1) CFL is closed under substitution

$\Rightarrow$ Let $G=(V,\Sigma,P,S)$ be grammar for CFL L

$\rightarrow$ Let $G_a=(V_a,T_a,P_a,S_a)$ be the grammar for each $a\in\Sigma$ with $V\cup V_a=\phi$

$G'=\langle V',T',P',S'\rangle$ for $S(L)$ where

$V'=V\cup V_a$

$T'$ = union of $T_a$ for all $a\in\Sigma$

$P'$ consists of all productions in any $P_a$ for $a\in\Sigma$

in Productions of P, each terminal a is replaced by $S_a$

$G'\Rightarrow$ represents the grammar for substitution

$\therefore$ By construction, we prove CFL is closed under substitution

2) CFL is closed under inverse homomorphism

$\Longrightarrow$ Let L be a CFL and h be a homomorphism corresponding to it

. Let P be PDA such that $L(P)=L$

We consider another PDA $P'$ such that $L(P')=h^{-1}(L)$

$P = \langle Q, \Sigma, T, \delta, s, \perp, F \rangle$

$P' = \langle Q', \Sigma', T', \delta', s', \perp, F' \rangle$

$Q' : Q \times B$

i.e. any $q' \in Q$, $q' = (q, b)$ where

$q$ is state of $P$

and $b$ is a suffix of $h(a)$

for symbol $a \in \Sigma$

Thus only a finite number of values of $b$ and hence $q'$ are possible

$s' = (s, \epsilon)$

$F'$ consists of $(q, \epsilon)$ $\forall q' \in F$

$\Sigma' = \bigcup_{a \in \Sigma} h(a)$

Define $\delta'$:

$\delta'((q, a), a, x) = \{(q, h(a)), x\}$ $\forall a \in \Sigma'$ and $\forall x \in T$

$\delta'([q, bw], \epsilon, x) = \{(p, w), \alpha\}$ if

$(p, \alpha) \in \delta(q, b, x)$ where $b \in \Sigma \cup \{\epsilon\}$

simulate $b$ from buffer

Clearly $P'$ accepts ~~the~~ only the $h^{-1}(L)$

and so, $h^{-1}(L)$ is a CFL