The History and Concept of Mathematical Proof

Steven G. Krantz[1]

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A mathematician is a master of critical thinking, of analysis, and of deductive reasoning. These skills travel well, and can be applied in a large variety of situations—and in many different disciplines. Today, mathematical skills are being put to good use in medicine, physics, law, commerce, Internet design, engineering, chemistry, biological science, social science, anthropology, genetics, warfare, cryptography, plastic surgery, security analysis, data manipulation, computer science, and in many other disciplines and endeavors as well.

The unique feature that sets mathematics apart from other sciences, from philosophy, and indeed from all other forms of intellectual discourse, is the use of rigorous proof. It is the proof concept that makes the subject cohere, that gives it its timelessness, and that enables it to travel well. The purpose of this discussion is to describe proof, to put it in context, to give its history, and to explain its significance.

There is no other scientific or analytical discipline that uses proof as readily and routinely as does mathematics. This is the device that makes theoretical mathematics special: the tightly knit chain of reasoning, following strict logical rules, that leads inexorably to a particular conclusion. It is *proof* that is our device for establishing the absolute and irrevocable truth of statements in our subject. This is the reason that we can depend on mathematics that was done by Euclid 2300 years ago as readily as we believe in the mathematics that is done today. No other discipline can make such an assertion.

[1]It is a pleasure to thank the American Institute of Mathematics for its hospitality and support during the writing of this paper.

Discourse = discussion or dialogue

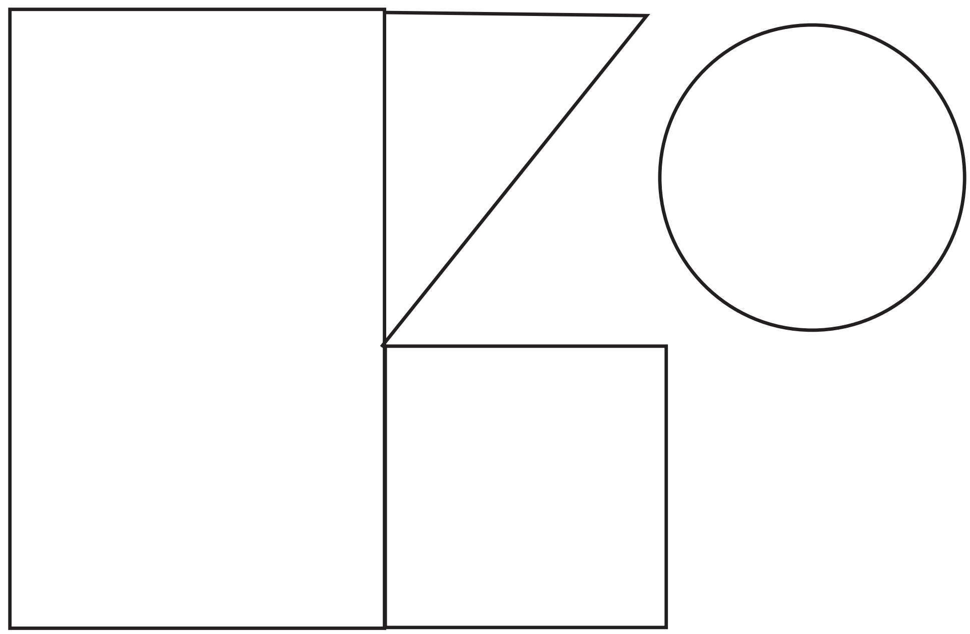

Figure 1: Mathematical constructions from surveying.

1 The Concept of Proof

The tradition of mathematics is a long and glorious one. Along with philosophy, it is the oldest venue of human intellectual inquiry. It is in the nature of the human condition to want to understand the world around us, and mathematics is a natural vehicle for doing so. Mathematics is also a subject that is beautiful and worthwhile in its own right. A scholarly pursuit that had intrinsic merit and aesthetic appeal, mathematics is certainly worth studying for its own sake.

In its earliest days, mathematics was often bound up with practical questions. The Egyptians, as well as the Greeks, were concerned with surveying land. Refer to Figure 1. Thus it was natural to consider questions of geometry and trigonometry. Certainly triangles and rectangles came up in a natural way in this context, so early geometry concentrated on these constructs. Circles, too, were natural to consider—for the design of arenas and water tanks and other practical projects. So ancient geometry (and Euclid's axioms for geometry) discussed circles.

The earliest mathematics was phenomenological. If one could draw a

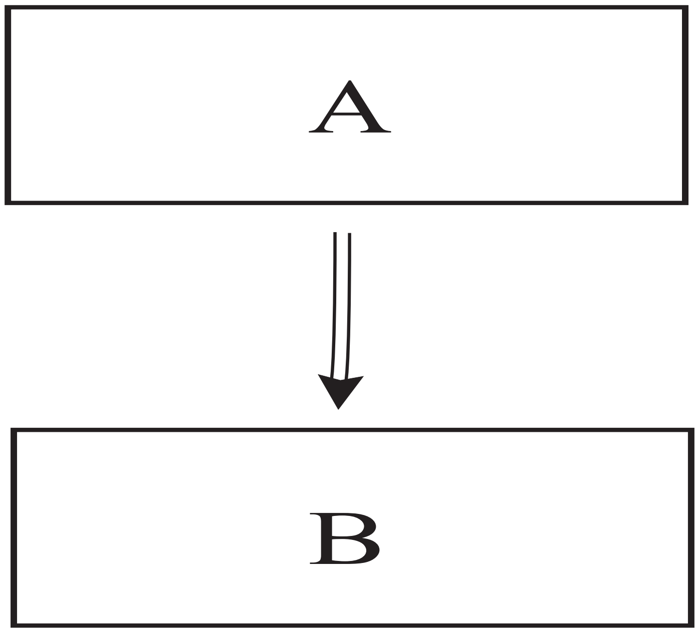

Figure 2: Logical derivation.

plausible picture, or give a compelling description, then that was all the justification that was needed for a mathematical "fact". Sometimes one argued by analogy. Or by invoking the gods. The notion that mathematical statements could be *proved* was not yet an idea that had been developed. There was no standard for the concept of proof. The logical structure, the "rules of the game", had not yet been created.

Thus we are led to ask: What is a proof? Heuristically, a proof is a rhetorical device for convincing someone else that a mathematical statement is true or valid. And how might one do this? A moment's thought suggests that a natural way to prove that something new (call it **B**) is true is to relate it to something old (call it **A**) that has already been accepted as true. Thus arises the concept of *deriving* a new result from an old result. See Figure 2. The next question then is, "How was the old result verified?" Applying this regimen repeatedly, we find ourselves considering a chain of reasoning as in Figure 3. But then one cannot help but ask: "Where does the chain begin?" And this is a fundamental issue.

It will not do to say that the chain has no beginning: it extends infinitely far back into the fogs of time. Because if that were the case it would undercut

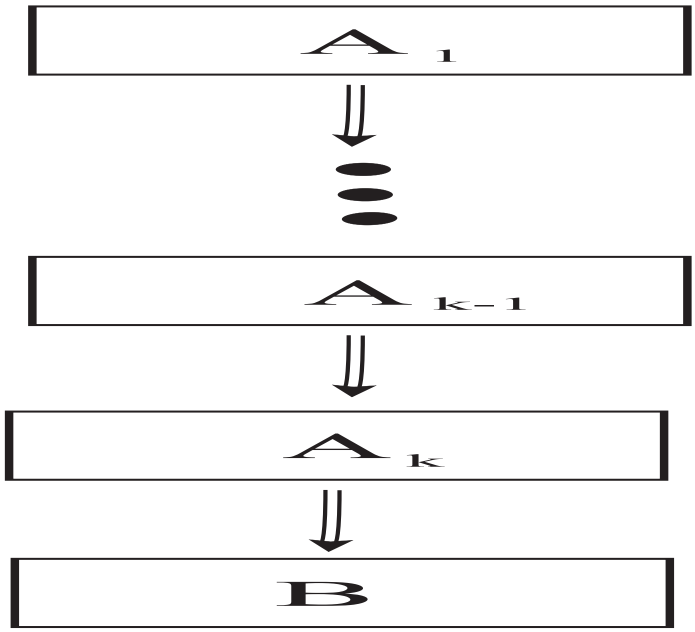

Figure 3: A chain of reasoning.

our thinking of what a proof should be. We are endeavoring to justify new mathematical facts in terms of old mathematical facts. But if the reasoning regresses infinitely far back into the past, then we cannot in fact ever grasp a basis or initial justification for our reasoning. As we shall see below, the answer to these questions is that the mathematician puts into place definitions and axioms before beginning to explore the firmament, determine what is true, and then to prove it. Considerable discussion will be required to put this paradigm into context.

As a result of these questions, ancient mathematicians had to think hard about the nature of mathematical proof. Thales (640 B.C.E.–546 B.C.E.), Eudoxus (408 B.C.E.–355 B.C.E.), and Theaetetus of Athens (417 B.C.E.–369 B.C.E.) actually formulated theorems. Thales definitely proved some theorems in geometry (and these were later put into a broader context by Euclid). A theorem is the mathematician's formal enunciation of a fact or truth. But Eudoxus fell short in finding means to prove his theorems. His work had a distinctly practical bent, and he was particularly fond of calculations.

It was Euclid of Alexandria who first formalized the way that we now

Endeavoring: achieve something or try hard to do something.

think about mathematics. Euclid had definitions and axioms and then theorems—in that order. There is no gainsaying the assertion that Euclid set the paradigm by which we have been practicing mathematics for 2300 years. This was mathematics done right. Now, following Euclid, in order to address the issue of the infinitely regressing chain of reasoning, we begin our studies by putting into place a set of *Definitions* and a set of *Axioms*.

What is a definition? A definition explains the meaning of a piece of terminology. There are logical problems with even this simple idea, for consider the first definition that we are going to formulate. Suppose that we wish to define a *rectangle*. This will be the first piece of terminology in our mathematical system. What words can we use to define it? Suppose that we define rectangle in terms of points and lines and planes and right angles. That begs the questions: What is a point? What is a line? What is a plane? How do we define "angle"? What is a right angle?

Thus we see that our *first* definition(s) must be formulated in terms of commonly accepted words that require no further explanation. It was Aristotle (384 B.C.E.–322 B.C.E.) who insisted that a definition must describe the concept being defined in terms of other concepts already known. This is often quite difficult. As an example, Euclid defined a *point* to be that which has no part. Thus he is using words *outside of mathematics*, that are a commonly accepted part of everyday argot, to explain the precise mathematical notion of "point".[2] Once "point" is defined, then one can use that term in later definitions—for example, to define "line". And one will also use everyday language that does not require further explication. That is how we build up our system of definitions.

The definitions give us then a language for doing mathematics. We formulate our results, or *theorems*, by using the words that have been established in the definitions. But wait, we are not yet ready for theorems. Because we have to lay cornerstones upon which our reasoning can develop. That is the purpose of axioms.

What is an axiom? An axiom[3] (or postulate[4]) is a mathematical state-

[2]It is quite common, among those who study the foundations of mathematics, to refer to terms that are defined in non-mathematical language—that is, which cannot be defined in terms of other mathematical terms—as *undefined terms*. The concept of "point" is an undefined term.

[3]The word "axiom" derives from the Greek *axios*, meaning "something worthy".

[4]The word "postulate" derives from a medieval Latin word *postulatus* meaning "to nominate" or "to demand".

ment of fact, formulated using the terminology that has been defined in the definitions, that is taken to be self-evident. An axiom embodies a crisp, clean mathematical assertion. One does not *prove* an axiom. One takes the axiom to be given, and to be so obvious and plausible that no proof is required.

Generally speaking, in any subject area of mathematics, one begins with a brief list of definitions and a brief list of axioms. Once these are in place, and are accepted and understood, then one can begin proving theorems.[5] And what is a proof? A proof is a rhetorical device for convincing another mathematician that a given statement (the theorem) is true. Thus a proof can take many different forms. The most traditional form of mathematical proof is that it is a tightly knit sequence of statements linked together by strict rules of logic. But the purpose of the present article is to discuss and consider the various forms that a proof might take. Today, a proof could (and often does) take the traditional form that goes back 2300 years to the time of Euclid. But it could also consist of a computer calculation. Or it could consist of constructing a physical model. Or it could consist of a computer *simulation* or *model*. Or it could consist of a computer algebra computation using `Mathematica` or `Maple` or `MatLab`. It could also consist of an agglomeration of these various techniques.

2 What Does a Proof Consist Of?

Most of the steps of a mathematical proof are applications of the elementary rules of logic. This is a slight oversimplification, as there are a great many proof techniques that have been developed over the past two centuries. These include proof by mathematical induction, proof by contradiction, proof by exhaustion, proof by enumeration, and many others. But they are all built on one simple rule: *modus ponendo ponens*. This rule of logic says that if we know that "**A** implies **B**", and if we know "**A**", then we may conclude **B**. Thus a proof is a sequence of steps linked together by *modus ponendo ponens*.[6]

It is really an elegant and powerful system. *Occam's Razor* is a logi-

[5] The word "theorem" derives from the Greek *theōrein*, meaning "to look at."

[6] One of the most important proof techniques in mathematics is "proof by contradiction". With this methodology, one assumes in advance that the desired result is false and shows that that leads to an untenable position. But in fact proof by contradiction is nothing other than a reformulation of *modus ponendo ponens*.

Knit: unite, unify or merge.

cal principle posited in the fourteenth century (by William of Occam (1288 C.E.–1348 C.E.)) which advocates that your proof system should have the smallest possible set of axioms and logical rules. That way you minimize the possibility that there are internal contradictions built into the system, and also you make it easier to find the source of your ideas. Inspired both by Euclid's *Elements* and by Occam's Razor, mathematics has striven for all of modern time to keep the fundamentals of its subject as streamlined and elegant as possible. We want our list of definitions to be as short as possible, and we want our collection of axioms or postulates to be as concise and elegant as possible. If you open up a classic text on group theory—such as Marshall Hall's masterpiece [HAL], you will find that there are just three axioms on the first page. The entire 434-page book is built on just those three axioms.[7] Or instead have a look at Walter Rudin's classic *Principles of Mathematical Analysis* [RUD]. There the subject of real variables is built on just twelve axioms. Or look at a foundational book on set theory like Suppes [SUP] or Hrbacek and Jech [HRJ]. There we see the entire subject built on eight axioms.

3 The Purpose of Proof

The experimental sciences (physics, biology, chemistry, for example) tend to use laboratory experiments or tests to check and verify assertions. The benchmark in these subjects is the *reproducible experiment with control*. In their published papers, these scientists will briefly describe what they have discovered, and how they carried out the steps of the corresponding experiment. They will describe the *control*, which is the standard against which the experimental results are compared. Those scientists who are interested can, on reading the article, then turn around and replicate the experiment in their own labs. The really classic, and fundamental and important, experiments become classroom material and are reproduced by students all over the world. Most experimental science is *not* derived from fundamental principles (like axioms). The intellectual process is more empirical, and the verification procedure is correspondingly practical and direct.

Mathematics is quite a different sort of intellectual enterprise. In mathe-

[7] In fact there has recently been found a way to enunciate the premises of group theory using just *one* axiom, and *not* using the word "and". References for this work are [KUN], [HIN], and [MCC].

Posit, posited: assume as a fact, assert, postulate or put forward.

matics we set our definitions and axioms in place *before* we do anything else. In particular, *before we endeavor to derive any results* we must engage in a certain amount of preparatory work. Then we give precise, elegant formulations of statements and we prove them. Any statement in mathematics which lacks a proof has no currency. Nobody will take it as valid. And nobody will use it in his/her own work. The proof is the final test of any new idea. And, once a proof is in place, that is the end of the discussion. Nobody will ever find a counterexample, nor ever gainsay that particular mathematical fact.

Another special feature of mathematics is its timelessness. The theorems that Euclid and Pythagoras proved 2500 years ago are still valid today; and we use them with confidence because we know that they are just as true today as they were when those great masters first discovered them. Other sciences are quite different. The medical or computer science literature of even three years ago is considered to be virtually useless. Because what people thought was correct a few years ago has already changed and migrated and transmogrified. Mathematics, by contrast, is here forever.

What is marvelous is that, in spite of the appearance of some artificiality in the mathematical process, mathematics provides beautiful models for nature (see the lovely essay [WIG], which discusses this point). Over and over again, and more with each passing year, mathematics has helped to explain how the world around us works. Just a few examples illustrate the point:

- Isaac Newton derived Kepler's three laws of planetary motion from just his universal law of gravitation and calculus.
- There is a complete mathematical theory of the refraction of light (due to Isaac Newton, Willebrord Snell, and Pierre de Fermat).
- There is a mathematical theory of the propagation of heat.
- There is a mathematical theory of electromagnetic waves.
- All of classical field theory from physics is formulated in terms of mathematics.
- Einstein's field equations are analyzed using mathematics.
- The motion of falling bodies and projectiles is completely analyzable with mathematics.

- The technology for locating distant submarines using radar and sonar waves is all founded in mathematics.

- The theory of image processing and image compression is all founded in mathematics.

- The design of music CDs is all based on Fourier analysis and coding theory, both branches of mathematics.

The list could go on and on.

The key point to be understood here is that *proof* is central to what modern mathematics is about, and what makes it reliable and reproducible. No other science depends on proof, and therefore no other science has the bulletproof solidity of mathematics. But mathematics is *applied* in a variety of ways, in a vast panorama of disciplines. And the applications are many and varied. Other disciplines often like to reduce their theories to mathematics—or at least explain them in mathematical terms—because it gives the subject a certain elegance and solidity. And it looks really sophisticated. Such efforts meet with varying success.

4 The History of Mathematical Proof

In point of fact the history of the proof concept is rather inchoate. It is unclear just when mathematicians and philosophers conceived of the notion that mathematical assertions required justification. This was quite a new idea. Then it was another considerable leap to devise methods for *constructing* such a justification. In the present section we shall outline what little is known about the development of the proof concept.

Perhaps the first mathematical "proof" in recorded history is due to the Babylonians. They seem (along with the Chinese) to have been aware of the Pythagorean theorem (discussed in detail below) well before Pythagoras.[8] The Babylonians had certain diagrams that indicate why the Pythagorean theorem is true, and tablets have been found to validate this fact.[9] They also

[8]Although it must be stressed that they did not have Pythagoras's sense of the structure of mathematics, of the importance of rigor, or of the nature of formal proof.

[9]We stress that the Babylonian effort was not a proof by modern standards. But it was at least an effort to provide logical justification for a mathematical fact.

had methods for calculating Pythagorean triples—that is, triples of integers (or whole numbers) a, b, c that satisfy

$$a^2 + b^2 = c^2$$

as in the Pythagorean theorem.

4.1 Pythagoras

Pythagoras (569–500 B.C.E.) was both a person and a society (i.e., the *Pythagoreans*). He was also a political figure and a mystic. He was special in his time, among other reasons, because he involved women as equals in his activities. One critic characterized the man as "one tenth of him genius, nine-tenths sheer fudge." Pythagoras died, according to legend, in the flames of his own school fired by political and religious bigots who stirred up the masses to protest against the enlightenment which Pythagoras sought to bring them.

The Pythagoreans embodied a passionate spirit that is remarkable to our eyes:

> Bless us, divine Number, thou who generatest gods and men.

and

> Number rules the universe.

Note that Pythagoras lived *before* Euclid. Thus his contributions should be thought of as feeding into Euclid's seminal creation. The Pythagoreans are remembered for two monumental contributions to mathematics. The first of these was establishing the importance of, and the necessity for, *proofs* in mathematics: that mathematical statements, especially geometric statements, must be verified by way of rigorous proof. Prior to Pythagoras, the ideas of geometry were generally rules of thumb that were derived empirically, merely from observation and (occasionally) measurement. Pythagoras also introduced the idea that a great body of mathematics (such as geometry) could be derived from a small number of postulates. The second great contribution was the discovery of, and proof of, the fact that not all numbers are commensurate. More precisely, the Greeks prior to Pythagoras believed with a profound and deeply held passion that everything was built on the whole numbers. Fractions arise in a concrete manner: as ratios of the sides of

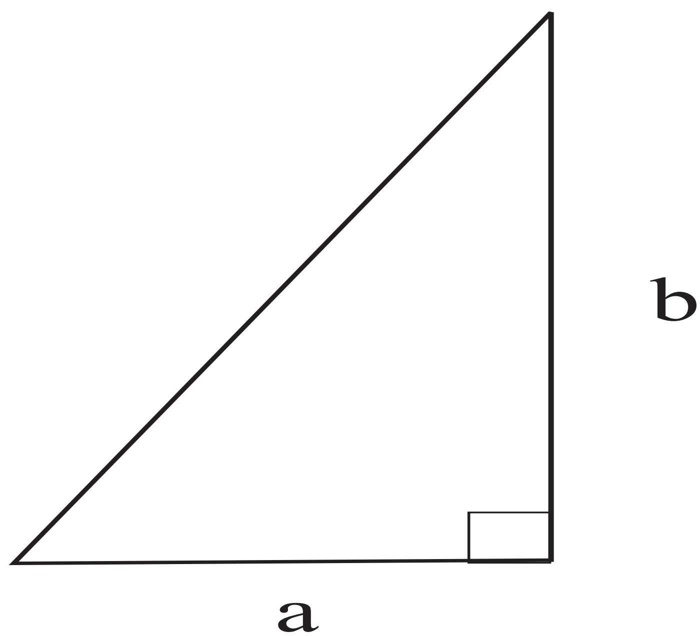

Figure 4: The fraction $\frac{b}{a}$.

triangles with integer length (and are thus *commensurable*—this antiquated terminology has today been replaced by the word "rational")—see Figure 4.

Pythagoras proved the result that we now call *the Pythagorean theorem*. It says that the legs a, b and hypotenuse c of a right triangle (Figure 5) are related by the formula

$$a^2 + b^2 = c^2 \,. \tag{$\star$}$$

This theorem has perhaps more proofs than any other result in mathematics—well over fifty altogether. And in fact it is one of the most ancient mathematical results. There is evidence that the Babylonians and the Chinese knew this theorem at least 500 years before Pythagoras.

Now Pythagoras noticed that, if $a = 1$ and $b = 1$, then $c^2 = 2$. He wondered whether there was a rational number c that satisfied this last identity. His stunning conclusion was this:

Theorem: *There is no rational number c such that $c^2 = 2$.*

Put in other words, if there is a number whose square is two then that number cannot be rational. This result caused considerable upset and confusion in Greek philosophical circles. It had been a rigidly held belief that

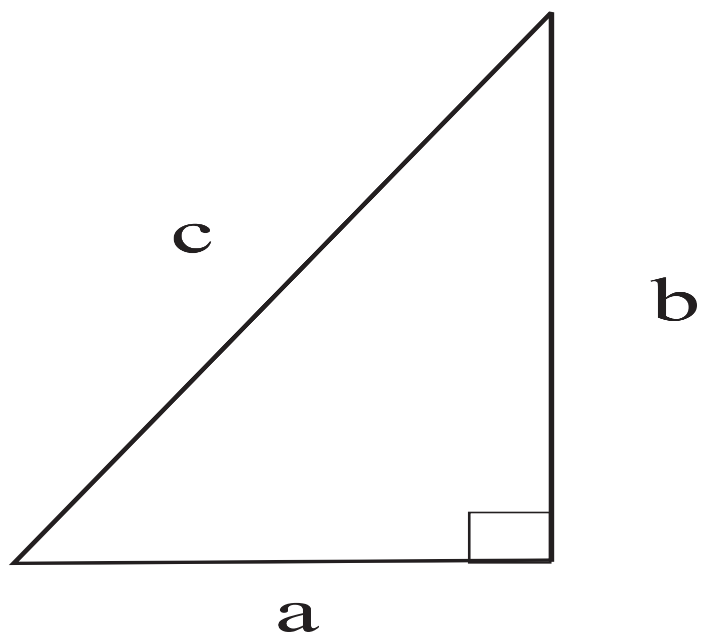

Figure 5: The Pythagorean theorem.

all numbers—at least numbers that one encountered in real life—were rational. Now it was found that there were other numbers (the irrationals) that must be dealt with. It would take two thousand years for scholars to fully understand and incorporate these new ideas into the infrastructure of mathematics.

4.2 Eudoxus and the Concept of Theorem

It was Eudoxus (408 B.C.E–355 B.C.E.) who began the grand tradition of organizing mathematics into theorems. Eudoxus was one of the first to use the word "theorem" in the context of mathematics.

In fact Eudoxus was a man of many interests and many talents. He knew a good deal about astronomy and number theory. He developed the theory of proportions, and built on the ideas of Pythagoras to devise methods to compare irrational numbers. This in turn enabled him to develop his *method of exhaustion*, which is a precursor of the modern integration theory (part of calculus) that is used to calculate areas and volumes.

What Eudoxus gained in the rigor and precision of his mathematical formulations, he lost because he did not prove anything. Formal proof was not yet the tradition in mathematics. As we have noted elsewhere, mathematics in its early days was a largely heuristic and empirical subject. It had never occurred to anyone that there was any need to prove anything.

4.3 Euclid the Geometer

Euclid (325 B.C.E.–265 B.C.E.) is hailed as the first scholar to systematically organize mathematics (i.e., a substantial portion of the mathematics that went before him), formulate definitions and axioms, and prove theorems. This was a monumental achievement, and a highly original one.

Although Euclid is not known so much (as were Archimedes and Pythagoras) for his original and profound mathematical insights, and although there are not many theorems named after Euclid,[10] he has had an incisive effect on human thought. After all, Euclid wrote a treatise (consisting of thirteen Books)—now known as Euclid's *Elements*—which has been continuously available for over 2000 years and has been through a large number of editions. It is still studied in detail today, and continues to have a substantial influence over the way that we think about mathematics.

As often happens with scientists and artists and scholars of immense accomplishment, there is disagreement, and some debate, over exactly who or what Euclid actually was. The three schools of thought are these:

- Euclid was an historical character—a single individual—who in fact wrote the *Elements* and the other scholarly works that are commonly attributed to him.

- Euclid was the leader of a team of mathematicians working in Alexandria. They all contributed to the creation of the complete works that we now attribute to Euclid. They even continued to write and disseminate books under Euclid's name after his death.

- Euclid was not an historical character at all. In fact "Euclid" was a *nom de plume* adopted by a group of mathematicians working in Alexandria. They took their inspiration from Euclid of Megara (who *was* in fact an historical figure), a prominent philosopher who lived about 100 years before Euclid the mathematician is thought to have lived.

[10]But we must note that the *Euclidean algorithm* and the proof that there are *infinitely many prime integers* are original creations of Euclid, and are of fundamental importance.

Most scholars today subscribe to the first theory—that Euclid was certainly a unique person who created the *Elements*. But we acknowledge that there is evidence for the other two scenarios. Certainly Euclid had a vigorous school of mathematics in Alexandria, and there is little doubt that his students participated in his projects.

It is thought that Euclid must have studied in Plato's (430 B.C.E.–349 B.C.E.) Academy in Athens, for it is unlikely that there would have been another place where he could have learned the geometry of Eudoxus and Theaetetus on which the *Elements* is based.

What is important about Euclid's *Elements* is the paradigm it provides for the way that mathematics should be studied and recorded. He begins with several definitions of terminology and ideas for geometry, and then he records five important postulates (or axioms) of geometry. A version of these postulates is as follows:

P1 Through any pair of distinct points there passes a line.

P2 For each segment $\overline{AB}$ and each segment $\overline{CD}$ there is a unique point E (on the line determined by A and B) such that B is between A and E and the segment $\overline{CD}$ is congruent to $\overline{BE}$ (Figure 6).

P3 For each point C and each point A distinct from C there exists a circle with center C and radius CA.

P4 All right angles are congruent.

These are the standard four axioms which give our Euclidean conception of geometry. The fifth axiom, a topic of intense study for two thousand years, is the so-called parallel postulate (in Playfair's formulation):

P5 For each line ℓ and each point P that does not lie on ℓ there is a unique line ℓ' through P such that ℓ' is parallel to ℓ (Figure 7).

The fifth axioim, **P5**, has a fascinating history. For two thousand years people suspected that it was *not* independent of the other axioms—that in fact it could be derived from **P1**–**P4**. There were mighty struggles to provide such a derivation, and many famous mistakes made (see [GRE] for some of the history). But, in 1826, Janos Bolyai and Nikolai Lobachevsky showed independently that the Parallel Postulate can never be proved. There are

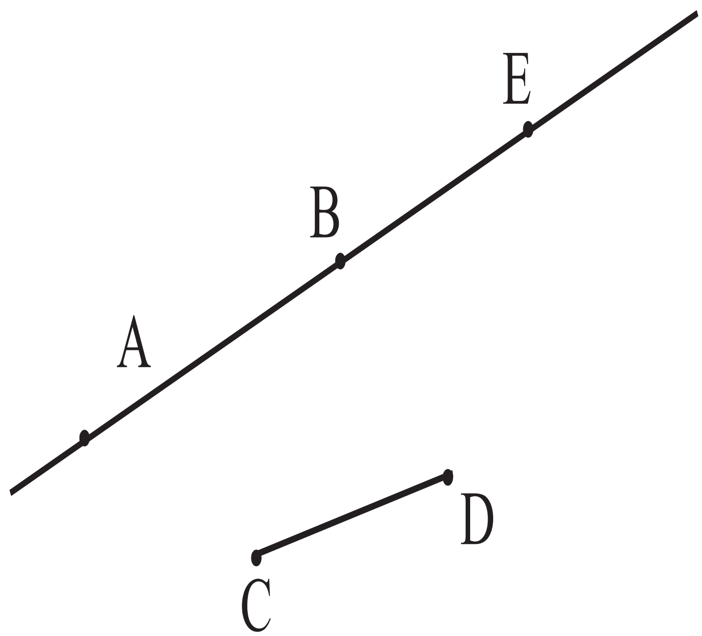

Figure 6: Euclid's Axiom **P2**.

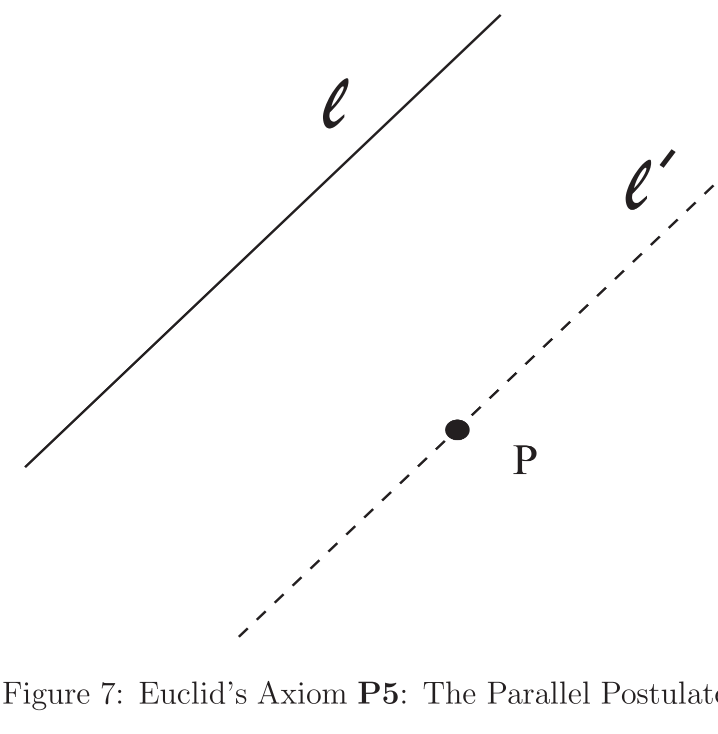

Figure 7: Euclid's Axiom **P5**: The Parallel Postulate.

models for geometry in which all the other axioms of Euclid are true yet the Parallel Postulate is false. So the Parallel Postulate now stands as one of the axioms of our most commonly used geometry.

Of course, prior to this enunciation of his celebrated five axioms, Euclid had defined “point”, “line”, “circle”, and the other terms that he uses. Although Euclid borrowed freely from mathematicians both earlier and contemporaneous with himself, it is generally believed that the famous “Parallel Postulate”, that is Postulate **P5**, is of Euclid’s own creation.

5 The Middle Ages

The Middle Ages were also called the Dark Ages, and not without good reason. This was a long period (over 1000 years by some measures) of intellectual stagnation. True, the Arabs developed some of their seminal ideas in algebra during this time. Some other cultures, including the Africans and the Incas and the Chinese, made some mathematical progress during this period (from about 500 C.E. to 1500 C.E.). But very little was done to develop the idea of mathematical proof. This is a very sophisticated concept—one of the pinnacles of human thought. And it awaited a fertile time in Europe to see the next major steps in the development.

6 The Golden Age of the Nineteenth Century

Nineteenth-century Europe was a haven for brilliant mathematics. So many of the important ideas in mathematics today grew out of ideas that were developed at that time. We list just a few of these:

- Jean Baptiste Joseph Fourier (1768–1830) developed the seminal ideas for Fourier series and created the first formula for the expansion of an arbitrary function into a trigonometric series. He developed applications to the theory of heat.

- Evariste Galois (1812–1832) and Augustin Louis-Cauchy (1789–1857) laid the foundations for abstract algebra by inventing group theory.

- Bernhard Riemann (1826–1866) established the subject of differential geometry, defined the version of the integral (from calculus) that we use today, and made profound contributions to complex variable theory and Fourier analysis.

- Augustin-Louis Cauchy laid the foundations of complex variable theory and partial differential equations. He also did seminal work in geometric analysis.

- Carl Jacobi (1804–1851), Ernst Kummer (1810–1893), Niels Henrick Abel (1802–1829), and numerous other mathematicians from many countries developed number theory.

- Joseph Louis Lagrange (1736–1813), Cauchy and others were laying the foundations of the calculus of variations, classical mechanics, the implicit function theorem, and many other important ideas in modern geometric analysis.

- Karl Weierstrass (1815–1897) laid the foundations for rigorous analysis with numerous examples and theorems. He made seminal contributions both to real and to complex analysis.

This list could be expanded considerably. The nineteenth century was a fecund time for European mathematics, and communication among mathematicians was at an all-time high. There were several prominent mathematics journals, and important work was widely disseminated. The many great universities in Italy, France, Germany, and England (England's was driven by physics) had vigorous mathematics programs and many students. This was an age when the foundations for modern mathematics were laid.

And certainly the seeds of rigorous discourse were being sown at this time. The language and terminology and notation of mathematics was not quite yet universal, the definitions were not well established, and even the methods of proof were in development. But the basic methodology was in place and the mathematics of that time traveled reasonably well among countries and to the twentieth century and beyond. As we shall see below, Bourbaki and Hilbert set the tone for rigorous mathematics in the twentieth century. But the work of the many nineteenth-century geniuses paved the way for those pioneers.

7 Hilbert and the Twentieth Century

Along with Henri Poincaré (1854–1912) of France, David Hilbert (1862–1943) of Germany was the spokesman for early twentieth century mathematics. Hilbert is said to have been one of the last mathematicians to be conversant with the entire subject—from differential equations to geometry to logic to algebra. He exerted considerable influence over all parts of mathematics, and he wrote seminal texts in many of them. Hilbert had an important and profound vision for the rigorization of mathematics (one that was later dashed by work of Bertrand Russell, Kurt Gödel, and others), and he set the tone for the way that mathematics was to be practiced and recorded in our time.

Hilbert had many important students, ranging from Richard Courant (1888–1972) to Theodore von Kármán (1881–1963) (the father of modern aeronautical engineering) to Hugo Steinhaus (1887–1972) to Hermann Weyl (1885-1955). His influence was felt widely, not just in Germany but around the world. He certainly helped to establish Göttingen as one of the world centers for mathematics, and it continues to be so today.

One of Hilbert's real coups was to study the subject of algebraic invariants and to prove that there was a basis for these invariants. For several decades people had sought to prove this result by constructive means—by actually *writing down* the basis.[11] Hilbert established the result nonconstructively, essentially with a proof by contradiction. This was quite controversial at the time—even though proof by contradiction had been around at least since the time of Euclid. Hilbert's work put a great many mathematicians out of business, and established him rather quickly as a force to be reckoned with. Certainly Hilbert is remembered today for a great many mathematical innovations, one of which was his *Nullstellensatz*—one of the key algebraic tools that he developed for the study of invariants.

Certainly David Hilbert was considered to be one of the premiere intellectual leaders of European mathematics. Just as an indication of his pre-eminence, he was asked to give the keynote address at the second International Congress of Mathematicians that was held in Paris in 1900. What Hilbert did at that meeting was earthshaking—from a mathematical point of view. He formulated twenty-three problems that he thought should serve as beacons in the mathematical work of the twentieth century. On the ad-

[11] A "basis" is a minimal generating set for an algebraic system.

vice of Hurwitz and Minkowski, Hilbert abbreviated his remarks and only presented ten of these problems in his lecture. But soon thereafter a more complete version of Hilbert's ideas was published in several countries. For example, in 1902 the *Bulletin of the American Mathematical Society* published an authorized translation by Mary Winston Newson[12] (1869–1959). This version described all twenty-three of the unsolved mathematics problems that Hilbert considered to be of the first rank, and for which it was of the greatest importance to find a solution.

Of course Hilbert's name carried considerable clout, and the mathematicians in attendance paid careful attention to the great savant's admonitions. They took the problems home with them and in turn disseminated them to their peers and colleagues. We have noted that Hilbert's remarks were written up and published, and thereby found their way to universities all over the world. It rapidly became a matter of great interest to solve a Hilbert problem, and considerable praise and encomia were showered on anyone who did so. Today most of the Hilbert problems are solved, but there are a few particularly thorny ones that remain. The references [GRA] and [YAN] give a detailed historical accounting of the colorful history of the Hilbert problems.

One of Hilbert's overriding passions was logic, and he wrote an important treatise in the subject [HIA]. Since Hilbert had a universal and comprehensive knowledge of mathematics, he thought carefully about how the different parts of the subject fit together. And he worried about the axiomatization of the subject. Hilbert believed fervently that there ought to be a universal (and rather small) set of axioms for mathematics, and that all mathematical theorems should be derivable from those axioms.[13] But Hilbert was also fully cognizant of the rather uneven history of mathematics. He knew all too well that much of the literature was riddled with errors and inaccuracies and inconsistencies.

[12]Newson was the first American woman to earn the Ph.D. degree at the university in Göttingen.

[13]We now know, thanks to work of Kurt Gödel, that in fact Hilbert's dream cannot be fulfilled. At least not in the literal sense. But it is safe to say that most working mathematicians take Hilbert's program seriously, and most of us approach our subject with this ideal in mind.

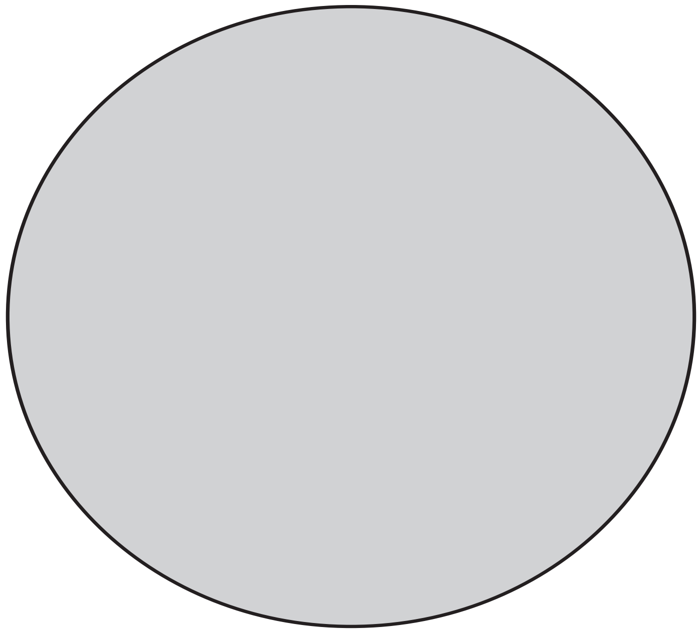

Figure 8: The closed unit disc.

7.1 L. E. J. Brouwer and Proof by Contradiction

L. E. J. Brouwer (1881–1966) was a bright young Dutch mathematician whose chief interest was in topology. Now topology was quite a new subject in those days (the early twentieth century). Affectionately dubbed “rubber sheet geometry”, the subject concerns itself with geometric properties of surfaces and spaces that are preserved under continuous deformation (i.e., twisting and bending and stretching). In his studies of this burgeoning new subject, Brouwer came up with a daring new result, and he found a way to prove it.

Known as the “Brouwer Fixed-Point Theorem”, the result can be described as follows. Consider the closed unit disc $\overline{D}$ in the plane, as depicted in Figure 8. This is a round, circular disc—including the boundary circle as shown in the picture. Now imagine a function $\varphi : \overline{D} \to \overline{D}$ that maps this disc continuously to itself, as shown in Figure 9. Brouwer’s result is that the mapping φ must have a fixed point. That is to say, there is a point $P \in \overline{D}$ such that $\varphi(P) = P$. See Figure 10.

This is a technical mathematical result, and its rigorous proof uses profound ideas such as homotopy theory. But, serendipitously, it lends itself rather naturally to some nice heuristic explanations. Here is one popular

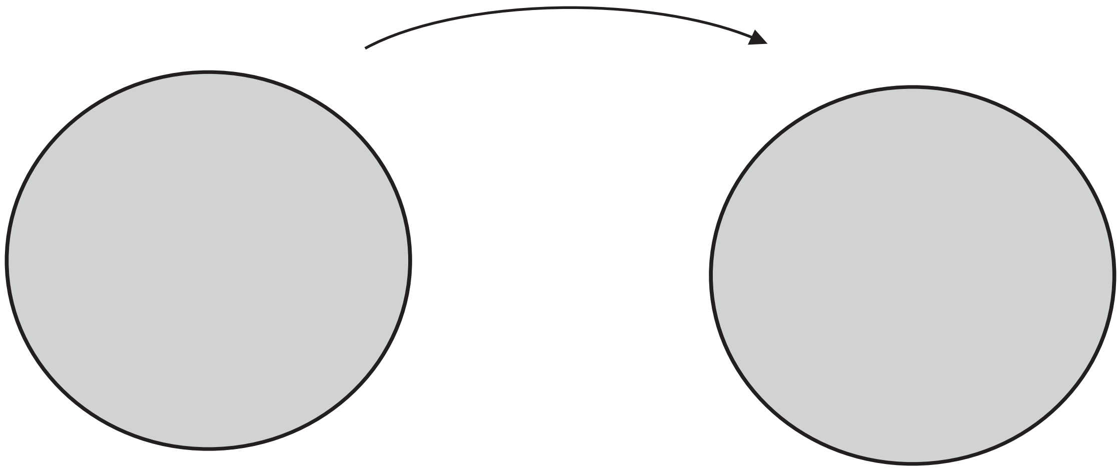

Figure 9: A continuous map from the disc to the disc.

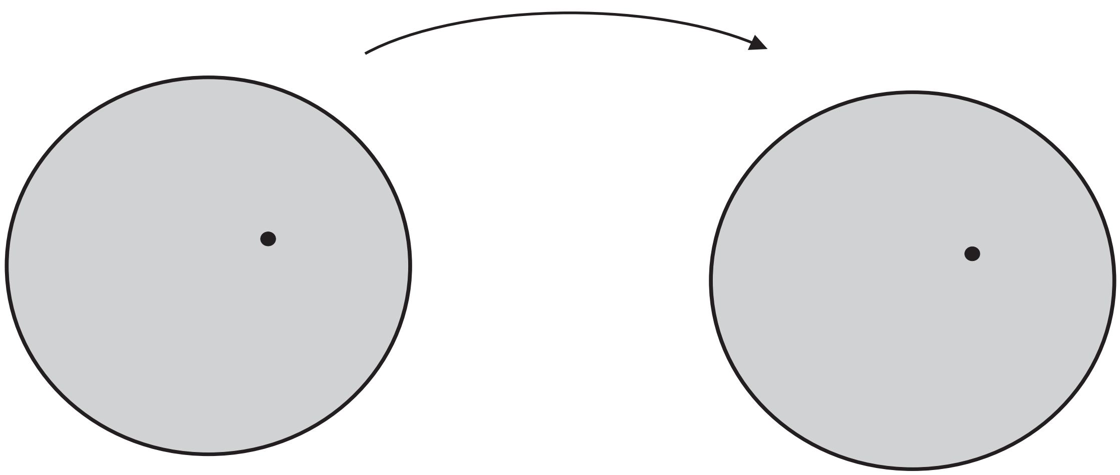

Figure 10: A fixed point of the mapping.

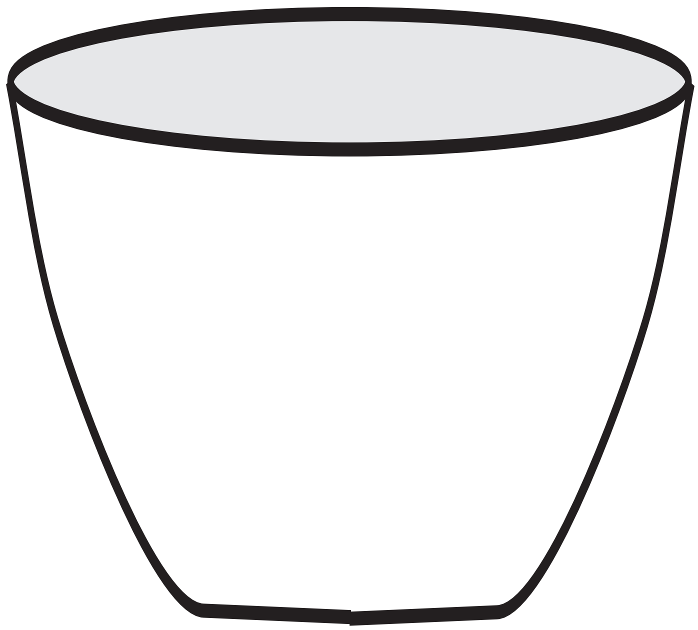

Figure 11: Eating a bowl of soup.

interpretation. Imagine that you are eating a bowl of soup—Figure 11. You sprinkle grated cheese uniformly over the surface of the soup (see Figure 12). And then you stir up the soup. We assume that you stir the soup in a civilized manner so that all the cheese remains on the surface of the soup (refer to Figure 13). Then some grain of cheese remains in its original position (Figure 14).

The soup analogy gives a visceral way to think about the Brouwer fixed-point theorem. Both the statement and the proof of this theorem—in the year 1909—were quite dramatic. In fact it is now known that the Brouwer fixed-point theorem is true in every dimension (Brouwer himself proved it only in dimension 2).

The Brouwer fixed-point theorem is one of the most fascinating and important theorems of twentieth-century mathematics. Proving this theorem established Brouwer as one of the pre-eminent topologists of his day. But he refused to lecture on the subject, and in fact he ultimately rejected this (his own!) work. The reason for this strange behavior is that L. E. J. Brouwer had become a convert to *constructivism* or *intuitionism*. He rejected the Aristotelian dialectic (that a statement is either true or false and there is no

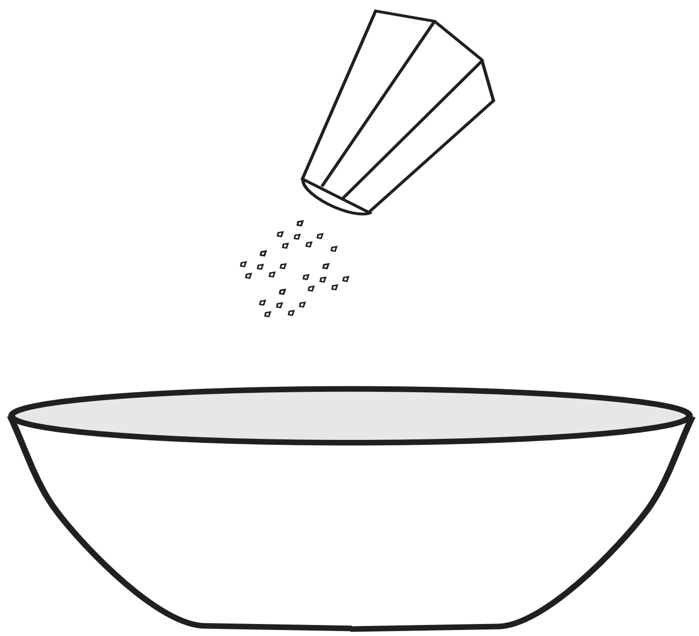

Figure 12: Distributing cheese uniformly over the soup.

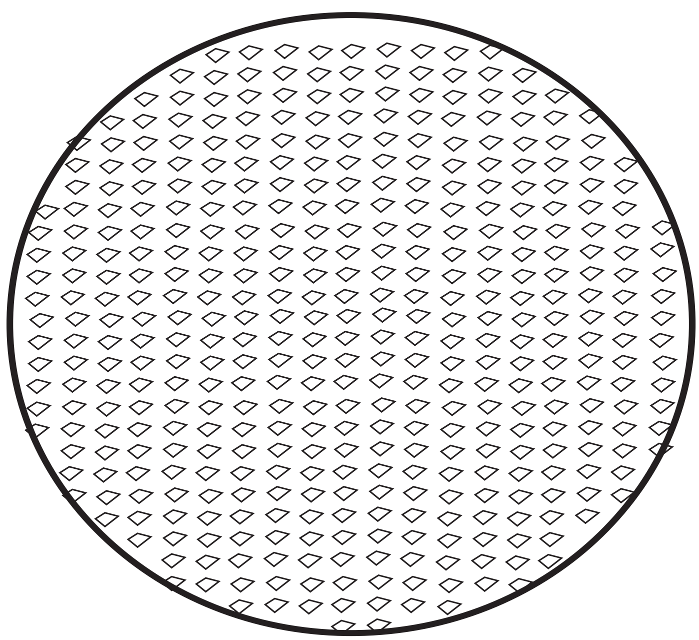

Figure 13: Stirring the soup while keeping the cheese on the surface.

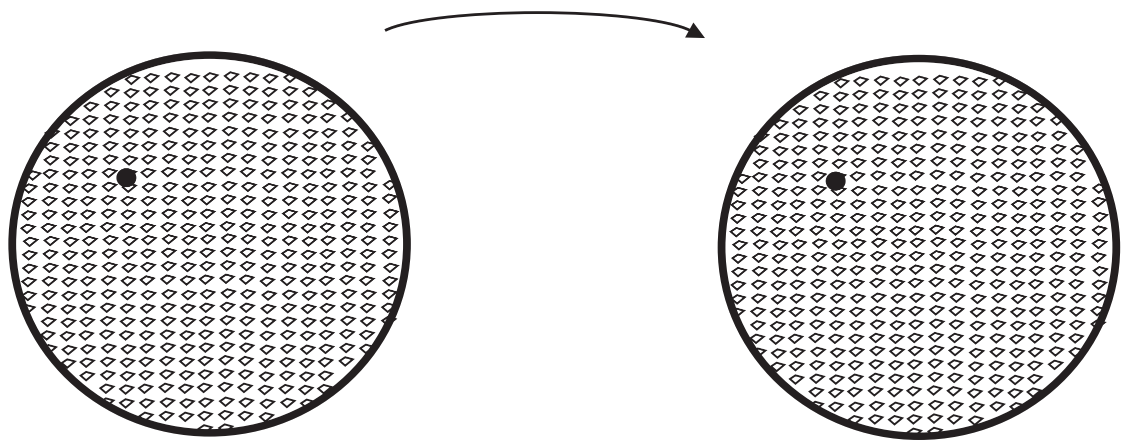

Figure 14: One grain of cheese remains in its original position.

alternative), and therefore rejected the concept of “proof by contradiction”. Brouwer had come to believe that the only valid proofs—at least when one is proving *existence* of some mathematical object (like a fixed point!) and when infinite sets are involved—are those in which we *construct* the asserted objects being discussed.[14] Brouwer's school of thought became known as “intuitionism”, and it has made a definite mark on twentieth century mathematics.

7.2 Errett Bishop and Constructive Analysis

Errett Bishop was one of the great geniuses of mathematical analysis in the 1950s and 1960s. He made his reputation by devising devilishly clever proofs about the structure of spaces of functions. Many of his proofs were indirect proofs—that is to say, proofs by contradiction.

Bishop underwent some personal changes in the mid- to late-1960s. He was a Professor of Mathematics at U. C. Berkeley and he was considerably troubled by all the political unrest on campus. After a time, he felt that he could no longer work in that atmosphere. So he arranged to transfer to U. C. San Diego. At roughly the same time, Bishop became convinced that proofs by contradiction were fraught with peril. He wrote a remarkable and rather poignant book [BIS] which touts the philosophy of constructivism—similar

[14]In fact, for the constructivists, the phrase “there exists” must take on a rigorous new meaning that exceeds the usual rules of formal logic.

in spirit to L. E. J. Brouwer's ideas from fifty years before. Unlike Brouwer, Bishop really put his money where his mouth was. In the pages of his book, Bishop is able to actually develop most of the key ideas of mathematical analysis without resort to proofs by contradiction. Thus he created a new field of mathematics called "constructive analysis".

A quotation from Bishop's Preface to his book gives an indication of how that author himself viewed what he was doing:

> Most mathematicians would find it hard to believe that there could be any serious controversy about the foundations of mathematics, any controversy whose outcome could significantly affect their own mathematical activity.

In a perhaps more puckish mood, Bishop elaborates:

> Mathematics belongs to man, not to God. We are not interested in properties of the positive integers that have no descriptive meaning for finite man. When a man proves a positive integer to exist, he should show how to find it. If God has mathematics of His own that needs to be done, let Him do it Himself.

But our favorite Errett Bishop quotation, and the one that bears most closely on the theme of this article, is

> A proof is any completely convincing argument.

Bishop's arguments in *Methods of Constructive Analysis* [BIS] were, as was characteristic of Bishop, devilishly clever. The book had a definite impact, and certainly caused people to reconsider the methodology of modern analysis. Bishop's acolyte and collaborator D. Bridges produced the revised and expanded version [BIB] of his work (published after Bishop's death), and there the ideas of constructivism are carried even further.

7.3 Nicolas Bourbaki

There had long been a friendly rivalry between French mathematics and German mathematics. Although united by a common subject that everyone loved, and by a shared geographical border, these two ethnic groups practiced mathematics with different styles and different emphases and different priorities. The French certainly took David Hilbert's program for mathematical rigor very seriously, but it was in their nature then to endeavor to create their own home-grown program. This project was ultimately initiated and carried out by a remarkable figure in the history of modern mathematics. His name was Nicolas Bourbaki.

Jean Dieudonné, the great raconteur of twentieth-century French mathematics, tells of a custom at the École Normale Supérieure in France to subject first-year students in mathematics to a rather bizarre rite of initiation. A senior student at the university would be disguised as an important visitor from abroad; he would give an elaborate and rather pompous lecture in which several "well-known" theorems were cited and proved. Each of the theorems would bear the name of a famous or sometimes not-so-famous French general, and each was wrong in some very subtle and clever way. The object of this farce was for the first-year students to endeavor to spot the error in each theorem, or perhaps not to spot the error but to provide some comic relief.

In the mid-1930's, a cabal of French mathematicians—ones who were trained at the notorious École Normale Supérieure—was formed with the purpose of writing definitive texts in the basic subject areas of mathematics. They ultimately decided to publish their books under the *nom de plume* Nicolas Bourbaki. In fact the inspiration for their name was an obscure French general named Charles Denis Sauter Bourbaki. This general, so it is told, was once offered the chance to be King of Greece but (for unknown reasons) he declined the honor. Later, after suffering an embarrassing retreat in the Franco-Prussian War, Bourbaki tried to shoot himself in the head—but he missed. Certainly Bourbaki's name had been used in the tomfoolery at the École Normale. Bourbaki was quite the buffoon. When the young mathematicians André Weil (1906–1998), Jean Delsarte (1903–1968), Jean Dieudonné (1906–1992), Lucien de Possel (1905–1974), Claude Chevalley (1909–1984), and Henri Cartan (1904–), decided to form a secret organization (named Nicolas Bourbaki) that was dedicated to writing definitive texts in the basic subject areas of mathematics, they decided to name themselves after someone completely ludicrous. For what they were doing was of the utmost importance for their subject. So it seemed to make sense to give their work a thoroughly ridiculous byline.

The Nicolas Bourbaki group was formed in the 1930s. Each of the founding members of the organization was himself a prominent and accomplished mathematician. Each had a broad view of the subject, and a clear vision of what Bourbaki was meant to be and what it set out to accomplish. Even though the books of Bourbaki became well known and widely used throughout the world, the identity of the members of Bourbaki was a closely guarded secret. Their meetings, and the venues of those meetings, were kept under wraps. The inner workings of the group were not leaked by anyone.

The membership of Bourbaki was dedicated to the writing of the fundamental texts—in all the basic subject areas—in modern mathematics. Bourbaki's method for producing a book was as follows:

- The first rule of Bourbaki is that they would not write about a mathematical subject unless **(i)** it was basic material that any mathematics graduate student should know and **(ii)** it was mathematically "dead". This second desideratum meant that the subject area must no longer be an active area of current research in mathematics. Considerable discussion was required among the Bourbaki group to determine which were the proper topics for the Bourbaki books.

- Next there would be extensive and prolonged discussion of the chosen subject area: what are the important components of this subject, how do they fit together, what are the milestone results, and so forth. If there were several different ways to approach the subject (and often in mathematics that will be the case), then due consideration was given to which approach the Bourbaki book would take. The discussions we are describing here often took a long weekend, or several long weekends. The meetings were punctuated by long and sumptuous meals at good French restaurants.

- Finally someone would be selected to write the first draft of the book. This of course was a protracted affair, and could take as long as a year or more. Jean Dieudonné, one of the founding members of Bourbaki, was famous for his skill and fluidity at writing. Of all the members of Bourbaki, he was perhaps the one who served most frequently as the scribe. Dieudonné was also a prolific mathematician and writer in his own right.

- After a first draft had been written, copies would be made for the members of the Bourbaki group. And they would read every word—assiduously and critically. Then the group would have another meeting or series of meetings—punctuated as usual by sumptuous repasts at elegant French restaurants—in which they would go through the book page by page or even line by line. The members of Bourbaki were good friends, and had the highest regard for each other as scholars, but they would argue vehemently over particular words or particular sentences in the Bourbaki text. It would take some time for the group to work together through the entire first draft of a future Bourbaki book.

- After the group got through that first draft, and amassed a copious collection of corrections and revisions and edits, then a second draft would be created. This task could be performed by the original author of the first draft, or by a different author. And then the entire cycle of work would repeat itself.

It would take several years, and many drafts, for a new Bourbaki book to be created. The first Bourbaki book, on set theory, was published in 1939; Bourbaki books, and new editions thereof, have appeared as recently as 2005. So far there are thirteen volumes in the monumental series *l'Éléments de Mathématique*. These compose a substantial library of modern mathematics at the level of a first or second year graduate student. Topics covered range from abstract algebra to point-set topology to Lie groups to real analysis. The writing in the Bourbaki books is crisp, clean, and precise. Bourbaki has a very strict notion of mathematical rigor. For example, *no Bourbaki books contain any pictures!* That is correct. Bourbaki felt that pictures are an intuitive device, and have no place in a proper mathematics text. If the mathematics is written correctly then the ideas should be clear—at least after sufficient cogitation. The Bourbaki books are written in a strictly logical fashion, beginning with definitions and axioms and then proceeding with lemmas and propositions and theorems and corollaries. Everything is proved rigorously and precisely. There are few examples and little explanation. Mostly just theorems and proofs. There are no "proofs omitted", no "sketches of proofs", and no "exercises left for the reader".

The Bourbaki books have had a considerable influence in modern mathematics. For many years, other textbook writers sought to mimic the Bourbaki style. Walter Rudin was one of these, and he wrote a number of influential texts without pictures and adhering to a strict logical formalism. Certainly, in the 1950s and 1960s and 1970s, Bourbaki ruled the roost. This group of dedicated French mathematicians with the fictitious name had set a standard to which everyone aspired. It can safely be said that an entire generation of mathematics texts danced to the tune that was set by Bourbaki.

But fashions change. It is now a commonly held belief in France that Bourbaki caused considerable damage to the French mathematics enterprise. How could this be? Given the value system for mathematics that we have been describing in this article, given the passion for rigor and logic that is part and parcel of the subject, it would seem that Bourbaki would be our hero for some time to come. But no. There are other forces at play.

One feature of Bourbaki is that the books were only about *pure* mathematics. There are no Bourbaki books about applied partial differential equations, or control theory, or systems science, or theoretical computer science, or cryptography, or any of the other myriad areas where mathematics is applied. Another feature of Bourbaki is that it rejects intuition of any kind.[15] Certainly one of the main messages of the present book is that we *record* mathematics for posterity in a strictly rigorous, axiomatic fashion. This is the mathematician's version of the reproducible experiment with control used by physicists and biologists and chemists. But we *learn* mathematics, we *discover* mathematics, we *create* mathematics using intuition and trial and error. Certainly we draw pictures. Certainly we try things and twist things around and bend things to try to make them work. Unfortunately, Bourbaki does not teach any part of this latter process.

Thus, even though Bourbaki has been a role model for what recorded mathematics ought to be, even though it is a shining model of rigor and the axiomatic method, it is not necessarily a good and effective teaching tool. So, in the end, Bourbaki has not necessarily completed its grand educational mission. Whereas, in the 1960s and 1970s, it was quite common for Bourbaki books to be used as texts in courses all over the world, now the Bourbaki books are rarely used anywhere in classes. They are still useful references, and helpful for self-study. But, generally speaking, there are much better texts written by other authors. We cannot avoid saying, however, that those "other authors" certainly learned from Bourbaki. Bourbaki's influence is still considerable.

[15]In this sense Bourbaki follows a grand tradition. The master mathematician Carl Friedrich Gauss used to boast that an architect did not leave up the scaffolding so that people could see how he constructed a building. Just so, a mathematician does not leave clues as to how he constructed or found a proof.

8 Computer-Generated Proofs

8.1 The Difference Between Mathematics and Computer Science

When the average person learns that someone is a mathematician, he or she often supposes that that person works on computers all day. This conclusion is both true and false.

Computers are a pervasive aspect of all parts of modern life. The father of modern computer design was John von Neumann, a mathematician. He worked with Herman Goldstine, also a mathematician. Today most every mathematician uses a computer to do e-mail, to typeset his or her papers and books, and to post material on the WorldWide Web. A significant number (but well less than half) of mathematicians use the computer to conduct *experiments*. They calculate numerical solutions of differential equations, they calculate propagation of data for dynamical systems and differential equations, they perform operations research, they engage in the examination of questions from control theory, and many other activities as well. But the vast majority of (academic) mathematicians still, in the end, pick up a pen and write down a *proof*. And that is what they publish.

The design of the modern computer is based on mathematical ideas—the Turing machine, coding theory, queuing theory, binary numbers and operations, high-level languages, and so forth. Certainly operating systems, high-level computing languages (like `Fortran`, `C++`, `Java`, etc.), central processing unit (CPU) design, memory chip design, bus design, memory management, and many other components of the computer world are mathematics-driven. The computer world is an effective and important implementation of the mathematical *theory* that we have been developing for 2500 years. But the computer *is not mathematics*. It is a device for manipulating data.

Still and all, exciting new ideas have come about that have altered the way that mathematics is practiced. The earliest computers were little more than glorifid calculators; they could do little more than arithmetic. Slowly, over time, the idea developed that the computer could carry out *routines*. Ultimately, because of work of John von Neumann, the idea of the stored-program computer was developed. In the 1960s, a group at MIT developed the idea that a computer could perform high-level algebra and geometry and calculus computations. Their product was called `Macsyma`. It could only run on a very powerful computer, and its programming language was very complex and difficult.

Today, thanks to Stephen Wolfram (1959–)[16] and the Maple group at the University of Waterloo[17] and the MathWorks group in Natick, Massachusetts,[18], and many others, we have *computer algebra systems*. A computer algebra system is a high-level computer language that can do calculus, solve differential equations, perform elaborate algebraic manipulations, graph very complicated functions, and perform a vast array of sophisticated mathematical operations. And these software products will run on a personal computer! A great many mathematicians and engineers and other mathematical scientists conduct high-level research using these software products. Many Ph.D. theses present results that are based on explorations using `Mathematica` or `Maple` or `MatLab`. Important new discoveries have come about because of these new tools.

8.2 How a Computer Can Search a Set of Axioms for the Statement and Proof of a New Theorem

With modern, high-level computing languages, it is possible to program into a computer the definitions and axioms of a logical system. And by this we do not simply mean the *words* with which the ideas are conveyed. In fact the machine is given information about how the ideas fit together, what implies what, what are the allowable rules of logic, and so forth. The programming language (such as `Otter`) has a special syntax for entering all this information. Equipped with this data, the computer can then search for valid chains of reasoning (following the hardwired rules of logic, and using only the axioms that have been programmed in) leading to new, valid statements—or *theorems*.

This theorem-proving software can run in two modes: **(i)** interactive mode, in which the machine halts periodically so that the user can input further instructions, and **(ii)** batch mode, in which the machine runs through the entire task and presents a result at the end. In either mode, the purpose is for the computer to find a new mathematical truth and create a logical chain of thought that leads to it.

Some branches of mathematics, such as real analysis, are rather synthetic.

[16]His famous product is `Mathematica`.

[17]Their famous product is `Maple`.

[18]Their famous product is `MatLab`.

Real analysis involves estimates and subtle reasoning that does not derive directly from the twelve axioms in the subject. Thus this area does not lend itself well to computer proofs, and computer proofs have pretty well passed this area by.

Other parts of mathematics are more formalistic. There is still insight and deep thought, but many results can be obtained by fitting the ideas and definitions and axioms together in just the right way. The computer can try millions of combinations in just a few minutes, and its chance of finding something that no human being has ever looked at is pretty good. The Robbins conjecture from Boolean algebra is a vivid example of such a discovery.

There still remain aesthetic questions. After the computer has discovered a new "mathematical truth"—complete with a proof—then some human being or group of human beings will have to examine it and determine its significance. Is it interesting? Is it useful? How does it fit into the context of the subject? What new doors does it open?

One would also wish that the computer reveal its chain of reasoning so that it can be recorded and verified and analyzed by a human being. In mathematics, we are not simply after the result. Our ultimate goal is *understanding*. So we want to see and learn and understand the *proof*.

Computers have been used effectively to find new theorems in Boolean algebra, projective geometry and other classical parts of mathematics. Even some new theorems in Euclidean geometry have been found (see [CHO]). Results in algebra have been obtained by Stickel [STI]. New theorems have also been found in set theory, lattice theory, and ring theory. One could argue that the reason these results were never found by a human being is that no human being would have been interested in them. Only time can judge that question. But certainly the positive resolution of the Robbins conjecture is of great interest for theoretical computer science and logic.

9 Closing Thoughts

9.1 Why Proofs are Important

Before proofs, about 2600 years ago, mathematics was a heuristic and phenomenological subject. Spurred largely (though not entirely) by practical considerations of land surveying, commerce, and counting, there seemed to be no real need for any kind of theory or rigor. It was only with the advent of abstract mathematics—or mathematics for its own sake—that it began to become clear why proofs are important. Indeed, proofs are central to the way that we view our discipline.

Today, there are tens of thousands of mathematicians all over the world. Just as an instance, the *Notices of the American Mathematical Society* has a circulation of about 30,000. [This is the news organ, and the journal of record, for the American Mathematical Society.] And abstract mathematics is a well-established discipline. There are few with any advanced knowledge of mathematics who would argue that proof no longer has a place in our subject. Proof is at the heart of the subject; it is what makes mathematics tick. Just as hand-eye coordination is at the heart of hitting a baseball, and practical technical insight is at the heart of being an engineer, and a sense of color and aesthetics is at the heart of being a painter, so an ability to appreciate and to create proofs is at the heart of being a mathematician.

If one were to remove "proof" from mathematics then all that would remain is a descriptive language. We could examine right triangles, and congruences, and parallel lines and attempt to learn something. We could look at pictures of fractals and make descriptive remarks. We could generate computer printouts and offer witty observations. We could let the computer crank out reams of numerical data and attempt to evaluate those data. We could post beautiful computer graphics and endeavor to assess them. *But we would not be doing mathematics.* Mathematics is **(i)** coming up with new ideas and **(ii)** validating those ideas by way of proof. The timelessness and intrinsic value of the subject come from the methodology, and that methodology is proof.

Proofs remain important in mathematics because they are our bellwether for what we can believe in, and what we can depend on. They are timeless and rigid and dependable. They are what hold the subject together, and what make it one of the glories of human thought.

9.2 What Will Be Considered a Proof in 100 Years?

It is becoming increasingly evident that the delinations among "engineer" and "mathematician" and "physicist" are becoming ever more vague. Today engineering and physics use mathematics at a very sophisticated level, and it is often difficult to tell where one subject ends and the other begins. The widely proliferated collaboration among these different groups is helping to erase barriers and to open up lines of communication. Although "mathematician" has historically been a much-honored and respected profession, one that represents the pinnacle of human thought, we may now fit that model into a broader context.

It seems plausible that in 100 years we will no longer speak of mathematicians as such but rather of *mathematical scientists*. This will include mathematicians to be sure, but also a host of others who use mathematics for analytical purposes. It would not be at all surprising if the notion of "Department of Mathematics" at the college and university level gives way to "Division of Mathematical Sciences".

In fact we already have a role model for this type of thinking at the California Institute of Technology (Caltech). For Caltech does not have departments at all. Instead it has divisions. There is a Division of Physics, Mathematics, and Astronomy—and these three rather different subjects peacefully coexist. There is a Division of Biology that includes Biology, Genetics, and several other fields. The philosophy at Caltech is that departmental divisions tend to be rather artificial, and tend to cause isolation and lack of communication among people who would benefit distinctly from cross-pollination. This is just the type of symbiosis that we have been describing for mathematics in the preceding paragraphs.

So what will be considered a "proof" in the next century? There is every reason to believe that the traditional concept of pure mathematical proof will live on, and will be designated as such. But there will also be computer proofs, and proofs by way of physical experiment, and proofs by way of numerical calculation. This author has participated in a project—connected with NASA's space shuttle program—that involved mathematicians, engineers, and computer scientists. The contributions from the different groups—some numerical, some analytical, some graphical—reinforced each other, and the end result was a rich tapestry of scientific effort. The end product is published in [CHE1] and [CHE2]. This type of collaboration, while rather the exception today, is likely to become ever more common as the field of applied mathematics grows, and as the need for interdisciplinary investigation proliferates.

The Mathematics Department that is open to interdisciplinary work is one that is enriched and fulfilled in a pleasing variety of ways. Colloquium talks will cover a broad panorama of modern research. Visitors will come from a variety of backgrounds, and represent many different perspectives. Mathematicians will direct Ph.D. theses for students from engineering and physics and computer science and other disciplines as well. Conversely, mathematics students will find thesis advisors in many other departments. One already sees this happening with students studying wavelets and harmonic analysis and numerical analysis. The trend will broaden and continue.

So the answer to the question is that "proof" will live on, but it will take on new and varied meanings. The traditional idea of proof will prosper because it will interact with other types of verification and affirmation. And other disciplines, ones that do not traditionally use mathematical proof, will come to appreciate the value of this mode of intellectual discourse. The end result will be a richer tapstry of mathematical science and mathematical work. We will all benefit as a result.

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