Simple row operations

So now that we know how to transfer a system of equations into a matrix, how do we actually go about “solving” the matrix?

That’s where row operations come in. Once we have a linear system represented as a matrix or an augmented matrix, we can use row operations to manipulate and simplify the matrix. Eventually, we’ll be able to get the matrix into a form where the solution to the system just reveals itself in the matrix.

Here are the row operations we need to understand in order to be able to simplify matrices:

1. How to switch rows in the matrix
2. How to multiply (or divide) a row by a constant
3. How to add one row to (or subtract one row from) another

Switching two rows

You can switch any two rows in a matrix without changing the value of the matrix. In this matrix, we’ll switch rows 1 and 2, which we write as $R_1 \leftrightarrow R_2$.

$$\begin{bmatrix} 3 & 2 & | & 7 \\ 1 & -6 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -6 & | & 0 \\ 3 & 2 & | & 7 \end{bmatrix}$$

Keep in mind that you can also make multiple row switches. For instance, in this 3×3 matrix, you could first switch the second row with the third row, $R_2 \leftrightarrow R_3$,

$$\begin{bmatrix} 7 & 3 & 4 \\ 1 & 6 & 1 \\ 2 & 2 & 3 \end{bmatrix} \to \begin{bmatrix} 7 & 3 & 4 \\ 2 & 2 & 3 \\ 1 & 6 & 1 \end{bmatrix}$$

and then switch the first row with the second row, $R_1 \leftrightarrow R_2$.

$$\begin{bmatrix} 7 & 3 & 4 \\ 2 & 2 & 3 \\ 1 & 6 & 1 \end{bmatrix} \to \begin{bmatrix} 2 & 2 & 3 \\ 7 & 3 & 4 \\ 1 & 6 & 1 \end{bmatrix}$$

Realize that it's okay to switch rows in a matrix, since a matrix just represents a linear system. It's no different than rewriting the system

$$3x + 2y = 7$$

$$x - 6y = 0$$

as

$$x - 6y = 0$$

$$3x + 2y = 7$$

Switching the order of the equations in a list of equations representing a linear system is all that you're doing when you switch two rows in a matrix.

Example

Write the new matrix after $R_3 \leftrightarrow R_2$.

$$\begin{bmatrix} 2 & 3 & -1 & | & 11 \\ 1 & 7 & 4 & | & 6 \\ 0 & -1 & -8 & | & -3 \end{bmatrix}$$

The operation described by $R_3 \leftrightarrow R_2$ is switching row 2 with row 3. Nothing will happen to row 1. The matrix after $R_3 \leftrightarrow R_2$ is

$$\begin{bmatrix} 2 & 3 & -1 & | & 11 \\ 0 & -1 & -8 & | & -3 \\ 1 & 7 & 4 & | & 6 \end{bmatrix}$$

Multiplying a row by a constant

You can multiply any row in a matrix by any non-zero constant without changing the value of the matrix. We often call this value a **scalar** because it "scales" the values in the row. For instance, if we multiply through the first row of this matrix by 2, we don't actually change the value of the matrix.

$$\begin{bmatrix} 3 & 2 & | & 7 \\ 1 & -6 & | & 0 \end{bmatrix} \to \begin{bmatrix} 2 \cdot 3 & 2 \cdot 2 & | & 2 \cdot 7 \\ 1 & -6 & | & 0 \end{bmatrix} \to \begin{bmatrix} 6 & 4 & | & 14 \\ 1 & -6 & | & 0 \end{bmatrix}$$

How can it be true that multiplying a row by a constant doesn't change the value of the matrix? Aren't the entires in the matrix now different?

Remember that a row in a matrix represents a linear equation. For instance, the matrix

$$\begin{bmatrix} 6 & 4 & | & 14 \\ 1 & -6 & | & 0 \end{bmatrix}$$

could represent this linear system:

$$6x + 4y = 14$$

$$x - 6y = 0$$

But given $6x + 4y = 14$, we know we can divide through the equation by 2, and it doesn't change the value of the equation. Dividing through by 2 would just give us $3x + 2y = 7$.

So in the same way, we can divide the 2 back out of the matrix, undoing the operation from before,

$$\begin{bmatrix} 6 & 4 & | & 14 \\ 1 & -6 & | & 0 \end{bmatrix} \to \begin{bmatrix} \frac{1}{2} \cdot 6 & \frac{1}{2} \cdot 4 & | & \frac{1}{2} \cdot 14 \\ 1 & -6 & | & 0 \end{bmatrix} \to \begin{bmatrix} 3 & 2 & | & 7 \\ 1 & -6 & | & 0 \end{bmatrix}$$

and the matrix still has the same value.

Keep in mind that you're not limited to multiplying only one row of a matrix by a non-zero constant. You can multiply as many rows as you like by a constant, and the constants don't even have to be the same.

For example, we can multiply the first row of the matrix by 2 (which we write as $2R_1 \to R_1$), and multiply the second row of the matrix by 3 (which we write as $3R_2 \to R_2$),

$$\begin{bmatrix} 3 & 2 & | & 7 \\ 1 & -6 & | & 0 \end{bmatrix} \to \begin{bmatrix} 2 \cdot 3 & 2 \cdot 2 & | & 2 \cdot 7 \\ 3 \cdot 1 & 3 \cdot -6 & | & 3 \cdot 0 \end{bmatrix} \to \begin{bmatrix} 6 & 4 & | & 14 \\ 3 & -18 & | & 0 \end{bmatrix}$$

and we still won't have changed the value of the matrix, since those constants could be divided right back out again.

Example

Write the new matrix after $3R_1 \leftrightarrow 2R_3$.

$$\begin{bmatrix} 2 & 3 & -1 & | & 11 \\ 1 & 7 & 4 & | & 6 \\ 0 & -1 & -8 & | & -3 \end{bmatrix}$$

The operation described by $3R_1 \leftrightarrow 2R_3$ is multiplying row 1 by a constant of 3, multiplying row 3 by a constant of 2, and then switching those two rows. Nothing will happen to row 2. The matrix after $3R_1$ is

$$\begin{bmatrix} 6 & 9 & -3 & | & 33 \\ 1 & 7 & 4 & | & 6 \\ 0 & -1 & -8 & | & -3 \end{bmatrix}$$

The matrix after $2R_3$ is

$$\begin{bmatrix} 6 & 9 & -3 & | & 33 \\ 1 & 7 & 4 & | & 6 \\ 0 & -2 & -16 & | & -6 \end{bmatrix}$$

The matrix after $3R_1 \leftrightarrow 2R_3$ is

$$\begin{bmatrix} 0 & -2 & -16 & | & -6 \\ 1 & 7 & 4 & | & 6 \\ 6 & 9 & -3 & | & 33 \end{bmatrix}$$

Adding a row to another row

It's also acceptable to add one row to another. Keep in mind though that this doesn't consolidate two rows into one. Instead, we replace a row with the sum of itself and another row. For instance, in this matrix,

$$\begin{bmatrix} 3 & 2 & | & 7 \\ 1 & -6 & | & 0 \end{bmatrix}$$

we could replace the first row with the sum of the first and second rows, $R_1 + R_2 \to R_1$. When we perform that operation, we're replacing the entries in row 1, but row 2 stays the same.

$$\begin{bmatrix} 3+1 & 2-6 & | & 7+0 \\ 1 & -6 & | & 0 \end{bmatrix} \to \begin{bmatrix} 4 & -4 & | & 7 \\ 1 & -6 & | & 0 \end{bmatrix}$$

Of course, you can also replace a row with the difference of itself and another row. But subtracting a row from another is the same as adding the row, multiplied by -1, so because we know we can add rows, it's logical that we can also subtract rows.

Example

Write the new matrix after $R_1 + 4R_3 \to R_1$.

$$\begin{bmatrix} 2 & 3 & -1 & | & 11 \\ 1 & 7 & 4 & | & 6 \\ 0 & -1 & -8 & | & -3 \end{bmatrix}$$

The operation described by $R_1 + 4R_3 \to R_1$ is multiplying row 3 by a constant of 4, adding that resulting row to row 1, and using that result to replace row 1. The row R_3 is

$$\begin{bmatrix} 0 & -1 & -8 & | & -3 \end{bmatrix}$$

So $4R_3$ would be

$$\begin{bmatrix} 4(0) & 4(-1) & 4(-8) & | & 4(-3) \end{bmatrix}$$

$$\begin{bmatrix} 0 & -4 & -32 & | & -12 \end{bmatrix}$$

Then because $R_1 = \begin{bmatrix} 2 & 3 & -1 & | & 11 \end{bmatrix}$, $R_1 + 4R_3$ is

$$\begin{bmatrix} 2+0 & 3+(-4) & -1+(-32) & | & 11+(-12) \end{bmatrix}$$

$$\begin{bmatrix} 2 & -1 & -33 & | & -1 \end{bmatrix}$$

The matrix after $R_1 + 4R_3 \to R_1$, which is replacing row 1 with this row we just found, is

$$\begin{bmatrix} 2 & -1 & -33 & | & -1 \\ 1 & 7 & 4 & | & 6 \\ 0 & -1 & -8 & | & -3 \end{bmatrix}$$