Modeling Hurricane Velocity Field by Vector Decomposition

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Multivariable Calculus - G
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Introduction

Tropical Cyclones are large scale storms that occur in tropical conditions. A cyclone collects and carries large amounts of energy and water over its path on the ocean and can cause tremendous amounts of damage to life and property while it approaches the coast. It can also travel a long distance onshore, dissipating rainfall and causing significant damages.

Hurricanes are one such type of tropical cyclone which form over tropical or subtropical waters. Hurricanes are simply the regional name for tropical cyclones in the North Atlantic Ocean, the Northeast Pacific Ocean east of the dateline, or the South Pacific Ocean east of 160E.[1] In order for a hurricane to form, ocean waters must be at least 27℃ (80℉), an unstable atmosphere due to temperature differences, moist atmosphere, 200 miles from the Equator, and have little vertical wind shear. Vertical wind shear is defined as the rate wind changes in speed or direction with increasing altitude.

Understanding the velocity field distribution is an important component in assessing the strength and potential damage in hurricane research. Modeling a realistic velocity field distribution is fundamental to subsequent hurricane research such as lateral and vertical mass flow with the hurricane. This project is an effort of modeling a hurricane velocity field by decomposing the vector field into various simpler components. The overall velocity field can then be obtained by superimposing these simpler fields. We also demonstrate how this modeled velocity field can be used to calculate the mass influx at the base of a hurricane. Mathematica is used to demonstrate the modeling process.

Hurricane Components

A hurricane's anatomy consists of an eye, an eye wall, and rain bands. The eye (20-40 miles in diameter) is the open area in the center of the storm, where only 'light' winds blow and occasional clouds drift. The eye wall (10 miles thick) consists of a ring of thunderstorms that spiral around the eye; the highest wind speeds are located here. Around the eye wall, rain bands (hundreds of miles) contain bands of clouds, rain, thunderstorms, and sometimes tornadoes. Winds blow towards the center, creating a low pressure system, where the air condenses at the

[1] Tropical Cyclones have regional names; In North America, it's called a Hurricane while in Asia, it's called a Typhoon

center to form clouds that sometimes precipitate. Thus, higher wind speeds create a system of lower pressure at that area.

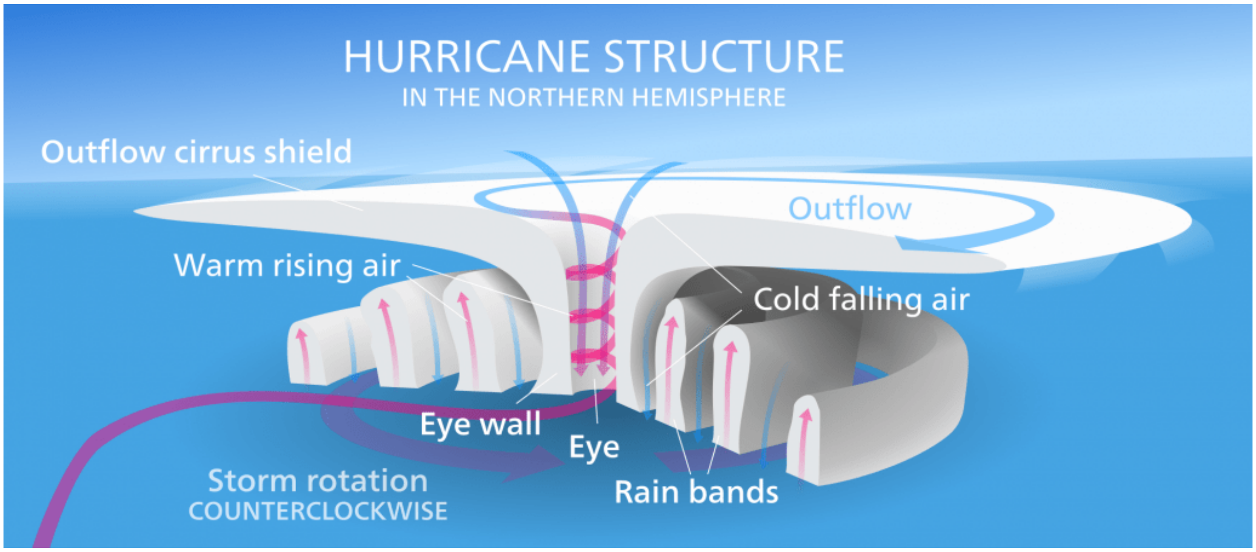


Fig. 1

The anatomy of a hurricane. The pink represents warm air and the blue represents cold air. The eye of the storm is located in the middle, and the eye wall is essentially the rain band next to the eye with the highest wind speed. The rain bands are concentric and can span hundreds of miles. *(Paul Webb).*

The directional rotation of a tropical cyclone depends on the hemisphere; Northern Hemisphere cyclones rotate in an anticlockwise (otherwise known as counterclockwise for Americans) direction, Southern Hemisphere cyclones rotate in a clockwise direction. This change is caused by the Coriolis Effect[2], where the rotation of the Earth deflects circulating air towards the right in the Northern Hemisphere and left in the Southern Hemisphere.

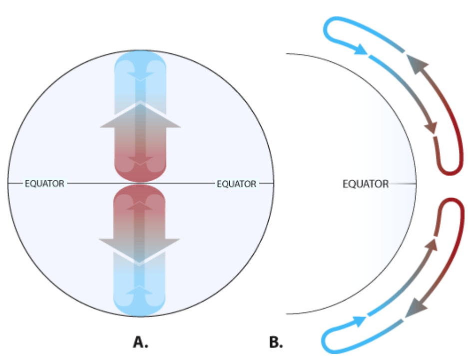


Fig. 2

Air circulation without taking into account the rotation of the Earth. Warm air rises near the equator and falls near the poles.
(NOAA, 2019).

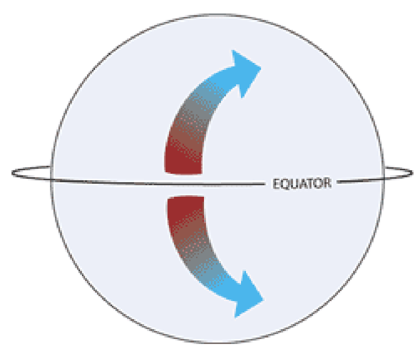


Fig. 3

Air circulation taking into account the rotation of the Earth. Since the Earth spins anticlockwise (eastward), the air circulation currents are deflected. This effect is known as the Coriolis Effect. *(NOAA, 2019).*

[2] An effect whereby a mass moving in a rotating system experiences a force (the Coriolis force) acting perpendicular to the direction of motion and to the axis of rotation. On the earth, the effect tends to deflect moving objects to the right in the northern hemisphere and to the left in the southern and is important in the formation of cyclonic weather systems.

Modeling the Velocity Field of a Hurricane

Firstly, hurricanes are a complex, 3D system, so we must simplify the entire system and the properties of fluid flow. To classify a moisture as an ideal fluid, the fluid must be incompressible and have a viscosity that can be ignored. The water vapor (air) involved in a hurricane is not enclosed within a closed container that would produce compression forces, so we can assume incompressibility for a basic hurricane model. Furthermore, water has nearly no viscosity at subsonic speeds, thus we can ignore viscosity. Furthermore, since we are only looking at a short period of time, hurricanes have a steady flow rate where the velocity does not fluctuate with time.

A counterclockwise vortex flow of an ideal fluid around the origin has four defining characteristics,

- The velocity vector at a point(x,y)is tangent to the circle that is centered at the origin and passes through the point(x,y).
- The direction of the velocity vector at a point(x,y)indicates a counterclockwise motion.
- The speed of the fluid is constant on circles centered at the origin.
- The speed of the fluid along a circle is inversely proportional to the radius of the circle

Knowing all that, we can now determine the simplified hurricane velocity vector field model. Taking a horizontal cross section, we can predict the model to resemble a rotational vector field, more specifically a spiral vector field. This motion for wind can be modeled by combining 2 simpler fields, an uniform inward "sink flow" field and a counterclockwise rotational "vortex" field.

Two Dimensional Flow

Considering a two dimensional flow in cartesian coordinates for any point in the cartesian plane for any time t, velocity will be:

$$\mathbf{v} = \langle v_x(x, y, z, t), v_y(x, y, z, t)\rangle$$

Considering a two dimensional flow in cylindrical coordinates (r, θ, z) for any point in the cylindrical plane will be

$$\mathbf{v} = \langle v_r(r, \theta, z, t), v_\theta(r, \theta, z, t)\rangle$$

Sink Flow

Sink flow is the opposite of source flow. The streamlines are radial, directed inwards to the line source. As we get closer to the sink, the area of flow decreases. In order to satisfy the continuity equation, the streamlines get bunched closer and the velocity increases as we get closer to the source. As with source flow, the velocity at all points equidistant from the sink is equal. The velocity of the flow around the sink can be given by

$$\mathbf{v} = -\, v_r(r)\mathbf{e_r}$$
$$v_r(r) = q \,/\, (2\pi r)$$

The stream function associated with sink flow is

$$\psi(r, \theta) = -q\theta \,/\, (2\pi)$$

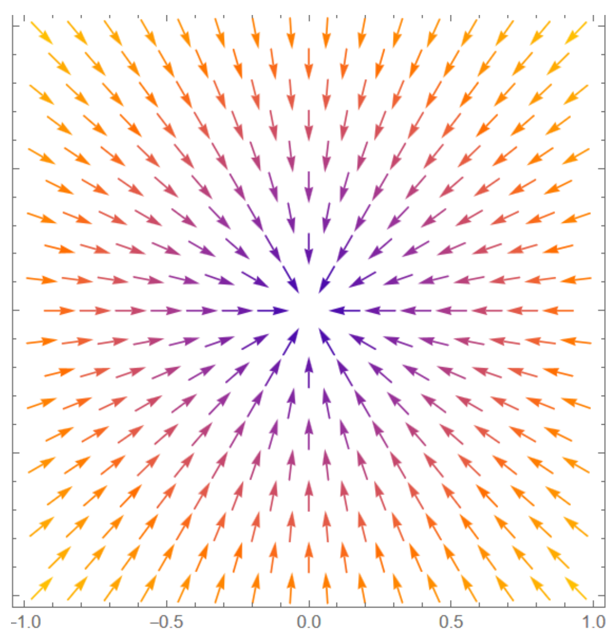
The flow around a line sink is irrotational and can be derived from the velocity potential. The velocity potential around a sink can be given by:

$$\phi(r, \theta) = -q*\ln(r)/2\pi$$

A sink flow can be modeled as vector field

$$\mathbf{G}(x,y) = q/2\pi*\langle -x, -y\rangle$$
$$= \langle -qx/2\pi, -qy/2\pi\rangle$$

(Example of **G**(x,y) shown with $q = 2\pi$)



Irrotational Vortex

A vortex is a region where the fluid flows around an imaginary axis. For an irrotational vortex, the flow at every point is such that a small particle placed there undergoes pure translation and does not rotate. Velocity varies inversely with radius in this case. Velocity will tend to infinity at r=0 that is the reason for the center being a singular point. The velocity is mathematically expressed as

$$\mathbf{v} = v_\theta \mathbf{e}_\theta$$
$$v_\theta = k/(2\pi r)$$

And because fluid flows around an axis

$$v_r = 0$$

The stream function for irrotational vortices is given by

$$\psi = - k*\ln(r) / (2\pi)$$

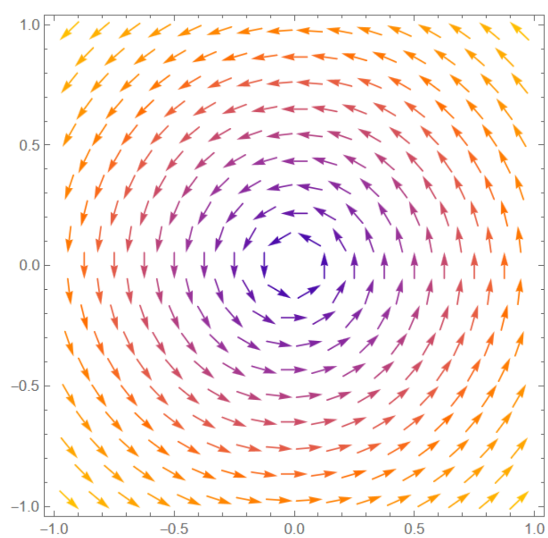
While the velocity potential is expressed as

$$\phi(r, \theta) = -k\theta/2\pi$$

For the closed curve enclosing the origin, circulation, which is the line integral of the velocity field, is equal to k and for any other closed curves, circulation = 0. Finally, the vector field can be modeled as a static rotating field

$$\mathbf{F}(x,y) = k/2\pi*\langle -y,x\rangle$$
$$= \langle -ky/2\pi, kx/2\pi\rangle$$

(Example of $\mathbf{F}$(x,y) shown with $k = 2\pi$)



Hurricane Vector Fields

As stated from before, the sink flow vector field can be modeled as

$$\mathbf{G}(x,y) = \langle -qx/2\pi, -qy/2\pi \rangle$$

And the vortex field can be modeled as

$$\mathbf{F}(x,y) = \langle -ky/2\pi, kx/2\pi \rangle$$

A hurricane vector field can be described as an inward spiraling vector field, which is a combination between the vortex field and sink flow field. When these two fields are added together, the resulting field is

$$\mathbf{P}(x,y) = \langle (-qx-ky)/2\pi, (kx-qy)/2\pi \rangle$$

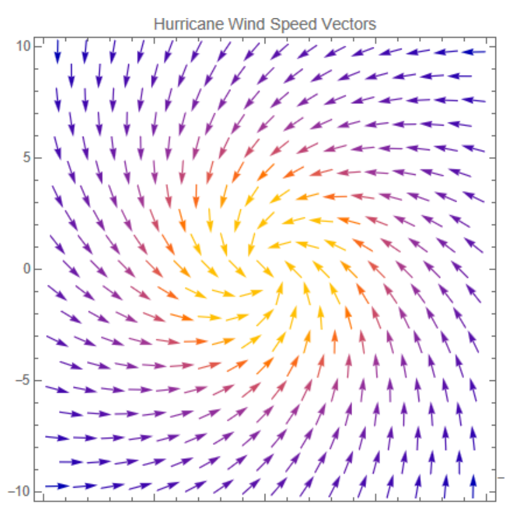
However, this vector field is incorrect because this vector field models the wind speed towards the center as the slowest and the wind speed farthest from the center as the fastest[3]. To describe this behavior, we divide by the distance formula for radius,

$$r^2 = x^2 + y^2$$

The denominator makes the distance correlate inversely to the magnitude on the vector field, thus causing the wind speed closer to the center to be faster while the wind speed towards the outer edges to be slower, thus resulting in the new vector field to be.

$$\mathbf{H}(x,y) = \langle (-qx-ky)/2\pi(x^2 + y^2), (kx-qy)/2\pi(x^2 + y^2) \rangle$$

(Example $\mathbf{H}(x,y)$ shown with $k = 2\pi$, $q = 2\pi$)



[3] See Mathematica

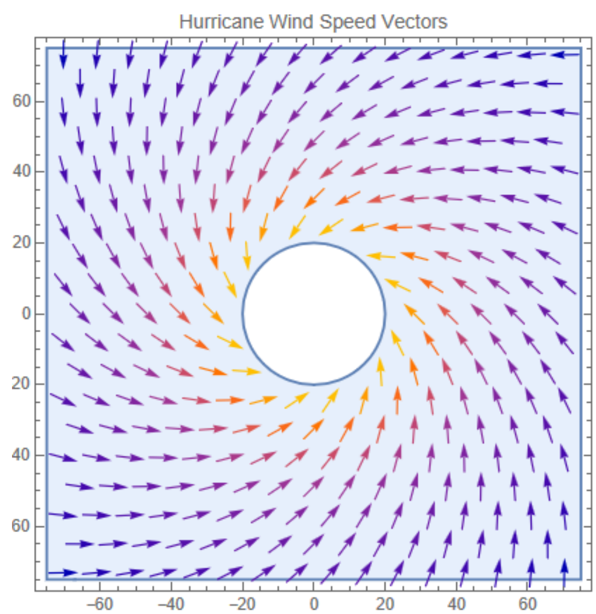
Modeling the Eye of the Storm

The eye of the storm for a static and ideal hurricane can be assumed to be a perfect circle. This can be described as:

$$r^2 = x^2 + y^2$$

The eye of the storm is relatively calm and will ideally return to normal air pressure and have no winds. When overlain on the hurricane vectors graph, there will be a hole in the middle of the field with radius r.

(Example shown with $k = 2\pi$, $q = 2\pi$, $r = 20$)

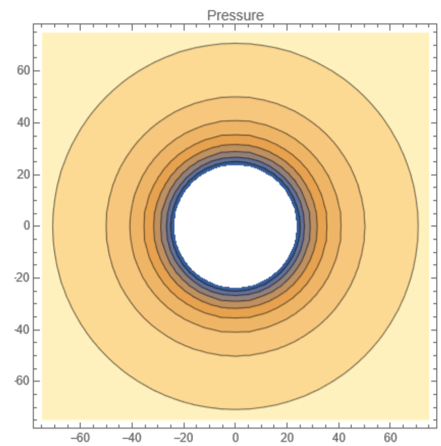


Air Pressure

To model air pressure, we already know that wind speed and air pressure have an inverse relationship, so higher velocity near the eye results in a lower pressure. Simply, this air pressure can be modeled as a multiplier divided by the negative of the distance formula, creating a contour field where the lowest pressure exists towards the center and the highest pressure exists at the edges. Note that the white region seen on the graph is such low pressure that it is treated as a discontinuity in mathematica, but the graph realistically still decreases in pressure until (x,y) = (0, 0), where there will be a discontinuity.

$$A(x,y) = p/-(x^2 + y^2)$$

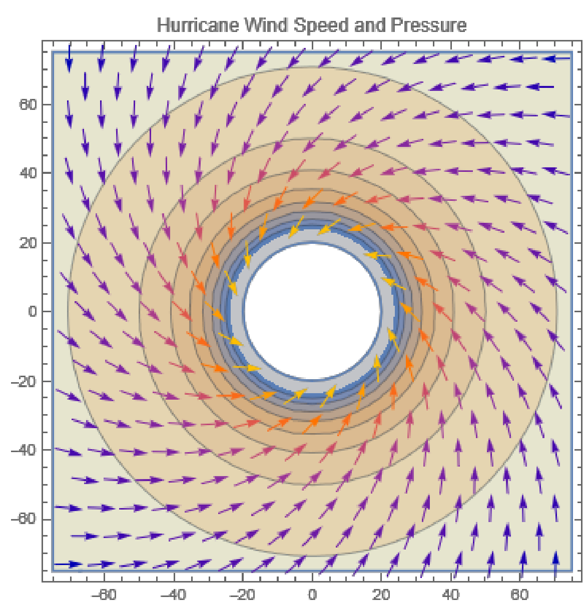
(Example shown with p = 1)



Combining Pressure and Vector Fields

When graph A(x,y) is overlain with vector field $\mathbf{H}(x,y) = \langle(-qx-ky)/2\pi(x^2 + y^2), (kx-qy)/2\pi(x^2 + y^2)\rangle$, the two dimensional model of the hurricane will look somewhat as below.

(Example shown with $k = 2\pi$, $q = 2\pi$, r = 20, p = 1)



Three Dimensional Implementation of Hurricane Vector Field

As stated before, our hurricane is modeled by vector field function

$$H(x,y) = \langle(-qx-ky)/2\pi(x^2 + y^2), (kx-qy)/2\pi(x^2 + y^2)\rangle$$

To extend this vector field into three dimensions, we can implement a z component described as

$$H_z = v/(x^2+y^2)$$
$$H(x,y,z) = \langle(-qx-ky)/(2\pi(x^2 + y^2)), (kx-qy)/(2\pi(x^2 + y^2)), v/(x^2+y^2)\rangle$$

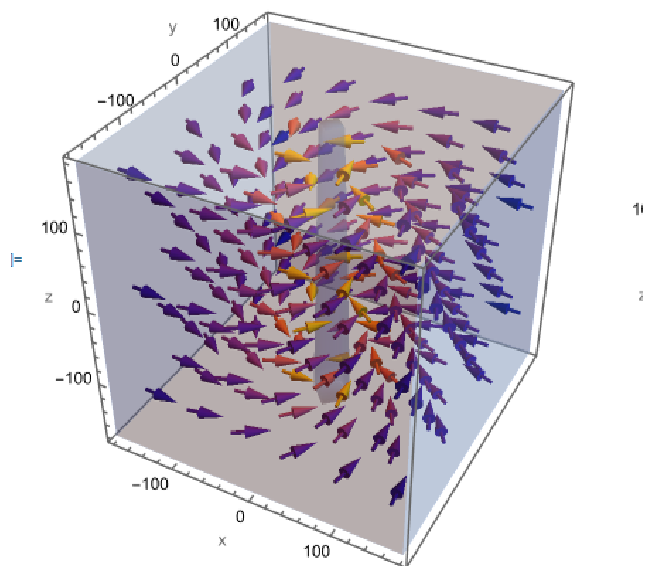
Which is an arbitrary coefficient (v) divided by the same distance formula from before. This causes our already existing two dimensional vector field to continue to spiral upwards at the speed of v/(x^2+y^2).

Three Dimensional Implementation of the Eye of the Storm

Similar to the two dimensional implementation of the Eye of the Storm, there will be a hole in the middle of the graph. Because this is in three dimensions, this hole appears in the form of a cylinder, described by equation

$$z = r^2 - x^2 - y^2$$

When plotted, the three dimensional vector field with the eye of the storm will be as shown:
(Example shown with k, q, v = 175/Sqrt[1/400 + 1/(40 \[Pi]^2)], explained in Mathematica)



Calculating Flux of Three Dimensional Hurricane

During a hurricane, air is constantly moved upwards and is calculated in how many $miles^3$ per hour it moves. This can be useful as it can calculate how much water is moved from an ocean surface into the storm or how much updraft there will be. This can be modeled through a flux integral, described as:

$$\iint_S F \cdot n \, dS.$$

Because ndS = (∇g/||∇g||)*||∇g||dA = ∇g dA, where g is the surface that the vector field flows through, we can substitute and create a new integral as:

$$\iint_D F \cdot \nabla g \, dA.$$

In this case, we assume that the ocean surface is g(x,y) = 0, bounded by the curve r^2 = x^2 + y^2 where r is equal to the hurricane radius, and F is the hurricane vector field. We convert to polar coordinates using the conversion:

$$x = r*\cos(\theta)$$
$$y = r*\sin(\theta)$$

Once plugged in, the integral will appear as

$$c\int_0^{2\,Pi}\int_{20}^{150} < (-r\cos(T) - r\sin(T)) / ((r\cos(T))^2 + (r\sin(T))^2), (r\cos(T) - r\sin(T)) / ((r\cos(T))^2 + (r\sin(T))^2), 1/(((r\cos(T))^2 + (r\sin(T))^2) > . \nabla g * r \, dr \, dT$$

∇g in this case would represent vector <0,0,1> as that is the normal of the surface we integrate over. The bounds are 20<r<150 and $0<\theta<2\pi$ because the radius goes from 20 miles (outer bound of the storm) to 150 miles (outer edge of the storm). There is also the multiplier c, explained in the mathematica code, which represents k, q, and v if assumed that k, q, and v are equal.

Mathematica Code:

Link to Github

Reflection:

Brayden: In this final project, Diane and I chose to do the hurricane project. We split up the work fairly evenly with Diane handling most of the background research and me handling most of the mathematica coding and mathematical derivations. Overall, I feel that Diane and I worked very well with each other as we were able to communicate effectively. For me, Diane was able to help me implement some mathematical aspects into my mathematica, such as how the pressure in the storm could not be modeled as the gradient of $f(x,y) = x^2+y^2$ but rather $1/(x^2+y^2)$ to represent the eye of the storm and als the dynamic air pressure when moving closer to the storm wall. I am very happy with my work on this project because I felt like I modeled the hurricane both accurately and effectively.

Diane: In this hurricane project, Brayden and I both split the work and completed it effectively and efficiently. I did the research that assisted Brayden's mathematical models, and he modeled them to the best of both of our understandings. I came across many new pieces of information I hadn't known, which were ironically on government-funded educational websites, and was able to both learn many things myself but also give necessary information to Brayden so that the models would be more accurate to an actual hurricane, such as the direction, size, and anatomy. Brayden in turn would confide his code and calculations with me, so our system of cross-checking worked really well. I thoroughly enjoyed this project and feel like I have learned much in terms of both science and math.

Citations:

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