Reading Course Notes on Complex Semisimple Lie Algebras

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Summer 2022

Preface

These notes grew out of the reading course MAT492H1 (Mathematics Reading Course) taken in the Summer of 2022 at the University of Toronto under the supervision of Michael Groechenig. The main goal of the course was to present an introductory exposition of highest weight theory for complex semisimple Lie algebras, culminating in the construction of Verma modules, in a way accessible to undergraduate students.

Many of the standard references, in particular the classic text by J.-P. Serre on complex semisimple Lie algebras, develop the theory in a very concise and formal style. While beautiful, these arguments often presuppose a level of mathematical maturity that can make the material difficult to approach on a first reading. One of the aims of this course—and of these notes—was therefore to repackage this theory in a more gradual and digestible exposition, while remaining faithful to the core ideas.

The notes are based on selected parts of the following four main texts:

1. W. Fulton, J. Harris, *Representation Theory: A First Course*.
2. B. C. Hall, *Lie Groups, Lie Algebras, and Representations*.
3. J.-P. Serre, *Complex Semisimple Lie Algebras*.
4. J. Hilgert, K.-H. Neeb, *Structure and Geometry of Lie Groups*.

The introductory material follows the first three chapters of Fulton–Harris, which provide a gentle entry point to basic representation-theoretic ideas. The core of the notes then follows Serre's book closely, and the chapter titles largely mirror those of the original text. The discussion of compact Lie groups draws on the exposition in Hall's book, and the final part of the notes is devoted to an introduction to Verma modules and highest weight representations.

These notes were written over an extended period from January to August 2022, starting with an introductory course on elementary Lie theory MAT305 at UTM and continuing through the more advanced material on semisimple Lie algebras. I was very fortunate to have the guidance of Prof. Michael Groechenig throughout this time, and I am deeply grateful to him for his support, patience, and encouragement of my interest in this beautiful subject.

Contents

Chapter 1

Modules over Finite Groups

For all what follows, assume vector spaces to be complex and finite dimensional.

1.1. Basic Definitions

Definition 1. *A G-*module *for a group G is a vector space V together with a homomorphism*

$$\rho : G \to \mathrm{GL}(V)$$

called a representation of G. *The module V is called* irreducible *if it contains no nontrivial G-invariant subspaces.*

Definition 2. *Given two G-modules V and W, define a G-action on the space* $\mathrm{Hom}(V, W)$ *by*

$$(g \cdot \phi)(v) = g \cdot (\phi(g^{-1} \cdot v)).$$

This action commutes with the following diagram:

$$\begin{array}{ccc} V & \xrightarrow{\phi} & W \\ g\downarrow & & \downarrow g \\ V & \xrightarrow{\phi} & W \end{array}$$

Remark 1. *If $W = \mathbb{C}$, then the commutativity condition above defines an action of G on V^*, the dual space of V. In this case,*

$$\rho^*(g) = (\rho(g^{-1}))^t.$$

Remark 2. *A map ϕ is a G-module homomorphism if and only if it is fixed by the above action of G.*

1.2. Maschke's Theorem; Schur's Lemma

Theorem 1 (Maschke)**.** *If W is a G-submodule of a G-module V of a finite group G, then there exists a complementary invariant subspace W' of V such that*

$$V = W \oplus W'.$$

Proof. The proof uses a Hermitian product H on V such that $H(gv, gw) = H(v, w)$ for all $v, w \in V$ and $g \in G$. Given any Hermitian product H_0, define

$$H(v, w) = \sum_{g \in G} H_0(gv, gw),$$

which makes it G-invariant and allows the definition of the orthogonal complement $W^\perp$.

□

Corollary 1. *Any G-module is a direct sum of irreducible G-modules.*

Lemma 1 (Schur's Lemma)**.** *If V and W are irreducible G-modules and $\phi : V \to W$ is a G-module homomorphism, then:*

- *either ϕ is an isomorphism or $\phi = 0$,*
- *if $V = W$, then $\phi = \lambda \operatorname{Id}$ for some $\lambda \in \mathbb{C}$.*

Proof. This follows from the fact that $\ker \phi$ and $\operatorname{Im} \phi$ are G-submodules. □

Proposition 1. *For any G-module V of a finite group G, there exists a decomposition*

$$V = V_1^{\oplus a_1} \oplus \cdots \oplus V_k^{\oplus a_k},$$

where the V_i are distinct irreducible G-modules. The decomposition, as well as the multiplicities a_i, are unique.

Remark 3. *Let V be a G-module for a finite group G. For a fixed $g \in G$, the map $\rho(g) : V \to V$ is G-linear if and only if $g \in Z(G)$. If G is abelian and V is irreducible, then by Schur's lemma, $\rho(g)$ acts as a scalar multiple of the identity, hence V is one-dimensional.*

1.3. Examples

Example 1 (Classification with the permutation group S_3)**.** *Consider the symmetric group S_3 acting on three elements. Let W be a G-module over $\mathbb{C}$ for $G = S_3$.*

Let $\tau = (123)$ be a 3-cycle generating the cyclic subgroup $C_3 < S_3$. Then C_3 is abelian, and by Schur's lemma, τ acts on each irreducible C_3-submodule of W as multiplication by a scalar.

Hence we can write

$$W = \bigoplus_{i \in I} V_i,$$

where I is some index set and each V_i is an irreducible C_3-submodule of W. Since $\tau^3 = e$, for every $i \in I$ there exists a nonzero vector $v_i \in V_i$ such that

$$V_i = \mathbb{C} \cdot v_i, \qquad \tau(v_i) = \omega^{x_i} v_i,$$

where $\omega = e^{2\pi i/3}$ and $x_i \in \{0, 1, 2\}$.

Now let $\sigma = (12)$ be a transposition in S_3. Observe that $\langle \sigma, \tau \rangle = S_3$ and

$$\sigma \tau \sigma = \tau^2.$$

Thus, for any eigenvector $v \in W$ of τ with eigenvalue ω^i, we have

$$\tau(\sigma(v)) = \sigma(\tau^2(v)) = \sigma(\omega^{2i} v) = \omega^{2i} \sigma(v).$$

Therefore, $\sigma(v)$ is an eigenvector of τ with eigenvalue ω^{2i}.

Remark 4. *The group S_3 admits one-dimensional representations where it acts:*

$$\rho_{\text{triv}}(g)x = x, \qquad \rho_{\text{alt}}(g)x = \operatorname{sign}(g)\, x,$$

for all $x \in V$ and $g \in S_3$. These are the trivial *and* alternating *representations.*

Case $\omega \neq 1$***: Construction of the Standard Representation.***

Suppose now that $V = \mathbb{C} \cdot v$ *is an irreducible* C_3*-module where* $\tau(v) = \omega^i v$ *with* $\omega \neq 1$*. Then* $\sigma(v)$ *is linearly independent from* v*. The subspace generated by* $\{v, \sigma(v)\}$ *is* S_3*-invariant, and defines a two-dimensional irreducible* S_3*-module called the* standard representation.

A convenient choice of basis is:

$$\alpha = (\omega, 1, \omega^2), \qquad \beta = (1, \omega, \omega^2),$$

so that

$$\tau(\alpha) = \omega\alpha, \quad \tau(\beta) = \omega^2\beta, \quad \sigma(\alpha) = \beta, \quad \sigma(\beta) = \alpha.$$

Case $\omega = 1$***.*** *In this case,* τ *acts trivially on* $V = \mathbb{C} \cdot v$*, so* $\sigma(v)$ *is also an eigenvector of* τ*. Two possibilities occur:*

(1) If $\sigma(v)$ *and* v *are linearly dependent, i.e.* $\sigma(v) = \pm v$*, then* S_3 *acts trivially (for* $+$*) or by sign (for* $-$*) on* V*, giving the one-dimensional trivial and alternating representations.*

(2) If $\sigma(v)$ *and* v *are linearly independent, then define*

$$w_\pm = \sigma(v) \pm v.$$

Each $w_\pm$ *is an eigenvector of both* τ *and* σ*, giving a decomposition*

$$V = \mathbb{C} \cdot v \oplus \mathbb{C} \cdot \sigma(v) = \mathbb{C} \cdot w_+ \oplus \mathbb{C} \cdot w_-,$$

where S_3 *acts trivially on one component and by sign on the other.*

Conclusion. *We have found exactly three distinct irreducible* S_3*-modules:*

$$\begin{aligned} \textit{Trivial:} &\quad \rho(g) = 1, \\ \textit{Alternating:} &\quad \rho(g) = \operatorname{sign}(g), \\ \textit{Standard:} &\quad \textit{the two-dimensional subspace spanned by } \{\alpha, \beta\}. \end{aligned}$$

Therefore, every S_3*-module* W *decomposes as a direct sum of these irreducible components:*

$$W \cong U_{\text{triv}}^{\oplus a} \oplus U_{\text{alt}}^{\oplus b} \oplus V_{\text{std}}^{\oplus c}.$$

Example 2 (Decomposition of $V \otimes V$ for the standard S_3-module)**.** *Let* V *denote the standard irreducible* S_3*-module generated by the basis vectors*

$$\alpha = (\omega, 1, \omega^2), \qquad \beta = (1, \omega, \omega^2),$$

where $\omega = e^{2\pi i/3}$ *and the group* S_3 *acts by permutation matrices on the standard basis* $\{e_1, e_2, e_3\}$ *of* $\mathbb{R}^3$*.*

Consider the generators of S_3*:*

$$\sigma = (12), \qquad \tau = (123).$$

Their action on the standard basis vectors of V *is given by:*

$$\tau(\alpha) = \omega\alpha, \quad \tau(\beta) = \omega^2\beta, \qquad \sigma(\alpha) = \beta, \quad \sigma(\beta) = \alpha.$$

Step 1. Tensor product basis. *The tensor product space* $V \otimes V$ *is* 4*-dimensional and spanned by*

$$\{\alpha \otimes \alpha,\ \beta \otimes \beta,\ \alpha \otimes \beta,\ \beta \otimes \alpha\}.$$

Since τ acts diagonally on α and β, we have:

$$\begin{aligned}\tau(\alpha\otimes\alpha) &= \omega^2\,(\alpha\otimes\alpha),\\ \tau(\beta\otimes\beta) &= \omega\,(\beta\otimes\beta),\\ \tau(\alpha\otimes\beta) &= 1\cdot(\alpha\otimes\beta),\\ \tau(\beta\otimes\alpha) &= 1\cdot(\beta\otimes\alpha).\end{aligned}$$

Thus, $\alpha\otimes\alpha$ and $\beta\otimes\beta$ are eigenvectors of τ with distinct eigenvalues, while $\alpha\otimes\beta$ and $\beta\otimes\alpha$ share the eigenvalue 1.

Step 2. Action of σ. *The transposition σ interchanges α and β, and therefore acts on the tensor basis as:*

$$\begin{aligned}\sigma(\alpha\otimes\alpha) &= \beta\otimes\beta,\\ \sigma(\beta\otimes\beta) &= \alpha\otimes\alpha,\\ \sigma(\alpha\otimes\beta) &= \beta\otimes\alpha,\\ \sigma(\beta\otimes\alpha) &= \alpha\otimes\beta.\end{aligned}$$

Step 3. Identification of invariant subspaces. *From the combined actions of τ and σ, we can identify the following invariant subspaces of $V\otimes V$:*

(1) The subspace spanned by $\{\alpha\otimes\alpha,\,\beta\otimes\beta\}$ is preserved by both σ and τ, and the action on it is identical to that of the standard representation V. Hence

$$\mathrm{span}\{\alpha\otimes\alpha,\,\beta\otimes\beta\}\cong V.$$

(2) The antisymmetric combination

$$\alpha\otimes\beta-\beta\otimes\alpha$$

is invariant under τ (since its eigenvalue is 1) and changes sign under σ. Therefore, it spans the alternating representation *U' of S_3.*

(3) The symmetric combination

$$\alpha\otimes\beta+\beta\otimes\alpha$$

is invariant under both σ and τ, and hence spans the trivial representation *U.*

Conclusion. *Combining all invariant subspaces, we obtain the decomposition*

$$V\otimes V\cong U\oplus V\oplus U',$$

where:

$$\begin{aligned}U &\quad \textit{is the trivial representation (1-dimensional),}\\ U' &\quad \textit{is the alternating representation (1-dimensional),}\\ V &\quad \textit{is the standard representation (2-dimensional).}\end{aligned}$$

Example 3 ($\mathrm{Sym}^2(V)$ for the standard S_3-module)**.** *Let $V=\langle\alpha,\beta\rangle$ be the standard irreducible S_3-module with*

$$\tau(\alpha)=\omega\,\alpha,\quad \tau(\beta)=\omega^2\,\beta,\qquad \sigma(\alpha)=\beta,\quad \sigma(\beta)=\alpha,$$

where $\omega=e^{2\pi i/3}$, $\tau=(123)$, $\sigma=(12)$.

The symmetric square $\mathrm{Sym}^2(V)$ *is spanned by the symmetric tensors*

$$\alpha\otimes\alpha,\quad \beta\otimes\beta,\quad s:=\alpha\otimes\beta+\beta\otimes\alpha.$$

Their τ-eigenvalues are

$$\tau(\alpha\otimes\alpha)=\omega^2(\alpha\otimes\alpha),\qquad \tau(\beta\otimes\beta)=\omega(\beta\otimes\beta),\qquad \tau(s)=s,$$

and σ acts by

$$\sigma(\alpha\otimes\alpha) = \beta\otimes\beta, \qquad \sigma(\beta\otimes\beta) = \alpha\otimes\alpha, \qquad \sigma(s) = s.$$

Hence:

- *$\langle s\rangle$ is fixed by both τ and σ: it affords the* trivial *representation U.*
- *$\langle \alpha\otimes\alpha,\ \beta\otimes\beta\rangle$ is S_3-stable, with τ having eigenvalues ω^2, ω and σ swapping the basis—this is isomorphic to the* standard *representation V.*

Therefore

$$\mathrm{Sym}^2(V) \cong U \oplus V.$$

Example 4 (Sym$^3(V)$ for the standard S_3-module)**.** *Keep $V = \langle \alpha, \beta\rangle$ as above. The symmetric cube* $\mathrm{Sym}^3(V)$ *is* 4*-dimensional with the convenient basis*

$$a := \alpha\otimes\alpha\otimes\alpha, \quad b := \beta\otimes\beta\otimes\beta, \quad u := \mathrm{sym}(\alpha, \beta, \beta), \quad v := \mathrm{sym}(\beta, \alpha, \alpha),$$

where sym *denotes the average over all permutations of the tensor factors.*

Action of τ. *Using $\tau(\alpha) = \omega\alpha$, $\tau(\beta) = \omega^2\beta$, we get*

$$\tau(a) = \omega^3 a = a, \qquad \tau(b) = (\omega^2)^3 b = b, \qquad \tau(u) = \omega^{1+2+2} u = \omega^2 u, \qquad \tau(v) = \omega^{2+1+1} v = \omega v.$$

Action of σ. *Since σ swaps α and β, we have*

$$\sigma(a) = b, \qquad \sigma(b) = a, \qquad \sigma(u) = v, \qquad \sigma(v) = u.$$

Decomposition into irreducibles.

- *The subspace $\langle a, b\rangle$ is S_3-stable. Define*

  $$a_+ := a + b, \qquad a_- := a - b.$$

  Then $\tau(a_\pm) = a_\pm$. Moreover, $\sigma(a_+) = a_+$ (so $\langle a_+\rangle$ is trivial *U) and $\sigma(a_-) = -a_-$ (so $\langle a_-\rangle$ is* alternating *U').*
- *The subspace $\langle u, v\rangle$ is S_3-stable with*

  $$\tau(u) = \omega^2 u, \quad \tau(v) = \omega v, \qquad \sigma(u) = v, \ \sigma(v) = u,$$

  which is isomorphic to the standard *representation V.*

Therefore

$$\mathrm{Sym}^3(V) \cong U \ \oplus \ U' \ \oplus \ V.$$

Example 5 (Regular representation)**.** *Consider the left-regular S_3-module*

$$\mathbb{C}[S_3] \ = \ \mathrm{span}_{\mathbb{C}}\{\, e, \ (12), \ (23), \ (13), \ (123), \ (132)\,\},$$

with the action $g \cdot h := gh$ (left multiplication). Let $\tau = (123)$ and $\sigma = (12)$, and put $\omega = e^{2\pi i/3}$.

Step 1. τ-eigenvectors. *Define*

$$\begin{aligned} u_1 &:= e + (123) + (132), & t_1 &:= (12) + (13) + (23), \\ u_2 &:= e + \omega^2(123) + \omega(132), & t_2 &:= (12) + \omega^2(13) + \omega(23), \\ u_3 &:= e + \omega(123) + \omega^2(132), & t_3 &:= (12) + \omega(13) + \omega^2(23). \end{aligned}$$

Then

$$\tau \cdot u_1 = u_1, \qquad \tau \cdot u_2 = \omega\, u_2, \qquad \tau \cdot u_3 = \omega^2\, u_3,$$

and one checks that each t_j lies in the τ-eigenspace with the same eigenvalue as the corresponding u_j (see Step 3 via pairing with σ).

Step 2. σ-pairing. *A direct computation in the regular action gives*

$$\sigma \cdot u_1 = t_1, \qquad \sigma \cdot t_1 = u_1, \qquad \sigma \cdot u_2 = t_3, \qquad \sigma \cdot t_3 = u_2, \qquad \sigma \cdot u_3 = t_2, \qquad \sigma \cdot t_2 = u_3.$$

Hence the three pairs

$$\operatorname{span}\{u_1, t_1\}, \qquad \operatorname{span}\{u_2, t_3\}, \qquad \operatorname{span}\{u_3, t_2\}$$

are S_3-invariant subspaces.

Step 3. Identifying the irreducible summands.

- $\operatorname{span}\{\mathbf{u_1}, \mathbf{t_1}\} \cong \mathbf{U} \oplus \mathbf{U'}$. *We have $\tau \cdot u_1 = u_1$ and (by class-multiplication) $\tau \cdot t_1 = t_1$. Since σ swaps u_1 and t_1, the vectors*

  $$w_+ := u_1 + t_1, \qquad w_- := u_1 - t_1$$

  diagonalize σ: $\sigma \cdot w_+ = +w_+$, $\sigma \cdot w_- = -w_-$, and both are fixed by τ. Thus $\langle w_+ \rangle \cong U$ (trivial) and $\langle w_- \rangle \cong U'$ (sign).

- $\operatorname{span}\{\mathbf{u_2}, \mathbf{t_3}\} \cong \mathbf{V}$ *and* $\operatorname{span}\{\mathbf{u_3}, \mathbf{t_2}\} \cong \mathbf{V}$. *Here σ interchanges $u_2 \leftrightarrow t_3$ and $u_3 \leftrightarrow t_2$, while*

  $$\tau \cdot u_2 = \omega u_2, \quad \tau \cdot u_3 = \omega^2 u_3.$$

  A short calculation shows τ acts on each 2D subspace with eigenvalues $\{\omega, \omega^2\}$ and σ swaps the eigenlines. This matches exactly the character and structure of the 2D standard *irreducible representation V.*

Conclusion (Regular representation decomposition). *Putting the three invariant blocks together,*

$$\mathbb{C}[S_3] \;\cong\; U \;\oplus\; U' \;\oplus\; V \;\oplus\; V,$$

i.e. each irreducible of S_3 appears with multiplicity equal to its dimension (a hallmark of the regular representation).

Example 6 (Recursive decomposition of $\operatorname{Sym}^k(V)$ for $k \geq 6$)**.** *Let V be the standard irreducible S_3-module. We now describe how to decompose $\operatorname{Sym}^k(V)$ for arbitrary $k \geq 6$ in terms of $\operatorname{Sym}^j(V)$ with $j \equiv k \pmod 6$.*

It is known that $\operatorname{Sym}^k(V)$ is isomorphic to the space $H_k[x,y]$ of homogeneous polynomials in two variables x, y of total degree k,

$$H_k[x,y] = \operatorname{span}\{x^m y^n \mid m + n = k\}, \qquad \dim H_k[x,y] = k+1.$$

Step 1. Embedding and invariant complement. *Consider the natural linear map*

$$\psi : H_k[x,y] \to H_{k+6}[x,y], \qquad p(x,y) \mapsto x^6 p(x,y).$$

By a previous proposition, for finite groups there exists a complementary invariant subspace W of $H_{k+6}[x,y]$ such that

$$H_{k+6}[x,y] \cong x^6 H_k[x,y] \oplus W.$$

Hence $\operatorname{Sym}^{k+6}(V) \cong \operatorname{Sym}^k(V) \oplus W$ as S_3-modules.

Step 2. Description of the complementary subspace. *We identify*

$$W \cong H_{k+6}[x,y]/x^6 H_k[x,y] = \operatorname{span}\{y^{k+6},\, y^{k+5}x,\, y^{k+4}x^2,\, y^{k+3}x^3,\, y^{k+2}x^4,\, y^{k+1}x^5\}.$$

Label these basis elements as $w_0, w_1, \ldots, w_5$.

Step 3. Identification with the group algebra $\mathbb{C}[S_3]$***.*** *Define a linear bijection*

$$\phi : W \longrightarrow \mathbb{C}[S_3]$$

by

$$\phi(w_0) = e, \quad \phi(w_1) = (12), \quad \phi(w_2) = (13), \quad \phi(w_3) = (23), \quad \phi(w_4) = (123), \quad \phi(w_5) = (132).$$

We extend ϕ linearly. Through this correspondence, we can transfer the multiplication structure from $\mathbb{C}[S_3]$ to W by defining

$$u \times v := \phi^{-1}(\phi(u) \cdot \phi(v)), \qquad u, v \in W.$$

This gives W the algebra structure (and hence the module structure) of the regular representation $\mathbb{C}[S_3]$.

Step 4. Decomposition of the complement. *Since the regular representation decomposes as*

$$\mathbb{C}[S_3] \cong U \oplus U' \oplus V \oplus V,$$

we conclude that

$$W \cong U \oplus U' \oplus V \oplus V.$$

Step 5. Recursive formula. *Combining the above results, we obtain the recursive relation*

$$\mathrm{Sym}^{k+6}(V) \cong \mathrm{Sym}^k(V) \oplus U \oplus U' \oplus V \oplus V,$$

which can be iterated to express $\mathrm{Sym}^k(V)$ for all $k \geq 6$ in terms of the first six symmetric powers.

Exercise 1 (1.14)**.** *Let V be an irreducible representation of a finite group G. Then, up to scalar multiples, there exists a unique Hermitian inner product on V which is preserved by G.*

Proof. Let H_0 be an arbitrary Hermitian inner product on V. Define a new Hermitian form

$$H(v, w) = \sum_{g \in G} H_0(g \cdot v, g \cdot w),$$

for all $v, w \in V$. The sum is finite, so H is well-defined and Hermitian. Moreover, for any $h \in G$,

$$H(hv, hw) = \sum_{g \in G} H_0(g \cdot hv, g \cdot hw) = \sum_{g' \in G} H_0(g' \cdot v, g' \cdot w) = H(v, w),$$

where we used the substitution $g' = gh$ and the fact that G is a group. Thus, H is G-invariant.

Since H is Hermitian, there exists a positive-definite Hermitian operator X such that

$$H(v, w) = \langle Xv, w \rangle,$$

where $\langle \cdot, \cdot \rangle$ denotes the standard inner product on V. The G-invariance of H implies

$$\langle X(gv), gw \rangle = \langle Xv, w \rangle \quad \forall g \in G, \; v, w \in V,$$

which in turn means that

$$g^\dagger X g = X.$$

Now let β be an orthonormal basis of V with respect to H. With respect to β, we have $X = I$, and so $g^\dagger g = I$ for all $g \in G$. Thus, in this basis, the representation acts by unitary matrices.

Define a linear map

$$L(v) = \frac{1}{|G|} \sum_{g \in G} g^\dagger X(gv).$$

Then for any $h \in G$,

$$L(hv) = \frac{1}{|G|} \sum_{g \in G} g^\dagger X(ghv) = \frac{1}{|G|} \sum_{g \in G} (hg)^\dagger X(hgv) = hL(v),$$

so L is a G-module homomorphism. By Schur's Lemma, since V is irreducible, any G-module endomorphism of V must be a scalar multiple of the identity. Therefore $L = cI$ for some $c \in \mathbb{C}$, and so X is also a scalar multiple of the identity. Hence, the Hermitian inner product H is unique up to a scalar multiple. □

Remark 5. *Note that the property of being* unitary *depends on the choice of Hermitian inner product. The same representation may be unitary with respect to one inner product but not with respect to another.*

Exercise 2 (One-dimensional characters arise from the abelianization)**.** *Let G be a finite group, and let $G^{\mathrm{ab}} := G/[G,G]$ denote its abelianization. Show that*

$$\mathrm{Irr}_1(G) \;\cong\; \mathrm{Hom}(G,\mathbb{C}^\times) \;\cong\; \mathrm{Hom}\big(G^{\mathrm{ab}},\mathbb{C}^\times\big),$$

and hence that the number of one-dimensional irreducible characters of G equals $|G^{\mathrm{ab}}|$.

Proof. A one-dimensional complex representation of G is a homomorphism

$$\rho : G \longrightarrow \mathrm{GL}_1(\mathbb{C}) = \mathbb{C}^\times,$$

so the set of all such representations is precisely $\mathrm{Hom}(G,\mathbb{C}^\times)$. For each $g \in G$, its character is given by

$$\chi_\rho(g) = \rho(g).$$

Since $\mathbb{C}^\times$ is abelian, for all $g, h \in G$ we have

$$\rho([g,h]) = \rho(ghg^{-1}h^{-1}) = \rho(g)\rho(h)\rho(g)^{-1}\rho(h)^{-1} = 1.$$

Thus $[G,G] \subseteq \ker\rho$, meaning that ρ is constant on cosets of the commutator subgroup. Therefore ρ factors uniquely through a homomorphism

$$\overline{\rho} : G^{\mathrm{ab}} \to \mathbb{C}^\times \quad \text{such that} \quad \rho = \overline{\rho} \circ q,$$

where $q : G \to G^{\mathrm{ab}}$ is the canonical quotient map. This establishes a bijection

$$\mathrm{Hom}(G,\mathbb{C}^\times) \xrightarrow{\sim} \mathrm{Hom}\big(G^{\mathrm{ab}},\mathbb{C}^\times\big), \qquad \rho \longmapsto \overline{\rho}.$$

If G is finite, then G^{ab} is a finite abelian group. Each homomorphism $G^{\mathrm{ab}} \to \mathbb{C}^\times$ has image in the roots of unity, and the group of all such homomorphisms (the character group) has the same order as G^{ab}. Hence

$$\#\,\mathrm{Irr}_1(G) = \#\,\mathrm{Hom}\big(G^{\mathrm{ab}},\mathbb{C}^\times\big) = |G^{\mathrm{ab}}|.$$

□

Remark 6. *1. A linear character χ satisfies $\chi([g,h]) = 1$ for all $g, h \in G$.*

2. There are no nontrivial one-dimensional characters if and only if G is perfect*, i.e. $[G,G] = G$.*

3. The set of one-dimensional characters forms an abelian group under pointwise multiplication, canonically isomorphic to the Pontryagin dual $\widehat{G^{\mathrm{ab}}}$.

Example 7.

1. *For S_n with $n \geq 2$, we have $S_n^{\mathrm{ab}} \cong C_2$. Thus there are exactly two one-dimensional characters: the trivial and the sign character.*
2. *For A_n with $n \geq 5$, A_n is perfect, so $A_n^{\mathrm{ab}} = 0$. Therefore the only one-dimensional character is the trivial one.*
3. *For the dihedral group D_{2m},*

$$D_{2m}^{\mathrm{ab}} \cong \begin{cases} C_2 \times C_2, & m \text{ even}, \\ C_2, & m \text{ odd}, \end{cases}$$

giving 4 or 2 one-dimensional characters respectively.

Chapter 2

Characters

The previous chapter suggests that knowing the eigenvalues of each element in a group G is enough to uniquely identify a representation. In fact, the key observation is that it suffices to know just the *sum* of eigenvalues for each element of G, that is, the trace.

2.1. Recovering Eigenvalues from Traces

The values of $\operatorname{tr}(g^k)$ for all $k \geq 1$ allow us to determine all eigenvalues of g. This follows from the formula for the characteristic polynomial:

$$\det(tI - M) = t^n - (\operatorname{tr} M)t^{n-1} + \cdots + (-1)^n \det(M),$$

where each coefficient can be expressed through traces of powers $\operatorname{tr}(M^k)$.

Example: 2×2 diagonal matrix

Let $M = \operatorname{diag}(a, b)$. Then

$$\det(tI - M) = (t - a)(t - b) = t^2 - (a + b)t + ab = t^2 - (\operatorname{tr} M)t + \det M.$$

Moreover, $\operatorname{tr}(M^2) = a^2 + b^2 = (\operatorname{tr} M)^2 - 2 \det M$.

Example: 3×3 diagonal matrix

Let $M = \operatorname{diag}(a, b, c)$. Then

$$\det(tI - M) = t^3 - (a + b + c)t^2 + (ab + ac + bc)t - abc,$$

and the coefficients can be expressed in terms of $\operatorname{tr}(M)$, $\operatorname{tr}(M^2)$, and $\operatorname{tr}(M^3)$ as demonstrated below.

3×3: expressing $ab + ac + bc$ and abc via traces

Let the eigenvalues of M be a, b, c and set

$$p_1 = \operatorname{tr}(M) = a + b + c, \qquad p_2 = \operatorname{tr}(M^2) = a^2 + b^2 + c^2, \qquad p_3 = \operatorname{tr}(M^3) = a^3 + b^3 + c^3.$$

Write the characteristic polynomial as

$$\det(tI - M) = t^3 - (a + b + c)t^2 + (ab + ac + bc)\,t - abc = t^3 - e_1 t^2 + e_2 t - e_3,$$

so $e_1 = p_1,\ e_2 = ab + ac + bc,\ e_3 = abc.$

Coefficient of t: $ab + ac + bc$. Use the square identity:

$$(a + b + c)^2 = a^2 + b^2 + c^2 + 2(ab + ac + bc).$$

Hence

$$\boxed{ab + ac + bc \;=\; \frac{(a + b + c)^2 - (a^2 + b^2 + c^2)}{2} = \frac{(\operatorname{tr} M)^2 - \operatorname{tr}(M^2)}{2}.}$$

Constant term: abc. Use the cube identity:

$$(a + b + c)^3 \;=\; a^3 + b^3 + c^3 \;+\; 3(a + b + c)(ab + ac + bc) \;-\; 3abc.$$

Solve for abc:

$$abc \;=\; \frac{a^3 + b^3 + c^3 \;+\; 3(a + b + c)(ab + ac + bc) \;-\; (a + b + c)^3}{3}.$$

Substitute p_1, p_2, p_3 and the expression for $ab + ac + bc$ above:

$$\boxed{abc \;=\; \frac{(\operatorname{tr} M)^3 - 3 \operatorname{tr} M \cdot \operatorname{tr}(M^2) + 2 \operatorname{tr}(M^3)}{6}.}$$

Equivalently, in Newton-identity form:

$$e_2 = \frac{p_1^2 - p_2}{2}, \qquad e_3 = \frac{p_1^3 - 3p_1 p_2 + 2p_3}{6}.$$

2.2. Newton's Identities

Observation on symmetric polynomials

For a polynomial that splits completely,

$$P(t) = \prod_{j=1}^{n} (t - a_j) = t^n - e_1 t^{n-1} + e_2 t^{n-2} - \cdots + (-1)^n e_n,$$

each coefficient e_m is the *elementary symmetric polynomial* in the roots:

$$e_m = \sum_{1 \le i_1 < \cdots < i_m \le n} a_{i_1} \cdots a_{i_m}, \quad (e_0 := 1).$$

It is therefore natural to ask how these coefficients relate to the *power sums*

$$p_k = \sum_{j=1}^{n} a_j^k = \operatorname{tr}(M^k).$$

Generating function derivation

Define the generating function of elementary symmetric polynomials:

$$E(t) = \sum_{m=0}^{n} e_m t^m = \prod_{j=1}^{n} (1 + a_j t).$$

Although the right-hand side is a polynomial of degree n and the left-hand side is a finite series, this equality is understood as one between *formal power series* expanded about $t = 0$. Thus differentiation and termwise comparison are valid operations in the ring of formal series.

Differentiating $\log E(t)$, we obtain

$$\frac{E'(t)}{E(t)} = \sum_{j=1}^{n} \frac{a_j}{1 + a_j t} = \sum_{k \geq 1} (-1)^{k-1} p_k \, t^{k-1}.$$

Multiplying by $E(t)$ and by t gives

$$tE'(t) = \Big(\sum_{k \geq 1} (-1)^{k-1} p_k \, t^k \Big) E(t).$$

Expanding $E(t) = \sum_{m \geq 0} e_m t^m$ and comparing coefficients of t^m on both sides yields

$$m \, e_m = \sum_{i=1}^{m} (-1)^{i-1} e_{m-i} \, p_i, \quad m \geq 1.$$

These are the **Newton identities**, expressing each symmetric coefficient e_m in terms of power sums $p_1, \ldots, p_m$.

Examples

$$\begin{aligned} e_1 &= p_1, \\ e_2 &= \frac{p_1^2 - p_2}{2}, \\ e_3 &= \frac{p_1^3 - 3p_1p_2 + 2p_3}{6}. \end{aligned}$$

Application to 3×3 matrices

For a 3×3 matrix M with eigenvalues a, b, c, the characteristic polynomial

$$\det(tI - M) = t^3 - e_1 t^2 + e_2 t - e_3$$

can be expressed entirely through traces:

$$\boxed{\begin{aligned} e_1 &= \mathrm{tr}(M), \\ e_2 &= \frac{\mathrm{tr}(M)^2 - \mathrm{tr}(M^2)}{2}, \\ e_3 &= \frac{\mathrm{tr}(M)^3 - 3 \, \mathrm{tr}(M) \, \mathrm{tr}(M^2) + 2 \, \mathrm{tr}(M^3)}{6}. \end{aligned}}$$

2.3. Definition of Character

From the above, it is of interest to examine the trace for each element $g \in G$ acting on the module V.

Definition 3. *Let V be a G-module over $\mathbb{C}$. The **character** of V is the complex-valued function*

$$\chi_V(g) = \mathrm{Tr}(g|_V)$$

for each $g \in G$.
In particular, for any $h, g \in G$,

$$\chi_V(hgh^{-1}) = \chi_V(g),$$

and for the identity element $e \in G$,

$$\chi_V(e) = \mathrm{tr}(I) = \dim V.$$

Remark 7 (Conjugacy Invariance)**.** *Since character values are invariant under conjugation, all elements within the same conjugacy class share the same character value. However, this value may vary depending on the G-module V. Given a list of irreducible G-modules, each has a distinct character function, and the set of these characters uniquely determines the representation theory of G.*

Remark 8. *(The original fixed-point formula). If V is the permutation representation associated to the action of a group G on a finite set X, then $\chi_V(g) = tr(g|_V)$ is the number of elements of X fixed by g. By viewing g as a permutation matrix, we know that identity value in the diagonal entry implies a fixed element by permutation.*

Remark 9. *Since we define the G-module action by group homomorphism $\rho : G \to GL(V)$ then for any finite group G we have*

$$g^{ord(g)} = 1 \implies \rho^{ord(g)}(g) = I \implies diag(\rho(g))^{ord(g)} = I \implies \forall\lambda(g): \ \lambda(g)^{ord(g)} = 1$$

Definition 4. *The* ***character table*** *of a finite group G is a table whose rows correspond to irreducible G-modules and whose columns correspond to conjugacy classes of G. The entry in the row of V_i and column of class C_j is $\chi_{V_i}(g_j)$ for any representative $g_j \in C_j$.*

Proposition 2. *For G-modules V and W, the following relations hold by linearity of trace:*

$$\chi_{V\oplus W} = \chi_V + \chi_W, \qquad \chi_{V\otimes W} = \chi_V \chi_W, \qquad \chi_{V^*} = \overline{\chi_V}.$$

Moreover, for the exterior square of V,

$$\chi_{\wedge^2 V}(g) = \frac{1}{2}\left(\chi_V(g)^2 - \chi_V(g^2)\right).$$

Remark 10. *The proof is elementary, it is omitted and can be found in F-H.*

Let V, W be finite-dimensional G-modules over $\mathbb{C}$. A basic result from linear algebra establishes the following natural isomorphism:

$$\mathrm{Hom}(V, W) \ \cong \ V^* \otimes W.$$

Proof. Define the map

$$\Psi : V^* \otimes W \longrightarrow \mathrm{Hom}(V, W), \qquad \Psi(v^* \otimes w)(x) = v^*(x)\, w, \quad x \in V.$$

The map is linear in both arguments and injective: if $\Psi(v_1^* \otimes w) = \Psi(v_2^* \otimes w)$, then $v_1^*(x)w = v_2^*(x)w$ for all $x \in V$, hence $v_1^* = v_2^*$.

To see surjectivity, take bases

$$\beta_{V^*} = \{v_1^*, \dots, v_n^*\}, \qquad \beta_W = \{w_1, \dots, w_m\}.$$

Then the family $\{\Psi(v_i^* \otimes w_j)\}_{i,j}$ has $n \times m$ elements, each defined by

$$x \mapsto v_i^*(x)\, w_j.$$

These are linearly independent and span $\mathrm{Hom}(V, W)$, which also has dimension $\dim V^* \cdot \dim W = nm$. Hence Ψ is bijective.

As a corollary, taking characters gives

$$\boxed{\chi_{\mathrm{Hom}(V,W)} = \overline{\chi_V}\, \chi_W.}$$

∎

Example: Characters of Symmetric and Exterior Powers (Fulton–Harris, Exercise 2.3)

Let V be an N-dimensional complex vector space with ordered eigenbasis

$$\beta = \{e_1, e_2, \dots, e_N\}$$

for a linear operator $g \in G$, having corresponding eigenvalues

$$(\lambda_1, \lambda_2, \dots, \lambda_N).$$

Symmetric power. The space $\mathrm{Sym}^k(V)$ has eigenbasis

$$\beta_{\mathrm{sym}} = \{\, \mathrm{sym}(e_{j_1} \otimes \cdots \otimes e_{j_k}) \mid 1 \le j_1 \le \cdots \le j_k \le N \},$$

where "sym" denotes the symmetrization operator. For each basis vector,

$$g \cdot \mathrm{sym}(e_{j_1} \otimes \cdots \otimes e_{j_k}) = (\lambda_{j_1} \cdots \lambda_{j_k})\, \mathrm{sym}(e_{j_1} \otimes \cdots \otimes e_{j_k}).$$

Hence the character of g on $\mathrm{Sym}^k(V)$ is

$$\boxed{\chi_{\mathrm{Sym}^k(V)}(g) = \sum_{1 \le j_1 \le \cdots \le j_k \le N} \lambda_{j_1}\lambda_{j_2}\cdots\lambda_{j_k} = \sum_{\substack{x_1+\cdots+x_N=k \\ x_i \ge 0}} \lambda_1^{x_1}\cdots\lambda_N^{x_N}.}$$

Exterior power. The space $\wedge^k(V)$ has eigenbasis

$$\beta_{\mathrm{alt}} = \{\, \mathrm{alt}(e_{j_1} \otimes \cdots \otimes e_{j_k}) \mid 1 \le j_1 < \cdots < j_k \le N \},$$

where "alt" denotes antisymmetrization. In this case,

$$g \cdot \mathrm{alt}(e_{j_1} \otimes \cdots \otimes e_{j_k}) = (\lambda_{j_1} \cdots \lambda_{j_k})\, \mathrm{alt}(e_{j_1} \otimes \cdots \otimes e_{j_k}),$$

so that

$$\boxed{\chi_{\wedge^k(V)}(g) = \sum_{1 \le j_1 < \cdots < j_k \le N} \lambda_{j_1}\lambda_{j_2}\cdots\lambda_{j_k}.}$$

Remark. The distinction between the two cases lies in the allowed repetition of indices: for $\mathrm{Sym}^k(V)$, repeated eigenvectors contribute (hence the inequality $\le$); for $\wedge^k(V)$, the antisymmetry forces all indices to be distinct ($<$).

2.4. Character Table of S_3 and Applications

Character table of S_3

We now introduce the character table for the three irreducible S_3-modules:

$$W \in \{\, V_{\mathrm{triv}},\, V_{\mathrm{alt}},\, V_{\mathrm{std}} \,\}.$$

There are three conjugacy classes in S_3:

Class	Representative	Cycle type	Size
C_1	e	(1)(2)(3)	1
C_2	(12)	transpositions	3
C_3	(123)	3-cycles	2

Each character takes a single value on each conjugacy class. For the trivial and alternating representations:

$$\chi_{V_{\mathrm{triv}}} = (1, 1, 1), \qquad \chi_{V_{\mathrm{alt}}} = (1, -1, 1).$$

By viewing S_3 as a permutation group on the basis of 2-dimensional standard module V_{std}, we get:

$$\chi_{V_{\mathrm{std}}}(e) = 2, \qquad \chi_{V_{\mathrm{std}}}((12)) = 0, \qquad \chi_{V_{\mathrm{std}}}((123)) = -1.$$

This follows because if $\tau = (123)$, then its eigenvalues are $\omega = e^{2\pi i/3}$ and ω^2, giving

$$\mathrm{tr}(\tau) = \omega + \omega^2 = 2\cos\left(\frac{2\pi}{3}\right) = -1.$$

Hence the complete character table is:

	C_1	C_2	C_3
$\chi_{V_{\mathrm{triv}}}$	1	1	1
$\chi_{V_{\mathrm{alt}}}$	1	-1	1
$\chi_{V_{\mathrm{std}}}$	2	0	-1

Linear combinations of irreducibles

Every S_3-module W can be expressed as a direct sum of irreducibles:

$$W = V_{\text{std}}^{\oplus a} \oplus V_{\text{triv}}^{\oplus b} \oplus V_{\text{alt}}^{\oplus c}.$$

Its character is then

$$\chi_W = a\,\chi_{V_{\text{std}}} + b\,\chi_{V_{\text{triv}}} + c\,\chi_{V_{\text{alt}}}.$$

Since the irreducible characters are linearly independent, χ_W determines W up to isomorphism.

Tensor powers of the standard representation (Exercise 2.7)

For fixed $n \geq 2$, consider $V_{\text{std}}^{\otimes n}$. Since $\chi_{V_{\text{std}}^{\otimes n}} = (\chi_{V_{\text{std}}})^n$, we can compute its values on the three conjugacy classes:

$$\chi_{V_{\text{std}}^{\otimes n}} = (2^n,\, 0,\, (-1)^n).$$

Expressing this as a linear combination of the three irreducible characters:

$$(2^n,\, 0,\, (-1)^n) = a(2, 0, -1) + b(1, 1, 1) + c(1, -1, 1).$$

Solving for a, b, c gives

$$\boxed{a = \frac{2^n - (-1)^n}{3}, \qquad b = c = \frac{2^{n-1} + (-1)^n}{3}.}$$

Projection onto G-invariants

For any G-module V, define the subspace of G-invariant vectors:

$$V^G = \{\, v \in V : g \cdot v = v \text{ for all } g \in G \,\}.$$

Define the averaging operator

$$\phi = \frac{1}{|G|} \sum_{g \in G} g.$$

Then for every $v \in V$,

$$g \cdot \phi(v) = \phi(v) \quad \text{for all } g \in G,$$

and for $v \in V^G$, we have $\phi(v) = v$. Hence $\phi^2 = \phi$ and ϕ is a projection onto V^G.

Dimension of V^G and orthogonality relations

Taking the trace of ϕ:

$$m = \dim V^G = \operatorname{tr}(\phi) = \frac{1}{|G|} \sum_{g \in G} \operatorname{tr}(g|_V) = \frac{1}{|G|} \sum_{g \in G} \chi_V(g).$$

More generally, for two G-modules V and W,

$$\operatorname{Hom}_G(V, W) = \{\, f : V \to W \text{ such that } f(gv) = gf(v) \,\}.$$

If V and W are irreducible, Schur's lemma implies

$$\dim \operatorname{Hom}_G(V, W) = \begin{cases} 1, & V \cong W, \\ 0, & V \ncong W. \end{cases}$$

Equivalently,

$$\frac{1}{|G|} \sum_{g \in G} \chi_{\operatorname{Hom}(V,W)}(g) = \begin{cases} 1, & V \cong W, \\ 0, & V \ncong W. \end{cases}$$

2.5. Class Functions, Orthonormality of Characters, and Regular Representations

Class functions

Definition 5. *A **class function** on a finite group G is a complex-valued function*

$$f : G \to \mathbb{C}$$

that is constant on conjugacy classes; that is,

$$f(hgh^{-1}) = f(g) \quad \text{for all } g, h \in G.$$

Examples.

- The *character* $\chi_V(g) = \mathrm{tr}(g|_V)$ of a representation V is a class function.
- The determinant map $g \mapsto \det(g|_V)$ is also a class function.

Orthonormality of irreducible characters

Let V, W be finite-dimensional G-modules. From previous results we know:

$$\frac{1}{|G|} \sum_{g \in G} \overline{\chi_V(g)}\, \chi_W(g) = \begin{cases} 1, & V \cong W, \\ 0, & V \not\cong W. \end{cases}$$

We can define a Hermitian inner product on the space of class functions on G by

$$(\alpha, \beta) = \frac{1}{|G|} \sum_{g \in G} \overline{\alpha(g)}\, \beta(g).$$

Theorem 2. *With respect to this inner product, the characters of the irreducible G-modules form an orthonormal set:*

$$(\chi_V, \chi_W) = \delta_{VW}.$$

Decomposition and multiplicity formula

Let

$$V = V_1^{\oplus a_1} \oplus V_2^{\oplus a_2} \oplus \cdots \oplus V_n^{\oplus a_n}$$

be a decomposition of a G-module into irreducible components V_i.

Then:

$$\chi_V = \sum_{i=1}^{n} a_i \chi_{V_i}.$$

By orthonormality,

$$(\chi_V, \chi_V) = \sum_{i=1}^{n} a_i^2.$$

In particular, V is irreducible if and only if $(\chi_V, \chi_V) = 1$.

Regular representation and the fixed-point formula

For any finite group G, define its **regular representation**:

$$R = \Big\{ \sum_{g \in G} c_g g \;\Big|\; c_g \in \mathbb{C} \Big\},$$

where G acts by left multiplication:

$$h \cdot \Big(\sum_{g \in G} c_g g \Big) = \sum_{g \in G} c_g (hg).$$

If a finite group G acts on a finite set X, the corresponding permutation representation $V = \mathbb{C}[X]$ has character

$$\chi_V(g) = \#\{\, x \in X \mid g \cdot x = x \,\},$$

i.e. the number of fixed points of g.

For the regular representation R, each $g \in G$ acts by permuting the basis $\{e_h : h \in G\}$. Only the identity element fixes every basis vector, hence:

$$\boxed{\chi_R(g) = \begin{cases} |G|, & g = e, \\ 0, & g \neq e. \end{cases}}$$

Decomposition of the regular representation

Suppose

$$R = \bigoplus_{i=1}^{n} V_i^{\oplus a_i}$$

is the decomposition of the regular representation into distinct irreducibles V_i.

Then

$$a_i = (\chi_R, \chi_{V_i}) = \frac{1}{|G|} \sum_{g \in G} \overline{\chi_{V_i}(g)}\, \chi_R(g) = \frac{1}{|G|}\, \overline{\chi_{V_i}(e)}\, |G| = \chi_{V_i}(e) = \dim V_i.$$

Hence each irreducible V_i appears in the regular representation, and it does so precisely $\dim V_i$ times, thus making

$$\boxed{\chi_R = \sum_{i=1}^{n} (\dim V_i)\, \chi_{V_i}.}$$

Equivalently, *any G-module W is isomorphic to a direct sum of submodules of $R = \mathbb{C}\,[G]$*

Fourier inversion formulas for finite groups

These two relations summarize the decomposition of the regular representation:

$$\boxed{\begin{aligned} \chi_R(e) &= |G| = \dim R = \sum_{i=1}^{n} (\dim V_i)^2, \\ \chi_R(g) &= 0 = \sum_{i=1}^{n} (\dim V_i)\, \chi_{V_i}(g), \quad g \neq e. \end{aligned}}$$

These identities are often viewed as the **Fourier inversion formulas** for finite groups.

2.6. Example: Classification of Irreducible S_4-Modules

Let $G = S_4$, the symmetric group on four elements. We will classify all irreducible G-modules by examining their characters.

1. The natural permutation representation and its decomposition

Let $V = \mathrm{span}_{\mathbb{C}}\{e_1, e_2, e_3, e_4\}$, where S_4 acts by permuting the basis vectors.

Define the following two invariant subspaces:

$$U = \mathrm{span}\{e_1 + e_2 + e_3 + e_4\}, \qquad W = \{(x_1, x_2, x_3, x_4) \in \mathbb{C}^4 \mid x_1 + x_2 + x_3 + x_4 = 0\}.$$

Then

$$V = U \oplus W,$$

where U carries the **trivial representation** and W carries the **standard representation** of S_4.

2. Conjugacy classes in S_4

The group S_4 has five conjugacy classes:

Cycle type	Representative	Size
(1)(2)(3)(4)	e	1
(12)	transpositions	6
(123)	3-cycles	8
(1234)	4-cycles	6
(12)(34)	double transpositions	3

3. Known irreducible characters

We already know three irreducible representations:

$$V_{\mathrm{triv}}, \quad V_{\mathrm{alt}}, \quad V_{\mathrm{std}}.$$

Their characters over the five conjugacy classes are:

$$\chi_{V_{\mathrm{triv}}} = (1, 1, 1, 1, 1),$$
$$\chi_{V_{\mathrm{alt}}} = (1, -1, 1, -1, 1).$$

For the permutation representation $\mathbb{C}^4$, each group element acts by permuting the basis $\{e_i\}$. The trace of such a permutation matrix equals the number of fixed basis vectors. Hence:

$$\chi_{\mathbb{C}^4} = (4, 2, 1, 0, 0).$$

Using the decomposition $\mathbb{C}^4 = V_{\mathrm{triv}} \oplus V_{\mathrm{std}}$, we find

$$\boxed{\chi_{V_{\mathrm{std}}} = \chi_{\mathbb{C}^4} - \chi_{V_{\mathrm{triv}}} = (3, 1, 0, -1, -1).}$$

4. Checking irreducibility of the standard representation

Compute the norm:

$$\|\chi_{\mathrm{std}}\|^2 = \frac{1}{|S_4|} \sum_{g \in S_4} |\chi_{\mathrm{std}}(g)|^2 = \frac{1}{24}\big(1 \cdot 3^2 + 6 \cdot 1^2 + 8 \cdot 0^2 + 6 \cdot (-1)^2 + 3 \cdot (-1)^2\big) = 1.$$

Thus V_{std} is irreducible.

5. Counting dimensions of all irreducibles

We know that for any finite group:

$$|G| = \sum_{\text{irreducible } V} (\dim V)^2.$$

For S_4:

$$24 = 1^2 + 1^2 + 3^2 + \sum_{\text{remaining}} (\dim V)^2 \quad \Rightarrow \quad \sum_{\text{remaining}} (\dim V)^2 = 13 = 2^2 + 3^2.$$

Hence there are two more irreducible representations of dimensions 2 and 3.

6. Constructing a new irreducible module

Fulton–Harris suggests considering
$$V_{\mathrm{prod}} = V_{\mathrm{std}} \otimes V_{\mathrm{alt}}.$$
Since $\chi_{V_{\mathrm{alt}}}$ only changes the sign of odd permutations, we obtain:
$$\chi_{\mathrm{prod}} = \chi_{\mathrm{std}} \cdot \chi_{\mathrm{alt}} = (3, -1, 0, 1, -1).$$
Compute its norm:
$$\|\chi_{\mathrm{prod}}\|^2 = \frac{1}{24}\big(1 \cdot 3^2 + 6 \cdot (-1)^2 + 8 \cdot 0^2 + 6 \cdot 1^2 + 3 \cdot (-1)^2\big) = 1.$$
Hence V_{prod} is irreducible.

7. Determining the final irreducible representation

Let V_{last} be the final unknown irreducible S_4-module. From the dimension formula above, $\dim V_{\mathrm{last}} = 2$. To determine its character, use the relation
$$0 = \sum_i (\dim V_i)\, \chi_{V_i}(g), \quad \text{for all } g \neq e.$$
Substituting all known irreducibles gives:
$$\boxed{\chi_{V_{\mathrm{last}}} = (2, 0, -1, 0, 2).}$$

8. Complete character table of S_4

	(1)	(12)	(123)	(1234)	(12)(34)
$\chi_{V_{\mathrm{triv}}}$	1	1	1	1	1
$\chi_{V_{\mathrm{alt}}}$	1	-1	1	-1	1
$\chi_{V_{\mathrm{std}}}$	3	1	0	-1	-1
$\chi_{V_{\mathrm{prod}}}$	3	-1	0	1	-1
$\chi_{V_{\mathrm{last}}}$	2	0	-1	0	2

We have thus obtained all five irreducible representations of S_4, whose squared dimensions sum to $1^2 + 1^2 + 3^2 + 3^2 + 2^2 = 24 = |S_4|$.

2.7. Example: Classifying the Irreducible Modules of A_4

Let $G = A_4$ (order $|G| = 12$). The conjugacy classes are:

$C_1 = \{e\}$ (1), $\quad C_2 = \{$3-cycles of type (123)$\}$ (4 elements), $\quad C_3 = \{$3-cycles of type (132)$\}$ (4 elements),

$$C_4 = \{\text{double transpositions } (12)(34)\} \text{ (3 elements)}.$$

The one–dimensional characters

Since $A_4/V_4 \cong C_3$, there are three 1-dimensional characters:
$$\chi_1 = (1,\, 1,\, 1,\, 1), \quad \chi_\omega = (1,\, \omega,\, \omega^2,\, 1), \quad \chi_{\overline{\omega}} = (1,\, \omega^2,\, \omega,\, 1),$$
where $\omega = e^{2\pi i/3}$ and $\omega + \omega^2 = -1$.

Dimension count

By $\sum(\dim V_i)^2 = |G| = 12$, with three linear characters we must have one remaining irreducible W of $\dim W = 3$. Thus

$$\chi_W(C_1) = \chi_W(e) = 3.$$

Solving for χ_W on all classes

Impose orthogonality of χ_W with each linear character using

$$\langle \alpha, \beta \rangle = \frac{1}{|G|} \sum_{\text{classes } C} |C|\ \alpha(C)\ \overline{\beta(C)}.$$

Let $X = (x_1, x_2, x_3, x_4)^\mathsf{T}$ with $x_j = \chi_W(C_j)$. We solve the 4×4 system

$$\underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 4 & 4 & 3 \\ 1 & 4\overline{\omega} & 4\overline{\omega}^{\,2} & 3 \\ 1 & 4\overline{\omega}^{\,2} & 4\overline{\omega} & 3 \end{pmatrix}}_{A} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \underbrace{\begin{pmatrix} 3 \\ 0 \\ 0 \\ 0 \end{pmatrix}}_{Y}.$$

(Here the first row fixes $x_1 = \chi_W(e) = 3$; the other rows are $\langle \chi_W, \chi_1 \rangle = 0$, $\langle \chi_W, \chi_\omega \rangle = 0$, $\langle \chi_W, \chi_{\overline{\omega}} \rangle = 0$.)

Quick elimination. Subtract the last two equations:

$$4(\overline{\omega} - \overline{\omega}^{\,2})(x_2 - x_3) = 0 \ \Rightarrow\ x_2 = x_3.$$

Use $\omega + \omega^2 = -1$ in either complex equation to get $x_1 - 4x_2 + 3x_4 = 0$. Combine with the real equation $x_1 + 8x_2 + 3x_4 = 0$ to deduce $x_2 = 0$, hence $x_3 = 0$. Finally, with $x_1 = 3$ the real equation gives $3 + 3x_4 = 0 \Rightarrow x_4 = -1$.

$$\boxed{\chi_W = (\,3,\,0,\,0,\,-1\,) \quad \text{on } (C_1, C_2, C_3, C_4).}$$

Character table of A_4

	$C_1 = e$	$C_2 : (123)$	$C_3 : (132)$	$C_4 : (12)(34)$
χ_1	1	1	1	1
χ_ω	1	ω	ω^2	1
$\chi_{\overline{\omega}}$	1	ω^2	ω	1
χ_W	3	0	0	-1

One checks $\sum_i(\dim V_i)^2 = 1^2 + 1^2 + 1^2 + 3^2 = 12$ and orthogonality holds.

2.8. Class Functions, Completeness of Irreducible Characters, and Projections

A class function yields a G-map on every module

Proposition 3. *Let $\alpha : G \to \mathbb{C}$ be any function and, for each G-module V with action $\rho : G \to \mathrm{GL}(V)$, define*

$$\Phi_{\alpha,V} \ = \ \sum_{g \in G} \alpha(g)\, \rho(g) \ :\ V \to V.$$

Then $\Phi_{\alpha,V}$ is G-linear (i.e. $\Phi_{\alpha,V}(\rho(h)v) = \rho(h)\Phi_{\alpha,V}(v)$ for all $h \in G$, $v \in V$) for every *V if and only if α is a class function.*

Remark 11. *If α is a class function, then*

$$\rho(h)\Phi_{\alpha,V}\rho(h)^{-1} = \sum_{g\in G}\alpha(g)\,\rho(hgh^{-1}) = \sum_{g\in G}\alpha(hgh^{-1})\,\rho(g) = \Phi_{\alpha,V}\,,$$

so $\Phi_{\alpha,V}$ commutes with $\rho(h)$ for all h and is thus G-linear. Conversely, if α fails to be constant on some conjugacy class, one can choose a representation where the above commutation fails, so G-linearity does not hold for all V.

Completeness of irreducible characters

Theorem 3. *For a finite group G, the number of isomorphism classes of irreducible G-modules equals the number of conjugacy classes of G. Equivalently, the set of irreducible characters $\{\chi_V\}$ is an orthonormal basis of the space $\mathbb{C}_{\text{class}}(G)$ of class functions (with inner product $\langle\alpha,\beta\rangle = \frac{1}{|G|}\sum_{g\in G}\overline{\alpha(g)}\,\beta(g)$).*

Proof sketch. Let α be a class function with $\langle\alpha,\chi_V\rangle = 0$ for every irreducible V. For any irreducible V, Schur's lemma implies $\Phi_{\alpha,V} = \lambda_V\,\mathrm{Id}_V$, hence

$$\lambda_V \;=\; \frac{1}{\dim V}\,\mathrm{tr}(\Phi_{\alpha,V}) = \frac{1}{\dim V}\sum_{g\in G}\alpha(g)\,\chi_V(g) = \frac{|G|}{\dim V}\,\overline{\langle\alpha,\chi_V\rangle} = 0.$$

Thus $\Phi_{\alpha,V} = 0$ for every irreducible V. By additivity, $\Phi_{\alpha,R} = 0$ on the regular representation R. But R has basis $\{e_g\}_{g\in G}$ on which $\rho(h)$ permutes basis vectors, so

$$\Phi_{\alpha,R}(e_e) = \sum_{g\in G}\alpha(g)\,e_g = 0 \quad\Longrightarrow\quad \alpha(g) = 0 \text{ for all } g.$$

Hence the only class function orthogonal to all irreducible characters is 0, so the χ_V span $\mathbb{C}_{\text{class}}(G)$. Together with orthonormality, they form an orthonormal basis, and their number equals the dimension of $\mathbb{C}_{\text{class}}(G)$, i.e. the number of conjugacy classes. ☐

We are now in the position to give the second orthogonality relation for group characters.

Theorem 4 (Column orthogonality of characters)**.** *Let G be a finite group with conjugacy classes*

$$C_1,\ldots,C_r$$

and let $\chi_1,\ldots,\chi_r$ be the irreducible characters of G. Then for $1\le m,n\le r$ we have

$$\sum_{i=1}^{r}\chi_i(C_m)\,\overline{\chi_i(C_n)} = \begin{cases}\dfrac{|G|}{|C_m|}, & \text{if } C_m = C_n,\\ 0, & \text{if } C_m \neq C_n.\end{cases}$$

Proof. Recall the standard inner product on class functions

$$\langle\varphi,\psi\rangle := \frac{1}{|G|}\sum_{g\in G}\varphi(g)\,\overline{\psi(g)}.$$

With this inner product, the irreducible characters are orthonormal:

$$\langle\chi_i,\chi_j\rangle = \delta_{ij}.$$

Since characters are constant on conjugacy classes, we can rewrite this using the conjugacy classes $C_1,\ldots,C_r$:

$$\langle\chi_i,\chi_j\rangle = \frac{1}{|G|}\sum_{m=1}^{r}\sum_{g\in C_m}\chi_i(g)\,\overline{\chi_j(g)} = \frac{1}{|G|}\sum_{m=1}^{r}|C_m|\,\chi_i(C_m)\,\overline{\chi_j(C_m)}.$$

Thus
$$\frac{1}{|G|}\sum_{m=1}^{r} |C_m|\,\chi_i(C_m)\,\overline{\chi_j(C_m)} = \delta_{ij}. \tag{2.1}$$

Now define the $r \times r$ matrix $A = (a_{im})$ by

$$a_{im} := \sqrt{\frac{|C_m|}{|G|}}\,\chi_i(C_m), \qquad 1 \le i, m \le r.$$

Then (2.1) can be rewritten as
$$\sum_{m=1}^{r} a_{im}\,\overline{a_{jm}} = \delta_{ij},$$
which is exactly the statement that
$$A\,A^* = I_r,$$
where A^* is the conjugate transpose of A. Thus the rows of A are orthonormal.

By the general theory of characters, the number of irreducible characters equals the number of conjugacy classes, so r is both the number of rows and the number of columns of A. Hence A is a square matrix, and the condition $AA^* = I_r$ implies that A is unitary. Therefore also

$$A^*A = I_r.$$

Now look at the (m, n)-entry of A^*A:

$$(A^*A)_{mn} = \sum_{i=1}^{r} \overline{a_{im}}\,a_{in} = \frac{\sqrt{|C_m||C_n|}}{|G|}\sum_{i=1}^{r} \overline{\chi_i(C_m)}\,\chi_i(C_n).$$

Since $A^*A = I_r$, we have
$$(A^*A)_{mn} = \delta_{mn},$$
so
$$\frac{\sqrt{|C_m||C_n|}}{|G|}\sum_{i=1}^{r} \overline{\chi_i(C_m)}\,\chi_i(C_n) = \delta_{mn}.$$

Rearranging, we obtain
$$\sum_{i=1}^{r} \chi_i(C_n)\,\overline{\chi_i(C_m)} = \frac{|G|}{\sqrt{|C_m||C_n|}}\,\delta_{mn}.$$

In particular, when $m = n$ this gives

$$\sum_{i=1}^{r} |\chi_i(C_m)|^2 = \frac{|G|}{|C_m|},$$

and when $m \neq n$ it gives
$$\sum_{i=1}^{r} \chi_i(C_n)\,\overline{\chi_i(C_m)} = 0.$$

This is exactly the desired statement. □

Proof. □

Character-theoretic projection onto an isotypic component

Corollary 2 (Canonical projector onto the W-isotypic component)**.** *Let V be a G-module and W an irreducible G-module. Suppose $V \cong U \oplus W^{\oplus n}$. Then the projection $\pi_W : V \to W^{\oplus n}$ is given by*

$$\boxed{\pi_W \;=\; \frac{\dim W}{|G|} \sum_{g\in G} \overline{\chi_W(g)}\, \rho(g).}$$

Verification on irreducibles. Since χ_W is a class function, π_W is G-linear. If S is irreducible, Schur's lemma gives $\pi_W|_S = \lambda_S \operatorname{Id}_S$ with

$$\lambda_S = \frac{1}{\dim S}\operatorname{tr}(\pi_W|_S) = \frac{\dim W}{|G|\,\dim S}\sum_{g\in G}\overline{\chi_W(g)}\,\chi_S(g) = \frac{\dim W}{\dim S}\,\langle \chi_W, \chi_S\rangle = \begin{cases} 1, & S \cong W,\\ 0, & S \not\cong W.\end{cases}$$

Thus π_W is the identity on each copy of W and zero on every irreducible inequivalent to W, hence it is the desired projection $V \to W^{\oplus n}$. □

Exercise 3. *Let G be a finite group. For a (unitary) representation $\rho : G \to U(V)$ on a d-dimensional complex Hilbert space V with orthonormal basis $\{e_1, \ldots, e_d\}$, define the* matrix coefficients

$$\rho_{kl}(g) \;=\; \langle \rho(g)e_l,\, e_k\rangle \qquad (1 \le k, l \le d,\; g \in G).$$

Consider the inner product on $\mathbb{C}^G$ (all complex-valued functions on G):

$$\langle f, h\rangle \;=\; \frac{1}{|G|}\sum_{g\in G} f(g)\,\overline{h(g)}.$$

We need to show that the matrix entries of these representations form an orthogonal basis for the space of all functions on G

Lemma 2 (Great Orthogonality)**.** *Let $\rho_i : G \to U(V_i)$ and $\rho_j : G \to U(V_j)$ be irreducible unitary representations of dimensions d_i and d_j, respectively, with matrix entries $(\rho_i(g))_{kl}$ and $(\rho_j(g))_{mn}$ in orthonormal bases. Then*

$$\sum_{g\in G} \big(\rho_i(g)\big)_{kl} \big(\rho_j(g^{-1})\big)_{mn} \;=\; \frac{|G|}{d_i}\,\delta_{ij}\,\delta_{km}\,\delta_{ln}.$$

Equivalently (using unitarity so that $\rho_j(g^{-1})_{mn} = \overline{\rho_j(g)_{nm}}$),

$$\frac{1}{|G|}\sum_{g\in G} \rho_i(g)_{kl}\,\overline{\rho_j(g)_{mn}} \;=\; \frac{1}{d_i}\,\delta_{ij}\,\delta_{km}\,\delta_{ln}.$$

Proof. Fix indices i, j, k, l, m, n. Consider the linear operator

$$T \;=\; \sum_{g\in G} \rho_i(g)\,E_{lm}\,\rho_j(g)^{-1},$$

where $E_{lm} : V_j \to V_i$ is the rank-one operator $E_{lm}(e_m^{(j)}) = e_l^{(i)}$ and $E_{lm}(e_r^{(j)}) = 0$ for $r \ne m$. A direct computation shows that, for every $h \in G$,

$$\rho_i(h)\,T = \sum_{g} \rho_i(hg)E_{lm}\rho_j(g)^{-1} = \sum_{g'} \rho_i(g')E_{lm}\rho_j(h^{-1}g')^{-1} = T\,\rho_j(h).$$

Thus T is an intertwiner from ρ_j to ρ_i. By Schur's Lemma, $T = 0$ if $i \ne j$, and $T = c\operatorname{Id}_{V_i}$ for some $c \in \mathbb{C}$ if $i = j$.

Now compute the (k, n)-entry of T:

$$T_{kn} = \sum_{g\in G} \big(\rho_i(g)\,E_{lm}\,\rho_j(g)^{-1}\big)_{kn} = \sum_{g\in G}\sum_{a,b} \rho_i(g)_{ka}\,(E_{lm})_{ab}\,\rho_j(g^{-1})_{bn} = \sum_{g\in G} \rho_i(g)_{kl}\,\rho_j(g^{-1})_{mn}.$$

Hence, if $i \neq j$, then $T = 0$ and the sum is 0. If $i = j$, then $T = c\,\mathrm{Id}$, so $T_{kn} = c\,\delta_{kn}$. Taking traces,

$$\mathrm{tr}(T) = \sum_{g\in G} \mathrm{tr}\big(\rho_i(g)E_{lm}\rho_i(g)^{-1}\big) = \sum_{g\in G} \mathrm{tr}(E_{lm}) = |G|\,\delta_{lm},$$

while $\mathrm{tr}(T) = c\,\mathrm{tr}(\mathrm{Id}) = c\,d_i$. Therefore $c = \dfrac{|G|}{d_i}\delta_{lm}$ and

$$\sum_{g\in G} \rho_i(g)_{kl}\,\rho_i(g^{-1})_{mn} = \frac{|G|}{d_i}\,\delta_{lm}\,\delta_{kn}.$$

This gives the displayed formula for all i, j (zero when $i \neq j$). Using unitarity, $\rho_j(g^{-1})_{mn} = \overline{\rho_j(g)_{nm}}$, yielding the equivalent orthogonality relation. □

Theorem 5 (Orthogonality and completeness of matrix coefficients)**.** *Let $\{\rho_i\}_{i=1}^r$ be a complete set of pairwise nonisomorphic irreducible unitary representations of G, with* $\dim V_i = d_i$*. For each i and $1 \leq k, l \leq d_i$, let*

$$\phi_{kl}^{(i)}(g) := \rho_i(g)_{kl} \quad (g \in G).$$

Then the family $\{\phi_{kl}^{(i)}\}_{i,k,l}$ is an orthogonal set in $\mathbb{C}^G$ with respect to

$$\langle f, h\rangle \;=\; \frac{1}{|G|}\sum_{g\in G} f(g)\,\overline{h(g)}.$$

Moreover, it is an orthogonal basis *of $\mathbb{C}^G$.*

Proof. Orthogonality is exactly the second equality in Lemma 2:

$$\left\langle \phi_{kl}^{(i)}, \phi_{mn}^{(j)} \right\rangle = \frac{1}{|G|}\sum_{g\in G} \rho_i(g)_{kl}\,\overline{\rho_j(g)_{mn}} = \frac{1}{d_i}\,\delta_{ij}\,\delta_{km}\,\delta_{ln}.$$

Thus distinct coefficients are orthogonal, and each $\phi_{kl}^{(i)}$ is nonzero.

To prove completeness, note that the number of these functions is

$$\sum_{i=1}^r d_i^2.$$

On the other hand, by the decomposition of the regular representation,

$$\sum_{i=1}^r d_i^2 \;=\; |G| \;=\; \dim_{\mathbb{C}} \mathbb{C}^G.$$

Therefore an orthogonal set of nonzero vectors of total cardinality $|G|$ in a complex inner product space of dimension $|G|$ must be a basis. □

Corollary 3 (Orthonormal normalization)**.** *If we set $\psi_{kl}^{(i)}(g) := \sqrt{\dfrac{d_i}{|G|}}\,\rho_i(g)_{kl}$, then*

$$\left\langle \psi_{kl}^{(i)}, \psi_{mn}^{(j)} \right\rangle = \delta_{ij}\,\delta_{km}\,\delta_{ln},$$

so $\{\psi_{kl}^{(i)}\}_{i,k,l}$ is an orthonormal basis of $\mathbb{C}^G$.

Remark 12 (Where unitarity enters)**.** *The exercise (1.14) from previous chapter ensures that each irreducible module admits a G-invariant Hermitian inner product, so we may and do choose bases making $\rho_i(g)$ unitary for all g. Unitarity is used at the key step $\rho_j(g^{-1}) = \rho_j(g)^\dagger$, which turns the sum over g^{-1} into the complex conjugate of the matrix entries at g, yielding orthogonality in the inner product on functions.*

Exercise 4. *Show that if V is a faithful representation of G, i.e., $\rho : G \to GL(V)$ is injective, then any irreducible representation of G is contained in some tensor power $V^{\otimes n}$ of V.*

Proof. Let χ_V and χ_W be the characters of V and W. For each $n \geq 1$,

$$\chi_{V^{\otimes n}}(g) = \big(\chi_V(g)\big)^n \qquad (g \in G).$$

Denote the standard inner product of class functions by

$$(\varphi, \psi) \ := \ \frac{1}{|G|} \sum_{g \in G} \varphi(g)\, \overline{\psi(g)}.$$

The multiplicity of W in $V^{\otimes n}$ equals $(\chi_{V^{\otimes n}}, \chi_W)$, so it suffices to show that this inner product is > 0 for some n.

Step 1: A uniform spectral gap for V. Let $\rho : G \to \mathrm{GL}(V)$ be the representation. For each $g \in G$, the eigenvalues of $\rho(g)$ are roots of unity, hence lie on the unit circle, and

$$\chi_V(g) = \sum_{j=1}^{\dim V} \lambda_j(g), \qquad |\lambda_j(g)| = 1.$$

If $g \neq e$ and V is faithful, then $\rho(g) \neq I$, so not all eigenvalues equal 1, hence $|\chi_V(g)| < \chi_V(e) = \dim V$. Set

$$\alpha \ := \ \max_{g \neq e} |\chi_V(g)| \quad \text{so that} \quad 0 \leq \alpha < \dim V.$$

Step 2: A lower bound for the inner product. For any finite-dimensional complex representations, we may assume without loss of generality that both V and W are unitary. Then

$$\overline{\chi_W(g)} = \chi_W(g^{-1}) \quad \text{for all } g \in G.$$

Hence,

$$(\chi_{V^{\otimes n}}, \chi_W) = \frac{1}{|G|} \sum_{g \in G} \chi_W(g^{-1})\big(\chi_V(g)\big)^n = \frac{1}{|G|}\Big(\dim W\,(\dim V)^n + \sum_{g \neq e} \chi_W(g^{-1})\big(\chi_V(g)\big)^n\Big).$$

Taking real parts and using the triangle inequality gives

$$\mathrm{Re}\big((\chi_{V^{\otimes n}}, \chi_W)\big) \geq \frac{1}{|G|}\Big(\dim W\,(\dim V)^n - \sum_{g \neq e} \big|\chi_W(g^{-1})\big|\,\big|\chi_V(g)\big|^n\Big).$$

Since $|\chi_W(g^{-1})| = |\chi_W(g)| \leq \chi_W(e) = \dim W$ and $|\chi_V(g)| \leq \alpha < \dim V$ for $g \neq e$, we obtain

$$\mathrm{Re}\big((\chi_{V^{\otimes n}}, \chi_W)\big) \geq \frac{\dim W}{|G|}\Big((\dim V)^n - (|G|-1)\alpha^n\Big).$$

Step 3: Choosing n. Since $\alpha < \dim V$, we can choose n such that

$$\Big(\frac{\dim V}{\alpha}\Big)^n \ > \ |G| - 1 \quad \iff \quad (\dim V)^n - (|G|-1)\alpha^n \ > \ 0.$$

Equivalently, any

$$n \ > \ \frac{\log(|G|-1)}{\log\big(\frac{\dim V}{\alpha}\big)}$$

will do. Thus, for large enough n, the real part of $(\chi_{V^{\otimes n}}, \chi_W)$ is strictly positive, implying that the inner product itself is nonzero. Therefore W occurs as a subrepresentation of $V^{\otimes n}$.

Remark. In general, the argument may fail to work since the inequality $|\chi_V(g)| \leq \dim V - 1$ need not hold (the trace can be arbitrarily close to $\dim V$ when many eigenvalues are 1), but we always have $\alpha < \dim V$ by faithfulness, which is exactly what the proof needs. ☐

Exercise 5. *Show that the dimension of an irreducible representation of G divides the order of G.*

Proof. Decomposition of the regular representation ☐

Chapter 3

Examples; Induced Representations; Group Algebras; Real Representations

3.1. Classifying irreducible representations of S_5

Let $G = S_5$ denote the symmetric group on five elements. We define a (complex) representation of G as a group homomorphism

$$\rho : G \to GL(V)$$

where V is a finite-dimensional complex vector space.

The group S_5 has order $|S_5| = 5! = 120$.

1. Conjugacy Classes

Conjugacy classes in S_5 are determined by cycle type. The seven conjugacy classes and their sizes are:

Cycle type	Representative	Size
1^5	()	1
2.1^3	(12)	10
3.1^2	(123)	20
4.1	(1234)	30
5	(12345)	24
$2^2.1$	(12)(34)	15
3.2	(12)(345)	20

Thus, S_5 has 7 conjugacy classes and hence 7 irreducible representations.

2. Known Irreducible Representations

We begin with the obvious ones:

- The *trivial representation* U_{triv} of dimension 1.
- The *alternating (sign) representation* U_{alt} of dimension 1.

3. The Standard Representation

Let $\mathbb{C}^5$ denote the permutation representation of S_5, acting by permuting coordinates of the standard basis $\{e_1, \ldots, e_5\}$. Define the subspace

$$V_{\text{std}} = \{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{C}^5 \mid x_1 + x_2 + x_3 + x_4 + x_5 = 0\}.$$

Then

$$\mathbb{C}^5 = U_{\text{triv}} \oplus V_{\text{std}},$$

so V_{std} is a 4-dimensional representation.

Its character is given by

$$\chi_{\mathbb{C}^5} = (5, 3, 2, 1, 0, 1, 0), \quad \chi_{\text{std}} = (4, 2, 1, 0, -1, 0, -1).$$

We verify that

$$\frac{1}{120} \sum_C |C| \, |\chi_{\text{std}}(C)|^2 = 1,$$

so V_{std} is irreducible.

4. Tensor with the Sign Representation

Tensoring with the alternating representation yields another 4-dimensional irrep:

$$V_{\text{alt}} \otimes V_{\text{std}},$$

whose character is

$$\chi_{\text{std}} \cdot \chi_{\text{alt}} = (4, -2, 1, 0, -1, 0, 1).$$

This is irreducible and corresponds to the partition $(2, 1, 1, 1)$.

5. Identification of the Remaining Irreducible Representations of S_5

We recall that the tensor square of the standard representation decomposes as

$$V_{\text{std}} \otimes V_{\text{std}} = \wedge^2 V_{\text{std}} \oplus \text{Sym}^2(V_{\text{std}}).$$

I. The Exterior Square $\wedge^2 V_{\text{std}}$

From the general formula

$$\chi_{\wedge^2 V}(g) = \tfrac{1}{2}\big(\chi_V(g)^2 - \chi_V(g^2)\big),$$

and the known values of $\chi_{V_{\text{std}}}$, we obtain

$$\chi_{\wedge^2 V_{\text{std}}} = (6, 0, 0, 0, 1, -2, 0).$$

Computing the inner product

$$\langle \chi_{\wedge^2 V_{\text{std}}}, \chi_{\wedge^2 V_{\text{std}}} \rangle = \frac{1}{120} \sum_C |C| \, |\chi_{\wedge^2 V_{\text{std}}}(C)|^2 = 1,$$

we see that $\wedge^2 V_{\text{std}}$ is irreducible. We therefore label this 6-dimensional irrep as

$$V_{(3,1,1)}.$$

II. The Symmetric Square $\text{Sym}^2(V_{\text{std}})$

The character of the symmetric square is given by

$$\chi_{\text{Sym}^2 V}(g) = \tfrac{1}{2}\big(\chi_V(g)^2 + \chi_V(g^2)\big),$$

so that

$$\chi_{\text{Sym}^2 V_{\text{std}}} = (10, 4, 1, 0, 0, 2, 1).$$

2. The Symmetric Square $\mathrm{Sym}^2(V_{\mathrm{std}})$

The character of the symmetric square is given by

$$\chi_{\mathrm{Sym}^2 V}(g) = \tfrac{1}{2}\big(\chi_V(g)^2 + \chi_V(g^2)\big),$$

so that

$$\chi_{\mathrm{Sym}^2 V_{\mathrm{std}}} = (10, 4, 1, 0, 0, 2, 1).$$

Since $\dim(\mathrm{Sym}^2 V_{\mathrm{std}}) = 10$, this representation may contain smaller irreducible components. To determine them, we use the inner product formula

$$\langle \chi_U, \chi_V \rangle = \frac{1}{|S_5|} \sum_C |C| \, \overline{\chi_U(C)} \, \chi_V(C),$$

which gives the multiplicity of the irreducible module with character χ_U inside V.

(a) Checking for the Trivial Representation. Let $\chi_{\mathrm{triv}} = (1, 1, 1, 1, 1, 1, 1)$. Then

$$\langle \chi_{\mathrm{triv}}, \chi_{\mathrm{Sym}^2 V_{\mathrm{std}}} \rangle = \frac{1}{120}(1 \cdot 10 + 10 \cdot 4 + 20 \cdot 1 + 30 \cdot 0 + 24 \cdot 0 + 15 \cdot 2 + 20 \cdot 1) = 1.$$

Hence the trivial representation occurs once in $\mathrm{Sym}^2(V_{\mathrm{std}})$.

(b) Checking for the Standard Representation. Now let $\chi_{\mathrm{std}} = (4, 2, 1, 0, -1, 0, -1)$. We compute

$$\langle \chi_{\mathrm{std}}, \chi_{\mathrm{Sym}^2 V_{\mathrm{std}}} \rangle = \frac{1}{120}\big(1 \cdot 4 \cdot 10 + 10 \cdot 2 \cdot 4 + 20 \cdot 1 \cdot 1 + 30 \cdot 0 \cdot 0 + 24 \cdot (-1) \cdot 0 + 15 \cdot 0 \cdot 2 + 20 \cdot (-1) \cdot 1\big) = 1.$$

Therefore, the standard representation also appears exactly once.

(c) Decomposition. Since the inner products with χ_{triv} and χ_{std} are each 1, we conclude that

$$\mathrm{Sym}^2(V_{\mathrm{std}}) \simeq U_{\mathrm{triv}} \oplus V_{\mathrm{std}} \oplus W,$$

where W is a third (possibly irreducible) representation orthogonal to both.

III. Determining the Character of W

By linearity of characters,

$$\chi_{\mathrm{Sym}^2 V_{\mathrm{std}}} = \chi_{\mathrm{triv}} + \chi_{\mathrm{std}} + \chi_W.$$

Thus,

$$\chi_W = \chi_{\mathrm{Sym}^2 V_{\mathrm{std}}} - \chi_{\mathrm{triv}} - \chi_{\mathrm{std}}.$$

Substituting,

$$\chi_W = (10, 4, 1, 0, 0, 2, 1) - (1, 1, 1, 1, 1, 1, 1) - (4, 2, 1, 0, -1, 0, -1) = (5, 1, -1, -1, 0, 1, 1).$$

Computing its inner product gives

$$\langle \chi_W, \chi_W \rangle = 1,$$

so W is irreducible of dimension 5.

We denote this fifth irreducible module by

$$V_{(3,2)}.$$

IV. Completing the Classification

At this stage, we have obtained six distinct irreducible representations:

$$U_{\mathrm{triv}}, \quad U_{\mathrm{alt}}, \quad V_{\mathrm{std}}, \quad V_{\mathrm{std}} \otimes U_{\mathrm{alt}}, \quad V_{(3,1,1)}, \quad V_{(3,2)}.$$

Since S_5 has seven conjugacy classes, one more irrep must exist.

Tensoring $V_{(3,2)}$ with the alternating representation produces a new, inequivalent representation:

$$V_{(2,2,1)} := V_{(3,2)} \otimes U_{\mathrm{alt}},$$

whose character is obtained by multiplying by the sign character:

$$\chi_{(2,2,1)} = \chi_{(3,2)} \cdot \chi_{\mathrm{alt}} = (5, -1, -1, 1, 0, 1, -1).$$

Conclusion

Hence the final two irreducible representations of S_5 are:

Partition	Dimension	Character
$(3,2)$	5	$(5, 1, -1, -1, 0, 1, 1)$
$(2,2,1)$	5	$(5, -1, -1, 1, 0, 1, -1)$

Together with the earlier five, these exhaust all seven irreducible representations of S_5.

$$1^2 + 1^2 + 4^2 + 4^2 + 5^2 + 5^2 + 6^2 = 120 = |S_5|.$$

Complete Character Table of S_5

	1	(12)	(123)	(1234)	(12345)	(12)(34)	(12)(345)
Size of class	1	10	20	30	24	15	20
$\chi_{(5)}$	1	1	1	1	1	1	1
$\chi_{(1^5)}$	1	-1	1	-1	1	1	-1
$\chi_{(4,1)}$	4	2	1	0	-1	0	-1
$\chi_{(2,1,1,1)}$	4	-2	1	0	-1	0	1
$\chi_{(3,2)}$	5	1	-1	-1	0	1	1
$\chi_{(2,2,1)}$	5	-1	-1	1	0	1	-1
$\chi_{(3,1,1)}$	6	0	0	0	1	-2	0

Each row corresponds to an irreducible representation indexed by a partition of 5, and each column corresponds to a conjugacy class of S_5. The orthogonality of rows (and columns) confirms that these seven characters form a complete orthonormal basis for the space of class functions on S_5.

$$\boxed{\dim(\text{Class functions on } S_5) = 7 = \text{Number of conjugacy classes.}}$$

3.2. Classification of irreducible representations of A_5

Recall that A_5 is the alternating subgroup of S_5 consisting of all even permutations. We begin by classifying its conjugacy classes.

Step 1. Cycle types in S_5

Every element of S_5 is conjugate to a permutation of the same cycle type. Hence, the possible cycle types for elements of A_5 are those corresponding to even permutations:

$$(1), \quad (3), \quad (5), \quad (2,2).$$

Thus, the elements of A_5 consist of:

- The identity element.
- 3-cycles, such as $(1\,2\,3)$.
- 5-cycles, such as $(1\,2\,3\,4\,5)$.
- Double transpositions, such as $(1\,2)(3\,4)$.

Step 2. Splitting criterion for conjugacy classes in A_n

Since A_n is a normal subgroup of index 2 in S_n, each conjugacy class of S_n either remains a single class in A_n or splits into two distinct classes of equal size. In A_5, the 5-cycles (1 2 3 4 5) and (1 3 5 2 4) are not conjugate, which implies that this class splits. The general criterion is described as follows:

Proposition 4 (Splitting Criterion)**.** *Let $g \in A_n$ have disjoint cycle decomposition with cycle lengths $b_1, b_2, \ldots, b_r$. Then the conjugacy class of g in S_n splits into two distinct conjugacy classes in A_n if and only if all the integers $b_1, b_2, \ldots, b_r$ are* odd *and* distinct*. Otherwise, the conjugacy class of g in S_n remains a single class in A_n.*

Proof. Can be found in Appendix. ☐

Step 3. Applying the criterion

1. **Identity.** The identity element forms its own class.
2. **3-cycles.** All 3-cycles in S_5 are conjugate, and their centralizers consist of even permutations. Hence, the 3-cycles form a single conjugacy class in A_5. There are $\frac{5\cdot4\cdot3}{3} = 20$ such elements.
3. **Double transpositions.** Each element of the form $(a\,b)(c\,d)$ is an even permutation, and all such elements are conjugate in S_5. Conjugation by an odd permutation does not change their type, so the class remains one in A_5. There are $\frac{1}{2}\binom{5}{2}\binom{3}{2} = 15$ such elements.
4. **5-cycles.** All 5-cycles are conjugate in S_5, but an odd permutation sends a 5-cycle to its inverse. Since a 5-cycle is not conjugate to its inverse within A_5, the class of 5-cycles in S_5 splits into two distinct classes in A_5, each containing 12 elements.

Step 4. Conjugacy Classes of A_5

The even conjugacy classes we retain from the S_5 table are:

Class in A_5	1	(123)	(12)(34)	5-A	5-B
Size	1	20	15	12	12

Here the 24 five-cycles split into two A_5-classes of size 12 each (denoted 5-A and 5-B).

Step 5. Irreducible representations of A_5 via restriction from S_5

We determine which irreducible S_5-modules remain irreducible (or split) when restricted to A_5, by computing the norms of their restricted characters using the S_5 character table you provided.

Inner product on A_5. For a character χ of S_5, its restriction $\chi|_{A_5}$ has squared norm

$$\langle \chi|_{A_5}, \chi|_{A_5} \rangle_{A_5} = \frac{1}{|A_5|}\left(1 \cdot |\chi(1)|^2 + 20 \cdot |\chi(123)|^2 + 15 \cdot |\chi(12)(34)|^2 + 12 \cdot |\chi(\text{5-A})|^2 + 12 \cdot |\chi(\text{5-B})|^2\right),$$

with $|A_5| = 60$. For characters coming from S_5, the two 5-cycle values coincide, so the last two terms combine to $24 \cdot |\chi(12345)|^2$.

Using the given S_5 table:

χ	$\chi(1)$	$\chi(123)$	$\chi\big((12)(34)\big)$	$\chi(12345)$
$\chi_{(5)}$	1	1	1	1
$\chi_{(1^5)}$	1	1	1	1
$\chi_{(4,1)}$	4	1	0	-1
$\chi_{(2,1,1,1)}$	4	1	0	-1
$\chi_{(3,2)}$	5	-1	1	0
$\chi_{(2,2,1)}$	5	-1	1	0
$\chi_{(3,1,1)}$	6	0	-2	1

Norm computations.

- $\chi_{(5)}$ (trivial of S_5):
$$\frac{1}{60}\big(1\cdot 1^2 + 20\cdot 1^2 + 15\cdot 1^2 + 24\cdot 1^2\big) = \frac{60}{60} = 1.$$
Hence irreducible on A_5 (the trivial rep of A_5).

- $\chi_{(1^5)}$ (sign of S_5): on A_5, sgn $\equiv 1$, so the same calculation gives norm 1. Thus the trivial and sign of S_5 both restrict to the *same* 1-dimensional irreducible of A_5.

- $\chi_{(4,1)}$ (and $\chi_{(2,1,1,1)} = \chi_{(4,1)} \otimes \mathrm{sgn}$):
$$\frac{1}{60}\big(16 + 20\cdot 1 + 15\cdot 0 + 24\cdot 1\big) = \frac{60}{60} = 1.$$
So the 4-dimensional S_5-rep remains irreducible on A_5, and its sign-twist restricts to the same A_5-irrep (since $\mathrm{sgn}|_{A_5} \equiv 1$).

- $\chi_{(3,2)}$ (and $\chi_{(2,2,1)} = \chi_{(3,2)} \otimes \mathrm{sgn}$):
$$\frac{1}{60}\big(25 + 20\cdot 1 + 15\cdot 1 + 24\cdot 0\big) = \frac{60}{60} = 1.$$
So the 5-dimensional S_5-rep remains irreducible on A_5, and again both sign-twists coincide when restricted.

- $\chi_{(3,1,1)}$ (the 6-dimensional S_5-rep):
$$\frac{1}{60}\big(36 + 20\cdot 0 + 15\cdot 4 + 24\cdot 1\big) = \frac{120}{60} = 2.$$
Hence it *splits* over A_5 as a sum of two non-isomorphic irreducibles:
$$(3,1,1)\big|_{A_5} \;\cong\; Y \oplus Z, \qquad \dim Y = \dim Z = 3.$$

Count of degrees. Since A_5 has 5 conjugacy classes, it has 5 irreducible characters. We have found degrees $1, 4, 5$, and two more degrees Y, Z coming from the split of the 6-dimensional S_5-irrep. Using
$$60 \;=\; 1^2 + 4^2 + 5^2 + Y^2 + Z^2 \;=\; 1 + 16 + 25 + Y^2 + Z^2,$$
we get $Y^2 + Z^2 = 18$. The only possibility in positive integers is $Y = Z = 3$.

Completing the A_5 character table. Let $\varphi = \frac{1+\sqrt{5}}{2}$ and $\bar{\varphi} = \frac{1-\sqrt{5}}{2}$. Write the irreducibles of A_5 in the order $\mathbf{1}$, $\mathbf{3}$, $\mathbf{3'}$, $\mathbf{4}$, $\mathbf{5}$, and the classes as 1, (123), (12)(34), 5-A, 5-B. Orthogonality (together with $\chi_{(3,1,1)} = \chi_{\mathbf{3}} + \chi_{\mathbf{3'}}$) yields:

	1	(123)	(12)(34)	5-A	5-B
Size	1	20	15	12	12
$\chi_{\mathbf{1}}$	1	1	1	1	1
$\chi_{\mathbf{3}}$	3	0	-1	φ	$\bar{\varphi}$
$\chi_{\mathbf{3'}}$	3	0	-1	$\bar{\varphi}$	φ
$\chi_{\mathbf{4}}$	4	1	0	-1	-1
$\chi_{\mathbf{5}}$	5	-1	1	0	0

Here:

- $\chi_{\mathbf{4}}$ is the restriction of $\chi_{(4,1)}$ (and of $\chi_{(2,1,1,1)}$).
- $\chi_{\mathbf{5}}$ is the restriction of $\chi_{(3,2)}$ (and of $\chi_{(2,2,1)}$).
- $\chi_{\mathbf{3}} + \chi_{\mathbf{3'}} = \chi_{(3,1,1)}|_{A_5}$, giving values $6, 0, -2, 1, 1$ as required.

Summary. Under restriction $S_5 \downarrow A_5$:

$$\begin{aligned} (5)\downarrow &\cong \mathbf{1}, \qquad (1^5)\downarrow \cong \mathbf{1}, \\ (4,1)\downarrow &\cong \mathbf{4} \ \text{(irreducible)}, \quad (2,1,1,1)\downarrow \cong \mathbf{4}, \\ (3,2)\downarrow &\cong \mathbf{5} \ \text{(irreducible)}, \quad (2,2,1)\downarrow \cong \mathbf{5}, \\ (3,1,1)\downarrow &\cong \mathbf{3} \oplus \mathbf{3'}. \end{aligned}$$

Thus the irreducible degrees of A_5 are $1, 3, 3, 4, 5$, and the table above follows by orthogonality.

3.3. Classification of irreducible representations of D_{2n} group

Let $D_{2n} = \langle r, s \mid r^n = 1,\ s^2 = 1,\ srs = r^{-1} \rangle$ be the symmetry group of a regular n-gon (of order $2n$). Write $\Gamma = \langle r \rangle \cong C_n$ for the rotation subgroup.

Step 1: Γ-eigendecomposition. Let (ρ, V) be a (complex) representation of D_{2n}. Since Γ is cyclic and abelian, V decomposes as a direct sum of Γ-eigenspaces

$$V = \bigoplus_{k \in \mathbb{Z}/n\mathbb{Z}} V_k, \qquad V_k = \{v \in V : \ \rho(r)v = \zeta_n^{\,k} v\},$$

where $\zeta_n = e^{2\pi i/n}$. The relation $srs = r^{-1}$ implies

$$v \in V_k : \ \rho(s)\rho(r)v = \zeta_n^{\,k}\rho(s)v = \rho(r^{-1})\rho(s)v \implies \zeta_n^{-k}\rho(s)v = \rho(r)\rho(s)v \implies \rho(s)\, V_k \subseteq V_{-k}.$$

Hence for each k, the pair (V_k, V_{-k}) is s-stable.

Step 2: Irreducible building blocks. There are two types of indices $k \in \mathbb{Z}/n\mathbb{Z}$:

(a) *Generic pairs* $k \not\equiv -k \pmod n$, i.e. $k \neq 0$ and, if n is even, $k \neq \frac{n}{2}$. Then $V_k \oplus V_{-k}$ is D_{2n}-stable and $\rho(r)$ has eigenvalues ζ_n^k, ζ_n^{-k} on this sum, while $\rho(s)$ swaps the two lines. This yields a *2-dimensional* irreducible representation, which is a subspace of $V_k \oplus V_{-k}$.

(b) *Fixed points* of the involution $k \mapsto -k$: these are $k \equiv 0$ and (only when n is even) $k \equiv \frac{n}{2}$. In these cases V_k is s-stable, so s acts on V_k by a scalar ± 1. This yields *1-dimensional* representations, which is a subpsace of V_k.

We continue with the presentation

$$D_{2n} = \langle r, s \mid r^n = 1,\ s^2 = 1,\ srs = r^{-1} \rangle,$$

where r is a rotation and s a reflection.

Step 3. Conjugacy classes of D_{2n}. Case 1: n odd.

1. *Rotations.* For each k, $sr^ks^{-1} = r^{-k}$, hence r^k and r^{-k} are conjugate. They are distinct for $1 \le k \le (n-1)/2$. Thus the rotation classes are

$$\{e\}, \quad \{r^k, r^{-k}\} \text{ for } 1 \le k \le (n-1)/2.$$

2. *Reflections.* All reflections $s, sr, sr^2, \ldots, sr^{n-1}$ are conjugate, forming one class.

Hence there are

$$1 + \frac{n-1}{2} + 1 = \frac{n+3}{2}$$

conjugacy classes.

Case 2: n **even.**

1. *Rotations.* Again $r^k \sim r^{-k}$, but $r^{n/2}$ is self-inverse, so its own class is a singleton:

$$\{e\}, \quad \{r^{n/2}\}, \quad \{r^k, r^{-k}\} \ (1 \le k \le n/2 - 1).$$

2. *Reflections.* There are two distinct types:

$$\{sr^{2m}\} \quad \text{and} \quad \{sr^{2m+1}\},$$

each forming a conjugacy class of size $n/2$.

Hence D_{2n} has

$$\frac{n}{2} + 3$$

conjugacy classes when n is even.

Step 4. Character formulas. For $1 \le k \le \lfloor (n-1)/2 \rfloor$, define a 2-dimensional representation

$$\rho_k: \quad r \mapsto \begin{pmatrix} \zeta_n^k & 0 \\ 0 & \zeta_n^{-k} \end{pmatrix}, \qquad s \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \zeta_n = e^{2\pi i/n}.$$

Then $\rho_k(srs) = \rho_k(r^{-1})$ as required.

The corresponding character $\chi_k = \operatorname{Tr} \rho_k$ takes values

$$\chi_k(r^m) = \zeta_n^{km} + \zeta_n^{-km} = 2\cos\Big(\frac{2\pi km}{n}\Big), \qquad \chi_k(sr^m) = 0.$$

Thus each χ_k is constant on conjugacy classes and real-valued.

The one-dimensional characters arise from the abelianization

$$D_{2n}^{\text{ab}} \cong \begin{cases} C_2, & n \text{ odd}, \\ C_2 \times C_2, & n \text{ even}. \end{cases}$$

Therefore:

	$\chi(r)$	$\chi(s)$
n odd	1	± 1
n even	± 1	± 1

giving 2 one-dimensional representations for n odd, and 4 for n even.

Step 5. Verification on conjugacy classes.

1. For a rotation class $\{r^k, r^{-k}\}$:

$$\chi_k(r^k) = \chi_k(r^{-k}) = 2\cos\Big(\frac{2\pi k^2}{n}\Big),$$

so χ_k is constant on each rotation class.

2. For reflections, since each sr^m has trace 0, $\chi_k(sr^m) = 0$ for all m, giving constancy on reflection classes.
3. Orthogonality of characters:

$$\frac{1}{2n} \sum_{g \in D_{2n}} \chi_i(g)\overline{\chi_j(g)} = \delta_{ij}$$

holds, confirming the list is complete.

Summary.

Case	Conjugacy classes	Irreps & dimensions	Character values
n odd	$\{1\},\ \{r^k, r^{-k}\},$ one reflection class	2 of dim 1, $\frac{n-1}{2}$ of dim 2	$\chi_k(r^m) = 2\cos\frac{2\pi km}{n},\ \chi_k(sr^m) = 0$
n even	$\{1\}, \{r^{n/2}\}, \{r^k, r^{-k}\},$ two reflection classes	4 of dim 1, $\frac{n}{2} - 1$ of dim 2	$\chi_k(r^m) = 2\cos\frac{2\pi km}{n},\ \chi_k(sr^m) = 0$

Hence, the two-dimensional irreducible representations correspond to pairs (V_k, V_{-k}) with eigenvalues $(\zeta_n^k, \zeta_n^{-k})$, while reflections exchange these eigenspaces, giving rise to the characteristic 2-dimensional rotation–reflection modules of the dihedral group.

Conceptual summary via eigenspaces. Decompose V into Γ-eigenspaces V_k. The relation $sV_k = V_{-k}$ forces irreducibles to be either:

- one-dimensional, when $k \equiv -k$ (i.e. $k = 0$, and also $k = \frac{n}{2}$ when n is even); or
- two-dimensional, coming from the s-stable pair $V_k \oplus V_{-k}$ with $k \not\equiv -k$.

This yields the complete classification above.

Quaternion group Q_8: classification of irreducible representations

Let $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$ with relations $i^2 = j^2 = k^2 = ijk = -1$. Its conjugacy classes are

$$\{1\}, \quad \{-1\}, \quad \{\pm i\}, \quad \{\pm j\}, \quad \{\pm k\},$$

so there are 5 irreducible complex representations (hence 5 irreducible characters).

Degrees. By the class equation for degrees,

$$\sum_{r \in Irr(Q_8)} (\deg r)^2 \;=\; |Q_8| \;=\; 8.$$

The abelianization is $Q_8^{\mathrm{ab}} \cong C_2 \times C_2$, so there are exactly 4 one–dimensional characters. It follows that the remaining irreducible has degree 2. Thus the multiset of degrees is $(1, 1, 1, 1, 2)$.

The four linear characters. A linear character is trivial on the commutator subgroup $[Q_8, Q_8] = \{\pm 1\}$, hence sends $-1 \mapsto 1$. Identifying linear characters with homomorphisms $Q_8^{\mathrm{ab}} \to \mathbb{C}^\times$, we may choose the following 4 characters (written on the five conjugacy classes in the order $\{1\}, \{-1\}, \{\pm i\}, \{\pm j\}, \{\pm k\}$):

$$\begin{aligned}
\chi_1 &= (1,\, 1,\, 1,\, 1,\, 1), \\
\chi_2 &= (1,\, 1,\, 1,\, -1,\, -1), \\
\chi_3 &= (1,\, 1,\, -1,\, 1,\, -1), \\
\chi_4 &= (1,\, 1,\, -1,\, -1,\, 1).
\end{aligned}$$

Note that for 1-dimensional representations $\rho : Q_8 \to \mathbb{C}$ we have that

$$\forall x, y \in Q_8 : \; tr(\rho(xy)) = tr(\rho(x)\rho(y)) = tr(\rho(x)) tr(\rho(y))$$

Since $Q_8 =< i, j >$ then this observation allows us to compute the above characters.

The 2-dimensional character via orthogonality. Let χ be the character of the 2-dimensional irreducible representation, and write

$$\chi(1) = 2, \qquad \chi(-1) = a, \qquad \chi(\pm i) = b, \qquad \chi(\pm j) = c, \qquad \chi(\pm k) = d.$$

Orthogonality with the trivial character χ_1 gives

$$\frac{1}{8}\big(1 \cdot 2 + 1 \cdot a + 2 \cdot b + 2 \cdot c + 2 \cdot d\big) = 0 \implies 2 + a + 2b + 2c + 2d = 0.$$

Orthogonality with χ_2, χ_3, χ_4 yields, respectively,

$$\begin{aligned} 2 + a + 2b - 2c - 2d &= 0, \\ 2 + a - 2b + 2c - 2d &= 0, \\ 2 + a - 2b - 2c + 2d &= 0. \end{aligned}$$

Subtracting these equations pairwise gives $b = c = d = 0$, and then the first equation forces $a = -2$. Finally, $\langle \chi, \chi \rangle = 1$ holds:

$$\frac{1}{8}\left(|2|^2 + |-2|^2 + 2|0|^2 + 2|0|^2 + 2|0|^2\right) = \frac{1}{8}(4+4) = 1.$$

Character table of Q_8. With the conjugacy classes listed as columns and the irreducibles as rows, we obtain:

	$\{1\}$	$\{-1\}$	$\{\pm i\}$	$\{\pm j\}$	$\{\pm k\}$
sizes	1	1	2	2	2
χ_1 (1)	1	1	1	1	1
χ_2 (1)	1	1	1	-1	-1
χ_3 (1)	1	1	-1	1	-1
χ_4 (1)	1	1	-1	-1	1
χ_5 (2)	2	-2	0	0	0

All row inner products are 1 and distinct rows are orthogonal, and the degree squares sum $1+1+1+1+4 = 8$, completing the classification.

3.4. Classification of Irreducible Representations for $SL_2(\mathbb{Z}/3\mathbb{Z})$ group

We give a short and self-contained derivation of the character table of

$$G = \mathrm{SL}_2(\mathbb{F}_3) = \{A \in M_2(\mathbb{F}_3) \mid \det A = 1\},$$

working entirely within matrix algebra, without appealing to any quotient description such as A_4.

Step 1. Generators and relations

Let

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Then

$$S^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -I, \quad S^4 = I, \quad T^3 = I, \quad (ST)^3 = -I.$$

Hence

$$G = \langle S, T \mid S^4 = T^3 = (ST)^6 = I,\ S^2 = (ST)^3 \rangle, \qquad |G| = 24.$$

The center is $Z(G) = \{\pm I\}$, so $G/Z(G)$ has order 12.

Step 2. Conjugacy classes and element orders

By direct computation from these generators (or via trace over $\mathbb{F}_3$):

Class	1*A*	2*A*	3*A*	3*B*	4*A*	6*A*	6*B*
Size	1	1	4	4	6	4	4
Order	1	2	3	3	4	6	6

The two central elements are $\pm I$. There are 6 elements of order 4 (noncentral elements of the quaternion subgroup), two conjugacy classes of order 3, and their $-I$ multiples of order 6.

Step 3. Counting irreducible degrees

Since $|Z(G)| = 2$, G is not abelian. The commutator subgroup $[G, G]$ has index 3, so there are exactly three 1-dimensional representations.

Let the degrees of the irreducible representations be

$$1,\ 1,\ 1,\ d_4,\ d_5,\ d_6,\ d_7.$$

Using $\sum_i d_i^2 = |G| = 24$, we have

$$24 - (1^2 + 1^2 + 1^2) = 21.$$

The only possibility with integer dimensions is

$$d_4 = d_5 = d_6 = 2, \qquad d_7 = 3.$$

Hence G has three 2-dimensional and one 3-dimensional irreducible representations.

Step 4. Determining character values

- For all linear characters, $\chi(-I) = 1$, since $-I \in [G, G]$. They are trivial on $2A$ and $4A$, and take values 1, ω, ω^2 on the 3- and 6-classes.
- In any 2-dimensional representation, $\det \rho(g) = 1$ and $\rho(S)^2 = -I$, so $\chi(2A) = -2$, $\chi(4A) = 0$. For T of order 3, $\rho(T)$ has eigenvalues ω, ω^2, giving $\chi(3A) = \chi(3B) = 1$. For ST of order 6, $\chi(6A) = \chi(6B) = -1$.
- Twisting this representation by the nontrivial linear characters multiplies its values on the 3- and 6-classes by ω and ω^2, producing the other two 2-dimensional irreps.
- Finally, the 3-dimensional representation arises from the conjugation action on the quaternion subgroup Q_8 (on the span of $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$), giving $\chi_3(1A) = 3$, $\chi_3(2A) = 3$, $\chi_3(4A) = -1$, and χ_3 vanishes on the elements of order 3 and 6.

Step 5. Character table

Let $\omega = e^{2\pi i/3}$. The complete table is:

	$1A$	$2A$	$3A$	$3B$	$4A$	$6A$	$6B$
Class size	1	1	4	4	6	4	4
Element order	1	2	3	3	4	6	6
χ_1	1	1	1	1	1	1	1
χ_ω	1	1	ω	ω^2	1	ω	ω^2
χ_{ω^2}	1	1	ω^2	ω	1	ω^2	ω
χ_2	2	-2	1	1	0	-1	-1
$\chi_2\chi_\omega$	2	-2	ω	ω^2	0	$-\omega$	$-\omega^2$
$\chi_2\chi_{\omega^2}$	2	-2	ω^2	ω	0	$-\omega^2$	$-\omega$
χ_3	3	3	0	0	-1	0	0

Step 6. Verification

$$1^2 + 1^2 + 1^2 + 2^2 + 2^2 + 2^2 + 3^2 = 24 = |G|.$$

All orthogonality relations hold, confirming the completeness of the table.

Conclusion

This computation uses only:

1. explicit generators and relations of $SL_2(\mathbb{F}_3)$,
2. the size of its center and abelianization,
3. and the orthogonality and degree-sum constraints for characters.

No eigenvalue analysis or external identification with A_4 is needed.

3.5. Exterior Powers of the Standard Representation of S_n

Our main goal is to show the following theorem

Theorem 6. *Each exterior power $\bigwedge^k V$ of the standard representation V of S_n is irreducible for all $0 \leq k \leq n-1$.*

Lemma 3. *Let $\mathbb{C}^n = V \oplus U_{[\mathrm{triv}]}$ be the decomposition of the permutation representation of S_n into the standard representation V and the 1-dimensional trivial representation $U_{[\mathrm{triv}]}$. Then for every $1 \leq k \leq n$ there is an isomorphism of S_n–modules*

$$\bigwedge^k \mathbb{C}^n \;\cong\; \bigwedge^k V \;\oplus\; \bigwedge^{k-1} V.$$

Proof. In general, for a direct sum of vector spaces $X \oplus Y$ there is a natural isomorphism

$$\bigwedge^k (X \oplus Y) \;\cong\; \bigoplus_{i=0}^{k} \Big(\bigwedge^i X \otimes \bigwedge^{k-i} Y\Big),$$

functorial in X and Y, and compatible with any group action.

Apply this with $X = V$ and $Y = U_{[\mathrm{triv}]}$, where $U_{[\mathrm{triv}]}$ is 1-dimensional. Then

$$\bigwedge^j U_{[\mathrm{triv}]} = \begin{cases} \mathbb{C}, & j = 0, \\ U_{[\mathrm{triv}]}, & j = 1, \\ 0, & j \geq 2, \end{cases}$$

so the only non-zero summands in the above direct sum occur for $i = k$ and $i = k-1$:

$$\bigwedge^k \mathbb{C}^n = \bigwedge^k (V \oplus U_{[\mathrm{triv}]}) \;\cong\; \Big(\bigwedge^k V \otimes \bigwedge^0 U_{[\mathrm{triv}]}\Big) \;\oplus\; \Big(\bigwedge^{k-1} V \otimes \bigwedge^1 U_{[\mathrm{triv}]}\Big).$$

Since $\bigwedge^0 U_{[\mathrm{triv}]} \cong \mathbb{C}$ and $\bigwedge^1 U_{[\mathrm{triv}]} \cong U_{[\mathrm{triv}]}$, this simplifies to

$$\bigwedge^k \mathbb{C}^n \;\cong\; \bigwedge^k V \;\oplus\; \Big(\bigwedge^{k-1} V \otimes U_{[\mathrm{triv}]}\Big).$$

Because $U_{[\mathrm{triv}]}$ is the trivial representation of S_n, tensoring with it does not change the isomorphism class of a representation. Thus

$$\bigwedge^{k-1} V \otimes U_{[\mathrm{triv}]} \;\cong\; \bigwedge^{k-1} V$$

as S_n–modules, and we obtain

$$\bigwedge^k \mathbb{C}^n \;\cong\; \bigwedge^k V \;\oplus\; \bigwedge^{k-1} V.$$

For completeness, we can also describe an explicit S_n–equivariant isomorphism. Fix a nonzero vector u spanning $U_{[\mathrm{triv}]}$. Every wedge of k vectors in $\mathbb{C}^n = V \oplus U_{[\mathrm{triv}]}$ can be uniquely expressed as a sum of

$$\text{(i)}\;\; v_1 \wedge \cdots \wedge v_k \in \bigwedge^k V, \quad \text{(ii)}\;\; w_1 \wedge \cdots \wedge w_{k-1} \wedge u \in \bigwedge^{k-1} V \wedge U_{[\mathrm{triv}]}.$$

Then the map

$$\bigwedge^k V \oplus \bigwedge^{k-1} V \;\longrightarrow\; \bigwedge^k \mathbb{C}^n, \qquad (v,\, w) \;\mapsto\; v \;+\; w \wedge u$$

is a linear S_n–equivariant isomorphism with inverse given by separating out the u–factor. This realizes the above decomposition concretely. $\square$

1. Decomposition Step

Since

$$\mathbb{C}^n = V \oplus U_{\mathrm{triv}},$$

it follows that

$$\bigwedge^k \mathbb{C}^n = \bigwedge^k V \oplus \bigwedge^{k-1} V.$$

Hence $\bigwedge^k V$ is irreducible if and only if

$$(\chi_{\bigwedge^k \mathbb{C}^n},\ \chi_{\bigwedge^k \mathbb{C}^n}) = 2.$$

2. Action of S_n on $\bigwedge^k \mathbb{C}^n$

Let S_n act on the standard basis $\{e_1, \dots, e_n\}$ of $\mathbb{C}^n$ by permutation:

$$g \cdot e_i = e_{g(i)}.$$

This induces an action on the basis of $\bigwedge^k \mathbb{C}^n$ given by

$$\beta = \{\, e_B = e_{i_1} \wedge \cdots \wedge e_{i_k} \mid B = \{i_1, \dots, i_k\} \subset [n] \,\}.$$

3. Character Computation

For $g \in S_n$ and any subset $B = \{i_1, \dots, i_k\} \subset [n]$, two possibilities occur:

1. If $g(B) = B$, then

$$g \cdot (e_{i_1} \wedge \cdots \wedge e_{i_k}) = e_{g(i_1)} \wedge \cdots \wedge e_{g(i_k)} = \mathrm{sign}(g|_B)\, e_{i_1} \wedge \cdots \wedge e_{i_k}.$$

2. If $g(B) \neq B$, then g maps $e_{i_1} \wedge \cdots \wedge e_{i_k}$ to another distinct basis vector in β.

Thus the matrix of g on $\bigwedge^k \mathbb{C}^n$ is a signed permutation matrix, with eigenvalues ± 1 on fixed wedge-basis vectors. Therefore

$$\chi_{\bigwedge^k \mathbb{C}^n}(g) = \sum_B \mathrm{sign}(g|_B),$$

and hence

$$(\chi_{\bigwedge^k \mathbb{C}^n},\ \chi_{\bigwedge^k \mathbb{C}^n}) = \frac{1}{n!} \sum_{g \in S_n} \Big(\sum_B \mathrm{sign}(g|_B) \Big) \Big(\sum_C \mathrm{sign}(g|_C) \Big) = \frac{1}{n!} \sum_{B,C} \sum_{\substack{g \in S_n \\ g(B)=B,\ g(C)=C}} \mathrm{sign}(g|_B)\, \mathrm{sign}(g|_C).$$

4. Decomposition Relative to B and C

Fix $B, C \subset [n]$ with $|B| = |C| = k$ and let $|B \cap C| = l$. If $g(B) = B$ and $g(C) = C$, then g must preserve the following subsets:

- $C \cap B$ (size l),
- $C \setminus (C \cap B)$ (size $k - l$),
- $B \setminus (C \cap B)$ (size $k - l$),
- $A \setminus (C \cup B)$ (size $n - 2k + l$).

Hence every such g decomposes uniquely as

$$g = a \cdot b \cdot c \cdot d$$

with

$$\begin{aligned} a &\in S_{|C\cap B|} = S_l, \\ b &\in S_{|C\setminus(C\cap B)|} = S_{k-l}, \\ c &\in S_{|B\setminus(C\cap B)|} = S_{k-l}, \\ d &\in S_{|A\setminus(C\cup B)|} = S_{n-2k+l}. \end{aligned}$$

Thus

$$\operatorname{sign}(g|_B)\operatorname{sign}(g|_C) = \operatorname{sign}(a)\operatorname{sign}(b)\operatorname{sign}(a)\operatorname{sign}(c) = \operatorname{sign}^2(a)\operatorname{sign}(b)\operatorname{sign}(c).$$

Therefore

$$\sum_{\substack{g\in S_n \\ g(B)=B,\ g(C)=C}} \operatorname{sign}(g|_C)\operatorname{sign}(g|_B) = \sum_{a,b,c,d} \operatorname{sign}^2(a)\operatorname{sign}(b)\operatorname{sign}(c) = \Big(\sum_{a\in S_l} 1\Big)\Big(\sum_{d\in S_{n-2k+l}} 1\Big)\sum_{b\in S_{k-l}} \operatorname{sign}(b)\sum_{c\in S_{k-l}} \operatorname{sign}(c).$$

Hence

$$\sum_{\substack{g\in S_n \\ g(B)=B,\ g(C)=C}} \operatorname{sign}(g|_C)\operatorname{sign}(g|_B) = l!\,(n-2k+l)!\Big(\sum_{b\in S_{k-l}} \operatorname{sign}(b)\Big)\Big(\sum_{c\in S_{k-l}} \operatorname{sign}(c)\Big).$$

5. Simplifying Using Parity in S_k

For S_k with $k \geq 2$ we have $[S_k : A_k] = 2$, giving

$$\sum_{b\in S_m} \operatorname{sign}(b) = 0 \quad \text{for } m > 1.$$

Thus the only nonzero contributions occur when $k - l \leq 1$. Hence the inner product splits into two cases:

$$\frac{1}{n!}\sum_{B,C}\sum_{\substack{g(B)=B \\ g(C)=C}} \operatorname{sign}(g|_C)\operatorname{sign}(g|_B) = \frac{1}{n!}\sum_{B=C} \operatorname{sign}(g|_B)^2 + \frac{2}{n!}\sum_{|B\triangle C|=2} \operatorname{sign}(g|_B)\operatorname{sign}(g|_C).$$

Case 1: $B = C$**.** Then

$$\frac{1}{n!}\sum_{B=C}\sum_{g(B)=B} \operatorname{sign}(g|_B)^2 = \frac{1}{n!}\sum_{B} k!(n-k)! = \frac{k!(n-k)!}{n!}\binom{n}{k} = 1.$$

Case 2: $|B\triangle C| = 2$**.** Then $|B\cap C| = k-1$, so

$$\frac{2}{n!}\sum_{|B\triangle C|=2} (k-1)!(n-k-1)! = 2\frac{(k-1)!(n-k-1)!}{n!}\sum_{|B\triangle C|=2} 1.$$

The number of pairs with $|B\triangle C| = 2$ is

$$\sum_{|B\triangle C|=2} 1 = \binom{n}{k-1}\binom{n-k+1}{2} = \frac{n!}{(k-1)!(n-k+1)!}\cdot\frac{(n-k+1)(n-k)}{2}.$$

Thus

$$2\frac{(k-1)!(n-k-1)!}{n!}\cdot\frac{n!}{(k-1)!(n-k+1)!}\cdot\frac{(n-k+1)(n-k)}{2} = 1.$$

Conclusion

Combining the two cases yields
$$\|\chi_{\bigwedge^k \mathbb{C}^n}\|^2 = 2.$$

Hence $\bigwedge^k V$ is irreducible.

3.6. Induced Representations

3.6.1. Motivation, definitions, basic properties

Throughout this section G is a finite group.

Definition 6 (Restriction)**.** *Let $H \le G$ and let V be a G-module. The* restriction *of V to H is*

$$\mathrm{Res}_H^G V,$$

the same vector space equipped with the action of H obtained by restricting the action of G. Sometimes in the notation, we may replace the H-module V with its character χ_V, i.e.

$$\boldsymbol{Res}_H^G V = \boldsymbol{Res}_H^G \chi_V$$

Remark 13 (Character under restriction)**.** *Let H be a subgroup of G and let V be a G-module with representation $\rho : G \to \mathrm{GL}(V)$. For any element $h \in H$, the linear operator $\rho(h)$ acting on V is the same map regardless of whether we view V as a G-module or as an H-module obtained by restriction.*

Hence the character values agree:

$$\chi_V(h) = \mathrm{Tr}(\rho(h)) = \chi_{V|_H}(h).$$

Thus, restriction of a representation does not change the character values on elements of the subgroup.

Remark 14 (Motivation)**.** *Let H be a subgroup of G, and V a G-module. Then V becomes an H-module by restriction. The natural question is:*

Given an H-module W, how can we extend it to a G-module?

As a first step, suppose W is an H-invariant subspace of a G-module V. Consider

$$W^* = \bigoplus_{\sigma \in G/H} \sigma \cdot W.$$

Here σ denotes a left coset gH, and $g \cdot H$ does not depend on the representative g, since $H \subseteq \mathrm{Stab}_G(W)$.

This W^* is the smallest G-stable subspace of V containing W, and can be described as linear combinations

$$W^* = \left\{ \sum_{\substack{g_\sigma \in \sigma,\, \sigma \in G/H \\ w \in \beta(W)}} c_w\, g_\sigma \cdot w \;\middle|\; \textit{finite sums, } \beta(W) = \textit{basis of } W,\ c_w \in \mathbb{C} \right\}.$$

Definition 7 (Induced Module in a Submodule Setting)**.** *If every element of W^* can be written* uniquely *as a sum of elements from the distinct translates $\sigma \cdot W$, then we say W induces a G-module*

$$V = \boldsymbol{Ind}_H^G(W) = \boldsymbol{Ind}(W) = \bigoplus_{\sigma \in G/H} \sigma \cdot W.$$

The action of G is inherited from the action of G on the coset space G/H. Sometimes in the notation, we may replace the H-module V with its character χ_V, i.e.

$$\boldsymbol{Ind}_H^G V = \boldsymbol{Ind}_H^G \chi_V$$

Remark 15 (Directness Subtlety)**.** *In general, W being H-invariant does* not *guarantee that the sum*

$$\bigoplus_{\sigma\in G/H} \sigma \cdot W$$

is direct. If $g_1 \cdot W = g_2 \cdot W$, then

$$g_1^{-1} g_2 \in \mathrm{Stab}_G(W),$$

which may not equal H. Hence uniqueness of decomposition may fail.

Definition 8 (Abstract Construction of the Induced Module)**.** *Given an H-module W, define $\boldsymbol{Ind}_H^G(W)$ abstractly as the vector space of finite sums*

$$\sum_{\substack{g_\sigma\in\sigma,\, \sigma\in G/H \\ w\in\beta(W)}} c_w\, g_\sigma \cdot w,$$

formally generated by symbols $g_\sigma \cdot w$, where g_σ is a chosen representative of a coset σ and w ranges over a basis of W.

The G-action is defined as follows: if $g \in G$ and $gg_\sigma = g_\tau h$ with g_τ a representative of $\tau \in G/H$ and $h \in H$, then

$$g \cdot (g_\sigma \cdot w) = g_\tau \cdot (h \cdot w).$$

This gives a well-defined G-action satisfying

$$g \cdot (h \cdot v) = (gh) \cdot v.$$

Thus a G-module $\boldsymbol{Ind}_H^G(W)$ exists and is unique up to isomorphism.

Remark 16 (Basic Properties of **Ind**)**.** *We record three fundamental structural properties of induction and append the explicit explanations (as in the original notes).*

- ***Direct sums.*** *If $W = \bigoplus_i W_i$ as H-modules, then*

  $$\boldsymbol{Ind}(W) = \bigoplus_i \boldsymbol{Ind}(W_i).$$

  This is immediate because

  $$\boldsymbol{Ind}(W) = \bigoplus_{\sigma\in G/H} \sigma \cdot W = \bigoplus_{\sigma\in G/H} \sigma \cdot \left(\bigoplus_i W_i\right) = \bigoplus_i \big(\bigoplus_{\sigma\in G/H} \sigma \cdot W_i \big).$$

- ***Tensor products.*** *If U is a G-module, then*

  $$U \otimes \boldsymbol{Ind}(W) \cong \boldsymbol{Ind}\big(\mathrm{Res}_H^G(U) \otimes W\big).$$

  Explanation:

  $$U \otimes \boldsymbol{Ind}(W) = U \otimes \bigoplus_{\sigma\in G/H} \sigma \cdot W = \bigoplus_{\sigma\in G/H} U \otimes (\sigma \cdot W).$$

  But $U \otimes (\sigma \cdot W) = \sigma \cdot (U \otimes W)$ as vector spaces, since σ only permutes the summands. Thus

  $$U \otimes \boldsymbol{Ind}(W) = \bigoplus_{\sigma\in G/H} \sigma \cdot (U \otimes W) = \boldsymbol{Ind}(U \otimes W).$$

- ***Transitivity.*** *If $H \subset K \subset G$, then*

  $$\boldsymbol{Ind}_H^G(W) \cong \boldsymbol{Ind}_K^G\big(\boldsymbol{Ind}_H^K(W)\big).$$

Explanation (your file):

$$\begin{aligned}\boldsymbol{Ind}_K^G\big(\boldsymbol{Ind}_H^K(W)\big) &= \boldsymbol{Ind}_K^G\Big(\bigoplus_{\tau\in K/H}\tau\cdot W\Big)\\ &= \bigoplus_{\sigma\in G/K}\bigoplus_{\tau\in K/H}\sigma\cdot(\tau\cdot W)\\ &= \bigoplus_{\phi\in G/H}\phi\cdot W = \boldsymbol{Ind}_H^G(W).\end{aligned}$$

Here a pair (σ,τ) with $\sigma\in G/K$, $\tau\in K/H$ corresponds uniquely to $\phi\in G/H$.

3.6.2. Universal Property and Frobenius Reciprocity

Proposition 5 (Universal Property)**.** *Let W be an H-module and U a G-module. Then every H-module homomorphism*

$$\varphi : W \to \mathrm{Res}_H^G(U)$$

extends uniquely to a G-module homomorphism

$$\Phi : \boldsymbol{Ind}_H^G(W) \to U.$$

Consequently,

$$\mathrm{Hom}_H\big(W, \mathrm{Res}\,U\big) \cong \mathrm{Hom}_G\big(\boldsymbol{Ind}W, U\big).$$

Proof. One constructs a free G-module $F(W)$ on W and uses its universal property: any H-module map $W\to U$ extends uniquely to a G-module map $F(W)\to U$. The induced module $\mathbf{Ind}(W)$ is obtained as a quotient of $F(W)$, and the universal property forces the factorization to be unique. Thus the stated correspondence holds. □

Proposition 6 (Character of an Induced Representation)**.** *Let $V=\boldsymbol{Ind}(W)$. Then for any $g\in G$,*

$$\chi_{\boldsymbol{Ind}W}(g) = \sum_{\substack{\sigma\in G/H\\ g\sigma=\sigma}}\chi_W(s^{-1}gs),$$

where s is any representative of the coset σ.

Proof. Since

$$g\cdot(\sigma\cdot W) = (g\sigma)\cdot W,$$

the summand $\sigma\cdot W$ contributes to the trace only when $g\sigma=\sigma$. In that case, $s^{-1}gs\in H$ for all $s\in\sigma$, and the trace on this summand equals the trace of $s^{-1}gs$ on W.

Independence of the choice of representative follows because if $\sigma=sH$ and sh is another representative, then

$$\chi_W\big((sh)^{-1}g(sh)\big) = \chi_W\big(h^{-1}(s^{-1}gs)h\big) = \chi_W(s^{-1}gs),$$

as χ_W is a class function on H. Hence the expression is well-defined. □

Corollary 4 (Frobenius Reciprocity)**.** *If W is a representation of H, and V a representation of G, then*

$$(\chi_{\boldsymbol{Ind}\ W}, \chi_U)_G = (\chi_W, \chi_{\boldsymbol{Res(U)}})_H$$

Proof. It suffices by linearity to prove this when W and V are irreducible. The left-hand side is the number of times V appears in **Ind W**, which is the dimension of $\mathbf{Hom_G(IndW, U)}$. The right-hand side is the dimension of $\mathbf{Hom_H(W, Res(U))}$. By proposition on the universal property, the result follows. □

3.6.3. Exercise Interlude

My exercise Let H be a subgroup of G and let W be an H–module with character χ_W. We will show that

$$\chi_{\mathbf{Ind}\ W}(g) = \sum_{g\sigma=\sigma} \chi_W(s^{-1}gs) = \frac{1}{|H|} \sum_{\substack{s\in G \\ s^{-1}gs\in H}} \chi_W(s^{-1}gs)$$

where $\sigma \in G/H$.

Proof. Let Γ denote the set of distinct representatives for the cosets G/H and define its subset $\Gamma_0 = \{s \in \Gamma :\ g \cdot sH = sH \iff s^{-1}gs \in H\}$. Thus

$$\chi_{\mathbf{Ind}\ W}(g) = \sum_{g\sigma=\sigma} \chi_W(s^{-1}gs) = \sum_{s\in\Gamma_0} \chi_W(s^{-1}gs) = \sum_{s\in\Gamma_0} \sum_{sh\in sH} \frac{1}{|H|}\chi_W(s^{-1}gs) = \frac{1}{|H|} \sum_{\substack{s\in\Gamma_0 \\ sh\in sH}} \chi_W(s^{-1}gs)$$

where the condition of summation can be rewritten as

$$sh \in sH \quad where \quad s \in \Gamma_0 \iff s \in G \quad where \quad s^{-1}gs \in H$$

□

Exercise 3.19 (a) If C is a conjugacy class of G, and $C \cap H$ decomposes into conjugacy classes $D_1, ..., D_r$ of H, then show that

$$\chi_{\mathbf{Ind}\ W}(C) = \frac{|G|}{|H|} \sum_{i=1}^{r} \frac{|D_i|}{|C|} \chi_W(D_i)$$

Proof. Let H be a subgroup of G and let W be an H–module with character χ_W. Fix a set $\Gamma \subset G$ of left coset representatives for G/H, so each $g \in G$ can be written uniquely as $g = sh$ with $s \in \Gamma$ and $h \in H$.

For a fixed element $g \in G$, define

$$\Gamma_0 := \{\, s \in \Gamma : s^{-1}gs \in H \,\}.$$

As shown in the previous exercise using Γ and Γ_0, we have the induced character formula

$$\chi_{\mathbf{Ind}_H^G W}(g) = \frac{1}{|H|} \sum_{\substack{s\in G \\ s^{-1}gs\in H}} \chi_W(s^{-1}gs). \tag{3.1}$$

Now let $C \subset G$ be a conjugacy class, and assume

$$C \cap H = D_1 \sqcup \cdots \sqcup D_r$$

is the decomposition of $C \cap H$ into conjugacy classes in H. We want to compute $\chi_{\mathbf{Ind}_H^G W}(C)$, the value of the induced character on the class C, i.e.

$$\chi_{\mathbf{Ind}_H^G W}(C) := \frac{1}{|C|} \sum_{g\in C} \chi_{\mathbf{Ind}_H^G W}(g).$$

Using (3.1) for each $g \in C$,

$$\chi_{\mathbf{Ind}_H^G W}(C) = \frac{1}{|C|} \sum_{g\in C} \left(\frac{1}{|H|} \sum_{\substack{s\in G \\ s^{-1}gs\in H}} \chi_W(s^{-1}gs) \right) = \frac{1}{|C||H|} \sum_{g\in C} \sum_{\substack{s\in G \\ s^{-1}gs\in H}} \chi_W(s^{-1}gs).$$

We claim that the two sets

$$A := \{(g,s) \in C \times G : s^{-1}gs \in H\} \qquad \text{and} \qquad B := \{(h,s) \in (C \cap H) \times G\}$$

are in bijection.

Indeed, define a map

$$A \longrightarrow B, \qquad (g,s) \longmapsto (h,s) := (s^{-1}gs,\ s).$$

If $(g,s) \in A$, then $g \in C$ and $s^{-1}gs \in H$. Since $s^{-1}gs$ is conjugate to g, it also lies in C. Thus $h = s^{-1}gs \in C \cap H$, and so $(h,s) \in B$. Hence the map is well-defined.

Conversely, define a map

$$B \longrightarrow A, \qquad (h,s) \longmapsto (g,s) := (shs^{-1},\ s).$$

If $(h,s) \in B$, then $h \in C \cap H$. Since $g = shs^{-1}$ is conjugate to h, we have $g \in C$. Moreover,

$$s^{-1}gs = s^{-1}(shs^{-1})s = h \in H,$$

so $(g,s) \in A$. Thus this map is also well-defined.

Finally, the two maps are inverses of one another:

$$(g,s) \mapsto (s^{-1}gs,\ s) \mapsto (s(s^{-1}gs)s^{-1},\ s) = (g,s),$$

$$(h,s) \mapsto (shs^{-1},\ s) \mapsto (s^{-1}(shs^{-1})s,\ s) = (h,s).$$

Therefore the sets A and B are in bijection.

Hence

$$\sum_{g\in C} \sum_{\substack{s\in G \\ s^{-1}gs\in H}} \chi_W(s^{-1}gs) = \sum_{h\in C\cap H} \sum_{s\in G} \chi_W(h) = |G| \sum_{h\in C\cap H} \chi_W(h).$$

Therefore,

$$\chi_{\mathbf{Ind}_H^G W}(C) = \frac{1}{|C||H|} \cdot |G| \sum_{h\in C\cap H} \chi_W(h) = \frac{|G|}{|C||H|} \sum_{h\in C\cap H} \chi_W(h).$$

Now decompose $C \cap H$ into its H–conjugacy classes:

$$C \cap H = D_1 \sqcup \cdots \sqcup D_r.$$

Since χ_W is a class function on H, it is constant on each D_i, so

$$\sum_{h\in C\cap H} \chi_W(h) = \sum_{i=1}^{r} \sum_{h\in D_i} \chi_W(h) = \sum_{i=1}^{r} |D_i|\, \chi_W(D_i),$$

where $\chi_W(D_i)$ denotes the common value of χ_W on D_i.

Substituting this into the previous expression, we obtain

$$\chi_{\mathbf{Ind}_H^G W}(C) = \frac{|G|}{|C||H|} \sum_{i=1}^{r} |D_i|\, \chi_W(D_i) = \frac{|G|}{|H|} \sum_{i=1}^{r} \frac{|D_i|}{|C|}\, \chi_W(D_i).$$

This is the desired formula:

$$\boxed{\chi_{\mathbf{Ind}_H^G W}(C) = \frac{|G|}{|H|} \sum_{i=1}^{r} \frac{|D_i|}{|C|}\, \chi_W(D_i)}.$$

□

Exercise 3.19 (b) Given the same conditions as in part (a) and additionally if W is the trivial representation of H, then

$$\chi_{\mathbf{Ind}\ W}(C) = \frac{[G:H]}{|C|} \cdot |C \cap H|$$

Solution. Based on the result from (a) it is enough for us to show that

$$|C \cap H| = \sum_{i=1}^{r} |D_i| \chi_W(D_i)$$

Given conjugacy classes $D_1, ..., D_r$ under H such that $C \cap H$ is a disjoint union of these classes then

$$|C \cap H| = \sum_{i=1}^{r} |D_i|$$

Since W is a trivial representation of H then $\chi_W(D_i) = \dim W = 1$ which implies the result. □

My Exercise based on Exercise 3.19 (a) Let V be a G-module, H subgroup of G and W a subspace of V invariant under H-action, i.e. W is an H-module. Then using *Exercise 3.19 (a)* we can deduce that

$$\dim(\mathbf{Ind}_H^G\ W) = [G:H] \dim W$$

Solution. Let's apply the formula from Exercise 3.19 (a)

$$\chi_{\mathbf{Ind}_H^G W}(C) = \frac{|G|}{|H|} \sum_{i=1}^{r} \frac{|D_i|}{|C|}\, \chi_W(D_i)$$

to the conjugacy class $C = \{e\}$:

$$\dim(\mathbf{Ind}_H^G\ W) = [G:H] \sum_{i=1}^{r} |D_i|\, \chi_W(D_i)$$

where $D_i \subseteq C \cap H = \{e\} \implies r = 1, \quad D_1 = \{e\} \implies \sum_{i=1}^{r} |D_i|\, \chi_W(D_i) = \chi_W(e) = \dim W$ □

Example 3.21: Decomposition of $\mathbf{Ind}_{S_2}^{S_3} W$ Compute $\mathbf{Ind}_H^G W$ where $G = S_3$, $H = \langle (12) \rangle$, and $W = V_2$ is the standard (i.e. sign) representation of $H \cong S_2$.

Solution. • The subgroup $H = \langle (12) \rangle \cong S_2$ has two 1-dimensional irreducible representations:

$$U_{\text{triv}} = \mathbb{C} \cdot e_1 \qquad \text{and} \qquad U_{\text{alt}} = \mathbb{C} \cdot (e_1 - e_2),$$

with character values

$$\chi_{U_{\text{triv}}}(e) = 1,\ \chi_{U_{\text{triv}}}((12)) = 1, \qquad \chi_{U_{\text{alt}}}(e) = 1,\ \chi_{U_{\text{alt}}}((12)) = -1.$$

In our notation, $W = V_2 = U_{\text{alt}}$.

- The group S_3 has three irreducible representations:

  (1) $V_{\text{triv}} = \mathbb{C} \cdot e_1$, (2) $V_{\text{alt}} = \mathbb{C} \cdot (e_1 - e_2)$, (3) $V_{\text{std}} = \{(x_1, x_2, x_3) \in \mathbb{C}^3 \mid x_1 + x_2 + x_3 = 0\}$,

  where a convenient decomposition is

$$V_{\text{std}} = \mathbb{C} \cdot (e_1 - 2e_2 + e_3) \oplus \mathbb{C} \cdot (2e_1 - e_2 - e_3).$$

  Their characters on the conjugacy classes of S_3 are:

$$\chi_{V_{\text{triv}}} = (1, 1, 1), \quad \chi_{V_{\text{alt}}} = (1, -1, 1), \quad \chi_{V_{\text{std}}} = (2, 0, -1),$$

  corresponding to classes (e), transpositions, and 3-cycles, respectively.

- By Frobenius reciprocity,

$$(\chi_{\mathbf{Ind}\,W}, \chi_V)_{S_3} \;=\; (\chi_W, \chi_{\mathrm{Res}^G_H\,V})_H \quad \text{for each irreducible } V \text{ of } S_3.$$

  We compute these inner products over $H = \{e, (12)\}$ with $|H| = 2$.

  – For V_{triv}:

  $$(\chi_{\mathbf{Ind}\,W}, \chi_{V_{\text{triv}}})_{S_3} = (\chi_W, \chi_{\mathrm{Res}\,V_{\text{triv}}})_H = \frac{1}{2}\big(\chi_W(e)\overline{\chi_{V_{\text{triv}}}(e)}+\chi_W((12))\overline{\chi_{V_{\text{triv}}}((12))}\big) = \frac{1}{2}(1\cdot1+(-1)\cdot1) = 0.$$

  – For V_{alt} (the sign representation of S_3):

  $$\chi_{\mathrm{Res}\,V_{\text{alt}}}(e) = 1, \quad \chi_{\mathrm{Res}\,V_{\text{alt}}}((12)) = -1,$$

  so $\mathrm{Res}\,V_{\text{alt}} \cong W$ as H-modules and

  $$(\chi_{\mathbf{Ind}\,W}, \chi_{V_{\text{alt}}})_{S_3} = (\chi_W, \chi_{\mathrm{Res}\,V_{\text{alt}}})_H = \frac{1}{2}(1\cdot 1 + (-1)\cdot(-1)) = 1.$$

  – For V_{std}:

  $$\chi_{V_{\text{std}}}(e) = 2, \quad \chi_{V_{\text{std}}}((12)) = 0,$$

  hence

  $$(\chi_{\mathbf{Ind}\,W}, \chi_{V_{\text{std}}})_{S_3} = (\chi_W, \chi_{\mathrm{Res}\,V_{\text{std}}})_H = \frac{1}{2}(1\cdot 2 + (-1)\cdot 0) = 1.$$

- Thus, the multiplicity of V_{triv} in $\mathbf{Ind}^G_H W$ is 0, while both V_{alt} and V_{std} appear with multiplicity 1. Therefore,

  $$\mathbf{Ind}^G_H W \cong V_{\text{alt}} \oplus V_{\text{std}}.$$

  As a check,

  $$\dim(\mathbf{Ind}^G_H W) = [G : H]\cdot \dim W = \frac{6}{2}\cdot 1 = 3,$$

  and

  $$\dim(V_{\text{alt}} \oplus V_{\text{std}}) = 1 + 2 = 3,$$

  so dimensions agree.

□

Example 3.22: Decomposition of $\mathbf{Ind}^{S_4}_{S_3} W$ Let $G = S_4$ and let $H \cong S_3$ be the subgroup of permutations fixing 4, i.e.

$$H = \{\, \sigma \in S_4 : \sigma(4) = 4 \,\}.$$

Let $W = V^{S_3}_{\text{std}}$ be the standard 2-dimensional irreducible representation of S_3. Its character is

$$\chi_W = (2,\ 0,\ -1)$$

on the conjugacy classes of S_3:

$$(e), \quad (12), \quad (123).$$

Our goal is to decompose

$$\mathbf{Ind}^{S_4}_{S_3}(W)$$

into irreducible S_4-modules.

We use the following irreducible characters of S_4 (derived previously). Conjugacy classes are ordered as

$$(1),\ (12),\ (123),\ (1234),\ (12)(34).$$

	(1)	(12)	(123)	(1234)	(12)(34)
$\chi_{V_{\text{triv}}}$	1	1	1	1	1
$\chi_{V_{\text{alt}}}$	1	-1	1	-1	1
$\chi_{V_{\text{std}}}$	3	1	0	-1	-1
$\chi_{V_{\text{prod}}}$	3	-1	0	1	-1
$\chi_{V_{\text{last}}}$	2	0	-1	0	2

To apply Frobenius reciprocity, we need the restriction of each S_4-character to the subgroup

$$H = S_3 = \{e, (12), (23), (13), (123), (132)\}.$$

A class of S_4 restricts to S_3 as follows:

Class of S_4	restriction to S_3
(1)	(1)
(12)	(12)
(123)	(123)
(1234)	$\emptyset$ (no element of this type in S_3)
(12)(34)	$\emptyset$

Thus the restriction of any χ_V simply ignores values at (1234) and (12)(34).
We list:

$$\begin{aligned}
\chi_{\text{triv}}|_{S_3} &= (1, 1, 1), \\
\chi_{\text{alt}}|_{S_3} &= (1, -1, 1), \\
\chi_{\text{std}}|_{S_3} &= (3, 1, 0), \\
\chi_{\text{prod}}|_{S_3} &= (3, -1, 0), \\
\chi_{\text{last}}|_{S_3} &= (2, 0, -1).
\end{aligned}$$

By Frobenius reciprocity,

$$\langle \chi_{\mathbf{Ind}\, W},\ \chi_V \rangle_{S_4} = \langle \chi_W,\ \chi_{\mathrm{Res}^{S_4}_{S_3} V} \rangle_{S_3}.$$

We compute the inner products in S_3. Recall the inner product on S_3:

$$\langle \alpha, \beta \rangle_{S_3} = \frac{1}{6}\Big(\alpha(e)\overline{\beta(e)} + 3\,\alpha((12))\overline{\beta((12))} + 2\,\alpha((123))\overline{\beta((123))}\Big).$$

1 With V_{triv}:

$$\langle \chi_W,\ \chi_{\text{triv}}|_{S_3} \rangle = \frac{1}{6}\big(2 \cdot 1 + 3(0 \cdot 1) + 2((-1) \cdot 1)\big) = \frac{1}{6}(2 - 2) = 0.$$

2 With V_{alt}:

$$\langle \chi_W,\ \chi_{\text{alt}}|_{S_3} \rangle = \frac{1}{6}(2 \cdot 1 + 3(0 \cdot -1) + 2((-1) \cdot 1)) = 0.$$

3 With V_{std}:

$$\langle \chi_W,\ \chi_{\text{std}}|_{S_3} \rangle = \frac{1}{6}(2 \cdot 3 + 3(0 \cdot 1) + 2((-1) \cdot 0)) = 1.$$

4 With V_{prod}:

$$\langle \chi_W,\ \chi_{\text{prod}}|_{S_3} \rangle = \frac{1}{6}(2 \cdot 3 + 3(0 \cdot -1) + 2((-1) \cdot 0)) = 1.$$

5 With V_{last}:

$$\langle \chi_W,\ \chi_{\text{last}}|_{S_3} \rangle = \frac{1}{6}(2 \cdot 2 + 3(0 \cdot 0) + 2((-1) \cdot -1)) = \frac{1}{6}(4 + 2) = 1.$$

We have found:

$$\mathbf{Ind}_{S_3}^{S_4}(W) = V_{\mathrm{std}} \oplus V_{\mathrm{prod}} \oplus V_{\mathrm{last}}.$$

Since

$$\dim(\mathbf{Ind}\, W) = [S_4 : S_3] \dim W = 4 \cdot 2 = 8,$$

and

$$\dim(V_{\mathrm{std}}) + \dim(V_{\mathrm{prod}}) + \dim(V_{\mathrm{last}}) = 3 + 3 + 2 = 8,$$

the dimensions agree.

$$\boxed{\mathbf{Ind}_{S_3}^{S_4}(V_{\mathrm{std}}^{S_3}) \;\cong\; V_{\mathrm{std}} \oplus V_{\mathrm{prod}} \oplus V_{\mathrm{last}}.}$$

Exercise 3.23 (Second solution via Frobenius reciprocity) Find $\mathbf{Ind}_H^G W$ for two choices of H in $G = S_4$:

(a) $H = \langle (1234) \rangle$, and $W = \mathbb{C} \cdot v$ with

$$\rho\big((1234)\big) v = i\, v.$$

(b) $H = \langle (123) \rangle$, and $W = \mathbb{C} \cdot v$ with

$$\rho\big((123)\big) v = e^{\frac{2\pi i}{3}} v.$$

Assume the full character table of S_4 is already known from the previous exercise:

	(1)	(12)	(123)	(1234)	(12)(34)
$\chi_{V_{\mathrm{triv}}}$	1	1	1	1	1
$\chi_{V_{\mathrm{alt}}}$	1	-1	1	-1	1
$\chi_{V_{\mathrm{std}}}$	3	1	0	-1	-1
$\chi_{V_{\mathrm{prod}}}$	3	-1	0	1	-1
$\chi_{V_{\mathrm{last}}}$	2	0	-1	0	2

Solution using Frobenius reciprocity. We use Frobenius reciprocity in the form

$$\big\langle \chi_{\mathbf{Ind}_H^G W}, \chi_V \big\rangle_{S_4} \;=\; \big\langle \chi_W, \chi_{\mathrm{Res}_H^G V} \big\rangle_H$$

for each irreducible S_4-module V.

(a) Case $H = \langle (1234) \rangle \cong C_4$.

Here H is cyclic of order 4, generated by $h = (1234)$, with elements

$$H = \{e, h, h^2, h^3\}.$$

The representation W is 1-dimensional, determined by

$$\chi_W(e) = 1, \quad \chi_W(h) = i, \quad \chi_W(h^2) = i^2 = -1, \quad \chi_W(h^3) = i^3 = -i.$$

So

$$\chi_W = (1,\ i,\ -1,\ -i)$$

on $\{e, h, h^2, h^3\}$.

We view $h = (1234)$ as an element in the 4-cycle class of S_4, and $h^2 = (13)(24)$ as an element of the class $(12)(34)$; h^3 is another 4-cycle.

For an irreducible S_4-module V, its restriction to H has character

$$\chi_{\mathrm{Res}\, V}(h^k) = \chi_V(h^k), \quad k = 0, 1, 2, 3.$$

The inner product on H is

$$\langle \alpha, \beta \rangle_H = \frac{1}{|H|} \sum_{k=0}^{3} \alpha(h^k) \overline{\beta(h^k)} = \frac{1}{4} \sum_{k=0}^{3} \alpha(h^k) \overline{\beta(h^k)}.$$

We now compute the multiplicities

$$m_V := \langle \chi_W, \chi_{\mathrm{Res}\, V} \rangle_H$$

for each irreducible V of S_4.

1) $V = V_{\text{triv}}$. On H:

$$\chi_{\text{triv}}(e) = 1, \quad \chi_{\text{triv}}(h) = 1, \quad \chi_{\text{triv}}(h^2) = 1, \quad \chi_{\text{triv}}(h^3) = 1.$$

So

$$m_{V_{\text{triv}}} = \frac{1}{4}\big(1 \cdot 1 + i \cdot 1 + (-1) \cdot 1 + (-i) \cdot 1\big) = \frac{1}{4}(1 + i - 1 - i) = 0.$$

2) $V = V_{\text{alt}}$. From the table:

$$\chi_{\text{alt}}(e) = 1, \quad \chi_{\text{alt}}(h) = -1, \quad \chi_{\text{alt}}(h^2) = 1, \quad \chi_{\text{alt}}(h^3) = -1.$$

Hence

$$m_{V_{\text{alt}}} = \frac{1}{4}\big(1 \cdot 1 + i \cdot (-1) + (-1) \cdot 1 + (-i) \cdot (-1)\big) = \frac{1}{4}(1 - i - 1 + i) = 0.$$

3) $V = V_{\text{std}}$. From the table:

$$\chi_{\text{std}}(e) = 3, \quad \chi_{\text{std}}(h) = -1, \quad \chi_{\text{std}}(h^2) = -1, \quad \chi_{\text{std}}(h^3) = -1.$$

So

$$m_{V_{\text{std}}} = \frac{1}{4}\big(1 \cdot 3 + i \cdot (-1) + (-1) \cdot (-1) + (-i) \cdot (-1)\big) = \frac{1}{4}(3 - i + 1 + i) = \frac{4}{4} = 1.$$

4) $V = V_{\text{prod}}$. From the table:

$$\chi_{\text{prod}}(e) = 3, \quad \chi_{\text{prod}}(h) = 1, \quad \chi_{\text{prod}}(h^2) = -1, \quad \chi_{\text{prod}}(h^3) = 1.$$

Then

$$m_{V_{\text{prod}}} = \frac{1}{4}\big(1 \cdot 3 + i \cdot 1 + (-1) \cdot (-1) + (-i) \cdot 1\big) = \frac{1}{4}(3 + i + 1 - i) = 1.$$

5) $V = V_{\text{last}}$. From the table:

$$\chi_{\text{last}}(e) = 2, \quad \chi_{\text{last}}(h) = 0, \quad \chi_{\text{last}}(h^2) = 2, \quad \chi_{\text{last}}(h^3) = 0.$$

So

$$m_{V_{\text{last}}} = \frac{1}{4}\big(1 \cdot 2 + i \cdot 0 + (-1) \cdot 2 + (-i) \cdot 0\big) = \frac{1}{4}(2 - 2) = 0.$$

Thus, by Frobenius reciprocity, the induced representation decomposes as

$$\mathbf{Ind}_{\langle (1234) \rangle}^{S_4} W \cong V_{\text{std}} \oplus V_{\text{prod}}.$$

As a check,

$$\dim\big(\mathbf{Ind} W\big) = [S_4 : H] \cdot \dim W = \frac{24}{4} \cdot 1 = 6,$$

and

$$\dim(V_{\text{std}}) + \dim(V_{\text{prod}}) = 3 + 3 = 6,$$

so the dimensions agree.

(b) Case $H = \langle (123) \rangle \cong C_3$.

Now H is cyclic of order 3, generated by $h = (123)$, with elements

$$H = \{e, h, h^2\}, \quad h^2 = (132).$$

The representation W is again 1-dimensional, with

$$\chi_W(e) = 1, \quad \chi_W(h) = \omega, \quad \chi_W(h^2) = \omega^2, \quad \omega = e^{2\pi i/3}.$$

Thus
$$\chi_W = (1,\ \omega,\ \omega^2).$$

For an irreducible S_4-module V, the restriction $\chi_{\operatorname{Res} V}$ to H is obtained by evaluating χ_V on e and on the class of 3-cycles (123); both h and h^2 lie in that same conjugacy class of S_4.

The inner product on H is
$$\langle \alpha, \beta \rangle_H = \frac{1}{3}\big(\alpha(e)\overline{\beta(e)} + \alpha(h)\overline{\beta(h)} + \alpha(h^2)\overline{\beta(h^2)}\big).$$

We again compute
$$m_V := \langle \chi_W, \chi_{\operatorname{Res} V} \rangle_H$$
for each irreducible V.

1) $V = V_{\text{triv}}$. On H:
$$\chi_{\text{triv}}(e) = 1, \quad \chi_{\text{triv}}(h) = 1, \quad \chi_{\text{triv}}(h^2) = 1.$$
Thus
$$m_{V_{\text{triv}}} = \frac{1}{3}(1 \cdot 1 + \omega \cdot 1 + \omega^2 \cdot 1) = \frac{1}{3}(1 + \omega + \omega^2) = 0,$$
since $1 + \omega + \omega^2 = 0$.

2) $V = V_{\text{alt}}$. On 3-cycles the sign is $+1$, so $\chi_{\text{alt}}(e) = 1$, $\chi_{\text{alt}}(h) = 1$, $\chi_{\text{alt}}(h^2) = 1$. Hence the same computation as above gives
$$m_{V_{\text{alt}}} = 0.$$

3) $V = V_{\text{std}}$. From the table:
$$\chi_{\text{std}}(e) = 3, \quad \chi_{\text{std}}(h) = 0, \quad \chi_{\text{std}}(h^2) = 0.$$
Therefore
$$m_{V_{\text{std}}} = \frac{1}{3}(1 \cdot 3 + \omega \cdot 0 + \omega^2 \cdot 0) = 1.$$

4) $V = V_{\text{prod}}$. From the table:
$$\chi_{\text{prod}}(e) = 3, \quad \chi_{\text{prod}}(h) = 0, \quad \chi_{\text{prod}}(h^2) = 0,$$
so the same computation gives
$$m_{V_{\text{prod}}} = 1.$$

5) $V = V_{\text{last}}$. From the table:
$$\chi_{\text{last}}(e) = 2, \quad \chi_{\text{last}}(h) = -1, \quad \chi_{\text{last}}(h^2) = -1.$$
Hence
$$m_{V_{\text{last}}} = \frac{1}{3}\big(1 \cdot 2 + \omega \cdot (-1) + \omega^2 \cdot (-1)\big) = \frac{1}{3}\big(2 - (\omega + \omega^2)\big) = \frac{1}{3}\big(2 - (-1)\big) = \frac{3}{3} = 1,$$
using again $\omega + \omega^2 = -1$.

Thus
$$\mathbf{Ind}_{\langle (123) \rangle}^{S_4} W \cong V_{\text{std}} \oplus V_{\text{prod}} \oplus V_{\text{last}}.$$

As a check,
$$\dim\big(\mathbf{Ind} W\big) = [S_4 : H] \cdot \dim W = \frac{24}{3} \cdot 1 = 8,$$
and
$$\dim(V_{\text{std}}) + \dim(V_{\text{prod}}) + \dim(V_{\text{last}}) = 3 + 3 + 2 = 8,$$
so the dimensions again match the index.

Conclusion.
$$\boxed{\begin{aligned} &\mathbf{Ind}_{\langle (1234) \rangle}^{S_4} \mathbb{C}_i \cong V_{\text{std}} \oplus V_{\text{prod}}, \\ &\mathbf{Ind}_{\langle (123) \rangle}^{S_4} \mathbb{C}_\omega \cong V_{\text{std}} \oplus V_{\text{prod}} \oplus V_{\text{last}}, \end{aligned}}$$
where $\mathbb{C}_i$ and $\mathbb{C}_\omega$ denote the 1-dimensional representations given in parts *(a)* and *(b)*. $\square$

Exercise 3.24: Induction from A_5 to S_5 Let $G = S_5$ and $H = A_5 \trianglelefteq S_5$, of index $[G : H] = 2$. In the following exercise we want to obtain $\mathbf{Ind}_{A_5}^{S_5}\chi_V$ for every irreducible χ_V of $H = A_5$. We assume the following character tables of S_5 and A_5.

For S_5, we label the irreducible characters by partitions of 5:

	1	(12)	(123)	(1234)	(12345)	(12)(34)	(12)(345)
Size of class	1	10	20	30	24	15	20
$\chi_{(5)}$	1	1	1	1	1	1	1
$\chi_{(1^5)}$	1	-1	1	-1	1	1	-1
$\chi_{(4,1)}$	4	2	1	0	-1	0	-1
$\chi_{(2,1,1,1)}$	4	-2	1	0	-1	0	1
$\chi_{(3,2)}$	5	1	-1	-1	0	1	1
$\chi_{(2,2,1)}$	5	-1	-1	1	0	1	-1
$\chi_{(3,1,1)}$	6	0	0	0	1	-2	0

Let $\varphi = \dfrac{1+\sqrt{5}}{2}$ and $\bar{\varphi} = \dfrac{1-\sqrt{5}}{2}$. For A_5, we list irreps in the order

$$\mathbf{1},\ \mathbf{3},\ \mathbf{3'},\ \mathbf{4},\ \mathbf{5},$$

and conjugacy classes as

$$1,\ (123),\ (12)(34),\ 5\text{-A},\ 5\text{-B}.$$

The character table is:

	1	(123)	(12)(34)	5-A	5-B
Size	1	20	15	12	12
$\chi_{\mathbf{1}}$	1	1	1	1	1
$\chi_{\mathbf{3}}$	3	0	-1	φ	$\bar{\varphi}$
$\chi_{\mathbf{3'}}$	3	0	-1	$\bar{\varphi}$	φ
$\chi_{\mathbf{4}}$	4	1	0	-1	-1
$\chi_{\mathbf{5}}$	5	-1	1	0	0

Note that the S_5-class of 5-cycles splits in A_5 into two classes 5-A and 5-B (each of size 12), while the classes of 3-cycles and of double transpositions remain single classes in A_5.

To restrict an S_5-character to A_5, we simply ignore the odd conjugacy classes and duplicate the value on 5-cycles across the two A_5 classes 5-A and 5-B.

Thus for each irreducible character χ of S_5, its restriction $\mathrm{Res}_{A_5}^{S_5}\chi$ is determined on the A_5 classes as follows:

	1	(123)	(12)(34)	5-A	5-B
$\chi_{(5)}\vert_{A_5}$	1	1	1	1	1
$\chi_{(1^5)}\vert_{A_5}$	1	1	1	1	1
$\chi_{(4,1)}\vert_{A_5}$	4	1	0	-1	-1
$\chi_{(2,1,1,1)}\vert_{A_5}$	4	1	0	-1	-1
$\chi_{(3,2)}\vert_{A_5}$	5	-1	1	0	0
$\chi_{(2,2,1)}\vert_{A_5}$	5	-1	1	0	0
$\chi_{(3,1,1)}\vert_{A_5}$	6	0	-2	1	1

Let $\langle \cdot, \cdot \rangle_{A_5}$ denote the standard inner product of class functions on A_5:

$$\langle \alpha, \beta \rangle_{A_5} = \frac{1}{|A_5|} \sum_C |C|\, \alpha(C)\, \overline{\beta(C)} \quad (|A_5| = 60).$$

Orthogonality of A_5 characters and a direct computation yield:

$$
\begin{aligned}
\mathrm{Res}^{S_5}_{A_5}\chi_{(5)} &= \chi_{\mathbf{1}},\\
\mathrm{Res}^{S_5}_{A_5}\chi_{(1^5)} &= \chi_{\mathbf{1}},\\
\mathrm{Res}^{S_5}_{A_5}\chi_{(4,1)} &= \chi_{\mathbf{4}},\\
\mathrm{Res}^{S_5}_{A_5}\chi_{(2,1,1,1)} &= \chi_{\mathbf{4}},\\
\mathrm{Res}^{S_5}_{A_5}\chi_{(3,2)} &= \chi_{\mathbf{5}},\\
\mathrm{Res}^{S_5}_{A_5}\chi_{(2,2,1)} &= \chi_{\mathbf{5}},\\
\mathrm{Res}^{S_5}_{A_5}\chi_{(3,1,1)} &= \chi_{\mathbf{3}} + \chi_{\mathbf{3'}}.
\end{aligned}
$$

In words:

- The trivial and sign representations of S_5 both restrict to the trivial representation of A_5.
- The two 4-dimensional irreps of S_5 restrict to the same 4-dimensional irrep of A_5.
- The two 5-dimensional irreps of S_5 restrict to the same 5-dimensional irrep of A_5.
- The 6-dimensional irrep of S_5 restricts as $\mathbf{3} \oplus \mathbf{3'}$ of A_5.

Let V be an irreducible A_5-module with character ψ. Then for any irreducible S_5-module W with character χ, Frobenius reciprocity says

$$\langle \mathrm{Ind}^{S_5}_{A_5}\psi, \chi\rangle_{S_5} = \langle \psi, \mathrm{Res}^{S_5}_{A_5}\chi\rangle_{A_5}.$$

Thus the multiplicity of W in $\mathrm{Ind}^{S_5}_{A_5}V$ is exactly the multiplicity of V in the restriction of W.

From the decompositions in Step 2 we immediately read off the induced representations:

- For the trivial representation $\mathbf{1}$ of A_5:

  $$\langle \mathbf{1}, \mathrm{Res}\,\chi_{(5)}\rangle_{A_5} = 1, \qquad \langle \mathbf{1}, \mathrm{Res}\,\chi_{(1^5)}\rangle_{A_5} = 1,$$

  and $\mathbf{1}$ does not appear in the restrictions of any other irreps. Hence

  $$\mathrm{Ind}^{S_5}_{A_5}(\mathbf{1}) \;\cong\; \chi_{(5)} \;\oplus\; \chi_{(1^5)}.$$

  (Dimensions: $2 \cdot 1 = 1 + 1$.)

- For the 3-dimensional irrep $\mathbf{3}$ of A_5: it appears once in $\mathrm{Res}\,\chi_{(3,1,1)}$, and nowhere else. Thus

  $$\mathrm{Ind}^{S_5}_{A_5}(\mathbf{3}) \;\cong\; \chi_{(3,1,1)}.$$

  (Dimensions: $2 \cdot 3 = 6$.)

- For the $3'$-dimensional irrep $\mathbf{3'}$ of A_5: similarly, it appears once in $\mathrm{Res}\,\chi_{(3,1,1)}$, and nowhere else, so

  $$\mathrm{Ind}^{S_5}_{A_5}(\mathbf{3'}) \;\cong\; \chi_{(3,1,1)}.$$

  (Again $2 \cdot 3 = 6$.)

- For the 4-dimensional irrep $\mathbf{4}$: it appears once in each of $\mathrm{Res}\,\chi_{(4,1)}$ and $\mathrm{Res}\,\chi_{(2,1,1,1)}$, and nowhere else. Hence

  $$\mathrm{Ind}^{S_5}_{A_5}(\mathbf{4}) \;\cong\; \chi_{(4,1)} \;\oplus\; \chi_{(2,1,1,1)}.$$

  (Dimensions: $2 \cdot 4 = 4 + 4$.)

- For the 5-dimensional irrep $\mathbf{5}$: it appears once in each of $\mathrm{Res}\,\chi_{(3,2)}$ and $\mathrm{Res}\,\chi_{(2,2,1)}$, and nowhere else. Thus

  $$\mathrm{Ind}^{S_5}_{A_5}(\mathbf{5}) \;\cong\; \chi_{(3,2)} \;\oplus\; \chi_{(2,2,1)}.$$

  (Dimensions: $2 \cdot 5 = 5 + 5$.)

Writing the A_5-irreps as $\mathbf{1}, \mathbf{3}, \mathbf{3}', \mathbf{4}, \mathbf{5}$, we obtain:

$$\begin{aligned}
\operatorname{Ind}_{A_5}^{S_5}(\mathbf{1}) &\cong \chi_{(5)} \oplus \chi_{(1^5)}, \\
\operatorname{Ind}_{A_5}^{S_5}(\mathbf{3}) &\cong \chi_{(3,1,1)}, \\
\operatorname{Ind}_{A_5}^{S_5}(\mathbf{3}') &\cong \chi_{(3,1,1)}, \\
\operatorname{Ind}_{A_5}^{S_5}(\mathbf{4}) &\cong \chi_{(4,1)} \oplus \chi_{(2,1,1,1)}, \\
\operatorname{Ind}_{A_5}^{S_5}(\mathbf{5}) &\cong \chi_{(3,2)} \oplus \chi_{(2,2,1)}.
\end{aligned}$$

In particular, each induced representation has dimension $[S_5 : A_5] \cdot \dim V = 2 \dim V$, as expected.

3.7. The Group Algebra

3.7.1. Introduction

Definition and Basic Properties Let G be a finite group. The *complex group algebra* of G is defined as

$$\mathbb{C}G \coloneqq \left\{ \sum_{g \in G} \omega_g \, g \,\middle|\, \omega_g \in \mathbb{C} \right\}.$$

As a complex vector space, $\mathbb{C}G$ has basis $\{g : g \in G\}$, so

$$\dim_{\mathbb{C}}(\mathbb{C}G) = |G|.$$

Multiplication in $\mathbb{C}G$ is defined by linearly extending the group multiplication:

$$\Big(\sum_g \omega_g g\Big)\Big(\sum_h \eta_h h\Big) \;=\; \sum_{g,h} \omega_g \eta_h \,(gh).$$

Now let (ρ, V) be a (finite-dimensional) representation of G, i.e. a homomorphism

$$\rho : G \longrightarrow \mathbf{Aut}(V) = \mathbb{GL}(V).$$

We extend ρ linearly to an algebra homomorphism

$$\widehat{\rho} : \mathbb{C}G \longrightarrow \mathbf{End}(V), \qquad \widehat{\rho}\Big(\sum_g \omega_g g\Big) \coloneqq \sum_g \omega_g \, \rho(g).$$

Thus every representation of G naturally defines a module structure of $\mathbb{C}G$ on V.

The Regular Representation and the Group Algebra Consider the *left regular representation* of G on the vector space $\mathbb{C}^{|G|}$ with basis $\{e_h : h \in G\}$:

$$\rho_{\mathrm{reg}} : G \longrightarrow \mathbb{GL}(\mathbb{C}^{|G|}), \qquad \rho_{\mathrm{reg}}(g) \cdot e_h = e_{gh}.$$

Identifying $\mathbb{C}^{|G|}$ with $\mathbb{C}G$ via $e_h \leftrightarrow h$, we see that $\rho_{\mathrm{reg}}(g)$ is exactly left multiplication by g in $\mathbb{C}G$:

$$\rho_{\mathrm{reg}}(g)(h) = gh.$$

Thus the regular representation is nothing but the action of $\mathbb{C}G$ on itself by left multiplication.

3.7.2. Wedderburn Theorem and Decomposition of $\mathbb{C}G$

Theorem 7 (Wedderburn)**.** *Let R be a nonzero ring with* 1 *(not necessarily commutative). Then the following are equivalent:*

1. *Every left R-module is projective.*

2. Every left R-module is injective.

3. Every left R-module is completely reducible (i.e. every submodule has a complementary submodule).

4. The left R-module R decomposes as a finite direct sum

$$R = L_1 \oplus L_2 \oplus \cdots \oplus L_n$$

where each L_j is a simple left ideal. Moreover, each L_j is of the form

$$L_j = Re_j$$

for some idempotent $e_j \in R$ such that

- $e_j^2 = e_j$,
- $e_j e_k = 0$ *for* $j \neq k$,
- $\sum_{j=1}^{n} e_j = 1$.

For the purposes of this theorem we will use only the decomposition into simple left ideals as stated; the explicit description $L_j = Re_j$ with the properties of the e_j will be proved later in the special case $R = \mathbb{C}G$.

Proof. We will prove the cycle

$$(1) \Rightarrow (3) \Rightarrow (4) \Rightarrow (3) \Rightarrow (1), (2),$$

and then show (2) ⇒ (3), which completes the equivalence of all four statements.

Step 1: (1) ⇒ (3)**.**

Assume that every left R-module is projective. Let M be any left R-module and let $N \subseteq M$ be a submodule. Consider the short exact sequence

$$0 \longrightarrow N \xrightarrow{\iota} M \xrightarrow{\pi} M/N \longrightarrow 0.$$

By assumption, M/N is projective, so the epimorphism π admits a splitting: there exists an R-linear map

$$s : M/N \longrightarrow M$$

such that $\pi \circ s = \mathrm{id}_{M/N}$. Then $\iota(N)$ is a direct summand of M:

$$M = N \oplus s(M/N).$$

Thus every submodule of every module is a direct summand. By the usual characterization of completely reducible (semisimple) modules, this means every R-module is completely reducible, i.e. (3) holds.

Step 2: (2) ⇒ (3)**.**

Assume every left R-module is injective. Let M be any left R-module and $N \subseteq M$ a submodule. Again consider the short exact sequence

$$0 \longrightarrow N \xrightarrow{\iota} M \xrightarrow{\pi} M/N \longrightarrow 0.$$

By assumption, N is injective as an R-module, so the monomorphism ι splits: there exists an R-linear map

$$p : M \longrightarrow N$$

such that $p \circ \iota = \mathrm{id}_N$. This implies that M decomposes as

$$M = N \oplus \ker(p).$$

Again every submodule is a direct summand, so every module is completely reducible. Thus (3) holds.

Step 3: (3) ⇒ (4)**.**

Assume now that every left R-module is completely reducible. In particular, this applies to the left regular module ${}_R R$.

By hypothesis, the module R is a direct sum of simple submodules:

$$R = \bigoplus_{j \in J} L_j,$$

where each L_j is a simple left R-submodule of R, equivalently a simple left ideal.

We first show that the index set J is finite. Since R is generated by 1 as a left R-module, the element $1 \in R$ can be written as a *finite* sum

$$1 = x_{j_1} + x_{j_2} + \cdots + x_{j_m}$$

with $x_{j_k} \in L_{j_k}$ and distinct indices $j_1, \ldots, j_m \in J$. This is because in a direct sum each element has only finitely many nonzero components. Hence every element of R lies in the finite subsum

$$L_{j_1} \oplus \cdots \oplus L_{j_m},$$

and therefore

$$R = L_{j_1} \oplus \cdots \oplus L_{j_m}.$$

Renaming indices, we obtain a finite decomposition

$$R = L_1 \oplus \cdots \oplus L_n$$

with each L_j a simple left ideal, as required in (4).

(The additional statement that each L_j is of the form Re_j for an idempotent e_j with $e_j e_k = 0$ and $\sum_j e_j = 1$ is a standard fact in the general theory of semisimple rings; in our context we will later prove these properties in the special case $R = \mathbb{C}G$.)

Step 4: (4) $\Rightarrow$ (3)**.**

Assume now that

$$R = L_1 \oplus \cdots \oplus L_n$$

with each L_j a simple left ideal. This means that the regular module ${}_R R$ is completely reducible (a direct sum of simple modules), so R is a *semisimple ring* on the left.

We claim that under this assumption, *every* left R-module is completely reducible.

Let M be any left R-module. Choose an epimorphism from a free module onto M, i.e. there exists a set I and a surjective R-linear map

$$\pi : R^{(I)} \longrightarrow M,$$

where $R^{(I)}$ denotes the direct sum of copies of R indexed by I.

Since R is a direct sum of simple modules, so is any direct sum of copies of R. More precisely,

$$R^{(I)} \cong \bigoplus_{i \in I} R \cong \bigoplus_{i \in I} \big(L_1 \oplus \cdots \oplus L_n\big),$$

which is a direct sum of simple modules L_j. Thus $R^{(I)}$ is completely reducible.

Let $K = \ker(\pi)$. As a submodule of a completely reducible module, K is a direct summand of $R^{(I)}$ (this is precisely the defining property of complete reducibility). Hence

$$R^{(I)} = K \oplus N$$

for some submodule $N \subseteq R^{(I)}$.

Restricting π to N, we get a surjective homomorphism

$$\pi|_N : N \longrightarrow M.$$

But $K = \ker(\pi)$ implies that π is injective on N, so $\pi|_N$ is an isomorphism. Therefore

$$M \cong N.$$

As N is a direct summand of the completely reducible module $R^{(I)}$, it is itself completely reducible. Hence M is completely reducible. Since M was arbitrary, (3) holds.

Step 5: (3) $\Rightarrow$ (1) **and** (3) $\Rightarrow$ (2)**.**

Finally, assume (3): every left R-module is completely reducible. We show that then every module is both projective and injective.

(3) $\Rightarrow$ (1): every module is projective.

Let M be a completely reducible left R-module, and let

$$f : N \longrightarrow M$$

be a surjective homomorphism. We must show that f splits, i.e. there exists $s : M \to N$ with $f \circ s = \mathrm{id}_M$.

Write M as a direct sum of simple submodules,

$$M = \bigoplus_{\alpha \in A} S_\alpha.$$

For each α, the restriction

$$f^{-1}(S_\alpha) \longrightarrow S_\alpha$$

is a surjection onto the simple module S_α. A surjective homomorphism onto a simple module is either 0 or an isomorphism; since $S_\alpha \neq 0$, it is an isomorphism onto its image (which is S_α itself). Thus for each α we can choose a section

$$s_\alpha : S_\alpha \longrightarrow N$$

with $f \circ s_\alpha = \mathrm{id}_{S_\alpha}$.

Define

$$s : M = \bigoplus_\alpha S_\alpha \longrightarrow N, \qquad s\big((x_\alpha)_\alpha\big) = \sum_\alpha s_\alpha(x_\alpha),$$

where the sum is finite for each element of M. Then s is R-linear and satisfies

$$f \circ s = \mathrm{id}_M.$$

Hence M is projective. Since this applies to every module M, (1) holds.

(3) $\Rightarrow$ (2): every module is injective.

Let M be a completely reducible left R-module. Consider any embedding of R-modules

$$\iota : M \longrightarrow E.$$

We must show that ι splits, i.e. there exists $p : E \to M$ with $p \circ \iota = \mathrm{id}_M$.

Since E is a left R-module, it is completely reducible by (3), and hence $M = \iota(M)$ is a direct summand of E:

$$E = \iota(M) \oplus E'$$

for some submodule E'. Let $p : E \to \iota(M)$ be the projection onto the first summand, and identify $\iota(M)$ with M via ι. Then p is an R-linear map with $p \circ \iota = \mathrm{id}_M$, so M is injective. Since M was arbitrary, (2) holds.

Combining the implications proved in Steps 1–5, we see that

$$(1) \Longleftrightarrow (3) \Longleftrightarrow (4) \quad \text{and} \quad (2) \Longleftrightarrow (3),$$

so all four conditions are equivalent. $\square$

Decomposition of $\mathbb{C}G$ Let $\{W_j\}_{j=1}^n$ be a complete set of pairwise non-isomorphic irreducible (complex) representations of G. From the character theory of finite groups we know that

$$|G| \;=\; \sum_{j=1}^{n} \big(\dim W_j\big)^2.$$

Moreover, the regular representation decomposes as

$$\rho_{\mathrm{reg}} \cong \bigoplus_{j=1}^{n} W_j^{\oplus \dim W_j}$$

(as G-representations and hence as $\mathbb{C}G$-modules).

For each j, the representation $\rho_j : G \to \mathbb{GL}(W_j)$ extends to an algebra homomorphism

$$\widehat{\rho}_j : \mathbb{C}G \longrightarrow \mathbf{End}(W_j), \quad \widehat{\rho}_j\Big(\sum_g \omega_g g\Big) = \sum_g \omega_g\, \rho_j(g).$$

These assemble into a canonical algebra homomorphism

$$\Phi : \mathbb{C}G \longrightarrow \bigoplus_{j=1}^{n} \mathbf{End}(W_j), \qquad \Phi(x) = \big(\widehat{\rho}_1(x), \ldots, \widehat{\rho}_n(x)\big).$$

Proposition 7. *Let G be a finite group, and let $\{W_j\}_{j=1}^n$ be a full set of irreducible representations of G. Then there is an isomorphism of $\mathbb{C}$-algebras*

$$\mathbb{C}G \;\cong\; \bigoplus_{j=1}^{n} \boldsymbol{End}(W_j).$$

Proof. We first show that Φ is injective. Consider $\mathbb{C}G$ as a left $\mathbb{C}G$-module via left multiplication. As recalled above, this regular module decomposes as

$$\mathbb{C}G \;\cong\; \bigoplus_{j=1}^{n} W_j^{\oplus \dim W_j}.$$

The map Φ records, for each $x \in \mathbb{C}G$, the induced endomorphisms on each irreducible summand W_j. If $\Phi(x) = 0$, then x acts as 0 on each W_j and hence on their direct sum, that is, x acts as 0 on the whole regular representation. But the regular representation is faithful as a $\mathbb{C}G$-module: if $x \neq 0$, then left multiplication by x is non-zero. Thus $x = 0$, so Φ is injective.

Next, we compare dimensions. Since $\dim \mathbf{End}(W_j) = (\dim W_j)^2$, we have

$$\dim_{\mathbb{C}}\Big(\bigoplus_{j=1}^{n} \mathbf{End}(W_j)\Big) = \sum_{j=1}^{n} \dim \mathbf{End}(W_j) = \sum_{j=1}^{n} \big(\dim W_j\big)^2 = |G| = \dim_{\mathbb{C}}(\mathbb{C}G).$$

Therefore an injective linear map between vector spaces of the same finite dimension must be an isomorphism. Hence Φ is bijective as a linear map, and since it is an algebra homomorphism, it is an isomorphism of algebras. $\square$

3.7.3. Irreducible Representations as Minimal Left Ideals

A (left) $\mathbb{C}G$-module is exactly a complex representation of G. The following standard result describes irreducible representations in terms of left ideals of the group algebra.

Proposition 8. *Let G be a finite group and W an irreducible (finite-dimensional) representation of G, viewed as a simple left $\mathbb{C}G$-module. Then:*

1. There exists a minimal (non-zero) left ideal $L \subseteq \mathbb{C}G$ such that $L \cong W$ as $\mathbb{C}G$-modules.

2. *Every minimal left ideal I of $\mathbb{C}G$ is generated by an idempotent $e \in \mathbb{C}G$ (i.e. $e^2 = e$), and one has $I = \mathbb{C}Ge$.*

Proof. (1) *Constructing a minimal left ideal isomorphic to W.*

Let $0 \neq v \in W$ and consider the $\mathbb{C}G$-module homomorphism

$$\varphi : \mathbb{C}G \longrightarrow W, \qquad \varphi(x) = x \cdot v.$$

Then φ is surjective: indeed, $\mathbb{C}G \cdot v$ is a non-zero submodule of W, and W is irreducible. Let

$$I := \ker(\varphi),$$

which is a left ideal of $\mathbb{C}G$. By the first isomorphism theorem,

$$W \;\cong\; \mathbb{C}G/I$$

as $\mathbb{C}G$-modules. Since W is simple, $\mathbb{C}G/I$ is a simple left $\mathbb{C}G$-module, so I is a maximal left ideal.

Now we want to realize $\mathbb{C}G/I$ (and hence W) *as a submodule of* $\mathbb{C}G$. This is where we use Maschke's theorem:

- The left regular representation of G on $\mathbb{C}G$ makes $\mathbb{C}G$ into a $\mathbb{C}G$-module.
- The kernel I is a submodule (left ideal) of this regular module.
- By Maschke's theorem (finite group, characteristic 0), every subrepresentation has a G-invariant complement. Hence there exists a $\mathbb{C}G$-submodule $L \subseteq \mathbb{C}G$ such that

  $$\mathbb{C}G \;=\; I \oplus L$$

  as $\mathbb{C}G$-modules.

Consider now the canonical projection

$$\pi : \mathbb{C}G \longrightarrow \mathbb{C}G/I.$$

The restriction of π to L gives a $\mathbb{C}G$-homomorphism

$$\pi|_L : L \longrightarrow \mathbb{C}G/I.$$

Since $\mathbb{C}G = I \oplus L$, the kernel of $\pi|_L$ is

$$\ker(\pi|_L) = L \cap \ker(\pi) = L \cap I = \{0\},$$

so $\pi|_L$ is injective. Moreover, π is surjective, so $\pi(L)$ is a non-zero submodule of $\mathbb{C}G/I$. But $\mathbb{C}G/I$ is simple, hence

$$\pi(L) = \mathbb{C}G/I.$$

Therefore $\pi|_L : L \to \mathbb{C}G/I$ is an isomorphism of $\mathbb{C}G$-modules. Composing with the isomorphism $\mathbb{C}G/I \cong W$ obtained above, we have

$$L \;\cong\; \mathbb{C}G/I \;\cong\; W$$

as $\mathbb{C}G$-modules.

Finally, L is a non-zero left ideal in $\mathbb{C}G$ (it is a submodule of the regular module), and we claim it is minimal. Indeed, let $0 \neq J \subseteq L$ be a left ideal. Then J is a submodule of L, and under the isomorphism $L \cong W$ it corresponds to a submodule of the simple module W. Hence J must be all of L. Thus L is a minimal non-zero left ideal and $L \cong W$, as required.

(2) *Every minimal left ideal is generated by an idempotent, via a splitting argument.*

Let $I \subseteq \mathbb{C}G$ be a minimal non-zero left ideal. View $\mathbb{C}G$ as a left $\mathbb{C}G$-module via left multiplication. Then I is a simple submodule of this regular module.

Consider the short exact sequence of $\mathbb{C}G$-modules

$$0 \longrightarrow I \xrightarrow{\iota} \mathbb{C}G \xrightarrow{\pi} \mathbb{C}G/I \longrightarrow 0,$$

where ι is the inclusion and π is the quotient map.

By Maschke's theorem again, this exact sequence splits: there exists a $\mathbb{C}G$-module homomorphism

$$p : \mathbb{C}G \longrightarrow I$$

such that p is a projection onto I, i.e.

$$p|_I = \mathrm{id}_I.$$

Equivalently, we have a direct sum decomposition

$$\mathbb{C}G = I \oplus \ker(p)$$

as $\mathbb{C}G$-modules, and p is the projection onto the first summand.

Now define

$$e \coloneqq p(1) \in I \subseteq \mathbb{C}G.$$

We show two facts:

1. $I = \mathbb{C}G \cdot e$;
2. $e^2 = e$.

(i) $I = \mathbb{C}Ge$. Because p is $\mathbb{C}G$-linear and 1 is the multiplicative identity of $\mathbb{C}G$, we have, for any $a \in \mathbb{C}G$,

$$p(a) = p(a \cdot 1) = a \cdot p(1) = ae.$$

Thus the image of p is

$$\mathrm{Im}(p) = \{p(a) : a \in \mathbb{C}G\} = \{ae : a \in \mathbb{C}G\} = \mathbb{C}Ge.$$

On the other hand, by definition p is a projection onto I, so $\mathrm{Im}(p) = I$. Hence

$$I = \mathbb{C}Ge.$$

(ii) $e^2 = e$. We use the fact that p is a projection, i.e. $p^2 = p$. For any $a \in \mathbb{C}G$,

$$p^2(a) = p(a) \quad \Longrightarrow \quad p\big(p(a)\big) = p(a).$$

Using the description $p(a) = ae$, we get

$$p\big(p(a)\big) = p(ae) = (ae)e.$$

So the equality $p^2(a) = p(a)$ becomes

$$(ae)e = ae \quad \text{for all } a \in \mathbb{C}G.$$

In particular, setting $a = 1$ gives

$$(1 \cdot e)e = 1 \cdot e \quad \Longrightarrow \quad e^2 = e.$$

Thus e is an idempotent in $\mathbb{C}G$, and the minimal left ideal I is exactly $\mathbb{C}Ge$.

This completes the proof. □

Thus, up to isomorphism, irreducible representations of G correspond to minimal left ideals of $\mathbb{C}G$, and these in turn are generated by primitive idempotents.

3.7.4. Central Idempotents from Characters

Let W be an irreducible representation of G with character χ_W. Let $\rho_{\text{reg}} : G \to \mathbb{GL}(\mathbb{C}G)$ denote the regular representation, and let us extend it linearly to

$$\widehat{\rho}_{\text{reg}} : \mathbb{C}G \to \mathbf{End}(\mathbb{C}G).$$

Recall the standard projection operator onto the W-isotypic component of a G-representation. If V is any representation of G with action ρ_V, the map

$$\pi_W^{(V)} := \frac{\dim W}{|G|} \sum_{g \in G} \overline{\chi_W(g)}\, \rho_V(g)$$

is the G-equivariant projection $V \to V$ whose image is the direct sum of all subrepresentations of V isomorphic to W.

We are especially interested in the case $V = \mathbb{C}G$ with $\rho_V = \rho_{\text{reg}}$. In this case, the operator

$$\pi_W := \frac{\dim W}{|G|} \sum_{g \in G} \overline{\chi_W(g)}\, \rho_{\text{reg}}(g)$$

is a projection on $\mathbb{C}G$. Since $\rho_{\text{reg}}(g)$ is left multiplication by g, the operator π_W is given by left multiplication by the element

$$e_W := \frac{\dim W}{|G|} \sum_{g \in G} \overline{\chi_W(g)}\, g \;\in\; \mathbb{C}G.$$

Proposition 9. *For each irreducible representation W of G, the element e_W defined above satisfies:*

1. $e_W^2 = e_W$ (idempotent);

2. e_W lies in the center of $\mathbb{C}G$;

3. if W and W' are non-isomorphic irreducible representations, then $e_W e_{W'} = 0$;

4. $\sum_W e_W = 1$ in $\mathbb{C}G$, where the sum runs over a complete set of irreducible representations.

The e_W are precisely the primitive central idempotents of $\mathbb{C}G$.

Proof. By construction, π_W is a projection operator on the regular representation $\mathbb{C}G$:

$$\pi_W^2 = \pi_W.$$

Because π_W is given by left multiplication by e_W, this means

$$e_W^2 = e_W$$

inside $\mathbb{C}G$, proving (1).

The regular representation decomposes as

$$\mathbb{C}G \cong \bigoplus_j W_j^{\oplus \dim W_j}.$$

The operator π_W acts as the identity on all copies of W and as 0 on all other irreducible summands. Since this decomposition is G-equivariant and π_W commutes with the G-action, the corresponding element e_W must commute with every $g \in G$, hence with every element of $\mathbb{C}G$. Thus e_W is central, proving (2).

If $W \not\cong W'$, then the images of π_W and $\pi_{W'}$ lie in distinct isotypic components, hence

$$\pi_W \pi_{W'} = 0.$$

This translates to $e_W e_{W'} = 0$, proving (3).

Finally, the sum of all π_W over a complete set of irreducible W is the identity operator on the regular representation (each irreducible summand is covered exactly once). Therefore,

$$\sum_W e_W = 1$$

in $\mathbb{C}G$, proving (4). The fact that these e_W are primitive central idempotents follows from the Wedderburn decomposition $\mathbb{C}G \cong \bigoplus_j \mathbf{End}(W_j)$: each central idempotent corresponds to the identity on one matrix block and zero on all others, which is clearly primitive. $\square$

Combining Propositions 8 and 9, we see that:

- the minimal left ideals of $\mathbb{C}G$ are of the form $\mathbb{C}Ge_W$ for the primitive central idempotents e_W;
- each such left ideal is isomorphic, as a $\mathbb{C}G$-module, to a direct sum of copies of the corresponding irreducible representation W.

In this sense, *locating the irreducible representations of G is equivalent to finding the primitive central idempotents e_W in the group algebra* $\mathbb{C}G$, which can be constructed explicitly from the characters via the formula above.

3.7.5. Equivalence of $\mathbf{Ind}_H^G W$ and $\mathbb{C}G \otimes_{\mathbb{C}H} W$

Let $H \leq G$ be a subgroup of a finite group G, and let W be a (left) $\mathbb{C}H$-module. There are two standard constructions of the induced representation:

(i) the *classical coset model*

$$\mathbf{Ind}_H^G(W) = \bigoplus_{\sigma \in G/H} \sigma \cdot W,$$

where the summands are the G-translates of W sitting inside some ambient vector space W^*;

(ii) the *extension of scalars* model

$$\mathbb{C}G \otimes_{\mathbb{C}H} W,$$

where $\mathbb{C}G$ is viewed as a right $\mathbb{C}H$-module and W as a left $\mathbb{C}H$-module.

These two constructions produce *canonically isomorphic* G-modules. This point of view is extremely useful because the second construction is functorial and gives a direct route to Frobenius reciprocity.

The tensor-product construction. Write $S = \mathbb{C}G$ and $R = \mathbb{C}H$. Viewing S as a right R-module via restriction of multiplication, the tensor product

$$S \otimes_R W = \mathbb{C}G \otimes_{\mathbb{C}H} W$$

is again a left S-module via left multiplication on the first factor. One thinks of this as "extending scalars" from R to S.

Unraveling the definitions shows that this construction reproduces the coset decomposition:

$$\mathbb{C}G \otimes_{\mathbb{C}H} W \;\cong\; \bigoplus_{\sigma \in G/H} \sigma \cdot W \;=\; \mathbf{Ind}_H^G(W)$$

as G-modules.

Universal property. The construction $\mathbb{C}G \otimes_{\mathbb{C}H} W$ satisfies the universal property of *extension of scalars*. In particular, for any G-module U,

$$\mathbf{Hom}_H(W, \mathbf{Res}_H^G U) \;\cong\; \mathbf{Hom}_G(\mathbb{C}G \otimes_{\mathbb{C}H} W,\, U),$$

which is exactly Frobenius reciprocity. Thus the reciprocity theorem is nothing more than the standard change-of-rings adjunction

$$\mathbf{Hom}_R(W, U) \;\cong\; \mathbf{Hom}_S(S \otimes_R W, U),$$

specialized to the inclusion $R = \mathbb{C}H \subseteq S = \mathbb{C}G$.

The embedding map $i : W \to \mathbb{C}G \otimes_{\mathbb{C}H} W$. Define the R-module homomorphism

$$i : W \longrightarrow \mathbb{C}G \otimes_{\mathbb{C}H} W, \qquad i(w) = 1 \otimes w.$$

Standard properties of tensor products imply:

Proposition 10. *Let i be as above. Then the quotient $W/\ker(i)$ is the largest quotient of W that embeds into some $\mathbb{C}G$-module.*

Proof. This follows directly from the universal property of tensor products. A detailed proof appears in Dummit–Foote, *Abstract Algebra*, §10.4 (Tensor Products of Modules). $\square$

Thus the tensor-product model $\mathbb{C}G \otimes_{\mathbb{C}H} W$ should be viewed as the "freest" G-module generated by W, modulo exactly those relations imposed by the $\mathbb{C}H$-module structure of W.

3.7.6. Exercise Interlude

Exercise 3.30 Let $H \leq G$ be a subgroup and W an H-module (that is, a left $\mathbb{C}H$-module). Recall that the induced module $\mathbf{Ind}_H^G(W)$ can be defined as

$$\mathbf{Ind}_H^G(W) \;=\; \bigoplus_{\sigma \in G/H} \sigma \cdot W,$$

or equivalently as the extension of scalars

$$\mathbf{Ind}_H^G(W) \;\cong\; \mathbb{C}G \otimes_{\mathbb{C}H} W.$$

We now realize the same representation as a space of W-valued functions on G.

Proposition 11. *There is an isomorphism of G-modules*

$$\boldsymbol{Ind}_H^G(W) \;\cong\; \boldsymbol{Hom}_H(\mathbb{C}G, W) \;\cong\; \{f : G \to W \mid f(hg) = h\, f(g) \quad \forall\, h \in H,\ g \in G\},$$

where G acts on the function space by

$$(g' \cdot f)(g) = f(gg').$$

Proof. **Step 1: $\mathbf{Hom}_H(\mathbb{C}G, W)$ as H-equivariant functions.**

Consider $\mathbb{C}G$ as a left $\mathbb{C}H$-module via left multiplication. Any $\mathbb{C}H$-linear map

$$\varphi : \mathbb{C}G \longrightarrow W$$

is completely determined by its values on the basis $\{g : g \in G\}$: for each $g \in G$, set

$$f(g) \coloneqq \varphi(g) \in W.$$

The condition that φ is $\mathbb{C}H$-linear says that for all $h \in H$ and $g \in G$,

$$\varphi(hg) = h\, \varphi(g).$$

In terms of the function f, this is exactly

$$f(hg) = h\, f(g) \qquad \forall\, h \in H,\ g \in G.$$

Conversely, given any function $f : G \to W$ satisfying $f(hg) = hf(g)$, we can extend it linearly to an H-module homomorphism

$$\varphi_f : \mathbb{C}G \longrightarrow W, \qquad \varphi_f\Big(\sum_g \omega_g g\Big) = \sum_g \omega_g f(g).$$

The H-equivariance condition ensures that $\varphi_f(hx) = h\, \varphi_f(x)$ for all $h \in H$, $x \in \mathbb{C}G$.

Thus we obtain a vector space isomorphism

$$\mathbf{Hom}_H(\mathbb{C}G, W) \;\cong\; \{f : G \to W \mid f(hg) = hf(g)\}.$$

Step 2: The G-action on $\mathbf{Hom}_H(\mathbb{C}G, W)$.

We now describe the natural G-action on $\mathbf{Hom}_H(\mathbb{C}G, W)$. Remember that $\mathbb{C}G$ is a (G, H)-bimodule: G acts on the left, H on the right. So we can let G act on $\mathbf{Hom}_H(\mathbb{C}G, W)$ by "right translation on the argument": for $g' \in G$ and $\varphi \in \mathbf{Hom}_H(\mathbb{C}G, W)$, define a new map

$$(g' \cdot \varphi)(x) := \varphi(xg') \qquad (x \in \mathbb{C}G).$$

It is easy to check that $g' \cdot \varphi$ is still H-linear, so this does define a left G-module structure on $\mathbf{Hom}_H(\mathbb{C}G, W)$.

Under the identification with functions $f : G \to W$ above, this action becomes

$$(g' \cdot f)(g) = f(gg'),$$

since

$$(g' \cdot \varphi)(g) \;=\; \varphi(gg') \;=\; f(gg').$$

Step 3: Identifying $\mathbf{Hom}_H(\mathbb{C}G, W)$ with $\mathbf{Ind}_H^G(W)$.

Choose a set $T \subseteq G$ of representatives for the right cosets of H in G, so that

$$G = \bigsqcup_{t \in T} Ht.$$

Any function $f : G \to W$ satisfying $f(hg) = hf(g)$ is determined entirely by its values on T: for each $t \in T$ we choose $w_t := f(t) \in W$, and then define for $g = ht$ with $h \in H$,

$$f(g) = f(ht) := h\,w_t.$$

The condition $f(hg) = hf(g)$ ensures that this is well-defined and gives back f.

Thus we have a linear isomorphism

$$\{f : G \to W \mid f(hg) = hf(g)\} \;\cong\; \bigoplus_{t \in T} W,$$

by

$$f \longmapsto (f(t))_{t \in T}, \qquad (w_t)_{t \in T} \longmapsto f \text{ with } f(ht) = hw_t.$$

On the other hand, the coset model of induction is

$$\mathbf{Ind}_H^G(W) = \bigoplus_{\sigma \in G/H} \sigma \cdot W \cong \bigoplus_{t \in T} W,$$

and the G-action on $\bigoplus_{t \in T} W$ is obtained by transporting the natural action of G on cosets.

One checks directly that under the above identification $(w_t)_{t \in T} \leftrightarrow f$, the action of G on $\mathbf{Ind}_H^G(W)$ corresponds exactly to the action

$$(g' \cdot f)(g) = f(gg')$$

on functions. Hence

$$\mathbf{Ind}_H^G(W) \;\cong\; \mathbf{Hom}_H(\mathbb{C}G, W)$$

as G-modules.

This completes the proof. $\square$

Exercise 3.31 If $\mathbb{C}G$ is identified with the space of functions on G, the element

$$\sum_{g\in G} \phi(g)\, e_g \in \mathbb{C}G$$

corresponds to the function $\phi : G \to \mathbb{C}$. Show that the product in $\mathbb{C}G$ corresponds to convolution of functions:

$$(\phi * \psi)(g) \;=\; \sum_{h\in G} \phi(h)\, \psi(h^{-1}g).$$

Solution. Recall that $\mathbb{C}G$ is the complex vector space with basis $\{e_g : g \in G\}$, and the multiplication is determined by

$$e_g \cdot e_h \;=\; e_{gh}, \qquad g, h \in G,$$

and extended bilinearly.

Any element $x \in \mathbb{C}G$ can be written uniquely as

$$x = \sum_{g\in G} x(g)\, e_g,$$

so we can identify x with the function $x : G \to \mathbb{C}$ given by $x(g) =$ "coefficient of e_g in x". Under this identification, a function $\phi : G \to \mathbb{C}$ corresponds to the element

$$\sum_{g\in G} \phi(g)\, e_g \in \mathbb{C}G,$$

and similarly for ψ.

Now compute the product in $\mathbb{C}G$:

$$\Big(\sum_{g\in G} \phi(g)\, e_g\Big)\Big(\sum_{k\in G} \psi(k)\, e_k\Big) \;=\; \sum_{g\in G}\sum_{k\in G} \phi(g)\psi(k)\, e_{gk}.$$

We want to read off the coefficient of e_x for a fixed $x \in G$. The coefficient of e_x is

$$\sum_{\substack{g,k\in G\\ gk=x}} \phi(g)\psi(k).$$

For each $g \in G$ there is a unique $k = g^{-1}x$ such that $gk = x$, so we can rewrite the sum as

$$\sum_{g\in G} \phi(g)\, \psi(g^{-1}x).$$

Thus, under the identification of $\mathbb{C}G$ with the space of functions on G, the product of the elements corresponding to ϕ and ψ is the function $x \mapsto (\phi * \psi)(x)$, where

$$(\phi * \psi)(x) \;=\; \sum_{g\in G} \phi(g)\, \psi(g^{-1}x).$$

This is exactly the standard convolution formula, so the multiplication in $\mathbb{C}G$ corresponds to convolution of functions on G. $\square$

Chapter 4

Nilpotent Lie Algebras and Solvable Lie Algebras

4.1. Lower Central Series

Definition 9. *Let $\mathfrak{g}$ be a Lie algebra. We define its descending (lower) central series inductively by*

$$C^1(\mathfrak{g}) = \mathfrak{g} \quad \text{and} \quad C^{n+1}(\mathfrak{g}) = [\mathfrak{g}, C^n(\mathfrak{g})]$$

Proposition 12. *For every $n \in \mathbb{N}$ it follows that $C^{n+1}(\mathfrak{g})$ is an ideal.*

Proof. The proof is by induction on n.

1. If n=0 then $C^1(\mathfrak{g}) = \mathfrak{g}$ is an ideal.
2. Suppose that $C^n(\mathfrak{g})$ is an ideal for some $n \in \mathbb{N}$. Then want to show that $C^{n+1}(\mathfrak{g})$ is an ideal.
   - By definition $C^{n+1}(\mathfrak{g})$ is a linear subspace.
   - Since $C^n(\mathfrak{g})$ is an ideal by induction then for any $w, x \in \mathfrak{g}$ and $y \in C^n(\mathfrak{g})$ it follows that $[w, [x, y]] = [w, v]$ where $v \in C^n(\mathfrak{g})$ so $[\mathfrak{g}, C^{n+1}(\mathfrak{g})] \subseteq C^{n+1}(\mathfrak{g})$. Therefore $C^{n+1}(\mathfrak{g})$ is an ideal.

■

Remark 17. *From this we can conclude that $C^{n+1}(\mathfrak{g}) \subseteq C^n(\mathfrak{g})$ for every $n \in \mathbb{N}$. Also, inductively it can be shown that for any n,m $\implies$ $[C^n(\mathfrak{g}), C^m(\mathfrak{g})] \subset C^{n+m}(\mathfrak{g})$.*

Definition 10. *A Lie algebra $\mathfrak{g}$ is said to be nilpotent if there exists an integer n s.t. $C^n(\mathfrak{g}) = 0$. One says that $\mathfrak{g}$ is nilpotent of class $\leq n$ if $C^{n+1}(\mathfrak{g}) = 0$*

4.2. Definition of Nilpotent Lie Algebras (part 1)

Proposition 13. *The following conditions are equivalent:*

1. *$\mathfrak{g}$ is nilpotent of class $\leq r$*
2. *For all $x_0, ..., x_r \in \mathfrak{g}$ we have*

   $$[x_0, [x_1, [..., x_r]...]] = (ad\ x_0)(ad\ x_1)\cdots(ad\ x_{r-1})(x_r) = 0$$

   where for $x \in \mathfrak{g}$ we define $ad\ x : \mathfrak{g} \to \mathfrak{g}$ as $ad\ x(y) = [x, y]$ for all $y \in \mathfrak{g}$. In particular this result implies that $\forall x \in \mathfrak{g}$ it follows that $(ad\ x)^{r-1} = 0$
3. *There is a descending series of ideals*

   $$\mathfrak{g} = a_0 \supset a_1 \supset \cdots \supset a_r = 0$$

   such that $[\mathfrak{g}, a_i] \subset a_{i+1}$ and a_{i+1} is an ideal for $0 \leq i \leq r-1$.

Proof.

- It follows easily from definition that (1) $\iff$ (2)
- WTS that (2) implies (3). Let $a_i = C^{i+1}(\mathfrak{g})$ then the sequence $(a_i)_{0\leq i\leq r-1}$ forms a descending series of ideals where $C^{i+1}(\mathfrak{g}) \subseteq C^i(\mathfrak{g})$.
- WTS that (3) implies (2). Take any $x_0, ..., x_r \in \mathfrak{g}$ and suppose there is a descending series of ideals $\mathfrak{g} = a_0 \supset a_1 \supset \cdots \supset a_r = 0$ such that $[\mathfrak{g}, a_i] \subset a_{i+1}$ and a_{i+1} is an ideal for $0 \leq i \leq r-1$. Then from the definition of a_i it follows that for all $0 \leq i \leq r-1 \Rightarrow (ad\ x_i)\cdots(ad\ x_1)(x_0) \in a_i$ so $(ad\ x_r)\cdots(ad\ x_1)(x_0) \in a_r = \{0\}$

■

4.3. Definition of Nilpotent Lie Algebras (part 2)

Definition 11. *The center of Lie algebra $\mathfrak{g}$ is $\mathfrak{z}(\mathfrak{g}) = \{x \in \mathfrak{g} : \forall y \in \mathfrak{g} \Rightarrow [x, y] = 0\}$.*

Proposition 14. *Let $\mathfrak{g}$ be a Lie algebra.*

1. If $\mathfrak{g}$ is nilpotent, then all subalgebras and all homomorphic images of $\mathfrak{g}$ are nilpotent.

2. If $\mathfrak{a} < \mathfrak{z}(\mathfrak{g})$ and $\mathfrak{g}/\mathfrak{a}$ is nilpotent, then $\mathfrak{g}$ is nilpotent.

In particular (1) and (2) imply that $\mathfrak{g}$ is nilpotent $\iff$ $\mathfrak{g}/\mathfrak{a}$ is nilpotent.

Proof.

1. If $\mathfrak{h} < \mathfrak{g}$, then $[\mathfrak{h}, \mathfrak{h}] \subseteq [\mathfrak{g}, \mathfrak{g}]$. Inductively it follows that for $\forall n \in \mathbb{N} \Rightarrow\ C^n(\mathfrak{h}) \subseteq C^n(\mathfrak{g})$. Therefore each subalgebra of nilpotent Lie algebra is nilpotent. Given Lie homomorphism $\alpha : \mathfrak{g} \to \mathfrak{h}$, it follows that
$$\alpha(C^2(\mathfrak{g})) = \alpha([\mathfrak{g}, \mathfrak{g}]) = [\alpha(\mathfrak{g}), \alpha(\mathfrak{g})] = C^2(\alpha(\mathfrak{g}))$$
Inductively follows that $C^n(\alpha) = \alpha(C^n(\mathfrak{g}))$ for each $n \in \mathbb{N}$. Thus, if $C^n(g) = \{0\}$, then $C^n(im\ \alpha) = \{0\}$
2. If $\mathfrak{g}/\mathfrak{a}$ is nilpotent, then $\exists\ n \in \mathbb{N}$ with $C^n(\mathfrak{g}/\mathfrak{a}) = \{0\}$. Let $\pi : \mathfrak{g} \to \mathfrak{g}/\mathfrak{a}$ be a quotient homomorphism, then from the previous arguments follows that $C^n(\mathfrak{g}/\mathfrak{a}) = C^n(\pi(\mathfrak{g})) = \pi(C^n(\mathfrak{g})) = \{0\}$. Therefore $C^n \subseteq \mathfrak{a} \subseteq \mathfrak{z}(\mathfrak{g})$ and thus $C^{n+1}(\mathfrak{g}) \subseteq [\mathfrak{g}, \mathfrak{z}(\mathfrak{g})] = \{0\}$, so $\mathfrak{g}$ is nilpotent. The remark about if and only if statement follows since if $\mathfrak{g}$ is nilpotent then $im\pi = \mathfrak{g}/\mathfrak{a}$ is nilpotent by (1).

■

4.4. An example of Nilpotent Algebra

Definition 12. *We define flag $D = (V_i)$ of V to be a descending series of vector spaces $V = V_n \supset \cdots \supset V_1 \supset V_0 = \{0\}$ such that dim $V_i = i$.*

Consider the set $\mathfrak{n}(D) = \{x \in \mathfrak{gl}(V) : (\forall j)\ xV_j \subseteq V_{j-1}\}$ corresponding to a flag $D = (V_i)$ of V. Then $\forall j$ and for any $x, y \in \mathfrak{n}(D) \implies (x+y)V_j \subseteq V_{j-1}$ and $[x, y]V_j \subseteq V_{j-1}$ so $\mathfrak{n}(D)$ forms a Lie subalgebra of $\mathfrak{gl}(V)$. To visualize this subalgebra, for any $j \in \{1, ..., n\}$ choose in V_j a subspace W_j with $V_j \cong V_{j-1} \oplus W_j$ so that we have $V_j \cong \bigoplus_{k=1}^{j} W_k$ where follows that dim $W_j = 1$. If $x \in \mathfrak{n}(D)$ then $xV_j \subseteq V_{j-1}$ implies that $x : \bigoplus_{k=1}^{j} W_k \to \bigoplus_{k=1}^{j-1} W_k$ so x corresponds to a strictly upper triangular matrix:

$$A = \begin{pmatrix} 0 & a_{21} & \dots & a_{n-11} & a_{n1} \\ 0 & 0 & \dots & a_{n-12} & a_{n2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & a_{nn-1} \\ 0 & 0 & \dots & 0 & 0 \end{pmatrix}$$

Given any such strictly upper triangular $n \times n$ matrices A, B it follows that $rank(AB) = rank(BA) \leq n-2$ so thus $rank([A, B]) \leq n-2$. Inductively follows that $C^n(\mathfrak{sut}) = 0$ where $\mathfrak{sut}$ is the Lie algebra of $n \times n$ strictly upper triangular matrices. Therefore follows that $\mathfrak{n}(D)$ is nilpotent.

4.5. Engel's Theorems (part 1)

We already have seen in Proposition 2.1 that if $\mathfrak{g}$ is nilpotent then $\forall x \in \mathfrak{g} \implies$ *ad x* is nilpotent. The main aim of Engel's Theorems is to show that a finite-dimensional Lie algebra is nilpotent if and only if *ad x* is nilpotent for every x in this algebra. Thus obtaining a simple characterization of nilpotency. To prove these theorems we will start with lemma.

Lemma 4. *Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{h} < \mathfrak{g}$ be a subalgebra. Then:*

1. $\mathfrak{g}/\mathfrak{h}$ is a vector space.

2. The set $R = \{ad_{\mathfrak{g}/\mathfrak{h}}(x) : x \in \mathfrak{h}\}$ is a Lie subalgebra of $\mathfrak{gl}(\mathfrak{g}/\mathfrak{h})$ where $ad_{\mathfrak{g}/\mathfrak{h}}(x) : \mathfrak{h} \to \mathfrak{g}/\mathfrak{h}$ such that $ad_{\mathfrak{g}/\mathfrak{h}}(x)(y + \mathfrak{h}) = [x, y] + \mathfrak{h}$ and where $y \in \mathfrak{g}/\mathfrak{h}$

3. If $x \in \mathfrak{h}$ is nilpotent, then so $ad_{\mathfrak{g}/\mathfrak{h}}(x)$ is nilpotent.

Proof.

1. The first is obvious.

2. WTS that (2) is true. For any $x_1, x_2 \in \mathfrak{h}$ and $y \in \mathfrak{g}/\mathfrak{h}$:

   - $ad_{\mathfrak{g}/\mathfrak{h}}(x_1) + ad_{\mathfrak{g}/\mathfrak{h}}(x_2) = ad_{\mathfrak{g}/\mathfrak{h}}(x_1 + x_2) \in R$
   -
$$\begin{aligned}
[ad_{\mathfrak{g}/\mathfrak{h}}(x_1), ad_{\mathfrak{g}/\mathfrak{h}}(x_2)](y + \mathfrak{h}) &= (ad_{\mathfrak{g}/\mathfrak{h}}(x_1) \cdot ad_{\mathfrak{g}/\mathfrak{h}}(x_2) - ad_{\mathfrak{g}/\mathfrak{h}}(x_2) \cdot ad_{\mathfrak{g}/\mathfrak{h}}(x_1))(y + \mathfrak{h}) \\
&= ad_{\mathfrak{g}/\mathfrak{h}}(x_1)([x_2, y] + \mathfrak{h}) - ad_{\mathfrak{g}/\mathfrak{h}}(x_2)([x_1, y] + \mathfrak{h}) \\
&= [x_1, [x_2, y]] - [x_2, [x_1, y]] + \mathfrak{h} \\
&= -([y, [x_1, x_2]] + [x_2, [y, x_1]]) - [x_2, [x_1, y]] + \mathfrak{h} \\
&= [[x_1, x_2], y] + \mathfrak{h} \\
&= ad_{\mathfrak{g}/\mathfrak{h}}([x_1, x_2])(y + \mathfrak{h})
\end{aligned} \tag{4.1}$$

     Thus follows that $[ad_{\mathfrak{g}/\mathfrak{h}}(x_1), ad_{\mathfrak{g}/\mathfrak{h}}(x_2)] = ad_{\mathfrak{g}/\mathfrak{h}}([x_1, x_2]) \in R$, so R is a Lie subalgebra.

   - To show that (3) is true, for fixed $x \in \mathfrak{h}$ consider operators: $\rho_x : \mathfrak{g}/\mathfrak{h} \to \mathfrak{g}/\mathfrak{h}$ and $\lambda_x : \mathfrak{g}/\mathfrak{h} \to \mathfrak{g}/\mathfrak{h}$ such that $\rho_x(y + \mathfrak{h}) = xy + \mathfrak{h}$ and $\lambda_x(y + \mathfrak{h}) = yx + \mathfrak{h}$. Since $x \in \mathfrak{h}$ and $\mathfrak{h}$ is closed under multiplication operation then $x \cdot (y + \mathfrak{h})$ and $(y + \mathfrak{h}) \cdot x$ are independent of the representative element of $\mathfrak{h}$. Thus operators ρ_x and λ_x are well defined. Also, we can note that under composition $\rho_x \cdot \lambda_x = \lambda_x \cdot \rho_x$ for any $x \in \mathfrak{h}$. Then follows that $ad_{\mathfrak{g}/\mathfrak{h}}(x)(y + \mathfrak{h}) = (\rho_x - \lambda_x)(y + \mathfrak{h})$. If for some $n \in \mathbb{N} \implies x^n = 0$ then by definition follows that $\rho_x^n = \lambda_x^n = 0$. Therefore can conclude that $(ad_{\mathfrak{g}/\mathfrak{h}}(x))^{2n} = 0$ so $ad_{\mathfrak{g}/\mathfrak{h}}(x)$ is nilpotent.

■

4.6. Engel's Theorems (part 2)

Theorem 8 (Engel's Theorem of linear Lie Algebras)**.** *Let $V \neq \{0\}$ be a finite-dimensional vector space and $\mathfrak{g} \subseteq \mathfrak{gl}(V)$ a Lie subalgebra. If all $x \in \mathfrak{g}$ are nilpotent, i.e, $x^n = 0$ for some $n \in \mathbb{N}$, then there exists a non-zero $v_0 \in V$ with $\mathfrak{g}(v_0) = \{0\}$.*

Proof. We will prove this result by induction on $dim\mathfrak{g}$.

- For $dim\mathfrak{g} = 0$ the result follows trivially.
- Suppose the result is true if $dim\mathfrak{g} \leq N - 1$ and let $dim\mathfrak{g} = N$. Choose a proper subalgebra $\mathfrak{h} < \mathfrak{g}$ of maximal dimension.

  1. By above Lemma follows that $\forall x \in \mathfrak{h} : ad_{\mathfrak{g}/\mathfrak{h}}(x)$ is nilpotent. Thus applying induction on $\mathfrak{g}/\mathfrak{h}$ and $ad_{\mathfrak{g}/\mathfrak{h}}(\mathfrak{h})$, it follows that $\exists x_0 \in \mathfrak{g}/\mathfrak{h}$ s.t. it is nonzero and $ad_{\mathfrak{g}/\mathfrak{h}}(\mathfrak{h})(x_0) = \{0\}$. Therefore by defintion of $ad_{\mathfrak{g}/\mathfrak{h}} \implies [\mathfrak{h}, x_0] \subseteq \mathfrak{h}$.

2. Above implies that $\mathbb{F}x_0 + \mathfrak{h}$ is a subalgebra of $\mathfrak{g}$. By maximality of $\mathfrak{h}$ follows that $\mathfrak{g} = \mathfrak{h} + \mathbb{F}x_0$, so $\mathfrak{h}$ is ideal.
3. The induction hypothesis applied to $\mathfrak{h}$ implies that $V_0 = \{v \in V : \mathfrak{h}(v) = \{0\}\}$ is non-zero. Also, if $x \in \mathfrak{g}, y \in \mathfrak{h}, and w \in V_0$ then

$$yx(w) = xy(w) - [x,y](w) \in x\mathfrak{h}(w) + \mathfrak{h}(w) = \{0\}$$

Therefore $\mathfrak{g}(V_0) \subseteq V_0$

4. Since V_0 is a vector space s.t. $x_0|_{V_0}$ is nilpotent, then by induction on subalgebra $\mathbb{F}x_0$ and vector space V_0 it follows that $\exists v_0 \in V_0$ s.t. $x_0(v_0) = 0$.
5. Above implies that $\mathfrak{g}(v_0) = \mathfrak{h}(v_0) + \mathbb{F}x_0(v_0) = \{0\}$.

■

4.7. Engel's Theorems (part 3)

Definition 13. *Let $\mathfrak{g}$ be a Lie algebra and V be a vector space. Suppose that $g \times V \to V$, $(x,v) \to x \cdot v$ is a bilinear map. If for all $x, y \in \mathfrak{g}$ and $v \in V$ it follows that*

$$[x,y] \cdot v = x \cdot (y \cdot v) - y \cdot (x \cdot v)$$

then V is called a $\mathfrak{g}$-module.

Definition 14. *A representation of a Lie algebra $\mathfrak{g}$ on a vector space V over a field K is a Lie algebra homomorphism from $\mathfrak{g}$ to $\mathfrak{gl}(V)$.*

Remark 18. *Let $\phi : \mathfrak{g} \to \mathfrak{gl}(V)$ be a representation. Then a $\mathfrak{g}$-module structure on V is defined by $x \cdot v = \phi(x)v$. Converely, for every $\mathfrak{g}$-module V, the map $\phi : \mathfrak{g} \to \mathfrak{gl}(V)$ defined by $\phi(x)v = x \cdot v$ is a representation. THus these concepts are equivalent.*

4.8. Why Engel's Theorem does not generalize?

Engel's Theorem does not generalize to infinite-dimensional spaces. Consider the vector space $V = \mathbb{F}^{\mathbb{N}}$ with the basis $\{e_i : i \in \mathbb{N}\}$. Consider the set $\mathfrak{g} = span\{E_{ij} : i > j\}$ where $E_{ij} = |e_i\rangle\langle e_j|$ rank-one-operators. Then $\mathfrak{g}$ is a set of strictly lower triangular matrices and it is easy to see that $\mathfrak{g}$ is a Lie algebra.

1. For any $E \in \mathfrak{g} \implies \exists\{E_{ij}\}$ finite collection of rank-one-operators s.t. $E = \sum_{i,j} a_{ij}E_{ij}$. This implies that E corresponds to a strictly lower triangular matrix of finite rank and thus acts non-trivially only on some finite-dimensional subspace of V. From this can conclude that $\exists n \in \mathbb{N}$ s.t. $E^n = 0$. Thus $\mathfrak{g}$ consists of nilpotent endomorphisms of finite rank
2. Easy to observe that $\{v \in V : \mathfrak{g}(v) = \{0\}\} = \{0\}$, therefore the result from the first Engel's theorem does not hold.

4.9. Engel's Theorems (part 4)

Theorem 9. *Let V be a finite-dimensional vector space and $\mathfrak{g} < \mathfrak{gl}(V)$ a subalgebra such that all elements of $\mathfrak{g}$ are nilpotent. Then:*

- *There is a flag D of V s.t. $\mathfrak{g} \subset \mathfrak{n}(D)$.*
- *$\mathfrak{g}$ is nilpotent Lie algebra.*

Proof. The proof is by induction on dim V.

- If dim$V = 0$ then the result is trivial.
- Suppose the Theorem holds if dim$V = n - 1$. Let dim$V$=n.

1. By previous Theorem $\exists v_1 \in V$ s.t. $\mathfrak{g}(v_1) = \{0\}$. Define $V_1 = \mathbb{F}v_1$, then $\pi : \mathfrak{g} \to \mathfrak{gl}(V/V_1)$ s.t. $\pi(x)(v + V_1) = x(v) + V_1$ is a well defined representation of $\mathfrak{g}$ on V/V_1 and $\pi(\mathfrak{g})$ consists of nilpotent endomorphisms.
2. Applying the induction hypothesis on V/V_1 and subalgebra $\pi(\mathfrak{g}) < \mathfrak{gl}(V/V_1)$, it follows that there exists flag $D' = (W_i)_{i=1}^k$ of V/V_1 s.t. $\pi(\mathfrak{g}) \subset \mathfrak{n}(D')$ i.e. $\forall x \in \pi(\mathfrak{g})$ and $\forall i \implies xW_i \subseteq W_{i-1}$ where $W_k = V/V_1$ and $W_1 = V_1$.
3. If $V_i = q^{-1}(W_i)$ where $q : V \to V/V_1$ is a quotient map and $V_0 = \{0\}$ then
$$V = V_k \supseteq \cdots \supseteq V_1 \supseteq V_0 = \{0\}$$
so for all $i \geq 1$ follows that $\mathfrak{g}(V_i) \subseteq V_{i-1}$. Therefore $\mathfrak{g} \subseteq \mathfrak{n}(D)$.
4. Since $\mathfrak{n}(D)$ is nilpotent then follows that $\mathfrak{g}$ is nilpotent.

■

4.10. Engel's Theorems (part 5)

Theorem 10 (Engel's Characterization Theorem for Nilpotent Lie Algebras)**.** *Let $\mathfrak{g}$ be a finite-dimensional Lie algebra. Then $\mathfrak{g}$ is nilpotent if and only if for each $x \in \mathfrak{g}$ the operator $ad(x)$ is nilpotent.*

Proof. ($\Rightarrow$) If $\mathfrak{g}$ is nilpotent thenfor some $n \in \mathbb{N}$ it follows that $C^{n+1}(\mathfrak{g}) = \{0\}$, so that $\forall x \in \mathfrak{g} : (ad\ x)^n \mathfrak{g} \subseteq C^{n+1}(\mathfrak{g}) = \{0\}$. Therefore $\forall x \in \mathfrak{g}$, $ad\ x$ is nilpotent.
($\Leftarrow$) If $ad\ x$ is nilpotent for $\forall x \in \mathfrak{g}$, then Theorem 9.1 implies that the Lie algebra $ad(\mathfrak{g})$ is nilpotent. From the defintion of center $\mathfrak{z}(\mathfrak{g})$ and $ad(\mathfrak{g})$ it follows that $ad(\mathfrak{g}) \cong \mathfrak{g}/\mathfrak{z}(\mathfrak{g})$. Then by Proposition 3.1 follows that $\mathfrak{g}$ is nilpotent. ■

4.11. Derived series

Definition 15. *Let $\mathfrak{g}$ be a Lie algebra. The derived series of $\mathfrak{g}$ is defined by:*
$$D^1(\mathfrak{g}) = \mathfrak{g},\ \text{and}\ D^n = [D^{n-1}(\mathfrak{g}), D^{n-1}(\mathfrak{g})]$$
for $n \geq 2$. Easy induction implies that $D^n(\mathfrak{g})$ are ideals of $\mathfrak{g}$ and that for all $n \in \mathbb{N} : D^n(\mathfrak{g}) \subseteq D^{n-1}(\mathfrak{g})$.

Definition 16. *The Lie algebra $\mathfrak{g}$ is said to be solvable if there exists an integer n such that $D^n(\mathfrak{g}) = 0$. One says that $\mathfrak{g}$ is solvable of derived length $\leq r$ if $D^{r+1}(\mathfrak{g}) = \{0\}$.*

4.12. Examples of solvable algebras

1. If $\mathfrak{g}$ is nilpotent then it is solvable because by easy induction follows that $D^n(\mathfrak{g}) \subseteq C^{n+1}(\mathfrak{g})$.
2. But solvable does not imply nilpotent. Let $\mathfrak{g}$ be a 2 dimensional non-abelian Lie algebra with basis x,y. Then $0 \neq [x, y] = ax+by$ for $(a, b) \neq (0, 0)$. Assume w.l.o.g. that $b \neq 0$ and put $v = b^{-1}x$ and $w = ax+by$. Then follows that $[x, y] = [v, w] = w$. Thus for any $c \in \mathbb{F}$ it follows that
$$[cx, y] = [x, cy] = c[x, y] = c[v, w] = cw$$
This implies that $D^1(\mathfrak{g}) = [\mathfrak{g}, \mathfrak{g}] = \mathbb{F}w$ and $D^2(\mathfrak{g}) = \{0\}$, so $\mathfrak{g}$ is solvable. On the other hand, $C^n(\mathfrak{g}) = \mathbb{F}w$ for each $n > 1$, so that $\mathfrak{g}$ is not nilpotent.

4.13. Definition of solvable algebras (part 1)

Proposition 15. *The following statements hold:*

1. If $\mathfrak{g}$ is solvable, then all subalgebras and homomorphic images of $\mathfrak{g}$ are solvable.

2. *If $\mathfrak{i}$ is solvable ideal of $\mathfrak{g}$ and $\mathfrak{g}/\mathfrak{i}$ is solvable, then $\mathfrak{g}$ is solvable.*

Proof. 1. If $\mathfrak{h} \subseteq \mathfrak{g}$ is a subalgebra, then $D^n(\mathfrak{h}) \subseteq D^n(\mathfrak{g})$ follows from the definition, and if $\alpha : \mathfrak{g} \to \mathfrak{h}$ is a homomorphism of Lie algebras, then from easy application of induction follows that

$$D^n(\alpha(\mathfrak{g})) = [D^{n-1}(\alpha(\mathfrak{g})), D^{n-1}(\alpha(\mathfrak{g}))] = \alpha(D^n(\mathfrak{g}))$$

which implies (1).

2. Let $\pi : \mathfrak{g} \to \mathfrak{g}/\mathfrak{i}$ be the quotient map. From the previous follows that $\pi(D^n(\mathfrak{g})) = D^n(\pi(\mathfrak{g}))$ for each n. If $\mathfrak{g}/\mathfrak{i}$ is solvable, then $\pi(D^n(\mathfrak{g})) = \bar{0}$ for some $n \in \mathbb{N}$. This implies that $D^n(\mathfrak{g}) \subseteq ker(\pi) = \mathfrak{i}$, so that $D^{n+k}(\mathfrak{g}) \subseteq D^k(\mathfrak{i})$ for each $k \in \mathbb{N}$. If $\mathfrak{i}$ is solvable then follows that $\mathfrak{g}$ is solvable.

■

4.14. One more example of solvable algebra

Let $D = (D_i)$ be a flag of a vector space V, and let $\mathfrak{b}(D) = \{x \in \mathfrak{gl}(V) : (\forall i) x D_i \subseteq D_i\}$. It is easy to note that $\mathfrak{b}(D)$ forms a Lie algebra and that it corresponds to a set of upper triangular matrices. Now, we want to show that it is solvable.

1. Observe that $\mathfrak{n}(D) \triangleleft \mathfrak{b}(D)$, which follows since for any matrices A upper triangular and B strictly upper triangular, that AB and BA are strictly upper triangular. Therefore $[\mathfrak{n}(D), \mathfrak{b}(D)] \subset \mathfrak{n}(D)$.
2. $\mathbb{F}^n \cong \mathfrak{b}(D)/\mathfrak{n}(D)$ which follows because $\mathfrak{b}(D)/\mathfrak{n}(D)$ corresponds to the set of diagonal $n \times n$ matrices, so $\mathfrak{b}(D)/\mathfrak{n}(D)$ is abelian and therefore solvable. Since $\mathfrak{n}(D)$ is nilpotent and therefore solvable then the Proposition 13.1 (2) implies that $\mathfrak{b}(D)$ is solvable.

4.15. Definition of solvable algebras (part 2)

Proposition 16. *The following conditions are equivalent:*

1. *$\mathfrak{g}$ is solvable of derived length $\leq r$*
2. *There is a descending series of ideals of $\mathfrak{g}$:*

$$\mathfrak{g} = a_0 \supset a_1 \supset \cdots \supset a_r = 0$$

such that $[a_i, a_i] \subset a_{i+1}$ for $0 \leq i \leq r-1$ which is equivalent to saying that a_i/a_{i+1} is abelian.

Proof. • (1) implies (2): If (1) holds, then let $a_i = D^{i+1}(\mathfrak{g})$. It follows that $(a_i)_{i=0}^{r-1}$ is a descending series of ideals of $\mathfrak{g}$ such that $[D^{i+1}(\mathfrak{g}), D^{i+1}(\mathfrak{g})] = [a_i, a_i] \subset a_{i+1} = D^{i+2}(\mathfrak{g})$.

- (2) implies (1). If (2) holds, then $\forall i \implies D^{i+1}(\mathfrak{g}) \subset a_i$ so therefore $D^{r+1}(\mathfrak{g}) = \{0\}$ which implies that $\mathfrak{g}$ is solvable of derived length $\leq r$.

■

4.16. Lie's Theorem (intro)

One of major results associated with solvable algebras is Lie's Theorem. In summary, it says that if $\mathfrak{g}$ is solvable and $\pi : \mathfrak{g} \to \mathfrak{gl}(V)$ is its representation in $\mathfrak{gl}(V)$ then there is a basis for V such that all linear transformations in $\pi(\mathfrak{g})$ are represented by upper triangular matrices. For Lie's Theorem to work the vector space V must be over a field $\mathbb{F}$ which is algebraically closed and of characteristic zero (similar to $\mathbb{C}$). This is important because such field allows the characteristic polynomial of a matrix to have roots. Thus, in the proof we will use $\mathbb{F}$ to have this quality. Theorem can be rephrased in 3 equivalent ways (A, B, C). We will use special Lemma to prove one of them and then will proceed to justify their equivalence.

4.17. Lie's Theorem (Lemma)

Lemma 5. *Let $\mathfrak{g}$ be a Lie algebra, $\mathfrak{h} \triangleleft \mathfrak{g}$, and $\phi : \mathfrak{g} \to \mathfrak{gl}(V)$ a finite-dimensional linear representation of $\mathfrak{g}$. Let v be a nonzero element of V and let λ be a linear form on $\mathfrak{h}$ such that $\lambda(y)v = \phi(y)v$ for all $y \in \mathfrak{h}$. Then $\lambda([\mathfrak{g}, \mathfrak{h}]) = 0$.*

Proof. For the proof we will adopt the notation that for $x \in \mathfrak{g} \implies \phi(x)v = x \cdot v$.

1. Fix $x \in \mathfrak{g}$ and $k \in \mathbb{N}$. Let $y \in \mathfrak{h}$. Then consider the space

$$W^k = \mathbb{F}v + \mathbb{F}x \cdot (v) + \cdots + \mathbb{F}x^k \cdot (v)$$

   Since $y \cdot x^k(v) = x \cdot y(x^{k-1}v) - [x, y](x^{k-1}v)$ and $y \cdot v = \lambda(y)v$ for $y \in \mathfrak{h}$ where $[x, y] \in \mathfrak{h}$, then by induction on k follows that $\mathfrak{h}(W^k) \subseteq W^k$ for each $k \in \mathbb{N}$.

2. Consider the set $(x^k \cdot v)_{k\in\mathbb{N}}$. Choose $m \in \mathbb{N}$, maximal with respect to the property that $\beta = (x^k \cdot v)_{k=0}^m$ is a basis to the vector space $W^m = span\{(x^k \cdot v)_{k\in\mathbb{N}}\}$. Then $D = \{0, W^1, ..., W^m\}$ is a flag in W^m which is invariant under $\mathfrak{h}$

3. Above implies that each $y \in \mathfrak{h}$ corresponds to an upper triangular matrix (y_{ij}) with respect to the basis β. Since for every $y \in \mathfrak{h} \implies y \cdot v = \lambda(y)v$, then by induction on k and from the formula

$$y \cdot x^k(v) = x \cdot y(x^{k-1}v) - [x, y](x^{k-1}v)$$

   it follows that $y \cdot x^k(v) \in \lambda(y)x^k(v) + W^{k-1}$ for every $x^k(v) \in \beta$. Thus the diagonal entries $y_{ii} = \lambda(y)$ for all i.

4. Since x and y leave the space W^m invariant, then $[x, y]|_{W^m} = [x|_{W^m}, y|_{W^m}]$. Therefore $trace([x, y]|_{W^m}) = 0$. Since $[x, y] \in \mathfrak{h}$ then $[x, y]_{ii} = \lambda([x, y])$ by previous observation. Therefore $trace([x, y]|_{W^m}) = m\lambda([x, y])$. Combining all of these results together, it follows that for all $x \in \mathfrak{g}$ and $y \in \mathfrak{h} \implies \lambda([x, y]) = 0$, which by linearity of λ implies that $\lambda([\mathfrak{g}, \mathfrak{h}]) = 0$

■

4.18. Lie Theorem (Version A)

Theorem 11 (Version A)**.** *Let V be a nonzero finite-dimensional vector space. Let $\mathfrak{g}$ be a solvable subalgebra of $\mathfrak{gl}(V)$. Then there exists a non zero common eigenvector v for $\mathfrak{g}$, i.e., $\mathfrak{g}(v) \subseteq \mathbb{F}v$.*

Proof. W.l.o.g. assume that $\mathfrak{g} \neq \{0\}$. Then we will show the result by induction on dim$\mathfrak{g}$.

1. If $\mathfrak{g} = \mathbb{F}x$, then since the characteristic polynomial of x always has a root in $\mathbb{F}$, it follows that any eigenvector of x satisfies the theorem.

2. Let dim$\mathfrak{g} > 1$ and $\mathfrak{h}$ be a maximal proper subalgebra of $\mathfrak{g}$ which contains $D^2(\mathfrak{g}) = [\mathfrak{g}, \mathfrak{g}]$. Since $\mathfrak{g}$ is solvable then $D^2(\mathfrak{g})$ is a proper subspace so such subalgebra must exist. Since $[\mathfrak{g}, \mathfrak{g}] \subseteq \mathfrak{h}$ then follows that $\mathfrak{h}$ is an ideal. Note that $\mathfrak{h}$ is solvable as a subalgebra of $\mathfrak{g}$.

3. Applying the induction hypothesis on $\mathfrak{h}$ and V, $\exists v \neq \{0\}$ s.t. $\forall x \in \mathfrak{h} \implies x \cdot v = \lambda(x)v$ where $\lambda : \mathfrak{g} \to \mathbb{F}$ is linear. (Obvious to see)

4. Define $V_\lambda(\mathfrak{h}) = \{w \in V : (\forall x \in \mathfrak{h})x \cdot w = \lambda(x)w\}$. Note that $V_\lambda(\mathfrak{h})$ is a vector space and suppose that $V_\lambda(\mathfrak{h})$ is $\mathfrak{g}$-invariant. Since $\mathfrak{g}/\mathfrak{h}$ is solvable, then applying the induction hypothesis on $\mathfrak{g}/\mathfrak{h}$ and $V_\lambda(\mathfrak{h})$, it follows that there exists a nonzero eigenvector $v_0 \in V_\lambda(\mathfrak{h})$ for all $y \in \mathfrak{g}/\mathfrak{h}$.

5. By maximality of $\mathfrak{h}$, it follows that $\mathfrak{g}/\mathfrak{h} = \mathbb{F}y + \mathfrak{h}$ for some $y \in \mathfrak{g}$, where $(y + \mathfrak{h}) \cdot v = \lambda(y)(y + \mathfrak{h})$. Therefore $\mathfrak{g} = \mathbb{F}y + \mathfrak{h}$ where v_0 is the common eigenvector for $\mathfrak{g}$, so the proof is complete.

6. It remains to show that $V_\lambda(\mathfrak{h})$ is $\mathfrak{g}$-invariant. Since for $w \in V_\lambda(\mathfrak{h}), x \in \mathfrak{g}$ and $y \in \mathfrak{h}$, it follows that $yx \cdot w = xy \cdot w - [x, y] \cdot w = \lambda(y)x \cdot w - \lambda([x, y])w$ where $[x, y] \in \mathfrak{h}$, then by lemma can conclude that $yx \cdot w = \lambda(y)x \cdot w$ so $x \cdot w \in V_\lambda(\mathfrak{h})$ which implies the result.

■

4.19. Lie Theorem (Version B)

Theorem 12 (Version B)**.** *Let V be a nonzero finite-dimensional vector space. Let $\mathfrak{g}$ be a solvable subalgebra of $\mathfrak{gl}(V)$. Then there exists a $\mathfrak{g}-$ invariant flag in V.*

Proof. 1. Assume that V is nonzero, then by Theorem of Version A, there is a non-zero common $\mathfrak{g}$-eigenvector $v_1 \in V$. Let $V_1 = \mathbb{F}v_1$, then:

$$\alpha : \mathfrak{g} \to \mathfrak{gl}(V/V_1), /\alpha(x)(v + V_1) = x \cdot v + V_1$$

is a well defined representation of $\mathfrak{g}$ on the quotient space V/V_1. Also, solvability of $\mathfrak{g}$ is induced into algebra $\alpha(\mathfrak{g})$ (as image of solvable algebra).

2. Proceeding by induction on dim$V$, we may assume that there exists an $\alpha(\mathfrak{g})$-invariant flag in V/V_1. It follows that the preimage of V/V_1 in V, together with $\{0\}$ is a complete $\mathfrak{g}$-invariant flag in V.

■

Remark 19. *If apply Lie's Theorem from Version B to $V = \mathfrak{g}$ and $ad(\mathfrak{g})$, where $\mathfrak{g}$ is solvable, then we get a flag of ideals*

$$\{0\} = \mathfrak{g}_0 < \mathfrak{g}_1 < ... < \mathfrak{g}_n = \mathfrak{g}$$

of $\mathfrak{g}$ with dim$\mathfrak{g}_k$=k. This chain is commonly known as Holder series.

4.20. Lie's Theorem (Version C)

Definition 17. *The irreducible representation (π, V) is a non-zero representation that has no proper non-trivial subrepresentation $(\pi|_W, W)$, with $W \subset V$ closed under the action of $\{\pi(a) : a \in \mathfrak{g}\}$.*

Theorem 13 (Version C)**.** *Let V be a nonzero finite-dimensional vector space. Let $\mathfrak{g}$ be a solvable subalgebra of $\mathfrak{gl}(V)$. Then the only finite-dimensional linear irreducible representations of $\mathfrak{g}$ in $\mathfrak{gl}(V)$ are one-dimensional.*

Proof. Consider any subspace $V' \subseteq V$ and let (π, V') be a nonzero representation. Since $\mathfrak{g}$ is solvable, then by Lie's Theorem Version B it follows that there is a flag D of V' s.t. $\pi(\mathfrak{g}) \subset \mathfrak{b}(D)$. Since flag D contains a subspace of dimension one which is $\pi(\mathfrak{g})$ invariant then π is irreducible only if V' is one dimensional. ■

For any representation $\pi : \mathfrak{g} \to \mathfrak{gl}(V)$, Lie's Theorem Version C guarantees existence of one-dimensional subspace which will be invariant under action of $\mathfrak{g}$, which implies the result stated in the Version A. Thus Theorems A, B, and C are all equivalent formulations of the same result known as Lie's Theorem.

4.21. Cartan's Criterion

This criterion characterizes solvable Lie algrebras by properties of its elements.

Theorem 14. *Let V be a finite-dimensional vector space and $\mathfrak{g}$ a Lie subalgebra of $\mathfrak{gl}(V)$. Then the following are equivalent:*

1. $\mathfrak{g}$ is solvable

2. $trace(xy) = 0$ for all $x \in \mathfrak{g}$ and $y \in [\mathfrak{g}, \mathfrak{g}]$

Proof. We will show only ($\implies$) direction, which is a consequence of Lie's Theorem. Complexifying all the vector spaces involved, and since $trace(x_{\mathbb{C}}) = trace(x)$ for $x \in End(V)$, then w.l.o.g. we may assume that $\mathbb{F} = \mathbb{C}$. Then by Lie's Theorem there is a basis for V with respect to which all $x \in \mathfrak{g}$ are upper triangular matrices. In particular, all elements of $[\mathfrak{g}, \mathfrak{g}]$ are given by strictly upper triangular matrices. After multiplying an upper triangular with a strictly upper triangular matrix, we find a strictly upper triangular matrix which has zero trace. ■

Chapter 5

Semisimple Lie Algebras (General Theorems)

5.1. Proposition 1

Proposition 17. *If* $\mathfrak{i}, \mathfrak{j}$ *are solvable ideals in Lie algebra* $\mathfrak{g}$*, then* $\mathfrak{i} + \mathfrak{j}$ *is a solvable ideal in* $\mathfrak{g}$*.*

Proof.

1. Given $\mathfrak{i}, \mathfrak{j}$ solvable ideals, it follows that $\mathfrak{i} + \mathfrak{j}$ is a subalgebra in $\mathfrak{g}$ such that $\mathfrak{i} + \mathfrak{j}/\mathfrak{j} \cong \mathfrak{i}/\mathfrak{i} \cap \mathfrak{j}$.
2. Since $\mathfrak{i} \cap \mathfrak{j}$ is a solvable ideal of $\mathfrak{i}$, and since solvability is an extension property, then the homomorphic image $\mathfrak{i}/\mathfrak{i} \cap \mathfrak{j}$ of quotient map $q : \mathfrak{i} \to \mathfrak{i}/\mathfrak{i} \cap \mathfrak{j}$ is solvable.
3. Since $\mathfrak{j}$ is solvable and $\mathfrak{i} + \mathfrak{j}/\mathfrak{j}$ is solvable then by extension property, this implies that $\mathfrak{i} + \mathfrak{j}$.

■

Remark 20. *The above result shows that a finite-dimensional Lie algebra* $\mathfrak{g}$ *has a maximal solvable ideal.*

Definition 18. *The maximal solvable ideal of* $\mathfrak{g}$ *is called the radical of* $\mathfrak{g}$*, and it is denoted by* $\mathfrak{rad}(\mathfrak{g})$*.*

Definition 19. *Let* $\mathfrak{g}$ *be a finite-dimensional Lie algebra. Then* $\mathfrak{g}$ *is called semisimple if* $\mathfrak{rad}(\mathfrak{g}) = \{0\}$*. The Lie algebra is called simple, if it is not abelian and if it contains no ideals other than* $\mathfrak{g}$ *and* $\{0\}$*.*

Remark 21. *From the definition of simple Lie algebra, it is easy to see that the ideal* $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$*, so* $\mathfrak{rad}(\mathfrak{g}) = \{0\}$ *which shows that simple implies semisimple. In the course MAT305 we saw that* $\mathfrak{so}(3), (\mathbb{R}^3, \times)$ *are examples of simple Lie algebras. More examples were discussed in our course textbook, such as,* $\mathfrak{su}(n), \mathfrak{sl}(n, \mathbb{C})$ *for each* n*,* $\mathfrak{so}(n)$ *for* $n > 4$*. Therefore, all of them can be examples of semisimple Lie algebras.*

5.2. Theorem 1

Theorem 15. *Let* $\mathfrak{g}$ *be a finite-dimensional Lie algebra and* $\mathfrak{rad}(\mathfrak{g})$ *its radical. Then the following holds:*

1. $\mathfrak{g}/\mathfrak{rad}(\mathfrak{g})$ *is semisimple*
2. $\mathfrak{g}$ *contains a semisimple subalgebra* $\mathfrak{s} \leq \mathfrak{g}$ *such that* $\mathfrak{rad}(\mathfrak{g}) \ltimes \mathfrak{s}$

The second result is called Levi decomposition, which we will not prove but mention here to illustrate how important is the concept of semisimplicity.

Proof. Will prove only the result from (1). Let $q : \mathfrak{g} \to \mathfrak{g}/\mathfrak{rad}(\mathfrak{g})$ be the quotient homomorphism and $\mathfrak{a} \trianglelefteq \mathfrak{g}/\mathfrak{rad}(\mathfrak{g})$ be a solvable ideal. Then $\mathfrak{b} = q^{-1}(\mathfrak{a}) \trianglelefteq \mathfrak{g}$ is an ideal and it contains $\mathfrak{rad}(\mathfrak{g})$ such that $\mathfrak{a} \cong \mathfrak{b}/\mathfrak{rad}(\mathfrak{g})$ is solvable. Since solvability is an extension property, then $\mathfrak{b}$ is solvable so that by definition $\implies \mathfrak{b} \subseteq \mathfrak{rad}(\mathfrak{g})$. Thus follows that $\mathfrak{a} = \{0\}$, so $\mathfrak{rad}(\mathfrak{g}/\mathfrak{rad}(\mathfrak{g})) = \{0\}$, i.e. $\mathfrak{g}/\mathfrak{rad}(\mathfrak{g})$ is semisimple. ■

Let $\mathfrak{g}$ be a Lie algebra. A bilinear form $\beta : \mathfrak{g} \times \mathfrak{g} \to \mathbb{F}$ on $\mathfrak{g}$ is said to be invariant if we have

$$\beta([x,y],z) + \beta(y,[x,z]) = 0 \quad \forall x,y,z \in \mathfrak{g}$$

Definition 20. *We define Cartan-Killing form to be:*

$$\kappa_{\mathfrak{g}} : \mathfrak{g} \times \mathfrak{g} \to \mathbb{F}, \quad \kappa_{\mathfrak{g}}(x,y) = tr(ad(x)ad(y))$$

Remark 22. *Previously, we studied the Cartan solvability criterion, which stated that $\mathfrak{g}$ is solvable if and only if for all $x \in [\mathfrak{g},\mathfrak{g}]$ and all $y \in \mathfrak{g} \implies \kappa_{\mathfrak{g}}(x,y) = 0$.*

Proposition 18. *Cartan-Killing form satisfies invariance condition.*

Proof. Computing both parts:

$$\kappa_{\mathfrak{g}}([x,y],z) = tr(ad([x,y]),ad(z)) = tr([ad(x),ad(y)],ad(z)) = tr(ad(x)ad(y)ad(z)) - tr(ad(y)ad(x)ad(z))$$

$$\kappa_{\mathfrak{g}}(y,[x,z]) = tr(ad(y),ad([x,z])) = tr(ad(y),[ad(x),ad(z)]) = tr(ad(y)ad(x)ad(z)) - tr(ad(y)ad(z)ad(x))$$

We find that

$$\kappa_{\mathfrak{g}}([x,y],z) + \kappa_{\mathfrak{g}}(y,[x,z]) = tr([ad(x),ad(y)ad(z)]) = 0$$

where the trace of the commutator is always zero. ■

Remark 23. *In similar fashion we can show that for all $x,y,z \in \mathfrak{g} \implies \kappa_{\mathfrak{g}}([x,y],z) = \kappa_{\mathfrak{g}}(x,[y,z])$.*

5.3. Lemma 1

Lemma 6. *For any ideal $\mathfrak{i} \trianglelefteq \mathfrak{g}$, it follows that $\kappa_{\mathfrak{i}} = \kappa_{\mathfrak{g}}|_{\mathfrak{i}\times\mathfrak{i}}$*

Proof. If the image of $A \in End(\mathfrak{g})$ is contained in $\mathfrak{i}$, then we pick a basis for $\mathfrak{g}$ which starts with basis for $\mathfrak{i}$. With respect to this basis, we can write A as a block matrix:

$$A = \begin{pmatrix} A|_{\mathfrak{i}} & * \\ 0 & 0 \end{pmatrix}$$

and such that $tr(A) = tr(A|_{\mathfrak{i}})$. For $x,y \in \mathfrak{i}$ let $A = ad(x)ad(y)$. Since $\mathfrak{i} \trianglelefteq \mathfrak{g}$ then for any $z \in \mathfrak{g} \implies A \cdot z \in \mathfrak{i}$. Thus follows that the image of A is contained in $\mathfrak{i}$, and by above follows that $tr(ad(x)ad(y)) = tr(ad(x)|_{\mathfrak{i}}ad(y)|_{\mathfrak{i}}) = \kappa_{\mathfrak{i}}(x,y)$ ■

Definition 21. *Let V be a vector space and $\beta : V \times V \to \mathbb{F}$ be a symmetric bilinear form. Then the orthogonal set of a subspace W with respect to $\beta$is*

$$W^{\perp,\beta} = \{v \in V | (\forall w \in W)\beta(v,w) = 0\}$$

We define the set $\mathfrak{rad}(\beta) = V^{\perp,\beta}$ and β is called degenerate if $\mathfrak{rad}(\beta) \neq \{0\}$.

Remark 24. *Cartan solvability criterion for Lie algebra $\mathfrak{g}$ can be formulated as:*

1. *$\mathfrak{g}$ is solvable*
2. *$\forall y \in \mathfrak{g}, x \in [\mathfrak{g},\mathfrak{g}] \implies \kappa_{\mathfrak{g}}(x,y) = 0$*
3. *$[\mathfrak{g},\mathfrak{g}] \subseteq \mathfrak{rad}(\kappa_{\mathfrak{g}})$*

5.4. Lemma 2

Lemma 7. *For any ideal* $\mathfrak{j}$ *of a Lie algebra* $\mathfrak{g}$*, the following assertions hold:*

1. *Its orthogonal space* $\mathfrak{j}^\perp$ *with respect to* $\kappa_\mathfrak{g}$ *is also an ideal.*
2. $\mathfrak{j} \cap \mathfrak{j}^\perp$ *is a solvable ideal.*
3. *If j is semisimple, then g decomposes as a direct sum* $\mathfrak{g} = \mathfrak{j} \oplus \mathfrak{j}^\perp$ *of Lie algebras.*

Proof.

1. Sufficient to show that $\forall x \in \mathfrak{j}^\perp$ and $\forall z \in \mathfrak{g}$ it follows that $[x, z] \in \mathfrak{j}^\perp$. Let $y \in \mathfrak{j}$ then by symmetry property of $\kappa_\mathfrak{g}$ it follows that $\kappa_\mathfrak{g}([x, y], z) = \kappa_\mathfrak{g}(x, [y, z]) = 0$. Since $[y, z] \in \mathfrak{j}$ then the result follows.
2. By definition of Cartan-Killing form, for $\mathfrak{i} = \mathfrak{j} \cup \mathfrak{j}^\perp$, then $\kappa_\mathfrak{g}$ vanishes on $\mathfrak{i} \times \mathfrak{i}$. This implies that $\mathfrak{rad}(\kappa_\mathfrak{i}) = \mathfrak{i}$ where by previous lemma, $\mathfrak{rad}(\kappa_\mathfrak{i})$ is well defined. By above remark, it follows that $\mathfrak{i}$ is solvable.
3. If $\mathfrak{j}$ is semisimple, then (2) implies that $\mathfrak{j} \cap \mathfrak{j}^\perp = \{0\} \subseteq \mathfrak{rad}(\mathfrak{j})$. Since $\mathfrak{j}^\perp$ is the kernel of the linear map $\mathfrak{g} \to \mathfrak{j}^*$, $x \to \kappa_\mathfrak{g}(x, \cdot)$, then $dim(ker) + dim(range) \geq dim(\mathfrak{g})$, so therefore $\mathfrak{j} + \mathfrak{j}^\perp = \mathfrak{g}$. Since $\mathfrak{j}$ and $\mathfrak{j}^\perp$ are both ideals by (1), then $[\mathfrak{j}, \mathfrak{j}^\perp] \subseteq \mathfrak{j} \cap \mathfrak{j}^\perp = \{0\}$, which shows that $\mathfrak{j} \oplus \mathfrak{j}^\perp = \mathfrak{g}$.

∎

5.5. Cartan's Semisimplicity Criterion

Theorem 16. *A Lie algebra* $\mathfrak{g}$ *is semisimple if and only if* $\kappa_\mathfrak{g}$ *is nondegenerate, i.e.,* $\mathfrak{rad}(\kappa_\mathfrak{g})$ *= {0}.*

Proof.

- ($\Longrightarrow$) Suppose $\mathfrak{g}$ is semisimple. From the definition of $\mathfrak{rad}(\kappa_\mathfrak{g})$, it follows that $\mathfrak{g} \cap \mathfrak{g}^\perp = \mathfrak{rad}(\kappa_\mathfrak{g})$ and the previous lemma implies that $\mathfrak{rad}(\kappa_\mathfrak{g})$ is solvable, meaning that $\mathfrak{rad}(\kappa_\mathfrak{g}) \subseteq \mathfrak{rad}(\mathfrak{g}) = \{0\}$ (by semisimplicity).
- ($\Longleftarrow$) We will show that if $\mathfrak{g}$ is not semisimple then $\mathfrak{rad}(\kappa_\mathfrak{g}) \neq \{0\}$, which is equivalent to the converse direction. Since $\mathfrak{rad}(\mathfrak{g}) \neq \{0\}$ then let $m \in \mathbb{N}$ be maximal exponent such that $\mathfrak{h} = D^m(\mathfrak{rad}(\mathfrak{g})) \neq \{0\} \implies [\mathfrak{h}, \mathfrak{h}] = \{0\}$. Let $x \in \mathfrak{h}$ and $y \in \mathfrak{g}$, then since $\mathfrak{h}$ is an ideal it follows that $ad(x)ad(y)\mathfrak{g} \subseteq \mathfrak{h}$. Since $\mathfrak{h}$ is abelian then $(ad(x)ad(y))^2 = 0$. This implies that $\kappa_\mathfrak{g}(x, y) = tr(ad(x)ad(y)) = 0$. Above implies that $\exists x \in \mathfrak{h}$ such that $\forall y \in \mathfrak{g} \implies \kappa_\mathfrak{g}(x, y) = 0$, so $x \in \mathfrak{rad}(\mathfrak{g})$, and $\kappa_\mathfrak{g}$ is degenerate.

∎

5.6. Theorem 2

Theorem 17. *Let* $\mathfrak{g}$ *be a semisimple Lie algebra. Then there are simple ideals* $\mathfrak{g}_1, ..., \mathfrak{g}_k$ *of* $\mathfrak{g}$ *with*

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k.$$

Every ideal $\mathfrak{i} \trianglelefteq \mathfrak{g}$ *is semisimple and a direct sum* $\bigoplus_{j \in I} \mathfrak{g}_j$ *for some subset* $I \subseteq \{1, ..., k\}$*. Conversely, each direct sum of simple Lie algebras is semisimple.*

Proof.

1. The first part of the statement can be proven with induction on dimension of $\mathfrak{g}$.
   - Let $\mathfrak{i} \trianglelefteq \mathfrak{g}$. Since $\mathfrak{g}$ is semisimple, then $\mathfrak{i} \cap \mathfrak{i}^\perp = \{0\}$. By definition of $\kappa_\mathfrak{i} = \kappa_\mathfrak{g}|_{\mathfrak{i} \times \mathfrak{i}}$, this implies that $\kappa_\mathfrak{i}$ is non degenerate, therefore, $\mathfrak{i}$ is semisimple.
   - Since $\kappa_\mathfrak{g}$ is symmetric, then the above implies that $\mathfrak{i}^\perp$ is semisimple and that $\mathfrak{i} \oplus \mathfrak{i}^\perp = \mathfrak{g}$. Induction on dimension of $\mathfrak{g}$ completes the argument.
2. For the second part of the statement, let $\mathfrak{i} \neq \{0\}$ be an ideal of $\mathfrak{g}$ and let $\pi_j : \mathfrak{g} \to \mathfrak{g}_j$ be projections. For some j it follows that $\pi_j(\mathfrak{i}) \neq \{0\}$.

- Since π_j is surjective then $\pi_j(\mathfrak{i})$ is an ideal in $\mathfrak{g}_j$. Since $\mathfrak{g}_j$ is simple then $\pi_j(\mathfrak{i}) = \mathfrak{g}_j$. Therefore

$$\mathfrak{g}_j = [\mathfrak{g}_j, \mathfrak{g}_j] = [\mathfrak{g}_j, \pi_j(\mathfrak{i})] = \pi_j([\mathfrak{g}_j, \mathfrak{i}]) = [\mathfrak{g}_j, \mathfrak{i}] \subseteq \mathfrak{i}$$

Above shows that if $\mathfrak{g}_j$ is such that $\pi_j(\mathfrak{i}) \neq \{0\}$, then $\mathfrak{g}_j \subseteq \mathfrak{i}$. Thus, $\mathfrak{i}$ is a direct sum of such $\mathfrak{g}_j$. By direct computation it follows that for any such ideal, it holds that $\mathfrak{i} = [\mathfrak{i}, \mathfrak{i}]$. Thus no nonzero ideal of a direct sum of simple Lie algebras is solvable, hence any such direct sum is semisimple.

■

5.7. Corollary

Corollary 5. *For a semisimple Lie algebra $\mathfrak{g}$, the following assertions hold:*

1. $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$.
2. *Each ideal $\mathfrak{n} \trianglelefteq \mathfrak{g}$ is semisimple and there exists another ideal $\mathfrak{c}$ with $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{c}$.*
3. *All homomorphic images of $\mathfrak{g}$ are semisimple.*

Proof.

1. Let $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k$ be the decomposition of $\mathfrak{g}$. Since $\mathfrak{g}_j$ is simple then $\mathfrak{g}_j = [\mathfrak{g}_j, \mathfrak{g}_j]$. Thus,

$$[\mathfrak{g}, \mathfrak{g}] = \sum_{j=1}^{k} [\mathfrak{g}_j, \mathfrak{g}_j] = \sum_{j=1}^{k} \mathfrak{g}_j = \mathfrak{g}.$$

2. Follows directly from the above theorem. From the last lemma, $\mathfrak{c} = \mathfrak{n}^{\perp}$
3. For any homomorphism $\alpha : \mathfrak{g} \to \mathfrak{h}$ it follows that $\alpha(\mathfrak{g}) \cong \mathfrak{g}/ker(\alpha)$. Since $ker(\alpha) \trianglelefteq \mathfrak{g}$, then by the 2nd bullet above, it follows that $ker(\alpha)$ and $ker(\alpha)^{\perp}$ are semisimple ideals in $\mathfrak{g}$ such that $\mathfrak{g} = ker(\alpha) \oplus ker(\alpha)^{\perp}$ (by the last lemma). Since $ker(\alpha)^{\perp} \cong \mathfrak{g}/ker(\alpha) \cong \alpha(\mathfrak{g})$, then $\alpha(\mathfrak{g})$ is semisimple.

■

5.8. Theorem 3

Definition 22. *Let A denote algebra. We define a derivation of A to be a linear mapping $D : A \to A$ which satisfies the identity:*

$$D(x \cdot y) = (Dx) \cdot y + x \cdot (Dy)$$

A derivation D of $\mathfrak{g}$ is called inner if $D = ad(x)$ for some $x \in \mathfrak{g}$.

Theorem 18. *For a semisimple Lie algebra $\mathfrak{g}$ all derivations are inner, i.e.,*

$$ad(\mathfrak{g}) = der(\mathfrak{g}).$$

Proof. 1. First, we will show that $ad(\mathfrak{g}) \trianglelefteq der(\mathfrak{g})$. Since for any $x, y \in ad(\mathfrak{g})$ it holds that $ad([x, y]) = [ad(x), ad(y)]$, then it suffices to show that for any $\delta \in der(\mathfrak{g})$ and $x \in \mathfrak{g}$, it follows that $[\delta, ad(x)] \in ad(\mathfrak{g})$. Given $y \in \mathfrak{g}$, we compute

$$[\delta, ad(x)]y = (\delta \cdot ad(x))y - ad(x)\delta(y) = \delta([x, y]) - [x, \delta(y)] = [\delta(x), y] = ad(\delta(x))y$$

Thus follows that $[\delta, ad(x)] = ad(\delta(x)) \in ad(\mathfrak{g})$ showing the result.

2. Since $\mathfrak{g}$ is semisimple then $\mathfrak{z}(\mathfrak{g}) = \{0\}$, so that $ad(\mathfrak{g}) \cong \mathfrak{g}$. Therefore, $der(\mathfrak{g})$ decomposes as a direct sum $ad(\mathfrak{g}) \oplus \mathfrak{j}$ for the orthogonal complement $\mathfrak{j}$ of $ad(\mathfrak{g})$ with respect to the Cartan–Killing form of $der(\mathfrak{g})$. This implies that for any $x, y \in \mathfrak{g}$ and $\delta \in \mathfrak{j}$ it follows that
$$0 = [\delta, ad(x)]y = ad(\delta(x))y.$$
This implies that $\delta(x) \in \mathfrak{z}(\mathfrak{g}) = \{0\}$, therefore $\mathfrak{j} = \{0\}$, so $ad(\mathfrak{g}) = der(\mathfrak{g})$.

■

Remark 25. *From above can conclude that the mapping $ad : \mathfrak{g} \to der(\mathfrak{g})$ is an isomorphism.*

5.9. ad-semisimple

Definition 23. *Let $\mathfrak{g}$ be a semisimple Lie algebra, and let $x \in \mathfrak{g}$.*

- *x is said to be nilpotent if the endomorphism $ad(x)$ of $\mathfrak{g}$ is nilpotent.*
- *x is said to be semisimple if $ad(x)$ is semisimple i.e. diagonalizable upon extedning the ground field.*

Theorem 19. *If $\mathfrak{g}$ is semisimple, every element x of $\mathfrak{g}$ can be written uniquely in the form $x = s + n$, with n nilpotent, s semisimple, and $[s, n] = 0$*

Proof.

1. By standard Jordan decomposition, for any $x \in \mathfrak{g}$ it follows that $ad(x) = (ad(x))_s + (ad(x))_n$ where $\exists$ unique elements $(ad(x))_s$ diagonal, and $(ad(x))_n$ nilpotent.
2. Since $\mathfrak{g}$ is semisimple then $ad : \mathfrak{g} \to der(\mathfrak{g})$ is an isomorphism, therefore $\exists! x_s, x_n \in \mathfrak{g}$ such that $x = x_s + x_n$, $ad(x_s) = (ad(x))_s$, $ad(x_n) = (ad(x))_n$ and $[x_s, x_n] = 0$.

■

Theorem 20. *Let $\pi : \mathfrak{g} \to End(V)$ be a linear representation of a semisimple Lie algebra. If x is nilpotent (reps. semisimple), then so is the endomorphism $\pi(x)$.*

Proof.

- By previous results it follows that $\pi(\mathfrak{g})$ is semisimple.
- Since $\mathfrak{g}$ is spanned by the eigenvectors of $ad_{\mathfrak{g}}(s)$ then $\pi(\mathfrak{g})$ is spanned by the eigenvectors of $ad_{\pi(\mathfrak{g})}(\pi(s))$, so $\pi(s)$ is ad-semisimple.
- Since $ad_{\mathfrak{g}}(n) = \rho_n - \lambda_n$ where $\rho_n y = ny$ and $\lambda_n y = yn$ then $\pi(ad_{\mathfrak{g}}(n)) = \rho_{\pi(n)} - \lambda_{\pi(n)} = ad_{\pi(\mathfrak{g})}(\pi(n))$, so that if $ad_{\mathfrak{g}}(n)$ is nilpotent, then $ad_{\pi(\mathfrak{g})}(\pi(n))$ is nilpotent.

■

5.10. Weyl Theorem

Definition 24. *A linear representation $\pi : \mathfrak{g} \to End(V)$ is called irreducible if $V \neq 0$ and V does not have any invariant subspaces, other than $\{0\}$ and V. One says that π is completely reducible if it is a direct sum of irreducible representations.*

Theorem 21 (Weyl)**.** *Every finite-dimensional linear representation of a semisimple Lie algebra is completely reducible.*

5.11. Cartan-Killing Classification

Every complex finite-dimensional simple Lie algebra is isomorphic to exactly one of the following list:

- Classical Lie Algebras: $\{A_n : \mathfrak{sl}(n+1, \mathbb{C})\}, n \geq 1$; $\{B_n : \mathfrak{so}(2n+1, \mathbb{C})\}, n \geq 2$; $\{C_n : \mathfrak{sp}(2n, \mathbb{C})\}, n \geq 3$; $\{D_n : \mathfrak{so}(2n, \mathbb{C})\}, n \geq 4$.
- Exceptional Lie Algebras: $\{E_6, E_7, E_8, F_4, G_2\}$

5.12. The passage from Real to Complex

Definition 25. *Let $\mathfrak{g}$ be a real Lie Algebra. The complexification of $\mathfrak{g}$ is a complex Lie algebra $\mathfrak{g}_{\mathbb{C}}$ with respect to the complex bilinear Lie bracket, defined by:*

$$[x + iy, x' + iy'] = ([x, x'] - [y, y']) + i([x, y'] + [y, x'])$$

where $x, x', y, y' \in \mathfrak{g}$ and satisfying

$$[z \otimes v, z' \otimes v'] = zz' \otimes [v, v']$$

for $z, z' \in \mathbb{C}$ and $v, v' \in \mathfrak{g}$.

Remark 26. *Via elementary calculations it follows that $[\mathfrak{g}\mathbb{C}, \mathfrak{g}_{\mathbb{C}}] \cong [\mathfrak{g}, \mathfrak{g}]_{\mathbb{C}}$. As a consequence of this, the following theorem holds:*

Theorem 22. $\mathfrak{g}$ *is abelian (resp. nilpotent, solvable, semisimple) if and only if* $\mathfrak{g}_{\mathbb{C}}$ *is.*

Chapter 6

Cartan Subalgebras

For this chapter the ground field is $\mathbb{C}$.

6.1. Introduction

Definition 26. *Let (π, V) be a representation of the Lie algebra $\mathfrak{h}$. For a function $\lambda : \mathfrak{h} \to \mathbb{K}$, we define the corresponding weight space and the corresponding generalized weight space by*

$$V_{\lambda(\mathfrak{h})} = \cap_{x\in\mathfrak{h}} V_{\lambda(x)}(\pi(x)) \quad \textit{and} \quad V^{\lambda(\mathfrak{h})} = \cap_{x\in\mathfrak{h}} V^{\lambda(x)}(\pi(x))$$

where

$$V_{\lambda(x)}(\pi(x)) = ker(\pi(x) - \lambda(x)\cdot id_V) \quad \textit{and} \quad V^{\lambda(x)}(\pi(x)) = \cup_{n\in\mathbb{N}} ker(\pi(x) - \lambda(x)\cdot id_V)^n.$$

Any function $\lambda : \mathfrak{h} \to \mathbb{K}$ for which $V^\lambda \neq \{0\}$ is called a weight of the representation (π, V).

Lemma 8. *Let (π, V) be a finite-dimensional representation of the nilpotent Lie algebra $\mathfrak{h}$ such that every $\pi(x)$, $x \in \mathfrak{h}$, is split. Then each weight is linear and V decomposes as*

$$V = \bigoplus_{\lambda\in\mathfrak{h}^*} V^\lambda(\mathfrak{h})$$

Moreover, each generalized weight space $V^\lambda(\mathfrak{h})$ is $\mathfrak{h}$-invariant.

Definition 27. *If $\mathfrak{h}$ is a nilpotent subalgebra of the Lie algebra $\mathfrak{g}$, then the weights of the representation $\pi = ad|_\mathfrak{h}$ which are different from zero are called roots of $\mathfrak{g}$ with respect to $\mathfrak{h}$. The set of all roots is denoted $\Delta(g, h)$. The (generalized) weight spaces $g^\lambda(\mathfrak{h})$ are called root spaces. Sometimes we write $\mathfrak{g}^\lambda$ instead of $g^\lambda(\mathfrak{h})$. If $0 \neq \mu \in \mathfrak{h}^*$ is not a root, we put $\mathfrak{g}^\mu = \{0\}$.*

Proposition 19. *Let $\mathfrak{g}$ be a finite-dimensional Lie algebra and $\mathfrak{h}$ a nilpotent subalgebra of $\mathfrak{g}$. Let $x \in \mathfrak{g}$. Then*

1. $\mathfrak{g} = \bigoplus_\lambda \mathfrak{g}^\lambda(ad\ x)$
2. $[\mathfrak{g}^\lambda(\mathfrak{h}), \mathfrak{g}^\mu(\mathfrak{h})] \subseteq \mathfrak{g}^{\mu+\lambda}(\mathfrak{h})$ *for all* $\lambda, \mu \in \mathfrak{h}^*$.
3. $\mathfrak{g}^0(\mathfrak{h})$ *is a subalgebra of* $\mathfrak{g}$.

Proof. 1. Want to prove that $\mathfrak{g} = \bigoplus_\lambda \mathfrak{g}^\lambda(ad\ x)$: Since $\mathfrak{g}$ is a complex vector space then $ad(x)$ has an upper-triangular matrix representation i.e. we can assume that $ad(x) = D + N$ where D is diagonal and N is strictly upper-triangular matrix. This implies that

$$\mathfrak{g}^\lambda(ad\ x) = \cup_{n\in\mathbb{N}} ker(ad(x) - \lambda\cdot id)^n = ker(D - \lambda\cdot id)$$

Therefore if $\{\lambda_i\}$ is a complete set of eigenvalues for D then $\mathfrak{g} = \bigoplus_{i=1} ker(D - \lambda_i) = \bigoplus \mathfrak{g}^{\lambda_i}(ad(\ x))$

2. Want to prove that $[\mathfrak{g}^\lambda(\mathfrak{h}), \mathfrak{g}^\mu(\mathfrak{h})] \subseteq \mathfrak{g}^{\mu+\lambda}(\mathfrak{h})$ for all $\lambda, \mu \in \mathfrak{h}^*$. For $x \in \mathfrak{g}^\lambda, y \in \mathfrak{g}^\mu$ and $h \in \mathfrak{h}$, we have

$$(ad(h) - \lambda(h) - \mu(h))^n[x, y] = \sum_{k=0}^{n} \binom{n}{k} [(ad(h) - \lambda(h))^k x, (ad(h) - \mu(h))^{n-k} y]$$

which follows by induction. If n is sufficiently large, then for every summand either the left factor or the right factor in the bracket vanishes, so that the whole sum vanishes. This proves that $[x, y] \in \mathfrak{g}^{\lambda+\mu}$.

3. Want to prove that $\mathfrak{g}^0(\mathfrak{h})$ is a subalgebra of $\mathfrak{g}$: This follows from (2) for the case $\lambda = \mu = 0$.

■

Proposition 20. *For a subalgebra $\mathfrak{h}$ of a finite-dimensional Lie algebra $\mathfrak{g}$, the following are equivalent:*

1. $\mathfrak{h}$ is nilpotent and self-normalizing, i.e., $\mathfrak{h} = \mathfrak{n}_{\mathfrak{g}}(\mathfrak{h})$.

2. $\mathfrak{h} = \mathfrak{g}^0(\mathfrak{h})$

If these conditions are satisfied, then $\mathfrak{h}$ is a maximal nilpotent subalgebra of $\mathfrak{g}$.

Proof. 1. Want to show that (1) $\implies$ (2). Since $\mathfrak{h}$ is nilpotent, then $\mathfrak{h} \subseteq \mathfrak{g}^0(\mathfrak{h})$. Assume that $\mathfrak{g}^0(\mathfrak{h}) - \mathfrak{h} \neq \emptyset$. Consider the representation

$$\pi : \mathfrak{h} \to \mathfrak{gl}(\mathfrak{g}^0(\mathfrak{h})/\mathfrak{h}), \quad \pi(h)(x + \mathfrak{h}) = [h, x] + \mathfrak{h}$$

whose image consists of nilpotent endomorphisms. By one of Engel's theorems, there exists some $x \in \mathfrak{g}^0(\mathfrak{h})/\mathfrak{h}$ with $ad(\mathfrak{h})x \in \mathfrak{h}$ for all $h \in \mathfrak{h}$. This implies that $\exists x \in \mathfrak{n}_{\mathfrak{g}}(\mathfrak{h}) - \mathfrak{h}$, but this contradicts $\mathfrak{h} = \mathfrak{n}_{\mathfrak{g}}(\mathfrak{h})$.

2. Want to show that (2) $\implies$ (1). Since $\mathfrak{h} \subseteq \mathfrak{g}^0(\mathfrak{h})$ then from Engel's theorem it follows that $\mathfrak{h}$ is nilpotent. To see that $\mathfrak{h}$ is self-normalizing, let $x \in \mathfrak{n}_{\mathfrak{g}}(\mathfrak{h})$. Then $ad(h)x \in \mathfrak{h}$ for all $h \in \mathfrak{h}$, and therefore $ad(h)^n x = 0$ for sufficiently large $n \in \mathbb{N}$. Hence $x \in \mathfrak{n}_{\mathfrak{g}}(\mathfrak{h}) = \mathfrak{h}$.

3. Let $\mathfrak{n}$ be a nilpotent subalgebra such that $\mathfrak{h} \subseteq \mathfrak{n}$, then if (1) and (2) assertions hold, it follows that $\mathfrak{n} \subseteq \mathfrak{g}^0(\mathfrak{h}) = \mathfrak{h}$. This implies that $\mathfrak{h}$ is a maximal nilpotent subalgebra of $\mathfrak{g}$.

■

Definition 28. *Let $\mathfrak{g}$ be a finite-dimensional Lie algebra. A nilpotent sub- algebra $\mathfrak{h}$ is called a Cartan subalgebra of $\mathfrak{g}$ if it is self-normalizing, i.e., $\mathfrak{n}_{\mathfrak{g}}(\mathfrak{h}) = \mathfrak{h}$.*

Examples

- If $\mathfrak{g}$ is nilpotent then $\mathfrak{g}$ is its own Cartan subalgebra.
- Let $\mathbb{K} = \mathbb{R}$ or $= \mathbb{C}$. Since for any diagonal matrices $D_1, D_2 \in \mathfrak{gl}(\mathbb{K})$ and any non-diagonal matrix $x \in \mathfrak{gl}(\mathbb{K})$ it follows that
  1. $[D_1, D_2] = 0$
  2. $[D_1, X]$ is not diagonal in $\mathfrak{gl}(\mathbb{K})$, then follows that $D = \{d \in \mathfrak{gl}(\mathbb{K}) :\ d\ is\ diagonal\ matrix\}$ is Cartan subalgebra of $\mathfrak{gl}(\mathbb{K})$.
- Since $\mathfrak{su}(2)$ is generated by quaternions i, j, k then the only Cartan subalgebras of $\mathfrak{su}(2)$ are subspaces $\langle i \rangle, \langle j \rangle, \langle k \rangle$ (Due to the fact that $[i, j] = 2k$ etc.)

Definition 29. *Let $\mathfrak{g}$ be a finite-dimensional Lie algebra and $\mathfrak{h} < \mathfrak{g}$ be a Cartan subalgebra. We call $\mathfrak{h}$ a splitting Cartan subalgebra if $ad(x)$ is split for each $x \in \mathfrak{h}$. By Lemma 1.1 and Proposition 1.2, we then have the root space decomposition*

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\lambda \neq 0} \mathfrak{g}^\lambda(\mathfrak{h})$$

Definition 30. *1. The number*

$$rank(\mathfrak{g}) = min\{dim(\mathfrak{g}^0(ad(x)))|\ x \in \mathfrak{g}\}$$

is called a rank of $\mathfrak{g}$.

2. *An element $x \in \mathfrak{g}$ is called regular if $dim(\mathfrak{g}^0(ad(x))) = rank(\mathfrak{g})$. We write $reg(\mathfrak{g})$ for the set of regular elements in $\mathfrak{g}$.*

Remark 27. *Observe that $dim(\mathfrak{g}^0(ad(x)))$ is the multiplicity of 0 as a root of the characteristic polynomial of $ad(x)$*

$$\det(ad(x) - t \cdot id_{\mathfrak{g}}) = \sum_{k=0}^{n} p_k(x)t^k$$

Therefore $rank(\mathfrak{g}) = min\{k \in \mathbb{N} | p_k \neq 0\}$. As the determinant is a polynomial function on $End(\mathfrak{g})$, all the functions $p_k : \mathfrak{g} \to \mathbb{K}$ are polynomials.

Lemma 9. *The set $reg(\mathfrak{g})$ of regular elements has the following properties:*

1. *$\mathfrak{g}$ $reg(\mathfrak{g})$ is the zero-set of a nonconstant polynomial.*
2. *$reg(\mathfrak{g})$ is open and dense and it is connected.*
3. *$reg(\mathfrak{g})$ is invariant under the automorphism group $\mathbf{Aut}(\mathfrak{g})$ of $\mathfrak{g}$.*

Proof. 1. Want to show that $\mathfrak{g}$ $reg(\mathfrak{g})$ is the zero-set of a nonconstant polynomial. Let $r = rank(\mathfrak{g})$, then the result follows from the fact that the functions $p_k : \mathfrak{g} \to \mathbb{C}$ seen previously are polynomials and

$$reg(\mathfrak{g}) = \{x \in \mathfrak{g} : p_k(x) \neq 0\}$$

2. Want to show that $reg(\mathfrak{g})$ is open and dense and it is connected. Since p_r is continuous and $\mathbb{C}$ $\{0\}$ is open then $reg(\mathfrak{g}) = p_k^{-1}(\mathbb{C}^{\times})$. Also, the zero set of the nonconstant polynomial p_r contains no open subset, so that $reg(\mathfrak{g})$ is also dense in $\mathfrak{g}$.

3. Want to show that $reg(\mathfrak{g})$ is invariant under the automorphism group $\mathbf{Aut}(\mathfrak{g})$ of $\mathfrak{g}$. For $\gamma \in \mathbf{Aut}(\mathfrak{g})$ we have $ad\gamma(x) = \gamma \circ ad(x) \circ \gamma^{-1}$. Therefore, $ad\gamma(x)$ and $ad(x)$ have the same characteristic polynomial. We conclude that all polynomials p_k are invariant under $\mathbf{Aut}(\mathfrak{g})$. In particular, $reg(\mathfrak{g}) = p_r^{-1}(\mathbb{C}^{\times})$ is invariant.

■

Definition 31. *For $x, y \in \mathfrak{g}$ define $g = e^{ad(x)}$ such that if $\rho_x y = xy$ and $\lambda_x y = yx$ where $[\rho_x, \lambda_x] = 0$ then*

$$g(y) = e^{ad(x)}y = e^{\rho_x}e^{-\lambda_x}y = e^{\rho_x}\sum_{k=0}^{\infty}\frac{(-\lambda_x)^k}{k!}y = e^{\rho_x}\sum_{k=0}^{\infty}\frac{y(-x)^k}{k!} = e^{\rho_x} \cdot y \cdot e^{-x} = e^{x} \cdot y \cdot e^{-x}$$

where the last equality is obtained in the same way of expanding exponential of an operator. It follows then that $g \in Inn(\mathfrak{g})$. These automorphisms are called inner automorphisms.

For every $x \in \mathfrak{g}$, we consider the generalized eigenspace $\mathfrak{g}^0(ad\ x) = \cup_{n \in \mathbb{N}} ker(ad(x)^n)$. Clearly it is a vector space. If $a, b \in \mathfrak{g}^0(ad\ x)$ then $\exists n \in \mathbb{N}$ such that $ad(x)^n a = ad(x)^n b = 0$, which implies that $ad(x)^n[a, b] = [ad(x)^n a, ad(x)^n b] = 0$. Therefore $\mathfrak{g}^0(ad\ x) < \mathfrak{g}$. This can be shown in another way, similar to Proposition 1.1.

Lemma 10. *Let $\mathfrak{h} \subseteq \mathfrak{g}$ be a splitting Cartan subalgebra and $\Delta := \Delta(\mathfrak{g}, \mathfrak{h})$. Then the following assertions hold:*

1. *$rank(\mathfrak{g}) = dim(\mathfrak{h})$ and $reg(\mathfrak{g}) \cap \mathfrak{h} = \mathfrak{h} - \{\cup_{\lambda \in \Delta} ker(\lambda)\}$.*
2. *$Inn(\mathfrak{g})(reg(\mathfrak{g}) \cap \mathfrak{h})$ is an open subset of $\mathfrak{g}$.*

Proof. Proving the first assertion with multiple steps:

1. For the first step, we want to show that there exists $h \in \mathfrak{h}$ such that $\mathfrak{g}^0(ad\ h) = \mathfrak{h}$. Since $\mathfrak{h}$ is nilpotent with split adjoint representation, then by theorem from before we know that $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\lambda \in \Delta} \mathfrak{g}^{\lambda}(\mathfrak{h})$. This implies that for any $h \in \mathfrak{h} \implies \mathfrak{g}^0(ad\ h) = \mathfrak{h} + \sum_{\lambda(h)=0} \mathfrak{g}^{\lambda}(\mathfrak{h})$. Since $\mathfrak{g}$ is finite-dimensional then the set of λ in the decomposition of $\mathfrak{g}$ is finite. Since

$$\mathfrak{g}^0(\mathfrak{h}) = \cap_{h \in \mathfrak{h}} \mathfrak{g}^0(ad\ h) = \cap_{h \in \mathfrak{h}} \{\mathfrak{h} + \sum_{\lambda(h)=0} \mathfrak{g}^{\lambda}(\mathfrak{h})\} = \mathfrak{h} \ \ s.t. \ \ \cap_{h \in \mathfrak{h}} \sum_{\lambda(h)=0} \mathfrak{g}^{\lambda}(\mathfrak{h}) = \{0\}$$

then all of the above implies that $\cup_{\lambda \in \Delta} ker\lambda \neq \mathfrak{h}$. Therefore $\exists h \in \mathfrak{h}$ such that $\forall \lambda \in \Delta, \lambda(h) \neq 0$, i.e. $\sum_{\lambda(h)=0} \mathfrak{g}^{\lambda}(\mathfrak{h}) = \{0\}$ i.e. $\mathfrak{g}^0(ad\ h) = \mathfrak{h}$.

2. Previous implies that $rank(\mathfrak{g}) \geq dim(\mathfrak{h})$. Now want to show that $rank(\mathfrak{g}) = dim(\mathfrak{h})$. Will begin with considering a map

$$\Phi : \mathfrak{g} \times \mathfrak{h} \to \mathfrak{g}, \quad \Phi(a, b) = e^{ad(a)} b = e^{-a} b e^{a}$$

Given $v \in \mathfrak{g}, w \in \mathfrak{h}$ we compute:

$$\frac{\mathrm{d}}{\mathrm{d}t}|_{t=0} \Phi(tv.b + tw) = \frac{\mathrm{d}}{\mathrm{d}t}|_{t=0} e^{tv}(b + tw)e^{-tv} = w + v \cdot b - b \cdot v = [v, b] + w$$

so then can conclude that $d\Phi(0, b)(v, w) = [v, b] + w$. Therefore the image of this tangent map is

$$Im(d\Phi(0, b)) = [\mathfrak{g}, b] + \mathfrak{h}$$

3. Now we want to show that for $\lambda \in \Delta$ such that $\lambda(b) \neq 0$ it follows $\mathfrak{g}^{\lambda}(\mathfrak{h}) \subseteq [\mathfrak{g}, b] = ad(b)(\mathfrak{g})$ which together with previous point implies that

$$\mathfrak{h} + \sum_{\lambda(b) \neq 0} \mathfrak{g}^{\lambda}(\mathfrak{h}) \subseteq Im(d\Phi(0, b)).$$

If $\lambda(b) \neq 0$ then for any $x \in \mathfrak{g}^{\lambda}(\mathfrak{h})$ $\exists n \in \mathbb{N} : (ad(b) - \lambda(b) id_{\mathfrak{g}})^n(x) = 0$ s.t. $(ad(b) - \lambda(b) id_{\mathfrak{g}})^{n-1}(x) \neq 0$. This implies that $y = (ad(b) - \lambda(b) id_{\mathfrak{g}})^{n-1}(x) \in \mathfrak{g}^{\lambda}(\mathfrak{h})$ and $[b, y] = \lambda(b) y \in [b, \mathfrak{g}]$. Suppose $n > 1$ in the above, then if $z = (ad(b) - \lambda(b) id_{\mathfrak{g}})^{n-2}(x)$ it follows that

$$(ad(b) - \lambda(b) id_{\mathfrak{g}}) z = [b, z] - \lambda(b) z = y \in [b, \mathfrak{g}] \implies z \in [b, \mathfrak{g}].$$

Thus inductively can conclude that $x \in [b, \mathfrak{g}]$ and therefore $\mathfrak{g}^{\lambda}(\mathfrak{h}) \subseteq [\mathfrak{g}, b]$.

4. From above follows that $\mathfrak{h} = \mathfrak{g}^0(ad\ h)$ where $\forall \lambda \in \Delta \implies \lambda(h) \neq 0$ and therefore

$$\mathfrak{h} + \sum_{\lambda(h) \neq 0} \mathfrak{g}^{\lambda}(\mathfrak{h}) \subseteq Im(d\Phi(0, h)) \ \ i.e. \ \ \mathfrak{g} = \mathfrak{h} + \sum_{\lambda(h) \neq 0} \mathfrak{g}^{\lambda}(\mathfrak{h}) = Im(d\Phi(0, h))$$

This implies that $d\Phi(0, h)$ is surjective. By the Implicit Function Theorem it follows that the image of Φ is a neighborhood of $h = \Phi(0, h)$.

5. By lemma 3.1 it follows that the set $reg(\mathfrak{g})$ is dense in $\mathfrak{g}$ and therefore it is dense in the image of Φ. Then for fixed h as above $\exists a \in \mathfrak{g}$ and regular element x in the neighborhood of $h = \Phi(0, h)$ such that $x = \Phi(a, h) = e^{ad(a)} h$. Again by lemma 3.1 since $e^{-ad(a)} \in \mathbf{Aut}(\mathfrak{g})$ then it follows that $h = e^{-ad(a)} x$ is also regular. This finally implies that $dim(\mathfrak{h}) = \mathfrak{g}^0(ad\ h) = rank(\mathfrak{g})$ and $\mathfrak{h} - \{\cup_{\lambda \in \Delta} ker(\lambda)\} \subseteq reg(\mathfrak{g}) \cap \mathfrak{h}$. Since for any $h \in reg(\mathfrak{g}) \cap \mathfrak{h} \implies \mathfrak{g}^0(ad\ h) = \mathfrak{h} \iff h \in \mathfrak{h} - \{\cup_{\lambda \in \Delta} ker(\lambda)\}$ then it implies $reg(\mathfrak{g}) \cap \mathfrak{h} = \mathfrak{h} - \{\cup_{\lambda \in \Delta} ker(\lambda)\}$.

To prove the second assertion we will consider few short steps:

1. Fix any $h \in reg(\mathfrak{g}) \cap \mathfrak{h} = \mathfrak{h} - \{\cup_{\lambda \in \Delta} ker(\lambda)\}$, then the above arguments show that $d\Phi(0, h)$ is surjective.

2. Since $reg(\mathfrak{g})$ and $\mathfrak{h}$ are open, then so $reg(\mathfrak{g}) \cap \mathfrak{h}$ is open. By Implicit Function Theorem there exist neighbourhoods U_h of h in $reg(\mathfrak{g}) \cap \mathfrak{h}$ and V of 0 in $\mathfrak{g}$ such that $\Phi(V \times U_h) = e^{ad\ V} U_h$ is an open subset of $\mathfrak{g}$.

3. By lemma 3.1 it follows that $e^{ad\ V} U_h$ consists of regular elements. Together with previous point this implies that h belongs to the interior of $Inn(\mathfrak{g})(reg(\mathfrak{g}) \cap \mathfrak{h})$. Since h is arbitrary then $Inn(\mathfrak{g})(reg(\mathfrak{g}) \cap \mathfrak{h})$ is open.

■

Lemma 11. *Let $\mathfrak{g}$ be a finite-dimensional Lie algebra and $\mathfrak{h} < \mathfrak{g}$. If $x \in \mathfrak{h}$ is regular in $\mathfrak{g}$, then x is also regular in $\mathfrak{h}$, i.e.*

$$dim(\mathfrak{g}^0(ad\ x)) = min\{dim(\mathfrak{g}^0(ad\ b)) | \ \ b \in \mathfrak{g}\} \implies dim(\mathfrak{h}^0(ad\ x)) = min\{dim(\mathfrak{h}^0(ad\ b)) | \ \ b \in \mathfrak{h}\}$$

Proof. For better understanding of the proof, it is good to recall that for a regular element x in a Lie algebra $\mathfrak{a}$, if $rank(\mathfrak{a}) = k = min\{e \in \mathbb{N} | p_e \neq 0\}$ where p_e are polynomials in x from $\det(ad(x) - t \cdot id_{\mathfrak{a}}) = \sum_{e=0} p_e(x)t^e$, then $p_k(x) \neq 0$.

1. For $h \in \mathfrak{h}$ and $V = \mathfrak{g}/\mathfrak{h}$, consider two linear maps:
   - $A(h) : V \to V, \quad y + \mathfrak{h} \to [h, y] + \mathfrak{h}$ such that its char. polynomial is denoted as $\det(A(h) - t \cdot id_V) = \sum_{k=0}^{m} a_k(h)t^k$. Let $d_A(h) = dim(V^0(A(h)))$ and $r_A = min_{h\in\mathfrak{h}} d_A(h)$.
   - $B(h) = ad(h)|_{\mathfrak{h}} : \mathfrak{h} \to \mathfrak{h}$ such that its char. polynomial is denoted as $\det(B(h) - t \cdot id_{\mathfrak{h}}) = \sum_{j=0}^{n} b_j(h)t^j$. Let $d_B(h) = dim(\mathfrak{h}^0(B(h)))$ and $r_B = min_{h\in\mathfrak{h}} d_B(h)$.
2. From above definitions it follows that $r_A = d_A(h)$ if and only if $a_{r_A}(h) \neq 0$. Similarly, $r_B = d_B(h)$ if and only if $b_{r_B}(h) \neq 0$. Consider the set

$$S = \{h \in \mathfrak{h} | a_{r_A}(h) \cdot b_{r_B}(h) \neq 0\} = \{h \in \mathfrak{h} | r_A = d_A(h), r_B = d_B(h)\}$$

   Since b_{r_B} and a_{r_A} are non-zero polynomials then follows that S cannot be empty. Every element of S is clearly regular in $\mathfrak{h}$, so to prove our result it suffices to show that the considered x is in S.
3. If we identify $V = \mathfrak{g}/\mathfrak{h}$ with a vector space complement of $\mathfrak{h}$ in $\mathfrak{g}$, then for $h \in \mathfrak{h}$ it is possible to express operator $ad\ h$ in the block form as

$$ad\ h = \begin{pmatrix} B(h) & * \\ 0 & A(h) \end{pmatrix}$$

   which has a char. polynomial

$$\det(ad(h) - t \cdot id_{\mathfrak{g}}) = \det(A(h) - t \cdot id_V) \det(B(h) - t \cdot id_{\mathfrak{h}}) = (\sum_{k=0}^{m} a_k(h)t^k)(\sum_{j=0}^{n} b_j(h)t^j)$$

4. If $h \in S \neq \emptyset$, then by the definition of $rank(\mathfrak{g})$ the above implies that $rank(\mathfrak{g}) \leq r_A + r_B$. If we consider the char. polynomial for $ad\ x$ where x is regular

$$\det(ad(x) - t \cdot id_{\mathfrak{g}}) = \det(A(x) - t \cdot id_V) \det(B(x) - t \cdot id_{\mathfrak{h}})$$

   then $rank(\mathfrak{g}) = d_A(x) + d_B(x) \geq r_A + r_B$, so therefore $rank(\mathfrak{g}) = r_A + r_B$ and thus the minimum exponent for char. polynomial of $ad\ x$ is $r = d_A(x) + d_B(x) = r_A + r_B$. This implies that $a_{r_A}(x) \cdot b_{r_B}(x) \neq 0$ and thus $x \in S$.

■

All of the previous lemmas were proven to establish the following between Cartan subalgebras and regular elements.

Theorem 23. *Let $\mathfrak{g}$ be a finite-dimensional Lie algebra. Then the following assertions hold:*

1. *For any regular element $x \in \mathfrak{g}$, $\mathfrak{g}^0(ad\ x)$ is a Cartan subalgebra of $\mathfrak{g}$.*
2. *Every Cartan subalgebra $\mathfrak{h}$ contains regular elements and if $x \in \mathfrak{h}$ is regular, then $\mathfrak{h} = \mathfrak{g}^0(ad\ x)$.*
3. *All Cartan subalgebras have the same dimension $rank(\mathfrak{g})$.*

Proof. 1. Want to prove that for any regular element $x \in \mathfrak{g}$, $\mathfrak{g}^0(ad\ x)$ is a Cartan subalgebra of $\mathfrak{g}$.

- Since $x \in \mathfrak{h} = \mathfrak{g}^0(ad\ x)$ is a subalgebra of $\mathfrak{g}$ and x is regular in $\mathfrak{g}$, then by lemma 3.3 it follows that x is regular in $\mathfrak{h}$.
- Since $\mathfrak{h} = \mathfrak{g}^0(ad\ x) = \cup_{n\in\mathbb{N}}\{y \in \mathfrak{g} | ad(x)^n y = 0\}$, then follows that

$$\mathfrak{h}^0(ad\ x) = \cup_{n\in\mathbb{N}}\{y \in \mathfrak{h} | ad(x)^n y = 0\} = \mathfrak{h},$$

  therefore we have $rank(\mathfrak{h}) = dim(\mathfrak{h}) = rank(\mathfrak{g})$.

- Since $dim(\mathfrak{h}) = rank(\mathfrak{h}) = min\{dim(\mathfrak{h}^0(ad\ y))|y \in \mathfrak{h}\}$ and $dim(\mathfrak{h}) \geq max\{dim(\mathfrak{h}^0(ad\ y))|y \in \mathfrak{h}\}$ then follows that $\forall y \in \mathfrak{h} \implies dim(\mathfrak{h}) = dim(\mathfrak{h}^0(ad\ y))$, i.e. $\forall y \in \mathfrak{h} \implies \mathfrak{h} = \mathfrak{h}^0(ad\ y)$.
- Previous implies that $\forall y \in \mathfrak{h} \implies ad(y)|_{\mathfrak{h}}$ is nilpotent, and therefore as a consequence of Engel's Theorem it follows that $\mathfrak{h}$ is nilpotent.
- Since $\mathfrak{h}$ is nilpotent then $\mathfrak{h} \subseteq \mathfrak{g}^0(ad(\mathfrak{h}))$ so that

$$\mathfrak{h} \subseteq \mathfrak{g}^0(ad(\mathfrak{h})) = \cap_{h\in\mathfrak{h}}\mathfrak{g}^0(ad(h)) \subseteq \mathfrak{g}^0(ad(x)) = \mathfrak{h}$$

Therefore $\mathfrak{h} = \mathfrak{g}^0(ad(\mathfrak{h}))$ so by Proposition 1.2 this implies that $\mathfrak{h}$ is a Cartan subalgebra.

2. Want to prove that every Cartan subalgebra $\mathfrak{h}$ contains regular elements and if $x \in \mathfrak{h}$ is regular, then $\mathfrak{h} = \mathfrak{g}^0(ad\ x)$.
   - Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. Since the ground field is $\mathbb{C}$, then $\mathfrak{h}$ is splitting and therefore by Lemma 3.2 it contains regular elements.
   - Let $x \in reg(\mathfrak{g}) \cap \mathfrak{h}$, then (1) implies that $\mathfrak{g}^0(ad\ x)$ is a Cartan subalgebra containing $\mathfrak{h}$. Since Cartan subalgebras are maximally nilpotent then follows that $\mathfrak{h} = \mathfrak{g}^0(ad\ x)$.
3. Want to prove that all Cartan subalgebras have the same dimension $rank(\mathfrak{g})$:
   - By (2) above, for any Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ there exists regular element $x \in \mathfrak{h}$ such that $\mathfrak{h} = \mathfrak{g}^0(ad\ x)$. This implies that $dim(\mathfrak{h}) = dim(\mathfrak{g}^0(ad\ x)) = rank(\mathfrak{g})$ for any Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$.

■

Definition 32. *Let $\mathfrak{g}$ be a finite-dimensional Lie algebra. Define an equivalence relation R on the set $reg(\mathfrak{g})$ of regular elements via*

$$x \sim y \iff \exists\ g \in Inn(\mathfrak{g}) \ \ s.t. \ \ g(\mathfrak{g}^0(ad\ x)) = \mathfrak{g}^0(ad\ y)$$

Lemma 12. *The equivalence classes of $\sim$ are open subsets of $reg(\mathfrak{g})$*

Proof. Consider any $x \in reg(\mathfrak{g})$ and define $\mathfrak{h} = \mathfrak{g}^0(ad\ x)$. By lemma 3.2, the set $Inn(reg(\mathfrak{g}) \cap \mathfrak{h})$ is an open neighbourhood of x. Each element in this set is of the form $g(y)$ for some $y \in reg(\mathfrak{g}) \cap \mathfrak{h}$. Since for any automorphism $g \in Inn(\mathfrak{g})$ *and* $x, y \in \mathfrak{g} \implies g([x, y]) = [g(x), g(y)]$ then $\mathfrak{g}^0(ad\ g(x)) = g(\mathfrak{g}^0(ad\ x))$. Also, by Theorem 3.4 for any $x, y \in reg(\mathfrak{g}) \implies \mathfrak{h} = \mathfrak{g}^0(ad\ x) = \mathfrak{g}^0(ad\ y)$. All together this implies that

$$\mathfrak{g}^0(ad\ g(y)) = g(\mathfrak{g}^0(ad\ y)) = g(\mathfrak{h}) = g(\mathfrak{g}^0(ad\ x))$$

which shows that $x \sim g(y)$ for all $g(y) \in Inn(reg(\mathfrak{g}) \cap \mathfrak{h})$. Therefore all equivalence classes of $\sim$ are open. ■

Theorem 24. *If $\mathfrak{g}$ a finite-dimensional complex Lie algebra, then the group $Inn(\mathfrak{g})$ acts transitively on the set of Cartan subalgebras of $\mathfrak{g}$.*

Proof. According to lemma 3.1, the set $reg(\mathfrak{g})$ is connected. On the other hand, by lemma 3.5 it is the disjoint union of the open equivalence classes (open sets) of the relation $\sim$. Hence only one such class exists. Since every Cartan subalgebra of $\mathfrak{g}$ is of the form $\mathfrak{g}^0(ad\ x)$ by Theorem 3.4, the assertion follows. ■

6.2. Semisimple case

Theorem 25. *Let $\mathfrak{h}$ be a Cartan subalgebra of a semisimple Lie algebra $\mathfrak{g}$. Then the following assertions hold:*

1. *The restriction of Cartan-Killing form of $\mathfrak{g}$ to $\mathfrak{h}$ is nondegenerate.*
2. *$\mathfrak{h}$ is abelian*
3. *The centralizer of $\mathfrak{h}$ is $\mathfrak{h}$.*
4. *Every element of $\mathfrak{h}$ is semisimple.*

Proof. 1. Want to prove that the restriction of Cartan-Killing form of $\mathfrak{g}$ to $\mathfrak{h}$ is nondegenerate.

- By Theorem 3.4 $\exists\ x \in \mathfrak{h}$ such that $\mathfrak{h} = \mathfrak{g}^0(ad\ x)$. From Proposition 1.1 follows that

$$\mathfrak{g} = \mathfrak{g}^0(ad\ x) \oplus \sum_{\lambda \neq 0} \mathfrak{g}^\lambda(ad\ x)$$

  is generalized eigenspace decomposition with respect to x.

- If $y \in \mathfrak{g}^0(ad\ x)$ and $z \in \mathfrak{g}^\lambda(ad\ x)$ for some nonzero λ, then after making some computations we can show that $\kappa_{\mathfrak{g}}(y,z) = trace(ad(y)ad(z)) = 0$. Therefore we have a decomposition of $\mathfrak{g}$ into mutually orthogonal subspaces $\mathfrak{g}^0(ad\ x)$ and $\sum_{\lambda \neq 0} \mathfrak{g}^\lambda(ad\ x)$ with respect to the Cartan-Killing form.

- Since $\mathfrak{g}$ is semisimple then $\kappa_{\mathfrak{g}}$ is nondegenerate, so is its restriction to each of these subspaces, in particular restriction to $\mathfrak{g}^0(ad\ x) = \mathfrak{h}$.

2. Want to prove that $\mathfrak{h}$ is abelian.

- Since $\mathfrak{h}$ is nilpotent and therefore solvable, then applying Cartan's criterion to $\mathfrak{h}$, we see that $trace(ad(x)ad(y)) = 0$ for $x \in \mathfrak{h}$ and $y \in [\mathfrak{h}, \mathfrak{h}]$. So that $[\mathfrak{h}, \mathfrak{h}] \subseteq \mathfrak{rad}(\kappa_{\mathfrak{h}})$.

- Since $\kappa_{\mathfrak{h}}$ is nondegenerate on $\mathfrak{h}$, i.e. $\mathfrak{rad}(\kappa_{\mathfrak{h}}) = \{0\}$ then this implies that $[\mathfrak{h}, \mathfrak{h}] = 0$.

3. Want to prove that the centralizer of $\mathfrak{h}$ is $\mathfrak{h}$. By (1) it follows $\mathfrak{h}$ is abelian so $\mathfrak{h} \subseteq \mathfrak{z}(\mathfrak{h})$. Therefore

$$\mathfrak{h} \subseteq \mathfrak{z}(\mathfrak{h}) \subseteq \mathfrak{n}(\mathfrak{h}) = \mathfrak{h}$$

   Thus $\mathfrak{h} = \mathfrak{z}(\mathfrak{h})$.

4. Want to prove that every element of $\mathfrak{h}$ is semisimple.

- Let $x \in \mathfrak{h}$, and let s (n resp.) be its semisimple (nilpotent resp.) component. If $y \in \mathfrak{h}$ then $[y, x] = 0$ and by result from the last chapter $[y, n] = [y, s] = 0$, thus implying that $n, s \in \mathfrak{z}(\mathfrak{h}) = \mathfrak{h}$.

- Since y and n commute and $ad(n)$ is nilpotent then this implies that $[ad(y), ad(n)] = 0$ and therefore $ad(y)ad(n)$ is nilpotent so $\kappa_{\mathfrak{h}} = trace(ad(y)ad(n)) = 0$. Since this holds for any $y \in \mathfrak{h}$, then n is orthogonal to every element of $\mathfrak{h}$. Since $n \in \mathfrak{h}$ then follows that $n = 0$. Thus follows that every $x \in \mathfrak{h}$ is semisimple.

■

Corollary 6. $\mathfrak{h}$ *is a maximal abelian subalgebra of* $\mathfrak{g}$.

Proof. Let $\mathfrak{a}$ be an abilian subalgebra such that $\mathfrak{h} \subseteq \mathfrak{a}$, then by (3) assertion in the Theorem 4.1 it follows that $\mathfrak{a} \subseteq \mathfrak{z}(\mathfrak{h}) = \mathfrak{h}$. ■

Corollary 7. *Every regular element of* $\mathfrak{g}$ *is semisimple.*

Proof. Since every regular element is contained in Cartan subalgebra then the result follows from the fourth assertion in the Theorem 4.1. ■

Chapter 7

The Lie algebra $\mathfrak{sl}_2$ and Its Representations

For this chapter we consider that the ground field is $\mathbb{C}$. The definition of this Lie algebra, we have seen in our course before and it goes like this:

$$\mathfrak{sl}_2 = \{M \in Mat_{2\times 2}(\mathbb{C}) |\ trace(M) = 0\}$$

For the further discussion we will label this Lie algebra as $\mathfrak{g}$. Given matrix $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, if $tr(M) = 0$ then $a = -d$ and thus can see that the basis for this vector space is

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

such that the following commutation relations hold

$$[X, Y] = H, \quad [H, X] = 2X, \quad [H, Y] = -2Y$$

From this we can observe the following:

1. If $I \triangleleft \mathfrak{g}$ is an ideal and $aX + bY + cH = \gamma \in I$, such that $(a, b, c) \neq (0, 0, 0)$ then the commutation relations imply that $X, Y, H \in I$, therefore $\mathfrak{g}$ is simple.
2. The endomorphism $ad(H)$ has three eigenvalues: -2, 0, 2. Thus H is semisimple. Considering $ad(X)$ and $ad(Y)$ it follows that $\mathfrak{g}$ has rank one.
3. Consider line $\mathfrak{h} = \mathbb{C} \cdot H$, then by above commutation relations it is self-normalizing in $\mathfrak{g}$ and $[\mathfrak{h}, \mathfrak{h}] = 0$. Therefore $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$.
4. Elements X and Y are nilpotent.
5. The subalgebra $\mathfrak{b}$ generated by H and X is solvable, and it is referred to as the Borel subalgebra of $\mathfrak{g}$.

7.1. Modules, Weights, Primitive elements

Let V be a $\mathfrak{g}$-module. If $\lambda \in \mathbb{C}$ is an eigenvalue of H in V then we denote V_λ as its corresponding eigenspace.

Proposition 21.

1. The sum $\sum_{\lambda \in \mathbb{C}} V_\lambda$ is direct.

2. If x corresponds to eigenvalue λ, then Xx has eigenvalue $\lambda + 2$ and Yx has eigenvalue $\lambda - 2$

Proof. 1. The first statement is a consequence of the fact that $V_\lambda \cap V_\mu = \{0\}$ if $\lambda \neq \mu$.

2. The second statement follows from commutations relations for X, Y, H. If $Hx = \lambda x$, then

$$HXx = [H, X]x + XHx = (2 + \lambda)Xx, \text{ and } HYx = [H, Y]x + YHx = (-2 + \lambda)Yx$$

■

Remark 28. *Since H is semisimple, then if V is finite dimensional then $V = \sum_{\lambda \in \mathbb{C}} V_\lambda$.*

Definition 33. *Let V be a $\mathfrak{g}$-module, and let $\lambda \in \mathbb{C}$. An element $e \in V$ is said to be a primitive element of weight λ if it is nonzero and if*

$$Xe = 0, \text{ and } He = \lambda e$$

Proposition 22. *For a non zero element e of the $\mathfrak{g}$-module V to be primitive, it is necessary and sufficient that the line it spans should be stable under the Borel algebra $\mathfrak{b}$.*

Proof. Necessity follows from definition of primitive. To argue sufficiency, we need to consider line $\mathbb{C} \cdot e$ which is stable under $\mathfrak{b}$, and deduce that $Xe = 0$, *and* $He = \lambda e$. ■

Proposition 23. *Every nonzero finite-dimensional $\mathfrak{g}$-module contains a primitive element.*

Proof. If V is $\mathfrak{g}$-module, then it is also $\mathfrak{b}$-module, which is a solvable subalgebra of $\mathfrak{g}$. Thus by Lie's theorem there is nonzero $x \in V$ such that $Hx = \lambda x$ and $Xx = \mu x$ for some $\lambda, \mu \in \mathbb{C}$, where by Proposition 2.1 follows that $HXx = (\lambda + 2)Xx = (\lambda + 2)\mu x = \mu\lambda x$, i.e. $\mu = 0$ so x is primitive. ■

7.2. Structure of the Submodule Generated by a Primitive Element

Theorem 26. *Let V be a $\mathfrak{g}$-module and $e \in V$ a primitive element of weight λ. Let us denote $e_n = Y^n e/n!$ for $n \geq 0$ and $e_{-1} = 0$. Then for all $n \geq 0$ the following holds:*

1. $He_n = (\lambda - 2n)e_n$
2. $Ye_n = (n + 1)e_{n+1}$
3. $Xe_n = (\lambda - n + 1)e_{n-1}$

Proof. 1. Directly follows from proposition 2.1.

2. Directly follows from the definition of e_n.
3. The third statement is proven by induction on n s.t. using commutation relations we establish the result, i.e. If by induction $Xe_{n-1} = (\lambda - n + 2)$ then

$$nXe_n = XYe_{n-1} = [X, Y]e_{n-1} + YXe_{n-1} = He_{n-1} + (\lambda - n + 2)Ye_{n-2} = n(\lambda - n + 1)e_{n-1}$$

which yields the required result.

■

Corollary 8. *Only two cases are possible:*

1. *the elements $(e_n)_{n \geq 0}$, are all linearly independent, or*
2. *the weight λ of e is an integer $m \geq 0$, the elements $\{e_o, ..., e_m\}$ are linearly independent, and $e_i = 0$ for $i > m$.*

Proof. Suppose (1) does not hold, then for some $m \in \mathbb{N}$ it follows that $\{e_o, ..., e_m\}$ are linearly independent and e_{m+1} can be expressed as their linear combination. Since e_{m+1} corresponds to an eigenvalue $\lambda - 2(m+1)$, distinct from the first m+1 eigenvalues $\{\lambda - 2n\}_{n=0}^{m}$, then $e_{m+1} = 0$ so that formula (2) from Theorem 3.1 implies that $e_{m+j} = 0$ for all $j \geq 1$. Applying formula (3) to $e_{m+1} = 0$ we find that

$$0 = Xe_{m+1} = (\lambda - m)e_m$$

which implies that $\lambda = m$. Therefore $(e_n)_{n \geq 0}$ either satisfies (1) or (2). ■

Corollary 9. *Suppose that V is finite dimensional. Then we are in case (2) of Corollary 3.1.1. The vector subspace W of V with basis $\{e_0, ..., e_m\}$ is stable under $\mathfrak{g}$; it is an irreducible $\mathfrak{g}$-module.*

Proof. •

- In the above condition we consider m to be a maximal integer such that the weight of e_0 is $\lambda = m$ and $Ye_m = 0$. Therefore from formulas (1),(2),(3) in Theorem 3.1. can conclude that W is stable under action of $\mathfrak{g}$.
- Note that $\{e_0, ..., e_m\}$ are eigenvectors of H with corresponding eigenvalues $m, m-2, m-4, ..., -m$ each of multiplicity one. If W' is a nonzero subspace of W, stable under $\mathfrak{g}$ then it contains one of the eigenvectors e_i and using formulas (2) and (3) from Theorem 3.1 we can conclude that $\{e_0, ..., e_m\} \subset W'$ i.e. $W' = W$ thus proving the irreducibility of W.

■

7.3. The Modules W_m

In what follows, we will consider previous constructions in reverse. Given m+1 dimensional vector space W_m with basis $\{e_0, ..., e_m\}$, we will define endomorphisms X, Y, H of W_m by the formulae

1. $He_n = (m - 2n)e_n$
2. $Ye_n = (n + 1)e_{n+1}$
3. $Xe_n = (m - n + 1)e_{n-1}$

A direct computation shows that the following are satisfied:

$$[H, X]e_n = 2Xe_n,\ [H, Y]e_n = -2Ye_n,\ [X, Y]e_n = He_n$$

Therefore the endomorphisms X, Y, H make W_m into a $\mathfrak{g}$-module.

Theorem 27.

1. *W_m is an irreducible $\mathfrak{g}$-module.*
2. *Every irreducible $\mathfrak{g}$-module of dimension $m + 1$ is isomorphic to W_m.*

Proof.

1. From the definition of endomorphisms H, X, Y it follows that W_m is generated by the primitive element e_0 of weight m, so then by Corollary 3.1.2 W_m is an irreducible $\mathfrak{g}$-module.
2. Let V be an irreducible $\mathfrak{g}$-module of dimension $m + 1$. By Prop. 2.3, V contains a primitive element e. Corollary 3.1.1 to Theorem 3.1 shows that the weight of e is an integer $m' \geq 0$, and that the $\mathfrak{g}$-submodule W of V generated by e has dimension $m' + 1$. Since V is irreducible, we must have $W = V$, so that $m' = m$, and the formulae of Theorem 3.1 show that V is isomorphic to W_m, as required.

■

7.4. Structure of the Finite-Dimensional $\mathfrak{g}$-Modules

Theorem 28. *Each finite-dimensional $\mathfrak{g}$-module is isomorphic to a direct sum of modules W_m.*

Proof. Since $\mathfrak{g}$ is simple Lie algebra then this result is a consequence of the Weyl's Theorem, meaning that such a module is a direct sum of irreducible modules. By Theorem 4.1 any finite-dimensional irreducible module is isomorphic to W_m for some m. ■

Theorem 29. *Let V be a finite-dimensional $\mathfrak{g}$-module. Then:*

1. *The endomorphism of V induced by H is diagonalizable. Its eigenvalues are integers. If $\pm n$ (with $n \geq 0$) are eigenvalues of H, then so are $n - 2, n - 4, ..., -n$.*

2. *If n is a non-negative integer, the linear maps*

$$Y^n : V_n \to V_{-n} \text{ and } X^n : V_{-n} \to V_n$$

are isomorphisms. In particular, V_{-n} and V_n have the same dimension.

Proof. By Theorem 5.1 finite-dimensional $\mathfrak{g}$-module V is isomorphic to a direct sum of irreducible $\mathfrak{g}$-modules W_m. In this case The endomorphism of V induced by H is diagonalizable, its eigenvalues are integers and if it has eigenvectors e_0, e_n with eigenvalues $n, -n$ resp. then obviously $e_1, ..., e_{n-1}$ are also its eigenvectors with resp. eigenvalues $n-2, n-4, ..., -n+2$. The second statement follows from n times applying Prop. 2.1 to eigenvectors from our direct sum of W_m. ■

7.5. Topological Properties of the Group SL_2

This is the group of complex matrices of order 2 and determinant equal to 1. Its topological properties can be summarized in the following theorem, parts of which we have proven in our previous course.

Theorem 30.

1. *SL_2 is isomorphic to $SU(2) \times \mathbb{R}^3$*
2. *$SU(2)$ is isomorphic (as a Lie group) to the group of quaternions of norm 1, which is itself homeomorphic to the sphere S^3.*
3. *$SU(2)$ and SL_2 are connected and simply connected.*

Proof. We will only prove the first statement of the theorem. Define vector space $P = i \cdot \mathfrak{su}_2$ and a map $\phi : SU(2) \times P \to SL_2$ as $(u, p) \to u \cdot e^p$. Function ϕ is a continuous group isomorphism between $SU(2) \times P$ and SL_2. Since P is isomorphic to $\mathbb{R}^3$ then the result follows. ■

Chapter 8

Root systems

In this chapter (apart from Sec. 17) the ground field is the field $\mathbb{R}$ of real numbers. The vector spaces considered are all finite dimensional.

8.1. Symmetries

Let V be a finite-dimensional vector space and α its element.

Definition 34. *An automorphism s_α of V is called a symmetry with vector α if it satisfies two conditions:*

1. $s_\alpha(\alpha) = -\alpha$

2. The set H of elements of V fixed by s_α is a hyperplane of V.

Therefore s_α corresponds to an operator of order two and it is completely determined by the line $\mathbb{R}\alpha$ and H. If $\alpha^* \in V^*$ is a unique element in the dual space which satisfies $\langle \alpha^*, \alpha \rangle = 2$ then $s_\alpha(x) = x - \langle \alpha^*, x \rangle \alpha$.

Lemma 13. *Let α be a non zero element of V, and let R be a finite subset of V which spans V. If there is symmetry with vector α which leaves R invariant, then it is unique.*

Proof. Suppose that s_1 and s_2 are two symmetries (order 2 automorphisms) with vector α which leave R invariant. Consider their composition $u = s_2 \circ s_1$, an automorphism which has the following properties:

1. $u(R) = R$, i.e. u is a permutation of R.

2. $u(\alpha) = \alpha$

3. u induces the identity on $V/\alpha\mathbb{R}$

The last two properties imply that u is an identity on $V/\alpha\mathbb{R}$ and $\alpha\mathbb{R}$. Since u is linear then it must be an identity on V. which implies that $s_1 = s_2$. ■

8.2. Definition of Root Systems

Definition 35. *The subset R of a vector space V is said to be a root system in V if the following conditions are satisfied:*

1. R is finite, spans V, and does not contain 0.

2. For each $\alpha \in R$, there is symmetry s_α, with vector α, leaving R invariant. (By the last Lemma it is unique)

3. For each $\alpha, \beta \in R$, it holds that $\langle \alpha^*, \beta \rangle \in \mathbb{Z}$, i.e. $s_\alpha(\beta) - \beta$ is an integer multiple of α. Here α^* satisfies $\langle \alpha^*, \alpha \rangle = 2$.

Definition 36. *The root system is said to be reduced if, for each $\alpha \in R$, α and $-\alpha$ are the only roots proportional to α.*

8.3. The Weyl Group

Definition 37. *Let R be a root system in a vector space V. The Weyl group of R is the subgroup W of $GL(V)$ generated by the symmetries $s_\alpha, \alpha \in R$.*

Let $Aut(R)$ be a group of automorphisms of V which leave R invariant. Then it is easy to observe that for any $s_\alpha \in W$ and $\phi \in Aut(R)$ there exists s_β such that $\phi \circ s_\alpha \circ \phi^{-1} = s_\beta$. Therefore W is a normal subgroup of $Aut(R)$.

Remark 29. *If R is a reduced root system of rank 2 which contains n pairs $(\alpha, -\alpha)$, then each $\alpha \in R$ corresponds to a vertex of 2n-gon and W is its set of symmetries. In other words, W is isomorphic to a dihedral group.*

8.4. Invariant Quadratic Forms

Proposition 24. *Let R be a root system in V. There is a positive definite symmetric bilinear form $(\cdot, \cdot)$ on V which is invariant under the Weyl group W of R.*

Proof. Let $B(x, y)$ be any positive definite symmetric bilinear form on V. Then define form

$$(x, y) = \sum_{w \in W} B(wx, wy)$$

where for any symmetry $s \in W$ it follows that

$$(s(x), s(y)) = \sum_{w \in W} B(w \cdot s(x), w \cdot s(y)) = \sum_{w \cdot s \in W} B(w \cdot s(x), w \cdot s(y)) = \sum_{w \in W} B(wx, wy) = (x, y)$$

Thus the form is invariant under W and $(x, x) > 0$ for all nonzero $x \in V$. ∎

This result implies that any symmetry $s_\alpha \in W$ has a form $s_\alpha(x) = x - 2\frac{(x,\alpha)}{(\alpha,\alpha)}x$ and condition (3) from the definition of root system is equivalent to $\alpha, \beta \in R \implies 2\frac{(\beta,\alpha)}{(\alpha,\alpha)} \in \mathbb{Z}$. Connecting this to the previous notation it follows that $\langle \alpha^*, \beta \rangle = 2\frac{(\beta,\alpha)}{(\alpha,\alpha)}$.

Remark 30. *We can define an isomorphism Φ between V and its dual counterpart V^* as follows:*

$$\Phi : V \to V^* \quad \forall \alpha \in V \implies \Phi : \alpha \to (\alpha, \cdot)$$

Then in the definition of symmetry s_α with vector α it follows that α^* functional corresponds to $(\gamma, \cdot)$ where $\gamma = 2\frac{\alpha}{(\alpha,\alpha)}$.

8.5. Inverse Systems

Let R be a root system of V.

Proposition 25. *The set R^* of inverse roots α^*, $\alpha \in R$, is a root system in V^*. Moreover $\alpha^{**} = \alpha$ for all $\alpha \in R$.*

Proof. This result follows from observing that each $\alpha^* \in R^*$ corresponds to a vector $2\frac{\alpha}{(\alpha,\alpha)} = \alpha' \in V$ such that $\forall x \in V \implies \langle \alpha^*, x \rangle = (\alpha', x)$. From this follows that R^* is finite and spans V^*. Given $\alpha \in R$ its corresponding symmetry is $s_\alpha(x) = x - \langle \alpha^*, x \rangle \alpha$ such that $s_\alpha(R) = R$, so for $\alpha^* \in R^*$ its symmetry is $s_{\alpha^*}(z) = z - \langle z, \alpha \rangle \alpha^* = z - \langle \alpha^{**}, z \rangle \alpha^*$, therefore follows that $\alpha^{**} = \alpha$ and $s_{\alpha^*}(R^*) = R^*$. Thus R^* is a root system. ∎

Remark 31. *We can observe that for a given $\alpha \in R$ its symmetry can be expressed as $s_\alpha = 1 - \alpha^* \otimes \alpha$. Then for the dual space V^*, the symmetry of α^* is $s_{\alpha^*} = 1 - \alpha \otimes \alpha^* = s_\alpha^T$.*

Remark 32. *If R^* is the dual root system then to define its corresponding Weyl group W^* we need to establish a correspondence which gives for each $w \in W$ a unique $w^* \in W^*$. More precisely we can define an isomorphism $\Phi : W \to W^*$ which satisfies the relation*

$$\forall w \in W,\ \lambda \in V^*,\ x \in V \implies \langle \Phi(w)(\lambda), x\rangle = \langle \lambda, w^{-1}(x)\rangle \iff \Phi(w)(\lambda)\cdot x = \lambda(w^{-1})\cdot x$$

1. *If $w = s_\alpha$ and $\lambda = \alpha^*$ then $\Phi(s_\alpha)(\alpha^*) = \alpha^*(s_\alpha)$ where for any $x \in V$ it follows that*

$$\alpha^*(s_\alpha)\cdot x = \alpha^*(x - \alpha^*(x)\alpha) = \alpha^*(x)(1 - \alpha^*(\alpha)) = -\alpha^*(x)$$

   therefore $\Phi(s_\alpha)(\alpha^*) = -\alpha^*$.

2. *It follows that $\forall \alpha \in S \implies \Phi(s_\alpha)(\lambda) = \lambda(s_\alpha)$ and $\Phi(s_\alpha) = s_{\alpha^*}$ i.e. $s_{\alpha^*}(\lambda) = \lambda(s_\alpha)$.*

8.6. Relative Position of Two Roots

For this section we will slightly modify our previous notation such that If α, β are two roots then

$$n(\beta,\alpha) = \langle \alpha^*, \beta\rangle = 2\frac{(\beta,\alpha)}{(\alpha,\alpha)}$$

where $n(\beta,\alpha) \in \mathbb{Z}$ by definition 2.1 and for given $\alpha \in R \implies s_\alpha(x) = x - n(x,\alpha)\alpha$. If for given root α its length is $|\alpha|^2 = (\alpha,\alpha)$ then for roots α,β with angle ϕ between them, it holds that $(\beta,\alpha) = |\beta|\cdot|\alpha|\cos\phi$ and therefore $n(\beta,\alpha) = 2\frac{|\beta|}{|\alpha|}\cos\phi$. From this we deduce the formula

$$n(\beta,\alpha)n(\alpha,\beta) = 4\cos^2\phi$$

Since $n(\beta,\alpha) \in \mathbb{Z}$, then $n(\beta,\alpha)n(\alpha,\beta) \in \{0,1,2,3,4\}$.

If α, β are proportional then $\phi = 0$ i.e. $\frac{|\beta|}{|\alpha|} \in \mathbb{N}$. Suppose that $\alpha = k\beta$, then since for any roots $\alpha,\beta \implies n(\alpha,\beta), n(\beta,\alpha) \in \mathbb{Z}$ it follows that $2\frac{|\beta|}{|\alpha|}, 2\frac{|\alpha|}{|\beta|} \in \mathbb{Z}$ i.e. $2k, \frac{2}{k} \in \mathbb{Z}$. Only one modulus for such k is possible and it is $|k| = 2$. Thus for a given root α the set of all its proportional roots in R is $\{-2\alpha, -\alpha, \alpha, 2\alpha\}$.

For non proportional roots seven possibilities (up to transposition of α and β) are available:

1. $n(\beta,\alpha) = 0$, $n(\alpha,\beta) = 0$, $\phi = \frac{\pi}{2}$
2. $n(\beta,\alpha) = 1$, $n(\alpha,\beta) = 1$, $\phi = \frac{\pi}{3}$, $|\beta| = |\alpha|$
3. $n(\beta,\alpha) = -1$, $n(\alpha,\beta) = -1$, $\phi = \frac{2\pi}{3}$, $|\beta| = |\alpha|$
4. $n(\beta,\alpha) = 1$, $n(\alpha,\beta) = 2$, $\phi = \frac{\pi}{4}$, $|\beta| = \sqrt{2}|\alpha|$
5. $n(\beta,\alpha) = -1$, $n(\alpha,\beta) = -2$, $\phi = \frac{3\pi}{4}$, $|\beta| = \sqrt{2}|\alpha|$
6. $n(\beta,\alpha) = 1$, $n(\alpha,\beta) = 3$, $\phi = \frac{\pi}{6}$, $|\beta| = \sqrt{3}|\alpha|$
7. $n(\beta,\alpha) = -1$, $n(\alpha,\beta) = -3$, $\phi = \frac{5\pi}{6}$, $|\beta| = \sqrt{3}|\alpha|$

Proposition 26. *Let α, β be two non proportional roots. If $n(\alpha,\beta) > 0$, then $\alpha - \beta$ is a root.*

Proof. If $n(\alpha,\beta) > 0$, then the above list of seven possibilities has only options (4) and (6) possible, so either of these two holds: $n(\alpha,\beta) = 1$ or (after transposing β and α) $n(\beta,\alpha) = 1$.

1. In the first case, $\alpha - \beta = -(\beta - n(\beta,\alpha)\alpha) = -s_\alpha(\beta) \in R$
2. In the second case, $\alpha - \beta = -(\alpha - n(\alpha,\beta)\beta) = s_\beta(\alpha) \in R$

■

8.7. Bases

Let R be a root system in V.

Definition 38. *The subset S of R is called a base for R if the following two conditions are satisfied:*

1. S is a basis for a vector space V.

2. Each $\beta \in R$ can be written as a linear combination

$$\beta = \sum_{\alpha \in S} m_\alpha \alpha$$

where m_α are integers with the same sign (i.e. all $m_\alpha \geq 0$ or all $m_\alpha \leq 0$).

Definition 39. *Let $t \in V^*$ such that for all $\alpha \in R$ it holds that $\langle t, \alpha \rangle \neq 0$. Let R_t^+ be the set of all $\alpha \in R$ such that $\langle t, \alpha \rangle > 0$ so that $R = R_t^+ \cup -R_t^+$.*

An element $\alpha \in R_t^+$ is called decomposable if there exists $\beta, \gamma \in R_t^+$ such that $\alpha = \beta + \gamma$. Otherwise α is called indecomposable.

Denote the set of indecomposable elements of R_t^+ as S_t.

Theorem 31. *There exists a base and more precisely S_t is a base of R. Conversely, if S is a base for R, and if $t \in V^*$ is such that $\langle t, \alpha \rangle > 0$ for all $\alpha \in S$, then $S = S_t$.*

To prove this theorem we will need three lemmas.

Lemma 14. *Each element of R_t^+ is a linear combination, with non-negative integer coefficients, of elements of S_t.*

Proof. Let I be the set of elements $\alpha \in R_t^+$ which fail the hypothesis of lemma. Then each such α is decomposable (because α is not in S_t) and there would an element $\alpha' \in I$ such that $\langle t, \alpha' \rangle$ is minimal. By definition $\exists \beta, \gamma \in R_t^+$ such that $\alpha' = \beta + \gamma$. Therefore

$$\langle t, \alpha' \rangle = \langle t, \beta \rangle + \langle t, \gamma \rangle$$

where $\langle t, \beta \rangle$ and $\langle t, \gamma \rangle$ are > 0, so there are strictly smaller then $\langle t, \alpha' \rangle$. Thus our assumption on minimality of $\langle t, \alpha' \rangle$ implies that γ, β don't belong to I, which implies that each of them is a linear combination, with non-negative integer coefficients, of elements of S_t. The same follows for α' which contradicts that $\alpha' \in I$. ∎

Lemma 15. *If $\alpha, \beta \in S_t$, then $(\alpha, \beta) \leq 0$.*

Proof. Suppose not, then $(\alpha, \beta) > 0$ and by Prop. 6.1 it follows that $\gamma = \alpha - \beta$ is a root. Then either $\gamma \in R_t^+$ which implies that $\alpha = \gamma + \beta$ is decomposable, contradicting that $\alpha \in S_t$. Or else, $-\gamma \in R_t^+$ which implies that $\beta = \alpha + (-\gamma)$ is decomposable, contradicting that $\beta \in S_t$. ∎

Lemma 16. *Let $t \in V^*$ and $A \subset V$ be such that:*

1. $\langle t, \alpha \rangle > 0$ for all $\alpha \in A$.

2. $(\beta, \alpha) \leq 0$ for all $\alpha, \beta \in A$.

Then the elements of A are linearly independent.

Proof.

- Suppose that the elements of A are not linearly independent, then there exists two disjoint subsets $B, C \subset A$ and two sets of non-negative coefficients $\{y_\beta\}_{\beta \in B}, \{z_\gamma\}_{\gamma \in C}$, which are not both identically zero, such that
$$\sum_{\beta \in B} y_\beta \beta = \sum_{\gamma \in C} z_\gamma \gamma$$
- Let $\lambda \in V$ be the element $\lambda = \sum_{\beta \in B} y_\beta \beta$. This implies that $(\lambda, \lambda) = \sum_{\beta \in B, \gamma \in C} y_\beta z_\gamma (\beta, \gamma)$ where $\forall \beta, \gamma \implies y_\beta z_\gamma \geq 0$. Therefore by (2) in the conditions that $(\lambda, \lambda) \leq 0$ and thus can deduce that $\lambda = 0$.

- Above conclusion implies that
$$0 = \langle t, \lambda \rangle = \sum_{\beta \in B} y_\beta \langle t, \beta \rangle$$
By (1) from the conditions it follows that $\forall \beta \in B \implies y_\beta = 0$. Similar conclusion follows for all $z_\gamma = 0$ where $\gamma \in C$.

- Since $\{y_\beta\}_{\beta \in B}, \{z_\gamma\}_{\gamma \in C} = \{0\}$ then contradiction follows and therefore all elements of A are linearly independent.

■

Proof. (of Theorem 7.1):

1. - From Lemma 7.2 follows that each element of R_t^+ is a linear combination, with non-negative integer coefficients, of elements of S_t. Same conclusion but with non-positive integer coefficients follows for elements of $-R_t^+$. Therefore each $\beta \in R$ can be written as a linear combination
$$\beta = \sum_{\alpha \in S} m_\alpha \alpha$$
where m_α are integers with the same sign (i.e. all $m_\alpha \geq 0$ or all $m_\alpha \leq 0$).
   - By Lemmas 7.3 and 7.4 follows that S_t is linearly independent therefore S_t is a basis of vector space V, which proves that S_t is a base of R.
2. - Conversely, let S be the base of R and let $t \in V^*$ such that $\langle t, \alpha \rangle > 0$ for all $\alpha \in S$, which implies that $\forall \beta \in R \implies \langle t, \beta \rangle \neq 0$.
   - Let R^+ denote the set of linear combinations with non-negative integer coefficients of elements of S, which implies that $R^+ \subset R_t^+$ and $(-R^+) \subset (-R_t^+)$. Since $R = -R^+ \cup R^+$ then follows that $R^+ = R_t^+$.
   - It follows that $S \subset R_t^+$ and all elements of S are indecomposable which implies that $S \subset S_t$.
   - Since $\dim(V) = |S_t| = |S|$ then follows that $S_t = S$.

■

Let $\dim(V) = 2$ and let $\{\alpha, \beta\}$ be a base of R. Then by Lemma 7.3 $n(\alpha, \beta) \leq 0$ so the relative angle between α, β can be only as in cases (1), (3), (5), (7) which correspond to $\phi = \frac{\pi}{2}, \frac{2\pi}{3}, \frac{3\pi}{4}, \frac{5\pi}{6}$ respectively. Possible root systems are of types $A_1 \times A_1, A_2, B_2, G_2$ which correspond to $\phi = \frac{\pi}{2}, \frac{2\pi}{3}, \frac{3\pi}{4}, \frac{5\pi}{6}$ respectively.

8.8. Some Properties of Bases

In the following section we admit the following notation:

- S denotes a base for the root system R.
- By R^+ we denote the set of roots which are linear combinations with non-negative integer coefficients, of elements of S.
- An element of R^+ is called a positive root.

Proposition 27. *Every positive root β can be written as*
$$\beta = \alpha_1 + ... + \alpha_k, \quad \text{with } \alpha_i \in S$$
in such a way that the partial sums
$$\alpha_1 + ... + \alpha_h, \quad 1 \leq h \leq k$$
are all roots.

Proof. - To prove the result, we first need to observe that for any positive root β there is $\alpha \in S$, such that $(\alpha, \beta) > 0$. Suppose otherwise, then by Lemma 7.4 it follows that β is linearly independent from all elements of S which is false.

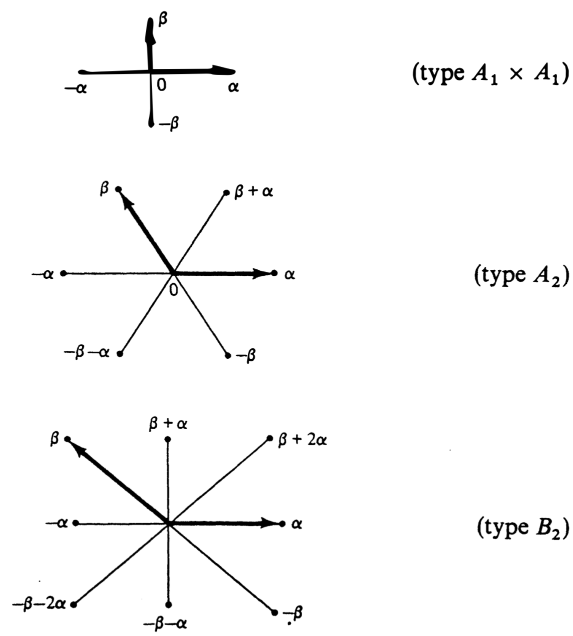

Figure 8.1: Types of roots: example 1

- Let $t \in V^*$ satisfy $\langle t, \alpha \rangle = 1$ for all $\alpha \in S$. Since β is a positive root then follows that $\langle t, \beta \rangle$ is a non-negative integer. We want to prove the result by induction on $k = \langle t, \beta \rangle$.
- By the first observation $\exists \alpha \in S$ such that $(\alpha, \beta) > 0$. If α and β are proportional then $\beta = \alpha$ or $\beta = 2\alpha$, and therefore the result is true.
- Suppose that α and β are not proportional, then Prop. 6.1 shows that $\gamma = \beta - \alpha$ is a root. If $\gamma \in -R^+$ then $\alpha = \beta + (-\gamma)$ would be decomposable, which gives contradiction. Hence $\gamma \in R^+$ and $\langle t, \gamma \rangle = \langle t, \beta \rangle - \langle t, \alpha \rangle = k - 1$. Applying the induction hypothesis to γ we obtain the result by labeling α as α_k.

■

Proposition 28. *Suppose that R is reduced, and let $\alpha \in S$. The symmetry s_α associated with α leaves $R^+ - \{\alpha\}$ invariant.*

Proof. • Let $\beta \in R^+ - \alpha$. Then follows that $\beta = \sum_{\gamma \in S} m_\gamma \gamma$, *where* $m_\gamma \geq 0$.

- Since R is reduced and $\beta \neq \alpha$ then β is not proportional to α, and $\exists \gamma' \neq \alpha$ such that $m_{\gamma'} \neq 0$.
- By Lemma 7.3 for any $\gamma, \alpha \in S \implies n(\gamma, \alpha) \leq 0$ and therefore $n(\beta, \alpha) = n'(\beta, \alpha) + m_\alpha n(\alpha, \alpha)$ where $n'(\beta, \alpha) = \sum_{\gamma \in S - \alpha} m_\gamma n(\gamma, \alpha) < 0$ and $n(\alpha, \alpha) = 2$.
- Since $\beta \in R^+ - \alpha$ then the coefficient of γ' in

$$s_\alpha(\beta) = \beta - n(\beta, \alpha)\alpha = \sum_{\gamma \in S - \alpha} m_\gamma \gamma + m_\alpha \alpha - n'(\beta, \alpha)\alpha - 2m_\alpha \alpha = \sum_{\gamma \in S - \alpha} m_\gamma \gamma - (n'(\beta, \alpha) + m_\alpha)\alpha$$

is $m_{\gamma'} > 0$. For any $\gamma \in S$ where $\gamma \neq \alpha$, this shows that $m_\gamma > 0$ or $m_\gamma = 0$. Since s_α is a symmetry and S is a base then all coefficients of $s_\alpha(\beta)$ are non negative and therefore $n'(\beta, \alpha) + m_\alpha < 0$. Therefore follows that $s_\alpha(\beta) \in R^+ - \{\alpha\}$, proving the proposition.

■

Corollary 10. *Let ρ be the sum of all positive roots multiplied by a factor of one-half. Then we have*

$$s_\alpha(\rho) = \rho - \alpha$$

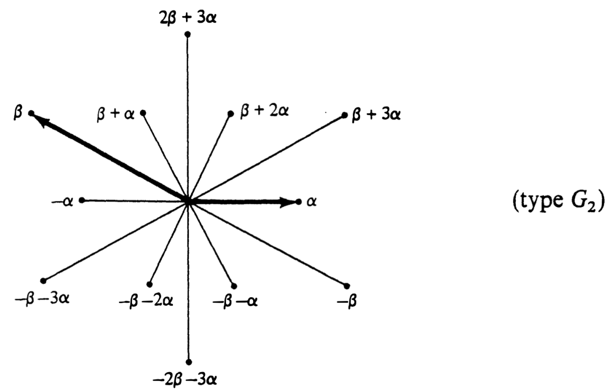

Figure 8.2: Types of roots: example 2

for all $\alpha \in S$.

Proof. Let ρ_α be the sum of the elements of $R^+ - \alpha$ multiplied by a factor of one-half. By Prop. 8.2 s_α acts on $R^+ - \{\alpha\}$ as a permutation and therefore $s_\alpha(\rho_\alpha) = \rho_\alpha$.

On the other hand, $\rho = \rho_\alpha + \alpha/2$ and since $s_\alpha(\alpha) = -\alpha$ then $s_\alpha(\rho) = \rho - \alpha$. ∎

Proposition 29. *Suppose that R is reduced. The set S^* of inverse roots of the elements of S is a base for R^*.*

Proof. Consider an isomorphism $\Psi : V \to V^*$ defined as $\Psi : \alpha \to (\alpha, \cdot)$. Let $R' = \{\frac{2\alpha}{(\alpha,\alpha)} | \alpha \in R\}$ and define $S' = \{\frac{2\alpha}{(\alpha,\alpha)} | \alpha \in S\}$. Since R is a root system then it follows that R' is a root system as well. To prove the result it is enough to show that S' forms a base for R', because $S^* = \Psi(S')$.

1. The set S' is linearly independent and spans the vector space V because so does the base S.
2. Since any root in R can be expressed as a linear combination of elements of $\alpha \in S$ with non-negative or non-positive integer coefficients, then the same relationship holds between roots of R' and S' which implies that S' is a base.

∎

8.9. Relations with the Weyl Group

We assume that R is reduced.

Theorem 32. *Let W be the Weyl group of R.*

1. *For each $t \in V^*$, there exists $w \in W$ such that $\langle w(t), \alpha \rangle \geq 0$ for all $\alpha \in S$.*
2. *If S' is a base for R, there exists $w \in W$ such that $w(S') = S$.*
3. *For each $\beta \in R$, there exists $w \in W$ such that $w(\beta) \in S$.*
4. *The group W is generated by symmetries $s_\alpha, \alpha \in S$.*

Remark 33. *In the above, the short hand notation is used $\langle w(t), x \rangle = \langle \Phi(w)(t), x \rangle$ where $\Phi : W \to W^*$ is an isomorphism discussed earlier.*

Proof.

1. Let W_S be a subgroup of W generated by the symmetries $s_\alpha, \alpha \in S$. Let $t \in V^*$, ρ be the sum of the positive roots multiplied by one half and choose $w \in W_S$ so that $\langle w(t), \rho \rangle$ is maximal i.e. $\forall w' \in W_S \implies$

$\langle w(t), \rho\rangle \geq \langle w'(t), \rho\rangle$. In particular this implies that $\forall \alpha \in S \implies \langle w(t), \rho\rangle \geq \langle s_\alpha w(t), \rho\rangle$. At the same time, from the isomorphism Φ it follows that

$$\langle s_\alpha w(t), \rho\rangle = \langle w(t), s_\alpha(\rho)\rangle = \langle w(t), \rho - \alpha\rangle$$

So the inequality implies that $\langle w(t), \alpha\rangle \geq 0$.

2. Let S' be a base of R and choose $t' \in V^*$ such that $\langle t', \alpha'\rangle > 0$ for all $\alpha' \in S'$, which implies that none of the roots are in the kernel of t'. By (1) there exists $w \in W_S$ s.t. $\langle w(t'), \alpha\rangle \geq 0$ for all $\alpha \in S$. Let $t = w(t')$ then the previous condition is equivalent to

$$\langle t, \alpha\rangle = \langle t', w^{-1}(\alpha)\rangle \geq 0$$

The fact that no root is in the kernel of t' implies that for a root $w^{-1}(\alpha) \implies \langle t, \alpha\rangle = \langle t', w^{-1}(\alpha)\rangle > 0$ for any $\alpha \in S$. Theorem 7.1 implies that

$$S_t = S \;\; and \;\; S_{t'} = S'$$

Moreover $w^{-1}(S_t) = \{w^{-1}(\alpha) : \alpha \in S_t \; and \; \langle t, \alpha\rangle = \langle t', w^{-1}(\alpha)\rangle > 0\}$ is a base for R and therefore by Theorem 7.1 follows that $w^{-1}(S_t) = S_{t'}$.

3. Let $\beta \in R$. To show the result, it is sufficient to find $t \in V^*$ such that $\beta \in S_t$ where S_t is a base in R. Then by (2) there exists $w \in W$ such that $w(S_t) = S$ which implies that $w(\beta) \in S$.

   - Let L be a hyperplane of V^* orthogonal to β, i.e. $\forall t \in L \implies \langle t, \beta\rangle = 0$.
   - Consider all hyperplanes of V^* associated with the roots other than $\pm\beta$. It follows that there are finitely many of them and they are all distinct from L. Hence $\exists\ t_0 \in L$ not contained in any of these hyperplanes.
   - Above observation implies that:

$$\langle t_0, \beta\rangle = 0 \;\; and \;\; \langle t_0, \gamma\rangle \neq 0 \;\; for \;\; \gamma \in R, \gamma \neq \pm\beta.$$

   - Sufficiently close to t_0 there exists $t \in V^*$ such that $0 < \langle t, \beta\rangle < \min\{|\langle t, \gamma\rangle| : \forall \gamma \neq \pm\beta\}$. This implies that $\beta \in R_t^+$ and by minimality it is indecomposable, and thus $\beta \in S_t$.

4. Let W_S be a subgroup of W generated by the symmetries $s_\alpha, \alpha \in S$. Want to show that $W_S = W$. Since W is generated by symmetries s_β, with $\beta \in R$, it is sufficient to show that $s_\beta \in W_S$. By (3), there exists $w \in W_S$ such that $\alpha = w(\beta)$ belongs to S. We have

$$s_\alpha = s_{w(\beta)} = w \cdot s_\beta \cdot w^{-1}$$

so that $s_\beta = w^{-1} \cdot s_\alpha \cdot w$ which shows that $s_\beta \in W_S$.

$\blacksquare$

Remark 34.

- *The element w given in (2) is unique.*
- *The set of elements $t \in V^*$ such that $\langle t, \alpha\rangle > 0$ for all $\alpha \in S$ is called the Weyl chamber associated with S. By (1) and (2), the Weyl chambers are the connected components of the complement in V^* of the hyperplanes orthogonal to the roots; the group W permutes them transitively.*

8.10. The Cartan Matrix

Definition 40. *The Cartan matrix of R (with respect to the chosen base S) is the matrix $(n(\alpha, \beta))_{\alpha,\beta\in S}$*

We recall that by definition 2.1 $n(\alpha, \beta) \in \mathbb{Z}$. From section 6 it follows that $n(\alpha, \alpha) = 2$, and if $\alpha \neq \beta$ then from Lemma 7.3 for $\alpha, \beta \in S \implies n(\alpha, \beta) \leq 0$. In that case, from section 6 we have that one of these four options is true: $n(\alpha, \beta) = 0, -1, -2, -3$.

8.10.1. Example

For the root system of G_2 type, it follows that for the base $S = \{\alpha, \beta\}$ it holds that the angle between its elements is $\phi = \frac{5\pi}{6}$ and

$$n(\beta, \alpha) = -1, \ n(\alpha, \beta) = -3, \quad |\beta| = \sqrt{3}|\alpha|$$

Therefore the Cartan Matrix of G_2 is $\begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$

Proposition 30. *A reduced root system is determined, up to isomorphism, by its Cartan matrix.*

More precisely:

Proposition 31. *Let R' be a reduced root system in a vector space V', let S' be a base for R', and let $\phi : S \to S'$ be a bijection such that $n(\phi(\alpha), \phi(\beta)) = n(\alpha, \beta)$ for all $\alpha, \beta \in S$. If R is reduced, then there is unique isomorphism $f : V \to V'$ which is an extension of ϕ and maps R onto R'.*

Proof.

1. To define f, we extend ϕ by linearity from S to V.
2. If $\alpha, \beta \in S$, we have

$$s_{\phi(\alpha)} \circ f(\beta) = s_{\phi(\alpha)}(\phi(\beta)) = \phi(\beta) - n(\phi(\alpha), n(\phi(\beta))\phi(\alpha)$$

   and

$$f \circ s_\alpha(\beta) = f(\beta - n(\beta, \alpha)\alpha) = \phi(\beta) - n(\beta, \alpha)\phi(\alpha)$$

   such that the condition on ϕ implies that $s_{\phi(\alpha)} \circ f = f \circ s_\alpha$ for all $\alpha \in S$.

3. If W denotes the Weyl group of R (resp. R'), then previous shows that $W' = f \circ W \circ f^{-1}$. Since $R = W(S)$ and $R' = W'(S')$ then from all the previous results we can deduce that $f(R) = R'$, as required.

■

In particular, let E be the group of permutations of S which leave the Cartan matrix invariant. By the above argument (where in the Prop. 10.2 $V' = V$), E can be identified with the group of automorphisms of R which leave the base S invariant.

Proposition 32. *The group $Aut(R)$ is the semidirect product of E and W.*

Proof. If $w \in W \cap E$, we have $w(S) = S$, so that w=1 by a result which will be proved later. Moreover, if $u \in Aut(R)$, $u(S)$ is a base for R, hence there exists $w \in W$ such that $w(u(S)) = S$. We therefore have $wu \in E$, showing that $Aut(R) = W \ltimes E$ ■

Corollary 11. *The group $Aut(R)/W$ is isomorphic to E.*

8.11. Coxeter Graph

Definition 41. *A Coxeter graph is a finite graph, each pair of distinct vertices being joined by 0, 1 , 2, or 3 edges. If R is the root system and S is its base then the Coxeter Graph of R with respect to S is defined as follows:*

The vertices of the graph are elements of S. For two distinct $\alpha, \beta \in S$ the number of edges between them is equal to $n(\alpha, \beta) \cdot n(\beta, \alpha) \in \{0, 1, 2, 3\}$.

Theorem 9.1 shows that the graphs associated with different bases of R are isomorphic.

8.11.1. Example

The Coxeter graphs from the section 7 are the following:

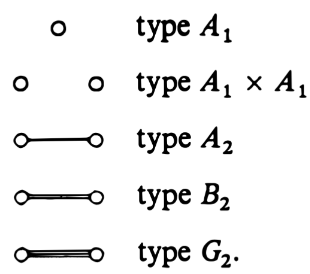

Figure 8.3: Coxeter graphs from the section 7

8.12. Irreducible Root Systems

Let R be a root system in V.

Proposition 33. *Suppose that V is the direct sum of two subspaces V_1 and V_2, and that R is contained in $V_1 \cup V_2$. Let $R_i = V_i \cap R,\ i = 1, 2$. Then*

1. *V_1 and V_2 are orthogonal.*
2. *R_i is a root system in V_i.*

Proof.

1. If $\alpha \in R_1$ and $\beta \in R_2$, then $\alpha \pm \beta$ are not contained in $V_1 \cup V_2$, therefore they are not roots. Proposition 6.1 implies that $(\alpha, \pm\beta) \leq 0$ which shows that $(\alpha, \beta) = 0$. Since R_i spans V_i then the first claim follows.
2. To show the second claim consider the symmetry s_α associated with an element α of R_1. It follows that s_α leaves invariant the hyperplane orthogonal to α. Since $V = V_1 \oplus V_2$ and $R \subset V_1 \cup V_2$ then s_α leaves invariant V_2 and the orthogonal complement of α in V_1. Thus s_α is a symmetry with vector α for R_1, and same holds for elements of R_2. Since R is a root system then all necessary properties satisfied for R_1, R_2 so that the second claim follows.

■

Definition 42. *If a root system R can not be a disjoint union of root systems then R is irreducible.*

An obvious result follows:

Proposition 34. *Every root system is a sum of irreducible systems.*

One can show that such a decomposition is unique. Now we want to connect the concepts of Coxeter graph and irreducibility of the root system.

Proposition 35. *For R to be irreducible, it is necessary and sufficient that its Coxeter graph should be connected and nonempty.*

Proof.

1. ($\Longrightarrow$) Will show that if R is reducible then its Coxeter graph is disconnected.
   - Suppose that R is the sum of two non-trivial subsystems R_1 and R_2 with corresponding bases S_1 and S_2. Let $S = S_1 \cup S_2$.
   - If $\alpha \in S_1$ and $\beta \in S_2$, then by Proposition 12.1 they are orthogonal and therefore not joined by the edge in the Coxeter Graph of S. From this we can conclude that the Coxeter Graph of S is disconnected.
2. ($\Longleftarrow$) Suppose that the Coxeter Graph of S is disconnected. Then S has a non-trivial partition

$$S = S_1 \cup S_2$$

such that every element of S_1 is orthogonal to every element of S_2, then the vector subspaces V_1 and V_2 spanned by S_1 and S_2 are orthogonal, and therefore are invariant under symmetries $s_\alpha, \alpha \in S$. Hence R is contained in $V_1 \cup V_2$ and $V = V_1 \oplus V_2$, therefore it is reducible.

■

8.13. Classification of Connected Coxeter Graphs

Theorem 33. *Every connected nonempty Coxeter graph which is attached to a root system is isomorphic to one of the following:*

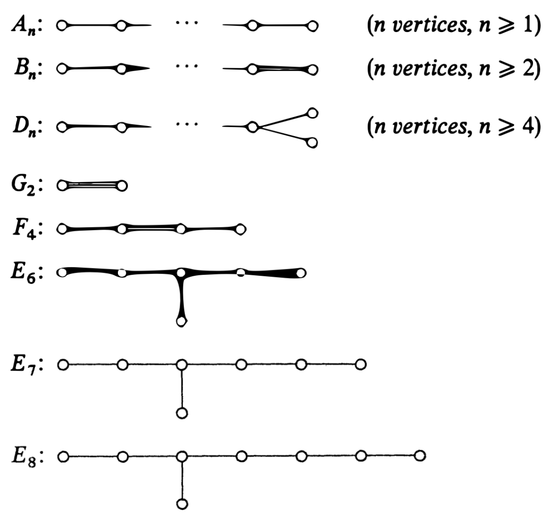

Figure 8.4: Types of nonempty Coxeter graphs

We will give only an overview of the proof strategy.

- Consider a nonempty connected Coxeter graph G, with vertex-set S.
- Associates with G a symmetric bilinear form $(\cdot,\cdot)$ on the space $\mathbb{R}^s$, with the basis $(e_\alpha)_{\alpha\in S}$, by defining

$$(e_\alpha, e_\alpha) = 1$$

  and $(e_\alpha, e_\beta) = \cos\left(\frac{\pi}{2}\right), \cos\left(\frac{2\pi}{3}\right), \cos\left(\frac{3\pi}{4}\right), \cos\left(\frac{5\pi}{6}\right)$ as α and β are joined by 0,1,2, or 3 edges.
- For G to be the Coxeter graph of a root system, it is necessary that this form should be positive definite. One then shows, by a series of ingenious reductions, that this positivity condition is sufficient to force an isomorphism between G and $A_n, B_n, ...$ or E_8.

8.14. Dynkin Diagrams

In this section we restrict our attention to root systems which are both reduced and irreducible.

The Coxeter graph is not sufficient to determine the Cartan matrix and hence the root system. This is because for given two roots $\alpha, \beta \in S$ the Coxeter graph only gives information about the relative angle between them and no information on which is the longer.

Definition 43. *Given the root system R and any of its base S, the Dynkin diagram of R is the Coxeter Graph of S where each vertex α is labeled with (α,α). Two Dynkin diagrams are identical if proportions of connected vertices are the same.*

Proposition 36. *Specifying a Dynkin diagram is equivalent to specifying a Cartan matrix. They determine the root system up to isomorphism.*

With the given Dynkin diagram we determine the Coxeter Matrix in the following way:

- If $\alpha = \beta$ then $n(\alpha, \beta) = 2$
- If $\alpha \neq \beta$ and α, β are not joined by an edge then $n(\alpha, \beta) = 0$.
- If $\alpha \neq \beta$, $|\alpha| \leq |\beta|$ and α, β are joined by one edge then from the list in section 6 (Relative Position of Two Roots) we have that $n(\alpha, \beta) = -1$.
- If $\alpha \neq \beta$, $|\alpha| \geq |\beta|$ and α, β are joined by i edges ($1 \leq i \leq 3$) then from the list in section 6 (Relative Position of Two Roots) we have that $n(\alpha, \beta) = -i$.

Theorem 34. *Each nonempty connected Dynkin diagram is isomorphic to one of the following:*

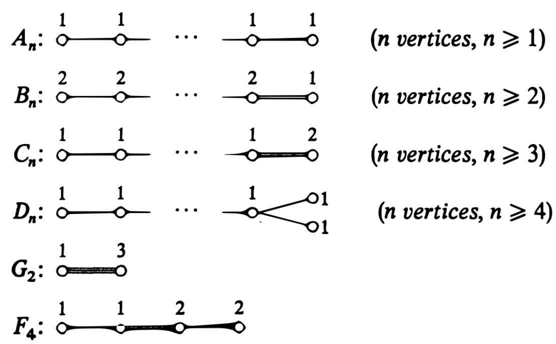

Figure 8.5: The nonempty connected Dynkin diagram from the theorem part 1

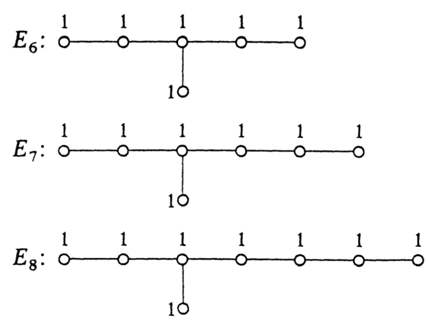

Figure 8.6: The nonempty connected Dynkin diagram from the theorem part 2

Theorem follows from the result on Classification of Connected Coxeter Graphs i.e. by Theorem 13.1.

Construction of Irreducible Root Systems

Let $e_1, ..., e_n$ be the canonical basis of $\mathbb{R}^n$, $(\cdot, \cdot)$ the standard scalar product in $\mathbb{R}^n$ for which $(e_i, e_j) = \delta_{ij}$, and Γ_n the lattice in $\mathbb{R}^n$ generated by the vectors $e_1, ..., e_n$.

Construction of A_n root system ($n \geq 1$)

Let $V = \{x \in \mathbb{R}^{n+1} | \langle x, \sum_{i=1}^{n+1} e_i \rangle = 0\}$ and define $A_n = \{\alpha \in V \cap \Gamma_n | (\alpha, \alpha) = 2\} = \{e_i - e_j | i \neq j,\ i, j = 1, ..., n+1\}$. It is easy to see that A_n makes a root system. For a base S we can take the set of all vectors $e_i - e_{i+1}$ for $1 \geq i \geq n$. The Weyl group can identified with the group of permutations of $e_1, ..., e_n$.

Construction of B_n root system $(n \geq 1)$

Define $B_n = \{\alpha \in \Gamma_n | (\alpha, \alpha) = 1 \ or \ 2\} = \{e_i : i = 1, ..., n\} \cup \ \{\pm e_i \pm e_j | i \neq j\}$. For the base of this root system we consider $S = \{e_i - e_{i+1} : 1 \leq i \leq n\} \cup \{e_n\}$. Its Weyl group is the set of all permutations and sign changes of the vectors e_i.

Construction of C_n root system $(n \geq 1)$

Define $C_n = \{\pm 2e_i : i = 1, ..., n\} \cup \ \{\pm e_i \pm e_j | i \neq j\}$. For the base of this root system we consider $S = \{e_i - e_{i+1} : 1 \leq i \leq n\} \cup \{2e_n\}$. Its Weyl group is the set of all permutations and sign changes of the vectors e_i.

Construction of D_n root system $(n \geq 2)$

Define $D_n = \{\alpha \in \Gamma_n : (\alpha, \alpha) = 2\} = \{\pm e_i \pm e_j | i \neq j, \ i, j = 1, ..., n\}$. It is a root system with base

$$S = \{e_i - e_{i+1} | 1 \leq i \leq n-1\} \cup \{e_{n-1} + e_n\}$$

Its Weyl Group is the set of permutations and sign changes of (an even number of) the vectors e_i.

G_2 **root system**

This root system was demonstrated previously, and it can be described as the set of algebraic integers of the cyclotomic field generated by a cubic root of unity with norm 1 or 3.

Construction of F_4 root system

In $\mathbb{R}^4$ we define the F_4 root system as

$$F_4 = \{\pm e_i | i \in [n]\} \cup \{\pm e_i \pm e_j | i \neq j\} \cup \frac{1}{2}(\pm e_1 \pm e_2 \pm e_3 \pm e_4)$$

The base of this root system can be taken as $S = \{e_4, e_2 - e_3, e_3 - e_4, \frac{1}{2}(e_1 - e_2 - e_3 - e_4)\}$.

Construction of E_8 root system

In $\mathbb{R}^8$ we define the E_8 root system as

$$E_8 = \{\pm e_i \pm e_j, \frac{1}{2}\sum_{i=1}^{8}(-1)^{m(i)} e_i : i \neq j, \ i, j = 1, ..., 8 \ with \ \sum_{i=1}^{8} m(i) \equiv 0 \mod 2\}$$

E_6 **and** E_7 **root systems**

A root system of type E_6 (E_7 resp.) can be obtained as the intersection of a root system of type E_8 with the subspace of $\mathbb{R}^8$ spanned by the first six (seven resp.) elements of the base $e_1, ..., e_6$ (resp. $e_1, ..., e_7$).

Nonreduced Root Systems

For each $n \geq 1$ there is exactly one nonreduced irreducible root system: it is the system BC_n obtained as the union of the systems B_n and C_n mentioned above.

8.15. Complex Root Systems

Let V be a finite-dimensional complex vector space, then all the previously stated results and definitions stay the same, in particular for all $\alpha^* \in R^*$ and $\beta \in R$ it holds that $\langle \alpha^*, \beta \rangle \equiv 0 \mod 1$. Below we want to establish the connection between real and complex root systems.

Let R be a root system in a real vector space V_0, and let V be the complexification $V_0 \otimes \mathbb{C}$ of V_0. The space V_0 is imbedded in V. By extending the symmetries s_α^0 of V_0 by linearity to V, we see that R is a root system in V.

Theorem 35. *Every complex root system can be obtained in the above way.*

More precisely:

Theorem 36. *Let R be a root system in a complex vector space V. Let V_0 be the $\mathbb{R}$-subspace of V spanned by R. Then:*

1. R is a root system in V_0.

2. The canonical mapping $i : V_0 \otimes \mathbb{C} \to V$ is an isomorphism.

3. If $\alpha \in R$, the symmetry s_α of V is the linear extension of the symmetry s_α^0 of V_0.

Proof.

1. Want to show that R is a root system in V_0.

   - First we note that R spans V_0.
   - To define the Weyl group of R in V_0, we observe that for any $\alpha \in R$, the symmetry s_α leaves invariant R, and hence V_0. Thus the Weyl group of R in V_0 can be defined as the set of s_α^0 which is a restriction of s_α to V_0.
   - By definition of R complex root system, if $\beta \in R$ then $s_\alpha^0(\beta) = \beta - \alpha^*(\beta)\alpha$ with $\alpha^*(\beta) \in \mathbb{Z}$. Thus all of the above shows that R is a root system in V_0.

2. Want to show that the canonical mapping $i : V_0 \otimes \mathbb{C} \to V$ is an isomorphism.

   - First note that for any inverse root α_0^* of α in V_0^* is the image of $\alpha^* \in V^*$ under the transpose of canonical map $i^t : V^* \to V_0^* \otimes \mathbb{C}$ for each $\alpha \in R$.
   - Because R spans V, the homomorphism $i : V_0 \otimes \mathbb{C} \to V$ is surjective.
   - Since the elements α_0^* span V_0^* then follows that i^t is surjective. Together with the previous this implies that i is bijective homomorphism i.e. an isomorphism.

3. The last part follows from the above parts.

■

Chapter 9

Structure of Semisimple Lie Algebras

Throughout this chapter, $\mathfrak{g}$ denotes a complex semisimple Lie algebra, and $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$. From lecture 3 notes (from Theorem 4.1) we know the following facts about Cartan subalgebra in a complex semisimple Lie algebra:

- The restriction of Cartan-Killing form of $\mathfrak{g}$ to $\mathfrak{h}$ is nondegenerate.
- $\mathfrak{h}$ is abelian
- The centralizer of $\mathfrak{h}$ is $\mathfrak{h}$.
- Every element of $\mathfrak{h}$ is semisimple i.e. $\forall x \in \mathfrak{h} \implies ad(x)$ is diagonalizable.

Definition 44. *Given a non-degenerate inner product $(\cdot,\cdot) : V \times V \to \mathbb{F}$, for the vector subspaces U and W of V to be dual to each other with respect to this inner product means that $\forall x \in U\ \exists y \in W \implies (x, y) \neq 0$. Non-degenerate means that $\forall x \neq 0 \in V\ \exists y \in V \implies (x, y) \neq 0$.*

9.1. Decomposition of $\mathfrak{g}$

First we will refresh the definitions stated in the lecture 3 notes.

- If $\alpha \in \mathfrak{h}^*$, denote $\mathfrak{g}^\alpha = \cap_{h\in\mathfrak{h}} ker\{ad(h) - \alpha(h)\cdot id_{\mathfrak{g}}\} = \{x \in \mathfrak{g} : \forall h \in \mathfrak{h} \implies ad(h)\cdot x = \alpha(h)x\}$. An element of $\mathfrak{g}^\alpha$ is said to have weight α. Since Cartan subalgebra is a self-centralizer in semi-simple Lie algebra (Theorem 4.1 Lecture 3), then $\mathfrak{g}^0 = \mathfrak{h}$.
- We will say that any element $\alpha \in \mathfrak{h}^*$ such that $\alpha \neq 0$ and $\mathfrak{g}^\alpha \neq 0$ is called a root of $\mathfrak{g}$. The set of roots is denoted by R.

Theorem 37. *One has $\mathfrak{g} = \mathfrak{h} \oplus \sum_{\alpha\in R} \mathfrak{g}^\alpha$.*

We have already proved this result in the lecture 3 notes but here will give a simpler proof. The direct sum in this result is defined as the direct sum of subspaces, and not Lie algebras.

Proof. By Theorem 4.1 from Lecture 3, every element of $\mathfrak{h}$ is semisimple i.e. for $x \in \mathfrak{h}$ the endomorphism $ad(x)$ of $\mathfrak{g}$ is diagonalizable. By the same Theorem it also follows that $\mathfrak{h}$ is an abelian subalgebra of $\mathfrak{g}$. Therefore, by the Jacobi identity, for any $x, y \in \mathfrak{h}$ and $z \in \mathfrak{g}$, it implies that $ad(x)ad(y)z = ad(y)ad(x)z$. Since all endomorphisms $ad(x)$ for $x \in \mathfrak{h}$ commute with each other then they are simultaneously diagonalizable. Thus $\mathfrak{g}$ is a direct sum of eigenspaces as required, where each on each eigenspace a root can be defined. (If $\hat{e}$ is an eigenvector of all $ad(h) \in ad(\mathfrak{h})$ then can define $\alpha(h) = \lambda$ where $ad(h)\hat{e} = \lambda\hat{e}$. This functional will be linear and belong to $\mathfrak{h}^*$). ■

In what follows we use all the definitions from Lecture 4 notes. The subspaces $\mathfrak{g}^\alpha$ have the following properties:

Theorem 38.

1. R is a root system in $\mathfrak{h}^*$, and this system is reduced.

2. *Let $\alpha \in R$. Then the subspaces $\mathfrak{g}^\alpha$ and $\mathfrak{h}_\alpha = [\mathfrak{g}^\alpha, \mathfrak{g}^{-\alpha}]$ are one dimensional. There is a unique element $H_\alpha \in \mathfrak{h}_\alpha$ such that $\alpha(H_\alpha) = 2$; it is the inverse root of α.*
3. *Let $\alpha \in R$. For each nonzero element X_α of $\mathfrak{g}^\alpha$, there is a unique element Y_α of $\mathfrak{g}^{-\alpha}$ such that $[X_\alpha, Y_\alpha] = H_\alpha$. One has $[H_\alpha, X_\alpha] = 2X_\alpha$ and $[H_\alpha, Y_\alpha] = -2Y_\alpha$. The subalgebra $\mathfrak{s}_\alpha = \mathfrak{h}_\alpha \oplus \mathfrak{g}^\alpha \oplus \mathfrak{g}^{-\alpha}$ is isomorphic to $\mathfrak{sl}_2$.*
4. *If $\alpha, \beta \in R$ and $\alpha + \beta \neq 0$, then*
$$[\mathfrak{g}^\alpha, \mathfrak{g}^\beta] = \mathfrak{g}^{\alpha+\beta}$$

The proof will be given in Sec.3. In the following theorem, we let $(\cdot, \cdot)$ denote an invariant nondegenerate symmetric bilinear form on $\mathfrak{g}$ (for example, the Killing form).

Theorem 39.

1. *The subspaces $\mathfrak{g}^\alpha$ and $\mathfrak{g}^\beta$ are orthogonal if $\alpha + \beta \neq 0$. The subspaces $\mathfrak{g}^\alpha$ and $\mathfrak{g}^{-\alpha}$ are dual with respect to $(\cdot, \cdot)$. The restriction of $(\cdot, \cdot)$ to $\mathfrak{h}$ is non degenerate. Also, if $\alpha \in \mathfrak{h}^*$ then so $-\alpha \in \mathfrak{h}^*$.*
2. *If $x \in \mathfrak{g}^\alpha$, $y \in \mathfrak{g}^{-\alpha}$, and $H \in \mathfrak{h}$, then*
$$(H, [x, y]) = \alpha(H) \cdot (x, y)$$
3. *Let $\alpha \in R$, and let h_α be the element of $\mathfrak{h}$ corresponding to α under the isomorphism $\mathfrak{h} \to \mathfrak{h}^*$ associated with the chosen bilinear form. Then*
$$[x, y] = (x, y) h_\alpha \quad \forall x \in \mathfrak{g}^\alpha, \forall y \in \mathfrak{g}^{-\alpha}.$$

Proof.

1. If $x \in \mathfrak{g}^\alpha$, $y \in \mathfrak{g}^\beta$, and $H \in \mathfrak{h}$, by invariance property of $(\cdot, \cdot)$ we have
$$([H, x], y) + (x, [H, y]) = (ad(H) \cdot x, y) + (x, ad(H) \cdot y) = \alpha(H)(x, y) + \beta(H)(x, y) = 0$$
If $\alpha + \beta \neq 0$, we can choose H so that $\alpha(H) + \beta(H) \neq 0$, from which we conclude that $(x, y) = 0$, i.e. $\mathfrak{g}^\alpha$ and $\mathfrak{g}^\beta$ are orthogonal. This shows that
$$\mathfrak{g} = \mathfrak{h} \oplus \sum (\mathfrak{g}^\alpha + \mathfrak{g}^{-\alpha})$$
is a decomposition of $\mathfrak{g}$ into mutually orthogonal subspaces. Since $(\cdot, \cdot)$ is non-degenerate then its restriction to any subspace is also non-degenerate. Therefore the first statement follows.
To show that $\mathfrak{g}^\alpha$ and $\mathfrak{g}^{-\alpha}$ are dual, it is sufficient to show that $x \in \mathfrak{g}^\alpha$, $y \in \mathfrak{g}^{-\alpha} \implies [x, y] \neq 0$. Suppose otherwise. First observe that for any $x, y \in \mathfrak{g}^\alpha$ and any $h \in \mathfrak{h}$, by invariance of $(\cdot, \cdot)$ it holds that
$$([x, h], y) = -\alpha(h)(x, y) = (x, [h, y]) = \alpha(h)(x, y) \implies (x, y) = 0 \;\; \textit{and} \;\; (\mathfrak{g}^\alpha, \mathfrak{g}^\alpha) = 0$$
Therefore if $\forall x \in \mathfrak{g}^\alpha$, $\forall y \in \mathfrak{g}^{-\alpha} \implies [x, y] = 0$ then together with the above it follows that $(\mathfrak{g}^\alpha, \mathfrak{g}) = 0$ which contradicts non-degeneracy of $(\cdot, \cdot)$. This also implies that if $\alpha \in \mathfrak{h}^*$ then $-\alpha \in \mathfrak{h}^*$ because otherwise $(\mathfrak{g}^\alpha, \mathfrak{g}) = 0$.
2. The invariance of $(\cdot, \cdot)$ implies that
$$(H, [x, y]) = ([H, x], y) = \alpha(H) \cdot (x, y)$$
3. Let $\alpha \in R$, and let h_α be the element of $\mathfrak{h}$ corresponding to α under the isomorphism $\mathfrak{h} \to \mathfrak{h}^*$ associated with the chosen bilinear form. If $H \in \mathfrak{h}$, then by definition of h_α follows that $\alpha(H) = (H, h_\alpha)$. By (2) above, this implies that
$$(H, [x, y]) = \alpha(H) \cdot (x, y) = (H, (x, y) \cdot h_\alpha)$$
Since the restriction of $(\cdot, \cdot)$ to $\mathfrak{h}$ is non-degenerate, then we deduce from this that $[x, y] = (x, y) \cdot h_\alpha$.

∎

9.2. Proof of Theorem 2

This rests basically on Theorem 2.3 and the properties of the algebra $\mathfrak{sl}_2$ proved in the lecture notes on $\mathfrak{sl}_2$. One proceeds in stages:

Step 1: If $\alpha, \beta \in \mathfrak{h}^*$, then $[\mathfrak{g}^\alpha, \mathfrak{g}^\beta] \subset \mathfrak{g}^{\alpha+\beta}$.

This follows from the Jacobi identity

$$[H,[x,y]] = [[H,x]y] + [x,[H,y]] = (\alpha(H) + \beta(H))[x,y]$$

applied to $H \in \mathfrak{h}, x \in \mathfrak{g}^\alpha, y \in \mathfrak{g}^\beta$.

Step 2: R spans $\mathfrak{h}^*$.

Suppose otherwise. By isomorphism $\mathfrak{h} \to \mathfrak{h}^*$ for every $\alpha \in R$ *and* $\forall H' \in \mathfrak{h}\ \exists\ h_\alpha \in \mathfrak{h}$ so that $\alpha(H') = (H', h_\alpha)$. If there is an element $k \in \mathfrak{h}^*$ not in the span of R, then $\exists\ k_\alpha \in \mathfrak{h}$ corresponding to k that will not be in the span of $\{h_\alpha\}_{\alpha \in R}$. This would imply that there exists a non-zero $H \in \mathfrak{h}$ orthogonal to all $h_\alpha, \forall \alpha \in R$ i.e. $\alpha(H) = 0\ \forall \alpha \in R$. Since for all $\alpha \in R,\ x \in \mathfrak{g}^\alpha \implies ad(H) \cdot x = 0$ then by Theorem 2.1 follows that every $x \in \mathfrak{g}$ is in the kernel of $ad(H)$. Thus H is in the center of $\mathfrak{g}$, however this contradicts that $\mathfrak{g}$ is semisimple and its only solvable ideal is zero

Step 3: If $\alpha \in R$, the subspace $\mathfrak{h}_\alpha = [\mathfrak{g}^\alpha, \mathfrak{g}^{-\alpha}]$ of $\mathfrak{h}$ is 1-dimensional.

This result actually follows from Theorem 2.3 bullet (3), which states that $\mathfrak{h}_\alpha$ consists of the multiples of the element h_α.

Step 4: If $\alpha \in R$, then there is a unique element H_α of $\mathfrak{h}_\alpha$ such that $\alpha(H_\alpha) = 2$.

Given that $\mathfrak{h}_\alpha$ is a 1-dimensional subspace, by linearity of α it sufficient to show that $\alpha(\mathfrak{h}_\alpha) \neq 0$.

- Suppose otherwise. Let us choose $x \in \mathfrak{g}^\alpha$ and $y \in \mathfrak{g}^{-\alpha}$ such that $z = [x,y]$ is non-zero.
- Since we assume that $\alpha(z) = 0$, then by definition of $\mathfrak{g}^\alpha$ and $\mathfrak{g}^{-\alpha}$ it follows that

  $$ad(z) \cdot x = [z,x] = 0,\ ad(z) \cdot y = [z,y] = 0,\ [x,y] = z$$

  These formulae show that the subalgebra $\mathfrak{a}$ of $\mathfrak{g}$ generated by x, y, z is solvable (and even nilpotent).
- If $\rho : \mathfrak{a} \to End(V)$ is a finite-dimensional linear representation of $\mathfrak{a}$, then Lie's Theorem implies that there is flag $D = \{V_i | V_{i+1} \subset V_i,\ 1 \leq i \leq n\ s.t.\ V_0 = V\ and\ V_{n+1} = \{0\}\ and\ \dim V_i = i\}$ of V stable under $\rho(\mathfrak{a})$.
- If $\rho(\mathfrak{a})$ is a set of upper triangular matrices then $\rho([\mathfrak{a}, \mathfrak{a}])$ are strictly upper triangular i.e. nilpotent. Since $z \in [\mathfrak{a}, \mathfrak{a}]$ then $\rho(z)$ is a nilpotent matrix.
- Applying this result to the representation $ad : \mathfrak{a} \to End(\mathfrak{g})$ we see that z is nilpotent. By 3.1 it follows that $z \in [\mathfrak{g}^\alpha, \mathfrak{g}^{-\alpha}] \subset \mathfrak{g}^0$ so therefore as an element of Cartan subalgebra must be semi-simple. Nilpotent and semi-simple implies that $z = 0$ which contradicts the original assumption.

Step 5: Let $\alpha \in R$ and let X_α be a non-zero element of $\mathfrak{g}^\alpha$. There exists $Y_\alpha \in \mathfrak{g}^{-\alpha}$ such that $[X_\alpha, Y_\alpha] = H_\alpha$.

- Since $\mathfrak{g}^\alpha$ and $\mathfrak{g}^{-\alpha}$ are dual with respect to $(\cdot,\cdot)$, there exists $y \in \mathfrak{g}^{-\alpha}$ such that $(X_\alpha, y) \neq 0$, giving $[X_\alpha, y] \neq 0$ by Theorem 2.3 (3). Multiplying y by a suitable scalar we obtain Y_α.
- Since H_α is an element of Cartan subalgebra $\mathfrak{h}$, then by the fact that $X_\alpha \in \mathfrak{g}^\alpha$ and $Y_\alpha \in \mathfrak{g}^{-\alpha}$ we have the following relations

  $$[H_\alpha, X_\alpha] = \alpha(H_\alpha)X_\alpha = 2X_\alpha,\ [H_\alpha, Y_\alpha] = -\alpha(H_\alpha)Y_\alpha = -2Y_\alpha$$

- If $\mathfrak{s}_\alpha$ is the subalgebra of $\mathfrak{g}$ generated by $X_\alpha, Y_\alpha, H_\alpha$ then the mapping $(X, Y, H) \to (X_\alpha, Y_\alpha, H_\alpha)$ defines an isomorphism ϕ_α from $\mathfrak{sl}_2$ onto $\mathfrak{s}_\alpha$. Using the adjoint representation, we can regard $\mathfrak{g}$ as an $\mathfrak{sl}_2$-module.

Step 6: We have dim $\mathfrak{g}^\alpha = 1$ if $\alpha \in R$.

- Suppose $\dim \mathfrak{g}^\alpha \neq 1$, then since $\mathfrak{g}^\alpha, \mathfrak{g}^{-\alpha}$ are dual with respect to $(\cdot,\cdot)$, there exists a non-zero element $y \in \mathfrak{g}^{-\alpha}$ orthogonal to $X_\alpha \in \mathfrak{g}^\alpha$. (It can be made as a combination from the other two vectors in $\mathfrak{g}^\alpha$)
- Theorem 2.3 (3) implies that $[X_\alpha, y] = (X_\alpha, y)h_\alpha = 0$ and moreover, since $H_\alpha \in \mathfrak{g}^0$ then $[H_\alpha, y] = -\alpha(H_\alpha)y = -2y$. Thus if $\mathfrak{g}$ is regarded as a $\mathfrak{sl}_2$-module (by means of ϕ_α) then y would be a primitive element of weight -2 in $\mathfrak{g}$. By Corollary 3.1.1 in the notes about $\mathfrak{sl}_2$ Lie algebra, it follows that the weights for primitive elements must be positive integers, therefore the previous conclusion yields contradiction.

Step 7: $\mathfrak{s}_\alpha = \mathfrak{h}_\alpha \oplus \mathfrak{g}^\alpha \oplus \mathfrak{g}^{-\alpha}$

Since $\mathfrak{s}_\alpha$ is the subalgebra of $\mathfrak{g}$ generated by $X_\alpha, Y_\alpha, H_\alpha$ then this result follows from the previous two subsections.

Step 8: The element Y_α in 3.5 is unique

This follows from the fact that $\dim(\mathfrak{g}^{-\alpha}) = 1$.

Step 9: If α and β are roots, then $\beta(H_\alpha)$ is an integer and $\beta - \beta(H_\alpha)\alpha$ is a root.

- Let $y \in \mathfrak{g}^\beta, y \neq 0$, and let $p = \beta(H_\alpha)$. From this we have
$$[H_\alpha, y] = \beta(H_\alpha)y = p \cdot y$$
- If we view $\mathfrak{g}$ as $\mathfrak{sl}_2$-module by means of ϕ_α then this shows that y has weight p and by Theorem 5.2 in the notes on $\mathfrak{sl}_2$ Lie algebras it follows that p is an integer.
- We put
$$z = Y_\alpha^p y \;\; if\ p \leq 0 \;\; and \;\; z = X_\alpha^{-p} y \;\; if\ p \geq 0$$
then the same theorem shows that $z \neq 0$. This implies that
$$(1)\ [H_\alpha, z] = [H_\alpha, Y_\alpha^p y] = (\beta - p\alpha)z, \quad and \quad (2)\ [H_\alpha, z] = [H_\alpha, X_\alpha^{-p} y] = (\beta - p\alpha)z$$
where $\alpha(H_\alpha) = 2$ and
$$[H_\alpha, Y_\alpha^p y] = [H_\alpha, Y_\alpha^p]y + Y_\alpha^p[H_\alpha, y] \;\; and \;\; [H_\alpha, Y_\alpha^p] = [H_\alpha, Y_\alpha^{p-1}]Y_\alpha + Y_\alpha^{p-1}[H_\alpha, Y_\alpha] = -p\alpha Y_\alpha^p.$$
The same relation holds for $X_\alpha^{-p}y$. Therefore z has weight $\beta - p\alpha$ and this holds for any nonzero $y \in \mathfrak{g}^\beta$. Thus follows that $\beta - p\alpha$ is a root.

Step 10: R is a root system, and H_α is an inverse root of $\beta - \beta(H_\alpha)\alpha$

- By subsection 3.2 it follows that R spans $\mathfrak{h}^*$.
- If $\alpha \in R$ and $\beta \in \mathfrak{h}^*$, let $s_\alpha(\beta) = \beta - \beta(H_\alpha)\alpha$. Since $\alpha(H_\alpha) = 2$ then s_α is a symmetry with respect to α.
- By above subsection, if $\alpha, \beta \in R$ then $s_\alpha(\beta) \in R$ where $\beta(H_\alpha)$ is an integer for each $\beta \in R$. Thus all properties of the root system are satisfied.

Step 11: The root system R is reduced.

- Suppose otherwise. Then there exists an $\alpha \in R$ such that $2\alpha \in R$. Let y be a nonzero element of $\mathfrak{g}^{2\alpha}$, which implies that
$$[H_\alpha, y] = 2\alpha(H_\alpha)y = 4y$$
- The above relation together with the fact that $[H_\alpha, X_\alpha] = \alpha X_\alpha$ implies
$$[H_\alpha, [X_\alpha, y]] + [y, [H_\alpha, X_\alpha]] + [X_\alpha, [y, H_\alpha]] = 0 \implies [H_\alpha, [X_\alpha, y]] = 3\alpha[X_\alpha, y]$$
On the other hand 3α is a not a root which that $ad(X_\alpha)y = 0$.

- The formula $H_\alpha = [X_\alpha, Y_\alpha]$ shows that $ad(H_\alpha)y = ad(X_\alpha)ad(Y_\alpha)y$. Since $[Y_\alpha, y] \in \mathfrak{g}^\alpha$ where

$$[H_\alpha, [Y_\alpha, y]] + [y, [H_\alpha, Y_\alpha]] + [Y_\alpha, [y, H_\alpha]] = 0 \implies [H_\alpha, [Y_\alpha, y]] = \alpha[Y_\alpha, y]$$

then so it is a multiple of X_α and it is annihilated by $ad(X_\alpha)$. Thus $4y = ad(H_\alpha)y = 0$, arriving at contradiction.

End of Proof. ■

Connection to W_m modules

In the lecture notes on $\mathfrak{sl}_2$ Lie algebra, special W_m modules were defined of dimension $m + 1$, which were shown to be irreducible and such that every $\mathfrak{g}-$module of dimension $m + 1$ is isomorphic to W_m.

Proposition 37. *Let α and β be two non proportional roots. Let p (q reps.) be the greatest integer such that $\beta - p\alpha$ (resp. $\beta + q\alpha$) is a root. Let $E = \sum_k \mathfrak{g}^{\beta+k\alpha}$. Then E is an irreducible $\mathfrak{s}_\alpha-$module, of dimension $p + q + 1$. If $-p \le k \le q - 1$, the map*

$$ad(X_\alpha) : \mathfrak{g}^{\beta+k\alpha} \to \mathfrak{g}^{\beta+(k+1)\alpha}$$

is an isomorphism. We have $\beta(H_\alpha) = p - q$.

Proof.

1. It follows that $\sum_k \mathfrak{g}^{\beta+k\alpha}$ is a $\mathfrak{s}_\alpha-$submodule of $\mathfrak{g}$. If considered as an $\mathfrak{sl}_2-$module then the weights of E are the integers $\beta(H_\alpha) + \alpha(H_\alpha)k = \beta(H_\alpha) + 2k$ where $-p \le k \le q$ each having multiplicity 1.
2. If we check with the definition of W_m from lecture notes on $\mathfrak{sl}_2$ Lie algebra (section 4), then we see that the above properties of E are sufficient for it to be a W_m module which is irreducible and of dimension $m + 1$ and where $m = \beta(H_\alpha) + 2q = -\beta(H_\alpha) + 2p$ and so that $\beta(H_\alpha) = p - q$.
3. The fact that
$$d(X_\alpha) : \mathfrak{g}^{\beta+k\alpha} \to \mathfrak{g}^{\beta+(k+1)\alpha}, \quad -p \le k \le q - 1$$
is an isomorphism follows from the structure of the irreducible $\mathfrak{sl}_2$ modules.

■

If $\alpha, \beta, \beta + \alpha \in R$ then $[\mathfrak{g}^\alpha, \mathfrak{g}^\beta] = \mathfrak{g}^{\alpha+\beta}$.

With the notation of the previous subsection we have $q \ge 1$. Taking $k = 0$ we find that

$$d(X_\alpha) : \mathfrak{g}^{\beta+k\alpha} \to \mathfrak{g}^{\beta+(k+1)\alpha}$$

is an isomorphism.

Thus we have proven Theorem 2.2.

9.3. Borel Subalgebras

Let R be the root system associated with $(\mathfrak{g}, \mathfrak{h})$ and let us choose a base S of R. Let R^+ be the set of positive roots with respect to S (roots which can be expressed as linear combinations of S elements with non-negative integer coefficients). We put

$$\mathfrak{n}^+ = \sum_{\alpha\in R^+} \mathfrak{g}^\alpha, \ \mathfrak{n}^- = \sum_{\alpha\in R^+} \mathfrak{g}^{-\alpha}, \quad \mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}^+$$

Theorem 40.

- *One has $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+ = \mathfrak{n}^- \oplus \mathfrak{b}$*
- *$\mathfrak{n}^-$ and $\mathfrak{n}^+$ are subalgebras of $\mathfrak{g}$ consisting of nilpotent elements; they are nilpotent.*

- $\mathfrak{b}$ *is a solvable subalgebra of* $\mathfrak{g}$*; its derived algebra is* $\mathfrak{n}^+$.

Proof.

1. The first result follows from Theorem 2.1
2. Let $x \in \mathfrak{g}^{\alpha_1},\ y \in \mathfrak{g}^{\alpha_2}$ where $\alpha_1, \alpha_2 \in R^+$ then for any $h \in \mathfrak{h}$

$$[h, xy] = [h, x]y + x[h, y] = \alpha_1 xy + \alpha_2 xy,\ i.e.\ \ xy \in \mathfrak{n}^+$$

therefore $\mathfrak{n}^+$ is a subalgebra of $\mathfrak{g}$. Let $x \in \mathfrak{n}^+$ then for each integer $k > 0$ and $\beta \in \mathfrak{h}^*,\ y \in \mathfrak{g}^\beta$. Since for a given $\alpha \in R^+ \implies [\mathfrak{g}^\alpha, \mathfrak{g}^\beta] \subset \mathfrak{g}^{\alpha+\beta}$ then after iterating this process k times we have

$$ad(x)^k(y) \in \mathfrak{g}^{\beta+\alpha_1+...+\alpha_k}\ s.t.\ \ \alpha_i \in R^+$$

It follows that for every $k > 0$ the sum $\beta + \alpha_1 + ... + \alpha_k$ in the above will be expanding. Since $R \cup 0$ is a finite set then for some k we cannot have β and $\beta + \alpha_1 + ... + \alpha_k$ both in $R \cup 0$. Therefore for such k follows that $ad(x)^k = 0$ i.e. x is nilpotent. By Engel's theorem this implies that $\mathfrak{n}^+$ is nilpotent.

The case of $\mathfrak{n}^-$ is similar.

3. By the definition of direct sum for Lie algebras, by the fact that $\mathfrak{h}$ is abelian and since $[\mathfrak{n}^+, \mathfrak{n}^+], [\mathfrak{n}^+, \mathfrak{h}], [\mathfrak{h}, \mathfrak{n}^+] \subseteq \mathfrak{n}^+$ then we have

$$[\mathfrak{b}, \mathfrak{b}] = [\mathfrak{n}^+, \mathfrak{n}^+] \oplus [\mathfrak{n}^+, \mathfrak{h}] \oplus [\mathfrak{h}, \mathfrak{n}^+] = \mathfrak{n}^+$$

which implies the third statement.

■

The algebra $\mathfrak{b}$ is called the Borel subalgebra corresponding to $\mathfrak{h}$ and S.

Theorem 41. *(Borel-Morozov). Every solvable subalgebra of* $\mathfrak{g}$ *can be mapped to a subalgebra of* $\mathfrak{h}$ *by an inner automorphism of* $\mathfrak{g}$*. In particular,* $\mathfrak{b}$ *is a maximal solvable subalgebra of* $\mathfrak{g}$.

Corollary 12. *Every subalgebra of* $\mathfrak{g}$ *consisting of nilpotent elements can be mapped to a subalgebra of* $\mathfrak{n}$ *by an inner automorphism of* $\mathfrak{g}$.

Proof. This is a consequence of Theorem 4.1, and from the fact that each nilpotent element of $\mathfrak{g}$ contained in $\mathfrak{b}$ belongs to $\mathfrak{n}$. ■

9.4. Weyl Bases

Let $(\alpha_1, ..., \alpha_n)$ denote the chosen base S. Let $n = \dim(\mathfrak{h}) = rank(\mathfrak{g})$. For each i, we put $H_{\alpha_i} = H_i$, and we choose elements $X_i \in \mathfrak{g}^{\alpha_i}$ and $Y_i \in \mathfrak{g}^{-\alpha_i}$ such that $[X_i, Y_i] = H_i$.

Denote

$$n(i, j) = \alpha_j(H_i)\ \ corresp.\ \ to\ \ \alpha_i, \alpha_j \in S$$

The matrix formed by the numbers (integers) $n(i, j)$ is the Cartan matrix of the given root system. Since $\alpha_i, \alpha_j \in S$ then from lecture notes 4 follows that $n(i, j) \leq 0$ if $i \neq j$.

Theorem 42.

1. $\mathfrak{n}$ *is generated by the elements* X_i, $\mathfrak{n}^-$ *by the elements* Y_i*, and* $\mathfrak{g}$ *by the elements* X_i, Y_i, H_i.

2. These elements satisfy the relations (called the "Weyl relations")

$$[H_i, H_j] = 0,\ \ [X_i, Y_i] = H_i,\ \ [H_i, X_j] = n(i, j)X_j$$

$$[X_i, Y_j] = 0\ \ if\ \ i \neq j\ \ \ [H_i, Y_j] = -n(i, j)Y_j$$

3. They also satisfy the following relations:

$$(\theta_{ij})\ \ ad(X_i)^{-n(i,j)+1}(X_j) = 0\ \ (i \neq j)$$

$$(\theta_{ij}^-)\ \ ad(Y_i)^{-n(i,j)+1}(Y_j) = 0\ \ (i \neq j)$$

Proof.

1. Want to show that $\mathfrak{n}$ is generated by the elements X_i (The corresp. statement for Y_i is proven in the same way). Let $\alpha \in R^+$. By Proposition 8.1 in lecture notes 4, α can be decomposed as a sum of roots α_i
$$\alpha = \alpha_{i_1} + ... + \alpha_{i_k}$$
in such a way that the partial sums $\alpha_{i_1} + ... + \alpha_{i_h}$ for $h \leq k$ belong to R^+. Given such a decomposition we define
$$X_\alpha = ad(X_{i_k})ad(X_{i_{k-1}}) \cdots ad(X_{i_2})X_{i_1}$$
By Theorem 2.2 it follows that $X_\alpha \in \mathfrak{g}^\alpha = \mathfrak{g}^{\alpha_{i_1}+\dots+\alpha_{i_k}}$ is a non-zero element. Since for each $\alpha \in R^+ \implies \dim(\mathfrak{g}^\alpha) = 1$ and since $\mathfrak{n}$ is the sum of $\mathfrak{g}^\alpha,\ \alpha \in R^+$ then $\mathfrak{n}$ is indeed generated by the elements of X_i.
2. Most of these relations were already established with the exception of $i \neq j \implies [X_i, Y_j] = 0$. To show this recall that every root is a linear combination of roots α_j with coefficients of the same sign. Since
$$[H, [X_i, Y_j]] = (\alpha_i - \alpha_j)[X_i, Y_j]$$
where $\alpha_i - \alpha_j$ is not a root. Therefore the result follows.
3. The elements
$$\theta_{ij} = ad(X_i)^{-n(i,j)+1} X_j$$
has weight $\alpha_j + \alpha_i - \alpha_i n(i,j) = s_i(\alpha_j - \alpha_i)$ where s_i corresponds to symmetry associated with α_i. Since $\alpha_i - \alpha_j$ is not a root, neither is $s_i(\alpha_i - \alpha_j)$, so $\theta_{ij} = 0$. The other relation is proven in the same way.

■

Theorem 43.

1. *The algebra $\mathfrak{n}$ can be defined by the generators X_i and the relations $\theta_{ij}, i \neq j$.*
2. *The algebra $\mathfrak{g}$ can be defined by the generators X_i, Y_i, H_j, the Weyl relations, and the relations $\theta_{ij}, \theta_{ij}^-$.*

Proof. Proof is outlined in the appendix to this chapter in J-P Serre. ■

Remark 35. *Theorem 5.2 gives an explicit description of $\mathfrak{g}$ and of $\mathfrak{n}$ in terms of the Cartan matrix $(n(i,j))$*

Corollary 13. *There is an automorphism σ of $\mathfrak{g}$ which is equal to -1 on $\mathfrak{h}$, and which sends X_i to $-Y_i$, Y_i to $-X_i$, for all i. One has $\sigma^2 = 1$*

Proof. Let us put $H_i' = -H_i, X_i' = -Y_i, Y_i' = -X_i$. Clearly the elements X_i', Y_i', H_i' satisfy the Weyl relations and the relations $\theta_{ij}, \theta_{ij}^-$. Hence by the second statement from the above theorem it follows that there is a homomorphism $\sigma : \mathfrak{g} \to \mathfrak{g}$ which maps X_i, Y_i, H_i to X_i', Y_i', H_i'. Since σ^2 fixes X_i, Y_i, H_i then it is the identity as required. ■

9.5. Existence and Uniqueness Theorems

In the lecture notes 3 there was proven a conjugacy theorem for Cartan subalgebras which shows that the root system of a semi simple Lie algebra is independent (up to isomorphism) of the chosen Cartan subalgebra. This results into the following theorem.

Theorem 44. *Two semisimple Lie algebras corresponding to isomorphic root systems are isomorphic .*

Having established uniqueness we go over to the existence theorem.

Theorem 45. *Let R be a reduced root system. There exists a semi simple Lie algebra $\mathfrak{g}$ whose root system is isomorphic to R.*

Proof. Let $S = \{\alpha_1, ..., \alpha_n\}$ be a base for R, with $(n(i,j))$ being the corresponding Cartan matrix. Let $\mathfrak{g}$ be a Lie algebra defined by $3n$ generators X_i, Y_i, H_i and by the relations from Theorem 6. One can show (Appendix of this chapter in J-P Serre) that this Lie algebra is finite dimensional, semisimple, and has a root system isomorphic to R. ■

Corollary 14. *For $\mathfrak{g}$ to be simple, it is necessary and sufficient that R should be irreducible.*

9.6. Chevalley's Normalization

For each $\alpha \in R$ choose a non zero element $X_\alpha \in \mathfrak{g}^\alpha$. Then we have

$$[X_\alpha, X_\beta] = \begin{cases} N_{\alpha,\beta} X_{\alpha+\beta} & if\ \alpha+\beta \in R \\ 0 & if\ \alpha+\beta \notin R, \alpha+\beta \neq 0 \end{cases}$$

where $N_{\alpha,\beta}$ is non-zero scalar depend on the choice of X_α, X_β and they determine multiplication table of $\mathfrak{g}$.

Theorem 46. *One can choose the elements X_α so that*

$$[X_\alpha, X_{-\alpha}] = H_\alpha \quad \forall \alpha \in R$$

$$N_{\alpha,\beta} = -N_{-\alpha,-\beta} \quad where \quad \alpha, \beta, \alpha+\beta \in R$$

Proof.

- Let R^+ be the set of positive roots relative to a base S of R. By Corollary 5.2.1 there is an automorphism of $\mathfrak{g}$ which would be equal to -1 on $\mathfrak{h}$, and such that $\sigma^2 = 1$. From the Corollary 5.2.1 we have that $\sigma(\mathfrak{g}^\alpha) = \mathfrak{g}^{-\alpha}$.
- Let $\alpha \in R^+$ and let us choose nonzero $x_\alpha \in \mathfrak{g}^\alpha$. We have $[x_\alpha, \sigma(x_\alpha)] \in [\mathfrak{g}^\alpha, \mathfrak{g}^{-\alpha}]$, so there is a nonzero scalar t_α such that
$$[x_\alpha, \sigma(x_\alpha)] = t_\alpha H_\alpha$$
- Since we are considering complex spaces then let $u_\alpha = \sqrt{-t_\alpha}$, and let us put
$$X_\alpha = u_\alpha^{-1} x_\alpha, \quad X_{-\alpha} = -\sigma(X_\alpha)$$
- We now have $[X_\alpha, X_{-\alpha}] = H_\alpha$ and $X_{-\alpha} + \sigma(X_\alpha) = 0$. The identity
$$N_{\alpha,\beta} = -N_{-\alpha,-\beta}$$
is then obtained by writing
$$\sigma([X_\alpha, X_\beta] = [\sigma(X_\alpha), \sigma(X_\beta)])$$

■

Theorem 47. *Let the conditions of the previous theorem be satisfied. Let $\alpha, \beta \in R$ be such that $\alpha+\beta \in R$, and let p be the greatest integer such that $\beta - p\alpha \in R$. Then one has*

$$N_{\alpha,\beta} = \pm(p+1)$$

Proof. For proof see Bourbaki Chap. 8, Sec. 2, No. 4. ■

9.7. Appendix

Let R be a root system in a complex vector space V and for consistency with the previous sections we denote $V = \mathfrak{h}^*$. Let

- $S = \{\alpha_1, ..., \alpha_n\}$ be a base for R,
- $H_1, ..., H_n \in \mathfrak{h}$ be the inverse roots of $\alpha_1, ..., \alpha_n$
- $n(i,j) = \langle \alpha_j, H_i \rangle$ which form the Cartan matrix of R with respect to S.

In this section we want to prove the following theorem.

Theorem 48. *Let $\mathfrak{g}$ be a Lie algebra defined by $3n$ generators X_i, Y_i, H_i and by the relations*

$$[H_i, H_j] = 0, \quad [X_i, Y_i] = H_i, \quad [H_i, X_j] = n(i,j)X_j$$

$$[X_i, Y_j] = 0 \;\; if \;\; i \neq j \quad [H_i, Y_j] = -n(i,j)Y_j$$

$$(\theta_{ij}) \;\; ad(X_i)^{-n(i,j)+1}(X_j) = 0 \;\; and \;\; (\theta_{ij}^{-}) \;\; ad(Y_i)^{-n(i,j)+1}(Y_j) = 0 \;\; (i \neq j)$$

Then $\mathfrak{g}$ is semisimple Lie algebra, with the subalgebra $\mathfrak{h}$ generated by the elements H_i as a Cartan subalgebra; its root system is R.

Proof.

1. Consider first the algebra $\mathfrak{a}$ defined by $3n$ generators X_i, Y_i, H_i and the relations

$$[H_i, H_j] = 0, \quad [X_i, Y_i] = H_i, \quad [H_i, X_j] = n(i,j)X_j$$

$$[X_i, Y_j] = 0 \;\; if \;\; i \neq j \quad [H_i, Y_j] = -n(i,j)Y_j$$

Let $\mathfrak{x}, \mathfrak{y}, \mathfrak{h}$ be Lie algebras generated by the elements $\{X_i\}_{i=1}^n, \{Y_i\}_{i=1}^n, \{H_i\}_{i=1}^n$ respectively. It is proven in Bourbaki, Chap . 8, Sec. 4, No. 2 that the algebra in question is a direct sum of its vector subspaces

$$\mathfrak{a} = \mathfrak{y} \oplus \mathfrak{h} \oplus \mathfrak{x}$$

∎

Chapter 10

Linear Representations of Semisimple Lie Algebras

For this chapter we need to reinstate all the notations used in the previous chapters. Let $\mathfrak{g}$ denote a complex semisimple Lie algebra, $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$, and R the corresponding root system. We choose a base $S = \{\alpha_1, ..., \alpha_n\}$ of R, and we denote by R^+ the set of positive roots (with respect to S).

For each $\alpha \in R^+$, we choose $X_\alpha \in \mathfrak{g}^\alpha, Y_\alpha \in \mathfrak{g}^{-\alpha}$ so that $H_\alpha = [X_\alpha, Y_\alpha]$. If $\alpha = \alpha_i \in S$ then $X_{\alpha_i} = X_i$ and the same notation holds for Y_i, H_i.

Denote $\mathfrak{n}^+ = \sum_{\alpha \in R^+} \mathfrak{g}^\alpha$, $\mathfrak{n}^- = \sum_{\alpha \in R^-} \mathfrak{g}^\alpha$, and $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}^+$

10.1. Weights

Let V be a $\mathfrak{g}$-module and let $\omega \in \mathfrak{h}^*$. Define the set

$$V^\omega = \{v \in V | \ \forall H \in \mathfrak{h} \implies Hv = \omega(H)v\}$$

which is vector subspace of V and where an element of V^ω is said to have weight ω. The multiplicity of ω in V is the dimension of V^ω. If $V^\omega \neq 0$ then ω is called a weight of V.

Proposition 38.

1. *Let $\omega \in \mathfrak{h}^*$ and $\alpha \in R$, then one has $\mathfrak{g}^\alpha V^\omega \subset V^{\omega+\alpha}$.*
2. *The sum $V^{'} = \sum_\omega V^\omega$ is direct and it is a $\mathfrak{g}$-submodule of V.*

Proof.

1. The first result follows from simple computation. Let $X_\alpha \in \mathfrak{g}^\alpha, v \in V^\omega$; if $H \in \mathfrak{h}$, then follows that

$$H(Xv) = X(Hv) + [H, X]v = (\omega(H) + \alpha(H))Xv$$

which shows that $Xv \in V^{\omega+\alpha}$ i.e. $V^\omega \subset V^{\omega+\alpha}$.

2. Since eigenvectors associated with different eigenvalues are linearly independent then the sum of the subspaces V^ω is direct. By the first clause follows that $V^{'}$ is invariant under $\mathfrak{g}^\alpha$ which implies that it is invariant under $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{n}^+ \oplus \mathfrak{n}^-$ i.e. it is a $\mathfrak{g}$-submodule of V.

■

10.2. Primitive elements

Let V be a $\mathfrak{g}$-module, $v \in V$ and $\omega \in \mathfrak{h}^*$. One says that v is a primitive element of weight ω if it satisfies the following two conditions:

1. v is non-zero and has weight ω.

2. One has $X_\alpha v = 0$ for all $\alpha \in R^+$.

For the future developments we need to introduce and define the concept of universal enveloping algebra for a given Lie algebra $\mathfrak{g}$.

Definition 45. *Let $\mathfrak{g}$ be a Lie algebra over a field F. An enveloping algebra of $\mathfrak{g}$ is a pair (ϕ, U), where U is a unital associative algebra and $\phi : \mathfrak{g} \to U_-$ is a Lie algebra homomorphism, where U_- stands for U with the bracket $[a, b] = ab - ba$.*

Example: Let $\phi : \mathfrak{g} \to End(V)$ be a representation of $\mathfrak{g}$ in a vector space V i.e. V is a $\mathfrak{g}$-module. Then the pair $(End(V), \phi)$ is an enveloping algebra of $\mathfrak{g}$.

Definition 46. *The universal enveloping algebra of $\mathfrak{g}$ is an enveloping algebra $(\Phi, U(\mathfrak{g}))$ which has the following universal mapping property: for any enveloping algebra (ϕ, U) of $\mathfrak{g}$ there exists a unique associative algebra homomorphism $f : U(\mathfrak{g}) \to U$ such that $\phi = f \circ \Phi$.*

Definition 47. *A generating set Γ of a module M over a ring R is a subset of M such that the smallest submodule of M containing Γ is M itself. This implies that if E is a $\mathfrak{g}$-submodule of V generated by v then $E = U(\mathfrak{g}) \cdot v$ where $(U(g), \Phi)$ is a universal enveloping algebra of $\mathfrak{g}$.*

Remark 36. *In what follows we will take the following result for granted. Let Lie algebra $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ then $U(\mathfrak{g}) \cong U(\mathfrak{g}_1) \otimes U(\mathfrak{g}_2)$.*

For the proposition below we need to have the definition of module E being indecomposable.

Definition 48. *A module E is called indecomposable if it is nontrivial and if, for each direct sum decomposition $E = E_1 \oplus E_2$ one has $E_1 = 0$ or $E_2 = 0$.*

Proposition 39. *Let V be a $\mathfrak{g}$-module and let $v \in V$ be a primitive element of V of weight ω; let E be $\mathfrak{g}$-submodule of V generated by v. Then:*

1. If $\beta_1, ..., \beta_k$ denote the different positive roots, E is spanned by the elements of the form

$$Y_{\beta_1}^{m_1} \cdots Y_{\beta_k}^{m_k} v \quad with \quad m_i \in \mathbb{N}$$

2. The weights of E have the form

$$\omega - \sum_{i=1}^{n} p_i \alpha_i$$

They have a finite multiplicity.

3. ω is a weight of E of multiplicity 1.

4. E is an indecomposable $\mathfrak{g}$-module.

Proof.

1. The proof of the first result rests on the observation that v is a primitive element of V so therefore the elements of $\mathfrak{b} = \mathfrak{n}^+ \oplus \mathfrak{h}$ leave it invariant.

   Let $A = U(\mathfrak{g}), B = U(\mathfrak{b}), C = U(\mathfrak{n}^-)$ be universal enveloping algebras of $\mathfrak{g}, \mathfrak{b}, \mathfrak{n}^-$ respectively. Then since $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{b}$, the remarks above imply that

$$E = A \cdot v = C \cdot B \cdot v = C \cdot v$$

   By the Birkhoff-Witt Theorem, the monomials

$$Y_{\beta_1}^{m_1} \cdots Y_{\beta_k}^{m_k} v \quad with \quad m_i \in \mathbb{N}$$

   form the basis of C which implies the first clause.

2. Proposition 2.1 (clause 1) shows that $Y_{\beta_1}^{m_1} \cdots Y_{\beta_k}^{m_k} v$ has weight $\omega - \sum m_j \beta_j$ where $m_j \in \mathbb{N}$ and β_j is a linear combination of the simple roots α_i with non-negative integer coefficients. This implies the second clause.

3. Since $v \in E$ then ω is a weight of E. Since $\omega - \sum m_j\beta_j$ is equal to ω if and only if $\forall j \implies m_j = 0$, then ω is a weight of E of multiplicity 1.

4. Suppose that E is a direct sum of two submodules E_1 and E_2. Then if

$$E^\omega = \{e \in E| \; \forall H \in \mathfrak{h} \implies He = \omega(H)e\}\}$$

it follows that $E^\omega = E_1^\omega \oplus E_2^\omega$. By the 3rd clause it follows that $\dim(E^\omega) = 1$ so that $E^\omega = E_1^\omega$ or $E^\omega = E_2^\omega$. Suppose that the first case holds, then $v \in E_1$, and since v generates E then this implies that $E = E_1$ and $E_2 = 0$ which implies the result.

■

10.3. Irreducible Modules with a Highest Weight

Theorem 49. *Let V be an irreducible $\mathfrak{g}$-module containing a primitive element v of weight ω. Then:*

1. The weights π of V have the form

$$\pi = \omega - \sum m_i\alpha_i \quad \textit{with} \quad m_i \in \mathbb{N}$$

They have finite multiplicity; ω has multiplicity 1. One has $V = \sum_\pi V^\pi$

2. v is the only primitive element of V (up to scalar multiplication); its weight ω is called the highest weight of V.

3. Two irreducible $\mathfrak{g}$-modules V_1 and V_2 with highest weights ω_1 and ω_2 are isomorphic if and only if $\omega_1 = \omega_2$.

Remark 37. *Statement (b) shows that the weights of V are dominated by ω, which justifies the terminology " highest weight".*

Proof.

1. Since V is irreducible $\mathfrak{g}$-module and $v \in V^\omega$ (i.e. $V^\omega \neq \varnothing$) then proposition 1 statement (2) implies that $V = \sum_\pi V^\pi$. Let E be a $\mathfrak{g}$-submodule of V generated by v. Since E is non-zero (because it contains v) and since V is an irreducible $\mathfrak{g}$-module then $V = E$. Proposition 2 statements (2) and (3) imply the remaining parts of the result.

2. Let v' be a primitive element of V, of weight ω'. By previous, ω' can be written as

$$\omega' = \omega - \sum m_i\alpha_i \quad \textit{with} \quad m_i \geq 0$$

Exchanging the roles of v and v' , we see that

$$\omega = \omega' - \sum m_i'\alpha_i \quad \textit{with} \quad m_i' \geq 0$$

Combining these two equations we see that they can hold only if $m_i' = m_i = 0$ which implies that $\omega' = \omega$ and therefore v must be proportional to v', giving (2).

3. Suppose that V_1 and V_2 are two isomorphic irreducible $\mathfrak{g}$-modules with highest weights ω_1 and ω_2. Then the isomorphism must map primitive element v_1 to a primitive element v_2 which implies by definition of module homomorphism that $\omega_1 = \omega_2$.

   Want to show that if $\omega_1 = \omega_2$ then two irreducible $\mathfrak{g}$-modules V_1 and V_2 with highest weights ω_1 and ω_2 are isomorphic.

   - Let v_1, v_2 be primitive elements of V_1, V_2 respectively of weight $\omega = \omega_1 = \omega_2$. Then this implies that $v = v_1 + v_2$ is a primitive element of weight ω for $\mathfrak{g}$-module $V = V_1 \oplus V_2$.
   - Let E be a $\mathfrak{g}$-module of V generated by v. The projection onto the V_2 component i.e. $pr_2 : V \to V_2$ induces a $\mathfrak{g}$-module homomorphism $f_2 : E \to V_2$. One has $f(v) = v_2$.

- Since v_2 generates V_2 then the previous implies that f is surjective. Moreover the kernel $N = V_1 \cap E$ of f_2 is a submodule of V_1.
- By proposition 3.1 the only elements of E of weight ω are the multiples of v, which implies that v_1 with weight ω is not contained in the $\mathfrak{g}$-submodule N. Therefore N is $\mathfrak{g}$-submodule of an irreducible $\mathfrak{g}$-module V_1 such that $N \neq V_1$, which implies that $N = 0$.
- Previous implies that $f_2 : E \to V_2$ is an isomorphism. Repeating all of the previous steps with $pr_1 : V \to V_1$ we would see that $V_1 \cong E \cong V_2$.

■

Theorem 50. *For each $\omega \in \mathfrak{h}^*$, there is an irreducible $\mathfrak{g}$-module with highest weight equal to ω.*

Remark 38. *Theorem 4.1 shows that such a module is unique up to isomorphism and therefore together these theorems give bijection between the elements $\omega \in \mathfrak{h}^*$ and the classes of irreducible $\mathfrak{g}$-modules with a highest weight.*

10.4. Finite-Dimensional Modules

Proposition 40. *Let V be a finite-dimensional $\mathfrak{g}$-module. Then one has*

1. $V = \sum_\pi V^\pi$

2. If $V \neq 0$, V contains a primitive element.

3. If V is generated by a primitive element, V is irreducible.

4. If π is a weight of V, $\pi(H_\alpha)$ is an integer for all $\alpha \in R$.

Proof.

1. Since $\mathfrak{g}$ is semisimple then by Theorem 4.1 from the lecture notes 3 it follows that $\forall h \in \mathfrak{h} \implies ad(h)$ are diagonalizable and they all commute with each other ($[\mathfrak{h}, \mathfrak{h}] = 0$). Therefore they can be diagonalized simultaneously which gives the decomposition of V into direct sum of eigenspaces.
2. Since borel subalgebra $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}$ is solvable then by Lie's theorem follows that $\exists\, v \in V$ such that $\mathfrak{b} \cdot v \subseteq \mathbb{C}v$. Suppose that $\alpha \in R^+$ and $\omega \in \mathfrak{h}^*$ such that

$$Hv = \omega(H)v \quad \textit{and} \quad X_\alpha v = \lambda_\alpha v$$

   Then since $[H_\alpha, X_\alpha]v = \alpha(H_\alpha)X_\alpha v = \alpha(H_\alpha)\lambda_\alpha v$ it holds that

$$[H_\alpha, X_\alpha]v = \alpha(H_\alpha)\lambda_\alpha v = (H_\alpha X_\alpha - X_\alpha H_\alpha)v = 0$$

   therefore $X_\alpha v = 0$ for all $\alpha \in R^+$.
3. By Weyl's theorem, every (finite-dimensional) linear representation of a semisimple algebra is completely reducible i.e. it is a direct sum of irreducible modules. By proposition 2 part (4) since V is generated by a primitive element then it is indecomposable. Therefore, by above, it is irreducible.
4. Let $\alpha \in R^+$ (sim. can choose $\alpha \in R^-$). Then V can be viewed as a module over the Lie algebra $\mathfrak{s}_\alpha$ generated by $X_\alpha, Y_\alpha, H_\alpha$. By applying Theorem 5.2 from the lecture notes on the lie algebra $\mathfrak{sl}_2$ to this module, one sees that the eigenvalues of H_α on V belong to $\mathbb{Z}$. Since these eigenvalues are none other than the values of $\pi(H_\alpha)$, one gets the last part of the proposition.

■

Corollary 15. *Every finite-dimensional irreducible $\mathfrak{g}$-module has a highest weight.*

Proof. By previous theorem every finite-dimensional nonzero $\mathfrak{g}$-module has a primitive element. Since $\mathfrak{g}$-module is irreducible then by theorem 4.1 this element is unique and it has the highest weight. ■

Theorem 51. *Let $\omega \in \mathfrak{h}^*$ and let E_ω be an irreducible $\mathfrak{g}$-module having ω as highest weight. For E_ω to be finite dimensional, it is necessary and sufficient that one has*

$$(*) \textit{ For all } \alpha \in R^+, \ \omega(H_\alpha) \textit{ is an integer} \geq 0.$$

Proof. It is sufficient to prove that for all $\alpha \in S$ base, $\omega(H_\alpha)$ is an integer ≥ 0.

1. The necessity of condition $(*)$ follows from the fact that, if v is a primitive element of E_ω for $\mathfrak{g}$, it is also a primitive element for the subalgebra $\mathfrak{s}_\alpha$ generated by $X_\alpha, Y_\alpha, H_\alpha$. By corollary 3.1.1 it follows that $\omega(H_\alpha)$ is a non-negative integer. Therefore if E_ω is a finite-dimensional irreducible $\mathfrak{g}$-module then the $(*)$ condition follows.

2. Now we want to show that condition $(*)$ is sufficient i.e. want to show that if the $(*)$ condition holds then an irreducible $\mathfrak{g}$-module E_ω is finite-dimensional.

   - Let v be a primitive element of E_ω and let i be an integer from 1 to n. Let us define

     $$m_i = \omega(H_i) \ \ and \ \ v_i = Y_i^{m_i+1} v$$

     We will show that $v_i = 0$. If $i \neq j$ then from Weyl's relations follows $[Y_i, X_j] = 0$. Therefore one has

     $$X_j v_i = Y_i^{m_i+1} X_j v = 0$$

     By Theorem 3.1 from lecture notes on $\mathfrak{sl}_2$ Lie algebra we see that

     $$\frac{1}{(m_i+1)!} X_i Y_i^{m_i+1} v = (\omega(H_i) + 1 - (m_i+1)) \frac{1}{m_i!} Y_i^{m_i} v = 0 \implies X_i v_i = 0$$

     and also by the same theorem it follows that $H_i v_i = (\omega(H_i) - 2(m_i+1)) v_i$. Since $[H_j, Y_i] = -\alpha_i(H_j) Y_i$ then from this recursive relation follows that

     $$\forall H_j \in \mathfrak{h} \implies H_j v_i = (\omega(H_j) - (m_i+1)\alpha_i(H_j)) v_i$$

     If $v_i \neq 0$ then the above shows that v_i is a primitive element of E_ω of weight $\omega - (m_i+1)\alpha_i$ which contradicts the uniqueness of v from Theorem 4.1 part (2). Therefore $v_i = 0$.
     Since X_i, Y_i, H_i satisfy Weyl's relations, then $W_{m_i} = span\{v, Y_i v, ..., Y_i^{m_i} v\}$ is a finite-dimensional $\mathfrak{s}_i$-submodule of E_ω.

   - Now, want to show that E_ω is a sum of finite-dimensional $\mathfrak{s}_i$-submodules. Let T_i be the set of all finite-dimensional $\mathfrak{s}_i$-submodules of E_ω, and E_i' their sum. Suppose $F \in T_i$, and consider a $\mathfrak{g}$-module $M = \mathfrak{g} \cdot F$ generated by F, which is a set $\{g_1 f_1 + ... + g_n f_n | g_i \in \mathfrak{g}, \ f_i \in F\}$. From Weyl's relations (making computations with commutators) it follows that M is a $\mathfrak{s}_i$-submodule of E_ω i.e. $\mathfrak{g} \cdot F \in T_i$. This implies that E_i' is a $\mathfrak{g}$-submodule of E_ω. Since E_ω is irreducible and E_i' non-empty (it contains W_{m_i}) we have $E_\omega = E_i'$. Thus the result follows.

   - Let P_ω be the set of weights of E_ω. Want to show that P_ω is invariant under the symmetry $s_i : \mathfrak{g}^* \to \mathfrak{g}^*$ associated with the root α_i ($s_i(\alpha_i) = -\alpha_i$). Let $\pi \in P_\omega$, and let y be a nonzero element of $E_\omega^\pi = \{v \in E_\omega | \ \forall H \in \mathfrak{h} \implies Hv = \omega(H) v\}$. By Proposition 5.1, $p_i = \pi(H_i)$ is an integer. Let us put

     $$x = Y_i^{p_i} y \text{ if } p_i \geq 0, \text{ and } x = X_i^{-p_i} y \text{ if } p_i \leq 0$$

     By Theorem 5.2 from the notes on $\mathfrak{sl}_2$ Lie algebras, applied to a finite-dimensional $\mathfrak{s}_i$-submodule of E_ω containing y, it follows that if $p_i \geq 0$ then $Y_i^{p_i} : E_\omega^{-p_i} \to E_\omega^{p_i}$ and if $p_i \leq 0$ then $X_i^{-p_i} : E_\omega^{p_i} \to E_\omega^{-p_i}$ are isomorphisms. Therefore the above defined x is nonzero and thus by proof of Theorem 2.2 from lecture notes 5 (sec. 3.9) its corresponding weight is equal to

     $$\pi - p_i \alpha_i = \pi - \pi(H_i)\alpha_i = s_i(\pi)$$

     The last equality follows from the definition of symmetries given in Theorem 2.2 from lecture 5 notes. This shows that for any $\pi \in P_\omega \implies s_i(\pi) \in P_\omega$, therefore the result follows.

- Want to show that P_ω is finite. Theorem 4.1 shows that π can be written as

$$\pi = \omega - \sum_{\alpha_i \in S} p_i \alpha_i, \quad p_i \in \mathbb{Z}_{\geq 0}$$

  To show the result we need to show that the coefficients p_i are bounded. Since $-S$ is a base of R root system, then by the result from lecture 5 notes, there is an element w of the Weyl group of R sending S to $-S$, and this element is the product of symmetries s_i.
  It follows that w as a product of symmetries s_i is invariant on P_ω, therefore $w(\pi) \in P_\omega$ and thus can be written as

$$w(\pi) = \omega - \sum_{\alpha_i \in S} q_i \alpha_i \quad q_i \geq 0$$

  Applying w^{-1} to this formula ($w^{-1} : S \to -S$), one finds that

$$\pi = w^{-1}(\omega) + \sum_{\alpha_i \in S} r_i \alpha_i \quad r_i \geq 0$$

  It follows that $\omega - w^{-1}(\omega) = \sum_{\alpha_i \in S} c_i \alpha_i = \sum_{\alpha_i \in S} (p_i + r_i)\alpha_i$ where c_i are finite non-negative integers which are independent from π and such that $p_i + r_i = c_i$. Therefore $\forall i \implies p_i \leq c_i$ so for all $\pi \in P_\omega$ the coefficients p_i are indeed bounded.
- Above shows that there are only finitely many weights of E_ω. By Theorem 4.1 they all have finite multiplicity and since E_ω is the sum of the corresponding eigenspaces, then E_ω is finite dimensional as required.

■

Definition 49. *Let (ω_i) be the basis of $\mathfrak{h}^*$ dual to the basis (H_i):*

$$\omega_i(H_i) = 1, \quad i \neq j \implies \omega_i(H_j) = 0$$

The ω_i are called the fundamental weights of the root system R (with respect to the chosen base S). The irreducible modules having the weights ω_i as highest weights are called the fundamental modules of $\mathfrak{g}$.

10.5. An Application to the Weyl Group

Definition 50. *Let G be a group acting on the set X. The Weyl group acts simply transitively on the set of bases of R. The action is said to be simply transitive if it is transitive and if $\forall x,\ y \in X$ there is a unique $g \in G$ such that $g \cdot x = y$.*

Proposition 41. *The Weyl group W acts simply transitively on the set of bases of R.*

Proof. From Theorem 9.1 in the lecture 4 notes, we know that W acts transitively on the set of bases of R. Thus we only need to show simplicity.

- If $\forall S_1,\ S_2 \in \{\textit{Bases of } R\}$ there is a unique $g \in W$ such that $g \cdot S_1 = S_2$, then can deduce that for any base $S \implies \exists!\ s.t.\ g \in W\ g \cdot S = S \iff g = 1$. Therefore it is sufficient to show that if $w(S) = S$, with $w \in W$, then $w = 1$.
- Let P be the set of fundamental weights. For any simple symmetry $s_i : \pi \to \pi - \pi(H_i)\omega_i$ it follows that $s_i(P) = P$. Since w as a product of symmetries s_i then $w(P) = P$.
- By proof of the previous Theorem it follows that if $\omega \in P$ then $w(\omega)$ is a weight of the fundamental module E_ω of the highest weight ω i.e. $w(\omega) \in P_\omega$. By Theorem 4.1 it follows that $\omega - w(\omega)$ is a linear combination of the simple roots α_i with coefficients ≥ 0. This applies for any $\omega \in P$.
- On the other hand we have

$$\sum_{\omega \in P} (\omega - w(\omega)) = \sum_{\omega \in P} \omega - \sum_{\omega \in P} w(\omega) = 0$$

  which is possible only if all summonds is zero. Since P is a basis for $\mathfrak{h}^*$ then this forces $w = 1$, as required.

■

10.6. Example: $\mathfrak{sl}_{n+1}$

- We define algebra $\mathfrak{sl}_{n+1}$ to consist from the set of square matrices of order $n+1$ and with zero trace. Its Cartan subalgebra $\mathfrak{h}$ consists of diagonal matrices $H = (\lambda_1, ..., \lambda_{n+1})$, with $\sum_i \lambda_i = 0$. The roots are linear forms $\alpha_{i,j}, i \neq j$ given by
$$\alpha_{i,j}(H) = \lambda_i - \lambda_j$$
- For a base, we take the roots $\alpha_i = \alpha_{i,i+1}$, $1 \leq i \leq n$. The element $H_i \in \mathfrak{h}$ corresponding to α_i has components $\lambda_i = 1, \lambda_{i+1} = -1, \lambda_j = 0$ if $i \neq j, i+1$. The fundamental weights are given by
$$\omega_i(H) = \lambda_1 + ... + \lambda_i$$

10.7. Characters

Let P be the subgroup of $\mathfrak{h}^*$ such that
$$P = \{\pi \in \mathfrak{h}^* | \forall \alpha \in R \implies \pi(H_\alpha) \in \mathbb{Z}\}$$
It follows that P is a free abelian group, with the basis consisting from the fundamental weights $\omega_1, ..., \omega_n$. Let the abelian group $A = \mathbb{Z}[P] = \{\sum_{\mu \in \mathfrak{h}^*} m_\mu e^\mu | m_\mu \in \mathbb{Z}\}$ then A is th4 group algebra of P with coefficients in $\mathbb{Z}$. By definition $(e^\pi)_{\pi \in P}$ is the basis of A where $e^\pi e^\mu = e^{\pi+\mu}$.

Definition 51. *Let V be a finite-dimensional $\mathfrak{g}$-module. The element*
$$ch(V) = \sum_{\pi \in P} \dim(V^\pi) e^\pi = \sum_\pi \dim(V^\pi) e^\pi$$
of the algebra A is called the character of V. The equality follows by Proposition 5.1 i.e. for V all weights belong to P.

Proposition 42.

1. $ch(V)$ is invariant under the Weyl group W.

2. One has
$$ch(V \oplus V') = ch(V') + ch(V) \quad \textit{and} \quad ch(V \otimes V') = ch(V') \cdot ch(V)$$

3. The two finite-dimensional $\mathfrak{g}$-modules V and V' are isomorphic if and only if $ch(V) = ch(V')$.

Proof.

1. Making some computations it can be shown that
$$\theta = e^{X_i} e^{-Y_i} e^{X_i} : V^\pi \to V^{s_i(\pi)}$$
is an isomoprhism so that $\dim(V^\pi) = \dim(V^{s_i(\pi)})$ and where $s_i \in W$ is a simple symmetry. Since any element $w \in W$ is a product of s_i symmetries then this implies that the two weights which are equivalent under W will have the same multiplicity. Question: Does this fully mean that
$$\forall w \in W \implies \sum_{w(\pi) \in \mathfrak{h}^*} \dim(V^{w(\pi)}) e^{w(\pi)} = \sum_\pi \dim(V^\pi) e^\pi ?$$
i.e. if $\dim(V^{w(\pi)}) = \dim(V^\pi)$ it is not always true that $e^{w(\pi)} = e^\pi$, unless $\pi - w(\pi) = 0$?
2. The two formulas follow from the definition of $ch(V \oplus V')$ and $ch(V \otimes V')$.
3. Isomorphic two finite-dimensional $\mathfrak{g}$-modules V and V' clearly have the same characters. Therefore want to show that $ch(V) = ch(V')$ implies that V and V' are isomorphic. For this we use inductive argument.
   - If $\dim(V) = 0$, then $ch(V) = 0$ and this implies that $\dim(V') = 0$.

- Let P_V be the set of weights of V which is the same as that for V' since the characters of V and V' are equal. Then $P_V \neq \emptyset$, and since it is finite then one can find $\omega \in P_V$ such that $\omega + \alpha_i$ does not belong to P_V for any i.
- If v is a nonzero element of V^ω, then the condition on ω implies that $X_\alpha v = 0$ for all $\alpha \in R^+$ i.e. v is primitive element. By proposition 5.1, the submodule V_1 of V generated by v is irreducible and has highest weight ω. By Weyl's Theorem one has $V = V_1 \oplus V_2$ where V_2 is a submodule of V.
- The same argument, applied to V', shows that $V' = V_1' \oplus V_2'$, where V_1' is irreducible and has the highest weight ω. (Since $P_V = P_{V'}$ then $\omega \in P_{V'}$ such that $\omega + \alpha_i$ does not belong to $P_{V'}$ for any i.)
- Since V' and V have the same highest weight, they are isomorphic by Theorem 4.1 so that $ch(V_1) = ch(V_1')$. Using the formula from (2) we see that $ch(V_2) = ch(V_2')$ and the induction hypothesis implies that V_2 is isomorphic to V_2'. Thus V and V' are isomorphic as required.

■

10.8. H. Weyl's Formula

This formula allows one to calculate the character of an irreducible $\mathfrak{g}$-module as a function of its highest weight. Introducing some notation:

1. If $w \in W$, then $\epsilon(w)$ denotes the determinant of w. It is $+1$ if w is the product of an even number of symmetries s_α, and -1 otherwise.
2. We put $\rho = \frac{1}{2}\sum_{\alpha \in R^+} \alpha$; one can show that $\rho(H_i) = 1$ for all i, so that $\rho \in P$.
3. We put
$$D = \Pi_{\alpha \in R^+}(e^{\alpha/2} - e^{-\alpha/2})$$
The product being evaluated in the algebra $\mathbb{Z}[\frac{1}{2}P]$. In fact it can be shown that
$$D = \sum_{w \in W} \epsilon(w) e^{w(\rho)}$$

Theorem 52. *Let E be a finite-dimensional irreducible $\mathfrak{g}$-module, and ω its highest weight. One has*

$$ch(V) = \frac{1}{D} \sum_{w \in W} \epsilon(w) e^{w(\rho + \omega)}$$

Corollary 16. *The dimension of E is given by the formula*

$$\dim(E) = \Pi_{\alpha \in R^+} \frac{\langle \omega + \rho, H_\alpha \rangle}{\langle \rho, H_\alpha \rangle}$$

Chapter 11

Introduction to Compact Lie Groups

At the last chapters of our notes we want to introduce some important facts and results from the Theory of Compact Lie Groups, mainly borrowed from Brian Hall's book. Results are stated without proof - for reference see Brian Hall's book - Lie Groups, Lie Algebras and Representations.

11.0.1. Torus theorem and its consequences

Definition 52. *A torus in a compact Lie group G is a compact, connected, abelian Lie subgroup of G (and therefore isomorphic to the standard torus $\mathbb{T}^n$). A maximal torus is one which is maximal among such subgroups.*

Let G be a compact, connected Lie group and let $\mathfrak{g}$ be the Lie algebra of G. The first main result is the torus theorem, which may be formulated as follows:

Theorem 53. *If $\mathbb{T}$ is one fixed maximal torus in G, then every element of G is conjugate to an element of $\mathbb{T}$.*

This theorem has the following consequences:

- All maximal tori in G are conjugate.
- All maximal tori have the same dimension, known as the rank of G.
- A maximal torus in G is a maximal abelian subgroup, but the converse need not hold.
- The maximal tori in G are exactly the Lie subgroups corresponding to the maximal abelian subalgebras of $\mathfrak{g}$ (Cartan subalgebras).
- Every element of G lies in some maximal torus; thus, the exponential map for G is surjective.
- For semisimple groups the rank is equal to the number of nodes in the associated Dynkin diagram.

11.0.2. Root systems

If $\mathbf{T}$ is a maximal torus in a compact Lie group G, one can define a root system as follows. The roots are the weights for the adjoint action of $\mathbf{T}$ on the complexified Lie algebra of G. To be more explicit, let $\mathfrak{t}$ denote the Lie algebra of $\mathbf{T}$, let $\mathfrak{g}$ denote the Lie algebra of G, and let $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \oplus i\mathfrak{g}$ denote the complexification of $\mathfrak{g}$. Then we say that an element $\alpha \in \mathfrak{t}$ is a root for G relative to $\mathbf{T}$ if $\alpha \neq 0$ and there exists a nonzero $X \in \mathfrak{g}_{\mathbb{C}}$ such that

$$Ad_{e^H}(X) = e^{i\langle \alpha, H \rangle} X$$

for all $H \in \mathfrak{t}$. Here $\langle \cdot, \cdot \rangle$ is a fixed inner product on $\mathfrak{g}$ that is invariant under the adjoint action of connected compact Lie groups.

11.0.3. Weyl group

For the future references we will give the following definition.

Definition 53. *The identity component of a topological group G is the connected component G^0 of G that contains the identity element of the group. It can be shown that G^0 is a closed normal subgroup of G.*

Fix a maximal torus $T = T_0$ in G. The Weyl group W of G with respect to maximal torus T_0 is defined as

$$W(T_0) = N(T_0)/T_0$$

The first major result about this definition of Weyl group is as follows.

Theorem 54. *The Weyl group is generated by reflections about the roots of the associated Lie algebra. Thus, the Weyl group of T is isomorphic to the Weyl group of the root system of the Lie algebra of G. Since maximal tori are conjugate, then the resulting quotient groups are all isomorphic.*

We now list some consequences of these main results.

- Two elements in T are conjugate if and only if they are conjugate by an element of W. That is, each conjugacy class of G intersects T in exactly one Weyl orbit. In fact, the space of conjugacy classes in G is homeomorphic to the orbit space T/W.
- The Weyl group acts by (outer) automorphisms on T (and its Lie algebra).
- The identity component of the normalizer of T is also equal to T. The Weyl group is therefore equal to the component group of $N(T)$.
- The Weyl group is finite.

11.1. Loop groups

Definition 54. *An infinite dimensional Lie group is a group Γ which is at the same time an infinite dimensional smooth manifold, and is such that the composition law $\Gamma \times \Gamma \to \Gamma$ and the operation of inversion $\Gamma \to \Gamma$ are given by smooth maps.*

We will start with considering the following example of an infinite dimensional Lie group. Let X be a compact space and G be a finite dimensional Lie group.

Definition 55. *Define $Map_{cts}(X;G)$ to be the group of all smooth maps from space X to G, where the group law of $Map_{cts}(X;G)$ is pointwise composition in G and the natural topology on $Map_{cts}(X;G)$ is the topology of uniform convergence.*

We want to show that this group is a smooth manifold i.e. that it has an atlas with smooth transition functions.

- Let U be an open neighbourhood of the identity element in G. By exponential map, it is homeomorphic to some open set $\hat{U}$ of the Lie algebra $\mathfrak{g}$ of G.
- Let $\mathcal{U} = Map_{cts}(X;U)$, then it is an open neighbourhood of the identity in $Map_{cts}(X;G)$ which is homeomorphic to the open set $\hat{\mathcal{U}} = Map_{cts}(X;\hat{U})$ of the Banach space $Map_{cts}(X;\mathfrak{g})$.
- If f is any element of $Map_{cts}(X;G)$, then $\mathcal{U}_f = \mathcal{U} \cdot f$ is a neighbourhood of f, which is also homeomorphic to $\hat{\mathcal{U}}$. The sets $\{\mathcal{U}_f\}_{f \in Map_{cts}(X;G)}$ provides a covering of $Map_{cts}(X;G)$.
- Let E be an infinite dimensional topological vector space, then for each open set $\mathcal{U}_f$ can define a smooth homeomorphism $\phi_f : \mathcal{U}_f \to E$. Therefore $\{(\mathcal{U}_f, \phi_f)\}_{f \in Map_{cts}(X;G)}$ is an atlas for $Map_{cts}(X;G)$ which implies that it is a smooth manifold and therefore

Definition 56. *Let G be a finite dimensional Lie group. We define the smooth loop group LG of G to be $Map_{smth}(S^1;G)$, the set of all smooth maps from circle S^1 to G. From above it follows that LG is an infinite dimensional Lie group.*

Remark 39. *For the future reference, we shall think of the circle as consisting interchangeably of real numbers θ modulo 2π or of complex numbers $x = e^{i\theta}$ of modulus one.*

Remark 40. *The Lie algebra of $Map(S^1; G)$ is $Map(S^1; \mathfrak{g})$ with the exponential map (induced by Lie group G)*

$$exp : Map(S^1; G) \to Map(S^1; \mathfrak{g})$$

Let L_0G be an identity component of LG, then the exponential map $exp : L\mathfrak{g} \to L_0G$ may not be surjective.

Example: Consider LG, where $G = SU_2$. Then G is simply connected, so LG is connected imlying that it is its own identity component. Consider the element $\gamma : S^1 \to LG$ defined by

$$z \to \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix}.$$

If $\gamma = \exp(\xi)$ for some $\xi \in L\mathfrak{g}$, then ξ must be a diagonal matrix: but there is no such diagonal matrix with smooth function θ on the circle in its entries such that $e^{i\theta} = z$.

Remark 41. *When G has a complexification $G_{\mathbb{C}}$ then $Map(X; G)$ has the complexification $Map(X; G_{\mathbb{C}})$.*

11.2. Some group theoretic properties of $Map(X; G)$

In this section G will be a compact connected Lie group, and X be a compact smooth manifold. For brevity we shall denote the group of smooth maps $Map(X; G)$ by MG.

A group is said to be perfect if it equals its own commutator subgroup. If G is a semisimple group then it is a direct product of non-abelian simple groups, which implies that it is perfect. Want to show the following result.

Proposition 43. *If G is semisimple then the identity component M_0G is perfect, and in fact $[G, M_0G] = M_0G$.*

Proof. Let us first consider the case $G = SU_2$. It is a simple connected Lie group, so therefore its identity component is perfect. If

$$e_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

is the usual basis for the Lie algebra of G and $\mathbb{T}_1, \mathbb{T}_2, \mathbb{T}_3$ are the circle subgroups they generate, then the multiplication $\mathbb{T}_1 \times \mathbb{T}_2 \times \mathbb{T}_3 \to G$ is surjective. The multiplication

$$M\mathbb{T}_1 \times M\mathbb{T}_2 \times M\mathbb{T}_3 \to MG$$

is therefore sujective in the neighbourhood of the identity.

Since the subgroups $\mathbb{T}_1, \mathbb{T}_2, \mathbb{T}_3$ are conjugate and since

$$\forall g \in G \implies [gGg^{-1}, gM_0Gg^{-1}] = g[G, M_0G]g^{-1} = [G, M_0G]$$

then if $M\mathbb{T}_3 \in [G, M_0G]$ this implies that $M\mathbb{T}_1, M\mathbb{T}_2 \in [G, M_0G]$ i.e. $[G, M_0G] = M_0G$. Consider the element of M_0G $\begin{pmatrix} \phi^{-\frac{1}{2}} & 0 \\ 0 & \phi^{\frac{1}{2}} \end{pmatrix}$ for $\phi \in S^1$. Then

$$\begin{pmatrix} \phi & 0 \\ 0 & \phi \end{pmatrix} = [\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} \phi^{-\frac{1}{2}} & 0 \\ 0 & \phi^{\frac{1}{2}} \end{pmatrix}] \in M\mathbb{T}_3 \cap [G, M_0G], \quad where \ \ [x, y] = xyx^{-1}y^{-1}$$

which generates $M\mathbb{T}_3$.

The result for a general semisimple group G follows from the particular case of SU_2. One can find number of homomorphisms $i_1, ..., i_n : SU_2 \to G$ corresponding to the positive roots of G, such that the mutliplication map

$$\Pi_{k=1}^{n} i_k : (SU_2)^n \to G$$

is locally surjective and so that the induced map $(MSU_2)^n \to MG$ is also locally surjective. ■

The group of diffeomorphisms of X acts on MG as a group of automorphisms. Apart from that, there are pointwise automorphisms of MG arising from smooth maps $X \to A$, where A is the group of automorphisms of G. If G is simple then we have the following decomposition.

Proposition 44. *If G is simple then the group of automorphisms of M_0G is the semidirect product*

$$Diff(X) \ltimes MA.$$

11.3. Subgroups of LG: polynomial loops

Definition 57. *Let $L_{pol}G$ be the group of loops whose matrix entries are finite Laurent polynomials in z and z^{-1} i.e. all finite series of the form*

$$\sum_{k=-N}^{N} \gamma_k z^k, \quad where \quad \gamma_k \quad are \quad matrices.$$

The associated Lie algebra is $\{\sum_{k=-N}^{N} \xi_k z^k | \xi \in \mathfrak{g}_{\mathbb{C}, N \in \mathbb{N}}\}$. The space $L_{pol}G$ is endowed with direct limit topology.

Proposition 45. *If G is semisimple, then $L_{pol}G$ is dense in LG.*

Proof.

1. Let H be the closure of $L_{pol}G$ in LG, and let V be the subset of $L\mathfrak{g}$ formed by the tangent vectors ξ such that the corresponding one parameter group γ_ξ belongs to H.

2. Want to show that V is a vector space. There exists a suitable neighbourhood U of identity in G, such that for a given $t > 0$ and $\xi, \eta \in V$ it holds by Trotter's formula that

$$\gamma_{\xi+\eta}(t) = \lim_{n\to\infty} \gamma_\xi(\frac{t}{n})\gamma_\eta(\frac{t}{n})^{\frac{1}{n}}$$

   converges in C^∞ topology. It is clear that V is a closed subspace of $L\mathfrak{g}$. Because the exponential function is locally surjective in LG, then it is enough to show that $V = L\mathfrak{g}$.

3. Consider first the case where $G = SU_2$. Then the elements

$$\xi_n = \begin{pmatrix} 0 & z^n \\ -z^{-n} & 0 \end{pmatrix} \quad and \quad \eta_n = \begin{pmatrix} 0 & iz^n \\ iz^{-n} & 0 \end{pmatrix}$$

   satisfy $\xi_n^2 = \eta_n^2 = -1$ which implies that $\exp(t\xi_n), \exp(t\eta_n)$ one-parameter subgroups lie in $L_{pol}G$, so therefore $\xi_n, \eta_n \in V$.

4. By linearity then, and because it is closed, V contains every element of the form

$$f \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + g \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

   where f and g are smooth real-valued functions on the circle. But V is invariant under conjugation by constant elements of SU_2, so we must have $V = L\mathfrak{g}$.

5. The general case follows from the fact that for any semisimple G there are a finite number of homomorphisms $SU_2 \to G$ for which the images of $\mathfrak{su}_2$ in $\mathfrak{g}$ span $\mathfrak{g}$.

$\blacksquare$

11.4. Maximal abelian subgroups of LG

In this section we want to define the maximal abelian subgroups of LG, and in the future notes we will establish the connection between these subgroups and the Weyl group.

- If A is any abelian subgroup of LG then for any point θ of the circle the subgroup $A(\theta)$ of G got by evaluating the loops in A at θ is abelian, and so is contained in maximal torus of G.
- Above implies that the most obvious maximal abelian subgroup of LG is LT, where T is a maximal torus of G. More generally, if λ is a map which assigns a maximal torus $T_{\lambda(\theta)}$ of G smoothly to each point θ of the circle, then the subgroup
$$A_\lambda = \{\gamma \in LG : \gamma(\theta) \in T_{\lambda(\theta)} \forall \theta\}$$
is a maximal abelian group.

11.5. Weyl group of Compact Lie groups

For this section, given compact Lie group K, we will consider the complexification of Lie algebra $\mathfrak{k}_{\mathbb{C}}$. To maintain clarity of discussion for this section, we need to use some results from Chapter 7 from Brian C. Hall book. If $\mathfrak{k}$ is the Lie algebra of K then denote any maximal commutative subalgebra of $\mathfrak{k}$ as $\mathfrak{t}$, where our inner product is real on $\mathfrak{t}$.

Proposition 46. *Let $\mathfrak{k}$ be a Lie algebra of compact Lie group K, then $\mathfrak{k} = \mathfrak{k}_1 \oplus \mathfrak{z}(\mathfrak{k})$ where $\mathfrak{g}_1 = (\mathfrak{k}_1)_{\mathbb{C}}$ is semisimple.*

Proposition 47. *If K is a simply connected compact matrix Lie group with Lie algebra $\mathfrak{k}$, then $\mathfrak{g} = \mathfrak{k}_{\mathbb{C}}$ is semisimple.*

Remark 42. *The above two propositions imply that given compact Lie group K with Lie algebra $\mathfrak{k}$, it is simply connected if and only if $\mathfrak{z}(\mathfrak{k})$ is trivial.*

Proposition 48. *Let $\mathfrak{g} = \mathfrak{k}_{\mathbb{C}}$ be a complex semisimple Lie algebra and let $\mathfrak{t}$ be any maximal commutative subalgebra of $\mathfrak{k}$. Define $\mathfrak{h} \subset \mathfrak{g}$ by*
$$\mathfrak{h} = \mathfrak{t}_{\mathbb{C}} = \mathfrak{t} + i\mathfrak{t}.$$
Then $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$.

Proposition 49. *Each root α belongs to $i\mathfrak{t} \subset \mathfrak{h}$.*

Definition 58. *If T is a maximal torus in K, then the normalizer of T , denoted $N(T)$, is the group of elements $x \in K$ such that $xTx^{-1} = T$. The quotient group*
$$W = N(T)/T$$
is the Weyl group of T.

Definition 59 (How Weyl group acts on Lie algebra of maximal Torus)**.** *Let $x \in N(T)$, and denote the Lie algebra of T as $\mathfrak{t}$. By surjectivity of exp map we have that for any $H \in \mathfrak{t} \implies x\exp(Ht)x^{-1} \in T$ so therefore $\frac{\mathrm{d}}{\mathrm{d}t}|_{t=0} x\exp(Ht)x^{-1} = xHx^{-1} = Ad_x(H) \in \mathfrak{t}$. If w is an element of Weyl group W represented by $x \in N(T)$ then we define an action of W on $\mathfrak{t}\mathfrak{t}$ by*
$$w \cdot H = Ad_x(H), \quad H \in \mathfrak{t}.$$

Definition 60. *An element α of $\mathfrak{t}$ is real root of $\mathfrak{g}$ with respect to $\mathfrak{t}$ if $\alpha \neq 0$ and there exists a nonzero element X of $\mathfrak{g}$ such that*
$$\forall H \in \mathfrak{t} \implies [H, X] = i\langle \alpha, H\rangle X, \quad \text{where} \quad \langle \alpha, H\rangle \in \mathbb{R}.$$
For each real root α we define the associated real coroot H_α as $H_\alpha = 2\frac{\alpha}{\langle \alpha,\alpha\rangle}$.

Let $R \subset \mathfrak{t}$ denote the set of real roots, $\triangle$ denote a fixed base for R; and R^+ denote the associated set of positive (real) roots. The set of real roots R may not span $\mathfrak{t}$.

Proposition 50. *For each $\alpha \in R$, there is an element x in $N(T)$ such that*

$$Ad_x(H_\alpha) = -H_\alpha$$

and such that

$$Ad_x(H) = H, \quad \forall H \in \mathfrak{t} : \langle \alpha, H \rangle = 0$$

Thus, the adjoint action of x on $\mathfrak{t}$ is the reflection s_α about the hyperplane orthogonal to α.

Proof. ∎

Theorem 55. *If T is a maximal torus in K, the following results hold.*

1. *$Z(T) = T$ where $Z(T)$ is the centralizer of T.*
2. *The Weyl group acts effectively on $\mathfrak{t}$ and this action is generated by the reflections $s_\alpha, \alpha \in R$ from previous Proposition.*

Proof. ∎

Remark 43. *There may exist maximal commutative subgroups of K that are not maximal tori.*

This Theorem can verified directly in the case of $U(n)$ and $SU(n)$.

11.6. Fundamental Group in theory of Lie manifolds

11.6.1. Some intro definitions and results about fundamental group

Definition 61. *Fundamental group $\pi_1(X, x_0)$ of a topological space X is the group of the equivalence classes under homotopy of the loops based at $x_0 \in X$ and contained in the space. The group operations is concatenation of loops.*

Remark 44. *Higher homotopy groups $\pi_k(X), k = 1, 2, 3...$ are defined as the set of of homotopy classes of maps of $\mathbb{S}^k$ into X.*

Proposition 51. *For a $d-$sphere $\mathbb{S}^d$, $\pi_k(\mathbb{S}^d)$ is trivial if $k < d$.*

Proof. This result is plausible because for $k < d$ the image of a "typical" continuous map of $\mathbb{S}^k$ into $\mathbb{S}^d$ will not be all of $\mathbb{S}^d$. However, if the image of the map omits even one point in $\mathbb{S}^d$ then we can remove that point and what is left of the sphere can be contracted continuously to a point. ∎

Definition 62. *If X and Y are Hausdorff topological space, a continuous map $\pi : Y \to X$ is a covering map if*

1. *π maps Y onto X.*
2. *For each $x \in X$, there is a neighborhood V of x such that $\pi^{-1}(V)$ is a disjoint union of open sets U_α where the restriction of π to each U is a homeomorphism of U onto V.*

Definition 63. *A cover of X is a pair (π, Y), where $\pi : Y \to X$ is a covering map. If (π, Y) is a cover of X and Y is simply connected, then (π, Y) is a universal cover of X.*

Definition 64. *If (π, Y) is a cover of X and $f : Z \to X$ is a continuous map, then a map $\hat{f} : Z \to Y$ is a lift of f if $\hat{f}$ is continuous and $\pi(\hat{f}) = f$.*

Proposition 52 (Path Lifting Property)**.** *Suppose (π, Y) is a cover of X and that $p : [0, 1] \to X$ is a (continuous) path with $p(0) = x$. Then for each $y \in \pi^{-1}(\{x\})$ there is a unique lift $\hat{p}$ of p for which $\hat{p}(0) = y$.*

Proposition 53 (Homotopy Lifting Property)**.** *Suppose that l is a loop in X, that a path p in Y is a lift of l, and that l_s is a homotopy of l in X with basepoint fixed. Then there is a unique lift of l_s to a homotopy p_s of p in Y with endpoints fixed.*

Definition 65. *Suppose that B and F are Hausdorff topological spaces. A fiber bundle with base B and fiber F is a pair (X, p) of Hausdorff topological space X together with a continuous map $p : X \to B$ called the projection map, having the following properties.*

1. *For each b in B; the preimage $p^{-1}(b)$ of b in X is homeomorphic to F.*
2. *For every b in B, there is a neighborhood U of b such that $p^{-1}(U)$ is homeomorphic to $U \times F$ in such a way that the projection map is simply projection onto the first factor.*

Remark 45. *If $X = U \times F$ then the fiber bundle (X, p_1) is called trivial.*

Theorem 56. *Suppose that X is a fiber bundle with base B and fiber F. If $\pi_1(B)$ and $\pi_2(B)$ are trivial, then $\pi_1(X)$ is isomorphic to $\pi_1(F)$.*

Proof. According to a standard topological result (Theorem 4.41 and Proposition 4.48 in Allen Hatcher) there is a long exact sequence of homotopy groups for a fiber bundle, which contains the portion

$$\pi_2(B) \to^f \pi_1(F) \to^g \pi_1(X) \to^h \pi_1(B)$$

where each map f, g, h is a homomorphism and the image of each map is equal to the kernel of the following map. Since $\pi_2(B)$ is trivial then the image of f is trivial so kernel of g is trivial. Since $\pi_1(B)$ is trivial then so the kernel of h must be $\pi_1(X)$, which means that the image of g is $\pi_1(X)$. Thus, g is an isomorphism of $\pi_1(F)$ with $\pi_1(X)$. ■

Proposition 54. *Suppose G is a matrix Lie group and H is a closed subgroup of G. Then G has the structure of a fiber bundle with base G/H and fiber H where the projection map $p : G \to G/H$ is given by $p(x) = [x]$; with $[x]$ denoting the coset $xH \in G/H$.*

Proof. Consequence of results proven about quotient manifolds in section 11.4 of the book. ■

Proposition 55. *Consider the map $p : SO(n) \to S^{n-1}$ given by*

$$p(R) = R \cdot e_n$$

where $e_n = (0, ..., 0, 1)$. Then $(SO(n), p)$ is a fiber bundle with base S^{n-1} and fiber $SO(n-1)$.

Proof.

1. We have proven previously that $SO(n-1)$ is a closed subgroup of $Gl(n)$ and therefore it is a closed subgroup of $SO(n)$. We think of $SO(n-1) \subset SO(n)$ as the set of matrices of the form

   $$\begin{pmatrix} R' & 0 \\ 0 & 1 \end{pmatrix} \quad s.t. \quad R' \in SO(n-1).$$

   By the above proposition $(SO(n), q)$ is a fiber bundle where $q : SO(n) \to SO(n)/SO(n-1)$ so that $SO(n)/SO(n-1)$ is the base and $SO(n-1)$ is fiber. If we can show that p descends to a homeomorphism $SO(n)/SO(n-1) \to S^{n-1}$ then this would imply the result.

2. It is easy to see that $SO(n)$ acts transitively on S^{n-1}, therefore $p : SO(n) \to S^{n-1}$ is onto. Since kernel of p is $SO(n-1)$ then p descends to a continuous bijection between $SO(n)/SO(n-1)$ and S^{n-1}. Since $SO(n)/SO(n-1)$ and S^{n-1} are compact then the map has a continuous inverse and therefore homeomorphism.

■

Proposition 56. *For all $n \geq 3$, the fundamental group of $SO(n)$ is isomorphic to $\mathbb{Z}/2$: Meanwhile,*

$$\pi_1(SO(2)) \cong \mathbb{Z}$$

.

Proof. Suppose that n is at least 4, so that $n-1$ is at least 3. By proposition 2.1 above we have $\pi_1(S^{n-1})$ and $\pi_2(S^{n-1})$ being trivial and so by Theorem 2.1 and Prop. 2.5 we have that $\pi_1(SO(n))$ is isomorphic to $\pi_1(SO(n-1))$ and therefore $\pi_1(SO(n)) \cong \pi_1(SO(3))$ for all $n \geq 4$. Noting that $SO(3)$ is homeomorphic to $\mathbb{RP}^3$ therefore

$$\pi_1(SO(3)) \cong \pi_1(\mathbb{RP}^3) \cong \mathbb{Z}/2$$

Finally, we observe that $SO(2)$ is homeomorphic to the unit circle S^1 so that $\pi_1(SO(2)) \cong \mathbb{Z}$. ■

Remark 46. *The algorithm for proving this proposition can be used to examine the fundamental group of other compact Lie groups $SU(n)$ and $U(n)$.*

Proposition 57. *The group $SU(n)$ is simply connected for all $n \geq 2$ For all $n \geq 1$ we have that $\pi_1(U(n)) \cong \mathbb{Z}$*

Proof.

1. For all $n \geq 3$ the group $SU(n)$ acts transitively on the sphere S^{2n-1}. By a small modification of the proof of Proposition 2.5, $SU(n)$ is a fiber bundle with base S^{2n-1} and fiber $SU(n-1)$. Since $2n-1 > 3$ for all $n \geq 2$ Theorem 2.1 and Proposition 2.1 tell us that $\pi_1(SU(n)) \cong \pi_1(SU(n-1))$. Since $\pi_1(SU(2)) \cong \pi_1(S^3)$ is trivial, we conclude that $\pi_1(SU(n))$ is trivial for all $n \geq 2$.

2. The analysis of the case of $U(n)$ is similar. The fiber bundle argument shows that $\pi_1(U(n)) \cong \pi_1(U(n-1))$ for all $n \geq 2$ Since $U(1)$ is just the unit circle S^1, we have that $\pi_1(U(1)) \cong \mathbb{Z}$. Thus, $\pi_1(U(n)) \cong \mathbb{Z}$ for all $n \geq 1$.

■

11.6.2. Fundamental group of Noncompact Classical Groups

Using the polar decomposition, we can reduce the computation of the fundamental group of certain noncompact groups to the computation of the fundamental group of one of the compact groups. For example, we know that $GL(n;\mathbb{C})$ is homeomorphic to $U(n) \times V$ for a vector space V of $n \times n$ self-adjoint matrices. Using similar decomposition for $SL(n;\mathbb{C})$ and $SL(n;\mathbb{R})$ we have that $\pi_1(SL(n;\mathbb{C})) \cong \pi_1(SU(n))$ and $\pi_1(SL(n;\mathbb{R})) \cong \pi_1(SO(n))$. Thus can conclude

1. $\pi_1(SL(n;\mathbb{R})) \cong \mathbb{Z}/2$
2. $\pi_1(GL(n;\mathbb{C})) \cong \pi_1(U(n)) \cong \mathbb{Z}$
3. $\pi_1(SL(2;\mathbb{R})) \cong \pi_1\mathbb{Z}$
4. $SL(n;\mathbb{C})$ is simply connected.

Chapter 12

Verma modules

Remark 47. *Throughout this paper, we consider $\mathfrak{g}$ to be a semisimple Lie algebra with $\mathfrak{h}$ denoting its Cartan's subalgebra.*

One of the most remarkable results that should be taken away from this course concerns the classification of the finite-dimensional irreducible representations of a semisimple Lie algebra. In particular it was established that **the irreducible representations are classified by their *highest weights.*** In one of the lectures the following results from the book by Jean-Pierre Serre were shown:

Theorem 57.

1. *Every irreducible, finite-dimensional representation of $\mathfrak{g}$ has a highest weight.*
2. *Two irreducible, finite-dimensional representations of g with the same highest weight are isomorphic.*
3. *Let $\omega \in \mathfrak{h}^*$ and let E_ω be an irreducible $\mathfrak{g}$-module having ω as highest weight. For E_ω to be finite dimensional, it is necessary and sufficient that ω is dominant integral, i.e.*

$$(*) \text{ For all } \alpha \in R^+,\ \omega(H_\alpha) \text{ is an integer} \geq 0.$$

However, due to my insufficient background, the proof of the following important result (Theorem 2 Ch. 7) was skipped.

Theorem 58. *For each $\omega \in \mathfrak{h}^*$, there is an irreducible $\mathfrak{g}$-module with highest weight equal to ω.*

The main goal of this paper is to fill this gap by completing the proof for this result, learning how to construct irreducible $\mathfrak{g}$-module with Verma modules.

Definition 66. *Let (π, V) be a representation of $\mathfrak{g}$. An element $\lambda \in \mathfrak{h}$ is a* **weight** *of $\mathfrak{g}$ if there exists a nonzero vector $v \in V$ such that*

$$\pi(H)v = \langle \lambda, H\rangle v \quad \text{which} \quad \text{holds} \quad \forall H \in \mathfrak{h}$$

The **weight space** *corresponding to λ is the set of all $v \in V$ satisfying the above and the* **multiplicity** *of λ is the dimension of the corresponding weight space.*

Definition 67. *Representation (π, V) of $\mathfrak{g}$ is highest weight cyclic with highest weight $\mu \in \mathfrak{h}$ if there exists a nonzero vector $v \in V$ such that*

1. *$\pi(H)v = \langle H, \mu\rangle v$ for all $H \in \mathfrak{h}$*
2. *$\pi(X)v = 0$ for all $X \in \mathfrak{g}_\alpha$ with $\alpha \in R^+$*
3. *The smallest invariant subspace containing v is V.*

In Chapter 4 of Brian C. Hall's book, it is shown that for any positive integer m there exists representation of $\mathfrak{sl}(2, \mathbb{C})$ with the highest weight m. It is defined on V_m the space of homogeneous polynomials of degree m in two complex variables. The method for proving existence of these irreducible representations of $\mathfrak{sl}(2, \mathbb{C})$ can be extended for more general case of $\mathfrak{sl}(k, \mathbb{C})$ s.t. $k \geq 2$, but it does not resolve the general case problem for an irreducible lie algebra $\mathfrak{g}$. Although there were found multiple ways of constructing irreducible $\mathfrak{g}-$modules with chosen highest weight, we will discuss the one which requires construction of special modules called Verma modules and study some of their properties.

12.1. Universal enveloping algebra

Definition 68. *For any Lie algebra $\mathfrak{g}$, we define $(U(\mathfrak{g}), i)$ to be its universal enveloping algebra if $U(\mathfrak{g})$ is an associative algebra with identity element, where the map $i : \mathfrak{g} \to U(\mathfrak{g})$ satisfies the following three properties:*

1. *For any $X, Y \in \mathfrak{g}$ the Lie bracket is preserved by i, i.e.*

$$i([X, Y]) = i(X)i(Y) - i(Y)i(X)$$

2. *The algebra $U(\mathfrak{g})$ is generated by elements of the form $i(X)$ where $X \in \mathfrak{g}$.*

3. *If $\mathcal{A}$ is an associative unital algebra and $j : \mathfrak{g} \to \mathcal{A}$ which satisfies property (1) above, then there exists unique algebra homomorphism $\phi : \mathcal{A} \to U(\mathfrak{g})$ such that $\phi(1) = 1$ and $\forall X \in \mathfrak{g} : \phi(j(X)) = i(X)$.*

One might ask to justify that given $\mathfrak{g}$ its UEA $U(\mathfrak{g})$ exists. This can be achieved by completing the following construction. First we need to note that $\mathfrak{g}$ is a vector space and therefore from it, can be constructed a tensor algebra $T(\mathfrak{g})$ over $\mathfrak{g}$ defined as

$$T(\mathfrak{g}) = \bigoplus_{k=0}^{\infty} \mathfrak{g}^{\otimes k} \quad \textit{where} \quad \mathfrak{g}^{\otimes k} = \mathfrak{g} \otimes \cdots \otimes \mathfrak{g} \quad (k \textit{ times})$$

We define $T(\mathfrak{g})$ to be an associative algebra with identity where

$$(u_1 \otimes \cdots \otimes u_n) \cdot (v_1 \otimes \cdots \otimes v_m) = u_1 \otimes \cdots \otimes u_n \otimes v_1 \otimes \cdots \otimes v_m$$

Based on $T(\mathfrak{g})$, the universal enveloping algebra of $\mathfrak{g}$ can be defined as $U(\mathfrak{g}) = T(\mathfrak{g})/\sim$, where the equivalence relation is given by $\forall a, b \in T(\mathfrak{g}) : a \sim b \iff [a, b] = a \otimes b - b \otimes a$.

12.1.1. PBW

The key structure result for universal enveloping algebras which will be used later is Poincare'-Birkhoff-Witt theorem or PBW for short.

Theorem 59 (PBW)**.** *If $\mathfrak{g}$ is a finite-dimensional Lie algebra with basis $X_1, ..., X_k$ then elements of the form*

$$i(X_1)^{n_1} \cdots i(X_k)^{n_k} \quad \textit{such that} \quad n_i \in \mathbb{Z}_{\geq 0}$$

span $U(\mathfrak{g})$ and are linearly independent. In particular, this implies that $i(X_1), ..., i(X_n)$ are linearly independent, so then $i : \mathfrak{g} \to U(\mathfrak{g})$ is injective.

This result will not be proven in this paper but it should be noted that, by definition of UEA of $\mathfrak{g}$ it follows that $U(\mathfrak{g})$ is spanned by elements of the form stated in the theorem. The hard part is to prove their linear independence. Since the map i is injective then for simplifying the notation we will denote $i(X)$ as X.

Remark 48. *Given $\mathfrak{h}$ subalgebra of finite-dimensional Lie algebra $\mathfrak{g}$, by PBW theorem there exists a natural injection of $U(\mathfrak{h})$ into $U(\mathfrak{g})$.*

12.2. Construction of Verma modules and their structure

Verma modules can be equivalently defined in two ways: (1) As a quotient of the enveloping algebra or (2) By extension of scalars. We will mainly consider the first formulation and will address the second at the end of the paper. Let R^+ as the set of positive roots. Define I_μ to be a left ideal generated by elements from the two sets

$$(1)\ S_1 = \{X \in \mathfrak{g}_\alpha,\ \alpha \in R^+\} \quad \textit{and} \quad (2)\ S_2 = \{H - \langle \mu, H \rangle \cdot 1, \quad H \in \mathfrak{h}\}$$

In particular $I_\mu = \{\sum_j \beta_j \alpha_j : \beta_j \in U(\mathfrak{g}) \ \textit{ and } \ \alpha_j \in S_1 \cup S_2\}$.

Definition 69. Verma module *with highest weight μ is the quotient space $W_\mu = U(\mathfrak{g})/I_\mu$.*

Define a representation π_μ of $U(\mathfrak{g})$ acting on W_μ as follows

$$\pi_\mu(\alpha)([\beta]) = [\alpha\beta] \quad where \quad \alpha \in U(\mathfrak{g}) \quad and \quad [\beta] \in W_\mu$$

This action does not depend on the representative of $[\beta] = \beta + I_\mu$. For example if $\beta' \in [\beta]$ then $\beta' = \beta + \gamma$ for some $\gamma \in I_\mu$ and since I_μ is left ideal then $\alpha\beta' \in \alpha\beta + I_\mu = [\alpha\beta]$. Checking that $\pi_\mu(\alpha\gamma)([\beta]) = [\alpha\gamma\beta] = \pi_\mu(\alpha)\pi_\mu(\gamma)([\beta])$ and $\pi_\mu([\alpha,\beta])([\gamma]) = [\pi_\mu(\alpha), \pi_\mu(\beta)]([\gamma])$ and restricting π_μ to $\mathfrak{g}$, it is then a representation of $\mathfrak{g}$ acting on W_μ.

12.2.1. Properties of W_μ

The basic structure of W_μ can be shown with two results. Recall that the semisimple Lie algebra can be decomposed as $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{b}$ where $\mathfrak{n}^-$ is the span of vectors $Y_\alpha \in \mathfrak{g}_{-\alpha}$ for $\alpha \in R^+$ and $\mathfrak{b}$ denotes the Borel subalgebra of $\mathfrak{g}$.

Theorem 60. *The element $v_0 = [1]$ of W_μ is non-zero and W_μ is a highest weight cyclic representation with highest weight μ and highest weight vector v_0.*

Proof. (Part one) Suppose that v_0 is non-zero. Then from the definition of π_μ, since $H - \langle H, \mu\rangle 1 \in I_\mu$

$$\pi_\mu(H - \langle H, \mu\rangle 1)v_0 = [H - \langle H, \mu\rangle 1] = 0 \implies \pi_\mu(H)v_0 = \langle H, \mu\rangle v_0, \qquad \forall H \in \mathfrak{h}$$

Similarly, since for any $\alpha \in R^+ : X_\alpha \in I_\mu$, then it holds that $\pi_\mu(X_\alpha)v_0 = [X_\alpha] = 0$.

Suppose now that U is an invariant subspace of W_μ under action of $\mathfrak{g}$ and which contains $v_0 = [1]$. For any $\alpha \in \mathfrak{g}$ we have that $\pi_\mu(\alpha)v_0 = [\alpha]$ which implies that $U = W_\mu$. Therefore, by our assumption on $v_0 \neq 0$, it follows that v_0 is a cyclic highest weight vector for W_μ with highest weight μ. ■

Remark 49. *Suppose that U_μ is an invariant subspace of W_μ such that $v_0 \notin U_\mu$, then the conclusion for the above theorem also holds for the quotient $V_\mu = W_\mu/U_\mu$.*

Remark 50. *The above theorem gives the most important for us property that Verma modules possess. It is that the weights of the Verma module with highest weight μ will consist of all elements λ that can be obtained from μ by subtracting integer combinations of positive roots.*

Theorem 61. *If $Y_1, ..., Y_k$ form a basis for $\mathfrak{n}^-$, then the elements*

$$\pi_\mu(Y_1)^{n_1} \cdots \pi_\mu(Y_k)^{n_k} v_0 \quad where \quad n_j \in \mathbb{Z}_{\geq 0}$$

form a basis of W_μ.

Remark 51. *Let's recall that if $Y_1, ..., Y_k$ form a basis for $\mathfrak{n}^-$, then by PBW the basis of $U(\mathfrak{n}^-)$ is of the form*

$$i(Y_1)^{n_1} \cdots i(Y_k)^{n_k} \quad such\ that \quad n_i \in \mathbb{Z}_{\geq 0}$$

Then from this last theorem it follows that the map $\alpha \to \pi_\mu(\alpha)v_0$ is an isomorphism between $U(\mathfrak{n}^-)$ and W_μ.

The hardest part in proving the first theorem, it is to show that v_0 is non-zero. Showing this will also allow us to prove the second theorem.

Lemma 17. *Let J_μ denote the left ideal in $U(\mathfrak{b}) \subset U(\mathfrak{g})$ generated by elements of the form from S_1 and S_2. Then 1 does not belong to J_μ.*

Proof.

1. Let $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}^+$ denote the Borel subalgebra. Consider the following one-dimensional homorphism of $\mathfrak{b}$ with $\mathbb{C}$:

   $$\sigma_\mu(X + H) = \langle \mu, H\rangle$$

   Let $Z_1, Z_2 \in \mathfrak{b}$, then follows that $[Z_1, Z_2] \in \mathfrak{n}^+$ so that $\sigma_\mu([Z_1, Z_2]) = 0$. Since σ_μ is one dimensional then obviously follows that $[\sigma(Z_1), \sigma(Z_2)] = 0$, so therefore σ_μ is actually a representation of .

2. It follows that σ_μ has a natural extension to be a representation $\hat{\sigma}_\mu$ of $U(\mathfrak{b})$ such that $\hat{\sigma}_\mu(1) = 1$. The kernel of $\hat{\sigma}_\mu$ is a left ideal of $U(\mathfrak{b})$ which contains elements of the form

$$(1)\ S_1 = \{X \in \mathfrak{g}_\alpha,\ \alpha \in R^+\} \quad and \quad (2)\ S_2 = \{H - \langle \mu, H \rangle \cdot 1, \quad H \in \mathfrak{h}\}$$

Therefore $ker(\hat{\sigma}_\mu)$ contains J_μ. Since $1 \notin ker(\hat{\sigma}_\mu)$ then $1 \notin ker(J_\mu)$.

■

With this lemma covered we now are able to complete the proof of Theorem 3.1 and give proof of Theorem 3.2. All the steps are followed similar to Brian C. Hall's book [1].

Proof. (Part two) Want to show that $v_0 = [1] \in W_\mu$ is nonzero.

1. We use the decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{b}$ to choose basis $Y_1, ..., Y_k, Z_1, ..., Z_m$ of $\mathfrak{g}$ consisting of basis $Y_1, ..., Y_k$ for $\mathfrak{n}^-$ and $Z_1, ..., Z_m$ of $\mathfrak{b}$.
2. By applying PBW we see that any basis element of $U(\mathfrak{g})$ is of the form $Y_1^{n_1} Y_2^{n_2} \cdots Y_k^{n_k} a_{n_1,\dots,n_k}$ where each $a_{n_1,\dots,n_k} \in U(\mathfrak{b})$ and $n_i \in \mathbb{Z}_{\geq 0}$, so that for any $\alpha \in U(\mathfrak{g})$ it has unique expansion

$$\alpha = \sum_{n_1,\dots,n_k=0}^{\infty} Y_1^{n_1} Y_2^{n_2} \cdots Y_k^{n_k} a_{n_1,\dots,n_k} \qquad (***)$$

3. Suppose that $\alpha \in I_\mu$ then by definition of I_μ it holds that α is a linear combination of

$$Y_1^{n_1} Y_2^{n_2} \cdots Y_k^{n_k} b_{n_1,\dots,n_k}(H - \langle H, \mu \rangle 1) \ \ and \ \ Y_1^{n_1} Y_2^{n_2} \cdots Y_k^{n_k} b_{n_1,\dots,n_k} X_\alpha, \quad \alpha \in R^+$$

where $b_{n_1,\dots,n_k} \in U(\mathfrak{b})$ and by definition J_μ it holds that $b_{n_1,\dots,n_k} X_\alpha,\ \ b_{n_1,\dots,n_k}(H - \langle H, \mu \rangle 1) \in J_\mu \subset U(\mathfrak{b})$. Therefore if $\alpha \in I_\mu$ then in the $(***)$ expression $a_{n_1,\dots,n_k} \in J_\mu$.

4. If $\alpha = 1$ then by uniqueness of expansion $(***)$ it follows that $a_{n_1,\dots,n_k} = 0$ for all n_i except for the only case where $n_1 = ... = n_k = 0$ and $a_{n_1,\dots,n_k} = 1$. If $1 \in I_\mu$ then the previous point implies that $1 \in J_\mu$ which is a contradiction.

■

Proof. (For Theorem 3.2) Want to show that if $Y_1, ..., Y_k$ form a basis for $\mathfrak{n}^-$, then the elements

$$\pi_\mu(Y_1)^{n_1} \cdots \pi_\mu(Y_k)^{n_k} v_0 \ \ where \ \ n_j \in \mathbb{Z}_{\geq 0}$$

are linearly independent. Since they also span W_μ then they form the basis of W_μ.

1. Suppose that a linear combination of these vectors is equal to zero. Then its preimage in $U(\mathfrak{g})$ is a linear combination

$$\alpha = \sum_{n_1,\dots,n_k=0}^{\infty} Y_1^{n_1} Y_2^{n_2} \cdots Y_k^{n_k} a_{n_1,\dots,n_k}$$

belongs to I_μ.

2. From the proof of Theorem 3.1 it follows that if $\alpha \in I_\mu$ then all coefficients $a_{n_1,\dots,n_k} \in J_\mu$. Since $1 \notin J_\mu$ by above lemma then $a_{n_1,\dots,n_k} = 0$ for any choice of n_i.

■

Decomposition of W_μ

The following two results are left as exercise 6 and 7 at the end of Ch. 9 by Brian C. Hall, and they both add another important information about properties of W_μ.

Problem. Let X be the subspace of W_μ consisting of all those vectors that can be expressed as finite linear combinations of weight vectors. Show that X contains v_0 and is invariant under the action of $\mathfrak{g}$ on W_μ. ■

Solution. Let $\beta = \{Y_1, ..., Y_k\}$ be the basis of $\mathfrak{n}^-$. Consider any basis element of W_μ as described in Theorem 3.2. We know that for any $v \in W_\mu$ and root α it holds that

$$\pi_\mu(H)\pi_\mu(Y_\alpha)v = \pi_\mu(Y_\alpha)\pi_\mu(H)v - \alpha(H)\pi_\mu(Y_\alpha)v \implies \pi_\mu(H)\pi_\mu(Y_\alpha)^n v = \pi_\mu(Y_\alpha)^n\pi_\mu(H)v - n\alpha(H)\pi_\mu(Y_\alpha)^n v$$

For any $H \in \mathfrak{h}$ and any basis element z, by above result and induction it follows that

$$\pi_\mu(H)z = \pi_\mu(H)\pi_\mu(Y_1)^{n_1} \cdots \pi_\mu(Y_k)^{n_k} v_0 = \langle \mu, H\rangle z - \sum_{i=1}^{k} n_i \langle \alpha_i, H\rangle z \quad n_i \geq 0$$

Which implies that every basis element of W_μ is a weight vector, including v_0. Therefore by definition of X it follows that $W_\mu = X$ which is invariant under the action of $\mathfrak{g}$. ■

Remark 52. *Since every vector of W_μ can be expressed as finite linear combinations of weight vectors then follows that W_μ is the direct sum of its weight spaces.*

Problem. Let μ be any element of $\mathfrak{h}$ and let W_μ be the associated Verma module. Suppose $\lambda \in \mathfrak{h}$ can be expressed in the form

$$\lambda = \mu - n_1\alpha_1 - ... - n_k\alpha_k \qquad (*)$$

where $\alpha_1, ..., \alpha_k$ are the positive roots and $n_1, ..., n_k$ are non-negative integers. Show that the multiplicity of λ in W_μ is equal to the number of ways that $\mu - \lambda$ can be expressed as a linear combination of positive roots with non-negative integer coefficients. That is to say, the multiplicity of λ is the number of k-tuples of non-negative integers $(n_1, ..., n_k)$ for which $(*)$ holds. ■

Solution. Recall that if S is a basis for the root system then R^+ w.r. to S is defined as the set of roots which are linear combinations with non-negative integer coefficients, of elements of S. Therefore the set $\alpha_1, ..., \alpha_k$ of positive roots is not necessarily linearly independent i.e. there maybe more than one way to write $\mu - \lambda$ in terms of $\alpha_1, ..., \alpha_k$.

Suppose that we fixed the choice of non-negative integers $(n_1, ..., n_k)$ such that $\mu - \lambda = n_1\alpha_1 + ... + n_k\alpha_k$. Let $z = \pi_\mu(Y_1)^{n_1} \cdots \pi_\mu(Y_k)^{n_k} v_0$ where $Y_i \in \mathfrak{g}_{-\alpha_i}$. By Theorem 3.2 it holds that z is the only basis vector of W_μ with weight λ which corresponds to this choice of coefficients $(n_1, ..., n_k)$. Since weight space V^λ is has dimension of basis vectors that it contains, then follows that the multiplicity of λ is the number of k-tuples of non-negative integers $(n_1, ..., n_k)$ for which $(*)$ holds. ■

12.3. From Verma module to irreducible V_μ module

So far we have learned that there exist W_μ Verma modules which are highest weight cyclic representation with highest weight μ and highest weight vector v_0. Our goal is to see how we can use them to construct irreducible representations of $\mathfrak{g}$ of highest weight μ.

Definition 70. *From previous we know that W_μ is a direct sum of its weight spaces and that v_0 is a weight vector. Given Verma module W_μ let U_μ be a subspace of W_μ consisting of all vectors v such that*

1. *The v_0-component of v is zero*
2. *The v_0-component of $\pi_\mu(X_1) \cdots \pi_\mu(X_N)v$ is zero for any collection of vectors $X_1, ..., X_N$ in $\mathfrak{n}^+$.*

Proposition 58. *The space U_μ is an invariant subspace for the action of $\mathfrak{g}$.*

Proof. Let $\beta = \{Y_1, ..., Y_n, H_1, ..., H_r, X_1, ..., X_n\}$ be the basis of $\mathfrak{g}$. Choose any $v \in U_\mu$, $Z \in \mathfrak{g}$ and any subset of $X_{i_1}, ..., X_{i_k}$ from the basis. Want to show that the vector $\pi_\mu(X_{i_1}) \cdots \pi_\mu(X_{i_k})\pi_\mu(Z)v$ has zero v_0-component.

By using commutation relations we can rewrite this vector into a sum of linear combinations of vector that have form

$$\pi_\mu(Y_{j_1}) \cdots \pi_\mu(Y_{j_l})\pi_\mu(H_{l_1}) \cdots \pi_\mu(H_{l_m})\pi_\mu(X_{n_1}^{\sim}) \cdots \pi_\mu(X_{n_p}^{\sim})v$$

By assumption, $\pi_\mu(X_{n_1}^{\sim}) \cdots \pi_\mu(X_{n_p}^{\sim})v$ has zero v_0-component. If $w \neq v_0$ then for any choice it follows that

$$\pi_\mu(Y_{j_1}) \cdots \pi_\mu(Y_{j_l})\pi_\mu(H_{l_1}) \cdots \pi_\mu(H_{l_m})w \neq v_0$$

Therefore the result follows. ■

Proposition 59. *The quotient space $V_\mu = W_\mu / U_\mu$ is an irreducible representation of $\mathfrak{g}$.*

Proof.

1. Want to show that there is any invariant subspace of W_μ which contains U_μ must be either U_μ or W_μ.

2. Let X be an invariant subspace that contains U_μ and a vector v such that $u = \pi_\mu(X_1)\cdots\pi_\mu(X_m)v$ has a nonzero v_0-component for some $X_1, ..., X_m \in \mathfrak{n}^+$. Want to show that $v_0 \in X$ i.e $X = W_\mu$.

3. By assumption we can decompose $u = a\cdot v_0 + b\cdot w$ where w is a finite sum of weight vectors i.e. $w = \sum_i c_i w_i$ where w_i is a weight vector corresponding to weight λ_i.

4. Consider operators $P_i = \pi_\mu(H) - \langle \lambda_i, H\rangle$ such that $P_i u = \langle \mu - \lambda_i, H\rangle v_0 + w^\sim$ where $w^\sim$ has one less vector (i.e. no w_i) in its decomposition. Repeating this process with different operators P_i we find that $v_0 \in X$, since X is invariant subspace.

5. By Theorem 1.1. part (3), it follows that if μ is integral dominant then V_μ is finite-dimensional. Using the remark after Theorem 3.1 and all of the above, we see that W_μ is the highest weight cyclic irreducible representation of $\mathfrak{g}$ with highest weight μ.

■

12.4. An explicit example of construction with $\mathfrak{sl}(2;\mathbb{C})$

For the special case of $\mathfrak{sl}(2;\mathbb{C})$ all of the previous results can be explicitly computed. Given complex number $\mu \in \mathbb{C}$ define W_μ to be the vector space of infinite sequences with finite number of nonzero entries. Consider $e_0, e_1, e_2, ...$ to be the standard basis of W_μ where $e_0 = (1, 0, 0, 0, ...), e_1 = (0, 1, 0, 0, ...)$, etc. and define the action of $\mathfrak{sl}(2;\mathbb{C})$ on W_μ by the same formulas we saw many times before:

$$(1)\ \pi_\mu(Y)e_i = e_{i+1} \quad (2)\ \pi_\mu(H)e_i = (\mu - 2i)e_i \quad (3)\ \pi_\mu(X)e_0 = 0 \quad (4)\ \pi_\mu(X)e_i = i(\mu - (i-1))v_{i-1}$$

Direct computations confirm that W_μ is a $\mathfrak{g}-$module. It follows that the highest weight of W_μ is μ with its corresponding highest weight vector is e_0 and that W_μ satisfies the definition of highest weight cyclic with highest weight μ. Therefore W_μ is Verma module for this example. Suppose that μ is dominant integral i.e. a non-negative integer m. Then by the formula (4) we get that

$$\pi_\mu(X)v_{m+1} = (m+1)(m-m)v_m = 0$$

It follows that $U_m = \{e_i : i \geq m+1\} \cup \{0\}$ is an invariant space under action of $\mathfrak{sl}(2;\mathbb{C})$ which does not contain e_0. Therefore can conclude that $V_\mu = W_\mu / U_\mu$ where $\mu = m$ is an irreducible $\mathfrak{sl}(2;\mathbb{C})-$module with weight m.

12.5. Bibliography

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