Exercises — Branching Process

3.8.1 A population begins with a single individual. In each generation, each individual in the population dies with probability $\frac{1}{2}$ or doubles with probability $\frac{1}{2}$. Let X_n denote the number of individuals in the population in the nth generation. Find the mean and variance of X_n.

3.8.2 The number of offspring of an individual in a population is $0, 1$, or 2 with respective probabilities $a > 0$, $b > 0$, and $c > 0$, where $a+b+c=1$. Express the mean and variance of the offspring distribution in terms of b and c.

3.8.3 Suppose a parent has no offspring with probability $\frac{1}{2}$ and has two offspring with probability $\frac{1}{2}$. If a population of such individuals begins with a single parent and evolves as a branching process, determine u_n, the probability that the population is extinct by the nth generation, for $n = 1, 2, 3, 4, 5$.

3.8.4 At each stage of an electron multiplier, each electron, upon striking the plate, generates a Poisson distributed number of electrons for the next stage. Suppose the mean of the Poisson distribution is λ. Determine the mean and variance for the number of electrons in the nth stage.

Problems

3.8.1 Each adult individual in a population produces a fixed number M of offspring and then dies. A fixed number L of these remain at the location of the parent. These local offspring will either all grow to adulthood, which occurs with a fixed probability β, or all will die, which has probability $1-\beta$. Local mortality is catastrophic in that it affects the entire local population. The remaining $N = M - L$ offspring disperse. Their successful growth to adulthood will occur statistically independently of one another, but at a lower probability $\alpha = p\beta$, where p may be thought of as the probability of successfully surviving the dispersal process. Define the random variable ξ to be the number of offspring of a single parent that survive to reach adulthood in the next generation. According to our assumptions, we may write ξ as

$$\xi = \nu_1 + \nu_2 + \cdots + \nu_N + (M-N)\Theta,$$

where $\Theta, \nu_1, \nu_2, \ldots, \nu_N$ are independent with $\Pr\{\nu_k = 1\} = \alpha$, $\Pr\{\nu_k = 0\} = 1-\alpha$, and with $\Pr\{\Theta = 1\} = \beta$ and $\Pr\{\Theta = 0\} = 1-\beta$. Show that the mean number of offspring reaching adulthood is $E[\xi] = \alpha N + \beta(M-N)$, and since $\alpha < \beta$, the mean number of surviving offspring is maximized by dispersing none ($N = 0$). Show that the probability of having no offspring surviving to adulthood is

$$\Pr\{\xi = 0\} = (1-\alpha)^N (1-\beta)$$

and that this probability is made smallest by making N large.

3.8.2 Let $Z = \sum_{n=0}^{x} X_n$ be the total family size in a branching process whose offspring distribution has a mean $\mu = E[\xi] < 1$. Assuming that $X_0 = 1$, show that $E[Z] = 1/(1-\mu)$.

3.8.3 Families in a certain society choose the number of children that they will have according to the following rule: If the first child is a girl, they have exactly one more child. If the first child is a boy, they continue to have children until the first girl, and then cease childbearing.

(a) For $k = 0, 1, 2, \ldots$, what is the probability that a particular family will have k children in total?

(b) For $k = 0, 1, 2, \ldots$, what is the probability that a particular family will have exactly k male children among their offspring?

3.8.4 Let $\{X_n\}$ be a branching process with mean family size μ. Show that $Z_n = X_n/\mu^n$ is a nonnegative martingale. Interpret the maximal inequality as applied to $\{Z_n\}$.

Exercises

3.9.1 Suppose that the offspring distribution is Poisson with mean $\lambda = 1.1$. Compute the extinction probabilities $u_n = \Pr\{X_n = 0 | X_0 = 1\}$ for $n = 0, 1, \ldots, 5$. What is u_∞, the probability of ultimate extinction?

Problems

3.9.1 One-fourth of the married couples in a far-off society have no children at all. The other three-fourths of couples have exactly three children, with each child equally likely to be a boy or a girl. What is the probability that the male line of descent of a particular husband will eventually die out?

3.9.2 One-fourth of the married couples in a far-off society have exactly three children. The other three-fourths of couples continue to have children until the first boy and then cease childbearing. Assume that each child is equally likely to be a boy or girl. What is the probability that the male line of descent of a particular husband will eventually die out?

3.9.3 Consider a large region consisting of many subareas. Each subarea contains a branching process that is characterized by a Poisson distribution with parameter λ. Assume, furthermore, that the value of λ varies with the subarea, and its distribution over the whole region is that of a gamma distribution. Formally, suppose that the offspring distribution is given by

$$\pi(k|\lambda) = \frac{e^{-\lambda}\lambda^k}{k!} \quad \text{for } k = 0, 1, \ldots,$$

where λ itself is a random variable having the density function

$$f(\lambda) = \frac{\theta^\alpha \lambda^{\alpha-1} e^{-\theta\lambda}}{\Gamma(\alpha)} \quad \text{for } \lambda > 0,$$

where θ and α are positive constants. Determine the marginal offspring distribution $p_k = \int \pi(k|\lambda) f(\lambda) d\lambda$.

3.9.5 At time 0, a blood culture starts with one red cell. At the end of 1 min, the red cell dies and is replaced by one of the following combinations with the probabilities as indicated:

Two red cells	$\frac{1}{4}$
One red, One white	$\frac{2}{3}$
Two white	$\frac{1}{12}$

Each red cell lives for 1 min and gives birth to offspring in the same way as the parent cell. Each white cell lives for 1 min and dies without reproducing. Assume that individual cells behave independently.

(a) At time $n + \frac{1}{2}$ min after the culture begins, what is the probability that no white cells have yet appeared?

(b) What is the probability that the entire culture eventually dies out entirely?