Simulation 1

Monte Carlo Simulation approach

$$\underset{\sim}{X} = (X_1, \ldots, X_n) \overset{\text{density}}{\sim} f(x_1, \ldots, x_n)$$

Compute $E[g(\underset{\sim}{X})] = \int \cdots \int g(x_1, \ldots, x_n)\, f(x_1, \ldots, x_n)\, dx_1 \cdots dx_n$

$$\underset{\sim}{X}^{(1)} = (\underset{\sim}{X}_1^{(1)}, \ldots, X_n^{(1)}) \overset{\text{density}}{\sim} f(x_1, \ldots, x_n)$$

Compute $Y^{(1)} = g(\underset{\sim}{X}^{(1)})$

$\underset{\sim}{X}^{(2)} \overset{\text{density}}{\sim} f(x_1, \ldots, x_n) \qquad Y^{(2)} = g(\underset{\sim}{X}^{(2)})$

$$Y^{(i)} = g(\underset{\sim}{X}^{(i)}), \quad i = 1, 2, \ldots, r$$

IID

$$\lim_{r \to \infty} \frac{Y^{(1)} + \cdots + Y^{(r)}}{r} = E(Y) = E[g(\underset{\sim}{X})]$$

—x—

initial seed X_0

$$X_{n+1} = (a X_n + c) \text{ modulo } m, \quad n \geq 0$$

PRN

$$U_n = \frac{X_n}{m}$$

Eg 1 $\quad X_i = (5 X_{i-1} + 1) \bmod 8 \qquad$ seed $X_0 = 0$

$X_1 = 1 \bmod 8 = 1 \qquad ; U_1 = 1/8$

$X_2 = 6 \bmod 8 = 6 \qquad ; U_2 = 6/8$

$8)\overline{1}(0$
$\underline{0}$
$1 = X_1$

$8)\overline{7}\overline{1}($

$X_3 = 31 \bmod 8 = 7$; $U_3 = 7/8$

$\frac{24}{7} = X_3$

$X_4 = 4$, $U_4 = 4/8$

Full period generator

$X_5 = 5$ $U_5 = 5/8$

$X_6 = 2$ $U_6 = 2/8$

$X_7 = 3$ $U_7 = 3/8$

$X_8 = 0$ $U_8 = 0$

(2) $X_i = (3X_{i-1} + 1) \bmod 7$, $X_0 = 3$

$X_1 = 3 = X_2 = \ldots$ Not full period

(3) generator $X_i = 16807 X_{i-1} \bmod (2^{31} - 1)$

seed $X_0 \neq 0$

is full period and used in many real world application and passes most statistical tests for uniformity and randomness.

Example (1) Monte Carlo integration

$$I = \int_a^b f(x)\,dx = (b-a)\int_0^1 f(a+(b-a)u)\,du$$

$x = a + (b-a)u$

$dx = (b-a)\,du$

$$\hat{I}_n = \frac{b-a}{n} \sum_{i=1}^{n} f(a + (b-a)u_i)$$

$U_i \sim U(0,1)$

density $f_{U_i}(u) = \begin{cases} 1 & , 0 < u < 1 \\ 0 & \text{ow} \end{cases}$

$$E(\hat{I}_n) = \frac{b-a}{n} \sum_{i=1}^{n} E(f(a+(b-a)U_i))$$

$$= \frac{b-a}{n} \sum_{i=1}^{n} \int_0^1 f(a+(b-a)u) \times 1 \, du$$

$$= (b-a) \int_0^1 f(a+(b-a)u) du = I$$

$\hat{I}_n$ UE for I.

$$I = \int_0^1 (1+\cos(\pi x)) dx \; (= 1)$$

$n=4$ $\quad U(0,1)$ $\quad \overset{U_1}{0.419}, \overset{U_2}{0.109}, \overset{U_3}{0.732}, \overset{U_4}{0.893}$

$$\hat{I}_4 = \frac{1-0}{4} \sum_{i=1}^{4} [1+\cos(\pi(0+(1-0)U_i))]$$

$$= 0.896$$

② i^{th} flip H if $U < 0.5$

T if $U \geq 0.5$

0.12	0.52	0.73	0.05	--
H	T	T	H	--

③ Single-server queue

6 customers arrive at a bank at the following times which have been generated from some appropriate prob. dist. $3, 4, 6, 10, 15, 20$ (A_i)

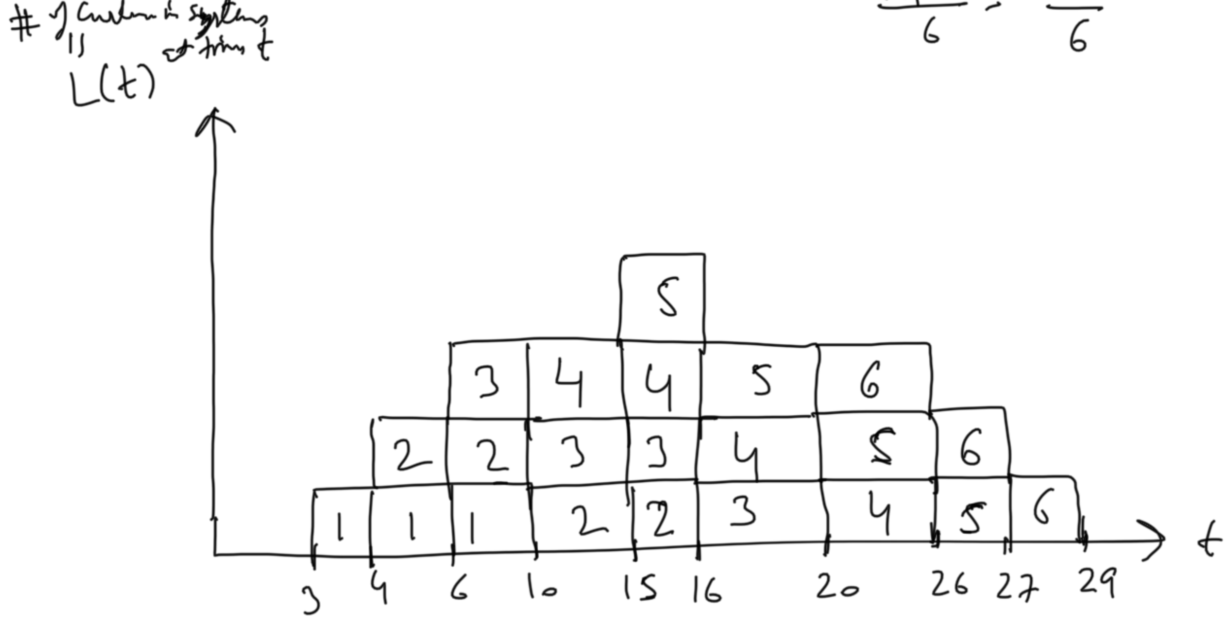
1 server FCFS service times corresponding to the arriving customer are $7, 6, 4, 6, 1, 2$ (S_i)

Customer i	arrival time A_i	begin service (B_i)	service time (S_i)	departure time $D_i = B_i + S_i$	Wait $W_i = B_i - A_i$
1	3	3	7	10	0

2	4	10	6	16	6
3	6	16	4	20	10
4	10	20	6	26	10
5	15	26	1	27	11
6	20	27	2	29	7

$\sum_i w_i = 44$

av. waiting time for six customers $= \frac{\sum_{i=1}^{6} w_i}{6} = \frac{44}{6}$



av. # of customers in the system $= \frac{1}{29}\int_0^{29} L(t)\,dt$

$= \frac{1}{29}\Big[\int_0^3 0\,dt + \int_3^4 1\,dt + \int_4^6 2\,dt + \int_6^{10} 3\,dt + \int_{10}^{15} 3\,dt$

$+ \int_{15}^{16} 4\,dt + \int_{16}^{20} 3\,dt + \int_{20}^{26} 3\,dt + \int_{26}^{27} 2\,dt + \int_{27}^{29} 1\,dt\Big]$

$= \frac{70}{29}$

(4) Estimate π

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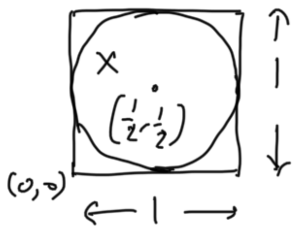
$$\frac{\text{circle area}}{\text{square area}} = \frac{\pi\left(\frac{1}{2}\right)^2}{1} = \frac{\pi}{4}$$



$\rightarrow$ prob. that a dart will land in the circle

$(U_{11}, U_{12}), (U_{21}, U_{22}), \ldots$

$$\left(U_{i1} - \frac{1}{2}\right)^2 + \left(U_{i2} - \frac{1}{2}\right)^2 \le \frac{1}{4} \quad (1)$$

(U_{i1}, U_{i2}) will fall in the circle if

$$X_i = \begin{cases} 1 & \text{if pair } (U_{i1}, U_{i2}) \text{ satisfy } (1) \\ 0 & \text{o.w.} \end{cases}$$

$$X = \sum_{i=1}^{n} X_i \sim Bin(n, p), \quad p = \frac{\pi}{4}$$

$\hat{p} = \frac{X}{n}$ is mle for $p = \frac{\pi}{4}$

mle of π is $4\hat{p}$

eg $n = 1000$, $X = 753$ dart are landing in the circle

$$\hat{\pi} = 4 \times \frac{753}{1000} = 3.12$$

—x—

Probability Integral Transform

cont. $X \overset{\text{cdf}}{\sim} F(x)$, then $U = F(X) \sim U(0,1)$

Sol, $F(x) = P(X \le x)$

cdf of U is for $0 \le u \le 1$

$$G(u) = P(U \le u) = P(F(X) \le u)$$
$$= P(X \le F^{-1}(u))$$
$$= F F^{-1}(u)$$
$$= u$$

pdf of U $\quad g(u) = \begin{cases} 1 & \text{if } 0 \le u \le 1 \\ 0 & \text{o.w.} \end{cases}$

$U \sim U(0,1)$

<u>Inverse transform method</u>

$X \overset{cdf}{\sim} F(\cdot) \qquad U = F(X) \sim U(0,1)$

$$\Rightarrow X = F^{-1}(U)$$

① Simulate expo. r.v. with mean 1.

cdf $X \sim F(x) = \begin{cases} 1 - e^{-x}, & x \ge 0 \\ 0 & \text{o.w.} \end{cases}$

$$u = F(x) = 1 - e^{-x} \Rightarrow x = -\ln(1-u)$$

$$X = -\ln(1-U) \sim \exp(1), \quad U \sim U(0,1)$$

$u_1 = 0.1, \quad x = -\ln(1-0.1)$ $\qquad$ <u>$X = -\ln U \sim \exp(1)$</u>

$$1 - U \overset{d}{=} U$$

$$P(1-U \le x) = P(U \ge 1-x) = 1 - P(U \le 1-x)$$
$$= 1 - (1-x) = x = P(U \le x)$$

② $X_1, \ldots, X_n \overset{i.i.d}{\sim} \exp(\lambda)$

n

$$Y = \sum_{i=1}^{n} X_i \sim \text{Gamma}(n, \lambda)$$

$$Y = -\sum_{i=1}^{n} \frac{1}{\lambda} \ln U_i = -\frac{1}{\lambda} \ln\left(\prod_{i=1}^{n} U_i\right) \sim \text{Gamma}(n,\lambda)$$

(3) generate a standard normal r.v. Z

$$\Phi(Z) = U \qquad Z \overset{cdf}{\sim} \Phi$$

$$\Rightarrow Z = \Phi^{-1}(U)$$

$u = 0.72$ $\qquad P(Z \le z) = 0.72$

$\Phi^{-1}(u) = z = 0.59$

(4) $Z \sim N(0,1)$; $X = \mu + \sigma Z \sim N(\mu, \sigma^2)$

$\mu = 1,\ \sigma^2 = 4$ $\quad X = 1 + 2 \times 0.59 \sim N(1,4)$

(5) $U_1, \ldots, U_n \sim$ IID PRN's

Using CLT

$$\frac{\sum_{i=1}^{n} U_i - \sum_{i=1}^{n} E(U_i)}{\sqrt{\sum_{i=1}^{n} Var(U_i)}} = \frac{\sum_{i=1}^{n} U_i - \frac{n}{2}}{\sqrt{\frac{n}{12}}} \simeq N(0,1)$$

$$n = 12 \qquad \sum_{i=1}^{12} U_i - 6 \simeq N(0,1)$$

Simulate from discrete dists:

simulate X pmf $P(X = x_j) = P_j$, $j = 1, 2, \ldots$; $\sum_j P_j = 1$

$U \sim U(0,1)$

$$\text{set } X = \begin{cases} x_1, & U < P_1 \\ x_2, & P_1 < U < P_1 + P_2 \\ \vdots & \\ x_j, & \sum_{i=1}^{j-1} P_i < U < \sum_{i=1}^{j} P_i \\ \vdots & \end{cases}$$

$$P(X = x_j) = P\left(\sum_{i=1}^{j-1} P_i < U < \sum_{i=1}^{j} P_i \right) = P_j$$

$\therefore$ X is the desired dis^n.

eg X pmf $p(x) = \begin{cases} 0.3, & x = -1 \\ 0.6, & x = 2.3 \\ 0.1, & x = 7 \\ 0 & \text{ow} \end{cases}$

x	$p(x)$	U
-1	0.3	$[0, 0.3)$
2.3	0.6	$[0.3, 0.9)$
7	0.1	$[0.9, 1]$

If $U = 0.4$, then $x = 2.3$

determining the number of runs

generate r i.i.d. r.v.s $Y^{(1)}, \dots, Y^{(r)}$ having mean μ, var σ^2.

$$\bar{Y}_r = \frac{Y^{(1)} + \dots + Y^{(r)}}{r}$$ as an estimate to μ

$$Var(\bar{Y}_r) = \frac{\sigma^2}{r}$$ Choose r s.t. $\frac{\sigma^2}{r}$ is small (acceptable)

$\sigma^2 \to$ unknown

Initially simulate k runs $Y^{(1)}, \ldots, Y^{(k)}$

$$\hat{\sigma}^2 = \frac{1}{k-1} \sum_{i=1}^{k} (Y^{(i)} - \bar{Y}_k)^2$$

Choose r s.t. $\frac{\hat{\sigma}^2}{r}$ is small (acceptable)

remaining $r-k$ simulate runs

—x—