2.3 Functions

Outline

Definition –
- Imprecise: An “assignment”
- Precise: Use Cartesian Product

Terminology:
- Domain
- Codomain
- Range

Types of Functions:
- Injective
- Surjective
- Bijective

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Informal definition of a function

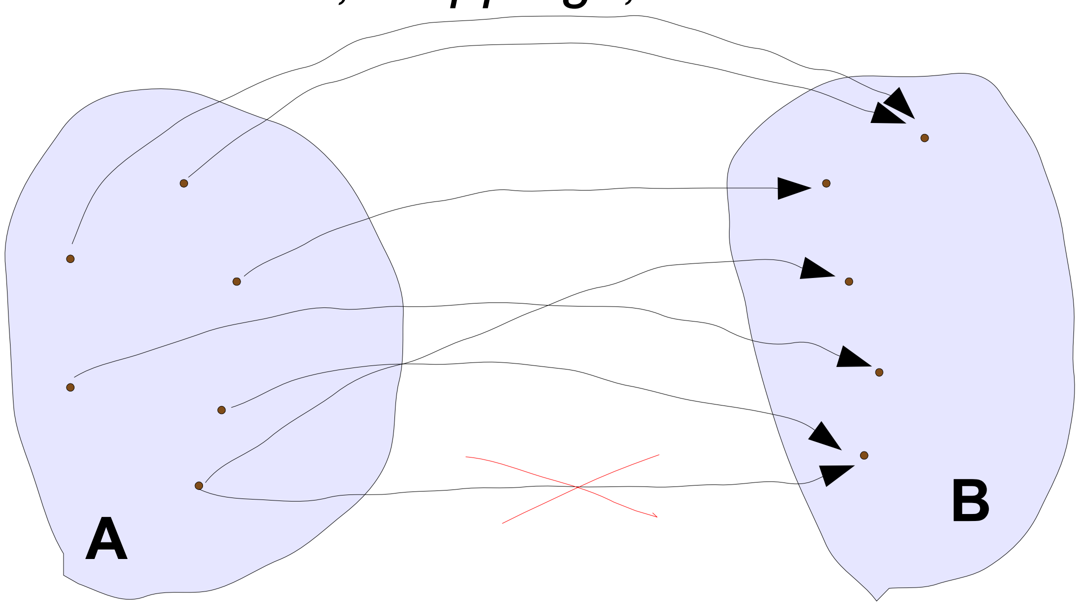
Let **A**, **B** be non-empty sets. A function *f* from **A** to **B** is an <u>assignment</u> of <u>exactly one</u> element of **B** to each element of **A**.

<u>denotations</u>:

$f(a) = b$ (if b is unique element from **B** assigned to a from **A** by f)

$f: A \to B$ f is a function from A to B

names: *functions, mappings, transformations*



f maps **A** to **B** (*f* is a function from **A** to **B**)

domain: set **A**

codomain: set **B**

if $f(a) = b$, then
b is the *image* of a,
a is a *preimage* of b

range of f is the set of all images of elements of A

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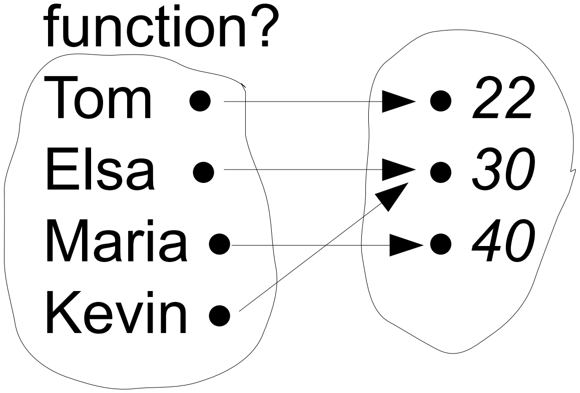
Precise definition of a function:

A relation from **A** to **B** is a subset of **A** $\times$ **B**. <u>denotation</u>: R

A relation from **A** to **B** that contains one and only one ordered pair *(a,b)* for every element $a \in$ **A**, defines a function *f* from **A** to **B**.

This function is defined by the assignment *f(a)* = *b*, where *(a,b)* is the unique ordered pair that has *a* as its first element.

Example:
R is the relation consisting of pairs: (Tom,22), (Elsa,30), (Maria,40), and (Kevin,30), where each pair consists of the person's name and the age of the person.
Can we define a function?
If we can, what is the domain, the codomain, and the range of this function?



Yes, this relation defines a function *f*, where *f*(Tom) = *22*, *f*(Elsa) = *30*, *f*(Maria) = *40*, and *f*(Kevin) = *30*.
domain: {Tom, Elsa, Maria, Kevin}
codomain: positive integers < *140*
range: {*22*,*30*,*40*}

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Let f be a function, f : **A** $\rightarrow$ **B**, and let **S** $\subseteq$ **A**. The *image of* **S** under the function f is the subset of **B** that consists of the images of the elements of **S, i.e.**

$$f(S) = \{ f(s) \mid s \in S \}$$

Example:

Let **A** = $\{a,b,c,d\}$, and **B** = $\{1,2,3\}$ with $f(a) = 1, f(b) = 3, f(c) = 2, f(d) = 3.$
Find the image of the set **S** = {b,c}.

<u>Solution</u>:
$f(b) = 3$, $f(c) = 2$, therefore $f(S) = \{2,3\}$

Let f be a function.

• If f never assigns the same value to two different domain elements, then it is called one-to-one or injective, i.e.
$f(a) \neq f(b)$ for all a and b, such that $a \neq b$; *or* $f(a) = f(b)$ implies that $a=b$

• *If for every element* $b \in$ ***B*** *there is an element* $a \in$ ***A****, such that* $f(a) = b$*, then function f is onto or surjective.*

$$\forall y \exists x\, (f(x)=y)$$

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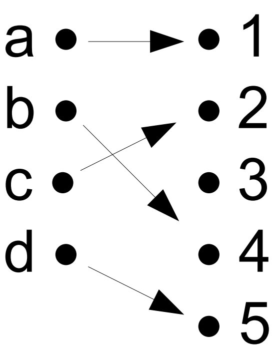
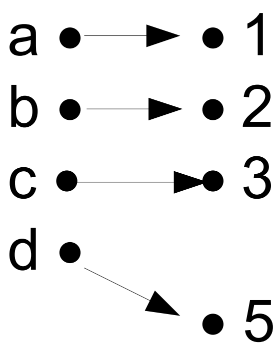
Let f be a function, f: $\mathbf{A} \rightarrow \mathbf{B}$



one-to-one,
but not onto

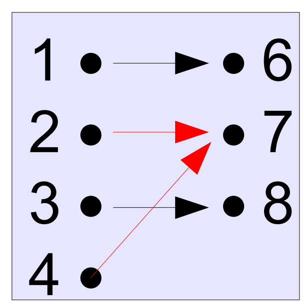


one-to-one, onto

Example:
determine whether the given functions are **one-to-one** and **onto**

a) $f(x) = x^3$; $\quad$ f: $\mathbf{Z} \rightarrow \mathbf{Z}$

b) f: $\{1,2,3,4\} \rightarrow \{6,7,8\}$,
with $f(1) = 6$, $f(2) = 7$, $f(3) = 8$, and $f(4) = 7$
not one-to-one, because of $f(2) = 7$ and $f(4) = 7$
onto



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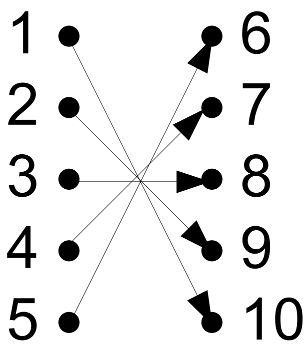
Function *f* is one-to-one correspondence or bijective if is both one-to-one and onto.

Example:
Let f:{$1,2,3,4,5$} → {$6,7,8,9,10$}, with $f(1) = 10$, $f(2) = 9$, $f(3) = 8$, $f(4) = 7$, $f(5) = 6$. Is f a bijection?

<u>Solution</u>:



We can see that for every element from the codomain {6,7,8,9,10} there is an element from the domain {1,2,3,4,5}, therefore it is onto.

We also can see that no two different elements from the domain have the same image, therefore it is one-to-one.

The given function is bijective.

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Suppose you are given what is *supposed to be* a function f: D → C. What kind of function do you have? A checklist:

1) Is it well-defined? (i.e. is it even a function?)
   a) Is it uniquely defined for every element of D?
   b) Are the outputs all in C?

2) Is it injective?
   a) For "yes": Prove it
   b) For "no": Find two different inputs which are counter-example

3) Is it onto?
   a) For "yes": Prove it
   b) For "no": Find an output value that is missed.

Practice Problems

From book, section 2.3:

1, 3, 5, 7, 15, 23