}{\pi^2}$$

CQ32. The probability of choosing an odd prime number is three times the probability of choosing an even prime number. Let five prime numbers P_1, P_2, P_3, P_4, P_5 are chosen independently. What is the probability that the number $N = P_1 \times P_2 \times P_3 \times P_4 \times P_5$ is odd?

(Answer should be correct up to three decimal places, error range: 0.005)

ANSWER : 0.2373

ERROR RANGE: 0.005

`ANS: Let` q `be the probability of choosing an even prime, then the probability of choosing an odd prime is` $3q$. `Hence` $q = \frac{1}{4}$. `Therefore the required probability is` $\left(\frac{3}{4}\right)^5 = 0.2373$

CQ33. Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined as follows. For $i = 0, 2, 4, 6, 8,$

$$f(x) = \begin{cases} k(x-i) & \text{for } \ i \leqslant x \leqslant i+1, \\ k(i+2-x) & \text{for } \ i+1 \leqslant x \leqslant i+2. \end{cases}$$

Determine the value of k for which the function $f(x)$ is a probability density function.

(Answer should be correct up to three decimal places, error range: 0.005)

ANSWER : 0.2

ERROR RANGE: 0.005

`ANS: Note that for` $i = 0$ `the graph of` $f(x)$ `is a triangle with vertices` $(0,0)$, $(2,0)$ `and` $(1,k)$. `The area of this triangle is` k. `Up to` $i = 8$, `there are five such triangles placed one after the other. The total area is` $5k$. `In order that` $f(x)$ `is the pdf, we require` $5k = 1$ `implying` $k = 0.2$

CQ21. Let the probability of finishing the examination in less than p hours is $p/2$ in an one-hour long examination, where $0 \leq p \leq 1$. Suppose a student is appearing the exam and continues still after 0.75 hours. Then find the probability that the student will use the full hour to finish the exam.

(answer should be correct up to three decimal places, error range: 0.005)

ANSWER : 0.8

ERROR RANGE: 0.005

`ANS: Let` A_p `denote the event that the student finishes the exam in less than` p `hours,` $0 \leq p \leq 1$, `and` B `be the event that the student uses full one hour for the exam. Then`

$$P(B) = P(A_1^c) = 1 - P(A_1) = 0.5.$$

`The desired probability is then`

$$P(B|A_{.75}^c) = \frac{B \cap A_{.75}^c}{P(A_{.75}^c)} = \frac{P(B}{1 - P(A_{.75})} = \frac{.5}{.625} = 0.8$$

CQ23. Let X and Y be two independent events of a random experiment such that $P(X) = 2P(Y)$ and $P(X \cap Y) = 0.15$. Then what is the value of $P(X^c \cap Y^c)$.

(Answer should be correct up to three decimal places, error range: 0.005)

ANSWER : 0.33

ERROR RANGE: 0.005

ANS: Since X and Y are independent, $0.15 = P(X \cap Y) = P(X)P(Y) = 2P(Y)^2$, so $P(Y) = \sqrt{0.075} = 0.273$, and $P(X) = 2P(Y) = 0.546$ Hence

$$P(X^c \cap Y^c) = P(X^c)P(Y^c) = (1 - 0.546)(1 - 0.273) = 0.454 \times 0.727 = 0.33$$

CQ25. Let a diagnostic test has probability 0.5 for giving correct test result. Let the probability of getting the disease is 0.005 in a population. Then determine the probability that a person with positive test result has the disease.

(Answer should be correct up to three decimal places, error range: 0.005)

ANSWER : 0.0545

ERROR RANGE: 0.005

ANS: let D be the event such that the person has the disease, and R denotes the event positive test. Then note that

$$P(R|D) = 0.95, P(D) = .005$$

Then the required probability is

$$P(D|R) = \frac{P(R|D)P(D)}{P(R)}.$$

Note that $P(R) = P(R \cap D) + P(R \cap D') = P(R|D)P(D) + P(R|D^c)P(D^c) = (0.95 \times 0.005) + (0.05 \times 0.995) = \frac{19}{218} = 0.087$ Then the required probability is

$$P(D|R) = \frac{.95 \times .005}{.087} = 0.0545$$

CQ52. Rakesh can solve 90 percent of the problems given in the book and Rohit can solve 75 percent. Then calculate the probability that atleast one of them will solve a problem selected at random from the book.

(Answer should be correct up to three decimal places, error range: 0.005)

ANSWER : 0.975
ERROR RANGE: 0.005

Solution: Let A be the event that Rakesh can solve a problem and B the event that Rohit can solve a problem. Then we need to find probability of $A \cup B$. So $P(A \cup B) = P(A) + P(B) - P(A \cap B)$. Since A and B are independent, $P(A \cap B) = P(A)P(B)$. Note that $P(A) = 0.9$ and $P(B) = 0.75$. Therefore $P(A \cup B) = 0.975$

CQ54. Consider the word SUCCESS. Let A be the event that the word is rearranged so that the three S's come consecutively. Then what is the probability of the event A?

(Answer should be correct up to three decimal places, error range: 0.005)

ANSWER : $1/7 = 0.14286$
ERROR RANGE: 0.005

Solution: The word can be rearranged in $\frac{7!}{3!2!} = 420$ ways. The three S can come together in 5 possible ways. Each case can be arranged in $\frac{4!}{2!} = 12$ ways. So $P(A) = 5 * 12/420 = \frac{60}{420} = 0.1429$

Prob-Stat/Question

SLOT D

CT1

DQ22. Suppose 50% of the emails that we receive in Gmail are spam emails. Let IIT Kgp have discovered a software which can detect 99% of the spam emails, however the probability that a non-spam email is detected as a spam email by the software is 5%. Assume that an email is detected as spam by the software, then calculate the probability that it is not a spam email.

(Answer should be correct up to three decimal places, error range: 0.005)

ANSWER : 0.048

ERROR RANGE: 0.005

ANS: Let A be the event that an email is detected as spam by the software, and B is the event that an email is spam. Then $P(B) = P(B^c = 0.5)$, $P(A|B) = .99$, $P(A|B^c) = 0.05$ The by Bayes formula the desired probability is

$$P(B^c|A) = \frac{P(A|B^c)P(B^c)}{P(A|B)P(B) + P(A|B^c)B(B^c)} = \frac{0.05 \times 0.5}{(0.05 \times 0.5) + (0.99 \times 0.5)} = \frac{5}{104} = 0.048$$

DQ24. Let us assume that 3 components are needed to pick randomly (all components are equally likely) from a collection of 100 components in order to build a machine. However, it is known that there are some defective components in that collection. The machine works if all 3 of the components should be non-defective. Determine the probability to build a working machine when it is known that there are 10 defective components in the collection.

(Answer should be correct up to three decimal places, error range: 0.005)

ANSWER : 0.726

ERROR RANGE: 0.005

ANS: Let A_i be the event that occur when component i is among those that are fully functional $i = 1, 2, 3$. Then the probability that the machine works $= P(A_1 \cap A_2 \cap A_3) = P(A_1)P(A_2|A_1)P(A_3|A_1 \cap A_2)$. Now, $P(A_1^c) = \dfrac{10}{100}$ which implies $P(A) = \dfrac{90}{100}$. Then $P(A_2|A_1) = \dfrac{89}{99}$, and hence $P(A_3|A_1 \cap A_2) = \dfrac{88}{98}$. Hence the desired probability is

$$= \frac{90}{100} \times \frac{89}{99} \times \frac{88}{98} = \frac{704880}{970200} = 0.726$$

DQ34. Let X be a random variable with the probability density function

$$f(x) = \begin{cases} \frac{2}{9}x^{-3} & \text{for } x \geqslant \frac{1}{3} \\ 0 & \text{otherwise} \end{cases}$$

Let $\alpha \geqslant 0$ be a given real number. Determine the maximum number $a > 0$ such that $E(X^\alpha)$ exists for every real number in the interval $[0, a)$ and further $E(X^\alpha)$ does not exist for any real number greater than or equal to a. Hence compute $E(X^{a/2})$.

(Answer should be correct up to three decimal places, error range: 0.005)

ANSWER : 0.6667

ERROR RANGE: 0.005

ANS: $E(x^\alpha) = \frac{2}{9}\int_{1/3}^{\infty} x^{\alpha-3}dx$. For this improper integral to be convergent, we know that $\alpha - 3 < 1$ implying $\alpha < 2$. Thus, $a = 2$. Further, $E(X) = 0.6667$

DQ51. An insurance company believes that people can be divided into two classes. Those that are accident prone and those that are not.Their statistics show that an accident-prone person will have an accident within a fixed one year period with probability 0.4 whereas this probability decreases to 0.2 for non-accident-prone person.If we assume that 30 percent of the population is accident prone, then what is the probability that a new policy holder will have an accident within a year of purchasing a policy?

(Answer should be correct up to three decimal places, error range: 0.005)

ANSWER : 0.26

ERROR RANGE: 0.005

Solution: Let A be the event that a new policy holder will have an accident within a year of purchasing a policy.

A_1 denotes the accident prone person

A_2 denotes the non-accident prone person.

So $P(A) = P(A|A_1)P(A_1) + P(A|A_2)P(A_2) = 0.4 \times o.3 + 0.2 \times 0.7 = 0.12 + 0.14 = 0.26$

DQ36. Denote by X the length of congruent sides in an isosceles right angle triangle. Let X follow the following probability mass function.

$X = x$	1	2	3	4	5	6
$P(X = x)$	$\frac{1}{24}$	$\frac{1}{3}$	$\frac{1}{24}$	$\frac{1}{12}$	$\frac{1}{6}$	$\frac{1}{3}$

What is the expected value of the length of the hypotenuse of this right angle triangle?
(Answer should be correct up to three decimal places, error range: 0.005)

ANSWER : 5.6569

ERROR RANGE: 0.005

`ANS: Here the pmf of the length` (Y) `of the hypotenuse is as follows.`

$Y = y$	$\sqrt{2}$	$2\sqrt{2}$	$3\sqrt{2}$	$4\sqrt{2}$	$5\sqrt{2}$	$6\sqrt{2}$
$P(Y = y)$	$\frac{1}{24}$	$\frac{1}{3}$	$\frac{1}{24}$	$\frac{1}{12}$	$\frac{1}{6}$	$\frac{1}{3}$

`Hence the required expected value is`

$$\sqrt{2} \times \frac{1}{24} + 2\sqrt{2} \times \frac{1}{3} + 3\sqrt{2} \times \frac{1}{24} + 4\sqrt{2} \times \frac{1}{12} + 5\sqrt{2} \times \frac{1}{6} + 6\sqrt{2} \times \frac{1}{3} \quad = \quad 4\sqrt{2}$$

DQ55. Let X be a discrete random variable and the probability mass function of X is $f(x) = \frac{x}{6}$ where $x = 1, 2, 3$. Calculate the standard deviation of X.

(Answer should be correct up to three decimal places, error range: 0.005)

ANSWER : 0.745

ERROR RANGE: 0.005

Solution: Mean $= \mu = 1 * \frac{1}{6} + 2 * \frac{2}{6} + 3 * \frac{3}{6} = \frac{7}{3}$.

$E(X^2) = 1^2 * \frac{1}{6} + 2^2 * \frac{2}{6} + 3^2 * \frac{3}{6} = 6$

Variance$= 6 - (\frac{7}{3})^2 = 5/9$.

Therefore the standard deviation is $\sqrt{5/9} = 0.745$.

DQ56. Let $c > 0$ be a constant and

$$f(x) = \begin{cases} cx(1-x); & 0 < x < 1 \\ 0; & \text{otherwise.} \end{cases}$$

Assume that the $f(x)$ is the probability density function with respect to the random variable X. Calculate $P\{X > 0.4\}$.

(Answer should be correct up to three decimal places, error range: 0.005)

ANSWER: 0.648

ERROR RANGE: 0.005

Solution: $\int_0^1 cx(1-x)dx = c/6$. Since $f(x)$ is a pdf, we have $c/6 = 1 \Rightarrow c = 6$. $P(X > 0.4) = 1 - P(X \leq 0.4) = 1 - 6[\int_0^{0.4} x(1-x)dx] = 81/125 = 0.648$

DQ57. Let X be a continuous random variable with the probability density function

$$f(x) = \begin{cases} e^{(-1-x)}; & -1 < x < \infty \\ 0; & \text{otherwise.} \end{cases}$$

Calculate $P\{X \geq 1\}$.

(Answer should be correct up to three decimal places, error range: 0.005)

ANSWER: 0.135

ERROR RANGE: 0.005

Solution: $P(X \geq 1) = \int_1^\infty e^{-x-1}dx = e^{-2} = 0.135$.