INDIAN INSTITUTE OF TECHNOLOGY
KHARAGPUR

	Stamp / Signature of the Invigilator

EXAMINATION (End Semester)		**SEMESTER (Autumn)**	
Roll Number		**Section**	**Name**
Subject Number	C S 2 1 0 0 1	**Subject Name**	*Discrete Structures*
Department / Center of the Student		**Additional sheets**	

Important Instructions and Guidelines for Students

1. You must occupy your seat as per the Examination Schedule/Sitting Plan.
2. Do not keep mobile phones or any similar electronic gadgets with you even in the switched off mode.
3. Loose papers, class notes, books or any such materials must not be in your possession, even if they are irrelevant to the subject you are taking examination.
4. Data book, codes, graph papers, relevant standard tables/charts or any other materials are allowed only when instructed by the paper-setter.
5. Use of instrument box, pencil box and non-programmable calculator is allowed during the examination. However, exchange of these items or any other papers (including question papers) is not permitted.
6. Write on both sides of the answer script and do not tear off any page. **Use last page(s) of the answer script for rough work.** Report to the invigilator if the answer script has torn or distorted page(s).
7. It is your responsibility to ensure that you have signed the Attendance Sheet. Keep your Admit Card/Identity Card on the desk for checking by the invigilator.
8. You may leave the examination hall for wash room or for drinking water for a very short period. Record your absence from the Examination Hall in the register provided. Smoking and the consumption of any kind of beverages are strictly prohibited inside the Examination Hall.
9. Do not leave the Examination Hall without submitting your answer script to the invigilator. **In any case, you are not allowed to take away the answer script with you.** After the completion of the examination, do not leave the seat until the invigilators collect all the answer scripts.
10. During the examination, either inside or outside the Examination Hall, gathering information from any kind of sources or exchanging information with others or any such attempt will be treated as **'unfair means'**. Do not adopt unfair means and do not indulge in unseemly behavior.

Violation of any of the above instructions may lead to severe punishment.

Signature of the Student

To be filled in by the examiner

Question Number	1	2	3	4	5	6	7	8	9	10	Total
Marks Obtained											

Marks obtained (in words)	Signature of the Examiner	Signature of the Scrutineer

CS21201/CS21001 Discrete Structures, Autumn 2022–2023

21-Nov-2022, 02:00pm–05:00pm | **End-Semester Test** | Maximum marks: 60

Instructions

- Write your answers in the question paper itself. Be brief and precise. Answer all questions.
- Write the answers only in the respective spaces provided. The last three blank pages may be used for rough work.
- If you use any theorem/result/formula covered in the class, just mention it, do not elaborate.
- Write all the proofs in mathematically precise language. Unclear and/or dubious statements would be severely penalized.

Do not write anything on this page.

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1. Suppose that you want to distribute 16 identical chocolates among 4 of your friends such that each of your friends receives at least 2 chocolates and no more than 6 chocolates. In how many ways can you do that? Solve this problem using generating functions. No other method will be given any credit. **(8)**

Solution The relevant generating function is

$$\begin{aligned} & \left(x^2+x^3+x^4+x^5+x^6\right)^4 \\ = \; & x^8\left(1+x+x^2+x^3+x^4\right)^4 \\ = \; & x^8\left(\frac{1-x^5}{1-x}\right)^4 \\ = \; & x^8\left(1-4x^5+6x^{10}-4x^{15}+x^{20}\right)\left(\sum_{n\geqslant 0}\binom{n+3}{3}x^n\right). \end{aligned}$$

The desired count is the coefficient of x^{16} in this expansion, that is,

$$\binom{8+3}{3}-4\times\binom{3+3}{3}=\binom{11}{3}-4\times\binom{6}{3}=165-4\times 20=85.$$

2. A string of decimal digits is considered to be a valid codeword if it contains an even number of 0 digits. For example, 02310023089 and 7254193776 are valid codewords, but 060796007620 is not valid. Let c_n denote the number of valid n-digit codewords.

(a) Derive, with clear justifications, a recurrence relation for c_n. Also supply the required number of initial conditions. **(6)**

Solution The initial condition is $c_0 = 1$ (the empty string contains zero (an even number) 0 digits). One may instead supply the initial condition $c_1 = 9$ (all length-one strings except 0 are valid codewords).

Now, take $n \geqslant 1$. A valid codeword W of length n can be obtained in two mutually exclusive ways.

(1) W starts with 0. Then the remaining $n-1$ digits form an invalid codeword of length $n-1$. The count of valid codewords W in this case is $10^{n-1} - c_{n-1}$.

(2) W starts with a digit other than 0 (there are nine possibilities). In this case, the remaining $n-1$ digits of W form a valid codeword of length $n-1$. The count of valid codewords W is this case is therefore $9c_{n-1}$.

Summing these counts gives

$$c_n = (10^{n-1} - c_{n-1}) + 9c_{n-1} = 8c_{n-1} + 10^{n-1}.$$

(b) Solve the recurrence relation of Part (a) to obtain a closed-form expression for c_n. For your derivation, you may use any method covered in the class. Clearly show all the steps. **(6)**

Solution This is an order-one recurrence, so we can simply unwind it to get

$$\begin{aligned}
c_n &= 8c_{n-1} + 10^{n-1} \\
&= 8(8c_{n-2} + 10^{n-2}) + 10^{n-1} \\
&= 8^2 c_{n-2} + \left[8 \times 10^{n-2} + 10^{n-1}\right] \\
&= 8^2(8c_{n-3} + 10^{n-3}) + 8 \times 10^{n-2} + 10^{n-1} \\
&= 8^3 c_{n-3} + \left[8^2 \times 10^{n-3} + 8 \times 10^{n-2} + 10^{n-1}\right] \\
&= \cdots \\
&= 8^n c_0 + \left[8^{n-1} + 8^{n-2} \times 10 + \cdots 8^2 \times 10^{n-3} + 8 \times 10^{n-2} + 10^{n-1}\right] \\
&= 8^n + 10^{n-1}\left[1 + \frac{8}{10} + \left(\frac{8}{10}\right)^2 + \cdots + \left(\frac{8}{10}\right)^{n-1}\right] \\
&= 8^n + 10^{n-1}\left[\frac{1 - \left(\frac{8}{10}\right)^n}{1 - \frac{8}{10}}\right] \\
&= 8^n + \frac{10^n - 8^n}{10 - 8} \\
&= \frac{10^n + 8^n}{2}.
\end{aligned}$$

3. Consider the following recursively defined sequence:

$$\begin{aligned} a_0 &= 2, \\ a_1 &= 8, \\ a_n &= 4a_{n-1} - 4a_{n-2} + n^2 - 5n + 2 \text{ for } n \geqslant 2. \end{aligned}$$

Derive a closed-form expression for a_n for all $n \geqslant 0$. You are not allowed to use generating functions in this exercise. **(10)**

Solution The characteristic equation of the recurrence is $r^2 - 4r + 4 = 0$, that is, $(r-2)^2 = 0$. Therefore the homogeneous solution will be of the form

$$a_n^{(h)} = (An+B)2^n.$$

The non-homogeneous part of the recurrence is $f(n) = n^2 - 5n + 2 = (n^2 - 5n + 2) \times 1^n$. Since 1 is not a characteristic root, the particular solution will be of the form

$$a_n^{(p)} = Un^2 + Vn + W.$$

Putting this in the recurrence gives

$$Un^2 + Vn + W = 4\Big(U(n-1)^2 + V(n-1) + W\Big) - 4\Big(U(n-2)^2 + V(n-2) + W\Big) + n^2 - 5n + 2.$$

Equating the coefficients of n^2, n^1, n^0 from the two sides gives

$$\begin{aligned} &U = 4U - 4U + 1, \text{ that is, } U = 1, \\ &V = -8U + 4V + 16U - 4V - 5, \text{ that is, } V = 8U - 5, \text{ that is, } V = 3, \\ &W = 4U - 4V + 4W - 16U + 8V - 4W + 2, \text{ that is, } W = -12U + 4V + 2, \text{ that is, } W = 2. \end{aligned}$$

We have therefore determined

$$a_n^{(p)} = n^2 + 3n + 2,$$

and so

$$a_n = a_n^{(h)} + a_n^{(p)} = (An+B)2^n + n^2 + 3n + 2.$$

The initial conditions give $a_0 = 2 = B + 2$, that is, $B = 0$, and $a_1 = 8 = 2(A+B) + 6 = 2A + 6$, that is, $A = 1$. Therefore

$$a_n = n2^n + n^2 + 3n + 2 \text{ for all } n \geqslant 0.$$

4. Consider the following C function. The parameters consist of a non-negative integer n and a bit $a \in \{0,1\}$.

```
int f ( int n, int a )
{
   if (n == 0) return (1 - a);
   if (a == 0) return f(n - 1, 0) + f(n - 1, 1);
   return f(n - 1, 0);
}
```

(a) Work out what $f(n,0)$ and $f(n,1)$ return for $n = 0,1,2,3,4,5$. Show your calculations. **(3)**

Solution These values are listed in the following table.

n	$f(n,0)$	$f(n,1)$
0	$f(0,0) = 1-0 = 1$	$f(0,1) = 1-1 = 0$
1	$f(1,0) = f(0,0) + f(0,1) = 1$	$f(1,1) = f(0,0) = 1$
2	$f(2,0) = f(1,0) + f(1,1) = 2$	$f(2,1) = f(1,0) = 1$
3	$f(3,0) = f(2,0) + f(2,1) = 3$	$f(3,1) = f(2,0) = 2$
4	$f(4,0) = f(3,0) + f(3,1) = 5$	$f(4,1) = f(3,0) = 3$
5	$f(5,0) = f(4,0) + f(4,1) = 8$	$f(5,1) = f(4,0) = 5$

(b) From Part (a), guess what $f(n,0)$ and $f(n,1)$ return for all $n \geqslant 0$. Prove your guess by induction on n. No credit for a wrong guess, or for a correct guess without a valid inductive proof. **(7)**

Solution An inspection of the return values for small n tends to indicate that $f(n,0) = F_{n+1}$ and $f(n,1) = F_n$ for all $n \geqslant 0$, where F_n is the n-th Fibonacci number. Let us prove this hunch by induction on n.

[Basis] For $n = 0$, we have $f(n,0) = 1 = F_1$ and $f(n,1) = 0 = F_0$.

[Induction] Take $n \geqslant 1$, and assume that $f(n-1,0) = F_n$ and $f(n-1,1) = F_{n-1}$. Then, $f(n,0) = f(n-1,0) + f(n-1,1) = F_n + F_{n-1} = F_{n+1}$, whereas $f(n,1) = f(n-1,0) = F_n$.

5. An infinite binary sequence $a_1a_2a_3\ldots$ (each a_i is a bit) is called *periodic* if there exists a positive integer p such that $a_{n+p} = a_n$ for all $n \geqslant 1$. The smallest such p is called the *period* of the sequence. For example, the sequence $011011011011\ldots = \overline{011} = \overline{011011}$ is periodic with period 3. Here, each horizontal line at the top indicates that the pattern beneath it is repeated infinitely often. An infinite binary sequence $a_1a_2a_3\ldots$ is called *eventually periodic* if there exist positive integers n_0 and p for which $a_{n+p} = a_n$ for all $n \geqslant n_0$. The smallest such p is again called the *period* of the sequence. The sequence $01110011011011011\ldots = 01110\overline{011} = 0111001\overline{101} = 011100\overline{110110}$ is eventually periodic of period 3.

(a) Prove that the set P of all periodic infinite binary sequences is countable. Your proof must contain the construction of an injective (not necessarily bijective) map from P to a suitable countable set S. **(5)**

Solution We know that the set S of all binary strings (finite sequences) is countable. Define the map $f : P \to S$ as follows. Take any periodic sequence $\overline{a_1a_2\ldots a_p}$ with p the period, and map this sequence to the (non-empty) string $a_1a_2\ldots a_p$. The map is clearly injective, so $|P| \leqslant |S|$, that is, P is countable.

(b) Is the set E of all eventually periodic infinite binary sequences countable? No credits without proper justifications. **(5)**

Solution *Countable*. Let S be as in Part (a). Take an eventually periodic infinite binary sequence $\sigma = a_1a_2a_3\ldots$ of period p. Consider the set

$$N_0 = \{n_0 \in \mathbb{N} \mid a_{n+p} = a_n \text{ for all } n \geqslant n_0\} \subseteq \mathbb{N}.$$

Since σ is eventually periodic, N_0 is non-empty. By the well-ordering principle, N_0 has a (unique) minimum element; call it n_0^*. We have $\sigma = a_1a_2\ldots a_{n_0^*-1}\overline{a_{n_0^*}a_{n_0^*+1}a_{n_0^*+2}\ldots a_{n_0^*+p-1}}$. Define a map $E \to S \times S$ that takes this σ to $(a_1a_2\ldots a_{n_0^*-1}, a_{n_0^*}a_{n_0^*+1}a_{n_0^*+2}\ldots a_{n_0^*+p-1})$. By the choice of n_0^*, this map is well-defined. Moreover, the map is clearly injective, so $|E| \leqslant |S \times S|$. But S is countable, and so $S \times S$ is countable too.

6. Let $\mathbb{R}$ be the set of real numbers, and $\pi = 3.1415926535\ldots$ (the ratio of the circumference to the diameter of any circle). Define two operations on $\mathbb{R}$ as

$$\begin{aligned} a \oplus b &= a + b + \pi, \\ a \odot b &= a + b + \frac{ab}{\pi}, \end{aligned}$$

where the expressions on the right-hand sides use the standard arithmetic of real numbers.

(a) Show that $(\mathbb{R}, \oplus, \odot)$ is a commutative ring. Verify all the axioms that define a commutative ring. Show all the steps clearly (despite that similar calculations were done in the class). **(6)**

Solution [Closure under $\oplus$] Obvious.

[Associativity of $\oplus$] $a \oplus (b \oplus c) = a \oplus (b + c + \pi) = a + (b + c + \pi) + \pi = a + b + c + 2\pi$, whereas $(a \oplus b) \oplus c = (a + b + \pi) \oplus c = (a + b + \pi) + c + \pi = a + b + c + 2\pi$.

[Commutativity of $\oplus$] Obvious.

[Additive identity] $a \oplus (-\pi) = a - \pi + \pi = a$, and so also $(-\pi) \oplus a = -\pi + a + \pi = a$. Therefore $-\pi$ is the additive identity.

[Additive inverse] $a \oplus b = -\pi$ requires $a + b + \pi = -\pi$, that is, $b = -(a + 2\pi)$, that is, $-(a + 2\pi)$ is the additive inverse of a.

[Closure under $\odot$] Obvious (division by π is legal in $\mathbb{R}$).

[Associativity of $\odot$] $a \odot (b \odot c) = a \odot (b + c + \frac{bc}{\pi}) = a + (b + c + \frac{bc}{\pi}) + \frac{a(b+c+\frac{bc}{\pi})}{\pi} = a + b + c + \frac{ab+bc+ca}{\pi} + \frac{abc}{\pi^2}$, whereas $(a \odot b) \odot c = (a + b + \frac{ab}{\pi}) \odot c = (a + b + \frac{ab}{\pi}) + c + \frac{(a+b+\frac{ab}{\pi})c}{\pi} = a + b + c + \frac{ab+bc+ca}{\pi} + \frac{abc}{\pi^2}$.

[Commutativity of $\odot$] Obvious.

[Distributivity of $\odot$ over $\oplus$] $a \odot (b \oplus c) = a \odot (b + c + \pi) = a + (b + c + \pi) + \frac{a(b+c+\pi)}{\pi} = 2a + b + c + \frac{ab+ac}{\pi} + \pi$. On the other hand, $(a \odot b) \oplus (a \odot c) = (a + b + \frac{ab}{\pi}) \oplus (a + c + \frac{ac}{\pi}) = (a + b + \frac{ab}{\pi}) + (a + c + \frac{ac}{\pi}) + \pi = 2a + b + c + \frac{ab+ac}{\pi} + \pi$.

(b) Prove that $(\mathbb{R}, \oplus, \odot)$ contains a multiplicative identity. What is it, and why? **(2)**

Solution 0 is the multiplicative identity, because $0 \odot a = a \odot 0 = a$.

(c) Find all the units of $(\mathbb{R}, \oplus, \odot)$. **(2)**

Solution $a \odot b = b \odot a = 0$ requires $a + b + \frac{ab}{\pi} = 0$, that is, $b = \frac{-a}{1+\frac{a}{\pi}}$. This b exists if and only if $1 + \frac{a}{\pi} \neq 0$, that is, $a \neq -\pi$. But $-\pi$ is the additive identity, that is, every element other than the additive identity is a unit in the ring. Indeed, $(\mathbb{R}, \oplus, \odot)$ is a field.