INDIAN INSTITUTE OF TECHNOLOGY, KHARAGPUR
Mid-Spring Semester 2022-23

Date of Exam: .02.23 | Session(FN/AN) | Duration: 2 hrs
Subject Number MA41017/MA60067 | Max. Marks: 30 | Department: Mathematics
Subject Name: Stochastic Processes/ Stochastic Process and Simulation | No. of students: 172

Specific Instructions: (I) Use of calculator is allowed. (II) Answer All questions.
(III) All notations are standard. (IV) All parts of a question Must Be answered at One Place.

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1. Examine the following statements and write **only answers on the first page** of your answer script against each part of questions. Detail working may be carried out on other pages. [1×10]

(a) Consider the Markov chain $\{X_n\}$ with the transition probability matrix (tpm) as

$$P = \begin{array}{c|cc} & 0 & 1 \\ \hline 0 & 1/3 & 2/3 \\ 1 & 2/3 & 1/3 \end{array}$$

Find $P(X_{100} = 0 | X_{99} = 1, X_{98} = 0)$.

(b) Consider the Markov chain $\{X_n\}$ with the tpm as of problem 1 (a). Find

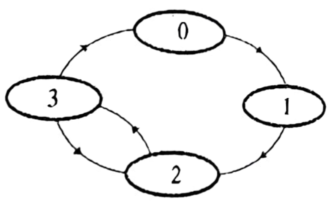
$$P(X_{100} = 1, X_{99} = 0 | X_{98} = 0).$$

(c) Consider the Markov chain $\{X_n\}$ with the tpm as of problem 1 (a). Let $P(X_0 = 0) = 1/4$ and $P(X_0 = 1) = 3/4$. Find $P(X_2 = 1, X_1 = 0, X_0 = 0)$.

(d) Consider the Markov chain $\{X_n\}$ with the tpm as of problem 1(a). Let $P(X_0 = 0) = 1/4$ and $P(X_0 = 1) = 3/4$. Find $P(X_1 = 0)$.

(e) A Markov chain $\{X_n,\ n = 0, 1, \ldots\}$ has TPM P of problem 1(a). What fraction of time, in long run, does the process spend in state 0?

(f) Consider a Markov chain with state space $S = \{0, 1, 2, 3\}$ and transition graph given below:



Find the period of state 1.

(g) Consider the Markov chain $\{X_n\}$ with the transition probability matrix (tpm) as:

$$P = \begin{array}{c|ccccccc} & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \hline 1 & 1/4 & 3/4 & 0 & 0 & 0 & 0 & 0 \\ 2 & 1/2 & 1/2 & 0 & 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 & 2/3 & 1/3 & 0 & 0 \\ 4 & 0 & 0 & 0 & 1/3 & 2/3 & 0 & 0 \\ 5 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 6 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 7 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}$$

Whether Markov chain $\{X_n\}$ is reducible or irreducible.

(h) For the tpm of problem 1(g), find the absorbing states.

(i) For the tpm of problem 1(g), find the aperiodic (period=1) states.

(j) For the tpm of problem 1(g), find the transient states.

2. (a) A spider lives in a rectangular box of which the sides are 3 cm and 4 cm long. It can only sit in one of the four corners marked with the numbers 1, 2, 3 and 4, as shown on the diagram:

From time to time the spider runs from the corner it occupies to another one, chosen at random with probabilities inversely proportional to the distances to the corner from the current position of the spider. Denote by X_n the corner number the spider is at after the nth run. (i) Find the tpm for the Markov chain X_n. (ii) Assume now that the spider has changed its tactics and, on any given transition, never returns directly back to the corner where it came from on the previous step. Whether the sequence X_n is a Markov chain now? If not suggest a Markov chain model for the modified system.

(b) A system can be in one of the states 1, 2, 3 and 4. If the system is at state k, $k < 4$, then, on the next step, it passes to state $k+1$. From state 4, the system passes either to 2 or to 3 with equal probabililties $P_{42} = P_{43} = 1/2$. (i) Draw a transition diagram for the Markov chain modelling the system. (ii) Classify the states of the Markov chain (recurrent/transient). (iii) Find the tpm P. (iv) Whether the Markov chain is regular.

[3+3]

3. (a) Consider the Markov chain whose TPM is given by

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \left\| \begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0.5 & 0.2 & 0.2 & 0.1 \\ 0.2 & 0.3 & 0.4 & 0.1 \\ 0 & 0 & 0 & 1 \end{array} \right\| \end{array}$$

Determine (i) the probability of absorption into state 0 starting from state 1. (ii) the mean time to absorption, strating from state 2. (iii) the mean time spent in each of the states 1 and 2 prior to absorption, starting from state 1.

(b) Consider the TPM P given in problem 1(g). Determine the limits, where they exists, as $n \to \infty$, of $P_{22}^{(n)}$, $P_{41}^{(n)}$ and $P_{67}^{(n)}$.

(c) If in a branching process $P_0 = 1/4$, $P_1 = 1/2$ and $P_2 = 1/4$, (i) then determine π_0, the probability that the population will eventually die out (under the assumption that initially there is single member in the population). Here P_j, $j \geq 0$ represents the probability that each individual, by the end of its lifetime, have produced j new offsprings. (ii) What is the probability that the population will die out if intially there are 10 individuals?

[3+2+1]

P.T.O.

4. (a) You have three coins: A (Heads probability 0.6), B (Heads probability 0.4) and C (Heads probability 0.2). Your plan is to toss one of the three coins each minute. Start by tossing coin A. Subsequently, if you toss Heads, you toss coin A next minute. If you toss Tails, you choose at random (with equal probability) coin B or C for your next toss. Similar strategy is used for coins B and C. (i) Determine the transition matrix of the Markov chain that keeps track of the coin you toss each minute. (ii) Determine the (long-run) proportion of minutes you toss coin A.

(b) For a Poisson process $\{N(t), t \geq 0\}$, the expected waiting time between events is 0.10 years. (i) What is the probability that 2 or fewer events occur during a 1-year time span?(ii) What is the probability that the waiting time between 2 consecutive events is at least 0.2 years? (iii) If $N(2) = 5$, what is the probability that exactly 3 events occur during (0, 1]?

(c) A rat is put into the linear maze a shown below:

0 Shock	1	2	3 Rat is here	4	5 food

At each step the rat moves one compartment to the right with probability 1/3 and to the left with probability 2/3. (i) What is the probability that the rat get shocked before finding food? (ii) What is the probability that the rat finds food before getting shocked?

[3+3+2]

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Q 1 (a) $\frac{2}{3}$

(b) $\frac{2}{9}$

(c) $\frac{1}{18}$

(d) $\frac{7}{12}$

(e) $\frac{1}{2}$

(f) 2

(g) reducible

(h) 7

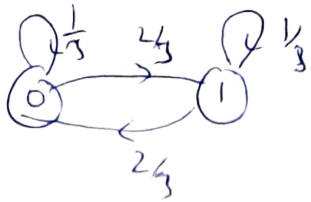
(i) 1, 2, 3, 4, 5, 7

(j) 6

[1×10m]

HW 1

Q1 (a)



$P(X_{100}=0 \mid X_{99}=1, \cancel{X_{98}=0}) =$

$= P(X_1=0 \mid X_0=1) = P_{10} = \frac{2}{3}$

(b)

$P(X_{100}=1, X_{99}=0 \mid X_{98}=0) = P(X_{100}=1 \mid X_{99}=0, \cancel{X_{98}=0})$

$\cdot P(X_{99}=0 \mid X_{98}=0)$

$= P_{01} P_{00} = \frac{2}{3} \times \frac{1}{3} = \frac{2}{9}$

(c) $P(X_2=1 \mid X_1=0, \cancel{X_0=0}) \; P(X_1=0 \mid X_0=0) \cdot P(X_0=0)$

$= P_{01} P_{00} \times \frac{1}{4} = \frac{\cancel{2}^{1}}{3} \times \frac{1}{3} \times \frac{1}{\cancel{4}_{2}} = \frac{1}{18}$

(d) $P(X_1=0)$ $\quad \underset{\sim}{p}^{(1)} = \underset{\sim}{p}^{(0)} P$

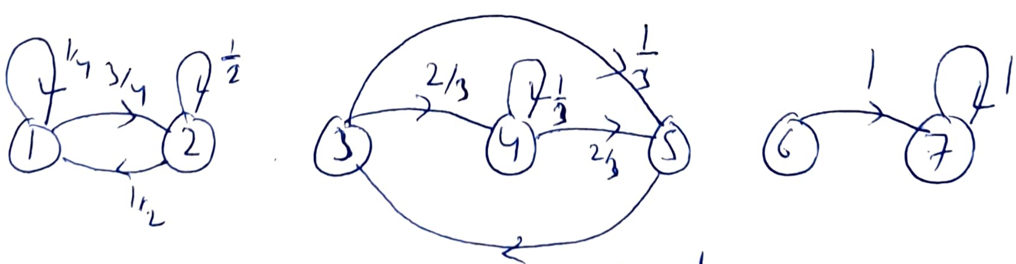
$\parallel$

$\frac{1}{4} \times \frac{1}{3} + \frac{3}{4} \times \frac{2}{3} = \frac{1+6}{12} = \frac{7}{12}$.

(e) doubly stochastic tpm $\quad \pi_0 = \frac{1}{2}, \pi_1 = \frac{1}{2}$

(f) $d(1) = \gcd\{4,6,8,\dots\} = 2$ $\quad p_{11}^{(n)} > 0$

(g)



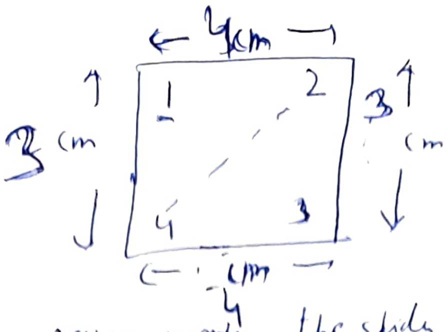
transient

Class (1,2) recurrent aperiodic , (3,4,5) recurrent aperiodic (6) transient, (7) absorbing/recurrent aperiodic

reducible

(h) absorbing 7 (i) aperiodic 1,2,3,4,5,7 (j) transient 6

Q2 (a)



$\sqrt{9+16} = \sqrt{25} = 5$

X_n corner number the spider is at after the nth turn

$X_n \in (1,2,3,4)$

(i)

$$\begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 0 & \frac{c}{4} & \frac{c}{5} & \frac{c}{3} \\ 2 & c/4 & 0 & c/3 & c/5 \\ 3 & c/5 & c/3 & 0 & c/4 \\ 4 & c/3 & c/5 & c/4 & 0 \end{array}$$

$$c = \frac{60}{15+12+20} = \frac{60}{47}$$

$$= \frac{1}{47}\begin{bmatrix} 0 & 15 & 12 & 20 \\ 15 & 0 & 20 & 12 \\ 12 & 20 & 0 & 15 \\ 20 & 12 & 15 & 0 \end{bmatrix}$$ [$1\frac{1}{2}$m]

(ii) (a)

$P(X_3=1 | X_2=2, X_1=1) = 0$

but

$P(X_3=1 | X_2=2, X_1=3) > 0$

But for M.C. they should be same (X_2)

Hence not a M.C. [$\frac{1}{2}$m]

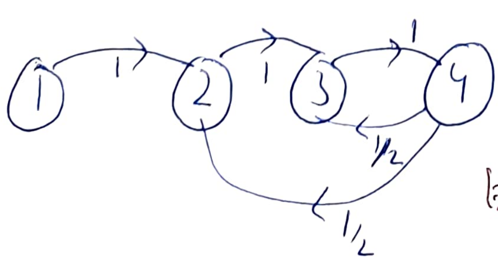
(ii) (b)

To get Markov chain consider [1m]

$Y_n = (X_{n-1}, X_n)$

(b) $S = (1,2,3,4)$

$$P = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 0 & 1 & 0 & 0 \\ 2 & 0 & 0 & 1 & 0 \\ 3 & 0 & 0 & 0 & 1 \\ 4 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \end{array}$$ [$\frac{1}{2}$m]



[$\frac{1}{2}$m]

$\{1\}$, $\{2,3,4\}$

transient [1M] recurrent, aperiodic

$d(3) = \gcd\{2,3,4,\dots\} = 1$

$N = 4$

$P^{N^2} = P^{16}$

$$\overset{P}{\begin{pmatrix} 0 & + & 0 & 0 \\ 0 & 0 & + & 0 \\ 0 & 0 & 0 & + \\ 0 & + & + & 0 \end{pmatrix}} \overset{P}{\begin{pmatrix} 0 & + & 0 & 0 \\ 0 & 0 & + & 0 \\ 0 & 0 & 0 & + \\ 0 & + & + & 0 \end{pmatrix}} = \overset{P^2}{\begin{pmatrix} 0 & 0 & + & 0 \\ 0 & 0 & + & + \\ 0 & 0 & 0 & 0 \\ 0 & 0 & + & + \end{pmatrix}} \overset{P^2}{\begin{pmatrix} 0 & 0 & + & 0 \\ 0 & 0 & + & + \\ 0 & 0 & 0 & 0 \\ 0 & 0 & + & + \end{pmatrix}}$$

(P2)

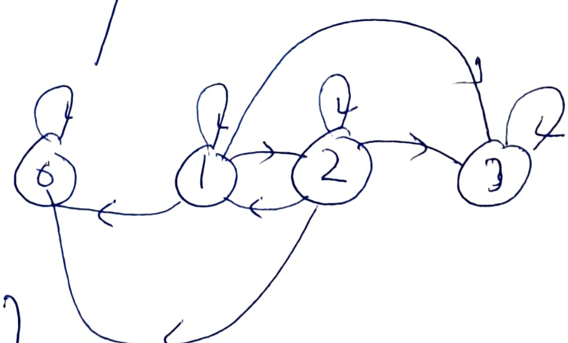
$$\begin{pmatrix} 0 & 0 & 0 & + \\ 0 & & & \\ & & & \\ & & & \end{pmatrix} = \underset{P^4}{\begin{pmatrix} 0 & 0 & + & + \\ 0 & 0 & + & + \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & + \end{pmatrix}} \underset{P^4}{\begin{pmatrix} 0 & 0 & + & + \\ 0 & 0 & + & + \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & + \end{pmatrix}}$$

$$P^{16} = \begin{pmatrix} 0 & + & + & + \\ 0 & + & + & + \\ 0 & + & + & + \\ 0 & + & + & + \end{pmatrix}$$

not regular

(1M)

Q3 (a)



$u_1 = 0.5 + 0.2u_1 + 0.2u_2$

$u_2 = 0.2 + 0.3u_1 + 0.4u_2$

$\Rightarrow$ $0.8u_1 - 0.2u_2 = 0.5$
$0.3u_1 - 0.6u_2 = -0.2$ (1M)

$u_1 = .8095 = 17/21$
$u_2 = .7381$

$v_1 = 1 + 0.2v_1 + 0.2v_2$

$v_2 = 1 + 0.3v_1 + 0.4v_2$

$\Rightarrow$ $0.8v_1 - 0.2v_2 = 1$
$0.3v_1 - 0.6v_2 = -1$ $\Rightarrow$ $v_1 = 1.905 = \frac{40}{21}$, $v_2 = 2.619 = \frac{55}{21}$ (1M)

(0) absorbing (1, 2) transient (3) absorbing

$$P_T = \begin{bmatrix} 0.2 & 0.2 \\ 0.3 & 0.4 \end{bmatrix} \qquad I - P_T = \begin{bmatrix} 0.8 & -0.2 \\ -0.3 & 0.6 \end{bmatrix}$$

$$S = (I-P_T)^{-1} = \begin{pmatrix} 60/42 & 20/42 \\ 30/42 & 80/42 \end{pmatrix} = \begin{pmatrix} 1.428 & 0.476 \\ 0.714 & 1.905 \end{pmatrix} = (s_{ij})$$

(1M)

Check
$\rightarrow 1.904 \rightarrow v_1$
$\rightarrow 2.619 \rightarrow v_2$

3(b)

$$\begin{array}{c|ccccccc} & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \hline 1 & \frac{1}{4} & \frac{3}{4} & 0 & 0 & 0 & 0 & 0 \\ 2 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 & \frac{2}{3} & \frac{1}{3} & 0 & 0 \\ 4 & 0 & 0 & 0 & \frac{1}{3} & \frac{2}{3} & 0 & 0 \\ 5 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 6 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 7 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}$$

$\pi_1 = \frac{1}{4}\pi_1 + \frac{1}{2}\pi_2 \Rightarrow \frac{3}{4}\pi_1 = \frac{1}{2}\pi_2 \Rightarrow \pi_2 = \frac{3}{2}\pi_1 \Rightarrow \pi_1 = \frac{2}{3}\pi_2$

$\pi_1 + \pi_2 = 1 \Rightarrow \frac{2}{3}\pi_2 + \pi_2 = 1 \Rightarrow \pi_2 = \frac{3}{5}$

$\lim_{n\to\infty} P_{22}^{(n)} = \pi_2 = \frac{3}{5}$ [1m]

$\lim_{n\to\infty} P_{41}^{(n)} = 0$ [$\frac{1}{2}$m] $\lim_{n\to\infty} P_{67}^{(n)} = 1 \times 1 = 1$ [$\frac{1}{2}$m]

3(c) (i) $\mu = \frac{1}{2} + 2\times\frac{1}{4} = 1 \Rightarrow \pi_0 = 1$ [$\frac{1}{2} + \frac{1}{2}$ m]

(ii) $(\pi_0)^{10} = 1^{10} = 1$

4(a) $X_n \in (A, B, C)$

$P =$

	A	B	C
A	0.6	0.2	0.2
B	0.3	0.4	0.3
C	0.4	0.4	0.2

[$1\frac{1}{2}$m]

$\pi_1 = 0.6\pi_1 + 0.3\pi_2 + 0.4\pi_3$

$\pi_2 = 0.2\pi_1 + 0.4\pi_2 + 0.4\pi_3$

$\pi_1 + \pi_2 + \pi_3 = 1$

$\Rightarrow \quad -0.4\pi_1 + 0.3\pi_2 + 0.4\pi_3 = 0$

$+0.2\pi_1 - 0.6\pi_2 + 0.4\pi_3 = 0$

$\pi_1 + \pi_2 + \pi_3 = 1$

$\Rightarrow \pi_1 = \frac{6}{13}, \quad \pi_2 = \frac{4}{13}, \quad \pi_3 = \frac{3}{13}$ $[1\frac{1}{2} m]$

4c

$i = 3, \; N = 5 \quad p = \frac{1}{3}, q = \frac{2}{3}; \; \frac{q}{p} = 2$

(i) $P(\text{Shocks}) = 1 - \frac{1-(\frac{q}{p})^i}{1-(\frac{q}{p})^N} = 1 - \frac{1-2^3}{1-2^5} = 1 - \frac{-7}{-31} = 1 - \frac{7}{31} =$

$= 1 - 0.2258 = 0.7742$

(ii) $0.2258 = \frac{1-2^3}{1-2^5} = P(\text{food})$ $[1+1m]$

4b

$N(t) \sim P.P.(\lambda)$

$T_n \sim \exp(\lambda) \quad E(T_n) = \frac{1}{\lambda} = 0.10 \Rightarrow \lambda = 10$

(i) $P(N(1) \le 2) = P(N(1) = 0) + P(N(1) = 1) + P(N(1) = 2)$

$= e^{-10\times 1}\left[1 + 10 + \frac{100}{2!}\right] = 61 e^{-10} = 2.76939 \times 10^{-3}$

$T_n > t \equiv N(t) = 0$

(ii) $P(T_n \ge 0.2) = P(N(0.2) = 0) = e^{-10\times 0.2} = e^{-2}$

$= 0.1353$

(iii) $P(N(1) = 3 \mid N(2) = 5) = \frac{P(N(1) = 3, N(1,2) = 2)}{P(N(2) = 5)}$

$[1+1+1m]$

$$= \frac{P(N(1)=3)\cdot P(N(1)=2)}{P(N(2)=5)}$$

$$= \frac{\dfrac{e^{-10}(10)^3}{3!} \times \dfrac{e^{-10}(10)^2}{2!}}{\dfrac{e^{-20}(20)^5}{5!}} = \frac{1\cancel{000} \times 1\cancel{00}}{\cancel{20} \times \cancel{20} \times 20 \times 20 \times 20} \times \frac{5 \times \cancel{4}}{2 \times 1}$$

$$= \frac{5}{16}$$

Alter.

$$\binom{5}{3}\left(\frac{1}{2}\right)^5 = \frac{5!}{3! \times 2!} \times \left(\frac{1}{2}\right)^5$$

$$= \frac{5 \times \cancel{4} \times \cancel{3} \times \cancel{2}}{\cancel{3} \times 2 \times 2} \times \frac{1}{2 \times 2 \times \cancel{2} \times \cancel{2} \times \cancel{2}}$$

$$= \frac{5}{16}$$