Mid-semester Examination (2024-25 Spring)

Computational Geometry (CS60064)

Answer all. **Duration = 2 hours.** **Full marks = 60.**

Note: Unless mentioned, the following should be assumed about an input.

- Points are all distinct, no three being collinear, and no four being concyclic.
- A polygon is simple, i.e., edges intersect each other only at their endpoints, and it is described as clockwise or counter-clockwise sequence of its vertices, starting from an arbitrary vertex.

1. Write the worst-case time complexities of the best-known algorithms for the following problems. Just write the complexities, no explanation required. $\boxed{1\frac{1}{2} \times 10 = 15 \text{ marks}}$

(i) Given a set S of n points, construct a polygon with S as the vertex set. $\boxed{O(n \log n)}$

(ii) Given a polygon with n vertices, check whether it is convex. $\boxed{O(n)}$

(iii) Given a polygon with n vertices, check whether it is monotone w.r.t. x-axis. $\boxed{O(n)}$

(iv) Given a convex polygon P with n vertices and a query point q, check whether q lies in P. $\boxed{O(\log n)}$

(v) Given a set S of n line segments, construct its convex hull. $\boxed{O(n \log n)}$

(vi) Given two non-intersecting convex polygons P and Q with m and n vertices respectively, compute their joint convex hull. $\boxed{O(m+n)}$

(vii) Given a convex polygon P with n vertices and a point q inside P, check whether q lies on a diameter of P. $\boxed{O(n)}$

(viii) Given two convex polygons P and Q with n vertices each, check whether P contains Q. $\boxed{O(m+n)}$

(ix) Make a triangulation of a polygon with n vertices.

$O(n \log n)$ [conventional] or $O(n)$ [unconventional, no implementation is found]. You get credit for writing either one.

(x) Given a polygon P with n vertices and two vertices s and t of P, compute the Euclidean shortest path from s to t, contained in P. $\boxed{O(n \log n)}$

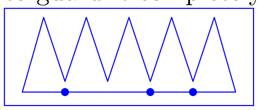
Sol.: Triangulate P, construct its dual graph $G = (V, E)$, then compute the shortest path (Dijkstra's algorithm) from the triangle of s to that of t. As G is planar, $|V| = |E| = O(n)$, and so the runtime will be $O(|E| + |V| \log |V|) = O(n \log n)$.

2. Answer in brief. $\boxed{2\frac{1}{2} \times 6 = 15 \text{ marks}}$

(i) Draw a set of 8 points such that its convex hull has exactly 3 vertices. $\boxed{\text{Easy}}$

(ii) Draw a polygon with 8 vertices such that it is both x-monotone and y-monotone. $\boxed{\text{Convex polygon}}$

(iii) Draw an art gallery (polygon) with at most 11 vertices such that 3 cameras are necessary to guard it completely.



(iv) Recall the algorithm for computing intersection points among n given line segments. Mention the data structures used in the algorithm, and state the total time complexity for all operations performed on each of them.

Sol.: **Event queue Q as a balanced BST:**
Initial construction: $O(n \log n)$ time for $2n$ endpoints;
$|Q| \leq 2n + O(n^2) = O(n^2) \implies$ any search / delete / insert time is $O(\log n) \implies$ total time is $\boxed{O(k \log n)}$, where $k = \#$intersections.

Sweep-line data structure T_λ as a balanced BST:
At most n segments when any event is processed.
$O(n + k)$ event points in total. Any operation takes log time. So, total time for search + delete + insert is $\boxed{O((n + k) \log n)}$.

(v) Is subdivision of a polygon into x-monotone polygons unique? Justify. $\boxed{\text{No, easy to show.}}$

(vi) How many different triangulations are possible for a convex hexagon?
Hint: A convex quadrilateral ABCD has two different triangulations: $\{\triangle\text{ABD}, \triangle\text{BCD}\}$ and $\{\triangle\text{ABC}, \triangle\text{ACD}\}$. A convex pentagon has 5 different triangulations. $\boxed{14}$

Sol.: Consider vertex 1. Denote by ij the line segment joining the vertices i and j. Denote by $f(D)$ the number of triangulations when D is a mandatory list of diagonals. Two cases:

i. $D = \{26\} \implies f(26) = 5$.

ii. $26 \notin D$, i.e., vertex 1 is an endpoint of a diagonal.
Applying principle of inclusion-exclusion, $f(26 \notin D) = f(13) + f(14) + f(15) - f(13, 14) - f(14, 15) - f(13, 15) + f(13, 14, 15) = 5 + 4 + 5 - 2 - 2 - 2 + 1 = 9$.

So, total count $= 5 + 9 = 14$.

3. Given a set S of n points on the plane, suggest an efficient algorithm to find a farthest pair of points in S.

Justify its correctness and deduce its time complexity. $\boxed{5 + 3 + 2 = 10 \text{ marks}}$

Sol.: Find the convex hull of S and then find its diagonal in linear time using the algorithm given in Assignment 3. Total time $= O(n \log n)$.

4. Suppose P is a polygon with $n \geq 4$ vertices, and let $T(P)$ be any triangulation of P. A triangle uvw in $T(P)$ is called a *pendant triangle* if its vertices u, v, w are consecutive along the boundary of P.

Prove or disprove: Every triangulation $T(P)$ contains at least two pendant triangles. $\boxed{10 \text{ marks}}$

Sol.: For $i = 0, 1, 2$, let n_i denote the number of triangles in $T(P)$, such that for each of them, each exactly i edges are edges of P and the rest are diagonals. If P has n vertices, then

$$\text{number of triangles in } T(P) = n_0 + n_1 + n_2 = n - 2.$$

Each diagonal is shared by exactly two triangles, and hence,

$$3n_0 + 2n_1 + n_2 = 2 \cdot \#\text{diagonals} = 2(n - 3) = 2n - 6.$$

Multiplying the first equation by 2 and subtracting the second from that,

$$-n_0 + n_2 = 2 \implies n_2 = n_0 + 2 \implies n_2 \geq 2.$$

5. Determine the diagonals in the x-monotone subdivision of the polygon given below, as per the algorithm discussed in the class. For clarity, vertex indices are shown below their projections on the x-axis in the diagram, with some omitted. These indices are according to their order of processing as the sweep-line moves from left to right.

You need not redraw the polygon on your answer sheet if it is time-consuming. Instead, list each diagonal as (i, j), where i and j are vertex indices, and justify its inclusion. Also explain how and from where the necessary information is obtained. $\boxed{\text{10 marks}}$