MATH 313: LINEAR ALGEBRA - HOMEWORK 4

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1. Problems

(1) Consider the matrices

$$A = \begin{pmatrix} 7 & 1 \\ 0 & -2 \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} 8 & -3 \\ -1 & 4 \end{pmatrix}.$$

Compute the following 2×2 matrices:

$$5A, \quad A+B, \quad 5A-6B, \quad AB, \quad BA, \quad AB-BA.$$

(2) Consider the matrices

$$C = \begin{pmatrix} 9 & -1 & 4 \\ 3 & 0 & 2 \end{pmatrix} \qquad \text{and} \qquad D = \begin{pmatrix} 1 & -13 \\ -2 & 5 \\ 0 & 8 \end{pmatrix}.$$

Compute CD and DC.

(3) (a) Consider the 3×3 matrices

$$G = \begin{pmatrix} 4 & 9 & 0 \\ -1 & -2 & 0 \\ 0 & 0 & -3 \end{pmatrix} \qquad H = \begin{pmatrix} 6 & 5 & 0 \\ 5 & 4 & 0 \\ 0 & 0 & 7 \end{pmatrix}$$

Compute GH.

(b) Consider the 2×2 matrices

$$G' = \begin{pmatrix} 4 & 9 \\ -1 & -2 \end{pmatrix} \qquad H' = \begin{pmatrix} 6 & 5 \\ 5 & 4 \end{pmatrix}$$

Compute $G'H'$.

(c) What do you observe when you compare GH and $G'H'$.

(4) Compute the row and column sums of the following matrices:

(a)

$$\begin{pmatrix} 2 & -7 & 3 & 0 \\ 11 & -10 & 0 & 1 \\ -8 & 4 & -1 & -2 \end{pmatrix}$$

(b)

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \\ 4 & 1 & 2 & 3 \end{pmatrix}$$

(5) The $m \times n$ matrix A is *row stochastic* if all of its row sums are equal to 1, that is $\text{rowsum}_i(A) = 1$ for all $i = 1, \ldots, m$.

The $m \times n$ matrix A is *column stochastic* if all of its column sums are equal to 1, that is $\text{colsum}_j(A) = 1$ for all $j = 1, \ldots, n$.

The $m \times n$ matrix A is *doubly stochastic* if it is both row stochastic and column stochastic.

Prove that if an $m \times n$ matrix is doubly stochastic, then $m = n$.

(6) Prove that if A is an $m \times n$ matrix and if B and C are $n \times p$ matrices, then

$$A(B + C) = AB + AC.$$

Prove that if A and B are $m \times n$ matrices and if C is an $n \times p$ matrices, then

$$(A + B)C = AC + BC.$$

(7) Let A be an $n \times n$ matrix. For every positive integer k, we define the kth power of A, denoted A^k, as the product of k copies of A. Thus,

$$\begin{aligned} A^1 &= A \\ A^2 &= A \cdot A \\ A^3 &= A \cdot A \cdot A \end{aligned}$$

and, in general,

$$A^k = \underbrace{A \cdots A}_{k \text{ factors}}.$$

Let

$$A = \begin{pmatrix} 0 & 4 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 5 & -9 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 5 & -9 & 2 \\ 0 & 0 & 4 & 3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Prove that $A^2 = 0$, $B^3 = 0$, and $C^4 = 0$.

(8) Let $x, y, z \in \mathbf{R}$. Prove that

$$\begin{pmatrix} 0 & x & z \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix}^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

(9) Define $f_0 = 0, f_1 = f_2 = 1$ and, for every $k \geq 2$, define

$$f_{k+1} = f_{k-1} + f_k.$$

The sequence of positive integers $(f_k)_{k=1}^{\infty}$ is called the *Fibonacci sequence*.

(a) Compute the first 11 terms of the Fibonacci sequence $f_0, f_1, \ldots, f_{10}$

(b) Prove that

$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} f_{k-1} \\ f_k \end{pmatrix} = \begin{pmatrix} f_k \\ f_{k+1} \end{pmatrix}$$

for all $k \geq 2$.

(c) Let $F = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$. Compute F^k for $k \in \{1, \ldots, 9\}$.

(d) Prove that

$$F^n = \begin{pmatrix} f_{n-1} & f_n \\ f_n & f_{n+1} \end{pmatrix}$$

for all $n \geq 1$.

2. Solutions to selected problems

(1)

$$5A = \begin{pmatrix} 35 & 5 \\ 0 & -10 \end{pmatrix}, \qquad A + B = \begin{pmatrix} 15 & -2 \\ -1 & 2 \end{pmatrix},$$
$$AB = \begin{pmatrix} 55 & -17 \\ 2 & -8 \end{pmatrix}, \qquad AB - BA = \begin{pmatrix} -1 & -31 \\ 9 & 1 \end{pmatrix}$$

(3)

$$G'H' = \begin{pmatrix} 69 & 56 \\ -16 & -13 \end{pmatrix}, \qquad GH = \begin{pmatrix} 69 & 56 & 0 \\ -16 & -13 & 0 \\ 0 & 0 & -21 \end{pmatrix}$$

(9) (a) Fibonacci sequence: $0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55$

(c) $F^2 = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}, \quad F^3 = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}, \quad F^9 = \begin{pmatrix} 21 & 34 \\ 34 & 55 \end{pmatrix}.$