Aufgabe 4:

1. prove or disprove $\sum_{k=0}^{\infty} \frac{k^2}{2^k} = O(1)$
   - applying ratio rule for series: let $S_k = \sum_{n=0}^{k} a_n \land \lim_{n\to\infty}\left(\frac{a_{n+1}}{a_n}\right) < 1 : \exists l \in \mathbb{R} : \lim_{k\to\infty}(S_k) = l$

     $$\Rightarrow \frac{\frac{(k+1)^2}{2^{k+1}}}{\frac{k^2}{2^k}} \Leftrightarrow \frac{(k+1)^2 * 2^k}{k^2 * 2^{k+1}} = \frac{(k+1)^2}{2k^2} \Leftrightarrow \frac{1}{2} * \left(\frac{k+1}{k}\right)^2 \Rightarrow \lim_{k\to\infty}\left(\frac{1}{2} * \left(\frac{k+1}{k}\right)^2\right) = \frac{1}{2} * 1 < 1$$

   $\Rightarrow \exists l \in \mathbb{R} : \lim_{k\to\infty}(S_k) = l \Rightarrow \sum_{k=0}^{\infty}\left(\frac{k^2}{2^k}\right) \le c * 1 \Rightarrow c \ge l \Rightarrow \sum_{k=0}^{\infty}\left(\frac{k^2}{2^k}\right) \in O(1)$

2. prove or disprove $n^m = O(\alpha^n)\ \forall m \in \mathbb{N} \land \alpha > 1$
   evaluate:

   $$\lim_{n\to\infty} \frac{n^m}{\alpha^n} \Leftrightarrow \text{using l'Hôpital: } \lim_{n\to\infty} \frac{\frac{\partial^m}{\partial n^m} n^m}{\frac{\partial^m}{\partial n^m} \alpha^n} = \lim_{n\to\infty} \frac{m!}{\alpha^n * \ln(\alpha)} = 0$$

   $$\Rightarrow n^m \in \Omega(\alpha^n) \Rightarrow \alpha^n \in O(n^m)$$

   other why

   $$e^x = \sum_{k=0}^{\infty}\left(\frac{x^k}{k!}\right)$$

   $$\text{show } n^m = O(\alpha^n) \text{ to show } n^m \le c\alpha^n\ \alpha^n = e^{n \ln \alpha} = \sum_{k=0}^{\infty} \frac{(n \ln \alpha)^k}{k!} \ge \frac{(n \ln \alpha)^m}{m!}$$

   $$\Rightarrow n^m \le \frac{m!}{(\ln \alpha)^m} \alpha^n$$

3. prove or disprove $n \ln n \in O\left(n^{\frac{3}{2}}\right)$
   from the definition of $O\left(n^{\frac{3}{2}}\right)$

   $$n \ln n \le c n^{\frac{3}{2}} \Leftrightarrow \ln n \le c n^{\frac{1}{2}} \Leftrightarrow c \ge \frac{\ln n}{\sqrt{n}}$$

   apply limit:

   $$\lim_{n\to\infty}\left(\frac{\ln n}{\sqrt{n}}\right) \Leftrightarrow \text{using l'Hôpital : } \lim_{n\to\infty}\left(\frac{\frac{1}{n}}{\frac{1}{2\sqrt{n}}}\right) = \lim_{n\to\infty}\left(2\frac{\sqrt{n}}{n}\right) = 0$$

   $$\Rightarrow \sqrt{n}^3 \in \Omega(n \ln n) \Rightarrow n \ln n \in O\left(\sqrt{n}^3\right)$$

4. prove or disprove $5^{\log_3 n} \in O(n^2)$
   from the definition of $O(n^2)$

   $$5^{\log_3 n} \le c n^2 \Leftrightarrow \frac{5^{\log_3 n}}{n^2} \le c \Leftrightarrow \frac{5^{\frac{\ln n}{\ln 3}}}{n^2} \le c \Leftrightarrow \frac{5^{\ln n}}{5^{\ln 3} n^2} \le c$$

   using the equality:

   $$a^x = e^{\ln(a)x}$$

   $$\Rightarrow \frac{5^{\ln n}}{5^{\ln 3} n^2} = \frac{e^{\ln(5)\ln(n)}}{5^{\ln 3} n^2} = \frac{n^{\ln(5)}}{5^{\ln 3} n^2} \le c$$

   taking the limit:

   $$\lim_{n\to\infty} \frac{n^{\ln(5)}}{5^{\ln 3} n^2} \approx \lim_{n\to\infty} \frac{n^{1.609}}{5^{\ln 3} n^2} = 0$$

   $$\Rightarrow 5^{\log_3 n} \in \Omega(n^2) \Rightarrow n^2 \in O\left(5^{\log_3 n}\right)$$