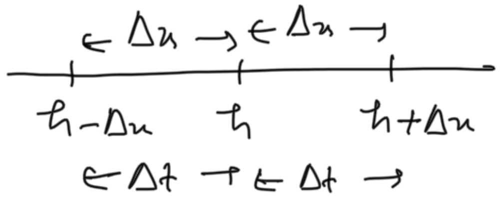
Brownian Motion process :



$X(t)$: position of particle at time t

$$= \Delta x \left(X_1 + X_2 + \cdots + X_{\left[\frac{t}{\Delta t}\right]}\right) = \Delta x \sum_{i=1}^{[t/\Delta t]} X_i \,,$$

$$X_i = \begin{cases} +1 & \text{right } i\text{th step} \\ -1 & \text{left } i\text{th step} \end{cases}$$

X_i's are indep

$$P(X_i = +1) = P(X_i = -1) = \frac{1}{2}$$

$$E(X_i) = 0 \;;\; V(X_i) = E(X_i^2) = 1$$

$$E(X(t)) = \Delta x \sum_{i=1}^{[t/\Delta t]} \underbrace{E(X_i)}_{=0} = 0$$

$$V(X(t)) = (\Delta x)^2 \sum_{i=1}^{\left[\frac{t}{\Delta t}\right]} \underbrace{V(X_i)}_{=1} = (\Delta x)^2 \left[\frac{t}{\Delta t}\right]$$

Let $\Delta x \to 0$, $\Delta t \to 0$

<u>Case I</u> $\Delta x = \Delta t$ $\qquad E(X(t)) = 0$, $V(X(t)) \to 0$

trivial case

<u>Case II</u> $\Delta x = \sigma\sqrt{\Delta t}$, $\sigma > 0$

$$E(X(t)) = 0 \,,\; V(X(t)) \to \sigma^2 t$$

<u>Intuitive properties</u>

$X(t)$ Brownian Motion (BM) process

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(i) $X(t)$ has stationary & independent increments

(ii) $X(t) \sim N(0, \sigma^2 t)$

(iii) $X(0) = 0$

$W(t)$ standard BM process or Weiner process

(i) $W(t)$ has stationary & indep. increment

(ii) $W(t) \sim N(0,t)$

(iii) $W(0) = 0$

— x —

$$W(t) = \frac{B(t)}{\sigma}$$

— x —

Weiner process

$W(t) \sim N(0,t)$ pdf $f_t(x) = \frac{1}{\sqrt{2\pi t}} e^{-x^2/2t}$, $-\infty < x < \infty$

$W(t_1) = x_1, \ldots, W(t_n) = x_n \equiv W(t_1) = x_1, W(t_2) - W(t_1) = x_2 - x_1, \ldots, W(t_n) - W(t_{n-1}) = x_n - x_{n-1}$

Joint density of $W(t_1), \ldots, W(t_n)$

$$f_{t_1, \ldots, t_n}(x_1, \ldots, x_n) = f_{t_1}(x_1) f_{t_2 - t_1}(x_2 - x_1) \cdots f_{t_n - t_{n-1}}(x_n - x_{n-1})$$

$$= \frac{\exp\left\{-\frac{1}{2}\left[\frac{x_1^2}{t_1} + \frac{(x_2 - x_1)^2}{t_2 - t_1} + \cdots + \frac{(x_n - x_{n-1})^2}{t_n - t_{n-1}}\right]\right\}}{(2\pi)^{n/2}\left[t_1(t_2 - t_1)\cdots(t_n - t_{n-1})\right]^{1/2}}$$

— x —

$[W(s) \mid W(t) = B]$, $s < t$

density $f_{s|t}(x|B) = \frac{f_{s,t}(x,B)}{f_t(B)} = \frac{f_s(x) f_{t-s}(B-x)}{f_t(B)}$

$\parallel$

$f_{W(s)|W(t)=B}(x)$

$$= k_1 \exp\left\{-\frac{1}{2}\left[\frac{x^2}{s} + \frac{(B-x)^2}{t-s}\right]\right\}$$

$$= k_2 \exp\left\{ -x^2\left(\frac{1}{2s} + \frac{1}{2(t-s)}\right) + \frac{Bx}{t-s} \right\}$$

$$= k_2 \exp\left\{ \frac{-t}{2s(t-s)}\left(x^2 - \frac{2Bsx}{t}\right) \right\}$$

$$= k_3 \exp\left\{ -\frac{\left(x - \frac{Bs}{t}\right)^2}{2s(t-s)/t} \right\}$$

$s<t$

$$[W(s) \mid W(t) = B] \sim N\Big(\underbrace{\frac{s}{t}B}_{E[W(s) \mid W(t)=B] \text{ for } s<t},\ \underbrace{\frac{s(t-s)}{t}}_{V(W(s) \mid W(t)=B) \text{ for } s<t}\Big)$$

<u>Example</u> : In a bicycle race between two competitors, let

$Y(t)$: amt of time (in secs) by which the racer that started in the inside position is ahead when $100t\%$ of the race has been completed, $\{Y(t), 0 \le t \le 1\}$ BM process var. param. σ^2

(a) If the inside racer is leading the race by σ sec's at the midpoint of the race, what is the prob. that she is the winner?

Sol.
$$P\left(Y(1) > 0 \mid Y\left(\tfrac{1}{2}\right) = \sigma\right) \qquad \underline{Y(t) \sim N(0, \sigma^2 t)}$$

$$= P\left(Y(1) - Y\left(\tfrac{1}{2}\right) > 0 - \sigma \mid Y\left(\tfrac{1}{2}\right) = \sigma\right)$$

$$= P\left(Y(1) - Y\left(\tfrac{1}{2}\right) > -\sigma\right)$$

$$= P\left(Y\left(\tfrac{1}{2}\right) > -\sigma\right) = P\left(\frac{Y\left(\frac{1}{2}\right) - 0}{\sigma \frac{1}{\sqrt{2}}} > \frac{-\sigma - 0}{\sigma/\sqrt{2}}\right)$$

$$= P(Z > -\sqrt{2}) \quad ; Z \sim N(0,1)$$

$$= 1 - P(Z \leq -\sqrt{2}) = 1 - \Phi(-\sqrt{2})$$

$$= \Phi(\sqrt{2}) \simeq 0.9213$$ (from standard normal distn table)

(b) If the inside racer wins the race by a margin of σ sec's, what is the prob. that she was ahead at the midpoint?

$$P\left(Y\left(\tfrac{1}{2}\right) > 0 \;\middle|\; Y(1) = \sigma\right)$$

$$s < t; \left[W(s) \mid W(t) = B\right] \sim N\left(\frac{s}{t}B, \frac{s(t-s)}{t}\right)$$

$$\sigma W(t) = Y(t)$$

$$\left[\sigma W(s) \mid Y(t) = \sigma B\right] \sim N\left(\sigma \frac{s}{t} B, \frac{s(t-s)}{t}\sigma^2\right)$$

$$\left[Y(s) \mid Y(t) = \sigma B\right] \sim N\left(\frac{s}{t}\sigma B, \frac{s(t-s)}{t}\sigma^2\right)$$

$$U = \left(Y\left(\tfrac{1}{2}\right) \mid Y(1) = \sigma\right) \sim N\left(\frac{1}{2}\sigma, \frac{1}{4}\sigma^2\right)$$

$$P(U > 0) = P\left(\frac{U - \sigma/2}{\sqrt{\sigma^2/4}} > \frac{-\sigma/2}{\sqrt{\sigma^2/4}}\right)$$

$$= P(Z > -1) \qquad Z = \frac{U - \sigma/2}{\sqrt{\sigma^2/4}} \sim N(0,1)$$

$= 1 - P(Z \leq -1)$

$= 1 - \Phi(-1)$

$= \Phi(1) \simeq 0.8413$

Geometric BM :

$Y(t)$ BM process with drift parameter μ and variance parameter σ^2 , $Y(t) \sim N(\mu t, \sigma^2 t)$.

$$X(t) = e^{Y(t)}$$

$\hookrightarrow$ Geometric BM.

$$E(X(t) \mid X(u),\ 0 \leq u \leq s)$$

$$= E(e^{Y(t)} \mid Y(u),\ 0 \leq u \leq s)$$

$$= E(e^{Y(t) - Y(s) + Y(s)} \mid Y(u),\ 0 \leq u \leq s)$$

$$= e^{Y(s)} E[e^{Y(t) - Y(s)} \mid Y(u),\ 0 \leq u \leq s]$$

$$= e^{Y(s)} E(e^{Y(t) - Y(s)})$$ $\because$ $Y(t)$ has indep. increment

$U = Y(t) - Y(s) \overset{d}{=} Y(t-s)$

$\sim N(\mu(t-s), \sigma^2(t-s))$

mgf $E(e^{aU}) = e^{\mu(t-s)a + \frac{1}{2}\sigma^2(t-s)a^2}$

$| \qquad u=1$

$$= e^{Y(s)}\, e^{\mu(t-s) + \frac{1}{2}\sigma^2(t-s)}$$

$$= X(s)\, e^{(t-s)(\mu + \sigma^2/2)}$$

$$E(X(t) \mid X(u), 0 \le u \le s) = X(s)\, e^{(t-s)(\mu+\sigma^2/2)}$$

$$E(X(t)) = E\big(E(X(t) \mid X(u),\ 0 \le u \le s)\big) = E(X(s))\, e^{(t-s)(\mu+\sigma^2/2)}$$