PRINT your name and student ID: ______________________________

**4. Stable Marriage**

Consider the set of men $M = \{1, 2, 3\}$ and the set of women $W = \{A, B, C\}$ with the following preferences.

| Men | Women | | |
|---|---|---|---|
| 1 | A | B | C |
| 2 | B | A | C |
| 3 | A | B | C |

| Women | Men | | |
|---|---|---|---|
| A | 2 | 1 | 3 |
| B | 1 | 2 | 3 |
| C | 1 | 2 | 3 |

**Run the male propose-and-reject algorithm on this example. How many days does it take and what is the resulting pairing?** (Show your work)

The algorithm takes 3 days to produce a matching. The resulting pairing is {(A, 1), (B, 2), (C, 3)}

| Woman | Day 1 | Day 2 | Day 3 |
|---|---|---|---|
| A | 1,3 | 1 | 1 |
| B | 2 | 2,3 | 2 |
| C | | | 3 |

**7. Stable Marriage**

Below are the observed proposals from the traditional male propose-and-reject algorithm.

| Day | Women | Men |
|---|---|---|
| 1 | A | - |
| | B | 1, 2 |
| | C | 3 |
| | D | 4 |
| 2 | A | - |
| | B | 1 |
| | C | 2, 3 |
| | D | 4 |
| 3 | A | - |
| | B | 1 |
| | C | 2 |
| | D | 3, 4 |
| 4 | A | - |
| | B | 1, 4 |
| | C | 2 |
| | D | 3 |
| 5 | A | - |
| | B | 4 |
| | C | 2 |
| | D | 1, 3 |
| 6 | A | - |
| | B | 3, 4 |
| | C | 2 |
| | D | 1 |
| 7 | A | - |
| | B | 3 |
| | C | 2, 4 |
| | D | 1 |
| 8 | A | 2 |
| | B | 3 |
| | C | 4 |
| | D | 1 |

After the algorithm terminates, only B is married to the man she likes the most. Also, it is known that A has the same preferences as B and every man likes C better than A.

**Reconstruct the complete preference lists of men and women given the information above.** (You do not have to show work.)

| Men | Preferences |
|---|---|
| 1 | > > > |
| 2 | > > > |
| 3 | > > > |
| 4 | > > > |

| Women | Preferences |
|---|---|
| A | > > > |
| B | > > > |
| C | > > > |
| D | > > > |

We know men propose starting from the top of their lists, therefore we can fill their preference lists with the series of women they proposed to. We also know that if a woman receives multiple proposals, she tells the man she likes the most to comeback the next day. So, if men 1 and 2 proposes to the same woman on one day, and 1 comes back to her the next day but 2 doesn't, it means the woman prefers 1 over 2. These reasonings give us the partially filled preference lists.

| Men | Preferences |
|---|---|
| 1 | $B > D > ? > ?$ |
| 2 | $B > C > A > ?$ |
| 3 | $C > D > B > ?$ |
| 4 | $D > B > C > ?$ |

| Women | Preferences |
|---|---|
| A | $? > ? > ? > ?$ |
| B | $3 > 4 > 1 > 2$ |
| C | $4 > 2 > 3$ |
| D | $1 > 3 > 4$ |

By process of elimination we know who are in 2, 3, and 4's last spots. C and A are both missing from 1's list, but since every man likes C better than A, 1 must too. For women, A has the same preferences as B so we can just copy B's list to A. 1 and 2 are missing from C and D's lists, respectively, and unlike men's preferences, these missing men can be anywhere in the women's lists. Fortunately, B being the only woman that is married to her top man means C and D must not like 4 and 1, the men they are married to, the most, and we can put the missing men in their top spots.

| Men | Preferences |
|---|---|
| 1 | $B > D > C > A$ |
| 2 | $B > C > A > D$ |
| 3 | $C > D > B > A$ |
| 4 | $D > B > C > A$ |

| Women | Preferences |
|---|---|
| A | $3 > 4 > 1 > 2$ |
| B | $3 > 4 > 1 > 2$ |
| C | $1 > 4 > 2 > 3$ |
| D | $2 > 1 > 3 > 4$ |

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**9. Does "No" Matter? (15 points)**

Consider an alternative to the Propose and Reject algorithm (with no rejections), where women take turns choosing the best available husband from the remaining unchosen men. On day 1, the oldest woman chooses her most preferred man, and marries him. On day $k$, the $k$-th eldest woman chooses her most preferred choice from the remaining unmarried men, and marries him. **No matter what the preferences are, this process always results in a stable matching.**

Mark one: TRUE or FALSE.

The statement is $\boxed{\text{FALSE}}$

Consider the set of men $M = \{1,2\}$ and women $W = \{A,B\}$ with the following preferences.

| Men | Women |
|---|---|
| 1 | B > A |
| 2 | A > B |

| Women | Men |
|---|---|
| A | 1 > 2 |
| B | 1 > 2 |

Running the alternative Propose and Reject algorithm on this set yields the pairings: $(A,1),(B,2)$. However, we can see that Man 1 prefers Woman B and vice versa, forming a rogue pair. Therefore, the pairing is not stable. We found a counterexample, so the statement is $\boxed{\text{FALSE}}$

**10.** ***n* Matchings (15 points)**

**For all positive $n$, it is always possible to construct a set of preferences for $n$ women and $n$ men such that at least $n$ distinct stable matchings are possible.**

Mark one: TRUE or FALSE.

The statement is $\boxed{\text{TRUE}}$

Consider preference lists where man $m$'s $j$-th preference (zero-based) is woman $(m+j) \bmod n$, and woman $w$'s $j$-th preference (zero-based) is man $(w+j+1) \bmod n$, $\forall m,w,j \in \mathbb{Z}, 0 \le m,w,j < n$. For example, the preference lists for $n=5$ are shown below.

| Men | Preferences |
|---|---|
| 0 | $0>1>2>3>4$ |
| 1 | $1>2>3>4>0$ |
| 2 | $2>3>4>0>1$ |
| 3 | $3>4>0>1>2$ |
| 4 | $4>0>1>2>3$ |

| Women | Preferences |
|---|---|
| 0 | $1>2>3>4>0$ |
| 1 | $2>3>4>0>1$ |
| 2 | $3>4>0>1>2$ |
| 3 | $4>0>1>2>3$ |
| 4 | $0>1>2>3>4$ |

**Claim:** A matching where all men are paired with the $j$-th women in their preference lists is stable.

**Proof:** We will call this matching $M_j$. Notice that all men are matched with their $j$-th woman, and all women are matched with their $(n-j-1)$-th man. Formally, we must show that there cannot be a rogue couple. From the way men's preferences are generated, we can find which man $m$ has woman $w$ in his $j$-th spot,

$$\begin{aligned} &(m+j) \bmod n = w \\ &\qquad m = (w-j) \bmod n. \end{aligned} \tag{1}$$

From this, we find a set $R_w$ of men who could form a rogue couple with a woman $w$, $0 \le w < n$. They must prefer $w$ over their partners in $M_j$. In other words, they must put $w$ in a spot $k < j$.

$$R_w = \{m' \in \mathbb{Z} \mid m' = (w-k) \bmod n, \forall k \in \mathbb{Z}, 0 \le k < j\}. \tag{2}$$

Similarly, we work on the women's preference formula to find the rank $r$ of a man $m$ in woman $w$'s list.

$$\begin{aligned} &(w+r+1) \bmod n = m \\ &\qquad r = (m-w-1) \bmod n \end{aligned} \tag{3}$$

Now, we substitute $m$ in Equation (3) with $(w-k) \bmod n$ to find out how woman $w$ ranks each $m' \in R_w$,

$$\begin{aligned} r_{m'} &= (((w-k) \bmod n) - w - 1) \bmod n \\ r_{m'} &= (w-k-w-1) \bmod n = (-k-1) \bmod n \\ r_{m'} &= n-1-k \qquad (0 \le k < j < n) \end{aligned} \tag{4}$$

Substituting $m$ in Equation (3) with $(w-j) \bmod n$, we get the rank of $w$'s current partner in her preference list,

$$\begin{aligned} r_m &= (((w-j) \bmod n) - w - 1) \bmod n \\ r_m &= n-1-j \end{aligned} \tag{5}$$

Since $r_m = n-1-j < r_{m'} = n-1-k, \forall k \in \mathbb{Z}, 0 \le k < j$, woman $w$ likes her partner better than all men who like her better than their partners. Therefore, a rogue couple cannot exists, and the matching $M_j$ is stable.

Because there are $n$ distinct $j$'s in range $0 \le j < n$, the proposed preference lists have at least $n$ possible stable matchings. $\square$

Alternatively, one could extend the Pairing Up example discussed in Discussion 3W, since it is guaranteed to have at least $2^{\lfloor n/2 \rfloor}$ stable matchings, which is greater than or equal to $n$ for all $n \ge 2$. To generalize the method to support odd $n$, it is sufficient to give preference lists for $n = 2$ and $n = 3$ and prove that we can get to higher $n$'s by dividing a person to a block of $2 \times 2$ anti-soulmates using strong induction on $n$. In this case, there will be at least $2^{\lfloor n/2 \rfloor}$ and $2^{\lfloor n/2 \rfloor} + 1$ matchings for even and odd $n$, respectively.

$n = 2$

| Men | Preferences |
|---|---|
| 0 | $0 > 1$ |
| 1 | $1 > 0$ |

| Women | Preferences |
|---|---|
| 0 | $1 > 0$ |
| 1 | $0 > 1$ |

$n = 3$

| Men | Preferences |
|---|---|
| 0 | $0 > 1 > 2$ |
| 1 | $1 > 0 > 2$ |
| 2 | $2 > 0 > 1$ |

| Women | Preferences |
|---|---|
| 0 | $1 > 0 > 2$ |
| 1 | $0 > 1 > 2$ |
| 2 | $2 > 0 > 1$ |

# Section 3: Free-form Problems (65 points)

**11. Bieber Fever (35 points)**

In this world, there are only two kinds of people: people who love Justin Bieber, and people who hate him. We are searching for a stable matching for everyone. The situation is as follows:

- For some $n \geq 5$, there are $n$ men, $n$ women, and one Justin Bieber[1].
- Men can be matched with women; or anyone can be matched with Justin Bieber.
- Everyone is either a Hater or a Belieber. Haters want to be matched with anyone but Justin Bieber. Beliebers really want to be matched with Justin Bieber but don't mind being matched with other people.
- Men and women still have preference lists, as usual, but if they are a Belieber, Justin Bieber is always in the first position. If they are a hater, Justin Bieber is always in the last position.
- Justin Bieber desires to have 10 individuals matched with him (to party forever). As Justin Bieber is a kind person and wishes to be inclusive, he wishes to have exactly 5 women and 5 men in his elite club.
- Justin Bieber also has a preference list containing all $2n$ men and women.

A stable matching is defined as follows:

- Justin Bieber has 10 partners, of which 5 are men and 5 are women.
- All men and women not matched up with Justin Bieber are married to someone of the opposite gender.
- No rogue couples exist; i.e., there is no man M and woman W such that M prefers W to his current wife, and W prefers M to her current husband.
- No Hater is matched with Justin Bieber.
- There is no man who (1) is not matched with Justin Bieber; and (2) who is preferred by Justin Bieber over one of his current male partners; and (3) who prefers Justin Bieber over his wife. And similarly for women vis-a-vis Justin Bieber relative to their husbands and Justin Bieber's female partners.

(a) (5 points) **Show that there does not necessarily exist a stable matching.**
Suppose that there are 5 men and 5 women, all haters. Then Justin Bieber certainly cannot be matched with 5 men and 5 women without being matched with a hater. Therefore a stable matching does not exist.

[1] For the purposes of this problem, Justin Bieber is neither male nor female.

(b) (10 points) **Provide an "if-and-only-if" condition for whether a stable matching exists.** (*No need to prove anything in this part. That comes in later parts of this question.*)

A stable matching exists if and only if there are at least 5 male Beliebers and at least 5 female Beliebers.

(c) (5 points) **Is Justin Bieber guaranteed to always get his Bieber-optimal group if a stable matching exists?** (Bieber-optimal means that he gets the best possible group that could be matched to him in any stable matching.)

Yes. Consider the group which consists of Justin Bieber's favorite 5 male Beliebers, and favorite 5 female Beliebers. If a stable matching exists, Justin Bieber must be matched with this group. Suppose towards a contradiction that he is matched with some other set of 5 male Beliebers. Then he is matched with some $M$ who is not in his top 5 (i.e. M is ranked 6th or worse in Justin Bieber's preferences among male Beliebers), and he is not matched with some $M^*$ who is in his top 5. But Justin prefers $M^*$ to $M$, and $M^*$ is a Belieber, violating condition 5. Thus this matching is not stable. The same reasoning follows symmetrically for females; therefore, there is only one group that Justin Bieber can possibly be matched with, and thus any stable matching is Justin Bieber-optimal.

(d) (10 points) **Give an algorithm which finds a stable matching if the condition you gave in (b) holds. Argue why this algorithm works.**

First, take Justin Bieber's 5 favorite male Beliebers and his 5 favorite female Beliebers, and match them with him. For example, suppose that Justin Bieber's preference list is A B 1 2 C D E F 3 4 5 G 6 H 7 I 8 9, and that A, B, D, E, G, H, I, 1, 3, 4, 7, 8, and 9 are Beliebers. Then we would match Justin Bieber with A, B, D, E, G and 1, 3, 4, 7, 8. Remove Justin Bieber and his crew from all remaining preference lists, and run the propose and reject algorithm on everyone else ($n-5$ men and $n-5$ women). Note that Justin Bieber's "favorite 5 male Beliebers" must exist, because there are at least 5 male Beliebers, and similarly for females. Now let us check the conditions. Certainly Justin Bieber is matched with 5 men and 5 women, none of whom are Haters, by construction (conditions 1 and 4). Because we matched Justin Bieber with his 5 favorite male Beliebers, there is no Belieber who is not matched with Justin Bieber who Justin Bieber prefers over one of his current partners (condition 5), and similarly for women. Finally, because we ran the propose-and-reject algorithm on the remaining people, we are guaranteed to have a matching with no rogue couples (conditions 2 and 3). Note that the definition of rogue couple can only apply to people who are not matched with Justin Bieber, so we don't need to concern ourselves with rogue couples involving anyone in Justin Bieber's crew.

For this problem, we saw several examples of incorrect algorithms. Algorithms that had two stages (first Justin Bieber collecting his crew, and then the rest running the propose-and-reject algorithm) generally worked. The algorithms which tried to run the propose-and-reject algorithm on Justin Bieber and everyone else "in parallel" were often incorrect in subtle ways. We also saw many algorithms which were not precisely stated. For example, students had men proposing to Justin Bieber without specifying how Justin Bieber should process those proposals, or other things like this.

In addition, although many students gave correct algorithms, many students did not give adequate arguments for stability. In particular, trying to prove stability for a single-stage algorithm is quite involved, because the old proofs no longer apply (one would essentially have to prove everything from scratch), and very few students gave correct arguments for such an algorithm. There were also other errors; for example, some students tried to use part c) to prove that Justin Bieber always gets his optimal group, but this is circular reasoning (this is a stable matching, therefore Justin Bieber gets his optimal group, therefore this is a stable matching).

(e) (5 points) **Prove that a stable matching cannot exist if the condition you gave in (b) does not hold.** Suppose, without loss of generality, that there are less than 5 male Beliebers. Then, if Justin Bieber is matched with 5 males, at least one must be a hater, so the matching is unstable; and otherwise, Justin Bieber is not matched with 5 males, so the matching is also unstable. Therefore a stable matching does not exist.

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**13. (optional) Matchmaking Cruise (20 points)**

Mr. and Mrs. Matchmaker are sponsoring a series of matchmaking cruises for single women. There are $n$ women and $n$ men with preference lists for each other. Assume $n$ is even. The Matchmakers guarantee a spot on the ship for all $n$ women, but can only fit $n/2$ men at a time. Since space is limited, the Matchmakers decide to let all $n$ women aboard but the men are divided into two groups of $n/2$ men, Group A and Group B.

For the first "week," men from Group A are allowed to come aboard and start proposing to the $n$ women through the male Propose and Reject algorithm, until no man receives any more rejections. (Just assume that a "week" is long enough for this to happen.) For the second week, Group A leaves and Group B starts proposing until no rejections are received. On the third week, Group A returns and this process continues to repeat until no man from either Group A or Group B is rejected.

During this process, if a woman still had an active proposal in hand from man $a$ from Group A at the end of a particular week, then the next week, she will reject man $b$ from Group B if she prefers $a$ over $b$. On the other hand, if she prefers $b$ over $a$, she will say "maybe" to $b$ and reject $a$ when he returns to the cruise the next week and re-proposes to her.

**State and prove an Improvement Lemma for this scenario.**

(This Improvement Lemma should be sufficiently powerful to be able to be used to eventually get a proof that the Matchmakers will end up with a stable pairing using their cruises.)

**Improvement Lemma**: If M proposes to W on the $k$-th day, then on every subsequent day she has someone M$'$ on a string whom she likes at least as much as M. The definition of W having M$'$ "on a string" is as follows: M$'$ is the latest man whom W said "maybe" to.

Proof: By the well-ordering principle, suppose towards a contradiction that the $j$-th day for $j > k$ is the first counterexample where W has either nobody or some M* inferior to M on a string. On day $j-1$, she has M$'$ on a string and likes M$'$ at least as much as M. According to the algorithm, we will have two cases:

1. On the $j$-th day, M$'$ is of the group proposing to W at that week. In this case, M$'$ will still propose to W on the $j$-th day. On the $j$-th day, W's best choice is at least as good as M$'$, and according to the algorithm, she will choose him over M*. This contradicts our initial assumption.
2. On the $j$-th day, M$'$ is not of the group proposing to W at that week. According to the algorithm, W will say "maybe" to somebody else on the $j$-th day only if W prefers that person over M$'$. Therefore if W has M* on the string on the $j$-th day, that means W prefers M* over M$'$ thus prefers M* over M. This contradicts our initial assumption. QED.

Weaker lemmas such as "If M proposes to W on the first day of the $k$-th week, then on the first day of every subsequent week she has someone M$'$ whom she likes at least as much as M on a string" is not strong enough to prove that the algorithm will end up with a stable pairing. It has to be combined with another improvement lemma within each week. We only gave partial credit to weaker improvement lemmas and proofs.