MATH 313: LINEAR ALGEBRA
HOMEWORK 3

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(1) Determine if the following sequences of vectors $\mathbf{v}_1, \mathbf{v}_2$ are linearly dependent or linearly independent.

(a)
$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \qquad \mathbf{v}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

(b)
$$\mathbf{v}_1 = \begin{pmatrix} -4 \\ 7 \end{pmatrix} \qquad \mathbf{v}_2 = \begin{pmatrix} -4 \\ 7 \end{pmatrix}.$$

(2) Determine if the following sequences of vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly dependent or linearly independent.

(a)
$$\mathbf{v}_1 = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 0 \\ 5 \\ 0 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 0 \\ 0 \\ 10 \end{pmatrix}.$$

(b)
$$\mathbf{v}_1 = \begin{pmatrix} 5 \\ -2 \\ 1 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ -2 \\ 5 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 1 \\ 1 \\ -4 \end{pmatrix}.$$

(3) Consider the vectors
$$\mathbf{v}_1 = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}, \qquad \mathbf{v}_2 = \begin{pmatrix} -4 \\ -3 \\ -1 \end{pmatrix}, \qquad \mathbf{v}_3 = \begin{pmatrix} 6 \\ 9 \\ 8 \end{pmatrix}, \qquad \mathbf{v}_4 = \begin{pmatrix} 10 \\ 21 \\ 9 \end{pmatrix}$$
in $\mathbf{R}^3$. Compute scalars x_1, x_2, x_3, x_4 not all 0 such that
$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 + x_4\mathbf{v}_4 = \mathbf{0}.$$

(4) Find two distinct representations of the vector $\begin{pmatrix} 7 \\ -3 \\ 2 \end{pmatrix}$ as a linear combination of the vectors $\begin{pmatrix} 5 \\ -2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \\ 5 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -4 \end{pmatrix}$.

(5) Prove that the set of vectors $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly independent if and only if the set of vectors $\{\mathbf{v}_1 + \mathbf{v}_2, \mathbf{v}_1 - \mathbf{v}_2\}$ is linearly independent

(6) Let $\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2$ be three vectors in $\mathbf{R}^n$ such that the two vectors $\mathbf{v}_1 - \mathbf{v}_0$ and $\mathbf{v}_2 - \mathbf{v}_0$ are linearly independent. Prove that the two vectors $\mathbf{v}_0 - \mathbf{v}_1$ and $\mathbf{v}_2 - \mathbf{v}_1$ are linearly independent.

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(7) Let
$$\mathbf{w}_1 = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \qquad \text{and} \qquad \mathbf{w}_2 = \begin{pmatrix} -3 \\ 2 \end{pmatrix}.$$
Let
$$\mathbf{v}_1 = 2\mathbf{w}_1 + 5\mathbf{w}_2, \qquad \mathbf{v}_2 = 3\mathbf{w}_1 - \mathbf{w}_2, \qquad \mathbf{v}_3 = \mathbf{w}_1 + 7\mathbf{w}_2.$$
Compute scalars x_1, x_2, x_3 such that
$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 = \mathbf{0}.$$

(8) Let $V = \mathbf{R}^n$. Let W_1 and W_2 be subspaces of V. Prove that $W_1 \cap W_2$ is a subspace of V.

(9) Let W_1 be the subspace of $\mathbf{R}^2$ generated by the vector $\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, and let W_2 be the subspace of $\mathbf{R}^2$ generated by the vector $\mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.
  (a) Compute $W_1 \cap W_2$.
  (b) Prove that $W_1 \cup W_2$ is not a subspace of $\mathbf{R}^2$.

(10) For each of the following subspaces of $\mathbf{R}^2$, construct a basis and determine the dimension.
  (a) The subspace generated by the set
$$S = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}.$$
  (b) The subspace generated by the set
$$S = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}.$$
  (c) The subspace generated by the set
$$S = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\}.$$

(11) For each of the following subspaces of $\mathbf{R}^3$, construct a basis and determine the dimension.
  (a) The subspace generated by the set
$$S = \left\{ \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 5 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} \right\}.$$
  (b) The subspace generated by the set
$$S = \left\{ \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 5 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 3 \\ 0 \end{pmatrix} \right\}.$$
  (c) The subspace generated by the set
$$S = \left\{ \begin{pmatrix} 6 \\ 9 \\ -3 \end{pmatrix}, \begin{pmatrix} -4 \\ -6 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \right\}.$$

(12) Let $\mathbf{v}$ be a nonzero vector in $\mathbf{R}^n$. Prove that $\{x\mathbf{v} : x \in \mathbf{R}\}$ is a one-dimensional subspace of $\mathbf{R}^n$.