Exercises - Classification of states

4.3.1 A Markov chain has a transition probability matrix

$$
\begin{array}{c|cccccccc|}
 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
\hline
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
2 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
3 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
4 & 0.5 & 0 & 0 & 0 & 0 & 0.5 & 0 & 0 \\
5 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
7 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}.
$$

Find the equivalence classes. For which integers $n = 1, 2, \ldots, 20$, is it true that

$$P_{00}^{(n)} > 0?$$

What is the period of the Markov chain?

Hint: One need not compute the actual probabilities. See Section 4.1.1.

4.3.2 Which states are transient and which are recurrent in the Markov chain whose transition probability matrix is

$$
\begin{array}{c|cccccc|}
 & 0 & 1 & 2 & 3 & 4 & 5 \\
\hline
0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 & 0 & \frac{1}{3} \\
1 & \frac{1}{2} & \frac{1}{4} & \frac{1}{4} & 0 & 0 & 0 \\
2 & 0 & 0 & 0 & 0 & 1 & 0 \\
3 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 & 0 & \frac{1}{4} \\
4 & 0 & 0 & 1 & 0 & 0 & 0 \\
5 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}?
$$

4.3.3 A Markov chain on states $\{0, 1, 2, 3, 4, 5\}$ has transition probability matrix

$$(a)\quad \left\|\begin{matrix} \frac{1}{3} & 0 & \frac{2}{3} & 0 & 0 & 0 \\ 0 & \frac{1}{4} & 0 & \frac{3}{4} & 0 & 0 \\ \frac{2}{3} & 0 & \frac{1}{3} & 0 & 0 & 0 \\ 0 & \frac{1}{5} & 0 & \frac{4}{5} & 0 & 0 \\ \frac{1}{4} & \frac{1}{4} & 0 & 0 & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \end{matrix}\right\|.$$

$$(b)\quad \left\|\begin{matrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{3}{4} & \frac{1}{4} & 0 & 0 & 0 \\ 0 & \frac{1}{8} & \frac{7}{8} & 0 & 0 & 0 \\ \frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{8} & \frac{3}{8} & 0 \\ \frac{1}{3} & 0 & \frac{1}{6} & \frac{1}{4} & \frac{1}{4} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{matrix}\right\|.$$

Find all communicating classes; which classes are transient and which are recurrent?

4.3.4 Determine the communicating classes and period for each state of the Markov chain whose transition probability matrix is

$$\begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array}\begin{array}{c} \begin{matrix} 0 & 1 & 2 & 3 & 4 & 5 \end{matrix} \\ \left\|\begin{matrix} \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} \end{matrix}\right\| \end{array}.$$

Problems

4.3.1 A two-state Markov chain has the transition probability matrix

$$\mathbf{P} = \begin{array}{c|cc} & 0 & 1 \\ \hline 0 & 1-a & a \\ 1 & b & 1-b \end{array}.$$

( ) Determine the first return distribution

$$f_{00}^{(n)} = \Pr\{X_1 \neq 0, \ldots, X_{n-1} \neq 0, X_n = 0 | X_0 = 0\}.$$

4.3.1(a) Show that $p_{ii}^{(n)} = \sum_{k=0}^{n} f_{ii}^{(k)} p_{ii}^{(n-k)}, n \geq 1$

4.3.2 Show that a finite-state aperiodic irreducible Markov chain is regular and recurrent.

4.3.3 Recall the first return distribution (Section 4.3.3),

$$f_{ii}^{(n)} = \Pr\{X_1 \neq i, X_2 \neq j \ldots, X_{n-1} \neq i, X_n = i | X_0 = i\} \quad \text{for } n = 1, 2, \ldots,$$

with $f_{ii}^{(0)} = 0$ by convention. Using equation (4.1[illegible]etermine $f_{00}^{(n)}, n = 1, 2, 3, 4, 5$, for the Markov chain whose transition probability matrix is

$$\begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 1 & 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 0 & 1 \\ 3 & \frac{1}{2} & 0 & 0 & \frac{1}{2} \end{array}.$$