Poisson Process & related distributions:

Poisson Process $N(t) \sim P.P.(\lambda)$

(i) $N(t)$ has indep increment

(ii) $N(t)$ " stationary "

(iii) $P(N(h)=1) = \lambda h + o(h)$

$h \to$ small $\quad P(N(h) \geq 2) = o(h)$

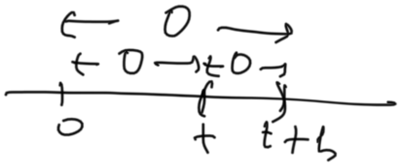
$P(N(h)=0) = 1 - \lambda h + o(h)$

$$P_n(t) = P(N(t)=n) = \frac{e^{-\lambda t}(\lambda t)^n}{n!}, \quad n=0,1,2,\ldots$$

Sol $\quad h \to$ small

$$P_0(t+h) = P(N(t+h)=0)$$

$$= P(\{N(t)=0\} \cap \{N(t,t+h]=0\})$$



$$= P(N(t)=0)\, P(N(t,t+h]=0)$$

$N(t,t+h] = N(t+h) - N(t)$

| indep increment

$$= P(N(t)=0)\, P(N(h)=0)$$

$$= (1 - \lambda h + o(h))\, P_0(t)$$

$$\Rightarrow \quad \frac{P_0(t+h) - P_0(t)}{h} = -\frac{\lambda \cancel{h} P_0(t)}{\cancel{h}} + P_0(t)\frac{o(h)}{h}$$

Take limit $h \to 0$

$$\frac{dP_0(t)}{dt} = -\lambda P_0(t)$$

$$P_0(t) = C\, e^{-\lambda t}$$

$$P_0(t) = e^{-\lambda t}$$

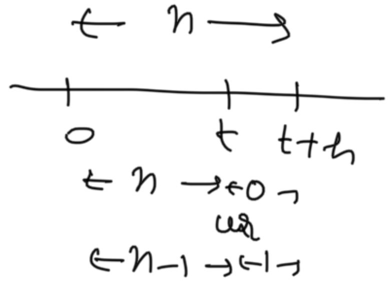
$P_0(0) = 1$

$\Rightarrow C = 1$

$$P_n(t+h) = P(N(t+h) = n)$$

$$= P\big(\{N(t) = n\} \cap \{N(t, t+h] = 0\}$$

$$\text{or } \{N(t) = n-1\} \cap \{N(t, t+h] = 1\}\big)$$



$$= P_n(t) \times \big(1 - \lambda h + o(h)\big) + P_{n-1}(t)\,\big(\lambda h + o(h)\big)$$

$$\Rightarrow \frac{P_n(t+h) - P_n(t)}{h} = -\lambda\,\big(P_n(t) - P_{n-1}(t)\big) + (\ast)\frac{o(h)}{h}$$

Limit $h \to 0$

$$\Rightarrow \frac{dP_n(t)}{dt} = -\lambda\,\big(P_n(t) - P_{n-1}(t)\big)$$

$$P_0(t) = e^{-\lambda t}$$

assume $\quad P_{n-1}(t) = \dfrac{e^{-\lambda t}(\lambda t)^{n-1}}{(n-1)!}$

$$\rightarrow \frac{dP_n(t)}{dt} = -\lambda\left(P_n(t) - \frac{e^{-\lambda t}(\lambda t)^{n-1}}{(n-1)!}\right)$$

$$e^{\lambda t}\frac{dP_n(t)}{dt} + \lambda e^{\lambda t}P_n(t) = \frac{\lambda^n t^{n-1}}{(n-1)!}$$

$$\frac{d\left(e^{\lambda t}P_n(t)\right)}{dt} = \frac{\lambda^n}{(n-1)!}t^{n-1}$$

$$e^{\lambda t}P_n(t) = \frac{\lambda^n}{(n-1)!}\int t^{n-1}\,dt \quad \rightarrow \frac{t^n}{n}$$

$$e^{\lambda t}P_n(t) = \frac{(\lambda t)^n}{n!}$$

$$\Rightarrow P_n(t) = \frac{e^{-\lambda t}(\lambda t)^n}{n!}$$

— x —

Interarrival and Waiting time distn :

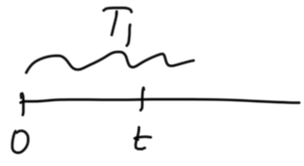
$N(t)$ = count # of events occurring in $(0,t]$

$N(t) \sim P.P(\lambda)$ $\quad P(N(t)=n) = \frac{e^{-\lambda t}(\lambda t)^n}{n!}$, $n=0,1,2,\ldots$

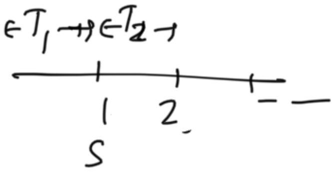
T_1 : time of the first event

T_n : time elapsed between $(n-1)^{st}$ and n^{th} event.

$T_1 > t \equiv N(t)=0$



$$P(T_1 > t) = P(N(t) = 0)$$
$$= e^{-\lambda t}$$
$$\Rightarrow \quad T_1 \sim \exp(\lambda)$$



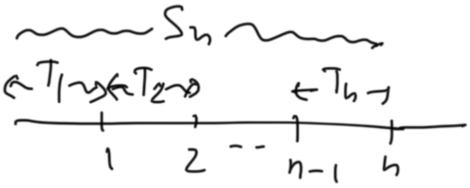
$$T_2 > t \mid T_1 = s \equiv N(s, t+s] = 0$$

$$P(T_2 > t \mid T_1 = s) = P(N(s, t+s] = 0)$$
$$= P(N(t) = 0)$$
$$= e^{-\lambda t}$$

$$\rightarrow P(X > t) = E[P(X > t \mid Y)]$$

$$P(T_2 > t \mid T_1) = e^{-\lambda t}$$

$$P(T_2 > t) = E(P(T_2 > t \mid T_1)) = e^{-\lambda t}$$
$$T_2 \sim \exp(\lambda)$$



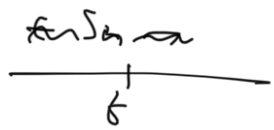
similarly $\quad T_n \sim \exp(\lambda)$

$$S_n = T_1 + T_2 + \dots + T_n \sim \text{Gamma}(n, \lambda)$$

$$M_{T_n}(t) = \left(1 - \frac{t}{\lambda}\right)^{-1}$$

$$M_{S_n}(t) = \prod_{i=1}^{n} M_{T_i}(t) = \left(1 - \frac{t}{\lambda}\right)^{-n}$$

$$S_n > t \equiv N(t) \leq n-1$$



$$P(S_n > t) = P(N(t) \leq n-1)$$

$P(S_n > t) = P(N(t) \leq n-1)$

$$= \sum_{i=0}^{n-1} P(N(t)=i)$$

cdf of S_n

$$F_{S_n}(t) = P(S_n \leq t) = 1 - e^{-\lambda t} \sum_{i=0}^{n-1} \frac{(\lambda t)^i}{i!}$$

pdf of S_n

$$f_{S_n}(t) = \frac{d}{dt} F_{S_n}(t) = \frac{\lambda^n}{(n-1)!} e^{-\lambda t} t^{n-1}, \quad t>0, \lambda>0, n=0,1,2,\ldots$$

$$E(S_n) = \frac{n}{\lambda}, \quad V(S_n) = \frac{n}{\lambda^2}.$$

—x—

P1

$$M_{X_i}(t) = e^{\lambda_i(e^t-1)}, \quad i=1,2$$

$\text{Poiss}(\lambda_1) \sim \underbrace{X_1, X_2}_{\text{indep}} \sim \text{Poiss}(\lambda_2)$

$$M_{X_1+X_2}(t) = M_{X_1}(t) M_{X_2}(t) = e^{(\lambda_1+\lambda_2)(e^t-1)}$$

$$X_1 + X_2 \sim \text{Poiss}(\lambda_1+\lambda_2)$$

—x—

P2 If $N(t) \sim P.P(\lambda)$ $\quad 1 \to p$, $\quad 2 \to 1-p=q$

$$N(t) = N_1(t) + N_2(t)$$

then $\Rightarrow$ indep.

$$N_1(t) \sim P.P(\lambda p)$$

$$N_2(t) \sim P.P(\lambda q)$$

Sol $P(N_1 = n, N_2 = m)$

$$= \sum_i P(N_1 = n, N_2 = m \mid N = i) P(N=i)$$

$$= P(N_1 = n, N_2 = m \mid N = n+m) P(N = n+m)$$

$$= {}^{n+m}C_n\, p^n q^m \times \frac{e^{-\lambda t}(\lambda t)^{n+m}}{(n+m)!}$$

$$= \frac{\cancel{(n+m)!}}{n!\, m!} \frac{e^{-\lambda(p+q)t}(\lambda p t)^n (\lambda q t)^m}{\cancel{(n+m)!}}$$

$$= \frac{e^{-\lambda p t}(\lambda p t)^n}{n!} \times \frac{e^{-\lambda q t}(\lambda q t)^m}{m!}$$

—x—

Binomial distribution also arises in context of PP:

$$N(t) \sim P.P.(\lambda)$$

Then for $0 < u < t$ and $0 \le k \le n$,

$$[N(u) \mid N(t) = n] \sim Bin\left(n, \frac{u}{t}\right)$$

Sol For $0 \le k \le n$

$$P(N(u) = k \mid N(t) = n) = \frac{P(N(u) = k, N(t) = n)}{P(N(t) = n)}$$

$$= \frac{P(N(0,u] = k, N(u,t] = n-k)}{P(N(t) = n)}$$

$$= \frac{\frac{\cancel{e^{-\lambda u}}(\lambda u)^k}{k!} \times \frac{\cancel{e^{-\lambda(t-u)}}(\lambda(t-u))^{n-k}}{(n-k)!}}{\frac{\cancel{e^{-\lambda t}}(\lambda t)^n}{n!}}$$

$$= \frac{n!}{k!\,(n-k)!} \quad \frac{u \qquad (t-u)}{t^n}$$

$$= \binom{n}{k} \left(\frac{u}{t}\right)^k \left(1-\frac{u}{t}\right)^{n-k}, \quad k = 0, 1, \dots, n.$$

— x —

indep $\begin{cases} X \sim \text{Gamma}(\alpha, \lambda) \\ Y \sim \text{Gamma}(\beta, \lambda) \end{cases}$ show indep $\quad U = \frac{X}{X+Y}, \; V = X+Y$

, then $U \sim \text{beta}(\alpha, \beta)$, i.e., $f_U(u) = \frac{\Gamma(\alpha+\beta)}{\Gamma\alpha\,\Gamma\beta} u^{\alpha-1} (1-u)^{\beta-1}$, $0 < u < 1$

Sol (Hint)

$$u = \frac{x}{x+y}, \quad v = x+y$$

$$J = \frac{\partial(x,y)}{\partial(u,v)} = v \qquad |J| = v$$

$$f_{U,V}(u,v) = f_{X,Y}(x,y)\,|J|$$

— x —

indep $\begin{cases} S_m = \sum_{i=1}^{m} T_i \sim \text{Gamma}(m, \lambda) \\ S_n - S_m \sim \text{Gamma}(n-m, \lambda) \end{cases}$ $, \quad S_n = S_m + (S_n - S_m)$

$$U = \frac{S_m}{S_n} = \frac{S_m}{S_m + (S_n - S_m)} \sim \text{beta}(m, n-m)$$

U and S_n are indep.

— x —

Example: Suppose customer streams into a drug store at a constant avg. rate of 15 per hr. (PP)

The pharmacy opens its door at 8 a.m. ...

... opens ... at 8 am and closes at 8 pm. Given that the 100th customer on a particular day walked in at 2pm, we want to know what is the prob. that 50th customer came before noon. $m = 50 \quad n=100$

Sol

$$P(S_m < 4 \mid S_n = 6)$$

$$= P\left(\frac{S_m}{S_n} < \frac{4}{6} \,\middle|\, S_n = 6 \right)$$

$$= P\left(\frac{S_m}{S_n} < \frac{4}{6} \right)$$

| $\frac{S_m}{S_n}$ and S_n are indep

$$\overset{\text{C.L.T.}}{\simeq} P\left(Z < \frac{\frac{4}{6} - \frac{1}{2}}{\sqrt{0.0025}} \right)$$

$$= P(Z < 3.33)$$

$$= 0.9997$$

$\frac{S_m}{S_n} \sim \text{beta}(50, 100-50)$

$\equiv \text{beta}(50, 50)$

$\sim \text{beta}(\alpha, \beta)$

mean $= \frac{\alpha}{\alpha+\beta} = \frac{1}{2}$

var $= \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$

$= 0.0025$

$Z \sim N(0,1)$

— x —

→ $N(t) \sim$ P.P. (λ) and one event take place in interval $[0, t]$. Let Y r.v. describing the time of occurence of this Poisson event, has a continuous uniform dist on the interval $(0, t]$

Sol

$$Y = [T_1 \mid N(t) = 1]$$

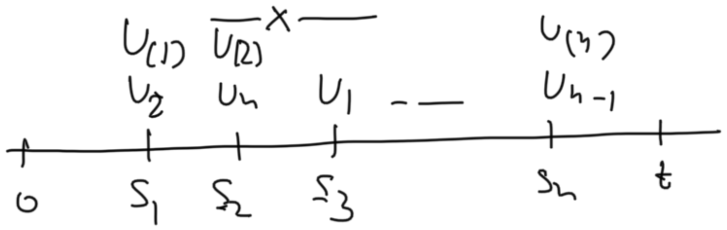
For exact

$$P(Y \le x) = P(T_1 \le x \mid N(t)=1)$$

$$= \frac{P(N(0,x]=1,\ N(x,t]=0)}{P(N(t)=1)}$$

$$= \frac{e^{-\lambda x}\,\lambda x \times e^{-\lambda(t-x)}}{e^{-\lambda t}\,\lambda t}$$

$$= \frac{x}{t}, \qquad 0 < x < t$$



complete randomness property

(*) Give $N(t)=n$, n arrival times $S_1, \ldots, S_n$ have the same dist^n as the order statistics corresponding to n indep. rv's which are uniformally distributed in $(0,t)$.

$Y_1, \ldots, Y_n$ n rv's i.i.d. cont rv's

$Y_{(1)}, \ldots, Y_{(n)}$ are order statistics corresponding to $Y_1, \ldots, Y_n$

$y_1 < y_2 \cdots < y_n$

$$f_{Y_{(1)}, \ldots, Y_{(n)}}(y_1, \ldots, y_n) = n! \prod_{i=1}^{n} f(y_i)$$

$y_1 < y_2$

$$f_{Y_{(1)}, Y_{(2)}}(y_1, y_2)$$

$$= f_{Y_1,Y_2}(y_1,y_2) + f_{Y_2,Y_1}(y_1,y_2)$$

U_i

$|||$

If $Y_i \sim U(0,t)$

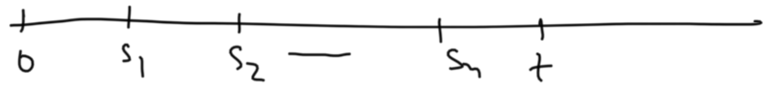
$$f_{U_{(1)},\dots,U_{(n)}}(y_1,\dots,y_n) = \frac{n!}{t^n}, \quad 0<y_1<y_2<\dots<y_n<t$$

proof

$$[S_1, S_2, \dots, S_n \mid N(t)=n]$$

density $\quad s_1<s_2<s_3<\dots<s_n<t$

$$f(s_1,s_2,\dots,s_n \mid t) = \frac{f(s_1,\dots,s_n,t)}{P(N(t)=n)}$$



$S_1 = s_1,\ S_2 = s_2, \dots, S_n = s_n,\ N(t)=n$

$\equiv\ T_1 = s_1,\ T_2 = s_2 - s_1, \dots, T_n = s_n - s_{n-1},\ T_{n+1} > t - s_n$

T_i's are i.i.d.

$$= \frac{\lambda e^{-\lambda s_1}\, \lambda e^{-\lambda(s_2-s_1)} \cdots \lambda e^{-\lambda(s_n-s_{n-1})}\, e^{-\lambda(t-s_n)}}{\dfrac{e^{-\lambda t}(\lambda t)^n}{n!}}$$

$$= \frac{n!}{t^n}$$

—x—

Ex. For $U(0,1)$ dist, the rv's $V_1 = \dfrac{U_{(i)}}{U_{(j)}}$ and

$V_2 = U_{(j)}$, $1 \le i \le j \le n$ are statistically indep. , with

$$V_1 \sim \text{beta}(i, j-i)$$

$$V_2 \sim \text{beta}(j, n-j+1)$$

— x —

Example 1 btw 9 AM – 5 pm, you recieve emails at the av constant rate of one per 10 min. You left for lunch at 12 noon, and when you returned at 1 pm, you found 9 emails. Assume that emails arrive according to a P.P. $\lambda = 6$ per hr

1) What is the prob. that 5 email arrived before 12:30 pm.

Sol $\quad [S_5 \mid N(1) = 9] = U_{(5)} \sim \text{beta}(5, 9-5+1)$
$\equiv \text{beta}(5,5)$

$$P(S_5 \le \tfrac{1}{2} \mid N(1) = 9) = P(U_{(5)} \le \tfrac{1}{2})$$

$$\overset{CLT}{=} P(Z \le 0) = \frac{1}{2}$$

2) What is the expected gap btw the times that the 3rd and 7th email arrived.

$$E(S_7 - S_3 \mid N(1) = 9)$$

$$= E(S_7 \mid N(1) = 9) - E(S_3 \mid N(1) = 9)$$

$$= E(U_{(7)}) - E(U_{(3)})$$

$$= \frac{7}{10} - \frac{3}{10} = 0{,}4 \text{ hrs}$$

$$= 24 \text{ min}$$

$U_{(7)} \sim \text{beta}(7,3)$

$U_{(3)} \sim \text{beta}(3,7)$

—X—

Compound P.P. !

$$N(t) \begin{cases} \boxed{\text{Bus 1}} \; Y_1 \\ \vdots \\ \boxed{\text{Bus}} \; Y_{N(t)} \end{cases}$$

form $X(t)$

$$X(t) = \sum_{i=1}^{N(t)} Y_i$$

compound P.P.

indep. $\left\{ \begin{array}{l} Y_i\text{'s are i.i.d} \\ N(t) \sim \text{P.P.}(\lambda) \end{array} \right.$

$$E(X(t)) = E\left(\sum_{i=1}^{N(t)} Y_i\right)$$

$$= E\left[E\left(\sum_{i=1}^{N(t)} Y_i \,\middle|\, N(t)\right)\right]$$

$$\boxed{\begin{aligned} E\left(\sum_{i=1}^{N(t)} Y_i \,\middle|\, N(t) = n\right) &= E\left(\sum_{i=1}^{n} Y_i\right) \\ &= n\,E(Y_1) \end{aligned}}$$

$$= E\left(N(t)\,E(Y_1)\right)$$

$$= E(Y_1)\, E(N(t)) \rightarrow \lambda t$$

$$= \lambda t \, E(Y_1)$$

$$\operatorname{Var}(X(t)) = E\big(\underbrace{\operatorname{Var}(X(t) \mid N(t))}\big) + \operatorname{Var}\big(\underbrace{E(X(t) \mid N(t))}\big)$$

$$\operatorname{Var}\Big(\sum_{i=1}^{N(t)} X_i \,\Big|\, N(t)\Big) \qquad\qquad N(t)\, E(Y_1)$$

$$\downarrow$$

$$N(t) \operatorname{Var}(Y_1)$$

$$= E(N(t)) \operatorname{Var}(Y_1) + E^2(Y_1) \operatorname{Var}(N(t))$$

$$\lambda t \qquad\qquad\qquad\qquad \lambda t$$

$$= \lambda t \left[\operatorname{Var}(Y_1) + E^2(X_1) \right]$$

$$= \lambda t \left[E(Y_1^2) - \cancel{E^2(X_1)} + \cancel{E^2(Y_1)} \right]$$

$$= \lambda t \, E(Y_1^2)$$

Example: (1) Customers arrive at the ATM in accordance with a P.P. with a rate 12 per hr. The amt of money withdrawn on each

... is a r.v. with mean \$ 30 and sd \$ 50. The machine is in use for 15 hr daily. Approximate the prob. that the total daily withdrawal is less than \$ 6000.

Sol iid. Y_i $\quad E(Y_1) = 30$, $V(Y_1) = (50)^2$

$$E(Y_1^2) = (50)^2 + (30)^2$$

$$X(15) = \sum_{i=1}^{N(15)} Y_i \qquad \lambda = 12$$

$$E(X(15)) = 12 \times 15 \times 30$$

$$V(X(15)) = 12 \times 15 \times [(50)^2 + (30)^2]$$

$$P(X(15) \leq 6000)$$

$$\overset{CLT}{=} P\left(Z \leq \frac{6000 - 5400}{\sqrt{612000}}\right)$$

$$= P(Z \leq 0.767)$$

$$= \Phi(0.767) = 0.78$$

(2) Suppose that families migrate to an area at a Poisson rate $\lambda = 2$ per week. If the # of people in each family is indep and takes on values 1, 2, 3, 4 with resp. prob $\frac{1}{6}, \frac{1}{3}, \frac{1}{3}, \frac{1}{6}$. Find the expected value and var. of the ...

individual migrating to the area during a fixed five week period.

Sol

$$E(X_1) = 1 \times \frac{1}{6} + 2 \times \frac{1}{3} + 3 \times \frac{1}{3} + 4 \times \frac{1}{6}$$

$$E(X_1^2) = 1^2 \times \frac{1}{6} + 2^2 \times \frac{1}{3} + 3^2 \times \frac{1}{3} + 4^2 \times \frac{1}{6}$$

$$E(X(5)) = 2 \times 5 \times \frac{5}{2} = 25$$

$$V(X(5)) = 2 \times 5 \times \frac{43}{6} = \frac{215}{3}$$

$$P(X(5) \geq 15) \overset{CLT}{=} P\left(Z \geq \frac{15-25}{\sqrt{\frac{215}{3}}}\right)$$

$$= 1 - \Phi\left(\frac{15-25}{\sqrt{\frac{215}{3}}}\right)$$

—x—