1.4/1.5 Predicates and Quantifiers

Can we express the following statement in **propositional logic**?

"Every computer has a CPU"

No

Outline:

- Predicates
- Quantifiers
- Domain of Predicate
- Translation:
  - Natural Languauge → Predicate Logic
  - Predicate Logic → Natural Language
- Negation of Quantifiers
- Multiple Quantifiers (repeat translation and negation)

1.4 Predicates and Quantifiers

Can we express the following statement in **propositional logic**? No

"Every computer has a CPU"

Let's see the new type of logic: Predicate Logic (or First Order Logic)

consider the statement: "$\underbrace{}_{}$*x* $\underbrace{\text{is greater than 3}}_{}$" - it has two parts:

variable x *predicate*

(subject of a statement) (refers to a property the subject of a statement can have)

Denotation: *P(x)*: "*x* is greater than 3"

- this kind of statement is neither true nor false when the value of variable is not specified.

Once *x* is assigned a value, *P(x)* becomes a proposition that has a truth value.

1.4 Predicates and Quantifiers

Example 1:
Let $P(x)$ denote "$x<10$". What are the truth values of $P(11)$ and $P(6)$?

$P(11)$: "$11<10$" False
$P(6)$: "$6<10$" True

Example 2:
Let $P(y,z)$ denote statement "$y = z-17$". What are the truth values of $P(10,11)$ and $P(10,27)$?

$P(10,11)$: "$10 = 11-17$" if we simplify the equation: "$10 = -6$" False
$P(10,27)$: "$10 = 27-17$" if we simplify the equation: "$10 = 10$" True

1.4 Predicates and Quantifiers

How to express words “all”, “any”, “some”, ... ?
- quantification

“all”, “any”: $\forall$ - universal quantification
“some”, “an”: $\exists$ - existential quantification

$\forall xP(x)$ is “for all values of x from the domain, P(x) is true”
“for any element x in the domain, P(x) is true”

$\exists xP(x)$ is “there exists value of x, such that P(x) is true”

Recall the question I asked in the beginning:

Can we express the following statement in **propositional logic**?

“Every computer has a CPU”

1.4 Predicates and Quantifiers

$\forall xP(x)$ is true when $P(x)$ is true for every x (from the domain)

$\forall xP(x)$ is false when there is an x, for which $P(x)$ is false

Example 3:
Let $Q(x)$ be "$2 \cdot x \geq x$". Is $\forall xQ(x)$ true? (domain: all real numbers)

Example 4: Let $Q(x)$ be "$2 + x \geq x$". Is $\forall xQ(x)$ true? (domain: all real numbers)

Example 5: What's the truth value of $\forall xP(x)$, where P(x) is the statement "$x^2 \leq 16$" and the domain consists of the positive integers not exceeding 4?

What's the domain?

What do we need to check?

What is the **domain** of a predicate?

Example:
For each of these statements find a domain for which the statement is true and a domain for which the statement is false.

a) Everyone is studying discrete math

b) Every two people have the same father

1.4 Predicates and Quantifiers

$\exists x P(x)$ is true, if $P(x)$ is true for at least one x

$\exists x P(x)$ is false, if $P(x)$ is false for all x from the domain

Example 6:

Let $P(x)$ denote "$x=x-3$". Domain: all real numbers. Is $\exists x P(x)$ true?

Example 7:

Let $P(x)$ stand for "$x > 10$". Domain: all real numbers. Is $\exists x P(x)$ true?

Example 8:

Let $P(x)$ be "$x^2 \leq 16$". Domain consists of positive integers between 4 and 7, including. Is $\exists x P(x)$ true?

Reasoning:

What is the domain?

What do we need to check?

1.4 Predicates and Quantifiers

Quantifiers with restricted domain

Let's assume that our domain is all real numbers.

Examples:

$\forall x<0\ P(x)$ “for all x less than *0*, *P(x)* holds”

$\forall x<0\ (x^2>0)$ “for all x less than *0*, x^2 is greater than *0*”

$\exists z>0\ (z^2=2)$ “there exists *z* grater than *0*, such that z^2 is equal to *2*”

Translating from English into Logical Expressions

Example:
Let $P(x)$ be the statement “x took a discrete math course”,
Let $Q(x)$ be the statement “x knows the computer language Python”. Express each of these sentences in terms of $P(x)$, $Q(x)$, quantifiers and logical connectives. Let the domain for quantifiers consist of all students from Mathematics, CS, and Engineering majors.

a) There is a student who took a discrete math course.

$\exists x P(x)$

b) There is a student who took a discrete math course, but doesn't know Python.

$\exists x\,(P(x) \wedge \neg Q(x))$

c) Every student either took a discrete math course or knows Python.

$\forall x\,(P(x) \vee Q(x))$

d) There is no student that took discrete math and knows Python.

$\neg\exists x(P(x) \wedge Q(x))$

Logical Equivalences Involving Quantifiers

Statements involving predicates and quantifiers are *logically equivalent* if and only if (*iff*) they have the same truth values no matter which predicates are substituted into these statements and which domain is used for the variables in these propositional functions.

$S \equiv T$ (same notation as before, with propositions only)

Negating Quantified Expressions - De Morgan's Laws

$\neg\exists x P(x) \equiv \forall x \neg P(x)$

Reading: Let P(x): "x has taken Calculus", and domain: all students from our class,then

"It is not the case that there exists a student that has taken Calculus"

is logically equivalent to saying

"All the students have not taken Calculus"

$\neg\forall x P(x) \equiv \exists x \neg P(x)$

Reading: Let P(x) be the same, then

"It is not the case that all students have taken Calculus"

is similar to saying

"There is a student that has not taken Calculus"

Multiple Quantifiers

Example:

a) $\forall x \, \exists y \, (x+y = 0)$ - *additive inverse*

Here is a reading: "For any x, there exists y such that x+y is 0"

By the way, what is the domain?

b) $\forall x \, \forall y \, (x+y = y+x)$ - *commutativity of addition*

Reading: "For any x and y, x+y is equal to y+x"

c) $\forall x \, \forall y \, ((x>0 \wedge y<0) \rightarrow xy<0)$

Reading: "For any x and y, if x is greater than 0 and y is less than 0, then their product is less than 0"

Example: Let $Q(x,y)$ denote "$x-y=0$". What are the truth values of the quantifications $\exists y\ \forall x\ Q(x,y)$ and $\forall x\ \exists y\ Q(x,y)$.

1) $\exists y\ \forall x\ Q(x,y)$ true?

2) $\forall x\ \exists y\ Q(x,y)$ true?

on page 53 (book), there is a nice table which shows quantifications of two variables:

Statement	**When True?**	**When False?**
$\forall y\ \forall x\ P(x,y)$ $\forall x\ \forall y\ P(x,y)$	P(x,y) is true for any pair x, y	there is a pair x, y for which P(x,y) is False
$\forall x\ \exists y\ P(x,y)$	For any x, there is a y, for which P(x,y) s true	There is an x, for which P(x,y) is false for any y
$\exists x\ \forall y\ P(x,y)$	There exists an x, for which P(x,y) is true for any y	There is no x, such that P(x,y) is true for any y
$\exists x\ \exists y\ P(x,y)$ $\exists y\ \exists x\ P(x,y)$	There is a pair x, y for which P(x,y) is true	P(x,y) is false for every pair x, y

Translatation: Natural Language → Predicate Logic

Example (page 67/20):
Express each of these statements using predicates, quantifiers, logical connectives, and mathematical operators where domain consists of all integers.

a) The product of two negative integers is positive.

$\forall x{<}0\ \forall y{<}0\ (x*y>0)$ or $\forall x \forall y\ (\ (x<0) \wedge (y<0) \rightarrow x*y>0\ )$

c) The difference of two negative integers is not necessarily negative.

$\exists x{<}0\ \exists y{<}0\ ((x-y) \geq 0\ )$ or $\exists x \exists y\ (\ (x<0) \wedge (y<0) \wedge (x-y) \geq 0\ )$

d) The absolute value of the sum of two integers does not exceed the sum of the absolute values of these integers.

$\forall x \forall y\ (\ |x+y| \leq |x|+|y|\ )$

Translatation: Predicate Logic → Natural Language

Example (page 65/7):
Let T(x,y) mean "student x likes cuisine y". Where the domain for x consists of all students at your school and the domain for y consists of all cuisines. Express each of these statements by a simple English sentence.

d) $\forall x \forall z \exists y \; ((x \neq z) \rightarrow \neg (T(x,y) \wedge T(z,y))$
Let's re-write it a little bit: $\forall x \forall z \exists y \; ((x \neq z) \rightarrow (\neg T(x,y) \vee \neg T(z,y))$
"For any two different students there exists a cuisine such that at least one of them dislikes it."

e) $\exists x \exists z \forall y \; (T(x,y) \leftrightarrow T(z,y))$ "There is a pair of students with the same tastes in cuisines: they like/dislike the same cuisines."

Negating Multiple Quantifiers

Example:
Negate the given statements, then re-write them so that negations appear only within predicates (i.e. no negation is outside a quantifier or an expression involving logical connectives)

a) $\forall x \exists y\ (x * y = 3)$

$\neg\ \forall x \exists y\ (x * y = 3) \equiv \exists x \forall y\ \neg(x * y = 3) = \exists x \forall y\ (x * y \neq 3)$

b) $\exists x \forall a \exists y\ (F(x,y) \wedge A(y,a))$

$\neg\ \exists x \forall a \exists y\ (F(x,y) \wedge A(y,a)) \equiv \forall x \exists a \forall y\ \neg(F(x,y) \wedge A(y,a))$

$\equiv\ \forall x \exists a \forall y\ (\neg F(x,y) \vee \neg A(y,a))$