1. (a) Yes, (b) no, (c) no.

2. (a) S_n is Poisson with mean $n\mu$.

   (b) $P\{N(t) = n\}$

$$= P\{N(t) \geq n\} - P\{N(t) \geq n+1\}$$

$$= P\{S_n \leq t\} - P\{S_{n+1} \leq t\}$$

$$= \sum_{k=0}^{[t]} e^{-n\mu}(n\mu)^k/k!$$

$$- \sum_{k=0}^{[t]} e^{-(n+1)\mu}[(n+1)\mu]^k/k!$$

   where $[t]$ is the largest integer not exceeding t.

3. By the one-to-one correspondence of $m(t)$ and F, it follows that $\{N(t), t \geq 0\}$ is a Poisson process with rate 1/2. Hence,

$$P\{N(5) = 0\} = e^{-5/2}$$

4. (a) No! Suppose, for instance, that the interarrival times of the first renewal process are identically equal to 1. Let the second be a Poisson process. If the first interarrival time of the process $\{N(t), t \geq 0\}$ is equal to 3/4, then we can be certain that the next one is less than or equal to 1/4.

   (b) No! Use the same processes as in (a) for a counter example. For instance, the first interarrival will equal 1 with probability $e^{-\lambda}$, where λ is the rate of the Poisson process. The probability will be different for the next interarrival.

   (c) No, because of (a) or (b).

5. The random variable N is equal to $N(I) + 1$ where $\{N(t)\}$ is the renewal process whose interarrival distribution is uniform on (0, 1). By the results of Example 2c,

$$E[N] = a(1) + 1 = e$$

6. (a) Consider a Poisson process having rate λ and say that an event of the renewal process occurs whenever one of the events numbered $r, 2r, 3r, \ldots$ of the Poisson process occur. Then

   $P\{N(t) \geq n\}$

$$= P\{nr \text{ or more Poisson events by } t\}$$

$$= \sum_{i=nr}^{\infty} e^{-\lambda t}(\lambda t)^i/i!$$

$1 \le n < \infty$

$nr \le i < \infty$

   (b) $E[N(t)]$

$$= \sum_{n=1}^{\infty} P\{N(t) \geq n\} = \sum_{n=1}^{\infty}\sum_{i=nr}^{\infty} e^{-\lambda t}(\lambda t)^i/i!$$

$r \le nr \le i$

$$= \sum_{i=r}^{\infty}\sum_{n=1}^{[i/r]} e^{-\lambda t}(\lambda t)^i/i! = \sum_{i=r}^{\infty} [i/r] e^{-\lambda t}(\lambda t)^i/i!$$

$1 \le n \le$

$n \le [i/r]$

7. Once every five months.

8. (a) The number of replaced machines by time t constitutes a renewal process. The time between replacements equals

   T, if lifetime of new machine is $\geq T$

   x, if lifetime of new machine is $x, x < T$.

   Hence,

   E[time between replacements]

$$= \int_0^T xf(x)dx + T[1 - F(T)]$$

   and the result follows by Proposition 3.1.

   (b) The number of machines that have failed in use by time t constitutes a renewal process. The mean time between in-use failures, $E[F]$, can be calculated by conditioning on the lifetime of the initial machine as

$$E[F] = E[E[F|\text{lifetime of initial machine}]]$$

   Now

   $E[F|\text{lifetime of machine is } x]$

$$= \begin{cases} x, & \text{if } x \leq T \\ T + E[F], & \text{if } x > T \end{cases}$$

Hence,

$$E[F] = \int_0^T xf(x)dx + (T + E[F])[1 - F(T)]$$

or

$$E[F] = \frac{\int_0^T xf(x)dx + T[1 - F(T)]}{F(T)}$$

and the result follows from Proposition 3.1.

9. A job completion constitutes a renewal. Let T denote the time between renewals. To compute $E[T]$ start by conditioning on W, the time it takes to finish the next job:

$E[T] = E[E[T|W]]$

Now, to determine $E[T|W = w]$ condition on S, the time of the next shock. This gives

$$E[T|W = w] = \int_0^\infty E[T|W = w, S = x]\lambda e^{-\lambda x}dx$$

Now, if the time to finish is less than the time of the shock then the job is completed at the finish time; otherwise everything starts over when the shock occurs. This gives

$$E[T|W = w, S = x] = \begin{cases} x + E[T], & \text{if } x < w \\ w, & \text{if } x \geq w \end{cases}$$

Hence,

$E[T|W = w]$

$$= \int_0^w (x + E[T])\lambda e^{-\lambda x}dx + w\int_w^\infty \lambda e^{-\lambda x}dx$$

$$= E[T][1 - e^{-\lambda w}] + 1/\lambda - we^{-\lambda w} - \frac{1}{\lambda}e^{-\lambda w} - we^{-\lambda w}$$

Thus,

$E[T|W] = (E[T] + 1/\lambda)(1 - e^{-\lambda W})$

Taking expectations gives

$E[T] = (E[T] + 1/\lambda)(1 - E[e^{-\lambda W}])$

and so

$$E[T] = \frac{1 - E[e^{-\lambda W}]}{\lambda E[e^{-\lambda W}]}$$

In the above, W is a random variable having distribution F and so

$$E[e^{-\lambda W}] = \int_0^\infty e^{-\lambda w}f(w)dw$$

10. Yes, ρ/μ

11. $$\frac{N(t)}{t} = \frac{1}{t} + \frac{\text{number of renewals in } (X_1, t)}{t}$$

Since $X_1 < \infty$, Proposition 3.1 implies that

$$\frac{\text{number of renewals in } (X_1, t)}{t} - \frac{1}{\mu} \text{ as } t - \infty.$$

12. Let X be the time between successive d-events. Conditioning on T, the time until the next event following a d-event, gives

$$E[X] = \int_0^d x\lambda e^{-\lambda x}dx + \int_d^\infty (x + E[X]\lambda e^{-\lambda x}dx$$

$$= 1/\lambda + E[X]e^{-\lambda d}$$

Therefore, $E[X] = \dfrac{1}{\lambda(1 - e^{-\lambda d})}$

(a) $\dfrac{1}{E[X]} = \lambda(1 - e^{-\lambda d})$

(b) $1 - e^{-\lambda d}$

13. (a) N_1 and N_2 are stopping times. N_3 is not.

(b) Follows immediately from the definition of I_i.

(c) The value of I_i is completely determined from $X_1, \ldots, X_{i-1}$ (e.g., $I_i = 0$ or 1 depending upon whether or not we have stopped after observing $X_1, \ldots, X_{i-1}$). Hence, I_i is independent of X_i.

(d) $\sum_{i=1}^\infty E[I_i] = \sum_{i=1}^\infty P\{N \geq i\} = E[N]$

(e) $E[X_1 + \cdots + X_{N_1}] = E[N_1]E[X]$

But $X_1 + \cdots + X_{N_1} = 5, E[X] = p$ and so

$E[N_1] = 5/p$

$E[X_1 + \cdots + X_{N_2}] = E[N_2]E[X]$

$E[X] = p, E[N_2] = 5p + 3(1 - p) = 3 + 2p$

$E[X_1 + \cdots + X_{N_2}] = (3 + 2p)p$

14. (a) It follows from the hint that $N(t)$ is not a stopping time since $N(t) = n$ depends on X_{n+1}.

Now $N(t) + 1 = n(\Leftrightarrow)N(t) = n - 1$

$(\Leftrightarrow)X_1 + \cdots + X_{n-1} \leq t,$

$X_1 + \cdots + X_n > t,$

and so $N(t) + 1 = n$ depends only on $X_1, \ldots, X_n$. Thus $N(t) + 1$ is a stopping time.