Polygon Triangulation

O’Rourke, Chapter 1

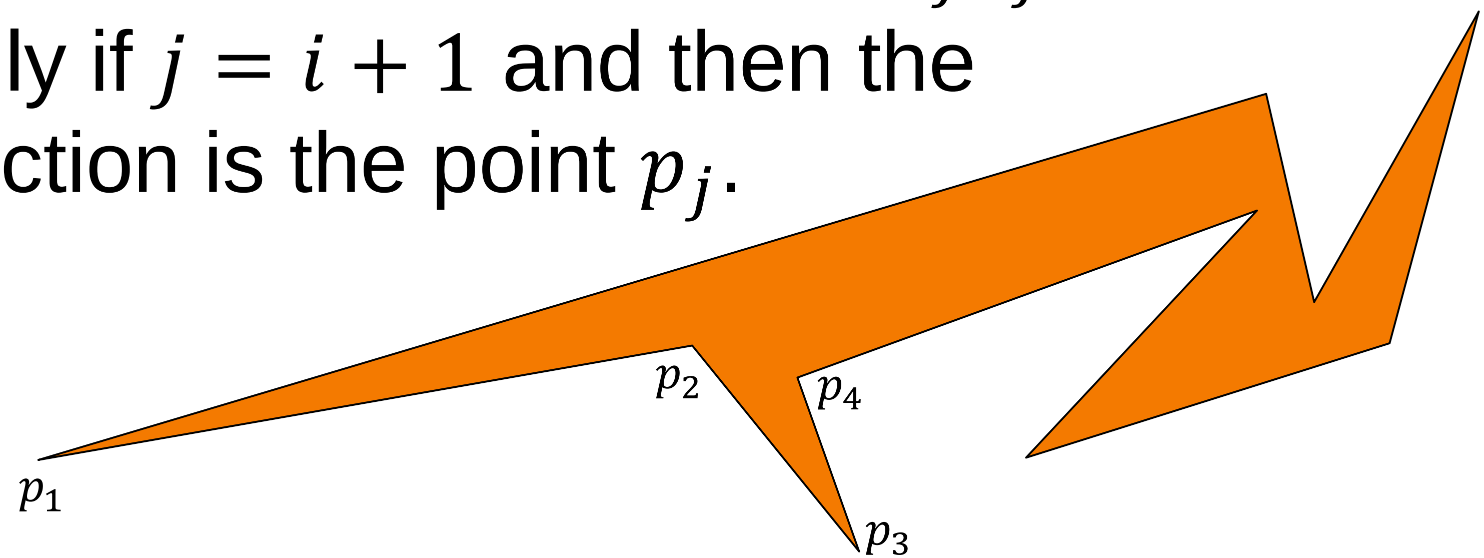
Outline

- Triangulation
- Duals
- Three Coloring
- Art Gallery Problem

Definition

A (*simple*) *polygon* is a region of the plane bounded by a finite collection of line segments forming a simple closed curve.

In practice, it is given by $\{p_1, \ldots, p_n\} \subset \mathbb{R}^2$ with the property that $\overline{p_i p_{i+1}} \cap \overline{p_j p_{j+1}} \neq \emptyset$ if and only if $j = i + 1$ and then the intersection is the point p_j.



Definition

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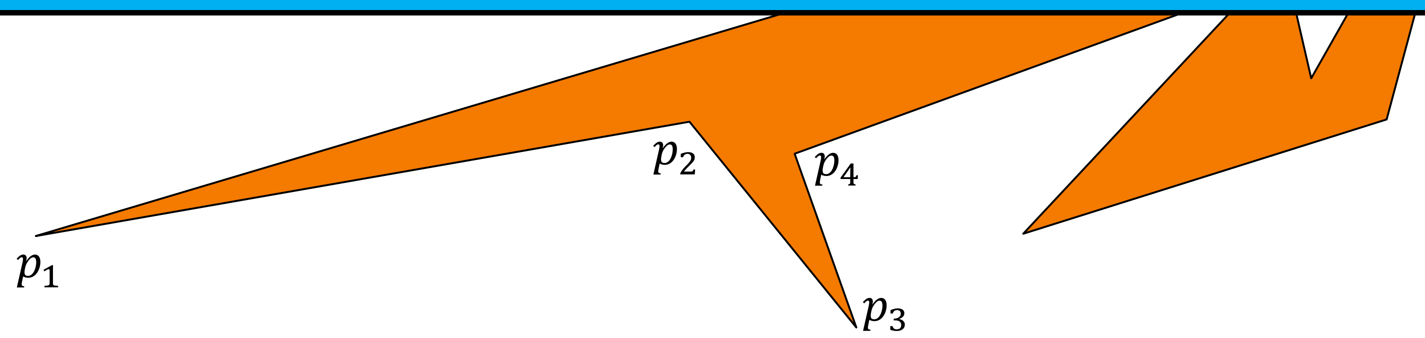
We will assume that vertices are given in CCW order, so that the interior of the polygon is on the left side of the edges.

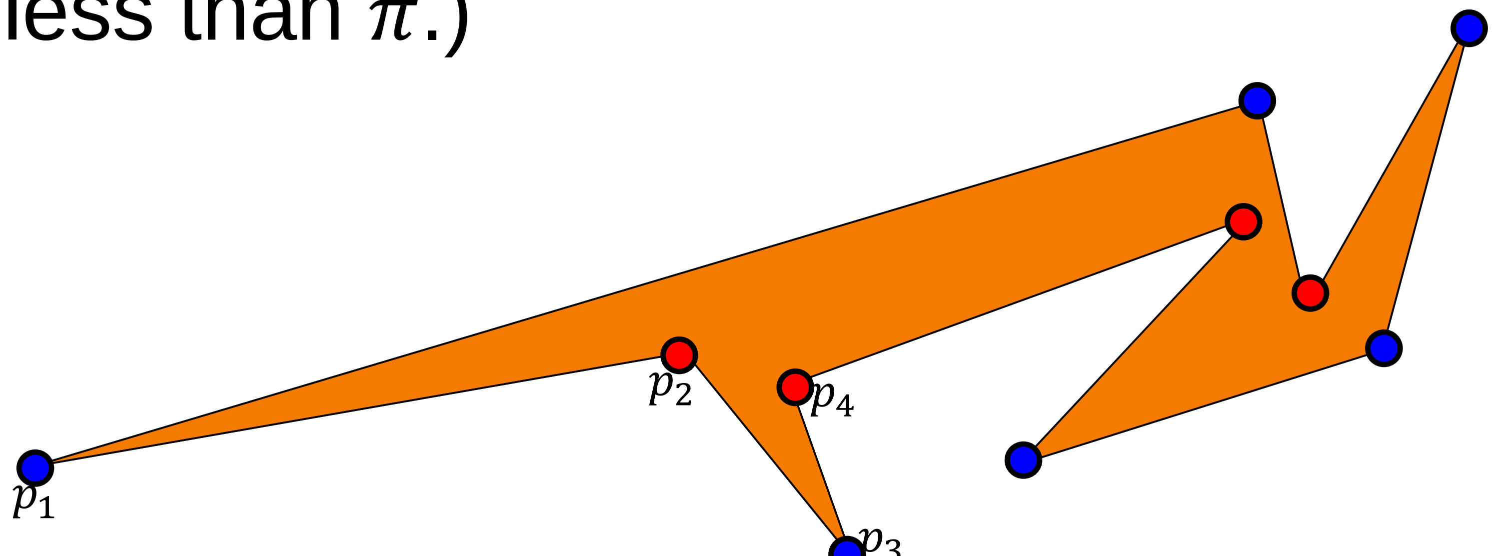


Definition

A vertex of a polygon is a *reflex vertex* if its interior angle is greater than π.

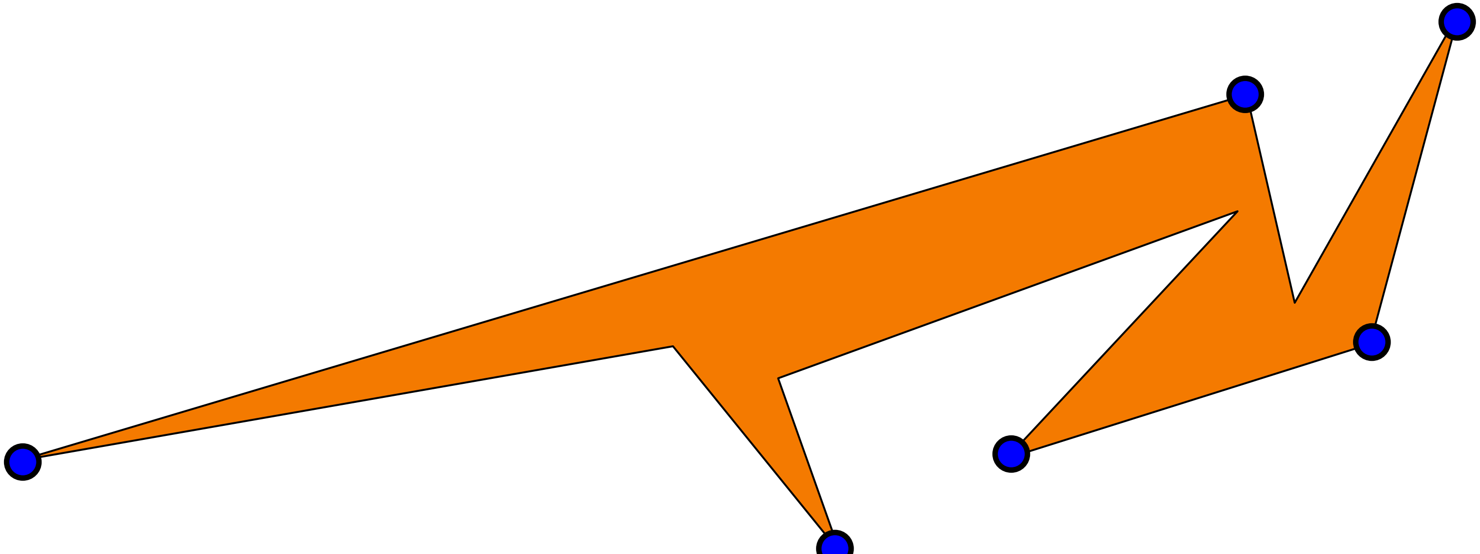
Otherwise it is a *convex vertex*.

(It is *strictly convex* if the interior angle is strictly less than π.)



Claim

Every polygon has at least one strictly convex vertex.

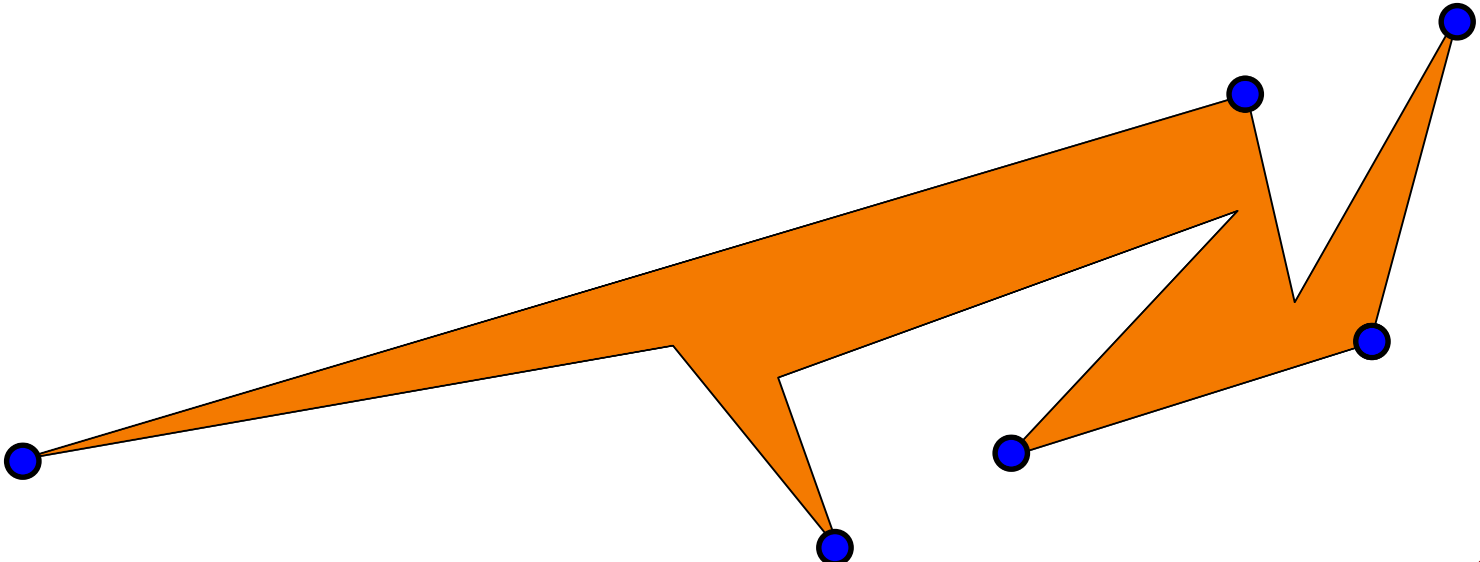


Proof 1

The sum of the interior angles is:

$$\pi \cdot (n-2)$$

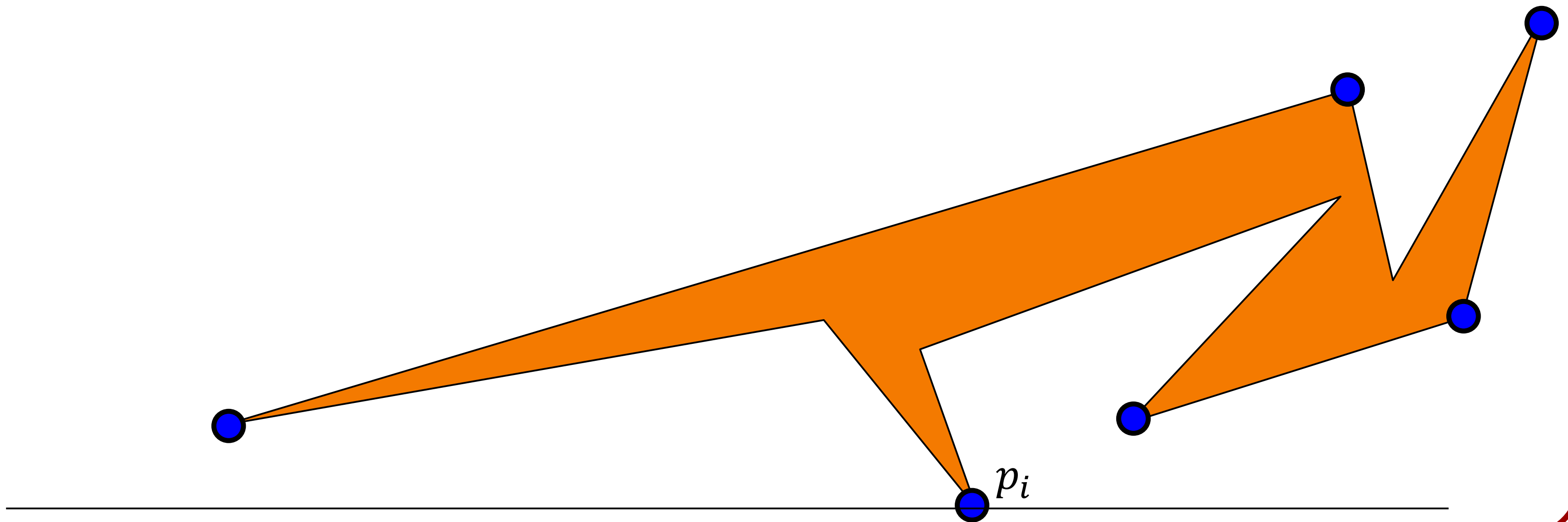
$\Rightarrow$ Some interior angle has to be less than π, otherwise the sum is at least $n \cdot \pi$.



Proof 2

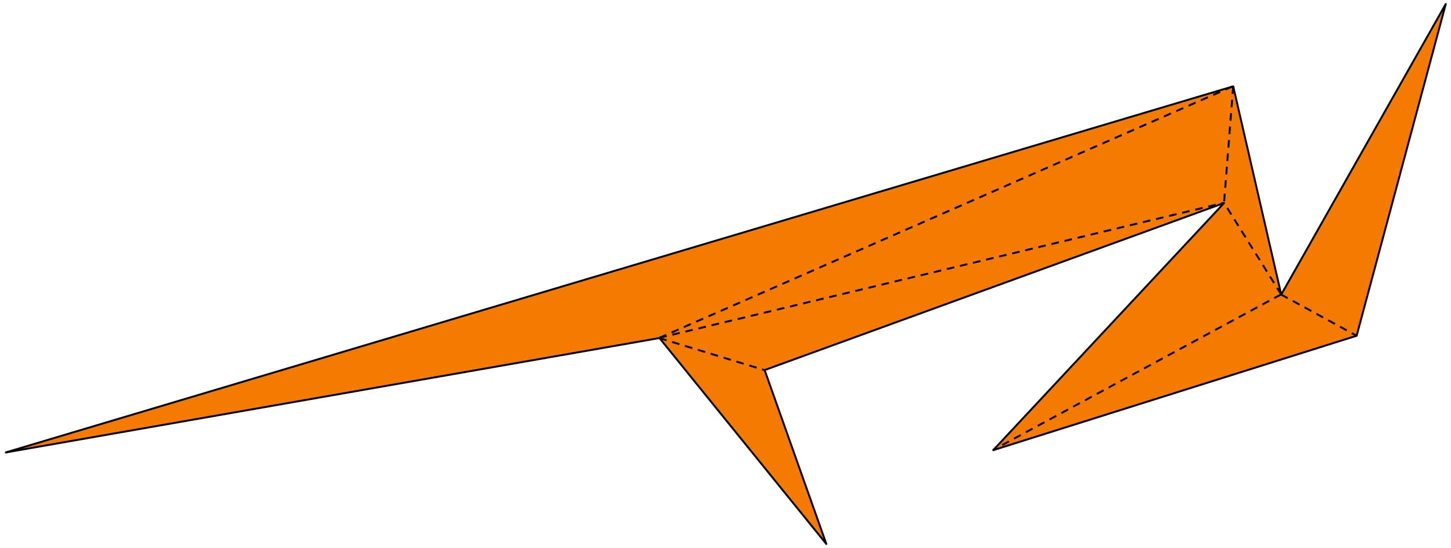
Find the lowest (right-most in case of tie) vertex of the polygon.

- $\Rightarrow$ The interior angle is (strictly) above the horizontal.
- $\Rightarrow$ The interior angle is smaller than π.



Goal

Given a polygon, compute a triangulation.



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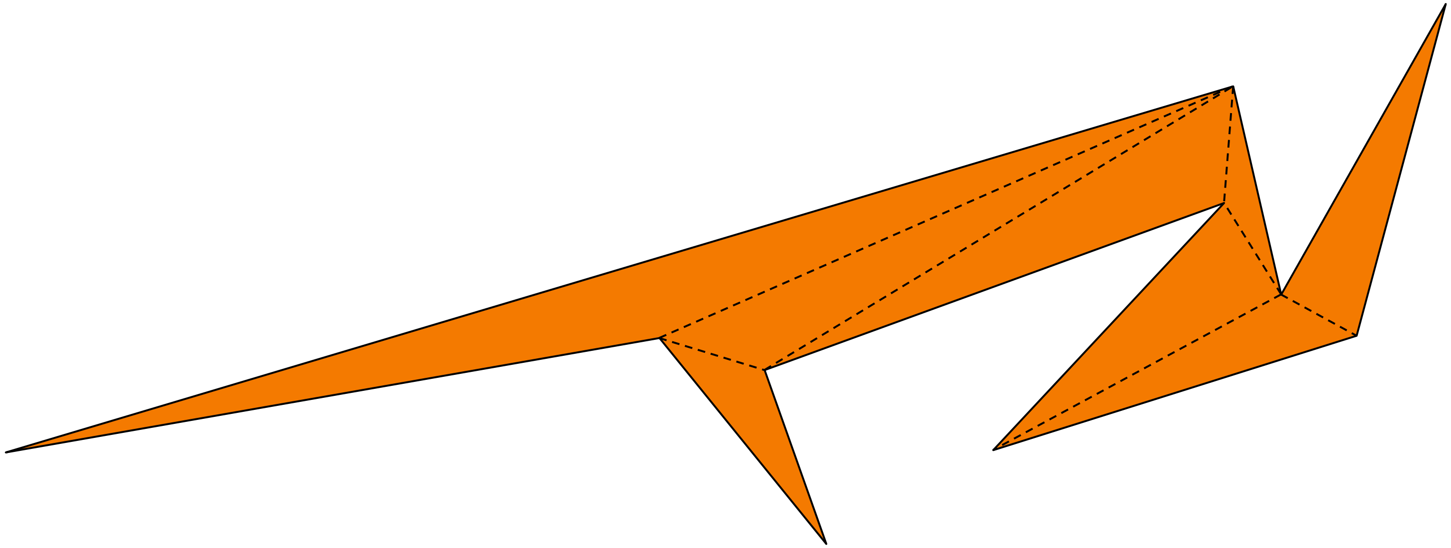
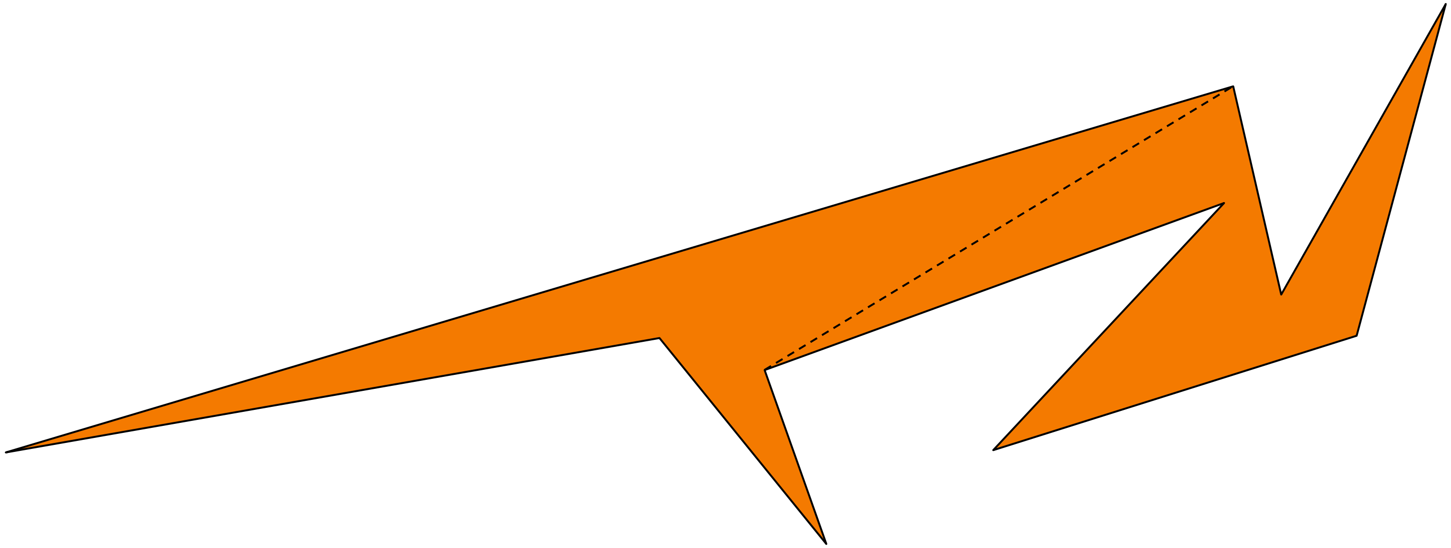


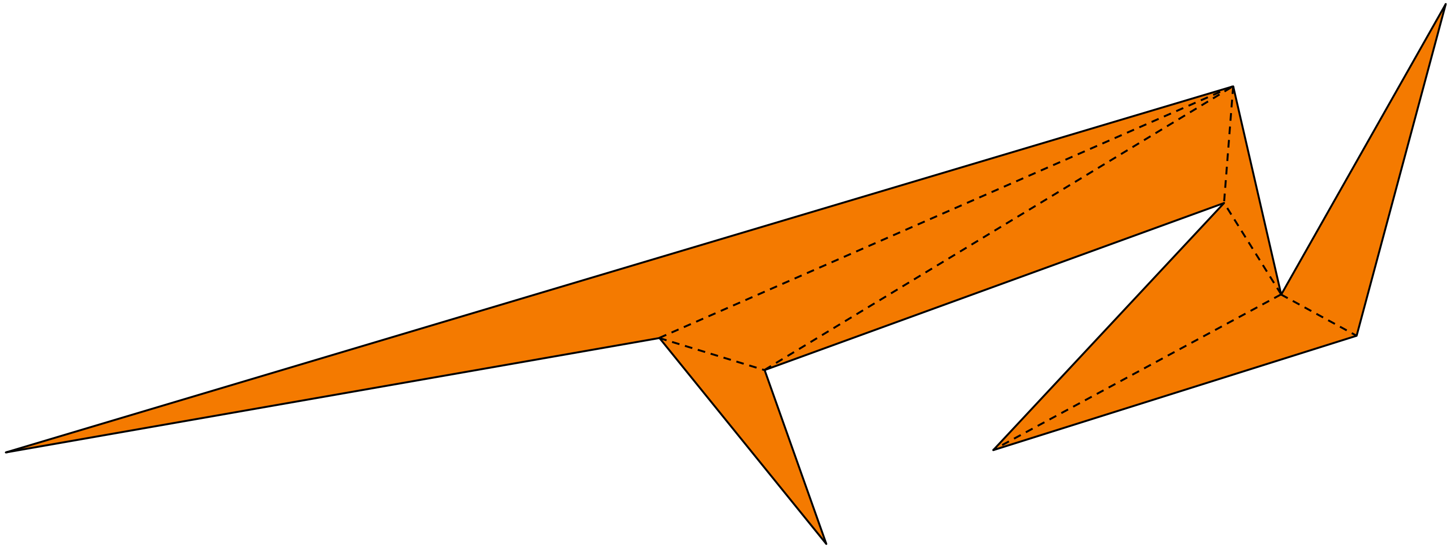
Definition

Given a polygon, a *diagonal* is a line segment between two vertices which does not intersect the polygon (aside from at the vertices).



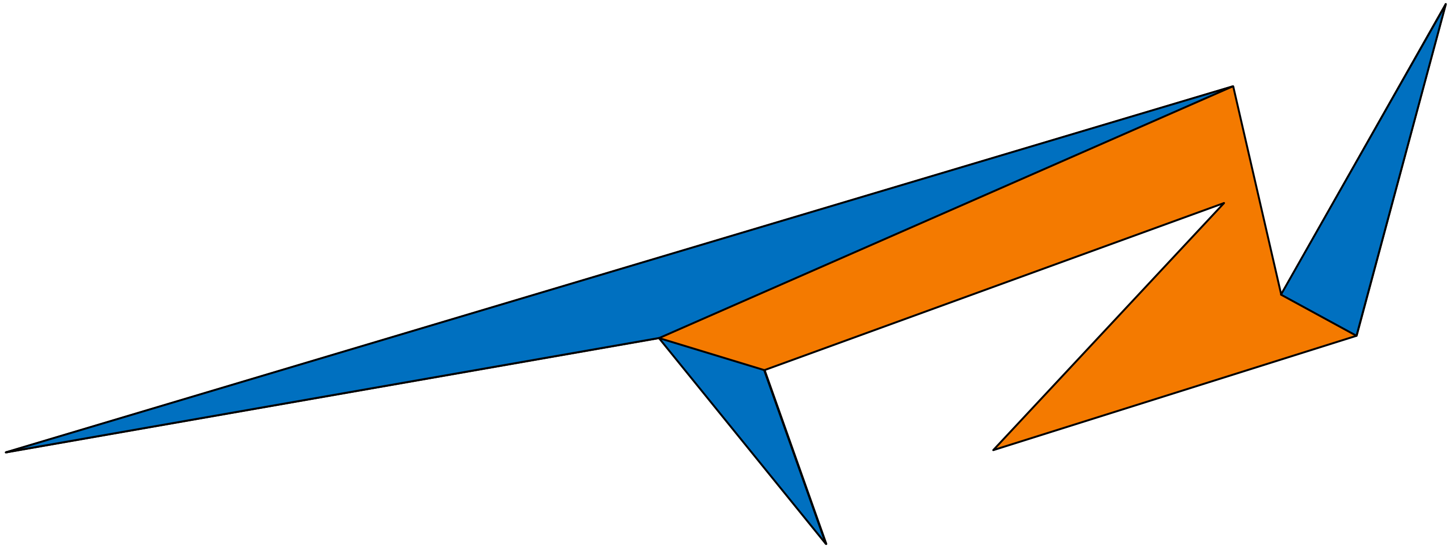
Definition

A *triangulation* of a polygon is a partition of the interior of the polygon into triangles whose edges are non-crossing diagonals.



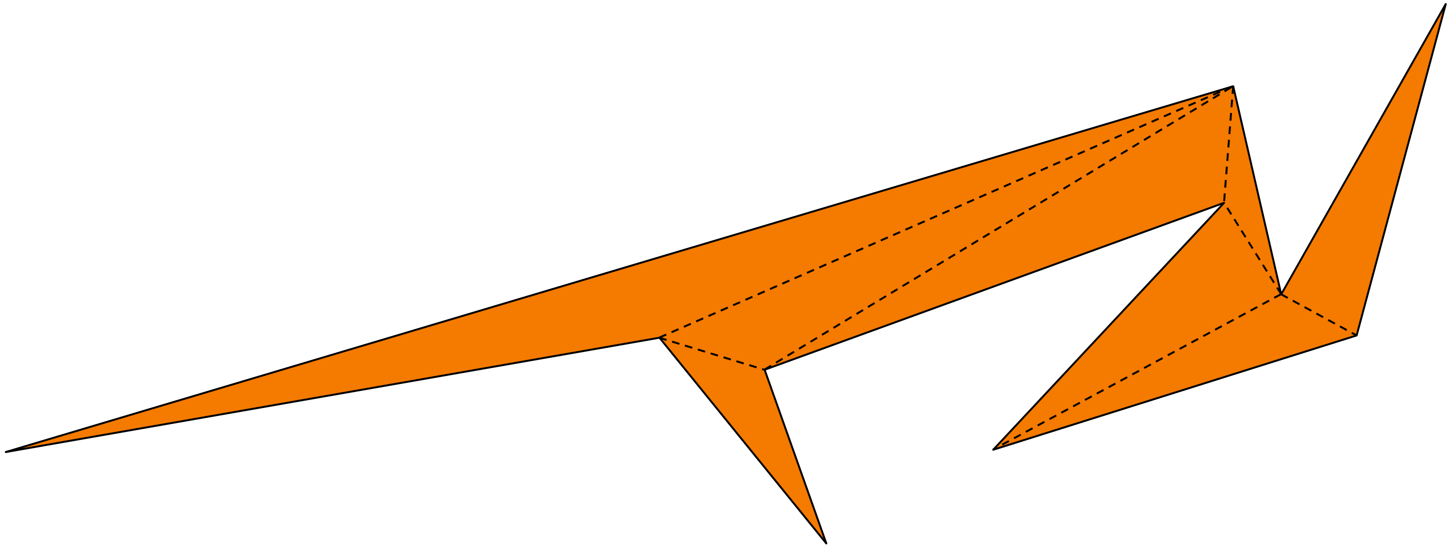
Definition

Three consecutive vertices, p_{i-1}, p_i, p_{i+1} of a polygon form an *ear* if the edge $\overline{p_{i-1}p_{i+1}}$ is a diagonal.



Claim

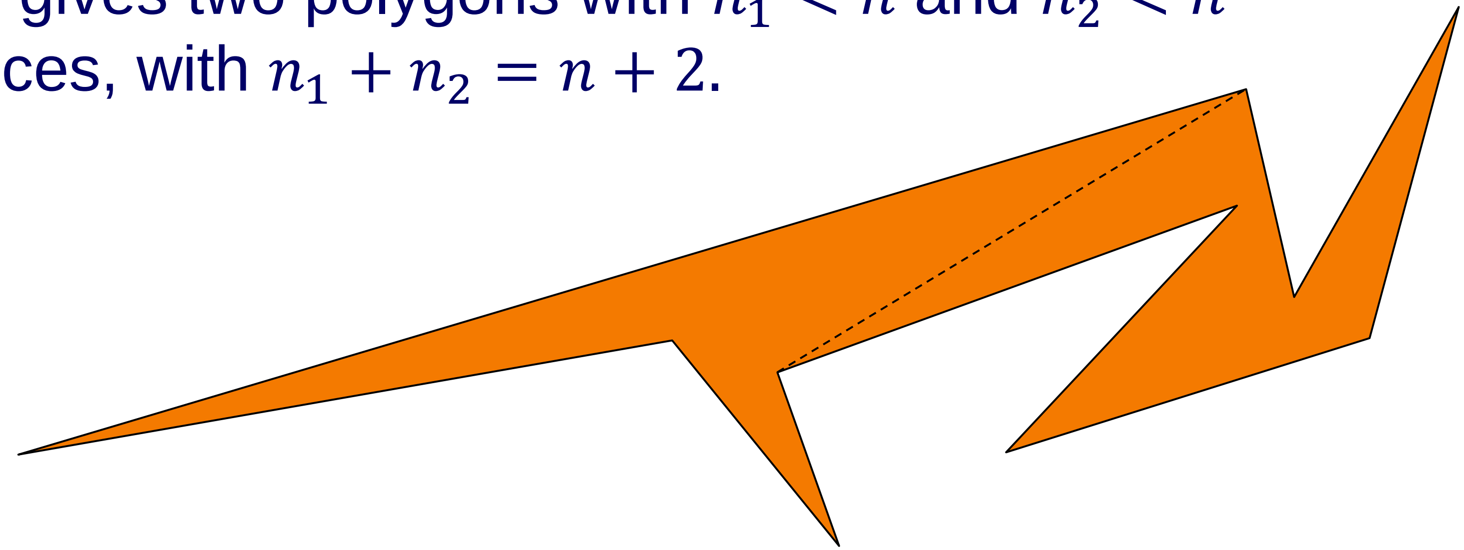
A polygon with n vertices can always be triangulated and will have $n - 2$ triangles and will require the introduction of $n - 3$ diagonals.



Proof

By induction:

- If $n = 3$, then we are done.
- If $n > 3$, add a diagonal to break the polygon into two smaller polygons.
  - This gives two polygons with $n_1 < n$ and $n_2 < n$ vertices, with $n_1 + n_2 = n + 2$.



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  - This gives two polygons with $n_1 < n$ and $n_2 < n$ vertices, with $n_1 + n_2 = n + 2$.

    $\Rightarrow$ They will have $n_1 - 2$ and $n_2 - 2$ triangles each.

    $\Rightarrow$ This gives $n_1 + n_2 - 4 = n - 2$ triangles.

Proof

By induction:

- If $n = 3$, then we are done.
- If $n > 3$, add a diagonal to break the polygon into two smaller polygons.
  - This gives two polygons with $n_1 < n$ and $n_2 < n$ vertices, with $n_1 + n_2 = n + 2$.
    - $\Rightarrow$ They will require $n_1 - 3$ and $n_2 - 3$ diagonals.
    - $\Rightarrow$ This gives $n_1 + n_2 - 6 + 1 = n - 3$ diagonals.

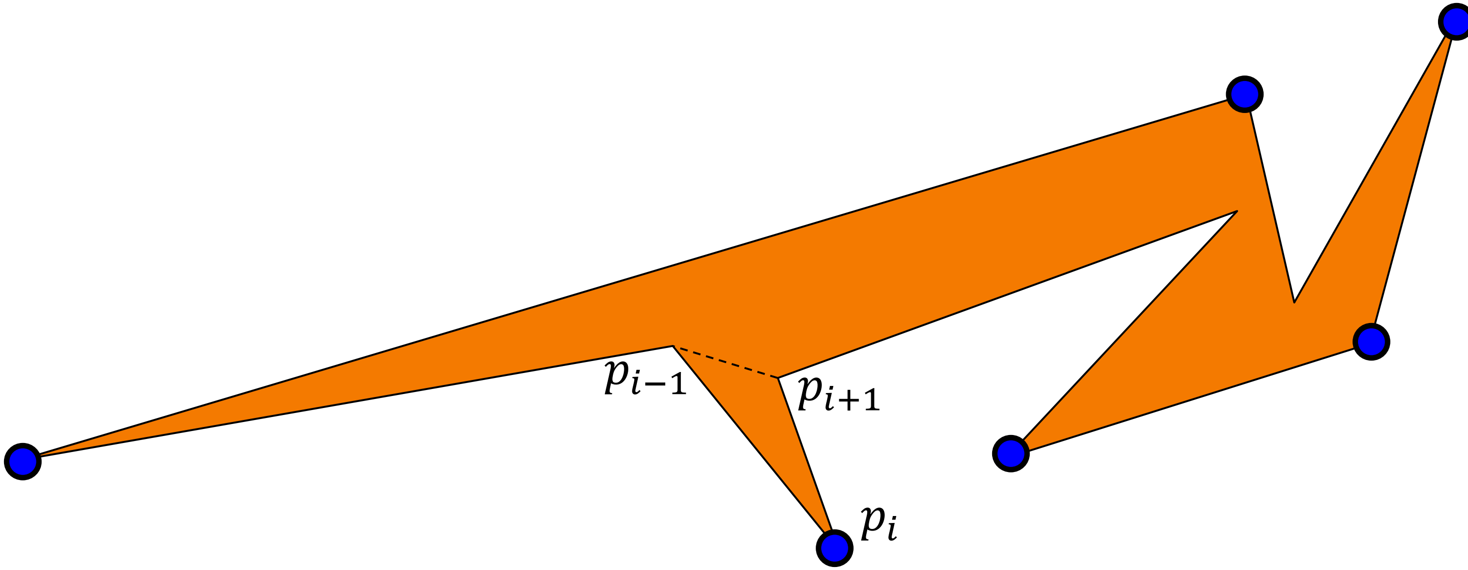
Sub-Claim

Given a polygon with $n > 3$ vertices, we can always find at least one diagonal.

Proof

Let p_i be a strictly convex vertex, and consider the line segment $\overline{p_{i-1}p_{i+1}}$.

If the line segment is a diagonal, we are done.



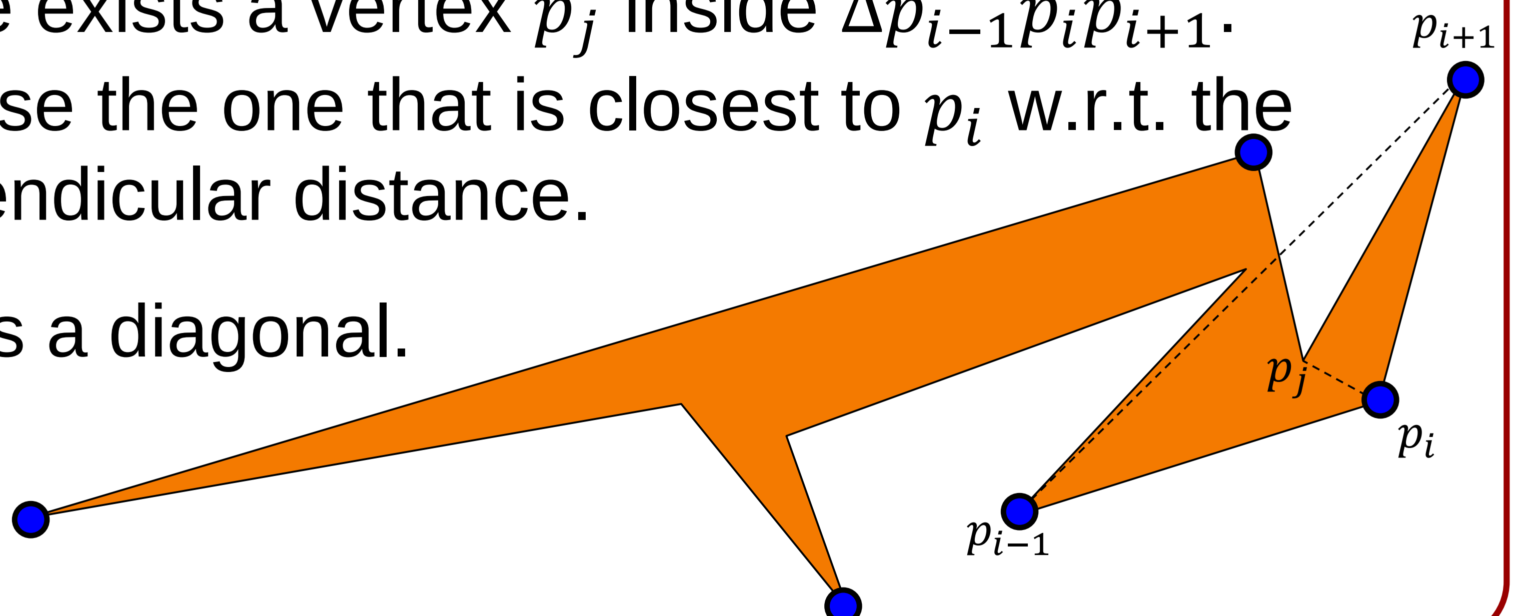
Proof

Let p_i be a strictly convex vertex, and consider the line segment $\overline{p_{i-1}p_{i+1}}$.

Otherwise, either the line segment is outside the polygon, or it intersects one of the edges.

$\Rightarrow$ There exists a vertex p_j inside $\Delta p_{i-1}p_ip_{i+1}$. Choose the one that is closest to p_i w.r.t. the perpendicular distance.

$\Rightarrow$ $\overline{p_ip_j}$ is a diagonal.



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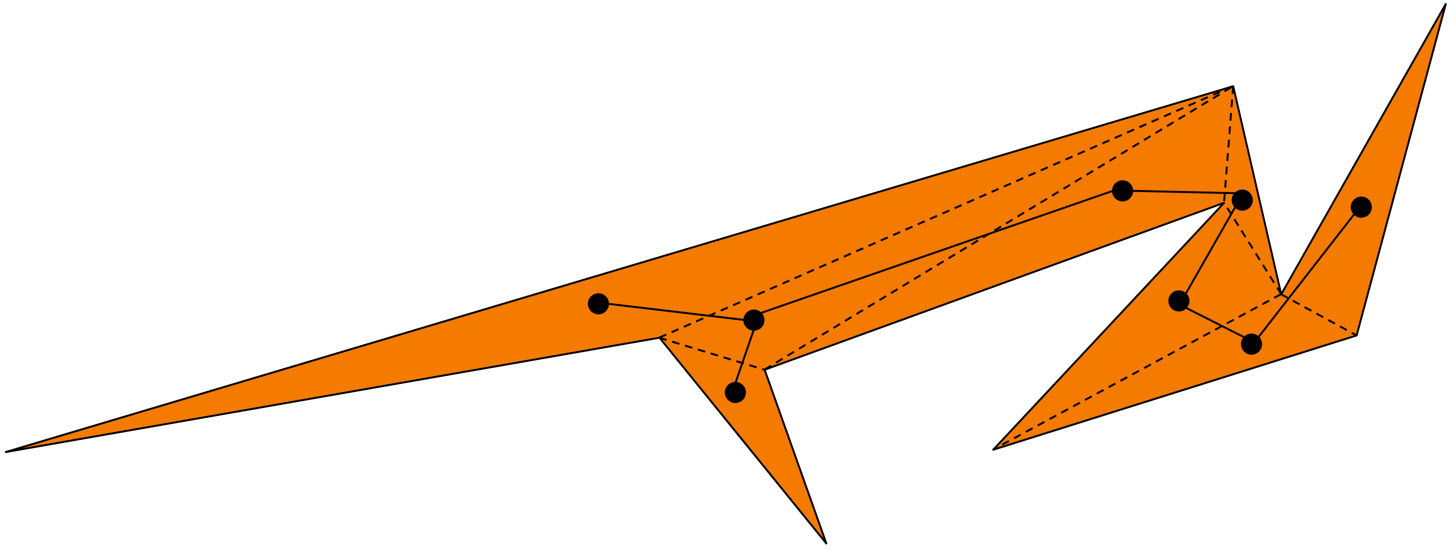
Definition

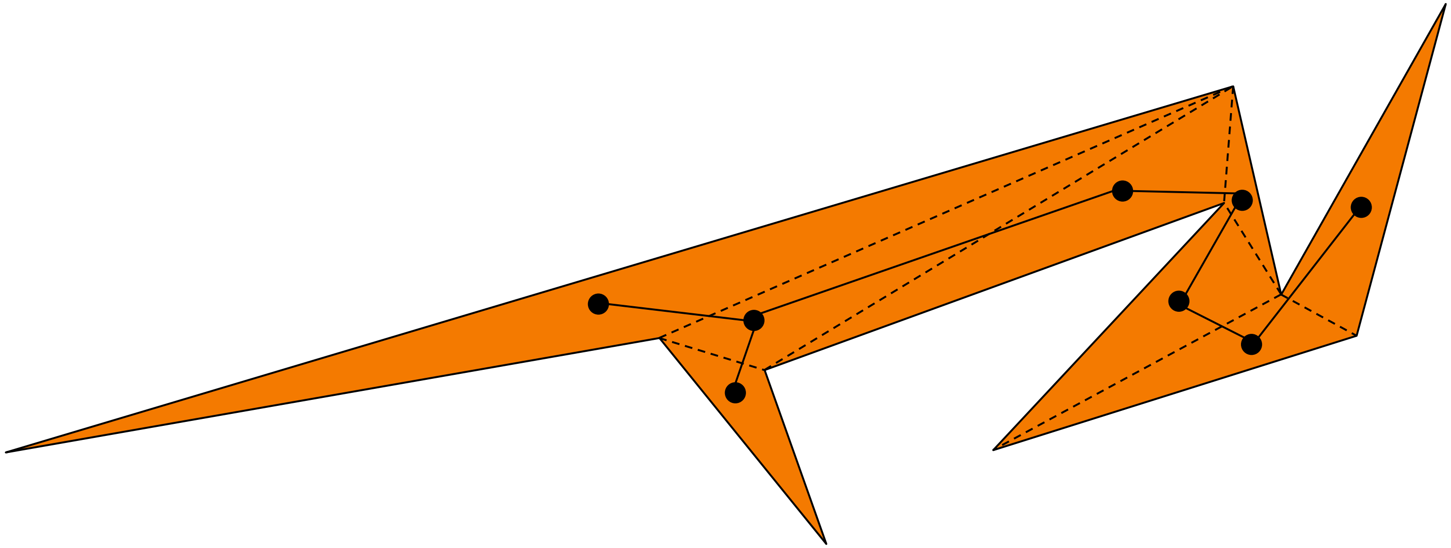
Given a triangulation of a polygon, the *dual* is the graph with:

- A node associated to each triangle
- An edge between nodes if the corresponding triangles share an edge.



Claim

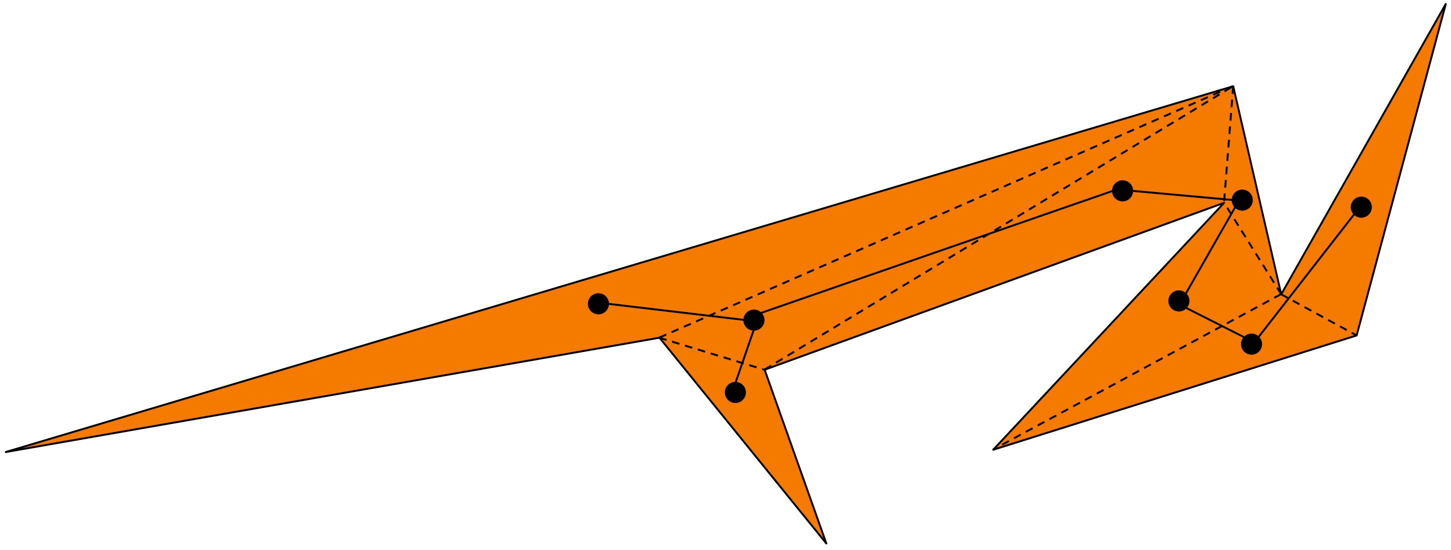
The triangulation dual is an acyclic graph with each node of degree at most three.



Proof

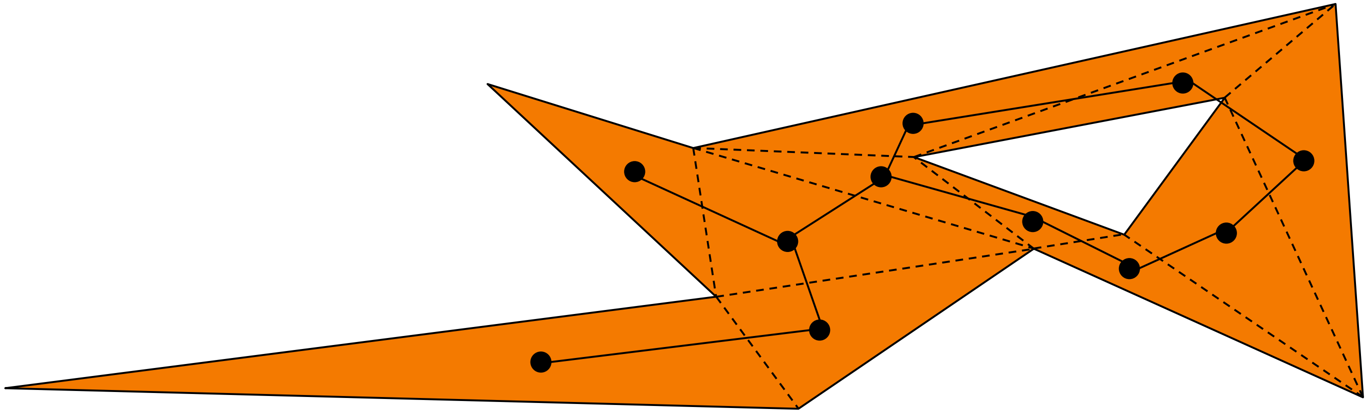
"…degree at most three":

This follows from the fact that each triangle has three edges.



Proof

"…acyclic graph…":

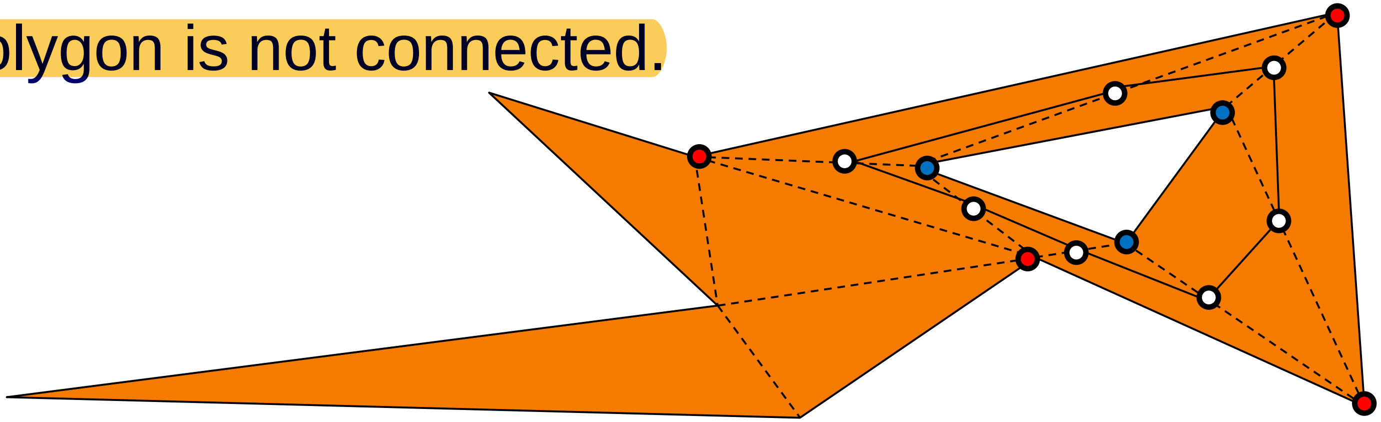


Proof

"…acyclic graph…":

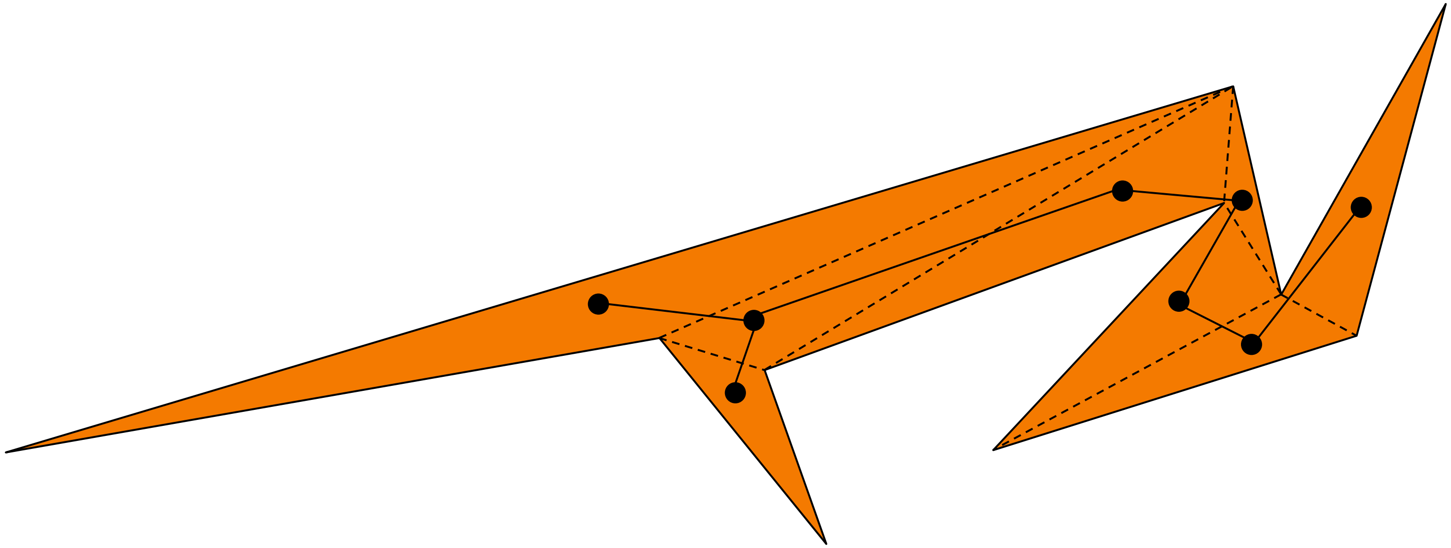
If the graph has a cycle, consider the curve connecting the mid-points of the (primal) edges of the cycle.

- ⇒ The curve is inside the polygon and encloses a subset of the vertices.
- ⇒ The polygon is not connected.



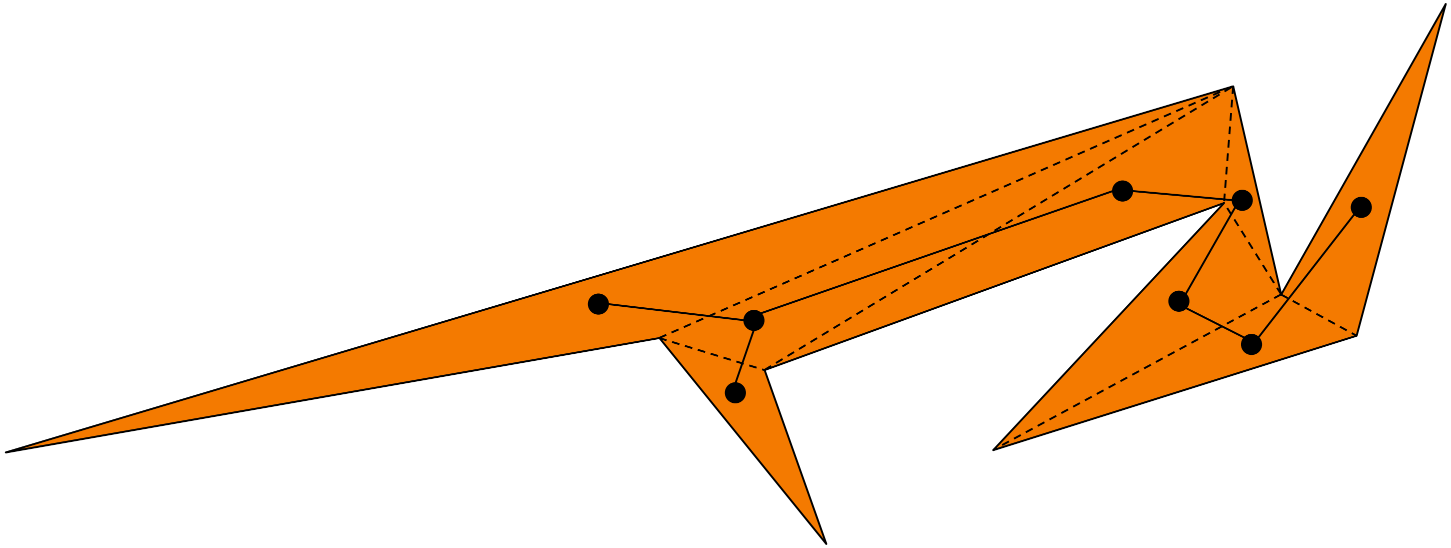
Note

The triangulation dual is a binary tree when rooted at a node of degree one or two.



Meisters's Two Ears Theorem

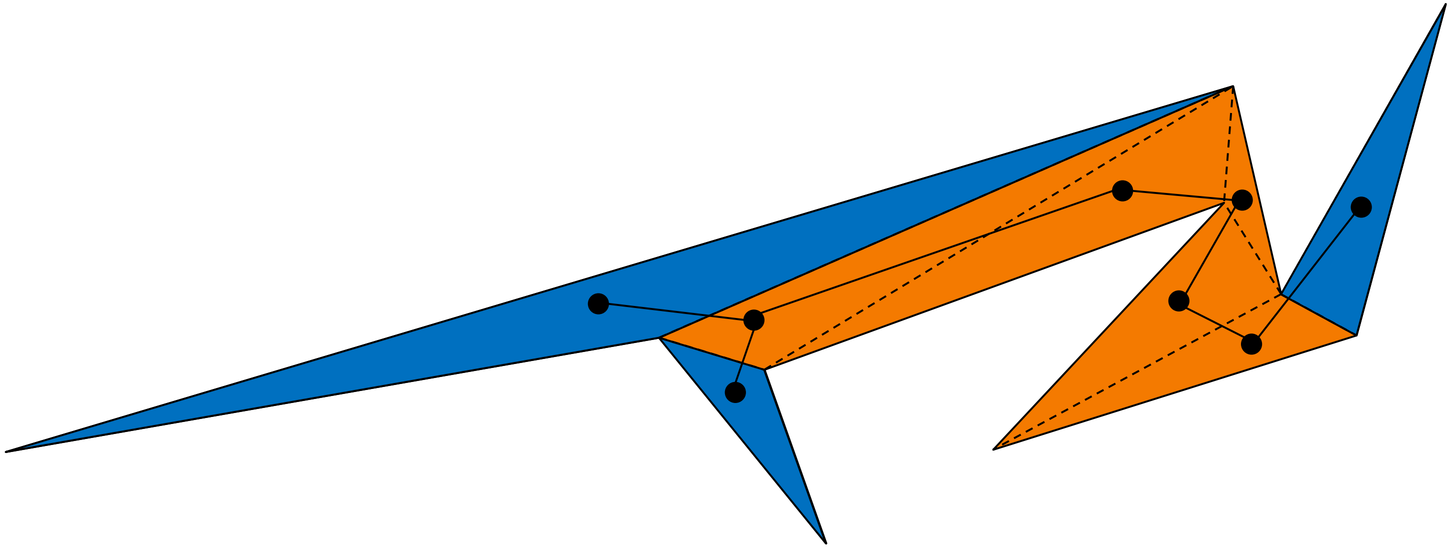
Every polygon with $n > 3$ vertices has at least two non-overlapping ears.



Proof

Compute a triangulation of the polygon and then take the triangulation dual.

- ⇒ A leaf of the graph must be an ear.
- ⇒ A binary tree with two or more nodes has at least two leaves.



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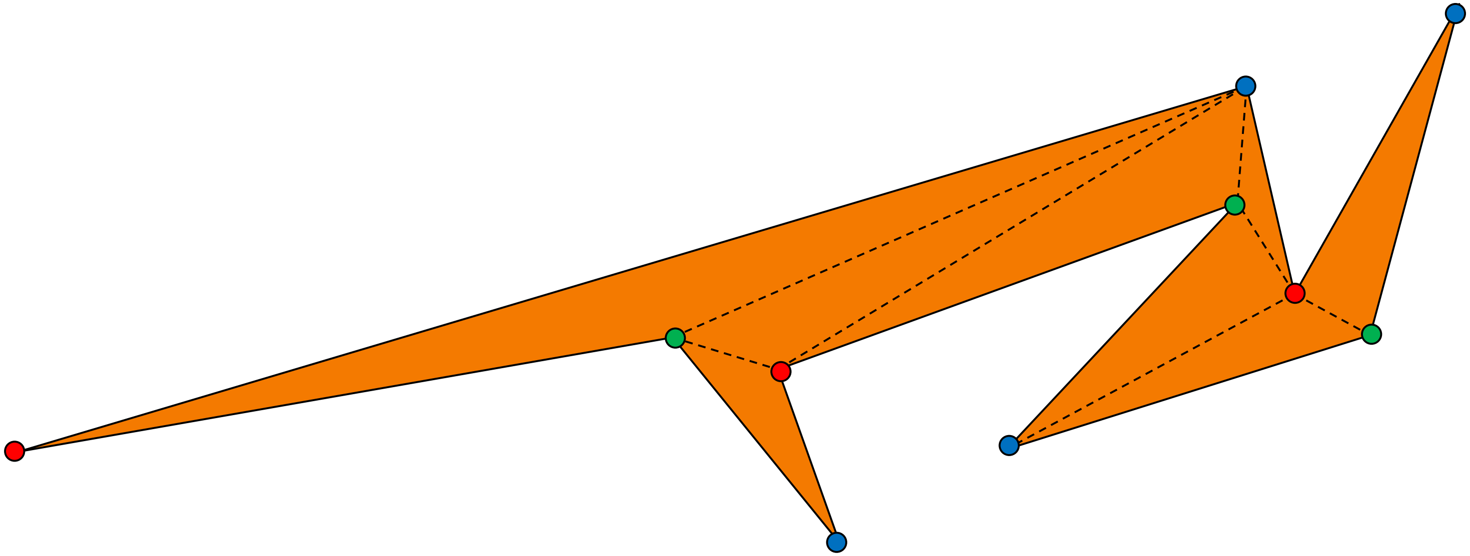
Claim

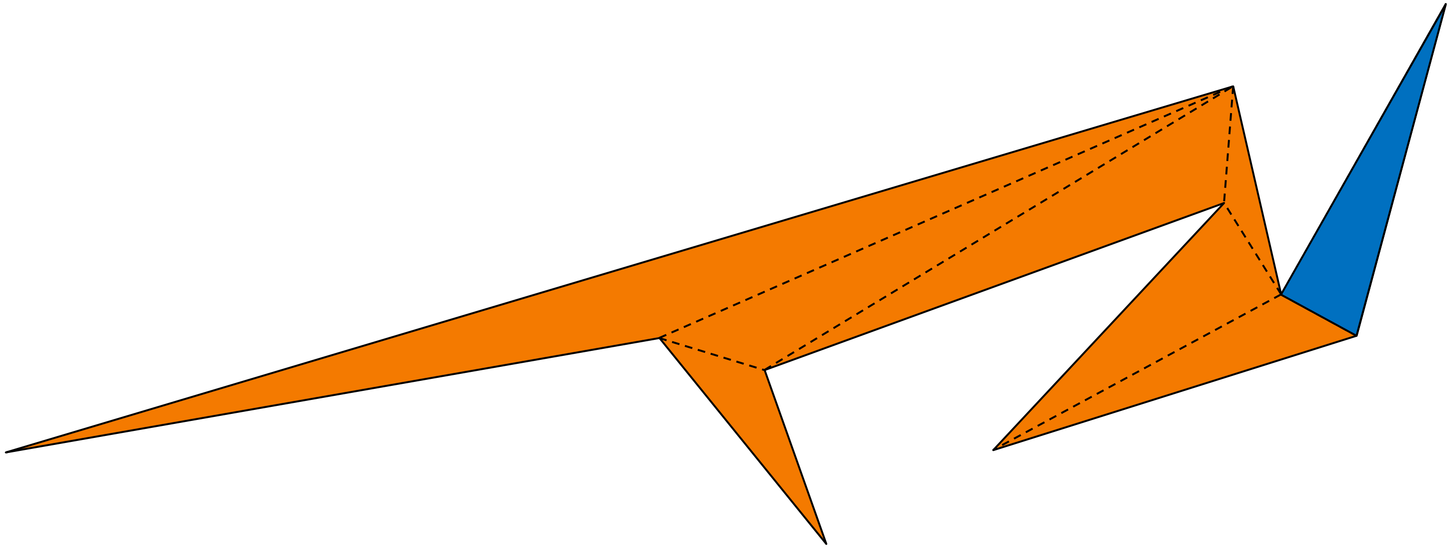
The triangulation graph of a polygon can be 3-colored.



Proof

By induction:

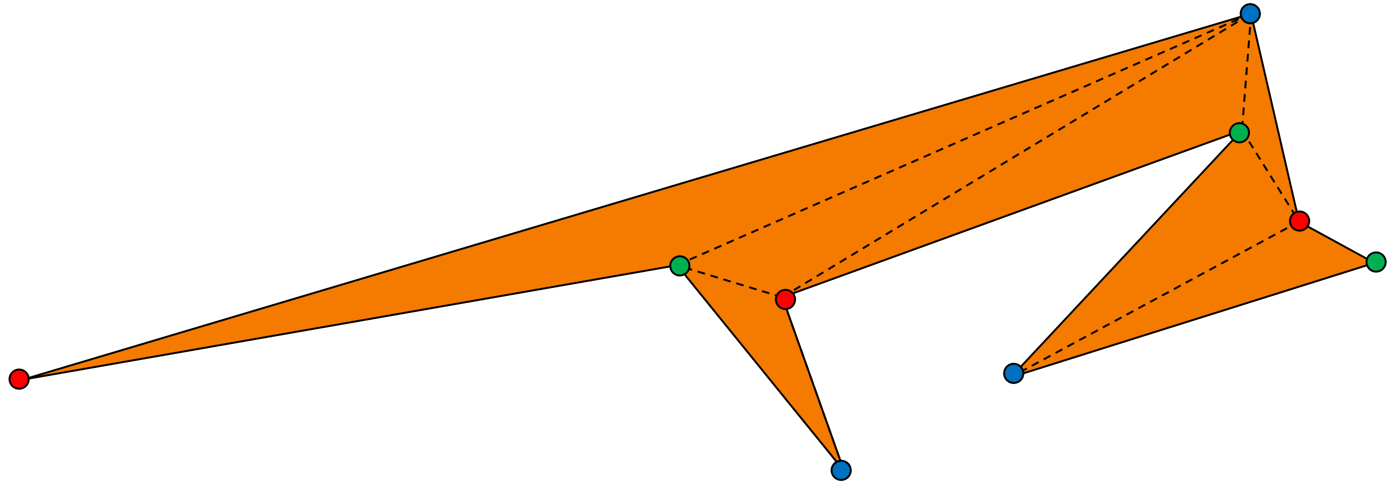
- If $n = 3$ we are done.
- Otherwise, the polygon has an ear.



Proof

By induction.

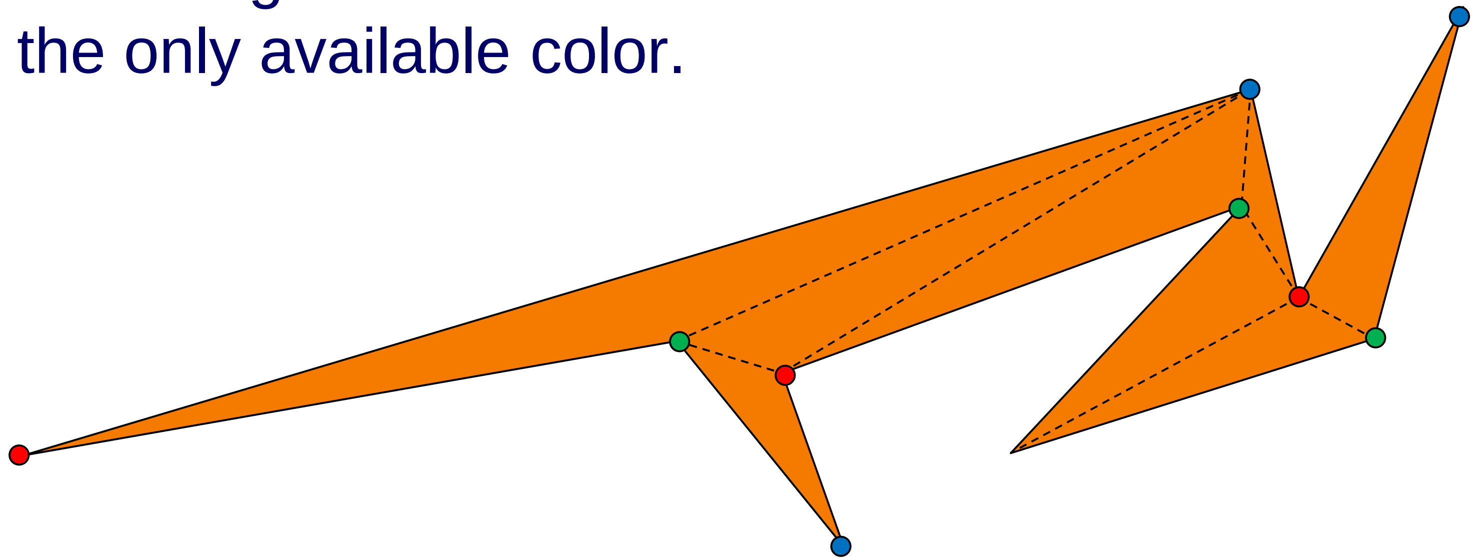
- If $n = 3$ we are done.
- Otherwise, the polygon has an ear.
  - Remove the ear and 3-color (induction hypothesis)



Proof

By induction.

- If $n = 3$ we are done.
- Otherwise, the polygon has an ear.
  - Remove the ear and 3-color (induction hypothesis)
  - Add the triangle back in and color the new vertex with the only available color.



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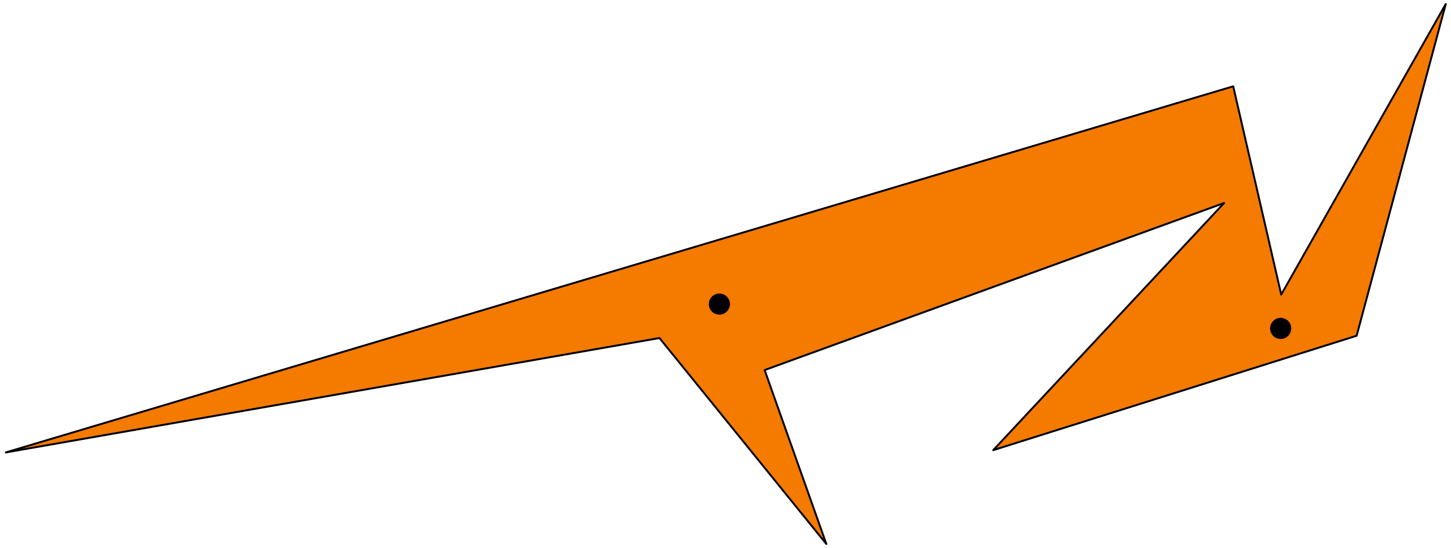
Art Gallery Problem

Given a polygonal room, what is the smallest number of (stationary) guards required to cover the room?

-- Klee (1976)



Art Gallery Problem

Given a polygonal room, what is the smallest number of (stationary) guards required to cover the room?

-- Klee (1976)

Formally:

- guard $\Leftrightarrow$ point
- A guard sees a point if the segment from the point to the guard doesn't intersect the polygon's interior.
- The polygon is covered if each point is seen by some guard.

Claim

Given a polygon with n vertices, $\lfloor n/3 \rfloor$ guards is necessary and sufficient.

Necessity:

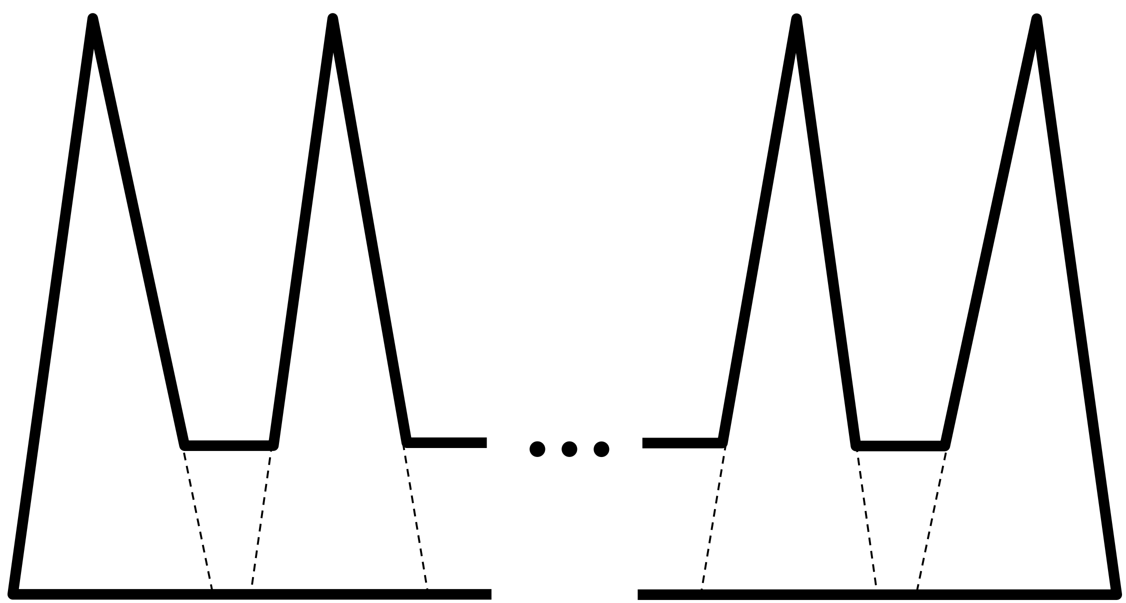
We can always choose n vertices of the polygons so that $\lfloor n/3 \rfloor$ guards are necessary.

Sufficiency:

We cannot choose n vertices so that more than $\lfloor n/3 \rfloor$ guards are necessary.

Necessity

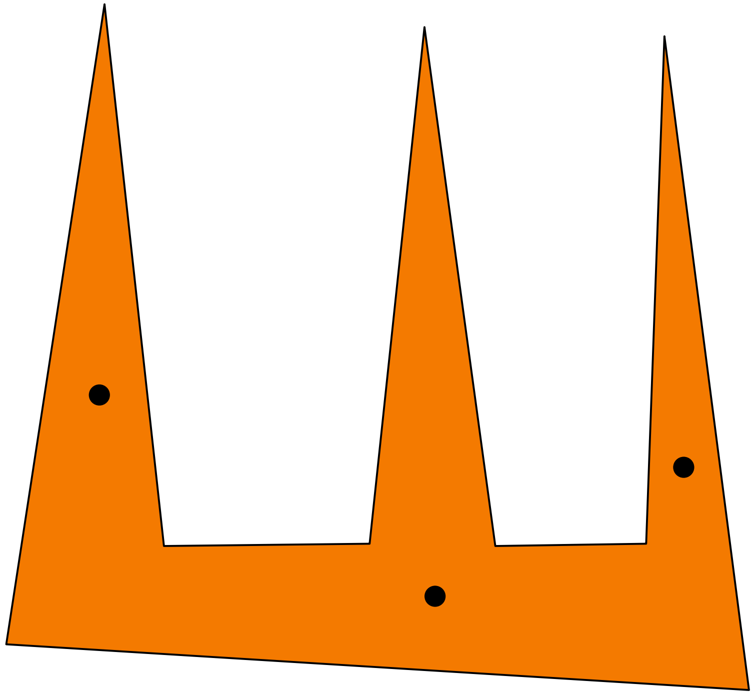
Given any value of n, we can always construct a polygon that requires at least $\lfloor n/3 \rfloor$ guards.



k prongs $\Rightarrow n = 3k$ vertices

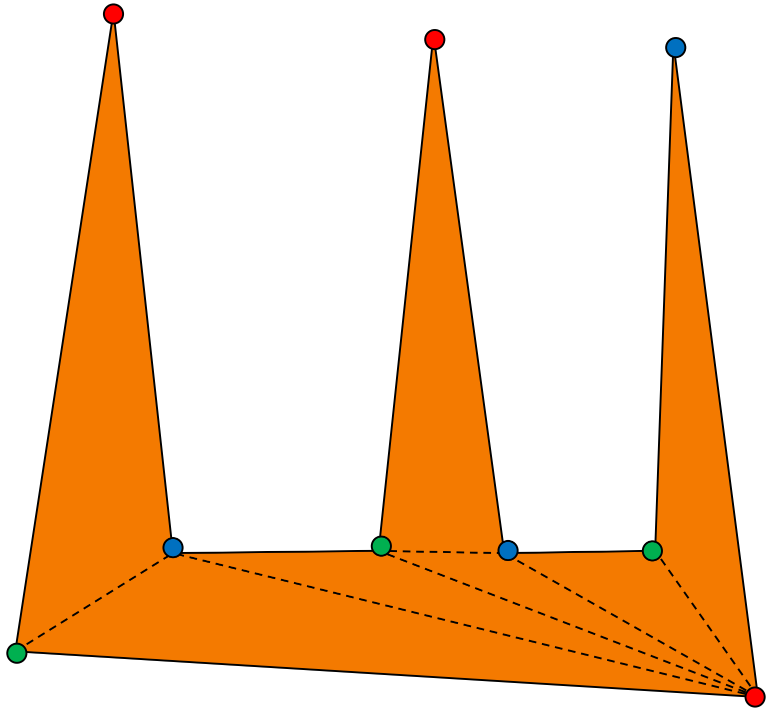
Sufficiency

For any polygon with n vertices, we can always cover with $\lfloor n/3 \rfloor$ guards.



Proof

- Triangulate the polygon.
- 3-color the vertices.
- Find the color occurring least often and place a guard at each associated vertex.
- By the pigeon-hole principal, there won't be more than $\lfloor n/3 \rfloor$ guards.



Tetrahedralization

Note that in three dimensions, not every polyhedron P can be tetrahedralized.

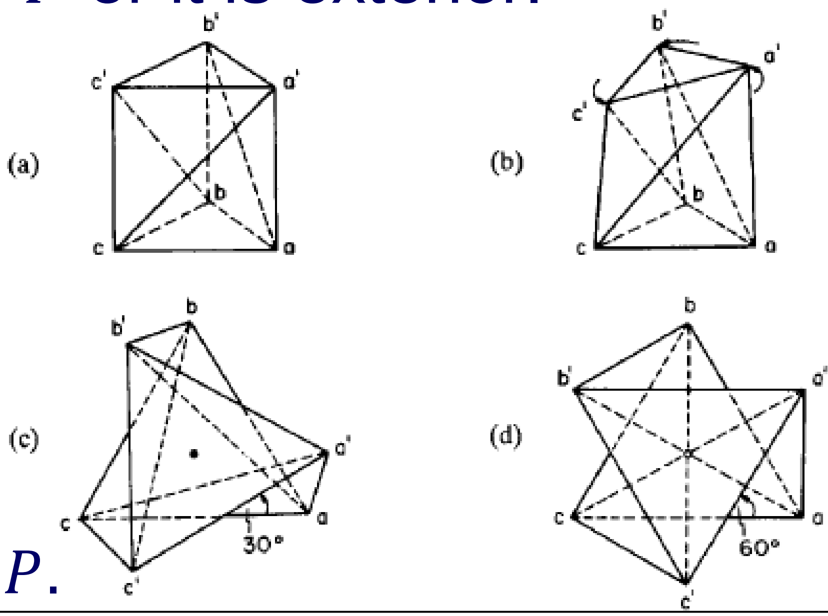
Claim:

1. Either $\overline{p_i p_j}$ is an edge of P or it is exterior.
2. Triangles whose edges are on P are faces of P.

$\Rightarrow$ Any interior tetrahedron has edges belonging to P.

$\Rightarrow$ Any interior tetrahedron has faces belonging to P.

$\Rightarrow$ Any interior tetrahadron is P.



Art Gallery Theorems and Algorithms, O'Rourke (1987)