CS3150 - Homework — Chapter 6 (6.14)*

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1 Problem 6.2

(a) Prove that for every integer n, there exists a coloring of the edges of the complete graph K_n by two colors so that the total number of monochromatic copies of K_a is at mist $\binom{n}{4}2^{-5}$.
(b) Give a randomized algorithm for finding a coloring with at most $\binom{n}{4}2^{-5}$ monochromatic copies of K_4 that runs in expected time polynomial in n.
(c) Show how to construct such a coloring deterministically in polynomial time using the methods of conditional expectations.

Proof:
(a) Each edge has 1/2 probablity to be colored in each of the two colors. There are 6 edges in each K_4, so the probability that a K_4 is monochromatic is $2 \cdot \frac{1}{2}^6 = 2^{-5}$.

The expected number of monochromatic K_4 is thus $\binom{n}{4} \cdot 2^{-5}$, so that there exists such coloring such that the number of monochromatic K_4 is at most $\binom{n}{4}2^{-5}$.

(b) By fliping a coin, if it is head, we use color 1, otherwise we use color 2. Assuming we have probability p to get such a coloring, then:

$$\binom{n}{4}2^{-5} \geq (1-p)(\binom{n}{4}2^{-5}+1)$$

So,

$$\begin{aligned} 1-p &\leq \frac{\binom{n}{4}/32}{\binom{n}{4}/32+1} \\ p &\geq 1-\frac{\binom{n}{4}/32}{\binom{n}{4}/32+1} \\ &= \frac{1}{\binom{n}{4}/32+1} \end{aligned} \tag{1}$$

Simply repeat the process, with success probability of p. The expected number of trials before a success is thus $1/p$, where

$$\frac{1}{p} = \binom{n}{4}/32+1 = 1+O(n^4) = O(n^4) \tag{2}$$

(c)

2 Problem 6.14

Consider a graph in $G_{n,p}$, with $p = 1/n$. Let X be the number of triangles in the graph, where a triangle is a clique with tjree edeges. Show that

$$Pr(X \geq 1) \leq 1/6$$

and that

$$lim_{n\to\infty} Pr(X \geq 1) \geq 1/7 \tag{3}$$

(it Hint: Use the conditional expecation inequality.

Proof:
By using the conditional expectation inequality, we have

$$Pr(X \geq 1) = Pr(X > 0) = \sum_{i=1}^{\binom{n}{3}} \frac{Pr(X_i = 1)}{E[X|X_i = 1]} \tag{4}$$

For any triangle, $Pr(X_i = 1) = p^3$, and

$$E[X|X_i = 1] = 1 + \binom{n-3}{3} p^3 + \binom{n-3}{2} p^3 + \binom{n-3}{1} p^2 \tag{5}$$

So,

$$\begin{aligned}
Pr(X \geq 1) &= \sum_{i=1}^{\binom{n}{3}} \frac{p^3}{1 + \binom{n-3}{3} p^3 + \binom{n-3}{2} p^3 + \binom{n-3}{1} p^2} \\
&= \frac{\binom{n}{3} p^3}{1 + \binom{n-3}{3} p^3 + \binom{n-3}{2} P p^3 + \binom{n-3}{1} p^2} \\
&\leq \binom{n}{3} p^3 \\
&\leq \frac{n^3}{6} p^3
\end{aligned} \tag{6}$$

Since $p = 1/n$, we have $Pr(X \geq 1) \leq (n^3 p^3)/6$.

We also have:

$$\begin{aligned}
\lim_{n\to\infty} Pr(X \geq 1) &= \lim_{n\to\infty} \frac{\binom{n}{3}p^3}{1+\binom{n-3}{3}p^3+\binom{n-3}{2}p^3+\binom{n-3}{1}p^2} \\
&= \lim_{n\to\infty} \frac{\binom{n}{3}/n^3}{1+\binom{n-3}{3}/n^3+\binom{n-3}{2}/n^3+\binom{n-3}{1}/n^3} \\
&= \lim_{n\to\infty} \frac{n(n-1)(n-2)/(6n^3)}{1+(n-3)(n-4)(n-5)/(6n^3)+(n-3)(n-4)/(2n^3)+(n-3)/n^3} \\
&= \lim_{n\to\infty} \frac{1/6}{1+1/6+0+0} \\
&= \frac{1/6}{7/6} \\
&= \frac{1}{7} \qquad (7)
\end{aligned}$$

3 Problem 6.16

Use the lovatz local lemma to show that if

$$4\binom{k}{2}\binom{n}{k-2}2^{1-\binom{k}{2}} \leq 1 \qquad (8)$$

then it is possible to color the edges of K_n with two colors so that is has no monochromatic K_k subgraph.

Proof:
Using lovatz local lemma, we need to $4dp \leq 1$.
Define the events set E as

$$E = \{E_i \mid E_i \in K_k, E_i \textit{ is monochromatic}\} \qquad (9)$$

By fliping coins, we can color the graph with two colors, each color has the same probability of $1/2$. The probability for each E_i is thus $2 \cdot (\frac{1}{2})^{\binom{k}{2}} = 2^{1-\binom{k}{2}}$, since there are $\binom{k}{2}$ edges in each K_k.

Next we need to prove that $d = \binom{k}{2}\binom{n}{k-2}$.

Two K_k are dependent means they should have at least two common edges, or say three common vertices. To bound the degree of dependence, we do the following. Choose one edge from the current K_k, choose one or more vertices from this K_k, and the rest from all vertices not in this K_k. Any such a selection is an instance of choose $k-2$ vertices from all the n vertices. So the degree of dependence is bounded by

$$d = \binom{k}{2}\binom{n}{k-2} \qquad (10)$$

We have $4dp \leq 1$, so we have

$$Pr(\bigcup_{i=1}^{n} \bar{E_i}) > 0 \qquad (11)$$

which means that it is possible to color the graph wuch that neither K_k is monochromatic.

For any one K_k, to construct a graph which is dependent on it, we can choose 2 vertices from this graph, and the rest from all vertices which are not in this K_k. Any such an instance must be an instance that