Linear Algebra Workbook Solutions

Operations on two matrcies

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MATRIX ADDITION AND SUBTRACTION

1. Add the matrices.

$$\begin{bmatrix} 7 & 6 \\ 17 & 9 \end{bmatrix} + \begin{bmatrix} 0 & 8 \\ -2 & 5 \end{bmatrix}$$

Solution:

To add matrices, you simply add together entries from corresponding positions in each matrix.

$$\begin{bmatrix} 7 & 6 \\ 17 & 9 \end{bmatrix} + \begin{bmatrix} 0 & 8 \\ -2 & 5 \end{bmatrix}$$

$$\begin{bmatrix} 7+0 & 6+8 \\ 17+(-2) & 9+5 \end{bmatrix}$$

$$\begin{bmatrix} 7 & 14 \\ 15 & 14 \end{bmatrix}$$

2. Add the matrices.

$$\begin{bmatrix} 8 & 3 \\ -4 & 7 \\ 6 & 0 \\ 1 & 13 \end{bmatrix} + \begin{bmatrix} 6 & 7 \\ 2 & -3 \\ 9 & 11 \\ 7 & -2 \end{bmatrix}$$

Solution:

To add matrices, you simply add together entries from corresponding positions in each matrix.

$$\begin{bmatrix} 8 & 3 \\ -4 & 7 \\ 6 & 0 \\ 1 & 13 \end{bmatrix} + \begin{bmatrix} 6 & 7 \\ 2 & -3 \\ 9 & 11 \\ 7 & -2 \end{bmatrix}$$

$$\begin{bmatrix} 8+6 & 3+7 \\ -4+2 & 7+(-3) \\ 6+9 & 0+11 \\ 1+7 & 13+(-2) \end{bmatrix}$$

$$\begin{bmatrix} 14 & 10 \\ -2 & 4 \\ 15 & 11 \\ 8 & 11 \end{bmatrix}$$

■ 3. Subtract the matrices.

$$\begin{bmatrix} 7 & 9 \\ 4 & -1 \end{bmatrix} - \begin{bmatrix} 3 & 8 \\ 12 & -3 \end{bmatrix}$$

Solution:

To subtract matrices, you simply subtract entries from corresponding positions in each matrix.

$$\begin{bmatrix} 7 & 9 \\ 4 & -1 \end{bmatrix} - \begin{bmatrix} 3 & 8 \\ 12 & -3 \end{bmatrix}$$

$$\begin{bmatrix} 7-3 & 9-8 \\ 4-12 & -1-(-3) \end{bmatrix}$$

$$\begin{bmatrix} 4 & 1 \\ -8 & 2 \end{bmatrix}$$

■ 4. Subtract the matrices.

$$\begin{bmatrix} 8 & 11 & 2 & 9 \\ 6 & 3 & 16 & 8 \end{bmatrix} - \begin{bmatrix} 6 & 11 & 7 & -4 \\ 5 & 8 & 1 & 15 \end{bmatrix}$$

Solution:

To subtract matrices, you simply subtract entries from corresponding positions in each matrix.

$$\begin{bmatrix} 8 & 11 & 2 & 9 \\ 6 & 3 & 16 & 8 \end{bmatrix} - \begin{bmatrix} 6 & 11 & 7 & -4 \\ 5 & 8 & 1 & 15 \end{bmatrix}$$

$$\begin{bmatrix} 8-6 & 11-11 & 2-7 & 9-(-4) \\ 6-5 & 3-8 & 16-1 & 8-15 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 & -5 & 13 \\ 1 & -5 & 15 & -7 \end{bmatrix}$$

■ 5. Solve for M.

$$\begin{bmatrix} 6 & 5 \\ 9 & -9 \end{bmatrix} + \begin{bmatrix} 3 & 7 \\ 1 & 6 \end{bmatrix} = M + \begin{bmatrix} 7 & 12 \\ -3 & -1 \end{bmatrix} - \begin{bmatrix} 1 & 8 \\ 4 & -7 \end{bmatrix}$$

Solution:

Let's start with the matrix addition on the left side of the equation and the matrix subtraction on the right side of the equation.

$$\begin{bmatrix} 6 & 5 \\ 9 & -9 \end{bmatrix} + \begin{bmatrix} 3 & 7 \\ 1 & 6 \end{bmatrix} = M + \begin{bmatrix} 7 & 12 \\ -3 & -1 \end{bmatrix} - \begin{bmatrix} 1 & 8 \\ 4 & -7 \end{bmatrix}$$

$$\begin{bmatrix} 6+3 & 5+7 \\ 9+1 & -9+6 \end{bmatrix} = M + \begin{bmatrix} 7-1 & 12-8 \\ -3-4 & -1-(-7) \end{bmatrix}$$

$$\begin{bmatrix} 9 & 12 \\ 10 & -3 \end{bmatrix} = M + \begin{bmatrix} 6 & 4 \\ -7 & 6 \end{bmatrix}$$

To isolate M, we'll subtract the matrix on the right from both sides in order to move it to the left.

$$\begin{bmatrix} 9 & 12 \\ 10 & -3 \end{bmatrix} - \begin{bmatrix} 6 & 4 \\ -7 & 6 \end{bmatrix} = M$$

$$\begin{bmatrix} 9-6 & 12-4 \\ 10-(-7) & -3-6 \end{bmatrix} = M$$

$$\begin{bmatrix} 3 & 8 \\ 17 & -9 \end{bmatrix} = M$$

The conclusion is that the value of M that makes the equation true is this matrix:

$$M = \begin{bmatrix} 3 & 8 \\ 17 & -9 \end{bmatrix}$$

6. Solve for N.

$$\begin{bmatrix} 4 & 12 \\ 9 & 8 \end{bmatrix} - \begin{bmatrix} 0 & 3 \\ 9 & 9 \end{bmatrix} = N - \begin{bmatrix} 6 & 3 \\ 5 & 11 \end{bmatrix} + \begin{bmatrix} 7 & -4 \\ -18 & 1 \end{bmatrix}$$

Solution:

Let's start with the matrix subtraction on the left side of the equation.

$$\begin{bmatrix} 4 & 12 \\ 9 & 8 \end{bmatrix} - \begin{bmatrix} 0 & 3 \\ 9 & 9 \end{bmatrix} = N - \begin{bmatrix} 6 & 3 \\ 5 & 11 \end{bmatrix} + \begin{bmatrix} 7 & -4 \\ -18 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 4-0 & 12-3 \\ 9-9 & 8-9 \end{bmatrix} = N - \begin{bmatrix} 6 & 3 \\ 5 & 11 \end{bmatrix} + \begin{bmatrix} 7 & -4 \\ -18 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 9 \\ 0 & -1 \end{bmatrix} = N - \begin{bmatrix} 6 & 3 \\ 5 & 11 \end{bmatrix} + \begin{bmatrix} 7 & -4 \\ -18 & 1 \end{bmatrix}$$

To isolate N, we'll move the matrices on the right side over to the left side, then flip the equation.

$$\begin{bmatrix}4 & 9\\0 & -1\end{bmatrix}+\begin{bmatrix}6 & 3\\5 & 11\end{bmatrix}-\begin{bmatrix}7 & -4\\-18 & 1\end{bmatrix}=N$$

$$N=\begin{bmatrix}4 & 9\\0 & -1\end{bmatrix}+\begin{bmatrix}6 & 3\\5 & 11\end{bmatrix}-\begin{bmatrix}7 & -4\\-18 & 1\end{bmatrix}$$

Simplify the right side to solve the equation for N.

$$N=\begin{bmatrix}4+6 & 9+3\\0+5 & -1+11\end{bmatrix}-\begin{bmatrix}7 & -4\\-18 & 1\end{bmatrix}$$

$$N=\begin{bmatrix}10 & 12\\5 & 10\end{bmatrix}-\begin{bmatrix}7 & -4\\-18 & 1\end{bmatrix}$$

$$N=\begin{bmatrix}10-7 & 12-(-4)\\5-(-18) & 10-1\end{bmatrix}$$

$$N=\begin{bmatrix}3 & 16\\23 & 9\end{bmatrix}$$

SCALAR MULTIPLICATION

■ 1. Use scalar multiplication to simplify the expression.

$$\frac{1}{4}\begin{bmatrix} 12 & 8 & 3 \\ 2 & -16 & 0 \\ 1 & 5 & 7 \end{bmatrix}$$

Solution:

The scalar $1/4$ is being multiplied by the matrix. Distribute the scalar across every entry in the matrix.

$$\frac{1}{4}\begin{bmatrix} 12 & 8 & 3 \\ 2 & -16 & 0 \\ 1 & 5 & 7 \end{bmatrix}$$

$$\begin{bmatrix} \frac{1}{4}(12) & \frac{1}{4}(8) & \frac{1}{4}(3) \\ \frac{1}{4}(2) & \frac{1}{4}(-16) & \frac{1}{4}(0) \\ \frac{1}{4}(1) & \frac{1}{4}(5) & \frac{1}{4}(7) \end{bmatrix}$$

$$\begin{bmatrix} 3 & 2 & \frac{3}{4} \\ \frac{1}{2} & -4 & 0 \\ \frac{1}{4} & \frac{5}{4} & \frac{7}{4} \end{bmatrix}$$

■ 2. Solve for Y.

$$4\begin{bmatrix} 2 & 9 \\ -5 & 0 \end{bmatrix} + Y = 5\begin{bmatrix} 1 & -3 \\ 6 & 8 \end{bmatrix}$$

Solution:

Apply the scalars to the matrices.

$$\begin{bmatrix} 4(2) & 4(9) \\ 4(-5) & 4(0) \end{bmatrix} + Y = \begin{bmatrix} 5(1) & 5(-3) \\ 5(6) & 5(8) \end{bmatrix}$$

$$\begin{bmatrix} 8 & 36 \\ -20 & 0 \end{bmatrix} + Y = \begin{bmatrix} 5 & -15 \\ 30 & 40 \end{bmatrix}$$

Subtract the matrix on the left from both sides of the equation in order to isolate Y.

$$Y = \begin{bmatrix} 5 & -15 \\ 30 & 40 \end{bmatrix} - \begin{bmatrix} 8 & 36 \\ -20 & 0 \end{bmatrix}$$

$$Y = \begin{bmatrix} 5-8 & -15-36 \\ 30-(-20) & 40-0 \end{bmatrix}$$

$$Y = \begin{bmatrix} -3 & -51 \\ 50 & 40 \end{bmatrix}$$

■ 3. Solve for N.

$$-2\begin{bmatrix} 6 & 5 \\ 0 & 11 \end{bmatrix} = N - 4\begin{bmatrix} 2 & 4 \\ -1 & 9 \end{bmatrix}$$

Solution:

Apply the scalars to the matrices.

$$\begin{bmatrix} -2(6) & -2(5) \\ -2(0) & -2(11) \end{bmatrix} = N - \begin{bmatrix} 4(2) & 4(4) \\ 4(-1) & 4(9) \end{bmatrix}$$

$$\begin{bmatrix} -12 & -10 \\ 0 & -22 \end{bmatrix} = N - \begin{bmatrix} 8 & 16 \\ -4 & 36 \end{bmatrix}$$

Add the matrix on the right to both sides of the equation in order to isolate N.

$$\begin{bmatrix} -12 & -10 \\ 0 & -22 \end{bmatrix} + \begin{bmatrix} 8 & 16 \\ -4 & 36 \end{bmatrix} = N$$

$$\begin{bmatrix} -12+8 & -10+16 \\ 0+(-4) & -22+36 \end{bmatrix} = N$$

$$\begin{bmatrix} -4 & 6 \\ -4 & 14 \end{bmatrix} = N$$

$$N = \begin{bmatrix} -4 & 6 \\ -4 & 14 \end{bmatrix}$$

■ 4. Solve the equation for M.

$$-4M = \begin{bmatrix} -5 & 0 & 4 \\ 1 & -8 & -2 \\ -4 & 12 & 3 \end{bmatrix}$$

Solution:

Multiply both sides of the matrix equation by the scalar $-1/4$ in order to isolate M.

$$-\frac{1}{4}(-4M) = -\frac{1}{4}\begin{bmatrix} -5 & 0 & 4 \\ 1 & -8 & -2 \\ -4 & 12 & 3 \end{bmatrix}$$

$$M = \begin{bmatrix} -\frac{1}{4}(-5) & -\frac{1}{4}(0) & -\frac{1}{4}(4) \\ -\frac{1}{4}(1) & -\frac{1}{4}(-8) & -\frac{1}{4}(-2) \\ -\frac{1}{4}(-4) & -\frac{1}{4}(12) & -\frac{1}{4}(3) \end{bmatrix}$$

$$M = \begin{bmatrix} \frac{5}{4} & 0 & -1 \\ -\frac{1}{4} & 2 & \frac{1}{2} \\ 1 & -3 & -\frac{3}{4} \end{bmatrix}$$

■ 5. Use scalar multiplication to simplify the expression.

$$-5A + \frac{1}{3}B$$

$$A = \begin{bmatrix} \frac{2}{5} & -\frac{1}{5} \\ 3 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} -\frac{1}{3} & 0 \\ 6 & -2 \end{bmatrix}$$

Solution:

Apply the scalars to the matrices.

$$-5A + \frac{1}{3}B$$

$$-5\begin{bmatrix} \frac{2}{5} & -\frac{1}{5} \\ 3 & 0 \end{bmatrix} + \frac{1}{3}\begin{bmatrix} -\frac{1}{3} & 0 \\ 6 & -2 \end{bmatrix}$$

$$\begin{bmatrix} -5(\frac{2}{5}) & -5(-\frac{1}{5}) \\ -5(3) & -5(0) \end{bmatrix} + \begin{bmatrix} \frac{1}{3}(-\frac{1}{3}) & \frac{1}{3}(0) \\ \frac{1}{3}(6) & \frac{1}{3}(-2) \end{bmatrix}$$

$$\begin{bmatrix} -2 & 1 \\ -15 & 0 \end{bmatrix} + \begin{bmatrix} -\frac{1}{9} & 0 \\ 2 & -\frac{2}{3} \end{bmatrix}$$

$$\begin{bmatrix} -2-\frac{1}{9} & 1+0 \\ -15+2 & 0-\frac{2}{3} \end{bmatrix}$$

$$\begin{bmatrix} -\frac{19}{9} & 1 \\ -13 & -\frac{2}{3} \end{bmatrix}$$

■ 6. Solve the equation for X.

$$2X - \frac{1}{2}\begin{bmatrix} 0 & -2 & 6 \\ 4 & -1 & 2 \\ 8 & 6 & 0 \end{bmatrix} = \begin{bmatrix} 6 & 3 & 7 \\ 0 & -\frac{3}{2} & 1 \\ 6 & 5 & 4 \end{bmatrix}$$

Solution:

Apply the scalar to the matrix.

$$2X - \begin{bmatrix} \frac{1}{2}(0) & \frac{1}{2}(-2) & \frac{1}{2}(6) \\ \frac{1}{2}(4) & \frac{1}{2}(-1) & \frac{1}{2}(2) \\ \frac{1}{2}(8) & \frac{1}{2}(6) & \frac{1}{2}(0) \end{bmatrix} = \begin{bmatrix} 6 & 3 & 7 \\ 0 & -\frac{3}{2} & 1 \\ 6 & 5 & 4 \end{bmatrix}$$

$$2X - \begin{bmatrix} 0 & -1 & 3 \\ 2 & -\frac{1}{2} & 1 \\ 4 & 3 & 0 \end{bmatrix} = \begin{bmatrix} 6 & 3 & 7 \\ 0 & -\frac{3}{2} & 1 \\ 6 & 5 & 4 \end{bmatrix}$$

Add the matrix on the left to both sides of the equation in order to isolate $2X$.

$$2X=\begin{bmatrix}6&3&7\\0&-\frac{3}{2}&1\\6&5&4\end{bmatrix}+\begin{bmatrix}0&-1&3\\2&-\frac{1}{2}&1\\4&3&0\end{bmatrix}$$

$$2X=\begin{bmatrix}6+0&3-1&7+3\\0+2&-\frac{3}{2}-\frac{1}{2}&1+1\\6+4&5+3&4+0\end{bmatrix}$$

$$2X=\begin{bmatrix}6&2&10\\2&-2&2\\10&8&4\end{bmatrix}$$

Multiply both sides of the equation by the scalar $1/2$ in order to isolate X.

$$\frac{1}{2}\cdot 2X=\frac{1}{2}\begin{bmatrix}6&2&10\\2&-2&2\\10&8&4\end{bmatrix}$$

$$X=\begin{bmatrix}\frac{1}{2}(6)&\frac{1}{2}(2)&\frac{1}{2}(10)\\\frac{1}{2}(2)&\frac{1}{2}(-2)&\frac{1}{2}(2)\\\frac{1}{2}(10)&\frac{1}{2}(8)&\frac{1}{2}(4)\end{bmatrix}$$

$$X=\begin{bmatrix}3&1&5\\1&-1&1\\5&4&2\end{bmatrix}$$

ZERO MATRICES

1. Add the zero matrix to the given matrix.

$$\begin{bmatrix} 8 & 17 \\ -6 & 0 \end{bmatrix}$$

Solution:

Adding the zero matrix to any other matrix doesn't change the value of the matrix, so

$$\begin{bmatrix} 8 & 17 \\ -6 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 8 & 17 \\ -6 & 0 \end{bmatrix}$$

2. Find the opposite matrix.

$$\begin{bmatrix} 6 & 8 & 0 \\ 2 & -3 & 11 \\ 4 & 12 & 9 \end{bmatrix}$$

Solution:

To get the opposite of a matrix, multiply it by a scalar of -1. Then the opposite of the given matrix is

$$(-1)\begin{bmatrix} (-1)6 & (-1)8 & (-1)0 \\ (-1)2 & (-1)(-3) & (-1)11 \\ (-1)4 & (-1)12 & (-1)9 \end{bmatrix}$$

$$\begin{bmatrix} -6 & -8 & 0 \\ -2 & 3 & -11 \\ -4 & -12 & -9 \end{bmatrix}$$

■ 3. Multiply the matrix by a scalar of 0.

$$\begin{bmatrix} 14 & -1 & 7 & 5 \\ 3 & 7 & 18 & -4 \end{bmatrix}$$

Solution:

Multiplying any matrix by a scalar of 0 results in a zero matrix.

$$(0)\begin{bmatrix} 14(0) & -1(0) & 7(0) & 5(0) \\ 3(0) & 7(0) & 18(0) & -4(0) \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

■ 4. Add the opposite of A to A.

$$A = \begin{bmatrix} 1 & -5 & 7 \\ -3 & 2 & 8 \end{bmatrix}$$

Solution:

The opposite matrix of A is

$$-A=(-1)\begin{bmatrix}1 & -5 & 7\\ -3 & 2 & 8\end{bmatrix}$$

$$-A=\begin{bmatrix}-1(1) & -1(-5) & -1(7)\\ -1(-3) & -1(2) & -1(8)\end{bmatrix}$$

$$-A=\begin{bmatrix}-1 & 5 & -7\\ 3 & -2 & -8\end{bmatrix}$$

Add the opposite matrices.

$$A+(-A)=\begin{bmatrix}1 & -5 & 7\\ -3 & 2 & 8\end{bmatrix}+\begin{bmatrix}-1 & 5 & -7\\ 3 & -2 & -8\end{bmatrix}$$

$$A+(-A)=\begin{bmatrix}1-1 & -5+5 & 7-7\\ -3+3 & 2-2 & 8-8\end{bmatrix}$$

$$A+(-A)=\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$$

■ 5. Solve the equation for X.

$$X+\begin{bmatrix}-1 & 2 & 5\\ 7 & -4 & 3\\ 1 & -2 & 4\end{bmatrix}=\begin{bmatrix}1 & 0 & -5\\ 0 & 7 & 3\\ -4 & 0 & -1\end{bmatrix}+\begin{bmatrix}-1 & 0 & 5\\ 0 & -7 & -3\\ 4 & 0 & 1\end{bmatrix}$$

Solution:

Add the matrices on the right side.

$$X+\begin{bmatrix}-1 & 2 & 5\\ 7 & -4 & 3\\ 1 & -2 & 4\end{bmatrix}=\begin{bmatrix}1-1 & 0+0 & -5+5\\ 0+0 & 7-7 & 3-3\\ -4+4 & 0+0 & -1+1\end{bmatrix}$$

$$X+\begin{bmatrix}-1 & 2 & 5\\ 7 & -4 & 3\\ 1 & -2 & 4\end{bmatrix}=\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$$

Subtract the matrix on the left from both sides of the equation in order to isolate X.

$$X=\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}-\begin{bmatrix}-1 & 2 & 5\\ 7 & -4 & 3\\ 1 & -2 & 4\end{bmatrix}$$

$$X=\begin{bmatrix}0-(-1) & 0-2 & 0-5\\ 0-7 & 0-(-4) & 0-3\\ 0-1 & 0-(-2) & 0-4\end{bmatrix}$$

$$X=\begin{bmatrix}1 & -2 & -5\\ -7 & 4 & -3\\ -1 & 2 & -4\end{bmatrix}$$

■ 6. Solve the equation for A.

$$\begin{bmatrix}-1&5&4\\-2&0&-3\\5&7&-9\end{bmatrix}-A=0\begin{bmatrix}-2&3&0\\-1&5&-2\\-7&0&4\end{bmatrix}+\begin{bmatrix}2&4&-7\\8&0&-5\\-1&4&3\end{bmatrix}-\begin{bmatrix}3&-1&-11\\10&0&-2\\-6&-3&12\end{bmatrix}$$

Solution:

Apply the scalar 0 to the matrix.

$$\begin{bmatrix}-1&5&4\\-2&0&-3\\5&7&-9\end{bmatrix}-A=\begin{bmatrix}0(-2)&0(3)&0(0)\\0(-1)&0(5)&0(-2)\\0(-7)&0(0)&0(4)\end{bmatrix}+\begin{bmatrix}2&4&-7\\8&0&-5\\-1&4&3\end{bmatrix}-\begin{bmatrix}3&-1&-11\\10&0&-2\\-6&-3&12\end{bmatrix}$$

$$\begin{bmatrix}-1&5&4\\-2&0&-3\\5&7&-9\end{bmatrix}-A=\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}+\begin{bmatrix}2&4&-7\\8&0&-5\\-1&4&3\end{bmatrix}-\begin{bmatrix}3&-1&-11\\10&0&-2\\-6&-3&12\end{bmatrix}$$

Adding the zero matrix does not change the value of the equation, so we can cancel it. Subtract the remaining matrices on the right side.

$$\begin{bmatrix}-1&5&4\\-2&0&-3\\5&7&-9\end{bmatrix}-A=\begin{bmatrix}2&4&-7\\8&0&-5\\-1&4&3\end{bmatrix}-\begin{bmatrix}3&-1&-11\\10&0&-2\\-6&-3&12\end{bmatrix}$$

$$\begin{bmatrix}-1&5&4\\-2&0&-3\\5&7&-9\end{bmatrix}-A=\begin{bmatrix}2-3&4-(-1)&-7-(-11)\\8-10&0-0&-5-(-2)\\-1-(-6)&4-(-3)&3-12\end{bmatrix}$$

$$\begin{bmatrix}-1&5&4\\-2&0&-3\\5&7&-9\end{bmatrix}-A=\begin{bmatrix}-1&5&4\\-2&0&-3\\5&7&-9\end{bmatrix}$$

Subtract the matrix on the left from both sides of the equation in order to isolate $-A$.

$$-A = \begin{bmatrix} -1 & 5 & 4 \\ -2 & 0 & -3 \\ 5 & 7 & -9 \end{bmatrix} - \begin{bmatrix} -1 & 5 & 4 \\ -2 & 0 & -3 \\ 5 & 7 & -9 \end{bmatrix}$$

$$-A = \begin{bmatrix} -1-(-1) & 5-5 & 4-4 \\ -2-(-2) & 0-0 & -3-(-3) \\ 5-5 & 7-7 & -9-(-9) \end{bmatrix}$$

$$-A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Multiply both sides by -1 to solve for A.

$$A = (-1)\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

MATRIX MULTIPLICATION

■ 1. If matrix A is 3×3 and matrix B is 3×4, say whether AB or BA is defined, and give the dimensions of any product that's defined.

Solution:

Line up the dimensions for the products AB and BA, and compare the middle terms, which represent the columns from the first matrix and the rows from the second matrix.

$$AB: \quad 3 \times \boxed{3 \qquad 3} \times 4$$

$$BA: \quad 3 \times \boxed{4 \qquad 3} \times 3$$

The middle numbers match for AB, so that product is defined. For BA, the middle numbers don't match, so that product isn't defined.

The dimensions of AB are given by the outside numbers, which are the rows from the first matrix and the columns from the second matrix.

$$AB: \quad \boxed{3} \times 3 \qquad 3 \times \boxed{4}$$

So the dimensions of AB will be 3×4.

■ 2. Find the product of matrices A and B.

$$A = \begin{bmatrix} 2 & 6 \\ -3 & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} -2 & 0 \\ 5 & -4 \end{bmatrix}$$

Solution:

Multiply matrix A by matrix B.

$$AB = \begin{bmatrix} 2 & 6 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} -2 & 0 \\ 5 & -4 \end{bmatrix}$$

$$AB = \begin{bmatrix} 2(-2)+6(5) & 2(0)+6(-4) \\ -3(-2)+1(5) & -3(0)+1(-4) \end{bmatrix}$$

$$AB = \begin{bmatrix} 26 & -24 \\ 11 & -4 \end{bmatrix}$$

■ 3. Find the product of matrices A and B.

$$A = \begin{bmatrix} 5 & -1 \\ 0 & 11 \\ 7 & -2 \end{bmatrix}$$

$$B = \begin{bmatrix} 6 & 1 & 8 \\ -3 & 0 & 4 \end{bmatrix}$$

Solution:

Multiply matrix A by matrix B.

$$AB = \begin{bmatrix} 5 & -1 \\ 0 & 11 \\ 7 & -2 \end{bmatrix} \begin{bmatrix} 6 & 1 & 8 \\ -3 & 0 & 4 \end{bmatrix}$$

$$AB = \begin{bmatrix} 5(6)+(-1)(-3) & 5(1)+(-1)(0) & 5(8)+(-1)(4) \\ 0(6)+11(-3) & 0(1)+11(0) & 0(8)+11(4) \\ 7(6)+(-2)(-3) & 7(1)+(-2)(0) & 7(8)+(-2)(4) \end{bmatrix}$$

$$AB = \begin{bmatrix} 33 & 5 & 36 \\ -33 & 0 & 44 \\ 48 & 7 & 48 \end{bmatrix}$$

■ 4. Find the product of matrices A and B.

$$A = \begin{bmatrix} 3 & -2 \\ 1 & 8 \\ 0 & 3 \end{bmatrix}$$

$$B = \begin{bmatrix} 5 & 2 \\ 4 & 8 \end{bmatrix}$$

Solution:

Multiply matrix A by matrix B.

$$AB = \begin{bmatrix} 3 & -2 \\ 1 & 8 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 5 & 2 \\ 4 & 8 \end{bmatrix}$$

$$AB = \begin{bmatrix} 3(5) + (-2)(4) & 3(2) + (-2)(8) \\ 1(5) + 8(4) & 1(2) + 8(8) \\ 0(5) + 3(4) & 0(2) + 3(8) \end{bmatrix}$$

$$AB = \begin{bmatrix} 7 & -10 \\ 37 & 66 \\ 12 & 24 \end{bmatrix}$$

■ 5. Use the distributive property to find $A(B+C)$.

$$A = \begin{bmatrix} 2 & 0 \\ 4 & -2 \end{bmatrix}$$

$$B = \begin{bmatrix} 3 & 1 \\ 5 & 4 \end{bmatrix}$$

$$C = \begin{bmatrix} 6 & 1 \\ 3 & -1 \end{bmatrix}$$

Solution:

Applying the distributive property to the initial expression, we get

$$A(B+C) = AB + AC$$

Use matrix multiplication to find $AB + AC$.

$$AB + AC = \begin{bmatrix} 2 & 0 \\ 4 & -2 \end{bmatrix} \cdot \begin{bmatrix} 3 & 1 \\ 5 & 4 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} 6 & 1 \\ 3 & -1 \end{bmatrix}$$

$$AB + AC = \begin{bmatrix} 2(3) + (0)(5) & 2(1) + 0(4) \\ 4(3) + (-2)(5) & 4(1) + (-2)(4) \end{bmatrix}$$

$$+ \begin{bmatrix} 2(6) + 0(3) & 2(1) + 0(-1) \\ 4(6) + (-2)(3) & 4(1) + (-2)(-1) \end{bmatrix}$$

$$AB + AC = \begin{bmatrix} 6 & 2 \\ 2 & -4 \end{bmatrix} + \begin{bmatrix} 12 & 2 \\ 18 & 6 \end{bmatrix}$$

Now use matrix addition.

$$AB + AC = \begin{bmatrix} 6 + 12 & 2 + 2 \\ 2 + 18 & -4 + 6 \end{bmatrix}$$

$$AB + AC = \begin{bmatrix} 18 & 4 \\ 20 & 2 \end{bmatrix}$$

So the value of the original expression is

$$A(B + C) = \begin{bmatrix} 18 & 4 \\ 20 & 2 \end{bmatrix}$$

■ 6. Find the product of matrices A and B.

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 6 & -2 & 8 & 1 \\ 7 & 3 & 5 & 2 \end{bmatrix}$$

Solution:

Multiply matrix A by matrix B.

$$AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 6 & -2 & 8 & 1 \\ 7 & 3 & 5 & 2 \end{bmatrix}$$

$$AB = \begin{bmatrix} 0(6)+0(7) & 0(-2)+0(3) & 0(8)+0(5) & 0(1)+0(2) \\ 0(6)+0(7) & 0(-2)+0(3) & 0(8)+0(5) & 0(1)+0(2) \\ 0(6)+0(7) & 0(-2)+0(3) & 0(8)+0(5) & 0(1)+0(2) \\ 0(6)+0(7) & 0(-2)+0(3) & 0(8)+0(5) & 0(1)+0(2) \end{bmatrix}$$

$$AB = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

IDENTITY MATRICES

■ 1. Write the identity matrix I_4.

Solution:

We always call the identity matrix I, and it's always a square matrix, like 2×2, 3×3, 4×4, etc. For that reason, it's common to abbreviate I_{2x2} as just I_2, or I_{3x3} as just I_3, etc. So, I_4 is the 4×4 identity matrix.

$$I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

■ 2. If we want to find the product IA, where I is the identity matrix and A is 4×2, then what are the dimensions of I?

Solution:

Start by setting up the equation $I \cdot A = A$, then substitute the dimensions for A into the equation.

$$I \cdot A = A$$

$$I \cdot 4 \times 2 = 4 \times 2$$

Break down the dimensions of the identity matrix as rows $\times$ columns, or $R \times C$.

$$R \times C \cdot 4 \times 2 = 4 \times 2$$

For matrix multiplication to be defined, you need the same number of columns in the first matrix as rows in the second matrix.

$$R \times 4 \cdot 4 \times 2 = 4 \times 2$$

The dimensions of the product come from the rows in the first matrix and the columns in the second matrix, so

$$4 \times 4 \cdot 4 \times 2 = 4 \times 2$$

Therefore, the identity matrix in this case is I_4.

$$I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

■ 3. If we want to find the product IA, where I is the identity matrix and A is a 3×4, then what are the dimensions of I?

Solution:

Start by setting up the equation $I \cdot A = A$, then substitute the dimensions for A into the equation.

$$I \cdot A = A$$

$$I \cdot 3 \times 4 = 3 \times 4$$

Break down the dimensions of the identity matrix as rows $\times$ columns, or $R \times C$.

$$R \times C \cdot 3 \times 4 = 3 \times 4$$

For matrix multiplication to be defined, you need the same number of columns in the first matrix as rows in the second matrix.

$$R \times 3 \cdot 3 \times 4 = 3 \times 4$$

The dimensions of the product come from the rows in the first matrix and the columns in the second matrix, so

$$3 \times 3 \cdot 3 \times 4 = 3 \times 4$$

Therefore, the identity matrix in this case is I_3.

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

■ 4. If we want to find the product IA, where I is the identity matrix and A is given, then what are the dimensions of I? What is the product IA?

$$A = \begin{bmatrix} 2 & 8 \\ -2 & 7 \\ 3 & 5 \end{bmatrix}$$

Solution:

Start by setting up the equation $I \cdot A = A$, then substitute the dimensions for A into the equation.

$$I \cdot A = A$$

$$I \cdot 3 \times 2 = 3 \times 2$$

Break down the dimensions of the identity matrix as rows $\times$ columns, or $R \times C$.

$$R \times C \cdot 3 \times 2 = 3 \times 2$$

For matrix multiplication to be defined, you need the same number of columns in the first matrix as rows in the second matrix.

$$R \times 3 \cdot 3 \times 2 = 3 \times 2$$

The dimensions of the product come from the rows in the first matrix and the columns in the second matrix, so

$$3 \times 3 \cdot 3 \times 2 = 3 \times 2$$

Therefore, the identity matrix in this case is I_3.

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The product IA is

$$IA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 8 \\ -2 & 7 \\ 3 & 5 \end{bmatrix}$$

$$IA = \begin{bmatrix} 1(2) + 0(-2) + 0(3) & 1(8) + 0(7) + 0(5) \\ 0(2) + 1(-2) + 0(3) & 0(8) + 1(7) + 0(5) \\ 0(2) + 0(-2) + 1(3) & 0(8) + 0(7) + 1(5) \end{bmatrix}$$

$$IA = \begin{bmatrix} 2 & 8 \\ -2 & 7 \\ 3 & 5 \end{bmatrix}$$

As we expected, we get back to matrix A after multiplying it by the identity matrix I_3.

■ 5. If we want to find the product IA, where I is the identity matrix and A is given, then what are the dimensions of I? What is the product IA?

$$A = \begin{bmatrix} 7 & 1 & 3 & -2 \\ 5 & 5 & 2 & 9 \end{bmatrix}$$

Solution:

Start by setting up the equation $I \cdot A = A$, then substitute the dimensions for A into the equation.

$$I \cdot A = A$$

$$I \cdot 2 \times 4 = 2 \times 4$$

Break down the dimensions of the identity matrix as rows $\times$ columns, or $R \times C$.

$$R \times C \cdot 2 \times 4 = 2 \times 4$$

For matrix multiplication to be defined, you need the same number of columns in the first matrix as rows in the second matrix.

$$R \times 2 \cdot 2 \times 4 = 2 \times 4$$

The dimensions of the product come from the rows in the first matrix and the columns in the second matrix, so

$$2 \times 2 \cdot 2 \times 4 = 2 \times 4$$

Therefore, the identity matrix in this case is I_2.

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The product IA is

$$IA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 7 & 1 & 3 & -2 \\ 5 & 5 & 2 & 9 \end{bmatrix}$$

$$IA = \begin{bmatrix} 1(7)+0(5) & 1(1)+0(5) & 1(3)+0(2) & 1(-2)+0(9) \\ 0(7)+1(5) & 0(1)+1(5) & 0(3)+1(2) & 0(-2)+1(9) \end{bmatrix}$$

$$IA = \begin{bmatrix} 7 & 1 & 3 & -2 \\ 5 & 5 & 2 & 9 \end{bmatrix}$$

As we expected, we get back to matrix A after multiplying it by the identity matrix I_2.

6. If A is a 2×4 matrix, what are the dimensions of the identity matrix that make the equation true?

$$AI = A$$

Solution:

Set up the equation $AI = A$, then substitute the dimensions for A into the equation.

$$A \cdot I = A$$

$$2 \times 4 \cdot I = 2 \times 4$$

Break up the dimensions of I as $R \times C$.

$$2 \times 4 \cdot R \times C = 2 \times 4$$

The number of rows in the second matrix must be equal to the number of columns from the first matrix.

$$2 \times 4 \cdot 4 \times C = 2 \times 4$$

The dimensions of the product come from the rows of the first matrix and the columns of the second matrix, so

$$2 \times 4 \cdot 4 \times 4 = 2 \times 4$$

So the identity matrix is I_4, the 4×4 identity matrix.

$$I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

THE ELIMINATION MATRIX

1. Find a single 2×2 elimination matrix E that accomplishes the given row operations.

1. $(1/3)R_1 \to R_1$

2. $-2R_1 + R_2 \to R_2$

Solution:

The row operation $(1/3)R_1 \to R_1$ means we'll put a $1/3$ in $E_{1,1}$.

$$E_1 = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & 1 \end{bmatrix}$$

The row operation $-2R_1 + R_2 \to R_2$ means we'll put a 1 in $E_{2,2}$ and a -2 in $E_{2,1}$.

$$E_1 = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$$

Consolidate these two row operations into one elimination matrix by multiplying E_2 by E_1.

$$E = E_2 E_1$$

$$E = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & 1 \end{bmatrix}$$

$$E = \begin{bmatrix} 1\left(\frac{1}{3}\right) + 0(0) & 1(0) + 0(1) \\ -2\left(\frac{1}{3}\right) + 1(0) & -2(0) + 1(1) \end{bmatrix}$$

$$E = \begin{bmatrix} \frac{1}{3} & 0 \\ -\frac{2}{3} & 1 \end{bmatrix}$$

■ 2. Find a single 3×3 elimination matrix E that accomplishes the given row operations.

1. $-3R_1 + R_3 \to R_3$

2. $5R_2 + R_1 \to R_1$

3. $-R_3 \to R_3$

Solution:

The row operation $-3R_1 + R_3 \to R_3$ means we'll put a 1 in $E_{3,3}$ and a -3 in $E_{3,1}$.

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}$$

The row operation $5R_2 + R_1 \to R_1$ means we'll put a 1 in $E_{1,1}$ and 5 in $E_{1,2}$.

$$E_2 = \begin{bmatrix} 1 & 5 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The row operation $-R_3 \to R_3$ means we'll put a -1 in $E_{3,3}$.

$$E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Consolidate these three row operations into one elimination matrix by multiplying E_3 by E_2 by E_1.

$$E = E_3E_2E_1$$

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 5 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}$$

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1(1)+5(0)+0(-3) & 1(0)+5(1)+0(0) & 1(0)+5(0)+0(1) \\ 0(1)+1(0)+0(-3) & 0(0)+1(1)+0(0) & 0(0)+1(0)+0(1) \\ 0(1)+0(0)+1(-3) & 0(0)+0(1)+1(0) & 0(0)+0(0)+1(1) \end{bmatrix}$$

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 5 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}$$

$$E = \begin{bmatrix} 1(1)+0(0)+0(-3) & 1(5)+0(1)+0(0) & 1(0)+0(0)+0(1) \\ 0(1)+1(0)+0(-3) & 0(5)+1(1)+0(0) & 0(0)+1(0)+0(1) \\ 0(1)+0(0)-1(-3) & 0(5)+0(1)-1(0) & 0(0)+0(0)-1(1) \end{bmatrix}$$

$$E = \begin{bmatrix} 1 & 5 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & -1 \end{bmatrix}$$

■ 3. Find a single 2×2 elimination matrix E that accomplishes the given row operations.

1. $-R_1 \to R_1$

2. $5R_1 + R_2 \to R_2$

3. $-(1/7)R_2 \to R_2$

4. $R_2 + R_1 \to R_1$

Solution:

The row operation $-R_1 \to R_1$ means we'll put a -1 in $E_{1,1}$.

$$E_1 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

The row operation $5R_1 + R_2 \to R_2$ means we'll put a 1 in $E_{2,2}$ and a 5 in $E_{2,1}$.

$$E_2 = \begin{bmatrix} 1 & 0 \\ 5 & 1 \end{bmatrix}$$

The row operation $-(1/7)R_2 \to R_2$ means we'll put a $-1/7$ in $E_{2,2}$.

$$E_3 = \begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{7} \end{bmatrix}$$

The row operation $R_2 + R_1 \to R_1$ means we'll put a 1 in $E_{1,1}$ and a 1 in $E_{1,2}$.

$$E_4 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

Consolidate these four row operations into one elimination matrix by multiplying E_4 by E_3 by E_2 by E_1.

$$E = E_4E_3E_2E_1$$

$$E = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{7} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 5 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$E = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{7} \end{bmatrix} \begin{bmatrix} 1(-1) + 0(0) & 1(0) + 0(1) \\ 5(-1) + 1(0) & 5(0) + 1(1) \end{bmatrix}$$

$$E = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{7} \end{bmatrix} \begin{bmatrix} -1 & 0 \\ -5 & 1 \end{bmatrix}$$

$$E = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1(-1) + 0(-5) & 1(0) + 0(1) \\ 0(-1) - \frac{1}{7}(-5) & 0(0) - \frac{1}{7}(1) \end{bmatrix}$$

$$E = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ \frac{5}{7} & -\frac{1}{7} \end{bmatrix}$$

$$E=\begin{bmatrix}1(-1)+1\left(\frac{5}{7}\right) & 1(0)+1\left(-\frac{1}{7}\right)\\ 0(-1)+1\left(\frac{5}{7}\right) & 0(0)+1\left(-\frac{1}{7}\right)\end{bmatrix}$$

$$E=\begin{bmatrix}-\frac{2}{7} & -\frac{1}{7}\\ \frac{5}{7} & -\frac{1}{7}\end{bmatrix}$$

■ 4. Find the single elimination matrix E that puts A into reduced row-echelon form, where E accounts for the given set of row operations.

$$A=\begin{bmatrix}-3 & 6\\ 1 & 2\end{bmatrix}$$

1. $-\dfrac{1}{3}R_1 \to R_1$

2. $-R_1+R_2 \to R_2$

3. $\dfrac{1}{4}R_2 \to R_2$

4. $2R_2+R_1 \to R_1$

Solution:

The row operation $-(1/3)R_1 \to R_1$ means we'll put a $-(1/3)$ in $E_{1,1}$.

$$E_1 = \begin{bmatrix} -\frac{1}{3} & 0 \\ 0 & 1 \end{bmatrix}$$

The row operation $-R_1 + R_2 \to R_2$ means we'll put a 1 in $E_{2,2}$ and a -1 in $E_{2,1}$.

$$E_2 = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$$

The row operation $(1/4)R_2 \to R_2$ means we'll put a $1/4$ in $E_{2,2}$.

$$E_3 = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{4} \end{bmatrix}$$

The row operation $2R_2 + R_1 \to R_1$ means we'll put a 1 in $E_{1,1}$ and a 2 in $E_{1,2}$.

$$E_4 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

Consolidate these four row operations into one elimination matrix by multiplying E_4 by E_3 by E_2 by E_1.

$$E = E_4 E_3 E_2 E_1$$

$$E = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{4} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} -\frac{1}{3} & 0 \\ 0 & 1 \end{bmatrix}$$

$$E = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{4} \end{bmatrix} \begin{bmatrix} 1\left(-\frac{1}{3}\right) + 0(0) & 1(0) + 0(1) \\ -1\left(-\frac{1}{3}\right) + 1(0) & -1(0) + 1(1) \end{bmatrix}$$

$$E=\begin{bmatrix}1 & 2\\ 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & \frac{1}{4}\end{bmatrix}\begin{bmatrix}-\frac{1}{3} & 0\\ \frac{1}{3} & 1\end{bmatrix}$$

$$E=\begin{bmatrix}1 & 2\\ 0 & 1\end{bmatrix}\begin{bmatrix}1\left(-\frac{1}{3}\right)+0\left(\frac{1}{3}\right) & 1(0)+0(1)\\ 0\left(-\frac{1}{3}\right)+\frac{1}{4}\left(\frac{1}{3}\right) & 0(0)+\frac{1}{4}(1)\end{bmatrix}$$

$$E=\begin{bmatrix}1 & 2\\ 0 & 1\end{bmatrix}\begin{bmatrix}-\frac{1}{3} & 0\\ \frac{1}{12} & \frac{1}{4}\end{bmatrix}$$

$$E=\begin{bmatrix}1\left(-\frac{1}{3}\right)+2\left(\frac{1}{12}\right) & 1(0)+2\left(\frac{1}{4}\right)\\ 0\left(-\frac{1}{3}\right)+1\left(\frac{1}{12}\right) & 0(0)+1\left(\frac{1}{4}\right)\end{bmatrix}$$

$$E=\begin{bmatrix}-\frac{1}{6} & \frac{1}{2}\\ \frac{1}{12} & \frac{1}{4}\end{bmatrix}$$

■ 5. Find the single elimination matrix E that puts X into reduced row-echelon form, where E accounts for the given set of row operations.

$$X=\begin{bmatrix}1 & 0 & 0\\ 3 & 1 & -4\\ -2 & -1 & 5\end{bmatrix}$$

1. $-3R_1+R_2\to R_2$

2. $2R_1 + R_3 \to R_3$

3. $R_2 + R_3 \to R_3$

4. $4R_3 + R_2 \to R_2$

Solution:

The row operation $-3R_1 + R_2 \to R_2$ means we'll put a 1 in $E_{2,2}$ and a -3 in $E_{2,1}$.

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The row operation $2R_1 + R_3 \to R_3$ means we'll put a 1 in $E_{3,3}$ and a 2 in $E_{3,1}$.

$$E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

The row operation $R_2 + R_3 \to R_3$ means we'll put a 1 in $E_{3,3}$ and a 1 in $E_{3,2}$.

$$E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

The row operation $4R_3 + R_2 \to R_2$ means we'll put a 1 in $E_{2,2}$ and a 4 in $E_{2,3}$.

$$E_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$$

Consolidate these four row operations into one elimination matrix by multiplying E_4 by E_3 by E_2 by E_1.

$$E = E_4E_3E_2E_1$$

$$E = \begin{bmatrix}1&0&0\\0&1&4\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\0&1&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\2&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\-3&1&0\\0&0&1\end{bmatrix}$$

$$E = \begin{bmatrix}1&0&0\\0&1&4\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\0&1&1\end{bmatrix}\begin{bmatrix}1(1)+0(-3)+0(0) & 1(0)+0(1)+0(0) & 1(0)+0(0)+0(1)\\0(1)+1(-3)+0(0) & 0(0)+1(1)+0(0) & 0(0)+1(0)+0(1)\\2(1)+0(-3)+1(0) & 2(0)+0(1)+1(0) & 2(0)+0(0)+1(1)\end{bmatrix}$$

$$E = \begin{bmatrix}1&0&0\\0&1&4\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\0&1&1\end{bmatrix}\begin{bmatrix}1&0&0\\-3&1&0\\2&0&1\end{bmatrix}$$

$$E = \begin{bmatrix}1&0&0\\0&1&4\\0&0&1\end{bmatrix}\begin{bmatrix}1(1)+0(-3)+0(2) & 1(0)+0(1)+0(0) & 1(0)+0(0)+0(1)\\0(1)+1(-3)+0(2) & 0(0)+1(1)+0(0) & 0(0)+1(0)+0(1)\\0(1)+1(-3)+1(2) & 0(0)+1(1)+1(0) & 0(0)+1(0)+1(1)\end{bmatrix}$$

$$E = \begin{bmatrix}1&0&0\\0&1&4\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\-3&1&0\\-1&1&1\end{bmatrix}$$

$$E = \begin{bmatrix}1(1)+0(-3)+0(-1) & 1(0)+0(1)+0(1) & 1(0)+0(0)+0(1)\\0(1)+1(-3)+4(-1) & 0(0)+1(1)+4(1) & 0(0)+1(0)+4(1)\\0(1)+0(-3)+1(-1) & 0(0)+0(1)+1(1) & 0(0)+0(0)+1(1)\end{bmatrix}$$

$$E = \begin{bmatrix}1&0&0\\-7&5&4\\-1&1&1\end{bmatrix}$$

■ 6. Find the single elimination matrix E that puts B into reduced row-echelon form, where E accounts for the given set of row operations.

$$B = \begin{bmatrix} 1 & 0 & -5 \\ 3 & 2 & -9 \\ 1 & -2 & -10 \end{bmatrix}$$

1. $-3R_1 + R_2 \to R_2$

2. $-R_1 + R_3 \to R_3$

3. $\frac{1}{2}R_2 \to R_2$

4. $2R_2 + R_3 \to R_3$

5. $-3R_3 + R_2 \to R_2$

6. $5R_3 + R_1 \to R_1$

Solution:

The row operation $-3R_1 + R_2 \to R_2$ means we'll put a 1 in $E_{2,2}$ and a -3 in $E_{2,1}$.

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The row operation $-R_1 + R_3 \to R_3$ means we'll put a 1 in $E_{3,3}$ and a -1 in $E_{3,1}$.

$$E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

The row operation $(1/2)R_2 \to R_2$ means we'll put a $1/2$ in $E_{2,2}$.

$$E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Let's consolidate what we have for $E_3E_2E_1$ so far.

$$E_3E_2E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E_3E_2E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1(1)+0(-3)+0(0) & 1(0)+0(1)+0(0) & 1(0)+0(0)+0(1) \\ 0(1)+1(-3)+0(0) & 0(0)+1(1)+0(0) & 0(0)+1(0)+0(1) \\ -1(1)+0(-3)+1(0) & -1(0)+0(1)+1(0) & -1(0)+0(0)+1(1) \end{bmatrix}$$

$$E_3E_2E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$$E_3E_2E_1 = \begin{bmatrix} 1(1)+0(-3)+0(-1) & 1(0)+0(1)+0(0) & 1(0)+0(0)+0(1) \\ 0(1)+\frac{1}{2}(-3)+0(-1) & 0(0)+\frac{1}{2}(1)+0(0) & 0(0)+\frac{1}{2}(0)+0(1) \\ 0(1)+0(-3)+1(-1) & 0(0)+0(1)+1(0) & 0(0)+0(0)+1(1) \end{bmatrix}$$

$$E_3E_2E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{3}{2} & \frac{1}{2} & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

The row operation $2R_2 + R_3 \to R_3$ means we'll put a 1 in $E_{3,3}$ and a 2 in $E_{3,2}$.

$$E_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$$

The row operation $-3R_3 + R_2 \to R_2$ means we'll put a 1 in $E_{2,2}$ and a -3 in $E_{2,3}$.

$$E_5 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix}$$

The row operation $5R_3 + R_1 \to R_1$ means we'll put a 1 in $E_{1,1}$ and a 5 in $E_{1,3}$.

$$E_6 = \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Let's consolidate what we have for $E_6E_5E_4$.

$$E_6E_5E_4 = \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$$

$$E_6E_5E_4 = \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1(1)+0(0)+0(0) & 1(0)+0(1)+0(2) & 1(0)+0(0)+0(1) \\ 0(1)+1(0)-3(0) & 0(0)+1(1)-3(2) & 0(0)+1(0)-3(1) \\ 0(1)+0(0)+1(0) & 0(0)+0(1)+1(2) & 0(0)+0(0)+1(1) \end{bmatrix}$$

$$E_6E_5E_4 = \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -5 & -3 \\ 0 & 2 & 1 \end{bmatrix}$$

$$E_6E_5E_4 = \begin{bmatrix} 1(1)+0(0)+5(0) & 1(0)+0(-5)+5(2) & 1(0)+0(-3)+5(1) \\ 0(1)+1(0)+0(0) & 0(0)+1(-5)+0(2) & 0(0)+1(-3)+0(1) \\ 0(1)+0(0)+1(0) & 0(0)+0(-5)+1(2) & 0(0)+0(-3)+1(1) \end{bmatrix}$$

$$E_6E_5E_4 = \begin{bmatrix} 1 & 10 & 5 \\ 0 & -5 & -3 \\ 0 & 2 & 1 \end{bmatrix}$$

Then the elimination matrix is

$$E = E_6E_5E_4E_3E_2E_1$$

$$E = \begin{bmatrix} 1 & 10 & 5 \\ 0 & -5 & -3 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -\frac{3}{2} & \frac{1}{2} & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$$E = \begin{bmatrix} 1(1) + 10\left(-\frac{3}{2}\right) + 5(-1) & 1(0) + 10\left(\frac{1}{2}\right) + 5(0) & 1(0) + 10(0) + 5(1) \\ 0(1) - 5\left(-\frac{3}{2}\right) - 3(-1) & 0(0) - 5\left(\frac{1}{2}\right) - 3(0) & 0(0) - 5(0) - 3(1) \\ 0(1) + 2\left(-\frac{3}{2}\right) + 1(-1) & 0(0) + 2\left(\frac{1}{2}\right) + 1(0) & 0(0) + 2(0) + 1(1) \end{bmatrix}$$

$$E = \begin{bmatrix} -19 & 5 & 5 \\ \frac{21}{2} & -\frac{5}{2} & -3 \\ -4 & 1 & 1 \end{bmatrix}$$