INDIAN INSTITUTE OF TECHNOLOGY, KHARAGPUR

End-Spring Semester 2022-23

Date of Exam: .04.23

Subject Number MA41017/MA60067 Session(FN/AN) Duration: 3 hrs

Max Marks: 50 Department: Mathematics

Subject Name: Stochastic Processes/ Stochastic Process and Simulation No. of students: 172

Instructions:

(i) Use of calculator and Statistical tables is allowed.
(ii) All answers of numerical questions must be in **at least two decimal places**.
(iii) All the notations are standard. If any data/statement is missing, identify it on your answer script.
(iv) Answer All questions.
(v) All parts of a question Must Be answered at One Place.

1. For this question write only answers on the first page of your answer script in the given tabular form. Detail working may be carried out on other pages. [1×10]

Q. No. 1	a	b	c	d	e	f	g	h	i	j
Answer										

Consider the discrete time Markov chain $\{X_n\}$ on states 0, 1, 2, 3 with transtion probability martix (TPM) as

$$P = \begin{array}{c c} & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \left\|\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 1/4 & 1/4 & 1/4 & 1/4 \\ 1/6 & 1/2 & 1/6 & 1/6 \\ 0 & 0 & 0 & 1 \end{array}\right\| \end{array}$$

Determine

(a) Whether the Markov chain is reducible or irreducible?

(b) the period of the state 2.

(c) Which states are transient or recurrent or absorbing.

(d) If it is known that the process starts in state $X_0 = 1$. Determine the probability $P(X_1 = 2, X_2 = 1, X_3 = 1)$.

(e) What is $P(X_2 = 3 | X_0 = 1)$?

(f) the probability of absorption into state 0 starting from state 1.

(g) the mean time spent in state 1 prior to absorption, given that starting state is 2.

(h) the mean time to absorption, starting from state 1.

(i) the limit, where they exists, as $n \to \infty$, of $P_{00}^{(n)}$.

(j) the limit, where they exists, as $n \to \infty$, of $P_{12}^{(n)}$.

P.T.O.

2. For this question write only answers on the second page of your answer script in the given tabular form. Detail working may be carried out on other pages. [1×10]

Q. No. 2	a	b	c	d	e	f	g	h	i	j
Answer										

(a) Suppose in a branching process the offspring distribution is Poisson with mean 0.5. Compute the probability of ultimate extinction.

(b) Let $\{W_t\}_{t\geq 0}$ be a Wiener process. Then for $0 < s < t$, calculate $Var(2W_s + W_t)$.

(c) Suppose that the interarrival distribution for a renewal process is Poisson distribution with mean 2. Let the waiting time for the 10th renewal is S_{10}. Determine $Var(S_{10})$.

(d) Consider a stable queueing system with arrival rate $\lambda_a = 2$ and $W = 2$, then find L.

(e) If the mean-value function of the renewal process $\{N(t),\ t \geq 0\}$ is given by $m(t) = t/3,\ t \geq 0$, what is $P(N(2) = 1)$?

(f) Interarrival times of customers at a ATM are independent exponential, with an average time of 20 minutes. The length of time spent to get service at ATM is assumed to be exponentially distributed with mean of 6 minutes. Find the average queue length.

(g) A barber shop has 2 chairs and one barber. Customers arrive according to a Poisson process of rate 2 and they will not enter if both chairs are taken. Suppose each haircut takes exponential amount of time with rate 3. Let $X(t)$ denote the number of customers in barber shop at time t. In steady state, write down the transition diagram mentioning the birth and death rates.

(h) John has a radio that works on a single battery. As soon as the battery in use fails, John immediately replaces it with a new battery. If the lifetime of a battery (in hours) is distributed uniformly over the interval (0,10), then at what rate does John have to change batteries?

(i) Let the pseudorandom numbers (PRN) is $U = 0.96$. Then simulate an exponential random variable with mean 2.

(j) Consider the PRN generator $X_i = (4X_{i-1} + 3) \mod 7$, with seed $X_0 = 0$. Calculate the PRN U_2.

3. (a) Let $\{N(t),\ t \geq 0\}$ be a Poisson process with rate λ, and suppose that, for fixed t, we are given that $N(t) = n$. For $1 \leq i \leq n$, let S_i be the time of the i^{th} event. Then find the joint probability density function of $S_1, S_2, \ldots, S_n$, given that $N(t) = n$.

(b) Suppose that jobs arrive at a file server at a Poisson rate 3 per minute. If two jobs arrived within one minute, between 11:00 and 11:01, what is the probability that the first job arrived before 20 seconds past 11:00 and the second job arrived before 40 seconds past 11:00?

[3+3]

4. (a) In New York, a man is stationed at a specific port and can be hired to give sightseeing tours with his boat. If the man is free, it takes an interested tourist a time period, exponentially distributed with mean 1/2, to negotiate the price and the type of tour. Suppose that the probability is 0.4 that a tourist does not reach an agreement and leaves. For those who decide to take a tour, the duration of the tour is exponentially distributed with mean 1/3. Suppose the tourists arrive at this businessman's station according to a Poisson process with parameter $\lambda = 2$ and request service only if he is free. They leave the station otherwise. Consider the states of continuous time Markov chain (CTMC) $X(t)$ as follows
$X(t) = 0$, if businessman is free;
$X(t) = 1$, if he is negotiating;

P.T.O.

$X(t) = 2$, if he is giving sightseeing tour.
If the negotiation times, the duration of the tours, and the arrival times of the tourists at the station are independent random variable, find, in steady state, the proportion of time the businessman if free.

(b) A salesman travels among three cities 1, 2 and 3. He spends an exponential time in each city, of means 2, 3 and 4 repectively, and then goes to another city. Suppose from city 1 or 2, he goes to one of two other cities with a probability of 1/2, but from city 3, he goes to 1 or 2 or remains at 3 with a probability of 1/3. Let $X(t)$ be his location at time t. Then $X(t)$ is a CTMC. Determine

(i) the generator Q;

(ii) the probability that he is in city 1 at time 10 given he is in city 2 at time 6.

[3+3]

5. (a) Explain $M/M/1/k$ queuing system. Find p_n and the parameters L_q, W_q, L, W, L_s, W_s of $M/M/1/k$ queuing system. Also find the cummulative distribution function (cdf) of the time spend by the customer in the queueing system, i.e., calculate $W(t) = P(w \leq t)$.

(b) A supermarket has two exponential checkout counters, each operating at rate $\mu = 2$. Arrivals are Poisson at rate $\lambda = 1$. The counters operate in the following way:
(I) One queue feeds both counters.
(II) One counter is operated by a permanent checker and the other by a stock clerk who instantaneously begins checking whenever there are two or more customers in the system. The clerk returns to stocking whenever he completes a service, and there are fewer than two customers in the system.

(i) Find P_n, proportion of time there are n in the system.

(ii) At what rate does the number in the system go from 0 to 1? From 2 to 1?

(c) Suppose that X has probability density function (pdf) $f(x) = e^{-2|x|}$, $-\infty < x < \infty$. Use probability intergal transfom method to develope a technique to generate a realization of X. Demonstrate your technique using $U = 0.6$.

[3+3+3]

6. (a) For the renewal process $N(t)$ whose interarrival times are uniformly distributed on (0,2), determine the renewal function $m(t) = E(N(t))$.

(b) Passengers arrive at a tram station according to a renewal process with mean cycle time $\mu = 5$. A tram leaves the station whenever there are n passangers waiting. The station incurs a cost Rs. 2 for waiting customers per unit time and an additional cost Rs 500 when a tram is dispatched. Find the value of n that minimizes the long-run cost per unit time.

(c) Consider a stochastic process that evolves according to the following law: If $X_n = 0$, then $X_{n+1} = 0$, whereas if $X_n > 0$, then

$$X_{n+1} = \begin{cases} X_n + 1, & \text{with probability } 1/2 \\ X_n - 1, & \text{with probability } 1/2 \end{cases}.$$

Whether X_n is a martingale? Prove or disprove.

[3+3+3]

***

1 (a) reducible

(b) 1

(c) Transient - 1,2
Absorbing - 0,3
Recurrent - 0,3

(d) 0.03125 $\left(\frac{1}{32}\right)$

(e) 0.3541 $\left(\frac{17}{48}\right)$

(f) 0.5

(g) 1

(h) 2.167 $\left(\frac{13}{6}\right)$

(i) 1

(j) 0

10

(2)

10

(a) 1

(b) $t+85$

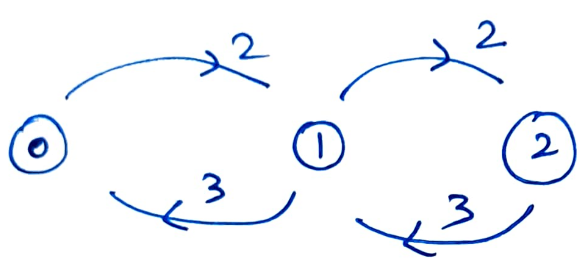
(c) 20

(d) 4

(e) 0.342

(f) 0.129 $\left(\frac{9}{70}\right)$

(g)



(h) $\frac{1}{5}$ [1 in every 5 hours]

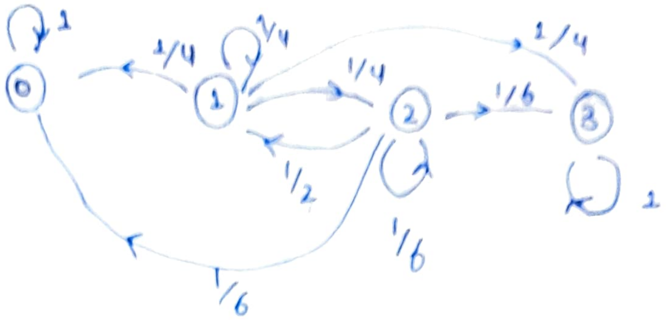
(i) 6.438

(j) 0.143 $\left(\frac{1}{7}\right)$

4.

1 min ~~...~~ P1

(1) (a)



we cannot reach anywhere from 0.

So, reducible

(b)

period of state 2 = gcd (1, 2, 3, 4 ----)

$$\boxed{\text{Period of state 2} = 1}$$

(c) Absorbing → 0, 3

~~Recu~~ $f_{00} = 1$

$f_{33} = 1$

Recurrent → 0, 3

Transient → 1, 2

1 ↔ 2

$$f_{11} = \frac{1}{4} + \frac{1}{4}\times\frac{1}{2} + \frac{1}{4}\times\frac{1}{2}\times\frac{1}{6} + \frac{1}{4}\times\frac{1}{2}\times\left(\frac{1}{6}\right)^2 + ---$$

$$= \frac{1}{4} + \frac{1}{4}\times\frac{1}{2}\times\frac{1}{1-\frac{1}{6}}$$

$$= \frac{1}{4} + \frac{1}{\cancel{8}\,4}\times\frac{\cancel{6}\,3}{5}$$

$$= \frac{1}{4}\left(\frac{8}{5}\right) = \frac{2}{5} < 1$$

(d) $P(X_0 = 1) = 1$

$$P(X_1 = 2, X_2 = 1, X_3 = 1) = P(X_3 = 1, X_2 = 1, X_1 = 2) \quad f)$$

$$= P(X_3 = 1 / X_2 = 1, X_1 = 2) \times P(X_2 = 1, X_1 = 2)$$

$$= P(X_3 = 1 / X_2 = 1) \times P(X_2 = 1 / X_1 = 2) \times P(X_1 = 2)$$

$$= \frac{1}{4} \times \frac{1}{2} \times P(X_1 = 2) = \frac{1}{4} \times \frac{1}{2} \times \frac{1}{4} = \boxed{\frac{1}{32}}$$

$$\left\{ P(X_1 = 2) = P(X_1 = 2 \mid X_0 = 1) \times P(X_0 = 1) + 0 \right.$$

$$\left. = \frac{1}{4} \times 1 = \frac{1}{4} \right\}$$

(e) $P(X_2 = 3 \mid X_0 = 1) = \cancel{P_{13}^{(2)}} \; \cancel{= \frac{1}{4} \times \frac{1}{4} + \frac{1}{6} \times \frac{1}{4}}$

$$\cancel{= \frac{1}{4}\left(\frac{3}{12} + \frac{2}{12}\right)}$$

$$\cancel{= \boxed{\frac{5}{48}}} \quad \cancel{0.1041}$$

$$P^2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ - & - & \frac{1}{4}\times\frac{1}{4}+\frac{1}{6}\times\frac{1}{4} & - \\ - & - & - & - \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$P_{13}^{(2)} = \frac{1}{4} \times 0 + \frac{1}{4} \times \frac{1}{4} + \frac{1}{4} \times \frac{1}{6} + \frac{1}{4} \times 1$$

$$= \frac{3 + 2 + 12}{48} = \boxed{0.3541}$$

(f) $u_1 = P(X_T = 0 \mid X_0 = 1)$

$u_2 = P(X_T = 0 \mid X_0 = 2)$

$$u_1 = \frac{1}{4} + \frac{1}{4}u_1 + \frac{1}{4}u_2$$

$$4u_1 = 1 + u_1 + u_2$$

$$3u_1 - u_2 = 1$$

$$u_2 = \frac{1}{6} + \frac{1}{2}u_1 + \frac{1}{6}u_2$$

$$6u_2 = 1 + 3u_1 + u_2$$

$$5u_2 - 3u_1 = 1$$

$$\begin{array}{rl} & 3u_1 - u_2 = 1 \\ + & 5u_2 - 3u_1 = 1 \\ \hline & 4u_2 = 2 \end{array}$$

$$u_2 = \frac{1}{2} \qquad 3u_1 = 1 + \frac{1}{2}$$

$$\boxed{u_1 = \frac{1}{2}}$$

(g)

$$P_T = \begin{array}{c c} & \begin{array}{cc} 1 & 2 \end{array} \\ \begin{array}{c} 1 \\ 2 \end{array} & \begin{bmatrix} 1/4 & 1/4 \\ 1/2 & 1/6 \end{bmatrix} \end{array}$$

$$S = (I - P_T)^{-1} = \left(\begin{bmatrix} 3/4 & -1/4 \\ -1/2 & 5/6 \end{bmatrix} \right)^{-1} = \begin{bmatrix} 1.6666 & 0.5 \\ 1 & 1.5 \end{bmatrix}$$

$$\text{Answer} = S_{21} = \boxed{1}$$

5/16

1/9

(h) $v_1 = E(T \mid X_0 = 1)$

$v_2 = E(T \mid X_0 = 2)$

$$v_1 = 1 + \frac{1}{4} v_1 + \frac{1}{4} v_2$$

$$v_2 = 1 + \frac{1}{2} v_1 + \frac{1}{6} v_2$$

$$3v_1 - v_2 = 4$$

$$5v_2 - 3v_1 = 6$$

(+)

$$4v_2 = 10$$

$$v_2 = 5/2$$

$$3v_1 - 5/2 = 4$$

$$3v_1 = \frac{13}{2}$$

$$\boxed{v_1 = \frac{13}{6}}$$

(i) $n \to \infty \quad P_{00}^{(n)} = 1$

(j) $n \to \infty \quad P_{12}^{(n)} = 0$

2 (a) $\mu = 0.5$

$\underline{\underline{\mu < 1}}$

So, $\boxed{\pi_0 = 1}$

(b) $Var(2W_s + W_t)$ $\quad s < t$

Independent

$2W_s + W_t = W_t - W_s + 3W_s$

$= N(0, t-s) + N(0, 9s)$

$= N(0, t+8s)$

$$\boxed{Var(2W_s + W_t) = t + 8s}$$

(c)

$S_{10} = \sum_{i=1}^{10} X_i$

X_i – Poisson distribution with mean 2

$\boxed{\lambda t = 2}$

S_{10} is poisson distribution with parameter 10λ

$Var(S_{10}) = 10\lambda \times t$

$= 10 \times 2 = \boxed{20}$

(d) $\lambda a = 2$, $W = 2$

$$L = \lambda a \times W$$

$$L = 2 \times 2$$

$$\boxed{L = 4}$$

(e) $m(t) = t/3$

$$m(t) = F(t) + \int_0^t m(t-x)\, f(x)\, dx$$

$$t/3 = f(t) + \frac{1}{3}\int_0^t f(x)\, dx$$

$$f'(t) = -\frac{1}{3} f(t)$$

$$f(t) = k e^{-t/3}$$

Interarrival time is exponential.

So, $N(t)$ is poisson process with $\lambda = \frac{1}{3}$

$$P(N(2)=1) = \frac{e^{-2/3}}{1!} \times \left(\frac{2}{3}\right)^1$$

$$= \frac{2}{3} \times e^{-2/3} = \boxed{0.342}$$

(f) $\lambda = \frac{1}{20}$

L_q

$\mu = \frac{1}{6}$

M/M/1

$\rho = \frac{\lambda}{\mu} = \frac{6}{20} = 0.3$

$\lambda_a = \lambda = \frac{1}{20}$

$L = \frac{\rho}{1-\rho} = \frac{3}{7}$

$W_s = \frac{1}{\mu} = 6$

$L_s = \lambda_a \times W_s = 0.3$

$L = L_q + L_s$

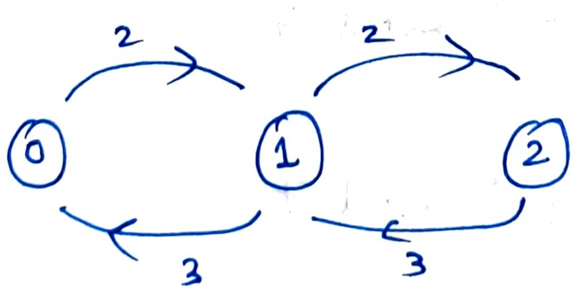
$L_q = \frac{3}{7} - 0.3$

$= \frac{3}{7} - \frac{3}{10} = \frac{9}{70}$

$$\boxed{L_q = \frac{9}{70} = 0.129}$$

(g) 2 chairs 1 barber

capacity = 2 server = 1 $\mu = 3$



State → no. of customers in the shop

$\lambda_0 = 2, \lambda_1 = 2$

$\mu_1 = 3, \mu_2 = 3$

(h)

$$\mu = \frac{(0+10)}{2} = 5$$

Answer as $t \to \infty$ $\dfrac{N(t)}{t} = \dfrac{1}{\mu} = \boxed{\dfrac{1}{5}}$

1 in every 5 hours

(i)

$U = 0.96$

$\lambda = \frac{1}{2}$

$\cancel{\lambda = 2}$

$\cancel{x = -\frac{1}{2} \ln(1-u)}$

$\cancel{u = 0.96}$

$\cancel{x = -\frac{1}{2} \ln(0.04)}$

$\cancel{\boxed{x = 1.609}}$

$$x = -\frac{1}{\lambda} \ln(1-u)$$

$$x = -2 \ln(1-u)$$

$$u = 0.96$$

$$x = -2 \ln(0.04)$$

$$\boxed{x = 6.438}$$

(j)

$$X_i = (4X_{i-1} + 3) \bmod 7 \qquad X_0 = 0$$

$$X_1 = (4X_0 + 3) \bmod 7$$

$$X_1 = 3$$

$$X_2 = (4 \times 3 + 3) \bmod 7$$

$$X_2 = 1$$

$$\boxed{U_2 = \frac{1}{7}}$$

(3) (a) $\{N(t), t \geq 0\}$ Poisson process parameter λ

Given $\rightarrow$ $N(t) = n$

$S_i =$ time of the ith event

To find $\rightarrow$ $f(S_1, S_2 \ldots, S_n / N(t) = n)$

$$f(S_1, S_2, \ldots, S_n \mid N(t) = n)$$

$$= P(S_1 = x_1, S_2 = x_2, \ldots, S_n = x_n \mid N(t) = n)$$

$$= \frac{P(S_1 = x_1, S_2 = x_2, \ldots S_n = x_n, N(t) = n)}{P(N(t) = n)}$$

$$= \frac{P(S_1 = x_1, S_2 - S_1 = x_2 - x_1, \ldots S_n - S_{n-1} = x_n - x_{n-1}, N(t) = n)}{P(N(t) = n)}$$

$$= \frac{P(S_1 = x_1, S_2 - S_1 = x_2 - x_1, \ldots S_n - S_{n-1} = x_n - x_{n-1}, N(x_n, t) = 0)}{P(N(t) = n)}$$

$$S_n = \sum_{i=1}^{n} X_i \qquad X_n = S_n - S_{n-1}$$

$$S_{n-1} = \sum_{i=1}^{n-1} X_i$$

$$= \frac{P(X_1 = x_1,\ X_2 = x_2 - x_1,\ \ldots\ X_n = x_n - x_{n-1},\ N(x_n, t) = 0)}{P(N(t) = n)}$$

$$= \frac{\lambda e^{-\lambda x_1} \times \lambda \times e^{-\lambda(x_2 - x_1)} \ldots \times \lambda e^{-\lambda(x_n - x_{n-1})} \times e^{-\lambda(t - x_n)}}{\frac{e^{-\lambda t} \times (\lambda t)^n}{n!}}$$

$$= \frac{\lambda^n \times e^{-\lambda(\cancel{x_1} + \cancel{x_2} - \cancel{x_1} + \cancel{x_3} - \cancel{x_2} + \ldots + \cancel{x_n} - \cancel{x_{n-1}} + t - \cancel{x_n})} \times n!}{e^{-\lambda t} \times \lambda^n \times t^n}$$

$$= \frac{\cancel{\lambda^n e^{-\lambda t}} \times n!}{\cancel{e^{-\lambda t} \times \lambda^n} \times t^n}$$

3

So, $\boxed{f(S_1, S_2, \ldots S_n \mid N(t) = n) = \frac{n!}{t^n}}$

$$\frac{e^{-\frac{1}{20}\times 60}\, 3^2}{2!}$$

$S_n < t \equiv N(t) \geq n$

3(b)

$P(S_1 < 20, S_2 < 40 \mid N(60) = 2)$

$$f_{S_1,S_2}(s_1,s_2 \mid N(t)=2) = \frac{n!}{t^n}, \quad 0<s_1<s_2<60 = \frac{2}{3600}, \quad 0<s_1<s_2<60$$

$$\int_0^{20}\int_{s_1}^{40} \frac{2}{3600}\, ds_2\, ds_1 = \int_0^{20} \frac{1}{1800}(40 - s_1)\, ds_1$$

$$= \frac{1}{1800}\left(40\times 20 - \frac{400}{2}\right) =$$

$$= \frac{1}{1800}(800 - 200) = \frac{600}{1800} = \frac{1}{3}$$

Alter

$\lambda = 3$ per min

$= \frac{3}{60}$ per sec $= \frac{1}{20}$ per sec

$P(N(20) \geq 1, N(0,40) \leq 2 \mid N(60)=2)$

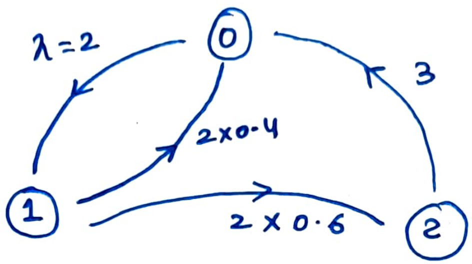
$$= \frac{P(N(20)=2, N(20)=0, N(20)=0) + P(N(20)=1, N(20)=1, N(20)=0)}{P(N(60)=2)}$$

$$= \frac{\frac{e^{-\frac{1}{20}\times 20}\times e^{-1}\times e^{-1}}{2!} + e^{-1}\times e^{-1}\times e^{-1}}{\frac{e^{-\frac{1}{20}\times 60}\,3^2}{2!}}$$

$$= \frac{\cancel{e^{-3}}\left(1+\frac{1}{2}\right)}{\frac{\cancel{e^{-3}}\times 9}{2}} = \frac{\cancel{3}}{\cancel{2}}\times\frac{\cancel{2}}{\cancel{9}_3} = \frac{1}{3}$$

4 (a)

$X(t)$



$$Q = \begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & -2 & 2 & 0 \\ 1 & 2\times0.4 & -2 & 2\times0.6 \\ 2 & 3 & 0 & -3 \end{array}$$

π_0, π_1, π_2

$$\pi_0 + \pi_1 + \pi_2 = 1$$

$$\pi Q = 0$$

$$[\pi_0, \pi_1 \quad \pi_2] \begin{bmatrix} -2 & 2 & 0 \\ 0.8 & -2 & 1.2 \\ 3 & 0 & -3 \end{bmatrix} = [0 \quad 0 \quad 0]$$

$$\pi_0 + \pi_1 + \pi_2 = 1$$

$$2\pi_0 - 2\pi_1 = 0 \quad \Rightarrow \quad \pi_0 = \pi_1$$

$$1.2\pi_1 - 3\pi_2 = 0$$

$$\pi_1 = \frac{30}{12}\pi_2 = 2.5\pi_2$$

$$\pi_0 = \pi_1 = 2.5\pi_2$$

$$2.5\pi_2 + 2.5\pi_2 + \pi_2 = 1$$

$$6\pi_2 = 1$$

3

$$\pi_2 = \frac{1}{6}$$

$$\pi_0 = 2.5 \times \frac{1}{6} = \frac{5}{2} \times \frac{1}{6}$$

$$\pi_1 = \pi_0 = \frac{5}{12}$$

$$\boxed{\pi_0 = \frac{5}{12}}$$ Answer

b) $\cancel{q_{12} = \frac{1}{2}}$ $v_1 = \frac{1}{2}$, $v_2 = \frac{1}{2}$, $v_3 = \frac{1}{4}$ (Sojourn Rates)

$q_{12} = v_1 P_{12} = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$, $q_{13} = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$

$\Rightarrow q_{11} = -\frac{1}{2}$

$q_{21} = \frac{1}{2} \cdot \frac{1}{3} = \frac{1}{6}$, $q_{23} = \frac{1}{2} \cdot \frac{1}{3} = \frac{1}{6}$

$\Rightarrow q_{22} = -\frac{1}{3}$

$q_{31} = \frac{1}{3} \cdot \frac{1}{4} = \frac{1}{12}$, $q_{32} = \frac{1}{3} \cdot \frac{1}{4} = \frac{1}{12}$

$\Rightarrow q_{33} = -\frac{1}{6}$

i) $\boxed{Q = \begin{bmatrix} -\frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{6} & -\frac{1}{3} & \frac{1}{6} \\ \frac{1}{12} & \frac{1}{12} & -\frac{1}{6} \end{bmatrix}}$

ii) $|Q - kI| = 0$

$\Rightarrow \begin{vmatrix} -\frac{1}{2} - k & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{6} & -\frac{1}{3} - k & \frac{1}{6} \\ \frac{1}{12} & \frac{1}{12} & -\frac{1}{6} - k \end{vmatrix} = 0$

$\Rightarrow \left(-\frac{1}{2} - k\right)\left(\left(\frac{1}{3} + k\right)\left(\frac{1}{6} + k\right) - \frac{1}{72}\right) - \frac{1}{4}\left(\frac{1}{6}\left(-\frac{1}{6} - k\right) - \frac{1}{72}\right)$

$+ \frac{1}{4}\left(\frac{1}{72} + \left(\frac{1}{3} + k\right)\frac{1}{12}\right) = 0$

$$\Rightarrow \left(-\frac{1}{2}-k\right)\left(k^2+\frac{1}{2}k+\frac{1}{24}\right)+\frac{1}{4}\left(\frac{1}{24}+\frac{k}{6}\right)$$

$$+\frac{1}{4}\left(\frac{1}{24}+\frac{k}{12}\right)=0$$

$$\Rightarrow \left(-\frac{1}{2}-k\right)\left(k^2+\frac{k}{2}+\frac{1}{24}\right)+\frac{1}{4}\left(\frac{k}{4}+\frac{1}{12}\right)=0$$

$$\Rightarrow \cancel{48k^3} \text{ or}$$

$$(1+2k)(24k^2+12k+1)-(3k+1)=0$$

$$\cancel{\Rightarrow 48k^3+24k^2}$$

$$\Rightarrow 48k^3+48k^2+11k=0$$

$$\Rightarrow k(48k^2+48k+11)=0$$

$$k=0,\ \frac{-48\pm\sqrt{48^2-4\times 11\times 48}}{2\times 48}$$

$$=0,\ \cancel{-48}\ \frac{-1}{2}\pm\frac{1}{96}\sqrt{16^2\cdot 3^2-3\times 11\times 64}$$

$$\boxed{k=0,\ \frac{-1}{2}\pm\frac{\sqrt{3}}{12}}$$

$$\Rightarrow P_{ij}(t)=A_{ij}e^0+B_{ij}e^{\left(\frac{-1}{2}+\frac{\sqrt{3}}{12}\right)t}+C_{ij}e^{\left(\frac{-1}{2}-\frac{\sqrt{3}}{12}\right)t}$$

$\cancel{P_{ij}(0)=\delta_{ij}}$ To find: $P_{12}(4)$.

$$\Rightarrow \boxed{P_{12}(4)=A_{12}+B_{12}e^{-2+\frac{1}{\sqrt{3}}}+C_{12}e^{-2-\frac{1}{\sqrt{3}}}}$$

$$P_{12}(0)=0 \Rightarrow A_{12}+B_{12}+C_{12}=0 \quad \text{--- (1)}$$

$$P_{12}'(0)=Q_{12} \Rightarrow B_{12}\left(\frac{-1}{2}+\frac{\sqrt{3}}{12}\right)+C_{12}\left(\frac{-1}{2}-\frac{\sqrt{3}}{12}\right)=\frac{1}{4}$$

$$\Rightarrow -6(B_{12}+C_{12}) + \sqrt{3}(B_{12}-C_{12}) = 3$$

$$\Rightarrow -2\sqrt{3}(B_{12}+C_{12}) + (B_{12}-C_{12}) = \sqrt{3} \quad -(2)$$

$$Q^2 = \begin{bmatrix} \frac{5}{16} \\ -\frac{1}{8} \\ \end{bmatrix}$$

$$P_{12}''(0) = (Q^2)_{12} = \left(\frac{-1}{4}\right)\left(\frac{-1}{2}\right) + \left(\frac{1}{2}\right)\left(\frac{1}{4}\right) + \left(\frac{7}{6}\right)(A_1)$$

$$= \frac{-1}{8}$$

$$\Rightarrow \left(\frac{-1}{2} + \frac{\sqrt{3}}{12}\right)^2 B_{12} + \left(\frac{-1}{2} - \frac{\sqrt{3}}{12}\right)^2 C_{12} = \frac{-1}{8}$$

$$\Rightarrow \frac{13}{48}(B_{12}+C_{12}) - 2\left(\frac{1}{2}\right)\left(\frac{\sqrt{3}}{12}\right)(B_{12}-C_{12}) = \frac{-1}{8}$$

$$\Rightarrow 13(B_{12}+C_{12}) - 4\sqrt{3}(B_{12}-C_{12}) = -6 \quad -(3)$$

$4\sqrt{3}(2) + (3)$

3

$$\Rightarrow -11(B_{12}+C_{12}) = 6$$

$$B_{12}+C_{12} = \frac{-6}{11}, \quad B_{12}-C_{12} = \frac{-\sqrt{3}}{11}$$

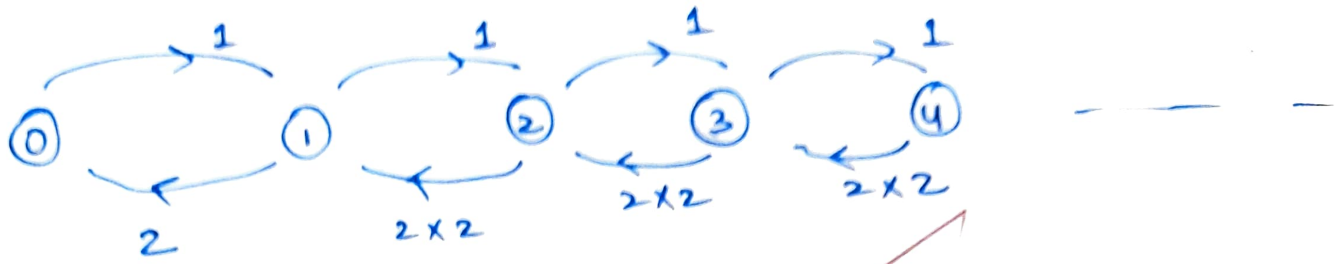
$$\Rightarrow B_{12} = \frac{-1}{22}(6+\sqrt{3}), \quad C_{12} = \frac{-1}{22}(6-\sqrt{3})$$

From (1): $A_{12} = \frac{6}{11}$

$$P_{12}(4) = \frac{6}{11} - \frac{1}{22}(6+\sqrt{3})e^{-2+\frac{1}{\sqrt{3}}} - \frac{1}{22}(6-\sqrt{3})e^{-2-\frac{1}{\sqrt{3}}}$$

$\boxed{P_{12}(4) = 0.44599}$: Probability that he is in city 1 at time 10 given he is in 2 at time 6.

⑤ (b)



(i) $S = 1 + \frac{1}{2} + \frac{1}{2} \times \frac{1}{4} + \frac{1}{2} \times \left(\frac{1}{4}\right)^2 + \frac{1}{2} \times \left(\frac{1}{4}\right)^3 + -$

$$S = 1 + \frac{1}{2}\left[1 + \frac{1}{4} + \left(\frac{1}{4}\right)^2 + \left(\frac{1}{4}\right)^3 + - - -\right]$$

$$S = 1 + \frac{1}{2} \times \frac{1}{1 - \frac{1}{4}}$$

$$S = 1 + \frac{1}{\cancel{2}_1} \times \frac{\cancel{4}2}{3}$$

$$S = 1 + \frac{2}{3} = \frac{5}{3}$$

$$P_0 = \frac{3}{5}$$

PNT

$P_n = C_n P_0$

$C_n = \frac{1}{2} \times \left(\frac{1}{4}\right)^{n-1}$

$P_n = \frac{3}{5} \times \frac{1}{2} \times \left(\frac{1}{4}\right)^{n-1}$ 3

$\boxed{P_n = \frac{3}{10} \times \left(\frac{1}{4}\right)^{n-1}}$

(ii)

from 0 to 1,

number ~~customers~~ go at a rate $\boxed{\lambda = 1}$

From 2 to 1

~~customer go~~

number go at a rate $\boxed{4}$

5 (c)

$f(x) = e^{-2|x|}$

$F(x) = \int_{-\infty}^{x} e^{-2|x|} \, dx$

if $x \leq 0$

$F(x) = \int_{-\infty}^{x} e^{2x} \, dx = \left[\frac{e^{2x}}{2} \right]_{-\infty}^{x} = \cancel{\frac{e^{2x}}{2}}$

$= \frac{e^{2x} - 0}{2} = \frac{1}{2} e^{2x}$

if $x > 0$

$$F(x) = \int_{-\infty}^{0} e^{-2|x|}\,dx + \int_{0}^{x} e^{-2|x|}\,dx$$

$$= \int_{-\infty}^{0} e^{2x}\,dx + \int_{0}^{x} e^{-2x}\,dx$$

$$= \left[\frac{e^{2x}}{2}\right]_{-\infty}^{0} + \left[\frac{e^{-2x}}{-2}\right]_{0}^{x}$$

$$= \frac{1}{2} + \left[\frac{1 - e^{-2x}}{2}\right]$$

$$= 1 - \frac{1}{2}e^{-2x}$$

$$F(x) = \begin{cases} \frac{1}{2}e^{2x}, & x \leq 0 \\ 1 - \frac{1}{2}e^{-2x}, & x > 0 \end{cases}$$

3

For $x \leq 0$,

$$u = \frac{1}{2}e^{2x}$$

$$2u = e^{2x}$$

$$x = \frac{1}{2}\ln(2u)$$

For $x > 0$

$$u = 1 - \frac{1}{2}e^{-2x}$$

$$\frac{1}{2}e^{-2x} = 1 - u$$

$$x = -\frac{1}{2}\ln(2(1-u))$$

$$F^{-1}(u) = \begin{cases} \frac{1}{2}\ln(2u), & u \leq \frac{1}{2} \\ -\frac{1}{2}\ln(2-2u), & u > \frac{1}{2} \end{cases}$$

for $u = 0.6$, $x = -\frac{1}{2}\ln(2-1.2) = -\frac{1}{2}\ln(0.8)$

$$\boxed{x = 0.11}$$

if $x > 0$

$$F(x) = \int_{-\infty}^{0} e^{-2|x|}\,dx + \int_{0}^{x} e^{-2|x|}\,dx$$

$$= \int_{-\infty}^{0} e^{2x}\,dx + \int_{0}^{x} e^{-2x}\,dx$$

$$= \left[\frac{e^{2x}}{2}\right]_{-\infty}^{0} + \left[\frac{e^{-2x}}{-2}\right]_{0}^{x}$$

$$= \frac{1}{2} + \left[\frac{1-e^{-2x}}{2}\right]$$

$$= 1 - \frac{1}{2}e^{-2x}$$

$$F(x) = \begin{cases} \frac{1}{2}e^{2x} & , x \leq 0 \\ 1-\frac{1}{2}e^{-2x} & , x > 0 \end{cases}$$

3

For $x \leq 0$,

$u = \frac{1}{2}e^{2x}$

$2u = e^{2x}$

$x = \frac{1}{2}\ln(2u)$

For $x > 0$

$u = 1 - \frac{1}{2}e^{-2x}$

$\frac{1}{2}e^{-2x} = 1-u$

$x = -\frac{1}{2}\ln(2(1-u))$

$$F^{-1}(u) = \begin{cases} \frac{1}{2}\ln(2u) & , u \leq \frac{1}{2} \\ -\frac{1}{2}\ln(2-2u) & , u > \frac{1}{2} \end{cases}$$

For $u = 0.6$, $x = -\frac{1}{2}\ln(2-1.2) = -\frac{1}{2}\ln(0.8)$

$$\boxed{x = 0.11}$$

6 ⓐ $m(t) = F(t) + \int_0^t m(t-x)\, f(x)\, dx$ For $t \le 2$

$X_i \sim U(0,2)$ $f(x) = \frac{1}{2}$, $0 < x < 2$.

$\int_0^x \frac{1}{2} dx$

$$F(x) = \begin{cases} 0, & x < 0 \\ \frac{x}{2}, & 0 \le x < 2 \\ 1 & x \ge 2 \end{cases}$$

$t - x = u$

$-dx = du$

$$m(t) = \frac{t}{2} + \int_0^t m(t-x) \frac{1}{2} dx$$

$$= \frac{t}{2} + \frac{1}{2}\int_0^t m(u)\, du$$

$$m'(t) = \frac{1}{2} + \frac{1}{2} m(t) = h(t) \quad \text{—} (1)$$

$$h'(t) = \frac{1}{2} m'(t) = \frac{1}{2} h(t)$$

$$\frac{h'(t)}{h(t)} = \frac{1}{2} \Rightarrow \ln(h(t)) = \frac{t}{2} + c$$

$$\Rightarrow h(t) = k e^{t/2}$$

Using (1), $\therefore$ $m(t) = 2\left(h(t) - \frac{1}{2}\right)$

$$= 2h(t) - 1$$

$$= 2ke^{t/2} - 1$$

$N(0) = 0 \Rightarrow m(0) = 0 \Rightarrow 0 = 2k - 1$

$$\Rightarrow k = \frac{1}{2}$$

$\therefore$ $m(t) = e^{t/2} - 1, \quad 0 \le t \le 2.$

(6) (b) We know $\lim_{t \to \infty} \frac{R(t)}{t} \to \frac{E(R)}{E(X)}$

$$E(X) = n\mu = 5n$$

$$E(R) = 500 + 2\left[(n-1)\mu + (n-2)\mu + \cdots + 2\mu + \mu + 0\right]$$

$$= 500 + 2\mu \times [1 + \cdots + (n-1)]$$

$$= 500 + 2\mu \times \frac{n(n-1)}{2}$$

$$= 500 + 5n(n-1)$$

We want to minimize $\dfrac{E(R)}{E(X)} = \dfrac{500 + 5n(n-1)}{5n}$

$$\frac{E(R)}{E(X)} = \frac{100 + n^2 - n}{n}$$

For minimum $\dfrac{d}{dn}\left(\dfrac{100 + n^2 - n}{n}\right) = 0$

$$\frac{d}{dn}\left(\frac{100}{n} + n - 1\right) = 0$$

3

$$\frac{-100}{n^2} + 1 = 0$$

$$\frac{100}{n^2} = 1$$

$$\boxed{n = 10}$$

- Value of n is 10 which minimize the long-run cost per unit time

6 (e) $X_n = 0 \Rightarrow X_{n+1} = 0$

$X_n > 0$ $\quad X_{n+1} = \begin{cases} X_n + 1 & \text{wp } \frac{1}{2} \\ X_n - 1 & \text{wp } \frac{1}{2} \end{cases}$

$E(X_{n+1} \mid X_0, \ldots, X_n) =$

$E(X_{n+1} \mid X_0 = x_0, \ldots, X_n = x) = (x+1) \times \frac{1}{2} + (x-1) \times \frac{1}{2}$

$= x$

$\therefore E(X_{n+1} \mid X_0, \ldots, X_n) = X_n$

$\therefore (X_n)$ is a martingale.