Topic: Linear independence in three dimensions

Question: Which vector set is linearly independent?

Answer choices:

A $\quad \vec{a} = \begin{bmatrix} 4 \\ 2 \\ 2 \end{bmatrix}, \vec{b} = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}, \vec{c} = \begin{bmatrix} -10 \\ -5 \\ 5 \end{bmatrix}$

B $\quad \vec{a} = \begin{bmatrix} 4 \\ 2 \\ 0 \end{bmatrix}, \vec{b} = \begin{bmatrix} -2 \\ -1 \\ 0 \end{bmatrix}, \vec{c} = \begin{bmatrix} -6 \\ 3 \\ 4 \end{bmatrix}$

C $\quad \vec{a} = \begin{bmatrix} 4 \\ 2 \\ 2 \end{bmatrix}, \vec{b} = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}, \vec{c} = \begin{bmatrix} -10 \\ 5 \\ 5 \end{bmatrix}$

D $\quad \vec{a} = \begin{bmatrix} 4 \\ 2 \\ 0 \end{bmatrix}, \vec{b} = \begin{bmatrix} -2 \\ -1 \\ 0 \end{bmatrix}, \vec{c} = \begin{bmatrix} -6 \\ -3 \\ 4 \end{bmatrix}$

Solution: A

All of the answer choices could be a linearly independent set in $\mathbb{R}^3$, since each set has three or fewer vectors.

Let's test answer choice A by setting up the vector equation.

$$c_1 \begin{bmatrix} 4 \\ 2 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} -10 \\ -5 \\ 5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Change the equation into an augmented matrix, then put the matrix into reduced row-echelon form.

$$\begin{bmatrix} 4 & -2 & -10 & | & 0 \\ 2 & 1 & -5 & | & 0 \\ 2 & 1 & 5 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -\frac{1}{2} & -\frac{5}{2} & | & 0 \\ 2 & 1 & -5 & | & 0 \\ 2 & 1 & 5 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -\frac{1}{2} & -\frac{5}{2} & | & 0 \\ 0 & 2 & 0 & | & 0 \\ 2 & 1 & 5 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -\frac{1}{2} & -\frac{5}{2} & | & 0 \\ 0 & 2 & 0 & | & 0 \\ 0 & 2 & 10 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -\frac{1}{2} & -\frac{5}{2} & | & 0 \\ 0 & 1 & 0 & | & 0 \\ 0 & 2 & 10 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -\frac{5}{2} & | & 0 \\ 0 & 1 & 0 & | & 0 \\ 0 & 2 & 10 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -\frac{5}{2} & | & 0 \\ 0 & 1 & 0 & | & 0 \\ 0 & 0 & 10 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -\frac{5}{2} & | & 0 \\ 0 & 1 & 0 & | & 0 \\ 0 & 0 & 1 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & | & 0 \\ 0 & 1 & 0 & | & 0 \\ 0 & 0 & 1 & | & 0 \end{bmatrix}$$

The rref form of the matrix gives the equations

$$c_1 = 0$$

$$c_2 = 0$$

$$c_3 = 0$$

These equations tell us that $(c_1, c_2 - c_3) = (0,0,0)$ is the only solution, which means the vectors in answer choice A are linearly independent.

Topic: Linear independence in three dimensions

Question: Which vector set is linearly independent?

Answer choices:

A $\quad \vec{u} = \begin{bmatrix} 7 \\ -1 \\ 4 \end{bmatrix}, \vec{v} = \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}, \vec{w} = \begin{bmatrix} 6 \\ 4 \\ -2 \end{bmatrix}, \vec{x} = \begin{bmatrix} -5 \\ 0 \\ 3 \end{bmatrix}$

B $\quad \vec{u} = \begin{bmatrix} 7 \\ -1 \\ 4 \end{bmatrix}, \vec{v} = \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}, \vec{w} = \begin{bmatrix} -6 \\ 4 \\ 2 \end{bmatrix}, \vec{x} = \begin{bmatrix} -5 \\ 0 \\ 3 \end{bmatrix}$

C $\quad \vec{u} = \begin{bmatrix} 7 \\ -1 \\ 4 \end{bmatrix}, \vec{v} = \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}, \vec{w} = \begin{bmatrix} 6 \\ 4 \\ -2 \end{bmatrix}$

D $\quad \vec{u} = \begin{bmatrix} 7 \\ -1 \\ 4 \end{bmatrix}, \vec{v} = \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}, \vec{w} = \begin{bmatrix} -6 \\ 4 \\ 2 \end{bmatrix}$

Solution: D

In $\mathbb{R}^n$ space, only vector sets with n or fewer vectors can be linearly independent. For instance, in $\mathbb{R}^3$, only vector sets with three or fewer vectors can be a linearly independent set, and any set with four or more vectors will be linearly dependent.

Because all of the vectors in these answer choices are in $\mathbb{R}^3$, a linearly independent set will include three or fewer vectors, which leaves only answer choices C and D as possibilities.

Let's test answer choice C by setting up the vector equation.

$$c_1 \begin{bmatrix} 7 \\ -1 \\ 4 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix} + c_3 \begin{bmatrix} 6 \\ 4 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Change the equation into an augmented matrix, then put the matrix into reduced row-echelon form.

$$\begin{bmatrix} 7 & 3 & 6 & | & 0 \\ -1 & 2 & 4 & | & 0 \\ 4 & -1 & -2 & | & 0 \end{bmatrix} \to \begin{bmatrix} -1 & 2 & 4 & | & 0 \\ 7 & 3 & 6 & | & 0 \\ 4 & -1 & -2 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & -4 & | & 0 \\ 7 & 3 & 6 & | & 0 \\ 4 & -1 & -2 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & -4 & | & 0 \\ 0 & 17 & 34 & | & 0 \\ 4 & -1 & -2 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & -4 & | & 0 \\ 0 & 17 & 34 & | & 0 \\ 0 & 7 & 14 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & -4 & | & 0 \\ 0 & 1 & 2 & | & 0 \\ 0 & 7 & 14 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & | & 0 \\ 0 & 1 & 2 & | & 0 \\ 0 & 7 & 14 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & | & 0 \\ 0 & 1 & 2 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

The rref form of the matrix gives the equations

$$c_1 = 0$$

$$c_2 + 2c_3 = 0 \quad \rightarrow \quad c_2 = -2c_3$$

This equation tells us that there are an endless number of solutions to the system. We can choose any value for c_3, and we'll get a different value for c_2, and all of those combinations will give us the zero vector. Because $(c_1, c_2, c_3) = (0,0,0)$ isn't the only solution, that tells us that the vectors in answer choice C are linearly dependent.

Which means answer choice D must be the correct choice, but let's verify that those vectors are, in fact, linearly independent. We'll test answer choice D by setting up the vector equation.

$$c_1 \begin{bmatrix} 7 \\ -1 \\ 4 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix} + c_3 \begin{bmatrix} -6 \\ -4 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Change the equation into an augmented matrix, then put the matrix into reduced row-echelon form.

$$\begin{bmatrix} 7 & 3 & -6 & | & 0 \\ -1 & 2 & 4 & | & 0 \\ 4 & -1 & 2 & | & 0 \end{bmatrix} \rightarrow \begin{bmatrix} -1 & 2 & 4 & | & 0 \\ 7 & 3 & -6 & | & 0 \\ 4 & -1 & 2 & | & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 & -4 & | & 0 \\ 7 & 3 & -6 & | & 0 \\ 4 & -1 & 2 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & -4 & | & 0 \\ 0 & 17 & 22 & | & 0 \\ 4 & -1 & 2 & | & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 & -4 & | & 0 \\ 0 & 17 & 22 & | & 0 \\ 0 & 7 & 18 & | & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 & -4 & | & 0 \\ 0 & 1 & \frac{22}{17} & | & 0 \\ 0 & 7 & 18 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -\frac{24}{17} & | & 0 \\ 0 & 1 & \frac{22}{17} & | & 0 \\ 0 & 7 & 18 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -\frac{24}{17} & | & 0 \\ 0 & 1 & \frac{22}{17} & | & 0 \\ 0 & 0 & \frac{152}{17} & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -\frac{24}{17} & | & 0 \\ 0 & 1 & \frac{22}{17} & | & 0 \\ 0 & 0 & 1 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & | & 0 \\ 0 & 1 & \frac{22}{17} & | & 0 \\ 0 & 0 & 1 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & | & 0 \\ 0 & 1 & 0 & | & 0 \\ 0 & 0 & 1 & | & 0 \end{bmatrix}$$

The rref form of the matrix gives the equations

$$c_1 = 0$$

$$c_2 = 0$$

$$c_3 = 0$$

These equations tell us that $(c_1, c_2, c_3) = (0,0,0)$ is the only solution, which means the vectors in answer choice D are linearly independent.

Topic: Linear independence in three dimensions

Question: Which vector set is linearly independent?

Answer choices:

A $\quad \vec{a}=\begin{bmatrix}-8\\4\\2\end{bmatrix}, \vec{b}=\begin{bmatrix}-3\\-6\\-9\end{bmatrix}, \vec{c}=\begin{bmatrix}16\\-8\\-4\end{bmatrix}$

B $\quad \vec{a}=\begin{bmatrix}-8\\4\\2\end{bmatrix}, \vec{b}=\begin{bmatrix}-3\\-6\\-9\end{bmatrix}, \vec{c}=\begin{bmatrix}-6\\1\\3\end{bmatrix}$

C $\quad \vec{a}=\begin{bmatrix}-8\\4\\2\end{bmatrix}, \vec{b}=\begin{bmatrix}-3\\-6\\-9\end{bmatrix}, \vec{c}=\begin{bmatrix}16\\-8\\-4\end{bmatrix}, \vec{d}=\begin{bmatrix}-8\\6\\-2\end{bmatrix}$

D $\quad \vec{a}=\begin{bmatrix}-8\\4\\2\end{bmatrix}, \vec{b}=\begin{bmatrix}-3\\-6\\-9\end{bmatrix}, \vec{c}=\begin{bmatrix}-6\\1\\3\end{bmatrix}, \vec{d}=\begin{bmatrix}-8\\6\\-2\end{bmatrix}$

Solution: B

In $\mathbb{R}^n$ space, only vector sets with n or fewer vectors can be linearly independent. For instance, in $\mathbb{R}^3$, only vector sets with three or fewer vectors can be a linearly independent set, and any set with four or more vectors will be linearly dependent.

Because all of the vectors in these answer choices are in $\mathbb{R}^3$, a linearly independent set will include three or fewer vectors, which leaves only answer choices A and B as possibilities.

Let's test answer choice A by setting up the vector equation.

$$c_1\begin{bmatrix}-8\\4\\2\end{bmatrix}+c_2\begin{bmatrix}-3\\-6\\-9\end{bmatrix}+c_3\begin{bmatrix}16\\-8\\-4\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

Change the equation into an augmented matrix, then put the matrix into reduced row-echelon form.

$$\begin{bmatrix}-8 & -3 & 16 & | & 0\\4 & -6 & -8 & | & 0\\2 & -9 & -4 & | & 0\end{bmatrix}\to\begin{bmatrix}1 & \frac{3}{8} & -2 & | & 0\\4 & -6 & -8 & | & 0\\2 & -9 & -4 & | & 0\end{bmatrix}\to\begin{bmatrix}1 & \frac{3}{8} & -2 & | & 0\\0 & -\frac{15}{2} & 0 & | & 0\\2 & -9 & -4 & | & 0\end{bmatrix}$$

$$\begin{bmatrix}1 & \frac{3}{8} & -2 & | & 0\\0 & -\frac{15}{2} & 0 & | & 0\\0 & -\frac{39}{4} & 0 & | & 0\end{bmatrix}\to\begin{bmatrix}1 & \frac{3}{8} & -2 & | & 0\\0 & 1 & 0 & | & 0\\0 & -\frac{39}{4} & 0 & | & 0\end{bmatrix}\to\begin{bmatrix}1 & 0 & -2 & | & 0\\0 & 1 & 0 & | & 0\\0 & -\frac{39}{4} & 0 & | & 0\end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -2 & | & 0 \\ 0 & 1 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

The rref form of the matrix gives the equations

$$c_1 - 2c_3 = 0 \quad \to \quad c_1 = 2c_3$$

$$c_2 = 0$$

These equations tell us that there are an endless number of solutions to the system. We can choose any value for c_3, and we'll get a different value for c_1, and all of those combinations will give us the zero vector. Because $(c_1, c_2, c_3) = (0,0,0)$ isn't the only solution, that tells us that the vectors in answer choice A are linearly dependent.

Which means answer choice B must be the correct choice, but let's verify that those vectors are, in fact, linearly independent. We'll test answer choice B by setting up the vector equation.

$$c_1 \begin{bmatrix} -8 \\ 4 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} -3 \\ -6 \\ -9 \end{bmatrix} + c_3 \begin{bmatrix} -6 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Change the equation into an augmented matrix, then put the matrix into reduced row-echelon form.

$$\begin{bmatrix} -8 & -3 & -6 & | & 0 \\ 4 & -6 & 1 & | & 0 \\ 2 & -9 & 3 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & \frac{3}{8} & \frac{3}{4} & | & 0 \\ 4 & -6 & 1 & | & 0 \\ 2 & -9 & 3 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & \frac{3}{8} & \frac{3}{4} & | & 0 \\ 0 & -\frac{15}{2} & -2 & | & 0 \\ 2 & -9 & 3 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & \frac{3}{8} & \frac{3}{4} & | & 0 \\ 0 & -\frac{15}{2} & -2 & | & 0 \\ 0 & -\frac{39}{4} & \frac{3}{2} & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & \frac{3}{8} & \frac{3}{4} & | & 0 \\ 0 & 1 & \frac{4}{15} & | & 0 \\ 0 & -\frac{39}{4} & \frac{3}{2} & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & \frac{13}{20} & | & 0 \\ 0 & 1 & \frac{4}{15} & | & 0 \\ 0 & -\frac{39}{4} & \frac{3}{2} & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & \frac{13}{20} & | & 0 \\ 0 & 1 & \frac{4}{15} & | & 0 \\ 0 & 0 & \frac{123}{30} & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & \frac{13}{20} & | & 0 \\ 0 & 1 & \frac{4}{15} & | & 0 \\ 0 & 0 & 1 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & | & 0 \\ 0 & 1 & \frac{4}{15} & | & 0 \\ 0 & 0 & 1 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & | & 0 \\ 0 & 1 & 0 & | & 0 \\ 0 & 0 & 1 & | & 0 \end{bmatrix}$$

The rref form of the matrix gives the equations

$$c_1 = 0$$

$$c_2 = 0$$

$$c_3 = 0$$

These equations tell us that $(c_1, c_2, c_3) = (0,0,0)$ is the only solution, which means the vectors in answer choice B are linearly independent.