3rd Assignment **Braulio Antonio**

Quantum and Nonlinear Optics

Problem 1 In class we defined our quadrature-squeezed state to be

$$|\alpha, \epsilon\rangle \equiv \hat{D}(\alpha)\hat{S}(\epsilon)|0\rangle, \tag{1}$$

This is a displaced, squeezed vacuum state. An alternative quadrature-squeezed state is the one defined y

$$|\epsilon, \alpha\rangle \equiv \hat{S}(\epsilon)\hat{D}(\alpha)|0\rangle \tag{2}$$

which is a squeezed coherent state.

(a) Show that the above two states are not the same by calculating the commutator, $\left[\hat{S}(\epsilon), \hat{D}(\alpha)\right]$, and showing that it is in general non-zero. This is difficult to do in general, so instead consider the special case that $|\epsilon| \ll 1$ so that we can write

$$\hat{S} \simeq 1 + \frac{1}{2}(\epsilon^* \hat{a}^2 - \epsilon \hat{a}^{\dagger 2})$$

Answer

$$\begin{aligned}
[\hat{S}(\epsilon), \hat{D}(\alpha)] &\simeq [1 + \frac{1}{2}(\epsilon^* \hat{a}^2 - \epsilon \hat{a}^{\dagger 2}), \hat{D}(\alpha)] \\
&= \frac{1}{2}\epsilon^*[\hat{a}^2, \hat{D}(\alpha)] - \frac{1}{2}\epsilon[\hat{a}^{\dagger 2}, \hat{D}(\alpha)] \\
&= \frac{1}{2}\epsilon[\hat{D}(\alpha), \hat{a}^{\dagger 2}] - \frac{1}{2}\epsilon^*[\hat{D}(\alpha), \hat{a}^2] \\
&= \frac{1}{2}\epsilon\left([\hat{D}(\alpha), \hat{a}^\dagger]\hat{a}^\dagger + \hat{a}^\dagger[\hat{D}(\alpha), \hat{a}^\dagger]\right) - \frac{1}{2}\epsilon^*\left([\hat{D}(\alpha), \hat{a}]\hat{a} + \hat{a}[\hat{D}(\alpha), \hat{a}]\right) \\
&= \frac{1}{2}\epsilon^*\left([\hat{a}, \hat{D}(\alpha)]\hat{a} + \hat{a}[\hat{a}, \hat{D}(\alpha)]\right) - \frac{1}{2}\epsilon\left([\hat{a}^\dagger, \hat{D}(\alpha)]\hat{a}^\dagger + \hat{a}^\dagger[\hat{a}^\dagger, \hat{D}(\alpha)]\right) \\
&= \frac{1}{2}\epsilon^*\left(\frac{\partial\hat{D}}{\partial\hat{a}^\dagger}\hat{a} + \hat{a}\frac{\partial\hat{D}}{\partial\hat{a}^\dagger}\right) + \frac{1}{2}\epsilon\left(\frac{\partial\hat{D}}{\partial\hat{a}}\hat{a}^\dagger + \hat{a}^\dagger\frac{\partial\hat{D}}{\partial\hat{a}}\right) \\
&= \frac{1}{2}\epsilon^*\alpha\left(\hat{D}(\alpha)\hat{a} + \hat{a}\hat{D}(\alpha)\right) - \frac{1}{2}\epsilon\alpha^*\left(\hat{D}(\alpha)\hat{a}^\dagger + \hat{a}^\dagger\hat{D}(\alpha)\right) \\
&= \frac{1}{2}\epsilon^*\alpha\left(2\hat{a}\hat{D}(\alpha) - \frac{\partial\hat{D}}{\partial\hat{a}^\dagger}\right) - \frac{1}{2}\epsilon\alpha^*\left(2\hat{a}^\dagger\hat{D}(\alpha) + \frac{\partial\hat{D}}{\partial\hat{a}}\right) \\
&= \frac{1}{2}\epsilon^*\alpha\left(2\hat{a}\hat{D}(\alpha) - \alpha\hat{D}(\alpha)\right) - \frac{1}{2}\epsilon\alpha^*\left(2\hat{a}^\dagger\hat{D}(\alpha) - \alpha^*\hat{D}(\alpha)\right) \\
&= \left[\epsilon^*\alpha\hat{a} - (\epsilon^*\alpha\hat{a})^\dagger\right]\hat{D}(\alpha) - i\,\mathrm{Im}(\epsilon^*\alpha^2)\hat{D}(\alpha)
\end{aligned}$$

Suppose that $[\hat{S}, \hat{D}] = 0$, then in particular $\langle\alpha|[\hat{S}, \hat{D}]|0\rangle = 0$, for any value of α. Calculating explicitly the previous expectation value

$$
\begin{aligned}
\langle\alpha| [\hat{S},\hat{D}] |0\rangle &= \langle\alpha| \left(\left[\epsilon^*\alpha\hat{a} - (\epsilon^*\alpha\hat{a})^\dagger\right] \hat{D}(\alpha) - i\,\mathrm{Im}(\epsilon^*\alpha^2)\hat{D}(\alpha) \right) |0\rangle \\
&= \langle\alpha| \left([\epsilon^*\alpha\hat{a} - (\epsilon^*\alpha\hat{a})^\dagger] |\alpha\rangle - i\,\mathrm{Im}(\epsilon^*\alpha^2) |\alpha\rangle \right) \\
&= \epsilon^*\alpha^2 - (\epsilon^*\alpha^2)^* - i\,\mathrm{Im}(\epsilon^*\alpha^2) \\
&= 2i\,\mathrm{Im}(\epsilon^*\alpha^2) - i\,\mathrm{Im}(\epsilon^*\alpha^2) = i\,\mathrm{Im}(\epsilon^*\alpha^2) \neq 0\,!
\end{aligned}
$$

Which proves[1] by contradiction that the commutator is non-zero[2]

(b) Using the results of Eqs. 3.59 and 3.60 in the notes, give the expressions for the following expectation values for our new squeezed coherent state (Hint: you should obtain different values from what we obtained for the displaced squeezed vacuum state):

i. $\langle\epsilon,\alpha| \hat{a} |\epsilon,\alpha\rangle$

Answer

$$
\begin{aligned}
\langle\epsilon\alpha| \hat{a} |\epsilon\alpha\rangle &= \langle 0| \hat{D}^\dagger \hat{S}^\dagger \hat{a} \hat{S} \hat{D} |0\rangle \\
&= \langle 0| \hat{D}^\dagger \left[\hat{a}\cosh r - e^{i\theta}\hat{a}^\dagger \sinh r\right] \hat{D} |0\rangle \\
&= \cosh r \langle\alpha| \hat{a} |\alpha\rangle - e^{i\theta}\sinh r \langle\alpha| \hat{a}^\dagger |\alpha\rangle \\
&= \alpha\cosh r - \alpha^* e^{i\theta}\sinh r \blacksquare
\end{aligned}
$$

ii.$\langle\epsilon,\alpha| \hat{a}^2 |\epsilon,\alpha\rangle$

Answer

$$
\begin{aligned}
\langle\epsilon\alpha| \hat{a}^2 |\epsilon\alpha\rangle &= \langle\alpha| \hat{S}^\dagger \hat{a}^2 \hat{S} |\alpha\rangle \\
&= \langle\alpha| \hat{S}^\dagger \hat{a} \hat{S} \hat{S}^\dagger \hat{a} \hat{S} |\alpha\rangle \\
&= \langle\alpha| \hat{a}^2 \cosh^2 r + e^{2i\theta}\hat{a}^{\dagger 2}\sinh^2 r - (2\hat{a}^\dagger\hat{a}e^{i\theta}\cosh r\sinh r + e^{i\theta}\cosh r\sinh r) |\alpha\rangle \\
&= \alpha^2\cosh^2 r + e^{2i\theta}\alpha^{*2}\sinh^2 r - 2|\alpha|^2 e^{i\theta}\cosh r\sinh r - e^{i\theta}\cosh r\sinh r \\
&= \alpha^2\cosh^2 r + e^{2i\theta}\alpha^{*2}\sinh^2 r - \cosh r\sinh r\left(2|\alpha|^2 + 1\right) \blacksquare
\end{aligned}
$$

iii. $\langle\epsilon,\alpha| \hat{a}^\dagger\hat{a} |\epsilon,\alpha\rangle$

Answer

$$
\begin{aligned}
\langle\epsilon\alpha| \hat{a}^\dagger\hat{a} |\epsilon\alpha\rangle &= \langle\alpha| \hat{S}^\dagger \hat{a}^\dagger \hat{a} \hat{S} |\alpha\rangle \\
&= \langle\alpha| \hat{S}^\dagger \hat{a}^\dagger \hat{S} \hat{S}^\dagger \hat{a} \hat{S} |\alpha\rangle \\
&= (\cosh^2 r + \sinh^2 r) \langle\alpha| \hat{a}^\dagger\hat{a} |\alpha\rangle - e^{i\theta}\sinh r\cosh r \langle\alpha| \hat{a}^{\dagger 2} |\alpha\rangle + \\
&\quad - e^{-i\theta}\sinh r\cosh r \langle\alpha| \hat{a}^2 |\alpha\rangle + \sinh^2 r \langle\alpha|\alpha\rangle \\
&= |\alpha|^2(\cosh^2 r + \sinh^2 r) - \sinh r\cosh r\left(\alpha^{*2}e^{i\theta} + \alpha^2 e^{-i\theta}\right) + \sinh^2 r \blacksquare
\end{aligned}
$$

(c) Using the result from part (b), find the expressions for $\langle\epsilon,\alpha| \hat{X}_1 |\epsilon,\alpha\rangle$ and $\langle\epsilon,\alpha| \hat{X}_2 |\epsilon,\alpha\rangle$ for the special case that both ϵ is real and positive and $\alpha = |\alpha|e^{i\psi}$. You should find that they are different than for a displaced squeezed vacuum state.

[1] Under the $|\epsilon| \ll 1$ approximation

[2] The displaced squeezed vacuum state and the quadrature-squeezed state are the same if $2\theta - \psi = 2\pi n$, where θ and ψ are the phases of α and ϵ respectively.

Answer

$$
\begin{aligned}
\langle\epsilon\alpha|\,\hat{X}_1\,|\epsilon\alpha\rangle &= \frac{1}{2}\Big(\langle\epsilon\alpha|\,\hat{a}\,|\epsilon\alpha\rangle + \langle\epsilon\alpha|\,\hat{a}^\dagger\,|\epsilon\alpha\rangle\Big)\\
&= \frac{1}{2}\Big(\alpha\cosh r - \alpha^* e^{i\theta}\sinh r + \alpha^*\cosh r - \alpha e^{-i\theta}\sinh r\Big)\\
&= \frac{1}{2}|\alpha|\left(2\cos\psi\cosh r - 2\cos\psi\sinh r\right)\\
&= |\alpha|\cos\psi(\cosh r - \sinh r) = |\alpha|\cos\psi e^{-r}\;\blacksquare
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
\langle\epsilon\alpha|\,\hat{X}_2\,|\epsilon\alpha\rangle &= \frac{1}{2i}\Big(\langle\epsilon\alpha|\,\hat{a}\,|\epsilon\alpha\rangle - \langle\epsilon\alpha|\,\hat{a}^\dagger\,|\epsilon\alpha\rangle\Big)\\
&= |\alpha|\sin\psi(\cosh r + \sinh r) = |\alpha|\sin\psi e^{r}\;\blacksquare
\end{aligned}
$$

(d) Show, using the results from part (b), that for the special case that ϵ is real and positive and $\alpha = |\alpha|e^{i\psi}$, then for state $|\epsilon,\alpha\rangle$

$$
\begin{aligned}
\Delta X_1 &= \frac{1}{2}e^{-r}\\
\Delta X_2 &= \frac{1}{2}e^{r}.
\end{aligned}
$$

Thus, even though these squeezed state are different than the ones discussed in class, they have the same quadrature variances (see notes on page 3.31).

Answer

$$
\begin{aligned}
\hat{X}_1^2 &= \frac{1}{4}\Big(\hat{a}^2 + \hat{a}^{\dagger 2} + 2\hat{a}^\dagger\hat{a} + 1\Big)\\
\hat{X}_2^2 &= -\frac{1}{4}\Big(\hat{a}^2 + \hat{a}^{\dagger 2} - \Big(2\hat{a}^\dagger\hat{a} + 1\Big)\Big)
\end{aligned}
$$

So for $\hat{X}_1$,

$$
\begin{aligned}
\left\langle X_1^2\right\rangle &= \frac{1}{4}\left[\left\langle\hat{a}^2\right\rangle + \left\langle\hat{a}^{\dagger 2}\right\rangle + 2\left\langle\hat{a}^\dagger\hat{a}\right\rangle + 1\right]\\
&= \frac{1}{4}\left[\left\langle\hat{a}^2\right\rangle + \left\langle\hat{a}^2\right\rangle^* + 2\left\langle\hat{a}^\dagger\hat{a}\right\rangle + 1\right]\\
&= \frac{1}{4}\left[2\,\mathrm{Re}\{\left\langle\hat{a}^2\right\rangle\} + 2\left\langle\hat{a}^\dagger\hat{a}\right\rangle + 1\right]\\
&= \frac{1}{2}\Big[|\alpha|^2\cos 2\psi(\cosh^2 r + \sinh^2 r) - \cosh r\sinh r(2|\alpha|^2 + 1) +\\
&+ |\alpha|^2\cosh^2 r + (|\alpha|^2 + 1)\sinh^2 r - 2|\alpha|^2\cos 2\psi\sinh r\cosh r + \frac{1}{2}\Big]\\
&= \frac{1}{2}\Big[|\alpha|^2\cos 2\psi\left(\cosh^2 r - 2\sinh r\cosh r + \sinh^2 r\right) +\\
&+ |\alpha|^2\left(\cosh^2 r - 2\sinh r\cosh r + \sinh^2 r\right) +\\
&+ \sinh^2 r - \cosh r\sinh r + \frac{1}{2}\Big]\\
&= \frac{1}{2}\left[|\alpha|^2(\cos 2\psi + 1)(\cosh r - \sinh r)^2 + \frac{1}{2}(\cosh 2r - 1) - \frac{1}{2}\sinh 2r + \frac{1}{2}\right]\\
&= \frac{1}{2}\left[2|\alpha|^2\cos^2\psi e^{-2r} + \frac{1}{2}e^{-2r}\right] = |\alpha|^2\cos^2\psi e^{-2r} + \frac{1}{4}e^{-2r}
\end{aligned}
$$

So,

$$
\begin{aligned}
(\Delta X_1)^2 &= |\alpha|^2 \cos^2\psi e^{-2r} + \frac{1}{4}e^{-2r} - |\alpha|^2 \cos^2\psi e^{-2r} \\
&= \frac{1}{4}e^{-2r} \\
\Rightarrow \Delta X_1 &= \frac{1}{2}e^{-r} \, \square
\end{aligned}
$$

For $\hat{X}_2$,

$$
\begin{aligned}
\left\langle X_2^2 \right\rangle &= -\frac{1}{2}\left[\mathrm{Re}\{\langle \hat{a}^2 \rangle\} - \left\langle \hat{a}^\dagger \hat{a} \right\rangle - \frac{1}{2}\right] \\
&= -\frac{1}{2}\left[|\alpha|^2 \cos 2\psi \left(\cosh^2 r + 2\sinh r \cosh r + \sinh^2 r\right) + \right. \\
&- |\alpha|^2 \left(\cosh^2 r + 2\sinh r \cosh r + \sinh^2 r\right) + \\
&\left. - \sinh^2 r - \cosh r \sinh r - \frac{1}{2}\right] \\
&= -\frac{1}{2}\left[|\alpha|^2 (\cos 2\psi - 1)(\cosh r + \sinh r)^2 - \frac{1}{2}\left(\cosh 2r - 1 + \sinh 2r + 1\right)\right] \\
&= -\frac{1}{2}\left[-2|\alpha|^2 \sin^2\psi e^{2r} - \frac{1}{2}e^{2r}\right] = |\alpha|^2 \sin^2\psi e^{2r} + \frac{1}{4}e^{2r}
\end{aligned}
$$

So,

$$
\begin{aligned}
(\Delta X_2)^2 &= |\alpha|^2 \sin^2\psi e^{2r} + \frac{1}{4}e^{2r} - |\alpha|^2 \sin^2\psi e^{2r} \\
&= \frac{1}{4}e^{2r} \\
\Rightarrow \Delta X_2 &= \frac{1}{2}e^{r} \, \blacksquare
\end{aligned}
$$

Problem 2. In this question we examine the evolution of the expectation value of the atomic dipole moment driven by a single-mode coherent-state field using the Jaynes-Cummings Hamiltonian.

(a) We take the initial state of the system to be

$$|\Psi(t=0)\rangle = |e\rangle\,|\alpha\rangle\,, \tag{3}$$

i.e. the atom is in the excited state and the field is in a single-mode coherent state. Using the result from section 5.6. in the notes, give the expression for $|\Psi(t)\rangle$ for $t>0$ as an expansion in the basis of states, $|\epsilon\rangle\,|n\rangle$ and $|g\rangle\,|n+1\rangle$.

Answer

Using the initial condition to find the c_n coefficients,

$$\begin{aligned}|\psi(0)\rangle &= \sum_n c_n\,|\psi_1(n)\rangle\\ &= \sum_n e^{-|\alpha|^2/2}\frac{\alpha^n}{\sqrt{n!}}\,|\psi_1(n)\rangle\end{aligned}$$

it is straightforward to see that $c_n = e^{-|\alpha|^2/2}\alpha^n/\sqrt{n!}$ Thus by substituting the dressed states by the $|\psi_i(n)\rangle$ kets in equation 136 from the notes:

$$\begin{aligned}|\psi(t)\rangle =& \sum_n e^{-|\alpha|^2/2}\frac{\alpha^n}{\sqrt{n!}}\Big[\Big(\cos^2(\Phi_n/2)e^{-iE_+^{(n)}t/\hbar}+\sin^2(\Phi_n/2)e^{-iE_-^{(n)}t/\hbar}\Big)\,|\psi_1(n)\rangle +\\ &+\Big(\cos(\Phi_n/2)\sin(\Phi_n/2)e^{-iE_+^{(n)}t/\hbar}-\cos(\Phi_n/2)\sin(\Phi_n/2)e^{-iE_-^{(n)}t/\hbar}\Big)\,|\psi_2(n)\rangle\Big]\\ =& \sum_n \frac{1}{2}e^{-|\alpha|^2/2}\frac{\alpha^n}{\sqrt{n!}}\Big[\cos(\Phi_n)\Big(e^{-iE_+^{(n)}t/\hbar}-e^{-iE_-^{(n)}t/\hbar}\Big)\,|\psi_1(n)\rangle +\\ &+\Big(e^{-iE_+^{(n)}t/\hbar}+e^{-iE_-^{(n)}t/\hbar}\Big)\,|\psi_1(n)\rangle +\\ &+\sin(\Phi_n)\Big(e^{-iE_+^{(n)}t/\hbar}-e^{-iE_+^{(n)}t/\hbar}\Big)\,|\psi_2(n)\rangle\Big]\ \blacksquare\end{aligned}$$

where,

$$\begin{aligned}|\psi_1(n)\rangle &= |e\rangle\,|n\rangle\\ |\psi_2(n)\rangle &= |g\rangle\,|n+1\rangle\\ \Phi_n &= \arctan\left[\frac{\Omega_n(0)}{\Delta}\right]\\ \Omega_n(\Delta) &= \left[\Delta^2+4\lambda^2(n+1)\right]^{1/2}\\ \Delta &= \omega_o-\omega\\ E_\pm^{(n)} &= \left(n+\frac{1}{2}\right)\hbar\omega \pm \frac{1}{2}\hbar\Omega_n(\Delta)\end{aligned}$$

(b) Using the result from part (a), give the expression for the expectation value of the dipole moment of the atom as a function of time for the initial state given in part (a) in the special case of resonant excitation ($\Delta=0$). Let $\alpha=|\alpha|e^{i\theta}$ and let $\mathbf{d}_{eg}\equiv\langle e|\,\mathbf{d}\,|g\rangle$

Answer

When $\Delta = 0$ the time dependent evolution simplifies to

$$|\psi(t)\rangle = \sum_n \frac{1}{2} e^{-|\alpha|^2/2} \frac{\alpha^n}{\sqrt{n!}} \left[\left(e^{-iE_+^{(n)}t/\hbar} + e^{-iE_-^{(n)}t/\hbar} \right) |\psi_1(n)\rangle + \right.$$
$$\left. + \left(e^{-iE_+^{(n)}t/\hbar} - e^{-iE_+^{(n)}t/\hbar} \right) |\psi_2(n)\rangle \right]$$
$$|\psi(t)\rangle = \frac{1}{2} e^{-|\alpha|^2/2} \sum_n \frac{\alpha^n}{\sqrt{n!}} e^{-i\epsilon_n t/\hbar} \left[\left(e^{-i\frac{1}{2}\Omega(n)t} + e^{i\frac{1}{2}\Omega(n)t} \right) |\psi_1(n)\rangle + \right.$$
$$\left. + \left(e^{-i\frac{1}{2}\Omega(n)t} - e^{i\frac{1}{2}\Omega(n)t} \right) |\psi_2(n)\rangle \right]$$
$$|\psi(t)\rangle = e^{-|\alpha|^2/2} \sum_n \frac{\alpha^n}{\sqrt{n!}} e^{-i\epsilon_n t/\hbar} \left[\cos\left(\lambda\sqrt{n+1}t\right) |e\rangle |n\rangle + \right.$$
$$\left. - i \sin\left(\lambda\sqrt{n+1}t\right) |g\rangle |n+1\rangle \right] \square$$

Where $\epsilon_n = \hbar\omega(n + \frac{1}{2})$. Assuming d_{eg} is real then,

$$\hat{d} = d_{e,g} \left(\hat{\sigma}_+ + \hat{\sigma}_- \right)$$

Notice that $\hat{\sigma}_+ |\psi_1(n)\rangle = |g\rangle |n\rangle$ and $\hat{\sigma}_- |\psi_2(n)\rangle = |e\rangle |n+1\rangle$ and the rest of the combinations are zero.

$$\begin{aligned} \frac{\left\langle \hat{d} \right\rangle}{d_{e,g}} &= \langle\psi(t)| \left(\hat{\sigma}_+ + \hat{\sigma}_- \right) |\psi(t)\rangle \\ &= \langle\psi(t)| \hat{\sigma}_+ |\psi(t)\rangle + \langle\psi(t)| \hat{\sigma}_- |\psi(t)\rangle \end{aligned}$$

Calculating the expectations values for $\hat{\sigma}_+$ and $\hat{\sigma}_-$,

$$\begin{aligned} \langle\psi(t)| \hat{\sigma}_+ |\psi(t)\rangle &= e^{-|\alpha|^2} \sum_{n,m} \frac{\alpha^{m*}\alpha^n}{\sqrt{m!n!}} e^{-i(\epsilon_n - \epsilon_m)t/\hbar} \times \\ &\times \left[\cos\left(\lambda\sqrt{m+1}t\right) \langle e| \langle m| + i \sin\left(\lambda\sqrt{m+1}t\right) \langle g| \langle m+1| \right] \\ &\left[\cos\left(\lambda\sqrt{n+1}t\right) \hat{\sigma}_+ |e\rangle |n\rangle - i \sin\left(\lambda\sqrt{n+1}t\right) \hat{\sigma}_- |g\rangle |n+1\rangle \right] \\ &= e^{-|\alpha|^2} \sum_{n,m} \frac{\alpha^{m*}\alpha^n}{\sqrt{m!n!}} e^{-i(\epsilon_n - \epsilon_m)t/\hbar} (-i) \sin\left(\lambda\sqrt{n+1}t\right) \cos\left(\lambda\sqrt{m+1}t\right) \langle m|n+1\rangle \\ &= e^{-|\alpha|^2} \sum_n \frac{|\alpha|^{2n}\alpha^*}{n!\sqrt{n+1}} e^{i\omega t} (-i) \sin\left(\lambda\sqrt{n+1}t\right) \cos\left(\lambda\sqrt{n+2}t\right) \end{aligned}$$

Similarly for $\hat{\sigma}_-$,

$$\begin{aligned} \langle\psi(t)| \hat{\sigma}_- |\psi(t)\rangle &= e^{-|\alpha|^2} \sum_{n,m} \frac{\alpha^{m*}\alpha^n}{\sqrt{m!n!}} e^{-i(\epsilon_n - \epsilon_m)t/\hbar} i \sin\left(\lambda\sqrt{m+1}t\right) \cos\left(\lambda\sqrt{n+1}t\right) \langle m+1|n\rangle \\ &= e^{-|\alpha|^2} \sum_m \frac{|\alpha|^{2m}\alpha}{m!\sqrt{m+1}} e^{-i\omega t} i \sin\left(\lambda\sqrt{m+1}t\right) \cos\left(\lambda\sqrt{m+2}t\right) \end{aligned}$$

Replacing m for n in the second expectation value and letting $\alpha = |\alpha|e^{i\theta}$ we have

$$\begin{aligned}
\frac{\left\langle \hat{d} \right\rangle}{d_{e,g}} &= e^{-|\alpha|^2} \sum_n \frac{|\alpha|^{2n}}{n!\sqrt{n+1t}} \sin\left(\lambda\sqrt{n+1}t\right)\cos\left(\lambda\sqrt{n+2}t\right)\left[-i\alpha^* e^{i\omega t} + i\alpha e^{-i\omega t}\right] \\
&= e^{-|\alpha|^2} \sum_n \frac{|\alpha|^{2n}}{n!\sqrt{n+1}} \sin\left(\lambda\sqrt{n+1}t\right)\cos\left(\lambda\sqrt{n+2}t\right) 2\,\mathrm{Re}(i\alpha e^{-i\omega t}) \\
&= 2e^{-|\alpha|^2} \sum_n \frac{|\alpha|^{2n+1}}{n!\sqrt{n+1}} \sin\left(\lambda\sqrt{n+1}t\right)\cos\left(\lambda\sqrt{n+2}t\right)\sin(\omega t - \theta) \blacksquare
\end{aligned}$$

(c) Plot the result from part (b) for $\alpha = 4$ and $\lambda = \omega/20$ as a function of ωt, where λ is the interaction parameter given in the notes. Run the plot for $\omega t = 0$ to $\omega t = 250$. If you use Maple (or something similar), please include your code sheets.

Answer

See part (d).

(d) Repeat part (c) but this time set $\lambda = \omega/2$. Comment on whether this result is valid.

Answer

The code used for plotting can be found here, with the name Dipole Expectation Value.ipynb .

In both cases we can relate the interaction time, t, to the period of the modes, T, as

$$t = \frac{250}{\omega} = \frac{250}{2\pi}\frac{2\pi}{\omega} \approx 39.8T,$$

which is good enough to say $t \gg T$ and thus that the rotating wave approximation is still valid. Problems would arise if we were exploring short time scales *i.e.* the fast dynamics of the interaction for $t \leq T$, since we ignored the terms proportional to $e^{\pm i(\omega_o+\omega)t}$ when deriving the Hamiltonian.

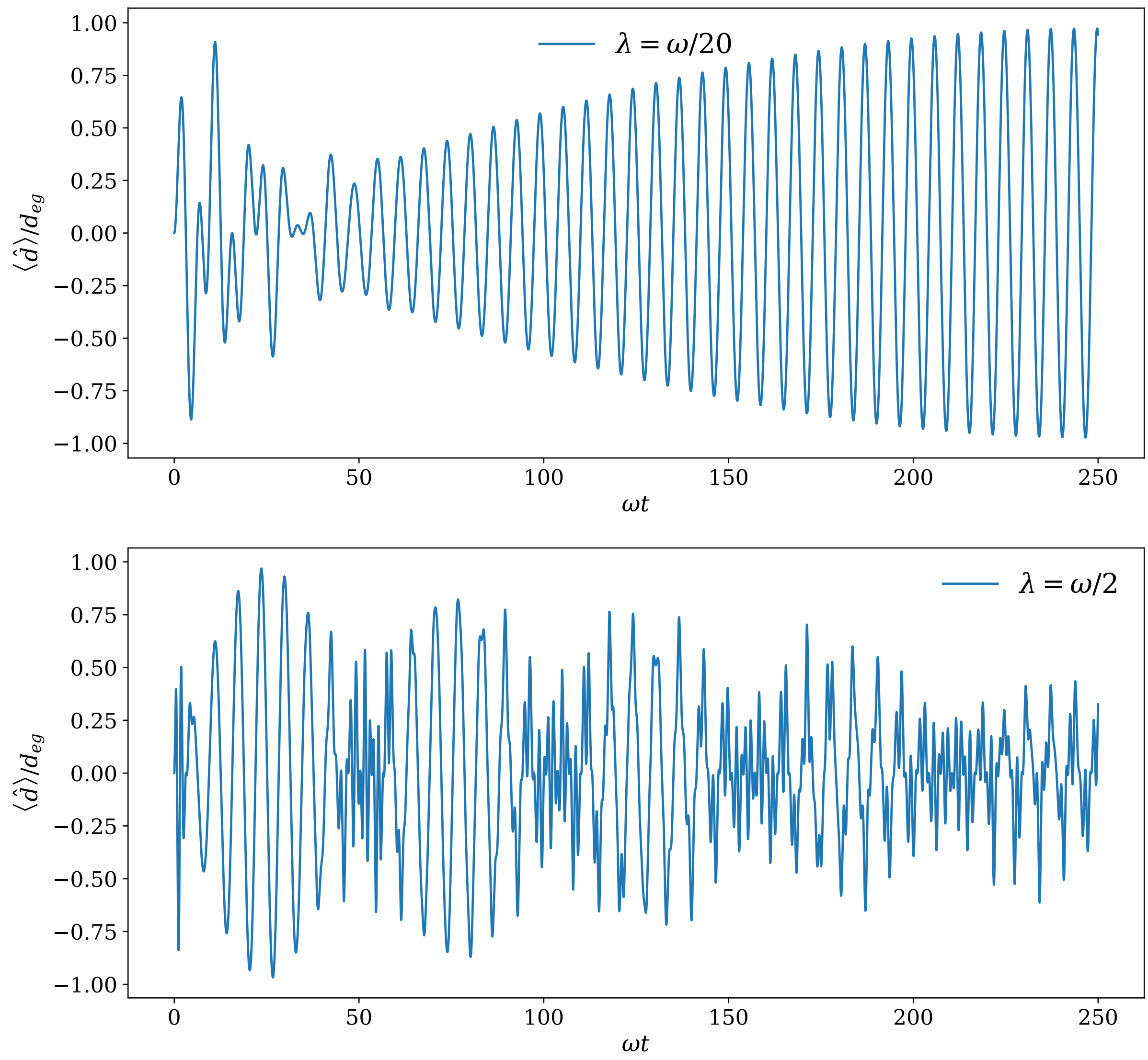

Figure 1: Dipole expectation value for two different interaction parameters λ

Problem 3. In this problem we apply Wigner-Weisskoph theory to the case of single-mode leaky cavity. As discussed in class, for a cavity, such as a defect in a photonic crystal slab, when ω is close to the atomic transition frequency, Ω_o, the shift-width function is given approximately by

$$W(\omega) = \frac{\Omega_o}{2\hbar\epsilon_o} \frac{|\mathbf{f}_\mu(\mathbf{r}_a) \cdot \mathbf{d}_{eg}|^2}{\omega - \omega_\mu + i\Gamma_\mu/2}, \tag{4}$$

where ω_μ is the resonance frequency of the cavity mode, $1/\Gamma_\mu$ is the photon lifetime in the cavity and $\mathbf{f}_\mu(\mathbf{r}_a)$ is the mode field at the atomic position.

(a) Using Eq. 156 from chapter 5 in the notes, show that the Fourier transform of the probability amplitude of finding the atom in the excited state (given that it was initially in the excited state with no photons present) is given by

$$\tilde{a}(\omega) = i\frac{(\omega - \omega_\mu + i\Gamma_\mu/2)}{(\omega - \tilde{\omega}_+)(\omega - \tilde{\omega}_-)}, \tag{5}$$

where $\tilde{\omega}_+$ and $\tilde{\omega}_-$ are two complex frequencies that are found as the solutions to a quadratic equation. Give the expressions for these frequencies. **Note:** Don't simple assume that you can replace $W(\omega)$ by $W(\omega_a)$.

Answer From equation 156,

$$\begin{aligned}\tilde{\alpha}(\omega) &= \frac{i}{\omega - \Omega_o - W(\omega)} = \frac{i}{\omega - \Omega_o - \frac{\Omega_o}{2\hbar\epsilon_o}\frac{|\mathbf{f}_\mu \cdot \mathbf{d}_{eg}|^2}{\omega - \omega_\mu + i\Gamma_\mu/2}} \\ &= i\frac{\omega - \omega_\mu + i\Gamma_\mu/2}{(\omega - \omega_\mu + i\Gamma_\mu/2)(\omega - \Omega_o) - \frac{\Omega_o}{2\hbar\epsilon_o}|\mathbf{f}_\mu \cdot \mathbf{d}_{eg}|^2}\end{aligned}$$

using the definition from the notes ($\hbar\lambda_\mu \equiv -i(\hbar\omega_\mu/2\epsilon_o)^{1/2}\mathbf{d}_{eg} \cdot \mathbf{f}_\mu(\mathbf{r}_a)$) In the denominator we have

$$\frac{\Omega_o}{2\hbar\epsilon_0}|\mathbf{f}_\mu \cdot \mathbf{d}_{eg}|^2 = \frac{\Omega_o}{2\hbar\epsilon_o}\frac{2\epsilon_o}{\hbar\omega_\mu}(\hbar\lambda_\mu)^2 = \frac{\Omega_o}{\omega_\mu}\lambda_\mu^2$$

which leads to the second order polynomial[3]

$$\omega^2 - \omega\left(\Omega_o + \omega_\mu - i\Gamma/2\right) + \Omega_o\left(\omega_\mu - i\Gamma/2\right) - \frac{\Omega_o}{\omega_\mu}\lambda_\mu^2$$

so having,

$$\tilde{\omega}_\pm = \frac{\Omega_o + \omega_\mu - i\Gamma/2}{2} \pm \sqrt{\left(\frac{\Omega_o + \omega_\mu - i\Gamma/2}{2}\right)^2 - \left(\Omega_o(\omega_\mu - i\Gamma/2) - \frac{\Omega_o}{\omega_\mu}\lambda_\mu^2\right)},$$

we can write

$$\tilde{\alpha}(\omega) = \frac{\omega - \omega_\mu + i\Gamma/2}{(\omega - \tilde{\omega}_+)(\omega - \tilde{\omega}_-)} \blacksquare$$

(b) Give the explicit and simplified expressions for $\tilde{\omega}_+$ and $\tilde{\omega}_-$ in the resonant case where $\Omega_o = \omega_\mu$.

Answer

In the resonant case

$$\begin{aligned}\tilde{\omega}_\pm &= \omega_\mu - i\Gamma/4 \pm \sqrt{(\omega_\mu - i\Gamma/4)^2 - \left(\omega_\mu^2 - i\Gamma\omega_\mu/2 - \lambda_\mu^2\right)} \\ &= \omega_\mu - i\Gamma/4 \pm \sqrt{(\omega_\mu - i\Gamma/4)^2 - \left[(\omega_\mu - i\Gamma/4)^2 + (\Gamma/4)^2 - \lambda_\mu^2\right]} \\ &= \omega_\mu - i\Gamma/4 \pm \sqrt{\lambda_\mu^2 - (\Gamma/4)^2} \blacksquare\end{aligned}$$

(c) Using the results for part (a) in the **resonant case**, use Cauchy's theorem to give an explicit expression for $\alpha(t)$ for this system. Hint: The form of Cauchy's theorem that you may want to use is:

$$\int_\gamma \frac{f(z)}{(z - z_0)} = -2\pi i f(z_0), \tag{6}$$

[3] By mistake I dropped the μ sub-index from Γ, however this does not affect in any way to the result.

where γ is a clockwise, closed curve in the complex plane that contains the pole at z_0. Answer
Notice that we can rewrite the numerator as

$$\begin{aligned}\omega - \omega_\mu + i\Gamma_\mu/2 &= \omega + \omega_\mu - (2\omega_\mu - i\Gamma/2)\\ &= \omega + \omega_\mu - (\tilde{\omega}_+ + \tilde{\omega}_-)\\ &= (\omega - \tilde{\omega}_+) + (\omega - \tilde{\omega}_-) + \omega_\mu - \omega\end{aligned}$$

This way is easier to see the partial fractions decomposition

$$\begin{aligned}\tilde{\alpha}(\omega) &= \frac{(\omega-\tilde{\omega}_+)+(\omega-\tilde{\omega}_-)}{(\omega-\tilde{\omega}_+)(\omega-\tilde{\omega}_-)} + \frac{\omega_\mu - \omega}{(\omega-\tilde{\omega}_+)(\omega-\tilde{\omega}_-)}\\ &= \frac{1}{\omega-\tilde{\omega}_+} + \frac{1}{\omega-\tilde{\omega}_-} + \frac{\omega_\mu-\tilde{\omega}_+}{\tilde{\omega}_+-\tilde{\omega}_-}\frac{1}{\omega-\tilde{\omega}_+} - \frac{\omega_\mu-\tilde{\omega}_-}{\tilde{\omega}_+-\tilde{\omega}_-}\frac{1}{\omega-\tilde{\omega}_-}\\ &= \left(1+\frac{\omega_\mu-\tilde{\omega}_+}{\tilde{\omega}_+-\tilde{\omega}_-}\right)\frac{1}{\omega-\tilde{\omega}_+} + \left(1-\frac{\omega_\mu-\tilde{\omega}_-}{\tilde{\omega}_+-\tilde{\omega}_-}\right)\frac{1}{\omega-\tilde{\omega}_-}\\ &= \left(\frac{\omega_\mu-\tilde{\omega}_-}{\tilde{\omega}_+-\tilde{\omega}_-}\right)\frac{1}{\omega-\tilde{\omega}_+} - \left(\frac{\omega_\mu-\tilde{\omega}_+}{\tilde{\omega}_+-\tilde{\omega}_-}\right)\frac{1}{\omega-\tilde{\omega}_-},\end{aligned}$$

thus taking the Fourier transform of $\tilde{a}(\omega)$,

$$\begin{aligned}\alpha(t) &= \frac{1}{2\pi}\int_\gamma d\omega\tilde{\alpha}(\omega)e^{-i\omega t}\\ &= \frac{1}{2\pi}\left(\frac{\omega_\mu-\tilde{\omega}_-}{\tilde{\omega}_+-\tilde{\omega}_-}\right)\int_\gamma \frac{e^{-i\omega t}d\omega}{\omega-\tilde{\omega}_+} + \frac{1}{2\pi}\left(\frac{\omega_\mu-\tilde{\omega}_+}{\tilde{\omega}_+-\tilde{\omega}_-}\right)\int_\gamma \frac{e^{-i\omega t}d\omega}{\omega-\tilde{\omega}_-}\\ &= -i\left(\frac{\omega_\mu-\tilde{\omega}_-}{\tilde{\omega}_+-\tilde{\omega}_-}\right)e^{-i\tilde{\omega}_+ t} + i\left(\frac{\omega_\mu-\tilde{\omega}_+}{\tilde{\omega}_+-\tilde{\omega}_-}\right)e^{-i\tilde{\omega}_- t} \blacksquare\end{aligned}$$

(d) Show from your results in part (c) that if $\Gamma_\mu \ll \Omega_{Ro}$, the probability of finding the atom in the excited state as a function of time is

$$P_e(t) \simeq e^{-\Gamma_\mu t/2}\cos^2(\Omega_R t/2)$$

where

$$\Omega_R \equiv \sqrt{\Omega_{Ro}^2 - \Gamma_\mu^2/4},$$

where

$$\Omega_{Ro} \equiv 2\sqrt{\frac{\Omega_o}{2\hbar\epsilon_0}}|\mathbf{f}_\mu(\mathbf{r}_a)\cdot\mathbf{d}_{eg}|$$

is the bare vacuum Rabi frequency of the system. These are called the **damped Vacuum Rabi Oscillations**

Answer

Notice that $\Omega_{Ro} = 2|\lambda_\mu|$ according to the definition I used above. In the regime $\Gamma \ll \Omega_{Ro}$, so the following values are approximately:

$$\begin{aligned}\omega_\mu - \tilde{\omega}_+ &= i\Gamma/4 - \sqrt{\lambda_\mu^2 - (\Gamma/4)^2}\\ &= \frac{1}{2}\left(i\Gamma/2 - \Omega_{Ro}\right) \simeq -\frac{1}{2}\Omega_R\\ \omega_\mu - \tilde{\omega}_- &= i\Gamma/4 + \sqrt{\lambda_\mu^2 - (\Gamma/4)^2}\\ &\simeq \frac{1}{2}\Omega_R\end{aligned}$$

So the terms multiplying $e^{-i\tilde{\omega}_+ t}$ and $e^{-i\tilde{\omega}_- t}$ in the expression for $\alpha(t)$ are approximately $-1/2$ and $1/2$. The following values are exact:

$$\begin{aligned}
\tilde{\omega}_+ - \tilde{\omega}_- &= 2\sqrt{\lambda_\mu^2 - (\Gamma/4)^2} \\
&= \sqrt{\Omega_{Ro}^2 - \Gamma^2/4} = \Omega_R
\end{aligned}$$

and (substituting Ω_o and Ω_{Ro} in $\tilde{\omega}_\pm$),

$$\tilde{\omega}_\pm = \omega_\mu + \frac{1}{2}\left(-i\Gamma/2 \pm \Omega_R\right) = \Omega_{Ro} \pm \frac{1}{2}\Omega_R - i\Gamma/4$$

which leads to

$$\begin{aligned}
\alpha(t) &\simeq -i\left[\frac{1}{2}e^{-i\tilde{\omega}_+} + \frac{1}{2}e^{-i\tilde{\omega}_-}\right] \\
&= -ie^{i^2\Gamma t/4}e^{-i\Omega_{Ro}t}\frac{1}{2}\left[e^{-i\frac{1}{2}\Omega_R t} + e^{i\frac{1}{2}\Omega_R t}\right] \\
&= -ie^{-\Gamma t/4}e^{-i\Omega_{Ro}t}\cos(\Omega_R t/2),
\end{aligned}$$

so,

$$\begin{aligned}
P_e(t) &= |\alpha(t)|^2 \\
&\simeq e^{-\Gamma t/2}\cos^2(\Omega_R t/2)\,\blacksquare
\end{aligned}$$