SECTION 4.1 Real Vector Spaces (Numeralization is from the book)

In this section we will extend the concept of a vector by using the basic properties of vectors in $\mathbb{R}^n$ as axioms, which if satisfied by a set of objects, guarantee that those objects behave like familiar vectors.

Vector Space Axioms

The following definition consists of ten axioms, eight of which are properties of vectors in $\mathbb{R}^n$ that are stated in Theorem 3.1.1. It is important to keep in mind that one does not prove axioms; rather, they are assumptions that serve as the starting point for proving theorems.

DEFINITION 1 Let V be an arbitrary nonempty set of objects on which two operations are defined: addition, and multiplication by numbers called scalars. By addition we mean a rule for associating with each pair of objects $\mathbf{u}$ and $\mathbf{v}$ in V an object $\mathbf{u}+\mathbf{v}$, called the sum of $\mathbf{u}$ and $\mathbf{v}$; by scalar multiplication we mean a rule for associating with each scalar k and each object $\mathbf{u}$ in V an object $k\mathbf{u}$, called the scalar multiple of $\mathbf{u}$ by k. If the following axioms are satisfied by all objects $\mathbf{u}, \mathbf{v}, \mathbf{w}$ in V and all scalars k and m, then we call V a vector space and we call the objects in V vectors.

1. If $\mathbf{u}$ and $\mathbf{v}$ are in V, then $\mathbf{u}+\mathbf{v}$ is in V.
2. $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$
3. $(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w})$
4. There is an object $\mathbf{0}$ in V, called a zero vector for V, such that $\mathbf{0}+\mathbf{u}=\mathbf{u}+\mathbf{0}=\mathbf{u}$ for all $\mathbf{u}$ in V.
5. For each $\mathbf{u}$ in V, there is an object $-\mathbf{u}$ in V, called a negative of $\mathbf{u}$, such that $\mathbf{u}+(-\mathbf{u})=(-\mathbf{u})+\mathbf{u}=\mathbf{0}$.
6. If k is any scalar and $\mathbf{u}$ is any object in V, then $k\mathbf{u}$ is in V.
7. $k(\mathbf{u}+\mathbf{v})=k\mathbf{u}+k\mathbf{v}$
8. $(k+m)\mathbf{u}=k\mathbf{u}+m\mathbf{u}$
9. $(km)\mathbf{u}=k(m\mathbf{u})$
10. $1\mathbf{u}=\mathbf{u}$

Observe that the definition of a vector space does not specify the nature of the vectors or the operations. Any kind of object can be a vector, and the operations of addition and scalar multiplication need not have any relationship to those on $\mathbb{R}^n$. The only requirement is that the ten vector space axioms be satisfied. In the examples that follow we will use four basic steps to show that a set with two operations is a vector space.

To Show That a Set with Two Operations Is a Vector Space

Step 1. Identify the set V of objects that will become vectors.

Step 2. Identify the addition and scalar multiplication operations on V.

Step 3. Verify Axioms 1 and 6; that is, adding two vectors in V produces a vector in V, and multiplying a vector in V by a scalar also produces a vector in V. Axiom 1 is called closure under addition, and Axiom 6 is called closure under scalar multiplication.

Step 4. Confirm that Axioms 2, 3, 4, 5, 7, 8, 9, and 10 hold.

EXAMPLE 1 The Zero Vector Space

Let V consist of a single object, which we denote by $\mathbf{0}$, and define $\mathbf{0}+\mathbf{0}=\mathbf{0}$ and $k\mathbf{0}=\mathbf{0}$ for all scalars k. It is easy to check that all the vector space axioms are satisfied. We call this the zero vector space.

Our second example is one of the most important of all vector spaces the familiar space $\mathbb{R}^n$. It should not be surprising that the operations on $\mathbb{R}^n$ satisfy the vector space axioms because those axioms were based on known properties of operations on $\mathbb{R}^n$.

EXAMPLE 2 $\mathbb{R}^n$ is a Vector Space

Let $V=\mathbb{R}^n$, and define the vector space operations on V to be the usual operations of addition and scalar multiplication of n-tuples; that is,

$$\mathbf{u}+\mathbf{v}=(u_1,\dots,u_n)+(v_1,\dots,v_n)=(u_1+v_1,\dots,u_n+v_n); \tag{1}$$

$$k\mathbf{u}=(ku_1,\dots,ku_n) \tag{2}$$

The set V is closed under addition and scalar multiplication because the foregoing operations produce n-tuples as their end result, and these operations satisfy Axioms 2, 3, 4, 5, 7, 8, 9, and 10 by virtue of Theorem 3.1.1.

Our next example is a generalization of $\mathbb{R}^n$ in which we allow vectors to have infinitely many components.

EXAMPLE 3 Vector Space of Infinite Sequences

Let V consist of objects of the form $\mathbf{u}=(u_1,\dots,u_n,\dots)$ where $u_1,\dots,u_n,\dots$ is an infinite sequence of real numbers. We define two infinite sequences to be equal if their corresponding components are equal, and we define addition and scalar multiplication componentwise alike to $\mathbb{R}^n$. In the exercises we ask you to confirm that V with these operations is a vector space. We will denote this vector space by the symbol $\mathbb{R}^\infty$.

REVIEW FROM YOUR BOOK!

1.) **EXAMPLE 4 The Vector Space of $\mathbf{2} \times \mathbf{2}$ Matrices**

2.) **EXAMPLE 5 The Vector Space of $\mathbf{m} \times \mathbf{n}$ Matrices**

3.) **EXAMPLE 6 The Vector Space of Real-Valued Functions**

This example is defined as follows. Let V the set of real-valued functions that are defined at each x in the interval $(-\infty, \infty)$. If $\mathbf{f} = f(x)$ and $\mathbf{g} = g(x)$ are two functions in V and if k is any scalar, then define the operations of addition and scalar multiplication by $(\mathbf{f} + \mathbf{g}) = f(x) + g(x)$ and $k\mathbf{f} = kf(x)$. Review the proof of that V is a vector space.

4.) **THEOREM 4.1.1** Let V be a vector space, $\mathbf{u}$ a vector in V, and k a scalar; then:

(a) $0\mathbf{u} = \mathbf{0}$

(b) $0\mathbf{0} = \mathbf{0}$

(c) $(-1)\mathbf{u} = -\mathbf{u}$

(d) If $k\mathbf{u} = \mathbf{0}$, then $k = 0$ or $\mathbf{u} = 0$.

Review the proof of this statement.

HOMEWORK(It is Exercise 1 in the book; You can find there the answer):

Let V the set of all ordered pairs of real numbers, and consider the following addition and scalar multiplication operations on $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$: $\mathbf{u} + \mathbf{v} = (u_1 + v_1, u_2 + v_2)$ and $k\mathbf{u} = (0, ku_2)$.

(a) Compute $\mathbf{u} + \mathbf{v}$ and $k\mathbf{u}$ for $\mathbf{u} = (-1, 2)$, and $\mathbf{v} = (3, 4)$.

(b) In words, explain why V is closed under addition and scalar multiplication.

(c) Since addition on V is the standard addition operation on $\mathbb{R}^2$, certain vector space axioms hold for V because they are known to hold for $\mathbb{R}^2$. Which axioms are they?

(d) Show that Axioms 7, 8, and 9 hold.

(e) Show that Axiom 10 fails and hence that V is not a vector space under the given operations.