

# Linear Regression Model

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## 1 Model Statement

### 1.1 Data Model

$$\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}$$

### 1.2 Process Model

$$\begin{aligned} \boldsymbol{\beta}|\sigma^2, \frac{1}{\gamma^2} &\sim N(0, \Sigma_{\boldsymbol{\beta}}) & \Sigma_{\boldsymbol{\beta}} &= \sigma^2 \mathbf{D}_{\gamma} & \mathbf{D}_{\gamma} &= \text{diag}(\gamma_1, \dots, \gamma_p) \\ \boldsymbol{\epsilon} &\sim N(0, \Sigma_{\boldsymbol{\epsilon}}) & \Sigma_{\boldsymbol{\epsilon}} &= \sigma_{\epsilon}^2 \mathbf{I} \end{aligned}$$

### 1.3 Parameter Model

$$\begin{aligned} \sigma_{\epsilon}^2 &\sim \text{IG}(\alpha_{\epsilon}, \beta_{\epsilon}) \\ \gamma_j &\sim \text{Exponential}(\lambda^2/2) \text{ for each } j = 1, \dots, p \\ \lambda^2 &\sim \text{Gamma}(\alpha_{\lambda}, \beta_{\lambda}) \end{aligned}$$

where  $p$  is the number of parameters in  $\boldsymbol{\beta}$  and  $\boldsymbol{\mu}_{\boldsymbol{\beta}} = 0$ . Equivalently,  $1/\gamma_j^2 \sim \text{IG}(1, \lambda^2/2)$ .

## 2 Posterior

$$[\boldsymbol{\beta}, \gamma^2, \sigma_{\epsilon}^2, \lambda^2 | \mathbf{y}] \propto [\mathbf{y} | \boldsymbol{\beta}, \sigma_{\epsilon}^2][\boldsymbol{\beta} | \sigma_{\epsilon}^2, \gamma^2][\sigma_{\epsilon}^2][\gamma][\lambda^2]$$

## 3 Full Conditionals

### 3.1 Full Conditional for $\boldsymbol{\beta}$

$$\begin{aligned} [\boldsymbol{\beta} | \cdot] &\propto [\mathbf{y} | \boldsymbol{\beta}, \sigma_{\epsilon}^2][\boldsymbol{\beta} | \sigma_{\epsilon}^2, \gamma^2] \\ &\propto e^{-\frac{1}{2}(\mathbf{y} - \mathbf{X}\boldsymbol{\beta})^T \Sigma_{\epsilon}^{-1}(\mathbf{y} - \mathbf{X}\boldsymbol{\beta})} e^{-\frac{1}{2}\boldsymbol{\beta}^T \Sigma_{\boldsymbol{\beta}}^{-1} \boldsymbol{\beta}} \\ &\propto e^{-\frac{1}{2}(\boldsymbol{\beta}^T (\mathbf{X}^T \Sigma_{\epsilon}^{-1} \mathbf{X} + \Sigma_{\boldsymbol{\beta}}^{-1}) \boldsymbol{\beta} - 2\boldsymbol{\beta}^T (\mathbf{X}^T \Sigma_{\epsilon}^{-1} \mathbf{y} + \Sigma_{\boldsymbol{\beta}}^{-1} \boldsymbol{\mu}_{\boldsymbol{\beta}}))} \end{aligned}$$

which is Normal with mean  $\mathbf{A}^{-1}\mathbf{b}$  and variance  $\mathbf{A}^{-1}$  where

$$\begin{aligned}\mathbf{A}^{-1} &= (\mathbf{X}^T \Sigma_\epsilon^{-1} \mathbf{X} + \Sigma_\beta^{-1})^{-1} \\ &= (\mathbf{X}^T \mathbf{X} + \mathbf{D}_\gamma)^{-1} \\ \mathbf{b} &= (\mathbf{X}^T \Sigma_\epsilon^{-1} \mathbf{y} + \Sigma_\beta^{-1} \mu_\beta) \\ &= \mathbf{X}^T \mathbf{y}\end{aligned}$$

### 3.2 Full Conditional for $\sigma_\epsilon^2$

$$\begin{aligned}[\sigma_\epsilon^2 | \cdot] &\propto [\mathbf{y} | \boldsymbol{\beta}, \sigma_\epsilon^2][\sigma_\epsilon^2] \\ &\propto (|\Sigma_\epsilon|^{-\frac{1}{2}}) e^{-\frac{1}{2}(\mathbf{y} - \mathbf{X}\boldsymbol{\beta})^T \Sigma_\epsilon^{-1} (\mathbf{y} - \mathbf{X}\boldsymbol{\beta})} (\sigma_\epsilon^2)^{-\alpha_\epsilon + 1} e^{-\frac{\beta_\epsilon}{\sigma_\epsilon^2}} \\ &\propto (\sigma_\epsilon^2)^{-\frac{n}{2} - \alpha_\epsilon - 1} e^{-\frac{1}{\sigma_\epsilon^2} (\frac{1}{2}(\mathbf{y} - \mathbf{X}\boldsymbol{\beta})^T (\mathbf{y} - \mathbf{X}\boldsymbol{\beta}) + \beta_\epsilon)}\end{aligned}$$

$\text{IG}(\alpha_\beta + \frac{n}{2}, \frac{1}{2}(\mathbf{y} - \mathbf{X}\boldsymbol{\beta})^T (\mathbf{y} - \mathbf{X}\boldsymbol{\beta}) + \beta_\beta)$  since the determinant  $|\Sigma_\epsilon| = (\sigma_\epsilon^2)^n$  and  $\Sigma_\epsilon^{-1} = \frac{1}{\sigma_\epsilon^2} \mathbf{I}$

### 3.3 Full Conditional for $\frac{1}{\gamma_j}$

$$\begin{aligned}[\frac{1}{\gamma_j^2} | \cdot] &\propto [\beta_j | \sigma_\epsilon^2, \gamma_j^2][\frac{1}{\gamma_j^2}] \\ &\propto |\gamma_j^2|^{-\frac{1}{2}} e^{-\frac{1}{\gamma_j^2} \frac{\beta_j^2}{2\sigma_\epsilon^2}} (\frac{1}{\gamma_j^2})^{-2} e^{\frac{\lambda^2/2}{\gamma_j^2}} \\ &\propto (\frac{1}{\gamma_j^2})^{-3/2} e^{-\frac{1}{2}(\frac{\beta_j^2}{\gamma_j^2 \sigma_\epsilon^2} + \frac{\lambda^2}{\gamma_j^2})}\end{aligned}$$

which is Inverse Gaussian( $\mu', \lambda'$ ) where  $\mu' = \sqrt{\frac{\lambda^2 \sigma_\epsilon^2}{\beta_j^2}}$  and  $\lambda' = \lambda^2$ . The inverse Gaussian distribution for  $x > 0, \mu' > 0$ , and  $\lambda > 0$

$$f(x) = (\frac{\lambda^2}{2\pi x^3})^{1/2} e^{-\frac{\lambda'(x - \mu')^2}{2\mu'^2 x}}$$

### 3.4 Full Conditional for $\lambda^2$

$$\begin{aligned}[\lambda^2 | \cdot] &\propto [\gamma^2 | \lambda^2][\lambda^2] \\ &\propto \prod_{j=1}^p \frac{\lambda^2}{2} e^{-\lambda^2 \frac{\gamma_j^2}{2}} (\lambda^2)^{\alpha_\lambda - 1} e^{-\beta_\lambda \lambda^2} \\ &\propto (\lambda^2)^{\alpha_\lambda + p + 1} e^{-\lambda^2 (\beta_\lambda + \sum_{j=1}^p \gamma_j^2)}\end{aligned}$$

which is Gamma( $\alpha_\lambda + p, \beta_\lambda + \frac{1}{2} \sum_{j=1}^p \gamma_j^2$ )

## 4 Posterior Predictive Distribution

The posterior predictive distribution for  $\mathbf{y}_t$  is sampled at each MCMC iteration  $k$  by

$$\mathbf{y}_t^{(k)} \sim N(\mathbf{H}_t \mathbf{X} \boldsymbol{\beta}_t^{(k)}, \Sigma^{(k)})$$