Marginal Distribution of an Element of a Matrix-T R.V.

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Let $X \sim MatT(\mu, R, Q, \nu)$, where X and μ are $p \times q$ matrices, R is a $p \times p$ matrix, and Q is a $q \times q$ matrix. The degrees of freedom is $\nu \geq p + q$. In the sequel, define $c = \nu - p - q + 1$.

Let a be a q-dimensional vector and let b be a p-dimensional vector. Define

$$t = \frac{ca'(X-\mu)'b}{(a'Qa)(b'Rb)} \tag{1}$$

From Kotz and Nadarajah (2004, p. 116), t has a Student's t distribution with c degrees of freedom. Therefore,

$$f(t) = \frac{\Gamma(\frac{c+1}{2})}{\Gamma(\frac{c}{2})\sqrt{c\pi}} \left(1 + \frac{t^2}{c}\right)^{-\frac{c+1}{2}} \tag{2}$$

Suppose a_j is all zeros, except for a 1 in the j^{th} element, and suppose b_i is all zeros, except for a 1 in the i^{th} element. That means that

$$\begin{aligned}
a'(X-\mu)'b &= x_{ij} - \mu_{ij} && (3)\\
a'Qa &= Q_{jj} && (4)\\
b'Rb &= R_{ii} && (5)\\
t &= \frac{c(x_{ij}-\mu_{ij})}{R_{ii}Q_{jj}} && (6)\\
\frac{dt}{dx_{ij}} &= \frac{c}{R_{ii}Q_{jj}} && (7)
\end{aligned}$$

The elements R_{ii} and Q_{jj} are diagonal elements from the covariance matrices, so they are always positive.

Let $\mathcal{K} = \dfrac{\Gamma(\frac{c+1}{2})\sqrt{c}}{\Gamma(\frac{c}{2})\sqrt{\pi}R_{ii}Q_{jj}}$. By changing variables and including the Jacobian,

$$f(x_{ij}) = \mathcal{K}\left(1 + \frac{c(x_{ij}-\mu_{ij})^2}{R_{ii}^2Q_{jj}^2}\right)^{-\frac{c+1}{2}} \tag{8}$$

This is a non-standardized Student's t distribution with mean μ_{ij} and scalaing parameter $\sigma = \frac{R_{ii}Q_{jj}}{c}$. Note that this is not the same as a non-central t distribution, which is what is in R. That means we need to compute the pdf and cdf ourselves.

We want to know the probability that $x_{ij} < 0$. Let $z = \frac{\mu\sqrt{c}}{R_{ii}Q_{jj}}$. Through the magic of Mathematica, we get

$$F(0) = \frac{1}{2} - \mathcal{K}\mu \, {}_2F_1\left(\frac{1}{2}, \frac{c+1}{2}; \frac{3}{2}; -z^2\right) \tag{9}$$

This is not so hard to compute, but we can simplify some more. Applying the transformation in Equation 15.8.1 in Olver et al. (2010).

$$\text{Prob}(x_{ij} < 0) = \frac{1}{2} - \mathcal{K}\mu \left(1 + z^2\right)^{-\frac{1}{2}} \, {}_2F_1\left(\frac{1}{2}, 1 - \frac{c}{2}; \frac{3}{2}; \frac{z^2}{z^2+1}\right) \tag{10}$$

Using Equation 8.17.7 in Olver et al. (2010), we can write this probability in terms of an incomplete beta function $\mathbb{B}(\cdot)$.

$$\begin{aligned}\text{Prob}(x_{ij} < 0) &= \frac{1}{2} - \mathcal{K}\mu \left(1 + z^2\right)^{-\frac{1}{2}} \, {}_2F_1\left(\frac{1}{2}, 1 - \frac{c}{2}; \frac{3}{2}; \frac{z^2}{z^2+1}\right) && (11)\\ &= \frac{1}{2} - \mathcal{K}\mu \left(1 + z^2\right)^{-\frac{1}{2}} \frac{1}{2}\left(\frac{z^2}{1+z^2}\right)^{-\frac{1}{2}} \mathbb{B}\left(\frac{z^2}{1+z^2}; \frac{1}{2}, \frac{c}{2}\right) && (12)\end{aligned}$$

When we substitute back $\mathcal{K}$ and z, and remember that $\sqrt{\pi} = \Gamma(\frac{1}{2})$, lots of stuff cancels out. The probability reduces to a *regularized* beta function $\widetilde{\mathbb{B}}(\cdot)$.

$$\text{Prob}(x_{ij} < 0) = \frac{1}{2}\left[1 - \text{sgn}(\mu)\widetilde{\mathbb{B}}\left(\frac{c\mu^2}{c\mu^2 + R_{ii}^2 Q_{jj}^2}; \frac{1}{2}, \frac{c}{2}\right)\right] \tag{13}$$

(The sgn function returns -1 if $\mu < 0$, 0 if $\mu = 0$ and 1 if $\mu > 0$). Note that the regularized beta function is equivalent to the cdf of a beta distribution. This function is available in most statistical packages.

For automatic differentiation, we will need the derivative of the incomplete beta function.

$$\frac{d\mathbb{B}(x; a, b)}{dx} = x^{a-1}(1-x)^{b-1} \tag{14}$$

Since c is fixed and known, we do not need the derivative with respect to the other arguments of the incomplete beta function. This makes life much easier.

It might be even faster to use a normal approximation to the non-standard t distribution. As c gets large, $f(x_{ij})$ approaches a normal distribution with standard deviation $\sigma = \frac{R_{ii}Q_{jj}}{c}$. Since c gets updated after each period, the normal approximation might work very well, very early. Whether this is advisable depends on how fast the system can compute the normal cdf.

1 Specifics

For updating H_t, we need to consider the posterior distribution of θ_{2t}:

$$\theta_{2t} \sim Mat\_T(M_{2,t-1}, C_{2,t-1}, \Omega_{t-1}, \nu_{t-1}) \tag{15}$$

The reason we use $t-1$ is that we are assuming advertising is purchased at the beginning of the period, and the effectiveness of the ads affects period t sales. Therefore, we cannot use period t sales to update the latent parameters. Besides, this is the only way to get the recursion to work.

Bayesian updating of the inverse Wishart for Σ gives us

$$\nu_t = \nu_{t-1} + N \tag{16}$$

$$\Omega_t = \Omega_{t-1} + (Y_t - f_t)' Q_t^{-1} (Y_t - f_t) \tag{17}$$

Because θ_{2t} has $1 + J + P$ rows and J columns, ν_0 must be greater than $P + 2J + 1$.

Define

$$S_t = \mathrm{diag}(C_{2t})\,\mathrm{diag}(\Omega_t)' \tag{18}$$

which is the outer product of the diagonals of C_{2t} and Ω_t.

Any element $\theta 2, ijt$ has a non-standardized Student t distribution with parameters $M_{2,ijt}$,S_{ijt} and $\nu_t - P - 2J$. Specifically,

$$f(\theta_{2,ijt}) = \frac{\Gamma\left(\frac{\nu_t - P - 2J + 1}{2}\right)}{\Gamma\left(\frac{\nu_t - P - 2J}{2}\right)\sqrt{\pi(\nu_t - P - 2J)}S_{ijt}} \left(1 + \frac{1}{\nu_t - P - 2J}\left(\frac{\theta_{2,ijt} - M_{2,ijt}}{S_{ijt}}\right)^2\right)^{-\frac{\nu_t - P - 2J + 1}{2}} \tag{19}$$

$$\mathrm{Prob}(\theta_{2,ijt} < 0) = \frac{1}{2}\left[1 - \mathrm{sgn}(M_{2.ijt})\widetilde{\mathbb{B}}\left(\frac{(\nu_t - P - 2J)\, M_{2,ijt}^2}{(\nu_t - P - 2J)\, M_{2,ijt}^2 + S_{ijt}^2}; \frac{1}{2}, \frac{\nu_t - P - 2J}{2}\right)\right] \tag{20}$$

When ν_t becomes large, we can use the normal approximation to the t.

$$\theta_{2,ijt} \sim N\left(M_{2,ijt}, \frac{S_{ijt}}{\nu_t - P - 2J}\right) \tag{21}$$

Writing the cdf at zero in terms of the error function,

$$\mathrm{Prob}(\theta_{2,ijt} < 0) = \frac{1}{2}\left[1 + \mathrm{erf}\left(\frac{-\,(\nu_t - P - 2J)\, M_{2,ijt}}{\sqrt{2}S_{ijt}}\right)\right] \tag{22}$$

References

Kotz, Samuel and Saralees Nadarajah (2004). *Multivariate t Distributions and Their Applications*. Cambridge, U.K.: Cambridge University Press.

Olver, Frank W J, Daniel W Lozier, Ronald F Boisvert, and Charles W Clark, eds. (2010). *NIST Handbook of Mathematical Functions*. New York: Cambridge University Press.