Computational Geometry (CS60064)
Spring 2024-25

Instructions

(a) The submission deadline is hard. There may be unforeseen glitches during submission. So, for safety, submit your files well ahead.

(b) All submissions should be on moodle only. No email submission will be accepted, excepting medical reasons.

(c) Do not forget to typeset your solutions. In particular, every mathematical expression must be properly typeset, e.g., the square of n must appear as n^2 and not as `n^2`. Improper typesetting may incur up to 25% deduction in marks.

You can use LaTeX for writing (that is what we recommend); else, typeset in Word and convert to pdf.

Handwritten text—converted to images or to pdf—will not be evaluated.

(d) You must submit all the source files and the final pdf as a single zip file. The name of the zip should be your roll number, followed by a hyphen, followed by the assignment number. For example, if your roll number is `XY190047`, then the zip file for the 1st assignment should be named as `XY190047-a1.zip`. For subsequent assignments, your zip files should be named as `XY190047-a2.zip`, `XY190047-a3.zip`, ...

If you typeset in LaTeX, then the zip should contain one tex file, image files if any, and the final pdf.

If you typeset in Word, then the zip should contain one odt/doc/docx file and the final pdf. Image files get embedded in a Word file, so image files are not needed.

Failing this, your assignment will not be evaluated.

Assignment 1

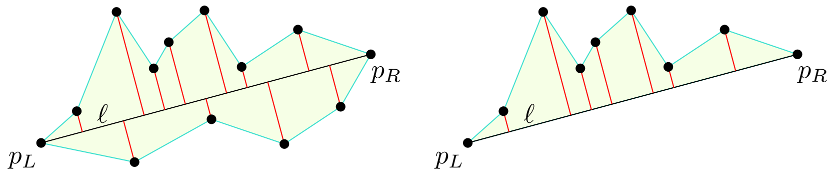
Submission deadline: 19-Jan-2025, 11:55 PM

1.1 Polygon Construction

Given n points on the xy-plane, design an algorithm to construct a simple polygon P such that all the given points serve as vertices of P, and no other points are included as vertices. Provide a proof of correctness for your algorithm and deduce its time complexity. (A *simple polygon* is defined as one in which no two edges intersect, except possibly at their endpoints.) $\boxed{4 + 3 + 3 = 10 \text{ marks}}$

Solution key:

Find the leftmost point p_L and the rightmost point p_R, and join them with a straight-line segment ℓ. Project all points that are above ℓ, on ℓ. Sort these footprints and connect the original points serially in the sorted order to form the upper chain of P; do the same for the lower chain. If one side of ℓ is empty, use ℓ as an edge of the polygon P (as shown in the figure on the right). This can be done in $O(n \log n)$ time.



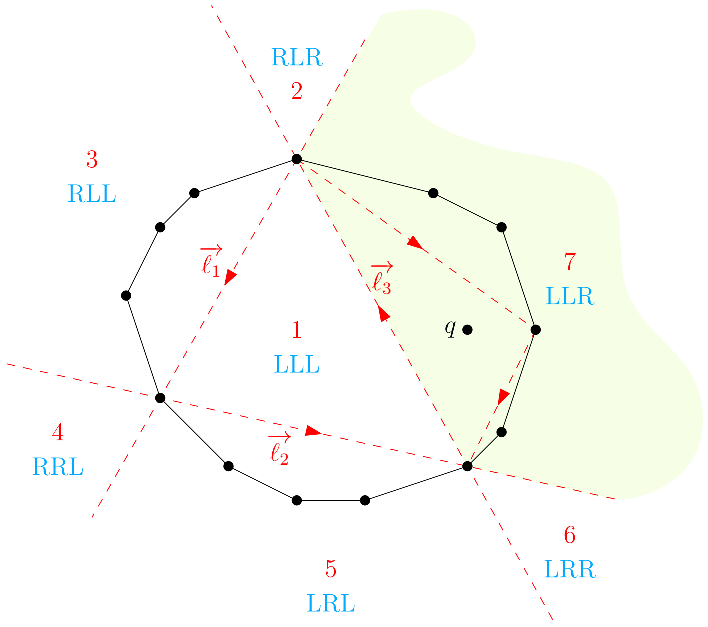
1.2 Point Location

A convex polygon P is provided as a counter-clockwise ordered sequence of n vertices, with their locations specified as (x, y) coordinates. Given a query point q, develop an algorithm to determine whether q lies inside P in $O(\log n)$ time, using $O(n)$ space, including any necessary preprocessing. Justify the time and space complexities of your algorithm. $\boxed{6 + 2 + 2 = 10 \text{ marks}}$

Solution key:

Choose three points on the boundary of P, which are almost equispaced—can be done in $O(1)$ time—via indexing. Construct three directed rays (cut-lines) $\overrightarrow{\ell_1}, \overrightarrow{\ell_2}, \overrightarrow{\ell_3}$ through these points—they partition the 2D-space into seven disjoint regions as shown. The location of the query point q w.r.t. these regions can be determined in $O(1)$ time via three orientation tests. Further refined partitioning can be done through $O(\log_3 n)$ steps, thus giving the precise location of q in $O(\log n)$ time, and in $O(n)$ space.

In the following example, q is initially identified to lie in Region 7, characterized by the unique 3-bit label LLR—indicating that q lies left of $\overrightarrow{\ell_1}$, left of $\overrightarrow{\ell_2}$, and right of $\overrightarrow{\ell_3}$. Subsequently, its position is evaluated with respect to $\overrightarrow{\ell_1}$ and two other rays, resulting in the label LLL, thereby confirming that q lies inside P.


RLR
2
3
RLL
$\overrightarrow{\ell_1}$
$\overrightarrow{\ell_3}$
7
LLR
q
1
LLL
4
RRL
$\overrightarrow{\ell_2}$
6
LRR
5
LRL

Assignment 2

Submission deadline: 26-Jan-2025, 11:55 PM

2.1 Point location w.r.t. line

For some algorithm, we have to test whether a point r lies to the left or right of the directed line $\overrightarrow{pq}$ through two points p and q. Let $p = (p_x, p_y)$, $q = (q_x, q_y)$, and $r = (r_x, r_y)$.

(a) Show that the sign of the determinant

$$D = \begin{vmatrix} 1 & p_x & p_y \\ 1 & q_x & q_y \\ 1 & r_x & r_y \end{vmatrix}$$

determines whether r lies to the left or right of the line.

(b) Show that $|D|$ is in fact twice the area of the triangle determined by p, q, and r.

(c) Why is this an attractive way to implement the basic test in any algorithm where the location of a point is determined w.r.t. a directed line? Provide arguments for both integer and floating-point coordinates. $\boxed{6 + 2 + 2 = 10 \text{ marks}}$

Solution key:

(a) Expanding the determinant:

$$D = \begin{vmatrix} 1 & p_x & p_y \\ 1 & q_x & q_y \\ 1 & r_x & r_y \end{vmatrix} = (q_x r_y - q_y r_x) + p_x(q_y - r_y) + p_y(r_x - q_x).$$

Let $\overrightarrow{u_x}$, $\overrightarrow{u_y}$, and $\overrightarrow{u_z}$ denote the respective unit vectors along the x-, y-, and z-axes. Then,

$$\begin{aligned}
\overrightarrow{pq} \times \overrightarrow{pr} &= \Big((q_x - p_x)\overrightarrow{u_x} + (q_y - p_y)\overrightarrow{u_y}\Big) \times \Big((r_x - p_x)\overrightarrow{u_x} + (r_y - p_y)\overrightarrow{u_y}\Big) \\
&= (q_x - p_x)(r_y - p_y)\overrightarrow{u_z} - (q_y - p_y)(r_x - p_x)\overrightarrow{u_z} \ \ [\text{since } \overrightarrow{u_x} \times \overrightarrow{u_y} = \overrightarrow{u_z} \text{ and } \overrightarrow{u_y} \times \overrightarrow{u_x} = -\overrightarrow{u_z}] \\
&= \Big((q_x - p_x)(r_y - p_y) - (q_y - p_y)(r_x - p_x)\Big)\overrightarrow{u_z} \\
&= D\overrightarrow{u_z}.
\end{aligned}$$

Now, consider an alternative way of evaluating the vector $\overrightarrow{pq} \times \overrightarrow{pr}$, as illustrated in Figure 2.1. Recall the cross-product formula that the vector $\overrightarrow{pq} \times \overrightarrow{pr}$ is given as $\big(|\overrightarrow{pq}| \cdot |\overrightarrow{pr}| \cdot \sin\theta\big)\overrightarrow{u_z}$, where,

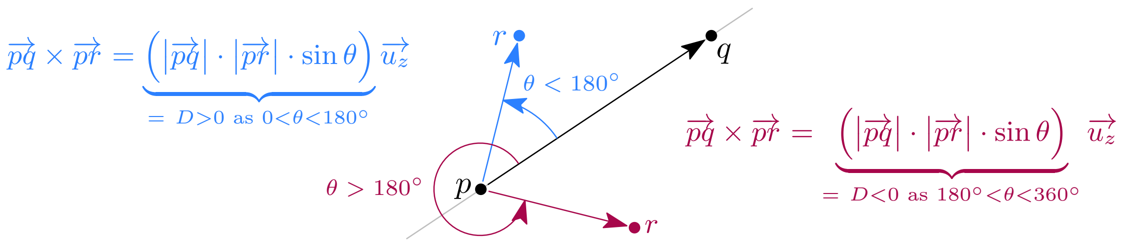

$\overrightarrow{pq} \times \overrightarrow{pr} = \underbrace{\left(|\overrightarrow{pq}| \cdot |\overrightarrow{pr}| \cdot \sin\theta\right)}_{=\ D>0 \text{ as } 0<\theta<180^\circ} \overrightarrow{u_z}$
r
$\theta < 180^\circ$
q
$\theta > 180^\circ$
p
r
$\overrightarrow{pq} \times \overrightarrow{pr} = \underbrace{\left(|\overrightarrow{pq}| \cdot |\overrightarrow{pr}| \cdot \sin\theta\right)}_{=\ D<0 \text{ as } 180^\circ<\theta<360^\circ} \overrightarrow{u_z}$

Figure 2.1: Determining the location of a point r w.r.t. the directed line p-to-q using the sign of D, which is basically the scalar value $|\overrightarrow{pq}| \cdot |\overrightarrow{pr}| \cdot \sin\theta$.

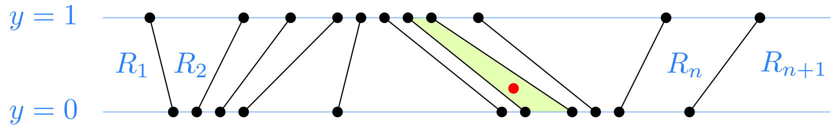


Figure 2.2: Determining the region of a query point in a strip.

θ is the angle measured counterclockwise from $\overrightarrow{pq}$ to $\overrightarrow{pr}$. Since either of the above two ways gives the same vector, we have

$$D = |\overrightarrow{pq}| \cdot |\overrightarrow{pr}| \cdot \sin\theta.$$

Hence, D is positive, negative, or zero depending on the signed value of $\sin\theta$. In other words, the sign of D determines the position of r, as follows:

(i) $D > 0 \implies 0^\circ < \theta < 180^\circ \implies r \in \text{left}(\overrightarrow{pq})$.
(ii) $D < 0 \implies 180^\circ < \theta < 360^\circ \implies r \in \text{right}(\overrightarrow{pq})$.
(iii) $D = 0 \implies \theta = 0^\circ \implies r$ lies on $\overrightarrow{pq}$.

(b) The area of a triangle formed by points p, q, and r is given by

$$\frac{1}{2}\left| |\overrightarrow{pr}| \sin\theta \cdot |\overrightarrow{pq}| \right| = \frac{|D|}{2}.$$

(c) **(i)** **Integer coordinates:** The determinant D is computed using only integer arithmetic, ensuring correct output.
(ii) **Floating-point coordinates:** When dealing with high-precision values (e.g., small numbers with numerous decimal places), the determinant computation may yield inaccurate results due to floating-point rounding errors.

2.2 Point location in strip

Let S be a set of n disjoint line segments whose upper endpoints lie on the line $y = 1$ and whose lower endpoints lie on the line $y = 0$. These segments partition the horizontal strip $[-\infty : \infty] \times [0 : 1]$ into $n + 1$ regions: $R_1, \ldots, R_{n+1}$. Give an $O(n \log n)$-time algorithm to build a binary search tree on the segments in S such that the region containing a query point can be determined in $O(\log n)$ time. Also, describe the query algorithm in full detail. $\boxed{5 + 5 = 10 \text{ marks}}$

Solution key:

See Figure 2.2. Any horizontal line within the strip intersects the segments of S in the same order as the order of their lower (or upper) endpoints. A query point q lies in R_i if and only if to its immediate left lies the $(i-1)$st segment. The sole exception is when q lies in R_1, in which case there is no segment to the left of q.

Using this observation, we construct a height-balanced binary search tree ordered by the x-coordinates of the lower endpoints of the n segments. (Write the steps in detail.) This construction can be completed in $O(n \log n)$ time.

To determine the region of $q = (q_x, q_y)$, use q_x as the search key to find the segment immediately to its left. (Write the steps in detail.) Since the height of the tree is $O(\log n)$, the time complexity for this search is $O(\log n)$.

Assignment 3

Submission deadline: 03-Feb-2025, 11:55 PM

3.1 Diameter of convex polygon

Diameter of a convex polygon is a/the longest line segment contained within it. Given a convex polygon P with n vertices in counterclockwise order, design an $O(n)$-time algorithm to find a diameter of P. Justify its correctness and why its time complexity is $O(n)$. $\boxed{5+3+2=10 \text{ marks}}$

To design the algorithm, you may use Observation 1 based on the notion of a tangent. Recall that a ***tangent*** to a convex polygon P is a straight line passing through a vertex or edge of P such that the interior of P lies entirely on one side of it. Two vertices u and v comprise an ***antipodal pair*** if there exist two tangents, say T_u and T_v, passing through u and v, such that T_u and T_v are parallel to each other and P is sandwiched between them.

Observation 1. *The endpoints of a diameter always comprise an antipodal pair. However, an antipodal pair may not form a diameter, as shown in Figure 3.1.*

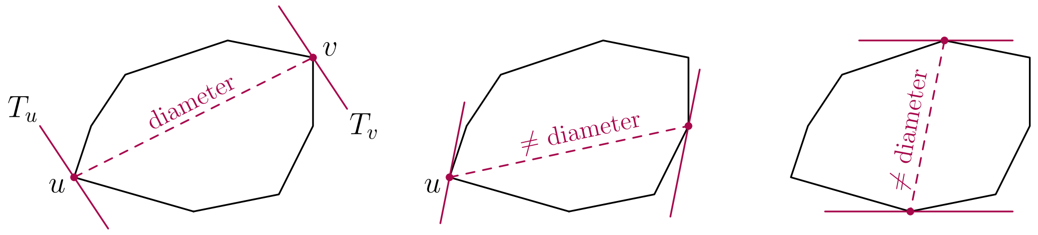


Figure 3.1: Left: $\overline{uv}$ is a diameter of a convex polygon. Middle and right: Not diameters because they are not longest.

3.2 Convex polygon containment

Let P and Q be two convex polygons with m and n vertices, respectively, given in counterclockwise order. Suggest an $O(m \log n)$-time algorithm to check whether P contains Q. Suggest another algorithm that will take time $O(m+n)$.

Derive the time complexities of both. $\boxed{(2+2)+(3+3)=10 \text{ marks}}$

Assignment 4

Submission deadline: 02-March-2025, 11:55 PM

4.1 Code for x-monotone polygon

Write a C/C++/python code to generate a simple x-monotone polygon P with n randomly-positioned vertices as follows:

(1) Generate n random vertices first for the polygon. During execution, the user will provide the value of n ($3 \leq n \leq 100$) as the input to your code.
(2) The vertex coordinates should be positive integers, in multiples of 10, with x and y values at most 800 and 600 respectively.
(3) Duplicate vertices should be discarded so that P has finally exactly n vertices.
(4) No three vertices are collinear.

You can use the idea of the algorithm in §1.1 The output should be two files:

(i) A simple text file named `monopoly.txt`. It should contain the value of n in the first line, followed by the counterclockwise list of vertices in n lines, with the coordinates of each vertex separated by a space in every new line. So, for example, if $n = 20$, the file should contain 21 lines.
(ii) An SVG file named `monopoly.svg` containing the polygon P.
The canvas size can be specified as:

```
<svg width="840px" height="640px" xmlns="http://www.w3.org/2000/svg" version="1.1">
```

The vertices and edges of the polygon to be colored black. The vertices can be small discs with `r="3" stroke="black" stroke-width="2"` and the edges can have `stroke-width:2`.

Note:

SVG files are simple text files and best for visual coding in 2D. Check with the file `test.svg` that I have shared on moodle. Open it on an Internet browser or an image viewer, and also in some text editor such as `gedit`. You will understand what is what :)

To know further about SVG, you can see:

- `https://www.w3schools.com/graphics/svg_intro.asp`
- `https://en.wikipedia.org/wiki/SVG`

$\boxed{\text{10 marks}}$

4.2 Code for triangulation of x-monotone polygon

Write a C/C++/python code for the following:

(1) User input: The name of a text file as input (e.g., `monopoly.txt` generated by your code in §4.1). This file will contain the n vertices of a polygon P: the value of n in the first line, followed by the counterclockwise list of vertices (all with integer coordinates), with the coordinates of each vertex separated by a space in a new line. Assume that it will be in the same folder as your code.

(2) Implement the linear-time triangulation algorithm discussed in the class to triangulate P and save the output in an SVG file named `monopolyTr.svg`. Recall that in the triangulation, a diagonal is added (D1) between two vertices in the same chain or (D2) between two vertices in two different chains. In the SVG, you have color the diagonals D1 in blue and D2 in brown, using `stroke-width:1` for both.

The polygon edges should be drawn after drawing the diagonals, using black color and `stroke-width:2`. The polygon vertices should be drawn at the end. Then the rendition will look nice. $\boxed{\text{10 marks}}$

Assignment 5

Submission deadline: 11-March-2025, 11:55 PM

5.1 Voronoi diagram

(a) Give an example of 9 sites such that their VD has 4 vertices. $\boxed{\text{3 marks}}$

(b) Give an example of 6 sites such that the plane sweep algorithm encounters the six site events before any of the circle events. The sites should lie in general position: no three sites on a line and no four sites on a circle. $\boxed{\text{3 marks}}$

(c) Give an example where the parabola defined by some site contributes more than one arc to the beach line.
Can you give an example where it contributes a linear number of arcs? $\boxed{2 + 2 \text{ marks}}$

5.2 Voronoi diagram time complexity

Show that $\Omega(n \log n)$ is a lower bound for computing Voronoi diagrams.
Hint: Reducing the sorting problem to the problem of computing Voronoi diagrams. You can assume that any Voronoi diagram algorithm always reports every Voronoi region as a sequence of its vertices in counterclockwise order. $\boxed{\text{10 marks}}$