Numerical methods for the Vlasov equation

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Outline

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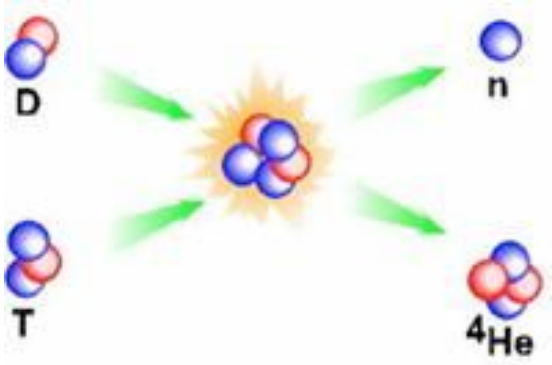
Plasmas

- When a gas is heated to a temperature of more than $10^4\,K$ electrons leave their orbit around the nucleus to which they belong.
- Gas becomes a globally neutral mixture of ions, electrons and neutrals which is called plasma.
- Plasmas are considered in addition to solid, liquid and gas to be the fourth state of matter.
- 99% of the mass of the universe consists of plasmas.
- Many applications: satellite amplifiers, plasma etching in micro-electronics, X-ray production, TVs, ..., thermonuclear fusion.

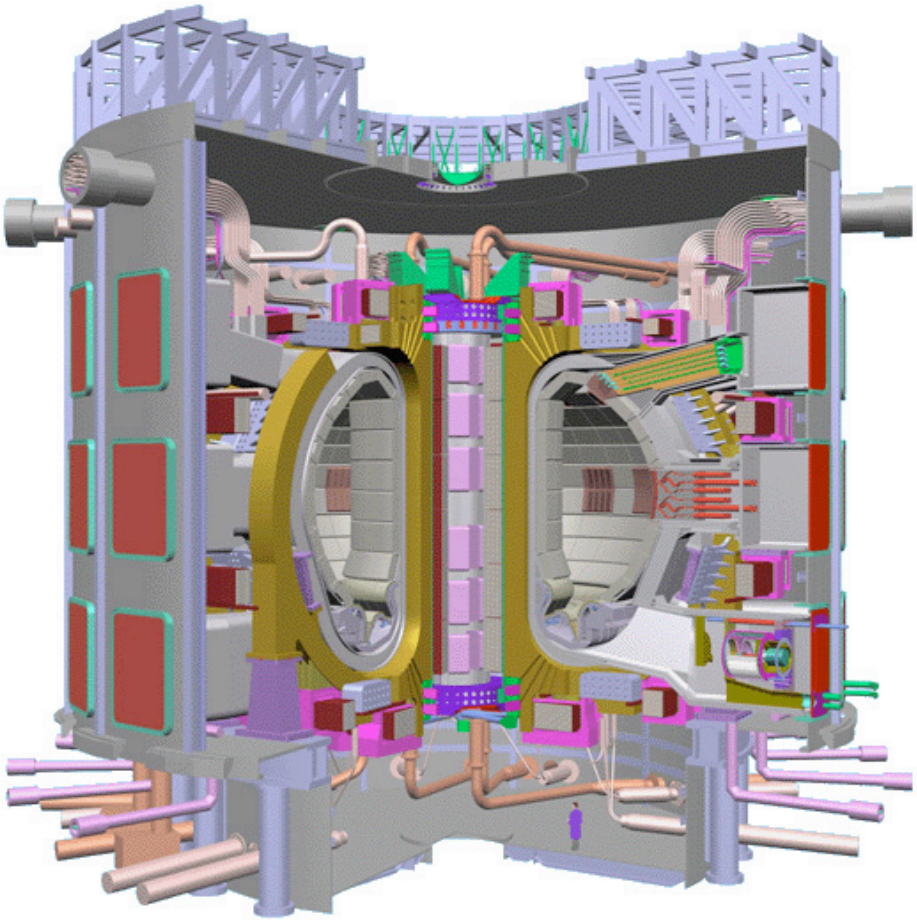
Controlled thermonuclear fusion

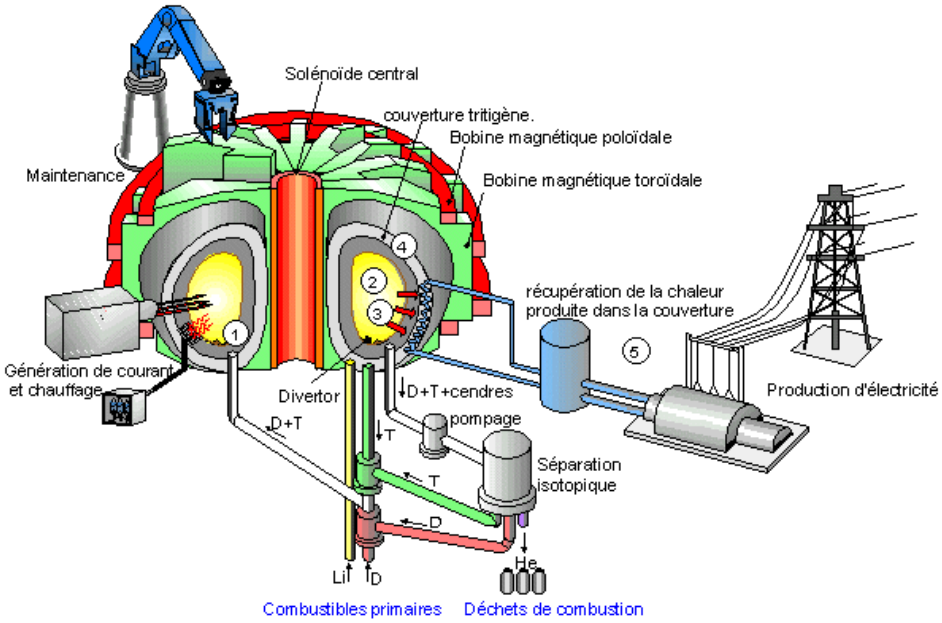


- Magnetic confinement (ITER)
- Inertial confinement
  - laser (LMJ)
  - heavy ions

The ITER project

International project involving European Union, China, India, Japan, South Korea, Russia and United States aiming to prove that magnetic fusion is viable source for energy.





Outline

The Vlasov-Maxwell system

$$\frac{\partial f_s}{\partial t} + \mathbf{v} \cdot \frac{\partial f_s}{\partial \mathbf{x}} + \frac{q_s}{m_s}(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \frac{\partial f_s}{\partial \mathbf{v}} = 0, \tag{1}$$

$$\begin{aligned} -\frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} + \nabla \times \mathbf{B} &= \mu_0 \mathbf{J}, \\ \frac{\partial \mathbf{B}}{\partial t} + \nabla \times \mathbf{E} &= 0, \\ \nabla \cdot \mathbf{E} &= \frac{\rho}{\varepsilon_0}, \\ \nabla \cdot \mathbf{B} &= 0. \end{aligned}$$

with

$$\rho(\mathbf{x}, t) = \sum_s q_s \int f_s(\mathbf{x}, \mathbf{v}, t)\, d\mathbf{v}, \qquad \mathbf{J}(\mathbf{x}, t) = \sum_s q_s \int f_s(\mathbf{x}, \mathbf{v}, t)\mathbf{v}\, d\mathbf{v}.$$

Outline

One species Vlasov-Maxwell system

We consider only one species of particles and set all physical constants to 1 for simpler computations. Domain periodic in $\mathbf{x}$ and whole space in $\mathbf{v}$
Results carry over to general model.

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla_x f + (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \nabla_v f = 0, \tag{2}$$

$$\begin{aligned} -\frac{\partial \mathbf{E}}{\partial t} + \nabla \times \mathbf{B} &= \mathbf{J}, & (3)\\ \frac{\partial \mathbf{B}}{\partial t} + \nabla \times \mathbf{E} &= 0, & (4)\\ \nabla \cdot \mathbf{E} &= \rho, & (5)\\ \nabla \cdot \mathbf{B} &= 0, & (6) \end{aligned}$$

with

$$\rho(\mathbf{x}, t) = \int f(\mathbf{x}, \mathbf{v}, t)\, d\mathbf{v}, \quad \mathbf{J}(\mathbf{x}, t) = \int f(\mathbf{x}, \mathbf{v}, t)\mathbf{v}\, d\mathbf{v}.$$

Characteristics of Vlasov equation

- The solution of the Vlasov equation can be expressed with help of characteristics which are the solutions of the differential system

$$\frac{d\mathbf{X}}{dt} = \mathbf{V}(t), \tag{7}$$

$$\frac{d\mathbf{V}}{dt} = \frac{q}{m}(\mathbf{E}(\mathbf{X}(t), t) + \mathbf{V}(t) \times \mathbf{B}(\mathbf{X}(t), t)). \tag{8}$$

- Characteristics are denoted by

$$(\mathbf{X}(t; \mathbf{x}, \mathbf{v}, s), \mathbf{V}(t; \mathbf{x}, \mathbf{v}, s))$$

- Solution of Vlasov equation then writes

$$f(\mathbf{x}, \mathbf{v}, t) = f_0(\mathbf{X}(0; \mathbf{x}, \mathbf{v}, t), \mathbf{V}(0; \mathbf{x}, \mathbf{v})).$$

Conservative form of Vlasov equation

- Vlasov equation can be written in conservative form

$$\frac{\partial f}{\partial t} + \nabla_{\mathbf{x},\mathbf{v}} \cdot (\mathbf{F} f) = 0, \tag{9}$$

  with $\mathbf{F} = (\mathbf{v}, \mathbf{E} + \mathbf{v} \times \mathbf{B})^T$ such that $\nabla_{\mathbf{x},\mathbf{v}} \cdot \mathbf{F} = 0$.
- Conservation of total number of particles

$$\int f(\mathbf{x}, \mathbf{v}, t)\, d\mathbf{x}\, d\mathbf{v}$$

  follows by integrating in phase space.

First conservation properties

Maximum principle

$$0 \leq f(\mathbf{x}, \mathbf{v}, t) \leq \max_{(\mathbf{x},\mathbf{v})}(f_0(\mathbf{x}, \mathbf{v})). \tag{10}$$

Follows from

$$f(\mathbf{x}, \mathbf{v}, t) = f_0(\mathbf{X}(0; \mathbf{x}, \mathbf{v}, t), \mathbf{V}(0; \mathbf{x}, \mathbf{v})).$$

Conservation of L^p norms

$$\frac{d}{dt}\left(\int (f(\mathbf{x}, \mathbf{v}, t))^p \, d\mathbf{x}\, d\mathbf{v}\right) = 0 \tag{11}$$

Multiply Vlasov eq. by f^{p-1} and integrate over phase space.

Conservation of phase space volume

For any volume V of phase space

$$\int_V f(\mathbf{x},\mathbf{v},t)\,d\mathbf{x}\,d\mathbf{v} = \int_{F^{-1}(V)} f_0(\mathbf{y},\mathbf{u})\,d\mathbf{y}\,d\mathbf{u}. \tag{12}$$

Jacobian of flow is equal to one because phase space divergence of advection field

$$(\mathbf{v}, \mathbf{E} + \mathbf{v} \times \mathbf{B})^T$$

vanishes.

Conservation of momentum and energy

- Momentum conservation

$$\frac{d}{dt}\left[\int \mathbf{v} f \, d\mathbf{x}d\mathbf{v} + \int \mathbf{E}\times\mathbf{B}\, d\mathbf{x}\right] = \frac{d}{dt}\left[\int \mathbf{J}\, d\mathbf{x} + \int \mathbf{E}\times\mathbf{B}\, d\mathbf{x}\right] = 0. \quad (13)$$

- Energy conservation

$$\frac{d}{dt}\left[\frac{1}{2}\int |\mathbf{v}|^2 f \, d\mathbf{x}d\mathbf{v} + \frac{1}{2}\int (E^2 + B^2)\, d\mathbf{x}\right] = 0. \quad (14)$$

Outline

Vlasov-Poisson model

- The one species Vlasov equation reads

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla_x f + \mathbf{E} \cdot \nabla_v f = 0. \tag{15}$$

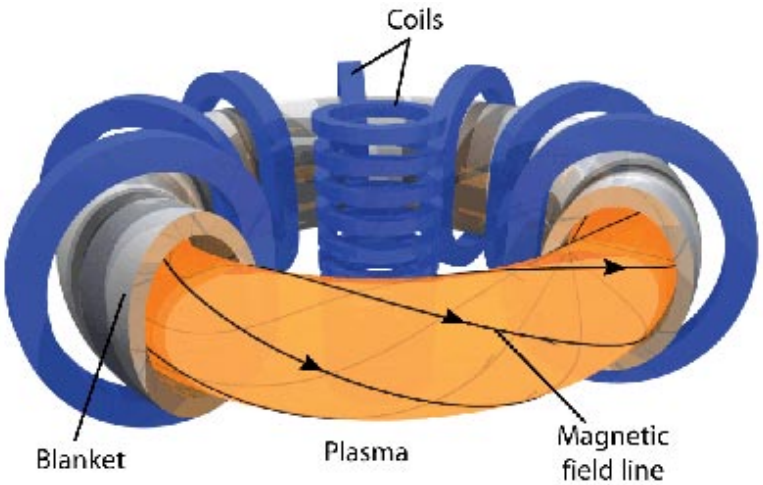
- It is coupled to the Poisson equation

$$-\Delta \phi = 1 - \rho = 1 - \int f(t, x, v)\, dv, \qquad E = -\nabla \phi.$$

- 1D Vlasov-Poisson equation first standard model.

Specificities of Tokamaks

- Large magnetic field.
- Particle trajectories mostly confined along magnetic field lines.
- Many instabilities of magnetized plasmas.
- Toroidal geometry



The drift-kinetic model (in slab geometry)

- Taking the limit of a large magnetic field we get a 4D model $(r, \theta, \phi, v_\parallel)$

$$\frac{\partial f}{\partial t} + v_D \cdot \nabla_x f + v_\parallel \cdot \nabla_\parallel f + \frac{q}{m} \mathbf{E}_\parallel \cdot \nabla_v f = 0,$$

with $v_D = \frac{\mathbf{E} \times \mathbf{B}}{B^2}$. Poisson's equation might be replaced by the quasi-neutrality relation

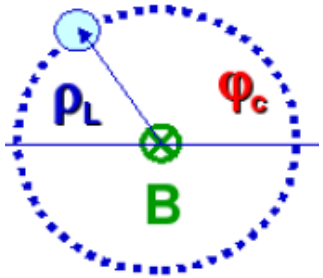
$$-\nabla_\perp \cdot \left(\frac{n_0(r)}{B\omega_c} \nabla_\perp \phi\right) + \frac{e\, n_0(r)}{T_e(r)} (\phi - \lambda \langle \phi \rangle) = n - n_0,$$

$\langle \phi \rangle$ representing the average of the potential over the magnetic flux lines.

- Guiding-center model as a starting point.

The gyrokinetic model

- It is a 5D model describing the evolution of the guiding center distribution $f(r, \theta, \phi, v_\parallel, \mu)$



$$\frac{\partial f}{\partial t} + v_D \cdot \nabla_x f + v_\parallel \cdot \nabla_\parallel f + \frac{q}{m} \mathbf{E}_\parallel \cdot \nabla_v f = 0,$$

with $v_D = -\frac{\nabla J(\phi) \times \mathbf{B}}{B^2}$, coupled with the quasi-neutrality equation

$$-\nabla_\perp \cdot (\frac{n_0(r)}{B\omega_c} \nabla_\perp \phi) + \frac{e\, n_0(r)}{T_e(r)} (\phi - \lambda \langle \phi \rangle) = \int J(f)\, dv_\parallel d\mu - n_0.$$

- The gyroaverage operator J transforms the guiding-center distribution onto the actual particle distribution enabling to take into account the finite Larmor radius.

Outline

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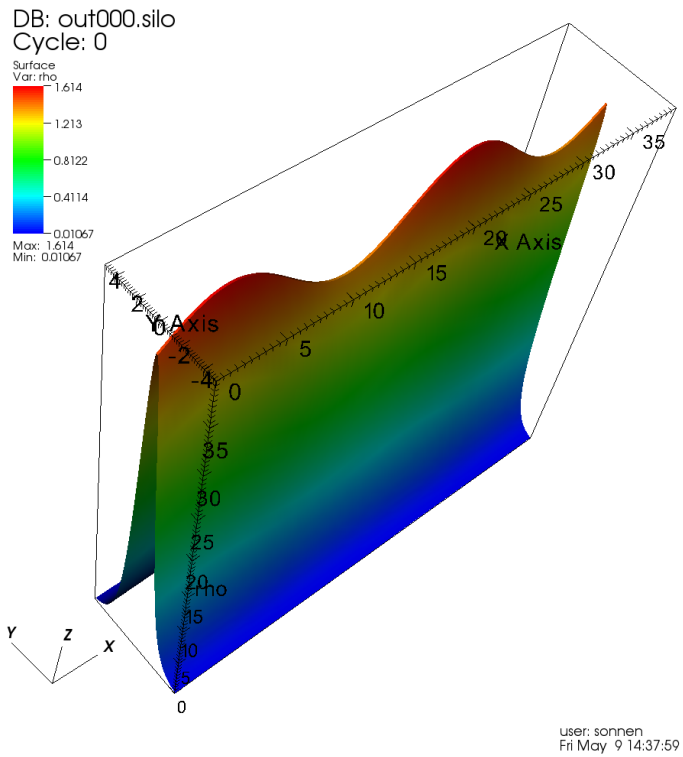
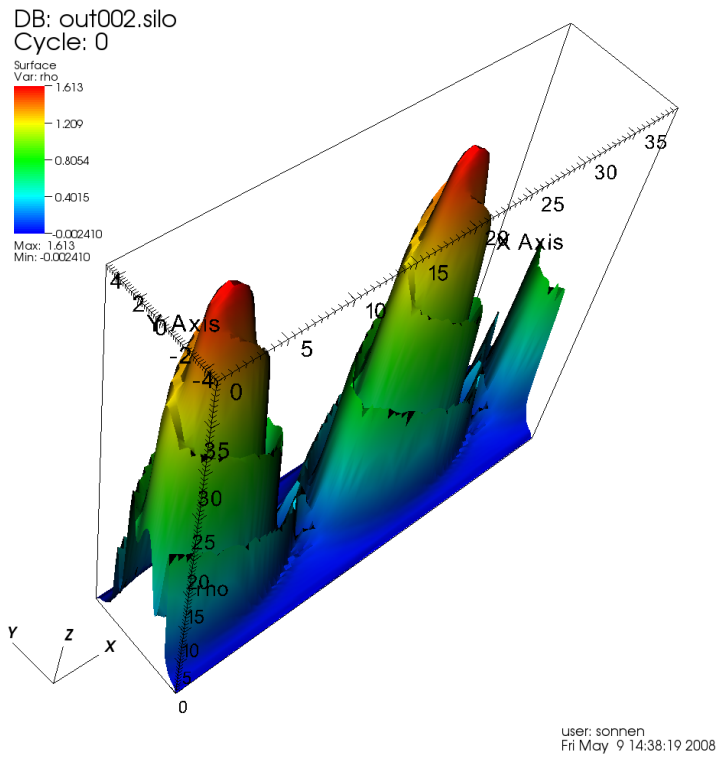
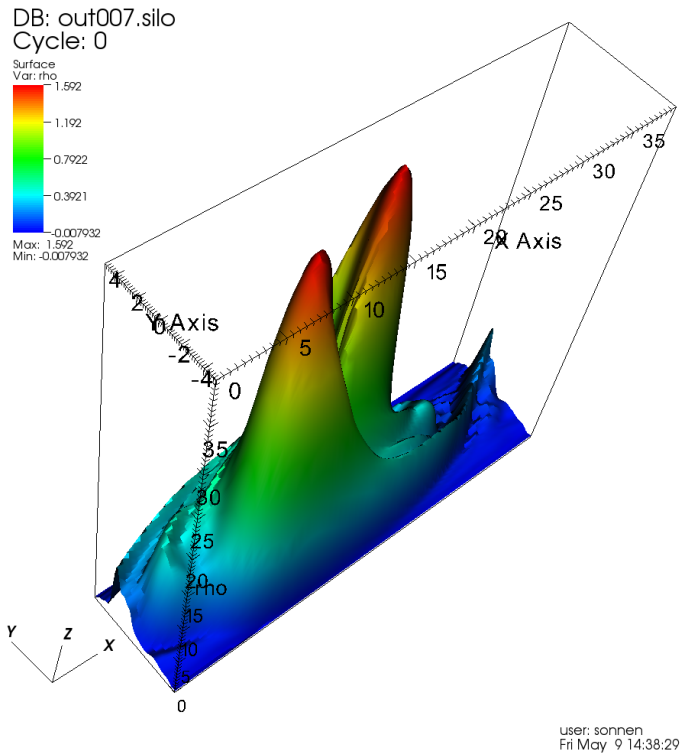
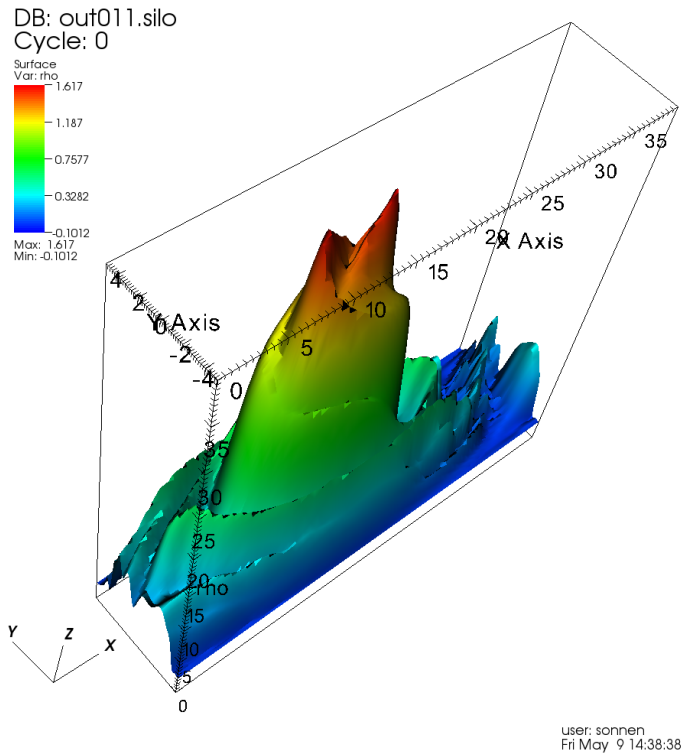
Kelvin-Helmoltz instability in magnetized plasma (1/3)



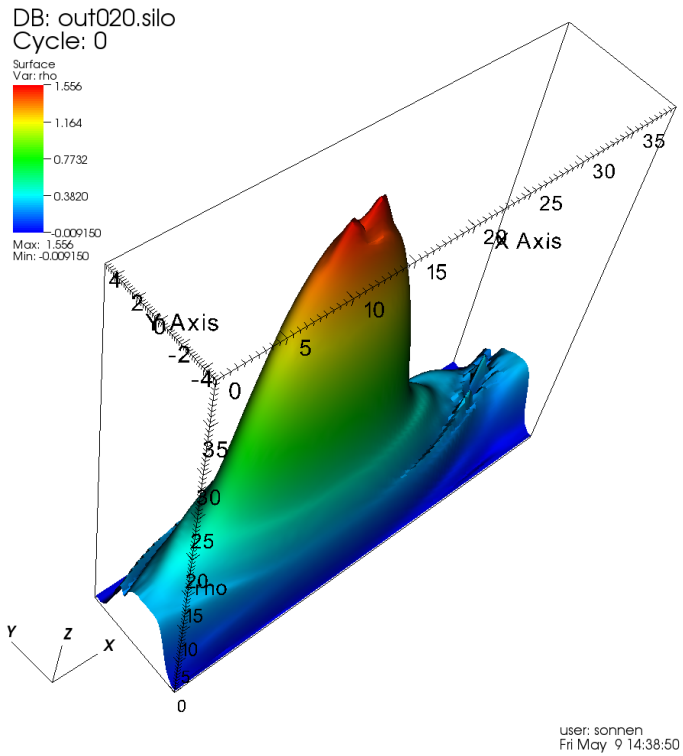


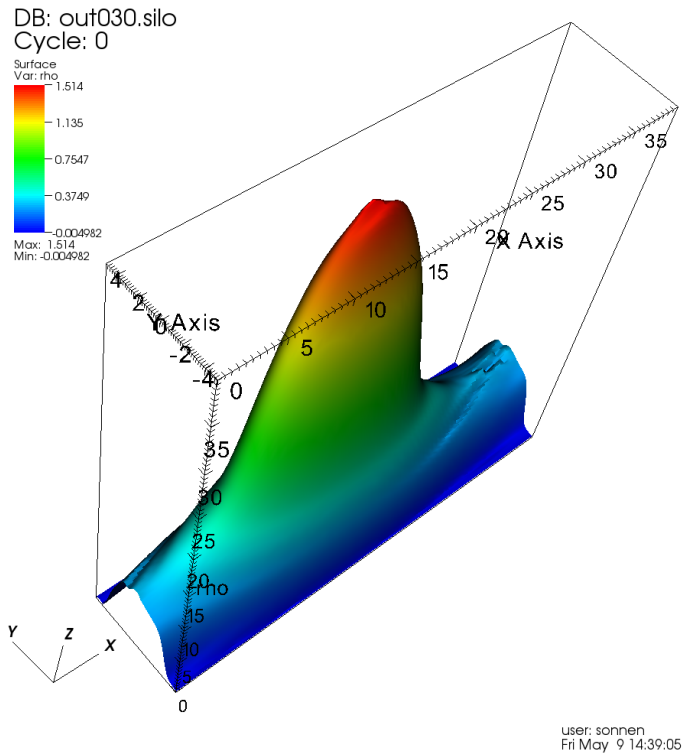
Kelvin-Helmoltz instability in magnetized plasma (2/3)





Kelvin-Helmoltz instability in magnetized plasma (3/3)





Numerical simulation of kinetic equations

- Difficulties:
  - defined in phase space (4D, 5D or 6D).
  - Appearance of small scales. Filamentation.
- Particle methods: more efficient in high dimensions.
  - Good qualitative results at relatively low cost
  - Numerical noise and slow convergence $\rightarrow$ difficult to have accurate results in some situations.
- Methods using a grid of phase space:
  - Huge amount of computations in more than 3 dimensions.
  - No numerical noise, but diffusion.
  - Small scales at some point will not be resolved by the grid.
    Need to dissipate them for stability. How? Subgrid model?
    Entropy cannot be conserved for long time simulations
- Fourier Transform in velocity space (Eliasson). Replace subgrid resolution problem by boundary conditions to evacuate high modes.

Outline

Particle Methods

- Particle-In-Cell (PIC) method :
  - Idea: Follow particle trajectories, use grid field solve.
  - Literature:
    - Physics (Birdsall-Langdon, Hockney-Eastwood)
    - Mathematical analysis (Neunzert-Wick, Cottet-Raviart, Victory-Allen)
- SPH type methods:
  - Idea: Compute interaction between finite sized macro-particles.
  - Literature: (Bateson-Hewett)

Discretization of the Vlasov equation by a particle method

- Particle approximation of the Vlasov equation. Distribution function is approximated by

$$f_h(x, v, t) = \sum_k w_k \delta(x - x_k(t))\delta(v - v_k(t)).$$

- Deterministic, pseudo-random or Monte-Carlo approximation of f_0.
- Once particles have been initialized, they are advanced using deterministic equations of motion

$$\frac{dx_k}{dt} = v_k, \quad \frac{dp_k}{dt} = q(E(x_k, t) + v_k \wedge B(x_k, t)).$$

Discrete particle motion

- Independent of mesh.
- Once fields at particle positions are known, equations of motion are generally discretized using a Leap-Frog scheme.

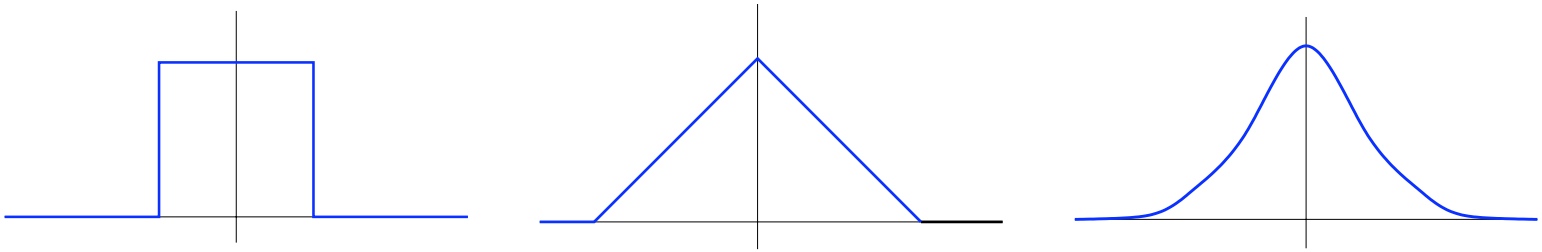
$$\frac{\mathbf{x}_k^{n+1} - \mathbf{x}_k^n}{\Delta t} = \mathbf{v}_k^{n+\frac{1}{2}} \qquad \frac{\mathbf{v}_k^{n+\frac{1}{2}} - \mathbf{v}_k^{n-\frac{1}{2}}}{\Delta t} = \frac{q}{m}(\mathbf{E}_k^n + \mathbf{v}_k^n \times \mathbf{B}_k^n)$$

- Use Boris trick to avoid acceleration by magnetic field

$$\begin{aligned}
\mathbf{v}^- &= \mathbf{v}_k^{n-\frac{1}{2}} + \frac{q}{2m}\mathbf{E}^n \\
\frac{\mathbf{v}^+ - \mathbf{v}^-}{\Delta t} &= \frac{q}{2m}(\mathbf{v}^+ + \mathbf{v}^-) \times \mathbf{B}^n \\
\mathbf{v}^{n+\frac{1}{2}} &= \mathbf{v}_k^+ + \frac{q}{2m}\mathbf{E}^n
\end{aligned}$$

Coupling particles with fields through shape functions

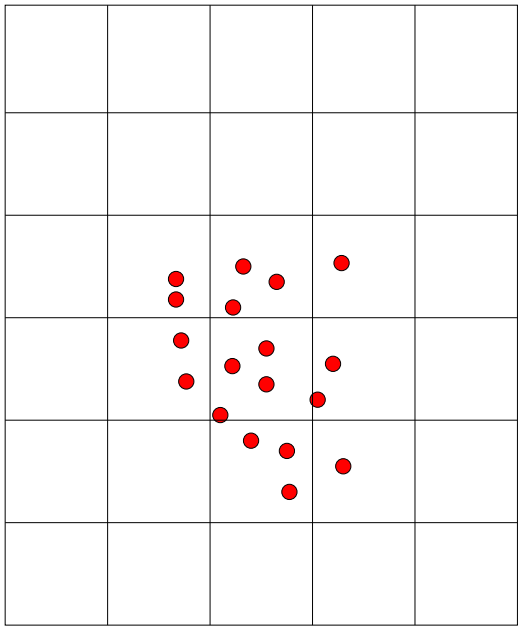
- Particle method defines point particles (Dirac masses).
- Regularization by convolution with a finite width smoothing kernel generally called weighting function.
  - Splines of different orders on are generally used on structured grids



NGP CIC order 2

  - P^1 Finite Element shape functions on unstructured grids. Generally fields obtained by Poisson or Maxwell field solver first interpolated at vertices of mesh
  - Truncated Gaussians have also been used by some authors (Jacobs-Hesthaven).

Coupling with the Field solver: The Particle-In-Cell Method



- Particle data scattered to surrounding grid points to compute charge and current densities using weight function:

$$\rho_i = q \sum_k S(x_k - x_i), \quad \mathbf{J}_i = q \sum_k S(x_k - x_i)\mathbf{v}_k.$$

- Field solve performed on grid: $\mathbf{E}(x_i)$, $\mathbf{B}(x_i)$ are computed using some grid based field solver.
- Fields are computed on particles using interpolation e.g. $\mathbf{E}(x_k) = \sum S(x_k - x_i)\mathbf{E}(x_i)$.

Noise reduction techniques

- Statistical noise reduction techniques can be applied:
- δf methods. Decompose $f = f^0 + \delta f$ and use particle approximation only for perturbation. Decreases noise where $\delta f << f^0$.

$$\begin{aligned}\mathbb{E}_f(S_i)) &= \mathbb{E}_{f^0}(S_i)) + \mathbb{E}_{f-f^0}(S_i)) \\ &= \int S_i(x) f^0(x,v)\,dx\,dv + \int S_i(x)(f(x,v) - f^0(x,v))\,dx\,dv\end{aligned}$$

  First part is known exactly, only second part is approximated by sample and its variance is a lot smaller if $\delta f << f^0$.
- Very efficient when f stays close to known equilibrium as in some tokamak plasma simulations.

Outline

The model

- An abstract form of a kinetic equation reads

$$\frac{\partial f}{\partial t} + \mathbf{a}(\mathbf{z}, t) \cdot \nabla f = 0,$$

  z are the phase space coordinates, $\mathbf{a}$ depends non linearly and non locally on f, through e.g. the electromagnetic field.

- The velocity field a is divergence free so that the equation can also be written in conservative form

$$\frac{\partial f}{\partial t} + \nabla \cdot (f\mathbf{a}) = 0.$$

- Mathematically for a given $\mathbf{a}$ the non conservative form is a transport equation which can be solved using the characteristics, which are the solution of the ODE

$$\frac{d\mathbf{Z}}{dt} = \mathbf{a}(\mathbf{Z}, t).$$

Operator splitting

- Consider e.g. the non relativistic Vlasov-Poisson equation

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla_x f + \frac{q}{m} \mathbf{E} \cdot \nabla_v f = 0.$$

- We decompose the equation into the two following steps.

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla_x f = 0, \tag{16}$$

with $\mathbf{v}$ fixed and

$$\frac{\partial f}{\partial t} + \frac{q}{m} \mathbf{E}(\mathbf{x}, t) \cdot \nabla_v f = 0, \tag{17}$$

with $\mathbf{x}$ fixed.

- We solve the two equations successively on one time step. At least dimension reduction and in our example constant coefficient advections for reduced equations.

Analysis of splitting error

- For constant coefficient differential operators A and B we consider

$$\frac{du}{dt} = (A + B)u$$

- Formal solution on one time step reads

$$u(t + \Delta t) = e^{\Delta t(A+B)} u(t).$$

- Split equations read

$$\frac{du}{dt} = Au,$$
$$\frac{du}{dt} = Bu.$$

with formal solutions on one time step

$$u(t + \Delta t) = e^{\Delta tA} u(t) \text{ et } u(t + \Delta t) = e^{\Delta tB} u(t).$$

Analysis of splitting error

- If operators A and B commute split solution

$$u(t+\Delta t) = e^{\Delta t B} e^{\Delta t A} u(t)$$

  exact.
- This is the case for constant coefficient transport not for general Vlasov.
- Alternate solution of A and of B on substeps to get any given order. Negative time advance coefficients starting from third order splitting.
- Coefficients are found by equating first terms of Taylor expansion of exponential.

First order splitting

Successive solution on one time step is first order in time.

$$e^{\Delta t(A+B)} = I + \Delta t(A+B) + \frac{\Delta t^2}{2}(A+B)^2 + O(\Delta t^3),$$

$$\begin{aligned} e^{\Delta tB}e^{\Delta tA} &= (I + \Delta tB + \frac{\Delta t^2}{2}B^2 + O(\Delta t^3))(I + \Delta tA + \frac{\Delta t^2}{2}A^2 + O(\Delta t^3)) \\ &= I + \Delta t(A+B)\frac{\Delta t^2}{2}(A^2 + B^2 + 2BA) + O(\Delta t^3)). \end{aligned}$$

If A and B do not commute we have
$(A+B)^2 = A^2 + AB + BA + B^2 \neq A^2 + B^2 + 2BA$.

Second order splitting

Strang splitting is second order: Solve A on half time step, B on full time step, again A on half time step.

$$e^{\Delta t(A+B)} = I + \Delta t(A+B) + \frac{\Delta t^2}{2}(A+B)^2 + O(\Delta t^3),$$

$$\begin{aligned} e^{\frac{\Delta t}{2}A}e^{\Delta tB}e^{\frac{\Delta t}{2}A} =& (I + \frac{\Delta t}{2}A + \frac{\Delta t^2}{4}A^2 + O(\Delta t^3)))(I + \Delta tB + \frac{\Delta t^2}{2}B^2 \\ &+ O(\Delta t^3))(I + \frac{\Delta t}{2}A + \frac{\Delta t^2}{4}A^2 + O(\Delta t^3))) \\ =& I + \Delta t(A+B) + \frac{\Delta t^2}{2}(A^2 + B^2 + BA + AB) + O(\Delta t^3). \end{aligned}$$

Development of methods based on phase space grid

- The semi-Lagrangian method (generalization of Cheng and Knorr splitting method).
- Vlasov solver based on WENO method (Carrillo-Vecil, Qiu-Chrislieb, Qiu-Shu, Banks-Hittinger).
- Discontinuous Galerkin (Qiu-Shu, Gamba-Morrison, Rossmanith-Seal, Crouseilles-Mehrenberger-Vecil)
- Conservative flux based methods : (Boris-Book, Fijalkow, Filbet-ES-Bertrand, Crouseilles-Mehrenberger-ES).
- Energy conserving Finite Difference (Arakawa).
- Fourier in velocity space (Eliasson).

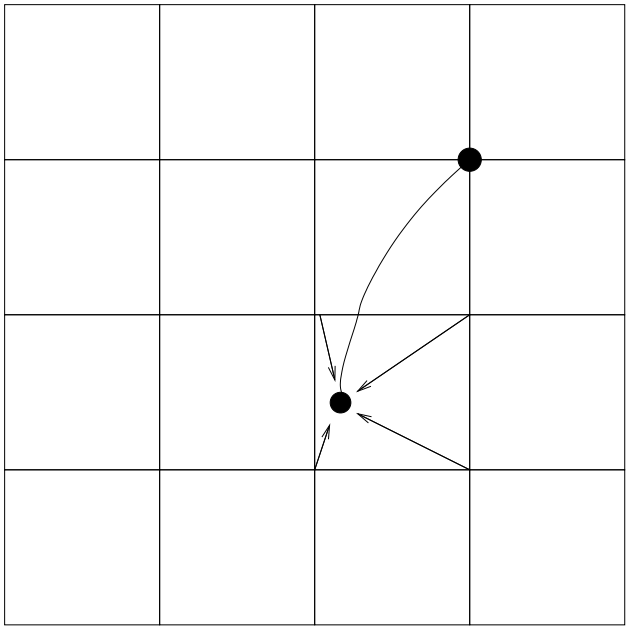
History of semi-Lagrangian schemes for Vlasov

- Cheng-Knorr (JCP 1976): splitting method for 1D Vlasov-Poisson. Cubic spline interpolation.
- ES-Roche-Bertrand-Ghizzo (JCP 1998): general semi-Lagrangian framework for Vlasov-type equations.
- Nakamura-Yabe (CPC 1999): semi-Lagrangian CIP method with Hermite interpolation
- Filbet-ES-Bertrand (JCP 2001) : semi-Lagrangian PFC method: positive and conservative
- N. Besse - ES (JCP 2003) : semi-Lagrangian solver on unstructured grids.
- V. Grandgirard et al. (JCP 2006): drift-kinetic SL scheme.
- Crouseilles-Respaud-ES (CPC 2009): Forward semi-Lagrangian
- Crouseilles-Mehrenberger-ES (JCP 2010): Equivalence of point based and conservative methods for Vlasov-Poisson + new class of filters.
- Qiu-Christlieb (JCP 2010): conservative SL WENO schemes.
- Qiu-Shu (JCP 2011): DG SL schemes.

Convergence of semi-Lagrangian schemes for Vlasov

- Filbet (SINUM 2001): PFC method for Vlasov-Poisson
- N. Besse (SINUM 2003): semi-Lagrangian method with linear interpolation for Vlasov-Poisson.
- Campos Pinto - Mehrenberger (Numer. Math. 2008): adaptative SL method for Vlasov-Poisson
- N. Besse - Mehrenberger (Math of Comp 2008): split SL method for Vlasov-Poisson for symmetric high order interpolation Lagrange and spline interpolation.
- N. Besse (SINUM 2008): SL method with Hermite interpolation and propagation of gradients.
- Respaud - ES (Numer math, 2011): Forward SL for Vlasov-Poisson with linear interpolation.

The backward semi-Lagrangian Method

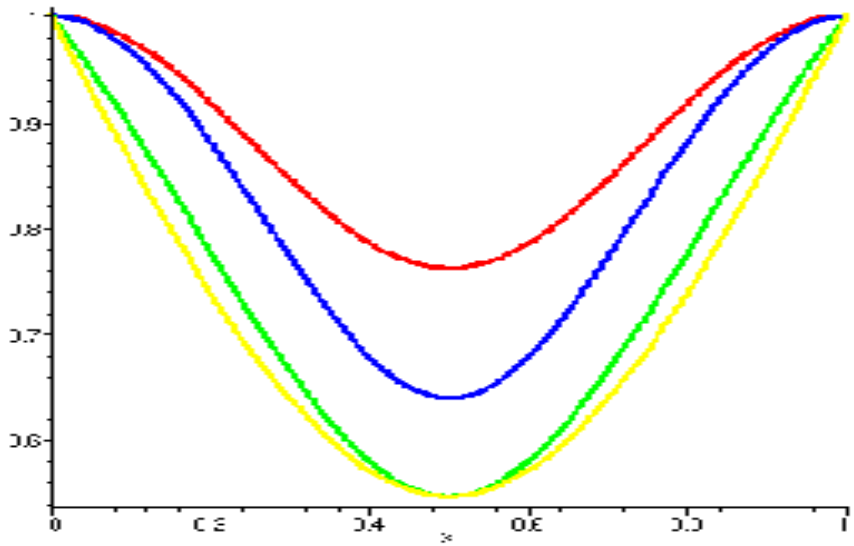


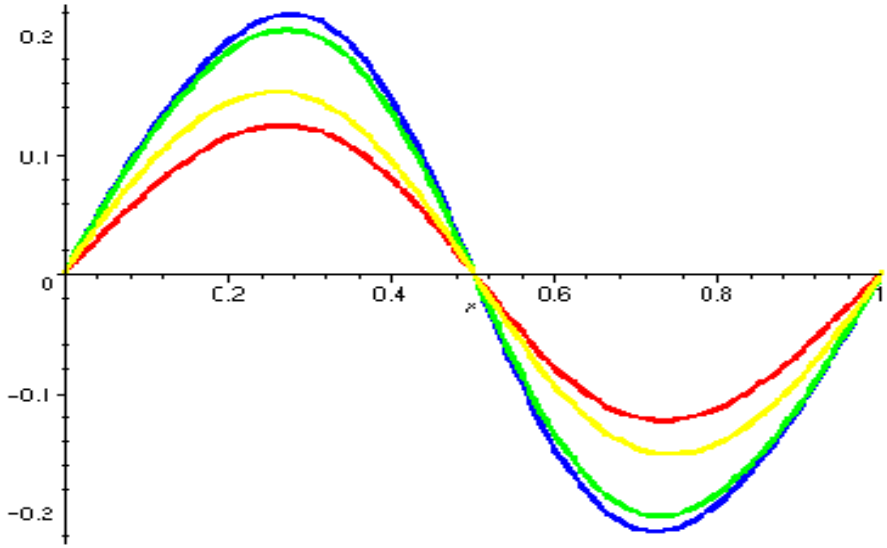
- f conserved along characteristics
- Find the origin of the characteristics ending at the grid points
- Interpolate old value at origin of characteristics from known grid values $\rightarrow$ High order interpolation needed

- Typical interpolation schemes.
  - Cubic spline (Cheng-Knorr)
  - Cubic Hermite with derivative transport (Nakamura-Yabe)

Comparison of interpolation operators

Amplification factor (left) and phase error (right)





Quadratic Lagrange (yellow), cubic Lagrange (green), cubic Hermite (blue), cubic spline (red).

Cubic spline interpolation

- Let $f \in C^k([a,b])$, $k \geq 0$ its cubic spline interpolant f_h on the knots $(x_i)_{i\in[0,N]}$ is defined by

$$f_h(x_i) = f(x_i) \text{ for } i = 0, \ldots, N, \quad f_h \in \mathbb{P}_3([x_i, x_{i+1}]), \quad f_h \in C^2([a,b]).$$

- f_h uniquely defined provided boundary conditions are given : Hermite (f' given), natural ($f'' = 0$) or periodic.
- f_h computed via its decomposition on $B-$splines. On uniform mesh

$$S^3(x) = \frac{1}{6h} \begin{cases} (2 - \frac{|x|}{h})^3 & \text{if } h \leq |x| < 2h, \\ 4 - 6\left(\frac{x}{h}\right)^2 + 3\left(\frac{|x|}{h}\right)^3 & \text{if } 0 \leq |x| < h, \\ 0 & \text{else.} \end{cases}$$

- Solution of triadiagonal matrix with additional terms on boundary.

Interpolation

- Cubic spline interpolation originally proposed by Cheng and Knorr is still our method of choice.
- Other methods have been tried: different variants of Lagrange, Hermite, higher order splines. None has proved superior to cubic splines for our applications.
- Features needed by interpolation: accuracy and robustness. Needs to degrade well when distribution is not resolved on the mesh.
- New implementations. Local splines
  - Series approximation of derivative on boundary: Crouseilles, Latu, ES: JCP 2007.
  - Fast algorithm by Unser IEEE Trans on Pattern Analysis and Machine Intelligence, vol 13 (3), 1991 for signal processing. Cholesky decomposition with constant coefficients on diagonal and off-diagonal. Iterations started with series using directly f.

Computation of the origin of the characteristics

- Transport equation
$$\frac{\partial f}{\partial t} + \mathbf{a} \cdot \nabla f = 0,$$
- Characteristics
$$\frac{dX}{dt} = \mathbf{a}$$
- Computation of the origin of the characteristics :
  - Explicit solution if $\mathbf{a}$ does not depend on x and t
  - Else, numerical algorithm needed.

Splitting for exact computation of characteristics

- In many cases splitting can enable to solve a constant coefficient advection at each split step. Ideal case.
- E.g. separable Hamiltonian $H(\mathbf{q},\mathbf{p}) = U(\mathbf{q}) + V(\mathbf{p})$.
  - Vlasov equation in canonical coordinates reads

$$\frac{\partial f}{\partial t} + \nabla_p H \cdot \nabla_q f - \nabla_q H \cdot \nabla_p f = 0.$$

  - Split equations then become

$$\frac{\partial f}{\partial t} + \nabla_p V \cdot \nabla_q f = 0, \qquad \frac{\partial f}{\partial t} - \nabla_q U \cdot \nabla_p f = 0,$$

    where U does not depend on $\mathbf{p}$ and V does not depend on $\mathbf{q}$
    $\Rightarrow$ characteristics can be solved explicitly.
  - Vlasov-Poisson falls into this category with $\mathbf{q} = \mathbf{x}$, $\mathbf{p} = \mathbf{v}$, $H(\mathbf{x},\mathbf{v}) = \frac{1}{2} m\mathbf{v}^2 + q\phi(\mathbf{x},t)$.

Backward computation of characteristics in general case.

- Consider general case. Characteristics defined by

$$\frac{d\mathbf{X}}{dt} = \mathbf{a}(\mathbf{X}, t).$$

- Backward solution: $\mathbf{X}^{n+1}$ is known and $\mathbf{a}^n$ known on the grid. Does not allow standard ODE integrator.
- Numerical method for EDO can be derived using a quadrature formula on RHS of system by integrating on one time step, e.g. left or right rectangle rule for 1st order:

$$\mathbf{X}^{n+1} - \mathbf{X}^n = \Delta t\, \mathbf{a}^n(\mathbf{X}^n) \quad \text{or} \quad \mathbf{X}^{n+1} - \mathbf{X}^n = \Delta t\, \mathbf{a}^{n+1}(\mathbf{X}^{n+1}).$$

- No explicit solution:
  - Fixed point procedure needed in first case (e.g. Newton).
  - Predictor-corrector method on $\mathbf{a}$ needed in second case.

A two step second order method

- Solve characteristics defined by $\frac{d\mathbf{X}}{dt} = \mathbf{a}(\mathbf{X}, t)$.
- Centered quadrature on two time steps:

$$\mathbf{X}^{n+1} - \mathbf{X}^{n-1} = 2\Delta t\, \mathbf{a}^n(\mathbf{X}^n), \quad \mathbf{X}^{n+1} + \mathbf{X}^{n-1} = 2\mathbf{X}^n + O(\Delta t^2).$$

- Use fixed point procedure to compute $\mathbf{X}^{n-1}$ such that

$$\mathbf{X}^{n+1} - \mathbf{X}^{n-1} = \Delta t\, \mathbf{a}^n\left(\frac{\mathbf{X}^{n+1} + \mathbf{X}^{n-1}}{2}\right).$$

- Problem: compute f^{n+1} from f^{n-1}. Even and odd order time approximations become decoupled after some time. Artificial coupling needs to be introduced.

A one step predictor-corrector second order method

- Solve characteristics defined by $\frac{d\mathbf{X}}{dt} = \mathbf{a}(\mathbf{X}, t)$.
- Centered quadrature on one time step:

$$\mathbf{X}^{n+1} - \mathbf{X}^n = \Delta t\, \mathbf{a}^{n+\frac{1}{2}}(\mathbf{X}^{n+\frac{1}{2}}), \quad \mathbf{X}^{n+1} + \mathbf{X}^n = 2\mathbf{X}^{n+\frac{1}{2}} + O(\Delta t^2).$$

- Now $\mathbf{a}^{n+\frac{1}{2}}$ is unknown. But can be computed with first order scheme (like in Runge-Kutta methods) for global second order accuracy. Requires two updates of distribution function per time step.
- Use fixed point procedure to compute $\mathbf{X}^n$ such that

$$\mathbf{X}^{n+1} - \mathbf{X}^n = \Delta t\, \mathbf{a}^{n+\frac{1}{2}}\left(\frac{\mathbf{X}^{n+1} + \mathbf{X}^n}{2}\right).$$

- In practice use linear interpolation for evaluation of $\mathbf{a}^{n+\frac{1}{2}}(X)$ to get explicit solution for $\mathbf{X}^n$.

Conservativity

- Consider abstract Vlasov equation where $\mathbf{z}$ are all the phase space variables
$$\frac{\partial f}{\partial t} + \mathbf{a}(\mathbf{z}, t) \cdot \nabla_z f = 0 \quad \text{with } \nabla \cdot \mathbf{a} = 0.$$
- The equation is conservative: $\frac{d}{dt} \int f \, d\mathbf{z} = 0$.
- Consider splitting the equations by decomposing the variables into $\mathbf{z}_1$ and $\mathbf{z}_2$. Then the split equations read
$$\frac{\partial f}{\partial t} + \mathbf{a}_1(\mathbf{z}, t) \cdot \nabla_{z_1} f = 0, \quad \text{and} \quad \frac{\partial f}{\partial t} + \mathbf{a}_2(\mathbf{z}, t) \cdot \nabla_{z_2} f = 0.$$
- We have $\nabla \cdot \mathbf{a} = \nabla_{z_1} \cdot \mathbf{a}_1 + \nabla_{z_2} \cdot \mathbf{a}_2 = 0$, but in general $\nabla_{z_1} \cdot \mathbf{a}_1$ and $\nabla_{z_2} \cdot \mathbf{a}_2$ do not vanish separately.
- One or more of the split equations may not be conservative.

Example 1: Vlasov-Poisson

- In this case the Vlasov equation reads

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla_x f + \mathbf{E} \cdot \nabla_v f = 0.$$

- So $\mathbf{a} = (\mathbf{v}, \mathbf{E}(\mathbf{x}, t))$
- Standard splitting yields:

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla_x f = 0 \quad \text{and} \quad \frac{\partial f}{\partial t} + \mathbf{E} \cdot \nabla_v f = 0.$$

- So that $\mathbf{a}_1 = \mathbf{v}$ and $\mathbf{a}_1 = \mathbf{E}(\mathbf{x}, t)$.
- In this case $\nabla_x \cdot \mathbf{a}_1 = 0$ and $\nabla_v \cdot \mathbf{a}_2 = 0$.
- Splitting yields two conservative equations.

Example 2: guiding-center model

- Classical model for magnetized plasmas. Describes motion in plane perpendicular to magnetic field.

$$\frac{\partial \rho}{\partial t} + \mathbf{v}_D \cdot \nabla \rho = 0, \qquad -\Delta \phi = \rho,$$

$$\mathbf{v}_D = \frac{-\nabla \phi \times \mathbf{B}}{B^2} = \begin{pmatrix} -\frac{\partial \phi}{\partial y} \\ \frac{\partial \phi}{\partial x} \end{pmatrix} \text{ if } \mathbf{B} = \mathbf{e}_z \text{ unit vector in direction } z.$$

- The model is conservative: $\nabla \cdot \mathbf{v}_D = -\frac{\partial^2 \phi}{\partial x \partial y} + \frac{\partial^2 \phi}{\partial y \partial x} = 0.$
- Split equations become

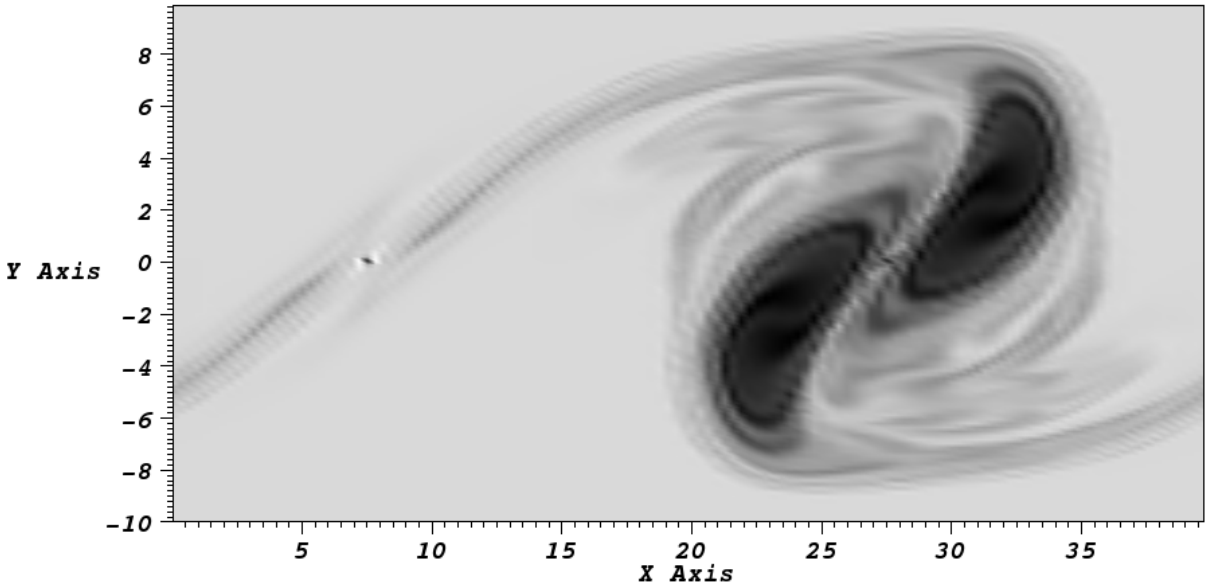
$$\frac{\partial \rho}{\partial t} - \frac{\partial \phi}{\partial y}\frac{\partial \rho}{\partial x} = 0, \qquad \frac{\partial \rho}{\partial t} + \frac{\partial \phi}{\partial x}\frac{\partial \rho}{\partial x} = 0.$$

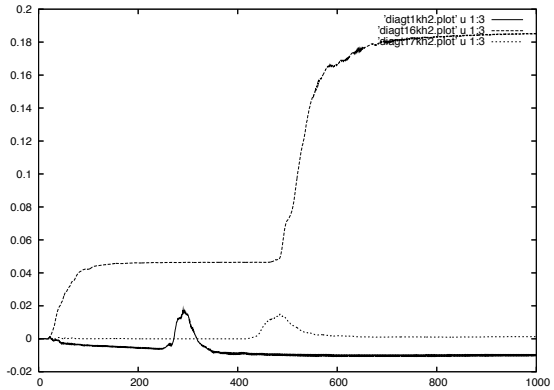
- In general $\frac{\partial^2 \phi}{\partial x \partial y} \neq 0.$
- The split equations are not conservative.

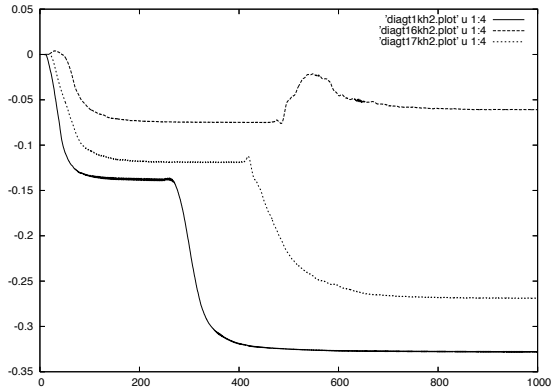
Problem with non conservative Vlasov solver

- When non conservative splitting is used for the numerical solver, the solver is not exactly conservative.
- Does generally not matter when solution is smooth and well resolved by the grid. The solver is still second order and yields good results.
- However: Fine structures develop in non linear simulations and are at some point locally not well resolved by the phase space grid.
- In this case a non conservative solvers can exhibit a large numerical gain or loss of particles which is totally unphysical.
- Lack of robustness.

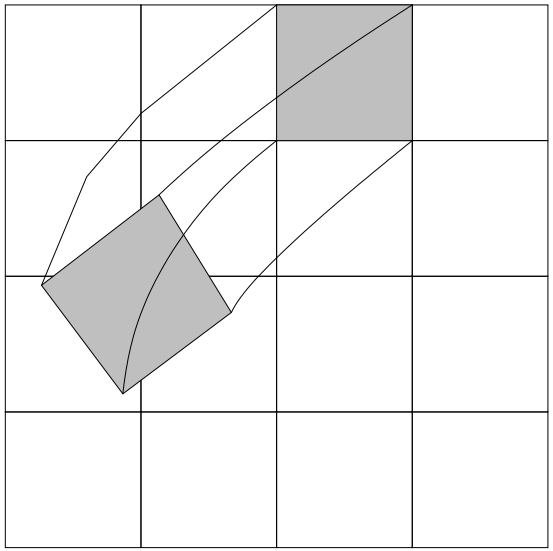
Vortex in Kelvin-Helmholtz instability







Conservative semi-Lagrangian method



- Start from conservative form of Vlasov equation

$$\frac{\partial f}{\partial t} + \nabla \cdot (f\mathbf{a}) = 0.$$

- $\int_V f \, dx \, dv$ conserved along characteristics
- Three steps:
  - High order polynomial reconstruction.
  - Compute origin of cells
  - Project (integrate) on transported cell.

- Efficient with splitting in 1D conservative equations as cells are then defined by their 2 endpoints. A lot more complex for 2D (or more) transport.
- Splitting on conservative form: always conservative.

High order polynomial reconstruction

- We only use the method with 1D splitting with equations in conservative form.
- Unknowns are cell averages: $f_j = \frac{1}{\Delta x} f(x)\,dx$.
- At time step t_n let f_j^n known average value of f^n on cell $[x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}]$ of length $h_j = x_{j+\frac{1}{2}} - x_{j-\frac{1}{2}}$.
- Construct polynomial $p_m(x)$ of degree m such that

$$\frac{1}{h_j} \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} p_m(x)\,dx = f_j^n.$$

Reconstruction by primitive

To this aim look for $\tilde{p}_m(x)$ such that $\frac{d}{dx}\tilde{p}_m(x) = p_m(x)$. Then

$$h_j f_j^n = \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} p_m(x)\,dx = \tilde{p}_m(x_{j+\frac{1}{2}}) - \tilde{p}_m(x_{j-\frac{1}{2}}).$$

Let $W(x) = \int_{x_{\frac{1}{2}}}^{x} \tilde{f}^n(x)\,dx$ a primitive of function $\tilde{f}^n$ with value f_j^n on $[x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}]$. Then $W(x_{j+\frac{1}{2}}) = \sum_{k=1}^{j} h_k f_k^n$ and

$$W(x_{j+\frac{1}{2}}) - W(x_{j-\frac{1}{2}}) = h_j f_j^n = \tilde{p}_m(x_{j+\frac{1}{2}}) - \tilde{p}_m(x_{j-\frac{1}{2}}).$$

Take for $\tilde{p}_m$ interpolating polynomial at points $x_{j+\frac{1}{2}}$ of W

$$\begin{aligned}\frac{1}{h_j}\int_{x_{\frac{1}{2}}}^{x_{j+\frac{1}{2}}} p_m(x)\,dx &= \frac{1}{h_j}(\tilde{p}_m(x_{j+\frac{1}{2}}) - \tilde{p}_m(x_{j-\frac{1}{2}})) \\ &= \frac{1}{h_j}(W(x_{j+\frac{1}{2}}) - W(x_{j-\frac{1}{2}})) = f_j^n,\end{aligned}$$

Choice of interpolation

- What interpolation should be chosen for primitive?
- Lagrange interpolation with centered stencil (used in PFC Filbet, ES, Bertrand JCP 2001).
- ENO type interpolation. Lagrange with varying stencil. Not efficient for Vlasov. WENO has proven a good choice.
- Cubic spline interpolation: cubic polynomial on each cell, globally C^2 $\rightarrow$ reconstructed function is then locally a quadratic polynomial and globally C^1. Linked to cubic spline interpolation for classical semi-Lagrangian method.

Origin of cells and projection

- Compute cell origins:
  - In 1D cell and its origin determined by end points. Compute origin of end points like in classical semi-Lagrangian method.
  - Need to make sure end points do not cross $\rightarrow$ restriction on time step.
- Compute average value of f^{n+1} on cells using

$$\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} f^{n+1}(x)\,dx = \int_{X(t_n;x_{i-\frac{1}{2}},t_{n+1})}^{X(t_n;x_{i+\frac{1}{2}},t_{n+1})} f^{n}(x)\,dx,$$

where $f^n(x)$ is the high order reconstruction.

Link between classical and conservative semi-Lagrangian methods

- For constant coefficient advections it can be shown that

$$\text{C-Lag(2d)} \Longleftrightarrow \text{SL-Lag(2d+1)}$$
$$\text{PSM} \Longleftrightarrow \text{SPL}$$

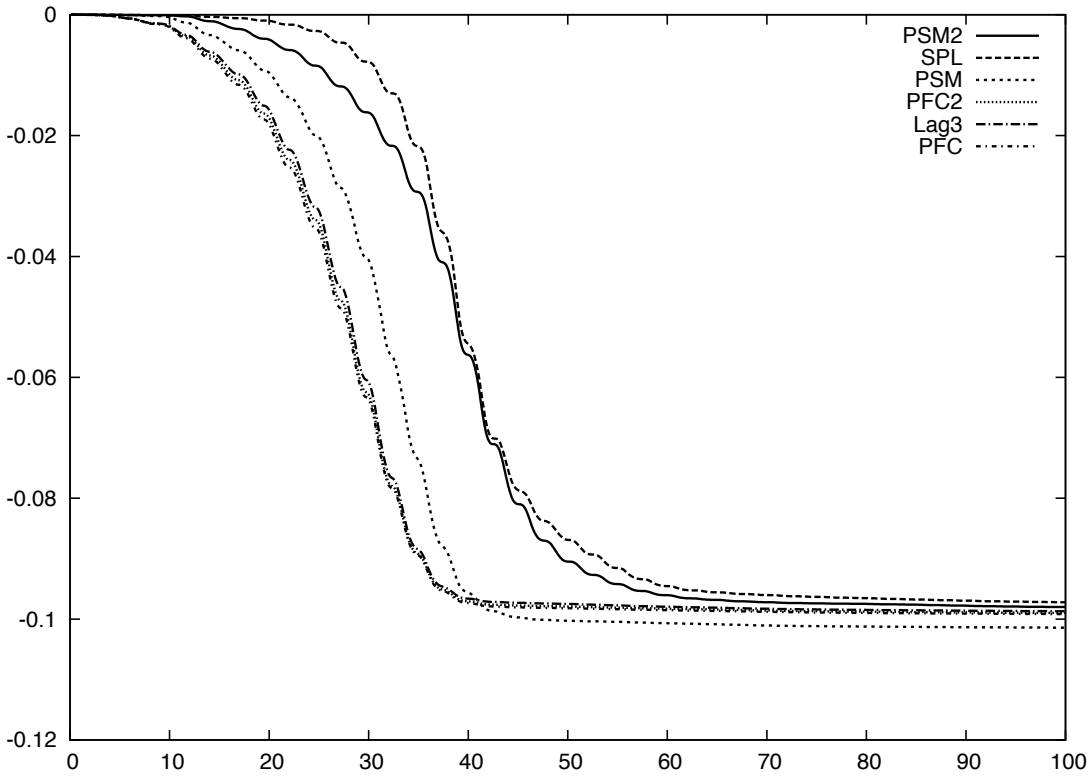
- Consequences :
  1. Classical and conservative semi-Lagrangian methods equivalent for constant coefficients split equations.
  2. The PFC method (Filbet-ES-Bertrand, JCP 2001) corresponds for the Vlasov-Poisson (or Vlasov-Maxwell) systems to a classical semi-Lagrangian method with cubic Lagrange interpolation.

Positive reconstruction

- Physical distribution function is always positive.
- High-order interpolation can lead to negative values in some zones.
- Reconstructed polynomial can be locally modified to remain positive.
- Performed for Lagrange reconstruction in PFC method.
- Introduces a little more dissipativity, but far less as monotonic reconstructions performed in fluid dynamics.

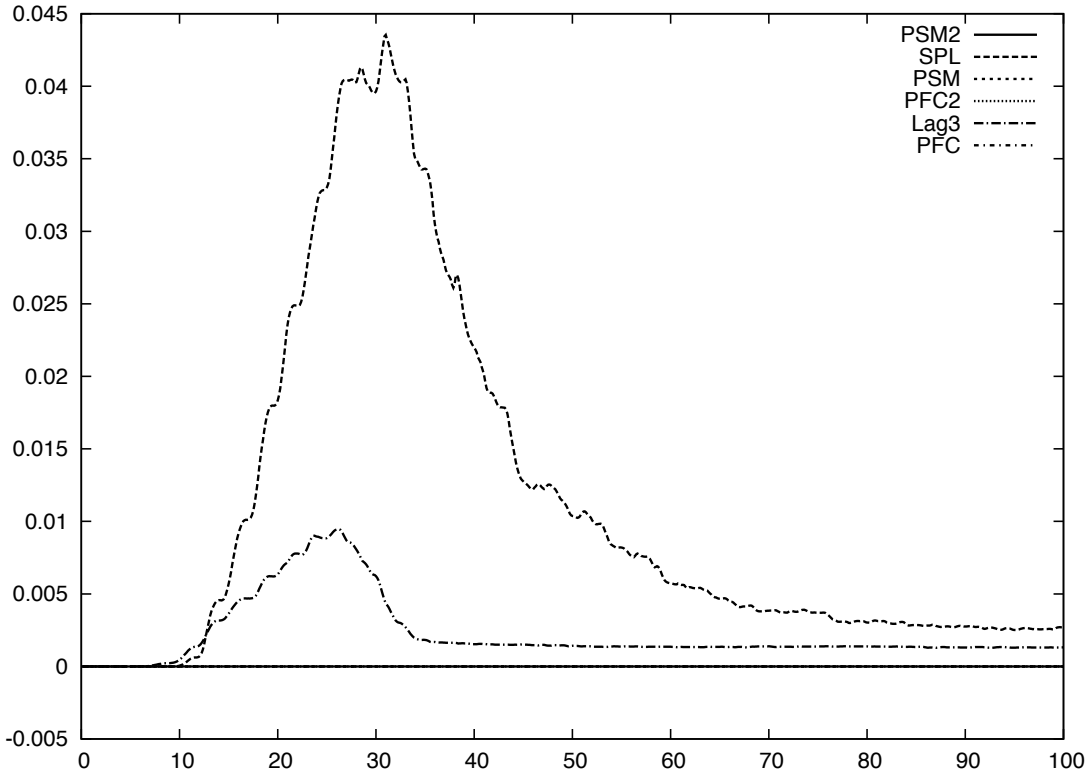
Simulations

Evolution of L^2 norm for $N = 128$ for SLD



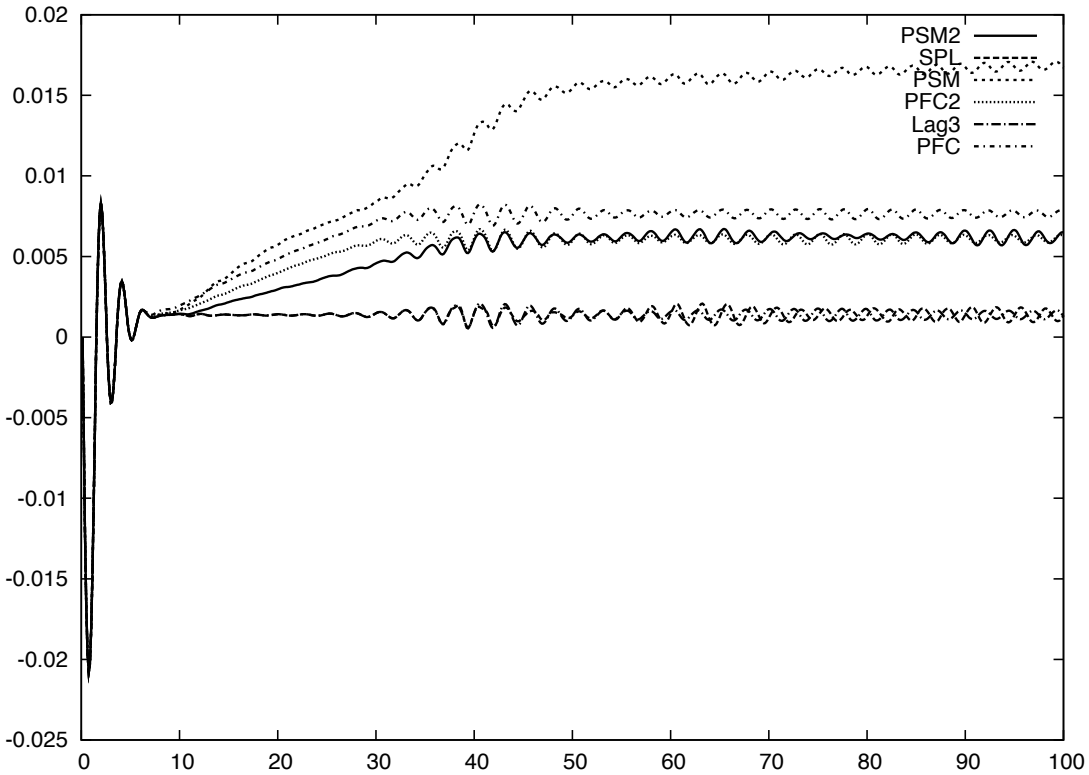
Simulations

Evolution of L^1 norm for $N = 128$ for SLD



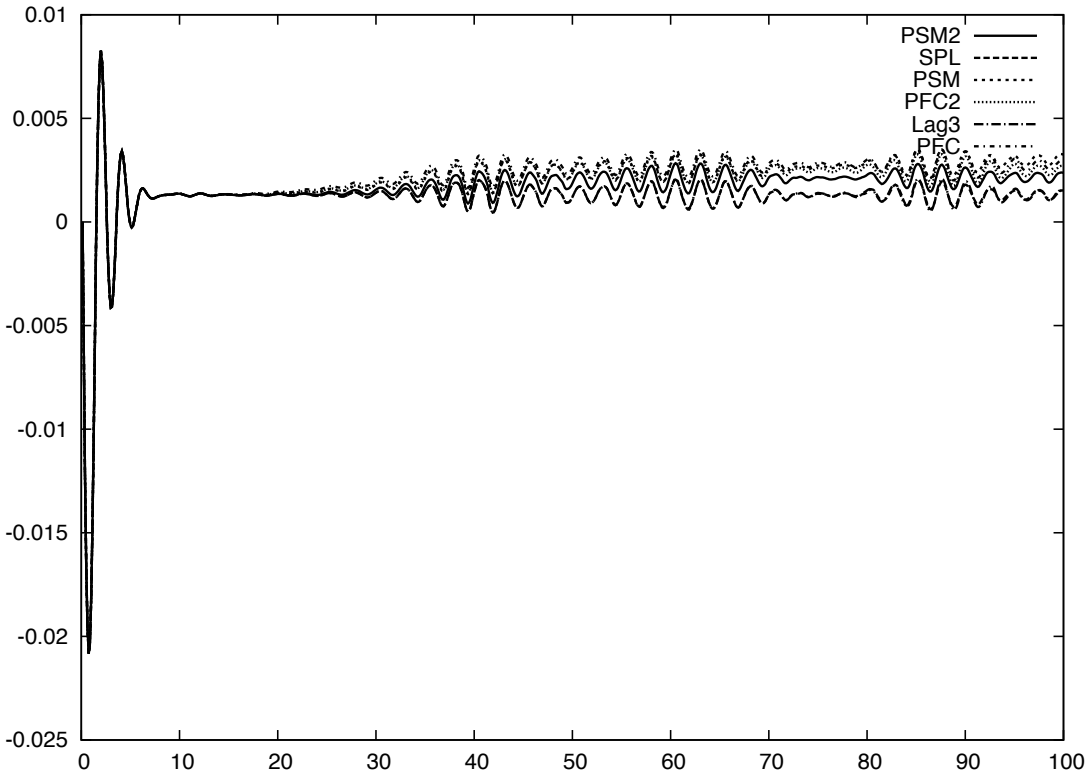
Simulations

Evolution of total energy for $N = 128$ for SLD

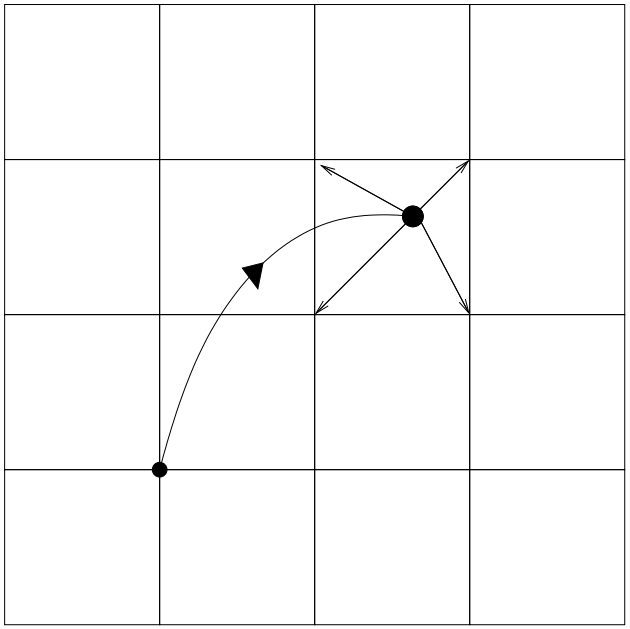


Simulations

Evolution of total energy for $N = 512$ for SLD



The forward semi-Lagrangian method



- f conserved along characteristics
- Characteristics advanced with same time schemes as in PIC method.
- Leap-Frog Vlasov-Poisson
- Runge-Kutta for guiding-center or gyrokinetic

- Values of f deposited on grid of phase space using convolution kernel.
- Identical to PIC deposition scheme but in whole phase space instead of configuration space only.
- Similar to PIC method with reconstruction introduced by Denavit (JCP 1972).

Discrete distribution function

- Function projected on partition of unity basis for conservativity.
- Linear B-splines very diffusive. Not useful in practice.
- Good choice is cubic B-splines.
- f is reprojected on mesh at each time step.
- Between t_n and t_{n+1}

$$f_h(x,v,t) = \sum_{i,j} w_{i,j} B(x - X(t; x_i, v_j, t_n)) B(v - X(t; x_i, v_j, t_n)).$$

The weight $w_{i,j}$ associated to the particle starting from grid point (x_i, v_j) at t_n is coefficient coefficient of spline satisfying interpolation conditions

$$f_h(x_k, v_l, t_n) = \sum_{i,j} w_{i,j} B(x_k - x_i) B(v_k - v_j).$$

- Projection on phase space mesh is obtained with formula

$$f^{n+1}(x_k, v_l) = \sum_{i,j} w_{i,j} B(x_k - X(t_{n+1}, x_i, v_j, t_n)) B(v_l - V(t_{n+1}, x_i, v_j, t_n)).$$

Time advance for Vlasov-Poisson

- As opposite to BSL, trajectories are advanced forward in time.
- Advection field is known at initial time. Standard EDO algorithms can be applied.
- For Vlasov-Poisson, we have $z(t^n) = (x^n, v^n)$, and $E(t^n, z^n) = E(t^n, x^n)$. Separable Hamiltonian.
- Natural scheme is Verlet algorithm, which is second order accurate in time

$$\text{Step1} : v^{n+\frac{1}{2}} - v^n = \frac{\Delta t}{2} E(t^n, x^n),$$

$$\text{Step2} : x^{n+1} - x^n = \Delta t \, v^{n+1/2},$$

$$\text{Step3} : v^{n+1} - v^{n+\frac{1}{2}} = \frac{\Delta t}{2} E(t^{n+1}, x^{n+1}).$$

Time advance for guiding-center

- Explicit Euler and Runge Kutta have been implemented.
- Second-order Runge Kutta method

$$\text{Step1} : X^{n+1} - X^n = \Delta t E^{\perp}(t^n, X^n)$$

$$\text{Step2} : \text{Compute } E^{\perp}(t^{n+1}, X^{n+1})$$

$$\text{Step3} : X^{n+1} - X^n = \frac{\Delta t}{2}\left[E^{\perp}(t^n, X^n) + E^{\perp}(t^{n+1}, X^{n+1})\right]$$

- Fourth order Runge-Kutta

$$\text{Step1} : k_1 = E^{\perp}(t^n, X^n)$$

$$\text{Step2} : \text{Compute } k_2 = E^{\perp}(t^{n+\frac{1}{2}}, X^n + \frac{\Delta t}{2} k_1)$$

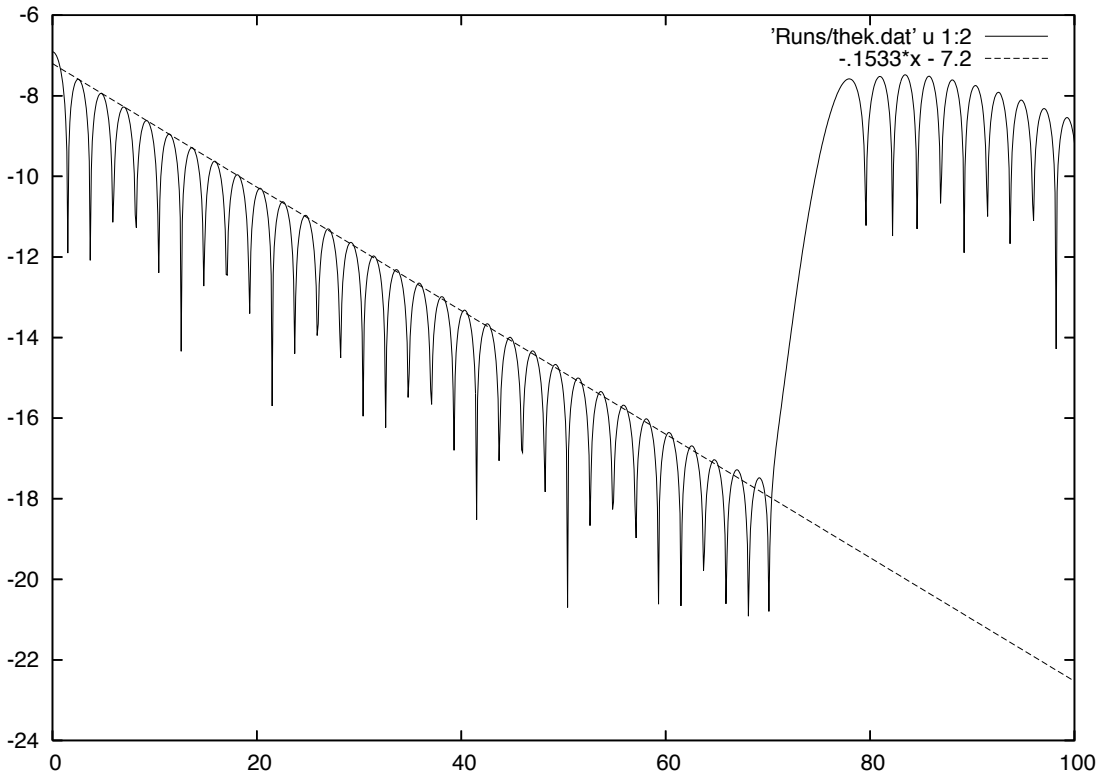
$$\text{Step3} : \text{Compute } k_3 = E^{\perp}(t^{n+\frac{1}{2}}, X^n + \frac{\Delta t}{2} k_2)$$

$$\text{Step4} : \text{Compute } k_4 = E^{\perp}(t^{n+1}, X^n + \Delta t k_3)$$

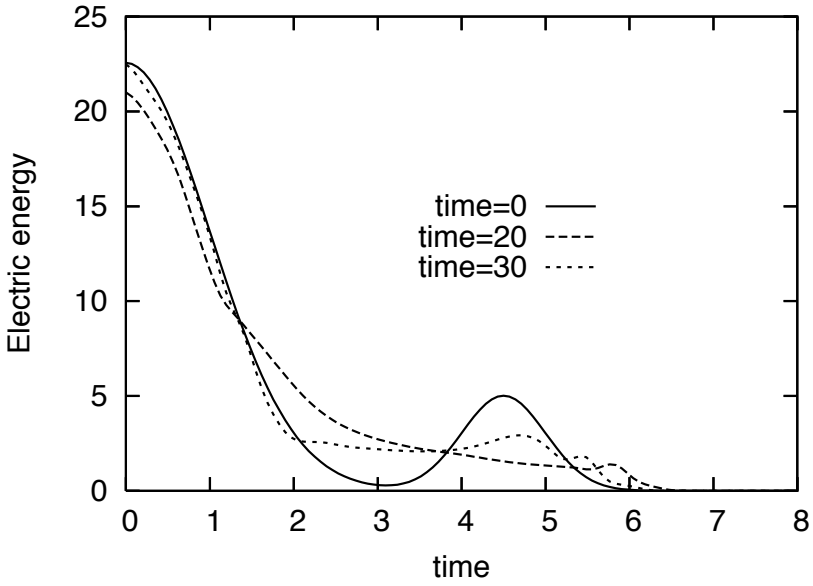
$$\text{Step5} : X^{n+1} - X^n = \frac{\Delta t}{6}\left[k_1 + 2k_2 + 2k_3 + k_4\right]$$

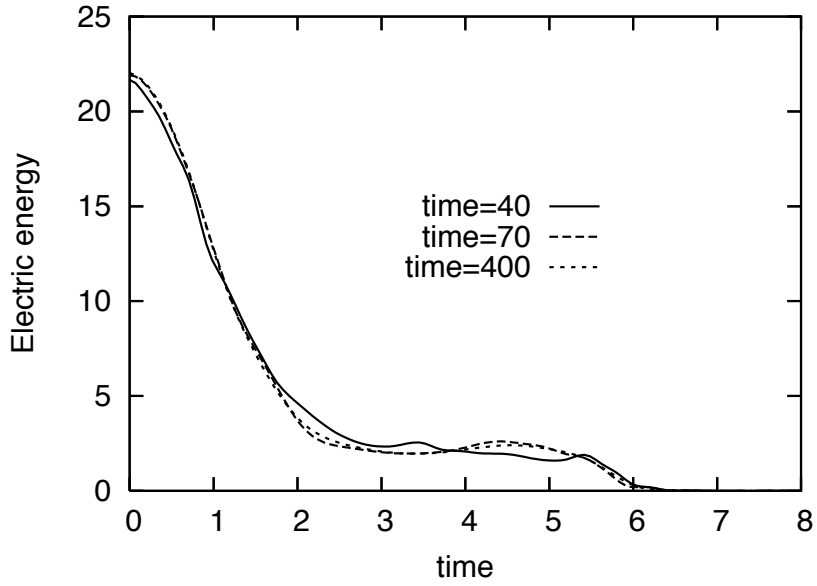
Landau damping

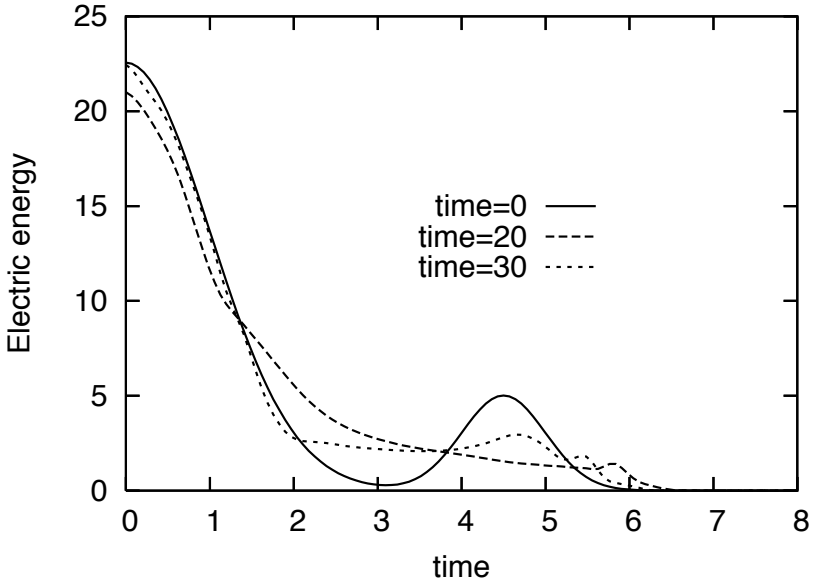
$$f_0(x,v) = (1 + 0.001\cos(kx))\frac{1}{\sqrt{2\pi}}e^{-\frac{v^2}{2}}, \quad L = 4\pi.$$

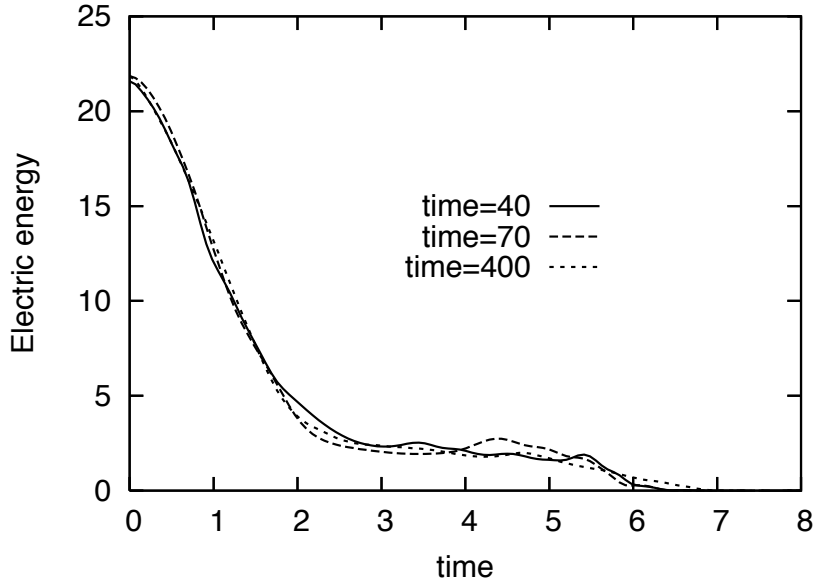


Bump on tail (BSL (top) vs. FSL (bottom))

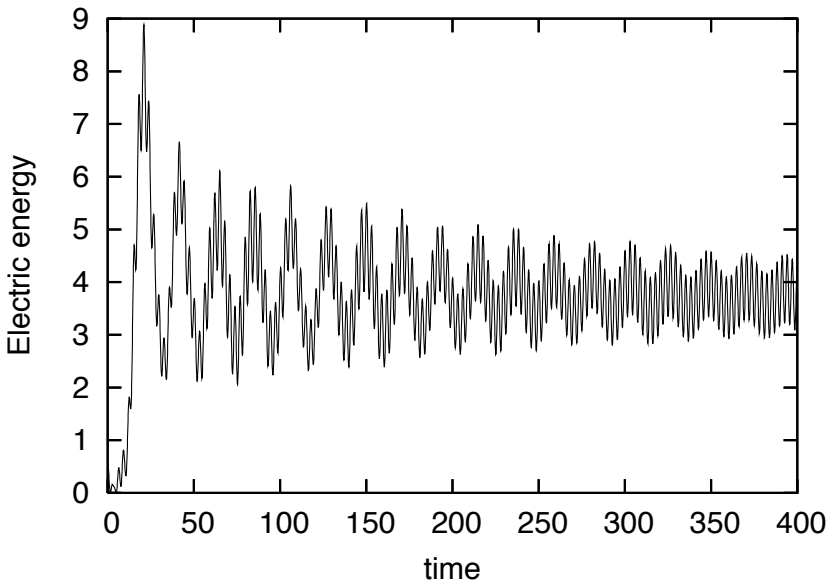


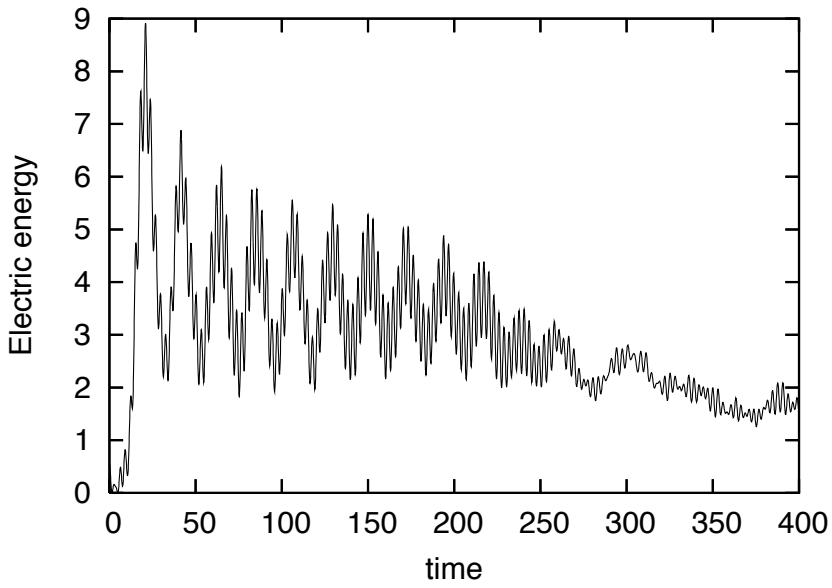




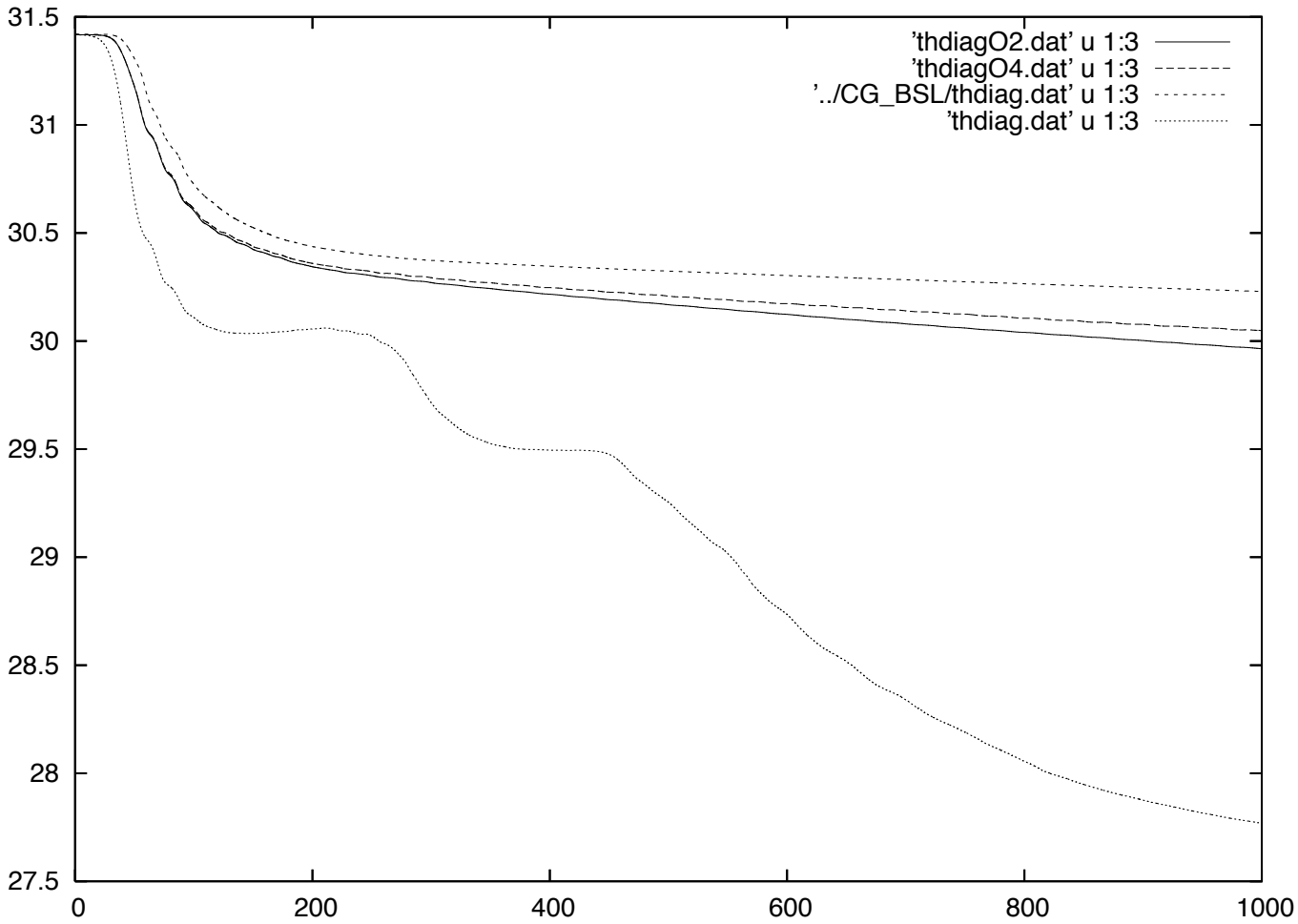


Bump on tail: potential energy (BSL (top) vs. FSL (bottom))





Energy conservation for Kelvin-Helmoltz instability



FSL vs. BSL

- Forward semi-Lagrangian method very promising.
- Accurate description of whole phase-space in particular tail of distribution function and small perturbations.
- A little more diffusive than BSL. Better with smaller time steps.
- Some advantages: classical explicit EDO solver can be used, in particular high order if needed. No need for predictor-corrector or fixed point algorithm.
- Can benefit from charge conserving PIC algorithms (Villasenor-Buneman).

A L^2-norm conserving Finite Difference scheme

- Introduced by Arakawa (1966) for equations of the form

$$\frac{\partial f}{\partial t} + J(\psi, f) = 0,$$

  for 1D Vlasov $\psi = \varphi - \frac{v^2}{2}$ and $J(\psi, f) = \frac{\partial \psi}{\partial x}\frac{\partial f}{\partial v} - \frac{\partial \psi}{\partial v}\frac{\partial f}{\partial x}$,
- Second and fourth order implemented:
  - Particle conservation: $\frac{d}{dt}\int_{\mathbf{R}^2} f(t)\,dx\,dv = 0$.
  - Energy conservation: $\frac{d}{dt}\int_{\mathbf{R}^2} f(t)\psi(t)\,dx\,dv = 0$.
  - Conservation of $\|f\|_{L^2}$.

Stabilization of the method

- Strong oscillations in presence of filamentation.
- Collision model of the form $\frac{\partial f}{\partial t} = \frac{\partial \mathcal{J}}{\partial v}$, where $\mathcal{J}$ is chosen so as to maximize entropy and conserve as many moments $\int f v^k \, dv$ as desired (Robert-Sommeria).
- A Lagrange multiplier technique yields (for $k = 2$):

$$\frac{\partial f}{\partial t} + J(\psi, f) = \alpha \frac{\partial}{\partial v}\left(\frac{\partial f}{\partial v} + A_1 f - A_2 f v\right),$$

with $A_1 = \frac{u_0}{\epsilon - u_0^2/n}$, and $A_2 = \frac{n}{\epsilon - u_0^2/n}$.

Evolution of L^1 and L^2 norms

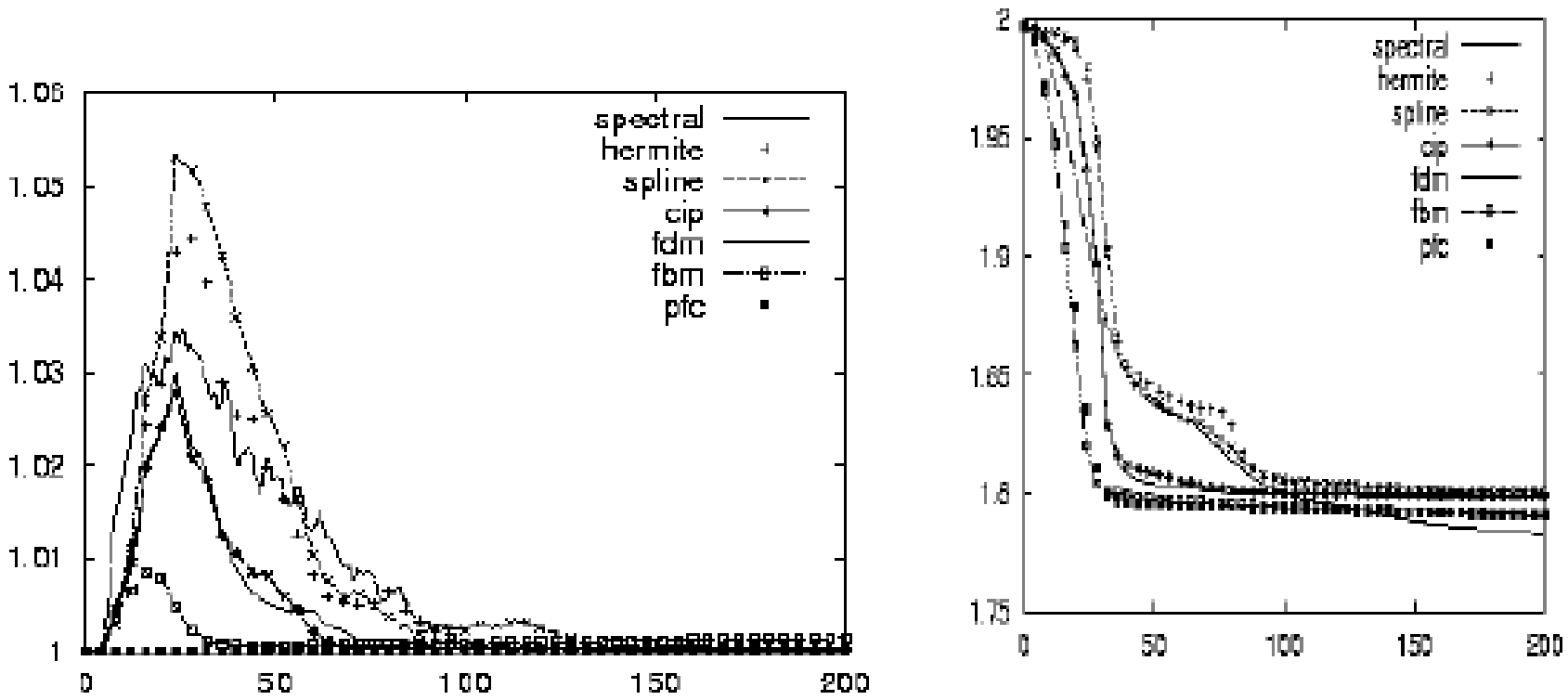


Figure: *Time development of numerical L^1 and L^2 norms for the non linear Landau damping test.*

Evolution of L^2 norm and kinetic entropy

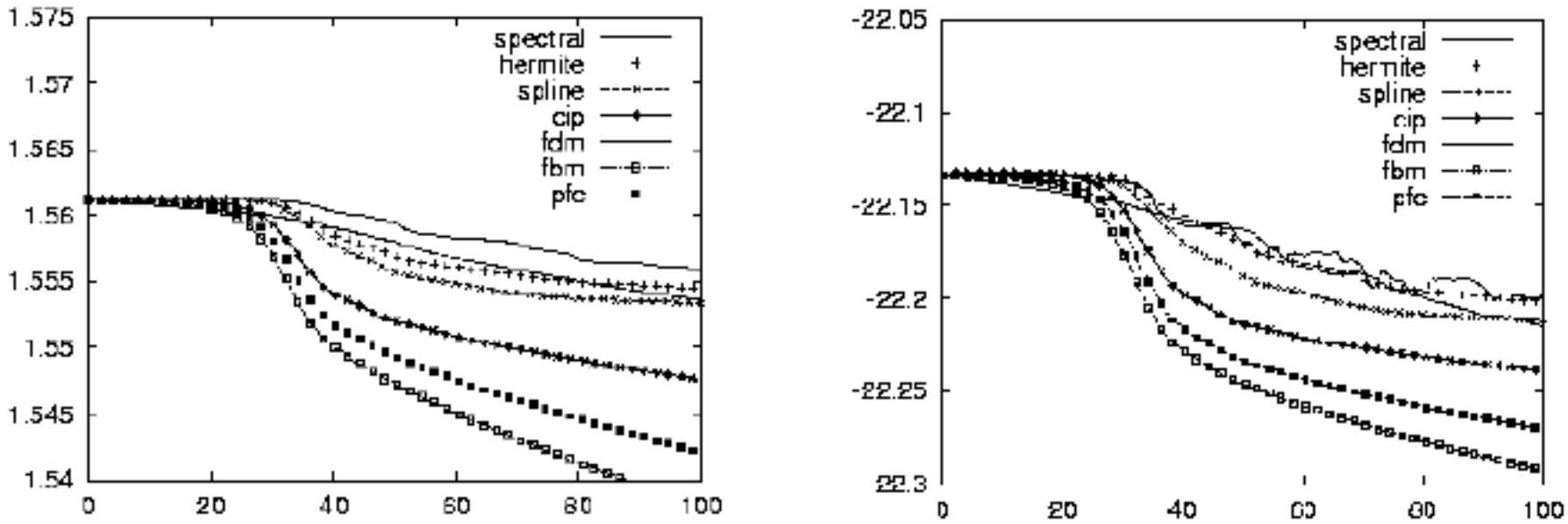


Figure: *Time development of numerical L^2 norm and entropy of $f(t)$ for the two stream instability test.*

Computation time

Num. method	32×32 pts	32×64 pts	32×128 pts
FBM	03.33 sec.	05.39 sec.	10.80 sec.
PFC	03.56 sec.	06.28 sec.	11.20 sec.
FDM	17.22 sec.	35.27 sec.	71.20 sec.
SPECTRAL	04.10 sec.	08.25 sec.	16.90 sec.
CIP	13.83 sec.	21.40 sec.	43.24 sec.
SL SPLINE	06.12 sec.	10.55 sec.	20.90 sec.
SL HERMITE	03.60 sec.	06.90 sec.	11.00 sec.

Outline

Outline

General ideas

- Monitor conservation of invariants:
  - Number of particles: $\int f(x,v,t)\,dxdv$ should be conserved exactly
  - Total energy
  - L^p norms: $\int |f(x,v,t)|^p\,dxdv$ in particular L^1, L^2, L^∞.
- Analytical solution of equations linearized around an equilibrium.
- Analytical solution of linear Vlasov equation: only external field, no self-consistent field.

Outline

Maxwellian equilibrium

- 1D Vlasov-Poisson. Periodic domain of length $L = 2\pi/k_0$ in x. Whole $\mathbb{R}$ in v. $-e$ charge of electron and m its mass.
- Equilibrium distribution solution of Vlasov-Poisson

$$\frac{\partial f^0}{\partial t} + v\frac{\partial f^0}{\partial x} - \frac{e}{m}E^0(x)\frac{df^0}{dv} = 0 \tag{18}$$

$$\frac{dE^0}{dx} = \frac{e}{\epsilon_0}(n_0 - \int_{-\infty}^{+\infty} f^0(x,v)\,dv) \tag{19}$$

with intial condition $f^0(x,v,t) = f_0(x,v)$ and where $n_0 = \frac{1}{L}\int_0^L \int_{-\infty}^{+\infty} f^0(x,v)\,dx\,dv$ is the uniform background neutralizing density.

- Equilibrium $f^0(x,v,t) = f_0(x,v) = f^0(v)$. Solution

$$f^0(v) = \frac{n_0}{2\pi v_{th}}\, e^{-\frac{v^2}{2v_{th}^2}}.$$

Linearization of Vlasov-Poisson around equilibrium

- Hilbert expansion around equilibrium
  $f(x,v,t) = f^0(x,v) + \epsilon f^1(x,v,t)$, $E(x,t) = E^0(x) + \epsilon E^1(x,t)$ (with $E^0(x) = 0$).
- Plug into Vlasov-Poisson, use fact that f^0 is solution and neglect $O(\epsilon^2)$ terms.
- Linearized Vlasov-Poisson equation around Maxwellian equilibrium reads

$$\frac{\partial f^1}{\partial t} + v\frac{\partial f^1}{\partial x} - \frac{e}{m}E^1(x)\frac{df^0}{dv} = 0, \tag{20}$$

$$\frac{dE^1}{dx} = -\frac{e}{\epsilon_0}\int_{-\infty}^{+\infty} f^1(x,v,t)\,dv, \tag{21}$$

with initial condition $f^1(x,v,0) = f_0^1(x,v)$.

Fourier series in x

- Consider $x \mapsto g(x)$ continuous and $L-$periodic. It can be expanded into a Fourier series defined by

$$g(x) = \sum_{k'=-\infty}^{+\infty} \hat{g}(k) e^{ikx}, \quad \hat{g}(k) = \frac{1}{L} \int_0^L g(x) e^{-ikx}\, dx, \quad k = \frac{2\pi}{L} k'.$$

- Apply to f^1. Multiply linearized Vlasov and Poisson by e^{-ikx} and integrate on one period:

$$\frac{\partial \hat{f}}{\partial t}(k, v, t) + ikv\hat{f}(k, v, t) - \frac{e}{m}\hat{E}(k, t)\frac{df^0}{dv} = 0, \tag{22}$$

$$ik\hat{E}(k, t) = -\frac{e}{\epsilon_0} \int_{-\infty}^{+\infty} \hat{f}(k, v, t)\, dv. \tag{23}$$

- Initial condition satisfies $\hat{f}(k, v, 0) = \hat{f}_0(k, v)$.

Laplace transform in time

- Let $\omega \in \mathbb{C}$ and define Laplace transform by

$$\tilde{f}(\omega) = \int_0^{+\infty} f(t) e^{i\omega t}\, dt \quad \text{for } \Im(\omega) > R, \tag{24}$$

- Inverse Laplace transform defined by

$$f(t) = \int_{-\infty+iu}^{+\infty+iu} \tilde{f}(\omega) e^{-i\omega t}\, d\omega \quad \forall u > R, \tag{25}$$

such that $\tilde{u}$ analytical in the half space $\Im(\omega) > R$

Application to linearized Vlasov-Poisson

- Multiplying $\frac{\partial \hat{f}}{\partial t}(k,v,t)$ by $e^{i\omega t}$ and integrating with respect to t between 0 and $+\infty$, we have

$$\int_0^{+\infty} \frac{\partial \hat{f}}{\partial t}(k,v,t)e^{i\omega t}\,dt = [\hat{f}(k,v,t)e^{i\omega t}]_0^{+\infty} - i\omega \int_0^{+\infty} \hat{f}(k,v,t)e^{i\omega t}\,dt$$
$$= -\hat{f}(k,v,0) - i\omega \tilde{f}(k,v,\omega),$$

  where we denote by $\tilde{f}(k,v,\omega)$ the Laplace transform of $\hat{f}(k,v,t)$.

- Whence, Laplace transform of Vlasov and Poisson

$$(-i\omega + ikv)\tilde{f}(k,v,\omega) - \frac{e}{m}\tilde{E}(k,\omega)\frac{df^0}{dv} = \hat{f}_0(k,v), \tag{26}$$

$$\tilde{E}(k,\omega) = \frac{ie}{k\epsilon_0}\int_{-\infty}^{+\infty} \tilde{f}(k,v,\omega)\,dv. \tag{27}$$

Expression of $\tilde{E}(k,\omega)$

- Plug the value of $\tilde{f}$ from Vlasov into Poisson

$$\tilde{E} = \frac{ie}{k\epsilon_0}\int_{-\infty}^{+\infty}\frac{\hat{f}_0(k,v)+\frac{e}{m}\tilde{E}\frac{df^0}{dv}}{-i\omega+ikv}\,dv =$$
$$\frac{ie^2}{k\epsilon_0 m}\tilde{E}\int_{-\infty}^{+\infty}\frac{\frac{df^0}{dv}}{-i\omega+ikv}\,dv+\frac{ie}{k\epsilon_0}\int_{-\infty}^{+\infty}\frac{\hat{f}_0(k,v)}{-i\omega+ikv}\,dv.$$

- Let

$$D(k,\omega)=1-\frac{e^2}{k^2\epsilon_0 m}\int_{-\infty}^{+\infty}\frac{\frac{df^0}{dv}}{v-\frac{\omega}{k}}\,dv,\quad N(k,\omega)=\frac{e}{k^2\epsilon_0}\int_{-\infty}^{+\infty}\frac{\hat{f}_0(k,v)}{v-\frac{\omega}{k}}\,dv.$$

- Then

$$\tilde{E}(k,\omega)=\frac{N(k,\omega)}{D(k,\omega)},\quad \tilde{f}(k,v,\omega)=i\frac{\frac{e}{m}\tilde{E}(k,\omega)\frac{df^0}{dv}+\hat{f}_0(k,v)}{(\omega-kv)}.$$

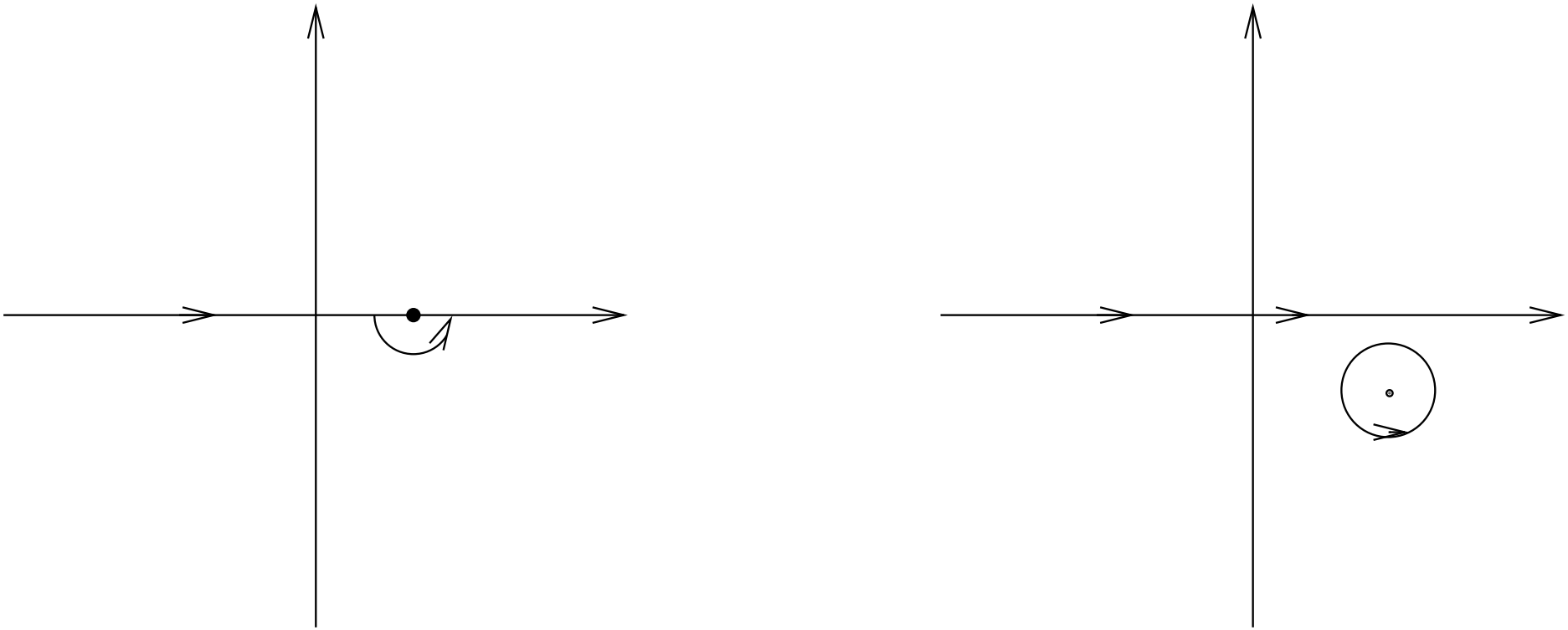
Computation of velocity integrals

- In the velocity integrals in D and N with have terms of the form

$$G(\omega) = \int_{-\infty}^{+\infty} \frac{g(v)}{v - \frac{\omega}{k}}\, dv.$$

  $G(\omega)$ is analytical (assuming g is) for $\Im(\omega) > 0$.
- Inverse Laplace transform well defined in this case.
- Pole for $\Im(\omega) = 0$. We need to extend the function to negative values of $\Im(\omega)$ by analytical continuation.
- Integral in complex plane does not depend on contour as long as there is no pole between chose contours $\rightarrow$ instead of choosing real line as contour for $\Im(\omega) > 0$ any line unerneath the real line can be chosen.
- For analytical continuation for $\Im(\omega) \leq 0$ just take integration line parallel to real line strictly below ω or any deformation of this contour staying underneath pole.

Examples of contours



Contour for pole on real line (left), underneath real line (right)

The plasma dispersion function

As plasma equilibria are strongly linked to Maxwellians, there is a function that naturally appears in calculation of dispersion equations for kinetic equations.
Fried and Conte called it the plasma dispersion function:

$$Z(\zeta) = \frac{1}{\sqrt{\pi}} \int_\gamma \frac{e^{-z^2}}{z-\zeta}\, dz, \tag{28}$$

where γ is any contour passing below the pole ζ.

Properties of Z function

Z verifies the following properties

$$Z(\zeta) = \frac{1}{\sqrt{\pi}} \left[\mathrm{Pr} \int_{-\infty}^{+\infty} \frac{e^{-(u+\zeta)^2}}{u} \, du + i\pi e^{-\zeta^2} \right], \tag{29}$$

$$= \sqrt{\pi} e^{-\zeta^2} [i - erfi(\zeta)], \tag{30}$$

where $erfi(\zeta) = \frac{2}{\sqrt{\pi}} \int_0^\zeta e^{t^2} \, dt$ is the complex error function. For $b \in \mathbb{R}$

$$\mathrm{Pr} \int_{-\infty}^{+\infty} \frac{g(u)}{u-b} \, du = \lim_{\delta \to 0} \left[\int_{-\infty}^{b-\delta} \frac{g(u)}{u-b} \, du + \int_{b+\delta}^{+\infty} \frac{g(u)}{u-b} \, du \right]$$

is the Cauchy principal value.
The derivatives of Z verify:

$$Z'(\zeta) = -2(1 + \zeta Z(\zeta)),$$
$$Z''(\zeta) = 4\zeta - 2(1 - 2\zeta^2) Z(\zeta).$$

Expression of $E(x,t)$

- We got an exact expression for N and D and therefore of E (and f).
- The electric field can be computed by inverse Laplace and Fourier transform of this expression.
- Inverse Laplace transform

$$\hat{E}(k,t) = \frac{1}{2i\pi} \int_{-\infty+iu}^{+\infty+iu} \tilde{E}(k,\omega) e^{-i\omega t}\, d\omega.$$

- Can be computed using the residue theorem assuming that $\tilde{E}(k,\omega)$ is analytical apart from a finite number of poles. Then

$$\hat{E}(k,t) = \sum_j \mathrm{Res}_{\omega=\omega_j}(\tilde{E}(k,\omega)) e^{-i\omega_j t},$$

the sum being made over the poles which can be computed numerically.

Computation of the residue

$$\hat{E}(k,t) = \sum_j Res_{\omega=\omega_j} \tilde{E}(k,\omega) e^{-i\omega t}, \text{ with } \tilde{E}(k,\omega) = \frac{N(k,\omega)}{D(k,\omega)}$$

and the ω_j are the roots of $D(k,\omega) = 0$ for fixed k. There are in general several roots of $D(k,\omega)$ for a given k. Only the one with largest imaginary part matters after some time.
The residue can be computed thanks to the Taylor expansion of $D(k,\omega)$ in a neighborhood of ω_j

$$D(k,\omega) = D(k,\omega_j) + (\omega - \omega_j)\frac{\partial D}{\partial \omega}(k,\omega_j) + O((\omega-\omega_j)^2),$$

and so, if ω_j is a simple root, we have $D(k,\omega_j) = 0$ and $\frac{\partial D}{\partial \omega}(k,\omega_j) \neq 0$. So that

$$Res_{\omega=\omega_j}\left(\frac{N(k,\omega)}{D(k,\omega)}e^{-i\omega t}\right) = \lim_{\omega\to\omega_j}((\omega-\omega_j)\frac{N(k,\omega)}{D(k,\omega)}e^{-i\omega t}) = \frac{N(k,\omega_j)}{\frac{\partial D}{\partial \omega}(k,\omega)}e^{-i\omega_j t}.$$

Landau damping

Initial condition

$$f_0(x,v) = (1+\epsilon\cos(kx))\frac{1}{\sqrt{2\pi}}e^{-\frac{v^2}{2}}, \quad L = 4\pi.$$

So that $f^0(v) = \frac{1}{\sqrt{2\pi}}e^{-\frac{v^2}{2}}$ and initial perturbation

$f_0^1(x,v) = \epsilon\cos(kx))\frac{1}{\sqrt{2\pi}}e^{-\frac{v^2}{2}}$ Charge density is

$$\rho_0^1(x) = \int f_0^1(x,v)\,dv = \epsilon\cos(kx) = \epsilon\frac{e^{ikx}+e^{-ikx}}{2}.$$

Only 1 and -1 modes are not 0. The electric field verifies $\frac{dE}{dx}(x,0) = \rho_0^1(x)$, so that $E(x,0) = \frac{\epsilon}{k}\sin(kx)$. We have

$$D(k,\omega) = 1 - \frac{1}{2}\frac{\omega_p^2}{k^2 v_{th}^2}Z'(\frac{\omega}{\sqrt{2}v_{th}k}),$$

$$N(k,\omega) = g(k)\frac{n_0 e}{k^2\epsilon_0}\frac{1}{\sqrt{2}v_{th}}Z(\frac{1}{\sqrt{2\pi}v_{th}}).$$

So that

$$\frac{N(k,\omega)}{\frac{\partial D}{\partial \omega}(k,\omega)} = -2g(k)\frac{m}{e}kv_{th}^2\frac{Z(\frac{\omega}{\sqrt{2}v_{th}k})}{Z''(\frac{\omega}{\sqrt{2}v_{th}k})} \tag{31}$$

$$= -g(k)\frac{m}{e}kv_{th}^2\frac{Z(\frac{\omega}{\sqrt{2}v_{th}k})}{2\frac{\omega}{\sqrt{2}v_{th}k} - (1-\frac{\omega^2}{v_{th}^2k^2})Z(\frac{\omega}{\sqrt{2}v_{th}k})}. \tag{32}$$

Dominant roots of dispersion relation

k	ω_j	$\frac{N(k,\omega_j)}{\frac{\partial D}{\partial \omega}(k,\omega_j)}$
0.5	$\pm 1.4156 - 0.1533i$	$0.3677\, e^{\pm i 0.536245}$
0.4	$\pm 1.2850 - 0.0661i$	$0.424666\, e^{\pm i 0.3357725}$
0.3	$\pm 1.1598 - 0.0126i$	$0.63678\, e^{\pm i 0.114267}$
0.2	$\pm 1.0640 - 5.510 \times 10^{-5} i$	$1.129664\, e^{\pm i 0.00127377}$

Denote by $\omega_r = \Re(\omega_j)$, $\omega_i = \Im(\omega_j)$, r the amplitude of $\frac{N(k,\omega_j)}{\frac{\partial D}{\partial \omega}(k,\omega_j)}$ and φ its phase.

Linear solution of Landau damping

- Maple code for dispersion relation

$$k := 0.2;$$
$$dd := w- > 1 + (1/k**2) * (1 + sqrt(Pi/2) * w/k$$
$$* exp(-w**2/(2 * k**2)) * (-erfi(w/(sqrt(2) * k)) + I));$$
$$root2 := fsolve(dd(w), w = 1, complex);$$

- Dominant solution for E

$$\begin{aligned}\hat{E}(k,t) &\approx re^{i\varphi}e^{-i(\omega_r+i\omega_i)t} + re^{-i\varphi}e^{-i(-\omega_r+i\omega_i)t},\\ &= re^{\omega_i t}(e^{-i(\omega_r t-\varphi)} + e^{i(\omega_r t-\varphi)}),\\ &= 2re^{\omega_i t}\cos(\omega_r t-\varphi).\end{aligned}$$

Then as $\hat{E}(-k,t) = -\hat{E}(k,t)$, we have

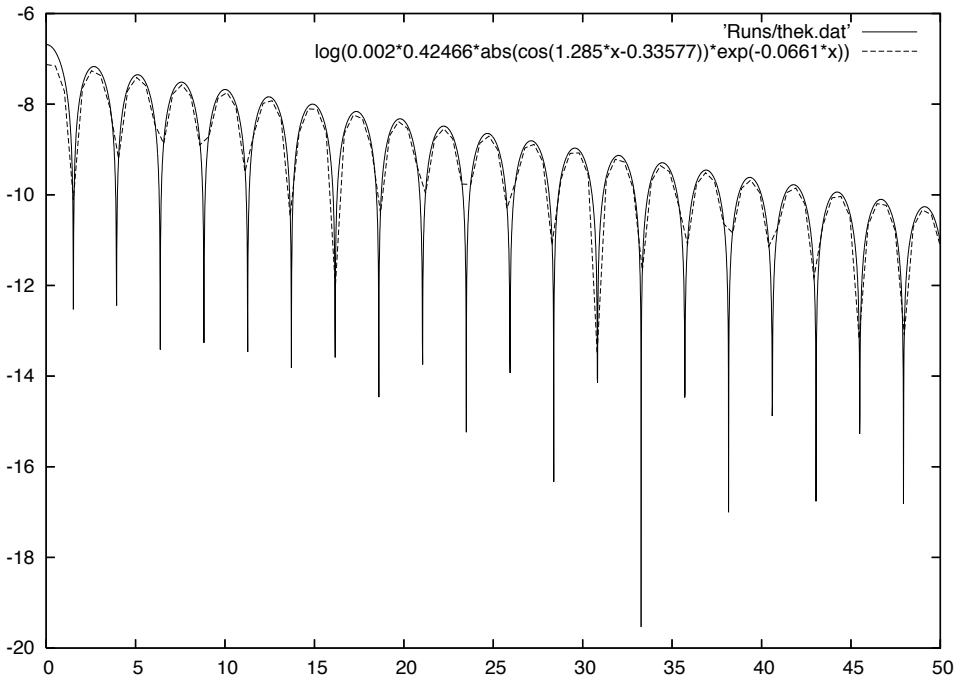
$$E(x,t) \approx 4\epsilon re^{\omega_i t}\sin(kx)\cos(\omega_r t-\varphi).$$

Analytical solution and computed solution for k=0.4

$$f_0(x,v) = (1 + 0.001\cos(kx))\frac{1}{\sqrt{2\pi}}e^{-\frac{v^2}{2}}, \quad L = 4\pi.$$

Dominant analytical solution

$$\hat{E}(k,t) = 0.002 \times 0.424666 e^{0.0661t}\cos(1.2850t - 0.3357725)$$



Two stream instability (1/2)

- Dispersion relation

$$D(k,\omega) = 1 + \frac{\omega_p^2}{2k^2 v_{th}^2}\,[2+$$
$$\sqrt{\frac{\pi}{2}}\left(\frac{\omega}{v_{th}k} - \frac{v0}{v_{th}}\right) e^{-\frac{(\frac{\omega}{k}-v_0)^2}{2v_{th}^2}} (i - \mathrm{erfi}(\frac{\frac{\omega}{k}-v_0}{\sqrt{2}v_{th}})$$
$$\sqrt{\frac{\pi}{2}}\left(\frac{\omega}{v_{th}k} + \frac{v0}{v_{th}}\right) e^{-\frac{(\frac{\omega}{k}+v_0)^2}{2v_{th}^2}} (i - \mathrm{erfi}(\frac{\frac{\omega}{k}+v_0}{\sqrt{2}v_{th}})\Big],$$

- Some roots for $k = 0.2$

v_0	ω	ω
1.3	$0.02115i$	$1.1648 - 0.00104i$
2.4	$0.2258i$	$1.3390 - 0.00242i$
3.0	$0.2845i$	$1.446 - 0.00299i$

Two stream instability (2/2)

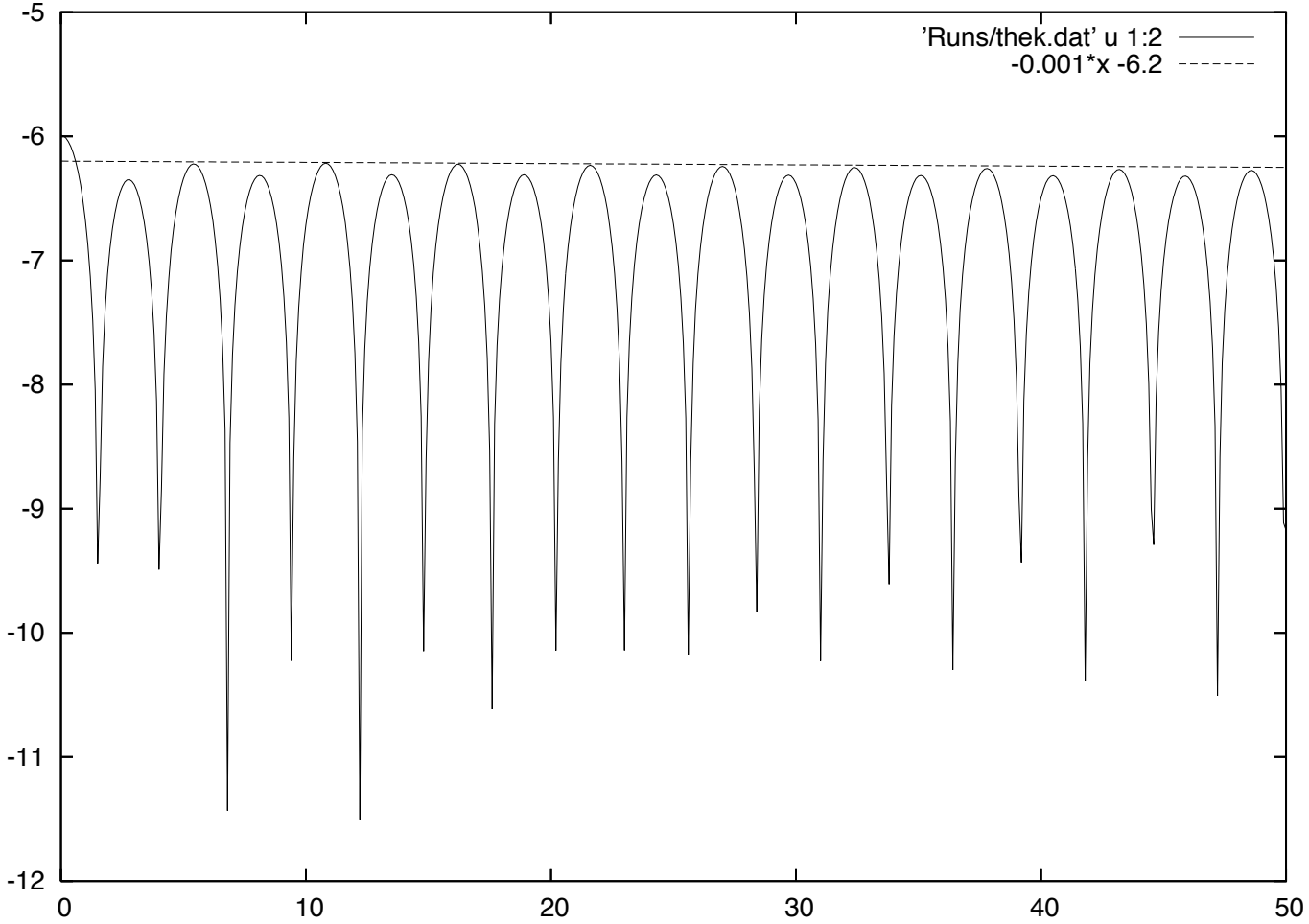
- Maple code

$$
\begin{aligned}
k &:= 0.2; v0 := 3; \\
dd &:= w->1+(1/(2*k**2))*((1+sqrt(Pi/2)*(w/k-v0) \\
&\quad *exp(-(w/k-v0)**2/2)*(-erfi((w/k-v0)/sqrt(2))+I)) \\
&\quad +(1+sqrt(Pi/2)*(w/k+v0)*exp(-(w/k+v0)**2/2) \\
&\quad *(-erfi((w/k+v0)/sqrt(2))+I))); \\
root2 &:= fsolve(dd(w), w=0, complex); \\
root3 &:= fsolve(dd(w), w=1, complex);
\end{aligned}
$$

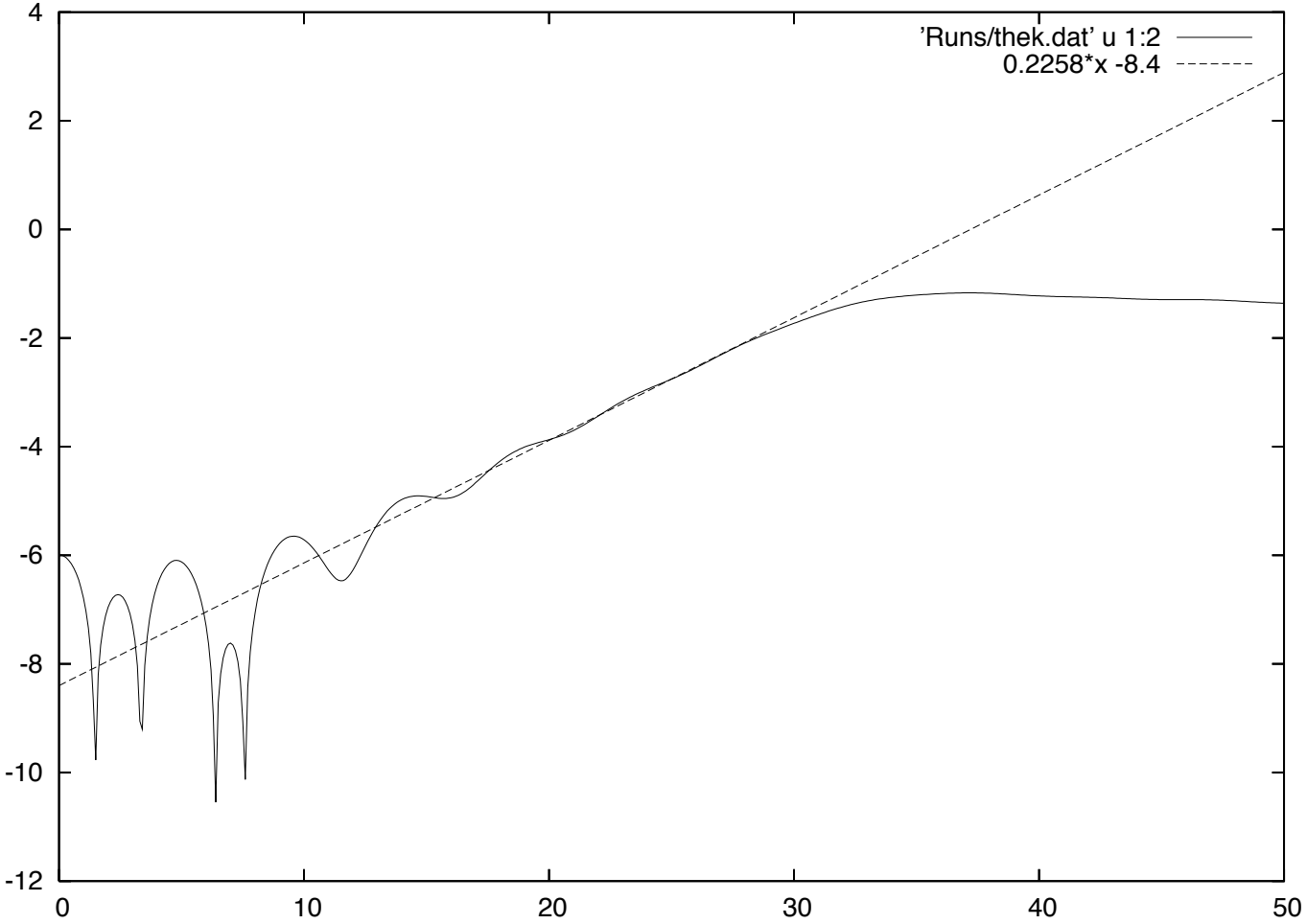
- Initial condition

$$
f_0(x,v) = (1+0.001\cos(kx))\frac{1}{2\sqrt{2\pi}}(e^{-\frac{(v-v0)^2}{2}}+e^{-\frac{(v+v0)^2}{2}}), \quad L=\frac{2\pi}{k}.
$$

k=0.2; v0=1.3



k=0.2; v0=2.4



k=0.2; v0=3