Probabilistic Methods

6.1, 6.2
https://www.fi.muni.cz/~xbouda1/teaching/current/IV111/cviceni/ex04.pdf
6.2, 6.3
https://www.cs.ox.ac.uk/people/varun.kanade/teaching/CS174-Fall2012/HW/HW6_sol.pdf
6.4
https://math.stackexchange.com/questions/2290220/2-players-game-finding-optimal-strategy
6.5
Consider partitions of equal sizes for an even number of nodes and with a size difference of 1 for an odd number of nodes.
6.6
https://people.seas.harvard.edu/~salil/pseudorandomness/basic.pdf
Algorithm 3.17 (deterministic MaxCut approximation).
6.7
https://math.stackexchange.com/questions/387592/dominating-set-in-an-r-uniform-hypergraph
6.8
https://people.cs.pitt.edu/~kirk/cs3150spring06/hw6.8.pdf

6.9

Problem 1

(a)

Note that for a given tournament and a given ranking, the edges that agree with the ranking are exactly those that disagree with the reverse of that ranking. Thus, for any tournament, we can choose an arbitrary ranking, and either that ranking or its reverse will disagree with at most 50% of the edges in the tournament.

(b)

Let $T = (V, E)$ be a tournament for which each edge's direction is chosen independently and uniformly at random. Let $|V| = n$ and $|E| = m$. Since T is a complete graph, $m = 2^{n(n-1)/2}$.

For a given ranking R, let X_R be the number of edges of T that disagree with R. We can express X_R as $\sum_{e \in E} X_R^e$, where X_R^e is the random variable equal to 1 if R disagrees with edge e and 0 otherwise. For each edge, there is a "correct" and "incorrect" orientation relative to R, and each occurs with probability 1/2. Thus $\mathbf{E}[X_R^e] = 1/2$ for all e, and $\mathbf{E}[X_R] = m/2$.

For any $0 < \delta < 1$, we can bound the probability that $X_R < (1 - \delta)(m/2)$ using a Chernoff bound:

$$\Pr(X_R \leq (1-\delta)\mathbf{E}[X_R]) \leq e^{-\mathbf{E}[X_R]\delta^2/2} = e^{-m\delta^2/4} .$$

We can use this to bound the probability that $X_R < (1 - \delta)(m/2)$ for *any* ranking R:

$$\Pr\left(\bigcup_R X_R \leq (1-\delta)(m/2)\right) \leq \sum_R \Pr(X_R \leq (1-\delta)(m/2)) \leq n! \cdot e^{-m\delta^2/4} .$$

Note that $m = 2^{n(n-1)/2}$. From Stirling's formula we can see that $\lim_{n\to\infty} n!/e^{m\delta^2/4} = 0$. Thus for any fixed $\delta \in (0, 1)$, for sufficiently large n, $\Pr(\bigcup_R X_R \leq (1-\delta)(m/2)) < 1$.

In particular, we can set $\delta = 0.02$, so that $(1 - \delta)(m/2) = 0.49m$. Thus we have shown that for sufficiently large n, the probability that, for a randomly chosen tournament T of size n, any ranking disagrees with less than 49% of the edges in T is less than 1. So there exists some tournament T^* such that every ranking disagrees with at least 49% of the edges in T^*. □

Wrong

6.10

https://www.cs.ox.ac.uk/people/varun.kanade/teaching/CS174-Fall2012/HW/HW7_sol.pdf

6.11

Doable

6.14

https://www.cs.cornell.edu/courses/cs683/2008sp/lecture%20notes/lec3.pdf (4)

6.15

https://www.cs.ox.ac.uk/people/varun.kanade/teaching/CS174-Fall2012/HW/HW7_sol.pdf

6.17
https://mail.tku.edu.tw/158778/research/randalg/exercises/exercise6.pdf
6.19
https://mail.tku.edu.tw/158778/research/randalg/exercises/exercise6.pdf

6.20
https://theory.stanford.edu/~jvondrak/MATH233A-2018/Math233-lec02.pdf