Chapter 4

MHD Equilibrium and Stability

Resistive Diffusion

Before discussing equilibrium properties let us first consider effects of electric resistivity. Using the resistive form of Ohm's law (with constant resistivity η) and the induction equation

$$\begin{aligned}\frac{\partial \mathbf{B}}{\partial t} &= -\nabla \times \mathbf{E} \\ &= \nabla \times \left[\mathbf{u} \times \mathbf{B} - \frac{\eta}{\mu_0}\nabla \times \mathbf{B}\right] \\ &= \nabla \times [\mathbf{u} \times \mathbf{B}] + \frac{\eta}{\mu_0}\Delta \mathbf{B}\end{aligned}$$

In cases where the velocity is negligible one obtains

$$\frac{\partial \mathbf{B}}{\partial t} = \frac{\eta}{\mu_0}\Delta \mathbf{B}$$

This equation indicates that the magnetic field evolves in the presence of a finite resistivity even in the absence of any plasma flow. This process is called resistive diffusion. Dimensional analysis yields for the typical time scale

$$\tau_{diff} \sim \frac{\mu_0 L^2}{\eta}$$

i.e., diffusion is fast for large values of η or small typical length scales. For the prior discussion of the frozen-in condition we require that the resistivity is 0 or at least so small that the diffusion time is long compared to the time scale for convection where the frozen-in condition is applied.

4.1 Basic Two-Dimensional Equilibrium Equations and Properties

Equilibrium requires $\partial/\partial t = 0$ and $\mathbf{u} = 0$. Thus the MHD equations lead to

$$
\begin{aligned}
-\nabla p + \mathbf{j} \times \mathbf{B} &= 0 \\
\nabla \times \mathbf{B} &= \mu_0 \mathbf{j} \\
\nabla \cdot \mathbf{B} &= 0
\end{aligned}
\tag{4.1}
$$

Taking the scalar product of the momentum equation with $\mathbf{B}$ and $\mathbf{j}$ yields

$$
\begin{aligned}
\mathbf{B} \cdot \nabla p &= 0 \\
\mathbf{j} \cdot \nabla p &= 0
\end{aligned}
$$

In other words the pressure is constant on magnetic field lines and on current lines.

Note: There is no equation for the plasma density. Only the pressure needs to be determined. From

$$
p = n k_B T
$$

it follows that only the product nT is fixed and either density or temperature can be chosen arbitrarily.

The momentum equation can also be expressed as

$$
\nabla \cdot \left[\left(p + \frac{B^2}{2\mu_0} \right) \underline{\underline{1}} - \frac{1}{\mu_0} \mathbf{B}\mathbf{B} \right] = 0
$$

or

$$
\nabla \left(p + \frac{B^2}{2\mu_0} \right) - \frac{1}{\mu_0} \mathbf{B} \cdot \nabla \mathbf{B} = 0
$$

This implies particularly simple equilibria for $\mathbf{B} \cdot \nabla \mathbf{B} = 0$ (a simple case of this condition is a magnetic field with straight magnetic field lines). In this case the equilibrium requires total pressure balance where the sum of thermal and magnetic pressure are 0, i.e.,

$$
\nabla \left(p + \frac{B^2}{2\mu_0} \right) = 0
$$

or

$$
p + \frac{B^2}{2\mu_0} = \mathrm{const} \tag{4.2}
$$

A particular example of this class of equilibria is the plain current sheet with the specific example of a so-called Harris sheet

4.2 Harris Sheet Equilibrium

The magnetotail is surprisingly stable for long periods of time. Typically convection is small and the tail configuration is well described by equilibrium solutions. Analytic equilibria are available for the section of the magnetotail where the variation along the magnetotail (or in the cross-tail direction y) is small compared to the variation perpendicular to the current sheet (weakly two- or three-dimensional) (e.g., [**?**, **?**, **?**, **?**]). Because of the variation in x the solutions are applicable only at sufficient distances ($\geq 10R_E$) from the Earth. These equilibria can be constructed as fully kinetic solutions. In the MHD approximation they solve the static MHD equations.

A simple example for this class of analytic solutions is the following solution which represents a two-dimensional modification of the classic Harris sheet configurations [**?**].

Kinetic equilibria can be constructed by assuming the distribution function as a function of the constants of motion. Specifically for the magnetotail the Harris solution can be derived by assuming

$$\frac{\partial}{\partial t} = 0 \quad \text{and} \quad \frac{\partial}{\partial y} = 0$$

Such that the constants of motion for the particle species s are

$$H_s = m_s v^2/2 + q_s\phi \quad \text{and} \quad P_{ys} = m_s v_y + q_s A_y$$

where ϕ and A_y are the electric potential and the y component of the vector potential. Any function of the constants of motion $f_s(\mathbf{r}, \mathbf{v}) = F_s(H_s, P_{ys})$ solves the collisionless Boltzmann equation. To obtain/specify distribution functions that are in local thermodynamics equilibrium one can choose

$$F_s(H_s, P_{ys}) = c_s \exp(-\alpha_s H_s + \beta_s P_{ys})$$

where α_s and β_s need to be specified to obtain the required distribution functions. The exponent $W_s = -\alpha_s H_s + \beta_s P_{ys}$can be re-written as

$$\begin{aligned} W_s &= -\alpha_s \left[\frac{m_s}{2}\left(v_x^2 + v_y^2 + v_z^2\right) + q_s\phi - \frac{\beta_s}{\alpha_s}\left(m_s v_y + q_s A_y\right)\right] \\ &= -\alpha_s \left[\frac{m_s}{2}\left(v_x^2 + \left(v_y - \frac{\beta_s}{\alpha_s}\right)^2 + v_z^2\right) - \frac{m_s}{2}\frac{\beta_s^2}{\alpha_s^2} + q_s\phi - \frac{\beta_s}{\alpha_s} q_s A_y\right] \end{aligned}$$

This illustrates that $\alpha_s = 1/(k_B T_s)$ to obtain a Maxwell distribution (local thermodynamic equilibrium). Further, β_s/α_s should be interpreted as a constant velocity that shifts the distribution in the v_y direction. Note also that $\phi = \phi(\mathbf{r})$ and $A_y = A_y(\mathbf{r})$ which need to be determined through the solution of Maxwell's equations (Poisson equation and Ampere's law). Defining $w_s = \beta_s/\alpha_s$ and $\widehat{c}_s = c_s \exp(\alpha_s m_s w_s^2/2)$ (note that all terms in the exponential in $\widehat{c}_s$ are constants such that we can just re-define the normalization constant c_s. The complete distribution function for species s can now be written as

$$f_s(\mathbf{r},\mathbf{v}) = \widehat{c}_s \exp\left\{\frac{m_s}{2k_BT_s}\left(v_x^2 + (v_y - w_s)^2 + v_z^2\right)\right\} \exp\left\{-\frac{q_s}{k_BT_s}\left(\phi(\mathbf{r}) - w_sA_y(\mathbf{r})\right)\right\}$$

Note that the first exponential depends only on velocity coordinates and the second depends only on spatial coordinates through ϕ and A_y. To obtain charge and current densities we need to integrate the distribution function over velocity space. The integral for the density also determines the normalization coefficient $\widehat{c}_s$. Let us abbreviate the second exponential function with

$$h(\phi(\mathbf{r}), A_y(\mathbf{r})) = \exp\left\{-\frac{q_s}{k_BT_s}\left(\phi(\mathbf{r}) - w_sA_y(\mathbf{r})\right)\right\}$$

Normalization: with the definition $n_s(\mathbf{r}) = n_{0s}h_s(\phi, A_y)$ and

$$\begin{aligned}
n_s(\mathbf{r}) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_s(\mathbf{r},\mathbf{v})\, d^3v \\
&= \widehat{c}_s h_s(\phi, A_y)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\left\{\frac{m_s}{2k_BT_s}\left(v_x^2 + (v_y - w_s)^2 + v_z^2\right)\right\} d^3v \\
&= \widehat{c}_s\left(\frac{2\pi k_BT_s}{m_s}\right)^{3/2} \exp\left\{-\frac{q_s}{k_BT_s}\left(\phi(\mathbf{r}) - w_sA_y(\mathbf{r})\right)\right\}
\end{aligned}$$

the normalization constant is

$$\widehat{c}_s = n_{0s}\left(\frac{m_s}{2\pi k_BT_s}\right)^{3/2}$$

Charge density ρ_c and current density $\mathbf{j}_s = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathbf{v} f_s(\mathbf{r},\mathbf{v})\, d^3v$ for species s are now obtained as

$$\begin{aligned}
\rho_{cs} &= q_sn_{0s}h_s(\phi, A_y) \\
\mathbf{j}_s &= q_sn_{0s}w_sh_s(\phi, A_y)\,\mathbf{e}_y
\end{aligned}$$

Note that current density in the x and z direction is 0 because of the symmetry. The current density along y can be formally computed from the above integral or it can be argued that the bulk velocity must be w_s because the distribution function is a Maxwellian (symmetric) shifted by the the velocity w_s in the v_y direction.

Assuming a plasma of single charged ions and electrons yields for the total charge and the current density

$$\begin{aligned}
\rho_c &= e\left\{n_{0i}\exp\left\{-\frac{e}{k_BT_i}\left(\phi - w_iA_y\right)\right\} - n_{0e}\exp\left\{\frac{e}{k_BT_e}\left(\phi - w_eA_y\right)\right\}\right\} \\
j_y &= e\left\{n_{0i}w_i\exp\left\{-\frac{e}{k_BT_i}\left(\phi - w_iA_y\right)\right\} - n_{0e}w_e\exp\left\{\frac{e}{k_BT_e}\left(\phi - w_eA_y\right)\right\}\right\}
\end{aligned}$$

With these sources we need to solve the Poisson equation and Ampere's law. Since we seek a solution on scales much larger than the Debye length we can assume a neutral plasma. Here is customary to assume either quasi-neutrality $\rho_c = 0$ or exact neutrality $\phi = 0$. Since the assumption of quasi-neutrality leads to basically the same result but requires a more complicated treatment let us start directly with exact neutrality $\phi = 0$. The condition $\rho_c = 0$ yields

$$n_{0i} \exp\left\{\frac{ew_i}{k_B T_i} A_y\right\} - n_{0e} \exp\left\{-\frac{ew_e}{k_B T_e} A_y\right\} = 0$$

this equation is solved by

$$\begin{aligned} n_{0e} = n_{0i} &= n_0 \\ w_e &= -\frac{T_e}{T_i} w_i \end{aligned}$$

Substitution in the equation for current density and using $\nabla \times \mathbf{B} = -\Delta \mathbf{A} = \mu_0 \mathbf{j}$ yields

$$\Delta A_y = \lambda \exp\left\{-A_y/\kappa\right\}$$

with $\lambda = -\mu_0 e n_0 w_i \left(1 + \frac{T_e}{T_i}\right)$ and $\kappa = -k_B T_i / (ew_i) = k_B T_e / (ew_e)$. The above equation is called the Grad-Shafranov equation. This equation is equivalent to $\Delta A_y = -\lambda \exp\left\{A_y/\kappa\right\}$ if we define λ and κ with opposite signs. This does not alter the physical solution. Note that there is also a relation to the plasma pressure. It is obvious that the plasma pressure has the same dependence on the flux function A_y as the number and current densities and it can in general be shown that $j(A_y) = dp(A_y)/dA_y$. This relation will be proven later also for the MHD solutions for two-dimensional equilibria.

There are various analytic solutions to this equation. Note that the specific form of the current density is due to the choice of the distribution function. Note, however, that our choice only permits two-dimensional solutions. To derive the Harris sheet let us consider a one-dimensional solution of the Grad-Shafranov equation.

$$\frac{d^2 A_y}{dx^2} = \lambda \exp\left(-A_y/\kappa\right)$$

Note that we can normalize this equation using $A_0 = 2\kappa$ and $A_0/L^2 = \lambda$ and with $\tilde{A} = A_y/A_0$, $\tilde{x} = x/L$ the normalized version becomes $d^2\tilde{A}/d\tilde{x}^2 = \exp\left(-2\tilde{A}\right)$. Multiplying this equation with dA_y/dx and re-arranging yields

$$\frac{d}{dx}\left(\frac{1}{2}\left(\frac{dA_y}{dx}\right)^2 + \lambda\kappa \exp\left(-A_y/\kappa\right)\right) = 0 \tag{4.3}$$

$$\text{or} \qquad \frac{1}{2}\left(\frac{dA_y}{dx}\right)^2 + \lambda\kappa \exp\left(-A_y/\kappa\right) = \text{const} \tag{4.4}$$

Note that $B_z = dA_y/dx$ such that the above expression can also be written as

$$\frac{B_z^2}{2\mu_0} + \frac{\lambda\kappa}{\mu_0}\exp\left(-A_y/\kappa\right) = \text{const}$$

which is the equation for total pressure balance where $B_z^2/2\mu_0$ is the magnetic and $\frac{\lambda\kappa}{\mu_0}\exp\left(-A_y/\kappa\right)$ the thermal plasma pressure with $p_0 = \lambda\kappa/\mu_0 = n_0 k_B \left(T_i + T_e\right)$. We can now integrate (4.4) once more by re-arranging it into

$$\frac{1}{\sqrt{2}}\int_{A_c}^{A_y}\left[p_{tot} - p_0 \exp\left(-2A_y'/A_0\right)\right]^{-1/2} dA_y' = \int_{x_c}^{x} dx'$$

which can be simplified and solved with the substitution $p' = p_0 \exp\left(-2A_y'/A_0\right)$ to obtain the solution for A_y as

$$A_y = A_0 \ln\cosh\frac{x}{L}$$

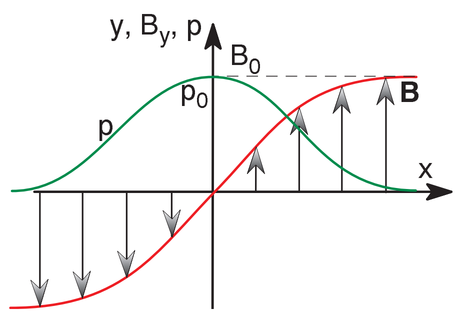
Expressing the length scale L and A_0 to typical plasma properties yields

$$\begin{aligned} L &= \lambda_i \frac{v_{thi}}{|w_i|}\sqrt{\frac{2T_i}{T_i + T_e}} \\ A_0 &= -2\frac{k_B T_i}{e w_i} \\ B_0 &= -\mathrm{sgn}\left(w_i\right)\sqrt{2\mu_0\left(p_{i0} + p_{e0}\right)} \end{aligned}$$

with $p_{s0} = n_0 k_B T_s$ and $v_{thi} = \sqrt{k_B T_i/m_i}$. These relations show that the magnetic field amplitude B_0 scale with the square root of the thermal pressure. The width L of the Harris sheet is proportional to the ion inertial length and scales with the ratio of ion thermal velocity v_{thi} to drift velocity w_i. The magnetic field, number density, current density, and pressure are

$$\mathbf{B} = B_0 \tanh\frac{x}{L}\,\mathbf{e}_y$$

with



$$\begin{aligned} p &= p_0 \cosh^{-2}\frac{x}{L} \\ \mathbf{j} &= \frac{B_0}{\mu_0 L}\cosh^{-2}\frac{x}{L}\,\mathbf{e}_z \\ \rho &= \rho_0 \cosh^{-2}\frac{x}{L} \end{aligned}$$

with $p_0 = B_0^2/(2\mu_0)$.

Note that the Harris solution is specific to the kinetic requirement of a local thermal equilibrium. It must satisfy th eMHD equilibrium condition (4.2). Within MHD one can add any constant to the pressure or the density. It is also straightforward to modify the magnetic field, for instance to an asymmetric configuration. the pressure is computed from (4.2) and only subject to the condition that it must be larger than 0 everywhere. However these modification do not represent necessarily a kinetic equilibrium. For instance, an additional constant pressure and density would require additional kinetic distribution for electrons and ions which in general will change the integrals that provide the charge and current density from the kinetic theory and hereby modify the Grad-Shafranov equation.

The magnetic and plasma configuration also determines how important pressure relative to magnetic forces are. It is common to use the so-called plasma β as a measure of the thermal pressure to magnetic field pressure (this is also a measure of the corresponding thermal and magnetic energy densities.). The plasma β is defined as

$$\beta = \frac{2\mu_0 p}{B^2}$$

and

4.3 Virial Theorem, General Conditions for Isolated Equilibria

To derive some general properties that are important for equilibria let us assume an isolated equilibrium in a Domain D such that the magnetic field outside of D vanishes and the thermal pressure assumes a value of p_0. Let us also assume that the magnetic configuration in D satisfies the magnetohydrostatic force balance conditions. In this case the normal magnetic field on the boundary ∂D must be 0 because otherwise there would be a magnetic field outside. Therefore the magnetic field can be locally 0 on both sides of the boundary or the boundary could be a tangential discontinuity across which $p + B^2/2\mu_0 = \mathrm{const}$. With

$$\begin{aligned} \underline{\underline{\mathbf{M}}} &= -\left(p + \frac{B^2}{2\mu_0}\right)\underline{\underline{1}} + \frac{1}{\mu_0}\mathbf{B}\mathbf{B} && (4.5)\\ and \qquad & \nabla \cdot \underline{\underline{\mathbf{M}}} = 0 && (4.6) \end{aligned}$$

we can multiply the force balance condition with an arbitrary vector field $\mathbf{K}\left(\mathbf{r}\right)$ and integrate the result over the domain D:

$$\int_D \mathbf{K} \cdot \left(\nabla \cdot \underline{\underline{\mathbf{M}}}\right) d^3r = 0 \tag{4.7}$$

Using Gauss theorem and integration by parts yields

$$\oint_{\partial D} \mathbf{K} \cdot \underline{\underline{\mathbf{M}}} \cdot \mathbf{n}_D ds - \int_D \underline{\underline{\mathbf{M}}} : \left(\nabla \mathbf{K}\right) d^3r = 0 \tag{4.8}$$

where : denotes the total contraction of the tensor multiplication. Substituting the tensor $\underline{\underline{\mathbf{M}}}$ yields

$$\begin{aligned} \mathbf{K} \cdot \underline{\underline{\mathbf{M}}} \cdot \mathbf{n}_D &= -\left(p + \frac{B^2}{2\mu_0}\right)\mathbf{K} \cdot \mathbf{n}_D + \frac{1}{\mu_0}\mathbf{K} \cdot \mathbf{B}\mathbf{B} \cdot \mathbf{n}_D && (4.9)\\ \underline{\underline{\mathbf{M}}} : \left(\nabla \mathbf{K}\right) &= -\left(p + \frac{B^2}{2\mu_0}\right)\nabla \cdot \mathbf{K} + \frac{1}{\mu_0}\mathbf{B} \cdot \left(\mathbf{B} \cdot \nabla \mathbf{K}\right) && (4.10) \end{aligned}$$

The surface terms can now be evaluated as follows using the properties that the magnetic field outside of D vanishes $\mathbf{B} \cdot \mathbf{n}_D$ and the pressure assumes p_0:

$$\begin{aligned}
\oint_{\partial D} \mathbf{K} \cdot \underline{\underline{\mathbf{M}}} \cdot \mathbf{n}_D ds &= -\oint_{\partial D} \left(p + \frac{B^2}{2\mu_0} \right) \mathbf{K} \cdot \mathbf{n}_D ds \\
&= -p_0 \oint_{\partial D} \mathbf{K} \cdot \mathbf{n}_D ds \\
&= -p_0 \int_D \nabla \cdot \mathbf{K} \cdot d^3 r
\end{aligned}$$

We can now collect all terms in the volume integral

$$\int_D \left[\left(p - p_0 + \frac{B^2}{2\mu_0} \right) \nabla \cdot \mathbf{K} - \frac{1}{\mu_0} \mathbf{B} \cdot (\mathbf{B} \cdot \nabla \mathbf{K}) \right] d^3 r = 0 \tag{4.11}$$

and choose the vector field in some manner. The significance of this is that the volume integral divided by the volume of D represents the average of the quantities in the integrant. The quantities can be selected to some degree by the choice of $\mathbf{K}$. Three examples for this choice are the following:

$$\begin{aligned}
\mathbf{K} = \mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} &\implies \left\langle \frac{B^2}{2\mu_0} \right\rangle_D = 3 \langle p_0 - p \rangle \\
\mathbf{K} = \mathbf{r} - z\mathbf{e}_z = \begin{pmatrix} x \\ y \\ 0 \end{pmatrix} &\implies \left\langle \frac{B_z^2}{2\mu_0} \right\rangle_D = \langle p_0 - p \rangle \\
\mathbf{K} = y\mathbf{e}_x = \begin{pmatrix} y \\ 0 \\ 0 \end{pmatrix} &\implies \langle B_x B_y \rangle_D = 0
\end{aligned}$$

where the brackets denote the average over the domain D. The results show that the average magnetic field energy density is determined by the average pressure difference between the outside and inside thermal pressure. The magnetic energy density is equally distributed into each component of the magnetic field and the average magnetic field crosproduct $B_x B_y$ is 0. It is also important to note that the magnetic field and pressure must be 0 if the outside pressure p_0 is 0. This implies that a magnetic field cannot be selfconfined, i.e., a static magnetic configuration cannot be maintained if the outside pressure is 0. However, this does not imply the existence of three-dimensional solutions of a static magnetic field confined in a limited region of space. This existence is not trivial. However, the situation changes if one allows that the magnetic field intersects with a portion of the three-dimensional domain boundary.

4.4 Two-Dimensional Equilibria

4.4.1 Basic Equations

A systematic approach to solve the equilibrium equations usually requires to represent the magnetic field through the vector potential.

$$\mathbf{B} = \nabla \times \mathbf{A}$$

Consider 2d system with $\partial/\partial z = 0$ such that $\mathbf{B} = \mathbf{B}(x, y)$. In this case the magnetic field is uniquely represented through the z component of the vector potential $A_z(x, y)$ and the $B_z(x, y)$ component:

$$\mathbf{B} = \nabla \times (A_z \mathbf{e}_z) + B_z \mathbf{e}_z = \nabla A_z \times \mathbf{e}_z + B_z \mathbf{e}_z$$

We only need two dependent variables because of $\nabla \cdot \mathbf{B} = 0$. Note also that this form of $\mathbf{B}$ always satisfies $\nabla \cdot \mathbf{B} = 0$. Denoting the field in the x, y plane as

$$\mathbf{B}_\perp = \nabla \times A_z(x, y)\mathbf{e}_z = \nabla A_z(x, y) \times \mathbf{e}_z \qquad (4.12)$$

it follows that $\mathbf{B}_\perp$ is perpendicular to ∇A_z and $\mathbf{e}_z$.

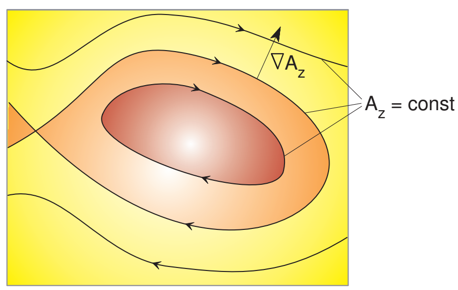


Figure 4.1: Representation of field lines by the the vector potential

Therefore lines of constant A_z (contour lines of A_z) are magnetic field lines projected into the x, y plane. The difference of the vector potential between two field lines is a direct measure of the magnetic flux bound by these field lines. The vector potential can be obtained by integrating

$B_x = \partial A_z/\partial y$ and $B_y = -\partial A_z/\partial x$.

Exercise: Consider the magnetic field $B_x = B_0 y/L$, $B_y = \epsilon B_0$. Compute the equations for magnetic field lines.

With $\nabla \times (\mathbf{A} \times \mathbf{B}) = \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A}) + (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B}$ the current density becomes

$$\begin{aligned}\mathbf{j} &= \frac{1}{\mu_0}\nabla \times [\nabla A_z \times \mathbf{e}_z + B_z \mathbf{e}_z] \\ &= -\frac{1}{\mu_0}\Delta A_z \mathbf{e}_z + \frac{1}{\mu_0}\nabla B_z \times \mathbf{e}_z\end{aligned}$$

such that the z component of the current density is

$$j_z = -\frac{1}{\mu_0}\Delta A_z$$

Substituting the current density in the force balance equation and using $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \cdot \mathbf{C})\,\mathbf{B} - (\mathbf{A} \cdot \mathbf{B})\,\mathbf{C}$

$$
\begin{aligned}
0 &= -\nabla p + \mathbf{j} \times \mathbf{B} \\
&= -\nabla p + \frac{1}{\mu_0}\left[-\Delta A_z \mathbf{e}_z + \nabla B_z \times \mathbf{e}_z\right] \times \left[\nabla A_z \times \mathbf{e}_z + B_z \mathbf{e}_z\right] \\
&= -\nabla p + \frac{1}{\mu_0}\left\{-\Delta A_z \nabla A_z - \left((\nabla B_z \times \mathbf{e}_z) \cdot \nabla A_z\right)\mathbf{e}_z - B_z \nabla B_z\right\} \\
&= -\nabla p + \frac{1}{\mu_0}\left\{-\Delta A_z \nabla A_z - B_z \nabla B_z + \nabla B_z \times \nabla A_z\right\}
\end{aligned}
$$

Here the term $\nabla B_z \times \nabla A_z$ has only a z component because ∇B_z and ∇A_z are both in the x, y plane. Since it is the only term in the z it follows that $\nabla B_z \times \nabla A_z = 0$ or $\nabla B_z \| \nabla A_z$. Thus we can in general express $B_z = B_z(A_z)$. This can be used to express

$$
B_z \nabla B_z = \frac{1}{2}\nabla B_z^2 = \frac{1}{2}\frac{dB_z^2}{dA_z}\nabla A_z
$$

In summary the force balance condition leads to

$$
\nabla p = \left(j_z - \frac{1}{2\mu_0}\frac{dB_z^2}{dA_z}\right)\nabla A_z
$$

Since the pressure gradient is along the gradient of A_z the pressure must be a function of A_z. It follows that

$$
\Delta A_z = -\mu_0 \frac{d}{dA_z}\left(p(A_z) + \frac{B_z(A_z)^2}{2\mu_0}\right)
$$

In this equation p represents the thermal pressure and $B_z^2/2\mu_0$ is the magnetic pressure due to the z component of the magnetic field. In other words in two dimensions the magnetic field along the invariant direction acts mostly as a pressure to maintain an equilibrium. Defining

$$
\widetilde{p}(A_z) = p + \frac{B_z^2}{2\mu_0}
$$

we have to seek the solution to

$$
\Delta A_z = -\mu_0 \frac{d}{dA_z}\widetilde{p}(A_z). \tag{4.13}
$$

Usually $\widetilde{p}(A_z)$ is defined as a relatively simple form. Most convenient for traditional solution methods is to define $\widetilde{p}(A_z)$ as a linear function of A_z. More realistic is a definition which requires a kinetic background. For the Harrsi equilibrium it is shown that local thermodynamic equilibrium implies $p(A_z) \propto \exp(-2A_z/A_c)$ where A_c is a constant which follows from the later kinetic treatment of the equilibrium. In general $j_z(A_z) = dp(A_z)/dA_z$ if $B_z = const$.

Exercise: Consider $p(A_z) = p_0 \exp(-2A_z/A_c)$ and $B_z = 0$. Show that a one-dimensional ($\partial/\partial y = 0$ and $\partial/\partial z = 0$) solution of equation (4.13) has the form $A_z = A_c \ln\cosh(x/L)$ and that the magnetic field in this case is the Harris sheet field.

Exercise: Consider one-dimensional solutions with $\partial/\partial x \neq 0$. Obtain the first integral of the equation (4.13) by multiplying the equation with dA_z/dx and integration. The resulting equation is the equation of total pressure balance. Interpret the term in the first integral in this manner.

Exercise: Consider a two-dimensional plasma $\partial/\partial z = 0$ with $B_z = 0$ and use the representation of the magnetic field through the vector potential. Replace the magnetic field in resistive Ohm's law $\mathbf{E} + \mathbf{u} \times \mathbf{B} = \frac{\eta}{\mu_0}\nabla \times \mathbf{B}$ through the vector potential and show for $\eta = \mathrm{const}$ that this yields for the z component $\partial A_z/\partial z + \mathbf{u}\cdot\nabla A_z = \frac{\eta}{\mu_0}\Delta A_z$. For $\eta = 0$ this equation directly demonstrates the frozen-in condition. Explain why.

Other coordinate systems: Note that the above discussion can be generalized to coordinate systems other than Cartesian. For Laboratory plasmas with an azimuthal invariance it is often convenient to use cylindrical coordinates with $\partial/\partial\varphi = 0$. If there is invariance along a cylinder axis one can also use $\partial/\partial z = 0$. Let us consider rotational invariance in cylindrical coordinates. In this case the magnetic field is

In spherical coordinates the use of a component of the vector potential is similar but slightly modified. Consider two-dimensionality with $\partial/\partial\varphi = 0$. In this case the magnetic field is expressed as

$$\begin{aligned} \mathbf{B}(r,z) &= \nabla \times A_\varphi(r,z)\mathbf{e}_\varphi + B_\varphi(r,z)\mathbf{e}_\varphi && (4.14)\\ &= \nabla\psi \times \nabla\varphi + \kappa\nabla\varphi && (4.15)\end{aligned}$$

with $\psi = rA_\varphi$ and $\kappa = rB_\varphi$. This decomposition is equivalent to the one used for cartesian coordinates. Since $\partial p/\partial\varphi = 0$ the φ component of the $\mathbf{j}\times\mathbf{B}$ force is 0 which yields $\kappa = \kappa(\psi)$ and evaluating force balance in cylinfrical coordinate with this invariance yields

$$-\frac{1}{\mu_0}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\psi}{\partial r^2}\right] = \frac{dp}{d\psi} + \frac{H}{\mu_0 r^2}\frac{dH}{d\psi} \tag{4.16}$$

It is fairly obvious that the procedure and the equation for two-dimensional equilibria is very similar to the case of cartesian coordinates.

Exercise: Demonstrate that magnetic field lines are determined by $\psi = const$ and that the $[\mathbf{j}\times\mathbf{B}]\cdot\mathbf{e}_\varphi = 0$ requires $\kappa = \kappa(\psi)$.

Exercise: Show that (4.16) represents the force balance equation in cylindrical coordinates.

Exercise: Repeat the above formulation for spherical coordinates with $\partial/\partial\varphi = 0$. Compute B_r and B_θ and demonstrate that magnetic field lines are defined by $f = r\sin\theta\, A_\varphi = const$ (using $\mathbf{B}\cdot\nabla f = 0$).

4.4.2 Liouville Solutions

Let us consider the Grad-Shafranov equation that was determined by the Harris sheet solution (for $\partial/\partial z = 0$)

$$\Delta A_z = -\mu_0 j_y = \lambda \exp\left\{-A_z/\kappa\right\}$$

with $\lambda = -\mu_0 e n_0 w_i \left(1 + \frac{T_e}{T_i}\right)$ and $\kappa = -k_B T_i / \left(e w_i\right) = k_B T_e / \left(e w_e\right)$ and $j_c = -\lambda/\mu_0$. With $j_z\left(A_z\right) = dp\left(A_z\right)/dA_z$

$$\begin{aligned} p\left(A_z\right) &= p_c \exp\left\{-A_z/\kappa\right\} \qquad \text{with} \qquad p_c = \lambda\kappa/\mu_0 \\ \Delta A_z &= -\mu_0 \frac{dp}{dA_z} = \mu_0 \frac{p_c}{\kappa} \exp\left\{-A_z/\kappa\right\} \end{aligned}$$

with $p_c = n_0 k_B \left(T_i + T_e\right)$.

Normalization: In the case of a kinetic background of the Grad Shafranoff equation the normalization of the flux function, length scales, and magnetic field can be based on the properties of the distribution function and the drift velocities. With $\hat{A} = A_z/A_0$, $\hat{x} = x/L_0$ we can use $A_0 = 2\kappa$ and $A_0/L_0^2 = \lambda$. This leads to

$$\begin{aligned} L_0 &= \sqrt{\frac{2\kappa}{\lambda}} = \sqrt{\frac{\epsilon_0 m_i 2 k_B T_i^2}{\mu_0 \epsilon_0 m_i n_0 e^2 w_i^2 \left(T_i + T_e\right)}} = \lambda_i \frac{v_{thi}}{|w_i|} \sqrt{\frac{2T_i}{T_i + T_e}} \\ A_0 &= -2 \frac{k_B T_i}{e w_i} \\ B_0 &= \frac{A_0}{L} = -\mathrm{sgn}\left(w_i\right) \sqrt{2\mu_0 \left(p_{i0} + p_{e0}\right)} \end{aligned}$$

And the normalized equations:

$$\begin{aligned} \hat{p}\left(\hat{A}\right) &= \frac{1}{2} \exp\left\{-2\hat{A}\right\} \qquad \text{with} \qquad \hat{p} = p/p_0 \quad and \quad p_0 = 2p_c \\ \hat{\Delta}\hat{A} &= \exp\left\{-2\hat{A}\right\} \end{aligned}$$

In the case of a MHD equilibrium the normalization is only based on macro variables such as typical magnetic field or pressure. MHD also allows to include a magnetic field component along the z direction

$$\widetilde{p}(A_z) = p\left(A_z\right) + \frac{B_z^2\left(A_z\right)}{2\mu_0} = \frac{1}{2} p_0 \exp\left(-2\frac{A_z}{A_0}\right)$$

with constants p_0 and A_0 and normalize the length scales to $L_0 = A_0/\sqrt{\mu_0 p_c}$ and the vectorpotential to A_0. In the following we will assume $B_z\left(A_z\right) = 0$ for most cases and in those cases the solution of the Grad-Shafranov equation can be identified with a kinetic equilibrium but $B_z\left(A_z\right) = 0$ is not a necessary condition for for these equilibria. With this normalization the Grad-Shafranov equation is

$$\Delta A = \exp\left(-2A\right) \tag{4.17}$$

where we omitted the hat to indicate normalized quantities. This equation also plays a role in fluid dynamics and Liouville had developed a method to generate an infinite number solutions to this equation once one solution is known. Consider a transformation from x and $y$coordinates to a complex coordinate $\zeta = x + iy$ and its complex conjugate ζ^*. Thus A becomes a function of ζ and ζ^* with $A'(\zeta, \zeta^*) = A(x(\zeta, \zeta^*), y(\zeta, \zeta^*))$. This transforms the Grad-Shafranov equation to

$$4\frac{\partial^2 A}{\partial\zeta\partial\zeta^*} = \exp(-2A)$$

With

$$\begin{aligned}
\zeta = x + iy \qquad & \zeta^* = x - iy \\
x = \frac{1}{2}(\zeta + \zeta^*) \qquad & y = \frac{1}{2i}(\zeta - \zeta^*) \\
\frac{\partial}{\partial\zeta} = \frac{\partial x}{\partial\zeta}\frac{\partial}{\partial x} + \frac{\partial y}{\partial\zeta}\frac{\partial}{\partial y} \qquad & \frac{\partial}{\partial\zeta^*} = \frac{\partial x}{\partial\zeta^*}\frac{\partial}{\partial x} + \frac{\partial y}{\partial\zeta^*}\frac{\partial}{\partial y} \\
= \frac{1}{2}\frac{\partial}{\partial x} + \frac{1}{2i}\frac{\partial}{\partial y} \qquad & = \frac{1}{2}\frac{\partial}{\partial x} - \frac{1}{2i}\frac{\partial}{\partial y} \\
\frac{\partial^2}{\partial\zeta\partial\zeta^*} &= \left(\frac{1}{2}\frac{\partial}{\partial x} + \frac{1}{2i}\frac{\partial}{\partial y}\right)\left(\frac{1}{2}\frac{\partial}{\partial x} - \frac{1}{2i}\frac{\partial}{\partial y}\right) \\
&= \frac{1}{4}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)
\end{aligned}$$

It is straightforward to show that any conformal mapping $\zeta' = \zeta'(\zeta) = x' + iy'$ generates a new solution $A'_y(x', y')$ with

$$A' = A + \ln\left|\frac{d\zeta'}{d\zeta}\right|$$

Another way to use this result is through analytic functions $f(\zeta) = h(x, y) + ig(x, y)$ and

$$A = \ln\frac{1 + |f(\zeta)|^2}{2|df/d\zeta|}$$

is a solution of the Grad-Shafranov equation.

4.4.3 Two-Dimensional Equilibrium Solutions

Harris Sheet and Kelvin's cat's eyes

A first explicit example of an equilibrium uses the normalized Grad-Shafranov equation (4.17) and employs a special Liouville solution defined by

$$f(\zeta) = \frac{p + \exp i\zeta}{\sqrt{1 - p^2}}$$

with p real and $\zeta = x + iy$. With

$$\begin{aligned}|f(\zeta)|^2 &= ff^* = \frac{1}{1-p^2}\left[(p + \cos x \exp(-y))^2 + \sin^2 x \exp(-2y)\right] \\ \frac{df(\zeta)}{d\zeta} &= \frac{i \exp i\zeta}{\sqrt{1-p^2}} = (i\cos x - \sin x)\frac{\exp(-y)}{\sqrt{1-p^2}}\end{aligned}$$

The solution for $0 \le p \le 1$ is given by

$$\begin{aligned}A &= \ln \frac{1 + |f(\zeta)|^2}{2\,|df/d\zeta|} = \ln\left(\frac{\cosh y + p\cos x}{\sqrt{1-p^2}}\right) \\ \text{with} \qquad & B_x = \partial_y A_z = \frac{\sinh y}{\cosh y + p\cos x} \\ & B_y = -\partial_x A_z = \frac{p \sin x}{\cosh y + p\cos x}\end{aligned}$$

For $p = 0$ we recover the Harris sheet. For $p > 0$ the solutions are periodic in x and behave like the Harris sheet at large distance from $y = 0$. A solution for $p = 0.25$ is shown in the figure below.

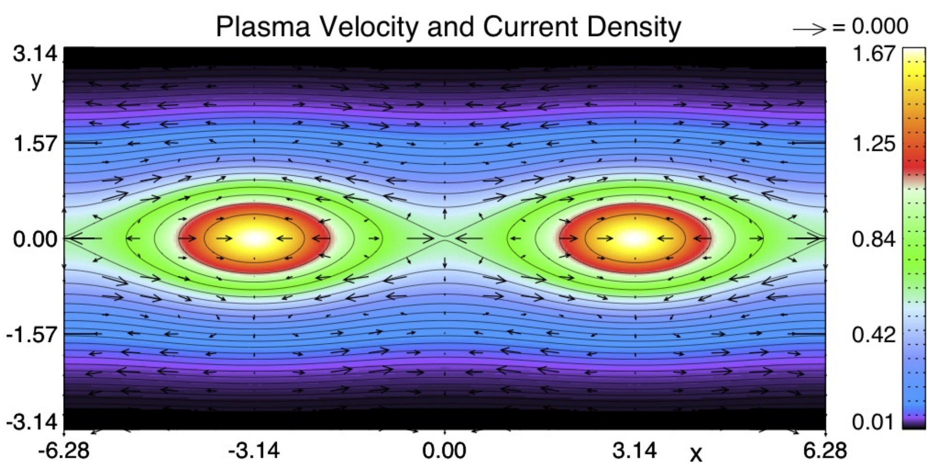


Figure 4.2: Magnetic field (lines), current density (color), and velocity (arrows) for the catseyes solution with $p = 0.25$.

The figure shows magnetic field lines and in color the current density. Arrows show the direction of the plasma velocity. The magnitude of the velocity is indicated by the arrow and the number in the upper right corner. Magnetic field, velocity and other quantities are normalized to typical values as explained above. This implies that velocity is measured in units of the Alfven speed $B_0/\sqrt{\mu_0\rho_0}$. The result in the figure is taken from a simulation after a short run of only 2 Alfven times $\tau_A = L_0/v_A$. After this short period any nonequilibrium perturbations show up as accelerated plasma. In this result the maximum velocity is $\le 10^{-4}$ in other words the acceleration is very small indicating that the configuration is an excellent equilibrium.

Two-Dimensional Magnetotail Models

i) Liouville solution: Equilibrium solutions resempling the magnetotail configuration are helpful in understanding properties of the magnetotail such as stability or the onset of magnetic reconnection. Here we use again the simple normalized Grad-Shafranov equation (4.17). Liouville solutions are difficult to formulate for a specific application and the typical approach is trial and error. For this application we consider

$$f(\zeta) = \exp\left[i\left(\zeta + \sqrt{\zeta/\varepsilon}\right)\right]$$

with $\zeta = x + iz$ and $\varepsilon \leq 1$. Typical magnetospheric coordinates (Geocentric Solar Magnetic coordinates) use x toward the sun, z in the northward direction, and y completes the coordinate system. Based on these coordinates, the magnetotail in the midnight meridian sector is mostly z dependent (crosstail current localized in z!) with some x dependence which is stronger closer to Earth. Here ε is a measure of the strength of the x dependence in the magnetotail (note that x points tailward here opposite to the GSM x direction).

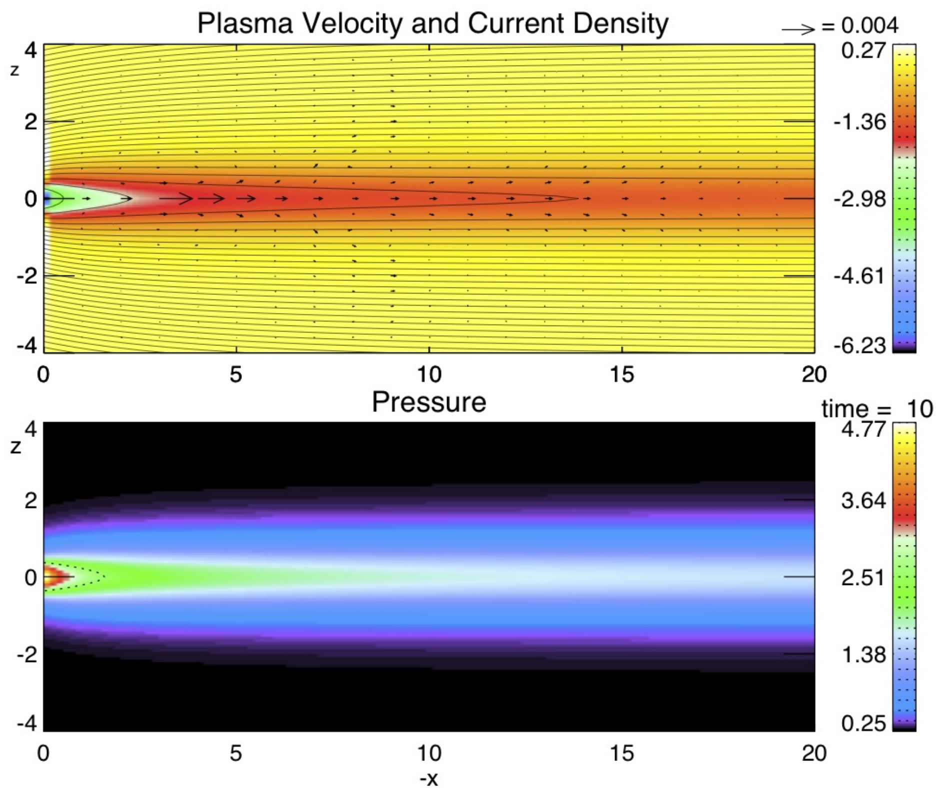


Figure 4.3: Magnetic field, current density, and velocity (upper plot) in the same format as Figure 4.2, and pressure (lower plot) for the Liouville tail solution.

In this case the flux function A actually represents A_y because close to the midnight meridian the y dependence is small and ignored in this approximation. The magnetic flux function in this case is given by

$$A_y = \ln \frac{\cosh\left\{\left[1 + (2r_1 + 2x_1)^{-1/2}\right] z\right\}}{\sqrt{1 + \left[(r_1 + x_1)/(2r_1^2)\right]^{1/2} + (4r_1)^{-1}}}$$

with $r_1 = \sqrt{x_1^2 + z_1^2}$, $x_1 = \varepsilon x$, and $z_1 = \varepsilon z$. This flux function has a singularity at the origin $x = z = 0$ which corresponds to a line current.

At large values of x the expressions in the square root and the square bracket in A_y approach 1 such that the configuration approaches a Harris sheet for large x which is also seen in the Figure . Using this flux function it is straightforward to compute the magnetic field, current density, and pressure. For instance pressure p and current density j_y are given by

$$p \sim j_y = \frac{1 + \left[(r_1 + x_1) / (2r_1^2)\right]^{1/2} + (4r_1)^{-1}}{\cosh^2 \left\{ \left[1 + (2r_1 + 2x_1)^{-1/2}\right] z \right\}}$$

This illustrates that pressure and current density decrease with increasing z values mostly due to the $\cosh^{-2}$ term. The width of this current sheet (current sheet half width) is small for small x values. For large x values the width approaches 1 and the magnitude of the current along the x axis approaches 1. Note, however, that there is only one parameter that can be used to modulate this magnitude of pressure and current along the x axis such that this solution is limited in matching an observed or statistical pressure profile to this solution. The simulation solution represents again an excellent equilibrium with maximum velocities $\approx 4 \cdot 10^{-4}$ after 10 Alfven times. In this cases a small background resisitivity is used corresponding to a magnetic Reynoldsnumber (Lundquistnumber) of 10^4 based on the unit length of the system. The flow seen in Figure 4.3 is actually due to this very small resistivity and applied over a very long period of time wou

ii) Asymptotic solution for weak x dependence: The Earth's magnetotail is a long drawn out structure generated by the solar wind. Close to Earth the magnetic field becomes approximately dipolar but at distances larger than about 15 R_E the dominant structure of the tail is that of a current sheet (similar to the Harris sheet) with a magnetic field component normal to the current sheet and with a weak variation of the current density along the tail axis. In other words there is a strong variation of the configuration in the z direction and a weak variation along the $x$direction (except for the dipolar region). This property can be taken into acount by using a flux function of teh form

$$A_y = A_y(x_1, z) \qquad \text{with} \qquad x_1 = \varepsilon x$$

This represent an approach which has been widely used in numerical modeling and stability analysis. It is noted that this approach can be formalized by representing the flux function in a power series expansion $A = \sum_k A_k(x_1, z)\,\varepsilon^k$ which inserted into the Grad-Shafranov equation and expanding $j(A)$ in powers of ε leads to a series of equations for the A_k. Using this form of A leads to

$$\varepsilon^2 \frac{\partial^2 A}{\partial x_1^2} + \frac{\partial^2 A}{\partial z^2} = -j(A) = -\mu_0 \frac{dp(A)}{dA}$$

Ignoring terms of order ε^2 the equation can be integrated with respect to z

$$\frac{1}{2}\left(\frac{\partial A}{\partial z}\right)^2 + p(A) = p_0(x_1)$$

A second integration can be performed like in the case of the Harris sheet leading to

$$z = z_0(x_1) + \int_{A_0(x_1)}^{A} \frac{dA'}{\sqrt{2(p_0(x_1) - p(A'))}} = \int_{p_0(x_1)}^{p'} \left(\frac{dp'}{dA'}\right) \frac{dp'}{\sqrt{2(p_0(x_1) - p')}}$$

For $p(A) = 0.5 p_0 \exp(-2A)$ and $A_0(x_1) = -0.5 \ln(2 p_0(x_1))$ the integral can be solved to yield

$$A(x_1, z) = \ln\left[l(x_1)\cosh(z/l(x_1))\right] \qquad with \qquad l(x_1) = (2p_0(x_1))^{-1/2}$$

In this formulation we can for instance choose $l(x_1)$ which represents a choice of the width of the current sheet as a function of x, or $p_0(x_1)$ which represents the total pressure as function of the distance along the tail. Decreasing pressure implies an increasing with of teh current sheet. It is straightforward to compute the magnetic field components from this formultion and it is noted that the x component also decreases with decreasing pressure (but also depends of the strength of the downtail gradient).

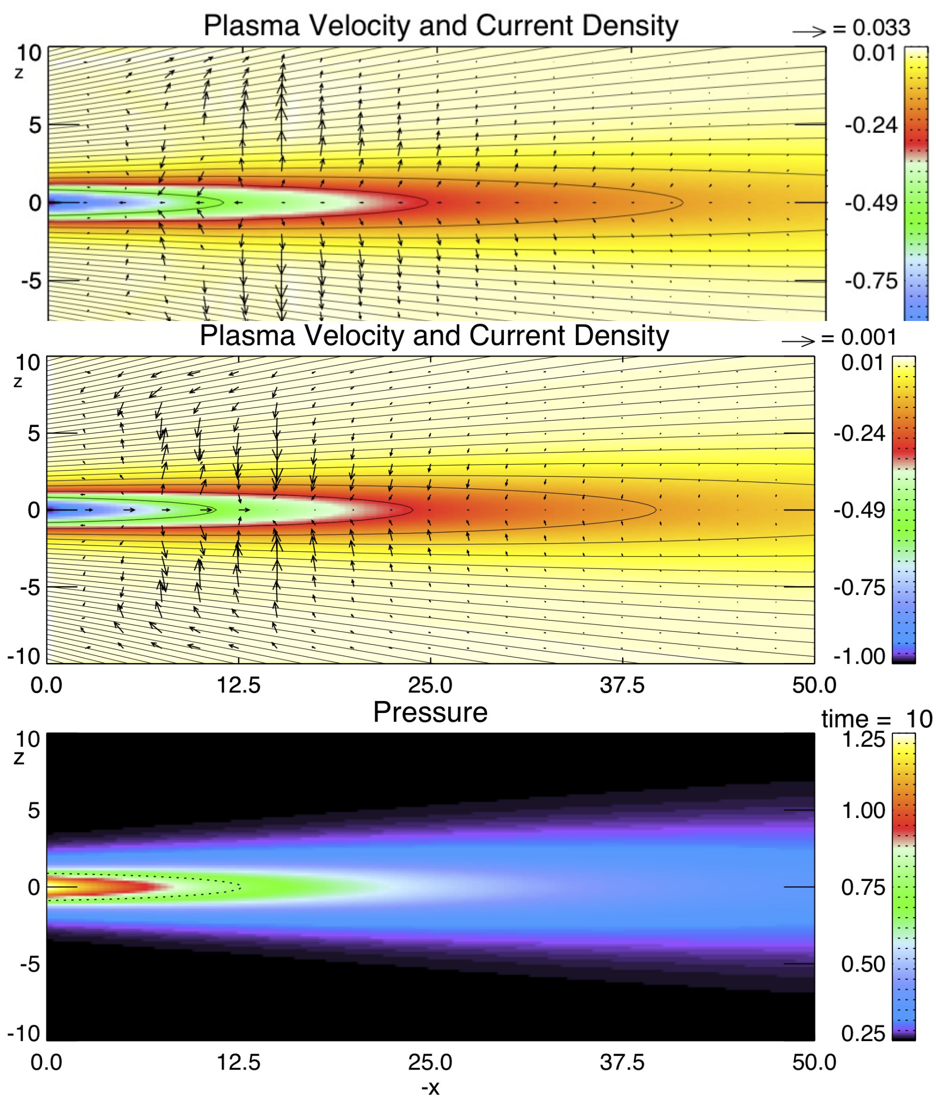


Figure 4.4: Asymptotic magnetotail equilibrium: Top - Magnetic field, current density, and velocity for a second order accurate asymtotic tail configuration; middle - the same plot for the 4th accurate configuration; and bottom - pressure for the 4th order configuration.

Figure 4.4 shows a solution of the asymtotic equilibrium for the pressure function

$$p_0(x) = a(\tilde{x}+1)^{-b} + c\tanh(d(\tilde{x}-1)) + e \qquad with \qquad \tilde{x} = x/x_{dist}$$

and the parameters $a = 1-p_{inf}+c(1+\tanh d)$; $b = 6$; $c = (1-p_{inf})/\left(2^{b+1}d/b - 1 - \tanh d\right)$; $d = 1$; and $e = p_{inf} - c$. Here the parameters are chosen that $p_0(0) = 1$; $p_0(\infty) = p_{inf}$; $\dot{p}_0(x_{dist}) = 0$ and they represent a reasonable approximation of a typical magnetotial pressure profile distribution. The first of these conditions provides the normalization and current sheet width of 1 at $x = 0$, the 2nd determines the pressure and sheet width at ∞ and the 3rd generates an X line at x_{dist} (the so-called distan X line). Here $x_{dist} = 100$ (R_E) and $p_{inf} = 0.1$. The figure demonstrates that the current density, magnetic field, and pressure decrease with increasing distance along the tail somewhat distinct from the Liouville solution. The plots in Figure 4.4 show 2nd and 4th order accurate solutions after 10 Alfven times into the simulation. Thers is no apparent difference in the magnetic field or current density distributions for the different accuracies indicating that 2nd order accuracy is not bad. However, acceleration by unbalanced forces is significantly higher in the 2nd order accurate configuration as documented by a maximum of about $3 \cdot 10^{-3}$ compared to a valu of 10^{-4} for the 4th order accurate configuration.

Solar Equilibrium Configurations

Solar magnetic fields are usually modelled force-free, i.e., the thermal pressure forcess are negected compared to the magnetic pressure. There are various equilibrium configurations that approximate different properties of the solar field. In a simple solution sequences of solar magnetic arcades have been modelled by Low (1977) with configurations like

$$A = \ln\left(1 + x^2 + y^2 - \sqrt{3}y\right)$$

where the force-free condition employs $B_z = \exp(-A)$. In this configuration the magnetic field lines are circles (in the x, y plane) with a center at $y = -\sqrt{3}/2$, i.e., below the photosphere. The B_z component implies that the field is sheared in the z direction. It is noted that this solution is obtained from a Liouville solution with $f(\zeta) = 2\zeta - \sqrt{3}i$.

4.5 Three-Dimensional Equilibria

In three dimensions we cannot make the simplification of using a single component of the vector potential. However there is a formulation which lends itself to a similar treatment.

Introducing so-called Euler potentials α and β: $\quad \mathbf{A} = \alpha\nabla\beta + \nabla\Xi \quad$ =>

$$\begin{aligned} \mathbf{B} = \nabla\times\mathbf{A} &= \nabla\times[\alpha\nabla\beta + \nabla\Xi] \\ &= \nabla\alpha\times\nabla\beta \end{aligned} \tag{4.18}$$

Note that Euler potentials imply $\mathbf{A} \cdot \mathbf{B} = 0$ which is not generally satisfied but it is always possible to find a gauge such that $\mathbf{B}$ is perpendicular to $\mathbf{A}$.

Using Euler potentials the magnetic field is perpendicular to $\nabla\alpha$ and $\nabla\beta$ or - in other words - magnetic field lines are the lines where isosurfaces of α and β intersect. The prior two-dimensional treatment is actually a special case of Euler coordinates with $\alpha = A_z$ and $\beta = z$.

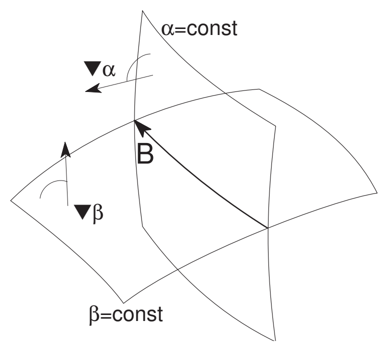


Figure 4.5: Sketch of the field interpretation of Euler potentials.

The current density is now given by

$\nabla \times (\mathbf{A} \times \mathbf{B}) = \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A}) + (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B}$

$$\begin{aligned}\mathbf{j} &= \frac{1}{\mu_0}\nabla \times [\nabla\alpha \times \nabla\beta] \\ &= \frac{1}{\mu_0}\left(\Delta\beta\nabla\alpha - \Delta\alpha\nabla\beta + \nabla\beta \cdot \nabla(\nabla\alpha) - \nabla\alpha \cdot \nabla(\nabla\beta)\right)\end{aligned}$$

Force balance: $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{A} \cdot \mathbf{B})\mathbf{C}$

$$\begin{aligned}0 &= -\nabla p + \mathbf{j} \times \mathbf{B} \\ &= -\nabla p + \frac{1}{\mu_0}[\nabla \times (\nabla\alpha \times \nabla\beta)] \times [\nabla\alpha \times \nabla\beta] \\ &= -\nabla p + \frac{1}{\mu_0}\{[\nabla\beta \cdot \nabla \times (\nabla\alpha \times \nabla\beta)]\nabla\alpha \\ &\qquad - [\nabla\alpha \cdot \nabla \times (\nabla\alpha \times \nabla\beta)]\nabla\beta\}\end{aligned}$$

This yields the basic dependencies of $p = p(\alpha\,\beta)$ and the equilibrium equations

$$\nabla\beta \cdot \nabla \times (\nabla\alpha \times \nabla\beta) = \mu_0 \frac{\partial p(\alpha, \beta)}{\partial\alpha} \tag{4.19}$$

$$\nabla\alpha \cdot \nabla \times (\nabla\alpha \times \nabla\beta) = -\mu_0 \frac{\partial p(\alpha, \beta)}{\partial\beta} \tag{4.20}$$

Note that in cases with a gravitational force the force balance equation is modified to

$$-\nabla p + \mathbf{j} \times \mathbf{B} - \rho \nabla \Phi = 0$$

where Φ is the gravitational potential. This is for instance important for solar applications of the equilibrium theory. In this case the pressure has to be also a function of Ψ and the equilibrium equations become

$$\begin{aligned}
\nabla \beta \cdot \nabla \times (\nabla \alpha \times \nabla \beta) &= \mu_0 \frac{\partial p\,(\alpha, \beta, \Phi)}{\partial \alpha} \\
\nabla \alpha \cdot \nabla \times (\nabla \alpha \times \nabla \beta) &= -\mu_0 \frac{\partial p\,(\alpha, \beta, \Phi)}{\partial \beta} \\
\rho &= -\frac{\partial p\,(\alpha, \beta, \Phi)}{\partial \Phi}
\end{aligned}$$

These equation are often referred to as the Grad Shafranov equations and they are commonly used to compute three-dimensional equilibrium configurations. While there are some analytic solutions these equations are mostly used with computational techniques. A numerical procedure to find solution usually starts with a straightforward initial solution. For instance, a vacuum magnetic field is always a solution to equations (4.19) and (4.20) for constant pressure (vacuum field refers to a magnetic field for which the current density is 0, i.e., there are no current carriers). Examples are constant magnetic fields, dipole or higher multipole fields or any magnetic field which is derived through a scalar potential $\mathbf{B} = -\nabla \Psi$. Note that Ψ has to satisfy $\Delta \Psi = 0$ otherwise $\nabla \cdot \mathbf{B} = 0$ is violated.

Magnetic dipole in spherical coordinates (r, θ, φ):

$$\Psi = -\frac{\mu_0 M_D}{4\pi} \frac{\cos\theta}{r^2} \tag{4.21}$$

Noting that

$$\nabla \Psi = \frac{\partial \Psi}{\partial r} \mathbf{e}_r + \frac{1}{r} \frac{\partial \Psi}{\partial \theta} e_\theta + \frac{1}{r \sin\theta} \frac{\partial \Psi}{\partial \varphi} \mathbf{e}_\varphi$$

the dipole magnetic field components become

$$\begin{aligned}
B_r &= -\frac{\mu_0 M_D}{2\pi} \frac{\cos\theta}{r^3} \\
B_\theta &= -\frac{\mu_0 M_D}{4\pi} \frac{\sin\theta}{r^3}
\end{aligned}$$

The formalism derived for spherical coordinates in two dimensions can then be used to derive Euler potentials for the start solution. In the course of the numerical solution of (4.19) and (4.20) the pressure is increased as a specified function of α and β. Also any changes in terms of boundary conditions etc. are applied in the iterative solution of the equations. Note that a nontrivial point of the system (4.19) and (4.20) is the existence of solution or possible multiple branches of solutions for the same boundary conditions.

Exercise: Determine the vector potential component A_ϕ for the dipole field and derive the equations for the magnetic field lines in spherical coordinates.

4.6 MHD Stability

The stability of plasmas both in laboratory and in the natural environment is of central importance to understand plasma systems. It is worth noting, however that the interest of laboratory plasma research is usually the stability of a configuration (for instance the stable confinement of plasma in a fusion device) while the interest in space plasma is clearly more in the instability of such systems. Either way plasma instability is a central issue in both communities.

A priory it is clear that a homogeneous plasma with all particle species moving a the same bulk velocity and all species having equal temperatures with Maxwell particle distributions is the ultimate stable system because it is in global thermal equilibrium. However, any deviation from this state has the potential to cause an instability. In this section we are interested in the stability of fluid plasma such that detail of the distribution function such as non-Maxwellian anisotropic distributions cannot be addressed here and will be left for later discussion.

In the case of fluid plasma the main driver for instability is spatial inhomogeneity, such as spatially vary magnetic field, current, and bulk velocity distributions. Most of the following discussion will assume that the bulk velocity is actually zero and the plasma is in an equilibrium state as discussed in the previous section.

There are several methods to address plasma stability/instability. First, modern numerical methods allow to carry out computer simulation if necessary on massive scales. Numerical studies have advantages and disadvantages. For instance, a computer simulation can study not only small perturbations but also nonlinear perturbations of the equilibrium and the basic formulation is rather straightforward. However, a simulation can test only one configuration, one set of boundary conditions, and one type of perturbation at a time. thus it may be cumbersome to obtain a good physical understanding of how stability properties change when system parameter change.

There are two basic analytic methods to study stability and instability of a plasma. The first methods uses a small perturbation and computes the characteristic evolution of the system. If all modes are have constant or damped amplitudes in time the plasma system is stable, if there are (exponentially) growing modes the system is unstable. This analysis is called the normal mode analysis. The second approach also uses small perturbations but considers a variational or energy principle. This is similar to a simple mechanical system where the potential has a local maximum or minimum. Thus this method attempts to formulate a potential for the plasma system which can be examined for stability.

Let us consider an MHD system state which is characterized by It will soon be clear that although the basic idea is the same, plasma systems are considerably more complex than simple mechanical examples.

4.6.1 Small oscillations near an equilibrium

Consider a configuration which satisfies the equilibrium force balance condition. For conditions $-\nabla p + \mathbf{j} \times \mathbf{B} = 0$. Equilibrium quantities are denoted by an index 0, i.e., p_0, $\mathbf{B}_0$, and $\mathbf{j}_0 = \frac{1}{\mu_0}\nabla \times \mathbf{B}_0$. All equilibrium plasma properties are a function of space only.

In the following we assume small perturbations from the equilibrium state. For these perturbations we use the index 1. The coordinate of a fluid element is $\mathbf{x}_0(t)$. Considering a small displacement of the plasma coordinate in the frame moving with the plasma (Lagrangian displacement) such that the new coordinate is

$$\mathbf{x}(t) = \mathbf{x}_0(t) + \boldsymbol{\xi}(\mathbf{x}_0, t)$$

which satisfies $\boldsymbol{\xi}(\mathbf{x}_0, 0) = 0$. A Taylor expansion of the fluid velocity yields

$$\mathbf{u}_1(\mathbf{x}, t) = \mathbf{u}_1(\mathbf{x}_0, t) + \left(\dot{\boldsymbol{\xi}} \cdot \nabla\right) \mathbf{u}_1(\mathbf{x}_0, t) + ...$$

with $\dot{\boldsymbol{\xi}} = \partial\boldsymbol{\xi}/\partial t$. However, since both $\boldsymbol{\xi}$ and $\mathbf{u}_1$ are perturbation quantities (and hence small) we can neglect the terms of the Taylor expansion except for the 0th order implying:

$$\mathbf{u}_1(\mathbf{x}, t) = \mathbf{u}_1(\mathbf{x}_0, t)$$

Here $\mathbf{u}_1(\mathbf{x}, t)$ represent the Eulerian velocity at the location $\mathbf{x}$ and $\mathbf{u}_1(\mathbf{x}_0, t)$ the Lagrangian description in the co-moving frame. Since $\mathbf{u}_1(\mathbf{x}_0, t) = \partial\boldsymbol{\xi}(\mathbf{x}_0, t)/\partial t$. Therefore we can replace replace the Eulerian velocity in the plasma equations by the Lagrangian by $\partial\boldsymbol{\xi}(\mathbf{x}_0, t)/\partial t$.

We now linearize the ideal MHD equations around the equilibrium state and substitute $\mathbf{u}_1$ with $\partial\boldsymbol{\xi}/\partial t$.

$$\begin{aligned}
\frac{\partial \rho_1}{\partial t} &= -\nabla \cdot \left(\rho_0 \dot{\boldsymbol{\xi}}\right) \\
\rho_0 \frac{\partial^2 \boldsymbol{\xi}}{\partial t^2} &= -\nabla p_1 + \frac{1}{\mu_0} (\nabla \times \mathbf{B}_1) \times \mathbf{B}_0 + \frac{1}{\mu_0} (\nabla \times \mathbf{B}_0) \times \mathbf{B}_1 \\
\frac{\partial \mathbf{B}_1}{\partial t} &= \nabla \times \left(\dot{\boldsymbol{\xi}} \times \mathbf{B}_0\right) \\
\frac{\partial p_1}{\partial t} &= -\dot{\boldsymbol{\xi}} \cdot \nabla p_0 - \gamma p_0 \nabla \cdot \dot{\boldsymbol{\xi}}
\end{aligned}$$

We can now integrate the continuity equation, the induction equation, and the pressure equation in time (assuming that the initial perturbation is zero) which yields

$$\begin{aligned}
\rho_1 &= -\nabla \cdot (\rho_0 \boldsymbol{\xi}) \\
\mathbf{B}_1 &= \nabla \times (\boldsymbol{\xi} \times \mathbf{B}_0) \equiv \mathbf{Q}_\xi \\
p_1 &= -\boldsymbol{\xi} \cdot \nabla p_0 - \gamma p_0 \nabla \cdot \boldsymbol{\xi}
\end{aligned}$$

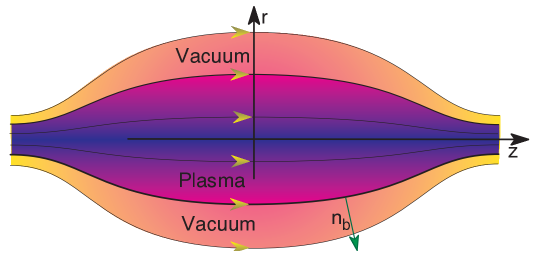
and substitute $\mathbf{u}_1$ with $\dot{\boldsymbol{\xi}}$. Take the time derivative of the momentum equation and replace the perturbed density, magnetic field, and pressure in the momentum equation.

$$\rho_0 \frac{\partial^2 \boldsymbol{\xi}}{\partial t^2} = \nabla (\boldsymbol{\xi} \cdot \nabla p_0 + \gamma p_0 \nabla \cdot \boldsymbol{\xi}) + \frac{1}{\mu_0} [(\nabla \times \mathbf{Q}_\xi) \times \mathbf{B}_0 + (\nabla \times \mathbf{B}_0) \times \mathbf{Q}_\xi]$$

To solve this equation it is necessary to provide initial conditions for $\boldsymbol{\xi}$ and $\dot{\boldsymbol{\xi}} = \mathbf{u}_1$, and boundary conditions for $\boldsymbol{\xi}$.

Boundary Conditions

Frequently used conditions for a laboratory system are that the plasma is confined to a region which is embedded in a vacuum which in turn is bounded by an ideal conducting wall. This is an assumption seldom realized but suitable for the mathematical treatment.



In the case of space plasma systems boundary conditions are even more difficult to formulate because they are not confined to a particular region. However, if the volume is taken sufficiently large surface contributions to the interior are small such that for instance an assumption that the perturbations are zero at the boundary is a suitable choice. In the case of solar prominences the field is anchored in the solar photosphere which is an almost ideal conductor which implies that the foot points of prominences are moving with the photospheric gas.

i) Conducting wall

If the boundary to the plasma system is an ideal conducting wall the boundary condition simplifies considerably. In this case it is necessary that the tangential electric field is zero because the wall has a constant potential:

$$\mathbf{n}_w \times \mathbf{E}_1 = 0$$

where $\mathbf{n}_w$ is the unit vector of the outward normal direction to the wall. With Ohm's law $\mathbf{E}_1 = \dot{\boldsymbol{\xi}} \times \mathbf{B}_0$ we obtain

$$\mathbf{n}_w \times \left(\dot{\boldsymbol{\xi}} \times \mathbf{B}_0\right) = \left(\mathbf{n}_w \cdot \mathbf{B}_0\right) \dot{\boldsymbol{\xi}} - \left(\mathbf{n}_w \cdot \dot{\boldsymbol{\xi}}\right) \mathbf{B}_0 = 0$$

If the magnetic field is tangential to the wall which is usually the case for fusion devices the first term vanishes and the boundary condition is

$$\mathbf{n}_w \cdot \dot{\boldsymbol{\xi}} = 0$$

which is satisfied if the perturbation is tangential to the boundary.

ii) Vacuum boundary:

Considering a very small section of the boundary one usually neglects curvature terms $\mathbf{B} \cdot \nabla \mathbf{B}$ in the force balance equation. In this case the local equilibrium is determined by total pressure balance at the plasma - vacuum boundary

$$p_0\left(\mathbf{x}_0\right) + \frac{B_0^2\left(\mathbf{x}_0\right)}{2\mu_0} = \frac{B_{v0}^2\left(\mathbf{x}_0\right)}{2\mu_0}$$

where $\mathbf{x}_0$ denotes a point on the unperturbed plasma - vacuum boundary and the index v denotes variables in the vacuum region. Note that this is an idealization because it implies a jump of the tangential magnetic field (which also implies a surface current). Note that the transition to vacuum implies that the boundary is a pressure boundary and from $\mathbf{B}_0 \cdot \nabla p_0 = 0$ we know that the magnetic field must be tangential to the boundary! If there were a magnetic field threading through the boundary it is obvious that the pressure is not constant on a field line and therefore that the equilibrium condition of $p_0 = \mathrm{const}$ on magnetic field lines is violated. A point on the perturbed boundary has the coordinate

$$\mathbf{x} = \mathbf{x}_0 + \mathbf{n}_b \xi_n$$

where $\mathbf{n}_b$ is the outward normal unit vector and ξ_n is the component of the displacement normal to the boundary. At the perturbed boundary the total pressure must also be continuous

$$p_0(\mathbf{x}) + p_1(\mathbf{x}) + \frac{(B_0(\mathbf{x}) + B_1(\mathbf{x}))^2}{2\mu_0} = \frac{(B_{v0}(\mathbf{x}) + B_{v1}(\mathbf{x}))^2}{2\mu_0}$$

Now we have to express this condition in terms of $\boldsymbol{\xi}$ which can be done by expanding equilibrium quantities in a Taylor expansion around $\mathbf{x}_0$ such as

$$\begin{aligned}
p_0(\mathbf{x}) &= p_0(\mathbf{x}_0) + \xi_n \mathbf{n}_b \cdot \nabla p_0(\mathbf{x}_0) + \ldots \\
p_1(\mathbf{x}) &= -\boldsymbol{\xi} \cdot \nabla p_0 - \gamma p_0 \nabla \cdot \boldsymbol{\xi} \\
B_0^2(\mathbf{x}) &= B_0^2(\mathbf{x}_0) + \xi_n \mathbf{n}_b \cdot \nabla B_0^2(\mathbf{x}_0) + \ldots \\
\mathbf{B}_0(\mathbf{x}) \cdot \mathbf{B}_1(\mathbf{x}) &= \mathbf{B}_0(\mathbf{x}_0) \cdot \mathbf{B}_1(\mathbf{x}_0) + \ldots
\end{aligned}$$

Substitution into the pressure balance equation yields

$$-\gamma p_0 \nabla \cdot \boldsymbol{\xi} + \frac{\mathbf{B}_0(\mathbf{x}_0) \cdot \mathbf{B}_1(\mathbf{x}_0)}{\mu_0} + \frac{\xi_n}{2\mu_0} \frac{\partial B_0^2(\mathbf{x}_0)}{\partial n} = \frac{\mathbf{B}_{v0}(\mathbf{x}_0) \cdot \mathbf{B}_{v1}(\mathbf{x}_0)}{\mu_0} + \frac{\xi_n}{2\mu_0} \frac{\partial B_{v0}^2(\mathbf{x}_0)}{\partial n}$$

where we have made use of $\xi_n \mathbf{n}_b$ being along ∇p_0.

A second boundary condition can be obtained from the fact that the electric field in a frame moving with the plasma velocity is zero $\mathbf{E}_1 + \mathbf{u}_1 \times \mathbf{B}_0 = 0$ and the tangential component must be continuous into the vacuum region, i.e.

$$\mathbf{n}_b \times (\mathbf{E}_{v1} + \mathbf{u}_1 \times \mathbf{B}_{v0}) = 0$$

This condition can be expressed as

$$\mathbf{n}_b \times \mathbf{E}_{v1} = (\mathbf{n}_b \cdot \mathbf{u}_1) \mathbf{B}_{v0} = u_n \mathbf{B}_{v0}$$

Introducing the vector potential for the perturbation in the vacuum region $\mathbf{A}$ the electric and magnetic fields are

$$\mathbf{E}_{v1} = -\frac{\partial \mathbf{A}}{\partial t} \qquad \mathbf{B}_1 = \nabla \times \mathbf{A}$$

which yields

$$\mathbf{n}_b \times \mathbf{A} = -\xi_n \mathbf{B}_{v0}$$

on the conducting wall the boundary condition for $\mathbf{A}$ is

$$\mathbf{n}_w \times \mathbf{A} = 0$$

i.e., the vector potential has only a component along the normal direction of the wall

4.6.2 Energy principle

We have derived an partial differential equation of the form

$$\rho_0 \frac{\partial^2 \boldsymbol{\xi}}{\partial t^2} = \mathbf{F}(\boldsymbol{\xi}) = -\underline{\underline{\mathbf{K}}} \cdot \boldsymbol{\xi}$$

where $\underline{\underline{\mathbf{K}}}$ is the differential operator

$$\begin{aligned} \underline{\underline{\mathbf{K}}} \cdot \boldsymbol{\xi} &= -\nabla\left(\boldsymbol{\xi} \cdot \nabla p_0 + \gamma p_0 \nabla \cdot \boldsymbol{\xi}\right) - \frac{1}{\mu_0}\left[\left(\nabla \times \mathbf{Q}_\xi\right) \times \mathbf{B}_0 + \left(\nabla \times \mathbf{B}_0\right) \times \mathbf{Q}_\xi\right] \\ \text{with} \quad & \mathbf{Q}_\xi = \nabla \times \left(\boldsymbol{\xi} \times \mathbf{B}_0\right) \end{aligned}$$

Assuming

$$\boldsymbol{\xi}(\mathbf{x}, t) = \boldsymbol{\xi}(\mathbf{x}) \exp(i\omega t)$$

the PDE becomes an Eigenvalue equation for ω^2:

$$\omega^2 \rho_0 \boldsymbol{\xi} = \underline{\underline{\mathbf{K}}} \cdot \boldsymbol{\xi}$$

and stability depends on the sign of ω^2. In the MHD case the operator $\underline{\underline{\mathbf{K}}}$ is self-adjoint, i.e.,

$$\int_V \boldsymbol{\eta} \cdot \underline{\underline{\mathbf{K}}} \cdot \boldsymbol{\xi} d\mathbf{x} = \int_V \boldsymbol{\xi} \cdot \underline{\underline{\mathbf{K}}} \cdot \boldsymbol{\eta} d\mathbf{x}$$

where the integrals are carried out over the plasma volume. The eigenvalues of a self-adjoint operator are always real that means either positive or negative. Explicitly the Eigenvalues are given by

$$\omega^2 = \frac{\int_V \boldsymbol{\xi} \cdot \underline{\underline{\mathbf{K}}} \cdot \boldsymbol{\xi} d\mathbf{x}}{\int_V \rho_0 \boldsymbol{\xi} \cdot \boldsymbol{\xi} d\mathbf{x}}$$

For $\omega^2 > 0$ the values for ω are real and the solution is oscillating but not growing in time, i.e., the solution is stable. However if there are negative eigenvalues $\omega^2 < 0$ there are solutions which are growing exponentially in time and are therefore unstable!

In the following it is demonstrated that $\underline{\underline{\mathbf{K}}}$ is self-adjoint:

$$
\begin{aligned}
U_{\xi\eta} = \int_V \boldsymbol{\eta}\cdot\underline{\underline{\mathbf{K}}}\cdot\boldsymbol{\xi}d\mathbf{x} &= -\int_V \boldsymbol{\eta}\cdot\nabla\left(\boldsymbol{\xi}\cdot\nabla p_0+\gamma p_0\nabla\cdot\boldsymbol{\xi}\right)d\mathbf{x} \\
&\quad -\frac{1}{\mu_0}\int_V \boldsymbol{\eta}\cdot\left[\left(\nabla\times\mathbf{Q}_\xi\right)\times\mathbf{B}_0+\left(\nabla\times\mathbf{B}_0\right)\times\mathbf{Q}_\xi\right]d\mathbf{x}
\end{aligned}
$$

Identities to be used:

$$
\begin{aligned}
\nabla\cdot(\boldsymbol{\eta}\phi) &= \boldsymbol{\eta}\cdot\nabla\phi+\phi\nabla\cdot\boldsymbol{\eta} \\
\nabla\cdot[\mathbf{A}\times\mathbf{B}] &= \mathbf{B}\cdot(\nabla\times\mathbf{A})-\mathbf{A}\cdot(\nabla\times\mathbf{B}) \\
\nabla\cdot[(\boldsymbol{\eta}\times\mathbf{B}_0)\times\mathbf{Q}_\xi] &= \mathbf{Q}_\xi\cdot(\nabla\times(\boldsymbol{\eta}\times\mathbf{B}_0))-(\boldsymbol{\eta}\times\mathbf{B}_0)\cdot(\nabla\times\mathbf{Q}_\xi) \\
&= \mathbf{Q}_\xi\cdot\mathbf{Q}_\eta+\boldsymbol{\eta}\cdot[(\nabla\times\mathbf{Q}_\xi)\times\mathbf{B}_0] \\
\text{with} \quad & \mathbf{Q}_\eta=\nabla\times(\boldsymbol{\eta}\times\mathbf{B}_0)
\end{aligned}
$$

which yields

$$
\begin{aligned}
U_{\xi\eta} &= \int_V \left\{(\nabla\cdot\boldsymbol{\eta})\left(\boldsymbol{\xi}\cdot\nabla p_0+\gamma p_0\nabla\cdot\boldsymbol{\xi}\right)\right. \\
&\quad \left.+\frac{1}{\mu_0}\mathbf{Q}_\xi\cdot\mathbf{Q}_\eta d\mathbf{x}-\frac{1}{\mu_0}\boldsymbol{\eta}\cdot\left[\left(\nabla\times\mathbf{B}_0\right)\times\mathbf{Q}_\xi\right]\right\}d\mathbf{x} \\
&\quad -\int_V \nabla\cdot\left[\boldsymbol{\eta}\left(\boldsymbol{\xi}\cdot\nabla p_0+\gamma p_0\nabla\cdot\boldsymbol{\xi}\right)-\frac{1}{\mu_0}\left(\boldsymbol{\eta}\times\mathbf{B}_0\right)\times\mathbf{Q}_\xi\right]d\mathbf{x} \\
&= \int_V \left\{\gamma p_0\nabla\cdot\boldsymbol{\eta}\nabla\cdot\boldsymbol{\xi}+\frac{1}{\mu_0}\mathbf{Q}_\xi\cdot\mathbf{Q}_\eta\right. \\
&\quad \left.+\boldsymbol{\xi}\cdot\nabla p_0\nabla\cdot\boldsymbol{\eta}-\frac{1}{\mu_0}\boldsymbol{\eta}\cdot\left[\left(\nabla\times\mathbf{B}_0\right)\times\mathbf{Q}_\xi\right]\right\}d\mathbf{x} \\
&\quad -\int_{S_V}\left(\boldsymbol{\xi}\cdot\nabla p_0+\gamma p_0\nabla\cdot\boldsymbol{\xi}-\frac{1}{\mu_0}\mathbf{B}_0\cdot\mathbf{Q}_\xi\right)\eta_n ds
\end{aligned}
$$

where we have used Gauss theorem for the surface integrals with $B_n=0$ on the plasma boundary. With the boundary conditions derived in the prior section

$$
\begin{aligned}
-\gamma p_0\nabla\cdot\boldsymbol{\xi}+\frac{\mathbf{B}_0\left(\mathbf{x}_0\right)\cdot\mathbf{B}_1\left(\mathbf{x}_0\right)}{\mu_0}+\xi_n\frac{\partial B_0^2\left(\mathbf{x}_0\right)}{\partial n} &= \frac{\mathbf{B}_{v0}\left(\mathbf{x}_0\right)\cdot\mathbf{B}_{v1}\left(\mathbf{x}_0\right)}{\mu_0}+\xi_n\frac{\partial B_{v0}^2\left(\mathbf{x}_0\right)}{\partial n} \\
\mathbf{B}_0 &= \frac{1}{\xi_n}\mathbf{n}_b\times\mathbf{A}_\xi \\
\mathbf{Q}_\xi=\mathbf{B}_1 &= \nabla\times\mathbf{A}_\xi
\end{aligned}
$$

For the surface terms we can use the boundary conditions

$$
\begin{aligned}
U_{\mathrm{S},\xi\eta} &= -\int_{S_V}\left(\boldsymbol{\xi}\cdot\nabla p_0+\gamma p_0\nabla\cdot\boldsymbol{\xi}-\frac{1}{\mu_0}\mathbf{B}_0\cdot\mathbf{Q}_\xi\right)\eta_n ds\\
&= -\int_{S_V}\left(\boldsymbol{\xi}\cdot\nabla p_0+\frac{\mathbf{B}_0\cdot\mathbf{Q}_\xi}{\mu_0}-\frac{\mathbf{B}_{v0}\cdot\mathbf{Q}_{v\xi}}{\mu_0}+\frac{\xi_n}{2\mu_0}\frac{\partial\left(B_0^2\left(\mathbf{x}_0\right)-B_{v0}^2\left(\mathbf{x}_0\right)\right)}{\partial n}\right)\eta_n ds\\
&\quad -\int_{S_V}\left(-\frac{1}{\mu_0}\mathbf{B}_0\cdot\mathbf{Q}_\xi\right)\boldsymbol{\eta}_n ds\\
&= -\int_{S_V}\left(\xi_n\frac{\partial p_0}{\partial n}-\frac{\mathbf{B}_{v0}\cdot\mathbf{Q}_{v\xi}}{\mu_0}+\frac{\xi_n}{2\mu_0}\frac{\partial\left(B_0^2\left(\mathbf{x}_0\right)-B_{v0}^2\left(\mathbf{x}_0\right)\right)}{\partial n}\right)\eta_n ds
\end{aligned}
$$

and the field terms in the vacuum region can be treated as follows

$$
\begin{aligned}
\mu_0 U_{\mathbf{B},\xi\eta} &= \int_{S_V}\mathbf{B}_{v0}\cdot\mathbf{Q}_\xi\eta_n ds=\int_{S_V}\left(\mathbf{n}_b\times\mathbf{A}_\eta\right)\cdot\left(\nabla\times\mathbf{A}_\xi\right)ds\\
&= \int_{S_V}\left(\mathbf{A}_\eta\times\nabla\times\mathbf{A}_\xi\right)d\mathbf{s}=\int_{V_{vacuum}}\nabla\cdot\left(\mathbf{A}_\eta\times\nabla\times\mathbf{A}_\xi\right)d\mathbf{r}\\
&= \int_{V_{vacuum}}\left(\nabla\times\mathbf{A}_\eta\right)\cdot\left(\nabla\times\mathbf{A}_\xi\right)d\mathbf{r}
\end{aligned}
$$

such that the sum of the surface terms can be written as

$$
U_{\mathrm{S},\xi\eta}=-\int_{S_V}\left(\frac{\partial p_0}{\partial n}+\frac{1}{2\mu_0}\frac{\partial\left(B_0^2\left(\mathbf{x}_0\right)-B_{v0}^2\left(\mathbf{x}_0\right)\right)}{\partial n}\right)\xi_n\eta_n ds+\frac{1}{\mu_0}\int_{V_{vacuum}}\left(\nabla\times\mathbf{A}_\eta\right)\cdot\left(\nabla\times\mathbf{A}_\xi\right)d\mathbf{r}
$$

Here it is clear that the surface contributions are symmetric in ξ and η.

Finally we have to demonstrate the symmetry of the remaining non-symmetric terms

$$
\widetilde{U}_{\xi\eta} = \int\boldsymbol{\xi}\cdot\nabla p_0\nabla\cdot\boldsymbol{\eta}-\frac{1}{\mu_0}\boldsymbol{\eta}\cdot\left[\left(\nabla\times\mathbf{B}_0\right)\times\mathbf{Q}_\xi\right]d\mathbf{x}
$$

Consider decomposition of

$$
\begin{aligned}
\boldsymbol{\xi}&=\boldsymbol{\xi}_\parallel+\boldsymbol{\xi}_\perp \quad \text{with} \quad \boldsymbol{\xi}_\parallel=\alpha\mathbf{B}_0\\
\boldsymbol{\eta}&=\boldsymbol{\eta}_\parallel+\boldsymbol{\eta}_\perp \quad \text{with} \quad \boldsymbol{\eta}_\parallel=\beta\mathbf{B}_0
\end{aligned}
$$

Considering that $\boldsymbol{\xi}_\parallel\cdot\nabla p_0=\alpha\mathbf{B}_0\cdot\nabla p_0=0$ and $\mathbf{Q}_{\xi_\parallel}=\nabla\times\left(\boldsymbol{\xi}_\parallel\times\mathbf{B}_0\right)=0$ the non-symmetric parts of the integral reduce to

$$
\widetilde{U}_{\xi\eta}=\int\boldsymbol{\xi}_\perp\cdot\nabla p_0\nabla\cdot\boldsymbol{\eta}-\frac{1}{\mu_0}\boldsymbol{\eta}\cdot\left[\left(\nabla\times\mathbf{B}_0\right)\times\mathbf{Q}_{\xi_\perp}\right]d\mathbf{x}
$$

The contributions from $\boldsymbol{\eta}_\parallel=\beta\mathbf{B}_0$ can also be symmetries. The algebra is somewhat more tedious and we note that the integrand for the $\boldsymbol{\eta}_\parallel$ may be reduced to a form $\nabla\cdot\left(\boldsymbol{\xi}\cdot\nabla p_0\boldsymbol{\eta}\right)$. This give a contribution of $\boldsymbol{\eta}_{\|,boundary} \propto B_{0n} = 0$ but for typical boundary conditions it is assumed that the normal magnetic field is 0 (i.e. the magnetic field is aligned with the boundary).

To show the symmetry of the perpendicular components it is necessary to decompose the perpendicular displacement into components along the equilibrium current and along the equilibrium pressure gradient:

$$\begin{aligned} \boldsymbol{\xi}_\perp &= \xi_1 \mu_0 \mathbf{j}_0 + \xi_2 \mathbf{e} \\ \boldsymbol{\eta}_\perp &= \eta_1 \mu_0 \mathbf{j}_0 + \eta_2 \mathbf{e} \\ \text{with} \qquad & \mathbf{e} = \frac{\nabla p_0}{|\nabla p_0|} \end{aligned}$$

With these definitions consider the term

$$\begin{aligned} -\boldsymbol{\eta}_\perp \cdot [\mathbf{j}_0 \times \mathbf{Q}_{\xi_\perp}] &= -\eta_2 \mathbf{e} \cdot [\mathbf{j}_0 \times (\nabla \times ((\xi_1 \mu_0 \mathbf{j}_0 + \xi_2 \mathbf{e}) \times \mathbf{B}_0))] \\ &= -\eta_2 \mathbf{e} \cdot [\mathbf{j}_0 \times (\nabla \times (\xi_1 \mu_0 \nabla p_0 + \xi_2 \mathbf{e} \times \mathbf{B}_0))] \\ &= -\eta_2 \mathbf{e} \cdot [\mathbf{j}_0 \times (\mu_0 \nabla \xi_1 \times \nabla p_0 + \xi_1 \mu_0 \nabla \times \nabla p_0)] \\ &\quad -\eta_2 \mathbf{e} \cdot [\mathbf{j}_0 \times (\nabla \xi_2 \times (\mathbf{e} \times \mathbf{B}_0) + \xi_2 \nabla \times (\mathbf{e} \times \mathbf{B}_0))] \\ &= -\mu_0 \eta_2 \mathbf{e} \cdot [(\mathbf{j}_0 \cdot \nabla p_0) \nabla \xi_1 - (\mathbf{j}_0 \cdot \nabla \xi_1) \nabla p_0] \\ &\quad -\eta_2 \mathbf{e} \cdot [\mathbf{j}_0 \cdot (\mathbf{e} \times \mathbf{B}_0) \nabla \xi_2 - (\mathbf{j}_0 \cdot \nabla \xi_2) (\mathbf{e} \times \mathbf{B}_0) + \xi_2 \mathbf{j}_0 \times \nabla \times (\mathbf{e} \times \mathbf{B}_0)] \\ &= \mu_0 \eta_2 (\mathbf{e} \cdot \nabla p_0) (\mathbf{j}_0 \cdot \nabla \xi_1) \\ &\quad \eta_2 (\mathbf{e} \cdot \nabla p_0) (\mathbf{e} \cdot \nabla \xi_2) - \eta_2 \xi_2 \mathbf{e} \cdot \mathbf{j}_0 \times \nabla \times (\mathbf{e} \times \mathbf{B}_0) \\ &= \boldsymbol{\eta}_\perp \cdot \nabla p_0 \nabla \cdot \boldsymbol{\xi}_\perp - \eta_2 \xi_2 \mathbf{e} \cdot \mathbf{j}_0 \times \nabla \times (\mathbf{e} \times \mathbf{B}_0) \end{aligned}$$

Thus the sum of the non-symmetric integrands becomes

$$\begin{aligned} \widetilde{u}_{\xi\eta} &= \boldsymbol{\xi} \cdot \nabla p_0 \nabla \cdot \boldsymbol{\eta} - \frac{1}{\mu_0} \boldsymbol{\eta} \cdot [(\nabla \times \mathbf{B}_0) \times \mathbf{Q}_\xi] \\ &= \boldsymbol{\xi} \cdot \nabla p_0 \nabla \cdot \boldsymbol{\eta} + \boldsymbol{\eta} \cdot \nabla p_0 \nabla \cdot \boldsymbol{\xi} - \eta_2 \xi_2 \mathbf{e} \cdot \mathbf{j}_0 \times \nabla \times (\mathbf{e} \times \mathbf{B}_0) \end{aligned}$$

Obviously this form is symmetric in $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ and therefore the operator $\underline{\underline{\mathbf{K}}}$ is self-adjoint implying that all Eigenvalues are real.

With this property we can now formulate a Lagrangian function with the kinetic energy

$$T_2 = \frac{1}{2} \int_V d\mathbf{r} \left(\frac{\partial \boldsymbol{\xi}}{\partial t} \right)^2$$

and the generalized potential energy

$$\begin{aligned} U_2 &= \frac{1}{2} \int_V \left[\gamma p_0 (\nabla \cdot \boldsymbol{\xi})^2 + \frac{1}{\mu_0} (\nabla \times (\boldsymbol{\xi} \times \mathbf{B}_0))^2 \right. \\ &\qquad \left. + \boldsymbol{\xi} \cdot \nabla p_0 \nabla \cdot \boldsymbol{\xi} - \frac{1}{\mu_0} (\boldsymbol{\xi} \times (\nabla \times \mathbf{B}_0)) \cdot \nabla \times (\boldsymbol{\xi} \times \mathbf{B}_0) \right] d\mathbf{x} \\ &\quad - \int_{S_V} \left(\frac{\partial p_0}{\partial n} + \frac{1}{2\mu_0} \frac{\partial \left(B_0^2 (\mathbf{x}_0) - B_{v0}^2 (\mathbf{x}_0) \right)}{\partial n} \right) \xi_n^2 ds + \frac{1}{\mu_0} \int_{V_{vacuum}} (\nabla \times \mathbf{A})^2 d\mathbf{r} \end{aligned} \qquad (4.22)$$

Here the operator $\underline{\underline{\mathbf{K}}}$ (and integration of volume play the role of a potential and the potential energy is the above expression for the small displacement ξ. This is analogous to the formulation in classical mechanics. Considering a kinetic energy T and a potential U one obtains the Lagrangian $L(q, \dot{q}, t) = T - U$ for the generalized coordinate q and velocities $\dot{q}$. The Lagrange equations of motion are

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0$$

An equilibrium point i.e., a point where the acceleration is zero is given by

$$\dot{P} = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = 0$$

which implies

$$\frac{\partial L}{\partial q} = -\left.\frac{\partial U}{\partial q}\right|_{q=q_0} = 0$$

We now define the coordinates as $\xi = q - q_0$ and assume the ξ to be small displacements from the equilibrium point. The potential can be expanded in terms of the displacement at the equilibrium point i.e.

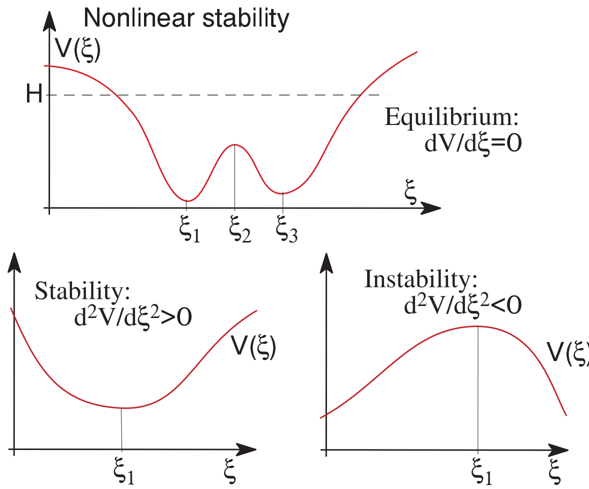
$$U\left(q_0 + \xi\right) = U\left(q_0\right) + \frac{\partial U\left(q_0\right)}{\partial q}\xi + \frac{1}{2}\frac{\partial^2 U\left(q_0\right)}{\partial q^2}\xi^2 + ..$$

Substituting ξ and in the Lagrangian with the kinetic energy $\frac{m}{2}\dot{\xi}^2$ yields the equation of motion

$$m\ddot{\xi} + \frac{\partial^2 U\left(q_0\right)}{\partial q^2}\xi = 0$$

with the solutions

$$\begin{aligned}
\xi = \exp\left(\pm i\omega t\right) \quad \text{with} \quad \omega^2 = \frac{1}{m}\frac{\partial^2 U\left(q_0\right)}{\partial q^2} \quad for \quad \frac{\partial^2 U\left(q_0\right)}{\partial q^2} > 0 \\
\xi = \exp\left(\pm \omega t\right) \quad \text{with} \quad \omega^2 = -\frac{1}{m}\frac{\partial^2 U\left(q_0\right)}{\partial q^2} \quad for \quad \frac{\partial^2 U\left(q_0\right)}{\partial q^2} < 0
\end{aligned}$$



Thus the solution is oscillatory and therefore stable if the potential has a local minimum and the solution is exponentially growing if the potential has a local maximum.

The energy principle for the MHD equations has to be interpreted in a similar way. Returning to our start point

$$\rho_0 \frac{\partial^2 \boldsymbol{\xi}}{\partial t^2} = \mathbf{F}(\boldsymbol{\xi}) = -\underline{\underline{\mathbf{K}}} \cdot \boldsymbol{\xi}$$

multiplication with $\dot{\boldsymbol{\xi}}$ and integrating over volume it follows that

$$\frac{d}{dt} \int_D \frac{1}{2} \rho_0 \dot{\boldsymbol{\xi}}^2 d^3 r = \int_D \dot{\boldsymbol{\xi}} \cdot \mathbf{F}(\boldsymbol{\xi})\, d^3 r$$

Using the fact that $\mathbf{F}$ is selfadjoint: $\int_D \boldsymbol{\xi}_1 \cdot \mathbf{F}(\boldsymbol{\xi}_2)\, d^3 r = \int_D \boldsymbol{\xi}_2 \cdot \mathbf{F}(\boldsymbol{\xi}_1)\, d^3 r$ we can re-write the time derivative as

$$\begin{aligned} \frac{d}{dt} \int_D \frac{1}{2} \rho_0 \dot{\boldsymbol{\xi}}^2 d^3 r &= \frac{1}{2} \int_D \dot{\boldsymbol{\xi}} \cdot \mathbf{F}(\boldsymbol{\xi})\, d^3 r + \frac{1}{2} \int_D \boldsymbol{\xi} \cdot \mathbf{F}\left(\dot{\boldsymbol{\xi}}\right) d^3 r \\ &= \frac{1}{2} \frac{d}{dt} \int_D \boldsymbol{\xi} \cdot \mathbf{F}(\boldsymbol{\xi})\, d^3 r \end{aligned}$$

with

$$\begin{aligned} T_2 &= \int_D \frac{1}{2} \rho_0 \dot{\boldsymbol{\xi}}^2 d^3 r \\ U_2 &= -\frac{1}{2} \frac{d}{dt} \int_D \boldsymbol{\xi} \cdot \mathbf{F}(\boldsymbol{\xi})\, d^3 r = \frac{1}{2} \frac{d}{dt} \int_D \boldsymbol{\xi} \cdot \underline{\underline{\mathbf{K}}} \cdot \boldsymbol{\xi} d^3 r \end{aligned}$$

We have derved total energy conservation to 2nd order accuracy:

$$W_2 = T_2 + U_2 = constant$$

An equilibrium is stable if the potential energy V_2 is positive for all small perturbations ξ and it is unstable if there are perturbations for which the potential energy becomes negative. Specifically this implies that T_2 is bounded in the case of stability.

If complex displacements are allowed the potential energy should be written as

$$\begin{aligned} U_2 \;=\; & \frac{1}{2}\int_V \left[\gamma p_0 \left|\nabla\cdot\boldsymbol{\xi}\right|^2 + \frac{1}{\mu_0}\left|\mathbf{Q}\right|^2 + \mathbf{j}_0\cdot\left(\boldsymbol{\xi}^*\times\mathbf{Q}_\xi\right)\right. \\ & \left. + \left(\boldsymbol{\xi}\cdot\nabla p_0\right)\nabla\cdot\boldsymbol{\xi}^* - \left(\boldsymbol{\xi}^*\cdot\nabla\psi_0\right)\nabla\cdot\left(\rho_0\boldsymbol{\xi}\right)\right] d\mathbf{x} \end{aligned} \tag{4.23}$$

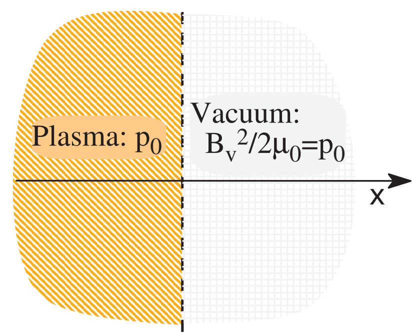
where we have ignored the surface terms and included a gravitational potential for generalization.

An equilibrium is stable if the potential energy is positive for all small perturbations ξ and it is unstable if there are perturbations for which the potential energy can become negative (Note that the equilibrium value of the potential is 0.

4.6.3 Applications of the energy principle

Stability of a plane plasma magnetic field interface

Consider the plane boundary at $x = 0$ between a homogeneous plasma (for $x < 0$) with constant pressure and density and a vacuum region at $x > 0$. Inside the plasma region the magnetic field is assumed to be 0. The plasma pressure is balanced by the magnetic field in the vacuum region. In this case the potential energy becomes



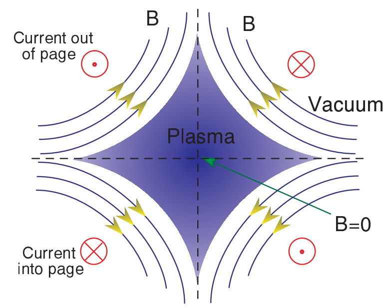
$$\begin{aligned} U \;=\; & \frac{1}{2}\int_V \gamma p_0 \left(\nabla\cdot\boldsymbol{\xi}\right)^2 d\mathbf{r} + \frac{1}{2\mu_0}\int_{V_{vacuum}} \mathbf{B}_{v1}^2 d\mathbf{r} \\ & + \frac{1}{2\mu_0}\int_{x=0} \frac{\partial B_{v0}^2\left(\mathbf{x}_0\right)}{\partial n}\xi_n^2 ds \end{aligned}$$

Properties:

- the $\gamma p_0 \left(\nabla\cdot\boldsymbol{\xi}\right)^2$ term requires compressible perturbations
- $\gamma p_0 \left(\nabla\cdot\boldsymbol{\xi}\right)^2 > 0$ always => stabilizing contribution for compressible perturbations!
- incompressible perturbations have a lower stability threshold
- the $\mathbf{B}_{v1}^2$ term is due to the perturbation of the vacuum field

- $\mathbf{B}_{v1}^2 > 0$ => always stabilizing
- $\partial B_{v0}^2(\mathbf{x}_0)/\partial n < 0$ can cause instability.
- all simple boundaries with $\partial B_{v0}^2(\mathbf{x}_0)/\partial n > 0$ are stable.

Magnetic cusp: This configuration is generated by two coils one in the upper half and one in the lower half of the system and the configuration is azimuthally symmetric with respect to the the vertical axis. Locally the plasma magnetic interface can be treated as a plane surface. On the larger system scale the magnetic field is curved into the plasma (as shown) such that $\partial B_{v0}^2(\mathbf{x}_0)/\partial n > 0$. The cause for this curvature is that the magnetic field increases closer to the coils. Therefore this configuration is always stable.



A similar result is obtained for the interchange instability. This is an instability where a flux tube from the plasma region is bulging out into the vacuum region and vice versa vacuum flux tube enters the plasma region. This instability also requires a magnetic field line curvature opposite to that shown for the cusp configuration.

The plasma can be unstable if $\partial B_{v0}^2(\mathbf{x}_0)/\partial n < 0$. Consider the same simple plane plasma vacuum boundary as before with $\mathbf{n}_b = \mathbf{e}_x$, and $\mathbf{B}_{v0} = B_{v0}\mathbf{e}_z$.

Consider a test function for the displacement and the vacuum perturbation field of the form

$$\begin{aligned}
\boldsymbol{\xi} &= \widetilde{\boldsymbol{\xi}} \exp i\left(k_x x + k_y y + k_z z\right) \\
\mathbf{B} &= \widetilde{\mathbf{b}} \exp i\left(k_x x + k_y y + k_z z\right)
\end{aligned}$$

where we dropped the indices v because we discuss exclusively the perturbed field in the vacuum region. The normal component of the perturbation field is given by

$$B_{1x} = \nabla \times (\boldsymbol{\xi} \times \mathbf{B}_0)|_x = i k_z \xi_x B_{v0}$$

The perturbed field in the vacuum region satisfies

$$\begin{aligned}
\nabla \times \nabla \times \mathbf{B}_1 &= \mathbf{k} \times \mathbf{k} \times \mathbf{B}_1 = 0 \\
\nabla \cdot \mathbf{B}_1 &= \mathbf{k} \cdot \mathbf{B}_1 = 0
\end{aligned}$$

which yields

$$\mathbf{k}^2 = 0$$

or $k_x = i\sqrt{k_y^2 + k_z^2}$. Since the current density in the vacuum region is zero we can use

$$j_{1x} = \frac{i}{\mu_0}(k_y B_{1z} - k_z B_{1y}) = 0$$

or $B_{1z} = (k_z/k_y)\, B_{1y}$ in $\nabla \cdot \mathbf{B}_1 = 0$ to obtain the y and z components of the perturbed field

$$\begin{aligned} B_{1y} &= -\frac{k_x k_y}{k_y^2 + k_z^2} B_{1x} \\ B_{1z} &= -\frac{k_x k_y}{k_y^2 + k_z^2} B_{1x} \end{aligned}$$

Thus we obtain

$$\begin{aligned} \mathbf{B}_1^2 &= \left(|B_{1x}|^2 + |B_{1y}|^2 + |B_{1z}|^2\right) \\ &= \left(1 + \frac{|k_x^2|}{k_y^2 + k_z^2}\right)|B_{1x}|^2 = 2\,|B_{1x}|^2 = 2\widetilde{b}_x^2 \exp\left(-2\sqrt{k_y^2 + k_z^2}x\right) \end{aligned}$$

Since the Magnetic field is exponentially decreasing the perturbation ξ_x has to take the also form

$$\xi_x = \widetilde{\xi}_x \exp\left(-\sqrt{k_y^2 + k_z^2}x\right) \exp i\,(k_y y + k_z z)$$

such that

$$\mathbf{B}_1^2 = 2k_z^2 \widetilde{\xi}_x^2 B_{v0}^2 \exp\left(-2\sqrt{k_y^2 + k_z^2}x\right)$$

and integration over the entire vacuum region from $x = 0$ to ∞ yields

$$\int_{V_{vacuum}} \mathbf{B}_{v1}^2 d\mathbf{r} = \int_{x=0} \frac{k_z^2 \widetilde{\xi}_x^2 B_{v0}^2}{\sqrt{k_y^2 + k_z^2}} ds$$

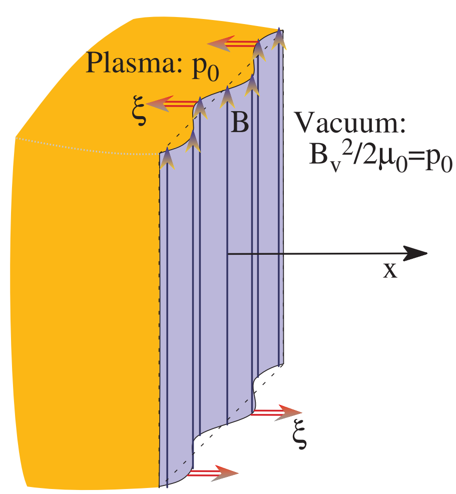
Assuming the most unstable, i.e., incompressible $\nabla \cdot \boldsymbol{\xi}$ perturbations we obtain for the potential energy

$$U = \frac{1}{2\mu_0} \int_{x=0} \left(\frac{2k_z^2}{\sqrt{k_y^2 + k_z^2}} B_{v0}^2 + \frac{\partial B_{v0}^2(\mathbf{x}_0)}{\partial n}\right) \widetilde{\xi}_x^2 ds$$

This result implies stability for

$$\frac{2k_z^2}{\sqrt{k_y^2 + k_z^2}} > -\frac{1}{B_{v0}^2} \frac{\partial B_{v0}^2(\mathbf{x}_0)}{\partial n}$$

Therefore configurations with $\partial B_{v0}^2(\mathbf{x}_0)/\partial n < 0$ are unstable. Instability occurs preferably for small values of k_z and the most unstable modes have $k_z = 0$. The perturbation moves magnetic field into a region where the plasma pressure is greater than the magnetic pressure



Stability of the Z pinch

In cylindrical geometry with a magnetic field in the θ direction and only radial dependence the force balance equation can be written as

$$\frac{dp_0}{dr} = -\frac{B_0}{\mu_0 r}\frac{d}{dr}(rB_0)$$

where B_0 is the θ component of the magnetic field. In this case we can derive an equilibrium for a constant current j_0 in the z direction by intergrating Ampere's law

$$\frac{1}{\mu_0}\frac{d}{dr}(rB_\theta) = j_z$$

and subsequently solve the pressure balance equation. Alternatively, we could specify B_0 then integrate the equation to obtain p_0 or vice versa specify p_0 to integrate and obtain the magnetic field. This is similar to the case of straight field lines in Cartesian coordinates, however, for the Z pinch field lines are curved and $\mathbf{B}\cdot\nabla\mathbf{B} \neq 0$. A straightforward solution for this case can be found by assuming the current density (along z) to be constant up to a radius a for which the pressure drops to 0. for $r > a$ the current density has to be 0 otherwise the pressure would be required negative which is unphysical.

Exercise: Assume a constant current j_0 along the z direction in a cylindrical coordinate system. Compute the magnetic field $B_\theta(r)$ and integrate the force balance equation to obtain $p(r)$. The pressure at $r = 0$ is p_0. Determine the critical radius for which the pressure decreases to 0.

The resulting configuration is a column or cylinder in which the current is flowing along the cylinder axis in the z direction and the magnetic field is in the azimuthal θ direction as illustrated in Figure 4.6. An equilibrium configuration which has some similarity with the Z pinch is the θ pinch in which the magnetic field is along the z direction and the current in the θ direction. To examine the stability of the Z pinch the perturbations for this configuration are chosen as

$$\boldsymbol{\xi}(r,\theta,z) = \boldsymbol{\xi}_0(r,z)\exp(im\theta)$$

where m is the wave number in the θ direction.

In cylindrical coordinates the perturbations contributing to the potential energy are

$$\begin{aligned}
\nabla\cdot\boldsymbol{\xi} &= \frac{1}{r}\frac{\partial}{\partial r}(r\xi_r) + \frac{1}{r}\frac{\partial\xi_\theta}{\partial\theta} + \frac{\partial\xi_z}{\partial z} \\
\mathbf{Q} = \nabla\times(\boldsymbol{\xi}\times\mathbf{B}_0) &= -B_0\left[\frac{1}{r}\frac{\partial\xi_r}{\partial\theta}\mathbf{e}_r - \left(\frac{1}{B_0}\frac{\partial}{\partial r}(B_0\xi_r) + \frac{\partial\xi_z}{\partial z}\right)\mathbf{e}_\theta + \frac{1}{r}\frac{\partial\xi_z}{\partial\theta}\mathbf{e}_z\right] \\
\boldsymbol{\xi}\times(\nabla\times\mathbf{B}_0) &= \frac{1}{r}\frac{\partial}{\partial r}(rB_0)(\xi_\theta\mathbf{e}_r - \xi_r\mathbf{e}_\theta)
\end{aligned}$$

We will assume that the boundary is a conducting wall at which the displacement is 0. With these relations the potential becomes

$$\begin{aligned}
U &= \frac{1}{2}\int_V \left[\gamma p_0 \left(\nabla\cdot\boldsymbol{\xi}\right)^2 + \frac{\partial p_0}{\partial r}\xi_r \nabla\cdot\boldsymbol{\xi} \right. \\
&\quad + \frac{B_0^2}{\mu_0}\left(\left(\frac{1}{B_0}\frac{\partial}{\partial r}\left(B_0\xi_r\right) + \frac{\partial \xi_z}{\partial z}\right)^2 + \frac{1}{r^2}\left(\frac{\partial \xi_r}{\partial \theta}\right)^2 + \frac{1}{r^2}\left(\frac{\partial \xi_z}{\partial \theta}\right)^2\right) \\
&\quad \left. - \frac{B_0}{\mu_0 r}\frac{\partial}{\partial r}\left(rB_0\right)\left(\frac{\xi_\theta}{r}\frac{\partial \xi_r}{\partial \theta} + \xi_r\left(\frac{1}{B_0}\frac{\partial}{\partial r}\left(B_0\xi_r\right) + \frac{\partial \xi_z}{\partial z}\right)\right)\right] d\mathbf{x} \qquad (4.24)
\end{aligned}$$

Azimuthally symmetric perturbations

In this case m is zero and all θ derivatives are 0. To evaluate the stability we bring the potential into the quadratic form using

$$\begin{aligned}
\frac{\partial \xi_z}{\partial z} &= \nabla\cdot\boldsymbol{\xi} - \frac{1}{r}\frac{\partial}{\partial r}\left(r\xi_r\right) = \nabla\cdot\boldsymbol{\xi} - \frac{\partial \xi_r}{\partial r} - \frac{\xi_r}{r} \\
\frac{d}{dr}\left(rB_0\right) &= -\frac{\mu_0 r}{B_0}\frac{dp_0}{dr}
\end{aligned}$$

The basic approach to evaluate the potential for the specific cylindrical geometry for the Z pinch is outlined below:

$$\begin{aligned}
U &= \frac{1}{2}\int_V \frac{B_0^2}{2\mu_0}\left[\gamma\frac{2\mu_0 p_0}{B_0^2}\left(\nabla\cdot\boldsymbol{\xi}\right)^2 + \frac{2\mu_0 p_0}{B_0^2}\frac{d\ln p_0}{dr}\xi_r\nabla\cdot\boldsymbol{\xi} \right. \\
&\quad + 2\left(\xi_r\frac{d\ln B_0}{dr} + \frac{\partial \xi_r}{\partial r} + \frac{\partial \xi_z}{\partial z}\right)^2 \\
&\quad \left. + 2\frac{1}{rB_0}\frac{drB_0}{dr}\xi_r\left(\frac{1}{B_0}\frac{\partial B_0\xi_r}{\partial r} + \frac{\partial \xi_z}{\partial z}\right)\right] d\mathbf{x} \\
&= \frac{1}{2}\int_V \frac{B^2}{2\mu_0}\left[\gamma\beta\left(\nabla\cdot\boldsymbol{\xi}\right)^2 + \beta\frac{\partial \ln p}{\partial \ln r}\frac{\xi_r}{r}\nabla\cdot\boldsymbol{\xi} \right. \\
&\quad + 2\left(\left(\left(\frac{d\ln B}{d\ln r} - 1\right)\frac{\xi_r}{r} + \nabla\cdot\boldsymbol{\xi}\right)^2\right) \\
&\quad \left. + \beta\frac{d\ln p}{d\ln r}\left(\left(\frac{d\ln B}{d\ln r} - 1\right)\frac{\xi_r^2}{r^2} + \frac{\xi_r}{r}\nabla\cdot\boldsymbol{\xi}\right)\right] d\mathbf{x} \\
&= \ldots
\end{aligned}$$

A more complete derivation is found in Appendix B. Collecting the coefficients for $(\nabla\cdot\boldsymbol{\xi})$ and ξ_r/r terms leads to:

$$U = \frac{1}{2}\int_V \left[a_{11}\left(\nabla\cdot\boldsymbol{\xi}\right)^2 + 2a_{12}\frac{\xi_r}{r}\nabla\cdot\boldsymbol{\xi} + a_{22}\frac{\xi_r^2}{r^2}\right] d\mathbf{x}$$

where the coefficients are

$$a_{11} = \gamma\beta + 2 \qquad \beta = \frac{2\mu_0 p_0}{B_0^2}$$
$$a_{12} = 2\left(\frac{d\ln B_0}{d\ln r} + \frac{\beta}{2}\frac{d\ln p_0}{d\ln r} - 1\right)$$
$$a_{22} = \left(\frac{d\ln B_0}{d\ln r} - 1\right) a_{12}$$

and stability requires

$$a_{11}a_{22} - a_{12}^2 > 0$$

Using the force balance equation one can simplify the stability equation to

$$-\frac{d\ln p_0}{d\ln r} = -\frac{r}{p_0}\frac{dp_0}{dr} < \frac{4\gamma}{2+\beta\gamma}$$

For the Z pinch the plasma pressure decreases with radial distance from the pinch axis such that at large distances $\beta \ll 1$. Here the rhs assumes a maximum such that stability requires

$$\frac{dp}{p} > -2\gamma\frac{dr}{r}$$

or

$$p >\sim r^{-2\gamma}$$

However to avoid plasma contact with the wall the density and pressure must be close to 0. Thus the Z pinch is unstable with respect to symmetric perturbations which lead to a periodic pinching of the plasma column. The resulting mode ($m = 0$) is often addressed as sausage mode because of the periodic pinching of the current collumn.

Azimuthally asymmetric perturbations

In this case the θ dependence is nonzero and on can assume a displacement of the following form

$$\begin{aligned}\xi_r &= \xi_r(r,z)\sin m\theta\\ \xi_\theta &= \xi_\theta(r,z)\cos m\theta\\ \xi_z &= \xi_z(r,z)\sin m\theta\end{aligned}$$

To simplify the analysis one usually assumes $\nabla\cdot\boldsymbol{\xi} = 0$. With this assumption the only change in the potential energy is that the term $\gamma p_0 (\nabla\cdot\boldsymbol{\xi})^2$ is replaced by

$$\frac{B_0^2}{\mu_0}\frac{m^2}{r^2}\left(\xi_r^2 + \xi_z^2\right)$$

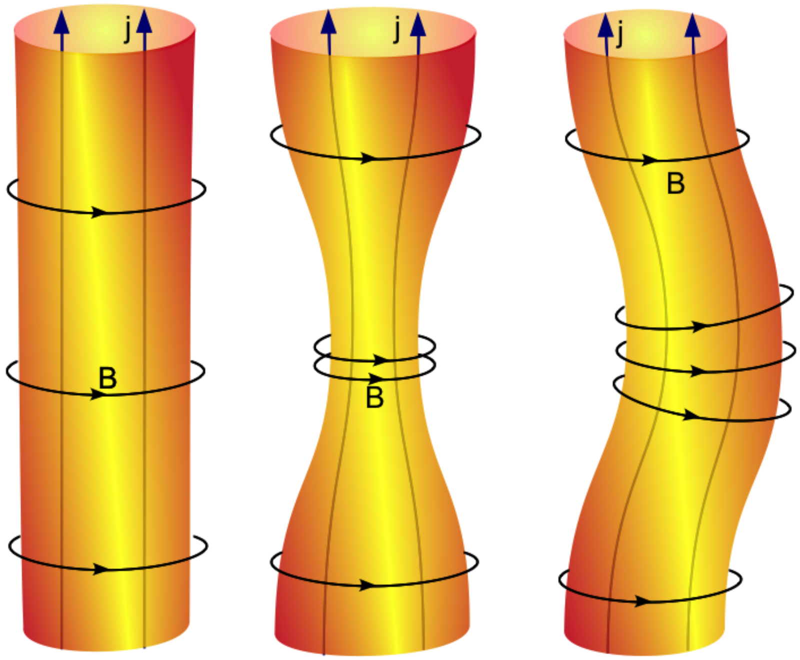


Figure 4.6: Illustration of the Z pinch (left), sausage mode (middle), and kink mode (right).

Stability is obtained for

$$-\frac{d\ln p_0}{d\ln r} = -\frac{r}{p_0}\frac{dp_0}{dr} < \frac{m^2}{\beta}$$

One can compare the stability limit for non-symmetric perturbation with that for symmetric perturbations. For small values of β the symmetric condition is always more unstable. For values with $\beta > 1$ the $m = 1$ or even $m = 2$ mode can be unstable even though the symmetric condition may imply stability.

The $m = 1$ mode is called kink instability (or corkscrew instability because of the form of the bending of the plasma column as shown in Figure 4.6.

Pressure gradients and parallel current:

Splitting the perturbation into parallel and perpendicular components relative to the equilibrium magnetic field allows to express the potential U_2 in the form [Schindler, 2007]

$$\begin{aligned} U_2 &= \frac{1}{2}\int_V \Big[\gamma p_0 \left(\nabla\cdot\boldsymbol{\xi}\right)^2 + \frac{1}{\mu_0}\mathbf{B}_{1\perp}^2 + \frac{1}{\mu_0}B_0^2\left(\nabla\cdot\boldsymbol{\xi}_\perp + 2\boldsymbol{\xi}_\perp\cdot\boldsymbol{\kappa}\right)^2 \\ &\qquad -2\boldsymbol{\xi}_\perp\cdot\boldsymbol{\kappa}\boldsymbol{\xi}_\perp\cdot\nabla p_0 - j_{0\parallel}\left(\boldsymbol{\xi}_\perp\times\mathbf{b}_0\right)\cdot\mathbf{B}_{1\perp}\Big]\,d\mathbf{x} \end{aligned} \tag{4.25}$$

with the unit magnetic field vector $\mathbf{b}_0$ and $\boldsymbol{\xi}_\perp = \mathbf{b}_0\times(\boldsymbol{\xi}\times\mathbf{b}_0)$, and $j_{0\parallel} = \mathbf{j}\cdot\mathbf{b}_0$. Further $\boldsymbol{\kappa}$ is the magnetic field curvature vector

$$\boldsymbol{\kappa} = \mathbf{b}_0\cdot\nabla\mathbf{b}_0$$

In the potential all except for the first term contain only the perpendicular perturbation. This allows for an explicit minimization for $\boldsymbol{\xi}_\parallel$ first. However, a complication of the procedure is the appearance of spatial averages in the resulting potential. Inspecting the terms in (4.25) demonstrates

- the first 3 terms are positive and cannot contribute to instability;
- the last two terms can cause instability if they are large enough to overcome the stabilization from the first terms;
- the 4th term involves pressure gradients and magnetic field curvature. This can cause instability if the perturbation is large in the direction perpendicular to the pressure gradient and along the radius of curvature for the field. Specifically favorable for instability is the situation where the direction of the field curvature $\boldsymbol{\kappa}$ is aligned with the direction of the pressure gradient;
- the last term can cause instability in the presence of large field-aligned current if the perturbed magnetic field $\mathbf{B}_{1\perp}$ maximizes perpendicular to the equilibrium magnetic field.

Two-dimensional cartesian equilibria:

In the following we consider two-dimensional equilibrium solutions for $\partial/\partial_y = 0$ as derived in section 4.4.3. In order to include gravity in the discussion we use the force balance equation including a gravitational potential

$$0 = -\nabla p + \left(j_y - \frac{1}{2\mu_0} \frac{dB_y^2}{dA_y} \right) \nabla A_y - \rho \nabla \psi$$

Abbreviating $A_y = A$ and asuming that ∇A and $\nabla \psi$ are not aligned everywhere the pressure has to be a function of the magnetic flux function and the gravitational potential $p = p\left(A_y, \psi\right)$ and from

$$0 = -\frac{\partial p}{\partial A} \nabla A - \frac{\partial p}{\partial \psi} \nabla \psi + \left(j_y - \frac{1}{2\mu_0} \frac{dB_z^2}{dA} \right) \nabla A - \rho \nabla \psi$$

Force balance requires

$$\begin{aligned} \frac{\partial p}{\partial A} &= j_z - \frac{1}{2\mu_0} \frac{dB_z^2}{dA} \\ \frac{\partial p}{\partial \psi} &= -\rho \end{aligned}$$

Following we sketch the manipulation of the ideal MHD potential into a more suitable form for the special case of a two-dimensional equilibrium and two-dimensional perturbations. Starting with the potential

$$\begin{aligned} U_2 &= \frac{1}{2} \int_V \left[\gamma p \left| \nabla \cdot \boldsymbol{\xi} \right|^2 + \frac{1}{\mu_0} \left| \mathbf{B}_1 \right|^2 + \mathbf{j} \cdot \left(\boldsymbol{\xi}^* \times \mathbf{B}_1 \right) \right. \\ &\qquad \left. + \left(\boldsymbol{\xi} \cdot \nabla p \right) \nabla \cdot \boldsymbol{\xi}^* - \left(\boldsymbol{\xi}^* \cdot \nabla \psi \right) \nabla \cdot \left(\rho \boldsymbol{\xi} \right) \right] d\mathbf{x} \end{aligned} \tag{4.26}$$

with

$$\mathbf{B}_1 = \nabla \times \left(\boldsymbol{\xi} \times \mathbf{B} \right)$$

Note, here p, j, and ρ are equilibrium quantities and $\boldsymbol{\xi}$, and $\mathbf{B}_1$ are perturbations.

Consider the 2D equilibrium with $\mathbf{B} = \nabla A \times \mathbf{e}_y + B_y \mathbf{e}_y$ where we address the x and z components ($\mathbf{B}_p = \nabla A \times \mathbf{e}_y$) as the poloidal magnetic field. The perturbed electric field can be expressed through the perturbation of the vectorpotential and ideal Ohm's law.

$$
\begin{aligned}
\mathbf{E}_1 &= -\frac{\partial \mathbf{A}_1}{\partial t} \\
&= -\dot{\boldsymbol{\xi}} \times (\nabla \times \mathbf{A}) = -\dot{\boldsymbol{\xi}} \times (\nabla A \times \mathbf{e}_y + B_y \mathbf{e}_y)
\end{aligned}
$$

Integrating this expression we obtain the perturbation of the vectorpotential and magnetic field through

$$
\begin{aligned}
\mathbf{A}_1 &= \boldsymbol{\xi} \times (\nabla A \times \mathbf{e}_y + B_{y0} \mathbf{e}_y) = \xi_y \nabla A - \boldsymbol{\xi} \cdot \nabla A \mathbf{e}_y + \boldsymbol{\xi} \times \mathbf{e}_y B_y \\
\mathbf{B}_1 &= \nabla \times \mathbf{A}_1 \\
&= \nabla \times (\xi_y \nabla A - \boldsymbol{\xi} \cdot \nabla A \mathbf{e}_y + \boldsymbol{\xi} \times \mathbf{e}_y B_y)
\end{aligned}
$$

This indicates that the perturbation of the vector potential is split into three component, one which is perpendicular to the polodial magnetic field along $\xi_y \nabla A$, one along the invariant y direction $-\boldsymbol{\xi} \cdot \nabla A$, and one which is along the magnetic field which is contained in the $\boldsymbol{\xi} \times \mathbf{e}_y B_y$ term. Further manipulation of the magnetic terms is illustrated in detail in Schindler et al. (Sol. Phys. 87, 1983).

Pressure and gravity terms can be treated as follows

Term $\nabla \cdot (\rho \boldsymbol{\xi})$:

$$
\begin{aligned}
\nabla \cdot (\rho \boldsymbol{\xi}) &= \rho \nabla \cdot \boldsymbol{\xi} + \boldsymbol{\xi} \cdot \nabla \rho \\
&= \rho \nabla \cdot \boldsymbol{\xi} + \frac{\partial \rho}{\partial \psi} \boldsymbol{\xi} \cdot \nabla \psi + \frac{\partial \rho}{\partial A} \boldsymbol{\xi} \cdot \nabla A
\end{aligned}
$$

Term $\boldsymbol{\xi} \cdot \nabla p$:

$$
\boldsymbol{\xi} \cdot \nabla p = \frac{\partial p}{\partial \psi} \boldsymbol{\xi} \cdot \nabla \psi + \frac{\partial p}{\partial A} \boldsymbol{\xi} \cdot \nabla A
$$

Combining pressure and gravity terms with $A_1 = \boldsymbol{\xi} \cdot \nabla A$ and $R_1 = \boldsymbol{\xi} \cdot \nabla \psi$

$$\begin{aligned}
U_p &= \gamma p \left|\nabla \cdot \boldsymbol{\xi}\right|^2 + \left(\boldsymbol{\xi} \cdot \nabla p\right) \nabla \cdot \boldsymbol{\xi}^* - \left(\boldsymbol{\xi}^* \cdot \nabla \psi\right) \nabla \cdot \left(\rho \boldsymbol{\xi}\right) \\
&= \gamma p \left|\nabla \cdot \boldsymbol{\xi}\right|^2 + \nabla \cdot \boldsymbol{\xi}^* \left(\frac{\partial p}{\partial \psi} \boldsymbol{\xi} \cdot \nabla \psi + \frac{\partial p}{\partial A} \boldsymbol{\xi} \cdot \nabla A\right) \\
&\quad - \left(\boldsymbol{\xi}^* \cdot \nabla \psi\right) \left(\rho \nabla \cdot \boldsymbol{\xi} + \frac{\partial \rho}{\partial \psi} \boldsymbol{\xi} \cdot \nabla \psi + \frac{\partial \rho}{\partial A} \boldsymbol{\xi} \cdot \nabla A\right) \\
&= \gamma p \left|\nabla \cdot \boldsymbol{\xi}\right|^2 - \rho R_1 \nabla \cdot \boldsymbol{\xi}^* + \frac{\partial p}{\partial A} A_1 \nabla \cdot \boldsymbol{\xi}^* \\
&\quad - \rho R_1^* \nabla \cdot \boldsymbol{\xi} - \frac{\partial \rho}{\partial \psi} \left|R_1\right|^2 - \frac{\partial \rho}{\partial A} R_1^* A_1 \\
&= \gamma p \left|\nabla \cdot \boldsymbol{\xi}\right|^2 + \frac{\partial \rho}{\partial \psi} \left(\frac{\rho}{\partial \rho / \partial \psi}\right)^2 \left|\nabla \cdot \boldsymbol{\xi}\right|^2 \\
&\quad - \frac{\partial \rho}{\partial \psi} \left[\left|R_1\right|^2 + \frac{\rho}{\partial \rho / \partial \psi} \left(R_1^* \nabla \cdot \boldsymbol{\xi} + R_1 \nabla \cdot \boldsymbol{\xi}^*\right) + \left(\frac{\rho}{\partial \rho / \partial \psi}\right)^2 \left|\nabla \cdot \boldsymbol{\xi}\right|^2\right] \\
&\quad + \frac{\partial p}{\partial A_y} A_1 \nabla \cdot \boldsymbol{\xi}^* - \frac{\partial \rho}{\partial A} R_1^* A_1 \\
&= \gamma p \left(\left|\nabla \cdot \boldsymbol{\xi}\right|^2 + \frac{\rho^2}{\partial \rho / \partial \psi} \left|\nabla \cdot \boldsymbol{\xi}\right|^2\right) - \frac{\partial \rho}{\partial \psi} \left[\boldsymbol{\xi} \cdot \nabla \psi + \frac{\rho}{\partial \rho / \partial \psi} \nabla \cdot \boldsymbol{\xi}\right] + ..
\end{aligned}$$

In summary one obtains the potential for 2D equilibrium and 2D perturbations ($\boldsymbol{\xi} = \boldsymbol{\xi}\left(x, z\right)$ and $\mathbf{B}_1 = \mathbf{B}_1\left(x, z\right)$) as

$$\begin{aligned}
U_2 &= \frac{1}{2} \int_V \left[\frac{1}{\mu_0} \left(\left|\nabla A_1\right|^2 - \frac{dJ}{dA} \left|A_1\right|^2 + \frac{1}{\mu_0} \left|B_y \nabla \cdot \boldsymbol{\xi}_p - \mathbf{B}_p \cdot \nabla \xi_y\right|^2\right)\right. \\
&\qquad \left. \gamma p \left(1 + \frac{\rho^2}{\gamma p \partial \rho / \partial \psi}\right) \left|\nabla \cdot \boldsymbol{\xi}\right|^2 - \frac{\partial \rho}{\partial \psi} \left|\boldsymbol{\xi} \cdot \nabla \psi + \frac{\rho}{\partial \rho / \partial \psi} \nabla \cdot \boldsymbol{\xi}\right|^2\right] dx dz \qquad (4.27)
\end{aligned}$$

Here we use the definition $A_1 = \boldsymbol{\xi} \cdot \nabla A$ and the index p denotes the poloidal components ($\mathbf{B}_p = \nabla A \times \mathbf{e}_y$ and $\boldsymbol{\xi}_p = \boldsymbol{\xi} - \xi \mathbf{e}_y$). Also,

$$J = \mu_0 j_y = \mu_0 \frac{\partial}{\partial A} \left(p + \frac{B_y^2}{2\mu_0}\right)$$

In using $\boldsymbol{\xi}$ or A_1 as test functions there is a constraint on A_1 since it depends on the flux function A which implies that

$$A_1 = 0 \qquad where \qquad \nabla A = 0 \qquad (4.28)$$

or $\mathbf{B}_p = 0$. In other words, the use of A_1 may provide provide a minimum of (4.27) with $A_1 \neq 0$ although it should be 0 based on $\boldsymbol{\xi}$ perturbations.

Most terms contribute to stabilization with the exception of the dj/dA term and pressure and gradient terms. Let us first ignore the magnetic terms and just consider stability by considering the gravitational terms. In the case the suffcient condition for the absence of gravitational instability is

$$1+\frac{\rho^2}{\gamma p\partial\rho/\partial\psi}\geq 0$$

For a better understanding of this condition consider the case

$$\frac{\partial\rho}{\partial\psi}=-\frac{\rho^2}{\gamma p}\qquad or\qquad -\gamma\frac{1}{\rho}\frac{\partial\rho}{\partial\psi}+\frac{1}{p}\frac{\partial p}{\partial\psi}=0$$

with the solution $p/\rho^\gamma = c(A)$ with c being an integration constant that dependents on the magnetic flux function. In other words for each field line the limiting pressure pressure and density distribution is the adiabatic pressure profile. Note that this is the limiting case for the convection instability. Whenever density decreases faster with height (or ψ) the corresponding gas or plasma is stable.

Ignoring the gravitational terms and pressure gradients leads to the potential

$$U_{2m}=\frac{1}{2}\int_V\left(|\nabla A_1|^2-\frac{dJ}{dA}|A_1|^2+\frac{1}{\mu_0}\left|B_y\nabla\cdot\boldsymbol{\xi}_p-\mathbf{B}_p\cdot\nabla\xi_y\right|^2\right)dxdz$$

Here it can be shown that all configurations with a nonvanishing (everywhere) cartesian component of the poloidal magnetic field. subject to two-dimensional perturbations $\boldsymbol{\xi}=\boldsymbol{\xi}(x,z)$ (which are 0 at the boundaries) are stable.

Sketch of proof:

Since A satisfies the Grad-Shafranov equation $\Delta A=-J(A)$ The magnetic field $\mathbf{B}=\nabla A\times\mathbf{e}_y$ satisfies

$$\Delta\mathbf{B}=-\frac{dJ}{dA}\mathbf{B}$$

If for instance the B_x component is nonzero everywhere and replacing $A_1=\kappa B_x$ we can use partial integration to show

$$\begin{aligned}U_{2m} &\geq F_{2m}=\frac{1}{2}\int_V\left(|\nabla A_1|^2+\frac{\Delta B_x}{B_x}|A_1|^2\right)dxdz\\ &= \frac{1}{2}\int_V B_x^2\,|\nabla\kappa|_x^2\,dxdz>0\end{aligned}$$

where κ can be any function of x and z. This demonstrates that U_{2m} is alway positive if we can find a solution with a cartesian magnetic field component which is $\neq 0$ everywhere.

Euler-Lagrange equations:

It is desireable to be able to determine the minimum value of the potential from an analytic procedure. This is possible by solving the Euler-Lagrange equations (derived from a variational principle). Assume a functional

$$I(y,y')=\int_D f(x,y,y')\,dx$$

with $y' = dy/dx$. The necessary condition for $I(y)$ to assume an extremum is the Euler-Lagrange equation:

$$\frac{d}{dx}\frac{\partial f}{\partial y'} - \frac{\partial f}{\partial y} = 0$$

In multiple (n) dependent variables y_1, y_2, .. y_n and the functional

$$I = \int_D f(x, y_1.., y_n, y_1'.., y_n')\, dx$$

with constraints $g_i(x, y_1, .., y_n) = 0$, $i = m < n$, the extremum is defined by a set of n Euler-Lagrange equations

$$\frac{d}{dx}\frac{\partial L}{\partial y_i'} - \frac{\partial L}{\partial y_i} = 0$$

with

$$L(x, y_1.., y_n, y_1'.., y_n') = f(x, y_1.., y_n, y_1'.., y_n') + \sum \lambda(x)\, g_i(x, y_1, .., y_n)$$

Stability of the Harris Sheet:

Considering $B_y = 0$ it is clear the the minimum of the potential for two-dimensional perturbations is determined for $\xi_y = 0$ by

$$U_{2m} = \frac{1}{2}\int_V \left(|\nabla A_1|^2 - \frac{dJ}{dA}|A_1|^2 \right) dxdz$$

For the Harris sheet this becomes

$$U_{2m} = \frac{1}{2}\int_V \left(|\nabla A_1|^2 - \frac{2}{\cosh^2 z}|A_1|^2 \right) dxdz$$

with boundary condition $A_1 = 0$ at $z = \pm\infty$. We can assume for the perturbation

$$A_1 = a(z)\exp(ikx)$$

Since the problem is linear and in order to avoid the trivial solution $A_1 = 0$ we normalize the perturbation by $\int_V |A_1|^2 / 2dxdz = 1$ and introduce a Lagrange multiplier λ to minimize

$$U_{2m} = \frac{1}{2}\int_V \left(|d_z a|^2 - \frac{2}{\cosh^2 z}|a|^2 - \left(\frac{\lambda}{2} - k^2\right)|a|^2 \right) dxdz$$

The resulting Euler-Lagrange equation is

$$\frac{d^2 a}{dz^2} - \frac{2}{\cosh^2 z}a = \Lambda a$$

with $\Lambda = \lambda - k^2$. The equation represents an Eigenvalue problem (Morse and Feshbach). The lowest Eigenvalue λ represents the minimum value of U_{2m}. This lowest Eigenvalue solution is

$$a_0(z) = \cosh^{-1} z \qquad with \qquad \Lambda_0 = -1$$

The solution satisfies the boundary conditions $a(\pm\infty) = 0$ and for $k < 1$ we find $\lambda = \Lambda + k^2 < 0$ such that U_{2m} is negative which implies instability.

However, inconsidering this solution we find that $a_0(0) > 0$ which violates the constraint (4.28). This implies that it should be rejected. The reason for this becomes clear when we look at the magnetic configuration for the instability which looks qualitatively similar to the catseyes solution (Figure 4.2), i.e, the equilibrium + perturbation generate a current sheet with a chain of periodic magnetic islands where the transverse size of teh islands is given by the magnitude of the perturbation. These island have a different magnetic topology, i.e., magnetic field has is closed into itself and thus has a different connection compared to the equilibrium. However, this is not possible within ideal MHD where the magnetic field is frozen to the plasma and cannot undergo a new connection. Therefore we conclude that absent another negative Eigenvalue of Λ the Harris sheet is stable in ideal MHD. Nevertheless this solution is a hint that the Harris sheet can be unstable if dissipation allows this new topology!

Stability and magnetic flux tube volume

In a plasma with a small plasma β the evolution is strongly dominated by the magnetic field such that magnetic flux tubes carry much of the energy. We had defined the magnetic flux tube volume as

$$V = \int \frac{dl}{B}$$

The thermal energy contained in a flux tube is pV which can be regarded as a potential energy of the flux tube. Since the flux tube volume changes during convection the energy associated with the flux tube changes. Using the pressure equation we have

$$p + \Delta p = p - \gamma p \nabla \cdot \mathbf{u} = p + \gamma p \frac{\Delta V}{V}$$

Note that $dp/dt = -\gamma p \nabla \cdot \mathbf{u}$ and $\nabla \cdot \mathbf{u} = -\frac{1}{n}\frac{dn}{dt} = -\frac{1}{V}\frac{dV}{dt}$ which is obtained from

$$\frac{dS}{dt} = \frac{dpV^\gamma}{dt} = 0$$

The pressure change in the vicinity of the flux tube is

$$p(V + \Delta V) = p + \frac{dp}{dV}\Delta V$$

Considering a small displacement from the equilibrium: If the pressure change in the flux tube is greater than the pressure in the surrounding tubes then lower energy state is the equilibrium and energy has to be brought into the system to achieve the change. In other words the configuration is stable for

$$\gamma p \frac{\Delta V}{V} + \frac{dp}{dV}\Delta V > 0$$

or

$$-\frac{V}{p}\frac{dp}{dV} < \gamma$$

or

$$-\frac{d\ln p}{d\ln V} < \gamma$$

Note that adiabatic convection yields

$$-\frac{d\ln p}{d\ln V} = \gamma$$

For the sheet pinch this yields

$$-\frac{d\ln p}{d\ln r} < 2\gamma$$

which is the small β approximation of our prior result.

Note, however that this is somewhat heuristic or intuitive and lacks the rigour or the prior discussion.