Note on differential equation for HDLM

October 21, 2015

Let's start with the following differential equation:

$$\frac{dq}{dt} = -\left[c + uA\right] q + r(1 - I(A))\left[b - q\right] \tag{1}$$

eq:de1

The symbols are defined as follows:

q	ad quality
A	ad spend
c	the rate at which quality attenuates towards zero in the absence of advertising
u	additional attenuation of quality that is proportional to ad spend
δ	a decay parameter
b	a constant
m	long-run asymptote

If there is no ad spend, so $A = 0$, then $\lim_{t\to\infty} q_t = \frac{\delta}{c+\delta} b$. If we define $m = \frac{\delta}{c+\delta} b$, then the differential equation is

$$\begin{aligned} \frac{dq}{dt} &= -\left[c + uA\right] q + \delta(1 - \mathbb{I}(A))\left[\frac{m(c+\delta)}{\delta} - q\right] && (2) \\ &= -\left[c + uA + (1 - \mathbb{I}(A))\,\delta\right] q + (1 - \mathbb{I}(A)) m(c+\delta) && (3) \end{aligned}$$

eq:de2

Thus, if there is no ad spend, then $\lim_{t\to\infty} q_t = m$.
The discrete time version is:

$$q_{t+1} = \left[1 - c - uA - (1 - \mathbb{I}(A))\,\delta\right] q_t + (1 - I(A)) m(c+\delta) \tag{4}$$

eq:de3

The evolution matrix is:

$$G_t = \begin{pmatrix} (1-\delta) & \tilde{g}(A_{1t}) & \cdots & \cdots & \tilde{g}(A_{Jt}) \\ 0 & (1 - c_1 - u_1 A_{1t}) + (1 - \mathbb{I}(A_{1t}))\delta & & & 0 \\ \vdots & & \ddots & & \vdots \\ \vdots & & & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & (1 - c_J - u_J A_{Jt}) + (1 - \mathbb{I}(A_{1t}))\delta \end{pmatrix} \tag{5}$$

eq:Gt

One component of the innovation matrix comes from the additive term in the differential equation (check subscripts for c_j).

$$H_{1t} = \begin{pmatrix} (1 - I(A_{1t}))(c_1 + \delta)m_{11} & \cdots & (1 - I(A_{Jt}))(c_J + \delta)m_{1J} \\ \vdots & & \vdots \\ (1 - I(A_{1t}))(c_1 + \delta)m_{J1} & \cdots & (1 - I(A_{Jt}))(c_J + \delta)m_{JJ} \end{pmatrix} \tag{6}$$

eq:H1t

If we add effects from creatives, then we add another component to the innovation matrix.

$$H_{2t} = \begin{pmatrix} E_{1t} & & & 0 \\ & E_{2t} & & \\ & & \ddots & \\ 0 & & & E_{Jt} \end{pmatrix} \begin{pmatrix} \phi_{11} & \cdots & \phi_{1J} \\ \vdots & \vdots & \vdots \\ \phi_{J1} & \cdots & \phi_{JJ} \end{pmatrix} \tag{7}$$

eq:H2t

Here is how this all relates to the flags in the code. If `replenish` is `TRUE`, then we compute H_{1t}. Otherwise, $H_{1t} = 0$. If `include.phi` is `TRUE`, then we compute H_{2t}. Otherwise, $H_{1t} = 0$.