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# The solution of Vlasov's equation for complicated plasma geometry. I. Spherical type

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**Abstract**

The asymptotic solution of Vlasov's equation for complicated plasma geometry (spherical type) using the computer algebra system **Maple** is presented. The approximation of small ion and electron Larmor radii (drift approximation) for a dipole magnetic field configuration modeling the earth's plasmasphere was used. The method of solution introduces elliptic integrals as variables, which makes it possible to obtain results in a relatively simple manner. A bounce resonance factor (connected with trapped particles) different from the one appearing in the Landau damping case is found.

## 1. Introduction

Most tokamak radio-frequency heating problems [1] and space plasma phenomena (for example, in the solar corona [2] and in the earth's magnetosphere [3]) are explained through the wave-particle interaction model. In this context, collisionless plasma problems are well described by the set of Vlasov–Maxwell equations

$$\frac{\partial f}{\partial t} + \boldsymbol{v} \cdot \frac{\partial f}{\partial \boldsymbol{r}} + \frac{e}{m}\left(\boldsymbol{E} + \frac{1}{c}(\boldsymbol{v} \times \boldsymbol{H})\right)\frac{\partial f}{\partial \boldsymbol{v}} = 0, \tag{1}$$

$$\nabla \times \boldsymbol{H} = \frac{4\pi}{c}\boldsymbol{J} + \frac{1}{c}\frac{\partial \boldsymbol{E}}{\partial t}, \qquad \nabla \times \boldsymbol{E} = -\frac{1}{c}\frac{\partial \boldsymbol{H}}{\partial t}, \tag{2}$$

where $f = f(t, \boldsymbol{r}, \boldsymbol{v})$ is the distribution function, $e$ and $m$ are the charge and mass of ions or electrons, and $\boldsymbol{r}$ and $\boldsymbol{v}$ are the mean position and velocity of the charged particles.

Now, it is possible to tackle these problems by finding analytic solutions for Vlasov's and Maxwell's equations separately through a linear approximation over the wave field amplitude. The required conditions for this separation are: to ignore the electron and ion drift motions across the magnetic surfaces, and to assume that the magnetic field, plasma density, and temperature inhomogeneity parameters are quite small in comparison with the electron and ion Larmor radii [3–5].


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For a homogeneous plasma in a straight magnetic field line configuration, the solution of Vlasov's equation through a linear approximation was found by Landau:

$$f = \frac{ieE_z}{m(k_z v_z - \Omega)} \left(\frac{\partial F_M}{\partial v_z}\right) \exp[i(k_z z - \Omega t)], \tag{3}$$

where $F_M$ is the standard equilibrium Maxwell distribution function, $k_z$ is the projection of the wave vector over the magnetic field, and the resonant denominator $(k_z v_z - \Omega)$ is responsible for the wave dissipation phenomenon known as Landau damping.

For a plasma immersed in a non-homogeneous magnetic field [1], the situation is more complicated, and includes the existence of turning points for the particles' velocity,

$$v_{\|} = \pm|v|\sqrt{1 - \Lambda|\boldsymbol{H}(l)|/|\boldsymbol{H}|_{\min}}, \tag{4}$$

where $l$ is a length along magnetic field lines, $|\boldsymbol{H}|_{\min}$ is the minimum of the magnetic field, $\Lambda$ is an adiabatic invariant, and $v_{\|}$ has turning points at the zeros of the square root, generating a trapped particle effect around $|\boldsymbol{H}|_{\min}$. As we shall see, this periodic motion generates an infinite number of bounce resonances, associated to a resonant denominator different from the one found by Landau for the straight magnetic field line case.

In a previous paper [6], we showed how to obtain the explicit form of Vlasov's equation for complicated plasma geometry (toroidal and spherical case) using the Maple computer algebra system. The importance of the computer approach resides in that the scheme can be adapted in a relatively simple manner to other geometries.

In this paper, we use the Maple system again and show the analytical solution of the linearized Vlasov equation for the case of a dipole magnetic field configuration (earth's magnetosphere [3]), starting from the explicit form of the equation, in spherical coordinates, obtained in Ref. [6]. The method here used is an extension of the one developed for the toroidal case [4,5,7].

The goals of this work can be summarized as:

- to develop a mathematical/computational strategy suitable both for the specific problem treated in this paper and (with few changes) for different geometric magnetic field configurations, using a personal computer and any computer algebra system [2];
- to obtain a (new) result in the specific case of the dipole configuration mentioned above.

The paper is organized as follows. In Section 2, the method of solution of Vlasov's equation in the drift approximation and the result obtained for the distribution function $f$, expressed as a sum of equilibrium and linearized parts, with an explicit display of the resonant denominator mentioned above, are shown. This section is divided into four subsections which make explicit the mathematical/computational strategy proposed for the problem. In this way, the line of reasoning can be followed by looking at the beginning of Section 2 and at the summarized explanations in each subsection. The reader interested only in the resulting expression for the distribution function $f$ can go straight to the end of Section 2.

Finally, the Conclusions contains some remarks about the computer approach used here, a discussion about the influence of the magnetic field geometry on plasma wave dissipation, and suggestions of possible extensions of this work.

Aside from this, a description of some special commands, built for and used along this work, are given in the Appendix.

[1] For example, in a mirror trap [3] or tokamak [4,5] magnetic field configuration.

[2] This work was thoroughly done using the **MAPLE 5.2** Computer Algebra system *for Windows* in a DOS platform.

## 2. Computer algebra solving of Vlasov's equation

The problem of finding the distribution function $f$ of Eq. (1) for a dipole magnetic field geometry [3] should be solved in the context of the complete set of Vlasov–Maxwell equations. Nevertheless, one could, as a simple approach (drift approximation), attempt to solve Vlasov's and Maxwell's equations separately [4,7], assuming that the magnetic field inhomogeneity parameter is of the order of the plasma radius. This approach will be used here, and what follows is the evaluation of the distribution function as the solution of Vlasov's equation (spherical geometry) alone.

The exposition appears split into four subsections, each one related to a different Maple session. Inside each subsection, an itemized explanation of its goals is found and, because of the great size of the expressions, only the most relevant information and the final results are displayed.

The departure point will be the explicit form of Vlasov's equation, obtained in Ref. [6], here presented as [4]

$$
\begin{aligned}
&\frac{\partial f(\mathrm{var})}{\partial t} + \left[h_\theta v_{\mathrm{pe}}\cos(\sigma) + h_r v_{\mathrm{par}}\right]\frac{\partial f(\mathrm{var})}{\partial r} + \left(-\frac{\cos(\sigma)h_r v_{\mathrm{pe}}}{r} + \frac{h_\theta v_{\mathrm{par}}}{r}\right)\frac{\partial f(\mathrm{var})}{\partial \theta} \\
&+ \frac{v_{\mathrm{pe}}\sin(\sigma)}{r\sin(\theta)}\frac{\partial f(\mathrm{var})}{\partial \phi} + \left\{\left[\left(\frac{\partial h_\theta/\partial\theta}{h_r r} + \frac{1}{r}\right)h_\theta + \frac{\partial h_\theta}{\partial r}\right]\cos(\sigma)v_{\mathrm{par}}^2\right. \\
&+ \left[\left(-\frac{\cos(\theta)h_\theta}{r\sin(\theta)} - \frac{h_r}{r}\right)\sin(\sigma)^2\right. \\
&\left.\left. + \left(\frac{h_\theta^2}{h_r r} + \frac{(\partial h_\theta/\partial r)\,h_\theta}{h_r} - \frac{\partial h_\theta/\partial\theta}{r} - \frac{1}{h_r r}\right)\cos(\sigma)^2\right] v_{\mathrm{pe}}v_{\mathrm{par}}\right\}\frac{\partial f(\mathrm{var})}{\partial v_{\mathrm{pe}}} \\
&+ \left\{\left(-\frac{h_\theta}{r} + \frac{\cos(\theta)h_r}{r\sin(\theta)}\right)\sin(\sigma)v_{\mathrm{pe}} + \left[\left(-\frac{\partial h_\theta/\partial r}{h_r} - \frac{\cos(\theta)}{r\sin(\theta)}\right)h_\theta + \frac{\partial h_\theta/\partial\theta}{r}\right]\sin(\sigma)\cos(\sigma)v_{\mathrm{par}}\right. \\
&\left. + \left[\left(-\frac{\partial h_\theta/\partial\theta}{h_r r} - \frac{1}{r}\right)h_\theta - \frac{\partial h_\theta}{\partial r}\right]\frac{\sin(\sigma)v_{\mathrm{par}}^2}{v_{\mathrm{pe}}}\right\}\frac{\partial f(\mathrm{var})}{\partial \sigma} + \left\{\left[\left(\frac{\cos(\theta)h_\theta}{r\sin(\theta)} + \frac{h_r}{r}\right)\sin(\sigma)^2\right.\right. \\
&\left. + \left(-\frac{h_\theta^2}{h_r r} - \frac{(\partial h_\theta/\partial r)\,h_\theta}{h_r} + \frac{\partial h_\theta/\partial\theta}{r} + \frac{1}{h_r r}\right)\cos(\sigma)^2\right] v_{\mathrm{pe}}^2 \\
&\left. + \left[\left(-\frac{\partial h_\theta/\partial\theta}{h_r r} - \frac{1}{r}\right)h_\theta - \frac{\partial h_\theta}{\partial r}\right]\cos(\sigma)v_{\mathrm{pe}}v_{\mathrm{par}}\right\}\frac{\partial f(\mathrm{var})}{\partial v_{\mathrm{par}}} \\
&+ \left[\left(-\frac{e\sin(\sigma)H_n}{mc} + \frac{e\cos(\sigma)H_\phi}{mc}\right)v_{\mathrm{par}} + \frac{e\cos(\sigma)E_n}{m} + \frac{eE_\phi\sin(\sigma)}{m}\right]\frac{\partial f(\mathrm{var})}{\partial v_{\mathrm{pe}}} \\
&+ \left(\frac{eH_{\mathrm{par}}}{mc} + \frac{\left[-e\cos(\sigma)H_n/mc - e\sin(\sigma)H_\phi/mc\right]v_{\mathrm{par}}}{v_{\mathrm{pe}}}\right. \\
&\left. + \frac{-e\sin(\sigma)E_n/m + eE_\phi\cos(\sigma)/m}{v_{\mathrm{pe}}}\right)\frac{\partial f(\mathrm{var})}{\partial \sigma} \\
&+ \left[\left(\frac{e\sin(\sigma)H_n}{mc} - \frac{e\cos(\sigma)H_\phi}{mc}\right)v_{\mathrm{pe}} + \frac{eE_{\mathrm{par}}}{m}\right]\frac{\partial f(\mathrm{var})}{\partial v_{\mathrm{par}}} = 0,
\end{aligned}
\tag{5}
$$

[3] This geometry is appropriate to represent the magnetic field configuration of the earth's plasmasphere.

[4] The equation appearing in Ref. [6] (as Eq. (38)) has two mistakes: (1) in line 7, the first term appears as $h_\theta/(rh_r)$ (wrong), instead of $h_\theta^2/(rh_r)$ (right); (2) in the same line, the next term appears as $h_\theta(\partial h_\theta/\partial r)/(rh_r)$ (wrong), instead of $h_\theta(\partial h_\theta/\partial r)/(h_r)$ (right).

where $r$ is the radial coordinate, $\theta$ and $\phi$ are the polar and azimuthal angles of the spherical coordinate system, $v_{pe}$ (perpendicular) and $\sigma$ are local-polar velocity coordinates, $v_{par}$ (parallel) is the projection of $\boldsymbol{v}$ on the $\boldsymbol{H}$ lines, var represents $(t, r, \theta, \phi, v_{pe}, \sigma, v_{par})$, and $h_r$ and $h_\theta$ are the components of the magnetic field unit vector $\boldsymbol{H}/|\boldsymbol{H}|$.

To begin the discussion, our general idea was to look for the distribution function as

$$\begin{aligned} f(\text{var}) &= F_{[\mathrm{M}]} + F_{[\mathrm{b}]}\sin(\sigma) + F_{[\mathrm{r}]}\cos(\sigma) \\ &\quad + \left[f_{-[0]} + f_{-[\mathrm{r}]}\cos(\sigma) + f_{-[\mathrm{b}]}\sin(\sigma)\right]\exp[i(n\phi - \Omega t)], \end{aligned} \tag{6}$$

where $F_{[\mathrm{M}]}$ represents the standard equilibrium Maxwell distribution function, and $F_{[\mathrm{b}]}$ and $F_{[\mathrm{r}]}$ represent the first order binormal and radial corrections of $F_{[\mathrm{M}]}$, related to the smallness of the drift parameter

$$\delta = v_{[T]}/a\omega_c \tag{7}$$

($v_{[T]}$ is the thermal velocity, $a$ is the minimal radius of the earth's plasmasphere and $\omega_c$ is the cyclotron frequency). All dependence on $t$ and $\phi$ appears through $\exp[i(n\phi - \Omega t)]$, due to the homogeneity of Eq. (5) over these variables, and the functions $f_{-[\mathrm{b}]}$, $f_{-[\mathrm{r}]}$ and $f_{-[0]}$ are unknowns for which we will rewrite the departure equation. High order $\sigma$ harmonics are disregarded due to the drift approximation ($\delta \ll 1$) we have already mentioned.

The *whole* strategy, as well as the distribution of the task along the four Maple sessions, can be summarized as:

- to determine the explicit form of $F_{[\mathrm{b}]}$ and $F_{[\mathrm{r}]}$ up to the first order (session *s1.ms*);
- to use Eq. (6) for $f(\text{var})$ together with Eq. (5) and build three equations relating $f_{-[\mathrm{b}]}$, $f_{-[\mathrm{r}]}$ and $f_{-[0]}$ (session *s2.ms*);
- to use the equations of the previous item to write $f_{-[\mathrm{b}]}$ and $f_{-[\mathrm{r}]}$ as functions of $f_{-[0]}$, eliminating them from the problem (session *s3.ms*);
- to solve the remaining (partial differential) equation for $f_{-[0]}$ (session *s4.ms*).

An explanation of how each item was realized is found in the following four subsections, while some technical aspects related to the **commands** used are discussed in the Appendix. Two general remarks about the strategy followed in the four *computer work-sheets* are:

- Each session was initialized with the reading of a preamble (an ASCII file prepared for that session) containing almost *all* the mathematical conventions, aliases, equivalencies and definitions valid within that session.
- Each session was closed with the saving of its *relevant results* in a file. This file, in turn, was read at the beginning of the following session, after the preamble, establishing the departure point of the new session.

### 2.1. Session s1.ms

The main goal of session *s1.ms* is to find an explicit form for the first order corrections $F_{[\mathrm{b}]}$ and $F_{[\mathrm{r}]}$ of the Maxwell distribution function $F_{[\mathrm{M}]}$ appearing in Eq. (6).

The mathematical identities used in this subsection take into account both that $h_{[\theta]}$ and $h_{[\mathrm{r}]}$ are (spherical) components of a unit vector (magnetic field) and that $F_{[\mathrm{M}]}$ has a standard symmetry, between its parallel and perpendicular derivatives in velocity space, given by

$$v_{pe}\left(\frac{\partial F_{[\mathrm{M}]}}{\partial v_{par}}\right) - v_{par}\left(\frac{\partial F_{[\mathrm{M}]}}{\partial v_{pe}}\right) = 0. \tag{8}$$

The steps of this session can be summarized as:

(i) throw away high order $\sigma$ harmonics ($\sin(n\sigma)$ and $\cos(n\sigma)$, with $n > 1$) from Eq. (5), since we are interested only in the zero and first order drift corrections of $F_{[\mathrm{M}]}$;

(ii) throw away, from Eq. (5), the $\phi$, parallel and normal components of $\boldsymbol{E}$, as well as the $\phi$ component of $\boldsymbol{H}$ and the $t$ and $\phi$ derivatives of $f(\text{var})$, due to the azimuthal symmetry of the corrections we are looking for;

(iii) introduce $f(\text{var})$ as given by Eq. (6), and express $H_{\text{par}}$ in terms of the cyclotron frequency $\omega_c$ and a general geometrical coefficient $C(r,\theta)$:

$$H_{\text{par}} = \frac{mc}{e}\omega_c C(r,\theta); \tag{9}$$

(iv) use the resulting coefficients of $\cos(\sigma)$ and $\sin(\sigma)$ as a pair of independent coupled equations for the unknowns $F_{[\text{b}]}$ and $F_{[\text{r}]}$;

(v) throw away all second order (in $\delta$) terms[5].

The results we found are:

$$F_{[\text{b}]} = \frac{v_{\text{pe}}}{r\,C(r,\theta)\,\omega_c}\left[h_{[\text{r}]}\left(\frac{\partial F_{[\text{M}]}}{\partial\theta}\right) - h_{[\theta]}\,r\left(\frac{\partial F_{[\text{M}]}}{\partial r}\right)\right], \tag{10}$$

$$F_{[\text{r}]} = 0, \tag{11}$$

which shows that, after discarding the second order terms, no $F_{[\text{r}]}$ correction is required.

### 2.2. Session s2.ms

In this session, we insert Eq. (6) into Eq. (5) and build three independent equations relating $f_{-[0]}$, $f_{-[\text{b}]}$ and $f_{-[\text{r}]}$. The mathematical identities used here are the same as those of the previous session, with the addition of Eqs. (10), (11).

The steps we followed can be summarized as:

(i) throw away high order $\sigma$ harmonics from Eq. (5) (drift approximation) and simplify the resulting expression, taking into account the relations between the squares and derivatives of squares of $(h_{[\text{r}]}, h_{[\theta]})$;

(ii) introduce $f(\text{var})$ (Eq. (6)) and the electric and magnetic field components as:

$$\begin{aligned}
&E_n = E_{[1]}\exp[i(n\phi - \Omega t)], && E_\phi = E_{[2]}\exp[i(n\phi - \Omega t)],\\
&E_{\text{par}} = E_{[3]}\exp[i(n\phi - \Omega t)], && H_n = H_{[1]}\exp[i(n\phi - \Omega t)],\\
&H_\phi = H_{[2]}\exp[i(n\phi - \Omega t)], && H_{\text{par}} = C(r,\theta)\frac{\omega_c c m}{e} + H_{[3]}\exp[i(n\phi - \Omega t)].
\end{aligned} \tag{12}$$

The introduction of these expressions turns Vlasov's equation into a polynomial of degree two with the exponential function as variable;

(iii) throw away the coefficient of the second power of the exponential function (linearized approximation) and split the result into three equations by selecting the coefficients of $\sin(\sigma)$ and $\cos(\sigma)$ from inside the coefficient of the first power of $\exp[i(n\phi - \Omega t)]$;

(iv) change variables from $(v_{\text{pe}}, v_{\text{par}}, r)$ to $(u_{\text{pe}}, u_{\text{par}}, x)$, according to

$$u_{\text{pe}} = \frac{v_{\text{pe}}}{v_{[T]}}, \qquad x = \frac{r}{a}, \qquad u_{\text{par}} = \frac{v_{\text{par}}}{v_{[T]}}. \tag{13}$$

This change of variables introduces the thermal velocity $v_{[T]}$ and the space parameter $a$ (see Eq. (7)). The new variables ($u_{\text{pe}}$, $u_{\text{par}}$, $x$) are dimensionless. The transformation is implemented using the **dchange** command [9] explained in the Appendix. The motivation for this change of variables is related to the introduction of the small drift parameter $\delta$ (see Ref. [4]) as an expansion parameter.

[5] Note that $1/\omega_c$ enters the expressions we are working on in dimensionless combinations proportional to $\delta$ (see Eq. (7)).

The session ends with the introduction of $\delta$, arriving at the set of three equations mentioned at the beginning. Due to the large size of each equation (36, 30 and 23 terms, respectively) we display here only the relevant information about their structures, given by

$$0 = \ldots(f_{-[0]}, f_{-[b]}, f_{-[r]})\ldots \tag{14}$$

$$f_{-[b]} = \ldots(f_{-[0]}, (\text{derivatives of } f_{-[b]}), f_{-[r]})\ldots \tag{15}$$

$$f_{-[r]} = \ldots(f_{-[0]}, (\text{derivatives of } f_{-[r]}), f_{-[b]})\ldots \tag{16}$$

where all derivatives of $(f_{-[b]}, f_{-[r]})$ appearing in Eqs. (15), (16) are of the second order (with respect to $\delta$).

### 2.3. Session s3.ms

In this session, $f_{-[b]}$ and $f_{-[r]}$ are eliminated from the problem by introducing them as functions of $f_{-[0]}$, using the three equations obtained in the previous session; this leads to a partial differential equation for $f_{-[0]}$.

The preamble for this session is that of the previous one plus some mathematical identities relating the first and second derivatives of $F_{[M]}$ (used in the simplification process), obtained from Eq. (8).

The steps of this session can be summarized as:

(i) introduce Eq. (16) into Eq. (15), expand the result in powers of the drift parameter $\delta$, and throw away all third order terms (eliminating $f_{-[r]}$), as well as all second order terms containing $f_{-[b]}$ and $f_{-[0]}$, since corrections to these functions already exist in the lower order terms;

(ii) introduce Eq. (15) into Eq. (16) doing the same thing as in item 1, but now eliminating $f_{-[b]}$;

(iii) introduce the resulting equations of the two previous items (they now appear as $f_{-[b]} = \ldots(f_{-[0]})\ldots$ and $f_{-[r]} = \ldots(f_{-[0]})\ldots$) into Eq. (14), eliminating $f_{-[b]}$ and $f_{-[r]}$ from the problem;

(iv) discard new third order (in $\delta$) terms, which appear as a consequence of the last item, and second order terms containing $f_{-[0]}$ (displayed below for future reference), arriving at a partial differential equation for $f_{-[0]}$ with $(\theta, u_{pe}, u_{par}, x)$ as differentiation variables;

(v) introduce a change of variables from $(u_{pe}, u_{par})$ to $(\Lambda, u)$ given by

$$\Lambda = \frac{u_{pe}^2}{(u_{pe}^2 + u_{par}^2)C(x,\theta)}, \qquad u_{pe} = u\sqrt{\Lambda C(x,\theta)},$$
$$u = \sqrt{u_{pe}^2 + u_{par}^2}, \qquad u_{par} = su\sqrt{1 - \Lambda C(x,\theta)}, \tag{17}$$

where $s = \pm 1$ and $\Lambda$ is a (first order) adiabatic invariant; with this, the number of differentiation variables is reduced from 4 to 3. The differentiation variables will now be $(x, \Lambda, \theta)$;

(vi) finally, $\partial f_{[0]}/\partial\Lambda$ cancels out after taking into account $\nabla \cdot \boldsymbol{H} = 0$, reducing the number of differentiation variables from three to two.

For reference, we display some relevant results obtained along these steps. To start with, the expressions for $f_{-[b]}$ and $f_{-[r]}$ obtained in the first and second steps are given, up to the first order in $\delta$, by

$$f_{-[b]} = u_{par}\delta\left[\frac{1}{xC}\left(h_{[\theta]} - \frac{\partial h_{[r]}}{\partial\theta}\right) - \frac{h_{[r]}}{Ch_{[\theta]}}\frac{\partial h_{[r]}}{\partial x}\right]\left(u_{pe}\frac{\partial f_{-[0]}}{\partial u_{par}} - u_{par}\frac{\partial f_{-[0]}}{\partial u_{pe}}\right)$$
$$+ \frac{u_{pe}\delta}{C}\left(\frac{h_{[r]}}{x}\frac{\partial f_{-[0]}}{\partial\theta} - h_{[\theta]}\frac{\partial f_{-[0]}}{\partial x}\right) - \frac{u_{pe}eE_{-[1]}}{u_{par}C\delta a\omega_c^2}\left(\frac{\partial F_{[M]}}{\partial u_{par}}\right),$$
$$f_{-[r]} = \frac{i\delta u_{pe}nf_{-[0]}}{xC\sin(\theta)} + \frac{u_{pe}eE_{-[2]}}{u_{par}Cm\delta a\omega_c^2}\left(\frac{\partial F_{[M]}}{\partial u_{par}}\right). \tag{18}$$

The second order terms containing $f_{-[0]}$, selected and thrown away in step four, are[6]

$$\left[\left(h_{[\theta]}-\frac{\partial h_{[r]}}{\partial\theta}-\frac{x h_{[r]}}{h_{[\theta]}}\frac{\partial h_{[r]}}{\partial x}\right)u_{\rm par}{}^2+\left(h_{[r]}\frac{\partial C}{\partial\theta}-x h_{[\theta]}\frac{\partial C}{\partial x}\right)\frac{u_{\rm pe}^2}{2C}\right]\frac{in\delta^2 f_{-[0]}}{x^2\sin(\theta)C^2}. \tag{19}$$

The partial differential equation for $f_{-[0]}$ obtained in step six is given by

$$-\frac{i\Omega f_{-[0]}}{C\omega_c}+\frac{i\delta u s\sqrt{\Lambda C-1}}{C}\left(h_{[r]}\frac{\partial f_{-[0]}}{\partial x}+\frac{h_{[\theta]}}{x}\frac{\partial f_{-[0]}}{\partial\theta}\right)$$
$$=-\frac{ies\sqrt{\Lambda C-1}E_{-[3]}}{\delta ma\omega_c^2 C}\left(\frac{\partial F_{[M]}}{\partial u}\right)-O(\delta^2), \tag{20}$$

where the second order terms are all given by

$$O(\delta^2)=\left(\frac{eE_{-[2]}}{C^2ma\omega_c^2}-\frac{ie\delta us\sqrt{\Lambda C-1}H_{-[1]}}{C^2m\omega_c c}\right)\left(\frac{h_{[r]}}{x}\frac{\partial F_{[M]}}{\partial\theta}-h_{[\theta]}\frac{\partial F_{[M]}}{\partial x}\right)$$
$$+\left[\frac{\Lambda ue}{2Cma\omega_c^2}\left(\frac{h_{[r]}E_{-[2]}}{xC}\frac{\partial C}{\partial\theta}-\frac{h_{[\theta]}E_{-[2]}}{C}\frac{\partial C}{\partial x}-\frac{h_{[r]}}{x}\frac{\partial E_{-[2]}}{\partial\theta}+h_{[\theta]}\frac{\partial E_{-[2]}}{\partial x}+\frac{h_{[\theta]}E_{-[2]}}{x}\right)\right.$$
$$+\frac{ueE_{-[2]}}{C^2ma\omega_c^2}(\Lambda C-1)\left(\frac{1}{x}\frac{\partial h_{[r]}}{\partial\theta}-h_{[\theta]}+\frac{h_{[r]}}{h_{[\theta]}}\frac{\partial h_{[r]}}{\partial x}\right)$$
$$\left.-\frac{\Lambda ue\left(h_{[r]}\cos(\theta)E_{-[2]}+inE_{-[1]}\right)}{2xC\sin(\theta)ma\omega_c^2}\right]\frac{\partial F_{[M]}}{\partial u}, \tag{21}$$

and $H_{-[1]}$ is related to the electric field components (through the Maxwell equations) by

$$H_{-[1]}=-\frac{ncE_{-[3]}}{\Omega ax\sin(\theta)}-\frac{ich_{[\theta]}}{\Omega ax}\frac{\partial E_{-[2]}}{\partial\theta}-\frac{ich_{[r]}}{\Omega a}\frac{\partial E_{-[2]}}{\partial x}-\left(\frac{h_{[\theta]}\cos(\theta)}{\sin(\theta)}+h_{[r]}\right)\frac{icE_{-[2]}}{\Omega ax}. \tag{22}$$

### 2.4. Session s4.ms

This session presents: a method for the solution of the first order terms of the partial differential equation Eq. (20), a final expression for $f_{-[0]}$, and a new resonant factor, different from the one associated to the Landau damping.

What motivated us to, as a first approach, solve for $f_{-[0]}$ only the first order terms, was that Eq. (20) contains a large number of second order terms, while the main part of the wave dissipation effects is connected with the parallel component of the electric field $E_{-[3]}$, which only appears in the first order part.

The preamble for this session is the same as that of the last one, and the steps we followed can be summarized as:

(i) introduce a change (the first of four changes) of variables from $x$ to $\rho$, related to the explicit form of the magnetic field structure (see next step). The transformation equation is given by

$$\rho=\frac{x}{\sin(\theta)^2}. \tag{23}$$

The main consequence of this change will be the reduction of the number of differentiation variables from 2 to 1 (the new variable $\rho$ will enter the equation as a parameter);

[6] These terms should be taken into account in the study of the influence of the final Larmor radii on wave-particle interaction.

(ii) introduce explicit expressions for $h_{[r]}$, $h_{[\theta]}$ and $C$, for the case of a dipole configuration [3], given by:

$$C(\rho,\theta) = \frac{\sqrt{3\cos(\theta)^2+1}}{2\,\rho^3\sin(\theta)^6}, \qquad h_{[r]} = \frac{\cos(\theta)}{\sqrt{\cos(\theta)^2+\frac{1}{4}\sin(\theta)^2}},$$

$$h_{[\theta]} = \frac{\sin(\theta)}{2\sqrt{\cos(\theta)^2+\frac{1}{4}\sin(\theta)^2}}, \tag{24}$$

arriving at an equation for $f_{-[0]}$ (Eq. (32)) with only $\theta$ as differentiation variable;

(iii) introduce a change (the second of four changes) of variables from $\theta$ to $\eta = \theta - \pi/2$ (where $\eta = 0$ represents the equatorial plane), as the first step of a general simplification of some trigonometric expressions in Eq. (32). This simplification (displayed below) is achieved by restricting the validity of the solution to the region where the trapped particles are most concentrated. This region is defined by the condition

$$\Lambda\, C(\rho,\theta) \le 1. \tag{25}$$

The result of this simplification is given by Eq. (38);

(iv) expand the electric field in a Fourier series,

$$E_{-[3]}(\rho,\eta) = \sum_{j=-\infty}^{\infty} E_{-[3,j]}(\rho)\exp[ij(\eta+\pi/2)] \tag{26}$$

and introduce a new change (the third of four changes) of variables from $\eta$ to $\xi$, where $\sin(\eta) = \sin(\xi)\kappa$, rearranging Eq. (32) in order to introduce elliptic integrals as variables;

(v) rewrite $f_{-[0]}$ in terms of a new indeterminate function $f_{-[1]}$ (first of three function redefinitions):

$$f_{-[0]}(\Lambda,u,\rho,\xi) = f_{-[1]}(\Lambda,u,\rho,\xi)\exp\left(\int_0^{\xi} \frac{i\rho^{5/2}\sqrt{2}\Omega\sin(\alpha)^2\kappa^3}{2\delta us\sqrt{-\Lambda+2\rho^3\omega_c\sqrt{1-\sin(\alpha)^2\kappa^2}}}\,\mathrm{d}\alpha\right), \tag{27}$$

reducing the non-linear (in $\xi$) terms, which appear in Eq. (32), to a form that allows that equation to be solved using elliptic integrals and Jacobi functions (see Eq. (39));

(vi) introduce a new change (the fourth of four changes) of variables from $\xi$ to w1, where w1 represents the elliptic integral of the first kind, the transformation equations appearing as:

$$\mathrm{w1}(\xi,\kappa) = \int_0^{\xi} \frac{1}{\sqrt{1-\sin(\rho)^2\kappa^2}}\,\mathrm{d}\rho, \qquad \xi = \mathbf{am}(\mathrm{w1}), \tag{28}$$

with **am** representing the *amplitude* [10]; introduce the Jacobi functions (cn, sn, dn), the elliptic integrals of the second kind,

$$\mathrm{w2}(\xi,\mathrm{w1}) = \int_0^{\xi} \sqrt{1-\sin(\rho)^2\kappa^2}\,\mathrm{d}\rho, \qquad \mathbf{E} = \mathrm{w2}(\pi/2,\mathrm{w1}), \tag{29}$$

**E** being the complete one, rewrite the *arcsin* function (see Eq. (39)) in terms of Jacobi functions, and rewrite both **am** and w2 turning their periodic parts pam and pw2 (see the Appendix) explicit, resulting in the splitting of the whole expression into a periodic ($P(...)$) and a non-periodic ($NP(...)$) part. The goal of such a procedure is to allow both the elimination of the $NP$ part and a Fourier expansion of the Jacobi functions of the $P$ part. This, in turn, will lead to the resonant factor in a simple manner;

(vii) rewrite $f_{-[1]}$, in terms of a new indeterminate function (second of three function redefinitions) $f_{-[2]}$, as

$$f_{-[1]}(\Lambda,u,\rho,\mathrm{w1}) = f_{-[2]}(\Lambda,u,\rho,\mathrm{w1})\exp\left(\frac{i\rho^{5/2}\Omega\kappa\mathrm{w1}\,[\mathbf{E}(\kappa)-\mathbf{K}(\kappa)]}{\sqrt{2}\,\delta us\omega_c\sqrt{-\Lambda+2\rho^3}\,\mathbf{K}(\kappa)}\right), \tag{30}$$

where $\mathbf{K}(\kappa) = \mathrm{w1}(\pi/2,\kappa)$ is the complete elliptic integral of the first kind, and the exponential contains all the *NP* terms mentioned in the last item. This function redefinition produces the cancellation of these *NP* terms (the exponential will appear as a common factor all along the expression), leading to a differential equation for $f_{-[2]}$ containing only periodic functions (see Eq. (40));

(viii) a final Fourier expansion of $f_{-[2]}$ (third of three function redefinitions),

$$f_{-[2]}(\Lambda,u,\rho,\mathrm{w1}) = \sum_{J=-\infty}^{\infty} f_{-[2,J]}(\Lambda,u,\rho)\exp\left(\frac{iJ\mathrm{w1}\pi}{2\mathbf{K}(\kappa)}\right), \tag{31}$$

leads to the desired solution of Eq. (32) in terms of $f_{-[2,J]}$ (see Eq. (43));

(ix) the resonant factor is evaluated and all the function redefinitions are restored, which leads to the expected result in terms of $f_{-[0]}$ (see Eqs. (44), (45)).

Due to the large volume of algebraic manipulations done in this session, we display here only some intermediate results, together with any necessary additional explanations. To start with, the differential equation in one variable obtained for $f_{-[0]}$ at the end of step two is given by

$$\frac{\partial f_{-[0]}}{\partial\theta} + \frac{2i\rho^{5/2}\sin(\theta)^4\sqrt{\cos(\theta)^2+\frac{1}{4}\sin(\theta)^2}\sqrt{2}\Omega f_{-[0]}}{\delta us\omega_c\sqrt{-\Lambda\sqrt{3\,\cos(\theta)^2+1}+2\rho^3\sin(\theta)^6}}$$
$$= -\frac{2eE_{-[3]}\rho\sin(\theta)\sqrt{\cos(\theta)^2+\frac{1}{4}\sin(\theta)^2}}{m\omega_c^2a\delta^2u}\left(\frac{\partial F_{[M]}}{\partial u}\right). \tag{32}$$

This equation does not contain a small parameter as in the toroidal case [4], and its solution using elliptic integrals is not possible in a direct manner because of the complicated dependence on trigonometric functions. This dependence is especially relevant in the left hand side (LHS), where the zeros of the denominator are the turning points for the trapped electrons.

The overcoming of this complicated trigonometric dependence is realized in step three, and the approximations we used to simplify the trigonometric dependence of Eq. (32) consist in substituting the undesired expressions by "quasi-equivalent" ones, checked through graph analysis, taking care to keep the main trapping effects, such as region of trapping (±45 degrees from the equatorial plane) and dissipated power, unchanged. The introduction of these "quasi-equivalent" terms will also cut the untrapped particle region out from our discussion [7].

To begin with, consider the undesired square root of the denominator in Eq. (32),

$$\sqrt{-\Lambda\sqrt{3\sin(\eta)^2+1}+2\rho^3\cos(\eta)^6}. \tag{33}$$

A suitable equivalent for this expression (displayed in the LHS below), as well as the format we are going to use for it (appearing in the RHS), is given by [8]

[7] In turn, this does not represent a problem since, in the earth magnetosphere, untrapped particles disappear in the atmosphere (aurora phenomena, etc.) without participating in wave-particle interaction.

[8] One could also have used $\cos(\eta)^6 \approx 1 - 3\sin(\eta)^2$ (Taylor expansion) instead of $1 - 2\sin(\eta)^2$ (which we used), obtaining for $\kappa$ an expression slightly different from Eq. (35): $\kappa = \sqrt{2}\sqrt{-\Lambda+2\rho^3}/\sqrt{3\Lambda+12\rho^3}$. The difference between this $\kappa$ and the one we used, for typical values of $\Lambda$ and $\rho$, is $|0.556 - 0.667| \approx 0.15$, which shows the equivalence between the two possible expansions.

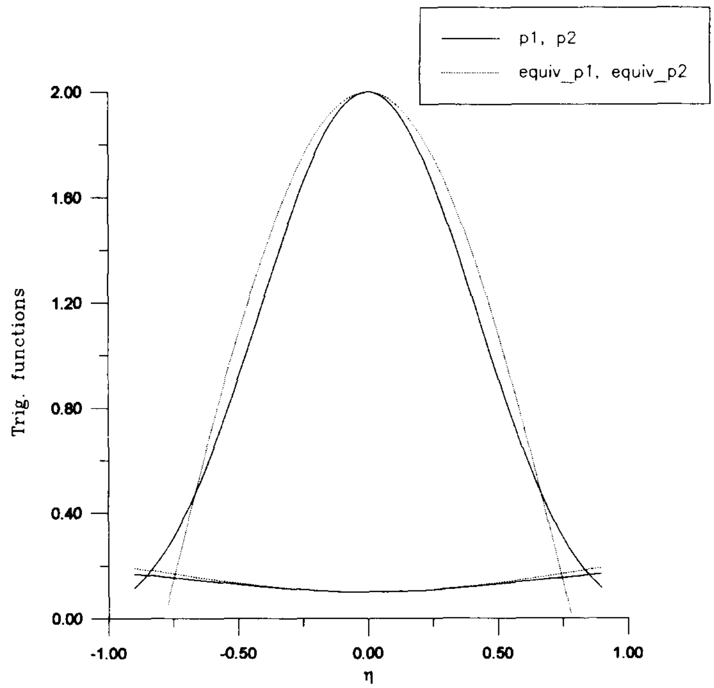


Fig. 1. A graph of the terms in the square root of the denominator shows well the equivalence between Eq. (33) and Eq. (34) inside the main trapped particle region, bounded by the exterior intersection of (solid) lines in the exact case, and by the interior intersection (dashed lines) in our approximation.

$$\sqrt{-\Lambda\left[1+\tfrac{3}{2}\sin(\eta)^2\right]+2\rho^3\left[1-2\sin(\eta)^2\right]}=\sqrt{-\Lambda+2\rho^3}\sqrt{1-\sin(\eta)^2/\kappa^2}, \tag{34}$$

where kappa is easily found to be

$$\kappa=\frac{\sqrt{2}\sqrt{-\Lambda+2\rho^3}}{\sqrt{3\Lambda+8\rho^3}}. \tag{35}$$

The advantage of the format of the RHS of Eq. (34) is that it has the appropriate form for working with elliptic functions. A plot for $\Lambda = 0.1$ and $\rho = 1$, is presented in Fig. 1. The four functions plotted (two pairs, one against the other) are

$$p1 = 0.1\sqrt{3\sin(\eta)^2+1}, \qquad equiv\_p1 = 0.1+0.15\sin(\eta)^2,$$
$$p2 = 2\cos(\eta)^6, \qquad equiv\_p2 = -4\sin(\eta)^2+2. \tag{36}$$

The other trigonometric functions appearing in Eq. (32) were approximated by

$$\sqrt{\sin(\eta)^2+\tfrac{1}{4}\cos(\eta)^2}\approx\tfrac{1}{2}+\tfrac{3}{4}\sin(\eta)^2, \qquad \cos(\eta)^4\approx 1-2\sin(\eta)^2, \qquad \sin(\eta)^4\approx 0, \tag{37}$$

leading to

$$\frac{\partial f_{-[0]}}{\partial \eta} - \frac{i\rho^{5/2}\sqrt{2}\Omega f_{-[0]}\left[\sin(\eta)^2 - 2\right]}{2\delta u s \omega_c \sqrt{-\Lambda + 2\rho^3}\sqrt{1-(1/\kappa^2)\sin(\eta)^2}}$$
$$= -\frac{e\rho\left[2+3\sin(\eta)^2\right]E_{-[3]}\cos(\eta)}{2ma\omega_c^2\delta^2 u}\left(\frac{\partial F_{[M]}}{\partial u}\right). \tag{38}$$

This equation should be the departure point for the study of the trapped particles. Steps four and five turn Eq. (38) into

$$\frac{\sqrt{1-\sin(\xi)^2\kappa^2}}{\kappa}\frac{\partial f_{-[1]}}{\partial \xi} + \frac{i\sqrt{2}\rho^{5/2}\Omega f_{-[1]}}{\delta\omega_c u s\sqrt{-\Lambda+2\rho^3}}$$
$$= -\sum_{j=-\infty}^{\infty} \frac{e\cos(\xi)\left[2+3\sin(\xi)^2\kappa^2\right]}{2ma\delta^2 u\omega_c^2}\rho\sqrt{1-\sin(\xi)^2\kappa^2}\left(\frac{\partial F_{[M]}}{\partial u}\right)$$
$$\times E_{-[3,j]}\exp\left(ij\{\arcsin[\sin(\xi)\kappa]+\pi/2\} - \int_0^{\xi}\frac{i\rho^{5/2}\sqrt{2}\Omega\sin(\alpha)^2\kappa^3}{2\delta u s\omega_c\sqrt{-\Lambda+2\rho^3}\sqrt{1-\sin(\alpha)^2\kappa^2}}\,d\alpha\right). \tag{39}$$

After introducing the Jacobi functions and splitting them into periodic and non-periodic parts (steps six and seven), we found

$$\frac{1}{\kappa}\frac{\partial f_{-[2]}}{\partial \mathrm{w1}} + \frac{i\rho^{5/2}\Omega\left[\mathbf{K}(\kappa)+\mathbf{E}(\kappa)\right]f_{-[2]}}{\sqrt{2}\,\delta u s\omega_c\sqrt{-\Lambda+2\rho^3}\,\mathbf{K}(\kappa)}$$
$$= -\sum_{j=-\infty}^{\infty}\left[1+\tfrac{3}{2}\kappa^2\mathrm{sn}(\mathrm{w1})^2\right]\frac{e\rho\,\mathrm{cn}(\mathrm{w1})\mathrm{dn}(\mathrm{w1})}{ma\delta^2 u\omega_c^2}\left(\frac{\partial F_{[M]}}{\partial u}\right)$$
$$\times E_{-[3,j]}\exp\left(\frac{ij\pi}{2} + \frac{i\rho^{5/2}\Omega\kappa\mathrm{pw2}(\mathrm{w1})}{\sqrt{2}\,\delta u s\omega_c\sqrt{-\Lambda+2\rho^3}} + ij\kappa\int_0^{\mathrm{w1}}\mathrm{cn}(\zeta)\,d\zeta\right). \tag{40}$$

Next (step 8), since all the functions in the RHS have the same periodicity, we expand them in a Fourier series over w1 with periodicity $\pi/2\mathbf{K}(\kappa)$. Two different expansions were realized:

$$\mathrm{cn}(\mathrm{w1})\mathrm{dn}(\mathrm{w1})\exp\left(\frac{i\rho^{5/2}\Omega\kappa\mathrm{pw2}(\mathrm{w1})}{\sqrt{2}\,\delta u s\omega_c\sqrt{-\Lambda+2\rho^3}} + ij\kappa\int_0^{\mathrm{w1}}\mathrm{cn}(\zeta)\,d\zeta\right)$$
$$= \sum_{J=-\infty}^{\infty} a_{[s,j,J]}(\rho,\Lambda,u)\exp\left(\frac{iJ\mathrm{w1}\pi}{2\mathbf{K}(\kappa)}\right),$$
$$\mathrm{sn}(\mathrm{w1})^2\mathrm{cn}(\mathrm{w1})\mathrm{dn}(\mathrm{w1})\exp\left(\frac{i\rho^{5/2}\Omega\kappa\mathrm{pw2}(\mathrm{w1})}{\sqrt{2}\,\delta u s\omega_c\sqrt{-\Lambda+2\rho^3}} + ij\kappa\int_0^{\mathrm{w1}}\mathrm{cn}(\zeta)\,d\zeta\right)$$
$$= \sum_{J=-\infty}^{\infty} b_{[s,j,J]}(\rho,\Lambda,u)\exp\left(\frac{iJ\mathrm{w1}\pi}{2\mathbf{K}(\kappa)}\right). \tag{41}$$

These equations introduce the coefficients $a_{[s,j,J]}(\rho,\Lambda,u)$ and $b_{[s,j,J]}(\rho,\Lambda,u)$, which can be evaluated, if desired, by inverting the series. Also, $f_{-[2]}$ on the LHS of Eq. (40) is expanded:

$$f_{-[2]}(\rho,\Lambda,u,\mathrm{w1}) = \sum_{J=-\infty}^{\infty} f_{-[2,j]}(\rho,\Lambda,u)\exp\left(\frac{iJ\mathrm{w1}\pi}{2\mathbf{K}(\kappa)}\right). \tag{42}$$

After equating the coefficients with the same $J$, we obtain a first solution for Eq. (38) in terms of $f_{-[2,J]}$:

$$\left(\frac{i\rho^{5/2}\Omega\left[\mathbf{K}(\kappa)+\mathbf{E}(\kappa)\right]}{\sqrt{2}\,\delta us\omega_{c}\sqrt{-\Lambda+2\rho^{3}}\mathbf{K}(\kappa)}+\frac{iJ\pi}{2\kappa\mathbf{K}(\kappa)}\right)f_{-[2,J]}$$
$$= -\sum_{j=-\infty}^{\infty}\frac{e\rho\left(2a_{[s,j,J]}+3\kappa^{2}b_{[s,j,J]}\right)E_{-[3,j]}}{2ma\delta^{2}u\omega_{c}^{2}}\left(\frac{\partial F_{[\mathrm{M}]}}{\partial u}\right)\exp\left(ij\pi/2\right). \tag{43}$$

The last step of this session evaluates the resonant factor $(J+\Delta_{[s]})$ and rewrites Eq. (43) in terms of the (original indeterminate) function $f_{-[0]}$, introduced in Eq. (6). For the $\Delta_{[s]}$ factor, we found[9]

$$\Delta_{[s]} = s\frac{\kappa\Omega\rho^{5/2}\sqrt{2}\left[\mathbf{K}(\kappa)+\mathbf{E}(\kappa)\right]}{\pi\delta u\omega_{c}\sqrt{-\Lambda+2\rho^{3}}}. \tag{44}$$

To conclude, our result for $f_{-[0]}$ as the solution of eq(38) appears as

$$f_{-[0]}(\Lambda,u,\rho,\mathrm{w1}) = \sum_{J=-\infty}^{\infty}\sum_{j=-\infty}^{\infty} ie\kappa\rho\mathbf{K}(\kappa)\left(\frac{\partial F_{[\mathrm{M}]}}{\partial u}\right)\frac{2a_{[s,j,J]}+3\,\kappa^{2}b_{[s,j,J]}}{\pi\left(J+\Delta_{[s]}\right)ma\delta^{2}u\omega_{c}^{2}}$$
$$\times E_{-[3,j]}(\rho)\exp\left(\frac{ij\pi}{2}+\frac{iJ\mathrm{w1}\pi}{2\mathbf{K}(\kappa)}-\frac{i\rho^{5/2}\Omega\kappa\mathrm{pw2}(\mathrm{w1})}{\sqrt{2}\,\delta us\omega_{c}\sqrt{-\Lambda+2\rho^{3}}}\right), \tag{45}$$

where $\rho$ was defined in Eq. (23) and $\kappa$ in Eq. (35); $F_{[\mathrm{M}]}$ and $\Omega$ were introduced in Eq. (6), $a_{[s,j,J]}$ and $b_{[s,j,J]}$ in Eq. (41), $\delta$ and $\omega_c$ in Eq. (7), $\Lambda$ and $u$ in Eq. (17), and $E_{-[3,j]}(\rho)$ in Eq. (26); all the elliptic and Jacobi functions (sn, cn, dn, w1, **K**, w2, **E**, etc.). appear in the Appendix.

## 3. Discussion and conclusions

In this paper, using the **Maple V** computer algebra system, we obtained the asymptotic solution of Vlasov's equation. First, this equation was reduced to the form of the drift kinetic equation (due to the smallness of the drift parameter), and its solution was found using elliptic and Jacobi functions, extending the method proposed in Ref. [4]. The approach used led us to simple expressions both for the distribution function $f_{-[0]}$ in the trapped electron region and for the (new) bounce resonance factor of the problem, $\Omega + J\omega_b$ (see Eqs. (44), (45)), where $\omega_b$ is the bounce frequency. For electrons with large perpendicular velocity, $\omega_b \approx 3\sqrt{2}u\,v_{[T]}/R$, where $R$ is the distance from a magnetic field line intersected by the equatorial plane to the origin. The solution obtained can be used for the calculation of the parallel oscillating current:

$$j_{\parallel} = 2\pi\sum_{\alpha} e_{\alpha}\int_{-\infty}^{+\infty} v_{\parallel}\,\mathrm{d}v_{\parallel}\int_{0}^{\infty} v_{\perp}f_{-[0]}^{(\alpha)}\,\mathrm{d}v_{\perp}. \tag{46}$$

This parallel component is the most important for an analysis of the wave dissipated power $P \approx \mathrm{Re}(j_{\parallel}E_{\parallel}^{*})$, whereas the other components (binormal and radial) can be written using Eqs. (18).

[9] Note the dependence on $s$.

Aside from this, the solution of Maxwell's set of equations (2) requires the above expressions for the components of the oscillating current, and may be obtained numerically (or analytically, using the geometric optics approximation). This task can be realized via computer algebra using the same approach, that is, taking into account the linear dependence of the equations and oscillating currents on the electric field components. The interesting feature of the computer approach is that it allows us to extend the solution here presented to cover equivalent problems, related to different geometrical magnetic field structures, in a relatively simple manner. We have also prepared the toroidal case for trapped and untrapped particles (representing the second part of this work), but due to the volume of calculations, the extended results will be presented in another article, as soon as possible.

Summarizing, the two important things to be pointed out in relation to the results Eqs. (44), (45) are:

– the oscillating part of the trapped particle distribution function $f_{-[0]}$ is coupled to all the Fourier harmonics (over the polar angle $\theta$) of the oscillating electric field, which results in additional wave dissipation effects. This mode coupling is a consequence of the spatial inhomogeneity of the dipole magnetic field configuration.

– the distribution function $f_{-[0]}$ has an infinite number of bounce resonances (zeros of the denominator in Eq. (45)) due to a resonant factor $(\Omega + J\omega_b)$, different from the Landau damping resonant factor $(\Omega - k_\| v_\|)$ [3]. These bounce resonances are related to the periodic motion of the particles along magnetic field lines in the trapped region (see Eq. (25)), and have a relevant contribution to the absorption of the waves.

Finally, one extension of this work would be to solve the same problem taking into account the second order corrections to the distribution function (the dependence of $f_{-[0]}$ on the binormal and radial component of the electric field, Eq. (21)). The main requirements are useful commands for finding Fourier coefficients and for careful manipulation of elliptic integrals. This subject is now under study and will be reported elsewhere.

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The authors wish to thank Katherina von Bülow[10] for fruitful comments and careful reading of this paper.

## Appendix

Some non-standard commands and a special file, containing the definitions of the Jacobi functions and simplification rules for them, were used along this work. A brief review, together with the code of these commands, is:

– the **asimplify** command implements a special combination of simplification procedures, useful for the kind of algebraic expression found in this work. Its code is given by:

```
> asimplify := proc(u)
>   combine(u,power):
>   indets(",{'exp'(anything),'Sum'(anything,anything)}):
>   collect("",[op(")],expand);
> end:
```

– the **selhas** command works in a similar manner as **select(has,...)**, but regards a monomial as being a sum of one operand, instead of a product of "*n*" operands, and allows for *nested* selection of terms. It was used, together with the **OOO** command (see below), when making the selection of terms (to be kept or discarded) along Section 2. Its code is

[10] From the Mathematical Institute at UERJ, Rio de Janeiro, Brazil.

```
> selhas := proc(a:{table,list,set,`=`,algebraic},b:{list,set,algebraic})
> local i;
>   if type(a,`=`) then map(selhas,a,b)
>   elif type(a,'table') then
>       table(map(u -> lhs(u) = selhas(rhs(u),args[2]),op(op(a)),b))
>   elif type(b,list) and 1<nops(b) then a; for i in b do selhas(",i) od
>   else proc(x) if has(args) then x else _identity fi end;
>       if type(a,{'list','set'}) then map(subs(_identity = NULL,"),args)
>       else
>           frontend(expand,[a]);
>           if type(",`+`) then map(subs(_identity = 0,""),",b)
>           else subs(_identity = 0,"")(args)
>           fi
>       fi
>   fi
> end:
```

– the **OOO** command was specially prepared for this problem and works splitting the RHS of a given expression into orders of $\delta$ or $1/\omega_c$. Its code is[11]

```
> OOO := proc(a) local u,i;
>   OO[0] := 0; OO[1] := 0; OO[2] := 0; OO[3] := 0;
>   if type(a,`=`) then u := expand(rhs(a)) else u := expand(a) fi;
>   for i to nops(u) do
>       if degree(op(i,u),delta)-degree(op(i,u),omega[c]) = 0 then
>           OO[0] := OO[0]+op(i,u)
>       elif degree(op(i,u),delta)-degree(op(i,u),omega[c]) = 1 then
>           OO[1] := OO[1]+op(i,u)
>       elif degree(op(i,u),delta)-degree(op(i,u),omega[c]) = 2 then
>           OO[2] := OO[2]+op(i,u)
>       elif degree(op(i,u),delta)-degree(op(i,u),omega[c]) = 3 then
>           OO[3] := OO[3]+op(i,u)
>       else ERROR(`Warning: the terms do not have the expected order!!`)
>       fi
>   od;
>   if nargs = 1 then map(indets,OO,Function)
>   else map(indets,OO,args[2])
>   fi
> end:
```

– the **dchange** command is part of a new package under development [9]. It realizes changes of variables in partial differential equations. A typical calling sequence is

```
> dchange(target,{transf_eqs},{inv_transf_eqs},[new_vars],simplif_proc)
```

[11] See footnote 5.

Here, {transf_eqs} and {inv_transf_eqs} are, respectively, sets of transformation and inverse transformation equations, containing only new (old) variables in the LHS and only old (new) variables in the RHS; [new_vars] is a specification of the variables which must be seen as the new ones, while simplif_proc is an optional argument allowing the indication of a simplification procedure (otherwise, the command returns a result without any simplification). Additionally, different criteria are applied to *known* and *unknown* mathematical functions. For instance, if the transformation equations contain $\theta \rightarrow n\xi/\pi$, the **dchange** command makes $\sin(\theta) \rightarrow \sin(n\xi/\pi)$, while for any unknown function it will return $f(\theta) \rightarrow f(\xi)$. The code is too large to be presented here; it is now under test but will be available soon.

Finally, all the definitions of the Jacobi functions and elliptic integrals were taken from Ref. [10] and introduced by loading an ASCII file (*elliptic.txt*), whose contents are given by

$$equivalence_{[1]} := \mathrm{w1} = \int_0^{\xi} \frac{1}{\sqrt{1-\sin(\rho)^2\kappa^2}}\,\mathrm{d}\rho,$$

$$equivalence_{[2]} := \mathrm{w2}(\mathrm{w1}) = \int_0^{\xi} \sqrt{1-\sin(\rho)^2\kappa^2}\,\mathrm{d}\rho,$$

$$equivalence_{[3]} := \int_0^{\xi} \frac{\sin(\rho)^2}{\sqrt{1-\sin(\rho)^2\kappa^2}}\,\mathrm{d}\rho = \frac{\mathrm{w1}}{\kappa^2} - \frac{\mathrm{w2}(\mathrm{w1})}{\kappa^2},$$

$$equivalence_{[4]} := \mathbf{K}(\kappa) = \int_0^{\pi/2} \frac{1}{\sqrt{1-\sin(\rho)^2\kappa^2}}\,\mathrm{d}\rho,$$

$$equivalence_{[5]} := \mathbf{E}(\kappa) = \int_0^{\pi/2} \sqrt{1-\sin(\rho)^2\kappa^2}\,\mathrm{d}\rho,$$

$$equivalence_{[6]} := \mathbf{am}(\mathrm{w1}) = \mathrm{pam}(\mathrm{w1}) + \frac{\pi \mathrm{w1}}{2\mathbf{K}(\kappa)},$$

$$equivalence_{[7]} := \mathrm{w2}(\mathrm{w1}) = \mathrm{pw2}(\mathrm{w1}) + \frac{\mathbf{E}(\kappa)\mathrm{w1}}{\mathbf{K}(\kappa)},$$

$$equivalence_{[8]} := \arcsin[\mathrm{sn}(\mathrm{w1})\kappa] = \int_0^{\mathrm{w1}} \mathrm{cn}(\zeta)\kappa\,\mathrm{d}\zeta, \tag{47}$$

$$sub\_set := \{\ \sin(\mathbf{am}(\mathrm{w1})) = \mathrm{sn}(\mathrm{w1}), \sin(2\mathbf{am}(\mathrm{w1})) = 2\mathrm{sn}(\mathrm{w1})\mathrm{cn}(\mathrm{w1}), \cos(\mathbf{am}(\mathrm{w1})) = \mathrm{cn}(\mathrm{w1})\ \},$$
$$(\ 1/\sqrt{1-\kappa^2\mathrm{sn}(\mathrm{w1})^2} = 1/\mathrm{dn}(\mathrm{w1}), \sqrt{1-\kappa^2\mathrm{sn}(\mathrm{w1})^2} = \mathrm{dn}(\mathrm{w1})\ ). \tag{48}$$

The *sub_set* of equations above was used as a *mask* for the **dchange** command in session *s4.ms*, and defines the Jacobi functions as well as some related identities. This mask introduces the desired Jacobi functions and related simplifications at each change of variables in an automatic manner. The code implementing the mask is quite simple:

```
> Dchange := () -> subs(sub_set,dchange(args)):
```

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