Linear Algebra Workbook Solutions

Operations on one matrix

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MATH

LINEAR SYSTEMS IN TWO UNKNOWNS

1. Find the unique solution to the system of equations.

$$-x + 2y = 6$$

$$3x = y - 10$$

Solution:

Solve for x in the second equation.

$$3x = y - 10$$

$$x = \frac{y - 10}{3}$$

Plug this value for x into the first equation, then solve for y.

$$-x + 2y = 6$$

$$-\frac{y - 10}{3} + 2y = 6$$

$$-y + 10 + 6y = 18$$

$$5y = 8$$

$$y = \frac{8}{5}$$

Plug $y = 8/5$ back into the equation we found for x.

$$x = \frac{y - 10}{3}$$

$$x = \frac{\frac{8}{5} - 10}{3}$$

$$x = \frac{\frac{8}{5} - \frac{50}{5}}{3}$$

$$x = -\frac{42}{5} \cdot \frac{1}{3}$$

$$x = -\frac{14}{5}$$

The unique solution to the system is

$$\left(-\frac{14}{5}, \frac{8}{5}\right)$$

2. Find the unique solution to the system of equations.

$$-5x + y = 8$$

$$y = 3x - 8$$

Solution:

Taking the value for y given in the second equation as $y = 3x - 8$, we'll substitute for y in the first equation.

$$-5x + y = 8$$

$$-5x + (3x - 8) = 8$$

$$-5x + 3x - 8 = 8$$

$$-2x = 16$$

$$x = -8$$

Now substitute $x = -8$ into the second equation to find a value for y.

$$y = 3x - 8$$

$$y = 3(-8) - 8$$

$$y = -32$$

The unique solution to the system is

$$(-8, -32)$$

3. Find the unique solution to the system of equations.

$$2x - y = 5$$

$$-3x + y = 7$$

Solution:

If we add the two equations together to eliminate y, we get

$$2x - y + (-3x + y) = 5 + (7)$$

$$2x - 3x = 12$$

$$-x = 12$$

$$x = -12$$

Plug $x = -12$ back into the second equation.

$$-3x + y = 7$$

$$-3(-12) + y = 7$$

$$y = -29$$

The solution to the system is

$$(-12, -29)$$

4. Find the unique solution to the system of equations.

$$x = 2y - 5$$

$$-3x + 6y = 15$$

Solution:

Multiplying the first equation by 3 gives

$$x = 2y - 5$$

$$3x = 6y - 15$$

Then adding $3x = 6y - 15$ to $-3x + 6y = 15$ gives

$$3x - 6y + (-3x + 6y) = -15 + (15)$$

$$3x - 6y - 3x + 6y = -15 + 15$$

$$-6y + 6y = -15 + 15$$

$$0 = 0$$

This is always true, so there are infinitely many solutions to the system of equations.

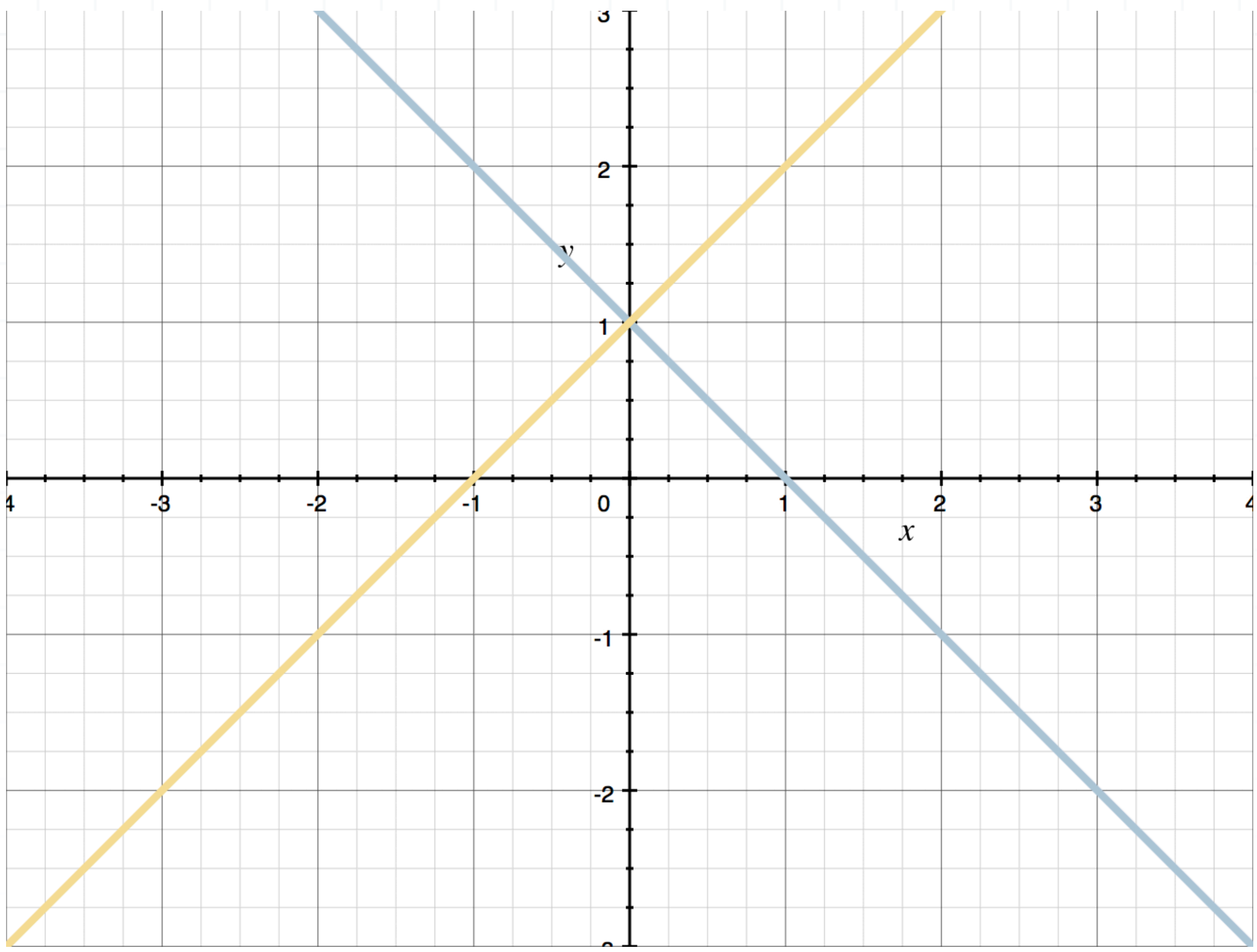
■ 5. Find the unique solution to the system of equations using the graphing method.

$$y - 2 = -(x + 1)$$

$$y = x + 1$$

Solution:

As you can see from the graphs of the two functions, the intersection point is at $(0,1)$, which means $(0,1)$ is the solution to the system of equations.



■ 6. Find the unique solution to the system of equations using the substitution method.

$$5y + x = 4$$

$$3y - 3x = 6$$

Solution:

Solve the first equation for x.

$$5y + x = 4$$

$$x = 4 - 5y$$

Substitute this into the second equation.

$$3y - 3x = 6$$

$$3y - 3(4 - 5y) = 6$$

$$3y - 12 + 15y = 6$$

$$18y = 18$$

$$y = 1$$

Plug $y = 1$ into the equation for x.

$$x = 4 - 5y$$

$$x = 4 - 5(1)$$

$$x = -1$$

Therefore the solution to the system of equation is

$$(-1,1)$$

LINEAR SYSTEMS IN THREE UNKNOWNS

■ 1. Find the unique solution to the system of equations.

$$2x + y - z = 3$$

$$x - y + z = 0$$

$$x - 2y - 3z = 4$$

Solution:

Let's number the equations to stay organized.

[1] $2x + y - z = 3$

[2] $x - y + z = 0$

[3] $x - 2y - 3z = 4$

Add equations [1] and [2] so that y and z will be eliminated.

$$(2x + y - z) + (x - y + z) = (3) + (0)$$

$$2x + y - z + x - y + z = 3$$

$$3x + y - y + z - z = 3$$

$$3x = 3$$

$$x = 1$$

Let's plug $x = 1$ into equations [2] and [3] to put them in terms of y and z.

$$x - y + z = 0$$

$$1 - y + z = 0$$

[4] $-y + z = -1$

and

$$x - 2y - 3z = 4$$

$$1 - 2y - 3z = 4$$

[5] $-2y - 3z = 3$

Multiply equation [4] by 3,

$$3(-y + z = -1)$$

$$-3y + 3z = -3$$

and then add this to equation [5] so that z can be eliminated.

$$(-3y + 3z) + (-2y - 3z) = (-3) + (3)$$

$$-3y + 3z - 2y - 3z = 0$$

$$-3y - 2y = 0$$

$$-5y = 0$$

$$y = 0$$

Plug $y = 0$ into equation [4] to solve for z.

$$-2y - 3z = 3$$

$$-2(0) - 3z = 3$$

$$-3z = 3$$

$$z = -1$$

Therefore, the solution to the system is $(x, y, z) = (1,0,-1)$.

2. Find the unique solution to the system of equations.

$$3x + y - z = -2$$

$$x - 2y + 3z = 23$$

$$2x + 3y + 2z = 5$$

Solution:

Let's number the equations to stay organized.

[1] $3x + y - z = -2$

[2] $x - 2y + 3z = 23$

[3] $2x + 3y + 2z = 5$

Multiply equation [1] by 2,

$$2(3x + y - z = -2)$$

$$6x + 2y - 2z = -4$$

and add this to equation [3] so that z can be eliminated.

$$(6x + 2y - 2z) + (2x + 3y + 2z) = (-4) + (5)$$

$$6x + 2y - 2z + 2x + 3y + 2z = 1$$

$$6x + 2y + 2x + 3y = 1$$

[4] $8x + 5y = 1$

Multiply equation [1] by 3,

$$3(3x + y - z = -2)$$

$$9x + 3y - 3z = -6$$

and add this to equation [2] so that z can be eliminated.

$$(9x + 3y - 3z) + (x - 2y + 3z) = (-6) + (23)$$

$$9x + 3y - 3z + x - 2y + 3z = 17$$

$$9x + 3y + x - 2y = 17$$

[5] $10x + y = 17$

Multiply equation [5] by -5,

$$-5(10x + y = 17)$$

$$-50x - 5y = -85$$

and then add this to equation [4] so that y will be eliminated.

$$(8x + 5y) + (-50x - 5y) = (1) + (-85)$$

$$8x + 5y - 50x - 5y = -84$$

$$8x - 50x = -84$$

$$-42x = -84$$

$$x = 2$$

Let's plug $x = 1$ into equation [5] to solve for y.

$$10x + y = 17$$

$$10(2) + y = 17$$

$$20 + y = 17$$

$$y = -3$$

Plug $x = 2$ and $y = -3$ into any of the original equations to solve for z. We'll use equation [1].

$$3x + y - z = -2$$

$$3(2) + (-3) - z = -2$$

$$6 - 3 - z = -2$$

$$3 - z = -2$$

$$-z = -5$$

$$z = 5$$

Therefore, the solution to the system is $(x, y, z) = (2, -3, 5)$.

■ 3. Find the unique solution to the system of equations.

$$5x - 3y + z = -8$$

$$2x + y - 2z = -6$$

$$-3x + 2y + 4z = 19$$

Solution:

Let's number the equations to stay organized.

[1] $5x - 3y + z = -8$

[2] $2x + y - 2z = -6$

[3] $-3x + 2y + 4z = 19$

Multiply equation [1] by 2,

$$2(5x - 3y + z = -8)$$

$$10x - 6y + 2z = -16$$

and then add this to equation [2] so that z will be eliminated.

$$(10x - 6y + 2z) + (2x + y - 2z) = (-16) + (-6)$$

$$10x - 6y + 2z + 2x + y - 2z = -22$$

$$10x - 6y + 2x + y = -22$$

[4] $12x - 5y = -22$

Multiply equation [1] by -4,

$$-4(5x - 3y + z = -8)$$

$$-20x + 12y - 4z = 32$$

and then add this to equation [3] so that z will be eliminated.

$$(-20x + 12y - 4z) + (-3x + 2y + 4z) = (32) + (19)$$

$$-20x + 12y - 4z - 3x + 2y + 4z = 51$$

$$-20x + 12y - 3x + 2y = 51$$

[5] $-23x + 14y = 51$

Solve for y in equation [4].

$$12x - 5y = -22$$

$$-5y = -12x - 22$$

[6] $y = \frac{12}{5}x + \frac{22}{5}$

Plug [6] into equation [5] to solve for x.

$$-23x + 14y = 51$$

$$-23x + 14\left(\frac{12}{5}x + \frac{22}{5}\right) = 51$$

$$-23x + \frac{168}{5}x + \frac{308}{5} = 51$$

$$\frac{53}{5}x + \frac{308}{5} = 51$$

$$53x + 308 = 255$$

$$53x = -53$$

$$x = -1$$

Plug $x = -1$ into equation [6] to solve for y.

$$y = \frac{12}{5}x + \frac{22}{5}$$

$$y = \frac{12}{5}(-1) + \frac{22}{5}$$

$$y = -\frac{12}{5} + \frac{22}{5}$$

$$y = \frac{10}{5}$$

$$y = 2$$

Plug $x = -1$ and $y = 2$ into any of the original equations to solve for z. We'll use equation [1].

$$5x - 3y + z = -8$$

$$5(-1) - 3(2) + z = -8$$

$$-5 - 6 + z = -8$$

$$-11 + z = -8$$

$$z = 3$$

Therefore, the solution to the system is $(x, y, z) = (-1,2,3)$.

■ 4. Find the unique solution to the system of equations.

$$-2x + 3y - 4z = 10$$

$$4x + 3y + 2z = 4$$

$$x - 6y + 4z = -19$$

Solution:

Let's number the equations to stay organized.

[1] $-2x + 3y - 4z = 10$

[2] $4x + 3y + 2z = 4$

[3] $x - 6y + 4z = -19$

Multiply equation [2] by 2,

$$2(4x + 3y + 2z = 4)$$

$$8x + 6y + 4z = 8$$

and then add this to equation [1] so that z will be eliminated.

$$(8x + 6y + 4z) + (-2x + 3y - 4z) = (8) + (10)$$

$$8x + 6y + 4z - 2x + 3y - 4z = 18$$

$$8x + 6y - 2x + 3y = 18$$

$$6x + 9y = 18$$

[4] $2x + 3y = 6$

Add equation [1] to equation [3] so that z will be eliminated.

$$(-2x + 3y - 4z) + (x - 6y + 4z) = (10) + (-19)$$

$$-2x + 3y - 4z + x - 6y + 4z = -9$$

$$-2x + 3y + x - 6y = -9$$

[5] $-x - 3y = -9$

Solve for x in equation [5].

$$-x - 3y = -9$$

$$-x = -9 + 3y$$

[6] $x = 9 - 3y$

Plug $x = 9 - 3y$ into equation [4] to solve for y.

$$2x + 3y = 6$$

$$2(9 - 3y) + 3y = 6$$

$$18 - 6y + 3y = 6$$

$$18 - 3y = 6$$

$$-3y = -12$$

$$y = 4$$

Let's plug $y = 4$ into equation [6] to solve for x.

$$x = 9 - 3y$$

$$x = 9 - 3(4)$$

$$x = 9 - 12$$

$$x = -3$$

Plug $x = -3$ and $y = 4$ into any of the original equations to solve for z. We'll use equation [2].

$$4x + 3y + 2z = 4$$

$$4(-3) + 3(4) + 2z = 4$$

$$-12 + 12 + 2z = 4$$

$$2z = 4$$

$$z = 2$$

Therefore, the solution to the system is $(x, y, z) = (-3,4,2)$.

■ 5. Find the unique solution to the system of equations.

$$2x - y + z = 9$$

$$4x - 2y + 2z = 18$$

$$-2x + y - z = -9$$

Solution:

Let's number the equations to stay organized.

[1] $2x - y + z = 9$

[2] $4x - 2y + 2z = 18$

[3] $-2x + y - z = -9$

Add equation [1] to equation [3] so that z will be eliminated.

$$(2x - y + z) + (-2x + y - z) = (9) + (-9)$$

$$2x - y + z - 2x + y - z = 0$$

$$-y + z + y - z = 0$$

$$z - z = 0$$

$$0 = 0$$

When all the variables eliminate and we get a true statement, it means all points (x, y, z) are a solution to the system. So far, this is the case with equations [1] and [3]. If this also happens with equation [2], then the whole system is an "identity" and there are infinite solutions.

Let's check the second equation to see if this is the case. Multiply equation [3] by 2,

$$2(-2x+y-z=-9)$$

$$-4x+2y-2z=-18$$

and then add it to equation [2].

$$(-4x+2y-2z)+(4x-2y+2z)=(-18)+(18)$$

$$-4x+2y-2z+4x-2y+2z=0$$

$$2y-2z-2y+2z=0$$

$$-2z+2z=0$$

$$0=0$$

Since all the variables eliminate and we get a true statement, the system is an identity and there are infinite solutions.

■ 6. Find the unique solution to the system of equations.

$$x+2y-z=9$$

$$3x+y-z=5$$

$$-x-4y+z=2$$

Solution:

Let's number the equations to stay organized.

[1] $x + 2y - z = 9$

[2] $3x + y - z = 5$

[3] $-x - 4y + z = 2$

Add equations [1] and [3] together so that x will be eliminated.

$$(x + 2y - z) + (-x - 4y + z) = (9) + (2)$$

$$x + 2y - z - x - 4y + z = 11$$

$$2y - 4y = 11$$

$$-2y = 11$$

[4] $y = -\dfrac{11}{2}$

Add equation [2] to equation [3] so that z will be eliminated.

$$(3x + y - z) + (-x - 4y + z) = (5) + (2)$$

$$3x + y - z - x - 4y + z = 7$$

$$3x + y - x - 4y = 7$$

[5] $2x - 3y = 7$

Plug equation [4] into equation [5] to solve for x.

$$2x - 3y = 7$$

$$2x - 3\left(-\frac{11}{2}\right) = 7$$

$$2x + \frac{33}{2} = 7$$

$$4x + 33 = 14$$

$$4x = -19$$

$$x = -\frac{19}{4}$$

Plug in $x = -19/4$ and $y = -11/2$ into any of the original equations to solve for z. We'll use equation [3].

$$-x - 4y + z = 2$$

$$-\left(-\frac{19}{4}\right) - 4\left(-\frac{11}{2}\right) + z = 2$$

$$\frac{19}{4} + \frac{44}{2} + z = 2$$

$$19 + 88 + 4z = 8$$

$$4z = -99$$

$$z = -\frac{99}{4}$$

Therefore, the solution to the system is $(x, y, z) = (-19/4, -11/2, -99/4)$.

MATRIX DIMENSIONS AND ENTRIES

■ 1. Give the dimensions of the matrix.

$$D = \begin{bmatrix} 11 & 9 \\ -4 & 8 \end{bmatrix}$$

Solution:

We always give the dimensions of a matrix as rows $\times$ columns. Matrix D has 2 rows and 2 columns, so D is a 2×2 matrix.

■ 2. Give the dimensions of the matrix.

$$A = \begin{bmatrix} 3 & 5 & -2 & 1 & 8 \end{bmatrix}$$

Solution:

We always give the dimensions of a matrix as rows $\times$ columns. Matrix A has 1 row and 5 columns, so A is a 1×5 matrix.

■ 3. Given matrix J, find $J_{4,1}$.

$$J = \begin{bmatrix} 6 \\ 2 \\ 7 \\ 1 \end{bmatrix}$$

Solution:

The value of $J_{4,1}$ is the entry in the fourth row, first column of matrix J, which is 1, so $J_{4,1} = 1$.

■ 4. Given matrix C, find $C_{1,2}$.

$$C = \begin{bmatrix} 3 & 12 \\ 1 & 4 \\ 9 & 5 \\ -3 & 2 \end{bmatrix}$$

Solution:

The value of $C_{1,2}$ is the entry in the first row, second column of matrix C, which is 12, so $C_{1,2} = 12$.

■ 5. Given matrix N, state the dimensions and find $N_{1,3}$.

$$N = \begin{bmatrix} 1 & 5 & 9 \\ 14 & -8 & 6 \end{bmatrix}$$

Solution:

We always give the dimensions of a matrix as rows $\times$ columns. Matrix N has 2 rows and 3 columns, so N is a 2×3 matrix.

The value of $N_{1,3}$ is the entry in the first row, third column of matrix N, which is 9, so $N_{1,3} = 9$.

■ 6. Given matrix S, state the dimensions and find $S_{3,4}$.

$$S = \begin{bmatrix} 3 & 6 & -7 & 1 & 0 \\ 0 & 9 & 15 & 3 & 4 \\ 4 & 0 & 2 & 11 & 8 \\ -5 & 8 & 7 & 9 & 2 \end{bmatrix}$$

Solution:

We always give the dimensions of a matrix as rows $\times$ columns. Matrix S has 4 rows and 5 columns, so S is a 4×5 matrix.

The value of $S_{3,4}$ is the entry in the third row, fourth column of matrix S, which is 11, so $S_{3,4} = 11$.

REPRESENTING SYSTEMS WITH MATRICES

■ 1. Represent the system with a matrix called A.

$$-2x + 5y = 12$$

$$6x - 2y = 4$$

Solution:

The system contains the variables x and y along with a constant. Which means the matrix will have two columns, one for each variable, plus a column for the constants, so three columns in total. Because there are two equations in the system, the matrix will have two rows. Plugging the coefficients and constants into a matrix gives

$$A = \begin{bmatrix} -2 & 5 & 12 \\ 6 & -2 & 4 \end{bmatrix}$$

Alternatively, it would be equally correct to express the matrix as

$$A = \left[\begin{array}{cc|c} -2 & 5 & 12 \\ 6 & -2 & 4 \end{array}\right]$$

■ 2. Represent the system with a matrix called D.

$$9y - 3x + 12 = 0$$

$$8 - 4x = 11y$$

Solution:

This system can be reorganized by putting each equation in order, with x and y on the left side, and the constant on the right side.

$$-3x + 9y = -12$$

$$4x + 11y = 8$$

The system contains the variables x and y along with a constant. Which means the matrix will have two columns, one for each variable, plus a column for the constants, so three columns in total. Because there are two equations in the system, the matrix will have two rows. Plugging the coefficients and constants into a matrix gives

$$D = \begin{bmatrix} -3 & 9 & -12 \\ 4 & 11 & 8 \end{bmatrix}$$

Alternatively, it would be equally correct to express the matrix as

$$D = \begin{bmatrix} -3 & 9 & | & -12 \\ 4 & 11 & | & 8 \end{bmatrix}$$

■ 3. Represent the system with an augmented matrix called H.

$$4a + 7b - 5c + 13d = 6$$

$$3a - 8b = -2c + 1$$

Solution:

The second equation can be reorganized by putting a, b, and c on the left side, and the constant on the right side. We also recognize that there is no d-term in the second equation, so we add in a 0 "filler" term.

$$4a + 7b - 5c + 13d = 6$$

$$3a - 8b + 2c + 0d = 1$$

The system contains the variables a, b, c, and d, along with a constant. Which means the matrix will have four columns, one for each variable, plus a column for the constants, so five columns in total. Because there are two equations in the system, the matrix will have two rows. Plugging the coefficients and constants into a matrix gives

$$H = \begin{bmatrix} 4 & 7 & -5 & 13 & 6 \\ 3 & -8 & 2 & 0 & 1 \end{bmatrix}$$

Alternatively, it would be equally correct to express the matrix as

$$H = \begin{bmatrix} 4 & 7 & -5 & 13 & | & 6 \\ 3 & -8 & 2 & 0 & | & 1 \end{bmatrix}$$

■ 4. Represent the system with a matrix called M.

$$-2x + 4y = 9 - 6z$$

$$7y + 2z - 3 = -3t - 9x$$

Solution:

Both equations can be reorganized by putting x, y, z, and t on the left side, and the constant on the right side. We also recognize that there is no t-term in the first equation, so we add in a 0 "filler" term.

$$-2x + 4y + 6z + 0t = 9$$

$$9x + 7y + 2z + 3t = 3$$

The system contains the variables x, y, z, and t, along with a constant. Which means the matrix will have four columns, one for each variable, plus a column for the constants, so five columns in total. Because there are two equations in the system, the matrix will have two rows. Plugging the coefficients and constants into a matrix gives

$$M = \begin{bmatrix} -2 & 4 & 6 & 0 & 9 \\ 9 & 7 & 2 & 3 & 3 \end{bmatrix}$$

Alternatively, it would be equally correct to express the matrix as

$$M = \begin{bmatrix} -2 & 4 & 6 & 0 & | & 9 \\ 9 & 7 & 2 & 3 & | & 3 \end{bmatrix}$$

5. Represent the system with a matrix called A.

$$3x - 8y + z = 7$$

$$2z = 3y - 2x + 4$$

$$5y = 12 - 9x$$

Solution:

The second and third equations can be reorganized by putting x, y, and z on the left side, and the constant on the right side. We also recognize that there is no z-term in the third equation, so we add in a 0 "filler" term.

$$3x - 8y + z = 7$$

$$2x - 3y + 2z = 4$$

$$9x + 5y + 0z = 12$$

The system contains the variables x, y, and z, along with a constant. Which means the augmented matrix will have three columns, one for each variable, plus a column for the constants, so four columns in total. Because there are three equations in the system, the matrix will have three rows. Plugging the coefficients and constants into a matrix gives

$$A = \begin{bmatrix} 3 & -8 & 1 & 7 \\ 2 & -3 & 2 & 4 \\ 9 & 5 & 0 & 12 \end{bmatrix}$$

Alternatively, it would be equally correct to express the matrix as

$$A = \begin{bmatrix} 3 & -8 & 1 & | & 7 \\ 2 & -3 & 2 & | & 4 \\ 9 & 5 & 0 & | & 12 \end{bmatrix}$$

6. Represent the system with a matrix called K.

$$-4b + 2c = 3 - 7a$$

$$9c = 4 - 2b$$

$$8a - 2c = 5b$$

Solution:

All three of these equations can be reorganized by putting a, b, and c on the left side, and the constant on the right side. We also recognize that there is no a-term in the second equation, and no constant in the third equation, so we add in 0 “filler” terms.

$$7a - 4b + 2c = 3$$

$$0a + 2b + 9c = 4$$

$$8a - 5b - 2c = 0$$

The system contains the variables a, b, and c, along with a constant. Which means the augmented matrix will have three columns, one for each variable, plus a column for the constants, so four columns in total. Because there are three equations in the system, the matrix will have three rows. Plugging the coefficients and constants into a matrix gives

$$K=\begin{bmatrix}7 & -4 & 2 & 3\\ 0 & 2 & 9 & 4\\ 8 & -5 & -2 & 0\end{bmatrix}$$

Alternatively, it would be equally correct to express the matrix as

$$K=\begin{bmatrix}7 & -4 & 2 & | & 3\\ 0 & 2 & 9 & | & 4\\ 8 & -5 & -2 & | & 0\end{bmatrix}$$

SIMPLE ROW OPERATIONS

■ 1. Write the new matrix after $R_1 \leftrightarrow R_2$.

$$\begin{bmatrix} 2 & 6 & -4 & 1 \\ 8 & 2 & 1 & -5 \end{bmatrix}$$

Solution:

The operation described by $R_1 \leftrightarrow R_2$ is switching row 1 with row 2. The matrix after $R_1 \leftrightarrow R_2$ is

$$\begin{bmatrix} 8 & 2 & 1 & -5 \\ 2 & 6 & -4 & 1 \end{bmatrix}$$

■ 2. Write the new matrix after $R_2 \leftrightarrow R_4$.

$$\begin{bmatrix} 1 & 2 & 7 & -3 \\ 6 & 1 & 5 & -4 \\ -7 & 7 & 0 & 3 \\ 9 & 2 & 8 & 3 \end{bmatrix}$$

Solution:

The operation described by $R_2 \leftrightarrow R_4$ is switching row 2 with row 4. Nothing will happen to rows 1 and 3. The matrix after $R_2 \leftrightarrow R_4$ is

$$\begin{bmatrix} 1 & 2 & 7 & -3 \\ 9 & 2 & 8 & 3 \\ -7 & 7 & 0 & 3 \\ 6 & 1 & 5 & -4 \end{bmatrix}$$

■ 3. Write the new matrix after $R_1 \leftrightarrow 3R_2$.

$$\begin{bmatrix} 9 & 2 & -7 \\ 1 & 6 & 4 \end{bmatrix}$$

Solution:

The operation described by $R_1 \leftrightarrow 3R_2$ is multiplying row 2 by a constant of 3 and then switching those two rows. The matrix after $3R_2$ is

$$\begin{bmatrix} 9 & 2 & -7 \\ 3 & 18 & 12 \end{bmatrix}$$

The matrix after $R_1 \leftrightarrow 3R_2$ is

$$\begin{bmatrix} 3 & 18 & 12 \\ 9 & 2 & -7 \end{bmatrix}$$

■ 4. Write the new matrix after $3R_2 \leftrightarrow 3R_4$.

$$\begin{bmatrix} 0 & 11 & 6 \\ 7 & -3 & 9 \\ 8 & 8 & 1 \\ 6 & 2 & 4 \end{bmatrix}$$

Solution:

The operation described by $3R_2 \leftrightarrow 3R_4$ is multiplying row 2 by a constant of 3, multiplying row 4 by a constant of 3, and then switching those two rows. Nothing will happen to rows 1 and 3. The matrix after $3R_2$ is

$$\begin{bmatrix} 0 & 11 & 6 \\ 21 & -9 & 27 \\ 8 & 8 & 1 \\ 6 & 2 & 4 \end{bmatrix}$$

The matrix after $3R_4$ is

$$\begin{bmatrix} 0 & 11 & 6 \\ 21 & -9 & 27 \\ 8 & 8 & 1 \\ 18 & 6 & 12 \end{bmatrix}$$

The matrix after $3R_2 \leftrightarrow 3R_4$ is

$$\begin{bmatrix} 0 & 11 & 6 \\ 18 & 6 & 12 \\ 8 & 8 & 1 \\ 21 & -9 & 27 \end{bmatrix}$$

■ 5. Write the new matrix after $R_1 + 2R_2 \to R_1$.

$$\begin{bmatrix} 6 & 2 & 7 \\ 1 & -5 & 15 \end{bmatrix}$$

Solution:

The operation described by $R_1 + 2R_2 \to R_1$ is multiplying row 2 by a constant of 2, adding that resulting row to row 1, and using that result to replace row 1. $2R_2$ is

$$\begin{bmatrix} 2(1) & 2(-5) & 2(15) \end{bmatrix}$$

$$[2 \quad -10 \quad 30]$$

The sum $R_1 + 2R_2$ is

$$[6+2 \quad 2-10 \quad 7+30]$$

$$[8 \quad -8 \quad 37]$$

The matrix after $R_1 + 2R_2 \to R_1$, which is replacing row 1 with this row we just found, is

$$\begin{bmatrix} 8 & -8 & 37 \\ 1 & -5 & 15 \end{bmatrix}$$

■ 6. Write the new matrix after $4R_2 + R_3 \to R_3$.

$$\begin{bmatrix} 13 & 5 & -2 & 9 \\ 8 & 2 & 0 & 6 \\ 4 & 1 & 7 & -3 \end{bmatrix}$$

Solution:

The operation described by $4R_2 + R_3 \to R_3$ is multiplying row 2 by a constant of 4, adding that resulting row to row 3, and using that result to replace row 3. $4R_2$ is

$$\begin{bmatrix} 4(8) & 4(2) & 4(0) & 4(6) \end{bmatrix}$$

$$\begin{bmatrix} 32 & 8 & 0 & 24 \end{bmatrix}$$

The sum $4R_2 + R_3$ is

$$\begin{bmatrix} 32+4 & 8+1 & 0+7 & 24-3 \end{bmatrix}$$

$$\begin{bmatrix} 36 & 9 & 7 & 21 \end{bmatrix}$$

The matrix after $4R_2 + R_3 \to R_3$, which is replacing row 3 with this row we just found, is

$$\begin{bmatrix} 13 & 5 & -2 & 9 \\ 8 & 2 & 0 & 6 \\ 36 & 9 & 7 & 21 \end{bmatrix}$$

PIVOT ENTRIES AND ROW-ECHELON FORMS

1. Use row operations to put the matrix into row-echelon form.

$$\begin{bmatrix} 3 & 6 & -7 \\ 1 & 2 & -1 \\ 1 & 2 & 1 \end{bmatrix}$$

Solution:

Start with $R_1 \leftrightarrow R_2$.

$$\begin{bmatrix} 1 & 2 & -1 \\ 3 & 6 & -7 \\ 1 & 2 & 1 \end{bmatrix}$$

After $-3R_1 + R_2 \to R_2$, we get

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & 0 & -4 \\ 1 & 2 & 1 \end{bmatrix}$$

After $-R_1 + R_3 \to R_3$, we get

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & 0 & -4 \\ 0 & 0 & 2 \end{bmatrix}$$

We'll use $-(1/4)R_2 \to R_2$ to get the pivot entry in the second row.

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

After $-2R_2 + R_3 \to R_3$, we get

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Now all the pivot entries are 1, the zeroed-out row is at the bottom, and the pivot entries follow a staircase pattern. Therefore, the matrix is in row-echelon form.

■ 2. Use row operations to put the matrix into reduced row-echelon form.

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 3 & 0 & -6 & 0 \end{bmatrix}$$

Solution:

Start with $(1/3)R_4 \to R_4$.

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & -2 & 0 \end{bmatrix}$$

After $R_1 \leftrightarrow R_4$, we get

$$\begin{bmatrix} 1 & 0 & -2 & 0 \\ 1 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

After $R_2 - R_1 \to R_2$, we get

$$\begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Use $R_3 \leftrightarrow R_4$ to move the zero row to the bottom.

$$\begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Now all the pivot entries are 1, the zeroed-out row is at the bottom, and the pivot entries follow a staircase pattern. All the pivot columns include only the pivot entry, and otherwise all zero entries. Therefore, the matrix is in reduced row-echelon form.

3. Use row operations to put the matrix into reduced row-echelon form.

$$\begin{bmatrix} 1 & 5 & 2 \\ 0 & -3 & 9 \\ 0 & 0 & 7 \end{bmatrix}$$

Solution:

Start with $-(1/3)R_2 \to R_2$.

$$\begin{bmatrix} 1 & 5 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 7 \end{bmatrix}$$

After $(1/7)R_3 \to R_3$, we get

$$\begin{bmatrix} 1 & 5 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix}$$

After $3R_3 + R_2 \to R_2$, we get

$$\begin{bmatrix} 1 & 5 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

After $-2R_3 + R_1 \to R_1$, we get

$$\begin{bmatrix} 1 & 5 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

After $-5R_2 + R_1 \to R_1$, we get

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Now all the pivot entries are 1, and follow a staircase pattern. All the pivot columns include only the pivot entry, and otherwise only 0 entries. Therefore, the matrix is in reduced row-echelon form.

■ 4. Use row operations to put the matrix into row-echelon form.

$$\begin{bmatrix} 3 & 2 & 0 & 9 \\ 2 & 4 & -3 & -1 \\ 2 & 12 & -12 & 1 \end{bmatrix}$$

Solution:

Start with $R_1 - R_2 \to R_1$.

$$\begin{bmatrix} 1 & -2 & 3 & 10 \\ 2 & 4 & -3 & -1 \\ 2 & 12 & -12 & 1 \end{bmatrix}$$

After $-R_2 + R_3 \to R_3$, we get

$$\begin{bmatrix} 1 & -2 & 3 & 10 \\ 2 & 4 & -3 & -1 \\ 0 & 8 & -9 & 2 \end{bmatrix}$$

After $-2R_1 + R_2 \to R_2$, we get

$$\begin{bmatrix} 1 & -2 & 3 & 10 \\ 0 & 8 & -9 & -21 \\ 0 & 8 & -9 & 2 \end{bmatrix}$$

After $-R_2+R_3 \to R_3$, we get

$$\begin{bmatrix} 1 & -2 & 3 & 10 \\ 0 & 8 & -9 & -21 \\ 0 & 0 & 0 & 23 \end{bmatrix}$$

After $(1/8)R_2 \to R_2$, we get

$$\begin{bmatrix} 1 & -2 & 3 & 10 \\ 0 & 1 & -\frac{9}{8} & -\frac{21}{8} \\ 0 & 0 & 0 & 23 \end{bmatrix}$$

After $(1/23)R_3 \to R_3$, we get

$$\begin{bmatrix} 1 & -2 & 3 & 10 \\ 0 & 1 & -\frac{9}{8} & -\frac{21}{8} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

In this matrix, the first non-zero entry in each row is a 1, and the pivots follow a staircase pattern that moves down and to the right. Therefore, this matrix is in row-echelon form.

5. Use row operations to put the matrix into reduced row-echelon form.

$$\begin{bmatrix} 1 & -2 \\ 3 & 1 \\ -3 & 0 \\ 2 & -3 \end{bmatrix}$$

Solution:

After $R_2 - 3R_1 \to R_2$, we get

$$\begin{bmatrix} 1 & -2 \\ 0 & 7 \\ -3 & 0 \\ 2 & -3 \end{bmatrix}$$

After $3R_1 + R_3 \to R_3$, we get

$$\begin{bmatrix} 1 & -2 \\ 0 & 7 \\ 0 & -6 \\ 2 & -3 \end{bmatrix}$$

After $-2R_1 + R_4 \to R_4$, we get

$$\begin{bmatrix} 1 & -2 \\ 0 & 7 \\ 0 & -6 \\ 0 & 1 \end{bmatrix}$$

After $R_2 \leftrightarrow R_4$, we get

$$\begin{bmatrix} 1 & -2 \\ 0 & 1 \\ 0 & 7 \\ 0 & -6 \end{bmatrix}$$

After $R_3 - 7R_2 \to R_3$, we get

$$\begin{bmatrix} 1 & -2 \\ 0 & 1 \\ 0 & 0 \\ 0 & -6 \end{bmatrix}$$

After $R_3 + 6R_2 \to R_3$, we get

$$\begin{bmatrix} 1 & -2 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

After $2R_2 + R_1 \to R_1$, we get

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Now all the pivot entries are 1, the zeroed-out rows are at the bottom, the pivot entries follow a staircase pattern, and all the pivot columns include only the pivot entry, and otherwise all 0 entries. Therefore, the matrix is in reduced row-echelon form.

6. Use row operations to put the matrix into row-echelon form.

$$\begin{bmatrix} 1 & 0 & -3 & 7 \\ 0 & 1 & -2 & 3 \\ -1 & 3 & -6 & -13 \\ -5 & -2 & 22 & -28 \end{bmatrix}$$

Solution:

Start with $R_3 + R_1 \to R_3$.

$$\begin{bmatrix} 1 & 0 & -3 & 7 \\ 0 & 1 & -2 & 3 \\ 0 & 3 & -9 & -6 \\ -5 & -2 & 22 & -28 \end{bmatrix}$$

After $5R_1 + R_4 \to R_4$, we get

$$\begin{bmatrix} 1 & 0 & -3 & 7 \\ 0 & 1 & -2 & 3 \\ 0 & 3 & -9 & -6 \\ 0 & -2 & 7 & 7 \end{bmatrix}$$

After $-3R_2 + R_3 \to R_3$, we get

$$\begin{bmatrix} 1 & 0 & -3 & 7 \\ 0 & 1 & -2 & 3 \\ 0 & 0 & -3 & -15 \\ 0 & -2 & 7 & 7 \end{bmatrix}$$

After $2R_2 + R_4 \to R_4$, we get

$$\begin{bmatrix} 1 & 0 & -3 & 7 \\ 0 & 1 & -2 & 3 \\ 0 & 0 & -3 & -15 \\ 0 & 0 & 3 & 13 \end{bmatrix}$$

After $-(1/3)R_3 \to R_3$, we get

$$\begin{bmatrix} 1 & 0 & -3 & 7 \\ 0 & 1 & -2 & 3 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 3 & 13 \end{bmatrix}$$

After $-3R_3 + R_4 \to R_4$, we get

$$\begin{bmatrix} 1 & 0 & -3 & 7 \\ 0 & 1 & -2 & 3 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & -2 \end{bmatrix}$$

After $-(1/2)R_4 \to R_4$, we get

$$\begin{bmatrix} 1 & 0 & -3 & 7 \\ 0 & 1 & -2 & 3 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

In this matrix, the first non-zero entry in each row is a 1, and the pivots follow a staircase pattern. Therefore, the matrix is in row-echelon form.

GAUSS-JORDAN ELIMINATION

■ 1. Use Gauss-Jordan elimination to find the solution to the linear system from the rref matrix.

$$x + 2y = -2$$

$$3x + 2y = 6$$

Solution:

The matrix for the system is

$$\begin{bmatrix} 1 & 2 & -2 \\ 3 & 2 & 6 \end{bmatrix}$$

After $3R_1 - R_2 \to R_2$, the matrix is

$$\begin{bmatrix} 1 & 2 & -2 \\ 0 & 4 & -12 \end{bmatrix}$$

The first column is done. After $(1/4)R_2 \to R_2$, the matrix is

$$\begin{bmatrix} 1 & 2 & -2 \\ 0 & 1 & -3 \end{bmatrix}$$

After $R_1 - 2R_2 \to R_1$, the matrix is

$$\begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & -3 \end{bmatrix}$$

The second column is done, and we can see that the solution to the linear system is $(x, y) = (4, -3)$.

■ 2. Use Gauss-Jordan elimination to find the solution to the linear system from the rref matrix.

$$2x + 4y = 22$$

$$3x + 3y = 15$$

Solution:

The matrix for the system is

$$\begin{bmatrix} 2 & 4 & 22 \\ 3 & 3 & 15 \end{bmatrix}$$

After $(1/2)R_1 \to R_1$ and $(1/3)R_2 \to R_2$the matrix is

$$\begin{bmatrix} 1 & 2 & 11 \\ 1 & 1 & 5 \end{bmatrix}$$

After $R_1 - R_2 \to R_2$, the matrix is

$$\begin{bmatrix} 1 & 2 & 11 \\ 0 & 1 & 6 \end{bmatrix}$$

The first column is done. After $R_1 - 2R_2 \to R_1$, the matrix is

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 6 \end{bmatrix}$$

The second column is done, and we can see that the solution to the linear system is $(x, y) = (-1,6)$.

■ 3. Use Gauss-Jordan elimination to find the solution to the linear system from the rref matrix.

$$x - 3y - 6z = 4$$

$$y + 2z = -2$$

$$-4x + 12y + 21z = -4$$

Solution:

The matrix for the system is

$$\begin{bmatrix} 1 & -3 & -6 & 4 \\ 0 & 1 & 2 & -2 \\ -4 & 12 & 21 & -4 \end{bmatrix}$$

After $4R_1 + R_3 \to R_3$, the matrix is

$$\begin{bmatrix} 1 & -3 & -6 & 4 \\ 0 & 1 & 2 & -2 \\ 0 & 0 & -3 & 12 \end{bmatrix}$$

The first column is done. After $3R_2 + R_1 \to R_1$, the matrix is

$$\begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 2 & -2 \\ 0 & 0 & -3 & 12 \end{bmatrix}$$

The second column is done. After $(-1/3)R_3 \to R_3$, the matrix is

$$\begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 2 & -2 \\ 0 & 0 & 1 & -4 \end{bmatrix}$$

After $R_2 - 2R_3 \to R_2$, the matrix is

$$\begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & -4 \end{bmatrix}$$

The third column is done, and we can see that the solution to the linear system is $(x, y, z) = (-2, 6, -4)$.

■ 4. Use Gauss-Jordan elimination to find the solution to the linear system from the rref matrix.

$$2y + 4z = 4$$

$$x + 3y + 3z = 5$$

$$2x + 7y + 6z = 10$$

Solution:

The matrix for the system is

$$\begin{bmatrix} 0 & 2 & 4 & 4 \\ 1 & 3 & 3 & 5 \\ 2 & 7 & 6 & 10 \end{bmatrix}$$

After $(1/2)R_1 \to R_1$, the matrix is

$$\begin{bmatrix} 0 & 1 & 2 & 2 \\ 1 & 3 & 3 & 5 \\ 2 & 7 & 6 & 10 \end{bmatrix}$$

Because the first entry in the first row is 0, swap it with the second row to get

$$\begin{bmatrix} 1 & 3 & 3 & 5 \\ 0 & 1 & 2 & 2 \\ 2 & 7 & 6 & 10 \end{bmatrix}$$

After $R_3 - 2R_1 \to R_3$, the matrix is

$$\begin{bmatrix} 1 & 3 & 3 & 5 \\ 0 & 1 & 2 & 2 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

The first column is done. After $R_1 - 3R_2 \to R_1$, the matrix is

$$\begin{bmatrix} 1 & 0 & -3 & -1 \\ 0 & 1 & 2 & 2 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

After $R_2 - R_3 \to R_3$, the matrix is

$$\begin{bmatrix} 1 & 0 & -3 & -1 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 2 & 2 \end{bmatrix}$$

The second column is done. After $(1/2)R_3 \to R_3$, the matrix is

$$\begin{bmatrix} 1 & 0 & -3 & -1 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

After $R_1 + 3R_3 \to R_1$, the matrix is

$$\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

After $R_2 - 2R_3 \to R_2$, the matrix is

$$\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

The third column is done, and we can see that the solution to the linear system is $(x, y, z) = (2,0,1)$.

■ 5. Use Gauss-Jordan elimination to find the solution to the linear system from the rref matrix.

$$3x + 12y + 42z = -27$$

$$x + 2y + 8z = -5$$

$$2x + 5y + 16z = -6$$

Solution:

The matrix for the system is

$$\begin{bmatrix} 3 & 12 & 42 & -27 \\ 1 & 2 & 8 & -5 \\ 2 & 5 & 16 & -6 \end{bmatrix}$$

After $(1/3)R_1 \to R_1$, the matrix is

$$\begin{bmatrix} 1 & 4 & 14 & -9 \\ 1 & 2 & 8 & -5 \\ 2 & 5 & 16 & -6 \end{bmatrix}$$

After $R_1 - R_2 \to R_2$, the matrix is

$$\begin{bmatrix} 1 & 4 & 14 & -9 \\ 0 & 2 & 6 & -4 \\ 2 & 5 & 16 & -6 \end{bmatrix}$$

After $2R_1 - R_3 \to R_3$, the matrix is

$$\begin{bmatrix} 1 & 4 & 14 & -9 \\ 0 & 2 & 6 & -4 \\ 0 & 3 & 12 & -12 \end{bmatrix}$$

The first column is done. After $(1/2)R_2 \to R_2$, the matrix is

$$\begin{bmatrix} 1 & 4 & 14 & -9 \\ 0 & 1 & 3 & -2 \\ 0 & 3 & 12 & -12 \end{bmatrix}$$

After $R_1 - 4R_2 \to R_1$, the matrix is

$$\begin{bmatrix} 1 & 0 & 2 & -1 \\ 0 & 1 & 3 & -2 \\ 0 & 3 & 12 & -12 \end{bmatrix}$$

After $R_3 - 3R_2 \to R_3$, the matrix is

$$\begin{bmatrix} 1 & 0 & 2 & -1 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 3 & -6 \end{bmatrix}$$

The second column is done. After $(1/3)R_3 \to R_3$, the matrix is

$$\begin{bmatrix} 1 & 0 & 2 & -1 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

After $R_1 - 2R_3 \to R_1$, the matrix is

$$\begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

After $R_2 - 3R_3 \to R_2$, the matrix is

$$\begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

The third column is done, and we can see that the solution to the linear system is $(x, y, z) = (3,4, -2)$.

6. Use Gauss-Jordan elimination to find the solution to the linear system from the rref matrix.

$$4x + 8y + 4z = 20$$

$$4x + 6y = 4$$

$$3x + 3y - z = 1$$

Solution:

The matrix for the system is

$$\begin{bmatrix} 4 & 8 & 4 & 20 \\ 4 & 6 & 0 & 4 \\ 3 & 3 & -1 & 1 \end{bmatrix}$$

After $(1/4)R_1 \to R_1$, the matrix is

$$\begin{bmatrix} 1 & 2 & 1 & 5 \\ 4 & 6 & 0 & 4 \\ 3 & 3 & -1 & 1 \end{bmatrix}$$

After $4R_1 - R_2 \to R_2$, the matrix is

$$\begin{bmatrix} 1 & 2 & 1 & 5 \\ 0 & 2 & 4 & 16 \\ 3 & 3 & -1 & 1 \end{bmatrix}$$

After $3R_1 - R_3 \to R_3$, the matrix is

$$\begin{bmatrix} 1 & 2 & 1 & 5 \\ 0 & 2 & 4 & 16 \\ 0 & 3 & 4 & 14 \end{bmatrix}$$

The first column is done. After $(1/2)R_2 \to R_2$, the matrix is

$$\begin{bmatrix} 1 & 2 & 1 & 5 \\ 0 & 1 & 2 & 8 \\ 0 & 3 & 4 & 14 \end{bmatrix}$$

After $R_1 - 2R_2 \to R_1$, the matrix is

$$\begin{bmatrix} 1 & 0 & -3 & -11 \\ 0 & 1 & 2 & 8 \\ 0 & 3 & 4 & 14 \end{bmatrix}$$

After $3R_2 - R_3 \to R_3$, the matrix is

$$\begin{bmatrix} 1 & 0 & -3 & -11 \\ 0 & 1 & 2 & 8 \\ 0 & 0 & 2 & 10 \end{bmatrix}$$

The second column is done. After $(1/2)R_3 \to R_3$, the matrix is

$$\begin{bmatrix} 1 & 0 & -3 & -11 \\ 0 & 1 & 2 & 8 \\ 0 & 0 & 1 & 5 \end{bmatrix}$$

After $R_1 + 3R_3 \to R_1$, the matrix is

$$\begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 2 & 8 \\ 0 & 0 & 1 & 5 \end{bmatrix}$$

After $R_2 - 2R_3 \to R_2$, the matrix is

$$\begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 5 \end{bmatrix}$$

The third column is done, and we can see that the solution to the linear system is $(x, y, z) = (4, -2, 5)$.

NUMBER OF SOLUTIONS TO THE LINEAR SYSTEM

1. Determine whether the system has one solution, no solutions, or infinitely many solutions.

$$2x - 8y = 18$$

$$-7x + 2y - 5z = -6$$

$$3x + 2z = 1$$

Solution:

Rewrite the system as an augmented matrix.

$$\begin{bmatrix} 2 & -8 & 0 & | & 18 \\ -7 & 2 & -5 & | & -6 \\ 3 & 0 & 2 & | & 1 \end{bmatrix}$$

Work toward putting the matrix into reduced row-echelon form, starting with finding the pivot entry in the first row.

$$\begin{bmatrix} 1 & -4 & 0 & | & 9 \\ -7 & 2 & -5 & | & -6 \\ 3 & 0 & 2 & | & 1 \end{bmatrix}$$

Zero out the rest of the first column.

$$\begin{bmatrix} 1 & -4 & 0 & | & 9 \\ 0 & -26 & -5 & | & 57 \\ 0 & 12 & 2 & | & -26 \end{bmatrix}$$

Find the pivot entry in the second row.

$$\begin{bmatrix} 1 & -4 & 0 & | & 9 \\ 0 & 1 & \frac{5}{26} & | & -\frac{57}{26} \\ 0 & 12 & 2 & | & -26 \end{bmatrix}$$

Zero out the rest of the second column.

$$\begin{bmatrix} 1 & 0 & \frac{10}{13} & | & \frac{3}{13} \\ 0 & 1 & \frac{5}{26} & | & -\frac{57}{26} \\ 0 & 0 & -\frac{4}{13} & | & \frac{4}{13} \end{bmatrix}$$

Find the pivot entry in the third row.

$$\begin{bmatrix} 1 & 0 & \frac{10}{13} & | & \frac{3}{13} \\ 0 & 1 & \frac{5}{26} & | & -\frac{57}{26} \\ 0 & 0 & 1 & | & -1 \end{bmatrix}$$

Zero out the rest of the third column.

$$\begin{bmatrix} 1 & 0 & 0 & | & 1 \\ 0 & 1 & 0 & | & -2 \\ 0 & 0 & 1 & | & -1 \end{bmatrix}$$

Therefore, there's one unique solution to the system, $(x, y, z) = (1, -2, -1)$.

2. Determine whether the system has one solution, no solutions, or infinitely many solutions.

$$-x + 3y - 5z - 8w = 2$$

$$4x - 8y + 4z + 4w = -44$$

$$3x + 5y - 16z + w = 18$$

$$-x + y - 3z - w = 6$$

Solution:

Rewrite the system as an augmented matrix.

$$\begin{bmatrix} -1 & 3 & -5 & -8 & | & 2 \\ 4 & -8 & 4 & 4 & | & -44 \\ 3 & 5 & -16 & 1 & | & 18 \\ -1 & 1 & -3 & -1 & | & 6 \end{bmatrix}$$

Work toward putting the matrix into reduced row-echelon form, starting with finding the pivot entry in the first row.

$$\begin{bmatrix} 1 & -3 & 5 & 8 & | & -2 \\ 4 & -8 & 4 & 4 & | & -44 \\ 3 & 5 & -16 & 1 & | & 18 \\ -1 & 1 & -3 & -1 & | & 6 \end{bmatrix}$$

Zero out the rest of the first column.

$$\begin{bmatrix} 1 & -3 & 5 & 8 & | & -2 \\ 0 & 4 & -16 & -28 & | & -36 \\ 0 & 14 & -31 & -23 & | & 24 \\ 0 & -2 & 2 & 7 & | & 4 \end{bmatrix}$$

Find the pivot entry in the second row.

$$\begin{bmatrix} 1 & -3 & 5 & 8 & | & -2 \\ 0 & 1 & -4 & -7 & | & -9 \\ 0 & 14 & -31 & -23 & | & 24 \\ 0 & -2 & 2 & 7 & | & 4 \end{bmatrix}$$

Zero out the rest of the second column.

$$\begin{bmatrix} 1 & 0 & -7 & -13 & | & -29 \\ 0 & 1 & -4 & -7 & | & -9 \\ 0 & 0 & 25 & 75 & | & 150 \\ 0 & 0 & -6 & -7 & | & -14 \end{bmatrix}$$

Find the pivot entry in the third row.

$$\begin{bmatrix} 1 & 0 & -7 & -13 & | & -29 \\ 0 & 1 & -4 & -7 & | & -9 \\ 0 & 0 & 1 & 3 & | & 6 \\ 0 & 0 & -6 & -7 & | & -14 \end{bmatrix}$$

Zero out the rest of the third column.

$$\begin{bmatrix} 1 & 0 & 0 & 8 & | & 13 \\ 0 & 1 & 0 & 5 & | & 15 \\ 0 & 0 & 1 & 3 & | & 6 \\ 0 & 0 & 0 & 11 & | & 22 \end{bmatrix}$$

Find the pivot entry in the fourth row.

$$\begin{bmatrix} 1 & 0 & 0 & 8 & | & 13 \\ 0 & 1 & 0 & 5 & | & 15 \\ 0 & 0 & 1 & 3 & | & 6 \\ 0 & 0 & 0 & 1 & | & 2 \end{bmatrix}$$

Zero out the rest of the fourth column.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & | & -3 \\ 0 & 1 & 0 & 0 & | & 5 \\ 0 & 0 & 1 & 0 & | & 0 \\ 0 & 0 & 0 & 1 & | & 2 \end{bmatrix}$$

Therefore, there is one unique solution to the system, $(x, y, z, w) = (-3, 5, 0, 2)$.

■ 3. How many solutions does the linear system have?

$$3x - 3y + 5z = -11$$

$$-2x + y - 2z = 5$$

$$x + y - z = 9$$

Solution:

Rewrite the system as an augmented matrix.

$$\begin{bmatrix} 3 & -3 & 5 & | & -11 \\ -2 & 1 & -2 & | & 5 \\ 1 & 1 & -1 & | & 9 \end{bmatrix}$$

Work toward putting the matrix into reduced row-echelon form, starting with finding the pivot entry in the first row.

$$\begin{bmatrix} 1 & -1 & \frac{5}{3} & | & -\frac{11}{3} \\ -2 & 1 & -2 & | & 5 \\ 1 & 1 & -1 & | & 9 \end{bmatrix}$$

Zero out the rest of the first column.

$$\begin{bmatrix} 1 & -1 & \frac{5}{3} & | & -\frac{11}{3} \\ 0 & -1 & \frac{4}{3} & | & -\frac{7}{3} \\ 0 & 2 & -\frac{8}{3} & | & \frac{38}{3} \end{bmatrix}$$

Find the pivot entry in the second row.

$$\begin{bmatrix} 1 & -1 & \frac{5}{3} & | & -\frac{11}{3} \\ 0 & 1 & -\frac{4}{3} & | & \frac{7}{3} \\ 0 & 2 & -\frac{8}{3} & | & \frac{38}{3} \end{bmatrix}$$

Zero out the rest of the second column.

$$\begin{bmatrix} 1 & 0 & \frac{1}{3} & | & -\frac{4}{3} \\ 0 & 1 & -\frac{4}{3} & | & \frac{7}{3} \\ 0 & 0 & 0 & | & 8 \end{bmatrix}$$

The third row tells us that $0=8$, which can't possibly be true. Therefore, the system has no solutions.

■ 4. How many solutions does the linear system have?

$$-x+6y+4z=-22$$

$$4x-22y-2z+2w=0$$

$$x-6y-5z+3w=5$$

$$-3y-22z=6$$

Solution:

Rewrite the system as an augmented matrix.

$$\begin{bmatrix} -1 & 6 & 4 & 0 & | & -22 \\ 4 & -22 & -2 & 2 & | & 0 \\ 1 & -6 & -5 & 3 & | & 5 \\ 0 & -3 & -22 & 0 & | & 6 \end{bmatrix}$$

Work toward putting the matrix into reduced row-echelon form, starting with finding the pivot entry in the first row.

$$\begin{bmatrix} 1 & -6 & -4 & 0 & | & 22 \\ 4 & -22 & -2 & 2 & | & 0 \\ 1 & -6 & -5 & 3 & | & 5 \\ 0 & -3 & -22 & 0 & | & 6 \end{bmatrix}$$

Zero out the rest of the first column.

$$\begin{bmatrix} 1 & -6 & -4 & 0 & | & 22 \\ 0 & 2 & 14 & 2 & | & -88 \\ 0 & 0 & -1 & 3 & | & -17 \\ 0 & -3 & -22 & 0 & | & 6 \end{bmatrix}$$

Find the pivot entry in the second row.

$$\begin{bmatrix} 1 & -6 & -4 & 0 & | & 22 \\ 0 & 1 & 7 & 1 & | & -44 \\ 0 & 0 & -1 & 3 & | & -17 \\ 0 & -3 & -22 & 0 & | & 6 \end{bmatrix}$$

Zero out the rest of the second column.

$$\begin{bmatrix} 1 & 0 & 38 & 6 & | & -242 \\ 0 & 1 & 7 & 1 & | & -44 \\ 0 & 0 & -1 & 3 & | & -17 \\ 0 & 0 & -1 & 3 & | & -126 \end{bmatrix}$$

Find the pivot entry in the third row.

$$\begin{bmatrix} 1 & 0 & 38 & 6 & | & -242 \\ 0 & 1 & 7 & 1 & | & -44 \\ 0 & 0 & 1 & -3 & | & 17 \\ 0 & 0 & -1 & 3 & | & -126 \end{bmatrix}$$

Zero out the rest of the third column.

$$\begin{bmatrix} 1 & 0 & 0 & 120 & | & -888 \\ 0 & 1 & 0 & 22 & | & -163 \\ 0 & 0 & 1 & -3 & | & 17 \\ 0 & 0 & 0 & 0 & | & -109 \end{bmatrix}$$

The third row tells that $0 = -109$, which can't be true. Therefore, the system has no solutions.

5. Determine whether the system has one solution, no solutions, or infinitely many solutions.

$$2x + 2y - 8z = 4$$

$$-3x - 5y + 6z = -4$$

$$5x - y - 38z = 16$$

Solution:

Rewrite the system as an augmented matrix.

$$\begin{bmatrix} 2 & 2 & -8 & | & 4 \\ -3 & -5 & 6 & | & -4 \\ 5 & -1 & -38 & | & 16 \end{bmatrix}$$

Work toward putting the matrix into reduced row-echelon form, starting with finding the pivot entry in the first row.

$$\begin{bmatrix} 1 & 1 & -4 & | & 2 \\ -3 & -5 & 6 & | & -4 \\ 5 & -1 & -38 & | & 16 \end{bmatrix}$$

Zero out the rest of the first column.

$$\begin{bmatrix} 1 & 1 & -4 & | & 2 \\ 0 & -2 & -6 & | & 2 \\ 0 & -6 & -18 & | & 6 \end{bmatrix}$$

Find the pivot entry in the second row.

$$\begin{bmatrix} 1 & 1 & -4 & | & 2 \\ 0 & 1 & 3 & | & -1 \\ 0 & -6 & -18 & | & 6 \end{bmatrix}$$

Zero out the rest of the second column.

$$\begin{bmatrix} 1 & 0 & -7 & | & 3 \\ 0 & 1 & 3 & | & -1 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

Since the entire last row has only zeros, the linear system has infinitely many solutions.

6. For the linear system below, determine whether it has one solution, no solutions, or infinitely many solutions.

$$x + y - z + 2w = 7$$

$$4x + 2y - 6z + 2w = 16$$

$$-3x + y + 7z + 6w = 3$$

$$-x - y + 4z + 3w = 8$$

Solution:

Rewrite the system as an augmented matrix.

$$\begin{bmatrix} 1 & 1 & -1 & 2 & | & 7 \\ 4 & 2 & -6 & 2 & | & 16 \\ -3 & 1 & 7 & 6 & | & 3 \\ -1 & -1 & 4 & 3 & | & 8 \end{bmatrix}$$

Work toward putting the matrix into reduced row-echelon form. First, zero out the rest of the first column.

$$\begin{bmatrix} 1 & 1 & -1 & 2 & | & 7 \\ 0 & -2 & -2 & -6 & | & -12 \\ 0 & 4 & 4 & 12 & | & 24 \\ 0 & 0 & 3 & 5 & | & 15 \end{bmatrix}$$

Find the pivot entry in the second row.

$$\begin{bmatrix} 1 & 1 & -1 & 2 & | & 7 \\ 0 & 1 & 1 & 3 & | & 6 \\ 0 & 4 & 4 & 12 & | & 24 \\ 0 & 0 & 3 & 5 & | & 15 \end{bmatrix}$$

Zero out the rest of the second column.

$$\begin{bmatrix} 1 & 0 & -2 & -1 & | & 1 \\ 0 & 1 & 1 & 3 & | & 6 \\ 0 & 0 & 0 & 0 & | & 0 \\ 0 & 0 & 3 & 5 & | & 15 \end{bmatrix}$$

Switch the third and fourth rows.

$$\begin{bmatrix} 1 & 0 & -2 & -1 & | & 1 \\ 0 & 1 & 1 & 3 & | & 6 \\ 0 & 0 & 3 & 5 & | & 15 \\ 0 & 0 & 0 & 0 & | & 0 \end{bmatrix}$$

Find the pivot entry in the third row.

$$\begin{bmatrix} 1 & 0 & -2 & -1 & | & 1 \\ 0 & 1 & 1 & 3 & | & 6 \\ 0 & 0 & 1 & \frac{5}{3} & | & 5 \\ 0 & 0 & 0 & 0 & | & 0 \end{bmatrix}$$

Zero out the rest of the third column.

$$\begin{bmatrix} 1 & 0 & 0 & \frac{7}{3} & | & 11 \\ 0 & 1 & 0 & \frac{4}{3} & | & 1 \\ 0 & 0 & 1 & \frac{5}{3} & | & 5 \\ 0 & 0 & 0 & 0 & | & 0 \end{bmatrix}$$

Since the entire last row has only zeros, the linear system has infinitely many solutions.

W W W . K R I S T A K I N G M A T H . C O M