Earthquake Cycling Produces Systematically Different Topography Than Steady-State Uplift When Erosion Is Nonlinear

1 Setup

Consider a 1D river profile $z(x,t)$ governed by the stream power law:

$$\frac{\partial z}{\partial t} = U(x,t) - KA(x)^m \left|\frac{\partial z}{\partial x}\right|^n \tag{1}$$

where $U(x,t)$ is the uplift rate, K is erodibility, $A(x)$ is drainage area, and m, n are positive exponents. The erosion rate at any point depends on the local slope $S = |\partial z/\partial x|$ through:

$$E(S) = KA^m S^n \tag{2}$$

We compare two uplift scenarios that deliver *identical total uplift* over any complete earthquake cycle.

1.1 Steady-state scenario

Uplift is constant in time:

$$U_{\text{ss}}(x,t) = V(x) \quad \text{for all } t \tag{3}$$

where $V(x)$ is the long-term structural velocity. This produces a steady-state slope field $S_{\text{ss}}(x)$ satisfying:

$$KA^m S_{\text{ss}}^n = V(x) \tag{4}$$

1.2 Earthquake cycle scenario

Uplift alternates between two phases within each cycle of period T:

- **Interseismic** (duration $\approx T$): uplift rate $U_{\text{inter}}(x)$, the creeping deformation that dominates the cycle.

- **Coseismic** (instantaneous): uplift $U_{\text{co}}(x) \cdot T$, releasing the accumulated slip deficit.

1.3 Moment conservation

The total uplift over one complete cycle is:

$$U_{\text{inter}}(x) \cdot T + U_{\text{co}}(x) \cdot T = V(x) \cdot T \tag{5}$$

This is simply the statement that slip deficit is zero over a full cycle: interseismic creep plus coseismic release equals the long-term structural rate. Both scenarios deliver the same total uplift $V(x) \cdot T$ over every cycle.

2 Slope oscillation

Because total uplift is identical (Eq. 5), the time-averaged slope over one complete cycle equals the steady-state slope:

$$\langle S(t) \rangle = S_{\text{ss}} \tag{6}$$

However, within the cycle the slope is *not* constant. During the interseismic phase, the spatially varying uplift $U_{\text{inter}}(x) \neq V(x)$ drives the slope away from S_{ss}. The coseismic event resets it. We decompose the slope into its mean and perturbation:

$$S(t) = S_{\text{ss}} + \delta S(t), \quad \text{where } \langle \delta S \rangle = 0 \tag{7}$$

The perturbation $\delta S(t)$ is *not* identically zero for any $T > 0$, because $U_{\text{inter}} \neq V$ at most locations. Therefore the slope variance is strictly positive:

$$\sigma^2 \equiv \langle \delta S^2 \rangle > 0 \quad \text{for } T > 0 \tag{8}$$

3 Qualitative result: Jensen's inequality

We first establish the *sign* of the erosion anomaly using Jensen's inequality.

Theorem 1 (Jensen's Inequality). *Let f be a twice-differentiable function and X a random variable (or time-varying quantity) with finite mean. Then:*

- *If f is **strictly convex** ($f'' > 0$): $\langle f(X) \rangle > f(\langle X \rangle)$*

- *If f is **strictly concave** ($f'' < 0$): $\langle f(X) \rangle < f(\langle X \rangle)$*

- *If f is **linear** ($f'' = 0$): $\langle f(X) \rangle = f(\langle X \rangle)$*

with strict inequality whenever $\text{Var}(X) > 0$.

The erosion rate $E(S) = KA^m S^n$ is a power-law function of slope. Its convexity is determined by:

$$\frac{d^2 E}{dS^2} = KA^m \, n(n-1) \, S^{n-2} \tag{9}$$

Since K, A^m, and S^{n-2} are all positive for $S > 0$, the sign depends only on $n(n-1)$:

n	Sign of $n(n-1)$	Convexity of $E(S)$	Jensen's result
$n > 1$	$+$	Strictly convex	$\langle E \rangle > E_{\text{ss}}$
$n = 1$	0	Linear	$\langle E \rangle = E_{\text{ss}}$
$0 < n < 1$	$-$	Strictly concave	$\langle E \rangle < E_{\text{ss}}$

Identifying $X = S(t)$ and $f = E$, and using $\langle S \rangle = S_{\text{ss}}$ from Eq. (6):

Proposition 1 (Qualitative Result — Sign of the Anomaly). *For any earthquake cycle with $T > 0$ and moment-conserving uplift:*

$$n > 1 : \quad \langle E \rangle > E_{\text{ss}} \quad \Rightarrow \quad \textit{EQ topography is \textbf{lower} than steady state} \tag{10}$$

$$n = 1 : \quad \langle E \rangle = E_{\text{ss}} \quad \Rightarrow \quad \textit{\textbf{no difference}} \tag{11}$$

$$n < 1 : \quad \langle E \rangle < E_{\text{ss}} \quad \Rightarrow \quad \textit{EQ topography is \textbf{higher} than steady state} \tag{12}$$

*This result is **exact**—it holds for any amplitude of δS, not just small perturbations.*

Jensen's inequality tells us the *direction* of the effect but not its *magnitude*. It also does not explicitly reveal how the effect depends on T. For that, we need the Taylor expansion.

4 Quantitative result: Taylor expansion

We now derive an explicit expression for the erosion anomaly $\Delta E \equiv \langle E \rangle - E_{\mathrm{ss}}$ by expanding $E(S)$ around S_{ss}.

4.1 Step 1: Taylor expand the erosion function

Expand $E(S) = K A^m S^n$ in a Taylor series around $S = S_{\mathrm{ss}}$:

$$E(S_{\mathrm{ss}} + \delta S) = E(S_{\mathrm{ss}}) + E'(S_{\mathrm{ss}}) \, \delta S + \frac{1}{2} \, E''(S_{\mathrm{ss}}) \, \delta S^2 + \mathcal{O}(\delta S^3) \tag{13}$$

Computing each derivative at $S = S_{\mathrm{ss}}$:

$$E(S_{\mathrm{ss}}) = K A^m \, S_{\mathrm{ss}}^n \tag{14}$$

$$E'(S_{\mathrm{ss}}) = K A^m \, n \, S_{\mathrm{ss}}^{n-1} \tag{15}$$

$$E''(S_{\mathrm{ss}}) = K A^m \, n(n-1) \, S_{\mathrm{ss}}^{n-2} \tag{16}$$

4.2 Step 2: Take the time average

Take $\langle \cdot \rangle$ of both sides of Eq. (13):

$$\langle E \rangle = E(S_{\mathrm{ss}}) + E'(S_{\mathrm{ss}}) \underbrace{\langle \delta S \rangle}_{=0} + \frac{1}{2} E''(S_{\mathrm{ss}}) \underbrace{\langle \delta S^2 \rangle}_{=\sigma^2} + \mathcal{O}(\langle \delta S^3 \rangle) \tag{17}$$

The first-order term vanishes because $\langle \delta S \rangle = 0$ (moment conservation, Eq. 7). The leading correction is therefore *second order*:

$$\boxed{\Delta E \; \equiv \; \langle E \rangle - E_{\mathrm{ss}} \; = \; \frac{n(n-1)}{2} \, K A^m \, S_{\mathrm{ss}}^{n-2} \, \sigma^2 \; + \; \mathcal{O}(\sigma^3)} \tag{18}$$

4.3 Step 3: Interpret the result

Equation (18) is the central quantitative result. Each factor has a clear physical meaning:

1. $\dfrac{n(n-1)}{2}$: the **nonlinearity coefficient**. This is the curvature of the erosion law at steady state. It controls the *sign*:

   - $n > 1 \Rightarrow n(n-1) > 0 \Rightarrow \Delta E > 0$ (more erosion $\Rightarrow$ lower topography)
   - $n = 1 \Rightarrow n(n-1) = 0 \Rightarrow \Delta E = 0$ (no effect)
   - $n < 1 \Rightarrow n(n-1) < 0 \Rightarrow \Delta E < 0$ (less erosion $\Rightarrow$ higher topography)

   This reproduces the Jensen's inequality result from Proposition 1, confirming internal consistency.

2. $K A^m \, S_{\mathrm{ss}}^{n-2}$: the **local sensitivity**. This prefactor depends on position x through drainage area $A(x)$ and steady-state slope $S_{\mathrm{ss}}(x)$. It determines *where* along the river the effect is strongest.

3. $\sigma^2 = \langle \delta S^2 \rangle$: the **slope variance**. This is the key factor that introduces T-dependence. Since δS is driven by the difference $U_{\mathrm{inter}} - V$ accumulating over the interseismic period, the variance scales as:

$$\sigma^2 \propto T^2 \tag{19}$$

(doubling the recurrence interval doubles the slope perturbation, which quadruples the variance). Therefore:

$$\Delta E \propto T^2 \tag{20}$$

4.4 Step 4: Cumulative elevation anomaly

The erosion anomaly ΔE (units: length/time) acts over every earthquake cycle of duration T. The net elevation change per cycle is:

$$\Delta z_{\mathrm{cycle}} = -\Delta E \cdot T = -\frac{n(n-1)}{2} \, K A^m \, S_{\mathrm{ss}}^{n-2} \, \sigma^2 \, T \tag{21}$$

(negative sign because more erosion produces lower topography). Over a total time t_{total} containing $N = t_{\mathrm{total}}/T$ cycles:

$$\Delta z_{\mathrm{total}} = N \cdot \Delta z_{\mathrm{cycle}} = -\frac{n(n-1)}{2} \, K A^m \, S_{\mathrm{ss}}^{n-2} \, \sigma^2 \, t_{\mathrm{total}} \tag{22}$$

Since $\sigma^2 \propto T^2$ (Eq. 19), the cumulative effect over fixed t_{total} grows as:

$$\boxed{\Delta z_{\mathrm{total}} \propto T^2 \cdot t_{\mathrm{total}}} \tag{23}$$

5 Limiting cases

The Taylor expansion cleanly recovers the correct behavior in all limiting cases:

$T \to 0$ **(infinitely frequent earthquakes).** The coseismic displacement per event is $U_{\mathrm{co}} \cdot T \to 0$, so $\delta S \to 0$ and $\sigma^2 \to 0$. From Eq. (18): $\Delta E \to 0$. Earthquake cycling becomes indistinguishable from steady state, as expected.

$n = 1$ **(linear erosion).** The nonlinearity coefficient $n(n-1)/2 = 0$, so $\Delta E = 0$ *exactly*, regardless of σ^2. This is not an approximation—S^1 is linear, so all higher-order terms in the Taylor expansion also vanish. The erosion PDE is linear, and superposition holds.

$n \neq 1$**,** $T > 0$**.** Both $n(n-1)/2 \neq 0$ and $\sigma^2 > 0$, so $\Delta E \neq 0$. Earthquake cycling *necessarily* produces different time-averaged erosion than steady state.

Increasing $|n - 1|$**.** The magnitude of $n(n-1)/2$ grows as n departs from 1, so the erosion anomaly is larger for more nonlinear erosion laws. For example, $n = 2$ gives a coefficient of $+1$, while $n = 1.5$ gives $+0.375$.

6 Intuition: the driving analogy

Consider two drivers covering the same total distance. Driver A maintains a constant speed $v_0 = 60$ km/h. Driver B alternates: speed $v_0 - \Delta v = 40$ km/h for half the time, then $v_0 + \Delta v = 80$ km/h for the other half. Both average the same speed: $\langle v \rangle = v_0 = 60$ km/h.

Fuel consumption depends on speed as $F(v) = c\,v^n$. What is the difference in total fuel consumed?

6.1 Jensen's inequality (qualitative)

$F(v) = c\,v^n$ is convex for $n > 1$ and concave for $n < 1$. Jensen's inequality immediately gives:

$$n > 1: \quad \langle F \rangle > F(\langle v \rangle) \quad \Rightarrow \quad \text{Driver B uses } \textbf{more} \text{ fuel} \tag{24}$$

$$n = 1: \quad \langle F \rangle = F(\langle v \rangle) \quad \Rightarrow \quad \text{same fuel} \tag{25}$$

$$n < 1: \quad \langle F \rangle < F(\langle v \rangle) \quad \Rightarrow \quad \text{Driver B uses } \textbf{less} \text{ fuel} \tag{26}$$

6.2 Taylor expansion (quantitative)

Expand $F(v_0 + \delta v)$ around v_0:

$$F(v_0 + \delta v) = c\,v_0^n + c\,n\,v_0^{n-1}\,\delta v + \frac{c\,n(n-1)}{2}\,v_0^{n-2}\,\delta v^2 + \cdots \tag{27}$$

Taking the time average ($\langle \delta v \rangle = 0$, $\langle \delta v^2 \rangle = \Delta v^2$):

$$\langle F \rangle - F(v_0) = \frac{c\,n(n-1)}{2}\,v_0^{n-2}\,\Delta v^2 \tag{28}$$

Now compute explicitly for $n = 2$, $\Delta v = 20$:

$$\langle F \rangle - F(v_0) = \frac{c \cdot 2 \cdot 1}{2} \cdot 60^0 \cdot 20^2 = 400\,c \tag{29}$$

Check directly: $\frac{1}{2}(c \cdot 40^2 + c \cdot 80^2) - c \cdot 60^2 = c \cdot (4000 - 3600) = 400\,c$. ✓

The Taylor expansion gives the *exact same answer* as the direct calculation, and reveals the structure: the fuel anomaly is proportional to Δv^2—the variance of speed. **If Driver B alternated between 50 and 70 km/h instead, the anomaly would be only** $(10/20)^2 = 1/4$ **as large.**

This is the T-dependence: bigger oscillations (longer earthquake cycles $\Rightarrow$ larger δS) produce a *quadratically* larger erosion anomaly.

7 Summary

The proof combines two mathematical tools:

1. **Jensen's inequality** (Section 3) establishes the *sign* of the effect. It is exact and requires no approximation:

   - $n > 1$: earthquake cycling produces *more* erosion $\Rightarrow$ lower topography
   - $n < 1$: earthquake cycling produces *less* erosion $\Rightarrow$ higher topography
   - $n = 1$: no effect (linear erosion obeys superposition)

5

2. **Taylor expansion** (Section 4) gives the *magnitude* and *scaling*:

$$\Delta E = \frac{n(n-1)}{2} K A^m S_{\text{ss}}^{n-2} \sigma^2 + \mathcal{O}(\sigma^3) \tag{18}$$

The anomaly is proportional to the slope variance σ^2, which scales as T^2. This explains why:

- The effect vanishes as $T \to 0$ (no oscillation $\Rightarrow$ no variance)
- The effect grows quadratically with recurrence interval T
- The cumulative topographic difference over fixed total time scales as $T^2 \cdot t_{\text{total}}$

Together, the three ingredients—**moment conservation** ($\langle S \rangle = S_{\text{ss}}$), **nonlinear erosion** ($n \neq 1$), and **slope oscillation** ($\sigma^2 > 0$)—guarantee that earthquake cycling produces systematically different topography than steady-state uplift.