Spherical Harmonics Expansion of the Vlasov-Poisson initial boundary value problem

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Abstract

We derive and analyze the 'SHE' (Spherical Harmonics Expansion) type system of equations coupled in energy. We also show that diffusive behavior occurs on long time and distance scales and we determine the diffusion tensor. The analysis is based on the governing kinetic equations arising in electron transport in semiconductors.

Key Words: Vlasov-Poisson system, Diffusion equation, Spherical Harmonics Expansion model, Semiconductor.
AMS Subject Classifications: 35Q20, 76P05, 82A70, 78A35, 41A60.

1 Introduction

This paper presents an extension of a previous work by P. Degond and S. Mancini [14] in which a diffusion model describing the evolution of an electron gas confined between two parallel planes by a strong magnetic field is derived. Electrons colliding against the plane are supposed to be reemitted following a combination of diffusive and specular elastic laws. When the distance between these planes goes to zero, the distribution function (solution of a Vlasov-Poisson equations) is proved to converge to the solution of the macroscopic model, the so-called SHE model which is a diffusion model for the energy distribution function. This situation arises in a certain kind of ion propellers for satellites. SHE is an acronym for Spherical Harmonics Expansion coming from its earlier derivation by physicists [16]. These models appeared to be a good compromise between a very accurate description of physical phenomena by kinetic models and less numerically expensive macroscopic models such as Drift-Diffusion or Energy-Transport models. Whereas kinetic equations deal with functions of seven variables (six variables in the phase and time) and classical macroscopic models deal with functions of four variables (three in the position, space and time), SHE models introduce an intermediate one dimensional variable replacing the velocity and which appears as the kinetic energy associated to the velocity of a particle. This makes of SHE models quite reliable models at a rather low numerical cost. The SHE model in literature has been derived from the Boltzmann equation first by P. Dmitruk, A. Saul, and L. Reyna [16].

The mathematical theory of the diffusion approximation started with the seminal papers [8], [5] in the context of neutron transport after the formal theory had been set up by Hilbert, Chapman, Enskog and co-workers (see for instance [11]). Their approaches were later extended by many authors. In this work, we are interested in a physical situation where the observation length scale is large compared to the mean free path while the observation time is large compared to the characteristic time evolution of the particle.

The mathematical study of the vanishing mean free path limit and of diffusion approximation is by now a classical problem with applications in various fields of physics. We refer among others to [3], [24] and for recent [15] and for SHE models [7], [14] and the references therein for details as the physical background concerning these models. Concerning SHE models this study has been derived from the Boltzmann equation first by N. Ben Abdallah and P. Degond [7] under the assumption that the dominant scattering mechanisms are elastic collisions.

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The present work has been inspired by [3], [15], [14], [13], but differs from them in the nature of collisions of electrons with the wall and the presence of a magnetic field. In [14] and [13], elastically diffusive collisions at the plates is considered: the particles are reemitted with the same energy as their incident one, and with a random velocity direction. The force field is the gradient of a smooth potential function which is assumed given, independent of time and of the mean free path, and amounts to assume vary over the macroscopic scale only. As a consequence, the large time behavior of the distribution function is given by an energy distribution function depending on the parallel components relative to the boundary of the domain and satisfying a diffusion equation in both position and energy. Here, we treat the self-consistent problem, the electric potential satisfying the Poisson equation. Hence, in contrast of [14] and [13], the presence of a quadratic term in Vlasov equation gives rise to some singular term which adds additional technical difficulties. The existence and regularity of the solution to the asymptotic model which is now constituted of the diffusion equation coupled to the Poisson equation, are not immediate, nevertheless verified. Our main contribution in this paper is to give a rigorous proof of this convergence.

In [3] the authors showed that collisions with a boundary can drive a system of neutral particles towards a diffusion regime and the diffusivity is infinite. It has been shown in [15] that a logarithmic time re-scaling restores a finite diffusivity. Nevertheless the divergence which appears in [3] and waived in [15] does not appear in our work because of the presence of a strong magnetic field directed transversally to the plates. Particle motion then does not occur along straight lines, but rather along the plates whose axes are parallel to the magnetic field lines.

We note also, that some results in the direction of this paper have been obtained, without the boundary conditions, and where collisions are taken into account through the non linear Pauli operator by T. Goudon et *al* in [19].

2 Setting up the problem and the main result

Let us recall the physical background carefully. Our starting point is a Vlasov equation for the electron distribution. The microscopic model describes the evolution of particles limited between two parallel plates. In the region of this space, the particles are submitted to a force field. The physical space where the electrons evolve is $\Omega \times \mathbb{R}^3$ where $\Omega = [0,1] \times \mathbb{R}^2$. The position vector is denoted by $X = (x, y, z)$ and we split X into its perpendicular $x \in [0,1]$ and parallel components $\underline{\xi} = (y, z) \in \mathbb{R}^2$ relative to the boundary Γ. We denote by $v = (v_x, v_y, v_z) \in \mathbb{R}^3$ the position and the velocity vectors of an electron between the planes. We decompose $v = (v_x, \underline{v})$, where v_x is the velocity component parallel to the x-axis and $\underline{v} = (v_y, v_z) \in \mathbb{R}^2$ is the component parallel to the plates. The electrons are subject to a magnetic field transverse to the plates $(B(\underline{\xi}), 0, 0)$ depending upon $\underline{\xi}$ and to a potential force field parallel to the plates, $(0, E_y(\underline{\xi}), E_z(\underline{\xi}))$ depending only upon $\underline{\xi}$ and satisfying $\underline{E} = -\nabla_{\underline{\xi}}\phi$, where $\nabla_{\underline{\xi}}$ denotes the 2-dimensional gradient with respect to $\underline{\xi}$ and ϕ is the potential. (cf. Fig.1). Therefore, they are supposed to move between the plates according to a collisional transport equation.

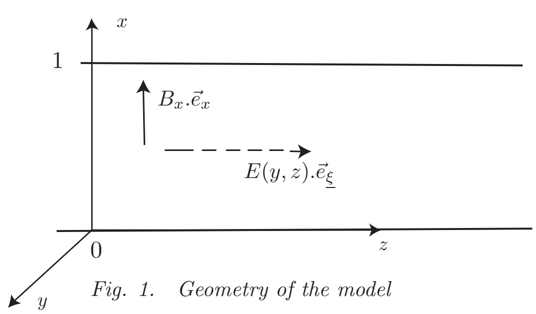

Fig. 1. Geometry of the model

Here $f(X, v, t)dXdv$ is understood to give the number of electrons that occupy any infinitesimal volume $dXdv$ at the point (X, v) which is given as the solution of the boundary value problem:

$$\partial_t f + \left(\underline{v} \cdot \nabla_{\underline{\xi}} + \underline{E} \cdot \nabla_{\underline{v}}\right) f + v_x \frac{\partial f}{\partial x} - (\underline{v} \times B) \cdot \nabla_{\underline{v}} f = 0 \tag{2.1}$$

where $(\underline{v} \times B) = (0, v_z B, -v_y B)$, while the electrostatic potential ϕ solves the Poisson equation

$$\underline{E} = -\nabla_{\underline{\xi}}\phi, \qquad -\Delta_{\underline{\xi}}\phi = \int_0^1\int_{\mathbb{R}^3} f dxdv - C(\underline{\xi}). \tag{2.2}$$

We shall assume that $C(\underline{\xi})$ is a given (nonnegative) function which is regular, say $C^\infty(\Omega)$.

Let us specify the proper boundary conditions to be considered. We introduce the set $\Theta = \Omega \times \mathbb{R}^3$, and consider its boundary $\Gamma = \gamma \times \mathbb{R}^3$, where $\gamma = \{0,1\} \times \mathbb{R}^2$; we denote $\mathbb{R}^3_{\pm} = \{v \in \mathbb{R}^3 \ : \ \pm v_x > 0\}$ and the following *incoming* and *outgoing* subsets of Γ (respectively representing incoming and outgoing particles to the domain Ω):

$$\Gamma_- = \left(\{0\} \times \mathbb{R}^3 \times \mathbb{R}^3_+\right) \cup \left(\{1\} \times \mathbb{R}^3 \times \mathbb{R}^3_-\right)$$

where for instance, $\left(\{0\} \times \mathbb{R}^3 \times \mathbb{R}^3_+\right)$ represents electrons entering the region $0 < x < 1$, through the boundary plane $x = 0$. The boundary Γ_+ is obtained by reversing the inequalities. We introduce the traces (or boundary values) of f on $\Gamma \times \mathbb{R}^3$ according to:

$$\gamma(f) = f|_{\{x=0,1\}}\,, \qquad \gamma^{\pm}(f) = f|_{\{x=0,1,\ \pm v_x>0\}}\,, \tag{2.3}$$

$\gamma^+(f)$ is the distributional function of the particles exiting the domain Ω at the boundary Γ, while $\gamma^-(f)$ is that of incoming particles. We suppose, finally that it is a function of outgoing trace through an operator $\mathcal{K}$ which expresses the interaction of the particles:

$$\gamma^-(f) = \mathcal{K}(\gamma^+(f)). \tag{2.4}$$

Introducing the operator $\mathcal{B}(\gamma(f)) = \gamma^-(f) - \mathcal{K}(\gamma^+(f))$, we have $\mathcal{B}(\gamma(f)) = 0$. The boundary condition for ϕ is:

$$\lim_{|\underline{\xi}|\to\infty} \phi(\underline{\xi}, t) = 0, \quad \text{a.e.} \ \ t > 0. \tag{2.5}$$

The homogeneous boundary condition for ϕ means that the system of electrons is in equilibrium at infinity. We refer to [9] for further properties of these boundary conditions and physical interpretations. In this work some important relations valid for these boundary conditions are derived.

Our model is based on the assumption that the distance between the plates is small compared to the characteristic length of the horizontal motion. Since we are looking for a diffusion process, this suggests to rescale the longitudinal coordinate according to $x' = \alpha x$, where $\alpha = O(1)$ is the small parameter. But then, the number of collisions between a typical particle and the plates per unit of unscaled time is large: if we assume that the collisions between the particles and the plates are a purely isotropic process, there is no reason to expect that, at a large scale, the particles would follow a horizontal drift. Hence, in order to observe a horizontal motion at a large scale, it is logical to rescale the time variable as $t' = \alpha^2 t$. After rescaling, (dropping the primes for the sake of clarity), the equation (2.1)-(2.2) is recast as:

$$\alpha \partial_t f^\alpha \ + \ \left(\underline{v} \cdot \nabla_{\underline{\xi}} + \underline{E}^\alpha \cdot \nabla_{\underline{v}}\right) f^\alpha + \frac{1}{\alpha}\left(v_x \frac{\partial}{\partial x} - (\underline{v} \times B) \cdot \nabla_{\underline{v}}\right) f^\alpha = 0, \tag{2.6}$$

$$\underline{E}^\alpha = -\nabla_{\underline{\xi}} \phi^\alpha, \quad -\Delta_{\underline{\xi}} \phi^\alpha = \int_0^1 \int_{\mathbb{R}^3} f^\alpha dx dv - C(\underline{\xi}). \tag{2.7}$$

It is clear that these scalings do not induce any modification in the boundary condition, namely, f^α still satisfies:

$$\gamma^-(f^\alpha) = \mathcal{K}(\gamma^+(f^\alpha)). \tag{2.8}$$

Finally, we prescribe an initial condition which is compatible with the expected asymptotic dynamics. This initial condition is

$$f(X, v, t = 0) \quad = \quad f_I(X, v) \quad \forall (X, v) \in \Omega \times \mathbb{R}^3. \tag{2.9}$$

Now, since we want to base our approximation of diffusion in position-energy space, it is relevant to interpret the velocity of particle in terms of its energy. For that purpose, let us introduce the spherical coordinates in velocity space:

$$\begin{cases} \ \omega = v/|v| \in \mathbb{S}^2 \,(\text{angular variable}); \\ \epsilon = |v|^2/2, (\text{energy variable}); \\ v = |v|\omega = \sqrt{2\epsilon}\omega = N(\epsilon)\omega. \end{cases} \tag{2.10}$$

We identify v with the pair (ε, ω). Hence, for C^1 function $\epsilon : \mathbb{R}^3 \to \mathbb{R}$, and any integrable function $\varphi \in \mathbb{R}^3$ we have (coarea formula [17]):

$$\begin{aligned} \int_{v\in\mathbb{R}^3} \varphi(v) dv \quad &= \quad \int_0^\infty \int_{\mathbb{S}^2} \varphi(\sqrt{2\epsilon}\,\omega)(2\epsilon)^{1/2} d\epsilon\, d\omega \\ &= \quad \int_{\epsilon>0} \int_{\omega\in\mathbb{S}^2} \varphi(\epsilon, \omega) N(\epsilon) d\epsilon d\omega \end{aligned} \tag{2.11}$$

where $d\omega$ denotes the normalized Euclidean measure on $\mathbb{S}^2$, so $dv = r^2 dr d\omega$, with $r = |v| = \sqrt{2\epsilon}$, and $dr = \frac{d\varepsilon}{\sqrt{2\epsilon}}$. In the coarea formula (2.11), $N(\epsilon)$ represents the energy-density of states.

The main contribution of this paper is to investigate the limit $\alpha \to 0$ of (2.6)-(2.9). We show that the limit f^0 as $\alpha \to 0$ is a function $F(\underline{\xi}, \varepsilon, t)$ of the longitudinal coordinate $\underline{\xi}$, of the energy $\varepsilon = |v|^2/2$ and of the time which obeys a diffusion equation in the position-energy space. In order to simplify the exposition, from now on, we will denote by (P) the problem (2.6), (2.7), (2.8), (2.9).

We are now ready to state our main result.

Theorem 2.1 *1. As α goes to zero, the solution f^α to the problem (P) formally converges to an equilibrium state $F(\underline{\xi}, |v|^2/2, t)$, solution to the following SHE model:*

$$4\pi\sqrt{2\epsilon}\frac{\partial F}{\partial t} + \left(\nabla_{\underline{\xi}} - \underline{E}\frac{\partial}{\partial \epsilon}\right)\cdot \underline{J} = 0, \tag{2.12}$$

$$\underline{J}(\underline{\xi}, \varepsilon, t) = -\mathbb{D}(\underline{\xi}, \varepsilon)\left(\nabla_{\underline{\xi}} - \underline{E}\frac{\partial}{\partial \epsilon}\right) F, \tag{2.13}$$

$$\underline{E} = -\nabla\phi, \quad -\Delta_{\underline{\xi}} = \int_0^1\int_{\mathbb{R}^3} f(\underline{\xi}, v, t)dxdv - C(\underline{\xi}), \tag{2.14}$$

$$F(\underline{\xi}, \varepsilon, t=0) = F_I(\underline{\xi}, \varepsilon), \tag{2.15}$$

$$\underline{J}(\underline{\xi}, \varepsilon = 0, t) = 0, \tag{2.16}$$

where F_I is a suitable initial condition, in the domain $(\xi, \epsilon) \in \mathbb{R}^2 \times (0, \infty)$. The 'diffusivity tensor' $\mathbb{D}$ is given by:

$$\mathbb{D}(\underline{\xi}, \varepsilon) = (2\epsilon)^{3/2}\int_0^1\int_{\mathbb{S}^2} \underline{\chi}(x, \omega; \underline{\xi}, \epsilon) \otimes \underline{\omega} dx d\omega, \tag{2.17}$$

where $\underline{\omega} = (\omega_y, \omega_z)$, $\underline{\chi} = (\chi_y, \chi_z)$, $\underline{\chi} \otimes \underline{\omega}$ is the tensor product $(\chi_i \omega_j)_{i,j\in\{y,z\}}$ and $\chi_i(x, \omega; \underline{\xi}, \frac{|v|^2}{2})$ is a solution of the problem

$$\begin{cases} -v_x \dfrac{\partial \chi_i}{\partial x} + (\underline{v} \times B)\cdot \nabla_{\underline{v}}\chi_i = \omega_i & \text{in } \Theta \\ \gamma^+(\chi_i) = \mathcal{K}^*(\gamma^-(\chi_i)) & \text{on } \Gamma, \end{cases} \tag{2.18}$$

satisfying $\int_0^1\int_{\mathbb{S}^2} \underline{\chi} dx d\omega = 0$, for all $(\underline{\xi}, \varepsilon)$, where $\mathcal{K}^*$ is the operator adjoint of $\mathcal{K}$.

2. Under hypothesis 3.1, 5.1 (to be specified later on) and 5.2, there exists a solution $(f^\alpha, \underline{E}^\alpha)$ in $L^\infty(0, T; L^2(\Theta)) \cap L^2(0, T; L^2_{loc}(\mathbb{R}^2_{\underline{\xi}}))$ for any $T \in \mathbb{R}^+$. f^α converges to f^0 in the weak star topology of $L^\infty(0, T; L^2(\Theta))$ for any $T > 0$, where $f^0(t, \underline{\xi}, \varepsilon) = F(t, \underline{\xi}, \varepsilon)$ and $F(t, \underline{\xi}, \varepsilon)$ is the weak solution of the problem (2.6)-(2.9).

The result is very close to [14]. Nevertheless, the proof will be different, though we shall use a lot of results developed in [14]. Existence results of Cauchy problem for Vlasov-Poisson system are now well known (see [27]) in dimension three without external potential and the boundary value problem for Vlasov-Poisson was studied in [20]. The theory of global weak solutions is due to [1]. The detailed mathematical study of (P) in the case of a constant magnetic field and a given electrostatic potential, which varies only over the macroscopic scale will be found in [14]. We shall not dwell on the existence and uniqueness of a solution for the Cauchy problem (P) which can be done by means of Leray-Schauder's fixed point Theorem; we shall solely give a material which makes it possible to obtain this solution and focus on the establishment of the limit model. The investigation of this limit when $\alpha \to 0$ proceeds in two steps. The first one consists in showing that f^α formally converges to a function of $f(x, \epsilon, t)$ only, solution of diffusion 'SHE' problem. Then we derive the continuity and current equation (2.12), (2.13). The second one corresponds to a rigorous convergence proof; we show that weak solutions of Vlasov-Poisson equations converge weakly (in an L^2 sense) towards weak solutions of the 'new' SHE model. To achieve these goals, two methods can be developed: the Hilbert expansion method [5] and the moment method [15]. We shall choose the latter because it involves more straightforward computations.

The outline of the paper is as follows. Section 3 is devoted to the preliminary materials regarding the functional setting of our problem and important properties of the boundary collision operator. We establish some mathematical results on this operator. In particular, we prove that the flux of particles is conserved at the boundary and there is an entropy dissipation (that is, a Darrozès-Guiraud-like inequality in gas surface interaction [12], [9]). This will be enough to allows us to derive the formal asymptotic limit which is achieved in Section 4. This asymptotic limit appears as a singular perturbation problem for Vlasov-Poisson equations. We solve this problem and derive our SHE model as the corresponding limit equation. A necessary step for the rigorous derivation is to establish the existence of a solution for the kinetic problem; this will be done in Section 5. Finally Section 6 is dedicated to the rigorous proof of convergence itself.

3 The boundary operator: assumptions and properties.

In the sequel, we consider an expression of $\mathcal{K}$ as follows

$$\mathcal{K}\phi(v) = \int_{\{\omega'\in\mathbb{S}^2,\ (\underline{\xi},v)\in\Gamma_+\}} K(\underline{\xi},|v|^2/2;\omega'\to\omega)\phi(|v|\omega')|\omega'_x|d\omega', \qquad \forall v\in\mathbb{R}^3, \tag{3.1}$$

where $\omega=(\omega_x,\omega_y,\omega_z)=\frac{v}{|v|}$, $\ |\omega|=1$ is the decomposition of v into spherical coordinates, $\mathbb{S}^2=\{\omega\in\mathbb{R}^3,\ |\omega|=1\}$ is the unit sphere. The operator $\mathcal{K}$ maps the outgoing trace $\gamma^+(f)$ to the incoming one $\gamma^-(f)$. The expression (3.1) models an elastic bounce against the planes with a random deflection of the velocity. Note that K is an integral kernel which describes the reflection law of the velocity direction. The quantity $K(\underline{\xi},|v|^2/2;\omega'\to\omega)|\omega_x|d\omega$ is the probability for a particle hitting the planes at point $\underline{\xi}$ with velocity $v'=|v|\omega'$ to be reflected with the same $|v|$ and velocity direction ω in the solid angle $d\omega$. Bearing in mind that $\mathcal{K}=\mathcal{K}(\underline{\xi},|v|^2/2)$ operates on the angular variable ω only while $\underline{\xi}$ and $|v|$ are mere parameters.

We give the adapted functional setting for the study of the collision operator. We set $\mathcal{S}_\pm(x),\ x=0,1$ to be the following half-spheres:

$$\mathcal{S}_+(0)=\mathcal{S}_-(1)=\{\omega\in\mathbb{S}^2,\ \omega_x<0\},\quad \mathcal{S}_-(0)=\mathcal{S}_+(1)=\{\omega\in\mathbb{S}^2,\ \omega_x>0\}. \tag{3.2}$$

We introduce the domain $\mathcal{S}=[0,1]\times\mathbb{S}^2$ with its associated *incoming* and *outgoing* boundaries defined by:

$$\mathcal{S}_-=(\mathcal{S}_-(0)\times\{0\})\cup(\mathcal{S}_-(1)\times\{1\}), \qquad \mathcal{S}_+=(\mathcal{S}_+(0)\times\{0\})\cup(\mathcal{S}_+(1)\times\{1\}). \tag{3.3}$$

We denote by $L^2(\mathcal{S}_\pm)$ the space of square integrable functions on $\mathcal{S}_\pm$ with respect to the measure $|\omega_x|d\omega$; by $(f,g)_\Theta,\ \ (f,g)_{\Gamma^\pm}$ the inner products on $L^2(\Theta)$ and on $L^2(\Gamma^\pm)$ respectively defined by:

$$(f,g)_\Theta=\int_\Theta fg\,d\theta, \qquad (f,g)_{\Gamma^\pm}=\int_{\Gamma^\pm} fg|v_x|d\Gamma$$

where $d\theta=dxd\underline{\xi}dv$ is the volume element in phase space, and $d\Gamma=\sum_{x=0,1}d\underline{\xi}dv$ is the surface element and by $(f,g)_{\mathcal{S}},\ (f,g)_{\mathcal{S}_\pm}$ the inner products on $L^2(\mathcal{S})$ and $L^2(\mathcal{S}_\pm)$ defined analogously:

$$\begin{aligned}(f,g)_{\mathcal{S}} &= \int_0^1\int_{\mathbb{S}^2}(fg)(x,\omega)dxd\omega,\\ (f,g)_{\mathcal{S}_\pm} &= \int_0^1\int_{\mathcal{S}_\pm(0)}(fg)(x,\omega)|\omega_x|dxd\omega+\int_0^1\int_{\mathcal{S}_\pm(1)}(fg)(x,\omega)|\omega_x|dxd\omega.\end{aligned}$$

and $|f|_{\mathcal{S}},\ |f|_{\mathcal{S}_\pm}$ the associated norms. We shall denote also $\Theta'=\mathbb{R}^2\times\mathbb{R}^{+*}$ with $d\theta'=d\underline{x}d\varepsilon$ its volume element. $\forall f=f(\omega),\ \langle f\rangle$ will denote the angular average of f on the sphere $\mathbb{S}^2$ and with respect to x i.e.

$$\langle f\rangle=\frac{1}{4\pi}\int_0^1\int_{\mathbb{S}^2}f(x,\omega)dxd\omega.$$

We denote by $\mathcal{C}^\pm=\{f\in L^2(\mathbb{R}^3_\pm):\ \ f(\epsilon,\omega)$ is constant with respect to $\omega\}$. We define the operator $Q^\pm$ as the orthogonal projection (for inner product $(\cdot,\cdot)_{\mathcal{S}_+}$) of $L^2(\mathcal{S}_+)$ on the space $\mathcal{C}^\pm$, i.e.

$$Q^\pm f(x,\omega)=\frac{1}{\pi}\int_{\mathcal{S}_\pm(x)}f(\omega)|\omega_x|d\omega, \qquad \omega\in\mathcal{S}_\pm,\quad x\in\{0,1\}, \tag{3.4}$$

and the operator $P^\pm$, as the orthogonal complement of $Q^\pm:\ \ P^\pm=I-Q^\pm$.

We shall list the required properties of the reflection operator K. They are summarized in the following

Hypothesis 3.1 *(i) Flux conservation:*

$$\int_{\mathcal{S}_-(x)} K(\omega' \to \omega)|\omega_x| d\omega = 1, \tag{3.5}$$

for almost all $(\omega, \omega') \in \mathcal{S}_-(x) \times \mathcal{S}_+(x), \quad x = 0, 1.$

(ii) Reciprocity principle:

$$K(\omega' \to \omega) = K(-\omega \to -\omega'), \qquad \forall (\omega, \omega') \in \mathcal{S}_-(x) \times \mathcal{S}_+(x), \quad x = 0, 1. \tag{3.6}$$

(iii) Positivity : $K(\omega' \to \omega) > 0$, *for almost all* $(\omega, \omega') \in \mathcal{S}_-(x) \times \mathcal{S}_+(x), \quad x = 0, 1.$
(iv) The operator $\mathcal{K}(\underline{\xi}, \varepsilon)$ *is a compact operator from* $L^2(\mathcal{S}_+)$ *onto* $L^2(\mathcal{S}_-)$.

The relation (3.5) expresses the conservation of the normal flux of particles at the boundary.
The reciprocity relation (3.6) is a macroscopic effect of the time reversibility of elementary particle-surface interactions. As a direct consequence of the flux conservation and reciprocity relation, by the change of ω into $-\omega$ in (3.5) and the use of (3.6), we get the following 'normalization' identity:

$$\int_{\mathcal{S}_+(x)} K(\omega' \to \omega)|\omega'_x| d\omega' = 1, \qquad x = 0, 1. \tag{3.7}$$

The assumption (iv) can be viewed as a regularity hypothesis for the integral kernel K. The adjoint of $\mathcal{K}(\underline{\xi}, \varepsilon)$ is obviously given by

$$\mathcal{K}^*(\phi)(x, \omega) = \int_{\mathcal{S}_-(x)} K(x, \omega \to \omega')\, \phi(x, \omega')\, |\omega'_x|\, d\omega', \qquad \omega \in \mathcal{S}_+(x), \quad x = 0, 1, \tag{3.8}$$

and is also a compact operator.

3.1 Flux conservation

Our first objective is to understand what the effects of the boundary condition (3.1) are on the fluxes of particles. For this, we need to define the incoming and outgoing normal fluxes associated to distribution function f.

$$\begin{aligned} J_x^-(\underline{\xi}, \varepsilon, t) &= \int_{\omega \in \mathcal{S}_-(x)} f(\underline{\xi}, x = 0, |v|\omega|)|v||\omega_x| d\omega \\ J_x^+(\underline{\xi}, \varepsilon, t) &= \int_{\omega' \in \mathcal{S}_+(x)} f(\underline{\xi}, x = 1, |v|\omega'|)|v||\omega'_x|\, d\omega'. \end{aligned}$$

We begin with the following the required identity:

Lemma 3.2 *Assume* $\mathcal{B}(\gamma(f^\alpha)) = 0$. *Then the normal flux is conserved:*

$$J_x^- = J_x^+. \tag{3.9}$$

Proof. The incoming normal flux J_x^- is obtained by integrating the left-hand side of the equation

$$\gamma^-(f^\alpha) = \mathcal{K}(\gamma^+(f^\alpha)). \tag{3.10}$$

Indeed, the coarea formula implies

$$\begin{aligned} J_x^-(\varepsilon) &= \int_{\omega \in \mathcal{S}_-(x)} \gamma^-(f^\alpha)(\varepsilon\omega)\sqrt{2\varepsilon}\, |\omega_x| d\omega \\ J_x^+(\varepsilon) &= \int_{\omega \in \mathcal{S}_+(x)} \gamma^-(f^\alpha)(\varepsilon\omega)\sqrt{2\varepsilon}\, |\omega_x| d\omega. \end{aligned}$$

Hypothesis (3.5) and Fubini's Theorem lead to

$$\begin{aligned} J_x^-(\varepsilon) &= \int_{\omega \in \mathcal{S}_-(x)} \left(\int_{\omega' \in \mathcal{S}_+(x)} K(\omega' \to \omega') f(|v|\omega'|)|\omega'_x|\, d\omega' \right) |v||\omega_x|\, d\omega \\ &= \int_{\omega' \in \mathcal{S}_+(x)} f(|v|\omega'|)|v||\omega'_x|\, d\omega' = J_x^+(\varepsilon). \end{aligned} \tag{3.11}$$

Finally, it is easy to check that the flux particles through $\Gamma_\pm$ vanished.
The hypothesis 3.1 is crucial for the validity of Theorem 2.1. From this, the following inequality bears similarities with Darrozès-Guiraud inequality in gas dynamics, which we can deduce from Jensen's inequality:

Lemma 3.3 *Let $\gamma^+(f) \in L^2(\mathcal{S}_+)$ and $\gamma^-(f) = \mathcal{K}(\gamma^+(f))$. Then*

$$\int_{\mathcal{S}_-(x)} |\gamma^-(f(x,\omega))|^2 |\omega_x|\, d\omega \leq \int_{\mathcal{S}_+(x)} |\gamma^+(f(x,\omega))|^2 \omega'_x |d\omega'. \tag{3.12}$$

Proof. Using Cauchy-Schwarz inequality and normalization identity (3.7), we have for $x = 0, 1$:

$$\begin{aligned} \int_{\mathcal{S}_-(x)} |\gamma^-(f(x,\omega))|^2 |\omega_x|\, d\omega &= \int_{\mathcal{S}_-(x)} |\mathcal{K}(\gamma^+(f(x,\omega)))|^2 |\omega_x| d\omega \\ &\leq \int_{\mathcal{S}_-(x)} \left| \int_{\mathcal{S}_+(x)} K(x,\omega' \to \omega)\, \gamma^+(f)(x,\omega')\, |\omega'_x|\, d\omega' \right|^2 |\omega_x|\, d\omega \\ &\leq \int_{\mathcal{S}_+(x)} |\gamma^+(f(x,\omega))|^2 \omega'_x |d\omega'. \end{aligned} \tag{3.13}$$

Therefore, the operator norm of $\mathcal{K}$ in the space $\mathcal{L}(L^2(\mathcal{S}_+), L^2(\mathcal{S}_+))$ of bounded operator from $L^2(\mathcal{S}_-)$ to $L^2(\mathcal{S}_+)$ is less or equal one: $\|\mathcal{K}\|_{\mathcal{L}(L^2(\mathcal{S}_+))} \leq 1$. Furthermore, from (3.7), if φ is constant over $\mathcal{S}_+$, then $\mathcal{K}\varphi$ is constant over $\mathcal{S}_-$. Hence, $\|\mathcal{K}\|_{\mathcal{L}(L^2(\mathcal{S}_+))} = 1$.

We introduce the specular reflection operator $\mathcal{J}$ which operates from $L^2(S_+)$ to $L^2(S_-)$ according to $\mathcal{J}\varphi(\omega) = \varphi(\omega^*)$, with $\omega_* = (-\omega_x, \omega_y, \omega_z)$. Its adjoint $\mathcal{J}^*$ is defined from $\phi \in L^2(\mathcal{S}_-)$ to $L^2(\mathcal{S}_+)$ and is also the mirror reflection operator, and we have the equalities $\mathcal{J}^*\mathcal{J} = I_{\mathcal{S}_+} = \mathcal{J}\mathcal{J}^* = I_{\mathcal{S}_-}$. Therefore the operator $\mathcal{K}\mathcal{J}^*$ and its adjoint $\mathcal{K}^*\mathcal{J}$ operate on $L^2(\mathcal{S}_-)$ while $\mathcal{K}^*\mathcal{J}$ and $\mathcal{K}\mathcal{J}^*$ operate on $L^2(\mathcal{S}_+)$.

In the derivation of the SHE model, we are interested in the characterization of the equilibria of operator $I - \mathcal{K}\mathcal{J}^*$ and $I - \mathcal{K}^*\mathcal{J}$ in $L^2(\mathcal{S}_\pm)$. By this hypothesis (iv), the operators $I - \mathcal{K}\mathcal{J}^*$, $I - \mathcal{K}^*\mathcal{J}$ are Fredholm operators. Using Krein-Rutman's Theorem and Fredholm's theory [23], we have the following lemma which can be easily adapted from that of [14], so the proof is omitted:

Lemma 3.4 *(i) The null-spaces $N(I - \mathcal{J}\mathcal{K}^*)$ and $N(I - \mathcal{J}^*\mathcal{K})$ are spanned by the constant functions on $\mathcal{S}_-$ and $\mathcal{S}_+$ respectively.*

(ii) $\mathcal{K}(x,\varepsilon)$ is of norm 1, i.e. $\|\mathcal{K}(x,\varepsilon)\varphi\|_{\mathcal{L}(L^2(\mathcal{S}_+))} \leq \|\varphi\|_{\mathcal{L}(L^2(\mathcal{S}_+))}$, for all $\varphi \in L^2(\mathcal{S}_+)$.

(iii) The range $R(I - \mathcal{K}^*\mathcal{J})$ is such that $R(I - \mathcal{K}^*\mathcal{J}) = N(I - \mathcal{J}^*\mathcal{K})^\perp$. Equivalently, the equation $(I - \mathcal{K}^*\mathcal{J})f = g$ has a solution f if and only if $\int_{\mathcal{S}_+} g(\omega)|\omega_x| d\omega = 0$. Then, the solution f is unique under the additional constraint $\int_{\mathcal{S}_+} f(\omega)|\omega_x| d\omega = 0$.

Property (ii) is reminiscent of Darrozès-Guiraud inequality for boundary conditions in gas dynamics. This property expresses that $\mathcal{K}(\underline{\xi}, \varepsilon)$ has a 'good' diffusion behavior. From (3.1) , $\mathcal{K}\varphi(\omega)$ for $\omega \in \mathcal{S}_-$ appears as a convex mean value of $\varphi(\omega')$ over $\mathcal{S}_+$.

Lemma 3.5 *The operator $\mathcal{K}$ satisfies*

$$\mathcal{K}Q^+ = Q^-\mathcal{K} = \mathcal{J}Q^+ = \mathcal{J}Q^-, \qquad \mathcal{K}P^+ = P^-\mathcal{K}. \tag{3.14}$$

The proof of this lemma is a straightforward adaptation from Ref. [14].

We close this section by giving the following assumption:

Hypothesis 3.6 *There exists $k_0 < 1$ such that, for $|v| \in \mathbb{R}^+$, $\underline{\xi} \in \mathbb{R}^2$*

$$\|\mathcal{K}P^+\|_{\mathcal{L}(L^2(\mathcal{S}_+), L^2(\mathcal{S}_-))} \leq k_0 < 1.$$

This assumption comes from the elementary operator theory. Indeed, for every $\underline{\xi} \in \mathbb{R}^2$ and $|v| \in \mathbb{R}^+$, there exists $k(\underline{\xi}, |v|)$ such that $\mathcal{K}P^+\| \leq k(\underline{\xi}, |v| < 1$. Obviously, the hypothesis 3.6 is satisfied in the case of the isotropic scattering.

4 Formal derivation of the macroscopic model

As a further purpose, we consider a sequence $(f^\alpha)_\alpha$ of solutions to the problem (P). Our goal is to study the asymptotic behavior of solutions (P) as $\alpha \to 0$. In this section, we assume that $f^\alpha \to f$ as $\alpha \to 0$ in a smooth way (in the sense where we will precise later). This approach will formally give rise to a set of diffusion equations to be viewed as a SHE system. In fact, this formal derivation will not be completed in this section since an 'auxiliary problem' - the resolution of which will be postponed to Section 6- will arise during this study.

We divide the formal asymptotic into several steps. First, we prove that the limit f is a function of total energy. Then we prove that the limit satisfies a continuity equation. An auxiliary problem will be considered which allows us to finally derive the current equation.

The following change of variables will be useful in the remainder of the paper. Since the velocity v satisfies the relation.

$$|v| = \sqrt{v_x^2 + v_y^2 + v_z^2}, \qquad \omega_j = \frac{v_j}{|v|}, \qquad j = x, y, z, \tag{4.1}$$

we can parameterize the sphere $\mathbb{S}^2$ in the direction $\underline{\omega} = (\omega_y, \omega_z)$, and ω_x is defined by $\omega_x = \sigma\sqrt{1 - \omega_y^2 - \omega_z^2}$, $\sigma = \pm 1$. Thus the sphere $\mathbb{S}^2$ is given by two local maps $\{(\sigma, \omega_y, \omega_z),\ \sigma = \pm 1\}$.

Hence, we have

Lemma 4.1 *Let the change of spherical coordinate be:*

$$(v_x, v_y, v_z) \longmapsto (v, \omega_x, \omega_y, \omega_z) \tag{4.2}$$

where $\omega_x = sgn(v_x)\sqrt{1 - \omega_y^2 - \omega_z^2}$, $\omega_y = \dfrac{v_y}{|v|}$, $\omega_z = \dfrac{v_z}{|v|}$. *Then, for* $sgn(v_x) = \pm 1$, *we have*

$$\begin{aligned}
\frac{\partial f}{\partial v_x} &= \frac{\partial f}{\partial |v|}\cdot \omega_x - \frac{\partial f}{\partial \omega_y}\cdot\frac{\omega_x\omega_y}{|v|} - \frac{\partial f}{\partial \omega_z}\cdot\frac{\omega_x\omega_z}{|v|}\\
\frac{\partial f}{\partial v_y} &= \frac{\partial f}{\partial |v|}\cdot \omega_y - \frac{\partial f}{\partial \omega_y}\cdot\frac{1-\omega_y^2}{|v|} - \frac{\partial f}{\partial \omega_z}\cdot\frac{\omega_y\omega_z}{|v|}\\
\frac{\partial f}{\partial v_z} &= \frac{\partial f}{\partial |v|}\cdot \omega_z - \frac{\partial f}{\partial \omega_z}\cdot\frac{\omega_z\omega_z}{|v|} - \frac{\partial f}{\partial \omega_z}\cdot\frac{1-\omega_z^2}{|v|}
\end{aligned} \tag{4.3}$$

Proof. - For $\mathrm{sgn}(v_x) = \pm 1$, we have

$$\frac{\partial f}{\partial v_j} = \frac{\partial f}{\partial v}\frac{\partial v}{\partial v_j} + \frac{\partial f}{\partial \omega_y}\frac{\partial \omega_y}{\partial v_j} + \frac{\partial f}{\partial \omega_y}\frac{\partial \omega_z}{\partial v_j}.$$

Therefore, for $j = x, y, z$, we get

$$\frac{\partial |v|}{\partial v_j} = \frac{v_j}{\sqrt{v_x^2 + v_y^2 + v_z^2}} = \omega_j. \tag{4.4}$$

Using (4.4) and the relation $v_j = |v|\omega_j$, we find

$$\begin{aligned}
\frac{\partial \omega_y}{\partial v_y} &= \frac{1-\omega_y^2}{|v|}, &\qquad \frac{\partial \omega_z}{\partial v_y} &= -\frac{\omega_y\omega_z}{|v|};\\
\frac{\partial \omega_y}{\partial v_z} &= -\frac{\omega_y\omega_z}{|v|}, &\qquad \frac{\partial \omega_z}{\partial v_z} &= \frac{1-\omega_z^2}{|v|};\\
\frac{\partial \omega_y}{\partial v_x} &= -\frac{\omega_y\omega_x}{|v|}, &\qquad \frac{\partial \omega_z}{\partial v_x} &= -\frac{\omega_z\omega_x}{|v|},
\end{aligned}$$

which proves the result.

4.1 The limit is a function of the energy

Taken formally the limit $\alpha \to 0$ in (2.6)-(2.8) shows that f is the solution to equations:

$$\begin{aligned}\left(v_x \frac{\partial}{\partial x} - (\underline{v} \times B) \cdot \nabla_{\underline{v}}\right) f &= 0, & (4.5)\\ \gamma^-(f) &= \mathcal{K}(\gamma^+(f)). & (4.6)\end{aligned}$$

We show the

Lemma 4.2 *The solution of (4.5)-(4.6) are functions of total energy only:*

$$f(x,v,t) = F(\underline{\xi}, \tfrac{1}{2}|v|^2, t). \tag{4.7}$$

Proof. The fact that the limit of f^α is a function of the energy only is an easy consequence of Lemma 3.4. According to Lemma 4.1, Eq. (4.5) is equal to

$$\frac{\partial f}{\partial x}(x,\omega) + \frac{B(\underline{\xi})}{|v|\omega_x}\frac{\partial f}{\partial \omega}(e_x \times \omega) = 0 \tag{4.8}$$

where $\dfrac{\partial f}{\partial \omega}$ denotes the derivatives of f with respect to $\omega \in \mathbb{S}^2$ of degree 1, $a \times b$ stands for the cross product of two vectors a and b and $e_x \times \omega$ denotes the tangent vector to $\mathbb{S}^2$ in x-axis. We recall that $\underline{\xi}$ and v are mere parameters in problem (4.5) and will be omitted in the remainder of the proof. For $f(x,\omega)$ we introduce the following change of variables:

$$\begin{cases} \theta = \dfrac{B(\underline{\xi})}{|v|\omega_x} = \dfrac{B(\underline{\xi})}{\sqrt{2\epsilon}\omega}, \quad \omega_x = \sigma\sqrt{1-\omega_y^2-\omega_z^2}, \quad \sigma \in \{-1,1\} \\ \omega_y^* = \omega_y \cos(\theta x) + \omega_z \sin(\theta x), \\ \omega_z^* = \omega_y \sin(\theta x) + \omega_z \cos(\theta x). \end{cases} \tag{4.9}$$

Since $\underline{\omega} = (\omega_y, \omega_z)$, with (4.9) we can write, $\underline{\omega}^* = (\omega_y^*, \omega_z^*) = \mathcal{R}^+_{x,\sigma}(\underline{\omega})$, where

$$\mathcal{R}^+_{(x,\sigma)}(\underline{\omega}) = \begin{pmatrix} \cos\theta x & -\sin\theta x \\ \sin\theta x & \cos\theta x \end{pmatrix} \tag{4.10}$$

is a rotation of $\underline{\omega}$ around the $x-$axis by an angle θx. Hence the function associated to this transformation is written as

$$f^*(x,\sigma,\underline{\omega}) = f(x,\sigma,\mathcal{R}^+_{(x,\sigma)}(\underline{\omega})). \tag{4.11}$$

Therefore, if f is solution of (2.6), f satisfies the problem:

$$|v|\omega_x \frac{\partial f^*}{\partial x} = 0, \qquad \gamma^-(f^*) = \mathcal{K}(\gamma^+(f^*)). \tag{4.12}$$

Integrating the first equation of (4.12) with respect to $x \in [0,1]$, yields

$$f^*(x,\sigma,\underline{\omega}) = f^*(0,\sigma,\underline{\omega}) = f^*(1,\sigma,\underline{\omega}), \tag{4.13}$$

or, in terms of the rotation, $\mathcal{R}^-_{(x,\sigma)}(\underline{\omega})$:

$$f(x,\sigma,\underline{\omega}) = f^*(0,\sigma,\mathcal{R}^-_{(x,\sigma)}(\underline{\omega})) = f(1,\sigma,\mathcal{R}^+_{(1-x,\sigma)}(\underline{\omega})). \tag{4.14}$$

This is equivalently written

$$\gamma^+(f) = \mathcal{K}^*\gamma^-(f). \tag{4.15}$$

Inserting (4.15) into the second equation of (4.6) leads to

$$(I - \mathcal{K}\mathcal{J}^*)(\gamma^-(f)) = 0 \tag{4.16}$$

which, by virtue of Lemma 3.4, implies that $\gamma^-(f) = F(\frac{1}{2}|v|^2)$.

4.2 The continuity equation

We are now concerned with the derivation of the continuity equation (2.12). Before, we introduce the following macroscopic quantities. For finite $\alpha > 0$, we define the average density F^α and current $\underline{J}^\alpha(\underline{\xi}, \epsilon, t) = (J_y^\alpha, J_z^\alpha)$ by

$$\begin{aligned} F^\alpha(\underline{\xi}, \varepsilon, t) &= \frac{1}{4\pi} \int_0^1 \int_{\mathbb{S}^2} f^\alpha(x, \underline{\xi}, |v|, \omega, t) dx d\omega, \qquad (4.17) \\ J^\alpha(\underline{\xi}, \varepsilon, t) &= \frac{|v|}{\alpha} \int_0^1 \int_{\mathbb{S}^2} \underline{v} f^\alpha(x, \underline{\xi}, |v|, t) dx d\omega := \frac{1}{\alpha} N(\varepsilon) \langle v f^\alpha(\varepsilon, \cdot) \rangle \\ &= \frac{2\epsilon}{\alpha} \int_0^1 \int_{\mathbb{S}^2} \underline{\omega} f^\alpha(x, \underline{\xi}, \varepsilon, \omega, t) dx d\omega. \qquad (4.18) \end{aligned}$$

According to reference [14], we have,

Lemma 4.3 *Let $\varphi(x, v)$ be a C^1 function. Then we have*

$$\frac{\partial J_\varphi}{\partial \varepsilon} = \sqrt{2\varepsilon} \int_0^1 \int_{\mathbb{S}^2} (\nabla_{\underline{v}} \varphi)(x, \sqrt{2\varepsilon}\omega) dx d\omega, \qquad J_\varphi(\varepsilon) = 2\varepsilon \int_0^1 \int_{\mathbb{S}^2} \underline{\omega} \varphi(x, \sqrt{2\varepsilon}\omega) dx d\omega. \qquad (4.19)$$

We prove:

Lemma 4.4 *We temporarily admit that $J^\alpha \to J$ as $\alpha \to 0$. Then F and J satisfy the continuity equation (2.12).*

Proof. Since from Lemma 4.2, f is a function of energy only (in velocity space), it is quite natural to expect that f^α be mainly given by a function of ϵ only at order O in α, for instance the macroscopic quantity $\langle f^\alpha \rangle$. We expect that there exists a function $g^\alpha(\underline{\xi}, v, t)$ such that:

$$f^\alpha(X, v, t) = F^\alpha(\underline{\xi}, \epsilon, t) + \alpha g^\alpha(\underline{\xi}, v, t).$$

We integrate (2.6) with respect to x and on a sphere of constant energy. We get

$$\begin{aligned} \int_0^1 \int_{\mathbb{S}^2} \left(\underline{v} \cdot \nabla_{\underline{\xi}} - \underline{E}^\alpha \cdot \nabla_{\underline{v}} \right) (F^\alpha + \alpha g^\alpha) dx d\omega &= \int_0^1 \int_{\mathbb{S}^2} \underline{v} d\omega \cdot \widetilde{\nabla} F^\alpha + \alpha \nabla_{\underline{\xi}} \cdot \int_0^1 \int_{\mathbb{S}^2} \underline{v} g^\alpha dx d\omega \\ &\quad - \alpha \underline{E}^\alpha \cdot \int_0^1 \int_{\mathbb{S}^2} \nabla_{\underline{v}} g^\alpha dx d\omega \\ &= \frac{4\pi\alpha}{N(\epsilon)} \nabla_{\underline{\xi}} \cdot J^\alpha - \alpha \underline{E}^\alpha \cdot \int_0^1 \int_{\mathbb{S}^2} \nabla_{\underline{v}} g^\alpha dx d\omega \end{aligned}$$

with $\widetilde{\nabla} = \nabla_{\underline{\xi}} + \underline{E} \partial / \partial \epsilon$ and for $\displaystyle\int_{\mathbb{S}^2} v d\omega = 0$. Moreover, for any test function $\psi(\epsilon)$, a straightforward computation yields:

$$\int_{\epsilon > 0} \psi \int_0^1 \int_{\mathbb{S}^2} \nabla_{\underline{v}} g^\alpha dx d\omega \, N(\epsilon) d\epsilon = 4\pi \int_{\epsilon > 0} \psi \frac{\partial J^\alpha}{\partial \epsilon} d\epsilon$$

which finally proves

$$\frac{1}{4\pi\alpha} \int (\underline{v} \cdot \nabla_{\underline{\xi}} - \underline{E}^\alpha \cdot \nabla_{\underline{v}}) f^\alpha dx d\omega N(\epsilon) = \widetilde{\nabla} \cdot J^\alpha. \qquad (4.20)$$

4.3 The current equation

Next, we have to find a current equation giving a relation between J^α and the density F^α. We prove:

Lemma 4.5 *The current equation (2.13) is satisfied.*

Proof. Using Green's formula (5.4) we get

$$\begin{aligned} S^\alpha &:= \int_0^1 \int_{\mathbb{R}^3} \left(v_x \frac{\partial}{\partial x} - (v \times B) \cdot \nabla_{\underline{v}} f^\alpha \right) \chi dx dv \qquad (4.21) \\ &= \left(\int_{\mathbb{R}^3} v_x f^\alpha \underline{\chi} dv \right) |_{x=1} - \left(\int_{\mathbb{R}^3} v_x f^\alpha \underline{\chi} dv \right) |_{x=0} + \int_0^1 \int_{\mathbb{R}^3} f^\alpha \left(v_x \frac{\partial f^\alpha}{\partial x} - (v \times B) \cdot \nabla_{\underline{v}} \right) \underline{\chi} dx dv. \end{aligned}$$

Now, using (2.8) and the second equation of (2.18) we get

$$\begin{aligned}\left(\int_{\mathbb{R}^3} v_x f^\alpha \underline{\chi} dv\right)\Big|_{x=1} &= |v|^2 \left(\int_{\mathcal{S}^+} \gamma^+(f^\alpha)\gamma^+(\underline{\chi})|\omega_x|d\omega - \int_{\mathcal{S}_-} \gamma^-(f^\alpha)\gamma^-(\underline{\chi})|\omega_x|d\omega\right)\\ &= |v|^2 \left(\int_{\mathcal{S}_+} \gamma^+(f^\alpha)\gamma^+(\underline{\chi})|\omega_x|d\omega - \int_{\mathcal{S}_-} \mathcal{K}(\gamma^+(f^\alpha))\gamma^-(\underline{\chi})|\omega_x|d\omega\right)\\ &= |v|^2 \int_{\mathcal{S}_+} \gamma^+(f^\alpha)\left(\gamma^+(\underline{\chi})|\omega_x|d\omega - \mathcal{K}^*(\gamma^-(\underline{\chi}))\right)|\omega_x|d\omega = 0. \end{aligned} \tag{4.22}$$

Similarly, we can prove the relation for $\left(\int_{\mathbb{R}^3} v_x f^\alpha \underline{\chi} dv)\right|_{x=0}$. With the first equation of (2.18) we get

$$\int_0^1\!\!\int_{\mathbb{R}^3} f^\alpha \left(v_x \frac{\partial}{\partial x} - (v\times B)\cdot\nabla_{\underline{v}}\right)\underline{\chi} dx dv \quad = \quad 2\epsilon \int_0^1\!\!\int_{\mathbb{R}^3} \underline{\omega} f^\alpha \underline{\chi} dx d\omega = \alpha J^\alpha. \tag{4.23}$$

Therefore, inserting (4.22) and (4.23) into (4.21), we deduce that $S^\alpha = \alpha J^\alpha$. This justifies the definition of $\underline{\chi}$. From (2.6) and (4.21), we deduce:

$$\underline{J}^\alpha = -\alpha \int_0^1\!\!\int_{\mathbb{R}^3} \partial_t f^\alpha \underline{\chi} dx dv - \int_0^1\!\!\int_{\mathbb{R}^3} \left(\underline{v}\cdot\nabla_{\underline{\xi}} - \nabla_{\underline{\xi}}\phi\cdot\nabla_{\underline{v}}\right) f^\alpha \underline{\chi} dx dv. \tag{4.24}$$

Taking the limit $\alpha \to 0$ and using (4.7), we obtain

$$\underline{J} = -\int_0^1\!\!\int_{\mathbb{R}^3} \left(\underline{v}\cdot\nabla_{\underline{\xi}} - \nabla_{\underline{\xi}}\phi\cdot\nabla_{\underline{v}}\right) F\underline{\chi} dx dv, \tag{4.25}$$

where $\underline{J} = \lim \underline{J}^\alpha$, $F = \lim F^\alpha$. Taking into account the relation

$$\left(\underline{v}\cdot\nabla_{\underline{\xi}} - \nabla_{\underline{\xi}}\phi\cdot\nabla_{\underline{v}}\right) F^\alpha = \underline{v}\left(\nabla_{\underline{\xi}} - \nabla_{\underline{\xi}}\phi\frac{\partial}{\partial_\epsilon}\right) F^\alpha$$

we get

$$\underline{J} = -\left(\int_0^1\!\!\int_{\mathbb{S}^2} \underline{\chi}(x,\omega;\xi,\epsilon)\otimes\underline{\omega} dx d\omega\right)\left(\nabla_{\underline{\xi}} - \nabla_{\underline{\xi}}\phi\frac{\partial}{\partial_\epsilon}\right) F, \tag{4.26}$$

which leads to equations (2.13). Finally, collecting (4.26) and (4.20), inserting in (4.5), multiplying by $\sqrt{2\varepsilon}$ and taking the limit $\alpha \to 0$ lead and second to (2.12), (2.17).

It only remains to prove the existence of $\underline{\chi}$ which will be achieved in section 6. The relevance of the model (2.12)-(2.13) relies on the positivity of the coefficient $\mathbb{D}$. We postpone this task to Section 6.5.

5 Existence of the solutions to the microscopic problem

In this section, we establish the existence of solutions to the microscopic problem (2.6), (2.8). To avoid the treatment of initial layers when we pass to the limit $\alpha \to 0$, we impose a compatibility condition on the initial data f_I:

Hypothesis 5.1 *There exists a smooth function F_I such that $f_I(x,\underline{\xi},v) = F_I(\underline{\xi},|v|^2/2)$ and that f_I satisfies: $f_I \in L^2(\Theta)$, $(\underline{v}\cdot\nabla_{\underline{\xi}} - \nabla_{\underline{\xi}}\phi\cdot\nabla_{\underline{v}})f_I \in L^2(\Theta)$.*

We define the transport operator by:

$$\mathcal{A}^\alpha f = \left(\underline{v}\cdot\nabla_{\underline{\xi}} - \nabla_{\underline{\xi}}\phi\cdot\nabla_{\underline{v}}\right) f + \frac{1}{\alpha}\left(v_x\frac{\partial}{\partial x} - (\underline{v}\times B)\cdot\nabla_{\underline{v}}\right) f \tag{5.1}$$

on the domain

$$D(\mathcal{A}^\alpha) = \{f \in L^2(\Theta), \quad \mathcal{A}^\alpha f \in L^2(\Theta), \quad \gamma^+(f)\in L^2(\Gamma_+), \quad \gamma^-(f) = \mathcal{K}\gamma^+(f)\}.$$

We shall denote by $\mathcal{A}$ the bare differential operator (5.1) when no indication of the domain is needed. Following [4] we define the space

$$H(\mathcal{A}^\alpha) \quad = \quad \{f \in L^2(\Theta), \quad \mathcal{A}^\alpha f \in L^2(\Theta)\}. \tag{5.2}$$

According to [4], it is well known that the regularity $f \in H(\mathcal{A}^\alpha)$ is not sufficient to guarantee the integrability of $|\gamma^+(f)|^2_{L^2(\Gamma_+)}$ and $|\gamma^-(f)|^2_{L^2(\Gamma_-)}$ over the boundary. But if one of these traces is integrable, the other one is also integrable. Therefore, we define the following space

$$H_0^\alpha(\mathcal{A}) \quad = \quad \{f \in H^\alpha(\mathcal{A}),\ \gamma^-(f) \in L^2(\Gamma_-)\} = \{f \in H^\alpha(\mathcal{A}),\ \gamma^+(f) \in L^2(\Gamma_+)\},$$

endowed with the graph norm and family of semi-norms, for $R > 0$

$$|f|^2_{H^\alpha(\mathcal{A}^\alpha)} = |f|^2_{L^2(\Theta)} + |\mathcal{A}^\alpha f|^2_{L^2(\Theta)}, \qquad |f|^2_{\Gamma_\pm, R} = \int_{\Gamma_\pm, |v| \le R} |v_x| |f|^2 d\Gamma. \tag{5.3}$$

Note that hypothesis 5.1 implies that $f_I \in D(\mathcal{A}^\alpha)$ for all $\alpha > 0$. For the solvability of the transport equation, we require the following hypothesis:

Hypothesis 5.2

(i) The first and second derivatives of the potential belong to the Sobolev space $W^{1,\infty}$ (in other words, $\nabla_{\underline{\xi}}\phi$ is bounded and globally lipschitz over $\mathbb{R}^2$).

(ii) The magnetic field $B = B(\underline{\xi}) \in C^1 \cap W^{1,\infty}(\mathbb{R}^2_{\underline{\xi}})$.

We give some results concerning the trace operators which are necessary in our setting. Since we will need to handle nonlinear functions of f and their traces on the boundary, we have to study for which nonlinear functions of f traces can be defined. The problem of existence of a trace is fundamental for the Cauchy problem with boundary conditions. This problem was investigate by many authors, such C. Bardos [4]. M. Cessenat in [10] studied it and applied it to neutron transport equation, also S. Ukai in [28] for the free transport operator, E. Beals et *al* in [6] for the abstract time-dependent linear kinetic equations and recently S. Mischler in [26] for Vlasov-Boltzmann equation.

According to Ref. [4], [6] and hypothesis 5.2 the following Green's type identity can be easily deduced.

Lemma 5.3 *For f and g in $H_0(\mathcal{A}^\alpha)$ compactly supported with respect to v, we have:*

$$(\mathcal{A}^\alpha f, g)_\Theta + (f, \mathcal{A}^\alpha g)_\Theta = \frac{1}{\alpha}\left((\gamma^+(f), \gamma^+(g))_{\Gamma_+} - (\gamma^-(f), \gamma^-(g))_{\Gamma_-}\right). \tag{5.4}$$

From that lemma we will deduce several estimates for the operator 5.1 and their traces on the boundary.

5.1 Resolution of an approximate kinetic problem

Applying Leray-Schauder's fixed point Theorem to the transport operator $\mathcal{A}^\alpha$ with domain $D(\mathcal{A}^\alpha)$ would be enough to prove the existence for the kinetic problem. Nevertheless, since the operator $\mathcal{K}$ acts like the identity on $\mathcal{C}^\pm$, we are lacking some estimates on the traces. Following [14], we introduce an approximate problem. Let $\eta > 0$ be a small approximating parameter; we approach the boundary operator by

$$\mathcal{K}_\eta = P^+\mathcal{K} + \frac{1}{1+\eta}\mathcal{J}Q^+. \tag{5.5}$$

We consider the associated initial value problem

$$\alpha\partial_t f^\alpha_\eta + \mathcal{A} f^\alpha_\eta = 0, \qquad f^\alpha_\eta|_{t=0} = F_\eta, \tag{5.6}$$

$$\underline{E}^\alpha = -\nabla_{\underline{\xi}}\phi^\alpha, \quad -\Delta_{\underline{\xi}}\phi^\alpha = \int_{\mathbb{R}^3} f^\alpha_\eta \, dv - C(\underline{\xi}), \tag{5.7}$$

$$\gamma^-(f^\alpha_\eta) = \mathcal{K}(\gamma^+(f^\alpha_\eta)) \tag{5.8}$$

with domain $D(\mathcal{A}^\alpha_\eta) = \{f \in H(\mathcal{A}^\alpha), \quad \gamma^+(f) \in L^2(\Gamma_+), \quad \gamma^-(f) = \mathcal{K}_\eta(\gamma^+(f))\}$. We denote by $\mathcal{A}^\alpha_\eta$ the transport operator $\mathcal{A}$ on the domain $D(\mathcal{A}^\alpha_\eta)$. Thanks to the Lemma 3.5, we have

$$\|\mathcal{K}_\eta P^+\|_{\mathcal{L}(L^2(\mathcal{S}_-), L^2(\mathcal{S}_+))} < 1, \qquad \forall\, \eta > 0. \tag{5.9}$$

We can now easily adapt Proposition 4.1 and Lemmas 4.2 and 4.3 in Ref. [14]. Namely we have:

Proposition 5.4 *(i) For all $\eta > 0$, and for all $F_\eta \in D(\mathcal{A}^\alpha_\eta)$, there exists a unique function $f^\alpha_\eta \in \mathcal{C}([0,T]; D(\mathcal{A}^\alpha_\eta)) \cap \mathcal{C}^1([0,T]; L^2(\Theta))$, that solves (5.6)-(5.7).*

(ii) We have also $|f^\alpha_\eta|_{L^2(\Theta)} \leq |F_\eta|_{L^2(\Theta)}$, $\quad |\alpha\partial_t f^\alpha_\eta|_{L^2(\Theta)} = \|\mathcal{A}^\alpha f^\alpha_\eta|_{L^2(\Theta)} \leq \|\mathcal{A}^\alpha F^\alpha_\eta|_{L^2(\Theta)}$.

(iii) Let F_I be as in hypothesis 5.1. There exists a sequence $(F_\eta)_{\eta>0}$ such that $F_\eta \in D(\mathcal{A}^\alpha_\eta)$ and $F_\eta \to F_I$, $\quad \mathcal{A}F_\eta \to \mathcal{A}F_I \quad$ in $L^2(\Theta)$ weak star, as $\eta \to 0$.

5.2 Traces estimates and existence for the kinetic problem

In this section, we assume that boundary conditions of the form (2.8) are satisfied and prove the existence of the traces on the boundary under suitable assumptions. We first establish that the projection P^+ of the trace at the boundary of a function of $D(\mathcal{A}^\alpha)$ is controlled by the graph norm. This is the key needed to prove the convergence of a solution to the problem (P).

We prove

Lemma 5.5 *We assume that ϕ satisfies hypothesis 5.2. If $f \in D(\mathcal{A}^\alpha_\eta)$, then there exists a constant $C > 0$ such that*

$$|P^-\gamma^-(f)|^2_{L^2(\Gamma_-)} \leq |P^+\gamma^+(f)|^2_{L^2(\Gamma_+)} \leq \frac{2\alpha}{1-k_0^2}(\mathcal{A}^\alpha f, f)_\Theta \leq C\alpha|f|^2_{\mathcal{A}^\alpha} \tag{5.10}$$

where k_0 is such that $\quad \|\mathcal{K}P^+\|_{\mathcal{L}(L^2(\mathcal{S}_+),L^2(\mathcal{S}_+))} < k_0$.

Proof. The outline of the proof, which is analogous of that of Lemma 4.4 in [14] is as follows. We apply Green's formula (5.4) to $f = g$ times a cutoff function $\rho_R(|v|^2/2) = \rho(|v|^2/2R))$ with $\rho \in C^\infty(\mathbb{R}^+)$ such that $0 \leq \rho \leq 1$, $\rho(u) = 1$ for $u < 1$ and $\rho(u) = 0$ for $u > 2$. We obtain, thanks to Lemma 3.4 and hypothesis 3.6:

$$2\alpha(\mathcal{A}^\alpha\rho_R, \rho_R f)_\Theta \geq (1-k_0^2)\int_{\Gamma_+} |v_x||P^+\gamma^+(f)|^2|\rho_R|^2 d\Gamma. \tag{5.11}$$

Using the boundedness of ϕ leads to

$$2\alpha(\mathcal{A}^\alpha\rho_R, \rho_R f)_\Theta \leq |\mathcal{A}^\alpha f|_{L^2(\Theta)} + |f|_{L^2(\Theta)} + \frac{C}{R}|f|_{L^2(\Theta)}. \tag{5.12}$$

Finally, taking the limit $R \to \infty$ in this estimate and recalling (5.11) allows us to conclude.

Next, we give some a priori estimates on the projections $\mathcal{C}^\pm$. Note that, for $f \in D(\mathcal{A}^\alpha_\eta)$, we have $\gamma^-(f) = \mathcal{K}_\eta(\gamma^+(f))$. By Lemma (3.14) and using the form (5.5) of $\mathcal{K}_\eta$, the orthogonal projection of the trace on $\mathcal{C}^-$ reads $Q^-\gamma^-(f) = \frac{1}{1+\eta}\mathcal{J}Q^+\gamma^+(f)$. Therefore, there exists a function $q(f) = q(x,\xi,|v|)$, $\quad x = 0,1$, $\xi \in \mathbb{R}^2$, $|v| > 0$, such that

$$q = (1+\eta)Q^-\gamma^-(f), \quad \text{on} \quad \Gamma_-, \qquad q = Q^+\gamma^+(f), \quad \text{on} \quad \Gamma_+. \tag{5.13}$$

We deduce

Lemma 5.6 *If $f \in D(\mathcal{A}^\alpha_\eta)$, then there exists a constant $C > 0$ such that*

$$|q(f)|^2_{\Gamma,R} \leq C\left(\alpha|f|^2_{\mathcal{A}^\alpha} + R|f|_{L^2(\Theta)}\right). \tag{5.14}$$

where, for $R > 0$, $|\cdot|^2_{\Gamma,R}$ is defined by $|\varphi|^2_{\Gamma,R} = \int_{\Gamma,\,|v|\leq R}|v_x||\varphi|^2 d\Gamma$.

Proof. The key of the proof is the control of $Q^-\gamma^-(f) = Q^+\gamma^+(f)$. Notice that, if $f \in D(\mathcal{A}^\alpha)$, then we can straightforwardly verify that $\mathrm{sgn}(v_x)\zeta(x)f \in H_0(\mathcal{A}^\alpha)$. We multiply $\mathcal{A}^\alpha f$ by $\rho_{_R}\mathrm{sgn}(v_x)\zeta(x)f$ with ζ a function such that $\zeta(1) = 1$, $\zeta(0) = -1$ and integrate on Γ. Thus we apply Green's formula (5.4) and take the limit $R \to \infty$ just as in the proof of Lemma 5.5. The remainder of the proof follows now straightforward from [14], so we omit the detail.

The existence for the kinetic problem can now be stated. It is given by following proposition, the proof of which can be found in Ref. [14].

Proposition 5.7 *Under hypothesis 3.1 and 5.2, there exists a solution f^α to problem (P), such that $f^\alpha \in L^\infty(0,T;L^2(\Theta))$, $\mathcal{A}f^\alpha \in L^\infty(0,T;L^2(\Theta))$, $P^+\gamma^+(f^\alpha) \in L^\infty(0,T;L^2(\Gamma_+))$, $Q^+\gamma^+(f^\alpha) \in L^\infty(0,T;L^2_R(\Gamma_+))$, for all $R > 0$, and the boundary condition is satisfied in the sense that $\quad P^-\gamma^-(f^\alpha) = \mathcal{B}P^+\gamma^+(f^\alpha)$; $\quad Q^-\gamma^-(f^\alpha) = Q^+\gamma^+(f^\alpha)$. Moreover, we have*

$$\int_0^T |P^+\gamma^+(f^\alpha)(t)|^2_{L^2(\Gamma_+)}dt \quad \leq \quad C\alpha^2|F_I|^2_{L^2(\Theta)}. \tag{5.15}$$

6 The rigorous derivation of the SHE model

In this section, we prove the convergence of the solutions of the Boltzmann-Poisson equation towards solutions of the SHE model (with coupled energies). The proof naturally falls into five steps. First, we recall the L^2 estimate obtaining in the previous section; secondly, we prove the weak convergence of f^α and establish the continuity equation. Thirdly, we prove that the current actually converges. Fourthly, we pass to the limit in the nonlinear term and we finish the section by investigating some properties of the diffusivity of $\mathbb{D}$.

6.1 L^2-estimates

We summarize the L^2 estimates deduced from the previous section. In the following estimates, C is a bound for different constants depending only on the initial data.

Lemma 6.1 *The following estimates are satisfied by solution f^α of the kinetic problem (P) constructed in the previous section under hypotheses 3.1 and 5.2;*

$$
\begin{aligned}
|f^\alpha|_{C^0(0,T;L^2(\Theta))} &\leq |F_I|_{L^2(\Theta)}, &(6.1)\\
|\mathcal{A}^\alpha f^\alpha|_{C^0(0,T;L^2(\Theta))} &\leq |\mathcal{A}^\alpha F_I|_{L^2(\Theta)}, &(6.2)\\
\int_0^T |P^+\gamma^+(f^\alpha)|^2_{L^2(\Gamma_+)}\,dt &\leq C\alpha^2|F_I|^2_{L^2(\Theta)}, &(6.3)\\
\int_0^T |P^-\gamma^-(f^\alpha)|^2_{L^2(\Gamma_-)}\,dt &\leq C\alpha^2|F_I|^2_{L^2(\Theta)}, &(6.4)\\
\int_0^T |q(f^\alpha)|^2_{\Gamma,R}\,dt &\leq C_R|F_I|^2_{H^\alpha(\mathcal{A}^\alpha)}, &(6.5)
\end{aligned}
$$

where C denotes generic constants independent of α and of the data and $q(f^\alpha)$ is defined by (5.13).

As a straight consequence of (6.1), $(f^\alpha)_\alpha$ admits a subsequence (still denoted by (f^α)) that converges to a function f^0 in $L^\infty(0,T;L^2(\mathbb{R}^3\times(0,\infty))$ weak star as $\alpha\to 0$. From the inequalities (6.3) and (6.4) we deduce that the convergence of the traces $P^+\gamma^+(f^\alpha)$ (resp. $P^-\gamma^-(f^\alpha)$) to 0 in $L^2(0,T;L^2(\Gamma_+))$ (resp. $L^2(0,T;L^2(\Gamma_-)))$ strongly towards zero. Furthermore, using the diagonal extraction process, (6.5) shows that $q(f^\alpha)$ converges to a function $q(x,\underline{\xi},|v|,t)$ with $x=0,1$ in $L^2(0,T;L^2(\gamma\times B_R))$ weak star for any R, where B_R is the ball centered at 0 and radius R in velocity space by properties of $H(div)$ spaces. In order to give a precise meaning to the limits of traces, we notice that the boundedness of $\mathcal{A}^\alpha f^\alpha$ in $H^{-1}(0,T;L^2(\Theta))$ by (2.6); this implies (according to [4], [18]) the boundedness of the sequence $(v_x f^\alpha)_\alpha|_\Gamma$ in $H^{-1}0,T;H^{1/2}(\gamma\times B_R)$. Thank to this estimate, we obtain the convergence in the distributional sense of the traces of f^α on $\Gamma\times\mathbb{R}^3$ to the traces of f^0. Finally, the traces $\gamma^\pm(f^0)$ on $\Gamma\times\mathbb{R}^3_\pm$ satisfy $P^-\gamma^-(f^0)=P^+\gamma^+(f^0)=0,\quad Q^-\gamma^-(f^0)=Q^+\gamma^+(f^0)=q$, so that

$$
f^0|_\Gamma = q, \tag{6.6}
$$

is independent of the angular part of the velocity variable.

6.2 Weak convergence of f^α and the continuity equation

From Green's formula (5.4) we deduce the following weak formulation of Eq. (2.6)-(2.8):

Lemma 6.2 *Let f^α be a solution to the kinetic problem (P) given by Proposition 5.7. Then, $\forall\,\psi\in C^1(\Theta'\times[0,T],\mathbb{R}^2)$ (i.e. twice continuosly differentiable and with compact support in $\mathbb{R}^2_{\underline{\xi}}\times\mathbb{R}^{+*}\times[0,T[$), such that $\psi(T,\cdot,\cdot)=0$, we have*

$$
\begin{aligned}
&\int_0^T\int_\Theta f^\alpha\left(\alpha\frac{\partial\psi}{\partial t}+(\underline{v}\cdot\nabla_{\underline{\xi}}-\underline{E}^\alpha\nabla_{\underline{v}})\psi+\frac{1}{\alpha}\left(v_x\frac{\partial}{\partial x}-(\underline{v}\times B)\cdot\nabla_{\underline{v}}\right)\psi\right)dtd\theta\\
&+\alpha\int_\Theta f_I\psi(0,\underline{\xi},\underline{v})\,d\theta \;=\; \frac{1}{\alpha}\left(\int_0^T\int_\Gamma^+ |v_x|\gamma^+(f^\alpha)\gamma^+(\psi)-\mathcal{B}^*\gamma^-(\psi)\,dtd\Gamma\right).
\end{aligned}\tag{6.7}
$$

Proof. Equation (6.7) is established by writing the weak formulation of the perturbed problem (5.5)-(5.8) and letting $\eta\to 0$ in this weak formulation.

As a consequence, we can prove that f^0 is a function of the total energy only.

Lemma 6.3 *The limit function f^0 is a function of $(\underline{\xi}, |v|, t)$ only, $f^0 = f^0(\underline{\xi}, \frac{1}{2}|v|^2, t)$.*

Proof. Using (6.7) with a test function ψ such that $\gamma^{\pm}(\psi) = 0$, we get

$$\alpha^2 \int_0^T \int_\Theta f^\alpha \frac{\partial \psi}{\partial t}\, dtd\theta + \alpha \int_0^T \int_\Theta f^\alpha (\underline{v} \cdot \nabla_{\underline{\xi}} \psi - \underline{E}^\alpha \nabla_{\underline{v}} \psi)\, dtd\theta \tag{6.8}$$
$$\int_0^T \int_\Theta f^\alpha \left(v_x \frac{\partial}{\partial x} \psi - (\underline{v} \times B) \cdot \nabla_{\underline{v}} \psi \right) dtd\theta + \int_\Theta f_I \psi|_{t=0}\, d\theta = 0.$$

Hence, when $\alpha \to 0$ in (6.8), using the fact that f^α is bounded in $L^\infty(0, T; L^2(\Theta))$, we get

$$\int_0^T \int_\Theta f^0 \left(v_x \frac{\partial}{\partial x} \psi - (\underline{v} \times B) \cdot \nabla_{\underline{v}} \psi \right) dtd\theta = 0. \tag{6.9}$$

Eq. (6.9) supplemented with (6.6), proves that f^0 is a distributional solution of equation

$$\left(v_x \frac{\partial}{\partial x} - (\underline{v} \times B) \cdot \nabla_{\underline{v}} \right) f^0 = 0, \qquad f^0|_\Gamma = q. \tag{6.10}$$

The remainder of the proof is a straightforward adaptation of Lemma 4.2 where we replace (4.6) by $\gamma^-(f) = q$.

It remains to prove that Eqs. (2.12)-(2.15) are satisfied in a weak sense. This is the object of the following

Lemma 6.4 *For any test function $\psi(\underline{\xi}, \varepsilon, t) \in C^2(\Theta' \times [0, T])$ with support compactly supported in $\mathbb{R}^2_{\underline{\xi}} \times \mathbb{R}^+ \times [0, T[$, we have*

$$\int_0^T \int_{\mathbb{R}^2 \times \mathbb{R}^{+*}} \left(4\pi\sqrt{2\varepsilon}\, F^\alpha \frac{\partial \psi}{\partial t} + \underline{J} \cdot \left(\nabla_{\underline{\xi}} - \underline{E} \frac{\partial}{\partial \epsilon} \right) \psi \right) dtd\theta'$$
$$+ \int_{\mathbb{R}^2 \times \mathbb{R}^{+*}} 4\pi\sqrt{2\varepsilon}\, F_I \psi|_{t=0}\, d\theta' = 0. \tag{6.11}$$

Proof. We apply (5.4) with ψ as a test function. Since ψ is even with respect to v, it is easy to check that the right-hand side in (6.7) vanishes. On the one hand,

$$\int_0^T \int_\Theta f^\alpha \frac{\partial \psi}{\partial t}\, dtd\theta \to \int_0^T \int_\Theta F \frac{\partial \psi}{\partial t}\, dtd\theta \tag{6.12}$$

for $f^\alpha \to F$ in $L^\infty(0, T; L^2(\Theta))$ weak-star and $\partial_t \psi \in L^1(0, T; L^2(\Theta))$. Moreover, on the grounds that $\int_\Theta f_I \psi(0, \underline{\xi}, \underline{v})\, d\theta = \int_{\mathbb{R}^2 \times \mathbb{R}^{+*}} 4\pi\sqrt{2\varepsilon} F_I \psi|_{t=0}\, d\theta'$ and $\left(v_x \frac{\partial}{\partial x} - (\underline{v} \times B) \cdot \nabla_{\underline{v}} \right) \psi = 0$, the result obviously follows from the coarea formula. The continuity equation (2.12) follows from (6.11) by taking the limit $\alpha \to 0$.

6.3 Equation for the current

Our goal in this section is to prove that the approximate current $\underline{J}^\alpha$ converges weakly towards a current $\underline{J}$ that satisfies (in a weak sense) the current equation (2.16) (with F the limit of $(f^\alpha)_\alpha$). We are first concerned with the consideration of the most general auxiliary problem of which (2.18) is a particular case. Given a function $g(x, \omega)$, find $\chi(x, \omega)$ such that:

$$\left(-v_x \frac{\partial}{\partial x} + (\underline{v} \times B) \nabla_{\underline{v}} \right) \chi(x, \omega) = g, \qquad (x, \omega) \in [0, 1] \times \mathbb{S}^2 \tag{6.13}$$
$$\gamma^+(\chi) = \mathcal{K}^*(\gamma^-(\chi)) \qquad (x, \omega) \in \mathcal{S}_+. \tag{6.14}$$

Once χ is determined, we deduce as a corollary, that there exist functions $\chi_i(x, \omega; \underline{\xi}, \epsilon)$, $(i = y, z)$, solutions of problem (6.13)-(6.14) with the right-hand side $g = \omega_i$, unique up to additive functions of $\underline{\xi}$ and $|v|$.

The existence of χ is provided by the following lemma which is shown by adapting the proof of Lemma 5.4 in [14] in a fairly straightforward way.

Lemma 6.5 *For all $g \in L^2([0,1] \times \mathbb{S}^2)$, the problem (6.13)-(6.14) has a solution if and only if g satisfies*

$$\int_0^1 \int_{\mathbb{S}^2} g(x,\omega) dx d\omega = 0. \tag{6.15}$$

Furthermore, if this condition is satisfied, the solution χ is unique under the condition

$$\int_0^1 \int_{\mathbb{S}^2} \chi(x,\omega) dx d\omega = 0 \tag{6.16}$$

and the set of solutions is the one-dimensional linear manifold $\{\chi + F(\varepsilon),\ \textit{with } F(\varepsilon) \textit{ arbitrary}\}$.

Corollary 6.6 *The function $g = \omega_i, \quad (i = y,z)$ satisfies the assumptions of Lemma 6.5.*

Therefore, the auxiliary function $\underline{\chi}$ defined by (2.18) exists and is unique under the constraint (6.16).

We require the following regularity assumptions on χ_i:

Hypothesis 6.7 *(i) χ_i, $(i = y,z)$, belongs to $L^2([0,1] \times \mathbb{S}^2)$ for almost every $(\underline{\xi}, \epsilon) \in \mathbb{R}^2_{\underline{\xi}} \times \mathbb{R}^+_\epsilon$ and are C^1 bounded functions on Θ away from the set $\{v_x = 0\}$.*

(ii) The functions $\omega_i \chi_j(x,\omega;\xi,\varepsilon)$ belongs to $L^1([0,1] \times \mathbb{S}^2)$ and $\int_0^1 \int_{\mathbb{S}^2} \omega_i \chi_j dx d\omega$ is a C^1 function of $(\underline{\xi}, \varepsilon) \in \mathbb{R}^2_{\underline{\xi}} \times \mathbb{R}^+_\varepsilon$, uniformly bounded on $\mathbb{R}^2_\xi \times [0,\infty[_\varepsilon$ and tending to 0 as $\varepsilon \to 0$.

In order to appreciate the importance of this hypothesis and, in particular, to realize that this hypothesis is not empty, we refer the reader to the example of isotropic scattering, where Ref. [14] have computed explicitly the diffusion coefficient.

From Hypothesis 6.7 (ii), we deduce that the diffusivity tensor (2.17) is defined and is C^1 function of $(\underline{\xi}, \varepsilon) \in \mathbb{R}^2_{\underline{\xi}} \times \mathbb{R}^+_\varepsilon$.

We are now in a position to establish the current equation. Actually we shall prove that J^α has a finite limit.

Lemma 6.8 *J^α is bounded in $L^2([0,T] \times \Theta)$. As $\alpha \to 0$, $J^\alpha \rightharpoonup J$ in the distributional sense and its limit satisfies a weak form of the current equation. More precisely: For any test function $\underline{\psi} = (\psi_y, \psi_z)$ in $C^1(\Theta' \times [0,T], \mathbb{R}^2)$ (i.e. twice continuosly differentiable and with compact support in $\mathbb{R}^2_{\underline{\xi}} \times \mathbb{R}^{+*} \times [0,T[)$, we have*

$$\int_0^T \int_{\mathbb{R}^2 \times \mathbb{R}^{+*}} \underline{J}^\alpha \cdot \underline{\psi}\, dt d\theta' \rightharpoonup \int_0^T \int_{\mathbb{R}^2 \times \mathbb{R}^{+*}} F \left(\nabla_{\underline{\xi}} - \underline{E} \frac{\partial}{\partial \varepsilon} \right) \cdot (\mathbb{D}^T \underline{\psi})\, dt d\theta', \tag{6.17}$$

as $\alpha \to 0$, where $\mathbb{D}^T$ denotes the transpose of $\mathbb{D}$. The right-hand side of equation (6.17) is the weak form of the current equation (2.13).

Proof. (i) Since J^α is a combination of $\gamma^+(J^\alpha)$ and $\gamma^-(J^\alpha)$, to prove the boundedness of the current in $L^2([0,T] \times \Theta)$, it is enough to prove that $\gamma^+(J^\alpha)$ and $\gamma^-(J^\alpha)$ separately are bounded in this space. We prove it for $\gamma^+(J^\alpha)$, the proof being similar for $\gamma^-(J^\alpha)$. First, we note that

$$\langle v \gamma^+(f^\alpha) \rangle = \langle v P^+ \gamma^+(f^\alpha) \rangle, \tag{6.18}$$

by symmetry. Then, using coarea formula and estimate (6.3) we get

$$\begin{aligned} |\gamma^+(J^\alpha)|^2_{L^2([0,T]\times\Theta)} &= \int_0^T \int_0^\infty \frac{1}{\alpha^2} \left| \langle v P^+ \gamma^+(f^\alpha) \rangle \right|^2 d\epsilon dt \\ &\leq \frac{C}{\alpha^2} \int_0^T \int_0^\infty \left| \int_{\Gamma_+} \omega P^+ \gamma^+(f^\alpha) d\Gamma \right|^2 d\epsilon dt \\ &\leq \frac{C}{\alpha^2} \int_0^T |P_+ \gamma^+(f^\alpha)|^2_{L^2(\Gamma_+)}\, dt \leq C |F_I|^2_{L^2([0,T]\times\Theta)}. \end{aligned} \tag{6.19}$$

The result follows straightforwardly.

(ii) Now, we choose ϕ as a test function $\phi(\underline{\xi},\varepsilon,t) = \sqrt{2\varepsilon}\underline{\psi}(\underline{\xi},\varepsilon,t)\cdot\underline{\chi}(\underline{\xi},\omega;\varepsilon,t)\mathbf{1}_\rho(v_x)$ where $\mathbf{1}_\rho(v_x)$ is given by

$$\mathbf{1}_\rho(v_x) = \left\{ \begin{array}{ll} 0 & |v_x| \le \rho \\ 1 & |v_x| \ge 2\rho. \end{array} \right. \tag{6.20}$$

The hypothesis 6.7 makes it possible to pass to the limit $\rho \to 0$ in (6.20). Because of (6.13)-(6.14) we have for $i \in \{y,z\}$:

$$\begin{aligned} \left(v_x\frac{\partial}{\partial x} - (\underline{v}\times B)\cdot\nabla_{\underline{v}}\right)(\sqrt{2\varepsilon}\psi_i\chi_i) &= \sqrt{2\varepsilon}\psi_i\left(v_x\frac{\partial}{\partial x} - (\underline{v}\times B)\cdot\nabla_{\underline{v}}\right)(\omega_i) \\ &= -\sqrt{2\varepsilon}\psi_i\chi_i, \qquad \text{in } \Theta \end{aligned} \tag{6.21}$$

and

$$\gamma^+(\sqrt{\varepsilon}\psi_i\chi_i) - \mathcal{K}^*\gamma^-(\sqrt{\varepsilon}\psi_i\chi_i) = \sqrt{\varepsilon}\psi_i[\gamma^+(\chi_i) - \mathcal{K}^*\gamma^-(\chi_i)] = 0, \qquad \text{on } \Gamma. \tag{6.22}$$

So that, using (6.21)-(6.22) together with coarera formula (2.11), we get

$$\begin{aligned} &\frac{1}{\alpha}\int_0^T\!\!\int_\Theta f^\alpha\left(v_x\frac{\partial}{\partial x} - (\underline{v}\times B)\cdot\nabla_{\underline{v}}\right)(\sqrt{2\varepsilon}\underline{\psi}\,\underline{\chi})\ dtd\theta \\ -\ &\frac{1}{\alpha}\int_0^T\!\!\int_{\Gamma^+}|v_x|f^\alpha(\gamma^+(\sqrt{2\varepsilon}\cdot\underline{\chi}\,\underline{\psi}) - \mathcal{K}^*\gamma^-(\sqrt{2\varepsilon})\cdot\underline{\chi}\,\underline{\psi})\ dtd\Gamma \\ =\ &-\frac{1}{\alpha}\int_0^T\!\!\int_\Theta f^\alpha\underline{\omega}\cdot\underline{\psi}\sqrt{2\varepsilon}dtd\theta = -\int_0^T\int_\Theta \underline{J}^\alpha\cdot\underline{\psi}dtd\theta', \end{aligned} \tag{6.23}$$

which justifies the introduction of the auxiliary function $\underline{\chi}$. Thus, the weak formulation (6.7) yields:

$$\begin{aligned} \int_0^T\!\!\int_{\Theta'}\underline{J}^\alpha\underline{\psi}dtd\theta' &= \alpha\int_0^T\!\!\int_\Theta\sqrt{2\varepsilon}f^\alpha\underline{\chi}\cdot\frac{\partial}{\partial t}\underline{\psi}dtd\theta + \alpha\int_\Theta\sqrt{2\varepsilon}f_I\underline{\chi}\cdot\underline{\psi}|_{t=0}d\theta \\ &+ \int_0^T\!\!\int_\Theta f^\alpha(\underline{v}\cdot\nabla_{\underline{\xi}} - \nabla_{\underline{\xi}}\phi\cdot\nabla_{\underline{v}})(\sqrt{2\varepsilon}\underline{\chi}\cdot\underline{\psi})dtd\theta. \end{aligned} \tag{6.24}$$

Now, by letting $\alpha \to 0$ in (6.24), since $\chi \in L^2([0,1]\times\mathbb{S}^2)$, for almost every $(\xi,\varepsilon) \in \mathbb{R}^2_{\underline{\xi}}\times\mathbb{R}^+_\varepsilon$, the first and second terms on the right hand side converge to 0. Then

$$\begin{aligned} \lim_{\alpha\to 0}\int_0^T\!\!\int_{\Theta'}\underline{J}^\alpha\underline{\psi}dtd\theta' &= \int_0^T\!\!\int_\Theta f^\alpha\underline{v}\cdot\nabla_{\underline{\xi}}(\sqrt{2\varepsilon}\underline{\chi}\cdot\underline{\psi})dtd\theta - \int_0^T\!\!\int_\Theta\nabla_{\underline{\xi}}\phi\cdot\nabla_{\underline{v}})(\sqrt{2\varepsilon}\underline{\chi}\cdot\underline{\psi})dtd\theta \\ &= L_1 - L_2. \end{aligned} \tag{6.25}$$

On one hand, using (4.7) we have

$$\begin{aligned} L_1 &= \int_0^T\!\!\int_\Theta f^0\cdot\nabla_{\underline{\xi}}(2\varepsilon\underline{\omega}(\underline{\chi}\cdot\psi))\ dtd\theta = \int_0^T\!\!\int_\Theta F\nabla_{\underline{\xi}}\cdot\left((2\varepsilon)^{3/2}\int_0^1\!\!\int_{\mathbb{S}^2}\underline{\omega}(\underline{\chi\psi})dxd\omega\right)dtd\theta \\ &= \int_0^T\!\!\int_{\Theta'}F\nabla_{\underline{\xi}}\cdot(\mathbb{D}^T\underline{\psi})\ dtd\theta'. \end{aligned} \tag{6.26}$$

On the other hand, using Lemma 4.19, we get:

$$\begin{aligned} L_2 &= \int_0^T\!\!\int_\Theta f^0(\nabla_{\underline{\xi}}\phi\nabla_{\underline{v}})(\sqrt{2\varepsilon}\underline{\chi}\cdot\underline{\psi})\ dtd\theta \\ &= \int_0^T\!\!\int_{\Theta'}F\nabla_{\underline{\xi}}\phi\cdot\left(\sqrt{2\varepsilon}\int_0^1\!\!\int_{\mathbb{S}^2}\nabla_{\underline{v}}(\sqrt{2\varepsilon}\underline{\chi}\cdot\underline{\psi})dxd\omega\right)\ dtd\theta' \\ &= \int_0^T\!\!\int_{\Theta'}F\nabla_{\underline{\xi}}\phi\cdot\frac{\partial}{\partial\varepsilon}\left((2\varepsilon)^{3/2}\int_0^1\!\!\int_{\mathbb{S}^2}\underline{\omega}(\underline{\chi})dxd\omega\right)\ dtd\theta' = \int_0^T\!\!\int_{\Theta'}F\nabla_{\underline{\xi}}\frac{\partial}{\partial\varepsilon}(\mathbb{D}^T\underline{\psi})\ dtd\theta'; \end{aligned} \tag{6.27}$$

so that, combining (6.26) and (6.27) to conclude.

6.4 Passing to the limit in the nonlinear term

Notice that the linear term of (6.7) converges because of the weak convergence of f^α. To pass to the limit in the nonlinear term of Eq. (6.7), we need the strong convergence of E^α. For this task, we use the Poisson equation to prove some compactness on the electric fields.

On the one hand, thanks to the classical regularizing properties of the Laplacian, $F^\alpha(\underline{\xi}, \varepsilon, t)$ lies in $L^\infty(0,T;L^2(\Theta'))$, implies that, since ϕ^α solves the Poisson equation, $\phi^\alpha(\underline{\xi}, t)$ belongs to $L^\infty(0,T;W^{2,p}(\mathbb{R}^2_{\underline{\xi}}))$, for all $\in [1,+\infty[$. In particular $\nabla_{\underline{\xi}}\phi^\alpha \in L^\infty(0,T;W^{1,p}(\mathbb{R}^2_{\underline{\xi}}))$, for some $p > 3$ and Sobolev's imbedding leads to $\nabla_{\underline{\xi}}\phi^\alpha \in L^\infty(\mathbb{R}^2_{\underline{\xi}})$.

On the other hand, by Poisson equation, we have

$$E^\alpha = \nabla_{\underline{\xi}}\Delta^{-1}_{\underline{\xi}}(F^\alpha - C(\underline{\xi})) \tag{6.28}$$

The relation (6.28) together the continuity equation give

$$\partial_t E^\alpha = \nabla_{\underline{\xi}}\Delta^{-1}_{\underline{\xi}} \cdot \int f^\alpha \, v dx dv. \tag{6.29}$$

Hence, as $\underline{J}^\alpha$ is bounded in $L^2(0,T;L^2(\Theta'))$ by Lemma 6.8, we conclude that $\partial_t F^\alpha$ is bounded in $L^2(0,T;W^{-1,2}(\Theta'))$ and from (6.29) the regularizing properties of the Laplacian now yield $\partial_t\phi^\alpha$ bounded in $L^\infty(0,T;W^{1,2}(\mathbb{R}^2_{\underline{\xi}}))$, which yields $\partial_t E^\alpha$ bounded in $L^\infty(0,T;L^2(\mathbb{R}^2_{\underline{\xi}}))$. Thus, as $W^{1,2}(\mathbb{R}^2_{\underline{\xi}}) \hookrightarrow L^2_{loc}(\mathbb{R}^2_{\underline{\xi}})$, the Aubin-Lions Lemma (see [2], [25]) yields that the functional space

$$\mathcal{M} = \{E \in L^\infty(0,T;W^{1,2}(\mathbb{R}^2_{\underline{\xi}})), \quad \partial_t E \in L^\infty(0,T;L^2(\mathbb{R}^2_{\underline{\xi}}))\},$$

provided with the usual product norm, is compactly embedded in $L^\infty(0,T;L^2_{loc}(\mathbb{R}^2_{\underline{\xi}}))$ and consequently, as E^α is bounded in $\mathcal{M}$, there exists a subsequence of E^α which converges strongly in $L^\infty(0,T;L^2_{loc}(\mathbb{R}^2_{\underline{\xi}}))$. Its remains to prove that E is the solution to the Poisson equation. To do so, we notice that, $\phi^\alpha \in L^\infty(0,T;W^{2,p}(\mathbb{R}^2_{\underline{\xi}}))$ and bounded in this space. Then, up an extraction, we have

$$\phi^\alpha \rightharpoonup \phi \quad L^\infty(0,T;W^{2,p}(\Theta')) \quad \text{weak-star}.$$

Thus, it is easy to notice that ϕ is solution of Poisson equation in the sense of distribution.

Hence, we have

Lemma 6.9 *Under hypotheses 5.1 and 5.2, let (f^α) be a family of a weak solution of the system (2.6)-(2.7). Then, extracting a subsequence, the family (E^α) satisfies*

$$E^\alpha \longrightarrow E \quad L^\infty(0,T;L^2_{loc}(\mathbb{R}^2_{\underline{\xi}})) \quad \textit{strong},$$

as $\alpha \to 0$.

6.5 Positivity of the diffusion tensor

We are now going to discuss some properties of the diffusivity of $\mathbb{D}$. We show that $\mathbb{D}$ is a positive definite tensor. Therefore (2.12)-(2.16) are well-posed. To this end, we prove

Lemma 6.10 *The diffusion tensor $\mathbb{D}$ defined by (2.17) is positive definite; this means that, for all $\underline{\xi} \in \mathbb{R}^2$ and all $\epsilon_0 > 0$, there exists $C = C(\epsilon_0) > 0$ such that:*

$$(\mathbb{D}Y, Y) = \sum_{i,j}^{2} \mathbb{D}_{ij} Y_i Y_j \geq C|Y|^2 = C \sum_{i=1}^{2} Y_i^2, \qquad \forall\, Y, \underline{\xi} \in \mathbb{R}^2, \qquad \forall\, \epsilon \geq \epsilon_0. \tag{6.30}$$

Proof. The proof is inspired from [14]. Let $Y = (y_1, y_2) \in \mathbb{R}^2$ such that $|Y| > 0$ and $\Phi(x,\omega) = \sum_{i=1}^2 y_i\chi_i(x,\omega)$. By virtue of (2.17), we get

$$(\mathbb{D}Y, Y) \quad = \quad (2\epsilon)^{3/2} \int_0^1 \int_{\mathbb{S}^2} \left(\sum_{i=1}^{2} \chi_i y_i\right) \left(\sum_{i=1}^{2} \omega_i y_i\right) dx d\omega \tag{6.31}$$

Inserting (2.18) into (6.31) yields

$$
\begin{aligned}
(\mathbb{D}Y, Y) &= (2\epsilon)^{3/2} \int_0^1 \int_{\mathbb{S}^2} \left(\sum_{i=1}^{2} \chi_i y_i \right) \left(-v_x \frac{\partial \chi_i}{\partial x} - (v \times B) \nabla_{\underline{v}}) \chi_i y_i \right) dx d\omega \\
&= (2\epsilon)^{3/2} (\mathcal{A}^{0*}\Phi, \Phi)_{\mathcal{S}}.
\end{aligned} \tag{6.32}
$$

On the other hand, Green's formula leads to

$$
\begin{aligned}
2(\mathbb{D}Y, Y) &= (2\epsilon)^{3/2} \left(\int_{\mathcal{S}_-} |\omega_x| |\gamma^-(\Phi)|^2 d\omega - \int_{\mathcal{S}_+} |\omega_x| |\gamma^+(\Phi)|^2 d\omega \right) \\
&\geq (2\epsilon_0)^{3/2} \left(|\gamma^-(\Phi)|^2_{L^2(\mathcal{S}_-)} - |\mathcal{B}^* \gamma^-(\Phi)|^2_{L^2(\mathcal{S}_+)} \right) \geq 0.
\end{aligned} \tag{6.33}
$$

On the grounds that $|\omega_x|$ is uniformly bounded by positive constant, we deduce that $\mathbb{D}$ is definite. Next, assume that $\mathbb{D}Y \cdot Y = 0$. Then we deduce that Φ does not depend on $\underline{\omega}$. So that $\mathcal{A}^{0*}\Phi = \underline{\omega} \cdot Y$. Since $\mathcal{A}^{0*}\Phi$ does not depend on $\underline{\omega}$, we have $\underline{\omega} \cdot Y$ independent of $\underline{\omega}$ for Ω in unit disk. This is only possible if $Y = 0$ and $\mathbb{D}$ is positive definite, and since $(\underline{\xi}, \epsilon) \longmapsto \mathbb{D}(\underline{\xi}, \epsilon)$ is smooth, we deduce that $\mathbb{D}$ is uniformly positive definite. This completes the proof.

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References

[1] A. A. Arsen'ev, Existence in the large of a weak solution of Vlasov's system of equations, Z. Vycisl. Math. i Mat. Fiz 15 (1975) 136–147, 276 (Russian).

[2] J.-P. Aubin, Un théorème de compacité, C. R. Acad. Sci. Paris 256 (1963), 5042-5044.

[3] H. Babovsky, C. Bardos, T. Platkwoski, Diffusion approximation for a Knudsen gas in thin domain with accommodation on the boundary, Asymptotic Analysis, 3, (1991), p. 265-289.

[4] C. Bardos, Problèmes aux limites pour les équations aux dérivées partielles du premier ordre à coefficients réels; Théorèmes d'approximations; application à l'équation de transport, Ann. Sci. Ec. Norm. Sup, 4, (1970), pp. 185-233.

[5] C. Bardos, C. Santos, R. Sentis, Diffusion, Approximation and Computation of the Critical Size". Trans. Amer. Math. Soc., 284 A (1984), 617–649.

[6] R. Beals, V. Protopopescu, Abstract Time Dependent Transport Problems, J. Math. Anal. Appl. (1987), 121, 370.

[7] N. Benabdallah, P. Degond, On a hierarchy of macrpscopic models for semiconductors, J. Maths. Phys. 37, (1996), 3306–3333.

[8] A. Bensoussan, J.L. Lions, G. C. Papanicolaou, Boundary layers and homogenization of transport processes. J. Publ. RIMS Kyoto Univ. 15, (1979), 53–157.

[9] C. Cercignani, R. Illner, M. Pulvirenti, The mathematical theory of dilute gases, Applied Mathematical Sciences, 106. Springer-Verlag, New York, 1994.

[10] M. Cessenat, Théorèmes de traces L^p pour les espaces de fonctions de la neutronique, C.R. Acad. sci. Paris. Ser. I, Math., 299 (1984), 831–834.

[11] S. Chapman, T.G. Cowling, The mathematical theory of non-uniform gases, Cambridge University Press, New-York, 1958.

[12] J. Darrozès, J. P. Guiraud, Généralisation formelle du théorème H en présence de parois, Applications, C.R. Acad. sci. Paris. Ser. A, 262, (1966) 1368–1371.

[13] P. Degond, V. Latocha, S. Mancini, A. Mellet, Diffusion dynamics of an electron gas between two plates, Methods Appl. Anal. 9 , (2002), 127–150.

[14] P. Degond, S. Mancini, Diffusion driven by collisions with the boundary. Asymptot. Anal. 27 (2001), no. 1, 47-73.

[15] C. Dogbe, Anomalous Diffusion Limit for a Knudsen Gas in Thin Domain with Accommodation on the Boundary. J. of Stat. Physics, Vol. 100, 3/4, (2000), 603-632.

[16] P. Dmitruk, A. Saúl, L. Reyna, High electric field approximation to charge transport in semiconductor devices. Appl. Math. Lett. 5 (1992), no. 3, 99–102.

[17] H. Federer, Geometric Measure Theory, Springer-Verlag, New York, 1969.

[18] V. Girault, P-A. Raviart, Finite Element Methods for Navier-Stokes Equations. Springer-Verlag, Berlin 1986.

[19] T. Goudon, A. Mellet, On fluid limit for the semiconductors Boltzmann equation, J. Differential Equations, J. Diff. Eq., 189, (2003), 17-45.

[20] Y. Guo, Regularity for the Vlasov equations in a half-space, Indiana Univ. Math. J. 43 (1994), no. 1, 255–320.

[21] O. Hansen, A. Jüngel, Analysis of Spherical harmonics expansion model of physics, Math Models and Meth. in Appl. Sci. vol 14, n 5, (2004) 759–774.

[22] A. Jüngel, Y-J. Peng, A hierarchy of hydrodynamic models for plasmas. Quasi-neutral limits in the drift-diffusion equations, Asymptotic Analysis, 28 (2001), 49–73.

[23] M. G. Krein, M. A. Rutman, Linear operator leaving invariant a cone in Banach space. Uspehi Mat. Nauk. 3, no 1 (123) 3-95 (1948). Translation by AMS Trans. ser. 1, Vol 10, 199-325.

[24] E. W. Larsen, J. B Keller, Asymptotic Solutions of Neutron Transport Problem, J. Math. Phys., 15 (1974), 75–81.

[25] J. L. Lions, Quelques méthodes de résolution de problèmes aux limites non linéaires. Dunod, Gauthier-Villars, 1969.

[26] S. Mischler, On the trace problem for solutions of Vlasov equation, Comm. in P.D.E., 25 (2000), 1415-1143.

[27] K. Plaffelmoser, Global classical solution of the Vlasov-Poisson system in three dimensions for general initial data, J. Diff. Eq. 95 (1992), 281-303.

[28] S. Ukai, Solutions of the Boltzmann equation, Patterns and Waves. Qualitative Analysis of Nonlinear Differential equations, North-Holland, Amsterdam, 1986, 37–96.

[29] E. Zeidler, Nonlinear functional analysis and its applications. I. Fixed-point theorems. Springer-Verlag, New York, 1986.