MA 41017/MA 60067

Stochastic Processes / Stochastic Process and Simulation

	Marks	
CT-1	8	(Fri) 3 Feb. 11:10 – 12:10
Mid	30	
CT-2	8	(Fri) 31 March 11:10 – 12:10
End	50	
Project/Attendance / Tutorial	4	
	100	

Books

1) Intro. to prob. models by S.M. Ross
2) Stochastic processes by S.M. Ross
3) An intro. to stochastic modeling by MA Pinsky, S. Karlin

N(t) indep increment $\quad t_2 > t_1$

$$P(N(t_2)-N(t_1) = x_2 - x_1 ,\ N(t_1) = x_1)$$

$$= P(N(t_2)-N(t_1) = x_2 - x_1)\ P(N(t_1) = x_1)$$

$$= P(N(t_2-t_1) = x_2 - x_1)\ P(N(t_1) = x_1)$$

N(t) stationary increment

$$E(X) = E(\overbrace{E(X|Y)}^{X})$$

$$E(X|Y=y) = \sum_x x\, p_{X|Y=y}(x) = \phi(y)$$

$$E(X|Y) = \phi(Y)$$

$$E(E(X|Y)) = E(\phi(Y)) = \sum_y \phi(y)\, p_Y(y)$$

$$= \sum_y \sum_x x\, p_{X|Y=y}(x)\, p_Y(y)$$

$$p_{X|Y=y}(x) = \frac{p(x,y)}{p_Y(y)}$$

$$= \sum_y \sum_x x\, p(x,y)$$

$$= \sum x \sum p(x,y) = \sum \ldots E(X)$$

$$= \sum_x x \underbrace{\sum_y p(x,y)}_{p_X(x)} = \sum_x x\, p_X(x) = E(X)$$

— x —

$X_i \sim \text{Poiss}(\lambda_i)$, $i=1,2$

$\underbrace{X_1, X_2}_{\text{indep}}$ $\qquad m_{X_i}(t) = e^{\lambda_i(e^t-1)}$, $i=1,2$

$S_2 = X_1 + X_2 \sim \text{Poiss}(\lambda_1+\lambda_2)$

$$m_{S_2}(t) = m_{X_1}(t)\, m_{X_2}(t)$$
$$= e^{(\lambda_1+\lambda_2)(e^t-1)}$$

— x —

Syllabus : DTMC, Poiss. Process and related distributions, CTMC, queueing theory, renewal process, martingales, Brownian Motion, simulation.

— x —

Stochastic Process (S.P.)

is a family of random variables (r.v.'s) $\{X(t), t \in T\}$, defined on a given probability space, indexed by the parameter t, $t \in T$ index set

values assumed by $X(t) \in S$ is called states

State space

T parameter space or time space

(1) discrete state, discrete parameter SP

(2) " " , continuous " "

(3) continuous " , " " "

(4) " " , discrete " "

Example Consider a queueing system with jobs arriving at random point in time, queueing for service and departing from the system ...

... departing from the system after service completion.

a) $X(t)$ # of jobs in the system at time t

$\{X(t), t \in T\}$

$X(t) \in \{0, 1, 2, \ldots\} = S$ discrete state,

$T = [0, \infty)$ continuous parameter SP.

b) W_k time that the k^{th} customer has to wait in the system before receiving service.

$\{W_k, k \in T\}$

$W_k \in [0, \infty] = S$ continuous state,

$T = \{1, 2, \ldots\}$ discrete parameter SP

c) $Y(t)$: cumulative service requirement (experience) of all jobs in the system at time t.

$Y(t) \in [0, \infty) = S$ $\quad T = [0, \infty)$

$\{Y(t)\}$ is continuous state, continuous parameter S.P.

d) N_k # of jobs in the system at the time of departure of the k^{th} customer (after service completion).

$N_k \in \{0, 1, 2, \ldots\} = S$

$T = \{1, 2, \ldots\}$

$\{N_k, k \in T\}$ is discrete state, discrete parameter S.P.

— x —

Discrete time Markov Chain: (DTMC)

S.P. $\{X_n, n = 0, 1, 2, \ldots\}$ that takes values on a finite or countable number of values

$\{0, 1, 2, \ldots\} = S \Rightarrow$ discrete state space

discrete parameter S.P. $\{X_n\}$

$i, j, i_0, i_1, \ldots \in S$

$P(X_{n+1} = j \mid X_n = i, X_{n-1} = i_{n-1}, \ldots, X_0 = i_0)$

$$P(X_{n+1}=j \mid X_n=i, X_{n-1}=i_{n-1}, \ldots, X_0=i_0)$$

$$= P(X_{n+1}=j \mid X_n=i)$$

$$= P_{ij}(n, n+1)$$

$$= \underset{\substack{\swarrow \quad \searrow \\ \text{initial state} \quad \text{final state}}}{P_{ij}^{(1)}}$$

stationary transition probability or Homogeneous M.C.

$$= P_{ij}$$

$P^{(1)} = P$ transition probability matrix (tpm)

$$= \begin{array}{c c} & \begin{matrix} j \to & 0 & 1 & 2 & \cdots \end{matrix} \\ \begin{matrix} i \downarrow \\ 0 \\ 1 \\ 2 \\ \vdots \end{matrix} & \begin{bmatrix} P_{00} & P_{01} & P_{02} & \cdots \\ P_{10} & P_{11} & P_{12} & \cdots \\ P_{20} & P_{21} & P_{22} & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix} \end{array}$$

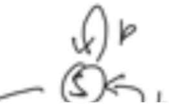
$$0 \le P_{ij} \le 1, \quad \forall i, \forall j$$

$$\sum_j P_{ij} = 1 \quad \text{for fixed } i$$

—x—

Example 1 Consider a game of ladder climbing. There are 5 levels in the game, level 1 is lowest (bottom) and level 5 is the highest (top). A player starts at the bottom. Each time, a fair coin is tossed. If it turns up heads, the player moves up one rung. If tails, the player moves down to the very bottom. Once at the top level, the player moves to the very bottom if tails turns up and stays at the top if head turns up. Let X_n be the level of the game in the nth step / transition. tpm of X_n

$p = q = \frac{1}{2}$



$$X_n \in \{1, 2, 3, 4, 5\} = S$$

X_n DTMC

$$P_{ij} = P(X_{n+1}=j \mid X_n = i)$$
$$= P(X_1 = j \mid X_0 = i)$$

tpm

$p+q=1$

$p=q=\frac{1}{2}$

$$P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} q & p & 0 & 0 & 0 \\ q & 0 & p & 0 & 0 \\ q & 0 & 0 & p & 0 \\ q & 0 & 0 & 0 & p \\ q & 0 & 0 & 0 & p \end{bmatrix} \end{array}$$

$$P(\underline{X_{n+1}=j} \mid \underline{X_n = i}, X_{n-1} = i_{n-1}, \dots, X_0 = i_0)$$
$$= P(\underline{X_{n+1}=j} \mid \underline{X_n=i})$$

—x—

<u>Example</u>: Let $\{X_n\}_{n=0,1,2,\dots}$ be a sequence of i.i.d. (independently & identically distributed) discrete r.v. with $\underset{p_j}{P(X_1=j)} = \left(\frac{1}{2}\right)^{j+1}$, $\forall j = 0,1,2,3,\dots$

Determine whether each of the following chain is Markovian or not. If so find its corresponding state space (S) and tpm

(i) $\{S_n\}_{n=0,1,2,\dots}$ where $S_n = \sum_{i=1}^{n} X_i$

(ii) $\{M_n\}_{n=0,1,2,\dots}$ where $M_n = \max\{X_1, X_2, \dots, X_n\}$

Sol (i) $S_n \in \{0,1,2,\dots\}$ $\qquad S_{n+1} = S_n + X_{n+1}$

$$P_{ij} = P(S_{n+1}=j \mid S_n = i)$$

$S_n=i$ $\quad S_{n+1}=j \rightarrow$ $0 \quad 1 \quad 2 \quad 3 \dots$

tpm

$$P = \begin{array}{c} 0 \\ 1 \\ 2 \\ \vdots \end{array} \begin{bmatrix} p_0 & p_1 & p_2 & p_3 & \cdots \\ 0 & p_0 & p_1 & p_2 & \cdots \\ 0 & 0 & p_0 & p_1 & \cdots \\ \cdots & \cdots & \cdots & & \\ \cdots & \cdots & & & \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{8} & \frac{1}{16} & \cdots \\ 0 & \frac{1}{2} & \frac{1}{4} & \frac{1}{8} & \cdots \\ 0 & 0 & \frac{1}{2} & \frac{1}{4} & \cdots \\ \cdots & \cdots & \cdots & & \end{bmatrix}$$

Example (Transformation of a process into M.C.)

Suppose that whether or not it rains today depends on previous weather conditions through the last two days. Suppose that if it has rained for the past two days, then it will rain tomorrow with prob. (wp) 0.7; if it has rained today but not yesterday, then it will rain tomorrow wp 0.5; if it has rained yesterday but not today, then it will rain tomorrow wp 0.4; if it has not rained in the past two days, then it will rain tomorrow wp 0.2.

Sol X_n state at any time is determined by the weather condition during both that day and the previous day

State X_n	Rained yesterday	Rained today
0	✓	✓
1	×	✓
2	✓	×
3	×	×

$S = \{0, 1, 2, 3\}$

tpm

$$X_{n+1} = j$$

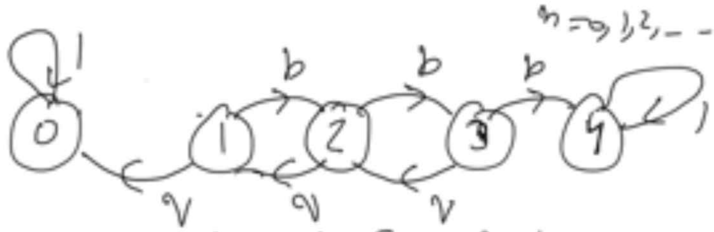
$$P = \begin{array}{ccc} & \text{yesterday} & \text{today} \\ & \downarrow & \downarrow \\ & 0 & ✓✓ \\ X_n = i & 1 & ×✓ \\ & 2 & [\text{illegible}] \end{array} \begin{array}{c} \begin{array}{cccc} \overset{\text{today}}{\downarrow}\ 0 \leftarrow \text{tomorrow} & 1 & 2 & 3 \\ ✓✓ & ×✓ & ✓× & ×× \end{array} \\ \begin{bmatrix} 0.7 & 0 & 0.3 & 0 \\ 0.5 & 0 & 0.5 & 0 \\ [\text{illegible}] & & & \end{bmatrix} \end{array}$$

$$P = \begin{array}{cc} 2 & \times\times \\ 3 & \times\times \end{array} \begin{bmatrix} 0 & 0.4 & 0 & 0.6 \\ 0 & 0.2 & 0 & 0.8 \end{bmatrix}$$

Example 1 Particle performs a random walk in states $\{0,1,2,3,4\}$. It remains in state 0 and 4 with probability 1. It moves from state r $(0<r<4)$ to $r+1$ with prob p; and from state r to $r-1$ with prob $q = 1-p$. $(0<r<4)$

Let X_n : position of particle at time/step n.

$X_n \in \{0,1,2,3,4\} = S$ $\quad \{X_n\}$ M.C. $\quad n = 0,1,2,\ldots$



0, 4 absorbing states

$$\text{tpm} \quad P = \begin{array}{c|c} & \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ q & 0 & p & 0 & 0 \\ 0 & q & 0 & p & 0 \\ 0 & 0 & q & 0 & p \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

n-step transition probability:

$i, j \in S \qquad \{X_n\}$ M.C. $\qquad S = \{0,1,2,\ldots\}$

$$P_{ij}^{(n)} = P(X_{m+n} = j \mid X_m = i) = P(X_n = j \mid X_0 = i)$$

$$\text{n step tpm} \rightarrow P^{(n)} = \left(\left(P_{ij}^{(n)}\right)\right)$$

$$S \; \begin{array}{c|c} & \begin{array}{cccc} 0 & 1 & 2 & \cdots \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ \vdots \end{array} & \begin{bmatrix} P_{00}^{(n)} & P_{01}^{(n)} & P_{02}^{(n)} & \cdots \\ P_{10}^{(n)} & P_{11}^{(n)} & P_{12}^{(n)} & \cdots \\ - & - & - & \\ & - & - & \\ & & & \end{bmatrix} \end{array}$$

$$\sum_j P_{ij}^{(n)} = 1 \quad \text{for fixed } i$$

$0 \le P_{ij} \le 1 \quad \forall i, \forall j$

Chapman Kolmogorov equation $\quad i, j, k \in S$

$$P_{ij}^{(m+n)} = \sum_k P_{ik}^{(m)} P_{kj}^{(n)} = \sum_k P_{ik}^{(n)} P_{kj}^{(m)}$$

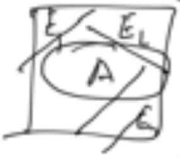
$(i,j)^{th}$ elt of $P^{(m+n)}$

Sol

$$P_{ij}^{(m+n)} = P(X_{m+n} = j \mid X_0 = i)$$

$$= \sum_k P(X_{m+n} = j, X_n = k \mid X_0 = i)$$

$$P(A) = P\left(\bigcup_i (A \cap E_i)\right)$$
$$= \sum_i P(A \cap E_i)$$



$$= \sum_k P(X_{m+n} = j \mid X_n = k, X_0 = i)\, P(X_n = k \mid X_0 = i)$$

$$P(AB \mid C) = P(A \mid BC)\, P(B \mid C)$$

$$= \sum_k P(X_{m+n} = j \mid X_n = k)\, P(X_n = k \mid X_0 = i)$$

$$= \sum_k P_{kj}^{(m)} P_{ik}^{(n)}$$

$$\underbrace{\begin{pmatrix} - & - \\ P_{i0}^{(n)} & P_{i1}^{(n)} \; - \; - \\ - & - \\ - & - \end{pmatrix}}_{P^{(n)}} \underbrace{\begin{pmatrix} - & P_{0j}^{(m)} & - \\ - & P_{1j}^{(m)} & - \\ - & | & - \\ - & | & - \end{pmatrix}}_{P^{(m)}}$$

$$= P^{(m+n)}$$

$$P^{(m+n)} = P^{(n)} P^{(m)} = P^{(m)} P^{(n)}$$

$$P^{(1)} = P$$

$$P^{(2)} = P^{(1)} P^{(1)} = P.P = P^2$$

$$p^{(n)} = P^n$$

—x—

<u>pmf of step X_n</u> $\quad i \in S = \{0,1,3,\ldots\}$

$$P(X_n = i) = p_i^{(n)}$$

pmf of X_n

$$\underset{\sim}{p}^{(n)} = (P(X_n=0), P(X_n=1), \ldots) = (p_0^{(n)}, p_1^{(n)}, \ldots)$$

pmf of X_0

$$\underset{\sim}{p}^{(0)} = (p_0^{(0)}, p_1^{(0)}, \ldots) \qquad P \leftarrow \text{tpm}$$

<u>Claim</u>

$$\underset{\sim}{p}^{(n)} = \underset{\sim}{p}^{(n-1)} P = \cdots = \underset{\sim}{p}^{(0)} P^n$$

$$p_i^{(n)} = p_0^{(n-1)} P_{0i} + p_1^{(n-1)} P_{1i} + \cdots$$

$$= \sum_k p_k^{(n-1)} P_{ki}$$

Sol

$$p_i^{(n)} = P(X_n = i)$$

$$= \sum_k P(X_n = i, X_{n-1} = k)$$

$$= \sum_k P(X_n = i \mid X_{n-1} = k) P(X_{n-1} = k)$$

$$= \underline{\sum_k P_{ki}\, p_k^{(n-1)}}$$

$$(p_0^{(n)}, p_1^{(n)}, \ldots p_1^{(n)}) = (\underline{p_0^{(n-1)}, p_1^{(n-1)}} \ldots) \begin{pmatrix} P_{00} & P_{01} & P_{01} \\ - & — & P_{11} \end{pmatrix}$$

$$\underset{\sim}{p}^{(n)} = \underset{\sim}{p}^{(n-1)} P$$

—x—

<u>Example</u>: $\{X_n\}$ M.C. $\quad S = \{1,2,3\}$

$$\begin{array}{c|ccc} & 1 & 2 & 3 \\ 1 & 0.1 & 0.5 & 0.4 \end{array}$$

$$\begin{array}{c} 2 \\ 3 \end{array}\begin{pmatrix} 0.6 & 0.2 & \underline{0.2} \\ 0.3 & 0.4 & \underline{0.3} \end{pmatrix}$$

$P(X_0=1)=0.7$, $\underline{P(X_0=2)=0.2}$, $P(X_0=3)=0.1$

ⓐ $P(X_0=2, X_2=3, X_1=3, X_3=2)$

$= P(X_3=2, X_2=3, X_1=3, X_0=2)$

$= P(X_3=2 \mid X_2=3, \cancel{X_1=3}, \cancel{X_0=2})\, P(X_2=3 \mid X_1=3, \cancel{X_0=2})$

$\cdot P(X_1=3 \mid X_0=2)\, P(X_0=2)$

$= P(X_3=2 \mid X_2=3)\, P(X_2=3 \mid X_1=3)\, P(X_1=3 \mid X_0=2)$
$P(X_0=2)$

$= P_{32}\, P_{33}\, P_{23}\, P(X_0=2) = 0.4 \times 0.3 \times 0.2 \times 0.2$
$= 0.0048$

ⓑ $P(X_2=3, X_1=3 \mid X_0=2)$

$= P(X_2=3 \mid X_1=3, \cancel{X_0=2})\, P(X_1=3 \mid X_0=2)$

$= P(X_2=3 \mid X_1=3)\ \ P(X_1=3 \mid X_0=2)$

$= P_{33}\, P_{23} = 0.3 \times 0.2$

ⓒ $P(X_3=2, X_0=2, X_1=3)$

$= P(X_3=2, X_1=3, X_0=2)$

$= P(X_3=2 \mid X_1=3, \cancel{X_0=2})\ P(X_1=3 \mid X_0=2)\, P(X_0=2)$

$= P(X_3=2 \mid X_1=3)\ P(X_1=3 \mid X_0=2)\ P(X_0=2)$

$= \underline{P^{(2)}_{32}}\ \ P_{23}\ \ P(X_0=2) = 0.35 \times 0.2 \times 0.2$
$= 0.014$

$\underline{P^{(2)} = P^2} = P.P \qquad k \in S = \{1,2,3\}$

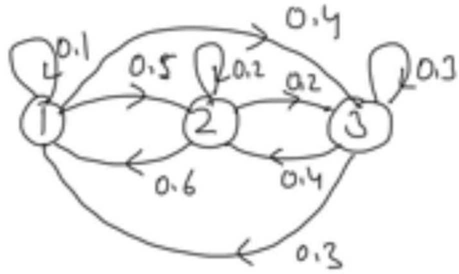
$P^{(2)}$ ...

$$P_{32}^{(2)} = \sum_k P_{3k} P_{k2}$$

$$= P_{31}P_{12} + P_{32}P_{22} + P_{33}P_{32}$$

$$= 0.3 \times 0.5 + 0.4 \times 0.2 + 0.3 \times 0.4$$

$$= 0.35$$



(d) $P(X_2 = 3) = p_3^{(2)}$ $\qquad P \leftarrow$ tpm

$$\underset{\sim}{p}^{(2)} = \underset{\sim}{p}^{(1)} P$$

$$\underset{\sim}{p}^{(1)} = \underset{\sim}{p}^{(0)} P \qquad \underset{\sim}{p}^{(0)} = (0.7, 0.2, 0.1)$$

$$\underset{\sim}{p}^{(1)} = (0.7, 0.2, 0.1)\begin{pmatrix} 0.1 & 0.5 & 0.4 \\ 0.6 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.3 \end{pmatrix}$$

$$= (0.22, 0.43, 0.35)$$

$$\underset{\sim}{p}^{(2)} = \underset{\sim}{p}^{(1)} P$$

$$p_3^{(2)} = 0.22 \times 0.4 + 0.43 \times 0.2 + 0.35 \times 0.3$$

$$= 0.279$$

<u>Example</u> Consider a two state M.C. (X_n) having state space $S = \{0, 1\}$ with tpm

$$Q = \begin{array}{c c} & \begin{array}{cc} 0 & 1 \end{array} \\ \begin{array}{c} 0 \\ 1 \end{array} & \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix} \end{array}$$

Whether $Z_n = (X_{n-1}, X_n)$ is a M.C.? If so determine state space and tpm of $\{Z_n\}$.

Sol. $\{Z_n\}$ is a M.C.

$S = \{(0,0), (0,1), (1,0), (1,1)\}$

	(0,0)	(0,1)	(1,0)	(1,1)
(0,0)	1/2	1/2	0	0
(0,1)	0	0	1/3	2/3
(1,0)	1/2	1/2	0	0
(1,1)	0	0	1/3	2/3

$$P_{(i,j),(k,l)} = P(\overset{(X_n, X_{n+1})}{Z_{n+1}} = (k,l) \mid \overset{(X_{n-1}, X_n)}{Z_n} = (i,j))$$

$i, j, k, l \in \{0,1\}$

$$P(AB \mid BC) = \frac{P(ABBC)}{P(BC)} = \frac{P(ABC)}{P(BC)} = P(A \mid BC)$$

$$P(Z_{n+1} = (1,1) \mid Z_n = (0,1)) =$$

$$= P(\overset{B}{X_n = 1}, \overset{A}{X_{n+1} = 1} \mid \overset{C}{X_{n-1} = 0}, \overset{B}{X_n = 1})$$

$$= P(X_{n+1} = 1 \mid X_n = 1, X_{n-1} = 0)$$

$$= P(X_{n+1} = 1 \mid X_n = 1) = Q_{11} = \frac{2}{3}$$

—x—

Classification of states:

$\{X_n\}$ m.c. $\quad S = \{0, 1, 2, 3, \ldots\}$

$i, j, k \in S$

Defn $i \to j$, state j is accessible from state i if $P_{ij}^{(n)} > 0$ for some n.

Defn $i \leftrightarrow j$, state i and j communicate with each other if $i \to j$ and $j \to i$

Result $i \leftrightarrow j$, $j \leftrightarrow k \Rightarrow i \leftrightarrow k$

Sol $\exists\, n, m$ s.t. $P_{ij}^{(n)} > 0$, $P_{jk}^{(m)} > 0$

$$P_{ik}^{(m+n)} = \sum_{l} P_{il}^{(n)} P_{lk}^{(m)} \quad (\text{CK equations})$$

$$\geq \tilde{P}_{ij}^{(n)} P_{jk}^{(m)}$$

$$> 0$$

$i \to k$. Similarly $k \to i$ $\therefore i \leftrightarrow k$

Def^n M.C $\{X_n\}$ is irreducible or (constable) if every state communicate with every other state otherwise reducible.

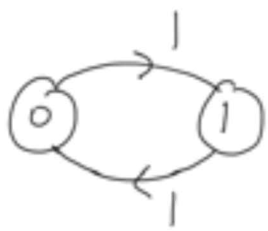
Def^n period of state i

$$d(i) = \gcd I_i^+ = \{1, 2, \ldots\} \; n \text{ s.t. } P_{ii}^{(n)} > 0$$

$$\left(\text{If } P_{ii}^{(n)} = 0 \;\; \forall n \geq 1, \text{ define } d(i) = 0 \right)$$

Example: ① $\{X_n\}$ $\qquad S = \{0, 1\}$

$$\begin{array}{c|cc} & 0 & 1 \\ \hline 0 & 0 & 1 \\ 1 & 1 & 0 \end{array}$$



Irreducible M.C

$0 \leftrightarrow 1$

Class $\{0, 1\}$

$$d(0) = \gcd \{2, 4, 6, \ldots\} \qquad P_{00}^{(m)} > 0$$

$$= 2 = d(1)$$

② $\{X_n\}$ M.C $\quad S = \{0, 1, 2\}$ having tpm

$$\begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{bmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 1/4 & 1/4 \\ 0 & 1/3 & 2/3 \end{bmatrix} \end{array}$$

Class $\{0,1\}$, $\{2\}$

$0 \leftrightarrow 1 \leftrightarrow 2$ Class $= \{0, 1, 2\}$

Irreducible M.C.

$d(0) = \gcd\{1,2,3,\ldots\} = 1$ $P_{00}^{(n)} > 0$

$= d(1) = d(2)$

—x—

$\{X_n\}$ M.C $S = \{0,1,2,\ldots\}$

For state $i \in S$

$f_{ii}^{(n)} = P(X_n = i,\ X_k \neq i, k = 1,2,\ldots,n-1 \mid X_0 = i)$

recurrence time prob — probability of first visit to state i in n transitions / steps, starting from state i

$f_{ii}^{(0)} = 1$

$f_{ii} = f_{ii}^{(1)} + f_{ii}^{(2)} + f_{ii}^{(3)} + \cdots$

↳ probability of ever visiting state i, starting from state i

→ $f_{ii} = 1$, i.e., return to state i is certain, starting from state i
i recurrent state

→ $f_{ii} < 1$, i.e., return to state i is uncertain
i transient state

—x—

$P_{ij}^{(n)} = P(X_{m+n} = j \mid X_m = i)$

Let $I_n = \begin{cases} 1 & \text{if } X_n = i \\ 0 & , X_n \neq i \end{cases}$

$\sum_{n=1}^{\infty} I_n$: # of time periods, the process is in state i

$$E\left(\sum_{n=1} I_n \mid X_0=i\right) = \sum_{n=1} E(I_n \mid X_0=i)$$

$$= \sum_{n=1}^{\infty}\left[1.\,P(X_n=i \mid X_0=i) + 0.\,P(X_n\neq i \mid X_0=i)\right]$$

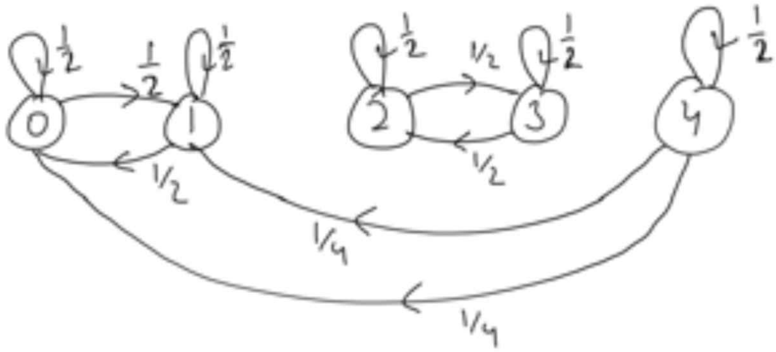
$$= \sum_{n=1}^{\infty} P_{ii}^{(n)}$$

i recurrent $\Leftrightarrow f_{ii}=1 \Leftrightarrow \sum_{n=1}^{\infty} P_{ii}^{(n)} = \infty$

i transient $\Leftrightarrow f_{ii}<1 \Leftrightarrow \sum_{n=1}^{\infty} P_{ii}^{(n)} < \infty$

Example Consider a M.C having states 0,1,2,3,4 and tpm

$$\begin{array}{c c} & \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{4} & \frac{1}{4} & 0 & 0 & \frac{1}{2} \end{bmatrix} \end{array}$$



$0 \leftrightarrow 1$, $2 \leftrightarrow 3$, 4 Reducible M.C.

Class $\{0,1\}$, $\{2,3\}$, $\{4\}$ → transient

recurrent

$$f_{00} = f_{00}^{(1)} + f_{00}^{(2)} + f_{00}^{(3)} + \cdots$$

$$= \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = \frac{1}{2}\left(1 + \frac{1}{2} + \frac{1}{4} + \cdots\right)$$

$$= \frac{1}{2} \times \frac{1}{1-\frac{1}{2}} = 1 \qquad 0 \text{ recurrent.}$$

$$f_{44} = f_{44}^{(1)} + f_{44}^{(2)} + \cdots = \frac{1}{2} + 0 + 0 + \cdots < 1$$

4 transient.

—x—

P1. $i \leftrightarrow j$, i recurrent $\Rightarrow$ j recurrent

Sol

Given
$$\begin{cases} i \leftrightarrow j \Rightarrow \exists\, n, m \text{ s.t. } P_{ij}^{(n)} > 0,\ P_{ji}^{(m)} > 0 \\ i \text{ recurrent} \Leftrightarrow \sum_{\nu} P_{ii}^{(\nu)} = \infty \end{cases}$$

$$P_{jj}^{(m+n+\nu)} \geq P_{ji}^{(m)} P_{ii}^{(\nu)} P_{ij}^{(n)} \quad \text{[Using CK eq's}$$

$$\sum_{\nu} P_{jj}^{(m+n+\nu)} \geq P_{ji}^{(m)} P_{ij}^{(n)} \underbrace{\sum_{\nu} P_{ii}^{(\nu)}}_{=\infty}$$

$$= \infty$$

$\Rightarrow$ j recurrent.

P2 $\quad i \leftrightarrow j$, i transient $\Rightarrow$ j transient

P3 $\quad$ In a finite state M.C. all states can not be transient.

P4 $\quad$ In a finite state, irreducible M.C. all states are recurrent.

Sol/ using P1 and P3.

Def^n let i recurrent

$$m_{ii} = \sum_{n=1}^{\infty} n f_{ii}^{(n)} \quad \text{mean recurrence time}$$

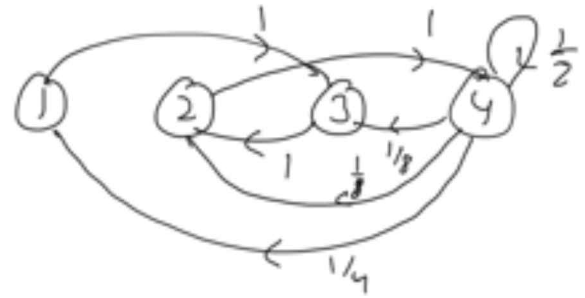
If $m_{ii} = \infty$, i null recurrent

If $m_{ii} < \infty$, i non-null recurrent/positive recurrent

<u>Example</u>: (X_n) M.C. $S = \{1,2,3,4\}$

tpm

$$P = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 0 & 1 \\ 3 & 0 & 1 & 0 & 0 \\ 4 & \frac{1}{4} & \frac{1}{8} & \frac{1}{8} & \frac{1}{2} \end{array}$$



$1 \leftrightarrow 2 \leftrightarrow 3 \leftrightarrow 4$ Irreducible M.C

Class $\{1,2,3,4\}$ finite state

all states are positive recurrent.

$$f_{44} = f_{44}^{(1)} + f_{44}^{(2)} + f_{44}^{(3)} + f_{44}^{(4)} + f_{44}^{(5)} + \cdots$$

$$= \frac{1}{2} + \frac{1}{8} + \frac{1}{8} + \frac{1}{4} + 0 + 0 + \cdots$$

$$= 1 \quad 4 \text{ recurrent}$$

$$m_{44} = \sum_{n=1}^{\infty} n f_{44}^{(n)} = 1 \times \frac{1}{2} + 2 \times \frac{1}{8} + 3 \times \frac{1}{8} + 4 \times \frac{1}{4} + 0 + 0 + \cdots$$

$$= \frac{17}{8} < \infty$$

→ Finite state, irreducible M.C. all states are +ve recurrent

→ Irreducible M.C., all states are either +ve recurrent or null recurrent or transient.

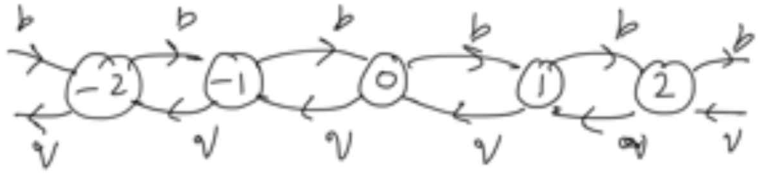
Example One - dimensional ...

... dimensional random walk

$$S = \{ \ldots, -2, -1, 0, 1, 2, \ldots \}$$

X_n position of particle at nth step

$$P_{i,i+1} = p \;;\; P_{i,i-1} = q = 1-p,\; P_{i,j} = 0,\; j \neq i-1, i+1$$



$$P_{ii}^{(n)} = \begin{cases} \binom{2m}{m} p^m (1-p)^m & n = 2m \\ 0 & n = 2m+1 \end{cases} \quad , m = 1,2,3,\ldots$$

$$= \begin{cases} a_m & , n = 2m \\ 0 & , n = 2m+1 \end{cases}$$

Ratio test

$$\lim_{m\to\infty} \frac{a_{m+1}}{a_m} = \begin{cases} <1, & \sum_m a_m \text{ converge} \\ >1, & \sum_m a_m \text{ diverge} \end{cases}$$

$$\frac{a_{m+1}}{a_m} = \frac{\binom{2m+2}{m+1} p^{m+1} (1-p)^{m+1}}{\binom{2m}{m} p^m (1-p)^m}$$

$$= \frac{(2m+2)(2m+1)}{(m+1)(m+1)} p(1-p)$$

$$\lim_{m\to\infty} \frac{a_{m+1}}{a_m} = 4p(1-p)$$

$$= \begin{cases} 1 & ; \quad p = \frac{1}{2} \\ <1 & \quad p \neq \frac{1}{2} \end{cases}$$

$p \neq \frac{1}{2} \Rightarrow \sum_n P_{ii}^{(n)} < \infty$, ie, i transient

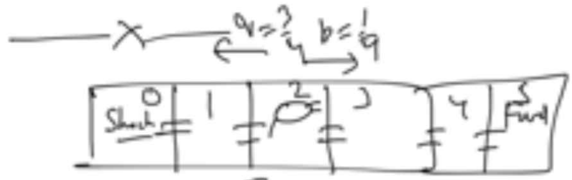
$p \neq \frac{1}{2}$ irreducible M.C all states are transients

show $p = \frac{1}{2}$ i recurrent ,, -- -- -- recurrent

Food

$\frac{q}{p} = 3$ $\quad P_2 = \frac{1-(3)^2}{1-3^5} =$

$i = 2, N = 5$



Gambler's ruin problem: $\quad$ Shark $\quad 1 - P_2$

initial capital Rs i $\qquad$ aim Rs N

$i = 0, 1, 2, \ldots, N$

$P(Z_i = +1) = p$; $P(Z_i = -1) = q = 1-p$

Z_i : i^{th} bet / step / transition.

X_n : player's fortune after n^{th} bet/steps

$\in \{0, 1, 2, \ldots, N\}$ $\qquad$ M.C

0, N recurrent/absorbing

$P_{00} = 1$, P_{NN} $\qquad$ $1, 2, \ldots, N-1$ transient

tpm

$$P = \begin{array}{c|cccccc} & 0 & 1 & 2 & \cdots & & N \\ \hline 0 & 1 & 0 & 0 & \cdots & & 0 \\ 1 & q & 0 & p & 0 & \cdots & 0 \\ 2 & 0 & q & 0 & p & \cdots & 0 \\ \vdots & \cdots & & & & & \\ N & 0 & 0 & 0 & \cdots & & 1 \end{array}$$

T_0 : time he broke

$T_0 = \inf\{n : X_n = 0\}$

T_N : time he has Rs. N

$T_N = \inf\{n : X_n = N\}$

$P_i = P(T_N < T_0)$: Prob. that starting with i, the gambler's fortune will reach N before reaching 0.

$$= P(T_N < T_0 | Z_1 = -1)\, \underline{P(Z_1=-1)}$$

$$+ P(\underline{T_N < T_0 | Z_1 = 1})\, \underline{P(Z_1=1)}$$

$$= q P_{i-1} + p P_{i+1}$$

$$\Rightarrow q P_i + p P_i = q P_{i-1} + p P_{i+1}$$

$$\Rightarrow P_{i+1} - P_i = \frac{q}{p}(P_i - P_{i-1})$$

$i=1$ $\quad \cancel{P_2} - P_1 = \frac{q}{p} P_1$ $\qquad P_0 = 0, P_N = 1$

$i=2$ $\quad \cancel{P_3} - \cancel{P_2} = \frac{q}{p}(P_2 - P_1) = \left(\frac{q}{p}\right)^2 P_1$

$\vdots$

$$P_i - \cancel{P_{i-1}} = \left(\frac{q}{p}\right)^{i-1} P_1$$

$$P_i - P_1 = P_1\left(\frac{q}{p} + \left(\frac{q}{p}\right)^2 + \cdots + \left(\frac{q}{p}\right)^{i-1}\right)$$

$$\Rightarrow P_i = P_1\left(1 + \frac{q}{p} + \left(\frac{q}{p}\right)^2 + \cdots + \left(\frac{q}{p}\right)^{i-1}\right)$$

$$P_i = \begin{cases} \dfrac{1-\left(\frac{q}{p}\right)^i}{1-\frac{q}{p}} P_1 & \text{if } \frac{q}{p} \neq 1 \\ i P_1 & \text{if } \frac{q}{p} = 1 \end{cases}$$

$$\because P_N = 1 \Rightarrow P_1 = \begin{cases} \dfrac{1-\frac{q}{p}}{1-\left(\frac{q}{p}\right)^N} & \text{if } \frac{q}{p} \neq 1 \\ \dfrac{1}{N} & \text{if } \frac{q}{p} = 1 \end{cases}$$

$$P_i = \begin{cases} \dfrac{1-\left(\frac{q}{p}\right)^i}{1-\left(\frac{q}{p}\right)^N} & \text{if } \frac{q}{p} \neq 1 \Leftrightarrow p \neq \frac{1}{2} \\ \dfrac{i}{N} & \text{if } \frac{q}{p} = 1 \Leftrightarrow p = \frac{1}{2} \end{cases}$$

$N \to \infty$

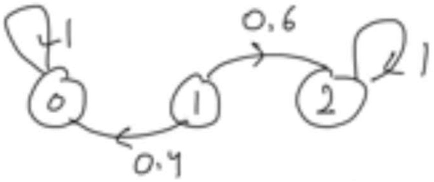
$$P_i = \begin{cases} 0, & p = \frac{1}{2} \\ 0, & p < \frac{1}{2} \Leftrightarrow \frac{q}{p} > 1 \\ 1 - \left(\frac{q}{p}\right)^i & p > \frac{1}{2} \Leftrightarrow \frac{q}{p} < 1 \end{cases}$$

—x—

Example | ① tpm

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 \\ 0.4 & 0 & 0.6 \\ 0 & 0 & 1 \end{bmatrix} \end{array}$$

Starting in 1, determine the prob. that M.C ends in state 0.



$p = 0.6$, $q = 0.4$ $\qquad p \neq \frac{1}{2}$

$i = 1, N = 2$ $\qquad \frac{q}{p} = \frac{2}{3}$

$$1 - P_1 = 1 - \frac{1 - \left(\frac{2}{3}\right)}{1 - \left(\frac{2}{3}\right)^2} = 0.4$$

② The probability of the thrower winning in the dice game called "Craps" is $p = 0.49$. Suppose Player A is the thrower and begins the game with \$5, and Player B, his opponent, begins with \$10. What is the probability that player A goes bankrupt before player B? Assume that the bet is \$1 per round.

$i = 5, N = 15$

$p = 0.49, q = 1 - p = 0.51$

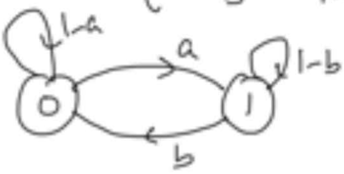
$$1 - P_5 = 1 - \frac{1 - \left(\frac{0.51}{0.49}\right)^5}{\quad}$$

$$1-\left(\frac{0.51}{0.49}\right)^{15}$$

—x—

Limiting prob.:

Thm
$$P=\begin{array}{c c} & \begin{array}{cc} 0 & 1 \end{array} \\ \begin{array}{c} 0 \\ 1 \end{array} & \begin{bmatrix} 1-a & a \\ b & 1-b \end{bmatrix} \end{array}, \quad 0\le a,b\le 1$$



then $|1-a-b|<1$

$a=b=0$ or $a=b=1$ we take separately

$$P^n=\begin{bmatrix} \dfrac{b+a(1-a-b)^n}{a+b} & \dfrac{a-a(1-a-b)^n}{a+b} \\[2ex] \dfrac{b-b(1-a-b)^n}{a+b} & \dfrac{a+b(1-a-b)^n}{a+b} \end{bmatrix}$$

Sol $P_{00}=1-a$, $P_{01}=a$, $P_{10}=b$, $P_{11}=1-b$

$$P_{00}^{(n)}=P_{00}^{(n-1)}P_{00}+P_{01}^{(n-1)}P_{10}$$
$$=(1-a)P_{00}^{(n-1)}+b\,P_{01}^{(n-1)} \qquad \because P_{00}^{(n-1)}+P_{01}^{(n-1)}=1$$
$$=(1-a)P_{00}^{(n-1)}+b(1-P_{00}^{(n-1)})$$
$$=b+(1-a-b)\underbrace{P_{00}^{(n-1)}}, \quad n\geq 1$$

$$=b+b(1-a-b)+b(1-a-b)^2+\cdots+b(1-a-b)^{n-2}$$
$$+P_{00}(1-a-b)^{n-1}$$
$$(1-a)$$

$$=b\sum_{k=0}^{n-2}(1-a-b)^k+\underline{(1-a)}(1-a-b)^{n-1}$$

$$\frac{1-(1-a-b)^{n-1}}{1-(1-a-b)}=\frac{1-(1-a-b)^{n-1}}{a+b}$$

$$= \frac{b}{a+b} + \frac{a(1-a-b)^n}{a+b}$$

$$(1-a-b)^{n-1}\left[1-a-\frac{b}{a+b}\right]$$

$$\frac{a+\cancel{b}-a^2-ab-\cancel{b}}{a+b}$$

$$= \frac{a(1-a-b)}{a+b}$$

$$\lim_{n\to\infty} P^n = \begin{pmatrix} \frac{b}{a+b} & \frac{a}{a+b} \\ \frac{b}{a+b} & \frac{a}{a+b} \end{pmatrix} = \begin{pmatrix} \pi_0 & \pi_1 \\ \pi_0 & \pi_1 \end{pmatrix}$$

$$\lim_{n\to\infty} p_i^{(n)} = \lim_{n\to\infty} P(X_n = i)$$

$$\lim_{n\to\infty} \underset{\sim}{p}^{(n)} = \underset{\sim}{p}^{(0)} P^n$$

$$\underline{\underline{\left(\frac{1}{3}, \frac{2}{3}\right)}} \begin{pmatrix} & \\ & \end{pmatrix} = \left(\frac{b}{a+b}, \frac{a}{a+b}\right)$$

$$= (\pi_0, \pi_1)$$

(*)

$$P = \begin{array}{c|cc} & 0 & 1 \\ \hline 0 & 0 & 1 \\ 1 & 1 & 0 \end{array}$$

$a=b=1$

periodic with period 2

$d(0)=2$

$$P^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$P^3 = P \;;\; P^4 = P^2$$

$\underset{\sim}{p}^{(0)}$

$(\alpha, 1-\alpha)$

$$\underset{\sim}{p}^{(1)} = \underset{\sim}{p}^{(0)} P = (1-\alpha, \alpha)$$

$$\underset{\sim}{p}^{(2)} = \underset{\sim}{p}^{(1)} P = (\alpha, 1-\alpha)$$

$\underset{\sim}{\pi} = \underset{\sim}{\pi} P$

$\sum \pi_i = 1$

$\pi_0 = \pi_1$

$\pi_0 + \pi_1 = 1$

$\underset{\sim}{\pi} = (\pi_0, \pi_1) \Rightarrow \pi_0 = \pi_1 = \frac{1}{2}$

limiting prob DNE.

(*)

$a=b=0$

$$P = \begin{array}{c|cc} & 0 & 1 \\ \hline 0 & 1 & 0 \\ 1 & 0 & 1 \end{array}$$

$$P^n = I$$

Finite state, irreducible, aperiodic (period=1), limiting prob. exist

$$\pi_j = \lim_{n\to\infty} p_j^{(n)} \qquad p_j^{(n)} = P(X_n = j)$$

$$\underline{p}^{(n)} = \underline{p}^{(n-1)} P \rightarrow \quad p_j^{(n)} = \sum_i p_i^{(n-1)} P_{ij}$$

Take limit

$$\left.\begin{aligned} \pi_j &= \sum_i \pi_i P_{ij} \\ \sum_i \pi_i &= 1 \end{aligned}\right] \longrightarrow \begin{aligned} \underline{\pi} &= \underline{\pi} P \\ \sum_i \pi_i &= 1 \end{aligned}$$

$$\underline{\pi} = (\pi_0, \pi_1, \ldots)$$

<u>Regular tpm:</u>

Given tpm P is regular if P^k has all element >0 for some k.

$$P = \begin{pmatrix} + & + & 0 \\ + & 0 & + \\ + & 0 & + \end{pmatrix} \qquad P^2 = \begin{pmatrix} + & + & 0 \\ + & 0 & + \\ + & 0 & + \end{pmatrix}\begin{pmatrix} + & + & 0 \\ + & 0 & + \\ + & 0 & + \end{pmatrix}$$

regular

$N=3 \qquad P^9$

$$= \begin{pmatrix} + & + & + \\ + & + & + \\ + & + & + \end{pmatrix}$$

states N

$P \qquad \underline{P^{N^2}}$

$N=2 \quad P^4$

$$P = \begin{pmatrix} 0 & + \\ + & 0 \end{pmatrix} ; P^2 = \begin{pmatrix} + & 0 \\ 0 & + \end{pmatrix}$$

not regular

$$P^4 = P^2 . P^2$$

$\Big[$ Show that finite state aperiodic irreducible MC. is regular and recurrent

<u>Thm:</u> (*) Let P regular tpm $S = \{0, 1, \ldots, N\}$. Then limiting pmf $\underline{\pi} = (\pi_0, \pi_1, \ldots, \pi_N)$ is <u>unique</u> soln to equations

$$\begin{aligned} \underline{\pi} &= \underline{\pi} P \\ \sum_{k=0}^{N} \pi_k &= 1 \end{aligned} \leftarrow \left\{ \begin{aligned} \pi_j &= \sum_{k=0}^{N} \pi_k P_{kj}, \quad j = 0, 1, \ldots, N \\ \sum_{k=0}^{N} \pi_k &= 1 \end{aligned} \right.$$

Sol MC is regular, we have limiting prob

[illegible]

$$\lim_{n\to\infty} P_{ij}^{(n)} = \pi_j \quad ; \quad \sum_{j=0}^{N} \pi_j = 1$$

$$P_{ij}^{(n)} = \sum_{k=0}^{N} P_{ik}^{(n-1)} P_{kj}$$

Take Limit as $n \to \infty$ $\quad P_{ij}^{(n)} \to \pi_j \; ; \; P_{ik}^{(n-1)} \to \pi_k$

$$\pi_j = \sum_{k=0}^{N} \pi_k P_{kj}$$

$$\sum_{j=0}^{N} \pi_j = 1$$

T.S. solⁿ is unique

$\exists \; x_0, \dots, x_N$ s.t.

$$x_j = \sum_{k=0}^{N} x_k P_{kj} \quad , \; j = 0, 1, \dots, N \qquad (1)$$

$$\sum_{k=0}^{N} x_k = 1$$

P_{jl}

$$\underbrace{\sum_{j=0}^{N} x_j P_{jl}}_{x_l \text{ using } (1)} = \sum_{j=0}^{N} \sum_{k=0}^{N} x_k P_{kj} P_{jl}$$

$$= \sum_{k=0}^{N} x_k \underbrace{\sum_{j=0}^{N} P_{kj} P_{jl}}_{P_{kl}^{(2)}}$$

$$\Rightarrow x_l = \sum_{k=0}^{N} x_k P_{kl}^{(2)}$$

$\vdots$

$$x_l = \sum_{k=0}^{N} x_k P_{kl}^{(n)} \quad , \; l = 0, 1, \dots, N$$

as $n \to \infty$ $\quad P_{kl}^{(n)} \to \pi_l$

$$x_l = \sum_{k=0}^{N} x_k \pi_l = \pi_l \underbrace{\sum_{k=0}^{N} x_k}_{1} = \pi_l$$

— x —

<u>Example 1</u> An NCD system has discount class E_0 (no discount), E_1 (20% discount) and E_2 (40% discount). Movement in the system is determined by the rule

[illegible]

... every one steps back one discount level (or stays in E_0) with one claim in a year, and return to a level of no discount if more than one claim is made. A claim free year results in a step up to a higher discount level (or one remains in class E_2 if already there).

NCD class	E_0	E_1	E_2
% discount	0	20	40
annual premium	100	80	60 ✓

If we suppose that for someone in this scheme the prob. of one claim in a year is 0.2 while the prob of two or more claims is 0.1 (i) Find tpm

(ii) In long run what proportion of time is the process is in each of the discount classes

(iii) Find the av annual premium paid.

$E_i \equiv i$

$i = 0,1,2$

$$P = \begin{array}{c c} & \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \end{array} & \begin{pmatrix} 0.3 & 0.7 & 0 \\ 0.3 & 0 & 0.7 \\ 0.1 & 0.2 & 0.7 \end{pmatrix} \end{array}$$

Class = {0,1,2} irreducible, aperiodic finite state

limiting prob. exist and same as stationary state prob.

$$\underset{\sim}{\Pi} P = \underset{\sim}{\Pi} \qquad \underset{\sim}{\Pi} = (\Pi_0, \Pi_1, \Pi_2)$$

$$\sum_{i=0}^{2} \Pi_i = 1$$

$$\Rightarrow \left\{ \Pi_0 = 0.3\Pi_0 + 0.3\Pi_1 + 0.1\Pi_2 \right.$$

$$\begin{cases} \pi_1 = 0.7\pi_0 + 0.2\pi_2 \\ \pi_0 + \pi_1 + \pi_2 = 1 \end{cases}$$

$\Rightarrow \pi_0 = 0.1860,\ \pi_1 = 0.2442,\ \pi_2 = 0.5698$

av. annual premium paid

$= 0.1860 \times 100 + 80 \times 0.2442 + 60 \times 0.5698$

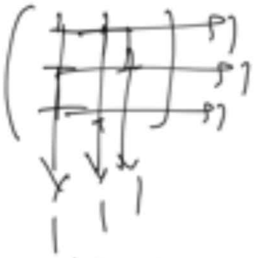
$= 72.324$

—x—

Doubly stochastic matrix

tpm P

$$\sum_k P_{ik} = \sum_i P_{ik} = 1$$



eg $P = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$

—x—

let P doubly stochastic, regular

$S = \{0, 1, \ldots, N-1\}$

limiting prob $\underset{\sim}{\pi} = \left(\frac{1}{N}, \frac{1}{N}, \ldots, \frac{1}{N}\right)$

$$\begin{cases} \pi_j = \sum_k \pi_k P_{kj} \\ \sum_k \pi_k = 1 \end{cases}$$

$$\frac{1}{N} = \sum_k \frac{1}{N} P_{kj} = \frac{1}{N}\underbrace{\left(\sum_k P_{kj}\right)}_{=1 \text{ since doubly stochastic tpm}}$$

$\therefore \underset{\sim}{\pi} = \left(\frac{1}{N}, \ldots, \frac{1}{N}\right)$ is unique soln using thm $(*)$

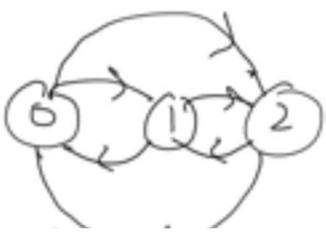
—x—

tpm

$S = \{0, 1, 2\}$

$N = 3$

$$\begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & 0 & 1/2 & 1/2 \\ 1 & 1/2 & 0 & 1/2 \\ 2 & 1/2 & 1/2 & 0 \end{array}$$



$$\underset{\sim}{\Pi} = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$$

irreducible, aperiodic, finite state M.C. doubly stochastic

eg doubly stochastic tpm may or may not be symmetric

not symmetric

$$P = \begin{pmatrix} 7/12 & 0 & 5/12 \\ 2/12 & 6/12 & 4/12 \\ 3/12 & 6/12 & 3/12 \end{pmatrix}, \quad P = \begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \end{array}$$

regular

not regular

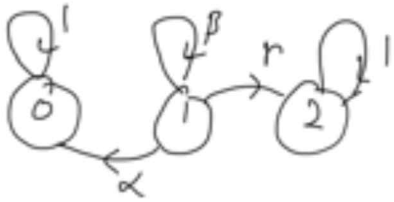
— x —

Simple first step analysis

$$P = \begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & 1 & 0 & 0 \\ 1 & \alpha & \beta & r \\ 2 & 0 & 0 & 1 \end{array}$$

$\alpha + \beta + r = 1$

$0 < \alpha, \beta, r < 1$



$T = \min\{n : X_n = 0 \text{ or } X_n = 2\}$

↳ time of absorption of the process

0,2 absorbing

1 transient

$S = \{0, 1, 2\}$

mean time to absorption, starting from state 1 $\to \nu = E(T \mid X_0 = 1)$

$u_1 = P(X_T = 0 \mid X_0 = 1)$

↳ Prob. of ultimate absorption into state 0 given that starting from state 1.

$$u_1 = P(X_T = 0 \mid X_0 = 1)$$

$$u_1 = \sum_{k \in S} P(X_T = 0, X_1 = k \mid X_0 = 1)$$

$$u_1 = \sum_{k \in S} P(X_T = 0 \mid X_1 = k, \cancel{X_0 = 1})\, P(X_1 = k \mid X_0 = 1)$$

$$u_1 = \sum_{k \in S} P(X_T = 0 \mid X_1 = k)\, P_{1k}$$

$$u_1 = \underbrace{P(X_T = 0 \mid X_1 = 0)}_{1}\, \underbrace{P_{10}}_{\alpha} + \underbrace{P(X_T = 0 \mid X_1 = 1)}_{u_1}\, \underbrace{P_{11}}_{\beta} + \underbrace{P(X_T = 0 \mid X_1 = 2)}_{0}\, \underbrace{P_{12}}_{\gamma}$$

$$u_1 = \alpha + \beta u_1 \Rightarrow (1-\beta) u_1 = \alpha \Rightarrow u_1 = \frac{\alpha}{1-\beta}$$

$$v = E(T \mid X_0 = 1)$$

$$= 1 + \underbrace{\alpha \times 0 + \gamma \times 0}_{X_1 = 0 \text{ or } X_1 = 2} + \underbrace{\beta \times v}_{X_1 = 1}$$

$$\Rightarrow v = 1 + \beta v \Rightarrow (1-\beta) v = 1 \Rightarrow v = \frac{1}{1-\beta}.$$

eg

$$P = \begin{array}{r c} & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{array}{r} 0 \\ u_1 = P(X_T=0 \mid X_0=1) \rightarrow 1 \\ u_2 = P(X_T=0 \mid X_0=2) \rightarrow 2 \\ 3 \end{array} & \begin{pmatrix} 1 & 0 & 0 & 0 \\ p_{00} & p_{11} & p_{12} & p_{13} \\ p_{20} & p_{21} & p_{22} & p_{23} \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

0, 3 absorbing

1, 2 transient

$0 < p_{ij} < 1$

$$\begin{cases} u_1 = p_{00} + p_{11} u_1 + p_{12} u_2 \\ u_2 = p_{20} + p_{21} u_1 + p_{22} u_2 \end{cases}$$

$$v_i = E(X_T \mid X_0 = i), \quad i = 1, 2$$

$$v_1 = 1 + p_{11} v_1 + p_{12} v_2$$

$$v_2 = 1 + p_{21} v_1 + p_{22} v_2$$

<u>Finite state M.C</u> $\quad S = \{0, 1, \to N\}$

$0, 1, \dots, r-1 \to$ transient

$r, \dots, N \to$ absorbing

tpm $P = \begin{pmatrix} Q & R \end{pmatrix}$

	r	$N-r+1$

$N+1$

$$\begin{pmatrix} 0 & I \end{pmatrix} \qquad \begin{array}{r|cc} & & \\ r & O_{r\times r} & R_{r\times N-r+1} \\ N+1 & & \\ N-r+1 & O_{N-r+1\times r} & I_{N-r+1\times N-r+1} \end{array}$$

$k \in \{r, \ldots, N\}$, $\quad i \in \{0, 1, \ldots, r-1\}$

$$u_{ik} = u_i = P(\text{absorbed in } k \mid X_0 = i)$$

$$= P_{ik} \times 1 + \sum_{\substack{j=r \\ j\neq k}}^{N} P_{ij} \times 0 + \sum_{j=0}^{r-1} P_{ij} u_j$$

$$= P_{ik} + \sum_{j=0}^{r-1} P_{ij} u_j \quad , i = 0, 1, \ldots, r-1.$$

$$v_i = 1 + \sum_{j=0}^{r-1} P_{ij} v_j \quad \text{for } i = 0, 1, \ldots, r-1$$

Example

0	1	7 Food
2	3	4
8 Shock	5	6

(7)

$u_0 = u_6$

$u_2 = u_5$

$u_1 = u_4$

$u_3 = \frac{1}{2}$

absorbing

tpm

	0	1	2	3	4	5	6	7	8
0		$1/2$	$1/2$						
1	$1/3$			$1/3$				$1/3$	
2	$1/3$			$1/3$					$1/3$
3		$1/4$	$1/4$		$1/4$	$1/4$			
4				$1/3$			$1/3$	$1/3$	
5				$1/3$			$1/3$		$1/3$
6					$1/2$	$1/2$			
7								1	
8									1

absorbing

$i = 0, 1, \ldots, 6$

$u_{i7} = u_i$

$u_0 = \frac{1}{2} u_1 + \frac{1}{2} u_2$

$u_0 = \frac{1}{2}$

$$u_1 = \frac{1}{3}u_0 + \frac{1}{3}u_3 + \frac{1}{3}$$
$$u_2 = \frac{1}{3}u_0 + \frac{1}{3}u_3$$

$\rightarrow u_1 = \frac{2}{3}$, $u_2 = \frac{1}{3}$, $u_3 = \frac{1}{2}$.

mean time spent in transient state:

Finite state M.C.

$T = \{1, 2, \ldots, t\}$ set of transient states

$$P_T = \begin{bmatrix} P_{11} & \cdots & P_{1t} \\ \vdots & & \vdots \\ P_{t1} & \cdots & P_{tt} \end{bmatrix}$$

$i, j \in T$

s_{ij} = expected # of time period the M.C. is in state j given that it starts in state i

$$I_{n,j} = \begin{cases} 1 & \text{if } X_n = j \\ 0 & \text{o.w.} \end{cases} \qquad \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{o.w.} \end{cases}$$

$i, j \in T$

$$s_{ij} = \delta_{ij} + E\left(\sum_{n=1}^{\infty} I_{n,j} \mid X_0 = i\right)$$
$$= \delta_{ij} + \sum_{n=1}^{\infty} E(I_{n,j} \mid X_0 = i)$$

$$1 \times P(X_n = j \mid X_0 = i) + 0 \times P(X_n \neq j \mid X_0 = i)$$

$$P_{ij}^{(n)}$$

$$= \delta_{ij} + \sum_{n=1}^{\infty} P_{ij}^{(n)} \qquad \text{——— } (**)$$

$$= \delta_{ij} + \sum_{n=1}^{\infty} \sum_{k} P_{ik} P_{kj}^{(n-1)} \qquad , k \in S, \; i, j \in T$$

$$= \delta_{ij} + \sum_{k} P_{ik} \sum_{n=1}^{\infty} P_{kj}^{(n-1)}$$

$$\rightarrow \delta_{kj} + \sum_{n=2}^{\infty} P_{kj}^{(n-1)}$$

$$\underbrace{\delta_{kj} + \sum_{n=1}^{\infty} P_{kj}^{(n)}}_{s_{kj}}$$

$$= \delta_{ij} + \sum_{k} P_{ik}\, s_{kj}$$

$$s_{ij} = \delta_{ij} + \sum_{k=1}^{t} P_{ik}\, s_{kj}, \longleftarrow$$

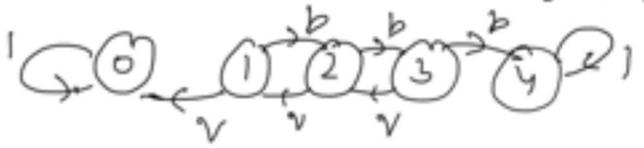
since it is impossible to go from a recurrent to a transient state $\Rightarrow s_{kj} = 0$, when k is recurrent state

$S = ((s_{ij}))$

$$S = I + P_T S$$

$$\Rightarrow (I - P_T) S = I \Rightarrow S = (I - P_T)^{-1}$$

<u>Example</u> Gambler ruin problem $p = 0.4$, $N = 4$



transient 1, 2, 3

absorbing 0, 4

$$P = \begin{array}{c|ccccc} & 0 & 1 & 2 & 3 & 4 \\ \hline 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & q & 0 & p & 0 & 0 \\ 2 & 0 & q & 0 & p & 0 \\ 3 & 0 & 0 & q & 0 & p \\ 4 & 0 & 0 & 0 & 0 & 1 \end{array} \qquad \begin{bmatrix} 0 & p & 0 \\ q & 0 & p \\ 0 & q & 0 \end{bmatrix} \rightarrow P_T$$

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}; \quad P_T = \begin{bmatrix} 0 & 0.4 & 0 \\ 0.6 & 0 & 0.4 \\ 0 & 0.6 & 0 \end{bmatrix}$$

$$(I - P_T) = \begin{bmatrix} 1 & -0.4 & 0 \\ -0.6 & 1 & 0 \\ 0 & -0.6 & 1 \end{bmatrix}$$

$$((s_{ij})) = S = (I - P_T)^{-1} = \begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & 1.46 & 0.76 & 0.31 \\ 2 & 1.15 & 1.92 & 0.76 \end{array}$$

$$3\,[\,0.69 \quad 1.15 \quad 1.46\,]$$

$$s_{2,3} = 0.76; \qquad \underline{s_{2,1} = 1.15}$$

f_{ij}: prob. that M.C ever makes a transition into state j given that it starts in state i

$$f_{2,1} = f_{2,1}^{(1)} + f_{2,1}^{(2)} + f_{2,1}^{(3)} + \cdots$$

$$= q + pq^2 + p^2q^3 + p^3q^4 + \cdots$$

$$= q + pq\left[q + pq^2 + p^2q^3 + \cdots\right]$$

$$= q + pq\, f_{2,1}$$

$$\Rightarrow f_{2,1} = \frac{q}{1-pq} = \frac{0.6}{1-0.4\times 0.6} = 0.78$$

$$f_{2,1} = 1 - \frac{1-\left(\frac{q}{p}\right)^1}{1-\left(\frac{q}{p}\right)^3} = 1 - \frac{1-\left(\frac{0.6}{0.4}\right)}{1-\left(\frac{0.6}{0.4}\right)^3} = 0.78$$

$$s_{ij} = E(\text{time in } j \mid \text{start } i)$$

$$= E(\text{time in } j \mid \text{start } i,\ \text{ever transit to } j)\cdot f_{ij}$$

$$+ \underbrace{E(\text{time in } j \mid \text{start } i,\ \text{never transit to } j)}_{\delta_{ij}}\cdot(1-f_{ij})$$

$$(\delta_{ij} + s_{jj})$$

$$= (\delta_{ij} + s_{jj})\cdot f_{ij} + \delta_{ij}(1-f_{ij})$$

$$= \delta_{ij} + s_{jj} f_{ij}$$

$$\Rightarrow f_{ij} = \frac{s_{ij} - \delta_{ij}}{s_{jj}}$$

$s_{2,1} = 1.15$, $\delta_{2,1} = 0$

$s_{1,1} = 1.46$

$$f_{2,1} = \frac{1.15 - 0}{1.46} = 0.78$$

Particular case:

$$P = \begin{pmatrix} Q & R \\ 0 & I \end{pmatrix}$$

$0, 1, \ldots, r-1$ transient

$r, \ldots, N$ absorbing

$$S = I + QS \Rightarrow S = (I - Q)^{-1}$$

$$s_{ij} = \delta_{ij} + \sum_{l=0}^{r-1} P_{il} s_{lj} \quad ; i, j = 0, 1, \ldots, r-1$$

$T \to$ time of absorption

$$T = \min\{n \; ; r \leq X_n \leq N\}$$

$$s_{ij} = E\left(\sum_{n=0}^{T-1} 1(X_n = j) \;\middle|\; X_0 = i\right)$$

$$v_i = E(T \mid X_0 = i)$$

$$\sum_{j=0}^{r-1} \sum_{n=0}^{T-1} 1(X_n = j) = \sum_{n=0}^{T-1} \underbrace{\sum_{j=0}^{r-1} 1(X_n = j)}_{1} = T$$

$$\sum_{j=0}^{r-1} s_{ij} = E\left(\sum_{j=0}^{r-1} \sum_{n=0}^{T-1} 1(X_n = j) \;\middle|\; X_0 = i\right)$$

$$= E(T \mid X_0 = i)$$

$$= v_i$$

$$\boxed{\sum_{j=0}^{r-1} s_{ij} = v_i}$$

Reducible M.C.

(1) tpm

$\underline{\pi} = (\pi_0, \pi_1)$

$\underline{\pi} = \underline{\pi} P$

$\pi_0 + \pi_1 = 1$

$\pi_0 = \frac{1}{2}\pi_0 + \frac{1}{4}\pi_1$

$\pi_0 + \pi_1 = 1$

$\Rightarrow \pi_0 = \frac{1}{3} \quad \pi_1 = \frac{2}{3}$

$$\begin{array}{c|c} & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \left[\begin{array}{cc|cc} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{4} & \frac{3}{4} & 0 & 0 \\ \hline 0 & 0 & \frac{1}{3} & \frac{2}{3} \\ 0 & 0 & \frac{2}{3} & \frac{1}{3} \end{array}\right] \end{array}$$

doubly stochastic matrix $\Rightarrow \pi_0^{(2)} = \frac{1}{2}, \pi_1^{(2)} = \frac{1}{2}$

Class $\{0,1\}$ $\{2,3\}$

$\{0,1\}$ → recurrent aperiodic; $\{2,3\}$ → recurrent aperiodic

$$\underset{\sim}{P} = \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix}, \quad P^2 = \begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix}\begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix} = \begin{pmatrix} P_1^2 & 0 \\ 0 & P_2^2 \end{pmatrix}$$

$$P^n = \begin{pmatrix} P_1^n & 0 \\ 0 & P_2^n \end{pmatrix}$$

$$\lim_{n\to\infty} P^n = \begin{pmatrix} \lim_{n\to\infty} P_1^n & 0 \\ 0 & \lim_{n\to\infty} P_2^n \end{pmatrix} = \begin{pmatrix} \underset{\sim}{\pi_0^{(1)}} \; \underline{\pi_1^{(1)}} & 0 \\ 0 & \underline{\pi_0^{(2)}} \; \underline{\pi_0^{(2)}} \end{pmatrix}$$

$$\lim_{n\to\infty} P^n = \begin{array}{c|c} & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & 0 & 0 \\ \frac{1}{3} & \frac{2}{3} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \end{array}$$

(2)

$\pi_0 = \frac{1}{2}\pi_0 + \frac{1}{4}\pi_1$

$\pi_0 + \pi_1 = 1$

$\Rightarrow \pi_0^{(1)} = 1/3$

$\pi_1^{(1)} = 2/3$

$$\begin{array}{c|c} & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \left[\begin{array}{cc|c|c} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{4} & \frac{3}{4} & 0 & 0 \\ \hline \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \end{array}$$

Classes $C_1 = \{0,1\}$, $C_2 = \{2\}$, $C_3 = \{3\}$

C_1 → recurrent aperiodic; C_2 → transient; C_3 → absorbing

C_1

$u_2 = \frac{1}{4} + \frac{1}{4} + \frac{1}{4}u_2$

$\Rightarrow u_2 = \frac{2}{3}$

$$\lim_{n\to\infty} P^n = \begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & \tfrac13 & \tfrac23 & 0 & 0 \\ 1 & \tfrac13 & \tfrac23 & 0 & 0 \\ 2 & \tfrac23\times\tfrac13 & \tfrac23\times\tfrac23 & 0 & \tfrac13 \\ 3 & 0 & 0 & 0 & 1 \end{array}$$

(3)

$$P = \begin{array}{c|cc|cc|cc} & 0 & 1 & 2 & 3 & 4 & 5 \\ \hline 0 & \tfrac12 & \tfrac12 & 0 & 0 & 0 & 0 \\ 1 & \tfrac13 & \tfrac23 & 0 & 0 & 0 & 0 \\ \hline 2 & \tfrac13 & 0 & 0 & \tfrac13 & \tfrac16 & \tfrac16 \\ 3 & \tfrac16 & \tfrac16 & \tfrac16 & 0 & \tfrac13 & \tfrac16 \\ \hline 4 & 0 & 0 & 0 & 0 & 0 & 1 \\ 5 & 0 & 0 & 0 & 0 & 1 & 0 \end{array}$$

$u_2 = \frac13 + \frac13 u_3$

$u_3 = \frac26 + \frac16 u_2$

$\Downarrow$

$u_2 = \frac{8}{17}$, $u_3 = \frac{7}{17}$

$C_1 = \{0,1\}$, $C_2 = \{2,3\}$, $C_3 = \{4,5\}$

recurrent, aperiodic; transient; recurrent period=2

$\pi_0 = \frac25$, $\pi_1 = \frac35$

$$\lim_{n\to\infty} P^n = \begin{array}{c|cccccc} & 0 & 1 & 2 & 3 & 4 & 5 \\ \hline 0 & \tfrac25 & \tfrac35 & 0 & 0 & 0 & 0 \\ 1 & \tfrac25 & \tfrac35 & 0 & 0 & 0 & 0 \\ 2 & \tfrac{8}{17}\times\tfrac25 & \tfrac{8}{17}\times\tfrac35 & 0 & 0 & X & X \\ 3 & \tfrac{7}{17}\times\tfrac25 & \tfrac{7}{17}\times\tfrac35 & 0 & 0 & X & X \\ 4 & 0 & 0 & 0 & 0 & X & X \\ 5 & 0 & 0 & 0 & 0 & X & X \end{array}$$

X DNE

For time av

$$\lim_{n\to\infty} \frac1n \sum_{m=0}^{n-1} P^m = \begin{array}{c|cccccc} & 0 & 1 & 2 & 3 & 4 & 5 \\ \hline 0 & \tfrac25 & \tfrac35 & 0 & 0 & 0 & 0 \\ 1 & \tfrac25 & \tfrac35 & 0 & 0 & 0 & 0 \\ 2 & \tfrac{8}{17}\times\tfrac25 & \tfrac{8}{17}\times\tfrac35 & 0 & 0 & \tfrac{9}{17}\times\tfrac12 & \tfrac{9}{17}\times\tfrac12 \\ 3 & \tfrac{7}{17}\times\tfrac25 & \tfrac{7}{17}\times\tfrac35 & 0 & 0 & \tfrac{10}{17}\times\tfrac12 & \tfrac{10}{17}\times\tfrac12 \end{array}$$

$$\begin{array}{c|cccccc|}
4 & 0 & 0 & 0 & 0 & 1/2 & 1/2 \\
5 & 0 & 0 & 0 & 0 & 1/2 & 1/2
\end{array}$$

Ergodic → irreducible, aperiodic, +ve recurrent

Branching process:

O j offsprings with probability p_j, independent of number produced by other individual

X_0 size of zeroth generation

X_1 size of first generation, i.e., all offsprings of zeroth generation

$\vdots$

$X_n \in \{0, 1, 2, \dots\}$

$p_0 > 0$

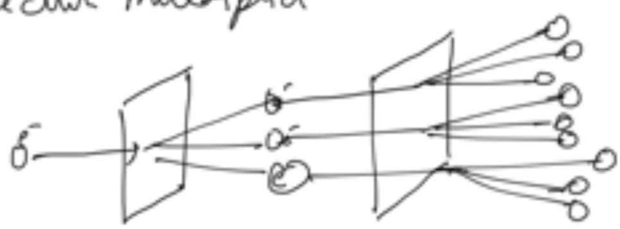
X_n size of nth generation, i.e., all offsprings of $(n-1)^{th}$ generation.

$E(X_n)$?

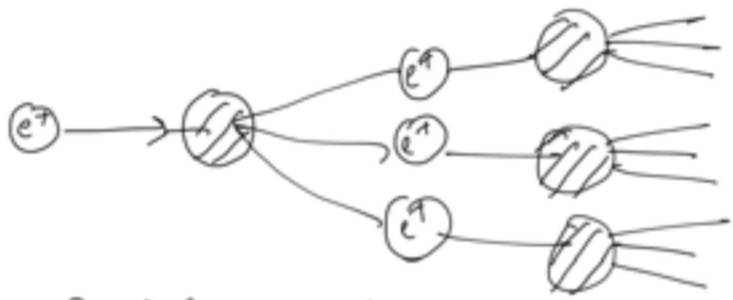
$V(X_n)$?

Applications

Electron multiplier



Neutron chain reaction



Suppose $X_0 = 1$ Survival of family name:

mean # of offsprings of a single individual $\mu = \sum_{j=0}^{\infty} j p_j$

var. " " " " $\sigma^2 = \sum_{j=0}^{\infty} (j-\mu)^2 p_j$

$(n-1)^{st}$ generation $\quad 1 \quad 2 \quad - - \quad X_{n-1}$

size of nth generation $X_n = Z_1 + Z_2 - - + Z_{X_{n-1}} = \sum_{i=1}^{X_{n-1}} Z_i$

Z_i # of offspring of ith individual of $(n-1)^{st}$ generation $E(Z_i) = \mu$, $V(Z_i) = \sigma^2$

$$E(X_n) = E\left(E(X_n \mid X_{n-1})\right)$$

$$= E\left(E\left(\sum_{i=1}^{X_{n-1}} Z_i \mid X_{n-1}\right)\right)$$

$$= E(X_{n-1}\mu)$$

$$\boxed{E\left(\sum_{i=1}^{X_{n-1}} Z_i \mid X_{n-1} = x\right) = E\left(\sum_{i=1}^{x} Z_i\right) = \sum_{i=1}^{x} \underbrace{E(Z_i)}_{\mu} = x\mu \checkmark}$$

$$= \mu E(X_{n-1})$$

$$= \mu^2 E(X_{n-2})$$

$$= \mu^n E(X_0)$$

$$= \mu^n$$

$$V(X_n) = E\big(\underbrace{V(X_n \mid X_{n-1})}_{X_{n-1}\sigma^2}\big) + V\big(\underbrace{E(X_n \mid X_{n-1})}_{X_{n-1}\mu}\big)$$

$$= E(X_{n-1}\sigma^2) + V(X_{n-1}\mu)$$

$$\boxed{V\left(\sum_{i=1}^{X_{n-1}} Z_i \mid X_{n-1} = x\right) = V\left(\sum_{i=1}^{x} Z_i\right) = \sum_{i=1}^{x} \underbrace{V(Z_i)}_{\sigma^2} = x\sigma^2}$$

$$= \sigma^2 E(X_{n-1}) + \mu^2 V(X_{n-1})$$

$$= \sigma^2\mu^{n-1} + \mu^2 V(X_{n-1}) \quad \longrightarrow \circledast$$

$$= \sigma^2\mu^{n-1} + \mu^2\left[\sigma^2\mu^{n-2} + \mu^2 V(X_{n-2})\right]$$

$$= \sigma^2\left[\mu^{n-1} + \mu^n\right] + \mu^4 V(X_{n-2})$$

$$= \sigma^2[\mu^{n-1}+\mu^n] + \mu^4[\sigma^2\mu^{n-3} + \mu^2 V(X_{n-3})]$$

$$= \sigma^2[\mu^{n-1}+\mu^n+\mu^{n+1}] + \mu^6 V(X_{n-3})$$

$$\vdots$$

$$= \sigma^2[\mu^{n-1}+\mu^n+\cdots+\mu^{2n-2}] + \mu^{2n}\underbrace{V(X_0)}_{\to 0}$$

$$= \sigma^2\mu^{n-1}[1+\mu+\cdots+\mu^{n-1}]$$

$$= \begin{cases} \sigma^2\mu^{n-1}\left(\dfrac{1-\mu^n}{1-\mu}\right) & \text{if } \mu \neq 1 \\ n\sigma^2 & \text{if } \mu = 1 \end{cases}$$

—x—

$$u_{n+1} = P(X_{n+1}=0) = \sum_{j=0}^{\infty} \underbrace{P(X_{n+1}=0 \mid X_1=j)}_{[P(X_n=0)]^j}\, p_j$$

$$\boxed{u_{n+1} = \sum_{j=0}^{\infty} u_n^j\, p_j}$$

—x—

π_0 prob. of ultimate extinction, i.e., prob. that the popln will eventually die out (under the assumption that $X_0=1$)

$$\pi_0 = \lim_{n\to\infty} P(X_n=0 \mid X_0=1)$$

$\rightarrow$ $\pi_0=1$ if $\underline{\mu<1}$

$$\mu^n = E(X_n) = \sum_{j=1}^{\infty} j\, P(X_n=j)$$

$$\geq \sum_{j=1}^{\infty} 1 \cdot P(X_n=j)$$

$$= P(X_n \geq 1)$$

$$\lim_{n\to\infty} P(X_n \geq 1) = 0 \Rightarrow \lim_{n\to\infty} P(X_n = 0) = 1$$

$$\Rightarrow \pi_0 = 1$$

→ $\underline{\pi_0 = 1 \text{ if } \mu = 1}$

→ When $\mu > 1$

$$\pi_0 = P(\text{popl}^n \text{ die out})$$

$$= \sum_{j=0}^{\infty} \underbrace{P(\text{popl}^n \text{ die out} \mid X_1 = j)}_{\pi_0^j} p_j$$

$$\Rightarrow \boxed{\pi_0 = \sum_{j=0}^{\infty} \pi_0^j p_j} \longrightarrow$$ π_0 is smallest +ve number which is the solution of this equation

Example: ① $X_0 = 1$ $p_0 = \frac{1}{2}$, $p_1 = \frac{1}{4}$, $p_2 = \frac{1}{4}$

$\pi_0 = 1$ $\qquad \mu = 0 \times \frac{1}{2} + 1 \times \frac{1}{4} + 2 \times \frac{1}{4} = \frac{3}{4} < 1$

$\underline{X_0 = n}$

$$\pi_0^n = 1^n = 1$$

② $\underline{X_0 = 1}$ $p_0 = \frac{1}{4}$, $p_1 = \frac{1}{4}$, $p_2 = \frac{1}{2}$ $\qquad \pi_0$

$$\mu = \frac{1}{4} + \frac{4}{4} = \frac{5}{4} > 1$$

$$\pi_0 = \sum_j \pi_0^j p_j$$

$$\Rightarrow \pi_0 = \frac{1}{4} + \frac{1}{4}\pi_0 + \frac{1}{2}\pi_0^2$$

$$\Rightarrow \pi_0 = 1, \frac{1}{2} \Rightarrow \boxed{\pi_0 = \frac{1}{2}}$$

$\underline{X_0 = n}$

$\pi_0^n = (1/2)^n$

$$\Pi_0 = \left(\frac{1}{2}\right)$$

—x—

$$x = f(x) \quad \text{— (1)}^*, \text{ where } f(x) = \sum_{j=0}^{\infty} x^j p_j$$

$$0 \leq x \leq 1.$$

$$f(0) = p_0 > 0$$

$$f(1) = \sum_{j=0}^{\infty} p_j = 1$$

$$f(x) = p_0 + x p_1 + x^2 p_2 + x^3 p_3 + \cdots$$

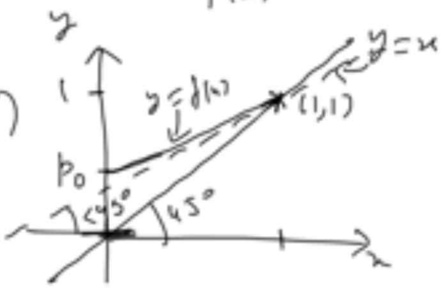
$$f'(x) = \underline{p_1} + \underline{2x p_2} + \underline{3x^2 p_3} + \cdots$$

$$f'(x) > 0 \ ; \ f''(x) > 0$$

$$f''(x) = 2p_2 + 6x p_3 + \cdots$$

$f(x)$ is increasing and convex. $\quad f'(1) = \mu$

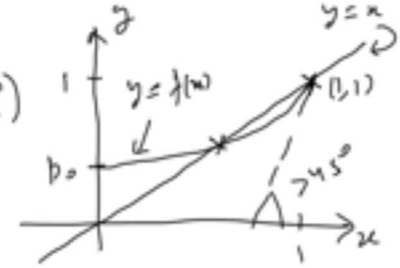
(I)



$$f(x) > x \ \forall x \in (0,1) \Leftrightarrow f'(1) \leq 1$$

$$\Leftrightarrow \mu \leq 1$$

(II)



$$f(x) = x \text{ for some } x \in (0,1)$$

$$\Leftrightarrow \mu > 1$$

Let $p_0 > 0$, $p_1 + p_2 < 1$. Let $\underline{\Pi \text{ satisfy } (1)^*}$

We will show by induction $\Pi \geq P(X_n = 0), \forall n$ — (**)

$$\Pi = \sum_{j=0}^{\infty} \Pi^j p_j \geq p_0 = P(X_1 = 0)$$

assume (**) holds for n, i.e., $\underline{\Pi \geq P(X_n = 0)}$

$$P(X_{n+1} = 0) = \sum_{j=0}^{\infty} \underbrace{P(X_{n+1} = 0 \mid X_1 = j)}_{[P(X_n = 0)]^j} p_j$$

$$= \sum_j \left(P(X_n = 0)\right)^j p_j$$

$$\leq \sum_j \Pi^j p_j$$

$$= \Pi$$

Using mathematical Induction $\pi \geq P(X_n=0), \forall n$

$$\pi \geq \lim_{n\to\infty} P(X_n=0) = \pi_0$$

— x —

Branching process & generating function:

r.v. $\xi \geq 0$, integer valued st $P(\xi = k) = p_k,\ k=0,1,2,\ldots$

generating function $\phi(s) = E(s^{\xi}) = \sum_{k=0}^{\infty} s^k p_k$

$$= p_0 + p_1 s + p_2 s^2 + \cdots \quad , \quad 0 \leq s \leq 1$$

$$\left.\frac{d\phi(s)}{ds}\right|_{s=0} = p_1 \quad ; \quad \frac{1}{2!}\left.\frac{d^2\phi(s)}{ds^2}\right|_{s=0} = p_2 \ ;$$

$$\frac{1}{k!}\left.\frac{d^k\phi(s)}{ds^k}\right|_{s=0} = p_k$$

indep. r.v ξ_i having generating function $\phi_i(s)$

$X = \sum_{i=1}^{n} \xi_i$ generating function

$$\phi_X(s) = E\left(s^{\sum_{i=1}^{n}\xi_i}\right)$$

$$= E(s^{\xi_1}) \cdots E(s^{\xi_n})$$

$$= \phi_1(s)\,\phi_2(s) \cdots \phi_n(s)$$

$$\left.\frac{d\phi(s)}{ds}\right|_{s=1} = p_1 + 2p_2 + 3p_3 + \cdots = E(\xi)$$

$$\left.\frac{d^2\phi(s)}{ds^2}\right|_{s=1} = E(\xi(\xi-1)) = E(\xi^2) - E(\xi)$$

$$V(\xi) = E(\xi^2) - (E(\xi))^2$$

$$= \left.\frac{d^2\phi(s)}{ds^2}\right|_{s=1} + \left.\frac{d\phi(s)}{ds}\right|_{s=1} - \left(\left.\frac{d\phi(s)}{ds}\right|_{s=1}\right)^2$$

Example : $\xi \sim \text{Poiss}(\lambda)$

$$p_k = P(\xi = k) = \frac{e^{-\lambda}\lambda^k}{k!}, \quad k = 0, 1, 2, \ldots$$

$$\phi(s) = E(s^{\xi}) = \sum_{k=0}^{\infty} s^k \frac{e^{-\lambda}\lambda^k}{k!}$$

$$= e^{-\lambda} \sum_{k=0}^{\infty} \frac{(\lambda s)^k}{k!}$$

$$= e^{-\lambda} e^{\lambda s}$$

$$= e^{-\lambda(1-s)} \quad \text{for } |s| < 1$$

$E(\xi) = \lambda$

$V(\xi) = \lambda$

—x—

Branching process $X_n \leftarrow$ at stage n, popln.

let offspring distn $p_k = P(\xi = k)$; $\underline{\phi(s) = E(s^{\xi}) = \sum_{k=0}^{\infty} s^k p_k}$

$$u_n = P(X_n = 0)$$

$$= \sum_{k=0}^{\infty} u_{n-1}^k p_k$$

$$= \phi(u_{n-1})$$

$u_0 = 0$, $u_1 = \phi(u_0)$, $u_2 = \phi(u_1)$, ——

π_0 smallest soln of $u = \phi(u)$

$$\pi_\infty = 1 - \pi_0$$

Ex. parent has no offspring w.p. $\frac{1}{4}$

2 " w.p. $\frac{3}{4}$

$$\phi(s) = E(s^{\xi}) = \sum_k s^k p_k$$

$$= p_0 + s^2 p_2 = \frac{1}{4} + \frac{3}{4} s^2$$

$$u_n = \phi(u_{n-1}) = \frac{1}{4} + \frac{3}{4} u_{n-1}^2$$

$$u_0 = 0$$

$$u_1 = \frac{1}{4}, \quad u_2 = \frac{1}{4} + \frac{3}{4}(u_1)^2 = \frac{1}{4} + \frac{3}{4} \times \frac{1}{16}$$