Number of solutions to the linear system

We already know how to solve linear systems using Gaussian elimination to put a matrix into reduced row-echelon form. But up to now, we've only looked at systems with exactly one solution. In fact, systems can have:

- one solution (called the unique solution), or
- no solutions, or
- infinitely many solutions.

The unique solution

We've seen that a unique solution is produced when our reduced row-echelon matrix turns out like this, as an example:

$$\begin{bmatrix} 1 & 0 & 0 & | & a \\ 0 & 1 & 0 & | & b \\ 0 & 0 & 1 & | & c \end{bmatrix}$$

In this form, we get the unique solution to the system $(x, y, z) = (a, b, c)$. There *is* a solution to the system, and there is only one solution to the system, and it's the point (a, b, c). This is an example with a three-dimensional system, but the same is true for any n-dimensional system.

Infinitely many or no solutions

Sometimes you'll get the matrix in reduced row-echelon form, and you'll end up with something like this:

$$\begin{bmatrix} 1 & 0 & 0 & | & a \\ 0 & 1 & 0 & | & b \\ 0 & 0 & 0 & | & c \end{bmatrix}$$

The difference here is that we have a row of zeros at the bottom of the matrix. Here, if c is nonzero, then you'll have a last row that tells you

$$0z = c$$

$$0 = c$$

Remember, we just said c is nonzero. But the equation

$$0 = \text{some non-zero value}$$

can never be true. Because it can never be true, you can conclude that the system has no solutions. In a two-dimensional system, that means you're looking at parallel lines; in a three-dimensional system, that means you're looking at parallel planes.

When there are infinitely many solutions to a system, you might end up with a reduced row-echelon matrix that looks something like this:

$$\begin{bmatrix} 1 & 0 & 0 & | & a \\ 0 & 1 & -3 & | & b \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

The zero row at the bottom (zeros across the entire last row) tells you that there will be infinitely many solutions to the system.

Pivot entries and free entries

You already know that the 1's in the first and second row of

$$\begin{bmatrix}1 & 0 & 0 & | & a \\ 0 & 1 & -3 & | & b \\ 0 & 0 & 0 & | & 0\end{bmatrix}$$

are called the **pivot entries** in the matrix. Any other non-zero values on the left side of the matrix, in this case the -3 in the second row, are called **free entries**.

You can almost think about the free entries as independent variables, in the sense that they can be set equal to anything, and you'll still be able to find a solution to the system. The values of the pivot entries are like dependent variables in the sense that their values will depend on the values you set for the free entries.

Example

Say whether the system has one solution, no solutions, or infinitely many solutions.

$$-x-5y+z=17$$

$$-5x-5y+5z=5$$

$$2x+5y-3z=-10$$

Rewrite the system as an augmented matrix.

$$\begin{bmatrix} -1 & -5 & 1 & | & 17 \\ -5 & -5 & 5 & | & 5 \\ 2 & 5 & -3 & | & -10 \end{bmatrix}$$

Work toward putting the matrix into reduced row-echelon form, starting with finding the pivot entry in the first row. To do that, multiply the first row by -1, or $-R_1 \to R_1$.

$$\begin{bmatrix} 1 & 5 & -1 & | & -17 \\ -5 & -5 & 5 & | & 5 \\ 2 & 5 & -3 & | & -10 \end{bmatrix}$$

Zero out the rest of the first column. First perform $5R_1 + R_2 \to R_2$, then $-2R_1 + R_3 \to R_3$.

$$\begin{bmatrix} 1 & 5 & -1 & | & -17 \\ 0 & 20 & 0 & | & -80 \\ 2 & 5 & -3 & | & -10 \end{bmatrix} \to \begin{bmatrix} 1 & 5 & -1 & | & -17 \\ 0 & 20 & 0 & | & -80 \\ 0 & -5 & -1 & | & 24 \end{bmatrix}$$

Find the pivot entry in the second row with $(1/20)R_2 \to R_2$.

$$\begin{bmatrix} 1 & 5 & -1 & | & -17 \\ 0 & 1 & 0 & | & -4 \\ 0 & -5 & -1 & | & 24 \end{bmatrix}$$

Zero out the rest of the second column. First perform $-5R_2 + R_1 \to R_1$, then $5R_2 + R_3 \to R_3$.

$$\begin{bmatrix} 1 & 0 & -1 & | & 3 \\ 0 & 1 & 0 & | & -4 \\ 0 & -5 & -1 & | & 24 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 & | & 3 \\ 0 & 1 & 0 & | & -4 \\ 0 & 0 & -1 & | & 4 \end{bmatrix}$$

Find the pivot entry in the third row with $-R_3 \to R_3$.

$$\begin{bmatrix} 1 & 0 & -1 & | & 3 \\ 0 & 1 & 0 & | & -4 \\ 0 & 0 & 1 & | & -4 \end{bmatrix}$$

Zero out the rest of the third column.

$$\begin{bmatrix} 1 & 0 & 0 & | & -1 \\ 0 & 1 & 0 & | & -4 \\ 0 & 0 & 1 & | & -4 \end{bmatrix}$$

Therefore, there is one unique solution to the system, which is $(x, y, z) = (-1, -4, -4)$.

Let's do another example where we get a different result.

Example

Say whether the system has one solution, no solutions, or infinitely many solutions.

$$3a - 3b + 4c = -23$$

$$a + 2b - 3c = 25$$

$$4a - b + c = 25$$

Rewrite the system as an augmented matrix.

$$\begin{bmatrix} 3 & -3 & 4 & | & -23 \\ 1 & 2 & -3 & | & 25 \\ 4 & -1 & 1 & | & 25 \end{bmatrix}$$

Work toward putting the matrix into reduced row-echelon form, starting with switching the first and second rows.

$$\begin{bmatrix} 1 & 2 & -3 & | & 25 \\ 3 & -3 & 4 & | & -23 \\ 4 & -1 & 1 & | & 25 \end{bmatrix}$$

Zero out the rest of the first column. First perform $-3R_1 + R_2 \to R_2$, then $-4R_1 + R_3 \to R_3$.

$$\begin{bmatrix} 1 & 2 & -3 & | & 25 \\ 0 & -9 & 13 & | & -98 \\ 4 & -1 & 1 & | & 25 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & -3 & | & 25 \\ 0 & -9 & 13 & | & -98 \\ 0 & -9 & 13 & | & -75 \end{bmatrix}$$

Perform $R_3 - R_2 \to R_3$.

$$\begin{bmatrix} 1 & 2 & -3 & | & 25 \\ 0 & -9 & 13 & | & -98 \\ 0 & 0 & 0 & | & 23 \end{bmatrix}$$

We don't have to go any further. The third row now tells us that $0 = 23$, which can't possibly be true. Therefore, the system has no solution.