PHYS 882: Problem Set 3

Due: Monday March 9 at 4:00 PM

1. In class we defined our quadrature-squeezed state to be

$$|\alpha, \epsilon\rangle \equiv \widehat{D}(\alpha)\,\widehat{S}(\epsilon)\,|0\rangle\,. \tag{1}$$

This is a displaced, squeezed vacuum state. An alternative quadrature-squeezed state is the one defined by

$$|\epsilon, \alpha\rangle \equiv \widehat{S}(\epsilon)\,\widehat{D}(\alpha)\,|0\rangle\,, \tag{2}$$

which is a squeezed coherent state.

(a) Show that the above two states are not the same by calculating the commutator, $\left[\widehat{S}(\epsilon), \widehat{D}(\alpha)\right]$, and showing that it is in general non-zero. This is difficult to do in general, so instead consider the special case that $|\epsilon| \ll 1$ so that we can write

$$\widehat{S}(\epsilon) \simeq 1 + \frac{1}{2}\left(\epsilon^{*}\widehat{a}^{2} - \epsilon\widehat{a}^{\dagger 2}\right). \tag{3}$$

(b) Using the results of Eqs. 3.59 and 3.60 in the notes, give the expressions for the following expectation values for our new squeezed coherent state (Hint: you should obtain different values from what we obtained for the displaced, squeezed vacuum state):

i. $\langle\epsilon, \alpha|\,\widehat{a}\,|\epsilon, \alpha\rangle$.

ii. $\langle\epsilon, \alpha|\,\widehat{a}^{2}\,|\epsilon, \alpha\rangle$.

iii. $\langle\epsilon, \alpha|\,\widehat{a}^{\dagger}\widehat{a}\,|\epsilon, \alpha\rangle$.

(c) Using the results from part (b), find the expressions for $\langle\epsilon, \alpha|\,\widehat{X}_1\,|\epsilon, \alpha\rangle$ and $\langle\epsilon, \alpha|\,\widehat{X}_2\,|\epsilon, \alpha\rangle$ for the special case that both ϵ is real and positive and $\alpha = |\alpha|\,e^{i\psi}$. You should find that they are different than for a displaced squeezed vacuum state.

(d) Show, using the results from part b, that for the special case that ϵ is real and positive and $\alpha = |\alpha|\,e^{i\psi}$, then for state $|\epsilon, \alpha\rangle$

$$\begin{aligned} \Delta X_1 &= \frac{1}{2}e^{-r} \\ \Delta X_2 &= \frac{1}{2}e^{r}. \end{aligned}$$

Thus, even though these squeezed states are different than the ones discussed in class, they have the same quadrature variances (see notes on page 3.31).

2. In this question, we examine the evolution of the expectation value of the atomic dipole moment driven by a single-mode coherent-state field using the Jaynes-Cummings Hamiltonian.

   (a) We take the initial state of the system to be

   $$|\Psi(t=0)\rangle = |e\rangle |\alpha\rangle , \tag{4}$$

   i.e. the atom is in the excited state and the field is in a single-mode coherent state. Using the results form section 5.5 in the notes, give the expression for $|\Psi(t)\rangle$ for $t>0$ as an expansion in the basis of states, $|e\rangle |n\rangle$ and $|g\rangle |n+1\rangle$.

   (b) Using the result from part (a), give the expression for the expectation value of the dipole moment of the atom as a function of time for the initial state given in part (a) in the special case of resonant excitation ($\Delta = 0$). Let $\alpha = |\alpha| e^{i\theta}$ and let $\mathbf{d}_{eg} \equiv \langle e| \mathbf{d} |g\rangle$.

   (c) Plot the result from part (b) for $\alpha = 4$ and $\lambda = \omega/20$ as a function of ωt, where λ is the interaction parameter given in the notes. Run the plot for $\omega t = 0$ to $\omega t = 250$. If you use Maple (or something similar), please include your code sheets.

   (d) Repeat part (c) but this time set $\lambda = \omega/2$. Comment on whether this result is valid.

3. In this problem we apply Wigner-Weisskoph theory to the case of a single-mode leaky cavity. As discussed in class, for a cavity, such as a defect in a photonic crystal slab, when ω is close to the atomic transition frequency, Ω_o, the shift-width function is given approximately by

   $$W(\omega) = \frac{\Omega_o}{2\hbar\epsilon_o} \frac{|\mathbf{f}_\mu(\mathbf{r}_a) \cdot \mathbf{d}_{eg}|^2}{\omega - \omega_\mu + i\Gamma_\mu/2}, \tag{5}$$

   where ω_μ is the resonance frequency of the cavity mode, $1/\Gamma_\mu$ is the photon lifetime in the cavity and $\mathbf{f}_\mu(\mathbf{r}_a)$ is the mode field at the atomic position.

   (a) Using Eq. (156) from chapter 5 in the notes, show that the Fourier-transform of the probability amplitude of finding the atom in the excited state (given that it was initially in the excited state with no photons present) is given by

   $$\widetilde{\alpha}(\omega) = i \frac{(\omega - \omega_\mu + i\Gamma_\mu/2)}{(\omega - \widetilde{\omega}_+)(\omega - \widetilde{\omega}_-)}, \tag{6}$$

   where $\widetilde{\omega}_+$ and $\widetilde{\omega}_+$ are two complex frequencies that are found as the solutions to a quadratic equation. Give the expressions for these frequencies. **Note:** Don't simple assume that you can replace $W(\omega)$ by $W(\omega_a)$.

   (b) Give the explicit and simplified expressions for $\widetilde{\omega}_+$ and $\widetilde{\omega}_+$ in the resonant case where $\Omega_o = \omega_\mu$.

(c) Using the results for part (a) in the **resonant case**, use Cauchy's theorem to give a explicit expression for $\alpha(t)$ for this system. Hint: The form of Cauchy's theorem that you may want to use is:

$$\int_\gamma \frac{f(z)}{(z - z_o)} = -2\pi i f(z_o), \tag{7}$$

where γ is a clockwise, closed curve in the complex plane that contains the pole at z_o.

(d) Show from your results in part c that if $\Gamma_\mu \ll \Omega_{R_o}$, the probability of finding the atom in the excited state as a function of time is

$$P_e(t) \simeq e^{-\Gamma_\mu t/2} \cos^2(\Omega_R t/2),$$

where

$$\Omega_R \equiv \sqrt{\Omega_{Ro}^2 - \Gamma_\mu^2/4},$$

where

$$\Omega_{Ro} \equiv 2\sqrt{\frac{\Omega_o}{2\hbar\epsilon_o}} \left|\mathbf{f}_\mu(\mathbf{r}_a) \cdot \mathbf{d}_{eg}\right|$$

is the bare vacuum Rabi frequency for this system. These are called **damped Vacuum Rabi Oscillations**.