CS 164 - Computational Geometry **Winter 2016**

HW 1 Solutions

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1. (Problem 1.1) The convex hull of a set S is defined to the be the intersection of all convex sets that contain S. For the convex hull of a set of points it was indicated that the convex hull is the convex set with smallest perimeter. We want to show that these are equivalent definitions.

   (a) Prove that the intersection of two convex sets is again convex. This implies that the intersection of a finite family of convex sets is convex as well.

   Let S_1 and S_2 be two convex sets. We will show that $S_1 \cap S_2$ is convex, that is, that for any two points in $S_1 \cap S_2$, the line segment between them is in $S_1 \cap S_2$.

   Let our two points $x, y \in S_1 \cap S_2$. Since $x, y \in S_1$, and S_1 is convex, the line segment $\overline{xy} \subset S_1$. Similarly, since $x, y \in S_2$, and S_2 is convex, the line $\overline{xy} \subset S_2$.

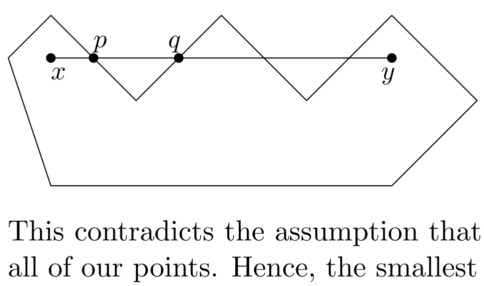
   Therefore, for any two points $x, y \in S_1 \cap S_2$, the line segment $\overline{xy} \subset S_1 \cap S_2$. So $S_1 \cap S_2$ is convex. By induction, any finite intersection of convex sets is convex. QED.

   (b) Prove that the smallest perimeter polygon $\mathcal{P}$ containing a set of points is convex.

   Intuitively, if a polygon is not convex, we can decrease its perimeter by short-cutting some of the sides.

   More formally, suppose by contradiction that $\mathcal{P}$ is the smallest perimeter polygon for a given set of points, and is not convex. Then for some x, y, the line segment $L = \overline{xy}$ crosses $\mathcal{P}$.

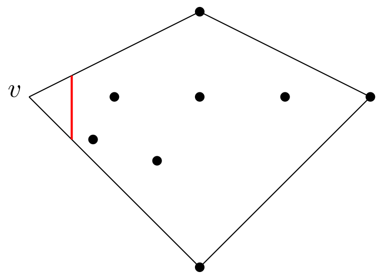
   L may cross in and out of $\mathcal{P}$ any even number of times. So take the two crossings closest to x, which we will call p and q. Then replace the boundary of $\mathcal{P}$ between p and q with the line segment L'. Since the shortest distance between two points is a straight line, this decreases the perimeter of our polygon.

   This contradicts the assumption that $\mathcal{P}$ was the smallest perimeter polygon containing all of our points. Hence, the smallest perimeter polygon must be convex. QED.

   (c) Prove that any convex set containing the set of points P contains the smallest perimeter polygon $\mathcal{P}$.

Suppose that the smallest perimeter polygon contains a vertex v not in P. Since P is finite, we can decrease the perimeter by chopping off that corner with a new line segment. By contradiction, this means that every vertex of $\mathcal{P}$ must be a point in P.



Hence every segment in $\mathcal{P}$ must be a line between two points in P. Any convex set containing P must contain all of these line segments. Hence, any convex set containing P must contain $\mathcal{P}$. QED.

Note: It may still be unclear how the three parts of this problem can be combined to produce the final result - that our two definitions for a convex hull are equivalent.

To show that two sets are equal, we usually show that each contains the other. Let P be our finite set of points, let $\mathcal{P}_1$ be the intersection of every convex set containing P, and let $\mathcal{P}_2$ be the smallest-perimeter polygon containing P. Our goal is to show that $\mathcal{P}_1 = \mathcal{P}_2$.

First, we show that $\mathcal{P}_1 \subseteq \mathcal{P}_2$. By part (b), the smallest-perimeter polygon containing a set of points is convex. Hence, $\mathcal{P}_2$ is one of the sets in the intersection that makes up $\mathcal{P}_1$. Hence, $\mathcal{P}_1 \subseteq \mathcal{P}_2$.

Second, we show that $\mathcal{P}_2 \subseteq \mathcal{P}_1$. First we need to show that $\mathcal{P}_1$ is convex. For this, we need a stronger version of part (a) that applies to infinite intersections of convex sets, rather than finite families.

Next, we know by part (c) that any convex set containing the set of points P also contains the smallest perimeter polygon. Hence, since $\mathcal{P}_1$ is convex, it contains $\mathcal{P}_2$. In other words, $\mathcal{P}_2 \subseteq \mathcal{P}_1$.

Since the sets produced by our two definitions both contains one another, they must be equal. QED.

2. Describe an $O(n \log n)$ time method for determining if two sets A and B of n points in the plane can be separated by a line.

First, note that A and B can be separated by a line if and only if their convex hulls do not overlap.

So our method will first compute the convex hulls of A and B. This takes $O(n \log n)$ time.

Then we compute all of the line segment intersections for the two convex hulls. If we find a single intersection, we stop. This also takes $O(n \log n)$ time. (See Chapter 2.) Therefore, our complete algorithm takes $O(n \log n)$ time. QED.

3. (Problem 1.4) For the convex hull algorithm we have to be able to test whether a point R lies left or right of the directed line through two points p and q. Let $p = (p_x, p_y), q = (q_x, q_y), r = (r_x, r_y)$.

   (a) Show that the sign of the determinant

$$D = \begin{vmatrix} 1 & p_x & p_y \\ 1 & q_x & q_y \\ 1 & r_x & r_y \end{vmatrix}$$

   determines whether r lies left or right of the line.

   We can simplify D by first subtracting the second row from the the third row, and then subtracting the first row from the second row. This gives

$$D = \begin{vmatrix} 1 & p_x & p_y \\ 0 & q_x - p_x & q_y - p_y \\ 0 & r_x - q_x & r_y - q_y \end{vmatrix} = (q_x - p_x)(r_y - q_y) - (r_x - q_x)(q_y - p_y)$$

   Next we show that this equation is the same one we get when we test whether r is above or below the line through p and q.

   Solution 1 (with vectors):
   Now suppose we have the three points p, q, r in the plane. The vector from p to q is $\bar{v} = \langle q_x - p_x, q_y - p_y \rangle$.

   To determine which side r is on, we need the orthogonal vector to $\bar{v}$. If a line has slope b/a, the orthogonal line has slope $-a/b$. So the orthogonal vector is $\bar{v}' = \langle -(q_y - p_y), q_x - p_x \rangle$.

   Next we take the dot product of this vector and the vector from q to r. If these vectors point in the same direction, then the dot product is positive, so r is on one side of q. But if they point in opposite directions, then the dot product is negative, and r is on the other side of q.

   The vector from q to r is $\bar{w} = \langle r_x - q_x, r_y - q_y \rangle$. The dot product is then

$$\bar{v}' \cdot w = (q_x - p_x)(r_y - q_y) - (r_x - q_x)(q_y - p_y)$$

   This is the same value as D.

   Solution 2 (Messy, but only high school algebra):
   First we find the line through our points p and q. Then we compute the y-coordinate on this line given r_x, and see whether r_y is above or below the line.

The slope of the line will be
$$m = \frac{q_y - p_y}{q_x - p_x}.$$
To find the intercept, we can take the standard equation $y = mx + b$, and plug in either one of our points. I'll plug in the coordinates for q, giving

$$\begin{aligned} b &= y - mx \\ &= q_y - \frac{(q_y - p_y)q_x}{q_x - p_x} \end{aligned}$$

Next we take our line and plug in the value for r_x to get the y-coordinate of our line.

$$\begin{aligned} y &= mx + b \\ &= \left(\frac{q_y - p_y}{q_x - p_x}\right) r_x + \left(q_y - \frac{(q_y - p_y)q_x}{q_x - p_x}\right) \end{aligned}$$

We have a left turn if $r_y > y$, i.e., if $r_y - y > 0$. We substitute our value for y, and then multiply through by the denominator to simplify the expression, and collect terms.

$$\begin{aligned} &r_y - y > 0 \\ &r_y - \left(\frac{q_y - p_y}{q_x - p_x}\right) r_x - \left(q_y - \frac{(q_y - p_y)q_x}{q_x - p_x}\right) > 0 \\ &r_y(q_x - p_x) - (q_y - p_y)r_x - q_y(q_x - p_x) + (q_y - p_y)q_x > 0 \\ &(q_x - p_x)(r_y - q_y) - (r_x - q_x)(q_y - p_y) > 0 \end{aligned}$$

This is also the same value as D.

Both of our solutions show that r is on the left side when D is positive, and on the right side when D is negative. Q.E.D.

(b) Show that $|D|$ is in fact twice the area of the triangle determined by p, q, and r.

Let $\overline{pq}$ be the base of our triangle. Using the same notation as in part (a), this has length $||\bar{v}||$. The height is given by the component of $\overline{qr}$ that is perpendicular to the base, which has length $\frac{|\bar{v}' \cdot w|}{||\bar{v}'||}$. Also note that $||\bar{v}|| = ||\bar{v}'||$. Therefore,

$$\begin{aligned} |D| &= |\bar{v}' \cdot w| \\ &= ||\bar{v}|| \frac{|\bar{v}' \cdot w|}{||\bar{v}'||} \\ &= \text{base} * \text{height} \end{aligned}$$

The area of the triangle is 1/2*base*height, so $|D|$ is indeed twice the area of this triangle.

(c) Why is this an attractive way to implement the basic test in algorithm ConvexHull? Give an argument for both integer and floating point coordinates.

This method is attractive both for computation time and precision. Finding $|D|$ involves addition, subtraction, and multiplication, which are cheap operations to perform on a processor, and unlikely to introduce large errors.

Other methods might require computing trig functions or inverse trig functions, which are very expensive, and can lead to large errors.

4. (Problem 1.6a) In many situations we need to compute convex hulls of objects other than points.

   Let S be a set of n line segments in the plane. Prove that the convex hull of S is exactly the same as the convex hull of the $2n$ endpoints of the segments.

   Let S' be the set of endpoints of our line segments. Let H_1 be the convex hull of our line segments, and let H_2 be the convex hull of the endpoints of those line segments. We will show that $H_2 \subseteq H_1$, and that $H_1 \subseteq H_2$.

   First, it is trivial to see that $H_2 \subseteq H_1$, since any point in S' is contained in one of the line segments in S.

   Next, we show that $H_1 \subseteq H_2$, by showing that each line segment in S is contained in H_2. Consider a line segment $l \in S$, with endpoints p, q. Since H_2 is a convex hull that contains the points p and q, it must contain the line segment between them, which is l. So every line segment in S is contained in H_2.

   Since $H_2 \subseteq H_1$ and $H_1 \subseteq H_2$, we conclude that $H_1 = H_2$. Q.E.D.

5. (Problem 1.8a) The $O(n \log n)$ algorithm to compute the convex hull of a set of n points in the plane that was described in the chapter is based on the paradigm of incremental construction: add the points one by one, and update the convex hull after each addition. In this exercise we shall develop an algorithm based on another paradigm, namely divide-and-conquer.

   Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be two disjoint convex polygons with n vertices in total. Give an $O(n)$ time algorithm that computes the convex hull of $\mathcal{P}_1 \cup \mathcal{P}_2$.

   We will show how to find the upper convex hull. The lower convex hull can be found similarly.

   We first find the leftmost point of $\mathcal{P}_1$, and then list the points in the upper half of $\mathcal{P}_1$ from left to right. We do the same for $\mathcal{P}_2$. Then we merge these two lists to get the list of all points in the upper hull from left to right. Finally, we use the plane sweep step of the Graham scan algorithm to compute the upper hull of this set of points.

   Every step in this algorithm takes time $O(n)$, so the overall time is still $O(n)$. Q.E.D.