Vector operations

Just like matrices, vectors can be added, subtracted, and multiplied.

Since vectors can be written as column matrices or row matrices, you'll be able to use what you've learned about matrix operations in order to perform the corresponding vector operations.

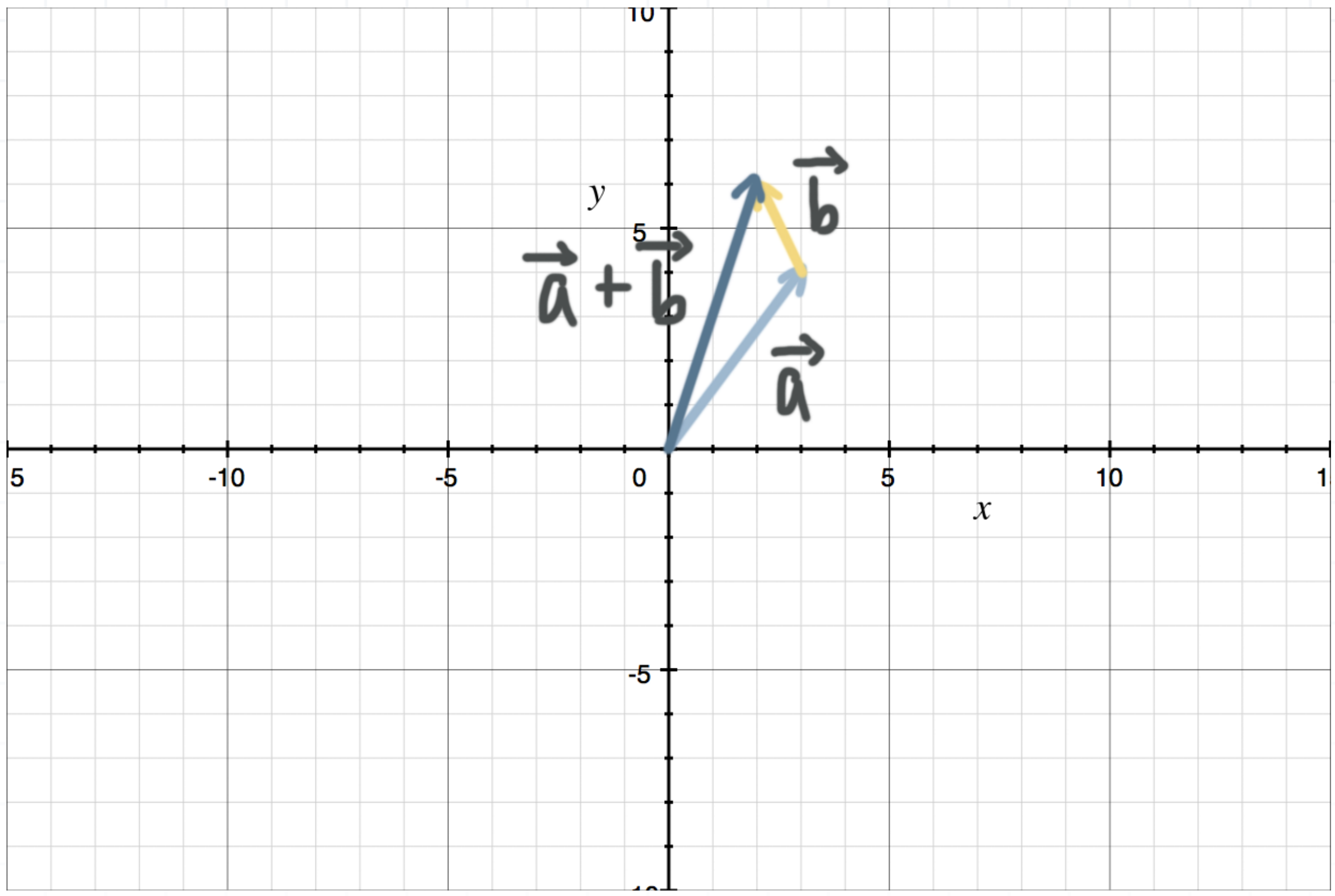
The sum of vectors

To add vectors, just add their corresponding components. Given $\overrightarrow{a} = (3,4)$ and $\overrightarrow{b} = (-1,2)$, the sum of the vectors

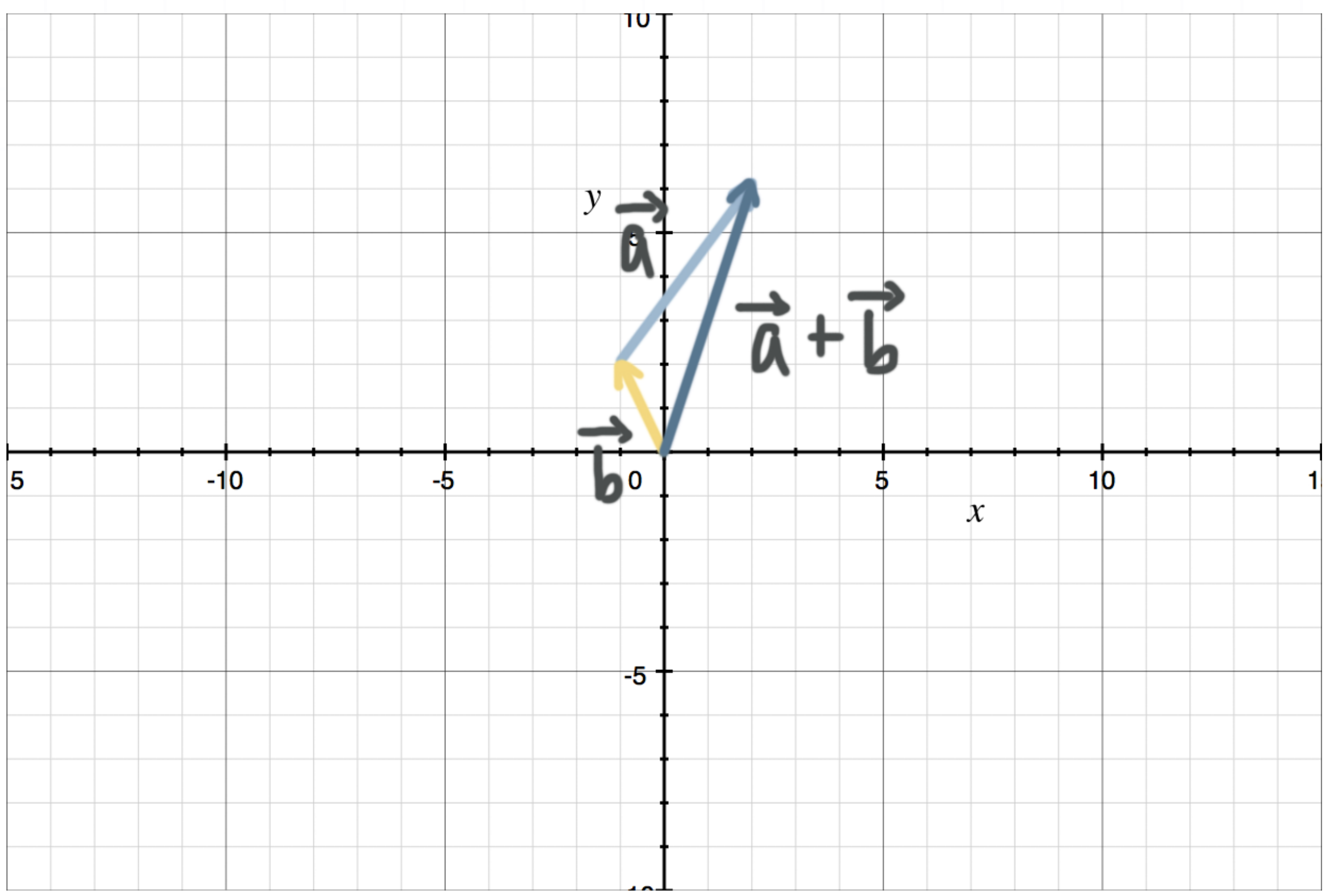
$$\overrightarrow{a} + \overrightarrow{b} = (3 + (-1), 4 + 2)$$

$$\overrightarrow{a} + \overrightarrow{b} = (2,6)$$

Graphically, we can see that adding vectors means connecting the terminal point of one to the tail of the other. If we start with $\overrightarrow{a} = (3,4)$ and add $\overrightarrow{b} = (-1,2)$ to it, that looks like this:



If we start with $\overrightarrow{b}=(-1,2)$ and add $\overrightarrow{a}=(3,4)$ to it, that looks like this:



Either way, we end up at $(2,6)$. Because matrix addition is commutative, it makes sense that we end up at the same point, regardless of the order in which we add the matrices. For instance, since $\overrightarrow{a}=(3,4)$ and $\overrightarrow{b}=(-1,2)$ could both be represented as column vectors (or row vectors), we really just have a matrix addition problem, and we know that matrices can be added in either order.

$$A+B=\begin{bmatrix}3\\4\end{bmatrix}+\begin{bmatrix}-1\\2\end{bmatrix}=\begin{bmatrix}2\\6\end{bmatrix}$$

$$B+A=\begin{bmatrix}-1\\2\end{bmatrix}+\begin{bmatrix}3\\4\end{bmatrix}=\begin{bmatrix}2\\6\end{bmatrix}$$

And of course, even though we've only added two vectors here, we could find the sum of any number of vectors that we'd like, and the result would always be the same, regardless of the order in which we add them (since matrix addition is commutative). Furthermore, while here we've added two-dimensional vectors, we could change these into n-dimensional vectors, and find the sum of any number of n-dimensional vectors.

The difference of vectors

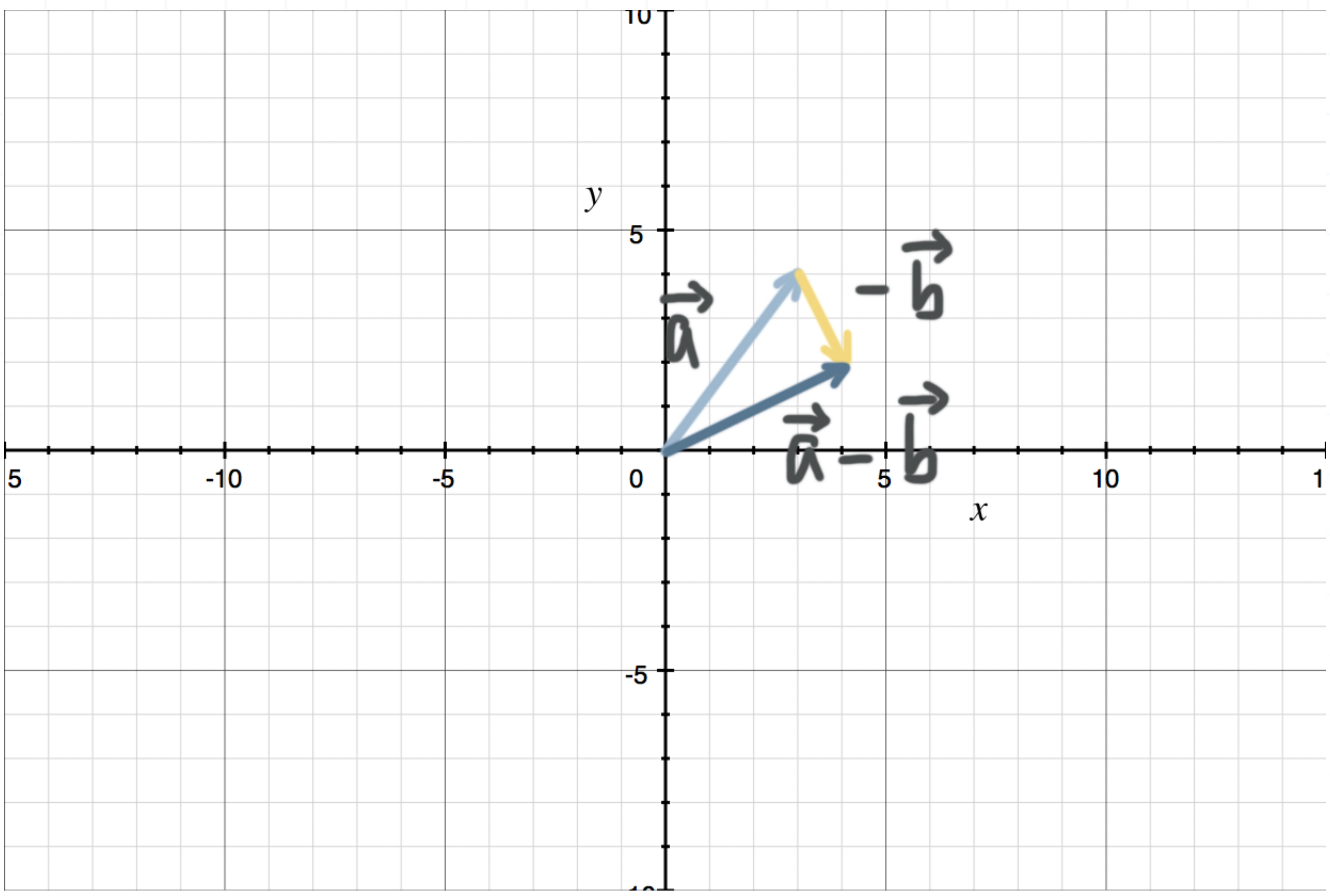
Remember that, unlike matrix addition, matrix subtraction is not commutative. Therefore, it makes a difference whether we subtract $\overrightarrow{b}=(-1,2)$ from $\overrightarrow{a}=(3,4)$ or $\overrightarrow{a}=(3,4)$ from $\overrightarrow{b}=(-1,2)$. We'll get a different result in each case.

But the operation is still simple. We just take the difference of corresponding components from each vector. Here are the differences $\overrightarrow{a}-\overrightarrow{b}$ and $\overrightarrow{b}-\overrightarrow{a}$.

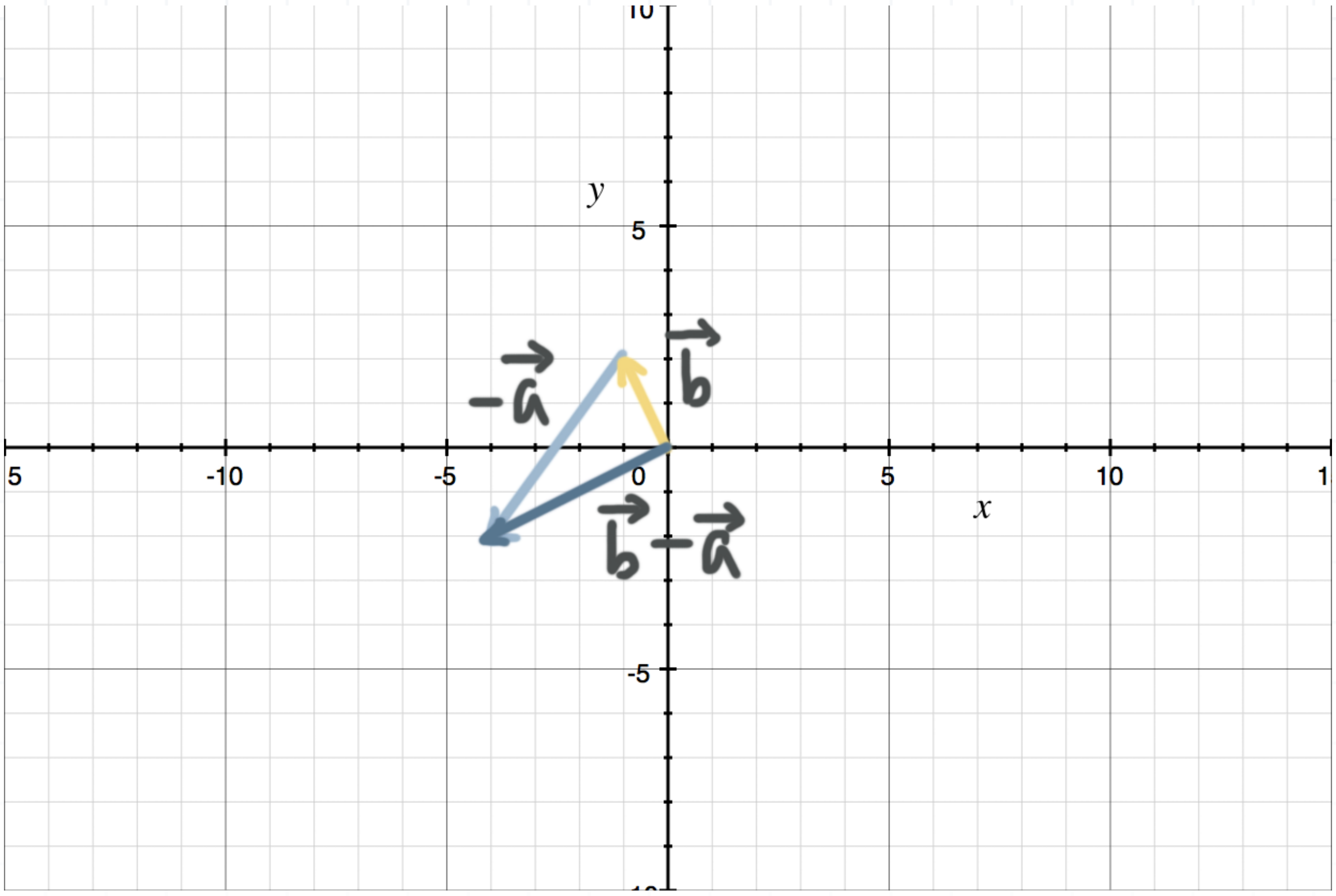
$$\vec{a} - \vec{b} = (3 - (-1), 4 - 2) = (4,2)$$

$$\vec{b} - \vec{a} = (-1 - 3, 2 - 4) = (-4, -2)$$

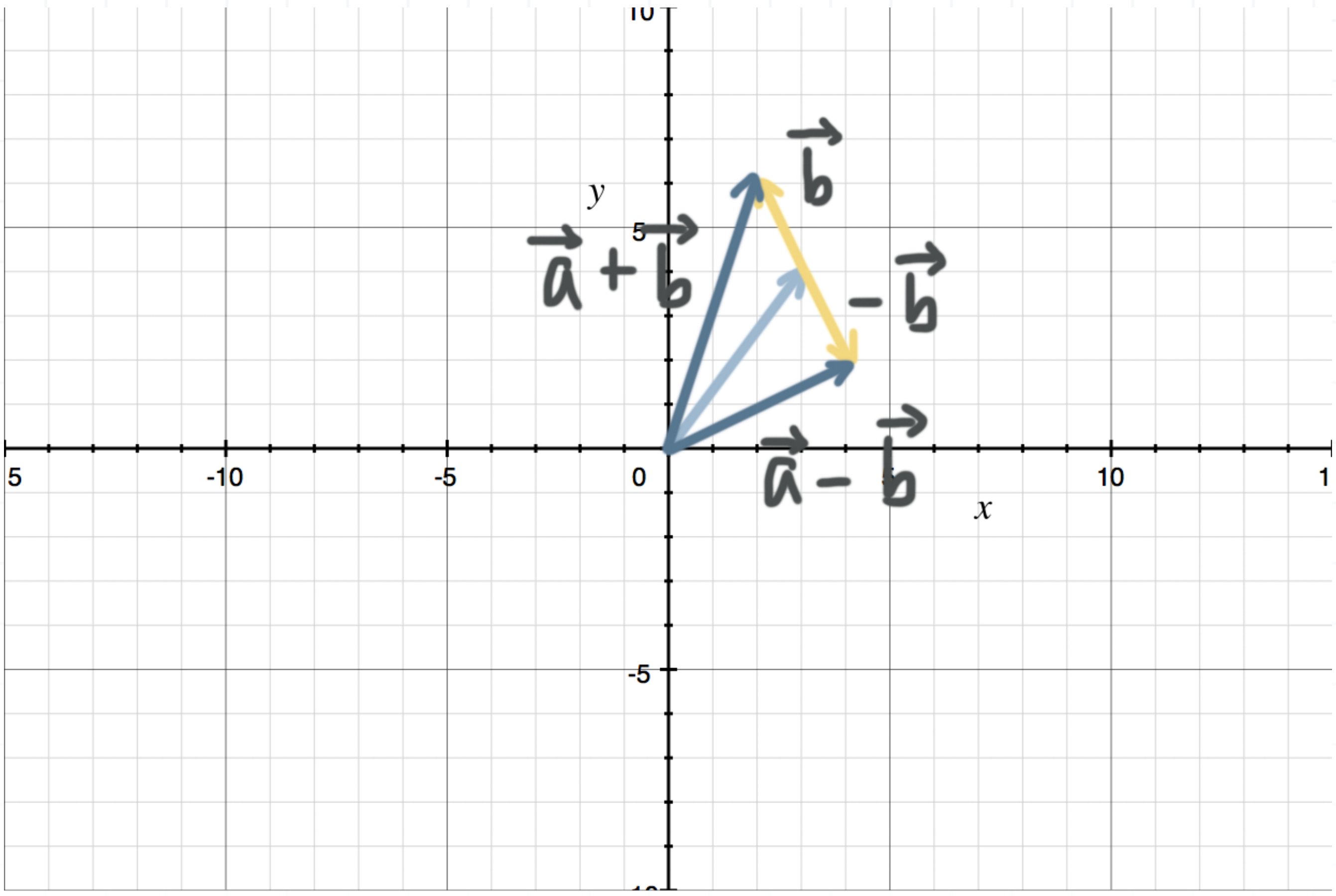
If we sketch the difference of the vectors, we can see that $\vec{a} - \vec{b}$ gets us to $\vec{a} - \vec{b} = (4,2)$,



and that $\vec{b} - \vec{a}$ gets us to $\vec{b} - \vec{a} = (-4, -2)$.



Notice also that $\vec{a}-\vec{b}$ is the same as $\vec{a}+(-\vec{b})$. The vector $-\vec{b}$ has the same length as $\vec{b}$, but moves in exactly the opposite direction. In other words, here are $\vec{a}$, $\vec{b}$, and $-\vec{b}$ sketched together. Notice how $\vec{b}$ and $-\vec{b}$ move in exactly opposite directions. And we can see the results $\vec{a}+\vec{b}$ and $\vec{a}-\vec{b}$.



Let's do an example with vector addition and subtraction.

Example

Find $\vec{a}+\vec{b}-\vec{c}+\vec{d}$.

$$\vec{a}=(3,4)$$

$$\vec{b}=(-1,2)$$

$$\vec{c}=(3,3)$$

$$\vec{d}=(-2,0)$$

Just like addition and subtraction with real numbers, we can work left to right. We've already seen the sum $\vec{a}+\vec{b}$.

$$\vec{a} + \vec{b} = (3 + (-1), 4 + 2)$$

$$\vec{a} + \vec{b} = (2,6)$$

When we subtract $\vec{c} = (3,3)$ from this, we get

$$\vec{a} + \vec{b} - \vec{c} = (2 - 3, 6 - 3)$$

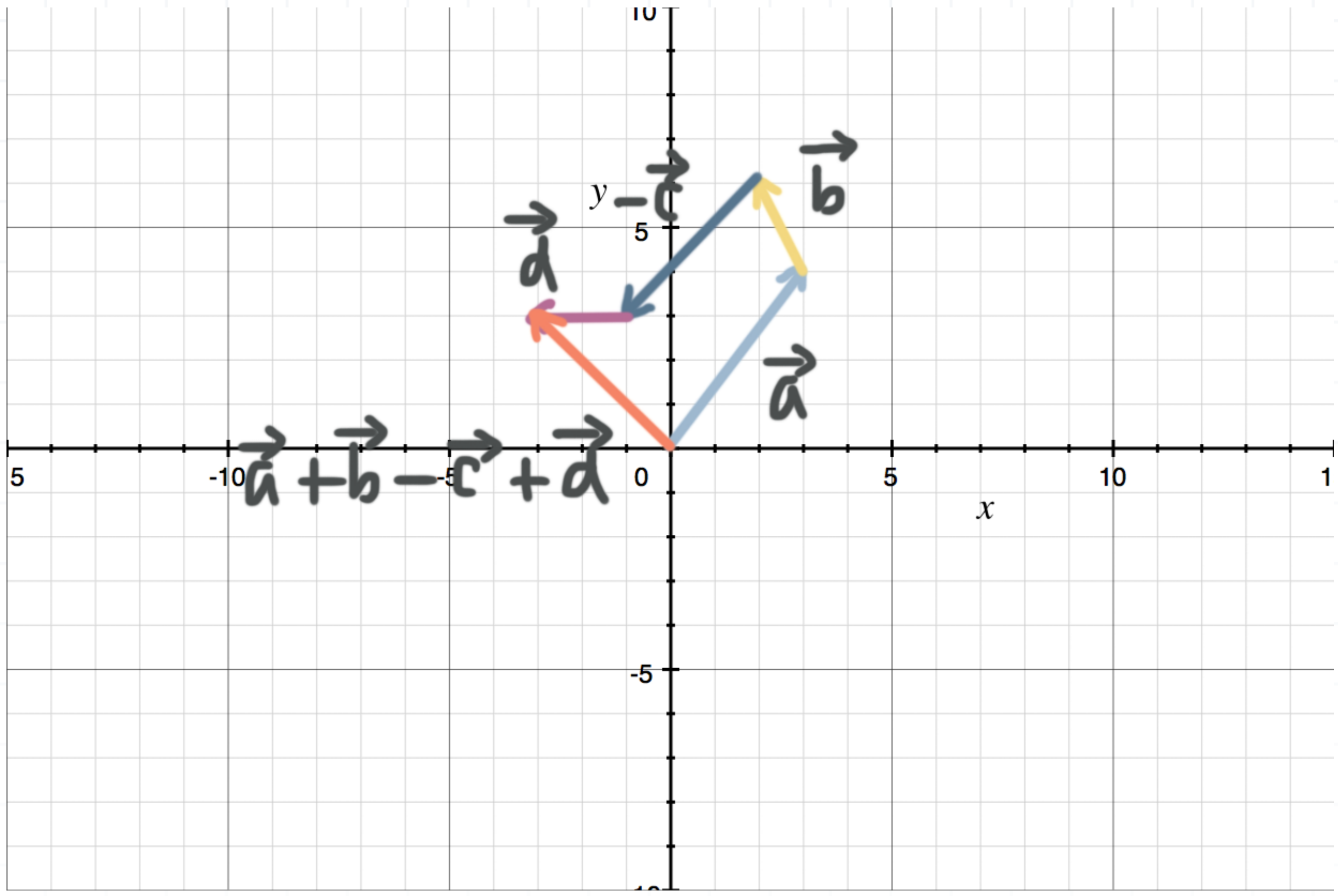
$$\vec{a} + \vec{b} - \vec{c} = (-1,3)$$

When we add $\vec{d} = (-2,0)$ to this, we get

$$\vec{a} + \vec{b} - \vec{c} + \vec{d} = (-1 + (-2), 3 + 0)$$

$$\vec{a} + \vec{b} - \vec{c} + \vec{d} = (-3,3)$$

We can sketch all the individual vectors by connecting the terminal point of each one to the tail of the one before it, plus the vector of the sum $\vec{a} + \vec{b} - \vec{c} + \vec{d}$, together in the same plane, and we see that the sum gets us to the same ending point.

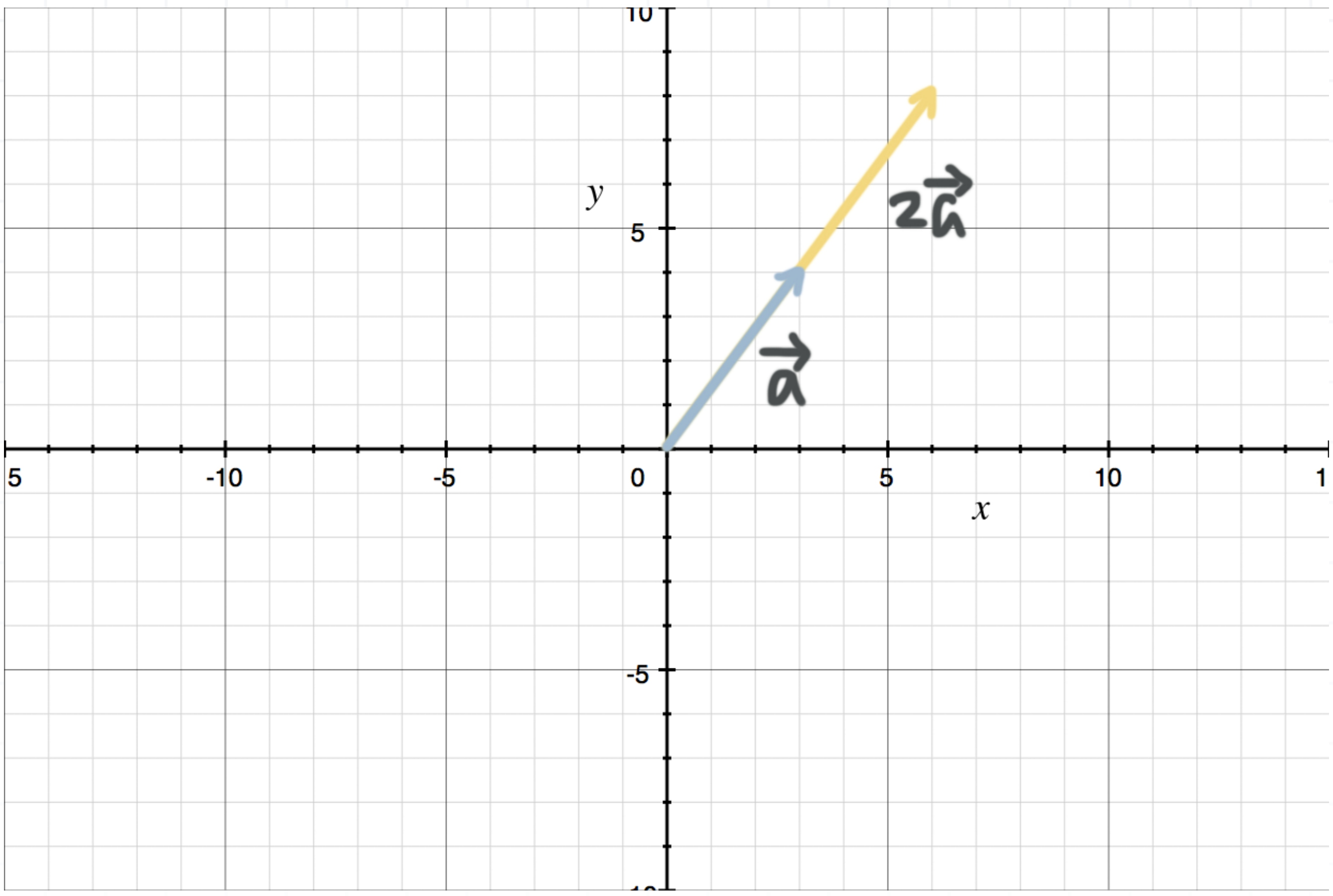


Multiplying a vector by a scalar

Multiplying a vector by a scalar is just like multiplying a column matrix by a scalar. For instance, if we multiply $\vec{a}=(3,4)$ by 2, we get

$$2\vec{a}=2\begin{bmatrix}3\\4\end{bmatrix}=\begin{bmatrix}2(3)\\2(4)\end{bmatrix}=\begin{bmatrix}6\\8\end{bmatrix}$$

Graphically, we can see that the resulting column vector has the same direction, but its magnitude is scaled by the absolute value of the scalar.

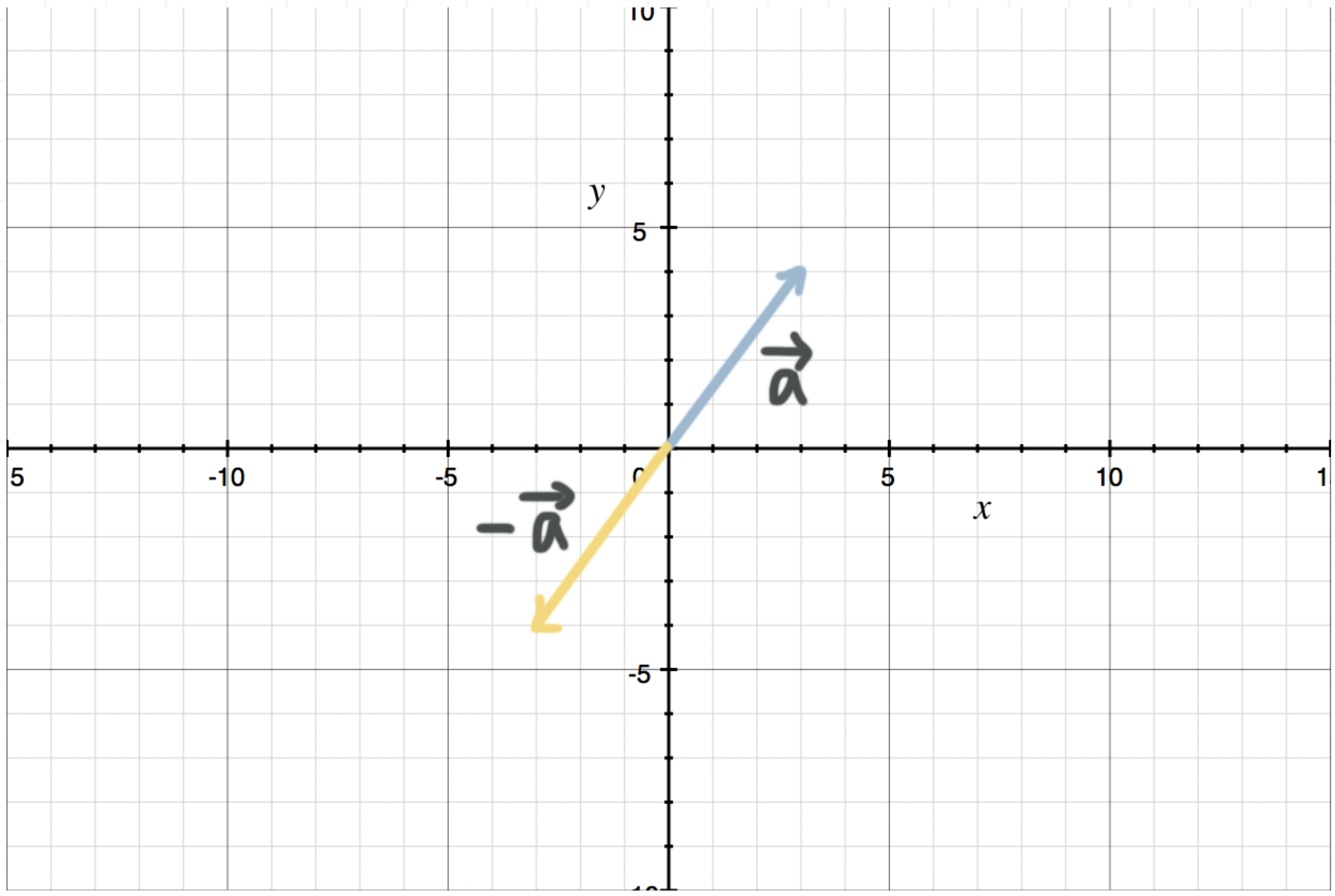


In the graph of these vectors, the yellow $2\vec{a}$ does not begin where $\vec{a}$ ends. The vectors $\vec{a}$ and $2\vec{a}$ both begin at $(0,0)$, and $2\vec{a}$ has double the length of $\vec{a}$.

If you multiply by a negative scalar, the vector will point in exactly the opposite direction, or you could say that its direction rotates 180°. So for $\vec{a}=(3,4)$,

$$-1\vec{a}=-1\begin{bmatrix}3\\4\end{bmatrix}=\begin{bmatrix}-1(3)\\-1(4)\end{bmatrix}=\begin{bmatrix}-3\\-4\end{bmatrix}$$

and $\vec{a}=(3,4)$ and $-\vec{a}=(-3,-4)$ sketched together looks like this:



Multiplying a vector by a vector

Beyond just multiplying a scalar by a vector, you can also multiply a vector by a vector. We'll talk about this more in a later section, but the product of two vectors is called the **dot product**, and we find it by summing the products of the individual components.

$$\vec{a} \cdot \vec{b} = (a_1, a_2) \cdot (b_1, b_2)$$

$$\vec{a} \cdot \vec{b} = a_1b_1 + a_2b_2$$

For example, the dot product of $\vec{a} = (3,4)$ and $\vec{b} = (-1,2)$ is

$$\vec{a} \cdot \vec{b} = (3,4) \cdot (-1,2)$$

$$\vec{a} \cdot \vec{b} = (3)(-1) + (4)(2)$$

$$\vec{a} \cdot \vec{b} = -3 + 8$$

$$\vec{a} \cdot \vec{b} = 5$$

We can also find the dot product when we write the vectors as matrices. So if we wrote $\vec{a}$ as the matrix A and $\vec{b}$ as the matrix B, then we could have written the dot product as

$$AB = \begin{bmatrix} 3 & 4 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

$$AB = [3(-1) + 4(2)]$$

$$AB = [-3 + 8]$$

$$AB = [5]$$

or as

$$BA = \begin{bmatrix} -1 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

$$BA = [-1(3) + 2(4)]$$

$$BA = [-3 + 8]$$

$$BA = [5]$$

When we express the vectors as matrices and then multiply them, it's important to multiply them as a row matrix first, multiplied by a column matrix second. That way, the dimensions are

$$R \times C \quad \times \quad R \times C$$

$$1 \times 2 \quad \times \quad 2 \times 1$$

and the resulting product will be a 1×1 matrix, whose only entry is the value of the dot product. If we instead multiply a column matrix by a row matrix, like

$$AB = \begin{bmatrix} 3 \\ 4 \end{bmatrix} \begin{bmatrix} -1 & 2 \end{bmatrix}$$

then the result isn't the dot product, it's something entirely different. We can tell from the dimensions that the result will be a 2×2 matrix. Because of the way the dimensions can get tricky, we usually just stick with the

$$\overrightarrow{a} \cdot \overrightarrow{b} = (a_1, a_2) \cdot (b_1, b_2)$$

form when we're finding the dot product of two vectors.