Gauss-Jordan elimination

Finally, we're at a point where we can start to solve a system using a matrix. To solve a system, our goal will be to use the simple row operations we learned earlier to transform the matrix into row-echelon form, or better yet, reduced row-echelon form.

Gaussian elimination

So we know that it's helpful to put a matrix into reduced row-echelon form, and we've said that we can use matrix row operations to do this, but is there any systematic, orderly way that we go about these row operations?

Yes! **Gauss-Jordan elimination** (or **Gaussian elimination**) is an algorithm (a specific set of steps that can be repeated over and over again) to get the matrix all the way down to reduced row-echelon form. These are the steps:

1. Optional: Pull out any scalars from each row in the matrix.

2. If the first entry in the first row is 0, swap it with another row that has a non-zero entry in its first column. Otherwise, move to step 3.

3. Multiply through the first row by a scalar to make the leading entry equal to 1.

4. Add scaled multiples of the first row to every other row in the matrix until every entry in the first column, other than the leading 1 in the first row, is a 0.

5. Go back step 2 and repeat the process until the matrix is in reduced row-echelon form.

Let's walk through an example of how to use Gauss-Jordan elimination to change an augmented matrix into reduced row-echelon form and then pull out the values of each variable to get the solution to the system.

Example

Use Gauss-Jordan elimination to solve the system.

$$\begin{bmatrix} -1 & -5 & 1 & | & 17 \\ -5 & -5 & 5 & | & 5 \\ 2 & 5 & -3 & | & -10 \end{bmatrix}$$

Remember first that this augmented matrix represents the linear system

$$-x - 5y + z = 17$$

$$-5x - 5y + 5z = 5$$

$$2x + 5y - 3z = -10$$

where the entries in the first column are the coefficients on x, the entries in the second column are the coefficients on y, and the entries in the third column are the coefficients on z. The entries in the fourth column are the constants.

Step 1:

Starting with the optional first step from Gauss-Jordan elimination, we could divide through the second row by 5, and that would reduce those values. After $(1/5)R_2 \to R_2$, the matrix is

$$\begin{bmatrix} -1 & -5 & 1 & | & 17 \\ -1 & -1 & 1 & | & 1 \\ 2 & 5 & -3 & | & -10 \end{bmatrix}$$

Step 2 (with the first row):

The first entry in the first row is non-zero, so there's no need to swap it with another row.

Step 3 (with the first row):

Multiply row 1 by -1 to get a leading 1 in the first row. After $-R_1 \to R_1$, the matrix is

$$\begin{bmatrix} -(-1) & -(-5) & -(1) & | & -(17) \\ -1 & -1 & 1 & | & 1 \\ 2 & 5 & -3 & | & -10 \end{bmatrix} \to \begin{bmatrix} 1 & 5 & -1 & | & -17 \\ -1 & -1 & 1 & | & 1 \\ 2 & 5 & -3 & | & -10 \end{bmatrix}$$

Step 4 (with the first row):

Replace row 2 with the sum of rows 1 and 2. After $R_1 + R_2 \to R_2$, the matrix is

$$\begin{bmatrix} 1 & 5 & -1 & | & -17 \\ 1-1 & 5-1 & -1+1 & | & -17+1 \\ 2 & 5 & -3 & | & -10 \end{bmatrix} \to \begin{bmatrix} 1 & 5 & -1 & | & -17 \\ 0 & 4 & 0 & | & -16 \\ 2 & 5 & -3 & | & -10 \end{bmatrix}$$

Replace row 3 with row 3 minus (2 times row 1). After $R_3 - 2R_1 \to R_3$, the matrix is

$$\begin{bmatrix} 1 & 5 & -1 & | & -17 \\ 0 & 4 & 0 & | & -16 \\ 2-2(1) & 5-2(5) & -3-2(-1) & | & -10-2(-17) \end{bmatrix} \to \begin{bmatrix} 1 & 5 & -1 & | & -17 \\ 0 & 4 & 0 & | & -16 \\ 0 & -5 & -1 & | & 24 \end{bmatrix}$$

We now have 1, 0, 0 in the first column, which is exactly what we want. It's time to go back to step 2, but this time with the second row.

Step 2 (with the second row):

The second entry in the second row is non-zero, so there's no need to swap it with another row.

Step 3 (with the second row):

Multiply row 2 by $1/4$ to get a leading 1 in the second row. After $(1/4)R_2 \to R_2$, the matrix is

$$\begin{bmatrix} 1 & 5 & -1 & | & -17 \\ \frac{1}{4}(0) & \frac{1}{4}(4) & \frac{1}{4}(0) & | & \frac{1}{4}(-16) \\ 0 & -5 & -1 & | & 24 \end{bmatrix} \to \begin{bmatrix} 1 & 5 & -1 & | & -17 \\ 0 & 1 & 0 & | & -4 \\ 0 & -5 & -1 & | & 24 \end{bmatrix}$$

Step 4 (with the second row):

Replace row 1 with the sum of (-5 times row 2) and row 1. After $-5R_2+R_1 \to R_1$, the matrix is

$$\begin{bmatrix} -5(0)+1 & -5(1)+5 & -5(0)-1 & | & -5(-4)-17 \\ 0 & 1 & 0 & | & -4 \\ 0 & -5 & -1 & | & 24 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 & | & 3 \\ 0 & 1 & 0 & | & -4 \\ 0 & -5 & -1 & | & 24 \end{bmatrix}$$

Replace row 3 with the sum of (5 times row 2) and row 3. After $5R_2+R_3 \to R_3$, the matrix is

$$\begin{bmatrix} 1 & 0 & -1 & | & 3 \\ 0 & 1 & 0 & | & -4 \\ 5(0)+0 & 5(1)-5 & 5(0)-1 & | & 5(-4)+24 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 & | & 3 \\ 0 & 1 & 0 & | & -4 \\ 0 & 0 & -1 & | & 4 \end{bmatrix}$$

We now have 0, 1, 0 in the second column, which is exactly what we want. It's time to go back to step 2, but this time with the third row.

Step 2 (with the third row):

The third entry in the third row is non-zero, so there's no need to swap it with another row.

Step 3 (with the third row):

Multiply row 3 by -1 to get a leading 1 in the third row. After $-R_3 \to R_3$, the matrix is

$$\begin{bmatrix} 1 & 0 & -1 & | & 3 \\ 0 & 1 & 0 & | & -4 \\ -(0) & -(0) & -(-1) & | & -(4) \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 & | & 3 \\ 0 & 1 & 0 & | & -4 \\ 0 & 0 & 1 & | & -4 \end{bmatrix}$$

Step 4 (with the third row):

Replace row 1 with the sums of rows 1 and 3. After $R_1 + R_3 \to R_1$, the matrix is

$$\begin{bmatrix} 1+0 & 0+0 & -1+1 & | & 3-4 \\ 0 & 1 & 0 & | & -4 \\ 0 & 0 & 1 & | & -4 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & | & -1 \\ 0 & 1 & 0 & | & -4 \\ 0 & 0 & 1 & | & -4 \end{bmatrix}$$

We now have 0, 0, 1 in the third column, which is exactly what we want. The matrix is now in reduced row-echelon form since the leading 1 is the first non-zero value in each row, and the leading 1 in each row is to the right of the leading 1 from all the rows above it, and all other values are 0.

From this resulting matrix, we get the solution set

$$1x + 0y + 0z = -1$$

$$0x + 1y + 0z = -4$$

$$0x + 0y + 1z = -4$$

or simplified, we get

$$x = -1$$

$$y = -4$$

$$z = -4$$

That's all it took to find that the solution to the system is $(x, y, z) = (-1, -4, -4)$.