Common page: Course number: **MA20205/ MA20104** Course name: **Probability and Statistics**

Special Instructions

1. There are total **7** questions in this paper.
2. There are total **20** number of pages.
3. Final answer to each question or sub-question must be clearly written in the box provided.
4. Total marks: 30
5. Students may take supplementary sheets for rough work.
6. Extra supplementary sheets, if any, must be attached and submitted with this question booklet.

Rough work

Q1 Let X be a normal random variable with mean μ and variance σ^2 such that

$$\text{Prob}\,(X > 4) = \text{Prob}\,(X < 2) = 0.1587.$$

Find the values of the followings.

(a) $\mu = \boxed{}$ [1 mark]

(b) $\sigma^2 = \boxed{}$ [1 mark]

(c) Let m be the median of X. Then

$E\left[(X-m)^4\right] = \boxed{}$ [2 marks]

Let $\mathbf{\Phi(x)}$ be the CDF of the standard normal random variable.
$\mathbf{\Phi}(1) = 0.8413$, $\mathbf{\Phi}(2) = 0.9772$, $\mathbf{\Phi}(3) = 0.9986$, $\mathbf{\Phi}(4) = 0.9999$
(Answers should be correct up to three decimal places, error range: 0.005)

Solution: Let $X \sim N(\mu, \sigma^2)$.

$$\begin{aligned} & \text{Prob}\,(X > 4) = \text{Prob}\,(X < 2) = 0.1587 \\ \Rightarrow\; & \text{Prob}\left(\frac{X-\mu}{\sigma} < \frac{2-\mu}{\sigma}\right) = \text{Prob}\left(\frac{X-\mu}{\sigma} > \frac{4-\mu}{\sigma}\right) = 0.1587 \end{aligned}$$

$$\Rightarrow \mathbf{\Phi}\left(\frac{2-\mu}{\sigma}\right) = 0.1587 \quad \text{and} \quad \mathbf{\Phi}\left(\frac{4-\mu}{\sigma}\right) = 1 - 0.1587 = 0.8413$$

From the above equation and symmetry of $N(0,1)$, it is clear that $\frac{2-\mu}{\sigma} < 0$ and $\frac{4-\mu}{\sigma} > 0$. Threfore,

$$\mathbf{\Phi}\left(-\frac{2-\mu}{\sigma}\right) = 1 - 0.1587 = 0.8413 \quad \text{and}\; \mathbf{\Phi}\left(\tfrac{4-\mu}{\sigma}\right) = 0.8413$$

$$\begin{aligned} \Rightarrow -\frac{2-\mu}{\sigma} &= 1 & \qquad \frac{4-\mu}{\sigma} &= 1 \\ \Rightarrow \mu - \sigma &= 2 & \mu + \sigma &= 4 \\ \Rightarrow \mu &= 3 & \sigma &= 1 \end{aligned}$$

Further, symmetry of $N(3,1)$ around $\mu = 3$ implies that the median $m = 3$. Then $X - 3 \sim N(0,1)$. Hence the forth raw moment of the standard normal random variable $(X-3)$ is given by $E(X-3)^4 = 3$.

Part Marking Scheme: Part 1) 1 mark (correct value of μ)
Part 2) 1 mark (correct value of σ^2)
Part 3) 1 mark for correctly finding m, the median and 1 mark for correctly finding out $E(X-m)^4$.

No marks for only for writing correct answer without any justification.

Q2. (a) Let X be a random variable with the PDF given as follows. For some $a > 0$ and $b > 0$,

$$f_X(x) = \left(\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}\right)^{-1} x^{a-1}\ (1-x)^{b-1} \quad \text{for } 0 \leqslant x \leqslant 1.$$

Further, assume that $E(X) = \dfrac{1}{2}$ and $\text{Var}(X) = \dfrac{1}{12}$. Then compute the following.

i. $f_X\left(\dfrac{1}{2}\right) =$ $\boxed{}$ [1 mark]

ii. The third quartile of X is $\boxed{}$ [1 mark]

iii. $E(X^4) =$ $\boxed{}$ [1 mark]

(b) Let X be a random variable with CDF F, $E(X) = 2$, $Var(X) = 2$. Let $X_1, \ldots, X_5$ be random variables with same CDF as X. Find $E(X_1^2) + \cdots + E(X_5^2)$.

Answer: $\boxed{}$ [3 marks]

(Answers should be correct up to three decimal places, error range: 0.005)

Solution: ANSWER(a): We have $f_X(x) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} x^{a-1}(1-x)^{b-1}$ for $0 \leq x \leq 1$. This gives us, $E(X) = \frac{a}{a+b} = 1/2$ and $V(X) = \frac{ab}{(a+b)^2(a+b+1)} = 1/12$. Solving these we get $\alpha = \beta = 1$ and putting these back we get that our distribution is uniform over $[0, 1]$.Now as uniform pdf has value 1 over $0 \leq x \leq 1$ we get the answer: (i) 1 and doing the appropriate integrals with respect to uniform distribution we get answers: (ii) 3/4 (iii) 1/5.

Grading Scheme: (i) 1 marks for correct calculation (ii) 1 marks for correct calculation (ii) 1 marks for correct calculation.

ANSWER (b) : As all X_is follow same distribution we have $E(X_i^k) = E(X), V(X_i) = V(X)$ for all i and for all k. We also not that $E(X^2) = (E(X))^+ V(X) = 2^2 + 2 = 6$. So the expected value of $E(X_1^2) + .. + E(X_5^2) = 5 * 6 = 30$.

Grading Scheme: 1 Marks for correct concept, 1 Marks for correct calculation/explanation + 1 Marks for correct answer

Q3. IITKGP buses arrive at Azad Hall at 10-minutes interval starting at 7:30 AM. A student arrives at the stop at a time that is uniformly distributed between 7:30 AM and 8:00 AM. Then the probability that the student waits

(a) less than 4 minutes for a bus is $\boxed{}$ [2 marks]

(b) at least 7 minutes for a bus is $\boxed{}$ [2 marks]

(Answers should be correct up to three decimal places, error range: 0.005)

Solution:
Let X denote the time in minutes past $7:30am$ that the student arrive at the stop. Since X is a uniform over the interval $(30, 60)$, it follows that the student will have to wait less than 4 minutes if he arrives between $7:36$ and $7:40$ or between $7:46$ and $7:50$ or between $7:56$ and $8:00$ Therefore, for (a), the probability is $4/30 + 4/30 + 4/30 = 12/30 = 0.4$ For (b), he waits at least 7 minutes if he arrives between $7:30$ and $7:33$ or between $7:40$ and $7:43$ or between $7:50$ and $7:53$. Therefore the probability is $3/30 + 3/30 + 3/30 = 9/30 = 0.3$

Marking Scheme:
(1) No marks are awarded if no justifications are provided even the correct answers were given in the box.
(2) For part (a), 2 marks are awarded if the solution is correct with proper justifications. Otherwise 0 marks are awarded.
(3) For part (b), 2 marks are awarded if the solution is correct with proper justifications. Otherwise 0 marks are awarded.
(4) If a student have written down the random variable X in minutes and have identified that X follows uniform distribution in the interval $(30, 60)$, then 1 mark is given for this much.

Q4. Let the moment generating function of a random variable X be given by

$$M(t) = \sum_{j=0}^{\infty} \frac{e^{tj-1}}{j!}, \quad t \in \mathbb{R}.$$

Then compute the following.

(a) Prob $(X = 2) =$ [ ] [3 marks]

(b) Mode of $X =$ [ ] [1 mark]

(Answer should be correct up to three decimal places, error range: 0.005)

Solution:

Obviously, the given random variable X is giben by $R_X = \{0, 1, 2, \ldots\}$. By definition,

$$M(t) = \sum_{j=0}^{\infty} e^{tj} f(j).$$

Thus,

$$M(t) = \sum_{j=0}^{\infty} \frac{e^{tj-1}}{j!} = \sum_{j=0}^{\infty} \frac{e^{-1}}{j!} e^{tj}.$$

Thus we have

$$f(j) = \frac{e^{-1}}{j!}, j = 0, 1, 2, \ldots \infty$$

Then

$$P(X = 2) = f(2) = \frac{e^{-1}}{2!} = \frac{1}{2e}.$$

(b)$P(X = 0) = 0.36787944 = P(X = 1) = 0.36787944 > P(X = 2) = 0.18393972 > P(X = 3) = 0.06131324$ Hence mode is 0 or 1

Part Marking Scheme: (a) Finding the pmf - 1 mark, finding the correct answer - 2 marks; (b) For correct answer 0 and/or 1 - 1 mark

Q5. A cricket club team is playing well and their chance to win any game is 55% and the results of games are independent of one another.

(a) What is the probability that they will have the 3^{rd} win in the 6^{th} game?

Answer: [2 marks]

(b) What is the expected value of the number of games they will take to win 4 games?

Answer: [1 mark]

(c) What is the standard deviation of the number of games they will take to win 4 games?

Answer: [1 mark]

(Answers should be correct up to three decimal places, error range: 0.005)

Solution: ANS:

(a) 3rd win in 6th game has probability $\binom{n-1}{x-1} \times p^{x-1} \times (1-p)^{n-x} \times p = \binom{5}{2} \times (.55)^3 \times (.45)^3 = .15161$.

Marking scheme for (a):- 2 marks for correct method and correct answer. If the method is correct, but the final answer is wrong due to some mistake in calculation, 1.5 marks have been given. If some small error is in the formula and wrong answer from that, .5 marks have been given. If final answer is given without details, 1 mark has been given. Otherwise 0 mark is given.

(b) Expectation is $\mu = \frac{x}{p} = \frac{4}{.55} = 7.27273$.

(c)Variance is $\frac{x(1-p)}{p^2} = \frac{4 \times .45}{(.55)^2}$. So, $\sigma = 2.43935$.

Marking scheme for (b) and (c):- 1 mark for correct method and correct answer. If the method is correct, but the final answer is wrong due to some mistake in calculation, 0.5 mark has been given. If some small error is in the formula and wrong answer from that, 0.5 marks have been given. If final answer is given without details, 0.5 mark has been given. Otherwise 0 mark is given.

Q6. Suppose there are 11 boxes numbered $1, 2, \ldots, 11$ and ten identical balls placed one each in the boxes 1 to 10. In one operation a ball is chosen at random and placed in the box which was already empty. Find the probability that the $5th$ box will remain empty after four such operations. Answer: $\boxed{}$ [4 Marks]

(Answer should be correct up to three decimal places, error range: 0.005)

Denote 5th box is occupied by O and empty by E after ith operation
$P(O_1O_2O_3E_4) + P(O_1E_2O_3E_4) + P(E_1O_2O_3E_4)$
$= P(O_1)P(O_2|O_1)P(O_3|O_2)P(E_4|O_3)$
$+P(O_1)P(E_2|O_1)P(O_3|E_2)P(E_4|O_3)$
$+P(E_1)P(O_2|E_1)P(O_3|O_2)P(E_4|O_3)$

$= (9/10) * (9/10) * (9/10) * (1/10)$ [1]
$+(9/10) * (1/10) * 1 * (1/10)$ [1]
$+(1/10) * 1 * (9/10) * (1/10)$ [1]
$= 0.0729 + 0.009 + 0.009 = 0.0909$ [1]

NOTE: If any one starts with an assumption that the initial probability of 5th box being empty is 1/11 and hence conclude/derive that after 4th operation the probability remains unchanged which is 1/11=0.09090909..... is an INCORRECT approach.

Q7. A portion of an electrical circuit is displayed in the following figure. The switches operate independently of each other and the probability that each switch is functional is displayed beside the switches in the figure.

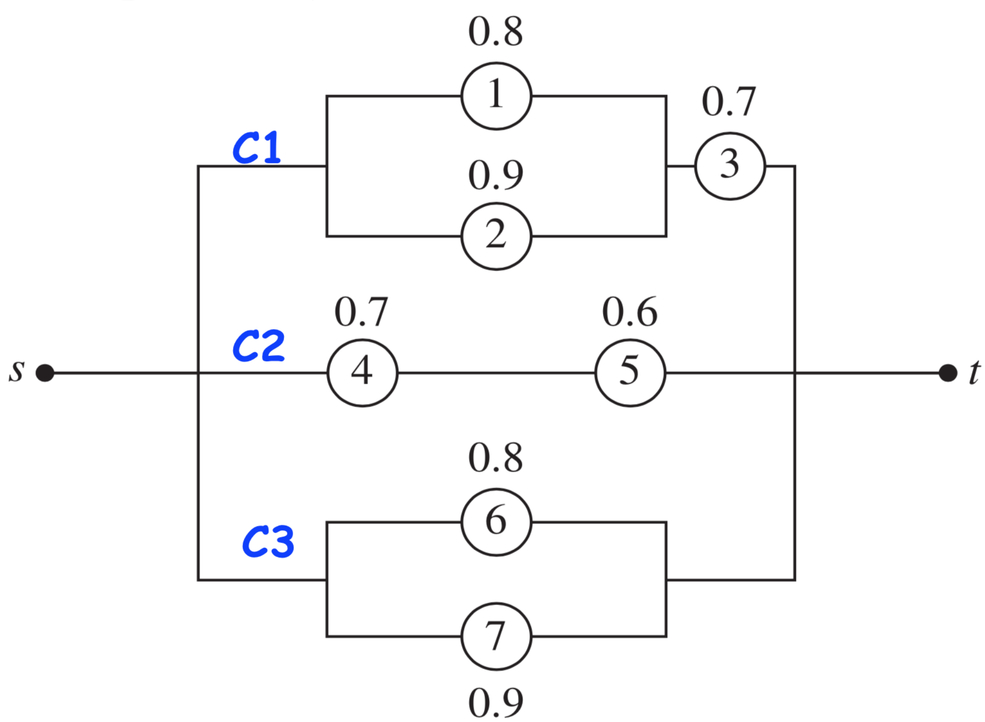

Figure 1: Circuit

(a) Find the probability the circuit C_1 is functional. [1 mark]

Answer:

(b) Find the probability the circuit C_2 is functional. [1 mark]

Answer:

(c) Find the probability the circuit C_3 is functional. [1 mark]

Answer:

(d) Find the the probability that current will flow from s to t.

Answer: [1 mark]

Solution: ANSWER. Denote that the ith gate is active is A_i
(a) $P(C_1 \text{active }) = P((A_1 \cup A_2) \cap A_3) = P(A_1 \cup A_2)P(A_3) = (0.8+0.9-0.8*0.9)*0.7 = 0.686$
(b) $P(C_2 \text{ active }) = P(A_4 \cap A_5) = P(A_4)P(A_5) = (0.7*0.6) = 0.42$
(c) $P(C_3 \text{ active }) = P(A_6 \cup A_7) = P(A_6) + P(A_7) - P(A_6)P(A_7) = 0.8+0.9-0.8*0.9 = 0.98$
(d) $P(C_1 \cup C_2 \cup C_3) = (0.686 + 0.42 + 0.98) - (0.686 * 0.42 + 0.42 * 0.98 + 0.686 * 0.98) + (0.686 * 0.42 * 0.98) = 0.9963576$

1 marks for each part