Linear independence in two dimensions

A set of vectors are **linearly dependent** when one vector in the set can be represented by a linear combination of the other vectors in the set. Put another way, if one or more of the vectors in the set doesn't add any new information or directionality, then the set is linearly dependent.

As we learned in the last section, only 2 two-dimensional linearly independent vectors are needed to span $\mathbb{R}^2$. Which means, given at set of 3 two-dimensional vectors, the set will always be linearly dependent, since at least one of the vectors could be made from some linear combination of the other two.

Linear dependence with two vectors

As an example, look at the vectors v_1 and v_2.

$$v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ and } v_2 = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$$

These vectors are linearly dependent, because we can multiply either one by a constant to get the other.

Multiplying v_1 by 4 gives v_2.

Multiplying v_2 by $1/4$ gives v_1.

When the only difference between two vectors is a scalar, then they lie on the same line, they're **collinear**, and we say that they're linearly dependent.

It's also helpful to think about linear dependence as the existence of one or more redundant vectors. In the example with v_1 and v_2, both vectors lie on the same line, which is the line $y = x$. But v_1 can define the entire line, because we can get any point on the line simply by multiplying v_1 by a scalar. But v_2 also defines the entire line for the same reason.

So having v_2 in addition to v_1 really doesn't help us; it doesn't give us any new information outside the line $y = x$. Similarly, having v_1 in addition to v_2 doesn't give us anything new that we didn't already have with v_2. So the two vectors are linearly dependent, because one of them is always redundant. They define the same thing together that they can each already define individually.

On the other hand, the vectors

$$w_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ and } w_2 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

are linearly independent. There's no scalar you can multiply by w_1 that'll give you w_2, and there's no scalar you can multiply by w_2 that'll give you w_1. Which means these vectors aren't collinear, and they therefore span $\mathbb{R}^2$ as a pair of linearly independent vectors.

Linear dependence with three vectors in $\mathbb{R}^2$

As we mentioned earlier, a set of three vectors (in two dimensions) will always be linearly dependent, because even if two of the vectors are linearly independent and span $\mathbb{R}^2$, the third vector will be redundant.

First, let's think about two vectors that aren't linearly dependent (they're linearly independent). Let's just take the basis vectors $\hat{i}$ and $\hat{j}$, but we'll call them v_1 and v_2 for now.

$$v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ and } v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

We can say that v_1 and v_2 are linearly independent. There's no combination of v_1 vectors that will give us v_2, and similarly no combination of v_2 vectors that will give us v_1, so they're linearly independent.

Spanning $\mathbb{R}^2$

In case you missed it, here's the amazing thing: Given *any* two linearly independent vectors, we can use them to define the entire real plane! It doesn't matter which two vectors we use, as long as they are linearly independent of one another. If they are, then we'll be able to define any point in the plane as a combination of the two linearly independent vectors.

Therefore, given two linearly independent vectors like v_1 and v_2, there's no other vector we can name in the same real plane that won't be a redundant addition to the set. Because every vector in the plane can already be defined by a linear combination of v_1 and v_2, so there's no new information we can get from a third vector in the same plane. For instance, because we know that v_1 and v_2 are linearly independent, the vector set

$$v_1 = \begin{bmatrix}1\\0\end{bmatrix} \text{ and } v_2 = \begin{bmatrix}0\\1\end{bmatrix} \text{ and } v_3 = \begin{bmatrix}3\\2\end{bmatrix}$$

must be linearly dependent. We can create v_3 as a linear combination of v_1 and v_2:

$$v_3 = 3\begin{bmatrix}1\\0\end{bmatrix} + 2\begin{bmatrix}0\\1\end{bmatrix}$$

$$v_3 = \begin{bmatrix}3(1)\\3(0)\end{bmatrix} + \begin{bmatrix}2(0)\\2(1)\end{bmatrix}$$

$$v_3 = \begin{bmatrix}3\\0\end{bmatrix} + \begin{bmatrix}0\\2\end{bmatrix}$$

$$v_3 = \begin{bmatrix}3+0\\0+2\end{bmatrix}$$

$$v_3 = \begin{bmatrix}3\\2\end{bmatrix}$$

So v_3 hasn't given us any new information. It's redundant when we already have v_1 and v_2, so the set of vectors $\{v_1, v_2, v_3\}$ is linearly dependent.

Testing for linear independence

Luckily, there's a reliable way to determine whether a set of vectors is linearly dependent or independent. We set up a system of equations that includes the vectors in the set, and one constant term for each vector. For example, given the vectors

$$v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ and } v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

we'd set up the equation

$$c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Notice how we took the sum of the vectors in the set, put a constant in front of each one, and set the sum equal to the zero vector. To determine linear (in)dependence, we'll always set up the equation this way. Then we can break the equation into a system of equations.

$$c_1(1) + c_2(0) = 0$$

$$c_1(0) + c_2(1) = 0$$

Then solve the system. In this case, we find $c_1 = 0$ and $c_2 = 0$. Whether the system is linearly dependent or independent is determined by the values of c_1 and c_2. If the only values that can make the system true are $c_1 = 0$ and $c_2 = 0$, then the vectors are linearly independent. But if either c_1 is nonzero and/or c_2 is nonzero, then the vectors are linearly dependent.

Example

Say whether the vectors are linearly dependent or linearly independent.

$$\begin{bmatrix} 1 \\ 4 \end{bmatrix} \text{ and } \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

Set up the equation first.

$$c_1 \begin{bmatrix} 1 \\ 4 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Then break the equation into a system of two linear equations.

$$c_1 + 3c_2 = 0$$

$$4c_1 + 2c_2 = 0$$

Multiply $c_1 + 3c_2 = 0$ by 4 to get

$$4(c_1 + 3c_2) = 4(0)$$

$$4c_1 + 12c_2 = 0$$

and then subtract this equation from $4c_1 + 2c_2 = 0$ to cancel c_1.

$$4c_1 + 2c_2 - (4c_1 + 12c_2) = 0 - 0$$

$$4c_1 + 2c_2 - 4c_1 - 12c_2 = 0$$

$$2c_2 - 12c_2 = 0$$

$$-10c_2 = 0$$

$$c_2 = 0$$

Substitute $c_2 = 0$ back into $c_1 + 3c_2 = 0$ to find the value of c_1.

$$c_1 + 3(0) = 0$$

$$c_1 + 0 = 0$$

$$c_1 = 0$$

The only solution set that makes the system true is $c_1 = 0$ and $c_2 = 0$. Because both values are 0, the given vectors are linearly independent.

Since this is Linear Algebra, we should of course mention that the system of linear equations from the last example,

$$c_1 + 3c_2 = 0$$

$$4c_1 + 2c_2 = 0$$

can always be solved using a matrix instead of using elimination and substitution. Just set up the matrix,

$$\begin{bmatrix} 1 & 3 & | & 0 \\ 4 & 2 & | & 0 \end{bmatrix}$$

then use Gaussian elimination to put the matrix into echelon form.

$$\begin{bmatrix} 1 & 3 & | & 0 \\ 0 & -10 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 & | & 0 \\ 0 & 1 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & | & 0 \\ 0 & 1 & | & 0 \end{bmatrix}$$

Because we get the identity matrix on the left, and all zero entries on the right, it means we've found $c_1 = 0$ and $c_2 = 0$, and can therefore conclude that the vectors

$$\begin{bmatrix} 1 \\ 4 \end{bmatrix} \text{ and } \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

are linearly independent.

Let's do another example where we solve the system using a matrix that we put into row-echelon form.

Example

Say whether the vectors are linearly dependent or linearly independent.

$$v_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 6 \\ -3 \end{bmatrix}$$

Set up an equation in which the sum of the linear combination is set equal to the zero vector.

$$c_1 \begin{bmatrix} -2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 6 \\ -3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Let's use a matrix to solve the system. As we saw in the previous lesson, we really only need the column vectors on the left to go into the matrix.

$$\begin{bmatrix} -2 & 6 \\ 1 & -3 \end{bmatrix}$$

Switch the rows, so that the first row starts with a pivot of 1.

$$\begin{bmatrix} 1 & -3 \\ -2 & 6 \end{bmatrix}$$

Zero out the first column below the pivot entry.

$$\begin{bmatrix} 1 & -3 \\ 0 & 0 \end{bmatrix}$$

This tells us that $(c_1, c_2) = (1, -3)$, which means we can get the zero vector using a combination of one of v_1 and -3 of v_2. So $(c_1, c_2) = (0,0)$ is not the only combination that gives the zero vector, which means that the vector set $\{v_1, v_2\}$ is linearly dependent. Which means that v_1 and v_2 can't span $\mathbb{R}^2$.