Lecture 19
Recursion

Objectives

- To understand recursion
- To understand when to use recursion
- Recursion vs iteration

Recursive Problem-Solving

Algorithm: Factorial – find the factorial for x

```
if n == 0:
    return the factorial of zero     [0! = 1]
else
    find factorial of (n-1)          [(n-1)!]
    return n * (n-1)!
```

- This version has no loop, and seems to refer to itself! What's going on??

Recursive Definitions

- A description of something that refers to itself is called a *recursive* definition.
- In the last example, the factorial algorithm uses its own description – a “call” to factorial “recurs” inside of the definition – hence the label “recursive definition.”

Recursive Definitions

- In mathematics, recursion is frequently used. The most common example is the factorial:

- For example, 5! = 5(4)(3)(2)(1), or
  5! = 5(4!)

$$n! = n(n-1)(n-2)...(1)$$

Recursive Definitions

- In other words, $n! = n(n-1)!$
- Or

$$n! = \begin{cases} 1 & \text{if } n = 0 \\ n(n-1)! & \text{otherwise} \end{cases}$$

- This definition says that 0! is 1, while the factorial of any other number is that number times the factorial of one less than that number.

Recursive Definitions

- Our definition is recursive, but definitely not circular. Consider 4!
  - *$4! = 4(4\text{-}1)! = 4(3!)$*
  - *What is 3!? We apply the definition again*
    $4! = 4(3!) = 4[3(3\text{-}1)!] = 4(3)(2!)$
  - *And so on…*
    $4! = 4(3!) = 4(3)(2!) = 4(3)(2)(1!) = 4(3)(2)(1)(0!) = 4(3)(2)(1)(1) = 24$
- Factorial is not circular because we eventually get to 0!, whose definition does not rely on the definition of factorial and is just 1. This is called a *base case* for the recursion.
- When the base case is encountered, we get a closed expression that can be directly computed.

Recursive Definitions

- All good recursive definitions have these two key characteristics:

    1. *There are one or more base cases for which no recursion is applied.*

    2. *All chains of recursion eventually end up at one of the base cases.*

    Put differently, each iteration must drive the computation toward a base case.

- The simplest way for these two conditions to occur is for each recursion to act on a smaller version of the original problem. A very small version of the original problem that can be solved without recursion becomes the base case.

Recursive Functions

- Factorial can be calculated using a loop accumulator.

```
def fact_loop(n):
    ans = 1
    for i in range(n,1,-1):
        ans *= i
    return ans
```

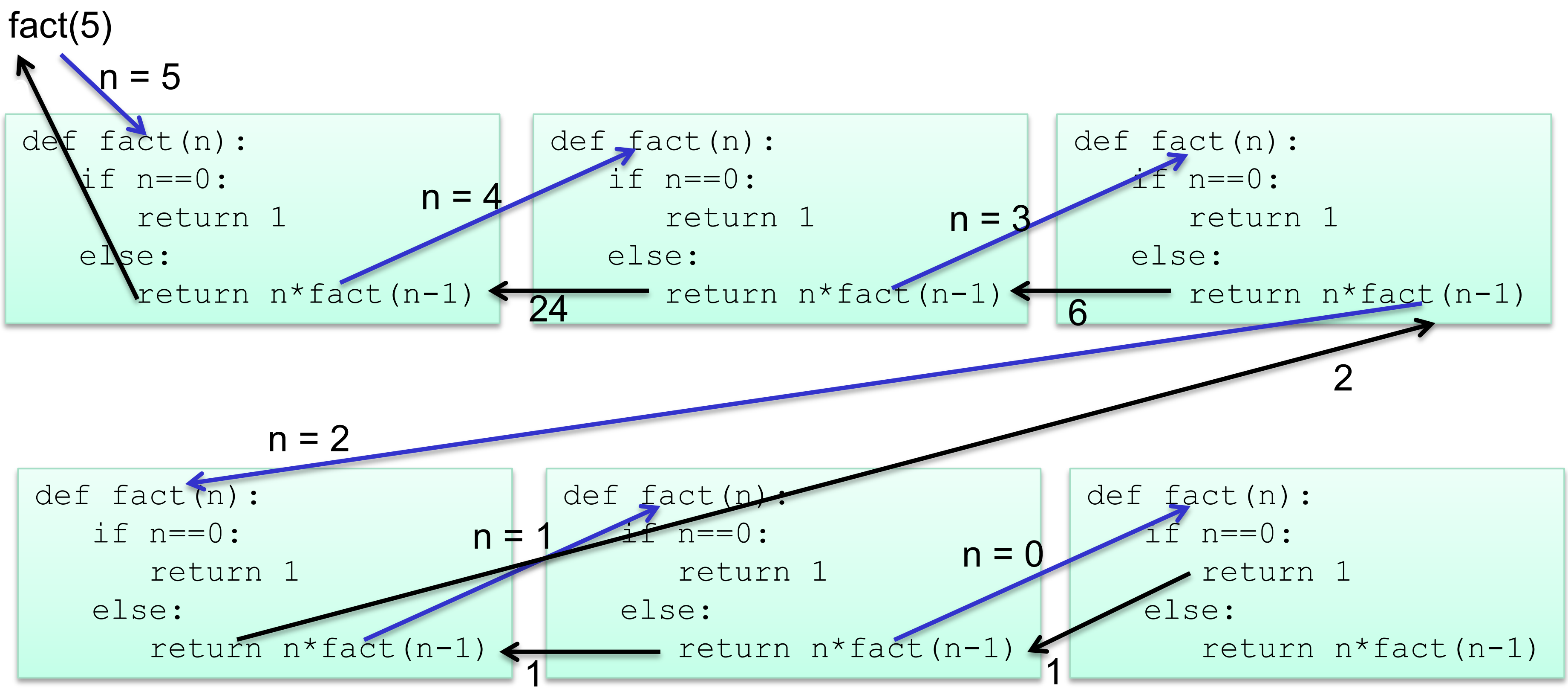
- If factorial is written as a separate recursive function:

```
def fact(n):
    if n == 0:
        return 1
    else:
        return n * fact(n-1)
```

Recursive Functions

- We've written a function that calls *itself*, i.e. a *recursive function*.
- The function first checks to see if we're at the base case `(n==0)`. If so, return `1`. Otherwise, return the result of multiplying n by the factorial of $n-1$, `fact(n-1)`.
- Remember that each call to a function starts that function anew, with its own copies of local variables and parameters.

Recursive Functions



Example: Binary Search

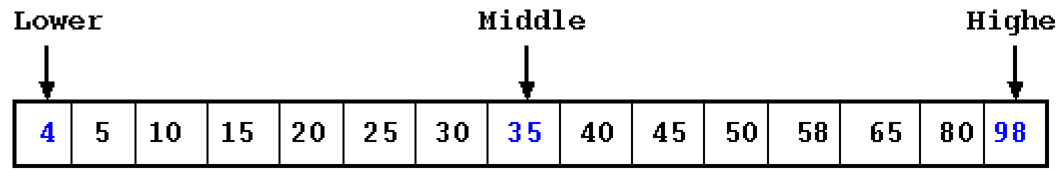
- In the last lecture, we learned how to perform binary search using a loop.
- If you haven't noticed already, we can perform binary search recursively.
- In binary search, we look at the middle value first, then we either search the lower half or upper half of the array.
- There are two base cases (to stop recursion/searching):
  - *when the target value is found*
  - *when we have run out of places to look.*

Example: Binary Search

- The recursive calls will cut the search in half each time by specifying the range of locations that are not searched and may contain the target value.
- Each invocation of the search routine will search the list between the given *low* and *high* parameters.

Example: Binary Search

```
def recBinSearch(x, nums, low, high):
    if low > high:           # No place left to look, return -1
        return -1
    mid = (low + high)//2
    item = nums[mid]
    if item == x:
        return mid
    if x < item:           # Look in lower half
        return recBinSearch(x, nums, low, mid-1)
                           # Look in upper half
    return recBinSearch(x, nums, mid+1, high)
```



We can then call the binary search with a generic search wrapping function

```
def search(x, nums):
 return recBinSearch(x,nums,0,len(nums)-1)
```

Recursion vs. Iteration

- There are similarities between iteration (looping) and recursion
- In fact, anything that can be done with a loop can be done with a simple recursive function!
  - *But some algorithms harder to set up with iteration*
- Some programming languages use recursion exclusively.
  - *Haskell, ML, Lisp (functional programming); Prolog (logic programming)*
- Some problems that are simple to solve with recursion are quite difficult to solve with loops.

Recursion vs. Iteration

- In the factorial and binary search problems, the looping and recursive solutions use roughly the same algorithms, and their efficiency is nearly the same.
- Lets take another example: Fast Exponentiation

Example: Fast Exponentiation

- One way to compute a^n for an integer n is to multiply a by itself n times.
- This can be done with a simple accumulator loop:

```
def loopPower(a, n):
    ans = 1
    for i in range(n):
        ans *= a
    return ans
```

Example: Fast Exponentiation

- We can also solve this problem using recursion and divide & conquer approach.
- Using the laws of exponents, we know that $2^8 = 2^4 \times 2^4$. If we know 2^4, we can calculate 2^8 using one multiplication.
- What's 2^4? $2^4 = 2^2 \times 2^2$, and $2^2 = 2 \times 2$.
- $2 \times 2 = 4$, $2^2 \times 2^2 = 16$, $2^4 \times 2^4 = 256 = 2^8$
- We've calculated 2^8 using only three multiplications!

Example: Fast Exponentiation

- We can take advantage of the fact that $a^n = a^{n//2}(a^{n//2})$
- This algorithm only works when *n* is even. How can we extend it to work when *n* is odd?
- $2^9 = 2^4 \times 2^4 \times 2^1$

$$a^n = \begin{cases} a^{n//2}(a^{n//2}) & \text{if } n \text{ is even} \\ a^{n//2}(a^{n//2})(a) & \text{if } n \text{ is odd} \end{cases}$$

Example: Fast Exponentiation

- This method relies on integer division (if n is 9, then $n//2 = 4$).
- To express this algorithm recursively, we need a suitable base case.
- If we keep using smaller and smaller values for n, n will eventually be equal to 0 ($1//2 = 0$), and $a^0 = 1$ for any value except $a = 0$.

Example: Fast Exponentiation

```
# raises a to the int power n
def recPower(a, n):
  if n ==  0:
    return 1
  factor = recPower(a, n//2)
  if n%2 == 0:    # n is even
    return factor * factor
   # n is odd
  return factor * factor * a
```

- Here, a temporary variable called `factor` is introduced so that we don't need to calculate $a^{n//2}$ more than once, simply for efficiency.

Recursion vs. Iteration

- In the exponentiation problem:
  - *The iterative version takes linear time to complete*
  - *The recursive version executes in log time.*
  - *The difference between them is like the difference between a linear and binary search.*
- So… will recursive solutions always be as efficient or more efficient than their iterative counterpart?
- It depends

Recursion vs. Iteration

- The Fibonacci sequence is the sequence of numbers 1,1,2,3,5,8,…
  - *The sequence starts with two 1's*
  - *Successive numbers are calculated by finding the sum of the previous two numbers.*

Recursion vs. Iteration

```
def loopfib(n):
    # returns the nth Fibonacci number
    curr = 1
    prev = 1
    for i in range(n-2):
        curr, prev = curr+prev, curr
    return curr
```

- Note the use of simultaneous assignment to calculate the new values of `curr` and `prev`.
- The loop executes only *n-2* times since the first two values have already been provided as a starting point.

Recursion vs. Iteration

- The Fibonacci sequence also has a recursive definition:

$$fib(n) = \begin{cases} 1 & \text{if } n < 3 \\ fib(n-1) + fib(n-2) & \text{otherwise} \end{cases}$$

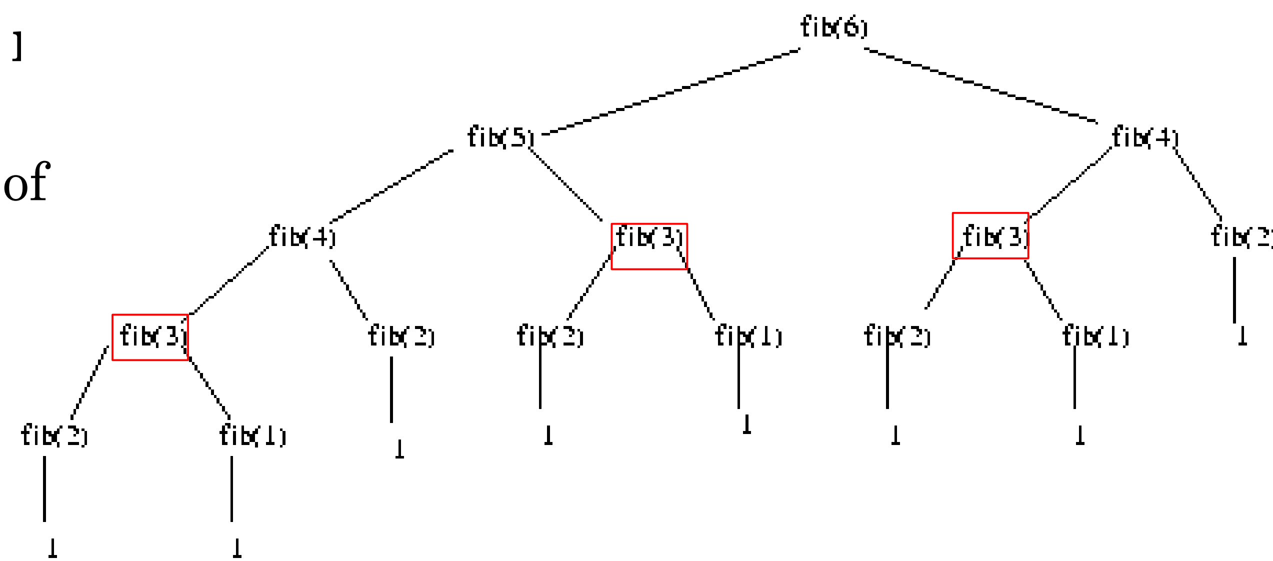
- This recursive definition can be directly turned into a recursive function!

```
def fib(n):
    if n < 3:
        return 1
    return fib(n-1)+fib(n-2)
```

Recursion vs. Iteration

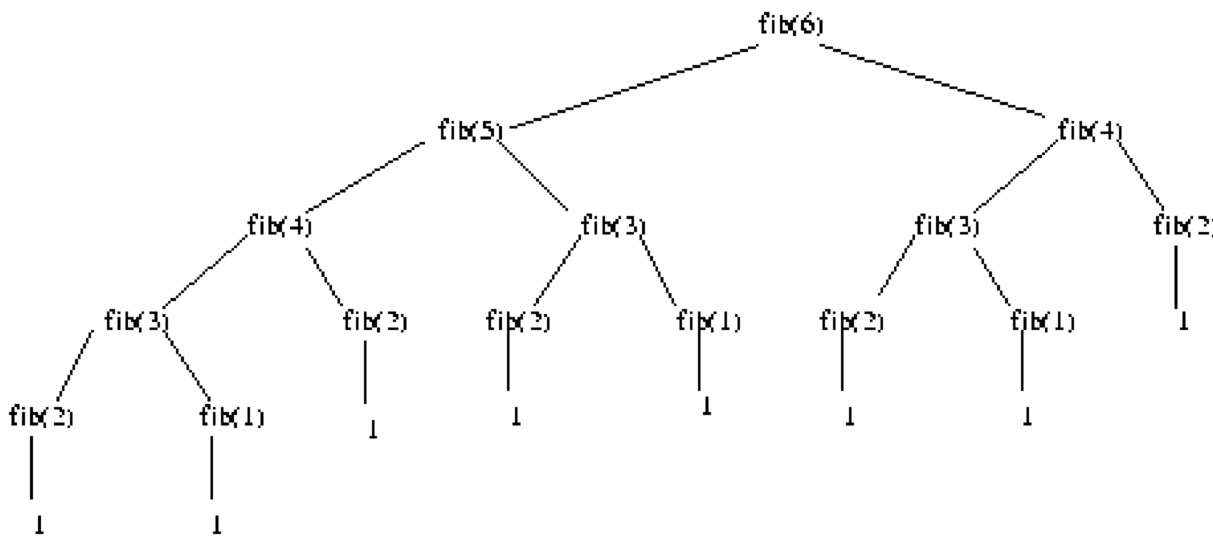
- This function obeys the rules that we've set out.
  - *The recursion is always based on smaller values.*
  - *There is a non-recursive base case.*
- So, this function will work great, won't it? – *Sort of…*
- The recursive solution is extremely inefficient, since it performs

Recomputing of fib(3) shown



Recursion vs. Iteration

- To calculate fib(6), fib(4)is calculated twice, fib(3)is calculated three times, fib(2)is calculated four times… For large numbers, this adds up!



Recursion vs. Iteration

- Recursion is another tool in your problem-solving toolbox.
- Sometimes recursion provides a good solution because it is more elegant or efficient than a looping version.
- At other times, when both algorithms are quite similar, the edge goes to the looping solution on the basis of speed and (generally) simplicity of programming
- Avoid the recursive solution if it is terribly inefficient, unless you can't come up with an iterative solution (which sometimes happens!)

Summary

- We learned the concept of recursion.
- We analyzed its performance and compared it to iterations (loops)
- We learned when to use recursion and when to use loops