Assignment - 6

Parameter ~~Optim~~ Estimation

Q. Let $(x_1, x_2 \ldots\ldots)$ be a random sample of size m taken from a Normal population with parameter mean $= \theta_1$ and Variance $= \theta_2$. Find Maximum Likelihood Estimates of these two parameters.

$$\theta_1 = \mu \qquad \theta_2 = \delta^2$$

$$f(x) = \frac{1}{\sqrt{2\pi\delta^2}} e^{-1/2 \, (x-\mu)^2/\delta^2} = \frac{1}{\sqrt{2\pi\delta^2}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\delta}\right)^2}$$

$$f(x, \theta_1, \theta_2) = \frac{1}{\sqrt{2\pi\theta_2}} e^{-1/2\left(\frac{x-\theta_1}{\theta_2}\right)}$$

Now, $\theta_1 \in (-\infty, \infty)$

$\theta_2 \in [0, \infty)$

$$L(\theta_1, \theta_2) = \prod_{i=1}^{m} f(x_i, \theta_1, \theta_2) = \prod_{i=1}^{m} \frac{1}{\sqrt{2\pi\theta_2}} e^{-\frac{1}{2}\left(\frac{x_i-\theta_1}{\theta_2}\right)}$$

$$= \theta_2^{-m/2} (2\pi)^{-m/2} \cdot e^{\frac{-1}{2\theta_2}\sum_{i=1}^{m}(x_i-\theta_1)^2}$$

$$\log L(\theta_1, \theta_2) = -\frac{m}{2}\log\theta_2 - \frac{m}{2}\log(2\pi) - \frac{\sum(x_i-\theta_1)^2}{2\theta_2}$$

Partial derivative with θ_1

$$\frac{d \log L(\theta_1, \theta_2)}{d\theta_1} = \frac{-2\sum(x_i-\theta_1)(-1)}{2\theta_2} = 0$$

$$\hat{\theta}_1 = \mu = \frac{\Sigma x_i}{m} = \bar{x}$$

WRT θ_2

$$\frac{d \operatorname{Log} L(\theta_1, \theta_2)}{d\theta_2} = \frac{-m}{2\theta_2} + \frac{\Sigma (x-\theta_1)^2}{2\theta_2^2} = 0$$

$$-m\theta_2 + \Sigma (x-\theta_1)^2 = 0$$

$$\hat{\theta}_2 = \hat{\delta}_2 = \frac{\Sigma (x_i - \bar{x})^2}{m}$$

$$\therefore \hat{\mu} = \frac{\Sigma x_i}{m} \quad \hat{\delta}^2 = \frac{\Sigma (x_i - \bar{x})^2}{m}$$

Let $X_1, X_2, \ldots X_m$ be a random sample from $B(m, \theta)$ distribution, where $\theta \in \Theta = (0,1)$ is unknown and 'm' is a known positive integer. Compute value of θ using the MLE.

$${}^{m}C_{x_i}\, \theta^{x_i} (1-\theta)^{m-x_i} = B(m, \theta)$$

$$L(\theta \mid x_1, x_2, \ldots x_m) = \prod_{i=1}^{m} \left({}^{m}C_{x_i}\right) \theta^{x_i} (1-\theta)^{m-x_i}$$

To compute log Likelihood

$$\operatorname{Log} L(\theta \mid x_1, \ldots x_m) = \sum_{i=1}^{m} \log\left({}^{m}C_{x_i}\right) + x_i \log\theta + (m - x_i) \operatorname{Log}(1-\theta)$$

$$\frac{dL}{d\theta} = \frac{\Sigma x_i}{\theta} + \frac{m - x_i}{1-\theta} = 0$$

$$\frac{\sum_{i=1}^{m} x_i - \theta . m}{\theta(1-\theta)} = 0$$

$$\sum_{i=1}^{m} \frac{x_i}{\theta(1-\theta)} - \frac{\Sigma \cancel{\theta}(m)}{\cancel{\theta}(1-\theta)} = 0$$

$$\sum_{i=1}^{m} \frac{x_i}{\theta \cancel{(1-\theta)}} = \frac{m . m}{\cancel{1-\theta}}$$

$$\theta = \frac{\sum_{i=1}^{m} x_i}{m . m}$$

$$\therefore \text{MLE} \Rightarrow B(m, \theta) = \frac{\sum_{i=1}^{m} x_i}{m . m}$$