ARFIMA-GARCH modeling of financial time series*

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Abstract. In this article we consider financial time series with long memory and variable variance. Such series are modeling by simultaneous use of the Autoregressive Fractionally Integrated Moving Average (ARFIMA) model and of the Generalized AutoRegressive Conditionally Heteroscedastic (GARCH) model. The approach is illustrated by the study of the structure of the currency basket. For the readers convenience we present in the Appendix a pseudocode of the algorithm oriented on the use of Phyton software.

Keywords. Financial time series, volatility, long memory, ARFIMA, GARCH.

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1 Introduction

The study of the time-structure of the series of different nature describing the processes in economics, tele-communication, astronomy etc. plays an important role in modeling and forecasting. In the recent decades so called time series with long memory attract an essential interest (see, e.g. [1, 10, 20]). These series are also known as series with long-range dependence or with strong dependence or persistence series. They can be as stationary as non-stationary in which dependence on time is decreasing fairly slow.

Different approaches are used to determine the dynamics of non-Gaussian processes, in particular such characteristics as presence/absence of a trend-resistance, presence/absence of a long memory, existence of quasi-cycles in different scales, neural networks, as well as the theory of chaos and fractal geometry

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(cf. [3, 16, 18, 19]). In fractal geometry it is used the classification of the fractal time-series due to Mandelbrot [16] based on the Hurst's R/S-method [12]. Hurst parameter H [12, 4] involves a minimal information on the considered process and can distinguish a random series (even the non-Gaussian one) from the non-random series. Hurst parameter is related to the so-called fractal dimension D: $D = 2 - H$ (see [15]).

Long time series are usually characterized by presence of the autocorrelation and have a large Hurst parameter. Such time series are the series with a long memory. The long memory or persistence is considered as a characteristic property of many economical and financial time series. Empirical relevance and theoretical problems in modeling and estimation lead to the necessity of the use of novel methods. Fractional integration (FI) of order d is one of the mostly used methods of determination of the presence of persistence in time series. The estimated values of d vary between -1/2 and 1 where $d = 1/2$ is a border between stationarity and non-stationarity. [1] In the modern literature there are some contradictions in application of this method. Thus, the question of forecasting taking into account the long memory is difficult to solve in general.

One of the useful realizations of fractional integration approach is an autoregressive model of fractional integrated moving average (ARFIMA) which shows its prognostic advantages in spite of the difficulties in the choice of the model and in the estimation of the fractional parameter d.

ARFIMA models describe the conditional mean of the data, however many economical and financial time series exhibit also changes in variance over time, with periods of large variability followed by periods of stability, suggesting heteroscedasticity (varying variance). These volatility clustering phenomena may be well described by conditional volatility models such as the Generalized AutoRegressive Conditionally Heteroscedastic (GARCH) (see e.g. [13] and references therein). It should by noted that in the recent decade an interest to the application of the ARFIMA-GARCH in different directions is growing rapidly. Since we are working with a specific type of time series, we decide not increase essentially the list of references. Probably, it is a real challenge for those who think about writing the survey paper on the application of the ARFIMA-GARCH model.

In this paper we discuss a peculiarity of application of the joint ARFIMA-GARCH approach to the study of financial time series. This approach is illustrated by the study of the formation of the currency basket of the National Bank of the Republic of Belarus.

The remainder of this paper is organized as follows. In the next section, we introduce the Hurst parameter, which is used here as a measure for long deviations from a random walk. In Section 3, we briefly review the long memory effects in economics. In Section 4, we provide a formalization of ARFIMA and GARCH modeling. Finally, in Appendix A, we present the results from applying the ARFIMA-GARCH approach to the study of the formation of the currency basket at the National Bank of the Republic of Belarus.

[1] A wider discussion of the values of the fractional parameter d is presented below in Subsec. 4.1.

For the readers convenience we present in Appendix B a pseudocode of the algorithm oriented on the use of Phyton software. It should be noted that other types of software are possible to apply for realization of the algorithm.

2 Hurst parameter

We can use the Hurst parameter (Hurst exponent) (H) as a measure for long-deviates from a random walk. Studies involving the Hurst parameter were originally developed in hydrology [12]. Hurst showed that many of natural processes including river flows, temperature, precipitations, sunspots are described by the skewered random walk (or the trendy processes with a noise). The trend strength and the level of the noise can be estimated by the behavior of the normalized range of variation with respect to time changing. Hurst's method is applicable in the study of time series in economics and business. It allows to understand whether these series are skewered random walks. Formally the Hurst parameter is defined as follows. In the time series x_t we consider samples of the length N and define the deflection span (range of variation) $R = max(x_t, N) - min(x_t, N)$, where $max(x_t, N)/min(x_t, N)$ is a maximum/minimum of x_t for all fixed samples of the length N. In order to compare different time series Hurst divided R by the standard deviation S. Such ratio Hurst represents in the form

$$(R/S) = (aN)^H , \tag{1}$$

where R/S is the normalized range of variation, N is the length of a sample, a is a constant, and H is the Hurst parameter. If the process is simply the random walk then the Hurst parameter is equal to 1/2. It means that the deflection range is changing proportionally to the square root of N. If $H \neq 1/2$ then the observations are not independent. Each observation has a memory on previous events. It is not a shot-time memory usually related to the Markovian property, but another type of memory, namely the long-term memory. In this case the recent events have a greater influence than the distant events, but the common influence of the latter is rather essential. In a large time scale the system with such value of the Hurst parameter is a result of the long stream of mutually connected events. There exist three characteristic interval for the Hurst parameter.

$H = 1/2$. No pronounced trend is presented and there is no hope that it will appear in the future. The events described by the time series are randomly distributed and non-correlated. The present state has no influence on the future. R/S-analysis can classify any series of such kind regardless of its distribution type.

$0 < H < 1/2$. This interval correspond to anti-persistent or ergodic series. Such systems are characterized by the return to the mean value. If the system has a growth in previous period, then in the next period it will most probably have a decay and vice versa. Stability of such anti-persistence behavior depends on closeness of H to zero. It has a negative correlation whenever H is close to zero. Such series are characterized by the greater volatility than a random series. In economics and finance it is quite rare situation.

$1/2 < H < 1$. Such series are trend-resistant or persistent. If the series has growth in previous period, it probably remains such tendency in the future. Trend-resistance is larger whenever H is close to 1. For H being close to $1/2$ there exists a greater noise in the series. Such behavior was recognized by Hurst in some processes in nature, but they are also relevant to some economical processes. The following formula for determination of the Hurst parameter is due to Hurst himself [12]

$$H = \frac{\log(R/S)}{\log n/2}, \tag{2}$$

where n is a number of observations. The Hurst parameter is not so much calculated as it is estimated. A variety of techniques exists for estimation of the Hurst parameter H and the process detailed here is both simple and highly data intensive. To estimate the Hurst parameter one must regress the rescaled range on the time span of observations (see, e.g. [4, 6] and references therein). Hurst parameter can be calculated/estimated by using special code in MatLab, Python or R computer systems.

3 Long memory effects in economical study

The modern economy is essentially based on statistics. In the theory of statistics most of the constructions are based on Gaussian distributions. It is characteristic for financial data. Thus 1-day return on stocks correspond to the gaussian curve. In this case the yesterday return has nothing with today return. But moving from from 1-day period of return to the larger period (say to 5-days, 10-days, 20-days) we arrive at another type of distribution. Larger time series have power-type tales, an autocorrelation and a non-random Hurst parameter. It shows that such long series describe processes with a long memory.

Mathematical definition of the long memory or persistence is given in terms of autocorrelations. Persistence characterized by the slow decay of autocorrelation makes it difficult to assess and modeling of the future behavior of time series. Fractional integration (FI) of the order d is one of the efficient models for determination of a long memory. The order d varies between -1/2 and 1, though $d = 1/2$ is a border between stationary and non-stationary case. It should be noted that the process of determination of the parameter d is fairly complicated, the level of the complexity is conditioned by large fluctuation of the slowly converging semi-parametric estimates.

As for empirical data, from the study of financial data follows a conclusion on prognostic advantage of FI models. These advantages are less evident when we are considering macroeconomic data. It is related to the larger information on financial data and thus leads to sharper estimate of the parameter of the long memory. FI models are widely used in modeling and forecasting of the realized volatility. Their efficiency with respect to concurrent models with a short memory is not so evident.

Previous investigations lead to the conclusion that overstatement losses of admissible values are not so essential. It happens even in the case when people do not take into account uncertainties in the estimates for d, but underestimation of d can lead to an essential losses [2].

The processes with the long memory can be defined rather rigorously formalizing their above described peculiarities. A stationary process $\mathbf{X}$ is called the process with the long memory if there exists a real number $\alpha, 0 < \alpha < 1$, and a constant $c, c > 0$, such that the following relation holds

$$\lim_{k \to \infty} \frac{p_k}{ck^{-\alpha}} = 1, \qquad (3)$$

where p is an autocorrelation function, k is a lag number. Therefore autocorrelations of the process satisfy the following asymptotic relation $p_k \sim ck^{-\alpha}$ as $k \to \infty$. Such hyperbolic decay is opposite to the exponential decay characteristic for the processes with a short memory (ARMA). The above definition of the long memory is asymptotic. This means that each autocorrelation could be rather small, but their sum is large. Thus for a recognition of the long memory it is not sufficient to find those lags for which autocorrelation is fairly large. More important is the convergence of autocorrelations to 0 at growing the lag. As larger is the level of persistence of the process as more slow is the convergence of autocorrelations to 0. It differs from the situation of the processes with a short memory for which usually exists one-two essential autocorrelations corresponding to certain small lags.

4 ARFIMA/GARCH modeling

4.1 ARFIMA model

All empirical studies need to make more strict description of changing of the values in series which can be obtained using ARFIMA processes (see pioneering works [10, 11]). First of all the order of integration of the series should be determined, i.e., the value of parameter d of the process $ARFIMA(p, d, q)$. The value $d = 0$ corresponds to a short memory, but the value $d = 1$ is related to infinite memory. Infinite memory means that each shock affects on the behavior of the series during infinitely long time. Vice versa existence of a short memory means that the influence of each shock disappears rather quick. The most interesting is an intermediate situation when the affects of the shocks are temporary but long lasting (i.e., the case of a long memory).

It is said that the series X_t corresponds to the process $ARFIMA(p, d, q)$ if the following relations hold:

$$\Phi(L)\,(1 - L)^d\,(x_t - \mu) = \Theta(l)\varepsilon_t, \qquad (4)$$

$$(1 - L)^d = \sum_{k=0}^{\infty} \frac{\Gamma(k - d)L^k}{\Gamma(-d)\Gamma(k + 1)}, \qquad (5)$$

where x_t are the values in time series with the mean value μ, L is the backward-shift operator, $(L) = 1 - \phi_1 L - \ldots - \phi_p L^p$ is a polynomial of order p, $\Theta(L) =$

$1 + \theta_1 L + \ldots + \theta_q L^p$ is a polynomial of order q, $\varepsilon_t \sim i.i.d.\ N(0, \sigma^2)$ is the white noise, d is the order of integration, $\Gamma(\cdot)$ is the Euler Gamma-function and the right-hand side of (5) is a special case of the so-called Mittag-Leffler function (see, e.g. [9]).

It is known (see [11]) that:

if $0 < d < 1/2$, then the process FI(d) is a long-memory process,

if $d < 0$, then the process FI(d) is an anti persistent process,

if $-1/2 < d < 0$, then the process FI(d) is not of long-memory, but it does not have the behavior of ARMA. This intermediate case called anti-persistent by Mandelbrot corresponds to alternations of increases and decreases in the process. This behavior is also called the "Joseph effect" by reference to the Bible. The process FI(d) being stationary is inverting for $-1/2 < d < 1/2$.

The process is non-stationary for $d \geq 1/2$, as it possesses infinite variance.

It is also well recognized that there is a relation between the values d in the ARFIMA processes and the Hurst parameter H, namely $d = H - 1/2$.

4.2 Long memory processes in dispersions for modeling of the volatility

Generalized Autoregressive Conditional Heteroskedasticity (GARCH-model) is an important tool in the analysis of time series data, particularly in financial applications developed by Engle in 1982 [7]. GARCH describes an approach to the estimate volatility and returns of stock and financial indices on the financial market.

Experts in financial analysis prefer GARCH modeling since it guarantees more real contest than other methods at forecasting of prices and courses of the financial instruments. Heteroskedasticity describes an irregular changing of error or variable in the statistical model. In fact, the presence of the heteroskedasticity shows that observations do not agree with linear models. An information obtained at GARCH modeling is used for determination of prices, asset valuation which potentially lead to a larger profitability, forecasting of the return on investments and asset allocation, hedging, risk control and portfolio optimization.

The usage of GARCH model consists of three steps. First, one has to evaluate the most suitable autoregressive model. Next, it is necessary to calculate autocorrelations corresponding to a chosen lag. At last, the significance control has to be provided. There exist also a widely used classical approaches to the evaluation and forecasting of the financial volatility, i.e., the historical volatility (VolSD) and the method of the exponentially weighted moving mean volatility (VolEWMA). GARCH processes differ from homoskedastic models which assume a constant volatility and use the method of the least squares (MLS). Since the asset profitability is changing in time and depends on previous dispersion then the usage of the homoskedastic models become non-optimal.

GARCH processes are autoregressive in nature, they depend on previous observations, previous dispersions having influence on the current dispersion.

GARCH models describe financial markets with changing volatility which becomes more unstable in the periods of financial crisis and world events. It is also less volatile in the periods of relatively calm situation and stable economical growth.

It has been shown that GARCH models with non-normal distributions are more reliable for the forecasting of volatility than other classical models. The most spread methods of modeling of the volatility are ARCH models (autoregressive conditional heteroskedasticity) [7] and GARCH models (generalized autoregressive conditional heteroskedasticity) [5]. GARCH model is a generalization of ARCH model which recognizes the difference between conditional and unconditional dispersion allowing the latter to change in time depending on previous errors. In the other words the classical assumption on constancy of the dispersion of errors fails in this case.

Let us consider the yield series $r_t = \mu + \varepsilon_t$, where μ is an expected (mean) profit, and ε_t is a white noise. We suppose that GARCH is written in a concrete parametric form, i.e., $\varepsilon_t = \sigma_t z_t$, where z_t has a standard normal distribution ($z_t \sim N(0,1)$), and

$$\sigma_t^2 = \omega + \alpha\varepsilon_{t-1}^2 + \beta\sigma_{t-1}^2. \tag{6}$$

Maximizing the logarithmic likelihood function

$$L(\theta) = \sum_{t=1}^{T} \frac{1}{2}\left(-\log 2\pi - \log\sigma_t^2 - \frac{\varepsilon_t^2}{\sigma_t^2}\right) \tag{7}$$

we can find the values of the coefficients ω, α, β. Here T is the length of the series. The above description means that we deal with GARCH(1, 1), i.e., the only first lag is used in modeling of the volatility of the yield series.

4.3 ARFIMA-GARCH modeling of the financial time series

The ARFIMA-GARCH modeling considered here allows the quantification of long range correlations and time-varying volatility. Several approaches were proposed to analyze the financial time series which cope with the non-stationarity of the data. One approach is time-variant AutoRegressive (AR) analysis using exponentially smoothed recursive least squares estimation, with fixed and varying forgetting factors. Another approach is based on the adaptive segmentation of the non stationary observations into approximately stationary group of data which are usually modeling by using short memory AR models. This procedure leads to linear parametric models for the conditional mean which, however, do not capture the long range correlations of the financial data. In [10, 11] it was proposed the use of Fractionally Integrated AutoRegressive Moving Average (ARFIMA) models, an extension of the well-known AutoRegressive Moving Average (ARMA) models, to represent both short and long term behaviours of financial series. ARFIMA models describe the conditional mean of the data, however financial time series exhibit also changes in variance over time, with periods of large variability followed by periods of stability, suggesting heteroscedasticity (varying variance). These volatility clustering phenomena may be well described by conditional volatility models such as the Generalized AutoRegressive Conditionally Heteroscedastic (GARCH) [7]. Consequently, some authors considered the joint modeling of long-memory and heteroscedasticity characteristics of time series using fractionally integrated ARFIMA models with GARCH innovations. In these works ARFIMA-GARCH modeling is used to capture and remove long-range correlation and estimate conditional volatility.

In the appendix below we illustrate ARFIMA-GARCH approach in the study of formation of the currency basket of the National Bank of the Republic of Belarus.

Appendix A: ARFIMA-GARCH model for formation of the currency basket

Some countries use so called currency basket in order to identify relevant changes in the formation of foreign exchange reserves made by national banks and to determine the real course of the national currency. Thus National Bank of the Republic of Belarus employs the basket consisting of three foreign currency US dollar (USD), Euro (EUR) and Russian Rouble (RUB). We use here the day exchange rate of these currency with respect to Belarusian Rouble (BYN) in the period from 01.01.1995 to 19.04.2022. Since during a certain time the course of BYN has been fixed then we decide to restrict our analysis to the period from 06.01.2015 to 19.04.2022. For modeling of the profitability in the foreign exchange market National Bank proposes the following formula

$$S_t = \ln \frac{P(t)}{P(t-1)}, \tag{8}$$

where $P(t)$ is an exchange rate at the moment t, and $P(t-1)$ is an exchange rate at previous period. Surely one should either exclude in time series the Sundays' and Mondays' values when National Bank does not conduct foreign exchange trading, or fix these values on Saturdays' level. Observing the exchange rate (https://www.nbrb.by/statistics/rates/ratesdaily.asp) for each pair USD-BYN, EUR-BYN and RUB-BYN we can conclude that the corresponding time series are trend-resistant.

To determine the presence of the long memory we calculate the corresponding values of the Hurst parameter (H) (for future use we include also the results for the pair CHY-BYN with China Yuan (CHY)) (https://www.nbrb.by/statistics).

The above calculated values of the Hurst parameter show a high level of persistence in time series and thus an essential influence of previous values in series on future values. We also calculate the descriptive statistics for the series represented a profit of each currency pair in order to define other properties of these series.

From the above values of asymmetry and kurtosis we can conclude that profit time series for each currency pair do not meet the normal distribution. Such conclusion is also true for many other financial actives. The largest value

Table 1. Hurst parameter (H) for time series represented a profit of an individual currency pair

Currency pair	Hurst parameter
USD-BYN	0.988227715
EUR-BYN	0.992567502
RUB-BYN	0.991329298
CHY-BYN	0.972145642

Table 2. Descriptive statistics for profit time series

Currency pair	Mean value	Standard deviation	Asymmetry	Kurtosis
USD-BYN	0.04717	0.609833	3.498762	32.866
EUR-BYN	0.04556	0.672639	2.282363	19.111
RUB-BYN	0.03223	0.706465	0.994421	9.878
CHY-BYN	0.04181	0.702631	4.012981	18.412

of the standard deviation (0.71) can be seen in RUB-BYN series, the smallest (0.61) is in USD-BYN series. By using the day standard deviation we calculate the parameter of the average annual volatility:

$$\sigma_T = \sigma\sqrt{T}. \tag{9}$$

Analyzing the obtained results on volatility and standard deviation we can conclude that the considered time series are similarly distributed. Since the basic currency in these series is BYN then the positive mean profits means an appreciation of each currency with respect to BYN.

For further modeling of the time series we can solve the stationarity question. The correct reduction of the considered time series to the stationary type was dona by using two unit root tests, namely Dickey-Fuller test (ADF-test) and Phillips-Perron test (PP-test). The calculations show that stationarity of the series is reached for $l = 1$, i.e., at the transition to lag-1 increments. This will be used in our further modeling.

For modeling of profit time series we provide a comparison of several models with short and long memory. The quality of this analysis was estimated by using the following criteria: Akaike information criterion (AIC), Bayesian (or Schwarz) information criterion (BIC) and Hannan-Quinn information criterion (HQC). Among the models with a short memory the most suitable is ARMA (0,8) excluding seasoning. But for USD-BYN series the most significant is the ARFIMA model. Thus ARFIMA(1,d,8) is characterized by the values HQC = 1.752, AIC = 1.746. This justifies previously made assumption on advantage of this model over short memory models and random walk model for the profit time series.

The most empirical and theoretical studies of the volatility of financial time series show existence of so called clustering phenomenon. The clustering of the volatility means that the high volatility of the previous period entails a high volatility in the next period and vice versa, a number of periods can form a cluster with high/low level of volatility.

For determination of a dependence of the volatility of its previous values (conditional heterockedasticity) in series of remainders in the model ARFIMA the Engle test (ARCH) was provided. This test is based on the method of the Lagrange multipliers with homosckedasticity as the null-hypothesis. The result of calculations is that all models contain ARCH structure in series of remainders. Applying to the series of remainders in the model ARFIMA (1, d,8) the model GARCH (1,1) we obtain for USD-BYN series the following values of criteria: BIC = 1.471, HQ = 1.48. This means an essential improvement of the significance and the quality of the model. Analogous results were obtained for other time series. Thus the best model in all considered cases is ARFIMA-GARCH (1, d, 8, 1, 1). For each pair d is defined according to Table 1 and the relation $d = H - 1/2$.

The prognostic quality of the models is estimated by using mean square errors (RMSE). The considered sample contains the data of the period from 06.01.2015 to 15.04.2022 with out-of-sample forecast for the period from 15.04.2022 to 01.05.2022.

Table 3. RMSE for different models

Currency pair	ARFIMA-GARCH	ARFIMA	Random Walk
USD-BYN	0.4943	0.523	0.546
EUR-BYN	0.521	0.533	0.578
RUB-BYN	0.607	0.717	0.726
CHY-BYN	0.781	0.814	0.962

The modern financial market is affected by numerous factors which increase volatility of the exchange rate of national currencies. No one currency does not immune to shocks. In spite the presence of enormous amount of available information and developed scientific approaches to the study of exchange rates, it is almost impossible to give an exact description of changing of exchange rates.

National Banks can use the method of currency basket for better understanding of changing of the worth of national currency. In this case the value of the domestic currency is calculated on the base of the combination of a number of foreign currencies. To evaluate the weight of each foreign currency in the currency basket one can use the following indicators:

1. A proportion of GDP of the country in the common GDP of all countries related to basket.

2. A proportion of the country in the trade turnover of all countries related to basket.

3. Integration of countries' economies.
4. Volatility of the national currency.

This is used in particular in our analysis of the formation of currency basket in the Republic of Belarus. National Bank of the Republic of Belarus revised the content of the basket two times. Before 2008 the domestic currency was tied to US dollar. For a short period it was changed for Russian rouble and then again return to binding to US dollar. In 2009 it was decided to tie the domestic currency to the basket of three currencies (USD, EUR, RUB) with their equal weight. Next in 2011 the proportions were slightly changes, namely, 0.4 for RUB, 0.3 for USD and EUR. Last changes were made in 2016 by fixing the following proportions: 0.5 for RUB, 0.3 for USD and 0.2 for EUR. Thus the course of Belarusian rouble is calculated according to formula

$$V_{BYN} = RUB^{0.5}USD^{0.3}EUR^{0.2}. \tag{10}$$

Our approach to formation of realistic currency basket is based on the above described model ARFIMA-GARCH (1, d, 8, 1, 1) for each currency pair and on the method of optimal portfolio. In the later we use the classical approach due to H.Markovitz [17]. Minimization of the dispersion in our portfolio is performed by using Eviews package and lead to the following formula:

$$V_{BYN} = RUB^{0.4291}USD^{0.3012}EUR^{0.1576}CHY^{0.1121}. \tag{11}$$

This result justify our hypothesis on the necessity to change the formation of the currency basket and to include China yuan as one of the world influenced currency.

Remark 1. When the paper was almost ready we have been informed that National Bank of the Republic of Belarus decided to include China yuan into the currency basket for the determination of the course of Belarusian rouble and proposed the following formula

$$V_{BYN} = RUB^{0.50}USD^{0.30}EUR^{0.10}CHY^{0.10}. \tag{12}$$

Appendix B: Pseudocode for ARFIMA-GARCH model application to time-series FX data

```
# Import necessary libraries
    import pandas as pd
    import numpy as np
    from statsmodels.tsa.statespace.sarimax import SARIMAX
    from arch import arch-model
    # Load the FX data
    data = pd.read-csv('fx-data.csv')
    fx-series = data['exchange-rate']
```

Step 1: Data Preprocessing
Check for missing values and handle them (e.g., interpolation, forward fill)
fx-series = fx-series.interpolate(method='linear')

Step 2: Estimate the order of differencing ('d') using fractional differencing
Implement a method to estimate fractional differencing (using Hurst parameter calculation) def hurst-exponent(ts, max-lag=100):
Calculate the Hurst exponent of a time series using the Rescaled Range (R/S) analysis.
Parameters: ts (array-like): Time series data max-lag (int): Maximum number of lags to consider for R/S analysis
Returns: float: Estimated Hurst exponent
lags = range(2, max-lag)
tau = [np.std(ts[lag:] - ts[:-lag]) for lag in lags]
poly = np.polyfit(np.log(lags), np.log(tau), 1)
return poly[0] * 2.0
# Calculate the Hurst exponent
H = hurst-exponent(fx-series)
Return H def estimate-fractional-differencing(series):
d = H - 1/2
return d

Step 3: Fit ARFIMA model
Define AR and MA orders (p, q) - these can be selected based on model selection criteria like AIC, BIC p = 1 q = 1
Fit ARFIMA model arfima-model = SARIMAX(fx-series, order=(p, d, q)).fit(disp=False)
Extract ARFIMA residuals residuals = arfima-model.resid

Step 4: Fit GARCH model to ARFIMA residuals
Specify GARCH model parameters (p, q) for GARCH(p, q) garch-model = arch-model(residuals, vol='Garch', p=1, q=1).fit(disp='off')

Step 5: Generate forecasts
Forecast future values using the ARFIMA model arfima-forecast = arfima-model.get-forecast(steps=10) forecasted-mean = arfima-forecast.predicted-mean
Forecast future volatility using the GARCH model garch-forecast = garch-model.forecast(horizon=10) forecasted-variance = garch-forecast.variance[-1:]

Step 6: Combine forecasts
Create a DataFrame to store the combined forecasts
combined-forecast = pd.DataFrame('Forecasted-Mean': forecasted-mean, 'Forecasted-Variance': forecasted-variance.values.flatten())

# Step 7: Output the forecasts
print("ARFIMA Forecast (Mean):", forecasted-mean)
print("GARCH Forecast (Variance):", forecasted-variance)

print("Combined Forecast:", combined-forecast)
# Optional: Visualize the forecasts import matplotlib.pyplot as plt
plt.figure(figsize=(12, 6))
plt.plot(fx-series.index, fx-series, label='Observed')
plt.plot(arfima-forecast.predicted-mean.index, forecasted-mean, label='ARFIMA Forecast')
plt.fill-between(forecasted-mean.index, forecasted-mean - 1.96 * np.sqrt(forecasted-variance.values.flatten()),
forecasted-mean + 1.96 * np.sqrt(forecasted-variance.values.flatten()),
color='gray', alpha=0.2, label='Forecast Interval')
plt.legend()
plt.title('ARFIMA-GARCH Forecast') plt.show()

Explanation:
1. **Data Preprocessing**: - The FX data is loaded and any missing values are handled.
2. **Estimate Fractional Differencing**: - A method is used to estimate the order of fractional differencing ‘d‘.
3. **Fit ARFIMA Model**: - The ARFIMA model is fit to the FX series using the estimated ‘d‘ and chosen ‘p‘ and ‘q‘.
4. **Fit GARCH Model**: - The residuals from the ARFIMA model are used to fit a GARCH model to capture volatility clustering.
5. **Generate Forecasts**: - Future values are forecasted using both ARFIMA for the mean and GARCH for the variance.
6. **Combine Forecasts**: - Forecasted mean and variance are combined into a single DataFrame.
7. **Output and Visualization**: - The forecasts are printed and optionally visualized using matplotlib to show observed data, forecasted mean, and forecast intervals.

This pseudocode provides a comprehensive step-by-step approach to implementing an ARFIMA-GARCH model on time-series FX data.

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