

(a) Show that an undirected graph has an Eulerian tour if and only if all its vertices have even degree.

Let the tour of Königsberg bridges be represented by the following set of vertices: {NB, BI, SB, SI}

Next, let D represent all the possible degrees that the vertices can have.

If all vertices are degree two, then each vertex is connected to two other vertices exactly once.

Since there is only one path between any two vertices, anyone crossing the edges need only to follow the open path,

or rather, the edge they did not just cross and in doing so closed off, and eventually, they will reach

a place that took total edges - 1 with only the edge back to the starting point with 4 total moves.

Next, let every vertex be degree = 4. Then for every possible path, we get an extra reversible path that

in a similar fashion, allows them to start and finish at the same vertex, while taking only unique edges,

this time, with 8 moves. We can do this an unlimited amount of times, which for every degree $2k$, then the Eulerian tour can be completed in $4k$ steps. However, adding an odd degree will prevent this tour to be completed because of the following. Since the mode of thinking is to go out an even amount of spacing, then reach the farthest point in k steps where you can trace back using k steps which will all be unique

Hence, adding an odd number will prevent our ability to trace back in k steps since we will either go $k+1$

(1 step too far), or $k-1$ steps (1 step too short) which will prevent a unique path back to the starting point. Hence, an undirected graph has an Eulerian tour if and only if all its vertices

have even degree. (No Eulerian tour of Königsberg Bridge)

(b) Give an iff characterization of which undirected graphs have Eulerian paths

An Eulerian path exists in an undirected graph \Leftrightarrow 2 vertices have some odd degree, and the rest all have even degrees

(c) Can you give an analog of part (a) for directed graphs?

For every vertex u_1, u_2, u_3, u_4 ,

for every outbound edge going from one vertex to another, there must be an incoming vertex going from that vertex, back to the original edge. (Or in other words, splitting the direction of the graph, the graph must still follow the even degree rule from (a))