MATH 313: LINEAR ALGEBRA - HOMEWORK 5

MELVYN B. NATHANSON

(1) Learn the precise definitions of the following:
   (a) Group
   (b) Abelian group
   (c) Ring
   (d) Commutative ring
   (e) Field
   (f) Vector space

(2) Consider the matrices
$$C = \begin{pmatrix} 9 & -1 & 4 \\ 3 & 0 & 2 \end{pmatrix} \qquad \text{and} \qquad D = \begin{pmatrix} 1 & -13 \\ -2 & 5 \\ 0 & 8 \end{pmatrix}.$$
Compute the following matrix transposes:
$$C^t, \quad D^t, \quad C^tD^t, \quad (CD)^t, \quad D^tC^t.$$

(3) Compute (if possible) the inverses of the following matrices:
$$\begin{pmatrix} 2 & 7 \\ 3 & 5 \end{pmatrix}, \quad \begin{pmatrix} 2 & 6 \\ 3 & 5 \end{pmatrix}, \quad \begin{pmatrix} 2 & 6 \\ 3 & 9 \end{pmatrix}$$

(4) The *integer special linear group* $SL_2(\mathbf{Z})$ is the set of all 2×2 matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that $a, b, c, d \in \mathbf{Z}$ and $ad - bc = 1$.
   (a) Prove that if $A, B \in SL_2(\mathbf{Z})$, then $AB \in SL_2(\mathbf{Z})$.
   (b) Prove that if $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbf{Z})$, then A is invertible and $A^{-1} \in SL_2(\mathbf{Z})$. Compute A^{-1}.

(5) Let $SL_2(\mathbf{R})$ be the set of all 2×2 matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Mat}_2(\mathbf{R})$ such that $ad - bc = 1$. For $\theta \in \mathbf{R}$, let
$$R(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.$$
   (a) Prove that $R(\theta) \in SL_2(\mathbf{R})$.
   (b) Prove that, for all $\theta_1, \theta_2, \theta \in \mathbf{R}$,
$$R(\theta_1)R(\theta_2) = R(\theta_1 + \theta_2)$$
   and
$$R(\theta)^{-1} = R(-\theta).$$

The *rotation group* $SO(2)$ is the set of matrices $\{R(\theta) : \theta \in \mathbf{R}\}$.

(6) The *affine group* $\mathcal{A}(\mathbf{R})$ is the set of all matrices of the form $\begin{pmatrix} r & x \\ 0 & 1 \end{pmatrix}$ with $r \in \mathbf{R} \setminus \{0\}$ and $x \in \mathbf{R}$.
   (a) Prove that if $A, B \in \mathcal{A}(\mathbf{R})$, then $AB \in \mathcal{A}$.
   (b) Prove that if $A = \begin{pmatrix} r & x \\ 0 & 1 \end{pmatrix} \in \mathcal{A}(\mathbf{R})$, then A is invertible and $A^{-1} \in \mathcal{A}(\mathbf{R})$. Compute A^{-1}.

(7) The *Heisenberg group* $H_3(\mathbf{R})$ is the set of all 3×3 matrices of the form $\begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}$ with $a, b, c \in \mathbf{R}$.
   (a) Prove that if $A, B \in H_3(\mathbf{R})$, then $AB \in H_3(\mathbf{R})$.
   (b) Prove that if $A \in H_3(\mathbf{R})$, then A is invertible and $A^{-1} \in H_3(\mathbf{R})$. Compute A^{-1}.

(8) Solve the following systems of linear equations by using elementary row operations to put the augmented matrix into reduced row echelon form.
   (a)
$$\begin{array}{rcrcr} 9x & - & 4y & = & 1 \\ -11x & + & 5y & = & 3 \end{array}$$
   (b)
$$\begin{array}{rcrcrcr} x & + & 4y & + & 3z & = & 2 \\ 2x & + & y & + & z & = & 3 \\ 3x & - & 2y & - & z & = & 4 \end{array}$$

(9) Compute the reduced row echelon form of the matrix
$$A = \begin{pmatrix} 2 & 0 & 5 & -1 \\ 1 & -1 & 10 & 5 \\ 3 & 1 & 0 & -7 \end{pmatrix}$$

(10) Let A be an $n \times n$ matrix. Let k and ℓ be postive integers. Use mathematical induction to prove that
$$A^k A^\ell = A^{k+\ell}.$$