Exercises — Markov Chain

3.1.1 A Markov chain $X_0, X_1, \ldots$ on states 0, 1, 2 has the transition probability matrix

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{ccc} 0 & 1 & 2 \\ \left\| \begin{array}{ccc} 0.1 & 0.2 & 0.7 \\ 0.9 & 0.1 & 0 \\ 0.1 & 0.8 & 0.1 \end{array} \right\| \end{array}$$

and initial distribution $p_0 = \Pr\{X_0 = 0\} = 0.3, p_1 = \Pr\{X_0 = 1\} = 0.4$, and $p_2 = \Pr\{X_0 = 2\} = 0.3$. Determine $\Pr\{X_0 = 0, X_1 = 1, X_2 = 2\}$.

3.1.2 A Markov chain $X_0, X_1, X_2, \ldots$ has the transition probability matrix

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{ccc} 0 & 1 & 2 \\ \left\| \begin{array}{ccc} 0.7 & 0.2 & 0.1 \\ 0 & 0.6 & 0.4 \\ 0.5 & 0 & 0.5 \end{array} \right\| \end{array}.$$

Determine the conditional probabilities

$$\Pr\{X_2 = 1, X_3 = 1 | X_1 = 0\} \quad \text{and} \quad \Pr\{X_1 = 1, X_2 = 1 | X_0 = 0\}.$$

3.1.3 A Markov chain $X_0, X_1, X_2, \ldots$ has the transition probability matrix

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{ccc} 0 & 1 & 2 \\ \left\| \begin{array}{ccc} 0.6 & 0.3 & 0.1 \\ 0.3 & 0.3 & 0.4 \\ 0.4 & 0.1 & 0.5 \end{array} \right\| \end{array}.$$

If it is known that the process starts in state $X_0 = 1$, determine the probability $\Pr\{X_0 = 1, X_1 = 0, X_2 = 2\}$.

3.1.4 A Markov chain $X_0, X_1, X_2, \ldots$ has the transition probability matrix

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{ccc} 0 & 1 & 2 \\ \left\| \begin{array}{ccc} 0.1 & 0.1 & 0.8 \\ 0.2 & 0.2 & 0.6 \\ 0.3 & 0.3 & 0.4 \end{array} \right\| \end{array}.$$

Determine the conditional probabilities

$$\Pr\{X_1 = 1, X_2 = 1 | X_0 = 0\} \quad \text{and} \quad \Pr\{X_2 = 1, X_3 = 1 | X_1 = 0\}.$$

3.1.5 A Markov chain $X_0, X_1, X_2, \ldots$ has the transition probability matrix

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{ccc} 0 & 1 & 2 \\ \left\| \begin{array}{ccc} 0.3 & 0.2 & 0.5 \\ 0.5 & 0.1 & 0.4 \\ 0.5 & 0.2 & 0.3 \end{array} \right\| \end{array}$$

and initial distribution $p_0 = 0.5$ and $p_1 = 0.5$. Determine the probabilities

$$\Pr\{X_0 = 1, X_1 = 1, X_2 = 0\} \quad \text{and} \quad \Pr\{X_1 = 1, X_2 = 1, X_3 = 0\}.$$

Problems

3.1.1 A simplified model for the spread of a disease goes this way: The total population size is $N = 5$, of which some are diseased and the remainder are healthy. During any single period of time, two people are selected at random from the population and assumed to interact. The selection is such that an encounter between any pair of individuals in the population is just as likely as between any other pair. If one of these persons is diseased and the other not, with probability $\alpha = 0.1$ the disease is transmitted to the healthy person. Otherwise, no disease transmission takes place. Let X_n denote the number of diseased persons in the population at the end of the nth period. Specify the transition probability matrix.

3.1.2 Consider the problem of sending a binary message, 0 or 1, through a signal channel consisting of several stages, where transmission through each stage is subject to a fixed probability of error α. Suppose that $X_0 = 0$ is the signal that is sent and let X_n be the signal that is received at the nth stage. Assume that $\{X_n\}$ is a Markov chain with transition probabilities $P_{00} = P_{11} = 1 - \alpha$ and $P_{01} = P_{10} = \alpha$, where $0 < \alpha < 1$.

    (a) Determine $\Pr\{X_0 = 0, X_1 = 0, X_2 = 0\}$, the probability that no error occurs up to stage $n = 2$.

    (b) Determine the probability that a correct signal is received at stage 2.

    Hint: This is $\Pr\{X_0 = 0, X_1 = 0, X_2 = 0\} + \Pr\{X_0 = 0, X_1 = 1, X_2 = 0\}$.

3.1.3 Consider a sequence of items from a production process, with each item being graded as good or defective. Suppose that a good item is followed by another good item with probability α and is followed by a defective item with probability $1 - \alpha$. Similarly, a defective item is followed by another defective item with probability β and is followed by a good item with probability $1 - \beta$. If the first item is good, what is the probability that the first defective item to appear is the fifth item?

3.1.4 The random variables $\xi_1, \xi_2, \ldots$ are independent and with the common probability mass function

$k =$	0	1	2	3
$\Pr\{\xi = k\} =$	0.1	0.3	0.2	0.4

Set $X_0 = 0$, and let $X_n = \max\{\xi_1, \ldots, \xi_n\}$ be the largest ξ observed to date. Determine the transition probability matrix for the Markov chain $\{X_n\}$.