MODEL PREDICTIVE CONTROL

HYBRID MODELS FOR MPC

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COURSE STRUCTURE

- ✔ Linear model predictive control (MPC)
- ✔ Linear time-varying and nonlinear MPC
- ✔ MPC computations: quadratic programming (QP), explicit MPC
- Hybrid MPC
- Stochastic MPC
- Data-driven MPC

MATLAB Toolboxes:

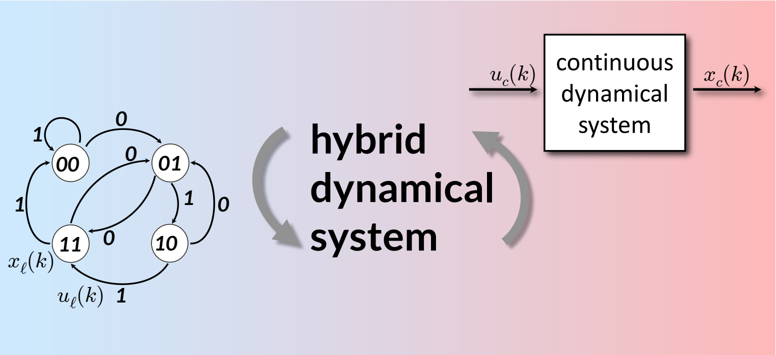
- **MPC Toolbox** (linear/explicit/parameter-varying MPC)
- **Hybrid Toolbox** (explicit MPC, hybrid systems)

Course page:

`http://cse.lab.imtlucca.it/~bemporad/mpc_course.html`

HYBRID MODELS

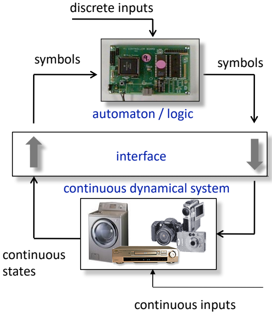
HYBRID DYNAMICAL SYSTEMS

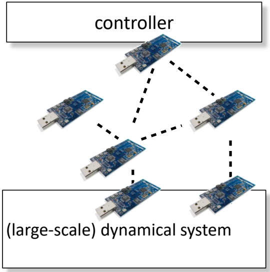


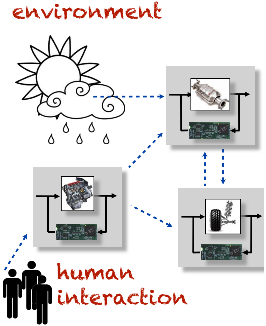
- Variables are **binary-valued**
  $x_\ell \in \{0,1\}^{n_\ell},\ u_\ell \in \{0,1\}^{m_\ell}$
- Dynamics = **finite state machine**
- **Logic constraints**

- Variables are **real-valued**
  $x_c \in \mathbb{R}^{n_c},\ u_c \in \mathbb{R}^{m_c}$
- **Difference/differential equations**
- **Linear inequality** constraints

TECHNOLOGICAL PUSH FOR STUDYING HYBRID SYSTEMS







embedded systems

networked control systems

cyber-physical systems

>1995 >2005 >2010

AN EXAMPLE OF "INTRINSICALLY HYBRID" SYSTEM

- Vehicle

continuous dynamical variables (speed, torque, ...)

continuous commands (brake & gas pedal)

+

discrete command (R,N,1,2,3,4,5)

KEY REQUIREMENTS FOR HYBRID MODELS

- **Descriptive** enough to capture the behavior of the system
  - **continuous** dynamics (physical systems)
  - **logic** components (switches, automata)
  - **interconnection** between logic and dynamics
- **Simple** enough for solving analysis and synthesis problems

$$\left\{\begin{array}{rcl} x' & = & Ax + Bu \\ y & = & Cx + Du \end{array}\right.$$

$\longleftrightarrow$ **linear hybrid systems** $\longleftrightarrow$

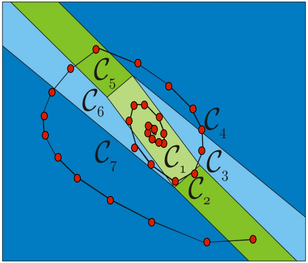
$$\left\{\begin{array}{rcl} x' & = & f(x,u,t) \\ y & = & g(x,u,t) \end{array}\right.$$

> "Perfection is achieved not when there is nothing more to add, but when there is nothing left to take away."

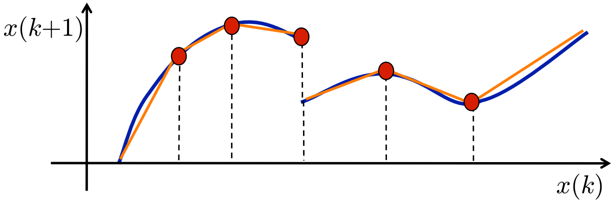
A. de Saint-Exupéry
(1900–1944)

PIECEWISE AFFINE SYSTEMS

$$\begin{aligned} x(k+1) &= A_{i(k)}x(k) + B_{i(k)}u(k) + f_{i(k)} \\ y(k) &= C_{i(k)}x(k) + D_{i(k)}u(k) + g_{i(k)} \\ i(k) \quad \text{s.t.} & \quad H_{i(k)}x(k) + J_{i(k)}u(k) \leq K_{i(k)} \end{aligned}$$

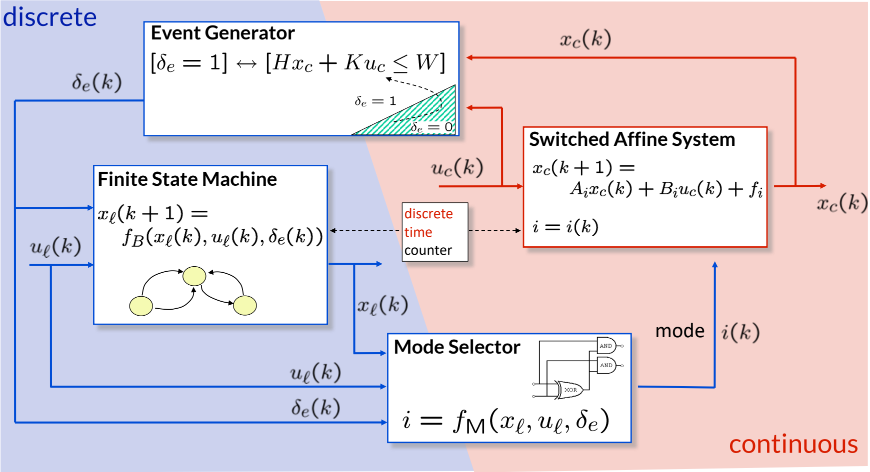


- PWA systems can approximate nonlinear dynamics arbitrarily well (even discontinuous ones)



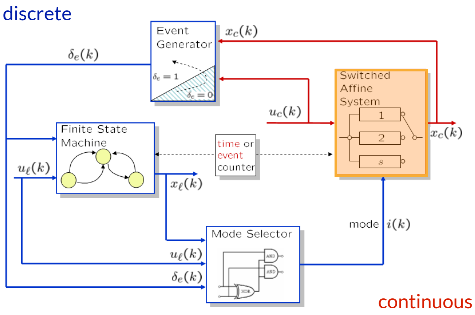
DISCRETE HYBRID AUTOMATON (DHA)

(Torrisi, Bemporad, 2004)



$x_\ell \in \{0,1\}^{n_\ell}$	$=$	**binary state**	$x_c \in \mathbb{R}^{n_c}$	$=$	**real-valued state**
$u_\ell \in \{0,1\}^{m_\ell}$	$=$	**binary input**	$u_c \in \mathbb{R}^{m_c}$	$=$	**real-valued input**
$\delta_e \in \{0,1\}^{n_e}$	$=$	**event variable**	$i \in \{1,\ldots,s\}$	$=$	**current mode**

SWITCHED AFFINE SYSTEM

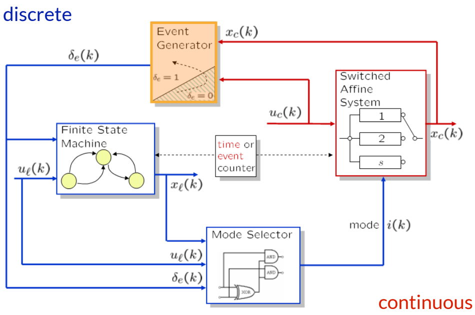


- The **affine dynamics** depend on the current mode $i(k)$:

$$x_c(k+1) = A_{i(k)} x_c(k) + B_{i(k)} u_c(k) + f_{i(k)}$$

$x_c \in \mathbb{R}^{n_c}$, $u_c \in \mathbb{R}^{m_c}$

EVENT GENERATOR



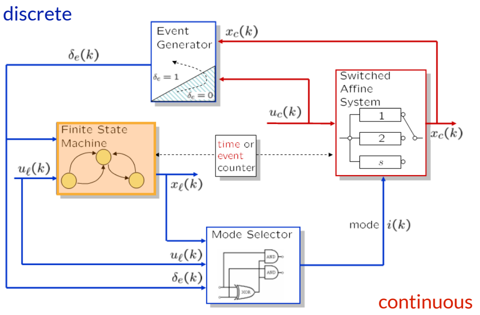
- **Event variables** are generated by **linear threshold conditions** over continuous states, continuous inputs, and time:

$$[\delta_e^i(k) = 1] \leftrightarrow [H^i x_c(k) + K^i u_c(k) \leq W^i]$$

$$x_c \in \mathbb{R}^{n_c}, \quad u_c \in \mathbb{R}^{m_c}$$
$$\delta_e \in \{0,1\}^{n_e}$$

- Example: $[\delta_e(k) = 1] \leftrightarrow [x_c(k) \geq 0]$

FINITE STATE MACHINE



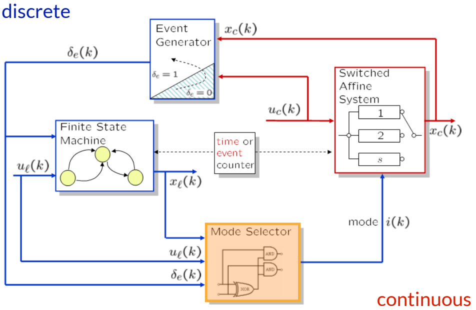
- The binary state of the **finite state machine** evolves according to a Boolean state update function $f_B : \{0,1\}^{n_\ell+m_\ell+n_e} \to \{0,1\}^{n_\ell}$:

$$x_\ell(k+1) = f_B(x_\ell(k), u_\ell(k), \delta_e(k))$$

$$x_\ell \in \{0,1\}^{n_\ell}, \quad u_\ell \in \{0,1\}^{m_\ell}$$
$$\delta_e \in \{0,1\}^{n_e}$$

- Example: $x_\ell(k+1) = \neg\delta_e(k) \vee (x_\ell(k) \wedge u_\ell(k))$

MODE SELECTOR



The mode selector can be seen as the output function of the discrete dynamics

- The active **mode** $i(k)$ is selected by a Boolean function of the current binary states, binary inputs, and event variables:

$$i(k) = f_M(x_\ell(k), u_\ell(k), \delta_e(k))$$

$$x_\ell \in \{0,1\}^{n_\ell}, \quad u_\ell \in \{0,1\}^{m_\ell}$$
$$\delta_e \in \{0,1\}^{n_e}$$

- Example:

$$i(k) = \begin{bmatrix} \neg u_\ell(k) \vee x_\ell(k) \\ u_\ell(k) \wedge x_\ell(k) \end{bmatrix}$$

u_ℓ / x_ℓ	0	1
0	$i = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$	$i = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$
1	$i = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$	$i = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

the system has 3 modes

CONVERSION OF LOGIC FORMULAS TO LINEAR INEQUALITIES

(Glover, 1975) (Williams, 1977) (Hooker, 2000)

- Key observation: $X_1 \vee X_2 = \texttt{true}$ ⟶ $\delta_1 + \delta_2 \geq 1, \delta_1, \delta_2 \in \{0,1\}$
- We want to impose the Boolean statement

$$F(X_1, \ldots, X_n) = \texttt{true}$$

- Convert the formula to **Conjunctive Normal Form (CNF)**

$$\bigwedge_{j=1}^{m} \left(\bigvee_{i \in P_j} X_i \bigvee_{i \in N_j} \bar{X}_i \right) = \texttt{true}, \quad P_j \cup N_j \subseteq \{1, \ldots, n\}$$

- Transform the CNF into the equivalent linear inequalities

$$\left\{ \begin{array}{rcl} \sum_{i \in P_1} \delta_i + \sum_{i \in N_1} (1 - \delta_i) & \geq & 1 \\ \vdots & & \vdots \\ \sum_{i \in P_m} \delta_i + \sum_{i \in N_m} (1 - \delta_i) & \geq & 1 \end{array} \right. \quad \Longrightarrow \quad A\delta \leq b, \ \delta \in \{0,1\}^n$$

polyhedron

Any logic proposition can be translated into integer linear inequalities

LOGIC $\rightarrow$ INEQUALITIES: SYMBOLIC APPROACH

- Example:

$$F(X_1, X_2, X_3) = [X_3 \leftrightarrow X_1 \wedge X_2]$$

- Convert Conjunctive Normal Form (CNF):

  (see e.g. `http://formal.cs.utah.edu:8080/pbl/PBL.php` or just google "CNF + converter" …)

$$(X_3 \vee \neg X_1 \vee \neg X_2) \wedge (X_1 \vee \neg X_3) \wedge (X_2 \vee \neg X_3)$$

- Transform into inequalities:

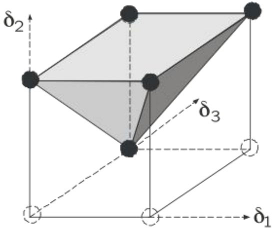
$$\left\{ \begin{array}{rcl} \delta_3 + (1-\delta_1) + (1-\delta_2) & \geq & 1 \\ \delta_1 + (1-\delta_3) & \geq & 1 \\ \delta_2 + (1-\delta_3) & \geq & 1 \end{array} \right.$$

- Consider the Boolean statement $F(X_1, \ldots, X_n) = \texttt{true}$ and collect the rows of the **truth table** $T(F)$ of F

> The **convex hull** $P = \{\delta \in \mathbb{R}^n : A\delta \leq b\}$ of the **rows in** $T(F)$ is the smallest polytope equivalent to the Boolean statement F

(Mignone, Bemporad, Morari, 1999)

$$T(F) : \begin{array}{c|c|c|c} X_1 & X_2 & \ldots & X_n \\ \hline 0 & 0 & \ldots & 1 \\ 0 & 1 & \ldots & 1 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & \ldots & 0 \end{array}$$



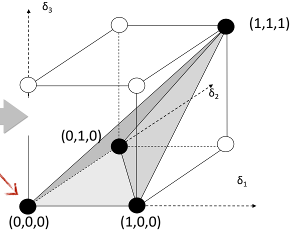
- Convex hull packages: `cdd`, `lrs`, `qhull`, `chD`, `Hull`, `Porto`
  `CDDMEX` package by K. Fukuda included in the Hybrid Toolbox

LOGIC $\longrightarrow$ INEQUALITIES: GEOMETRIC APPROACH

- Example: $F(X_1, X_2, X_3) = [X_3 \leftrightarrow X_1 \wedge X_2]$ (logic **and**)

$T(F)$:

X_1	X_2	X_3
0	0	0
0	1	0
1	0	0
1	1	1



- **Key idea:** white points cannot be inside the convex hull of black points

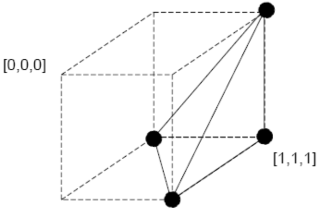
$$\mathrm{conv}\left(\begin{bmatrix}0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix}\right) = \left\{\delta \in \mathbb{R}^3 : \begin{array}{rcl} -\delta_1 + \delta_3 & \leq & 0 \\ -\delta_2 + \delta_3 & \leq & 0 \\ \delta_1 + \delta_2 - \delta_3 & \leq & 1 \end{array}\right\}$$

```
>> V=struct('V',[0 0 0;0 1 0;1 0 0;1 1 1]);
>> H=cddmex('hull',V);A=H.A,b=H.B
```

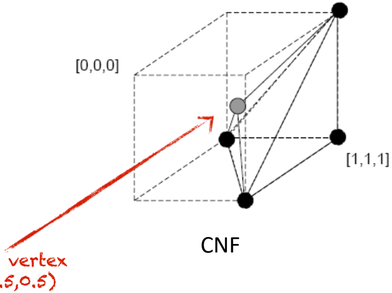
GEOMETRIC VS SYMBOLIC APPROACH

- The polyhedron obtained via convex hull is the smallest one
- The one obtained via CNF may be larger. Example:

$$(X_1 \vee X_2) \wedge (X_1 \vee X_3) \wedge (X_2 \vee X_3) = \texttt{true}$$



convex hull



CNF

- **Note:** no other example with 3 vars but
  $(X_1 \vee X_2) \wedge (X_1 \vee X_3) \wedge (X_2 \vee X_3) \wedge (\neg X_1 \vee \neg X_2 \vee \neg X_3)$ = $\texttt{true}$

BIG-M TECHNIQUE (IFF)

- Consider the **if-and-only-if** condition

$$\boxed{[\delta = 1] \leftrightarrow [a'x_c - b \leq 0]} \qquad \begin{array}{l} x_c \in \mathcal{X} \\ \delta \in \{0, 1\} \end{array}$$

- Assume $\mathcal{X} \subset \mathbb{R}^{n_c}$ bounded. Let M and m such that $\forall x_c \in \mathcal{X}$

$$\begin{array}{rcl} M & > & a'x_c - b \\ m & < & a'x_c - b \end{array}$$

- The if-and-only-if condition is equivalent to

$$\boxed{\left\{ \begin{array}{rcl} a'x_c - b & \leq & M(1 - \delta) \\ a'x_c - b & > & m\delta \end{array} \right.}$$

- We can replace the second constraint with $a'x_c - b \geq \epsilon + (m - \epsilon)\delta$ to avoid strict inequalities, where $\epsilon > 0$ is a small number (e.g., the machine precision)

BIG-M TECHNIQUE (IF-THEN-ELSE)

- Consider the **if-then-else** condition

$$z = \begin{cases} a_1' x_c - b_1 & \text{if } \delta = 1 \\ a_2' x_c - b_2 & \text{otherwise} \end{cases}$$

$$x_c \in \mathcal{X}$$
$$\delta \in \{0,1\}$$
$$z \in \mathbb{R}$$

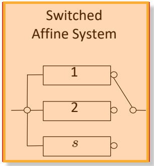
- Assume $\mathcal{X} \subset \mathbb{R}^{n_c}$ bounded. Let M_1, M_2 and m_1, m_2 such that $\forall x_c \in \mathcal{X}$

$$\begin{array}{ccccc} M_1 & > & a_1' x_c - b_1 & > & m_1 \\ M_2 & > & a_2' x_c - b_2 & > & m_2 \end{array}$$

- The if-then-else condition is equivalent to

$$\begin{cases} (m_1 - M_2)(1-\delta) + z & \leq & a_1' x_c - b_1 \\ (m_2 - M_1)(1-\delta) - z & \leq & -(a_1' x_c - b_1) \\ (m_2 - M_1)\delta + z & \leq & a_2' x_c - b_2 \\ (m_1 - M_2)\delta - z & \leq & -(a_2' x_c - b_2) \end{cases}$$

SWITCHED AFFINE SYSTEM



- The state-update equation of a SAS can be rewritten as

$$x_c(k+1) = \sum_{i=1}^{s} z_i(k) \qquad z_i(k) \in \mathbb{R}^{n_c}$$

with

$$\begin{aligned} z_1(k) &= \begin{cases} A_1 x_c(k) + B_1 u_c(k) + f_1 & \text{if } \delta_1(k) = 1 \\ 0 & \text{otherwise} \end{cases} \\ &\vdots \\ z_s(k) &= \begin{cases} A_s x_c(k) + B_s u_c(k) + f_s & \text{if } \delta_s(k) = 1 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

and with $\delta_i(k) \in \{0,1\}$ subject to the **exclusive or** condition

$$\sum_{i=1}^{s} \delta_i(k) = 1 \text{ or equivalently } \begin{cases} \sum_{i=1}^{s} \delta_i(k) & \geq & 1 \\ \sum_{i=1}^{s} \delta_i(k) & \leq & 1 \end{cases}$$

- Output eqs $y_c(k) = C_i x_c(k) + D_i u_c(k) + g_i$ admit similar transformation

TRANSFORMATION OF A DHA INTO LINEAR (IN)EQUALITIES

$X_1 \vee X_2 = \text{TRUE} \quad \longrightarrow \quad \delta_1 + \delta_2 \geq 1, \qquad \delta_1, \delta_2 \in \{0, 1\}$

Any logic statement $f(X) = \text{TRUE}$

$$\bigwedge_{j=1}^{m} \left(\vee_{i \in P_j} X_i \vee_{i \in N_j} \neg X_i \right) \quad \text{(CNF)}$$

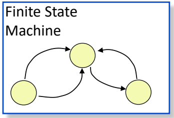
$N_j, P_j \subseteq \{1, \ldots, n\}$

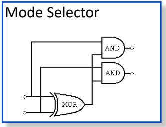
$$\longrightarrow \quad \begin{cases} 1 \leq \displaystyle\sum_{i \in P_1} \delta_i + \sum_{i \in N_1} (1 - \delta_i) \\ \vdots \\ 1 \leq \displaystyle\sum_{i \in P_m} \delta_i + \sum_{i \in N_m} (1 - \delta_i) \end{cases}$$

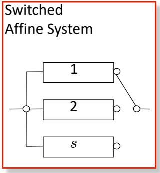
$$[\delta_e^i(k) = 1] \leftrightarrow [H^i x_c(k) \leq W^i] \quad \longrightarrow \quad \begin{cases} H^i x_c(k) - W^i \leq M^i(1 - \delta_e^i(k)) \\ H^i x_c(k) - W^i > m^i \delta_e^i(k) \end{cases}$$

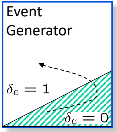
IF $[\delta = 1]$ THEN $z = a_1^T x + b_1^T u + f_1$
ELSE $z = a_2^T x + b_2^T u + f_2$

$$\longrightarrow \quad \begin{cases} (m_1 - M_2)(1 - \delta) + z \leq a_1 x + b_1 u + f_1 \\ (m_2 - M_1)(1 - \delta) - z \leq -a_1 x - b_1 u - f_1 \\ (m_2 - M_1)\delta + z \leq a_2 x + b_2 u + f_2 \\ (m_1 - M_2)\delta - z \leq -a_2 x - b_2 u - f_2 \end{cases}$$





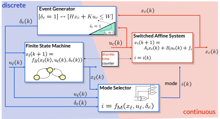




MIXED LOGICAL DYNAMICAL (MLD) SYSTEMS

(Bemporad, Morari, 1999)

- By converting logic relations into mixed-integer linear inequalities a DHA can be rewritten as the **Mixed Logical Dynamical (MLD)** system



$$\left\{\begin{array}{rcl} x(k+1) & = & Ax(k) + B_1u(k) + B_2\delta(k) + B_3z(k) + B_5 \\ y(k) & = & Cx(k) + D_1u(k) + D_2\delta(k) + D_3z(k) + D_5 \\ E_2\delta(k) & + & E_3z(k) \leq E_4x(k) + E_1u(k) + E_5 \end{array}\right.$$

$$x \in \mathbb{R}^{n_c} \times \{0,1\}^{n_b},\ u \in \mathbb{R}^{m_c} \times \{0,1\}^{m_b}$$
$$y \in \mathbb{R}^{p_c} \times \{0,1\}^{p_b},\ \delta \in \{0,1\}^{r_b},\ z \in \mathbb{R}^{r_c}$$

- The translation from DHA to MLD can be automatized, see e.g. the language **HYSDEL** (HYbrid Systems DEscription Language) (Torrisi, Bemporad, 2004)

- MLD models allow solving MPC, verification, state estimation, and fault detection problems via **mixed-integer programming**

A SIMPLE EXAMPLE OF MLD SYSTEM

- PWA system[1]: $$x(k+1) = \begin{cases} 0.8x(k) + u(k) & \text{if} \quad x(k) \geq 0 \\ -0.8x(k) + u(k) & \text{if} \quad x(k) < 0 \end{cases}$$

- Introduce event variable $[\delta(k) = 1] \leftrightarrow [x(k) \geq 0]$ and use big-M technique:

$$\longrightarrow \quad \begin{array}{l} x(k) \geq m(1 - \delta(k)) \\ x(k) \leq -\epsilon + (M + \epsilon)\delta(k) \end{array} \qquad \begin{array}{l} M = -m = 10 \\ \epsilon > 0 \text{ "small"} \end{array}$$

- Since $x(k+1) = 1.6\delta(k)x(k) - 0.8x(k) + u(k)$, introduce the aux variable

$$z(k) = \delta(k)x(k) \quad \longrightarrow \quad \begin{array}{rcl} z(k) & \leq & M\delta(k) \\ z(k) & \geq & m\delta(k) \\ z(k) & \leq & x(k) - m(1 - \delta(k)) \\ z(k) & \geq & x(k) - M(1 - \delta(k)) \end{array} \qquad \delta(k) \in \{0, 1\}$$

- Linear state update: $$x(k+1) = -0.8x(k) + 1.6z(k) + u(k)$$

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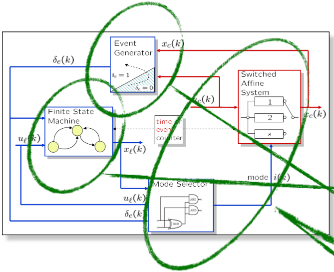
[1] This is the nonlinear system $x(k+1) = 0.8|x(k)| + u(k)$

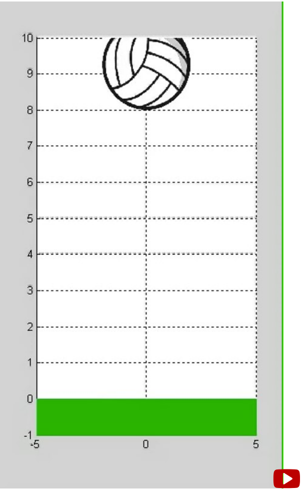
DHA AND HYSDEL MODELS




```
SYSTEM name {
   INTERFACE {
      STATE {
         REAL xc [xmin,xmax];
         BOOL xl; }
      INPUT {
         REAL uc [umin,umax];
         BOOL ul; }
      PARAMETER {
         REAL param1 = 1;}
   } /* end of interface */

   IMPLEMENTATION {
      AUX { BOOL d;
            REAL z; }

      AUTOMATA { xl = xl & ~ul; }

      AD { d = xc - 1 <= 0; }

      DA { z = { IF d THEN 2*xc ELSE -xc }; }

      CONTINUOUS {
         xc = z; }

      MUST {
         xc + uc <= 2;
         ~(xl & ul); }
   } /* end implementation */
} /* end system */
```

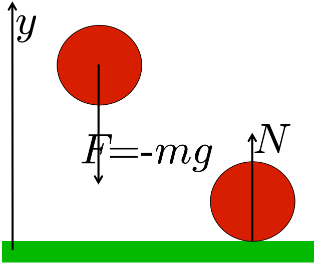
Additional relations constraining system's variables

BOUNCING BALL EXAMPLE



$$\ddot{y} = -g$$
$$y \leq 0 \;\Rightarrow \dot{y}(t^+) = -(1-\alpha)\dot{y}(t^-)$$

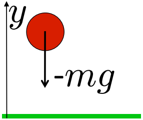
$$\alpha \in \{0, 1\}$$



How to model the bouncing ball as a discrete-time hybrid system ?

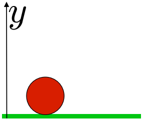
BOUNCING BALL – TIME DISCRETIZATION

- Case $y(k) > 0$ (ball falling): $$\begin{array}{rcl} v(k) & \approx & \frac{y(k)-y(k-1)}{T_s} \\ -g & = & \approx \frac{v(k)-v(k-1)}{T_s} \end{array}$$



$$\left\{\begin{array}{rcl} v(k+1) & = & v(k) - T_s g \\ y(k+1) & = & y(k) + T_s v(k+1) \\ & = & y(k) + T_s v(k) - T_s^2 g \end{array}\right.$$

- Case $y(k) \leq 0$ (ground level): $$\begin{array}{rcl} v(k) & = & -(1-\alpha)v(k-1) \\ y(k+1) & = & y(k-1) = y(k) - T_s v(k) \end{array}$$



$$\left\{\begin{array}{rcl} v(k+1) & = & -(1-\alpha)v(k) \\ y(k+1) & = & y(k) - T_s v(k) \end{array}\right.$$

- We need a binary variable $[\delta(k) = 1] \leftrightarrow [y(k) \leq 0]$

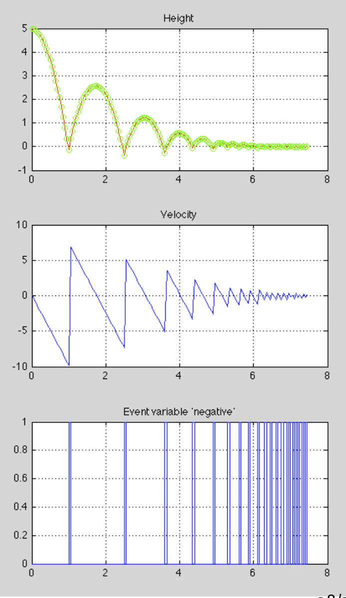
BOUNCING BALL - HYSDEL MODEL

```
SYSTEM bouncing_ball {
INTERFACE {
/* Description of variables and constants */
        STATE { REAL height [-10,10];
                REAL velocity [-100,100];    }

        PARAMETER {
                REAL g;
                REAL alpha;  /* 0=elastic, 1=completely anelastic */
                REAL Ts; }
}
IMPLEMENTATION {
        AUX {   BOOL negative;
                REAL hnext;
                REAL vnext;    }

        AD {    negative = height <= 0; }

        DA {    hnext = {  IF negative THEN height-Ts*velocity
                        ELSE height+Ts*velocity-Ts*Ts*g};
                vnext = {  IF negative THEN -(1-alpha)*velocity
                        ELSE velocity-Ts*g};   }
        CONTINUOUS {
                height   = hnext;
                velocity = vnext;}
}}
```

go to demo `demos/hybrid/bball.m`

BOUNCING BALL - SIMULATION

```
>> Ts=0.05;
>> g=9.8;
>> alpha=0.3;

>> S=mld('bouncing_ball',Ts);

>> N=150;
>> U=zeros(N,0);
>> x0=[5 0]';

>> [X,T,D]=sim(S,x0,U);
```

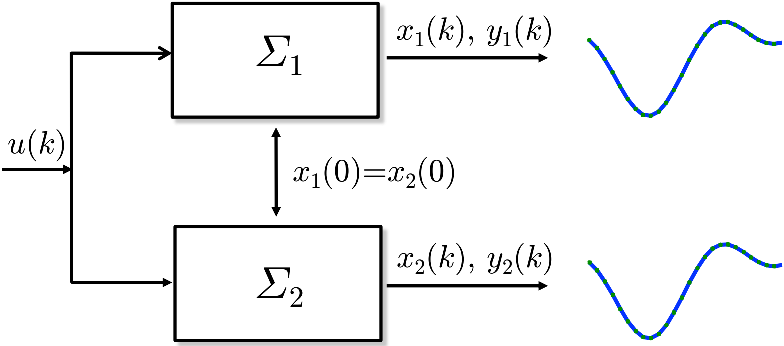


- **Note:** no **Zeno effect** in discrete time !

EQUIVALENCE OF HYBRID MODELS

EQUIVALENCE OF HYBRID MODELS

- Two hybrid models Σ_1, Σ_2 are **equivalent** if for all initial states $x_1(0) = x_2(0)$ and input excitations $u_1(k) \equiv u_2(k)$, the corresponding trajectories $x_1(k) \equiv x_2(k)$ and $y_1(k) \equiv y_2(k)$, $\forall k = 0, 1, \ldots$



EQUIVALENCE OF HYBRID MODELS

- **MLD** and **PWA** systems are equivalent (Bemporad, Ferrari-Trecate, Morari, 2000)

  Proof: For a given combination (x_ℓ, u_ℓ, δ) of an MLD model, the state and output equation are linear and valid in a polyhedron.

  Conversely, a PWA system can be modeled as MLD system (see next slide)

- Efficient **conversion algorithms** from MLD to PWA form exist

  (Bemporad, 2004) (Geyer, Torrisi, Morari, 2003)

- Further equivalences exist with other classes of hybrid dynamical systems, such as **Linear Complementarity (LC)** systems (Heemels, De Schutter, Bemporad, 2001)

MODELING A PWA SYSTEM IN MLD FORM

- PWA system with bounded states and inputs and s regions

$$\begin{array}{rcl} x(k+1) & = & A_{i(k)}x(k) + B_{i(k)}u(k) + f_{i(k)} \\ y(k) & = & C_{i(k)}x(k) + D_{i(k)}u(k) + g_{i(k)} \\ i(k) & = & \text{such that} \begin{bmatrix} x(k) \\ u(k) \end{bmatrix} \in \mathcal{C}_{i(k)} \end{array}$$

with $\mathcal{C}_i = \{\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] : H_i x + J_i u \leq K_i\}$, and $\mathring{\mathcal{C}}_i \cap \mathring{\mathcal{C}}_j = \emptyset$, $\forall i \neq j$, $i, j = 1, \ldots, s$ ($\{\mathcal{C}_i\}$ is a **polyhedral partition** of the set $\mathcal{C} \triangleq \cup_{i=1}^s \mathcal{C}_i$)

- Introduce s binary variables δ_i, $i = 1, \ldots, s$ and the logic constraints

$$[\delta_i = 1] \to [H_i x + J_i u \leq K_i] \qquad H_i x + J_i u \leq K_i + M_i(1 - \delta_i)$$

$$\bigoplus_{i=1}^{s} [\delta_i = 1] = \texttt{true} \qquad \Longrightarrow \qquad \sum_{i=1}^{s} \delta_i = 1$$

were the vector M_i of upper-bounds can be computed, e.g., via LP

MODELING A PWA SYSTEM IN MLD FORM

- Introduce auxiliary real vectors z_i, w_i defined by if-then-else rules

$$z_i = \begin{cases} A_i x + B_i u + f_i & \text{if } \delta_i = 1 \\ 0 & \text{otherwise} \end{cases} \qquad w_i = \begin{cases} C_i x + D_i u + g_i & \text{if } \delta_i = 1 \\ 0 & \text{otherwise} \end{cases}$$

and convert the relations above into mixed-integer inequalities

- Finally, write the state update and output equations

$$\begin{cases} x(k+1) &= \sum_{i=1}^{s} z_i(k) \\ y(k) &= \sum_{i=1}^{s} w_i(k) \end{cases}$$

PWA SYSTEM MODELED USING DISJUNCTIVE PROGRAMMING

(Balas, 1985)

- A PWA system with bounded states and inputs is equivalent to the **disjunction**

$$\bigvee_{i=1}^{s} \left[\begin{array}{c} H_i x(k) + J_i u(k) \le K_i \\ x(k+1) = A_i x(k) + B_i u(k) + f_i \end{array} \right] \qquad \begin{array}{c} x_{\ell b} \le x(k) \le x_{ub} \\ u_{\ell b} \le u(k) \le u_{ub} \end{array}$$

- Introduce s binary variables $\delta_1(k), \ldots, \delta_s(k)$ subject to $\sum_{i=1}^{s} \delta_i(k) = 1$
- Introduce the **convex hull relaxation** of the disjunction

$$x(k) = \sum_{i=1}^{s} v_i(k), \quad x_{\ell b}\delta_i(k) \le v_i(k) \le x_{ub}\delta_i(k)$$

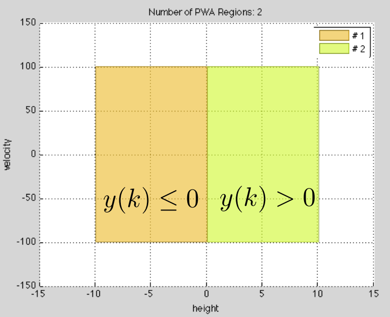
$$u(k) = \sum_{i=1}^{s} w_i(k), \quad u_{\ell b}\delta_i(k) \le w_i(k) \le u_{ub}\delta_i(k)$$

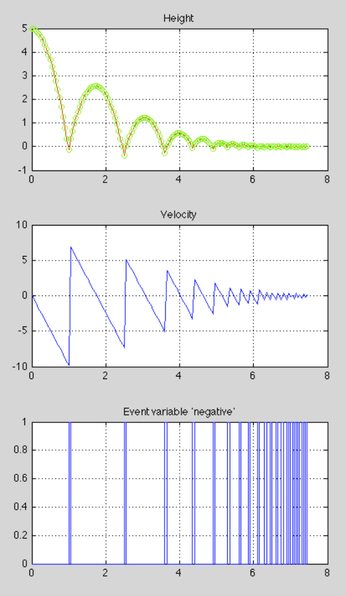
and impose

$$x(k+1) = \sum_{i=1}^{s} A_i v_i(k) + B_i w_i(k) + f_i \delta_i(k), \quad H_i v_i(k) + J_i w_i(k) \le K_i \delta_i(k)$$

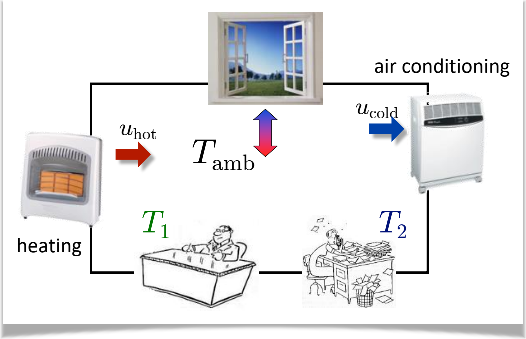
BOUNCING BALL - PWA EQUIVALENT

```
>> P=pwa(S);
>> plot(P)

>> [X,T,I]=sim(P,x0,U);
```





EXAMPLE: ROOM TEMPERATURE CONTROL



discrete dynamics

- #1 = cold $\rightarrow$ heater = on
- #2 = cold $\rightarrow$ heater = on **unless** #1 hot
- A/C activation has similar rules

continuous dynamics

$$\frac{dT_i}{dt} = -\alpha_i(T_i - T_{\rm amb}) + k_i(u_{\rm hot} - u_{\rm cold})$$

$i = 1, 2$

go to demo `demos/hybrid/heatcool.m`

EXAMPLE: ROOM TEMPERATURE CONTROL

```
SYSTEM heatcool {

INTERFACE {
    STATE { REAL T1 [-10,50];
            REAL T2 [-10,50];
          }
    INPUT { REAL Tamb [-10,50];
          }
    PARAMETER {
        REAL Ts, alpha1, alpha2, k1, k2;
        REAL Thot1, Tcold1, Thot2, Tcold2, Uc, Uh;
          }
}
IMPLEMENTATION {
        AUX { REAL uhot, ucold;
              BOOL hot1, hot2, cold1, cold2;
          }
        AD  { hot1 = T1>=Thot1;
              hot2 = T2>=Thot2;
              cold1 = T1<=Tcold1;
              cold2 = T2<=Tcold2;
          }
        DA  { uhot = {IF cold1 | (cold2 & ~hot1) THEN Uh ELSE 0};
              ucold = {IF hot1 | (hot2 & ~cold1) THEN Uc ELSE 0};
          }
        CONTINUOUS { T1 = T1+Ts*(-alpha1*(T1-Tamb)+k1*(uhot-ucold));
                     T2 = T2+Ts*(-alpha2*(T2-Tamb)+k2*(uhot-ucold));
          }
      }
}
```

```
>> S=mld('heatcoolmodel',Ts);
```

get the MLD model in MATLAB

```
>> [XX,TT]=sim(S,x0,U);
```

simulate the MLD model

EXAMPLE: ROOM TEMPERATURE CONTROL

- MLD model of the room temperature system

$$\begin{cases} x(k+1) & = & Ax(k) + B_1u(k) + B_2\delta(k) + B_3z(k) + B_5 \\ y(k) & = & Cx(k) + D_1u(k) + D_2\delta(k) + D_3z(k) + D_5 \\ E_2\delta(k) & + & E_3z(k) \leq E_4x(k) + E_1u(k) + E_5 \end{cases}$$

  - 2 continuous states (temperature T_1, T_2)
  - 1 continuous input (room temperature T_{amb})
  - 2 auxiliary continuous vars (power flows $u_{\mathrm{hot}}, u_{\mathrm{cold}}$)
  - 6 auxiliary binary vars (4 threshold events + 2 for the OR condition)
  - 20 mixed-integer inequalities

- In principle we have $2^6 = 64$ possible combinations of integer variables

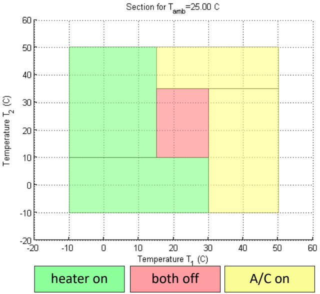
EXAMPLE: ROOM TEMPERATURE CONTROL

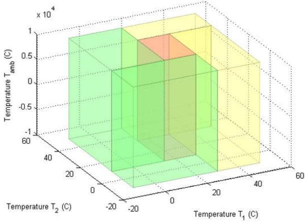
- PWA model of the room temperature system

$$
\begin{aligned}
x(k+1) &= A_{i(k)}x(k) + B_{i(k)}u(k) + f_{i(k)} \\
y(k) &= C_{i(k)}x(k) + D_{i(k)}u(k) + g_{i(k)}
\end{aligned}
$$

```
>> P=pwa(S);
```

$$
i(k) \quad \text{s.t.} \quad H_{i(k)}x(k) + J_{i(k)}u(k) \leq K_{i(k)}
$$





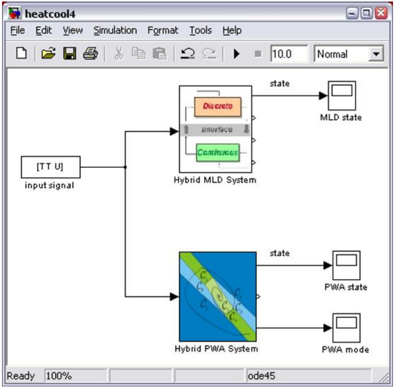
5 polyhedral regions

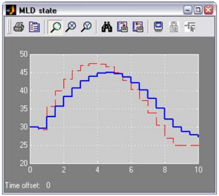
(partition does not depend on input)

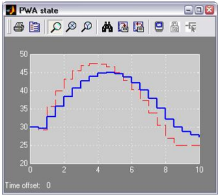
2 continuous states (T_1, T_2)

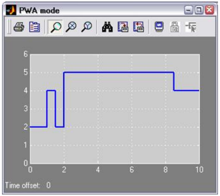
1 continuous input $(T_{\rm amb})$

EXAMPLE: ROOM TEMPERATURE CONTROL









- MLD and PWA models are equivalent, hence simulated states are the same

USING PWA EQUIVALENCE FOR MODEL ANALYSIS

- Assume plant + controller can be modeled as DHA:
  - **plant** = approximated as PWA system (e.g.: nonlinear switched model)
  - **controller** = switched linear controller (e.g: combination of threshold conditions, logic, linear feedback laws, …)
- Convert DHA to MLD form, then to PWA form
- The resulting closed-loop PWA model reveals how the closed-loop system behaves in different regions of the state-space
- Can analyze **closed-loop stability** analysis using **piecewise quadratic Lyapunov functions** (Johansson, Rantzer, 1998) (Mignone, Ferrari-Trecate, Morari, 2000)

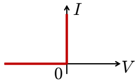
OTHER EXISTING HYBRID MODELS

(Heemels, De Schutter, Bemporad, 2001)

- **Linear complementarity (LC)** systems (Heemels, 1999)

$$\begin{array}{rcl} x(k+1) & = & Ax(k)+B_1u(k)+B_2w(k) \\ y(k) & = & Cx(k)+D_1u(k)+D_2w(k) \\ v(k) & = & E_1x(k)+E_2u(k)+E_3w(k)+E_4 \\ 0 & \leq & v(k)\perp w(k)\geq 0 \end{array}$$

Examples:
mechanical systems,
electrical circuits



- **Min-max-plus-scaling (MMPS)** systems (De Schutter, Van den Boom, 2000)

$$\begin{array}{rcl} x(k+1) & = & M_x(x(k),u(k),w(k)) \\ y(k) & = & M_y(x(k),u(k),w(k)) \\ 0 & \geq & M_c(x(k),u(k),w(k)) \end{array}$$

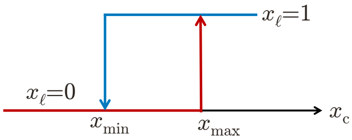
Example:
discrete-event system
k = event counter

where $M_{()}$ are MMPS functions defined by the grammar

$M \; := x_i|\alpha|\max(M_1,M_2)|\min(M_1,M_2)|M_1+M_2|\beta M_1$

Example: $x(k+1)=2\max(x(k),0)+\min(\frac{1}{2}u(k),1)$

MODELING HYSTERESIS



- Hysteresis between $x_{\min} \leq x_c(k) \leq x_{\max}$

- Introduce two binary variables

$$\begin{array}{rcl} [\delta_{\min}(k) = 1] & \leftrightarrow & [x_c(k) \leq x_{\min}] \\ [\delta_{\max}(k) = 1] & \leftrightarrow & [x_c(k) \geq x_{\max}] \end{array}$$

- Introduce logic state $x_\ell \in \{0, 1\}$ with dynamics

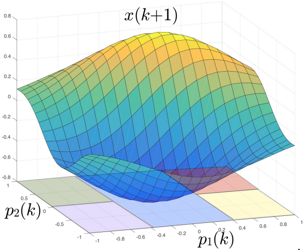
$$x_\ell(k+1) = (x_\ell(k) \wedge \neg\delta_{\min}(k)) \vee (\neg x_\ell(k) \wedge \delta_{\max}(k))$$

IDENTIFICATION OF HYBRID SYSTEMS

HYBRID SYSTEM IDENTIFICATION

- A hybrid model of the process may not be available from physical principles
- Therefore, a model must be either
  - **estimated from data** (model is unknown)
  - or **hybridized** (model is known but nonlinear)
- If one linear model is enough: easy problem (SYS-ID TBX) (Ljung, 1999)
- If switching sequence known: easy, just identify one linear model per mode
- If modes & dynamics must be identified simultaneously, we need **hybrid system identification** (or **piecewise affine regression**)

> In industrial MPC most effort is spent in **identifying (multiple) linear prediction models** from data

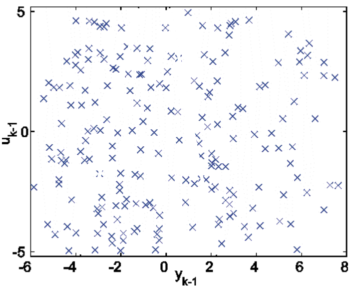


PWA IDENTIFICATION PROBLEM

Estimate from data **both** the **parameters** of the affine submodels and the **partition** of the PWA map

Example: Let the data be generated by the PWARX system

$$y_k = \begin{cases} \begin{bmatrix} -0.4 & 1 & 1.5 \end{bmatrix} \phi_k + \epsilon_k \\ \quad \text{if } \begin{bmatrix} 4 & -1 & 10 \end{bmatrix} \phi_k < 0 \\ \begin{bmatrix} 0.5 & -1 & -0.5 \end{bmatrix} \phi_k + \epsilon_k \\ \quad \text{if } \begin{bmatrix} -4 & 1 & 10 \\ 5 & 1 & -6 \end{bmatrix} \phi_k \leq 0 \\ \begin{bmatrix} -0.3 & 0.5 & -1.7 \end{bmatrix} \phi_k + \epsilon_k \\ \quad \text{if } \begin{bmatrix} -5 & -1 & 6 \end{bmatrix} \phi_k < 0 \end{cases}$$



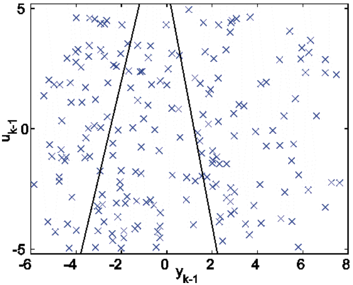
with $\phi_k = [y_{k-1}\ u_{k-1}\ 1]'$, $|u_k| \leq 5$, and $|\epsilon_k| \leq 0.1$

PWA IDENTIFICATION PROBLEM

Estimate from data **both** the **parameters** of the affine submodels and the **partition** of the PWA map

Example: Let the data be generated by the PWARX system

$$y_k = \begin{cases} \begin{bmatrix} -0.4 & 1 & 1.5 \end{bmatrix} \phi_k + \epsilon_k \\ \quad \text{if } \begin{bmatrix} 4 & -1 & 10 \end{bmatrix} \phi_k < 0 \\ \begin{bmatrix} 0.5 & -1 & -0.5 \end{bmatrix} \phi_k + \epsilon_k \\ \quad \text{if } \begin{bmatrix} -4 & 1 & 10 \\ 5 & 1 & -6 \end{bmatrix} \phi_k \leq 0 \\ \begin{bmatrix} -0.3 & 0.5 & -1.7 \end{bmatrix} \phi_k + \epsilon_k \\ \quad \text{if } \begin{bmatrix} -5 & -1 & 6 \end{bmatrix} \phi_k < 0 \end{cases}$$



with $\phi_k = [y_{k-1}\ u_{k-1}\ 1]'$, $|u_k| \leq 5$, and $|\epsilon_k| \leq 0.1$

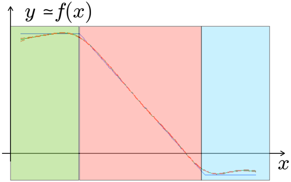
PWA REGRESSION PROBLEM

- **Problem:** Given input/output pairs $\{x(k), y(k)\}$, $k = 1, \ldots, N$ and number s of models, compute a **piecewise affine** (PWA) approximation $y \approx f(x)$

$$f(x) = \begin{cases} F_1 x + g_1 & \text{if } H_1 x \leq K_1 \\ \vdots \\ F_s x + g_s & \text{if } H_s x \leq K_s \end{cases}$$

- Need to learn **both** the parameters $\{F_i,\ g_i\}$ of the affine submodels **and** the partition $\{H_i,\ K_i\}$ of the PWA map from data (**off-line learning**)

- Possibly update model and partition as new data become available (**on-line learning**)



APPROACHES TO PWA IDENTIFICATION

- Mixed-integer linear or quadratic programming (Roll, Bemporad, Ljung, 2004)
- Partition of infeasible set of inequalities (Bemporad, Garulli, Paoletti, Vicino, 2005)
- K-means clustering in a feature space (Ferrari-Trecate, Muselli, Liberati, Morari, 2003)
- Bayesian approach (Juloski, Wieland, Heemels, 2004)
- Kernel-based approaches (Pillonetto, 2016)
- Hyperplane clustering in data space (Münz, Krebs, 2002)
- **Recursive multiple least squares & PWL separation** (Breschi, Piga, Bemporad, 2016)

PWA REGRESSION ALGORITHM

(Breschi, Piga, Bemporad, 2016)

1. Estimate models $\{F_i,\ g_i\}$ **recursively**. Let $e_i(k) = y(k) - F_i x(k) - g_i$ and only update model $i(k)$ such that

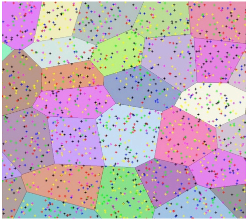
$$i(k) \leftarrow \arg\min_{i=1,\ldots,s} \underbrace{e_i(k)'\Lambda_e^{-1}e_i(k)}_{\text{one-step prediction error of model \#i}} + \underbrace{(x(k)-c_i)'\,R_i^{-1}(x(k)-c_i)}_{\text{proximity to centroid of cluster \#i}}$$

using **recursive LS** and **inverse QR decomposition** (Alexander, Ghirnikar, 1993)

This also splits the data points $x(k)$ in **clusters** $C_i = \{x(k) :\ i(k) = i\}$

2. Compute a polyhedral partition $\{H_i,\ K_i\}$ of the regressor space via **multi-category linear separation**

$$\phi(x) = \max_{i=1,\ldots,s} \{w_i' x - \gamma_i\}$$



PWA REGRESSION EXAMPLES

(Breschi, Piga, Bemporad, 2016)

- Identification of **piecewise-affine ARX** model

$$\begin{bmatrix} y_1(k) \\ y_2(k) \end{bmatrix} = \begin{bmatrix} -0.83 & 0.20 \\ 0.30 & -0.52 \end{bmatrix} \begin{bmatrix} y_1(k-1) \\ y_2(k-1) \end{bmatrix} + \begin{bmatrix} -0.34 & 0.45 \\ -0.30 & 0.24 \end{bmatrix} \begin{bmatrix} u_1(k-1) \\ u_2(k-1) \end{bmatrix}$$
$$+ \begin{bmatrix} 0.20 \\ 0.15 \end{bmatrix} + \max \left\{ \begin{bmatrix} 0.20 & -0.90 \\ 0.10 & -0.42 \end{bmatrix} \begin{bmatrix} y_1(k-1) \\ y_2(k-1) \end{bmatrix} \right.$$
$$\left. + \begin{bmatrix} 0.42 & 0.20 \\ 0.50 & 0.64 \end{bmatrix} \begin{bmatrix} u_1(k-1) \\ u_2(k-1) \end{bmatrix} + \begin{bmatrix} 0.40 \\ 0.30 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\} + e_o(k),$$

- **Quality of fit**: best fit rate (BFR) = $\max\left\{1 - \frac{\|y_{o,i} - \hat{y}_i\|_2}{\|y_{o,i} - \bar{y}_{o,i}\|_2}, 0\right\}, i = 1, 2$

		$N = 4000$	$N = 20000$	$N = 100000$
y_1	(Off-line) RLP	96.0 %	96.5 %	99.0 %
	(Off-line) RPSN	96.2 %	96.4 %	98.9 %
	(On-line) ASGD	86.7 %	95.0 %	96.7 %
y_2	(Off-line) RLP	96.2 %	96.9 %	99.0 %
	(Off-line) RPSN	96.3 %	96.8 %	99.0 %
	(On-line) ASGD	87.4 %	95.2 %	96.4 %

RLP = Robust linear programming
(Bennett, Mangasarian, 1994)

RPSN = Piecewise-smooth Newton method
(Bemporad, Bernardini, Patrinos, 2015)

ASGD = Averaged stochastic gradient descent
(Bottou, 2012)

- **CPU time for computing the partition**:

	$N = 4000$	$N = 20000$	$N = 100000$
(Off-line) RLP	0.308 s	3.227 s	112.435 s
(Off-line) RPSN	0.016 s	0.086 s	0.365 s
(On-line) ASGD	0.013 s	0.023 s	0.067 s

PWA REGRESSION EXAMPLES

(Breschi, Piga, Bemporad, 2016)

- Identification of **linear parameter varying ARX** model

$$\begin{bmatrix} y_1(k) \\ y_2(k) \end{bmatrix} = \begin{bmatrix} \bar{a}_{1,1}(p(k)) & \bar{a}_{1,2}(p(k)) \\ \bar{a}_{2,1}(p(k)) & \bar{a}_{2,2}(p(k)) \end{bmatrix} \begin{bmatrix} y_1(k-1) \\ y_2(k-1) \end{bmatrix} + \begin{bmatrix} \bar{b}_{1,1}(p(k)) & \bar{b}_{1,2}(p(k)) \\ \bar{b}_{2,1}(p(k)) & \bar{b}_{2,2}(p(k)) \end{bmatrix} \begin{bmatrix} u_1(k-1) \\ u_2(k-1) \end{bmatrix} + e_o(k)$$

$\bar{a}(p)$ = PWA function of p

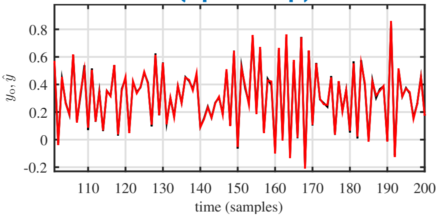
$\bar{b}(p)$ has **quadratic** and **sin** terms

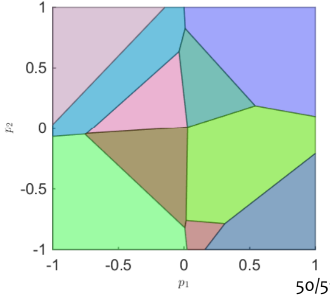
- **Quality of fit (BFR)**:

	y_1	y_2
PWA regression	87 %	84 %
parametric LPV*	80 %	70 %

* (Bamieh, Giarré, 2002)

- **Validation data (open-loop)**:

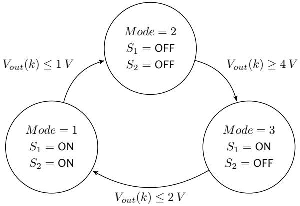




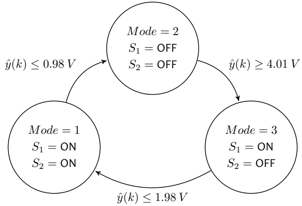
IDENTIFICATION OF HYBRID SYSTEMS WITH LOGIC STATES

(Breschi, Piga, Bemporad, CDC 2016)

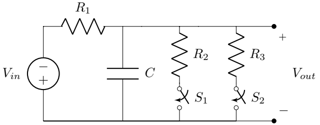
- Identification of a **hybrid model** with logic states

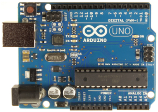


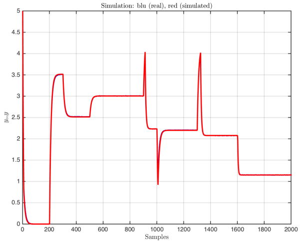
true system



identified system







Quality of fit: BFR=96.64 % (validation)

CPU time for identification: 78 ms

(2000 samples, MacBook Pro 2.8 GHz)