Linear Algebra & Matrices

Matrix Basics (Matrix Basix)

If we have an $m \times n$ matrix,

m # of rows

n # of columns

Ex:

$$\begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \end{bmatrix} \qquad \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

2x3 Matrix

2x1 matrix also column matrix

Matrix addition:

$$\begin{pmatrix} 2 & 3 \\ 4 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 3 \\ 3 & 5 \end{pmatrix} = \begin{pmatrix} 3 & 6 \\ 7 & 6 \end{pmatrix}$$

Matrices have to be the same size to add

Practice:

$$\begin{pmatrix} 4 & 5 \\ 3 & 6 \end{pmatrix} + \begin{pmatrix} 1 & 2 \\ 8 & 9 \end{pmatrix} = \begin{pmatrix} 5 & 7 \\ 11 & 15 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 3 & 2 \\ 4 & 6 & -3 \\ 2 & -1 & 1 \end{pmatrix} + \begin{pmatrix} 5 & 6 & 7 \\ 3 & 2 & 1 \\ 8 & 9 & 10 \end{pmatrix} = \begin{pmatrix} 6 & 9 & 9 \\ 7 & 8 & -2 \\ 10 & 8 & 11 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -2 & 3 \\ 7 & 8 & -9 \end{pmatrix} + \begin{pmatrix} 6 & 4 & 2 \\ 7 & -5 & -3 \end{pmatrix} = \begin{pmatrix} 7 & 2 & 5 \\ 14 & 3 & -12 \end{pmatrix}$$

Matrix Multiplication

Scalar: $4\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} = \begin{pmatrix} 4 & 12 \\ 8 & 16 \end{pmatrix}$

Two Matrices: $\begin{pmatrix} 2 & 1 & 2 \\ 3 & 2 & 3 \end{pmatrix} \begin{pmatrix} 5 & 1 \\ 3 & 1 \\ 1 & 1 \end{pmatrix} =$

$2 \times \underline{3}$ $\quad \underline{3} \times 2 = 2 \times 2$

To multiply matrices your # of columns in the first matrix must equal the # of rows in the second matrix.

The resulting matrix will be the # of rows in the first matrix by the number of columns in the second matrix.

$$\begin{pmatrix} 2 & 1 & 2 \\ 3 & 2 & 3 \end{pmatrix} \begin{pmatrix} 5 & 1 \\ 3 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 15 & 5 \\ 24 & 8 \end{pmatrix}$$

Practice:

$$\begin{pmatrix} 4 & 1 & 2 \\ 1 & 5 & 3 \\ 0 & 1 & 6 \end{pmatrix} \begin{pmatrix} 3 \\ 8 \\ 2 \end{pmatrix} = \begin{pmatrix} 24 \\ 49 \\ 20 \end{pmatrix}$$

$$\begin{pmatrix} 4 & 6 \\ 5 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 16 & 14 \\ 11 & 13 \end{pmatrix}$$

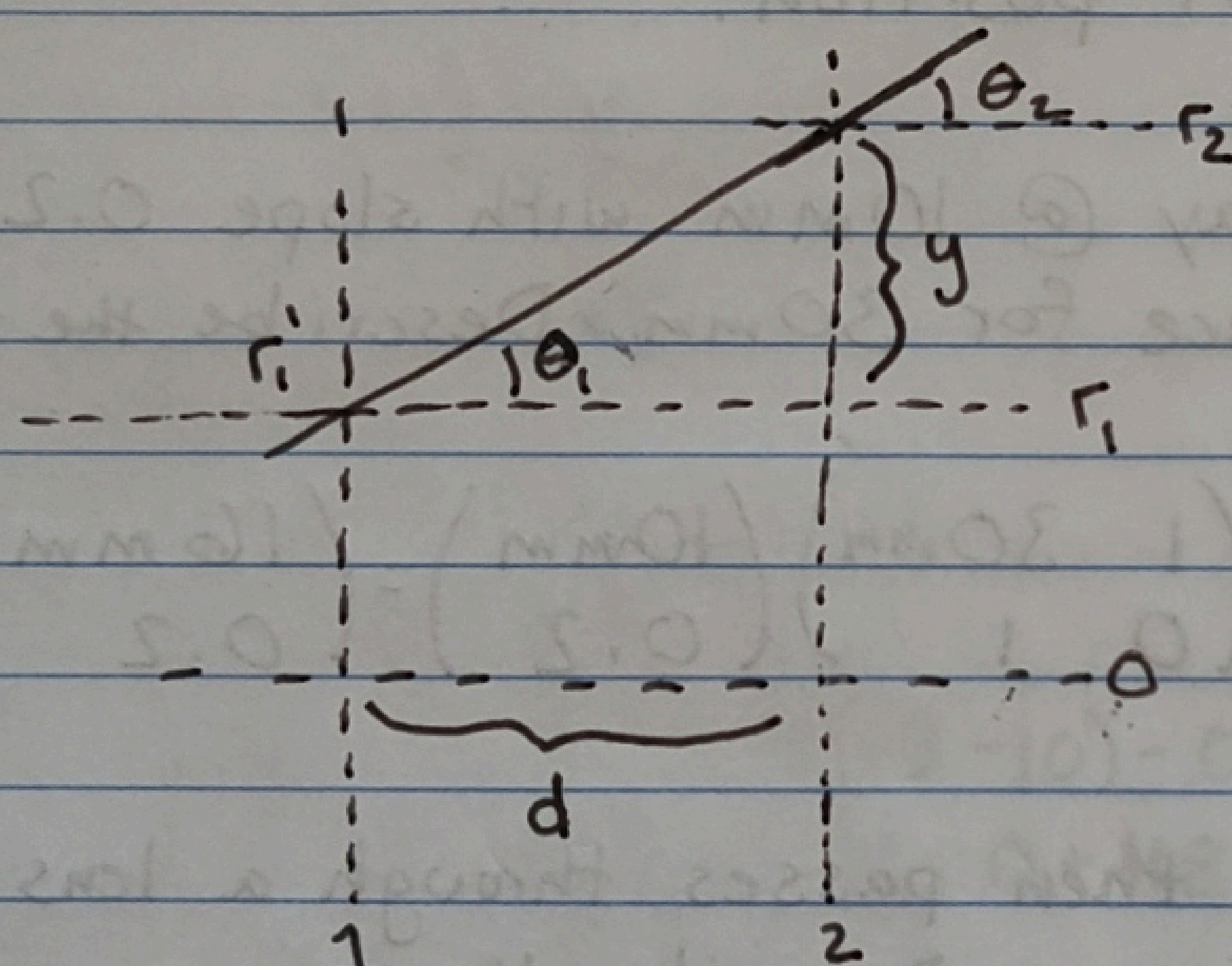
For two matrices,

$$AB \neq BA$$

$$(AB)C = A(BC)$$

Optics examples:

Rays - column matrix $\begin{pmatrix} r \\ r' \end{pmatrix} = \begin{pmatrix} \text{position} \\ \text{slope} \end{pmatrix}$



$\theta_1 = \theta_2$

$r_1' = r_2'$

$r_2 = r_1 + y$

$\tan\theta_1 = \frac{y}{d} = r_1' = r_2'$

$y = r_1' d$

$\therefore$

Ray through free space

$r_2 = r_1 + r_1' d$

$r_2' = 0\, r_1 + r_1'$

$$\begin{pmatrix} r_2 \\ r_2' \end{pmatrix} = \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix} \begin{pmatrix} r_1 \\ r_1' \end{pmatrix}$$

Free space matrix

Lens matrix: $\begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix}$ where f = focal length

Ray Example: ray @ 5mm with slope 0.1 rad represent after a lens of f = 100mm

$$\begin{pmatrix} r_2 \\ r_2' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\frac{1}{100mm} & 1 \end{pmatrix} \begin{pmatrix} 5mm \\ 0.1 \end{pmatrix} = \begin{pmatrix} 5mm \\ .05 \end{pmatrix}$$

Conclusion: Free space matrix changes position but not slope.

Lens matrix changes slope but not position.

Practice: A ray @ 10mm with slope 0.2 passes through free space for 30mm. Describe the ray.

$$\begin{pmatrix} r_2 \\ r_2' \end{pmatrix} = \begin{pmatrix} 1 & 30mm \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 10mm \\ 0.2 \end{pmatrix} = \begin{pmatrix} 16mm \\ 0.2 \end{pmatrix}$$

The same ray then passes through a lens with focal length 200mm. Describe the ray.

$$\begin{pmatrix} r_2 \\ r_2' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\frac{1}{200mm} & 1 \end{pmatrix} \begin{pmatrix} 16mm \\ 0.2 \end{pmatrix} = \begin{pmatrix} 16mm \\ 0.12 \end{pmatrix}$$

Matrix Determinants

Ex: $|A| = \begin{vmatrix} 1 & 2 & 5 \\ 0 & 1 & 2 \\ 3 & 1 & 6 \end{vmatrix}$

* can only find determinants of square matrices

$$|A| = 1(6-2) - 2(0-6) + 5(0-3)$$
$$= 4 + 12 - 15 = 1$$

Ex: $|B| = \begin{vmatrix} 7 & 2 \\ 3 & 4 \end{vmatrix}$

$$= 28 - 6 = 22$$

Practice:

$$\begin{vmatrix} 4 & 0 & 3 \\ 2 & -1 & 2 \\ 6 & 5 & 8 \end{vmatrix} = 4(-8-10) - 0 + 3(10+6)$$
$$= -72 - 0 + 48 = -24$$

$$\begin{vmatrix} -1 & 9 \\ -3 & 5 \end{vmatrix} = -5 + 27 = 22$$

Vectors in matrix form

Component notation:

$$\vec{v} = v_x\hat{x} + v_y\hat{y} + v_z\hat{z}$$

$$= v_i\hat{\imath} + v_j\hat{\jmath} + v_k\hat{k}$$

Column Matrix

$$\begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix}$$

Row Matrix

$$\begin{pmatrix} v_x & v_y & v_z \end{pmatrix}$$

Ex. $\vec{a} = \hat{x} + 2\hat{y} + 4\hat{z}$

$\vec{b} = -2\hat{x} + 4\hat{y} + 5\hat{z}$

$$\vec{a} + \vec{b} = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} + \begin{pmatrix} -2 \\ 4 \\ 5 \end{pmatrix} = \begin{pmatrix} -1 \\ 6 \\ 9 \end{pmatrix}$$

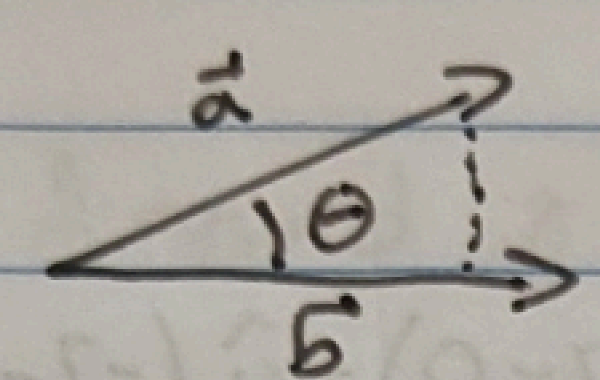
Multiplying Vectors

Dot Product -also- Scalar Product

$$\text{Work} = \vec{F} \cdot \vec{d} = Fd\cos\theta$$

$$\vec{a} \cdot \vec{b} = ab\cos\theta$$

* Perpendicular equals zero



* Dot products are a projection of $\vec{a}$ along $\vec{b}$

Ex. $\vec{a} = \hat{x} + 5\hat{y} + 2\hat{z}$
$\vec{b} = 2\hat{x} + 3\hat{y} - 4\hat{z}$

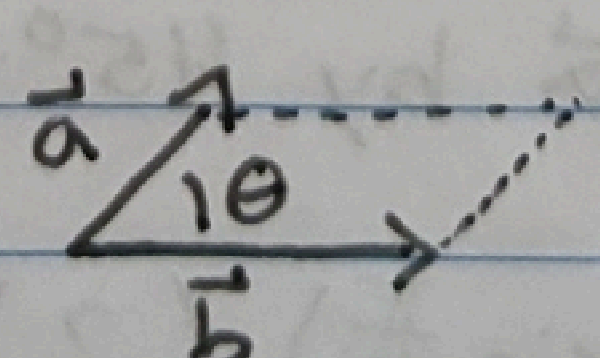
$$\vec{a} \cdot \vec{b} = \begin{pmatrix} 1 & 5 & 2 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix} = 2 + 15 - 8 = 9$$

Cross Product

$$\vec{\tau} = \vec{r} \times \vec{F}$$
$$\vec{F} = q\vec{v} \times \vec{B}$$

$$\vec{a} \times \vec{b} = ab\sin\theta\,\hat{n}$$

Right hand rule



$\vec{a} \times \vec{b}$ = area of parallelogram

Ex. $\vec{a} = \hat{x} + 5\hat{y} + 2\hat{z}$
$\vec{b} = 2\hat{x} + 3\hat{y} - 4\hat{z}$

* Parallel equals zero

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 1 & 5 & 2 \\ 2 & 3 & -4 \end{vmatrix} = \hat{x}(-20-6) - \hat{y}(-4-4) + \hat{z}(3-10)$$
$$= -26\hat{x} + 8\hat{y} - 7\hat{z}$$

Practice: Dot product, then Cross product

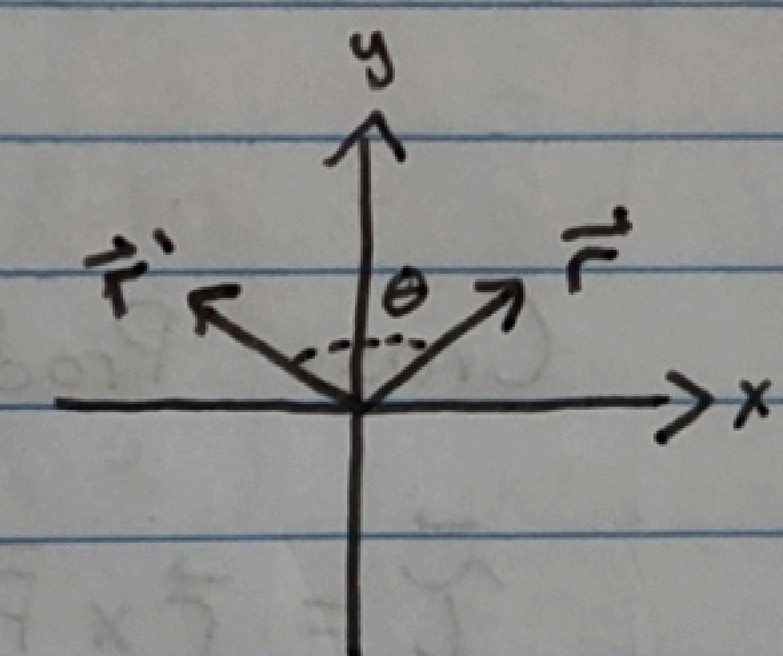
$$\vec{a} = -\hat{x} + \hat{y} + 5\hat{z}$$
$$\vec{b} = 4\hat{x} + 7\hat{z}$$

$$\vec{a}\cdot\vec{b} = \begin{pmatrix} -1 & 1 & 5 \end{pmatrix}\begin{pmatrix} 4 \\ 0 \\ 7 \end{pmatrix} = -4 + 0 + 35 = 31$$

$$\vec{a}\times\vec{b} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ -1 & 1 & 5 \\ 4 & 0 & 7 \end{vmatrix} = \hat{x}(7-0) - \hat{y}(-7-20) + \hat{z}(0-4)$$
$$= 7\hat{x} + 27\hat{y} - 4\hat{z}$$

Rotation Matrices

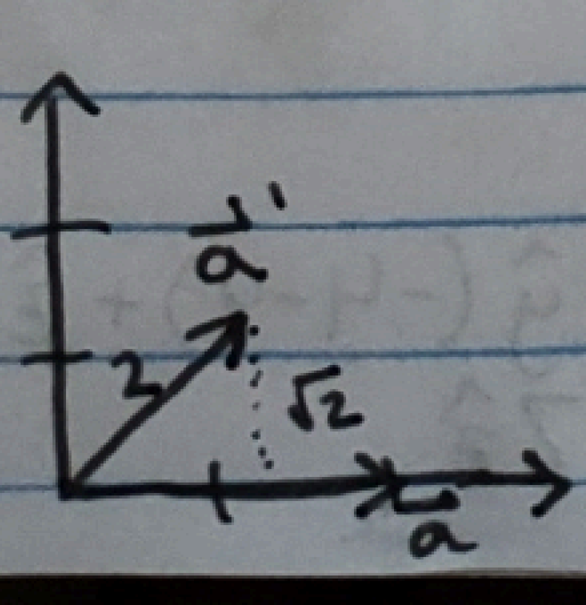
2D: $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$



Ex: Vector $\vec{a} = 2\hat{x}$
Rotate $\vec{a}$ by $45°\left(\frac{\pi}{4}\right)$

$$\cos\frac{\pi}{4} = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

$$\vec{a}' = \begin{pmatrix} \cos\frac{\pi}{4} & -\sin\frac{\pi}{4} \\ \sin\frac{\pi}{4} & \cos\frac{\pi}{4} \end{pmatrix}\begin{pmatrix} 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 2\cos\frac{\pi}{4} \\ 2\sin\frac{\pi}{4} \end{pmatrix} = \begin{pmatrix} \sqrt{2} \\ \sqrt{2} \end{pmatrix}$$



* The rotation matrix changes the direction of the vector, but not the magnitude

3D:

Around x-axis: $\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$

Around y-axis: $\begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}$

Around z-axis: $\begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$

* These are 2D rotations and identity matrices. The axis around which you rotate has identity and doesn't change, but the other two rotate.

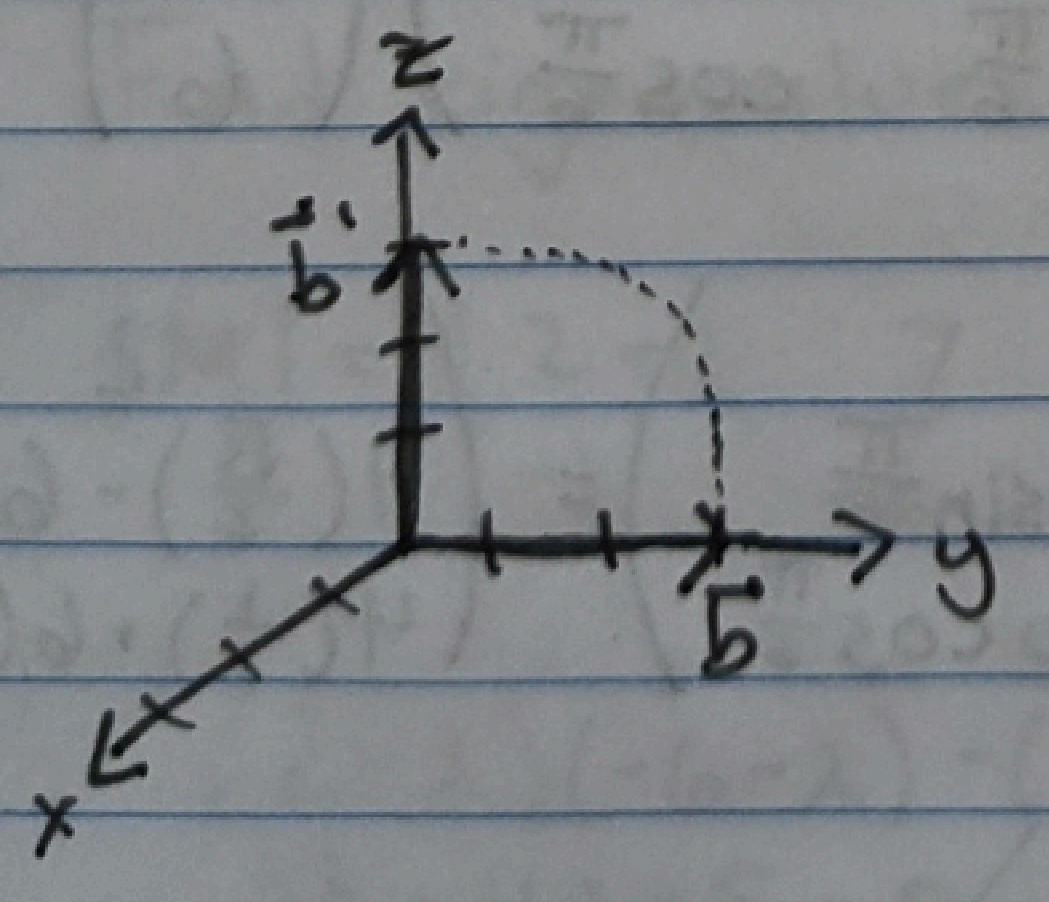
Ex: $\vec{b} = 3\hat{y}$

Rotate $\vec{b}$ by $90° (\frac{\pi}{2})$ around x-axis

$$\vec{b}' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\frac{\pi}{2} & -\sin\frac{\pi}{2} \\ 0 & \sin\frac{\pi}{2} & \cos\frac{\pi}{2} \end{pmatrix} \begin{pmatrix} 0 \\ 3 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 3\cos\frac{\pi}{2} \\ 3\sin\frac{\pi}{2} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix} = 3\hat{z}$$

$\cos\frac{\pi}{2} = 0$

$\sin\frac{\pi}{2} = 1$



Practice:

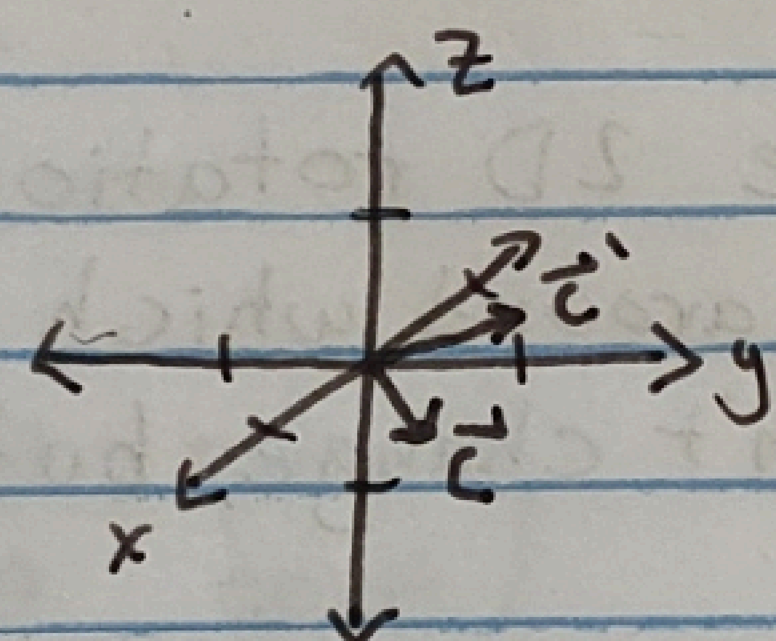
Rotate $\vec{c} = \hat{x} + \hat{y}$ $90^\circ \left(\frac{\pi}{2}\right)$ about z-axis

$$\vec{c}\,' = \begin{pmatrix} \cos\frac{\pi}{2} & -\sin\frac{\pi}{2} & 0 \\ \sin\frac{\pi}{2} & \cos\frac{\pi}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$\cos\frac{\pi}{2} = 0$

$\sin\frac{\pi}{2} = 1$

$$= \begin{pmatrix} \cos\frac{\pi}{2} - \sin\frac{\pi}{2} \\ \sin\frac{\pi}{2} + \cos\frac{\pi}{2} \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = -\hat{x} + \hat{y}$$



Rotate $\vec{d} = 2\hat{x} + 4\hat{y} + 6\hat{z}$ $30^\circ \left(\frac{\pi}{6}\right)$ about x-axis

$$\vec{d}\,' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\frac{\pi}{6} & -\sin\frac{\pi}{6} \\ 0 & \sin\frac{\pi}{6} & \cos\frac{\pi}{6} \end{pmatrix} \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix}$$

$$= \begin{pmatrix} 2 \\ 4\cos\frac{\pi}{6} - 6\sin\frac{\pi}{6} \\ 4\sin\frac{\pi}{6} + 6\cos\frac{\pi}{6} \end{pmatrix} = \begin{pmatrix} 2 \\ 4\left(\frac{\sqrt{3}}{2}\right) - 6\left(\frac{1}{2}\right) \\ 4\left(\frac{1}{2}\right) + 6\left(\frac{\sqrt{3}}{2}\right) \end{pmatrix}$$

$$= \begin{pmatrix} 2 \\ 2\sqrt{3} - 3 \\ 2 + 3\sqrt{3} \end{pmatrix} = 2\hat{x} + (2\sqrt{3} - 3)\hat{y} + (2 + 3\sqrt{3})\hat{z}$$

Eigenvalues and Eigenvectors

* Eigenvectors are special vectors whose direction is not changed under application of transformation matrices

* Eigenvalues are the amount by which the vector is scaled

$$M\vec{a} = \lambda\vec{a}$$

* 2x2 matrix has 2 Eigenvalues, 3x3 has 3...

↑ Eigenvalue, constant

We want to find what the Eigenvectors are and their corresponding Eigenvalues

$$(M-\lambda\mathbb{1})\vec{a} = 0$$

$$\left[M-\lambda\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\right]\vec{a} = 0$$

Ex: $M = \begin{pmatrix}2 & 7\\ -1 & -6\end{pmatrix}$

Find eigenvalues with determinant

$$|M| = \begin{vmatrix}2-\lambda & 7\\ -1 & -6-\lambda\end{vmatrix} = 0$$

$$(2-\lambda)(-6-\lambda)-(-7)=0$$
$$\lambda^2+4\lambda-5=0$$
$$(\lambda+5)(\lambda-1)=0 \therefore \lambda=-5, 1 \leftarrow \text{eigenvalues}$$

eigenvectors →

Find the eigenvectors

* Each eigenvalue corresponds to one eigenvector

$\lambda = 1$: $\begin{pmatrix} 2 & 7 \\ -1 & -6 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 1\begin{pmatrix} x \\ y \end{pmatrix}$

$$\left.\begin{aligned} 2x + 7y &= x \\ -x - 6y &= y \end{aligned}\right\} \rightarrow x = -7y = \begin{pmatrix} -7 \\ 1 \end{pmatrix} \text{ for } \lambda = 1$$

$\lambda = -5$: $\begin{pmatrix} 2 & 7 \\ -1 & -6 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = -5\begin{pmatrix} x \\ y \end{pmatrix}$

$$2x + 7y = -5x \rightarrow 7y = -7x = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \text{ for } \lambda = -5$$

Practice:

Find eigenvalues and eigenvectors for

$M = \begin{pmatrix} 1 & 3 \\ 2 & 2 \end{pmatrix}$

λ's: $\begin{vmatrix} 1-\lambda & 3 \\ 2 & 2-\lambda \end{vmatrix} = 0$

$$(1-\lambda)(2-\lambda) - 6 = 0$$
$$2 - \lambda - 2\lambda + \lambda^2 - 6 = 0$$
$$\lambda^2 - 3\lambda - 4 = 0$$
$$(\lambda+1)(\lambda-4) = 0 \rightarrow \lambda = -1, 4$$

eigen vectors $\longrightarrow$

$\lambda = -1$: $\begin{pmatrix} 1 & 3 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = -1\begin{pmatrix} x \\ y \end{pmatrix}$

$x + 3y = -x$

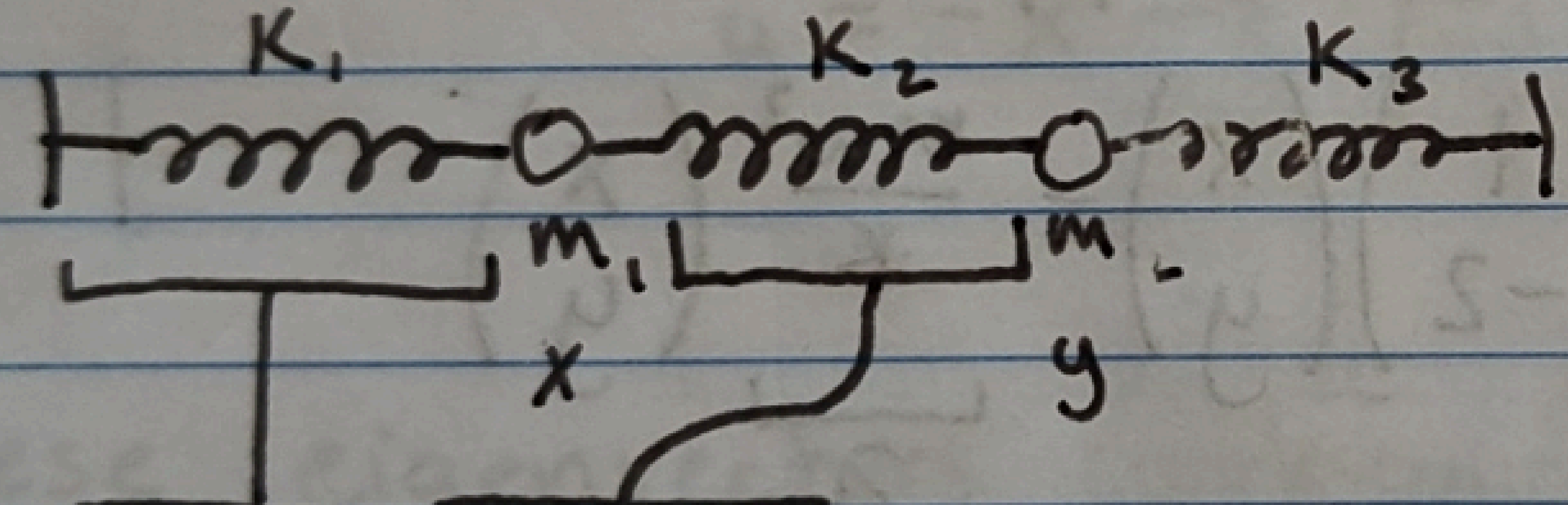
$3y = -2x \rightarrow x = -\frac{3}{2}y = \begin{pmatrix} -\frac{3}{2} \\ 1 \end{pmatrix}$ for $\lambda = -1$

$\lambda = 4$: $\begin{pmatrix} 1 & 3 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 4\begin{pmatrix} x \\ y \end{pmatrix}$

$x + 3y = 4x$

$3y = 3x \rightarrow x = y = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ for $\lambda = 4$

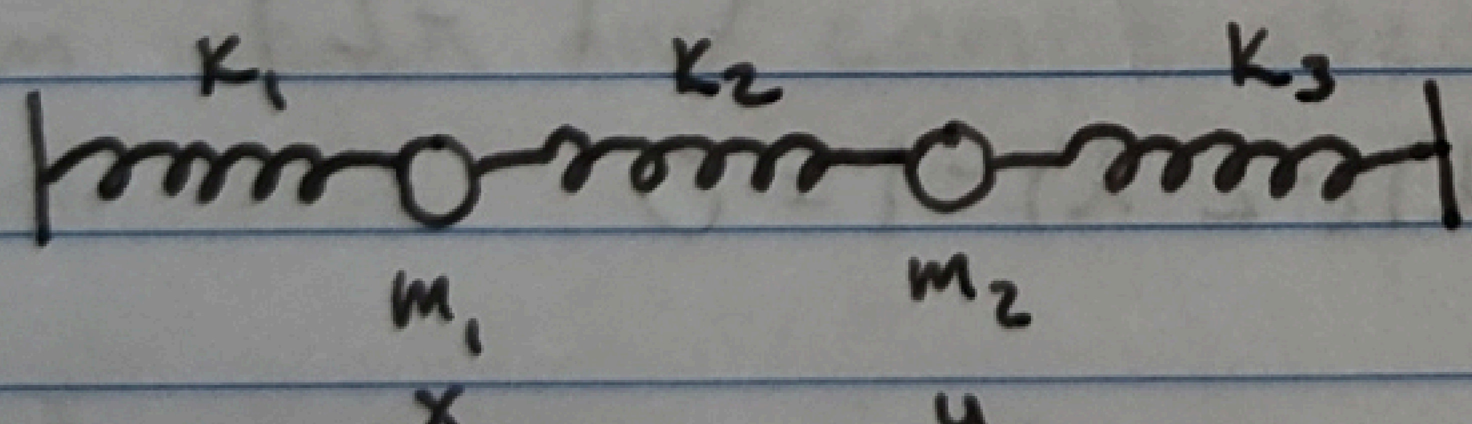
System of masses and springs



$m_1 = m_2 = m$

$k_1 = k_2 = k_3 = k$

$F_1 = -kx + k(y-x) = -2kx + ky$ = Force on m_1



$F_2 = -k(y-x) + ky = kx - 2ky$ = Force on m_2

N2: $F_1 = ma_x$ $\qquad$ $F_2 = ma_y$

$= m\frac{d^2x}{dt^2}$ $\qquad$ $= m\frac{d^2y}{dt^2}$

$-2kx + ky = m\frac{d^2x}{dt^2}$ $\qquad$ $kx - 2ky = m\frac{d^2y}{dt^2}$

* These are coupled differential equations

We want to solve this system of coupled differential equations.
We are working with springs, so we can assume our solutions will be oscillatory functions.

$x(t) = Ae^{i\omega t}$ $y(t) = Be^{i\omega t}$

$\frac{d^2x}{dt^2} = -A\omega^2 e^{i\omega t}$ $\frac{d^2y}{dt^2} = -B\omega^2 e^{i\omega t}$

$= -\omega^2 x$ $= -\omega^2 y$

$-2kx + ky = -m\omega^2 x$ $kx - 2ky = -m\omega^2 y$

$-2x + y = -\frac{m\omega^2}{k} x$ $x - 2y = -\frac{m\omega^2}{k} y$

$$\begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \underbrace{\frac{m\omega^2}{k}}_{\lambda}\begin{pmatrix} x \\ y \end{pmatrix}$$

Find eigenvalues for $M = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}$

$$\begin{vmatrix} -2 & 1 \\ 1 & -2 \end{vmatrix} = (-2-\lambda)(-2-\lambda) - 1 = 0$$

$$= 4 + 2\lambda + 2\lambda + \lambda^2 - 1 = 0$$

$$= \lambda^2 + 4\lambda + 3 = 0$$

$$(\lambda+1)(\lambda+3) = 0 \rightarrow \lambda = -1, -3$$

Use $\lambda = -\frac{m\omega^2}{k}$ to solve for ω

$$-1 = -\frac{m\omega^2}{k} \rightarrow \sqrt{\frac{k}{m}} = \omega, \quad -3 = -\frac{m\omega^2}{k} \rightarrow \sqrt{\frac{3k}{m}} = \omega$$

$$\omega = \sqrt{\frac{k}{m}}, \sqrt{\frac{3k}{m}}$$

Find the eigenvectors

$\lambda = -1: \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = -1 \begin{pmatrix} x \\ y \end{pmatrix}$

$-2x + y = -x$

$y = x \rightarrow \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ for $\lambda = -1$

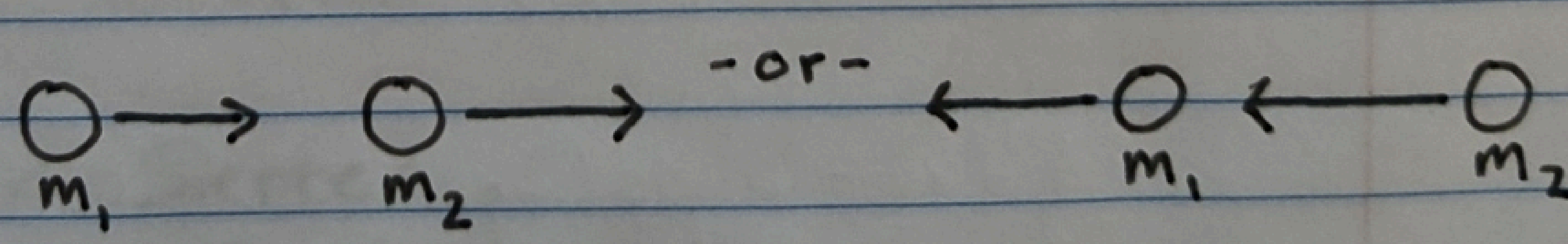
$\lambda = -3: \begin{pmatrix} -2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = -3 \begin{pmatrix} x \\ y \end{pmatrix}$

$-2x + y = -3x$

$y = -x \rightarrow \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ for $\lambda = -3$

These eigenvectors correspond to the normal modes of our spring system. Normal modes are the basis vectors from which any complicated vibration can be constructed.

$\lambda = -1$ $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$:
$w = \sqrt{\frac{k}{m}}$



$\lambda = 3$ $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$:
$w = \sqrt{\frac{3k}{m}}$