Vectors

Now that we have a basic understanding of simple matrices and some of their properties, we need to take a little step to the side and introduce vectors. Vectors and matrices are closely related concepts, and throughout the course we'll be using them together to solve problems.

A **vector** has two pieces of information contained within it:

1. the direction in which the vector points, and

2. the **magnitude** of the vector, which is just the length of the vector.

You can express a vector in several ways. Sometimes it's expressed kind of like a coordinate point, $\vec{a} = (3,4)$, or $\vec{b} = (3,4,5)$.

Row and column vectors

But these same vectors $\vec{a} = (3,4)$ and $\vec{b} = (3,4,5)$ can also be expressed as **column matrices** (also called **column vectors**),

$$\vec{a} = \begin{bmatrix}3\\4\end{bmatrix} \text{ and } \vec{b} = \begin{bmatrix}3\\4\\5\end{bmatrix}$$

or as a **row matrices** (also called **row vectors**),

$$\vec{a} = \begin{bmatrix}3 & 4\end{bmatrix} \text{ and } \vec{b} = \begin{bmatrix}3 & 4 & 5\end{bmatrix}$$

This is part of the reason why we say that matrices and vectors are closely related. A **column matrix** is a matrix with any number of rows, but exactly one column; a **row matrix** is a matrix with exactly one row, but any number of columns. And it's really common to write vectors as these kinds of matrices.

In fact, for any matrix, each column in the matrix is technically a **column vector**. For example, in matrix A,

$$A = \begin{bmatrix} 4 & -6 & 1 & -8 & 5 \\ 1 & 1 & -2 & 9 & 0 \end{bmatrix}$$

there are five column vectors:

$$a_1 = \begin{bmatrix} 4 \\ 1 \end{bmatrix}, a_2 = \begin{bmatrix} -6 \\ 1 \end{bmatrix}, a_3 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}, a_4 = \begin{bmatrix} -8 \\ 9 \end{bmatrix}, \text{ and } a_5 = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$$

Because each of these column vectors has two components, it means they're vectors in two-dimensional space, $\mathbb{R}^2$. In matrix B,

$$B = \begin{bmatrix} 7 & 3 & 4 \\ 1 & 6 & 1 \\ 2 & 2 & 3 \end{bmatrix}$$

there are three column vectors,

$$b_1 = \begin{bmatrix} 7 \\ 1 \\ 2 \end{bmatrix}, b_2 = \begin{bmatrix} 3 \\ 6 \\ 2 \end{bmatrix}, \text{ and } b_3 = \begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix}$$

Because each of these column vectors has three components, it means they're vectors in three-dimensional space, $\mathbb{R}^3$.

Similarly, we could also say that A has two **row vectors**,

$$a_1 = \begin{bmatrix} 4 & -6 & 1 & -8 & 5 \end{bmatrix} \text{ and } a_2 = \begin{bmatrix} 1 & 1 & -2 & 9 & 0 \end{bmatrix}$$

or that B has three row vectors,

$$b_1 = \begin{bmatrix} 7 & 3 & 4 \end{bmatrix} \text{ and } b_2 = \begin{bmatrix} 1 & 6 & 1 \end{bmatrix} \text{ and } b_3 = \begin{bmatrix} 2 & 2 & 3 \end{bmatrix}$$

When we look at a set of row vectors or column vectors, it's important to understand the space that the vectors occupy. There are two aspects we want to consider: first, the space $\mathbb{R}^n$ in which the vectors lie, and second, the dimension of the "surface" or "space" formed by the vectors specifically.

For instance, given the two vectors $a_1 = \begin{bmatrix} 4 & -6 & 1 & -8 & 5 \end{bmatrix}$ and $a_2 = \begin{bmatrix} 1 & 1 & -2 & 9 & 0 \end{bmatrix}$,

- because they each have 5 components, they're vectors in $\mathbb{R}^5$, and
- because there are 2 vectors, they form a two-dimensional plane in $\mathbb{R}^5$.

Or given the three vectors $b_1 = \begin{bmatrix} 7 & 3 & 4 \end{bmatrix}$ and $b_2 = \begin{bmatrix} 1 & 6 & 1 \end{bmatrix}$ and $b_3 = \begin{bmatrix} 2 & 2 & 3 \end{bmatrix}$,

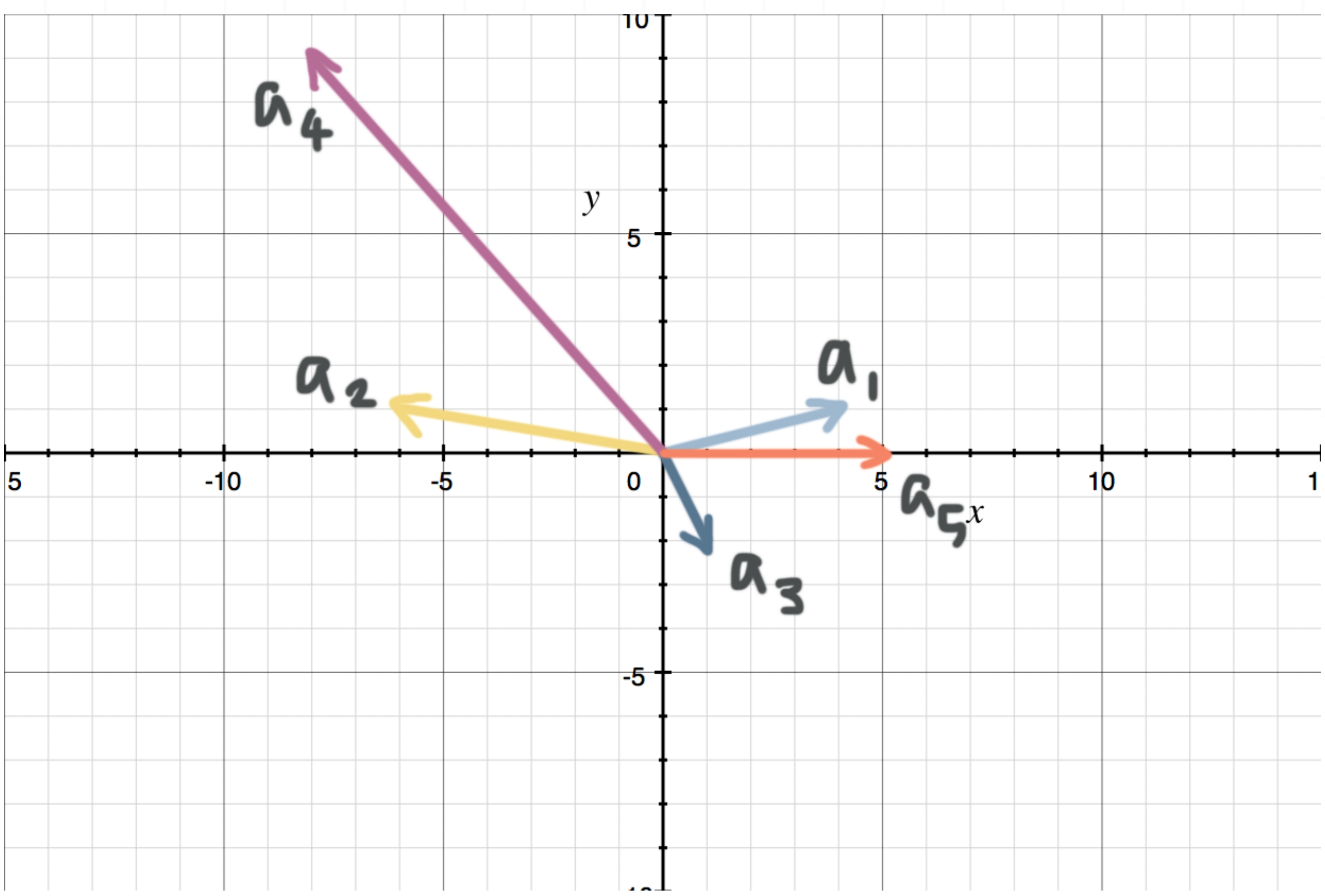
- because they each have 3 components, they're vectors in $\mathbb{R}^3$, and
- because there are 3 vectors, they form a three-dimensional space in $\mathbb{R}^3$ (in other words, they span all of $\mathbb{R}^3$ space).

Vectors can be moved

To sketch a vector, we often start at the origin and move out to the "coordinate point" that's expressed by the vector. Placing the starting point of the vector at the origin means that you're sketching the vector in **standard position**. For instance, the two-dimensional vectors

$$a_1 = \begin{bmatrix}4\\1\end{bmatrix}, a_2 = \begin{bmatrix}-6\\1\end{bmatrix}, a_3 = \begin{bmatrix}1\\-2\end{bmatrix}, a_4 = \begin{bmatrix}-8\\9\end{bmatrix}, \text{ and } a_5 = \begin{bmatrix}5\\0\end{bmatrix}$$

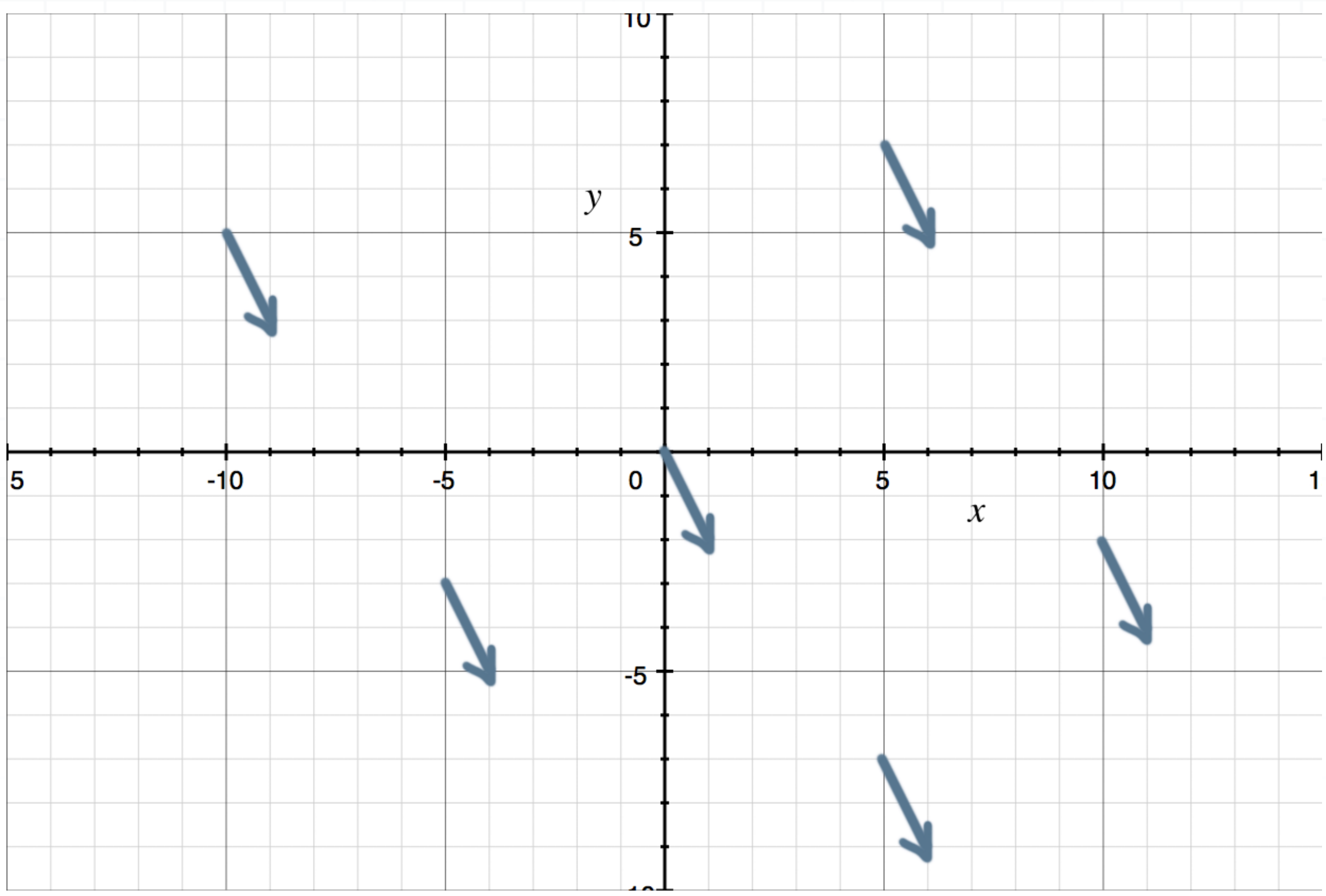
could all be sketched together in $\mathbb{R}^2$:



Each of these vectors has its **initial point** at the origin (each vector "starts" at $(0,0)$), and its **terminal point** at the location described by the vector (each vector "ends" at the coordinate point that you see in each vector).

The little arrow at the end of the vector indicates the terminal point, and then the other end of the vector is the initial point.

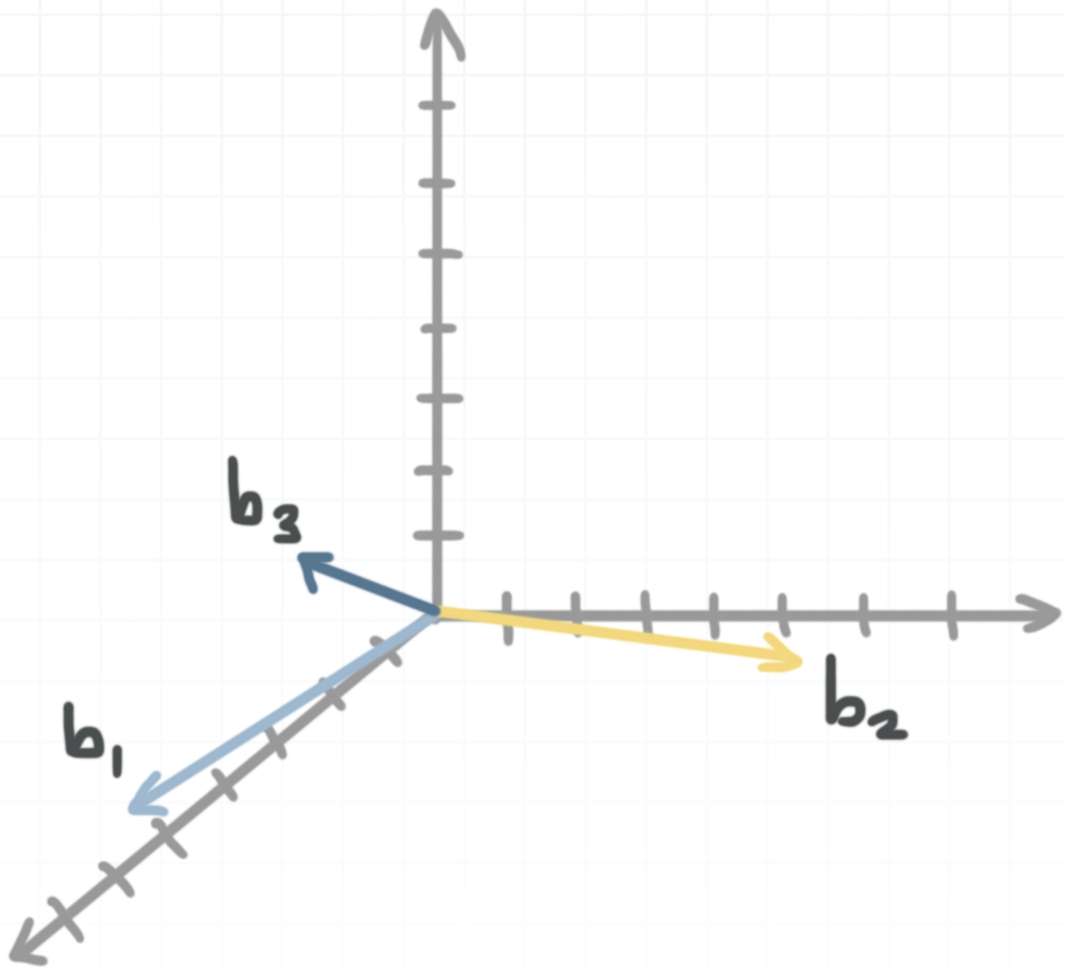
Remember that the information contained in a vector is only its direction and its length, which means vectors don't always have to start at the origin. We can move a vector parallel to itself anywhere in coordinate space, and it'll still be the same vector. Here are many different vectors, every one of which is $a_3 = [1 \quad -2]$.



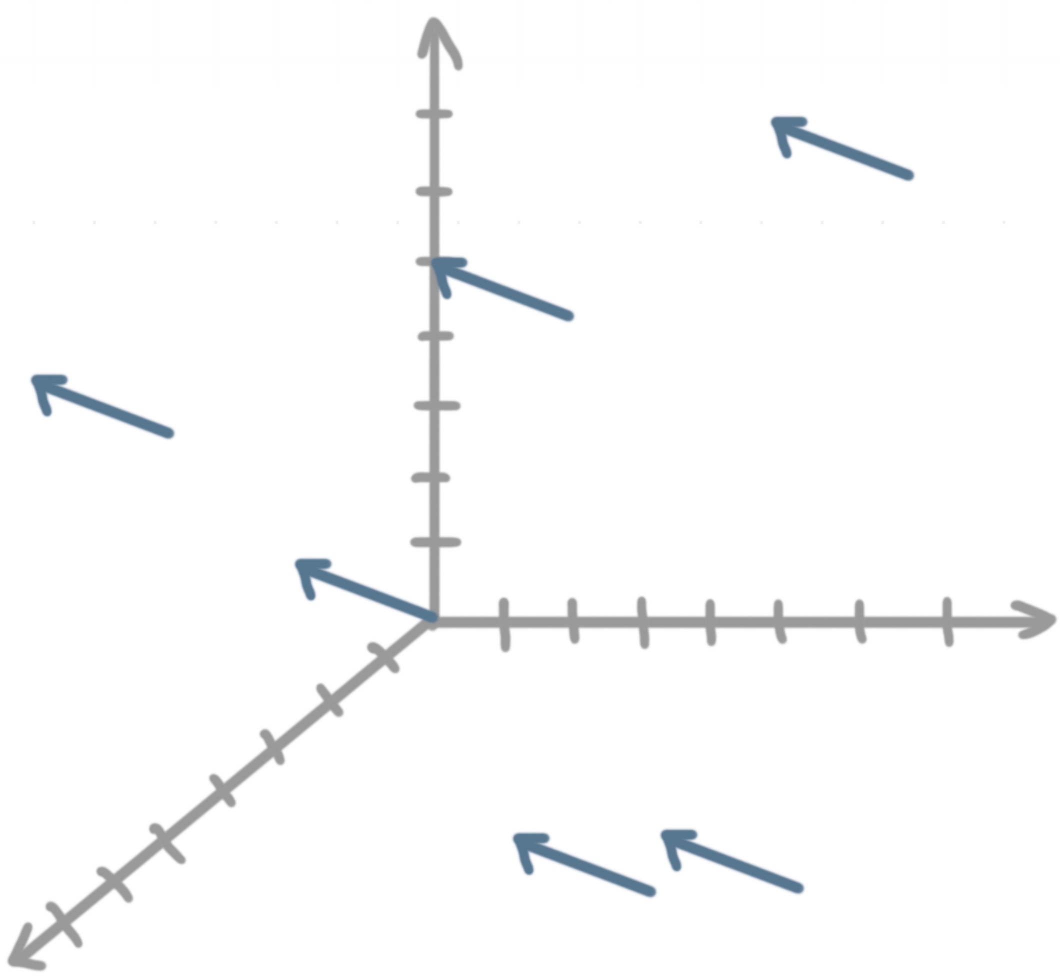
Similarly, the three-dimensional vectors

$$b_1 = \begin{bmatrix} 7 \\ 1 \\ 2 \end{bmatrix}, \; b_2 = \begin{bmatrix} 3 \\ 6 \\ 2 \end{bmatrix}, \text{ and } b_3 = \begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix}$$

could all be sketched together in $\mathbb{R}^3$:



And here are many different vectors, every one of which is $b_3 = [4 \quad 1 \quad 3]$.



Let's do an example of sketching vectors.

Example

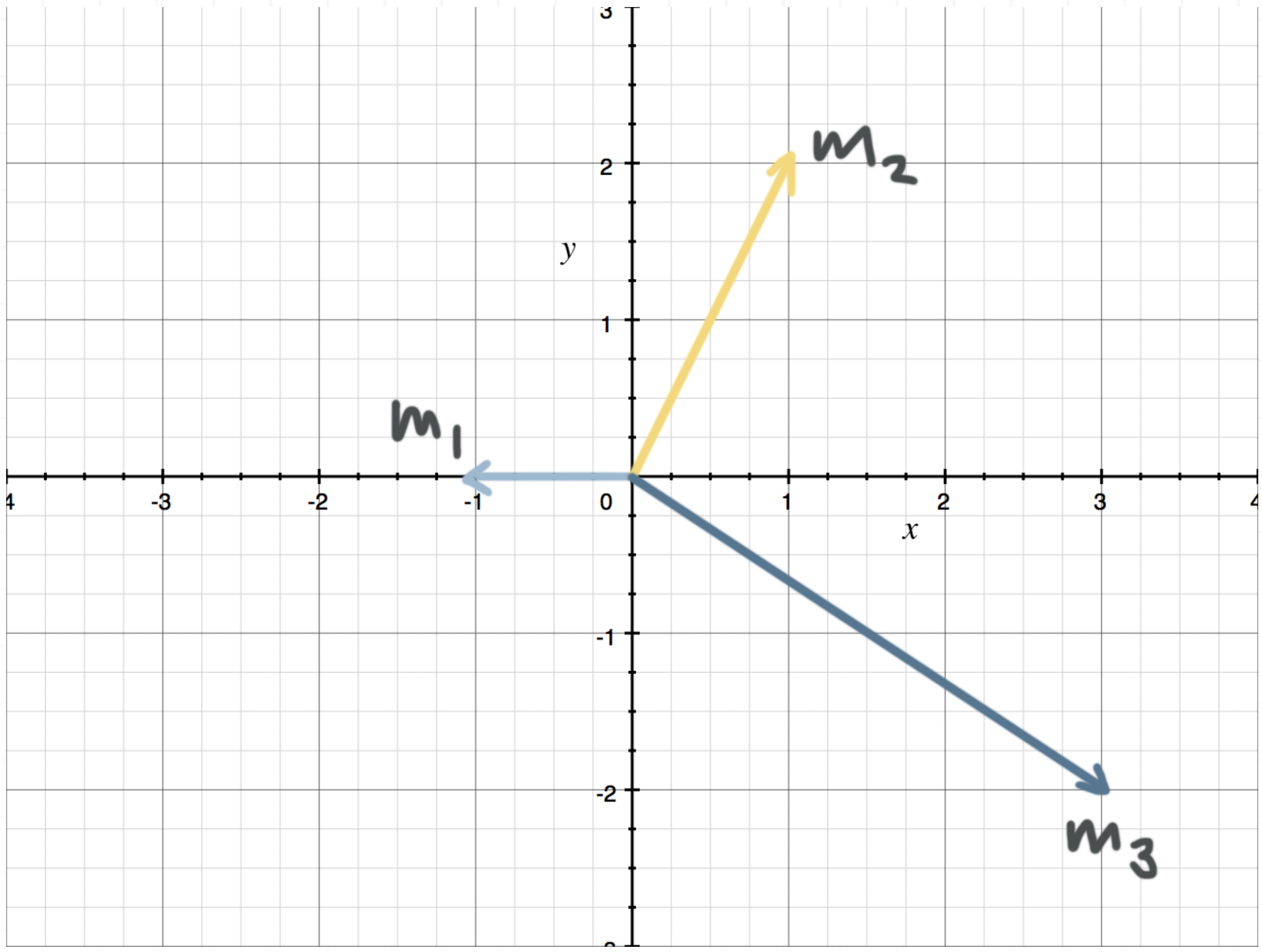
Sketch the column vectors of M in standard position in $\mathbb{R}^2$.

$$M = \begin{bmatrix} -1 & 1 & 3 \\ 0 & 2 & -2 \end{bmatrix}$$

The column vectors of M are

$$m_1 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, m_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \text{ and } m_3 = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$$

The vector m_1 in standard position will point from $(0,0)$ to $(-1,0)$; the vector m_2 in standard position will point from $(0,0)$ to $(1,2)$; the vector m_3 in standard position will point from $(0,0)$ to $(3,-2)$. The sketch of the three of them together in $\mathbb{R}^2$ is therefore



m1
m2
m3
x
y