Martingale:

SP $\{X_n, n=0,1,2,\ldots\}$ is a martingale if for $n=0,1,2,\ldots$

a) $E|X_n| < \infty$

b) $E(X_{n+1} \mid X_0, \ldots, X_n) = X_n$

Example: (1) $X_1, X_2, \ldots$ are indep. r.v.'s with mean 0

Let $Z_n = \sum_{i=1}^{n} X_i$. $\{Z_n\}$ martingale?

$$E(Z_{n+1} \mid Z_1, \ldots, Z_n) = E(Z_n + X_{n+1} \mid Z_1, \ldots, Z_n)$$

$$= E(Z_n \mid Z_1, \ldots, Z_n) + E(X_{n+1} \mid Z_1, \ldots, Z_n)$$

$$= Z_n + E(X_{n+1} \mid X_1, \ldots, X_n)$$

$$= Z_n + E(X_{n+1}) \to 0$$

$$= Z_n$$

$\{Z_n\}$ is a martingale.

(2) Let $X_1, X_2, \ldots$ are indep. r.v.'s with $E(X_i) = 1$.

Let $Z_n = X_1 X_2 \cdots X_n$

Z_n martingale ✓

$$E(Z_{n+1} \mid Z_1, \ldots, Z_n) = \underline{E(Z_n X_{n+1} \mid Z_1, \ldots, Z_n)} \qquad z_1, \ldots, z_n \in \mathbb{R}$$

$$= Z_n E(X_{n+1} \mid Z_1, \ldots, Z_n) \qquad \Big| \; E(\underline{Z_n} X_{n+1} \mid Z_1 = z_1, \ldots, Z_n = z_n)$$

$$= Z_n E(X_{n+1} \mid X_1, X_2, \ldots, X_n) \qquad \Big| \; = \underline{z_n} E(X_{n+1} \mid Z_1 = z_1, \ldots, Z_n = z_n)$$

$$= Z_n \, E(X_{n+1}) \xrightarrow{} 1$$

$E(Z_n X_{n+1} | Z_1, \dots, Z_n)$
$= Z_n E(X_{n+1} | Z_1, \dots, Z_n)$

$$= Z_n$$

$\therefore \{Z_n\}$ is a martingale.

(3) Consider a branching process $\{X_n\}$

X_n size of nth generation

$$X_{n+1} = \sum_{i=1}^{X_n} Z_i$$

$\underline{E(Z_i) = m}$

Z_i # of offsprings of ith individual in nth generation

$$U_{n+1} = \frac{X_{n+1}}{m^{n+1}}, \quad U_n = \frac{X_n}{m^n}$$

U_n is martingale?

$$E(U_{n+1} | U_1, \dots, U_n) = E\left(\frac{X_{n+1}}{m^{n+1}} \,\middle|\, X_1, \dots, X_n \right)$$

$$= \frac{1}{m^{n+1}} E\left(\sum_{i=1}^{X_n} Z_i \,\middle|\, X_1, \dots, X_n \right)$$

$$= \frac{1}{m^{n+1}} X_n \times m = \frac{X_n}{m^n} = U_n$$

$\therefore \{U_n\}$ martingale

(3) Random walk

$X_0 := 0, \; X_n := Y_1 + \dots + Y_n, \; n \geq 1$; $\underline{Y_i \text{ IID}}$ with $E|Y_1| < \infty$

$\{X_n\}$ is martingale?

Sol $\quad E|X_n| < \infty$

$E(X_{n+1} | X_0, \dots, X_n) = E(X_n + Y_{n+1} | \dots$

$$\ldots = E(X_n + Y_{n+1} \mid X_0, \dots, X_n)$$

$$= E(X_n \mid X_0, \dots, X_n) + E(Y_{n+1} \mid X_0, \dots, X_n)$$

$$= X_n + E(Y_{n+1} \mid Y_0, \dots, Y_n)$$

$$= X_n + E(Y_{n+1})$$

If $E(Y_1) = 0$, then (X_n) is martingale

(2) <u>Geometric Random walk</u>

$$X_n := X_0 \, e^{Y_1 + \dots + Y_n}, \quad n \geq 1$$

$X_0 :=$ const > 0, Y_j IID rvs

Sol (X_n) martingale?

$$E|X_n| = E(X_n) = X_0 \, E(e^{Y_1 + \dots + Y_n}) = X_0 \left(E(e^{Y_1})\right)^n < \infty$$

if $M_{Y_1}(1) = E(e^{Y_1}) < \infty$

$$E(X_{n+1} \mid X_0, \dots, X_n) = E(X_0 \, e^{Y_1 + \dots + Y_n} e^{Y_{n+1}} \mid X_0, \dots, X_n)$$

$$= E(X_n e^{Y_{n+1}} \mid X_0, \dots, X_n)$$

$$= X_n \, E(e^{Y_{n+1}} \mid Y_0, Y_1, \dots, Y_n)$$

$$= X_n \, E(e^{Y_{n+1}}) \to 1$$

if $M_{Y_1}(1) = E(e^{Y_1}) = 1$, then (X_n) martingale

mgf $M_{Y_1}(t) = E(e^{tY_1})$