Topic: Basis

Question: Does the span of V form a basis for $\mathbb{R}^2$?

$$V = \left\{ \begin{bmatrix} -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$$

Answer choices:

A Yes

B No, because the vectors in V don't span $\mathbb{R}^2$

C No, because the vectors in V aren't linearly independent

D No, because the vectors in V don't span $\mathbb{R}^2$ and aren't linearly independent

Solution: A

In order for V to form a basis for $\mathbb{R}^2$,

1. the vectors in V need to span $\mathbb{R}^2$, and

2. the vectors in V need to be linearly independent.

To span $\mathbb{R}^2$, we need to be able to get any vector in $\mathbb{R}^2$ using a linear combination of the vectors in the set. In other words,

$$c_1 \begin{bmatrix} -2 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

From this linear combination, we can build a system of equations.

$$-2c_1 + c_2 = x$$

$$-c_2 = y$$

We'll substitute $c_2 = -y$ into the first equation in the system, and then solve that equation for c_1.

$$-2c_1 - y = x$$

$$-2c_1 = y + x$$

$$c_1 = -\frac{1}{2}y - \frac{1}{2}x$$

From this process we can conclude that, given any vector $\vec{v} = (x, y)$ in $\mathbb{R}^2$, we can "get to it" using the values of c_1 and c_2 given by

$$c_1 = -\frac{1}{2}y - \frac{1}{2}x$$

$$c_2 = -y$$

It doesn't matter which vector we pick in $\mathbb{R}^2$. If we use the values of x and y that we want, and plug them into these equations for c_1 and c_2, we'll get the values of c_1 and c_2 that we need to use in the linear combination in order to arrive at the vector $\vec{v} = (x, y)$. These formulas for c_1 and c_2 won't break, regardless of which (x, y) we pick for the vector, so the vector set V spans $\mathbb{R}^2$.

Then to show that the vectors in V are linearly independent, we'll set $(x, y) = (0,0)$.

$$c_1 \begin{bmatrix} -2 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

When we do, we get

$$c_1 = -\frac{1}{2}(0) - \frac{1}{2}(0), \text{ or } c_1 = 0$$

$$c_2 = -0, \text{ or } c_2 = 0$$

Because the only values of c_1 and c_2 that give the zero vector are $c_1 = 0$ and $c_2 = 0$, we know that the vectors in V are linearly independent.

Therefore, because the vector set V spans all of $\mathbb{R}^2$, and because the vectors in V are linearly independent, we can say that V forms a basis for $\mathbb{R}^2$.

Topic: Basis

Question: Does the span of V form a basis for $\mathbb{R}^2$?

$$V = \left\{ \begin{bmatrix} 3 \\ -1 \end{bmatrix}, \begin{bmatrix} -6 \\ 2 \end{bmatrix} \right\}$$

Answer choices:

A Yes

B No, because the vectors in V don't span $\mathbb{R}^2$

C No, because the vectors in V aren't linearly independent

D No, because the vectors in V don't span $\mathbb{R}^2$ and aren't linearly independent

Solution: D

In order for V to form a basis for $\mathbb{R}^2$,

1. the vectors in V need to span $\mathbb{R}^2$, and

2. the vectors in V need to be linearly independent.

To span $\mathbb{R}^2$, we need to be able to get any vector in $\mathbb{R}^2$ using a linear combination of the vectors in the set. In other words,

$$c_1 \begin{bmatrix} 3 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} -6 \\ 2 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

From this linear combination, we can build a system of equations.

$$3c_1 - 6c_2 = x$$

$$-c_1 + 2c_2 = y$$

Multiply the second equation by 3 to get $-3c_1 + 6c_2 = 3y$. Add the equations.

$$3c_1 - 6c_2 + (-3c_1 + 6c_2) = x + (3y)$$

$$3c_1 - 6c_2 - 3c_1 + 6c_2 = x + 3y$$

$$-6c_2 + 6c_2 = x + 3y$$

$$0 = x + 3y$$

$$3y = -x$$

From this process we can conclude that we can't get to any vector $\overrightarrow{v} = (x, y)$ in $\mathbb{R}^2$. It doesn't matter which c_1 and c_2 we choose, the relationship between x and y is always given by $3y = -x$. For instance, if $y = 1$, x can only be $x = -3$. Which means we have no way of "getting to" any other $(x,1)$. So the vectors don't span $\mathbb{R}^2$.

Then to show that the vectors in V are linearly independent, we'll set $(x, y) = (0,0)$.

$$c_1 \begin{bmatrix} 3 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} -6 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

When we do, we get

$$3c_1 - 6c_2 = 0$$

$$-c_1 + 2c_2 = 0$$

If we divide $3c_1 - 6c_2 = 0$ by -3, we get $-c_1 + 2c_2 = 0$. Then when we subtract this from the second equation, we get

$$-c_1 + 2c_2 - (-c_1 + 2c_2) = 0 - (0)$$

$$-c_1 + 2c_2 + c_1 - 2c_2 = 0 - 0$$

$$0 = 0$$

This tells us that any values of c_1 and c_2 will satisfy the vector equation, not just $c_1 = 0$ and $c_2 = 0$. So we know that the vectors in V aren't linearly independent.

Topic: Basis

Question: Does the span of V form a basis for $\mathbb{R}^3$?

$$V = \left\{ \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -3 \\ 1 \end{bmatrix} \right\}$$

Answer choices:

A Yes

B No, because the vectors in V don't span $\mathbb{R}^3$

C No, because the vectors in V aren't linearly independent

D No, because the vectors in V don't span $\mathbb{R}^3$ and aren't linearly independent

Solution: A

In order for V to form a basis for $\mathbb{R}^3$,

1. the vectors in V need to span $\mathbb{R}^3$, and

2. the vectors in V need to be linearly independent.

To span $\mathbb{R}^3$, we need to be able to get any vector in $\mathbb{R}^3$ using a linear combination of the vectors in the set. In other words,

$$c_1\begin{bmatrix}1\\0\\-2\end{bmatrix}+c_2\begin{bmatrix}-1\\1\\0\end{bmatrix}+c_3\begin{bmatrix}0\\-3\\1\end{bmatrix}=\begin{bmatrix}x\\y\\z\end{bmatrix}$$

From this linear combination, we can build a system of equations.

$$c_1-c_2=x$$

$$c_2-3c_3=y$$

$$-2c_1+c_3=z$$

Solve the system with a matrix, where each column in the augmented matrix represents the "variables" c_1, c_2, and c_3. Then put the matrix into reduced row-echelon form.

$$\begin{bmatrix}1 & -1 & 0 & | & x\\0 & 1 & -3 & | & y\\-2 & 0 & 1 & | & z\end{bmatrix}\to\begin{bmatrix}1 & -1 & 0 & | & x\\0 & 1 & -3 & | & y\\0 & -2 & 1 & | & 2x+z\end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -3 & | & x+y \\ 0 & 1 & -3 & | & y \\ 0 & -2 & 1 & | & 2x+z \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -3 & | & x+y \\ 0 & 1 & -3 & | & y \\ 0 & 0 & -5 & | & 2x+2y+z \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -3 & | & x+y \\ 0 & 1 & -3 & | & y \\ 0 & 0 & 1 & | & -\frac{2}{5}x-\frac{2}{5}y-\frac{1}{5}z \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & | & -\frac{1}{5}x-\frac{1}{5}y-\frac{3}{5}z \\ 0 & 1 & -3 & | & y \\ 0 & 0 & 1 & | & -\frac{2}{5}x-\frac{2}{5}y-\frac{1}{5}z \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & | & -\frac{1}{5}x-\frac{1}{5}y-\frac{3}{5}z \\ 0 & 1 & 0 & | & -\frac{6}{5}x-\frac{1}{5}y-\frac{3}{5}z \\ 0 & 0 & 1 & | & -\frac{2}{5}x-\frac{2}{5}y-\frac{1}{5}z \end{bmatrix}$$

From this process we can conclude that, given any vector $\vec{v}=(x,y,z)$ in $\mathbb{R}^3$, we can "get to it" using the values of c_1, c_2, and c_3 given by

$$c_1 = -\frac{1}{5}x - \frac{1}{5}y - \frac{3}{5}z$$

$$c_2 = -\frac{6}{5}x - \frac{1}{5}y - \frac{3}{5}z$$

$$c_3 = -\frac{2}{5}x - \frac{2}{5}y - \frac{1}{5}z$$

It doesn't matter which vector we pick in $\mathbb{R}^3$. If we use the values of x, y, and z that we want, and plug them into these equations for c_1, c_2, and c_3, we'll get the values of c_1, c_2, and c_3 that we need to use in the linear combination in order to arrive at the vector $\vec{v}=(x,y,z)$. These formulas for

c_1, c_2, and c_3 won't break, regardless of which (x, y, z) we pick for the vector, so the vector set V spans $\mathbb{R}^3$.

Then to show that the vectors in V are linearly independent, we'll set $(x, y, z) = (0,0,0)$.

$$c_1 \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ -3 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

When we do, we get

$$\begin{bmatrix} 1 & -1 & 0 & | & 0 \\ 0 & 1 & -3 & | & 0 \\ -2 & 0 & 1 & | & 0 \end{bmatrix}$$

When we put the matrix into reduced row-echelon form, we get

$$\begin{bmatrix} 1 & 0 & 0 & | & 0 \\ 0 & 1 & 0 & | & 0 \\ 0 & 0 & 1 & | & 0 \end{bmatrix}$$

Which means $c_1 = 0$, $c_2 = 0$, and $c_3 = 0$. Because the only values of c_1, c_2, and c_3 that give the zero vector are $c_1 = 0$, $c_2 = 0$, and $c_3 = 0$, we know that the vectors in V are linearly independent.

Therefore, because the vector set V spans all of $\mathbb{R}^3$, and because the vectors in V are linearly independent, we can say that V forms a basis for $\mathbb{R}^3$.