Continuous Time Markov Chain (CTMC):

Def^n 1

$X(t)$ state at time t

$\{X(t), t \geq 0\}$ CTMC if $\forall\, s, t \geq 0$ and non-negative integers i, j, $x(u)$, $0 \leq u < s$,

$$P(X(t+s) = j \mid X(s) = i, X(u) = x(u), 0 \leq u < s)$$
$$= P(X(t+s) = j \mid X(s) = i)$$
$$= P(X(t) = j \mid X(0) = i) \quad \text{—— } (*)$$
$$= P_{ij}(t)$$

Let T_i sojourn time or the amt of time the process stays in state i before making a transition into a different state,

$$P(T_i > t+s \mid T_i > s) = P(T_i > t) \quad \Big| \text{using } (*)$$

$$T_i \sim \exp(\nu_i)$$

Def^n 2 (CTMC)

SP having the property that each time the process enters state i

(i) the amt of time it spends in that state before making a transition into different state $\sim$ exp. with mean $\frac{1}{\nu_i}$

(ii) When it leaves state i, it next enters state j with some prob, say, P_{ij}.

$P_{ii} = 0$, ν_i ...

$P_{ii} = 0, \forall i$, $\sum_j P_{ij} = 1$

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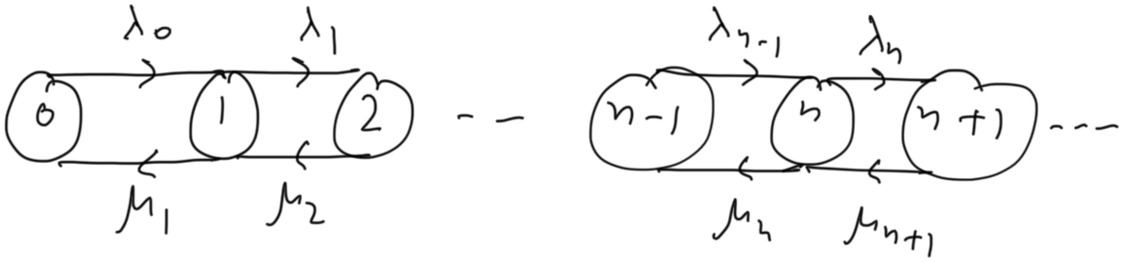
Example (1) Birth and death process (B&D process)

$$P(X(t+h) = n+1 \mid X(t) = n) = \lambda_n h + o(h)$$

$$P(X(t+h) = n-1 \mid X(t) = n) = \mu_n h + o(h)$$



$X(t) = n$ mdy $\Big\langle$ $X \sim \exp(\lambda_n)$, $Y \sim \exp(\mu_n)$

$$T = \min(X,Y) \sim \exp(\lambda_n + \mu_n)$$

$$P(\min(X,Y) > t) = P(X > t, Y > t)$$
$$= P(X>t)\, P(Y>t)$$
$$= e^{-\lambda_n t}\, e^{-\mu_n t}$$
$$= e^{-(\lambda_n + \mu_n) t}$$

$$\nu_n = \lambda_n + \mu_n$$

$$P_{01} = 1$$

$$P_{n,n+1} = P(X < Y) = \int_0^\infty P(Y > x)\, \lambda_n e^{-\lambda_n x}\, dx$$

$$= \int_0^\infty e^{-\mu_n x} \lambda_n e^{-\lambda_n x}\, dx = \frac{\lambda_n}{\lambda_n + \mu_n}$$

P μ_n

$$r_{n,n-1} = \frac{\mu_n}{\lambda_n+\mu_n}.$$

(2) P.P(λ) Poisson process

is B&D process $\lambda_n = \lambda \ \forall n = 0,1,2,\ldots$

$\mu_n = 0, \ \forall n = 1,2,\ldots$

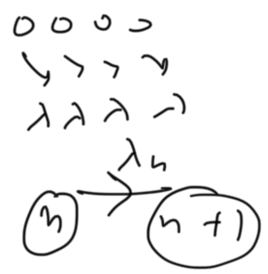
(3) Pure Birth process

is B&D process with λ_n, $n = 0,1,2,\ldots$

$\mu_n = 0, \ \forall n = 1,2,\ldots$

(4) Yule process or birth process with linear birth rate

$\lambda_n = n\lambda$, $n = 0,1,2,\ldots$



(5) A linear growth model with immigration

$\mu_n = n\mu$, $n = 1,2,\ldots$

$\lambda_n = n\lambda + \theta$, $n = 0,1,2,\ldots$

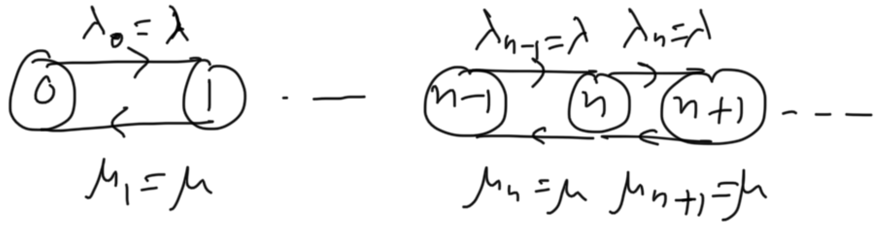
immigration

(6) M/M/1 queueing systems:

$X(t)$: # of programs /customers in queueing system at time t

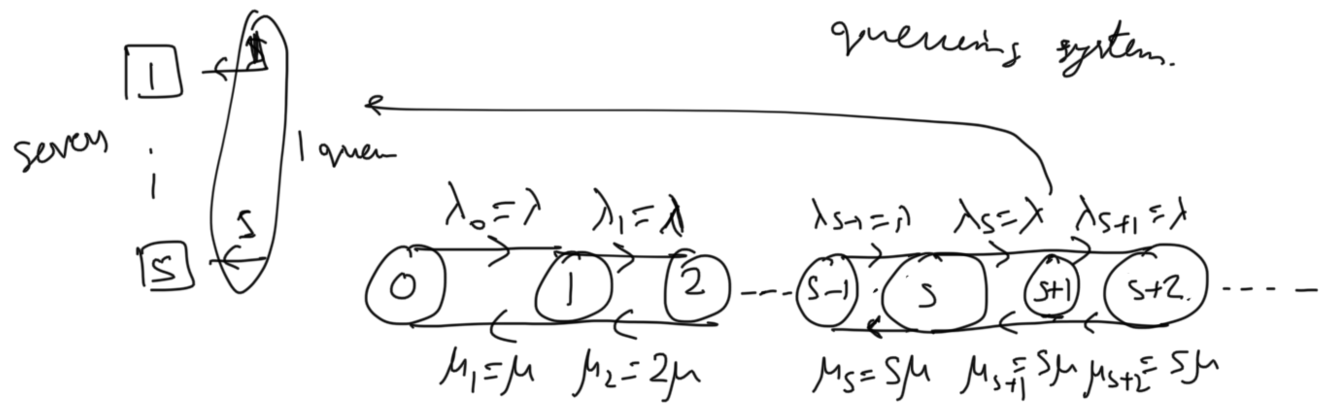
B&D process with $\lambda_n = \lambda$

$\mu_n = \mu$



(7) M/M/1 ...

(7) M/M/S queueing system or multiserver exponential queueing system.



B & D process

$$\lambda_n=\lambda \quad , \quad n=0,1,2,\ldots$$

$$\mu_n=\begin{cases} n\mu & , n=1,2,\ldots,s \\ s\mu & , n=s+1, s+2, \ldots \end{cases}$$

(8) A Shoeshine shop

customer $\longrightarrow$ P.P. (λ)

$T_i \sim exp(\lambda)$

$E(T_i)=\frac{1}{\lambda}$

Chair 1: Shoe are cleaned and polish is applied $\longrightarrow$ Chair 2: polish is buffed

$exp(\mu_1)$ $\qquad$ $exp(\mu_2)$

potential customer enters the system only if both Chairs are empty

customer in the system at time $\qquad$ [0, 1] customer

State	
0	system is empty
1	customer in Chair 1
2	" " Chair 2

$P_{01} = P_{12} = P_{20} = 1$

$v_0 = \lambda, \quad v_1 = \mu_1, \quad v_2 = \mu_2$ not B&D process.

$\because$ we can move from state 2 to 0 also

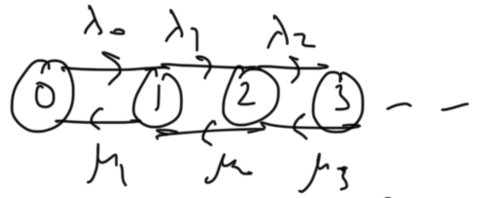
Expected time to go from state i to state j in B&D process

B& D process $\{\lambda_n\}_{n=0}^{\infty}$ $\{\mu_n\}_{n=1}^{\infty}$

T_i time, starting from state i, it takes for the process to enter state $i+1$, $i \geq 0$.

$T_0 \sim \exp(\lambda_0)$

$E(T_0) = \dfrac{1}{\lambda_0}$



$$I_i = \begin{cases} 1 & \text{as the first transition is from } i \text{ to } i+1 \\ 0 & \text{'' '' '' '' } i \text{ to } i-1 \end{cases}$$

$$E(T_i \mid I_i = 1) = \frac{1}{\lambda_i + \mu_i}$$

$$E(T_i \mid I_i = 0) = \frac{1}{\lambda_i + \mu_i} + E(T_{i-1}) + E(T_i)$$

$$E(T_i) = E(T_i \mid I_i = 1)\, P(I_i = 1) + E(T_i \mid I_i = 0)\, P(I_i = 0)$$

$$= \frac{1}{\lambda_i + \mu_i} \times \frac{\lambda_i}{\lambda_i + \mu_i} + \left(\frac{1}{\lambda_i + \mu_i} + E(T_{i-1}) + E(T_i)\right) \times \frac{\mu_i}{\lambda_i + \mu_i}$$

$E(T_0) = \dfrac{1}{\lambda_0}$, $\quad E(T_i) = \dfrac{1}{\lambda_i} + \dfrac{\mu_i}{\lambda_i} E(T_{i-1})$ — ①

②

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X_1 , X_2

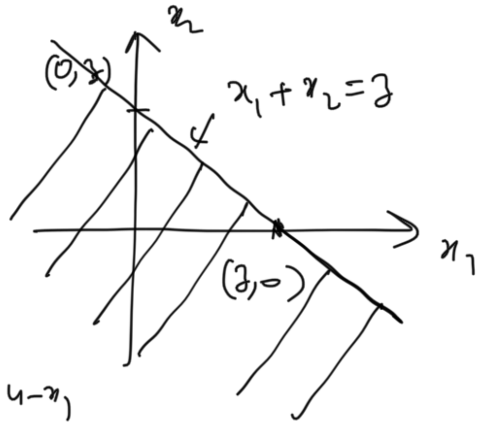
$$Z = X_1 + X_2$$

CDF of Z

$$F_Z(z) = P(Z \le z) = P(X_1 + X_2 \le z)$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{z-x_1} f(x_1, x_2)\, dx_2\, dx_1$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{z} f(x_1, u - x_1)\, du\, dx_1$$



$x_2 = u - x_1$
$u = x_2 + x_1$

pdf of Z

$$f_Z(z) = \frac{dF_Z(z)}{dz} = \int_{-\infty}^{\infty} f(x_1, z - x_1)\, dx_1 .$$

Hypoexponential distn:

$$\underbrace{X_1, \dots, X_n}_{\text{indep}} \quad \text{s.t. } X_i \sim exp(\lambda_i), i = 1, \dots, n$$

$$\sum_{i=1}^{n} X_i$$

pdf of $X_1 + X_2$

$$f_{X_1+X_2}(t) = \int_{-\infty}^{\infty} f_{X_1}(s)\, f_{X_2}(t-s)\, ds$$

$$= \int_0^t \lambda_1 e^{-\lambda_1 s} \lambda_2 e^{-\lambda_2 (t-s)}\, ds$$

$t - s > 0$

$$= \frac{\lambda_1 \lambda_2}{\lambda_1 - \lambda_2} e^{-\lambda_2 t} \underbrace{\int_0^t (\lambda_1 - \lambda_2)\, e^{-(\lambda_1-\lambda_2)s}\, ds}_{\to\ 1 - e^{-(\lambda_1-\lambda_2)t}}$$

$$= \frac{\lambda_1}{\lambda_1 - \lambda_2} \lambda_2 e^{-\lambda_2 t} + \frac{\lambda_2}{\lambda_2 - \lambda_1} \lambda_1 e^{-\lambda_1 t}$$

pdf

$$f_{\sum_{i=1}^n X_i}(t) = \sum_{i=1}^n C_{i,n} \lambda_i e^{-\lambda_i t}, \text{ where } C_{i,n} = \prod_{j \neq i} \frac{\lambda_j}{\lambda_j - \lambda_i}$$

reliability function of $\sum_{i=1}^n X_i$

$$\bar{F}_{\sum_{i=1}^n X_i}(t) = P\left(\sum_{i=1}^n X_i > t\right) = \int_t^\infty \sum_{i=1}^n C_{i,n} \lambda_i e^{-\lambda_i u} du$$

$$= \sum_{i=1}^n C_{i,n} e^{-\lambda_i t}$$

<u>Pure Birth process</u>: B&D process λ_i ; $\mu_i = 0, \forall i$

$X(t)$: state at time t

$$P_{ij}(t) = P(X(t+s) = j \mid X(s) = i)$$

Let X_k time the process spends in state k before making a transition into state $k+1$, $k \geq 1$.

Suppose the process is presently at state i ; $j > i$

Given $X(0) = i$

$$X(t) < j \equiv X_i + X_{i+1} + \cdots + X_{j-1} > t$$

$$P(X(t) < j \mid X(0) = i) = P\left(\sum_{k=i}^{j-1} X_k > t\right)$$

$$= \sum_{k=i}^{j-1} e^{-\lambda_k t} \prod_{r=i}^{j-1} \frac{\lambda_r}{\lambda_r - \lambda_k}$$

$X_k \sim \exp(\lambda_k)$

$X_1, X_2, \ldots$ indep

$\sum^{j-1} X_i$ ...

$n \neq k$ (**) $k=1$

$$P_{ij}(t) = P(X(t)=j \mid X(0)=i)$$

$$= P(X(t) < j+1 \mid X(0)=i) - P(X(t) < j \mid X(0)=i)$$

$$P_{ii}(t) = P(X_i > t) = e^{-\lambda_i t}$$

Ex. For Yule process show

$$P_{1j}(t) = e^{-\lambda t}(1-e^{-\lambda t})^{j-1} \simeq \text{Geo}(e^{-\lambda t})$$

Also $$P_{ij}(t) = \binom{j-1}{i-1} e^{-i\lambda t}(1-e^{-\lambda t})^{j-i}, \quad j \geq i \geq 1$$

$$\simeq NB(i, e^{-\lambda t})$$

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