Wishart Multivaraite Response Linear Regression

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Model

We begin with the model for the multivariate observation i

$$
\begin{aligned}
\mathbf{y}_i &\sim \mathrm{N}\left(X_i\boldsymbol{\beta}, \mathbf{Q}^{-1}\right) \\
\boldsymbol{\beta} &\sim \mathrm{N}\left(\mathbf{0}, \sigma^2_\beta \mathbf{I}\right), \\
\boldsymbol{Q} &\sim \mathrm{Wishart}\left(\nu, \nu\mathbf{S}\right),
\end{aligned}
$$

where $\boldsymbol{\beta}$ is the p-dimensional set of regression coefficients for covariate X_i. $\boldsymbol{\epsilon}_i$ is a correlated random error that accounts for the correlations between the p response variables for observation i that are not explained by the covariate X_i through the $p \times p$ covariance matrix $\mathbf{Q}^{-1}$.

Posterior

The posterior that we wish to sample is

$$
\begin{aligned}
[\boldsymbol{\beta}, \mathbf{Q} | \mathbf{y}] &\propto \left(\prod_{i=1}^N [\mathbf{y}_i | \boldsymbol{\beta}, \mathbf{Q}] \right) [\boldsymbol{\beta}] [\mathbf{Q}] \\
&\propto \left(\prod_{i=1}^N |\mathbf{Q}^{-1}|^{-\frac{1}{2}} \exp\left\{ -\frac{1}{2} \left(\mathbf{y}_i - X_i\boldsymbol{\beta}\right)' \mathbf{Q} \left(\mathbf{y}_i - X_i\boldsymbol{\beta}\right) \right\} \right) \\
&\quad \times \left(\left(\sigma^2_\beta\right)^{-\frac{p}{2}} \exp\left\{ -\frac{1}{2\sigma^2_\beta} \boldsymbol{\beta}'\boldsymbol{\beta} \right\} \right) \\
&\quad |\mathbf{Q}|^{\frac{\nu-p-1}{2}} \exp\left\{ -\frac{1}{2} \mathrm{tr}\left(\left(\nu\mathbf{S}\right)^{-1} \mathbf{Q} \right) \right\}
\end{aligned}
$$

The log posterior density is

$$
\begin{aligned}
\log\left[\boldsymbol{\beta}_1, \ldots, \boldsymbol{\beta}_p, \mathbf{Q} | \mathbf{y}\right] &\propto \left(\sum_{i=1}^N \log\left[\mathbf{y}_i | \boldsymbol{\beta}_1, \ldots, \boldsymbol{\beta}_p, \mathbf{Q}\right] \right) + \left(\sum_{j=1}^p \log\left[\boldsymbol{\beta}_j\right] \right) + \log[\mathbf{Q}] \\
&\propto \left(\sum_{i=1}^N -\frac{1}{2} \log|\mathbf{Q}^{-1}| - \frac{1}{2} \left(\mathbf{y}_i - \sum_{j=1}^p \mathbf{X}_i \boldsymbol{\beta}_j \right)' \mathbf{Q} \left(\mathbf{y}_i - \sum_{j=1}^p \mathbf{X}_i \boldsymbol{\beta}_j \right) \right) + \\
&\quad \left(\sum_{j=1}^p -\frac{p}{2} \log\left(\sigma^2_\beta\right) - \frac{1}{2\sigma^2_\beta} \boldsymbol{\beta}_j' \boldsymbol{\beta}_j \right) + \\
&\quad \frac{\nu - p - 1}{2} \log|\mathbf{Q}| - \frac{1}{2} \mathrm{tr}\left(\left(\nu\mathbf{S}\right)^{-1} \mathbf{Q} \right)
\end{aligned}
$$

Full Conditionals

Full Conditional for $\boldsymbol{\beta}_j$

$$
\begin{aligned}
\log\left[\boldsymbol{\beta}_j|\cdot\right] &\propto \sum_{i=1}^{N} \log\left[\mathbf{y}_i|\boldsymbol{\beta}_1,\ldots,\boldsymbol{\beta}_p,\mathbf{Q}\right] + \log\left[\boldsymbol{\beta}_j\right] \\
&\propto
\end{aligned}
$$

```
## libraries and functions
source('~/Linear-Model/dinvgamma.R')
source('~/Linear-Model/mcmc.lm.R')
source('~/Linear-Model/rMVN.R')
source("~/Linear-Model/wishartLinearRegression/mcmc.lm.R")
library(mvtnorm)

## Simulate some data

N <- 1000                                    ## sample size
p <- 4
beta <- matrix(seq(-3, 3, length=p^2), p, p)
s2 <- 0.25
Q <- matrix(rWishart(1, 10, diag(p)), p, p)

make.lm.data <- function(N, n, beta, s2, Q){
  p <- dim(beta)[2]
  X <- matrix(rnorm(N*p), nrow=N, ncol=p)
  Y <- X %*% beta + rmvnorm(N, rep(0, p), s2 * Q)
  list(Y=Y, X=X)
}

data <- make.lm.data(N, n, beta, s2, Q)

## Setup MCMC

# priors for beta
mu_beta <- matrix(rep(0, p^2), p, p)
s2_beta <- 100
# priors for s2
s2_lower <- 0
s2_upper <- 100
# priors for Q
nu <- p
S <- diag(p)
n_mcmc <- 5000

params <- list(n_mcmc=n_mcmc, mu_beta=mu_beta, s2_beta=s2_beta,
               s2_lower=s2_lower, s2_upper=s2_upper, nu=nu, S=S)

## Fit mcmc
out <- mcmc.lm(Y, X, params)
```