Matrix multiplication

We talked before about scalar multiplication, which is when we multiply a matrix by a real-number value. But **matrix multiplication** is what we do when we multiply two matrices together.

Based on what we've already learned about matrix addition and subtraction, you'd think that multiplying matrices is just a matter of multiplying corresponding entries, since matrix addition is just a matter of adding corresponding entries, and matrix subtraction is just a matter of subtracting corresponding entries.

But in fact, we follow an entirely different process to multiply matrices, and we'll walk through exactly what that is in this section.

Dimensions matter

First, when you multiply two matrices A and B together, the order matters. So $A \cdot B$ doesn't have the same result as $B \cdot A$.

This is different than real numbers. We know that $3(4)$ is the same as $4(3)$. That's because real numbers follow the commutative property of multiplication, which means that you can multiply them in any order, and you get the same result in both cases. For matrices, that's not the case; order matters.

The reason the order matters is because of the way we multiply the matrices, which really depends on the dimensions. Here's the thing to remember about dimensions:

The number of columns in the first matrix must be equal to the number of rows in the second matrix.

So for example, you can multiply a 3×2 matrix by any of these:

2×1

2×2

2×3

2×4

...

That's because, when we multiply one matrix by another, we multiply the rows in the first matrix by the columns in the second matrix. Let's say we want to multiply a 2×2 matrix called A by a 2×2 matrix called B.

$$A = \begin{bmatrix} 2 & 6 \\ 3 & -1 \end{bmatrix}$$

$$B = \begin{bmatrix} -4 & -2 \\ 1 & 0 \end{bmatrix}$$

If we call the first and second rows in A rows R_1 and R_2, and call the first and second columns in B columns C_1 and C_2,

$$A = \begin{array}{c} R_1 \rightarrow \\ R_2 \rightarrow \end{array} \begin{bmatrix} 2 & 6 \\ 3 & -1 \end{bmatrix}$$

$$B = \begin{array}{c} \begin{array}{cc} C_1 & C_2 \\ \downarrow & \downarrow \end{array} \\ \begin{bmatrix} -4 & -2 \\ 1 & 0 \end{bmatrix} \end{array}$$

then the product of A and B is

$$AB = \begin{bmatrix} R_1C_1 & R_1C_2 \\ R_2C_1 & R_2C_2 \end{bmatrix}$$

Let's look at each entry in the product:

$(AB)_{1,1}$ is the product of the first row and first column

$(AB)_{2,1}$ is the product of the second row and first column

$(AB)_{1,2}$ is the product of the first row and second column

$(AB)_{2,2}$ is the product of the second row and second column

The easy way to tell whether or not you can multiply matrices is to line up their dimensions. For instance, given matrix A is a 2×3 and matrix B is a 3×4, then line up the product AB this way:

$$AB: \ 2 \times \boxed{3 \quad 3} \times 4$$

If the middle numbers match like they do here (they're both 3), then you can multiply the matrices to get a valid result, because you have the same number of columns in the first matrix as rows in the second matrix. If you wanted to multiply B by A, you'd line up the product this way:

$$BA: \ 3\times\boxed{4 \quad 2}\times 3$$

Because those middle numbers aren't equal, you can't multiply B by A (even though A by B was a valid product; that's why order matters!). You don't have the same number of columns in the first matrix as rows in the second matrix, so the product isn't even defined.

Dimensions of the product

Now that you know how to determine whether or not the product of two matrices will be defined, let's talk about the dimensions of the product.

We said before that, because we have the same number of columns in the first matrix as rows in the second matrix, AB will be defined in this case:

$$AB: \ 2\times\boxed{3 \quad 3}\times 4$$

Once you know that the product AB is defined, you can also quickly know the dimensions of the resulting product. To get those dimensions, just take the number of rows from the first matrix by the number of columns from the second matrix.

$$AB: \ \boxed{2}\times 3 \quad 3\times\boxed{4}$$

So the dimensions of the product AB will be 2×4. In other words, a 2×3 matrix multiplied by a 3×4 matrix will always result in a 2×4 matrix.

Example

If matrix X is 2×2 and matrix Y is 4×2, say whether XY or YX is defined, and give the dimensions of the product if it is defined.

Line up the dimensions for the products XY and YX.

$$XY: \ 2 \times 2 \quad 4 \times 2$$

$$YX: \ 4 \times 2 \quad 2 \times 2$$

For XY, the middle numbers don't match, so that product isn't defined. For YX, the middle numbers match, so that product is defined.

The dimensions of YX are given by the outside numbers,

$$YX: \ \boxed{4} \times 2 \quad 2 \times \boxed{2}$$

so the dimensions of YX will be 4×2.

Using the dot product to multiply matrices

The **dot product** is the tool we'll use to multiply matrices. When you're calculating a dot product, you want to think about ordered pairs. For instance, we said before that when we take the product AB for

$$A = \begin{bmatrix} 2 & 6 \\ 3 & -1 \end{bmatrix}$$

$$B = \begin{bmatrix} -4 & -2 \\ 1 & 0 \end{bmatrix}$$

the first entry we'll need to find is the dot product of the first row in A and the first column in B. The first row in A is the ordered pair $(2,6)$, and the first column in B is the ordered pair $(-4,1)$.

To take the dot product of these ordered pairs, we take the product of the first values, and then add that result to the product of the second values. In other words, the dot product of $(2,6)$ and $(-4,1)$ is

$$2(-4) + 6(1)$$

$$-8 + 6$$

$$-2$$

So the product of two 2×2 matrices looks like this:

$$\begin{bmatrix} R_{1,1} & R_{1,2} \\ R_{2,1} & R_{2,2} \end{bmatrix} \begin{bmatrix} C_{1,1} & C_{1,2} \\ C_{2,1} & C_{2,2} \end{bmatrix} = \begin{bmatrix} R_{1,1}C_{1,1} + R_{1,2}C_{2,1} & R_{1,1}C_{1,2} + R_{1,2}C_{2,2} \\ R_{2,1}C_{1,1} + R_{2,2}C_{2,1} & R_{2,1}C_{1,2} + R_{2,2}C_{2,2} \end{bmatrix}$$

Therefore, to find the product of matrices A and B, we get

$$AB = \begin{bmatrix} 2 & 6 \\ 3 & -1 \end{bmatrix} \cdot \begin{bmatrix} -4 & -2 \\ 1 & 0 \end{bmatrix}$$

$$AB = \begin{bmatrix} 2(-4) + 6(1) & 2(-2) + 6(0) \\ 3(-4) + (-1)(1) & 3(-2) + (-1)(0) \end{bmatrix}$$

$$AB = \begin{bmatrix} -8+6 & -4+0 \\ -12+(-1) & -6+0 \end{bmatrix}$$

$$AB = \begin{bmatrix} -2 & -4 \\ -13 & -6 \end{bmatrix}$$

The little dot between the matrices in

$$AB = \begin{bmatrix} 2 & 6 \\ 3 & -1 \end{bmatrix} \cdot \begin{bmatrix} -4 & -2 \\ 1 & 0 \end{bmatrix}$$

indicates the dot product. So we could write the dot product of two points x_1 and x_2 as $x_1 \cdot x_2$, or the dot product of two matrices A and B as $A \cdot B$. The little dot tells you "the dot product of these elements."

Because matrix multiplication is always done with the dot product, we don't usually write the dot in between two matrices that we want to multiply, because it's assumed that we're taking the dot product. So you'll usually just see matrix multiplication indicated by two matrices directly next to each other, like this:

$$AB = \begin{bmatrix} 2 & 6 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} -4 & -2 \\ 1 & 0 \end{bmatrix}$$

Properties of matrix multiplication

When it comes to multiplication, matrices do not follow the same rules as real numbers.

We've already seen that multiplication **is not commutative**. The fact that it's not commutative means that you can't multiply matrices in a different order and still get the same answer.

$$AB \neq BA$$

Matrix multiplication **is associative**. The fact that it's associative means that you can shift around the parentheses and still get the same answer, as long as you don't change the order of the matrices:

$$(AB)C = A(BC)$$

Matrix multiplication **is distributive**. The fact that it's distributive means that you can distribute multiplication across another value.

$$A(B + C) = AB + AC$$

$$(B + C)A = BA + CA$$

$$A(B - C) = AB - AC$$

$$(B - C)A = BA - CA$$

When it comes to the zero matrix, it doesn't matter whether you multiply a matrix by the zero matrix, or multiply the zero matrix by a matrix; you'll get the O matrix either way. But the dimensions of the zero matrix may change, depending on whether it's the first or second matrix in the multiplication.

When $OA = O$, the zero matrix O must have the same number of columns as A has rows.

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Let's do an example with these properties of matrix multiplication.

Example

Use matrix multiplication to say whether or not the expression is defined.

$$\begin{bmatrix}5 & -1 & 0\\4 & -1 & 2\end{bmatrix}\left(\begin{bmatrix}-4 & -2\\1 & 0\\8 & -1\end{bmatrix}+\begin{bmatrix}1 & 2 & -1\\-6 & 0 & 4\\2 & -7 & 2\end{bmatrix}\right)$$

Let's first recognize that we have an expression with three matrices that looks like this:

$$A(B+C)$$

We can't add the matrices B and C inside the parentheses, because matrix addition is only defined when the matrix dimensions are the same. B is a 3×2 and C is a 3×3, so we can't do the addition.

We can, however, use the distributive property to distribute matrix A across B and C.

$$\begin{bmatrix}5 & -1 & 0\\4 & -1 & 2\end{bmatrix}\begin{bmatrix}-4 & -2\\1 & 0\\8 & -1\end{bmatrix}+\begin{bmatrix}5 & -1 & 0\\4 & -1 & 2\end{bmatrix}\begin{bmatrix}1 & 2 & -1\\-6 & 0 & 4\\2 & -7 & 2\end{bmatrix}$$

Now we have $AB + AC$. The dimensions of the product AB are

$$AB: \boxed{2}\times\boxed{3 \quad 3}\times\boxed{2}$$

The middle numbers match, so the product is defined, and AB will be 2×2.

$$\begin{bmatrix}5(-4)-1(1)+0(8) & 5(-2)-1(0)+0(-1)\\ 4(-4)-1(1)+2(8) & 4(-2)-1(0)+2(-1)\end{bmatrix}+\begin{bmatrix}5 & -1 & 0\\ 4 & -1 & 2\end{bmatrix}\begin{bmatrix}1 & 2 & -1\\ -6 & 0 & 4\\ 2 & -7 & 2\end{bmatrix}$$

$$\begin{bmatrix}-20-1+0 & -10-0+0\\ -16-1+16 & -8-0-2\end{bmatrix}+\begin{bmatrix}5 & -1 & 0\\ 4 & -1 & 2\end{bmatrix}\begin{bmatrix}1 & 2 & -1\\ -6 & 0 & 4\\ 2 & -7 & 2\end{bmatrix}$$

$$\begin{bmatrix}-21 & -10\\ -1 & -10\end{bmatrix}+\begin{bmatrix}5 & -1 & 0\\ 4 & -1 & 2\end{bmatrix}\begin{bmatrix}1 & 2 & -1\\ -6 & 0 & 4\\ 2 & -7 & 2\end{bmatrix}$$

The dimensions of the product AC are

$$AC: \boxed{2}\times\boxed{3 \quad 3}\times\boxed{3}$$

The middle numbers match, so the product is defined, and AC will be 2×3.

$$\begin{bmatrix}-21 & -10\\ -1 & -10\end{bmatrix}+\begin{bmatrix}5(1)-1(-6)+0(2) & 5(2)-1(0)+0(-7) & 5(-1)-1(4)+0(2)\\ 4(1)-1(-6)+2(2) & 4(2)-1(0)+2(-7) & 4(-1)-1(4)+2(2)\end{bmatrix}$$

$$\begin{bmatrix}-21 & -10\\ -1 & -10\end{bmatrix}+\begin{bmatrix}5+6+0 & 10-0+0 & -5-4+0\\ 4+6+4 & 8-0-14 & -4-4+4\end{bmatrix}$$

$$\begin{bmatrix}-21 & -10\\ -1 & -10\end{bmatrix}+\begin{bmatrix}11 & 10 & -9\\ 14 & -6 & -4\end{bmatrix}$$

Matrix addition is only defined when the matrices being added have the same dimensions. Here we're trying to add a 2×2 to a 2×3. The dimensions aren't the same, so the sum isn't defined, which means the original expression is not defined, either.