Exercises

5.1.1 Defects occur along the length of a filament at a rate of $\lambda = 2$ per foot.

(a) Calculate the probability that there are no defects in the first foot of the filament.

(b) Calculate the conditional probability that there are no defects in the second foot of the filament, given that the first foot contained a single defect.

5.1.2 Let $p_k = \Pr\{X = k\}$ be the probability mass function corresponding to a Poisson distribution with parameter λ. Verify that $p_0 = \exp\{-\lambda\}$, and that p_k may be computed recursively by $p_k = (\lambda/k)p_{k-1}$.

5.1.3 Let X and Y be independent Poisson distributed random variables with parameters α and β, respectively. Determine the conditional distribution of X, given that $N = X + Y = n$.

5.1.4 Customers arrive at a service facility according to a Poisson process of rate λ customer/hour. Let $X(t)$ be the number of customers that have arrived up to time t.

(a) What is $\Pr\{X(t) = k\}$ for $k = 0, 1, \ldots$?

(b) Consider fixed times $0 < s < t$. Determine the conditional probability $\Pr\{X(t) = n + k | X(s) = n\}$ and the expected value $E[X(t)X(s)]$.

5.1.5 Suppose that a random variable X is distributed according to a Poisson distribution with parameter λ. The parameter λ is itself a random variable, exponentially distributed with density $f(x) = \theta e^{-\theta x}$ for $x \geq 0$. Find the probability mass function for X.

5.1.6 Messages arrive at a telegraph office as a Poisson process with mean rate of 3 messages per hour.

(a) What is the probability that no messages arrive during the morning hours 8:00 A.M. to noon?

(b) What is the distribution of the time at which the first afternoon message arrives?

5.1.7 Suppose that customers arrive at a facility according to a Poisson process having rate $\lambda = 2$. Let $X(t)$ be the number of customers that have arrived up to time t. Determine the following probabilities and conditional probabilities:

(a) $\Pr\{X(1) = 2\}$.

(b) $\Pr\{X(1) = 2 \text{ and } X(3) = 6\}$.

(c) $\Pr\{X(1) = 2 | X(3) = 6\}$.

(d) $\Pr\{X(3) = 6 | X(1) = 2\}$.

5.1.8 Let $\{X(t): t \geq 0\}$ be a Poisson process having rate parameter $\lambda = 2$. Determine the numerical values to two decimal places for the following probabilities:

(a) $\Pr\{X(1) \leq 2\}$.

(b) $\Pr\{X(1) = 1 \text{ and } X(2) = 3\}$.

(c) $\Pr\{X(1) \geq 2 | X(1) \geq 1\}$.

5.1.9 Let $\{X(t): t \geq 0\}$ be a Poisson process having rate parameter $\lambda = 2$. Determine the following expectations:

(a) $E[X(2)]$.

(b) $E\left[\{X(1)\}^2\right]$.

(c) $E[X(1)X(2)]$.

Problems

5.1.1 Let $\xi_1, \xi_2, \ldots$ be independent random variables, each having an exponential distribution with parameter λ. Define a new random variable X as follows: If $\xi_1 > 1$, then $X = 0$; if $\xi_1 \leq 1$ but $\xi_1 + \xi_2 > 1$, then set $X = 1$; in general, set $X = k$ if

$$\xi_1 + \cdots + \xi_k \leq 1 < \xi_1 + \cdots + \xi_k + \xi_{k+1}.$$

Show that X has a Poisson distribution with parameter λ. (Thus, the method outlined can be used to simulate a Poisson distribution.)

Hint: $\xi_1 + \cdots + \xi_k$ has a gamma density

$$f_k(x) = \frac{\lambda^k x^{k-1}}{(k-1)!} e^{-\lambda x} \quad \text{for } x > 0.$$

Condition on $\xi_1 + \cdots + \xi_k$ and use the law of total probability to show

$$\Pr\{X = k\} = \int_0^1 [1 - F(1 - x)] f_k(x)\,dx,$$

where $F(x)$ is the exponential distribution function.

5.1.2 Suppose that minor defects are distributed over the length of a cable as a Poisson process with rate α, and that, independently, major defects are distributed over the cable according to a Poisson process of rate β. Let $X(t)$ be the number of defects, either major or minor, in the cable up to length t. Argue that $X(t)$ must be a Poisson process of rate $\alpha + \beta$.

5.1.3 The *generating function* of a probability mass function $p_k = \Pr\{X = k\}$, for $k = 0, 1, \ldots$, is defined by

$$g_X(s) = E\left[s^X\right] = \sum_{k=0}^{\infty} p_k s^k \quad \text{for } |s| < 1.$$

Show that the generating function for a Poisson random variable X with mean μ is given by

$$g_X(s) = e^{-\mu(1-s)}.$$

5.1.4 (Continuation) Let X and Y be independent random variables, Poisson distributed with parameters α and β, respectively. Show that the generating function of their sum $N = X + Y$ is given by

$$g_N(s) = e^{-(\alpha+\beta)(1-s)}.$$

Hint: Verify and use the fact that the generating function of a sum of independent random variables is the product of their respective generating functions. See Chapter 3, Section 3.9.2.

5.1.5 For each value of $h > 0$, let $X(h)$ have a Poisson distribution with parameter λh. Let $p_k(h) = \Pr\{X(h) = k\}$ for $k = 0, 1, \ldots$. Verify that

$$\lim_{h \to 0} \frac{1 - p_0(h)}{h} = \lambda, \quad \text{or } p_0(h) = 1 - \lambda h + o(h);$$

$$\lim_{h \to 0} \frac{p_1(h)}{h} = \lambda, \quad \text{or } p_1(h) = \lambda h + o(h);$$

$$\lim_{h \to 0} \frac{p_2(h)}{h} = 0, \quad \text{or } p_2(h) = o(h).$$

Here $o(h)$ stands for any remainder term of order less than h as $h \to 0$.

5.1.6 Let $\{X(t): t \geq 0\}$ be a Poisson process of rate λ. For $s, t > 0$, determine the conditional distribution of $X(t)$, given that $X(t + s) = n$.

5.1.7 Shocks occur to a system according to a Poisson process of rate λ. Suppose that the system survives each shock with probability α, independently of other shocks, so that its probability of surviving k shocks is α^k. What is the probability that the system is surviving at time t?

5.1.8 Find the probability $\Pr\{X(t) = 1, 3, 5, \ldots\}$ that a Poisson process having rate λ is odd.

5.1.9 Arrivals of passengers at a bus stop form a Poisson process $X(t)$ with rate $\lambda = 2$ per unit time. Assume that a bus departed at time $t = 0$ leaving no customers behind. Let T denote the arrival time of the next bus. Then, the number of passengers present when it arrives is $X(T)$. Suppose that the bus arrival time T is independent of the Poisson process and that T has the uniform probability density function

$$f_T(t) = \begin{cases} 1 & \text{for } 0 \leq t \leq 1, \\ 0 & \text{elsewhere.} \end{cases}$$

(a) Determine the conditional moments $E[X(T)|T = t]$ and $E\left[\{X(T)\}^2|T = t\right]$.
(b) Determine the mean $E[X(T)]$ and variance $\operatorname{Var}[X(T)]$.

Exercises

5.3.1 A radioactive source emits particles according to a Poisson process of rate $\lambda = 2$ particles per minute. What is the probability that the first particle appears after 3 min?

5.3.2 A radioactive source emits particles according to a Poisson process of rate $\lambda = 2$ particles per minute.

(a) What is the probability that the first particle appears some time after 3 min but before 5 min?

(b) What is the probability that exactly one particle is emitted in the interval from 3 to 5 min?

5.3.3 Customers enter a store according to a Poisson process of rate $\lambda = 6$ per hour. Suppose it is known that only a single customer entered during the first hour. What is the conditional probability that this person entered during the first 15 min?

5.3.4 Let $X(t)$ be a Poisson process of rate $\xi = 3$ per hour. Find the conditional probability that there were two events in the first hour, given that there were five events in the first 3 h.

5.3.5 Let $X(t)$ be a Poisson process of rate θ per hour. Find the conditional probability that there were m events in the first t hours, given that there were n events in the first T hours. Assume $0 \leq m \leq n$ and $0 < t < T$.

5.3.6 For $i = 1, \ldots, n$, let $\{X_i(t): t \geq 0\}$ be independent Poisson processes, each with the same parameter λ. Find the distribution of the first time that at least one event has occurred in every process.

5.3.7 Customers arrive at a service facility according to a Poisson process of rate λ customers/hour. Let $X(t)$ be the number of customers that have arrived up to time t. Let $W_1, W_2, \ldots$ be the successive arrival times of the customers. Determine the conditional mean $E[W_5|X(t) = 3]$.

5.3.8 Customers arrive at a service facility according to a Poisson process of rate $\lambda = 5$ per hour. Given that 12 customers arrived during the first two hours of service, what is the conditional probability that 5 customers arrived during the first hour?

5.3.9 Let $X(t)$ be a Poisson process of rate λ. Determine the cumulative distribution function of the gamma density as a sum of Poisson probabilities by first verifying and then using the identity $W_r \leq t$ if and only if $X(t) \geq r$.