Linear combinations and span

At the end of the last lesson, we took 6 of the basis vector $\hat{i}=(1,0)$ and 4 of the basis vector $\hat{j}=(0,1)$ to express the vector $\overrightarrow{a}=(6,4)$ as

$$\overrightarrow{a}=6\begin{bmatrix}1\\0\end{bmatrix}+4\begin{bmatrix}0\\1\end{bmatrix}$$

Notice how the vector $\hat{i}=(1,0)$ is multiplied by a scalar of 6, and the vector $\hat{j}=(0,1)$ is multiplied by a scalar of 4. In other words, to express $\overrightarrow{a}$, we've only done two operations: 1) we've multiplied vectors by scalars, and 2) we've added these scaled vectors together.

Any expression like this one, which is just the sum of scaled vectors, is called a **linear combination**. Linear combinations can sum any number of vectors, not just two. So $3\overrightarrow{a}-2\overrightarrow{b}$ is a linear combination, $-\overrightarrow{a}+0.5\overrightarrow{b}$ is a linear combination, $4.2\overrightarrow{a}-7\overrightarrow{b}+\pi\overrightarrow{c}$ is a linear combination, and so on.

Span of a vector set

The **span** of a set of vectors is the collection of all vectors which can be represented by some linear combination of the set.

That sounds confusing, but let's think back to the basis vectors $\hat{i}=(1,0)$ and $\hat{j}=(0,1)$ in $\mathbb{R}^2$. If you choose absolutely any vector, anywhere in $\mathbb{R}^2$, you can get to that vector using a linear combination of $\hat{i}$ and $\hat{j}$. If I choose $(13,2)$, I can get to it with the linear combination $\overrightarrow{a}=13\hat{i}+2\hat{j}$, or if I choose $(-1,-7)$, I can get to it with the linear combination $\overrightarrow{a}=-\hat{i}-7\hat{j}$. There's no vector you can find in $\mathbb{R}^2$ that you can't reach with a linear combination of $\hat{i}$ and $\hat{j}$.

And because you can get to any vector in $\mathbb{R}^2$ with a linear combination of $\hat{i}$ and $\hat{j}$, you can say specifically that $\hat{i}$ and $\hat{j}$ **span** $\mathbb{R}^2$. If a set of vectors spans a space, it means you can use a linear combination of those vectors to reach any vector in the space.

In the same way, I can get to any vector, anywhere in $\mathbb{R}^3$, using a linear combination of the basis vectors $\hat{i}$, $\hat{j}$, and $\hat{k}$, which means $\hat{i}$, $\hat{j}$, and $\hat{k}$ **span** $\mathbb{R}^3$, the entirety of three-dimensional space.

I could also write these facts as

$$\text{Span}(\hat{i}, \hat{j}) = \mathbb{R}^2$$

$$\text{Span}(\hat{i}, \hat{j}, \hat{k}) = \mathbb{R}^3$$

One other point: the span of the zero vector $\vec{O}$ is always just the origin, so $(0,0)$ in $\mathbb{R}^2$, $(0,0,0)$ in $\mathbb{R}^3$, etc.

Span, and linear independence

So our next step is to be able to determine when a vector set spans a space, and when it doesn't. In other words, how can we tell when every point in the space is, or is not, reachable by a linear combination of the vector set?

The answer has to do with whether or not the vectors in the set are linearly independent or linearly dependent. We'll talk about linear (in)dependence in the next lesson, but for now, let's just make three points:

- Any 2 two-dimensional linearly independent vectors will span $\mathbb{R}^2$. The two-dimensional basis vectors $\hat{i}$ and $\hat{j}$ are linearly independent, which is why they span $\mathbb{R}^2$.

- Any 3 three-dimensional linearly independent vectors will span $\mathbb{R}^3$. The three-dimensional basis vectors $\hat{i}$, $\hat{j}$, and $\hat{k}$ are linearly independent, which is why they span $\mathbb{R}^3$.

- Any n n-dimensional linearly independent vectors will span $\mathbb{R}^n$. The n-dimensional basis vectors are linearly independent, which is why they span $\mathbb{R}^n$.

So when is a set of vectors linearly dependent, such that they won't span the vector space $\mathbb{R}^2$, $\mathbb{R}^3$, or $\mathbb{R}^n$?

First, we should say that we can never span $\mathbb{R}^n$ with fewer than n vectors. In other words, we can't span $\mathbb{R}^2$ with one or fewer vectors, we can't span $\mathbb{R}^3$ with two or fewer vectors, and we can't span $\mathbb{R}^n$ with $n-1$ or fewer vectors.

Second, assuming we have enough vectors to span the space, generally speaking, those vectors need to be "different enough" from each other that they can cover the whole vector space. It's actually easier to think about when the vectors *won't* be "different enough" to span the vector space:

- When 2 two-dimensional vectors lie along the same line (or along parallel lines), they're called **collinear**, they're linearly dependent, and they won't span $\mathbb{R}^2$.

- When 3 three-dimensional vectors lie in the same plane, they're called **coplanar**, they're linearly dependent, and they won't span $\mathbb{R}^3$.

- When n n-dimensional vectors lie in the same $(n-1)$-dimensional space, they're linearly dependent, and they won't span $\mathbb{R}^n$.

Let's hold off until the next section on more detail about linear dependence and independence, and turn to an example.

Example

Prove that you can use a linear combination of the basis vectors $\mathbf{i} = (1,0)$ and $\mathbf{j} = (0,1)$ to get any vector $\vec{k} = (k_1, k_2)$ in $\mathbb{R}^2$.

We can set up a vector equation, then write the basis vectors as column vectors.

$$c_1\mathbf{i} + c_2\mathbf{j} = \vec{k}$$

$$c_1\begin{bmatrix}1\\0\end{bmatrix} + c_2\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}k_1\\k_2\end{bmatrix}$$

This matrix equation can be rewritten as a system of equations:

$$1c_1 + 0c_2 = k_1$$

$$0c_1 + 1c_2 = k_2$$

Simplifying the system leaves us with

$$c_1 = k_1$$

$$c_2 = k_2$$

So what have we shown? We realize that this system means we could pick any vector $\overrightarrow{k} = (k_1, k_2)$ in $\mathbb{R}^2$, and we'd get $k_1 = c_1$ and $k_2 = c_2$, which means our linear combination will simply be a k_1 number of $\mathbf{i}$'s, and a k_2 number of $\mathbf{j}$'s.

So if, for example, the vector we chose in $\mathbb{R}^2$ was $\overrightarrow{k} = (7,4)$, then the linear combination of the basis vectors is

$$k_1\mathbf{i} + k_2\mathbf{j} = \overrightarrow{k}$$

$$7\mathbf{i} + 4\mathbf{j} = \overrightarrow{k}$$

Which means we would need to use 7 of the $\mathbf{i}$ vectors and 4 of the $\mathbf{j}$ vectors in order to reach $\overrightarrow{k} = (7,4)$ from the origin.