Sin$\left[\frac{n\pi x}{L}\right]$ and **Cos**$\left[\frac{m\pi x}{L}\right]$ are mutually orthogonal on an interval $\left[\frac{-L}{2}, \frac{L}{2}\right]$ for all positive L.
But on the interval [0, L], specifically when m and n are of opposite parity, the inner product is nonzero and mutual orthogonality fails.

In[∘]:=
```
assumptions = {{m, n} ∈ Integers, {x, L} ∈ Reals, L > 0};
```

Integrated on $\left[\frac{-L}{2}, \frac{L}{2}\right]$, the modes are completely mutually orthogonal.

Case I, *m* = *n*:

In[∘]:=
```
case = m == n;
$Assumptions = Flatten[{case, assumptions}]
```

Out[∘]= {m == n, (m | n) ∈ ℤ, (x | L) ∈ ℝ, L > 0}

In[∘]:= $\frac{2}{L}\int_{\frac{-L}{2}}^{\frac{L}{2}} \text{Cos}\left[\frac{m\pi x}{L}\right] \text{Sin}\left[\frac{n\pi x}{L}\right] dx$

Out[∘]= 0

Case II, *m* ≠ *n:*

In[∘]:=
```
case = m ≠ n;
```

Sub-case I, m & n both even:

In[∘]:=
```
subcase = {Mod[m, 2] == 0, Mod[n, 2] == 0};
$Assumptions = Flatten[{case, subcase, assumptions}]
```

Out[∘]= {m ≠ n, Mod[m, 2] == 0, Mod[n, 2] == 0, (m | n) ∈ ℤ, (x | L) ∈ ℝ, L > 0}

In[∘]:= $\frac{2}{L}\int_{\frac{-L}{2}}^{\frac{L}{2}} \text{Cos}\left[\frac{m\pi x}{L}\right] \text{Sin}\left[\frac{n\pi x}{L}\right] dx$

Out[∘]= 0

Sub-case II, m & n both odd:

In[∘]:=
```
subcase = {Mod[m, 2] == 1, Mod[n, 2] == 1};
$Assumptions = Flatten[{case, subcase, assumptions}]
```

Out[∘]= {m ≠ n, Mod[m, 2] == 1, Mod[n, 2] == 1, (m | n) ∈ ℤ, (x | L) ∈ ℝ, L > 0}

In[∘]:= $$\frac{2}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} \text{Cos}\left[\frac{m\,\pi\,x}{L}\right]\,\text{Sin}\left[\frac{n\,\pi\,x}{L}\right]\,dx$$

Out[∘]= 0

Sub-case III, one even, one odd (WLOG, let m be even):

In[∘]:=
```
subcase = {Mod[m, 2] == 0, Mod[n, 2] == 1};
$Assumptions = Flatten[{case, subcase, assumptions}]
```

Out[∘]= {m ≠ n, Mod[m, 2] == 0, Mod[n, 2] == 1, (m | n) ∈ ℤ, (x | L) ∈ ℝ, L > 0}

In[∘]:= $$\int_{-\frac{L}{2}}^{\frac{L}{2}} \text{Cos}\left[\frac{m\,\pi\,x}{L}\right]\,\text{Sin}\left[\frac{n\,\pi\,x}{L}\right]\,dx$$

Out[∘]= 0

Integrated on [0, L], the modes are not completely mutually orthogonal.

Case I, *m* = *n*:

In[∘]:=
```
case = m == n;
$Assumptions = Flatten[{case, assumptions}]
```

Out[∘]= {m == n, (m | n) ∈ ℤ, (x | L) ∈ ℝ, L > 0}

In[∘]:= $$\frac{2}{L}\int_{0}^{L} \text{Cos}\left[\frac{m\,\pi\,x}{L}\right]\,\text{Sin}\left[\frac{n\,\pi\,x}{L}\right]\,dx$$

Out[∘]= 0

Case II, *m* ≠ *n*:

In[∘]:=
```
case = m ≠ n;
```

Sub-case I, m & n both even:

In[∘]:=
```
subcase = {Mod[m, 2] == 0, Mod[n, 2] == 0};
$Assumptions = Flatten[{case, subcase, assumptions}]
```

Out[∘]= {m ≠ n, Mod[m, 2] == 0, Mod[n, 2] == 0, (m | n) ∈ ℤ, (x | L) ∈ ℝ, L > 0}

In[∘]:= $$\frac{2}{L}\int_{0}^{L} \text{Cos}\left[\frac{m\,\pi\,x}{L}\right]\,\text{Sin}\left[\frac{n\,\pi\,x}{L}\right]\,dx$$

Out[∘]= 0

Sub-case II, m & n both odd:

In[∘]:=
```
subcase = {Mod[m, 2] == 1, Mod[n, 2] == 1};
$Assumptions = Flatten[{case, subcase, assumptions}]
```

Out[∘]= $\{m \neq n, \mathrm{Mod}[m, 2] == 1, \mathrm{Mod}[n, 2] == 1, (m \mid n) \in \mathbb{Z}, (x \mid L) \in \mathbb{R}, L > 0\}$

In[∘]:= $$\int_0^L \mathrm{Cos}\left[\frac{m\,\pi\,x}{L}\right] \mathrm{Sin}\left[\frac{n\,\pi\,x}{L}\right] dx$$

Out[∘]= 0

Sub-case III, one even, one odd (WLOG, let m be even):

In[∘]:=
```
subcase = {Mod[m, 2] == 0, Mod[n, 2] == 1};
$Assumptions = Flatten[{case, subcase, assumptions}]
```

Out[∘]= $\{m \neq n, \mathrm{Mod}[m, 2] == 0, \mathrm{Mod}[n, 2] == 1, (m \mid n) \in \mathbb{Z}, (x \mid L) \in \mathbb{R}, L > 0\}$

In[∘]:= $$\int_0^L \mathrm{Cos}\left[\frac{m\,\pi\,x}{L}\right] \mathrm{Sin}\left[\frac{n\,\pi\,x}{L}\right] dx$$

Out[∘]= $$-\frac{2\,L\,n}{m^2\,\pi - n^2\,\pi}$$