Exercises

4.1.1 A Markov chain $X_0, X_1, X_2, \ldots$ has the transition probability matrix

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array}\begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \left\| \begin{array}{ccc} 0.7 & 0.2 & 0.1 \\ 0 & 0.6 & 0.4 \\ 0.5 & 0 & 0.5 \end{array} \right\| \end{array}.$$

Determine the limiting distribution.

4.1.2 A Markov chain $X_0, X_1, X_2, \ldots$ has the transition probability matrix

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array}\begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \left\| \begin{array}{ccc} 0.6 & 0.3 & 0.1 \\ 0.3 & 0.3 & 0.4 \\ 0.4 & 0.1 & 0.5 \end{array} \right\| \end{array}.$$

Determine the limiting distribution.

4.1.3 A Markov chain $X_0, X_1, X_2, \ldots$ has the transition probability matrix

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array}\begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \left\| \begin{array}{ccc} 0.1 & 0.1 & 0.8 \\ 0.2 & 0.2 & 0.6 \\ 0.3 & 0.3 & 0.4 \end{array} \right\| \end{array}.$$

What fraction of time, in the long run, does the process spend in state 1?

4.1.4 A Markov chain $X_0, X_1, X_2, \ldots$ has the transition probability matrix

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array}\begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \left\| \begin{array}{ccc} 0.3 & 0.2 & 0.5 \\ 0.5 & 0.1 & 0.4 \\ 0.5 & 0.2 & 0.3 \end{array} \right\| \end{array}.$$

Every period that the process spends in state 0 incurs a cost of \$2. Every period that the process spends in state 1 incurs a cost of \$5. Every period that the process spends in state 2 incurs a cost of \$3. What is the long run cost per period associated with this Markov chain?

4.1.5 Consider the Markov chain whose transition probability matrix is given by

$$\mathbf{P} = \begin{array}{c|cccc|} & 0 & 1 & 2 & 3 \\ 0 & 0.1 & 0.5 & 0 & 0.4 \\ 1 & 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 0 & 1 \\ 3 & 1 & 0 & 0 & 0 \end{array}.$$

Determine the limiting distribution for the process.

4.1.6 Compute the limiting distribution for the transition probability matrix

$$\mathbf{P} = \begin{array}{c|ccc|} & 0 & 1 & 2 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 1 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 2 & \frac{1}{6} & \frac{1}{2} & \frac{1}{3} \end{array}.$$

4.1.7 A Markov chain on the states 0, 1, 2, 3 has the transition probability matrix

$$\mathbf{P} = \begin{array}{c|cccc|} & 0 & 1 & 2 & 3 \\ 0 & 0.1 & 0.2 & 0.3 & 0.4 \\ 1 & 0 & 0.3 & 0.3 & 0.4 \\ 2 & 0 & 0 & 0.6 & 0.4 \\ 3 & 1 & 0 & 0 & 0 \end{array}.$$

Determine the corresponding limiting distribution.

4.1.8 Suppose that the social classes of successive generations in a family follow a Markov chain with transition probability matrix given by

		Son's class		
		Lower	Middle	Upper
Father's class	Lower	0.7	0.2	0.1
	Middle	0.2	0.6	0.2
	Upper	0.1	0.4	0.5

What fraction of families are upper class in the long run?

4.1.9 Determine the limiting distribution for the Markov chain whose transition probability matrix is

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \left\| \begin{array}{ccc} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{3} & \frac{1}{2} & \frac{1}{6} \\ 0 & \frac{1}{4} & \frac{3}{4} \end{array} \right\| \end{array}.$$

4.1.10 A bus in a mass transit system is operating on a continuous route with intermediate stops. The arrival of the bus at a stop is classified into one of three states, namely

1. Early arrival;
2. On-time arrival;
3. Late arrival.

Suppose that the successive states form a Markov chain with transition probability matrix

$$\mathbf{P} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \left\| \begin{array}{ccc} 0.5 & 0.4 & 0.1 \\ 0.2 & 0.5 & 0.3 \\ 0.1 & 0.2 & 0.7 \end{array} \right\| \end{array}.$$

Over a long period of time, what fraction of stops can be expected to be late?

Problems

4.1.1 Five balls are distributed between two urns, labeled A and B. Each period, an urn is selected at random, and if it is not empty, a ball from that urn is removed and placed into the other urn. In the long run what fraction of time is urn A empty?

4.1.2 Five balls are distributed between two urns, labeled A and B. Each period, one of the five balls is selected at random, and whichever urn it's in, it is moved to the other urn. In the long run, what fraction of time is urn A empty?

4.1.3 A Markov chain has the transition probability matrix

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & 5 \end{array} \\ \left\| \begin{array}{cccccc} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \alpha_5 & \alpha_6 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array} \right\| \end{array},$$

where $\alpha_i \geq 0, i = 1, \ldots, 6$, and $\alpha_1 + \cdots + \alpha_6 = 1$. Determine the limiting probability of being in state 0.

4.1.4 A finite-state regular Markov chain has transition probability matrix $\mathbf{P} = \|P_{ij}\|$ and limiting distribution $\pi = \|\pi_i\|$. In the long run, what fraction of the *transitions* are from a prescribed state k to a prescribed state m?

4.1.5 The four towns A, B, C, and D are connected by railroad lines as shown in the following diagram:

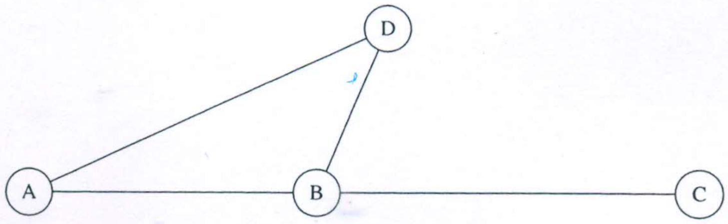

Figure 4.1 A graph whose nodes represent towns and whose arcs represent railroad lines.

Each day, in whichever town it is in, a train chooses one of the lines out of that town at random and traverses it to the next town, where the process repeats the next day. In the long run, what is the probability of finding the train in town D?

4.1.6 Determine the following limits in terms of the transition probability matrix $\mathbf{P} = \|P_{ij}\|$ and limiting distribution $\pi = \|\pi_j\|$ of a finite-state regular Markov chain $\{X_n\}$:

(a) $\lim_{n\to\infty} \Pr\{X_{n+1} = j | X_0 = i\}$.

(b) $\lim_{n\to\infty} \Pr\{X_n = k, X_{n+1} = j | X_0 = i\}$.

(c) $\lim_{n\to\infty} \Pr\{X_{n-1} = k, X_n = j | X_0 = i\}$.

4.1.7 Determine the limiting distribution for the Markov chain whose transition probability matrix is

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \left\| \begin{array}{cccc} \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{3} & \frac{1}{6} \\ 0 & 0 & 1 & 0 \end{array} \right\| \end{array}.$$

4.1.8 Show that the transition probability matrix

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \left\|\begin{array}{ccccc} 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 \end{array}\right\| \end{array}$$

is regular and compute the limiting distribution.

4.1.9 Determine the long run, or limiting, distribution for the Markov chain whose transition probability matrix is

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \left\|\begin{array}{cccc} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{array}\right\| \end{array}.$$

4.1.10 Consider a Markov chain with transition probability matrix

$$\mathbf{P} = \left\|\begin{array}{ccccc} p_0 & p_1 & p_2 & \cdots & p_N \\ p_N & p_0 & p_1 & \cdots & p_{N-1} \\ p_{N-1} & p_N & p_0 & \cdots & p_{N-2} \\ \vdots & \vdots & \vdots & & \vdots \\ p_1 & p_2 & p_3 & \cdots & p_0 \end{array}\right\|,$$

where $0 < p_0 < 1$ and $p_0 + p_1 + \cdots + p_N = 1$. Determine the limiting distribution.

4.1.11 Suppose that a production process changes state according to a Markov process whose transition probability matrix is given by

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \left\|\begin{array}{cccc} 0.3 & 0.5 & 0 & 0.2 \\ 0.5 & 0.2 & 0.2 & 0.1 \\ 0.2 & 0.3 & 0.4 & 0.1 \\ 0.1 & 0.2 & 0.4 & 0.3 \end{array}\right\| \end{array}.$$

It is known that $\pi_1 = \frac{119}{379} = 0.3140$ and $\pi_2 = \frac{81}{379} = 0.2137$.

(a) Determine the limiting probabilities π_0 and π_3.

(b) Suppose that states 0 and 1 are "In-Control" while states 2 and 3 are deemed "Out-of-Control." In the long run, what fraction of time is the process Out-of-Control?

(c) In the long run, what *fraction of transitions* are from an In-Control state to an Out-of-Control state?

4.1.12 Let $\mathbf{P}$ be the transition probability matrix of a finite-state regular Markov chain, and let Π be the matrix whose rows are the stationary distribution π. Define $\mathbf{Q} = \mathbf{P} - \mathbf{\Pi}$.

(a) Show that $\mathbf{P}^n = \mathbf{\Pi} + \mathbf{Q}^n$.

(b) When

$$\mathbf{P} = \begin{Vmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{Vmatrix}$$

obtain an explicit expression for $\mathbf{Q}^n$ and then for $\mathbf{P}^n$.

4.1.13 A Markov chain has the transition probability matrix

$$\mathbf{P} = \begin{array}{c|ccc} & 0 & 1 & 2 \\ 0 & 0.4 & 0.4 & 0.2 \\ 1 & 0.6 & 0.2 & 0.2 \\ 2 & 0.4 & 0.2 & 0.4 \end{array}.$$

After a long period of time, you observe the chain and see that it is in state 1. What is the conditional probability that the previous state was state 2? That is, find

$$\lim_{n\to\infty} \Pr\{X_{n-1} = 2 | X_n = 1\}.$$

Exercises

4.4.1 Determine the limiting distribution for the Markov chain whose transition probability matrix is

$$
\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array}
\begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\
\left\|\begin{array}{ccccc}
q & p & 0 & 0 & 0 \\
q & 0 & p & 0 & 0 \\
q & 0 & 0 & p & 0 \\
q & 0 & 0 & 0 & p \\
1 & 0 & 0 & 0 & 0
\end{array}\right\| \end{array},
$$

where $p > 0, q > 0$, and $p + q = 1$.

4.4.2 Determine the stationary distribution for the Markov chain whose transition probability matrix is

$$
\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array}
\begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\
\left\|\begin{array}{cccc}
0 & 0 & \frac{1}{2} & \frac{1}{2} \\
0 & 0 & \frac{1}{3} & \frac{2}{3} \\
\frac{1}{4} & \frac{3}{4} & 0 & 0 \\
\frac{1}{3} & \frac{2}{3} & 0 & 0
\end{array}\right\| \end{array}.
$$

4.4.3 Consider a random walk Markov chain on state $0, 1, \ldots, N$ with transition probability matrix

$$
\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ \vdots \\ N-1 \\ N \end{array}
\begin{array}{c} \begin{array}{ccccccccc} 0 & 1 & 2 & 3 & 4 & 5 & & N-1 & N \end{array} \\
\left\|\begin{array}{ccccccccc}
0 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
q_1 & 0 & p_1 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & q_2 & 0 & p_2 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & q_3 & 0 & p_3 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & p_{N-1} \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 0
\end{array}\right\| \end{array},
$$

where $p_i + q_i = 1, p_i > 0, q_i > 0$ for all i.

The transition probabilities from state 0 and N "reflect" the process back into state $1, 2, \ldots, N-1$. Determine the limiting distribution.

4.4.4 Let $\{\alpha_i : i = 1, 2, \ldots\}$ be a probability distribution, and consider the Markov chain whose transition probability matrix is

$$
\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ \vdots \end{array}
\begin{array}{c} \begin{array}{ccccccc} 0 & 1 & 2 & 3 & 4 & 5 & \cdots \end{array} \\
\left\|\begin{array}{ccccccc}
\alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \alpha_5 & \alpha_6 & \cdots \\
1 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 1 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 1 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 1 & 0 & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{array}\right\| \end{array}.
$$

What condition on the probability distribution $\{\alpha_i : i = 1, 2, \ldots\}$ is necessary and sufficient in order that a limiting distribution exist, and what is this limiting distribution? Assume $\alpha_1 > 0$ and $\alpha_2 > 0$ so that the chain is aperiodic.

4.4.2 Consider the Markov chain whose transition probability *matrix* is given by

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \left\|\begin{array}{cccc} 0 & 1 & 0 & 0 \\ 0.1 & 0.4 & 0.2 & 0.3 \\ 0.2 & 0.2 & 0.5 & 0.1 \\ 0.3 & 0.3 & 0.4 & 0 \end{array}\right\| \end{array}$$

(a) Determine the limiting probability π_0 that the process is in state 0.

(b) By pretending that state 0 is absorbing, use a first step analysis (Chapter 3, Section 3.4) and calculate the mean time m_{10} for the process to go from state 1 to state 0.

(c) Because the process always goes directly to state 1 from state 0, the mean *return* time to state 0 is $m_0 = 1 + m_{10}$. Verify equation (4.26), $\pi_0 = 1/m_0$.

4.4.3 Determine the stationary distribution for the periodic Markov chain whose transition probability matrix is

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \left\|\begin{array}{cccc} 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{4} & 0 & \frac{3}{4} & 0 \\ 0 & \frac{1}{3} & 0 & \frac{2}{3} \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \end{array}\right\| \end{array}.$$

Problems

4.4.1 Consider the Markov chain on $\{0, 1\}$ whose transition probability matrix is

$$\begin{array}{c} \\ 0 \\ 1 \end{array} \begin{array}{c} \begin{array}{cc} 0 & 1 \end{array} \\ \left\|\begin{array}{cc} 1-\alpha & \alpha \\ \beta & 1-\beta \end{array}\right\| \end{array}, \quad 0 < \alpha, \beta < 1.$$

(a) Verify that $(\pi_0, \pi_1) = (\beta/(\alpha+\beta), \alpha/(\alpha+\beta))$ is a stationary distribution.

(b) Show that the first return distribution to state 0 is given by $f_{00}^{(1)} = (1-\alpha)$ and $f_{00}^{(n)} = \alpha\beta(1-\beta)^{n-2}$ for $n = 2, 3, \ldots$.

(c) Calculate the mean return time $m_0 = \sum_{n=1}^{\infty} n f_{00}^{(n)}$ and verify that $\pi_0 = 1/m_0$.

4.4.5 Let P be the transition probability matrix of a finite-state regular Markov chain. Let $\mathbf{M} = \|m_{ij}\|$ be the matrix of mean *return* times.

(a) Use a first step argument to establish that

$$m_{ij} = 1 + \sum_{k \neq j} P_{ik} m_{kj}.$$

(b) Multiply both sides of the preceding by π_i and sum to obtain

$$\sum_i \pi_i m_{ij} = \sum_i \pi_i + \sum_{k \neq j} \sum_i \pi_i P_{ik} m_{kj}.$$

Simplify this to show (see equation (4.26))

$$\pi_j m_{jj} = 1, \quad \text{or } \pi_j = 1/m_{jj}.$$

4.4.6 Determine the period of state 0 in the Markov chain whose transition probability matrix is

$$\mathbf{P} = \begin{array}{c} \\ 3 \\ 2 \\ 1 \\ 0 \\ -1 \\ -2 \\ -3 \\ -4 \end{array} \begin{array}{c} \begin{array}{cccccccc} 3 & 2 & 1 & 0 & -1 & -2 & -3 & -4 \end{array} \\ \left\| \begin{array}{cccccccc} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{array} \right\| \end{array}.$$

4.4.7 An individual either drives his car or walks in going from his home to his office in the morning, and from his office to his home in the afternoon. He uses the following strategy: If it is raining in the morning, then he drives the car, provided it is at home to be taken. Similarly, if it is raining in the afternoon and his car is at the office, then he drives the car home. He walks on any morning or afternoon that it is not raining or the car is not where he is. Assume that, independent of the past, it rains during successive mornings and afternoons with constant probability p. In the long run, on what fraction of *days* does our man walk in the rain? What if he owns two cars?

4.4.8 A Markov chain on states $0, 1, \ldots$ has transition probabilities

$$P_{ij} = \frac{1}{i+2} \quad \text{for } j = 0, 1, \ldots, i, i+1.$$

Find the stationary distribution.

Exercises

4.5.1 Given the transition matrix

$$\mathbf{P} = \begin{array}{cc} & \begin{matrix} 0 & 1 & 2 & 3 & 4 \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{Vmatrix} \frac{1}{4} & \frac{3}{4} & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & \frac{2}{3} & 0 \\ 1 & 0 & 0 & 0 & 0 \end{Vmatrix} \end{array},$$

determine the limits, as $n \to \infty$, of $P_{i0}^{(n)}$ for $i = 0, 1, \ldots, 4$.

4.5.2 Given the transition matrix

$$\mathbf{P} = \begin{array}{cc} & \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{matrix} & \begin{Vmatrix} \frac{1}{3} & \frac{2}{3} & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{4} & \frac{3}{4} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{2}{3} & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \frac{1}{6} & 0 & \frac{1}{6} & \frac{1}{6} & 0 & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{Vmatrix} \end{array},$$

derive the following limits, where they exist:

(a) $\lim_{n\to\infty} P_{11}^{(n)}$ (e) $\lim_{n\to\infty} P_{21}^{(n)}$

(b) $\lim_{n\to\infty} P_{31}^{(n)}$ (f) $\lim_{n\to\infty} P_{33}^{(n)}$

(c) $\lim_{n\to\infty} P_{61}^{(n)}$ (g) $\lim_{n\to\infty} P_{67}^{(n)}$

(d) $\lim_{n\to\infty} P_{63}^{(n)}$ (h) $\lim_{n\to\infty} P_{64}^{(n)}$

Problems

4.5.1 Describe the limiting behavior of the Markov chain whose transition probability matrix is

$$\begin{array}{cc} & \begin{matrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{matrix} & \begin{Vmatrix} 0.1 & 0.1 & 0.2 & 0.2 & 0.1 & 0.1 & 0.1 & 0.1 \\ 0 & 0.1 & 0.1 & 0.1 & 0 & 0.3 & 0.2 & 0.2 \\ 0.6 & 0 & 0 & 0.1 & 0.1 & 0.1 & 0.1 & 0.0 \\ 0 & 0 & 0 & 0.3 & 0.7 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.7 & 0.3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.3 & 0.4 & 0.3 \\ 0 & 0 & 0 & 0 & 0 & 0.1 & 0 & 0.9 \\ 0 & 0 & 0 & 0 & 0 & 0.8 & 0.2 & 0 \end{Vmatrix} \end{array}.$$

Hint: First consider the matrices

$$
\mathbf{P}_A = \begin{array}{c|ccccc|}
 & 0 & 1 & 2 & 3\text{–}4 & 5\text{–}7 \\
\hline
0 & 0.1 & 0.1 & 0.2 & 0.3 & 0.3 \\
1 & 0 & 0.1 & 0.1 & 0.1 & 0.7 \\
2 & 0.6 & 0 & 0 & 0.2 & 0.2 \\
3\text{–}4 & 0 & 0 & 0 & 1 & 0 \\
5\text{–}7 & 0 & 0 & 0 & 0 & 1
\end{array}
$$

and

$$
\mathbf{P}_B = \begin{array}{c|cc|}
 & 3 & 4 \\
\hline
3 & 0.3 & 0.7 \\
4 & 0.7 & 0.3
\end{array}, \qquad
\mathbf{P}_C = \left\|\begin{array}{ccc}
5 & 6 & 7 \\
0.3 & 0.4 & 0.3 \\
0.1 & 0 & 0.9 \\
0.8 & 0.2 & 0
\end{array}\right\|.
$$

4.5.2 Determine the limiting behavior of the Markov chain whose transition probability matrix is

$$
\mathbf{P} = \begin{array}{c|cccccccc|}
 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
\hline
0 & 0.1 & 0.2 & 0.1 & 0.1 & 0.2 & 0.1 & 0.1 & 0.1 \\
1 & 0 & 0.1 & 0.2 & 0.1 & 0 & 0.3 & 0.1 & 0.2 \\
2 & 0.5 & 0 & 0 & 0.2 & 0.1 & 0.1 & 0.1 & 0 \\
3 & 0 & 0 & 0.3 & 0.7 & 0 & 0 & 0 & 0 \\
4 & 0 & 0 & 0.6 & 0.4 & 0 & 0 & 0 & 0 \\
5 & 0 & 0 & 0 & 0 & 0 & 0.3 & 0.4 & 0.3 \\
6 & 0 & 0 & 0 & 0 & 0 & 0.2 & 0.2 & 0.6 \\
7 & 0 & 0 & 0 & 0 & 0 & 0.9 & 0.1 & 0
\end{array}.
$$