Modeling Hurricane Velocity Field by Vector Decomposition

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Multivariable Calculus (Calculus III) - G
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On our honors, we have not given nor received any unauthorized aid on this work.
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Introduction

Hurricanes are a type of tropical cyclone that forms over tropical or subtropical waters, and tropical cyclones are an organized, rotating system of clouds and thunderstorms with closed, low-level circulation. Hurricanes are simply the regional name for tropical cyclones in the North Atlantic Ocean, the Northeast Pacific Ocean east of the dateline, or the South Pacific Ocean east of 160E. For a hurricane to form, ocean waters must be at least 27℃ (80℉), have an unstable atmosphere due to temperature differences, have a moist atmosphere, be 200 miles from the Equator, and have little vertical wind shear. Vertical wind shear is defined as the rate wind changes in speed or direction with increasing altitude. A lack of vertical wind shear is important since changing wind velocities (speed and direction given that velocity is a vector) would inhibit the formation of fast enough winds to cause a low-pressure system.

In order for a hurricane to form, there must be a prior disturbance, such as a tropical wave. As such a weather system moves along the tropicals, the warm ocean air rises into the storm, creating that low pressure under the storm, thus drawing in more air. The warm air cools as it rises, condensing into droplets that further release more heat to power the storm. This natural disaster can also travel a long distance onto the shore, dissipating rainfall and causing significant damage. The intensity of damage is measured based on the Saffir-Simpson scale, in which 74-95mph is Level 1, with dangerous winds and moderate damage. Such levels progress to extremely dangerous with extensive damage (Level 2, 96-110mph), devastating damage (Level 3, 111-129mph), and catastrophic damage for both Level 4 at 130-156mph and Level 5 at 157+mph. This means that potential hurricane damage ranges from snapped roofs to entire wall collapses and month-long power outages and uninhabitability.

Understanding the velocity field distribution is an important component in assessing the strength and potential damage in hurricane research. Modeling a realistic velocity field distribution is fundamental to subsequent hurricane research such as the lateral and vertical mass flow of a hurricane. This project is an effort in modeling a hurricane velocity field by decomposing the vector field into various simpler components. The overall velocity field can then be obtained by superimposing these simpler fields. We also demonstrate how this modeled velocity field can be used to calculate the mass influx at the base of a hurricane. Mathematica is used to demonstrate the modeling process.

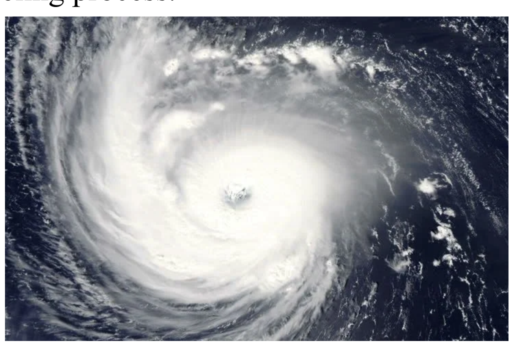


Fig. 1

The satellite view of Hurricane Florence. The first major hurricane in the 2018 Atlantic hurricane season, its highest winds of 150mph denote it as a Level 4 hurricane. *(Film Master, 2018).*

Hurricane Components

A hurricane's anatomy consists of an eye, an eyewall, and rain bands. The eye (~20-40 miles in diameter) is the open area in the center of the storm, where only 'light' winds blow and occasional clouds drift. The eyewall (~10 miles thick) consists of a ring of thunderstorms that spiral around the eye; the highest wind speeds are located here. Around the eyewall, rain bands (hundreds of miles thick) contain bands of clouds, rain, thunderstorms, and occasionally tornadoes. Winds blow towards the center of the hurricane, creating a low-pressure system with the eyewall, where the air condenses at the center to form clouds that sometimes precipitate. Thus, higher wind speeds create a system of lower pressure in that area.

The creation of an eye within a tropical weather system always signifies an increasingly organized and devastating storm. As convection, a circulation of currents (and, for hurricanes, low vertical wind shear), causes bands of water vapor (air) to begin rotating around a common point, creating the eyewall that rotates faster than the rain bands. As discussed prior, warm air rises and cool air falls. While most of the cool air dissipates outwards into the outflow, creating a positive feedback loop for intensity, some of the air falls back within the center of the storm. At some point, this cool, descending air in the center counteracts and equalizes the strength of the updrafts in the region, thus allowing for the air to slowly descend and create an eye of the storm. While the eyes may be the calmest parts of hurricanes on land, it is possibly the most dangerous when over the ocean, with waves as tall as 130 feet slamming into the eyewall in all directions.

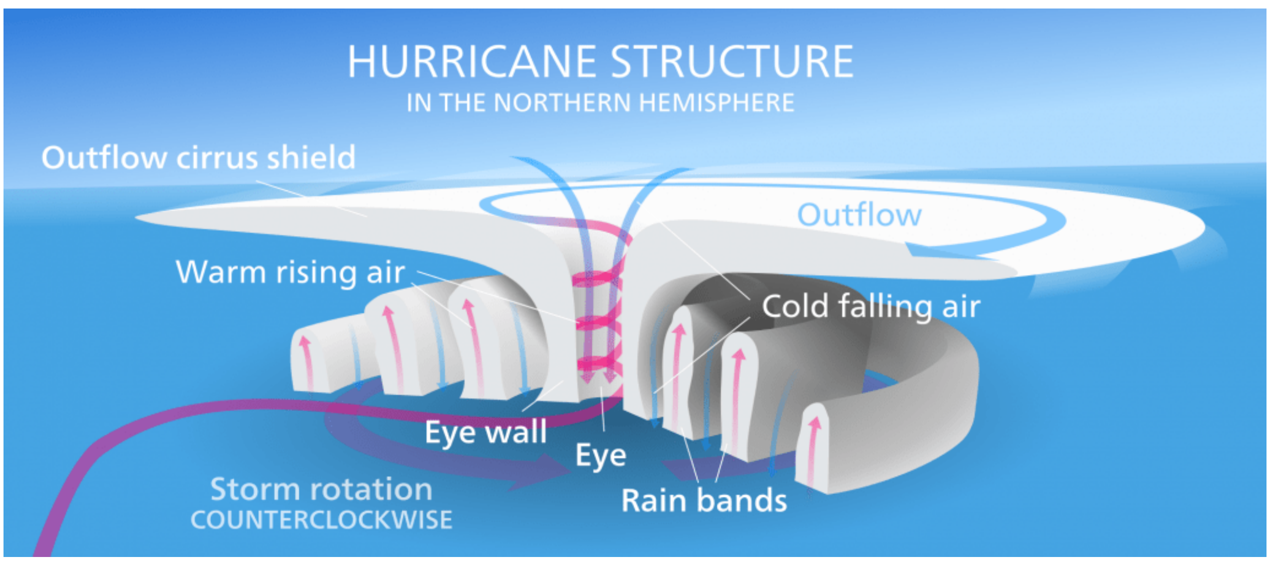


Fig. 2

The anatomy of a hurricane. The pink represents warm air and the blue represents cold air. The eye of the storm is located in the middle, and the eyewall is essentially the rain band next to the eye with the highest wind speed. The rainbands are concentric and can span hundreds of miles. *(Paul Webb, 2021).*

The directional rotation of a tropical cyclone depends on the hemisphere; Northern Hemisphere cyclones rotate in an anticlockwise direction and Southern Hemisphere cyclones rotate in a clockwise direction. This change is caused by the Coriolis Effect[1], where the rotation of the Earth deflects circulating air towards the right in the Northern Hemisphere and left in the Southern Hemisphere. Simply put, the Coriolis Effect is an effect where a mass moving in a rotating system experiences a force (the Coriolis force) acting perpendicular to the direction of motion and to the axis of rotation. On the earth, the effect deflects moving objects to the right in the northern hemisphere and to the left in the southern. The formation of cyclonic weather systems depends on this effect to determine rotation.

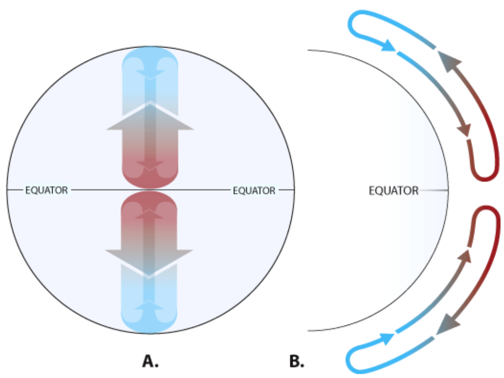


Fig. 3
Air circulation without taking into account the rotation of the Earth. Warm air rises near the equator and falls near the poles. *(NOAA, 2019).*

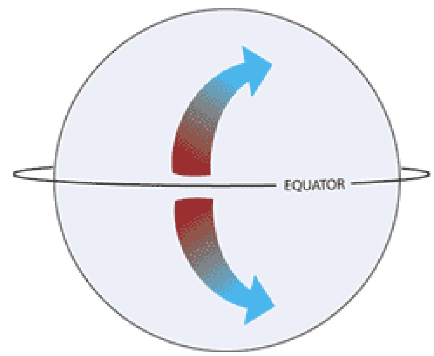


Fig. 4
Air circulation takes into account the rotation of the Earth. Since the Earth spins anticlockwise (eastward), the air circulation currents are deflected. This effect is known as the Coriolis Effect. *(NOAA, 2019).*

[1] an effect whereby a mass moving in a rotating system experiences a force (the Coriolis force) acting perpendicular to the direction of motion and to the axis of rotation. On the earth, the effect tends to deflect moving objects to the right in the northern hemisphere and to the left in the southern and is important in the formation of cyclonic weather systems.

Preface to Vector Mathematics and Mathematical Terms

Throughout the mathematical derivations for the hurricane vector fields, various mathematical terms are used, namely **flows**, **stream functions**, **velocity potential**, **flux**, and **circulation**.

A **Flow** formalizes the idea of the motion of particles in a fluid. The vector flow refers to a set of closely related concepts of the flow determined by a vector field. One can think of such a vector field as representing fluid flow in two dimensions so that F(x,y) gives the velocity of a fluid at the point (x,y). In this case, we may call F(x,y) the velocity field of the fluid. With this interpretation, the above example illustrates the clockwise circulation of fluid around the origin.

Stream functions are defined for two-dimensional flow and for three-dimensional axial symmetric flow. The stream function can be used to plot the streamlines of the flow and find the φvelocity. For two-dimensional flow the velocity components can be calculated in Cartesian coordinates by

$$u = -\partial\psi/\partial y$$
$$v = \partial\psi/\partial x$$

where u and v are the velocity vectors in the x and y directions, respectively, ψ is the stream function, r is the distance from the origin, and θ is the polar angle measured from the x-axis.

A **Velocity Potential** function is a scalar function of space and time. If 'phi' is the representation of the velocity potential function, then the velocity function for a steady fluid flow is given by the expression,

$$\phi = f(x,y,z)$$

The velocity potential is a scalar function, whose negative derivative, with respect to any direction, gives the velocity component in that direction.

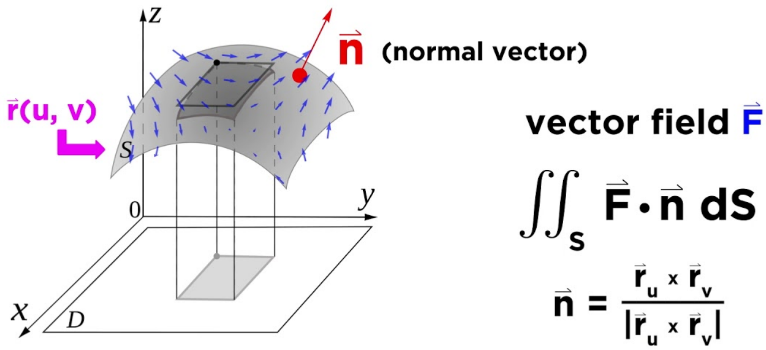
$$u = -\partial\varphi/\partial x$$
$$v = -\partial\varphi/\partial y$$
$$w = -\partial\varphi/\partial z$$

Here, u, v, and w are the velocity components of the fluid flow along x, y, and z directions.

Flux is a scalar value that defines how much of an arbitrary unit (be it water, air, bananas) that crosses some boundary per unit time. The total flux depends on the strength of the field, the size of the surface it passes through, and its orientation. It is primarily calculated through a modified line or surface integral, dotting the field with the unit normal of the surface or boundary that the "something" passes through. This flux in three dimensions can be defined as

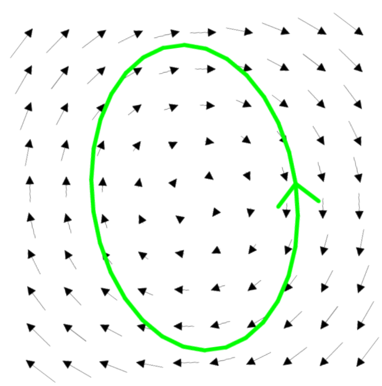
$$Flux = \iint_S F.ndS$$

Where F is the vector field flowing through surface S and n is the unit normal vector of the surface.



Circulation is the integral of a vector field along a path or curve. If the curve C is a closed curve, then the line integral indicates how much the vector field tends to circulate around the curve C. In fact, for an oriented closed curve C, we call the line integral the "circulation" of F around C. This can be given as:

$$\oint_C F.ds \;=\; Circulation\ of\ Vector\ Field\ F\ around\ Curve\ C$$



Modeling the Velocity Field of a Hurricane

Firstly, hurricanes are a complex, 3D system, so we must simplify the entire system and the properties of fluid flow. To classify moisture as an ideal fluid, the fluid must be incompressible and have a viscosity that can be ignored. The water vapor (air) involved in a hurricane is not enclosed within a closed container that would produce compression forces, so we can assume incompressibility for a basic hurricane model. Furthermore, water has nearly no viscosity at subsonic speeds, thus we can ignore viscosity. Furthermore, since we are only looking at a short period of time, hurricanes have a steady flow rate where the velocity does not fluctuate with time.

A counterclockwise vortex flow of an ideal fluid around the origin has four defining characteristics;

- The velocity vector at a point (x,y) is tangent to the circle that is centered at the origin and passes through the point (x,y).
- The direction of the velocity vector at a point (x,y) indicates a counterclockwise motion.
- The speed of the fluid is constant in circles centered at the origin.
- The speed of the fluid along a circle is inversely proportional to the radius of the circle.

Knowing all that, we can now determine the simplified hurricane velocity vector field, model. Taking a horizontal cross-section, we can predict the model to resemble a rotational vector field, more specifically a spiral vector field. This motion for wind can be modeled by combining 2 simpler fields, a uniform inward "sink flow" field and a counterclockwise rotational "vortex" field.

Two Dimensional Flow

Considering a two dimensional flow in cartesian coordinates for any point in the cartesian plane for any time t, velocity will be:

$$\mathbf{v} = \langle v_x(x, y, z, t), v_y(x, y, z, t)\rangle$$

Considering a two dimensional flow in cylindrical coordinates (r, θ, z) for any point in the cylindrical plane will be

$$\mathbf{v} = \langle v_r(r, \theta, z, t), v_\theta(r, \theta, z, t)\rangle$$

Sink Flow

Sink flow is the opposite of source flow. The streamlines are radial, directed inwards to the line source. As we get closer to the sink, the area of flow decreases. In order to satisfy the continuity equation, the streamlines become closer in proximity and the velocity increases as points are closer to the source. As with source flow, the velocity at all points equidistant from the sink is equal. The velocity of the flow around the sink can be given by

$$\mathbf{v} = - v_r(r)\mathbf{e_r}$$
$$v_r(r) = q / (2\pi r)$$

The stream function associated with sink flow is

$$\psi(r, \theta) = -q\theta / (2\pi)$$

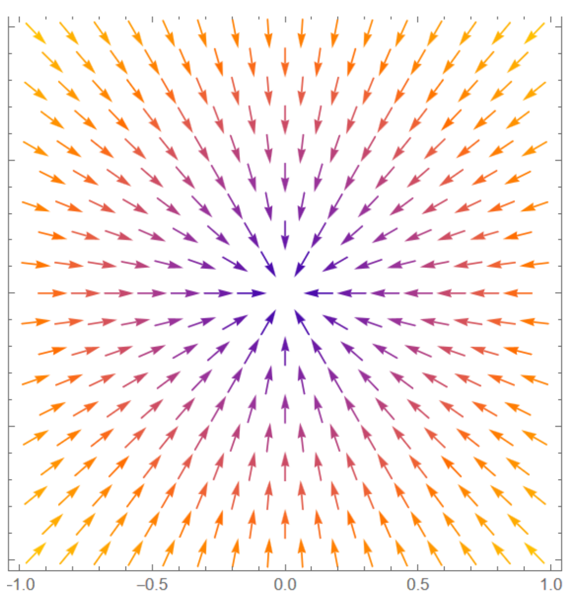
The flow around a line sink is irrotational and can be derived from the velocity potential. The velocity potential around a sink can be given by:

$$\phi(r, \theta) = -q*\ln(r)/2\pi$$

A sink flow can be modeled as vector field[2]

$$\mathbf{G}(x,y) = q/2\pi*\langle -x, -y\rangle$$
$$= \langle -qx/2\pi, -qy/2\pi\rangle$$

(Example of $\mathbf{G}(x,y)$ shown with $q = 2\pi$)



[2] In all vector fields, warmer colors (yellow) signify faster velocities, while cooler colors (blue) signify slower velocities.

Irrotational Vortex

A vortex is a region where the fluid flows around an imaginary axis. For an irrotational vortex, the flow at every point is such that a small particle placed there undergoes pure translation and does not rotate. Velocity varies inversely with the radius in this case. Velocity will tend to infinity at r=0. The velocity is mathematically expressed as

$$\mathbf{v} = v_\theta \mathbf{e}_\theta$$
$$v_\theta = k/(2\pi r)$$

And because fluid flows around an axis is

$$v_r = 0$$

The stream function for irrotational vortices is given by

$$\psi = - k*\ln(r) / (2\pi)$$

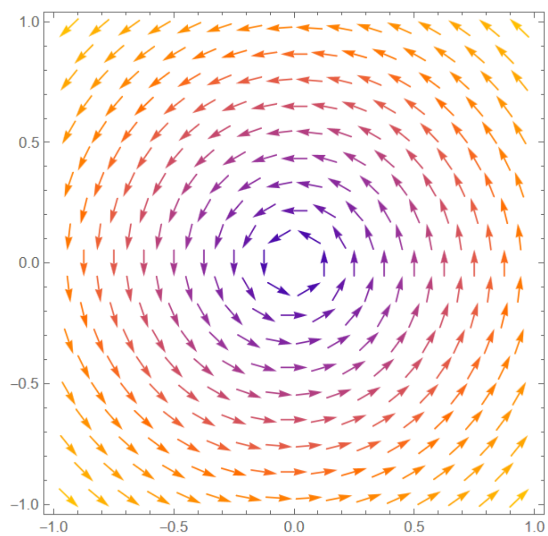
While the velocity potential is expressed as

$$\phi(r, \theta) = -k\theta/2\pi$$

For the closed curve enclosing the origin, circulation, the line integral of the velocity field, is equal to k. For any other closed curves, circulation = 0. Finally, the vector field can be modeled as a static rotating field

$$\mathbf{F}(x,y) = k/2\pi * \langle -y,x \rangle$$
$$= \langle -ky/2\pi,\ kx/2\pi \rangle$$

(Example of $\mathbf{F}(x,y)$ shown with $k = 2\pi$)



Hurricane Vector Fields

As stated before, the sink flow vector field can be modeled as

$$\mathbf{G}(x,y) = \langle -qx/2\pi, -qy/2\pi \rangle$$

And the vortex field can be modeled as

$$\mathbf{F}(x,y) = \langle -ky/2\pi,\ kx/2\pi \rangle$$

A hurricane vector field can be described as an inward spiraling vector field, which is a combination between the vortex field and sink flow field. When these two fields are added together, the resulting field is

$$\mathbf{P}(x,y) = \langle (-qx-ky)/2\pi,\ (kx-qy)/2\pi \rangle$$

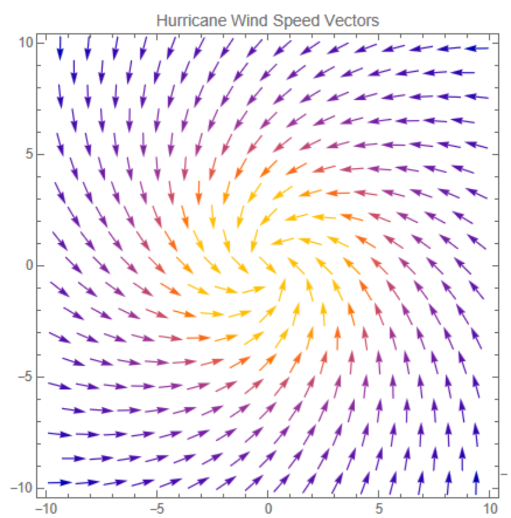
However, this vector field is incorrect because this vector field depicts the wind speed towards the center as the slowest and the wind speed farthest from the center as the fastest[3]. To describe this behavior, we divide by the distance formula for radius,

$$r^2 = x^2 + y^2$$

The denominator makes the distance correlate inversely to the magnitude on the vector field, thus causing the wind speed closer to the center to be faster while the wind speed towards the outer edges to be slower, thus resulting in the new vector field to be.

$$\mathbf{H}(x,y) = \langle (-qx-ky)/2\pi(x^2 + y^2),\ (kx-qy)/2\pi(x^2 + y^2) \rangle$$

(Example $\mathbf{H}(x,y)$ shown with $k = 2\pi$, $q = 2\pi$)



[3] See Mathematica.

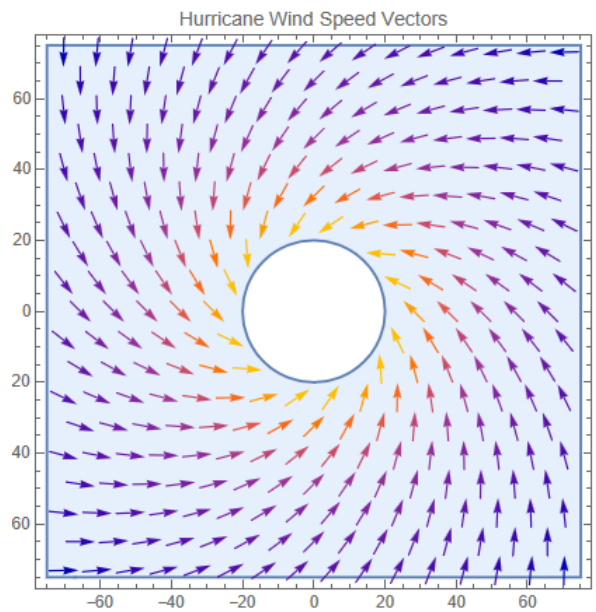
Modeling the Eye of the Storm

The eye of the storm for a static and ideal hurricane can be assumed to be a perfect circle. This can be described as

$$r^2 = x^2 + y^2$$

The eye of the storm is relatively calm and will ideally return to normal air pressure and have no winds. When overlain on the hurricane vectors graph, there will be a hole in the middle of the field with radius r.

(Example shown with $k = 2\pi$, $q = 2\pi$, $r = 20$)

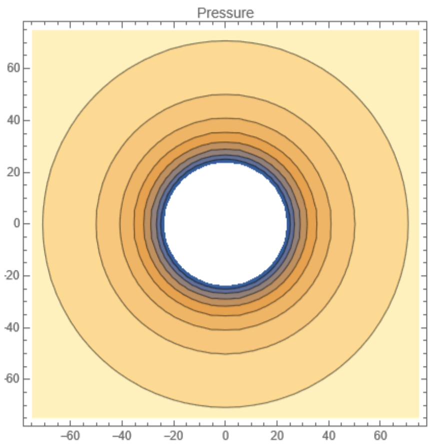


Air Pressure

To model air pressure, we already know that wind speed and air pressure have an inverse relationship, so higher velocity near the eye results in lower pressure. Simply, this air pressure can be modeled as a multiplier divided by the negative of the distance formula, creating a contour field where the lowest pressure exists towards the center and the highest pressure exists at the edges. Note that the white region seen on the graph is such low pressure that it is treated as a discontinuity in Mathematica, but the graph realistically still decreases in pressure until (x,y) = (0, 0), where there will be a discontinuity.

$$A(x,y) = p/-(x^2 + y^2)$$

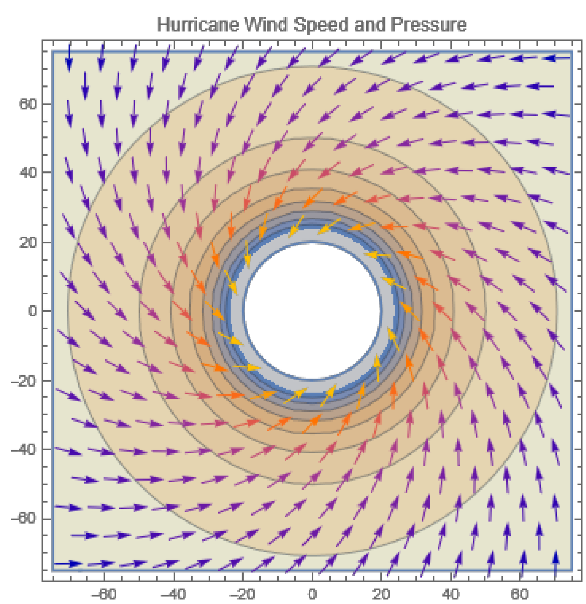
(Example shown with p = 1)



Combining Pressure and Vector Fields

When graph A(x,y) is overlain with vector field $\mathbf{H}(x,y) = \langle(-qx-ky)/2\pi(x^2 + y^2), (kx-qy)/2\pi(x^2 + y^2)\rangle$, the two dimensional model of the hurricane will look somewhat as below.

(Example shown with $k = 2\pi$, $q = 2\pi$, r = 20, p = 1)



Three Dimensional Implementation of Hurricane Vector Field

As stated before, our hurricane is modeled by vector field function

$$H(x,y) = \langle(-qx-ky)/2\pi(x^2 + y^2), (kx-qy)/2\pi(x^2 + y^2)\rangle$$

To extend this vector field into three dimensions, we can implement a z component described as

$$H_z = v/(x^2+y^2)$$
$$H(x,y,z) = \langle(-qx-ky)/(2\pi(x^2 + y^2)), (kx-qy)/(2\pi(x^2 + y^2)), v/(x^2+y^2)\rangle$$

Which is an arbitrary coefficient (v) divided by the same distance formula from before. This causes our already existing two dimensional vector field to continue to spiral upwards at the speed of $v/(x^2+y^2)$.

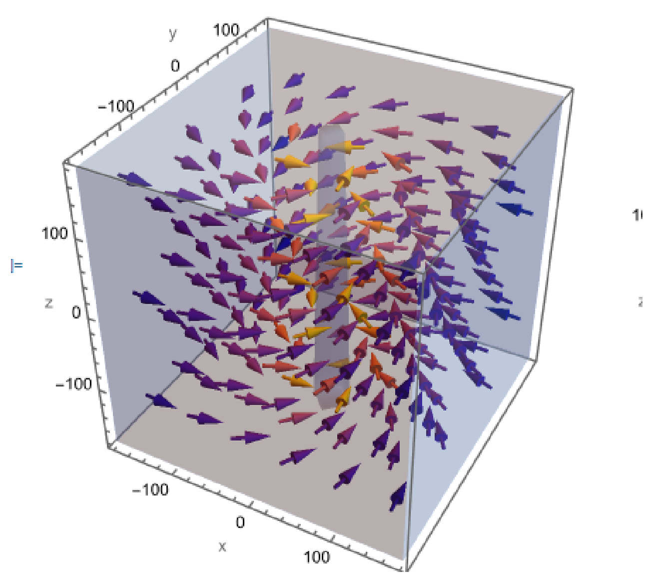
Three-Dimensional Implementation of the Eye of the Storm

Similar to the two-dimensional implementation of the Eye of the Storm, there will be a hole in the middle of the graph. Because this is in three dimensions, this hole appears in the form of a cylinder, described by equation

$$z = r^2 - x^2 - y^2$$

When plotted, the three-dimensional vector field with the eye of the storm will be as shown:

(Example shown with k, q, v = 175/Sqrt[1/400 + 1/(40 \[Pi]^2)], explained in Mathematica)



Calculating Flux of Three Dimensional Hurricane

During a hurricane, the air is constantly moved upwards and is calculated in how many cubic miles (mi^3) per hour it moves. This can be useful as it can calculate how much water is moved from an ocean surface into the storm or how much updraft there will be. This can be modeled through a flux integral, described as

$$\iint_S F \bullet n\, dS$$

Because $n\, dS = (\frac{\nabla g}{||\nabla g||}) \cdot ||\nabla g|| dA = \nabla g dA$ ndS = (∇g/||∇g||)*||∇g||dA = ∇g dA, where g is the surface that the vector field flows through, we can substitute and create a new integral as:

$$\iint_D F \bullet \nabla g\, dA$$

In this case, we assume that the ocean surface is g(x,y) = 0, bounded by the curve $r^2 = x^2 + y^2$ where r is equal to the hurricane radius, and F is the hurricane vector field. We convert to polar coordinates using the conversion:

$$x = r*\cos(\theta)$$
$$y = r*\sin(\theta)$$

Once plugged in, the integral will appear as

$$C \int_0^{2\pi} \int_{20}^{150} < (-\,rcos(T) - rsin(T))/(rcos(T))^2 + (rsin(T))^2,\ (rcos(T) - rsin(T))/((rcos(T))^2 + (rsin(T))^2),\ 1/(((rcos(T))^2 + (rsin(T))^2 >. \nabla g \cdot r\, dR\, dT$$

∇g, in this case, would represent vector <0,0,1> as that is the normal of the surface we integrate over. The bounds are 20<r<150 and 0<θ<2π because the radius goes from 20 miles (outer bound of the storm) to 150 miles (outer edge of the storm). There is also the multiplier c, explained in the Mathematica code, which represents k, q, and v if assumed that k, q, and v are equal.

Mathematica Code:

Link to Github

Reflection:

Brayden:

In this project, Diane and I chose to do the hurricane project. Overall, our teamwork went fairly well with the work being divided as me being in charge of the majority of the mathematics, Mathematica, and background math research and Diane being in charge of all the background scientific research. I do regret that we did not conduct an in-depth research phase over the constitution and behavior of a hurricane earlier as it did inhibit me slightly during my modeling as I would get vector fields inconsistent with the behavior of the hurricane. Some advice I would provide to any future students taking up this project is to definitely not jump directly into the mathematics as I did, but rather have an intensive research period over the scientific and natural aspects of the hurricane. Overall, this project took a significant chunk of time outside of school, especially during my mathematical research period and hurricane modeling period because of the sheer amount of research and/or learning I had to indulge in. Additionally, Diane and I spent some time outside of school to confer and discuss our findings on this project. Though there were not any strict deadlines that Diane and I set for ourselves during this project, we were consistently on top of our work, where Diane would provide background information so I could successfully translate such information into math, and hence a model. During this project, most programming and Mathematica were done by me because we agreed that I was more fluent in Mathematica and had more experience with computer languages. The Mathematica coding went without any hiccups and was fairly simple, though I must confess that I had to rely on some Stackoverflow help for some specific aspects I wanted to implement. Because I am a proficient programmer in other languages and have had extensive experience with Mathematica, there was no learning curve for me and the programming that I engaged in was fairly basic and straightforward.

Diane:

In this hurricane project, Brayden and I both split the work in ways that would maximize our productivity. Learning about hurricanes on a scientific level took a bit of time as we were rather unfamiliar with natural sciences, but surpassing that learning curve was rather easy after understanding the generalities. I must admit that I greatly enjoyed learning about these tropical storms, and may have included more information than needed in the introduction. However, Brayden and I should have conferred immediately after I understood the basics so that we would not have needed to constantly readjust the code to match a hurricane’s actions. Some advice I would give includes doing extensive research on not only hurricanes but also the mathematical portions of vector fields and fluid dynamics and mechanics. Learning about flows, stream functions, and navigating how each portion of knowledge intersected with others required a significant amount of effort to understand so that I could write the introduction. Furthermore, a more cohesive understanding of how hurricanes acted and how to model them would have saved me a lot of time, since I struggled in the beginning with how to explain certain tropical movements. Completing this project required quite a few hours outside of school in order to not only summarize research but also continue adjusting research and code in accordance to new things we learned. Our respective roles fit us well given our prior experiences as well as interests. Brayden worked on the coding parts while I provided the foundations of scientific and mathematical knowledge. Since Brayden coded, as he is far more familiar with coding, he required the knowledge I had learned (and cited) in order to accomplish his part, and we both completed our respective parts efficiently and to the best of our abilities.

Summary Project Comments

This project was incredibly desirable as it took already existing multivariable and mathematics skills and applied them to real-world phenomena, allowing the application of complex concepts, though it will still require significant learning with stream functions, flows, and velocity potentials. Similarly, the merit of having this final project is quite high, as it challenged us to accomplish much more difficult mathematics and learn more difficult concepts.

Citations:

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