Exercises

1. Is it true that
   (a) $N(t) < n$ if and only if $S_n > t$?
   (b) $N(t) \leqslant n$ if and only if $S_n \geqslant t$?
   (c) $N(t) > n$ if and only if $S_n < t$?
2. Suppose that the interarrival distribution for a renewal process is Poisson distributed with mean μ. That is, suppose

$$P\{X_n = k\} = e^{-\mu}\frac{\mu^k}{k!}, \quad k = 0, 1, \ldots$$

   (a) Find the distribution of S_n.
   (b) Calculate $P\{N(t) = n\}$.

*3. If the mean-value function of the renewal process $\{N(t), t \geqslant 0\}$ is given by $m(t) = t/2$, $t \geqslant 0$, what is $P\{N(5) = 0\}$?

4. Let $\{N_1(t), t \geqslant 0\}$ and $\{N_2(t), t \geqslant 0\}$ be independent renewal processes. Let $N(t) = N_1(t) + N_2(t)$.
   (a) Are the interarrival times of $\{N(t), t \geqslant 0\}$ independent?
   (b) Are they identically distributed?
   (c) Is $\{N(t), t \geqslant 0\}$ a renewal process?
5. Let $U_1, U_2, \ldots$ be independent uniform $(0, 1)$ random variables, and define N by

$$N = \min\{n : U_1 + U_2 + \cdots + U_n > 1\}$$

   What is $E[N]$?

*6. Consider a renewal process $\{N(t), t \geqslant 0\}$ having a gamma (r, λ) interarrival distribution. That is, the interarrival density is

$$f(x) = \frac{\lambda e^{-\lambda x}(\lambda x)^{r-1}}{(r-1)!}, \quad x > 0$$

   (a) Show that

$$P\{N(t) \geqslant n\} = \sum_{i=nr}^{\infty} \frac{e^{-\lambda t}(\lambda t)^i}{i!}$$

   (b) Show that

$$m(t) = \sum_{i=r}^{\infty} \left[\frac{i}{r}\right] \frac{e^{-\lambda t}(\lambda t)^i}{i!}$$

   where $[i/r]$ is the largest integer less than or equal to i/r.

   Hint: Use the relationship between the gamma (r, λ) distribution and the sum of r independent exponentials with rate λ to define $N(t)$ in terms of a Poisson process with rate λ.

7. Mr. Smith works on a temporary basis. The mean length of each job he gets is three months. If the amount of time he spends between jobs is exponentially distributed with mean 2, then at what rate does Mr. Smith get new jobs?

*8. A machine in use is replaced by a new machine either when it fails or when it reaches the age of T years. If the lifetimes of successive machines are independent with a common distribution F having density f, show that
   (a) the long-run rate at which machines are replaced equals

$$\left[\int_0^T x f(x)\, dx + T(1 - F(T))\right]^{-1}$$

   (b) the long-run rate at which machines in use fail equals

$$\frac{F(T)}{\int_0^T x f(x)\, dx + T[1 - F(T)]}$$

9. A worker sequentially works on jobs. Each time a job is completed, a new one is begun. Each job, independently, takes a random amount of time having distribution F to complete. However, independently of this, shocks occur according to a Poisson process with rate λ. Whenever a shock occurs, the worker discontinues working on the present job and starts a new one. In the long run, at what rate are jobs completed?

10. Consider a renewal process with mean interarrival time μ. Suppose that each event of this process is independently "counted" with probability p. Let $N_C(t)$ denote the number of counted events by time t, $t > 0$.
    (a) Is $N_C(t)$, $t \geqslant 0$ a renewal process?
    (b) What is $\lim_{t\to\infty} N_C(t)/t$?