Aufgabe 2

let $n \in \mathbb{N} \land c \in \mathbb{R}$

1. show or disprove "$13 + 37 + 4 \in O(1)$"

$$13 + 37 + 4 = 54 = 54 * 1 \Rightarrow 13 + 37 + 4 \leq 54 * 1 \Rightarrow 13 + 37 + 4 \in O(1)$$

2. show or disprove "$2n^3 + 4n^2 + 8n + 3 \in \Omega(n^3)$"

$$\begin{gathered} 2n^3 + 4n^2 + 8n + 3 \geq cn^3 \\ \Leftrightarrow \frac{2n^3 + 4n^2 + 8n + 3}{n^3} \geq c \\ \Leftrightarrow 2 + \frac{4}{n} + \frac{8}{n^2} + \frac{3}{n^3} \geq c \\ \Leftrightarrow c \leq 2 \\ \Rightarrow 2n^3 + 4n^2 + 8n + 3 \in O(n^3) \end{gathered}$$

3. show or disprove "$6^{-5}n^{1.25} \in \Theta(\sqrt{n})$"

$$\begin{gathered} \text{Assume } 6^{-5}n^{1.25} \in O(\sqrt{n}) \Rightarrow 6^{-5}n^{1.25} \leq c\sqrt{n} \\ \Leftrightarrow \frac{6^{-5}n^{1.25}}{\sqrt{n}} = 6^{-5}\frac{n^{\frac{5}{4}}}{n^{\frac{1}{2}}} = 6^{-5}n^{0.75} \leq c \\ \Leftrightarrow n \to \infty : c \to \infty \Rightarrow \nexists c : 6^{-5}n^{1.25} \leq c\sqrt{n} \\ \Rightarrow 6^{-5}n^{1.25} \notin O(\sqrt{n}) \Rightarrow 6^{-5}n^{1.25} \notin \Theta(\sqrt{n}) \end{gathered}$$

4. show or disprove "$4^{n+1} \in O(4^n)$"

$$4^{n+1} = 4 * 4^n \Rightarrow c = 4 \Rightarrow 4^{n+1} \in O(4^n)$$

5. show or disprove "$4^{2n} \in O(4^n)$"

$$4^{2n} = 4^n * 4^n \Rightarrow c = 4^n \Rightarrow n \to \infty : c \to \infty \Rightarrow \nexists c : 4^{2n} \leq c * 4^n \Rightarrow 4^{2n} \notin O(4^n)$$

6. show or disprove "$2 \log n! \in \Theta(n \log n)$"
   - case $1 : 2 \log n! \in O(n \log n)$

$$\begin{gathered} 2 \log n! = 2 \log(n * (n-1) * ... * 1) = 2(\log(n) + \log(n-1) + ... + \log(1)) \\ \Leftrightarrow 2 * \sum_{i=1}^{n} \log i < 2 \sum_{i=1}^{n} \log n = 2n \log n \Rightarrow c \geq 2 : 2 \log n! \leq cn \log n \end{gathered}$$

   - case $2 : 2 \log n! \in \Omega(n \log n)$

$$\begin{gathered} 2 \log n! = 2 \log(n * (n-1) * ... * 1) = 2(\log(n) + \log(n-1) + ... + \log(1)) \\ \Leftrightarrow 2 * \sum_{i=1}^{n} \log i > 2 \sum_{i=\frac{n}{2}}^{n} \log\left(\frac{n}{2}\right) = 2\frac{n}{2} \log\left(\frac{n}{2}\right) \end{gathered}$$

$$= n(\log n - \log 2) = n \log n - n \log 2 = n \log n - n$$
$$\Rightarrow c = 1 : 2 \log n! \geq cn \log n \Leftrightarrow 2 \log n! \in \Omega(n \log n)$$

$$\Rightarrow 2 \log n! \in O(n \log n) \land 2 \log n! \in \Omega(n \log n) \Rightarrow 2 \log n! \in \Theta(n \log n)$$

7. show or disprove "$2^n \in O(n!)$"

$$2^n \leq cn! \Leftrightarrow \frac{2^n}{n!} \leq c$$

Idea:
$\frac{2^n}{n!} = \frac{2*...*2}{n*(n-1)*...*1} = \frac{2}{n} * \frac{2}{n-1} * ... * \frac{2}{2} * \frac{2}{1}$ (n-2 Faktoren kleiner als 0 => für $n \to \infty : \frac{2^n}{n!} \to 0$)

Test:

n	1	2	3	4
$\frac{2^n}{n!}$	2	2	$\frac{4}{3}$	$\frac{2}{3}$

to prove: $\forall n \geq 4 : 2^n < n!$
- Induction start: $n = 4$

$$2^4 = 16 \leq 4! = 24$$

- Induction step: $n \to n + 1$

$$2^{n+1} = 2^n * 2 \leq n! * 2 \text{ [Induction Start]}$$

$$\Rightarrow n! * 2 \leq n! * (n+1) \ [n \geq 4]$$

$$\Rightarrow 2^{n+1} \leq (n+1)!$$

from the above it follows that $n! \in o(2^n) \Leftrightarrow 2^n \in O(n!)$

8. show or disprove "$n! \in O(n^n)$"

$$n! \leq c * n^n \Leftrightarrow \frac{n!}{n^n} \leq c$$

Idea: analogical to number 7

to prove: $\forall n \in \mathbb{N} : n! \leq n^n$
- Induction start: $n = 1$

$$1! = 1 \leq 1^1 = 1$$

- Induction step: $n \to n + 1$

$$(n+1)! = (n+1)n! \leq (n+1)n^n \text{ [Induction start]}$$

$$\Rightarrow (n+1)n^n \leq (n+1)(n+1)^n = (n+1)^{n+1} \text{ therefore } n! \in O(n^n)$$